Using the Fourier transform to identify the lag/position between identical signals

(Dated: September 21, 2020)

I. BACKGROUND

The Fast Fourier Transform (FFT) can be used to quickly identify the lag between two identical one-dimensional (1D) signals, or to find a smaller image which is contained in a bigger image which is the lag in two dimensions(2D).

The steps to identify the lag between two signals $s1$ and $s2$ is:

1. Take the FFT of $s_1(t)$ and $s_2(t)$ over time t to create the transformed vectors $\hat{s_1}(\omega)$ and $\hat{s_2}(\omega)$ for frequencies ω.
2. Take the negative conjugate of one of the signals, say $\hat{s_2}(\omega) \to -\overline{\hat{s_2}(\omega)}$.
3. Point-wise multiply $\hat{s_1}(\omega)$ and $-\overline{\hat{s_2}(\omega)}$: - $\hat{s_1}(\omega).\overline{\hat{s_2}(\omega)}$.
4. Take the inverse FFT to get the lag.

A. Deriving the lag between two identical 1D signals

I start by deriving the time-shift property of the Fourier Transform (FT), then use to show the steps above work.

1. Time-shift property of the Fourier transform

The FT acts on a signal $f(t)$ via the operator $\mathcal{F}$ such that a function over time $f(t)$ is mapped onto frequency space $\hat{\chi}(\omega)$:

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \tag{1}$$

$$= \hat{\chi}(\omega). \tag{2}$$

Now introduce a time-shift $t - t_0$ to Eqn. 2 and use a change of variables to obtain the time-shift property:

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}\,dt \tag{3}$$

$$= \int_{-\infty}^{\infty} f(u)e^{-i\omega(u+t_0)}\,du \tag{4}$$

$$\text{where } u = t - t_0$$

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du \tag{5}$$

$$= e^{-i\omega t_0}\hat{\chi}(\omega). \tag{6}$$

2. Obtaining the lag between two identical 1D signals

Using Eqn. 6 and following the steps outlined above we start with two signals $f_1(t)$ and $f_2(t)$ where $f_2(t) = f_1(t-t_0)$.

Step 1: Transform $f_1(t)$ and $f_2(t)$ into frequency space:

$$\mathcal{F}\{f_1(t)\} = \hat{\chi}_1(\omega), \tag{7}$$

and

$$\mathcal{F}\{f_2(t)\} = \hat{\chi}_2(\omega)$$

$$= e^{-i\omega t_0}\hat{\chi}_1(\omega). \tag{8}$$

Where Eqn. 8 is due to the time-shift property of the Fourier transform.

Step 2: Next take negative complex conjugate of one of the signals (say $\hat{\chi}_1(\omega)$) in Eqn. 7 and multiply by Eqn. 8:

$$\hat{\chi}_2(\omega)\hat{\chi}_1^*(\omega) = e^{-i\omega t_0}\hat{\chi}_1(\omega)\hat{\chi}_1^*(\omega) \tag{9}$$

$$(\text{time} - \text{shift property})$$

$$= e^{-i\omega t_0}|\hat{\chi}_1(\omega)|^2 \tag{10}$$

$$(\text{modulus of a complex function})$$

$$= e^{-i\omega t_0}E_1(\omega) \tag{11}$$

$$(\text{definition of energy spectral density})$$

$$= \int_{\infty}^{\infty} e^{-i\omega t_0}|f_1(t)|^2\,dt, \tag{12}$$

$$(\text{definition of energy spectral density})$$

where $E_1(\omega)$ is the energy spectral density of $\hat{\chi}(\omega)$ and is equal to $E_1(t)$.

Step 3: Now applying the inverse FT operator $\mathcal{F}^{-1}$, we have:

$$\mathcal{F}^{-1}\left\{\int_{\infty}^{\infty} e^{-i\omega t_0}f_1(t)\,dt\right\} = \tag{13}$$

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{-i\omega t_0}|f_1(t)|^2\,dt\,e^{i\omega t'}d\omega. \tag{14}$$

$$\tag{15}$$

Separating out the integrals:

$$\int_{-\infty}^{\infty}|f_1(t)|^2\,dt\int_{-\infty}^{\infty} e^{-i\omega(t'-t_0)}d\omega. \tag{16}$$

$$= S_f(f)\,\delta(t-t_0) \tag{17}$$

EquationXX means the value of $E_\chi\delta(t-t_0)$ except at t_0 which is the lag.

II. APPROACH

III. RESULTS

IV. IMPROVEMENTS