1)

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a)

$$P(X \geq a)$$

$$= P(X - t + t \geq a)$$

$$= P(X + t \geq a + t)$$

$$\leq P\left((X+t)^2 \geq (a+t)^2\right)$$

$$\leq \frac{E\left[(X+t)^2\right]}{(a+t)^2}$$

(By Markov's Inequality)

$$= \frac{\sigma^2 + t^2}{(a+t)^2}$$

(By Expectation)

$$\leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Consider: $\frac{\sigma^2 + t^2}{(a+t)^2}$

$$\frac{d}{dt}\left[\frac{\sigma^2 + t^2}{(a+t)^2}\right] = \frac{2(at - \sigma^2)}{(t+a)^3}$$

$\therefore \frac{\sigma^2 + t^2}{(a+t)^2}$ minimized when $t_* = \frac{\sigma^2}{a}$

$$\therefore \frac{\sigma^2 + t_*^2}{(a+t_*)^2} = \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{a^2 + 2\sigma^2 + \frac{\sigma^4}{a^2}} = \frac{a^2\sigma^2 + \sigma^4}{a^4 + 2a^2\sigma^2 + \sigma^4}$$

$$= \frac{\sigma^2(a^2 + \sigma^2)}{(a^2 + \sigma^2)^2} = \frac{\sigma^2}{(a^2 + \sigma^2)}$$

$$\therefore P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2} \quad a > 0$$

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b)

$P(Y - E(Y) \geq a)$

Let $Z = Y - E(Y)$, $E(Z) = 0$ & $Var(Z) = \sigma^2$

$\therefore P(Z \geq a)$

$= P(Z + \lambda \geq \lambda + a)$

$\leq P((Z+\lambda)^2 \geq (\lambda+a)^2)$

$$\leq \frac{E[(Z+\lambda)^2]}{(\lambda+a)^2}$$

$$= \frac{\sigma^2 + \lambda^2}{(\lambda+a)^2} \leq \frac{\sigma^2}{a^2+\sigma^2}$$

$$\frac{d}{d\lambda}\left(\frac{\sigma^2+\lambda^2}{(\lambda+a)^2}\right) = \frac{2(a\lambda - \sigma^2)}{(\lambda+a)^2}$$

$\therefore \frac{d}{d\lambda} = 0$, $\lambda = \frac{\sigma^2}{a}$

$$\therefore \min\left(\frac{\sigma^2+\lambda^2}{(\lambda+a)^2}\right) = \frac{\sigma^2 + \left(\frac{\sigma^2}{a}\right)^2}{\left(\frac{\sigma^2}{a}+a\right)^2}$$

$$= \frac{\sigma^2}{a^2+\sigma^2}$$

As λ is mimimized when $\lambda = \frac{\sigma^2}{a}$

$$\therefore P(Z \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$$

$$\therefore P(Y - E(Y) \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$$

$$\therefore P(Y - \mu \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$$

$$\therefore P(Y \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2+a^2}$$

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From (i) we know,

$\cancel{P(Y-E(Y)) \leq}$

$$P(Y - E(Y) \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$\therefore$

$$P(-Y - E(-Y) \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$E(-Y) = -\mu \quad Var(-Y) = \sigma^2$

$$\therefore P(-Y + \mu \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$$\therefore P(-Y \geq a - \mu) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$$\therefore P(Y \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

2) a) (i) Guru Shyaam Shankar - 46448608

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} f_X(x)$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$K_X(t) = \ln[M_X(t)]$$

$$= \ln\left[e^{\lambda(e^t-1)}\right]$$

$$= \lambda(e^t - 1)$$

$$K_X'(t) = \lambda e^t$$

$$K_X''(t) = \lambda e^t$$

$$K_X'''(t) = \lambda e^t$$

$$\therefore K_X^{(n)}(t) = \lambda e^t$$

$$\therefore K_X'''(0) = \lambda$$

$$\therefore K_X''''(0) = \lambda$$

$\therefore \kappa_1 = \lambda = E(X)$

$\kappa_2 = \lambda = var(X)$

$\kappa_3 = \lambda = \mu_3$

$\kappa_4 = \lambda = \mu_4 - 3\sigma^4$

$\therefore \mu_4 - 3(\lambda)^2 = \lambda$

$\therefore \mu_4 = 3\lambda^2 + \lambda$

$$\boxed{\begin{aligned} \therefore \mu_3 &= \lambda \\ \mu_4 &= 3\lambda^2 + \lambda \end{aligned}}$$

2) a) (ii) Guru Shyaam Shankar - 46448608

$$Y = \frac{X-\lambda}{\sqrt{\lambda}} = \frac{X}{\sqrt{\lambda}} - \sqrt{\lambda} = \frac{\sqrt{\lambda}}{\lambda}X - \sqrt{\lambda}$$

From Lectures,

$$M_{a+bX}(t) = e^{at}M_X(bt) \qquad M_X(t) = e^{\lambda(e^t-1)}$$

$$\therefore M_Y(t) = e^{-t\sqrt{\lambda}}M_X\left(\frac{\sqrt{\lambda}}{\lambda}t\right)$$

$$= e^{-t\sqrt{\lambda}}\left(e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}\right)$$

$$= e^{-t\sqrt{\lambda}}\left(e^{\lambda e^{\frac{t\sqrt{\lambda}}{\lambda}}-\lambda}\right)$$

$$= e^{\lambda e^{\frac{t}{\sqrt{\lambda}}}-t\sqrt{\lambda}-\lambda}$$

$$= \exp(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda)$$

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2) b)

$X \sim \text{Poisson}(\lambda = 36) \approx \text{Normal}(\mu = 36, \sigma = 6)$

$\therefore P(X \geq 45) \approx P\left(\frac{X-\mu}{\sigma} \geq \frac{45-36-0.5}{6}\right)$ By CLT theorem & continuity correction

$= P(Z \geq 1.41667\ldots) \approx 0.92171\ldots$

$= 0.92171$ (5 d.p.)

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c)

Poisson Process:

- Events are independant of one another
- Average rate (1) is constant
- Two events cannot occur at the same time.

$P(X > x) = e^{-x}$ which is an exponential ~~q~~ w/ rate 1

$P(X \leq x) = 1 - e^{-x}$

By definition, the exponential distribution is the probability distribution of time between events in a Poisson Process

$\therefore$ X ~~has~~ poisson distribution w/ parameter λ.

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3)

a)

Each value of X_n can be either 0 or 1, w/ probability $\frac{1}{2}$, at infinity

Hence, any X_n is a random variable $\sim B(\frac{1}{2})$

$\therefore X_n \xrightarrow{d}$ Bernoulli $(\frac{1}{2})$

$\therefore X_n \xrightarrow{d} X$

Because $Y = 1 - X$,

X & Y are symmetrical about each other

$\therefore X_n \xrightarrow{d} Y$

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b)

$$\lim_{n \to \infty} P(|X_n - Y| \geq \varepsilon)$$

$$= P(X_n = 1, Y = 0) + P(X_n = 0, Y = 1)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\neq 0$$

$$\therefore X_n \overset{P}{\not\to} Y$$

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If valid joint probability density function:

$$\iint_{\mathbb{R}^2} f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 = 1$$

$$\int_0^\infty \int_0^\infty 4x_1x_2e^{-x_1^2}\, dx_1\, dx_2$$

$$= \int_0^\infty 4x_2 \int_0^\infty \cancel{4}x_1e^{-x_1^2}\, dx_1\, dx_2$$

$$= \int_0^\infty 4x_2\left(\tfrac{1}{2}\right) dx_2$$

$$= \int_0^\infty 2x_2\, dx_2$$

$$= \left[(x_2)^2\right]_0^\infty$$

$$= \infty$$

$$\neq 1$$

Consider:

$$\int_0^\infty ve^{-v^2}\, dv$$

Let $u = -v^2$ $\quad dv = -\frac{1}{2v}\, du$

$$= \int_0^\infty -\tfrac{1}{2}e^u\, du$$

$$= -\tfrac{1}{2}\int_0^\infty e^u = -\tfrac{1}{2}\left[e^u\right]_0^\infty$$

$$= -\tfrac{1}{2}(0-1)$$

$$= \tfrac{1}{2}$$

∴ It is not a valid joint probability density function.

However for $x_1 > 0$, $x_2 > 0$:

$x_1x_2 > 0$, $e^{-x_1^2} > 0$

∴ $f_{X_1,X_2}(x_1,x_2) > 0$ for all $x_1 > 0$, $x_2 > 0$

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5)

a)

$$T_X(z) = \sum_{x=a}^{b-1} z^x \, P(X>x)$$

$$\sum_{x=0}^{b-1} z^{x+1} P(X>x)$$
$$= \sum_{x=a}^{b-1} \left(z^{x+1}\left(P(X>x+1) + P(X=x+1)\right)\right)$$

$$\therefore (1-z)T_X(z) = \sum_{x=a}^{b-1} z^x P(X>x) - \sum_{x=0}^{b-1} z^{x+1} P(X>x)$$

$$= \sum_{x=a}^{b-1} z^x P(X>x) - \sum_{x=a}^{b-1}\left(z^{x+1}\left(P(X>x+1) + P(X=x+1)\right)\right)$$

$$= \sum_{x=a}^{b-1} z^x P(X>x) - \sum_{x=a}^{b-1} P(X>x+1) z^{x+1} - \sum_{x=a}^{b-1} z^{x+1} P(X=x+1)$$

$$= \sum_{x=a}^{b-1} z^x P(X>x) - \sum_{x=a+1}^{b} z^x P(X>x) - \sum_{x=a}^{b-1} z^{x+1} P(X=x+1)$$

$$= z^a P(X>a) - z^b P(X>b) - \sum_{x=a}^{b-1} z^{x+1} P(X=x+1)$$

$$= z^a P(X>a) - \sum_{x=a+1}^{b} z^x P(X=x) \quad \left[P(X>b)=0\right]$$

$$= z^a P(X\geq a) - \sum_{x=a}^{b} z^x P(X=x) \quad \left[P(X\geq a = 1\right]$$

$$= z^a - G_X(z)$$

If X is a non-negative discrete random variable, $a=0$

$$\therefore T_X(z)(1-z) = z^0 - G_X(z)$$
$$= 1 - G_X(z)$$

5) b) Guru Shyaam Shankar - 46448608

$$T_X(1) = \sum_{x=0}^{b-1} P(X>x) = P(X>0) + P(X>1) + \dots + P(X>b-1)$$

$$P(X>0) = P(X=1) + P(X=2) + P(X=3) + \dots + P(X=b)$$

$$P(X>1) = P(X=2) + P(X-3) + \dots + P(X=b)$$

$$\vdots$$

$$P(X>b-1) = P(X=b)$$

$$\therefore T_X(1) = P(X=1) + 2P(X=2) + 3P(X=3) + \dots + bP(X=b)$$

$$= \sum_{x=0}^{b} x P(X=x) = E(X)$$ [Expectation of Discrete Random Variable]

$$T'_X(z) = \sum_{x=1}^{b-1} x z^{x-1} P(X>x)$$

$$\cancel{= z^0 P(X>0)+}$$

$$T'_X(1) = 1\left[P(X=2) + P(X=3) + \dots + P(X=b)\right] + 2\left[P(X=3) + \dots + P(X=b)\right] + \dots + (b-1)\left[P(X=b)\right]$$

$$\therefore T'_X(1) = 1P(X=2) + (1+2)P(X=3) + (1+2+3)P(X=4) + \dots + (1+2+\dots+b-1)P(X=b)$$

$$= \sum_{x=2}^{b} \frac{x(x-1)}{2} P(X=x)$$

$$\therefore 2T'_X(1) = \sum_{x=2}^{b} x(x-1) P(X=x)$$

5) b) Continued Guru Shyaam Shankar - 46448608

$$\therefore 2T'_X(1) = \sum_{x=2}^{b} (x^2 - x) P(X=x)$$

$$= \sum_{x=0}^{b} (x^2 - x) P(X=x) \quad ((x^2-x)=0, \text{ for } x = 0, 1 \text{ anyway})$$

$$= \sum_{x=0}^{b} x^2 P(X=x) - \sum_{x=0}^{b} x P(X=x)$$

$$= E(X^2) - E(X)$$

$$\therefore 2T'_X(1) + T_X(1) - (T_X(1))^2$$

$$= E(X^2) - E(X) + E(X) - (E(X))^2$$

$$= E(X^2) - (E(X))^2 = Var(X)$$

$$\therefore Var(X) = 2T'_X(1) + T_X(1) - (T_X(1))^2$$

5) c) Guru Shyaam Shankar - 46448608

$$P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_j = y_j) = \frac{1}{n_j} \text{ for} \in (y_1, \ldots, y_j)$$

$\therefore$ For $r \geq j$, $\sum_{i=1}^{j} Y_i = r$ has many solutions :

$$\begin{cases} \binom{r-j+j-1}{j-1} & r \geq j \\ \text{None} & \text{otherwise} \end{cases} = \begin{cases} \binom{r-1}{j-1} & r \geq j \\ \text{None} & \text{otherwise} \end{cases}$$

$$\therefore P(S_j = r) = \frac{\binom{r-1}{j-1}}{n^j} \text{ iff } r \geq j$$

$$\therefore P(S_j \leq r) = \sum_{r=j}^{n} \frac{\binom{r-1}{j-1}}{n^j} = \frac{1}{n^j} \sum_{r=j}^{n} \binom{r-1}{j-1}$$

$$= \frac{1}{n^j}\left[\binom{j-1}{j-1} + \binom{j}{j-1} + \binom{j+1}{j-1} + \cdots \binom{n-1}{j-1}\right]$$

$$= \frac{\binom{n}{j}}{n^j} \text{ (Binomial Expansion)}$$

5) d) Guru Shyaam Shankar - 46448608

Prove: $\tau_n \geq j+1$ iff $S_j \leq n$

$\tau_n \geq j+1 \therefore \tau_n > j$ iff $S_j \leq n$

$S_j \leq n \Rightarrow \tau_n > j \Rightarrow \tau_n \geq j+1$

$\therefore \tau_n \geq j+1 \iff S_j \leq n$

5) e) Guru Shyaam Shankar 46448608

$$T_{\tau_n}(z) = \sum_{j=0}^{n} z^j P(\tau_n > j)$$

$$= \sum_{j=0}^{n} z^j P(S_j \leq n) \quad \text{(From (d))}$$

$$= \sum_{j=0}^{n} z^j \binom{n}{j} n^{-j} \quad \text{(From (c))}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{z}{n}\right)^j$$

$$= \left(1 + \frac{z}{n}\right)^j \quad \text{(Binomial Exponsion)}$$

5) f) Guru Shyaam Shankar - 46448608

From (b)

$T_X(1) = E(X)$ $T_{\gamma_n}(z) = \left(1 + \frac{z}{n}\right)^j$ [From (e)]

$\therefore E(\gamma_n) = T_{\gamma_n}(1) = \left(1 + \frac{1}{n}\right)^j$

$$T'_{\gamma_n}(z) = \frac{j(z+n)^{j-1}}{n^j}$$

$$\therefore 2T'_{\gamma_n}(1) = 2\left[\frac{j(1+n)^{j-1}}{n^j}\right]$$

$var(X) = 2T'_X(1) + T_X(1) - (T_X(1))^2$ [From (b)]

$\therefore Var(\gamma_n)$

$$= 2T'_{\gamma_n}(1) + T_{\gamma_n}(1) - \left[T_{\gamma_n}(1)\right]^2$$

$$= \frac{2j(1+n)^{j-1}}{n^j} + \frac{(n+1)^j}{n^j} - \frac{(n+1)^{2j}}{n^{2j}}$$

$$= \frac{(n+1)^{j-1}\left[2jn^j + n^j(n+1) - (n+1)^{2j}\right]}{n^{2j}}$$

5) g) Guru Shyaam Shankar - 46448608

$$T_X(z)(1-z) = 1 - G_X(z)$$

$$\therefore G_{\gamma_n}(z) = 1 - T_{\gamma_n}(z) + z\,T_{\gamma_n}(z)$$

$$= 1 - \left(1+\frac{z}{n}\right)^j + z\left(1+\frac{z}{n}\right)^j$$

5) h) Guru Shyaam Shanker - 46448608

$$G_{\Upsilon_n}(z) = 1 - \left(1+\frac{z}{n}\right)^j + z\left(1+\frac{z}{n}\right)^j$$

$G_X(z)$ for Bernoulli $= 1 - p + pz$

$\therefore \Upsilon_n$ is a bernoulli distribution w/ $p = \left(1+\frac{z}{n}\right)^j$

$\therefore$ Probability Function:

$$= \cancel{\left(\left(1+\frac{z}{n}\right)^j\right)^k \left(1-\left(1+\frac{z}{n}\right)^j\right)^{1-k}}$$

$$= \left(\left(1+\frac{z}{n}\right)^j\right)^k \left(1-\left(1+\frac{z}{n}\right)^j\right)^{1-k} \quad \text{for } k \in \{0, 1\}$$

5) (i) Guru Shyaam Shanker - 46448608

$$\left(1-\left(1+\frac{z}{n}\right)^j\right) + \left(1+\frac{z}{n}\right)^j e^t = M_{Y_n}(t)$$

Let $\left(1+\left(\frac{z}{n}\right)^j\right) = p$ & $1-p=q$

$$q+pe^t = M_{Y_n}(t)$$

Let $Z_n = \dfrac{Y_n - np}{\sqrt{npq}}$

$$\therefore M_{Z_n}(t) = M_{\frac{Y_n}{\sqrt{npq}} - \frac{np}{\sqrt{npq}}}(t)$$

$$M_{a+bX}(t) = e^{at} M_X(bt)$$

$$\therefore M_{Z_n}(t) = e^{-\frac{np}{\sqrt{npq}}} M_{Y_n}\left(\frac{1}{\sqrt{npq}}t\right)$$

$$= e^{-\frac{np}{\sqrt{npq}}}\left(q+pe^{\frac{t}{\sqrt{npq}}}\right)$$

$$= qe^{-\frac{np}{\sqrt{npq}}} + pe^{\frac{t-np}{\sqrt{npq}}}$$

$$= \cancel{p}e^{-\frac{np}{\sqrt{npq}}} - pe^{-\frac{np}{\sqrt{npq}}} + pe^{\frac{t-np}{\sqrt{npq}}}$$

$\therefore Z_n$ is a Binomial Distribution

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