**Linear Diophantine equations**

A Diophantine equation is a polynomial equation whose solutions are restricted to integers. These types of equations are named after the ancient Greek mathematician Diophantus. A linear Diophantine equation is a first-degree equation of this type. Diophantine equations are important when a problem requires a solution in whole amounts.

Linear Diophantine equations come in the following format:

(\*)

where a, b and n are integers.

**Recall Bezout’s Identity.**  For nonzero integers a and b, (infinitely many) such that:

For the case of (\*), let d be the greatest common factor of a and b, gcd (a, b).

It follows that:

Therefore, if there exists a solution, the greatest common divisor must divide n.

**Theory.** For nonzero integers a and b, such that:

**Example.** Solve .

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Since the gcd(11, 13) = 1 | 369, there are infinitely many solutions.

Instead of guessing and checking, the Extended Euclidean algorithm should be used:

*The gcd is the last nonzero remainder which is 1.*

Starting from the last equation with nonzero remainder:

However, from the previous equation, therefore:

By comparing coefficients, x = 6 and y = -5

A bit more complicated example involving finding the general solution:

**Example.** Phoebe buys large shirts for $18 each and small shirts for $11 each. The shirts cost a total of $1188. What is the smallest total number of shirts she could have bought?

Let x be the number of large shirts and let y be the number of small shirts. Then

18x + 11y = 1188.

Since the gcd(18, 11) = 1 | 1188, there are infinitely many solutions.

By the Extended Euclidean algorithm:

*The gcd is the last nonzero remainder which is 1.*

By multiplying both sides by 1188:

(1)

One solution is (x, y) = (-3564, 5940).

The values can’t be negative; hence the general solution should be found:

Notice that lcm(11, 18) = 198.

(2)

By adding (1) and (2):

By doing simple calculation using inequalities,

Total number of shirts is x + y = (-3564 + 11k) + (5940 – 18k)

The value is the smallest when k = 330:

She bought 66 large shirts, no small shirts, and a total of 66 shirts.

**General formula for the solutions of a linear Diophantine equation.**

By generalizing the previous example, consider

(3)

Note that the lowest common multiple of a and b is L. lcm(a, b) = L

By factoring out a or b from each L:

Multiply both side by a variable k:

(4)

By adding equations (3) and (4):

Since :

**Recall.** For all integers a and b:

We get:

Hence, the general solution is:

Where x0 and y0 are particular solutions.

**Additional results:**

Consider the following linear Diophantine equation:

It is obvious that (0, 0) is a solution, therefore by using the formula for general solutions, we get:

**Corollary.**  For nonzero integers a and b, satisfying the following equation:

Consider a 3-variable linear Diophantine equation:

For some nonzero integers a, b, c and n.

Using similar argument, the previous equation has a solution if and only if the greatest common divisor of a, b and c divides n, in other words gcd(a, b, c) | n.

In fact, for a general m-variable Diophantine equation in the form:

(\*\*)

There exists a solution if and inly if the

**Proof.**

Using similar argument, let d = .

Using the fundamental theorem of arithmetic.

Therefore, there exists a solution if and inly if the .

Back to the 3-variable linear Diophantine equation, consider the following example:

It became a 2-variable linear Diophantine equation, it is easy to see that w = 3 and z = 1 is a solution, therefore the general solution is:

Substituting into w, we get:

For this equation, it is just equivalent to solving to linear Diophantine equation:

But for this example, it is easy to guess the solution since the numbers are small.

A particular solution is:

Therefore, the general solution is:

Hence, .

Try k = 253, t = -33