Bäcklund Transformation for the Restricted Modified Constrained KP Hierarchy

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October 18, 2020

1 Introduction

The Kadomtsev-Petviashvili Hierarchy (KP), which is a set of commuting flows, obtains its name from one of the flow contained in it and is a cornerstone in the integrable systems. Moreover, such hierarchy can be generated from a splitting of Lie algebra,[1] for example, the AKNS hierarchy and the KdV hierarchy.

It is also known that the Matrix modified constrained KP hierarchy (Matrix mcKP) comes from a Lie algebra splitting and has some interesting flows, for example, the NLS hierarchy and the mKdV hierarchy.[2] In this paper, we

(1) add the restriction to the reality condition of the Matrix mcKP hierarchy and construct the Matrix restricted modified constrained KP hierarchy (Matrix rmcKP),

(2) show that the Matrix rmcKP hierarchy, which is generated from a matrix Lax pair, is equivalent to the restricted modified constrained KP hierarchy (rmcKP), which is generated from a pseudodifferential Lax pair,

(3) derive the Bäcklund Transformation (BT) for the Matrix rmcKP hierarchy.

The organization of this paper is as follows: In Section 2, we construct pseudodifferential constrained hierarchy, for example, the rmcKP hierarchy. In Section 3, we construct the Matrix rmcKP hierarchy from a splitting of Lie algebra. In Section 4, we show the equivalence of the Matrix rmcKP hierarchy and the rmcKP hierarchy. In section 5 and 6, we construct the BT for the Matrix rmcKP hierarchy and obtain the nontrivial new solution for the explicit cases.

2 Various Kadomtsev-Petviashvili Hierarchy

Pseudodifferential Operators and Flows

Let $\mathcal{D}$ be the algebra of pseudodifferential operators and $\mathcal{D} = \{Y|Y(x) = \sum_{j\leq j_0} Y_j(x)\partial^j\}$. Let $\mathcal{D}_+ = \{Y \in \mathcal{D}|Y(x) = \sum_{j\geq 0} Y_j(x)\partial^j\}$ and $\mathcal{D}_- = \{Y \in \mathcal{D}|Y(x) = \sum_{j<0} Y_j(x)\partial^j\}$ be subalgebras of $\mathcal{D}$, then $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ is a direct sum of linear subspaces, which is called a splitting of $\mathcal{D}$.

Set

$$\mathcal{M} = \{L \in \mathcal{D}|L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots\}.$$

Then the flow on $\mathcal{M}$ is

$$L_{t_j} = [(L^j)_+, L]. \tag{2.1}$$

The flow generated from (2.1) is called the *Kadomtsev−Petviashvili hierarchy* (KP).[3] Combining the second and third flows in (2.1), we get the KP equation:[2]

$$3u_{1t_2t_2} = (\frac{4}{3}u_{1t_3} - \frac{1}{3}u_{1xxx} - 4u_1u_{1x})_x. \tag{2.2}$$

Moreover, a direct computation shows that

$$L^n_{t_j} = [(L^j)_+, L^n], \tag{2.3}$$

and the right hand side of (2.3) starts from the power ∂^{n-2}, so L^n can be written as

$$L^n = \partial^n + u_{n-2}\partial^{n-2} + \cdots. \tag{2.4}$$

It is known that invariant submanifolds $\mathcal{N}$ of $\mathcal{M}$ give new hierarchy, i.e., let θ be a vector field on a manifold $\mathcal{M}$, then $\theta(p) \in T\mathcal{N}_p$ for all $p \in \mathcal{N}$ and the flow on $\mathcal{M}$: $x_t = \theta(x(t))$ restricts to a flow on $\mathcal{N}$.[2] See the following examples for details.

2.1 Example

2.1.1 KdV and mKdV Hierarchy[4]

Let $\mathcal{N}_{KdV} = \{L \in \mathcal{M}|L^2 = (L^2)_+ = \partial^2 + u\}$. Then we get the $(2j-1)$-th flow and the third flow is the KdV equation:

$$u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x. \tag{2.5}$$

Let $\mathcal{N}_{mKdV} = \{L \in \mathcal{N}_{KdV}|L^2 = (\partial - v)(\partial + v) = \partial^2 + v_x - v^2\}$ and the third flow is the mKdV equation:

$$v_{t_3} = \frac{1}{4}v_{xxx} - \frac{3}{2}v^2v_x. \tag{2.6}$$

It follows that if v is a solution of the mKdV, then $u = v_x - v^2$ is a solution of the KdV (the Miura transformation).[5] Moreover, let $L^2 = (\partial + v)(\partial - v) = \partial^2 - v_x - v^2$, then $\widetilde{u} = -v_x - v^2$ is a new solution of the KdV.

2.1.2 Gelfand-Dickey Hierarchy (GD_n)[6]

$\mathcal{N}_n = \{L \in \mathcal{M} | L^n = (L^n)_+ = \partial^n + \sum_{i=1}^{n-1} u_i \partial^{i-1}\}$. Then we get the j-th ($j \not\equiv 0 \bmod n$) flow, which is called the GD_n hierarchy. If we factor L^n as $L^n = (\partial - v_n) \cdots (\partial - v_1)$, then (2.3) gives the modified hierarchy (mGD$_n$). If $v_1, \cdots, v_n$ is a solution of the mGD$_n$ hierarchy, then $u_0, \cdots, u_{n-1}$ is a solution of the GD$_n$ hierarchy (the Miura transformation). Moreover, if $L^n = (\partial - v_n) \cdots (\partial - v_1)$ is a solution of the j-th mGD$_n$ flow, then $\widetilde{L}^n = (\partial - v_{n-1}) \cdots (\partial - v_1)(\partial - v_n)$ is a new solution of (2.3) (the Bäcklund transformation).[7]

2.1.3 Modified Constrained KP Hierarchy (mcKP)[2]

Let $\mathcal{N}_{mcKP} = \{L \in \mathcal{M} | L^{k+1} = (\partial - u_{k+1})(\partial - u_k) \cdots (\partial - u_2)(\partial - u_1 - \sum_{i=1}^{m} q_i \partial^{-1} r_i)\}$, and (2.3) gives the modified constrained KP hierarchy (mcKP). $\mathcal{N}_{mcKP}$ is invariant under the KP flows, and next we will consider the restricted case of the mcKP hierarchy.

2.1.4 Restricted Modified Constrained KP Hierarchy (rmcKP)

Let $\mathcal{N}_{rmcKP} = \{L \in N_{mcKP} | L^{k+1} = (\partial + \bar{u}_2)(\partial + \bar{u}_3) \cdots (\partial - u_2)(\partial - u_1 + \sum_{i=1}^{m} q_i \partial^{-1} \bar{q}_i)\}$, where u_1 is a pure imaginary function, $u_i = -\bar{u}_{k+3-i}, 2 \le i \le k+1$, and $\sum_{i=1}^{k+1} u_i = 0$. In Section 4, we will use the algebra method to prove that $\mathcal{N}_{rmcKP}$ is also invariant under the KP flows and then (2.3) gives the $rmcKP$ hierarchy.

Next, we will consider the algebra structure of the rmcKP hierarchy.

3 Matrix rmcKP hierarchy

In this section, we use the Lie algebra splitting to construct the Matrix rmcKP hierarchy.

Let

$$C = \begin{pmatrix} I_m & 0 \\ 0 & D \end{pmatrix}, \quad D = diag(1, \alpha, \cdots, \alpha^k), \quad \alpha = e^{\frac{2\pi i}{k+1}},$$

$$S = \begin{pmatrix} I_m & 0 \\ 0 & H \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \\ 0 & 1 & & \end{pmatrix}_{k+1},$$

where m and k are nonnegative integers. Note that $C^{k+1} = I_{m+k+1}$ and $S^2 = I_{m+k+1}$. Let τ be the automorphism of $\mathcal{L}(sl(m+k+1))$ and $\mathcal{L} = \{A(\lambda) \in$

$\mathcal{L}(sl(m+k+1))|\tau(A(\alpha\lambda)) = CA(\alpha\lambda)C^{-1} = A(\lambda)\}$. An earlier paper shows that Matrix mcKP hierarchy is generated from the splitting of $\mathcal{L}$.[2] Consider the splitting:

$$\begin{cases} \mathcal{L}^\sigma = \{A \in \mathcal{L}|\sigma(A(-\lambda)) = -S(\overline{A(-\bar{\lambda})})^T S^{-1} = A(\lambda)\}, \\ \mathcal{L}_+^\sigma = \{A \in \mathcal{L}^\sigma|A(\lambda) = \sum\limits_{j\geq 0} A_j\lambda^j\}, \\ \mathcal{L}_-^\sigma = \{A \in \mathcal{L}^\sigma|A(\lambda) = \sum\limits_{j<0} A_j\lambda^j\}. \end{cases}$$

A direct computation shows that

$$\tau(A_j) = CA_jC^{-1},$$

$$\sigma(A_j) = -S\bar{A_j}^T S^{-1}.$$

then

$$sl(m+k+1) = \mathcal{G}_0^{'} \oplus \cdots \oplus \mathcal{G}_k^{'} = \mathcal{G}_0 \oplus \cdots \oplus \mathcal{G}_k,$$

where $\mathcal{G}_j^{'}$ is the eigenspace of τ with the eigenvalue α^{-j}, i.e.,

$$\mathcal{G}_j^{'} = \{A_j \in sl(m+k+1)|CA_jC^{-1} = \alpha^{-j}A_j, 0 \leq j \leq k\},$$

and where $\mathcal{G}_j$ is the eigenspace of τ with the eigenvalue α^{-j} while σ with the eigenvalue $(-1)^j$, i.e.,

$$\mathcal{G}_j = \{A_j \in sl(m+k+1)|CA_jC^{-1} = \alpha^{-j}A_j, -S\bar{A_j}^T S^{-1} = (-1)^j A_j, 0 \leq j \leq k\}.$$

Then we obtain the form of $\mathcal{G}_0^{'} = \left\{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right\}$, where $A_{11} \in \mathbb{C}^{m\times m}$, $A_{12} \in \mathbb{C}^{m\times(k+1)}$, $A_{21} \in \mathbb{C}^{(k+1)\times m}$, $A_{22} \in \mathbb{C}^{(k+1)\times(k+1)}$, satisfying

$$A_{12} = \begin{pmatrix} q & 0_{m\times k} \end{pmatrix}, A_{21} = \begin{pmatrix} r \\ 0_{k\times m} \end{pmatrix}, A_{22} = diag(a_1, \cdots, a_{k+1}),$$

and $\mathcal{G}_0 = \left\{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right\}$, $A_{11} = -\bar{A}_{11}^T, A_{12} = \begin{pmatrix} q & 0_{m\times k} \end{pmatrix}, A_{21} = \begin{pmatrix} -\bar{q}^T \\ 0_{k\times m} \end{pmatrix}, A_{22} = diag(a_1, \cdots, a_{k+1})$, where a_1 is a pure imaginary function.

Let $J = a\lambda$, where

$$a = \begin{pmatrix} 0_m & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{k+1}.$$

Then $\mathcal{J} = \{J^j|j \geq 1, j \not\equiv 0 \text{ mod } (k+1) \text{ and } j \text{ is odd}\}$ is a vacuum sequence. A direct computation shows that the phase space is

$$\mathcal{M} = \{(gJg^{-1})_+|g \in L_-^\sigma\} = a\lambda + \mathcal{U},$$

where

$$\mathcal{U}=\left\{U\middle|U_{11}=0,U_{12}=\begin{pmatrix}q & 0_{m\times k}\end{pmatrix},U_{21}=\begin{pmatrix}-\bar{q}^T\\ 0_{k\times m}\end{pmatrix},U_{22}=diag(u_1,\cdots,u_{k+1})\right\},$$

and $q=(q_1,\cdots,q_m)^T$ is a vector, u_1 is a pure imaginary function, $u_i=-\bar{u}_{k+3-i}, 2\le i\le k+1$. Moreover,

$$\mathcal{U}^{'}=\left\{U\middle|U_{11}=0,U_{12}=\begin{pmatrix}q & 0_{m\times k}\end{pmatrix},U_{21}=\begin{pmatrix}r\\ 0_{k\times m}\end{pmatrix},U_{22}=diag(u_1,\cdots,u_{k+1})\right\},$$

where $q=(q_1,\cdots,q_m)^T$, $r=(r_1,\cdots,r_m)^T$.

Given a smooth $U:\mathbb{R}\to\mathcal{M}$, there exists a unique $Q_1(U)\in\mathcal{L}$, s.t.

$$\begin{cases}[\partial_x-(a\lambda+U),Q_1(U)]=0,\\ Q_1(U)^{k+2-\delta_m}=Q_1(U)^{1-\delta_m}\lambda^{k+1}I_{m+k+1},\end{cases}\tag{3.1}$$

where

$$\delta_m=\begin{cases}1, & m=0,\\ 0, & m\neq 0.\end{cases}$$

Let $Q_1^j(U)$ be a polynomial of the form

$$Q_1^j(U)=a^j\lambda^j+\sum_{l<j}Q_{j,l}(U)\lambda^l,$$

then the flow generated by J^j is

$$\begin{aligned}U_{t_j}&=[\partial_x-(a\lambda+U),Q_1^j(U)_+]\\ &=[\partial_x-U,Q_{j,0}(U)].\end{aligned}\tag{3.2}$$

This gives the Matrix rmcKP hierarchy.

Next, we compute some examples:

(1) $k=0$, the phase space is

$$\mathcal{M}=\left\{a\lambda+\begin{pmatrix} & & & q_1\\ & 0_m & & \vdots\\ & & & q_m\\ -\bar{q}_1 & \cdots & -\bar{q}_m & 0\end{pmatrix}\right\}.$$

The third flow is

$$q_{it_3}=q_{ixxx}+3q_i\sum_{l=1}^m\bar{q}_lq_{lx}+3q_{ix}\sum_{l=1}^m|q_l|^2\ (1\le i\le m).$$

Note that $m=1$ gives the NLS:

$$q_{t_3}=q_{xxx}+6|q|^2q_x.\tag{3.3}$$

(2) $k=1$, the phase space is

$$\mathcal{M}=\left\{a\lambda+\begin{pmatrix} & & & q_1 & 0\\ & 0_m & & \vdots & 0\\ & & & q_m & 0\\ -\bar{q}_1 & \cdots & -\bar{q}_m & u & 0\\ 0 & \cdots & 0 & 0 & -u\end{pmatrix}\right\}.$$

The third flow is

$$\begin{cases} q_{it_3} = q_{ixxx}+\frac{3}{4}q_i(\sum_{l=1}^{m}(\bar{q}_l q_{lx}-q_l\bar{q}_{lx}-2|q_l|^2u)-(u_{xx}+2uu_x)) \\ \qquad +\frac{3}{2}q_{ix}(\sum_{l=1}^{m}|q_l|^2-(u_x+u^2))\ (1\le i\le m) \\ u_{t_3}=\frac{1}{4}u_{xxx}+\frac{3}{4}\sum_{l=1}^{m}(\bar{q}_l q_{lxx}-q_l\bar{q}_{lxx})-\frac{3}{2}u^2u_x. \end{cases},$$

Moreover, $m=0$ gives the complex mKdV:

$$u_{t_3}=\frac{1}{4}u_{xxx}-\frac{3}{2}u^2u_x. \tag{3.4}$$

4 Equivalence

Theorem 4.1: [2] *Let $P_U=\partial_x-(a\lambda+U), U\in\mathcal{U}'$ be a solution of the Matrix mcKP hierarchy, where a and $\mathcal{U}'$ are defined as in Section 3, then $L^{k+1}=(-1)^{k+1}(\partial+v_2)(\partial+v_3)\cdots(\partial+v_{k+1})(\partial+v_1-\sum_{i=1}^{m}q_i\partial^{-1}r_i)$ is a solution of the mcKP hierarchy, and vice versa.*

Theorem 4.2: *Let $P_U=\partial_x-(a\lambda+U), U\in\mathcal{U}$ be a solution of the Matrix rmcKP hierarchy (3.2), where a and $\mathcal{U}$ are defined as in Section 3, then $L^{k+1}=(-1)^{k+1}(\partial+v_2)(\partial+v_3)\cdots(\partial-\bar{v}_3)(\partial-\bar{v}_2)(\partial+v_1+\sum_{i=1}^{m}q_i\partial^{-1}\bar{q}_i)$ with v_1 a pure imaginary function, is a solution of the rmcKP hierarchy, and vice versa.*

Proof: Recall that the rmcKP hierarchy is the mcKP hierarchy with the restriction: v_1 is a pure imaginary function, $v_i=-\bar{v}_{k+3-i}, 2\le i\le k+1$, and $r_i=-\bar{q}_i, 1\le i\le m$. With this restriction, the Matrix mcKP hierarchy turns into the Matrix rmcKP hierarchy. Using Theorem 4.1, we finish the proof. □

5 Bäcklund Transformation

Given a smooth $U:\mathbb{R}\to\mathcal{M}$, then the following statements are equivalent:

(1) U is a solution of (3.2),

(2)

$$\begin{cases} E_x E^{-1} = a\lambda + U, \\ E_{t_j} E^{-1} = Q_1^j(U)_+, \end{cases} \tag{5.1}$$

is solvable for any initial data $E(0,0) = c \in L_+$.

The solution of (5.1) is called an *frame* for U, and the solution with $c = e$ is the *normalized frame*. Note that $U = 0$ is a solution for the j-th flow and its corresponding normalized frame is $E(x, t_j) = \exp(J_1 x + J^j t_j)$ (the *vacuum frame*).

Theorem 5.1: [8, 9] *Let U be a solution of the flow (3.2) generated by J^j and E be the frame of U. Given $f \in L_-$, then there is an open subset $\mathcal{O}$ of the origin in $\mathbb{R}^2$ such that*

$$E(x,t_j)f^{-1} = \widetilde{f}^{-1}(x,t_j)\widetilde{E}(x,t_j), \tag{5.2}$$

where $\widetilde{E}(x,t_j) \in L_+$ and $\widetilde{f}(x,t_j) \in L_-$ for $(x,t_j) \in \mathcal{O}$.

Moreover,

(1) $\widetilde{U}$ satisfying $\widetilde{E}_x\widetilde{E}^{-1} = a\lambda + \widetilde{U}$ is a solution of the flow (3.2) and $\widetilde{E}$ is the frame of $\widetilde{U}$,

(2) $f * U = \widetilde{U}$ defines an action of L_- on the space of solutions of the flow (3.2) and $f * E = \widetilde{E}$ defines an action of L_- on the corresponding frames,

(3) $f * J$ is given explicitly in terms of exponential functions and is a soliton solution,

(4) let $L(G)$ be the group of smooth maps f from S^1 to a complex Lie group G. If L is a subgroup of $L(G)$, $f \in L_-$ is a rational element with poles and $\widetilde{f}(x,t_j) \in L_-$ is rational and has the same poles as f, then $f * U$ can be computed explicitly using E and poles of f.

Recall that L is the group of smooth maps f from S^1 to G and satisfies $\tau(f(\alpha\lambda)) = Cf(\alpha\lambda)C^{-1} = f(\lambda)$, and L^σ is a subgroup of L and satisfies $\sigma(f(-\lambda)) = S(\overline{f(-\bar{\lambda})})^{-T}S^{-1} = f(\lambda)$, i.e., $L^\sigma = \{f : S^1 \to SL(m{+}k{+}1) | f(\lambda) = Cf(\alpha\lambda)C^{-1}, f(\lambda)S(\overline{f(-\bar{\lambda})})^T S^{-1} = I\}$. Let L_+^σ be the subgroup of $f \in L^\sigma$ that can be extended holomorphically to $|\lambda| < 1$, and L_-^σ be the subgroup of $f \in L^\sigma$ that can be extended holomorphically to $|\lambda| > 1$ and $f(\infty) = I$. Next, we construct the BT for the Matrix rmcKP hierarchy for $k = 0, 1$, and we can use the same argument for any postive k.

5.1 BT for $k = 0$

Theorem 5.2: *Let $L^\sigma = \{f : S^1 \to SL(m+1) | f(\lambda)(\overline{f(-\bar{\lambda})})^T = I\}$. Let $z \in \mathbb{C} \setminus \mathbb{R}i$, π be the Hermitian projection of $\mathbb{C}^{m+1}$ and $\pi^\perp = I - \pi$. Let*

$$f(\lambda) = f_{z,\pi}(\lambda) = \pi + \frac{\lambda + \bar{z}}{\lambda - z}\pi^\perp.$$

Then $f \in L_-^\sigma$.

Theorem 5.3: *Let $U \in \mathcal{U}$ be a solution of the Matrix rmcKP hierarchy with a Baker function, and E be the corresponding frame defined on an open neighborhood of $(0,0)$. Let z and π be defined as in Theorem 5.2. Let*

$$\widetilde{\pi}(x,t) \text{ be the Hermitian projection onto } E(x,t,z)(Im\pi).$$

Then

$$\begin{aligned}\widetilde{E}(x,t,\lambda) =& \widetilde{f}(x,t,\lambda)E(x,t,\lambda)f^{-1}(\lambda)\\ =& (\widetilde{\pi} + \frac{\lambda+\bar{z}}{\lambda-z}\widetilde{\pi}^\perp)E(x,t,\lambda)(\pi + \frac{\lambda-z}{\lambda+\bar{z}}\pi^\perp)\end{aligned}$$

is the corresponding frame for the new solution $\widetilde{U}$.

Next, we derive the new solution $\widetilde{U}$. Let $\widetilde{E}_x\widetilde{E}^{-1} = a\lambda + \widetilde{U}$. According to $\widetilde{E} = \widetilde{f}Ef^{-1}$ and $E_xE^{-1} = a\lambda + U$, we have $\widetilde{f}_x\widetilde{f}^{-1} + \widetilde{f}(a\lambda + U)\widetilde{f}^{-1} = a\lambda + \widetilde{U}$. Multiply both sides with $\widetilde{f}$ on the right, we have

$$\widetilde{f}_x + \widetilde{f}(a\lambda + U) = (a\lambda + \widetilde{U})\widetilde{f}.$$

Set $\lambda = z, -\bar{z}$ in the above equation, we have

$$\begin{cases} -\widetilde{\pi}_x + \widetilde{\pi}^\perp(az+U) = (az+\widetilde{U})\widetilde{\pi}^\perp, \\ \widetilde{\pi}_x + \widetilde{\pi}(-a\bar{z}+U) = (-a\bar{z}+\widetilde{U})\widetilde{\pi}. \end{cases}$$

Add the two equation, we have

$$\widetilde{U} = U + (z+\bar{z})[a, \widetilde{\pi}]. \tag{5.3}$$

Hence, we get $\widetilde{U}$.

5.2 BT for $k = 1$

For $k = 1$, the reality condition is $f(\lambda)(\overline{f(-\bar{\lambda})})^T = I$, $f(\lambda) = Cf(-\lambda)C^{-1}$. Note that if $f \in L_-^\sigma$ has two simple poles in $\mathbb{C} \setminus \mathbb{R} \cup \mathbb{R}i$, then it follows from the reality condition that the poles of f must be z, $-z$. Hence, we need two factors to generate f, i.e., let

$$f_{z,\pi}(\lambda) = \pi + \frac{\lambda+\bar{z}}{\lambda-z}\pi^\perp,$$

where $\pi^\perp = I - \pi$, then $f = f_{-z,\rho}f_{z,\pi}$ for some Hermitian projections π, ρ. To find such f that lies in L_-^σ, we need the permutability formula for G_-^m, where

$$G^m = \{f : S^1 \to SL(m+2) | f(\lambda)(\overline{f(-\bar{\lambda})})^T = I\}.$$

Lemma 5.4: [10] *(Permutability Theorem) Given* f_{z_j,π_j} *in* G_-^m *with* $z_j \in \mathbb{C}\setminus\mathbb{R}i$ *for* $j=1,2$ *and* $z_1 \neq z_2, -\bar{z}_2$, *let* ρ_1 *be the Hermitian projection of* $\mathbb{C}^{m+2}$ *onto* $f_{z_2,\pi_2}(z_1)(Im\pi_1)$ *and* ρ_2 *be the Hermitian projection onto* $f_{z_1,\pi_1}(z_2)(Im\pi_2)$. *Then*

$$f_{z_2,\rho_2} f_{z_1,\pi_1} = f_{z_1,\rho_1} f_{z_2,\pi_2}.$$

Moreover, such factorization is unique.

Theorem 5.5: *Let* $z \in \mathbb{C}\setminus\mathbb{R}\cup\mathbb{R}i$, π *be a Hermitian projection of* $\mathbb{C}^{m+2}$ *and* ρ *be the Hermitian projection onto* $f_{z,\pi}(-z)(ImC\pi C^{-1})$. *Let*

$$f(\lambda) = f_{-z,\rho}(\lambda) f_{z,\pi}(\lambda).$$

Then $f \in L_-^\sigma$.

Proof: Let $z_2 = -z_1 = -z$ and $C\pi_2 C^{-1} = \pi_1 = \pi$. By Lemma 5.4, we have

$$f_{-z,\rho_2} f_{z,\pi} = f_{z,\rho_1} f_{-z,C\pi C^{-1}},$$

where

$$Im\rho_1 = f_{-z,C\pi C^{-1}}(z)(Im\pi),$$
$$Im\rho_2 = f_{z,\pi}(-z)(ImC\pi C^{-1}).$$

Since $f_{z,\pi}(-z) = C f_{-z,C\pi C^{-1}}(z) C^{-1}$, we have

$$\rho_2 = C\rho_1 C^{-1} = \rho.$$

In other words, we have

$$f(\lambda) = f_{-z,\rho}(\lambda) f_{z,\pi}(\lambda) = f_{z,C\rho C^{-1}}(\lambda) f_{-z,C\pi C^{-1}}(\lambda),$$

which implies that f satisfies the Matrix rmcKP hierarchy reality condition. $\square$

Theorem 5.6: *Let* $U \in \mathcal{U}$ *be a solution of the Matrix rmcKP hierarchy with a Baker function, and* E *be the corresponding frame defined on an open neighborhood of* $(0,0)$. *Let* z, π *and* ρ *be defined as in Theorem 5.5. Let*

$\widetilde{\pi}(x,t)$ *be the Hermitian projection onto* $E(x,t,z)(Im\pi)$,

$\widetilde{\rho}(x,t)$ *be the Hermitian projection onto* $E_1(x,t,-z)(Im\rho)$,

where $E_1(x,t,\lambda) = f_{z,\widetilde{\pi}}(\lambda) E(x,t,\lambda) f_{z,\pi}^{-1}(\lambda)$. *Then*

$$\begin{aligned}
\widetilde{E}(x,t,\lambda) =& \widetilde{f}(x,t,\lambda) E(x,t,\lambda) f^{-1}(\lambda) \\
=& f_{-z,\widetilde{\rho}}(\lambda) f_{z,\widetilde{\pi}}(\lambda) E(x,t,\lambda) f_{z,\pi}^{-1}(\lambda) f_{-z,\rho}^{-1}(\lambda) \\
=& (\widetilde{\rho} + \frac{\lambda-\bar{z}}{\lambda+z}\widetilde{\rho}^\perp)(\widetilde{\pi} + \frac{\lambda+\bar{z}}{\lambda-z}\widetilde{\pi}^\perp) E(x,t,\lambda) \\
& (\pi + \frac{\lambda-z}{\lambda+\bar{z}}\pi^\perp)(\rho + \frac{\lambda+z}{\lambda-\bar{z}}\rho^\perp)
\end{aligned}$$

is the corresponding frame for the new solution $\widetilde{U}$.

Proof: First, we prove that $Im\widetilde{\rho} = f_{z,\widetilde{\pi}}(-z)(ImC\widetilde{\pi}C^{-1})$. From $Im\rho = f_{z,\pi}(-z)(ImC\pi C^{-1})$ and $Im\widetilde{\rho} = f_{z,\widetilde{\pi}}(-z)E(-z)f_{z,\pi}^{-1}(-z)(Im\rho)$, we know that $Im\widetilde{\rho} = f_{z,\widetilde{\pi}}(-z)E(-z)(ImC\pi C^{-1}) = f_{z,\widetilde{\pi}}(-z)E(-z)C(Im\pi)$. Since $E(z) = CE(-z)C^{-1}$ and $Im\widetilde{\pi} = E(x,t,z)(Im\pi)$, we have

$$Im\widetilde{\rho} = f_{z,\widetilde{\pi}}(-z)C(Im\widetilde{\pi}) = f_{z,\widetilde{\pi}}(-z)(ImC\widetilde{\pi}C^{-1}).$$

Second, we prove that $\widetilde{E}(x,t,\lambda)$ is holomorphic. Note that $\widetilde{E}$ is holomorphic in $\lambda \in \mathbb{C} \setminus \{\pm z, \pm\bar{z}\}$. From $z \in \mathbb{C} \setminus \mathbb{R} \cup \mathbb{R}i$, we know that $\widetilde{E}$ only has possible poles at $\lambda = \pm z, \pm\bar{z}$ with power 1. Compute the residue of $\widetilde{E}$ at $\lambda = \pm z, \pm\bar{z}$ respectively:

$$\begin{aligned}
Res(\widetilde{E}(z)) &= (z+\bar{z})(\widetilde{\rho} + \frac{z-\bar{z}}{2z}\widetilde{\rho}^{\perp})\widetilde{\pi}^{\perp}E(z)\pi(\rho + \frac{2z}{z-\bar{z}}\rho^{\perp}),\\
Res(\widetilde{E}(-z)) &= -(z+\bar{z})\widetilde{\rho}^{\perp}(\widetilde{\pi} + \frac{z-\bar{z}}{2z}\widetilde{\pi}^{\perp})E(-z)(\pi + \frac{2z}{z-\bar{z}}\pi^{\perp})\rho,\\
Res(\widetilde{E}(\bar{z})) &= (z+\bar{z})\widetilde{\rho}(\widetilde{\pi} + \frac{2\bar{z}}{\bar{z}-z}\widetilde{\pi}^{\perp})E(\bar{z})(\pi + \frac{\bar{z}-z}{2\bar{z}}\pi^{\perp})\rho^{\perp},\\
Res(\widetilde{E}(-\bar{z})) &= -(z+\bar{z})(\widetilde{\rho} + \frac{2\bar{z}}{\bar{z}-z}\widetilde{\rho}^{\perp})\widetilde{\pi}E(-\bar{z})\pi^{\perp}(\rho + \frac{\bar{z}-z}{2\bar{z}}\rho^{\perp}).
\end{aligned}$$

Since $E(z)(\overline{E(-\bar{z})})^T = I$, a direct computation shows that $Res(\widetilde{E}(\lambda))$ is equal to zero at $\lambda = \pm z, \pm\bar{z}$. Hence, $\widetilde{E}$ is holomorphic in $\lambda \in \mathbb{C}$.

Third, we prove that $\widetilde{E}(x,t,\lambda)$ is an frame. Let $\widetilde{E}_x\widetilde{E}^{-1} = a\lambda + \widetilde{U}$, and $M \in L_-^{\sigma}$ be a Baker function for U. Then $M(\partial_x - J)M^{-1} = \partial_x - (J+U)$ and $M(\partial_{t_j} - J^j)M^{-1} = \partial_{t_j} - (Q_1^j(U))_+$.[2] Hence, $\tilde{E}$ is the frame for $\widetilde{U}$.

Next, we derive the new solution $\widetilde{U}$. Recall that

$$\widetilde{f}_x + \widetilde{f}(a\lambda + U) = (a\lambda + \widetilde{U})\widetilde{f}.$$

Set $\lambda = \pm z, \pm\bar{z}$ in the above equation, we have

$$\begin{cases}
\quad (z+\bar{z})(\widetilde{\rho}_x\widetilde{\pi}^{\perp} - \widetilde{\rho}\widetilde{\pi}_x + \widetilde{\rho}\widetilde{\pi}^{\perp}(az+U)) + (z-\bar{z})(-\widetilde{\pi}_x + \widetilde{\pi}^{\perp}(az+U))\\
= (z+\bar{z})(az+\widetilde{U})\widetilde{\rho}\widetilde{\pi}^{\perp} + (z-\bar{z})(az+\widetilde{U})\widetilde{\pi}^{\perp},\\
\quad (z+\bar{z})(-\widetilde{\rho}_x\widetilde{\pi} + \widetilde{\rho}^{\perp}\widetilde{\pi}_x + \widetilde{\rho}^{\perp}\widetilde{\pi}(-az+U)) + (z-\bar{z})(-\widetilde{\rho}_x + \widetilde{\rho}^{\perp}(-az+U))\\
= (z+\bar{z})(-az+\widetilde{U})\widetilde{\rho}^{\perp}\widetilde{\pi} + (z-\bar{z})(-az+\widetilde{U})\widetilde{\rho}^{\perp},\\
\quad (z+\bar{z})(\widetilde{\rho}_x\widetilde{\pi} + \widetilde{\rho}\widetilde{\pi}_x + \widetilde{\rho}\widetilde{\pi}(a\bar{z}+U)) - 2\bar{z}(\widetilde{\rho}_x + \widetilde{\rho}(a\bar{z}+U))\\
= (z+\bar{z})(a\bar{z}+\widetilde{U})\widetilde{\rho}\widetilde{\pi} - 2\bar{z}(a\bar{z}+\widetilde{U})\widetilde{\rho},\\
\quad (z+\bar{z})(\widetilde{\rho}_x\widetilde{\pi} + \widetilde{\rho}\widetilde{\pi}_x + \widetilde{\rho}\widetilde{\pi}(-a\bar{z}+U)) - 2\bar{z}(\widetilde{\pi}_x + \widetilde{\pi}(-a\bar{z}+U))\\
= (z+\bar{z})(-a\bar{z}+\widetilde{U})\widetilde{\rho}\widetilde{\pi} - 2\bar{z}(-a\bar{z}+\widetilde{U})\widetilde{\pi}.
\end{cases}$$

Add the first two and the last two equation respectively, we have

$$\begin{cases} (\widetilde{\rho}_x - \widetilde{\pi}_x) + z[\widetilde{\rho}, a] - (z+\bar{z})[\widetilde{\rho}\widetilde{\pi}, a] + \dfrac{2\bar{z}^2 - z^2 + |z|^2}{z+\bar{z}}[\widetilde{\pi}, a] + \\ (\widetilde{\rho} - \widetilde{\pi} + \dfrac{z-\bar{z}}{z+\bar{z}}I)U = \widetilde{U}(\widetilde{\rho} - \widetilde{\pi} + \dfrac{z-\bar{z}}{z+\bar{z}}I), \\ (\widetilde{\rho}_x - \widetilde{\pi}_x) + z[\widetilde{\pi}, a] - (z+\bar{z})[\widetilde{\rho}\widetilde{\pi}, a] + \dfrac{2\bar{z}^2 - z^2 + |z|^2}{z+\bar{z}}[\widetilde{\rho}, a] + \\ (\widetilde{\rho} - \widetilde{\pi} - \dfrac{z-\bar{z}}{z+\bar{z}}I)U = \widetilde{U}(\widetilde{\rho} - \widetilde{\pi} - \dfrac{z-\bar{z}}{z+\bar{z}}I). \end{cases} \tag{5.4}$$

The first equation minus the second equation of (5.4), we have

$$\widetilde{U} = U + (z+\bar{z})[a, \widetilde{\pi} - \widetilde{\rho}]. \tag{5.5}$$

Hence, we get $\widetilde{U}$. □

For $k > 1$, let $z \in \mathbb{C} \setminus \mathbb{R}i$ and $\alpha^j z \neq -\alpha^l \bar{z}$ for $0 \leq j, l \leq k$. Let π_j be a Hermitian projection of $\mathbb{C}^{m+k+1}$ and $\pi_j^\perp = I - \pi_j$. Let

$$f_{z_j,\pi_j}(\lambda) = \pi_j + \frac{\lambda + \bar{z}}{\lambda - z}\pi_j^\perp.$$

Still, we can factor $f \in L_-^\sigma$ as

$$f(\lambda) = f_{\alpha^k z, \pi_k}(\lambda) \cdots f_{\alpha z, \pi_1}(\lambda) f_{z,\pi_0}(\lambda).$$

Use the permutability theorem (Lemma 5.4) repeatly, we can construct the BT to obtain the new solution, but the computation is much more complicated.

Theorem 5.7: [11] *(Scaling Transformation) Let $U(x,t)$ be a solution of the j-th Matrix rmcKP flow, and $E(x,t,\lambda)$ be the corresponding frame. Let $r \in \mathbb{C} \setminus 0$, and $G(r) = diag(I_m, 1, r, \cdots, r^k)$. Then*

(1) $\widetilde{U}(x,t) = rU(rx, r^j t)$ is a solution of the j-th Matrix rmcKP flow,

(2) $\widetilde{E}(x,t,\lambda) = G(r)E(rx, r^j t, r^{-1}\lambda)G(r)^{-1}$ is a frame of $\widetilde{U}$.

Hence, the scaling transformation gives the BT for the Matrix rmcKP with arbitrary non-zero parameter $r^{-1}\lambda$.

6 Example

In this section, we apply the BT for the explicit Matrix rmcKP hierarchy where $k = 0, 1$. We start from $U = 0$, so we need the corresponding normalized frame E.

Let $N = diag(e^{\lambda x + \lambda^j t}, e^{\alpha\lambda x + \alpha^j \lambda^j t}, \cdots, e^{\alpha^k \lambda x + \alpha^{kj}\lambda^j t})$,

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \alpha & \cdots & \alpha^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^k & \cdots & \alpha^{k^2} \end{pmatrix}.$$

Then the normalized frame $E(x,t,\lambda)$ for the Matrix rmcKP hierarchy is

$$E(x,t,\lambda) = \begin{pmatrix} I_m & \\ & V \end{pmatrix} \begin{pmatrix} I_m & \\ & N \end{pmatrix} \begin{pmatrix} I_m & \\ & V^{-1} \end{pmatrix}.$$

6.1 $k = 0$

The normalized frame is

$$E(x,t,\lambda) = \begin{pmatrix} I_m & \\ & e^{\lambda x + \lambda^j t} \end{pmatrix}.$$

Given $z \in \mathbb{C} \setminus \mathbb{R}i$ and $Im\pi = (\xi_1, \cdots, \xi_{m+1})^T$, $\xi_1, \cdots, \xi_{m+1} \neq 0$, a direct computation shows that

$$\begin{aligned} Im\widetilde{\pi} &= E(z)(Im\pi) = (\eta_1, \cdots, \eta_{m+1})^T, \\ \widetilde{\pi} &= \begin{pmatrix} |\eta_1|^2 & \eta_1\bar{\eta}_2 & \eta_1\bar{\eta}_3 & \cdots & \eta_1\bar{\eta}_{m+1} \\ \bar{\eta}_1\eta_2 & |\eta_2|^2 & \eta_2\bar{\eta}_3 & \cdots & \eta_2\bar{\eta}_{m+1} \\ \bar{\eta}_1\eta_3 & \bar{\eta}_2\eta_3 & |\eta_3|^2 & \cdots & \eta_3\bar{\eta}_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\eta}_1\eta_{m+1} & \bar{\eta}_2\eta_{m+1} & \bar{\eta}_3\eta_{m+1} & \cdots & |\eta_{m+1}|^2 \end{pmatrix} /n, \end{aligned}$$

where

$$\begin{aligned} &\eta_r = \xi_r \ (1 \leq r \leq m), \\ &\eta_{m+1} = \xi_{m+1} e^{zx + z^3 t}, \\ &n = \sum_{l=1}^{m+1} |\eta_l|^2. \end{aligned}$$

Hence, the new solution is

$$\widetilde{q}_r = -(z + \bar{z}) \frac{\eta_r \bar{\eta}_{m+1}}{n} \ (1 \leq r \leq m).$$

6.2 $k = 1$

The normalized frame is

$$E(x,t,\lambda) = \begin{pmatrix} I_m & \\ & \begin{pmatrix} \cosh(\lambda x + \lambda^j t) & \sinh(\lambda x + \lambda^j t) \\ \sinh(\lambda x + \lambda^j t) & \cosh(\lambda x + \lambda^j t) \end{pmatrix} \end{pmatrix}.$$

Given $z \in \mathbb{C} \setminus \mathbb{R} \cup \mathbb{R}i$ and and $Im\pi = (\xi_1, \cdots, \xi_{m+2})^T$, $\xi_1, \cdots, \xi_{m+2} \neq 0$, a direct computation shows that

$$Im\widetilde{\pi} = E(z)(Im\pi) = (\eta_1, \cdots, \eta_{m+2})^T,$$

$$\widetilde{\pi} = \begin{pmatrix} |\eta_1|^2 & \eta_1\bar{\eta}_2 & \eta_1\bar{\eta}_3 & \cdots & \eta_1\bar{\eta}_{m+2} \\ \bar{\eta}_1\eta_2 & |\eta_2|^2 & \eta_2\bar{\eta}_3 & \cdots & \eta_2\bar{\eta}_{m+2} \\ \bar{\eta}_1\eta_3 & \bar{\eta}_2\eta_3 & |\eta_3|^2 & \cdots & \eta_3\bar{\eta}_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\eta}_1\eta_{m+2} & \bar{\eta}_2\eta_{m+2} & \bar{\eta}_3\eta_{m+2} & \cdots & |\eta_{m+2}|^2 \end{pmatrix} /n,$$

$$\widetilde{\rho} = \begin{pmatrix} |\eta_1|^2 & \eta_1\bar{\eta}_2 & \eta_1\bar{\eta}_3 & \cdots & \eta_1\bar{\eta}_{m+2}\frac{b^2}{|b|^2} \\ \bar{\eta}_1\eta_2 & |\eta_2|^2 & \eta_2\bar{\eta}_3 & \cdots & \eta_2\bar{\eta}_{m+2}\frac{b^2}{|b|^2} \\ \bar{\eta}_1\eta_3 & \bar{\eta}_2\eta_3 & |\eta_3|^2 & \cdots & \eta_3\bar{\eta}_{m+2}\frac{b^2}{|b|^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\eta}_1\eta_{m+2}\frac{\bar{b}^2}{|b|^2} & \bar{\eta}_2\eta_{m+2}\frac{\bar{b}^2}{|b|^2} & \bar{\eta}_3\eta_{m+2}\frac{\bar{b}^2}{|b|^2} & \cdots & |\eta_{m+2}|^2 \end{pmatrix} /n,$$

where

$$\begin{aligned} &\eta_r = \xi_r \ (1 \le r \le m), \\ &\eta_{m+1} = \xi_{m+1}\cosh(zx + z^3t) + \xi_{m+2}\sinh(zx + z^3t), \\ &\eta_{m+2} = \xi_{m+1}\sinh(zx + z^3t) + \xi_{m+2}\cosh(zx + z^3t), \\ &n = \sum_{l=1}^{m+2} |\eta_l|^2, \\ &b = z\sum_{l=1}^{m+1} |\eta_l|^2 - \bar{z}|\eta_{m+2}|^2. \end{aligned}$$

Hence, the new solution is

$$\begin{cases} \widetilde{q}_r = (z+\bar{z})(\frac{b^2}{|b|^2} - 1)\frac{\eta_r\bar{\eta}_{m+2}}{n} \ (1 \le r \le m), \\ \widetilde{u} = (z+\bar{z})((\frac{b^2}{|b|^2} - 1)\frac{\eta_{m+1}\bar{\eta}_{m+2}}{n} - (\frac{\bar{b}^2}{|b|^2} - 1)\frac{\bar{\eta}_{m+1}\eta_{m+2}}{n}). \end{cases}$$

6.3 Conclusion

Next, we will derive the smooth soulution of the NLS and the complex mKdV.

Consider the case where $k = 0$, $m = 1$, $z = 1$ and $Im\pi = (1,1)^T$, then $\widetilde{q} = -\frac{1}{\cosh(x+t)}$ is the new solution of the NLS: $q_{t_3} = q_{xxx} + 6|q|^2 q_x$. Moreover, let $Im\pi = (1,i)^T$, then $\widetilde{q} = \frac{i}{\cosh(x+t)} \in C(\mathbb{R}, \mathbb{C})$ gives the solution of the complex mKdV: $q_{t_3} = q_{xxx} - 6q^2 q_x$.

Consider the case where $k = 1$, $m = 0$, $z = -1+i$ and $Im\pi = (1,2)^T$, then $\widetilde{u} = -4Im\frac{9e^{-2(x-2t)} - e^{2(x-2t)} - 6(\cos 1 + i\sin 1)\sinh 2(x+2t)}{9e^{-2(x-2t)} + e^{2(x-2t)} + 6(\sin 1 - i\cos 1)\cosh 2(x+2t)}$ is the new solution of the complex mKdV: $u_{t_3} = \frac{1}{4}u_{xxx} - \frac{3}{2}u^2 u_x$.

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