Fuzzy

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1 Introduction

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2 Problem Statement

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2.1 System Dynamics

The end velocity of the tooltip in the workspace and the joint-space angular velocity have the following form:

$$\dot{X} = J(q)\dot{q}, \tag{2.1}$$

where $\dot{X} \in \mathbb{R}^3$ is the actual end-effector Cartesian velocity, and the Jacobian matrix from the base to the end-effector is represented by $J(q) \in \mathbb{R}^{3\times n}$.

The dynamic equation of the robot system is:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau + \tau_{ext}, \tag{2.2}$$

where $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ is the Coriolis and centrifugal forces, $G(q) \in \mathbb{R}^n$ is the gravitational force, $\tau \in \mathbb{R}^n$ is the control input and $\tau_{ext} \in \mathbb{R}^n$ is the external torque exerted by the environment.

Proposition 2.1: *The Cartesian inertia matrix $M(q)$ defined in (2.2) is positive and symmetric, which is also bounded as:*

$$\lambda_1\|A\| \leq A^T M(q) A \leq \lambda_2\|A\|, \forall A \in \mathbb{R}^n,$$

where λ_1 and λ_2 are positive constants.

Proposition 2.2: *The matrix $C(q,\dot{q})$ and the time derivative of $M(q)$ satisfy the condition:*

$$A^T\left[\dot{M}(q) - 2C(q,\dot{q})\right]A = 0, \forall A \in \mathbb{R}^n.$$

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2.2 System Kinematics

The inverse kinematic equation is:

$$\dot{q}(t) = J^{+}(q)\dot{X}(t) + N(q)\xi(t), \tag{2.3}$$

where $J^{+}(q) \in \mathbb{R}^{n\times 3}$ is the right pseudo-inverse of $J(q)$ defined as $J^{+}(q) = J^{T}(q)\left[J(q)J^{T}(q)\right]^{-1}$, $N(q) \in \mathbb{R}^{n\times n}$ is the nullspace projection defined as $N(q) = Z^{T}(ZZ^{T})^{-1}Z$, where $Z = Z(q) \in \mathbb{R}^{3\times n}$ is a nullspace base of $J(q)$. Note that $\xi(t) \in \mathbb{R}^{n}$ is a nullspace auxiliary variable to be designed later.

A direct computation shows that:

$$\begin{cases} J(q)J^{+}(q) = I, \\ N^{T}(q) = N(q), \\ N^{2}(q) = N(q), \\ J(q)N(q) = 0, \\ N(q)J^{+}(q) = 0. \end{cases} \tag{2.4}$$

2.3 Objective

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$$\begin{cases} \lim\limits_{t\to+\infty} \left[X(t) - X_{d}(t)\right] = 0, \\ \lim\limits_{t\to+\infty} \left[n(t) - n_{d}(t)\right] = 0. \end{cases} \tag{2.5}$$

2.4 Control Design

Design the nullspace variable $\xi(t)$ in (2.3) as:

$$\xi(t) = \dot{q}(t) + Z^{T}\left[n(t) - n_{d}(t)\right].$$

Design the joint angle velocity as:

$$v_{r}(t) = J^{+}(q)\left\{\dot{X}_{d}(t) - K_{1}\left[X(t) - X_{d}(t)\right]\right\} + N(q)\xi(t), \tag{2.6}$$

where K_1 is a positive define matrix. Hence, we have the following theorem:

Theorem 2.3: *If* $\lim\limits_{t\to+\infty} \dot{q}(t) = v_{r}(t)$, *then the objective in (2.5) can be satisfied.*

Proof: First, we prove that $\lim\limits_{t\to+\infty} \left[X(t) - X_{d}(t)\right] = 0$: Left multiply $J(q)$ on both sides in (2.6), we have:

$$J(q)v_{r}(t) = \dot{X}_{d}(t) - K_{1}\left[X(t) - X_{d}(t)\right].$$

According to $\lim_{t\to+\infty} \dot{q}(t) = v_r(t)$ and (2.1), we have:

$$\dot{X}(t) - \dot{X}_d(t) + K_1 [X(t) - X_d(t)] = 0. \tag{2.7}$$

Let $e = \dot{X}(t) - \dot{X}_d(t)$, then (2.7) becomes $\dot{e} = -K_1 e$. Left mulxiply e^T on both sides, we have $e^T \dot{e} = -e^T K_1 e \le 0$, i.e., $\frac{1}{2}\frac{d}{dt} e^T e \le 0$. Integrate from 0 to t on both sides, we have $\frac{1}{2}\left(\|e(t)\|^2 - \|e(0)\|^2\right) \le 0$. Let $e(0) = 0$, then $e(t) = 0$, i.e., $\lim_{t\to+\infty} [X(t) - X_d(t)] = 0$.

Next, we prove that $\lim_{t\to+\infty} [n(t) - n_d(t)] = 0$: Left multiply $N(q)$ on both sides in (2.6), we have:

$$\begin{aligned} N(q)v_r(t) &= N(q)\xi(t) \\ &= N(q)\left\{\dot{q}(t) + Z^T [n(t) - n_d(t)]\right\}. \end{aligned}$$

Let $t \to +\infty$, we have $N(q)v_r(t) = N(q)v_r(t) + N(q)Z^T [n(t) - n_d(t)]$, i.e., $N(q)Z^T [n(t) - n_d(t)] = 0$. Recall that $N(q) = Z^T (ZZ^T)^{-1} Z$, then

$$Z^T [n(t) - n_d(t)] = 0.$$

According to $Rank(Z^T) = Rank(Z) = 3$, we have $[n(t) - n_d(t)] = 0$, i.e., $\lim_{t\to+\infty} [n(t) - n_d(t)] = 0$. $\square$

3 Stability Analysis

According to Theorem 2.3, as $t \to +\infty$, the objective in (2.5) can be satisfied. Next, we prove that $\lim_{t\to+\infty} \dot{q}(t) = v_r(t)$ can be achieved if the control input τ satisfies the defined condition, i.e, $v_r(t)$ is stable under the above τ.

Let

$$\eta_1(t) = X(t) - X_d(t), \eta_2(t) = \dot{q}(t) - v_r(t),$$

according to (2.6), we have:

$$\dot{q}(t) - v_r(t) - \dot{q}(t) = J^+(q)\left\{-X_d(t) + K_1 [X(t) - X_d(t)]\right\} - N(q)\xi(t).$$

Left multiply $J(q)$ on both sides, we have:

$$\dot{\eta}_1(t) = J(q)\eta_2(t) - K_1\eta_1(t). \tag{3.1}$$

According to the dynamic equation in (2.2), we have:

$$\ddot{q}(t) = M^{-1}(q)\left[\tau + \tau_{ext} - C(q,\dot{q})\dot{q}(t) - G(q)\right],$$

then

$$\dot{\eta}_2(t) = M^{-1}(q)\left[\tau + \tau_{ext} - C(q,\dot{q})\dot{q}(t) - G(q)\right] - \dot{v}_r(t). \tag{3.2}$$

Let

$$P\left[q, \dot{q}, v_r(t), \dot{v}_r(t)\right] = M(q)\dot{v}_r(t) + C(q,\dot{q})v_r(t) + G(q) - \tau_{ext},$$

then (3.2) becomes

$$M(q)\dot{\eta}_2(t) + C(q,\dot{q})\eta_2(t) = \tau - P\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right]. \tag{3.3}$$

Consider the following Lyapunov function:

$$V_1 = \frac{1}{2}\eta_1^T(t)\eta_1(t) + \frac{1}{2}\eta_2^T(t)M(q)\eta_2(t).$$

According to (3.1) and (3.3), a direct computation shows that the time derivative of V_1 is:

$$\dot{V}_1 = \eta_1^T(t)\left[J(q)\eta_2(t) - K_1\eta_1(t)\right] + \eta_2^T(t)\left\{\tau - P\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right]\right\}.$$

Design the control input τ as:

$$\tau = P\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right] - J^T(q)\eta_1(t) - K_2\eta_2(t),$$

where K_2 is a positive define matrix, then we have:

$$\dot{V}_1 = -\eta_1^T(t)K_1\eta_1(t) - \eta_2^T(t)K_2\eta_2(t). \tag{3.4}$$

Hence, $\dot{V}_1 \le 0$, i.e., $v_r(t)$ is stable as $t \to +\infty$.

However, as the control input τ includes unknown components $M(q), C(q,\dot{q}), G(q)$, Fuzzy Feedforward Approximation is introduced as below:

$$\begin{cases} P\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right] = \theta^{*T}\xi(X) + E, \\ \hat{P}\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right] = \hat{\theta}^T\xi(X) + \delta, \end{cases}$$

where l is the dimension of Fuzzy base function $\xi(X)$, $\theta^*, \hat{\theta} \in \mathbb{R}^{l\times n}$ and $E = (\epsilon_1,\cdots,\epsilon_n)^T, \delta = (\delta_1,\cdots,\delta_n)^T \in \mathbb{R}^{n\times 1}$ are the approximation error and the estimation error respectively, then

$$\hat{\tau} = \hat{P}\left[q,\dot{q},v_r(t),\dot{v}_r(t)\right] - J^T(q)\eta_1(t) - K_2\eta_2(t). \tag{3.5}$$

Subsitute (3.5) into (3.4), we have:

$$\dot{V}_1 = -\eta_1^T(t)K_1\eta_1(t) - \eta_2^T(t)K_2\eta_2(t) + \eta_2^T\tilde{\theta}\xi(X) - \eta_2^T(\delta + E), \tag{3.6}$$

where $\tilde{\theta} = \hat{\theta} - \theta^*$.

Next, we will clarify that $-\eta_2^T(\delta + E) \le 0$, then introduce the augmented Lyapunov function V_2 to state the stability of $v_r(t)$ even under Fuzzy Feedforward Approximation.

Let

$$\delta_i = \text{sign}\left[\eta_{2,i}(t)\right]s_i, i = 1,\cdots,n,$$

where s_i are positive constants satisfying $s_i \ge \bar{\epsilon}$ with $\bar{\epsilon}$ the upper bound of E, then

$$-\eta_2^T(\delta + E) \le \sum_{i=1}^{n}\left|\eta_{2,i}(t)\right|(\bar{\epsilon} - s_i) \le 0,$$

and (3.6) becomes:

$$\dot{V}_1 \leq -\eta_1^T(t)K_1\eta_1(t) - \eta_2^T(t)K_2\eta_2(t) + \eta_2^T\tilde{\theta}\xi(X).$$

Consider the augmented Lyapunov function as below:

$$V_2 = V_1 + \frac{1}{2}\sum_{i=1}^{n}\tilde{\theta}_i^T\tilde{\theta}_i,$$

where $\tilde{\theta}_i$ are row vectiors of $\tilde{\theta}$. Introduce the adaptive law for $\tilde{\theta}_i$ with $\beta_i > 0$:

$$\dot{\tilde{\theta}}_i = -\left[\eta_{2,i}(t)\xi(X) + \beta_i\tilde{\theta}_i\right],$$

then the time derivative of V_2 is:

$$\begin{aligned} \dot{V}_2 &\leq -\eta_1^T(t)K_1\eta_1(t) - \eta_2^T(t)K_2\eta_2(t) - \sum_{i=1}^{n}\beta_i\tilde{\theta}_i^T\hat{\theta} \\ &\leq -\eta_1^T(t)K_1\eta_1(t) - \eta_2^T(t)K_2\eta_2(t) - \sum_{i=1}^{n}\frac{\beta_i}{2}\|\tilde{\theta}_i\|^2 + \sum_{i=1}^{n}\frac{\beta_i}{2}\|\theta_i^*\|^2 \\ &\leq -aV_2 + b, \end{aligned} \tag{3.7}$$

where

$$\begin{cases} \lambda_{\min}(K_1) > 0, \lambda_{\min}\left(K_2M^{-1}\right) > 0, \\ a = \min\left[2\lambda_{\min}(K_1), 2\lambda_{\min}\left(K_2M^{-1}\right), \min(\beta_i)\right], \\ b = \sum\limits_{i=1}^{n}\frac{\beta_i}{2}\|\theta_i^*\|^2, \\ i = 1, \cdots, n. \end{cases}$$

Multiply e^{at} on both sides in (3.7), we have $\frac{d(e^{at}V_2)}{dt} \leq be^{at}$. Integrate from 0 to t on both sides, we have:

$$\begin{aligned} V_2 &\leq \frac{b}{a} + \left[V_2(0) - \frac{b}{a}\right]e^{-at} \\ &\leq \frac{b}{a} + V_2(0). \end{aligned} \tag{3.8}$$

Subsitute (3.8) into (3.7), we have $\dot{V}_2 \leq 0$, i.e., $v_r(t)$ is stable as $t \to +\infty$ under Fuzzy Feedforward Approximation.