Analysis of Queuing Model

Module 2

Operating Characteristics

- *Operating characteristics* are assumed to approach

Notation	Operating Characteristic
L	Average number of customers in the system (waiting and being served)
L_q	Average number of customers in the waiting line
W	Average time a customer spends in the system (waiting and being served)
W_q	Average time a customer spends waiting in line
P_0	Probability of no (i.e., zero) customers in the system
P_n	Probability of n customers in the system
ρ	Utilization rate; the proportion of time the system is in use

Basic Single-Server Model

- Assumptions
  - Poisson arrival rate
  - exponential service times
  - first-come, first-served queue discipline
  - infinite queue length
  - infinite calling population
- Computations
  - λ = mean arrival rate
  - μ = mean service rate
  - n = number of customers in line

Basic Single-Server Model (cont.)

- probability that no customers are in queuing system

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right)$$

- average number of customers in queuing system

$$L = \frac{\lambda}{\mu - \lambda}$$

- probability of n customers in queuing system

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

- average number of customers in waiting line

$$L_q = \frac{\lambda^2}{\mu\,(\mu - \lambda)}$$

Basic Single-Server Model (cont.)

- average time customer spends in queuing system

$$W = \frac{1}{\mu - \lambda} = \frac{L}{\lambda}$$

- average time customer spends waiting in line

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

- probability that server is busy and a customer has to wait (utilization factor)

$$\rho = \frac{\lambda}{\mu}$$

- probability that server is idle and customer can be served

$$I = 1 - \rho$$
$$= 1 - \frac{\lambda}{\mu} = P_0$$

Basic Single-Server Model Example

Arrival rate = 24 customers / hour
Service rate = 30 customers / hour

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = \left(1 - \frac{24}{30}\right)$$
$$= 0.20 \text{ probability of no customers in the system}$$

$$L = \frac{\lambda}{\mu - \lambda} = \frac{24}{30 - 24}$$
$$= 4 \text{ customers on the average in the queuing system}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(24)^2}{30(30 - 24)}$$
$$= 3.2 \text{ customers on the average in the waiting line}$$

Basic Single-Server Model Example (cont.)

$$W = \frac{1}{\mu - \lambda} = \frac{1}{30 - 24}$$
$$= 0.167 \text{ hour (10 minutes) average time in the system per customer}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{24}{30(30 - 24)}$$
$$= 0.133 \text{ hour (8 minutes) average time in the waiting line per customer}$$

$$\rho = \frac{\lambda}{\mu} = \frac{24}{30}$$
$$= 0.80 \text{ probability that the server will be busy and the customer must wait}$$

$$I = 1 - \rho = 1 - 0.80$$
$$= 0.20 \text{ probability that the server will be idle and a customer can be served}$$

Basic Multiple-Server Model

- single waiting line and service facility with several independent servers in parallel
- same assumptions as single-server model
- $s\mu > \lambda$
  - s = number of servers
  - servers must be able to serve customers faster than they arrive

Basic Multiple-Server Model (cont.)

- probability that there are no customers in system

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\left(\frac{s\mu}{s\mu - \lambda}\right)}$$

- probability of n customers in system

$$P_n = \begin{cases} \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{for } n > s \\ \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{for } n \le s \end{cases}$$

Basic Multiple-Server Model (cont.)

- probability that customer must wait

$$P_w = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda} P_0 \qquad L_q = L - \frac{\lambda}{\mu}$$

$$L = \frac{\lambda\mu\,(\lambda/\mu)^s}{(s-1)!\,(s\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu} \qquad W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda} \qquad \rho = \frac{\lambda}{s\mu}$$

Basic Multiple-Server Model Example

λ = 10 students per hour
μ = 4 students per hour per service representative
s = 3 service representatives
$s\mu = (3)(4) = 12 \ (> \lambda = 10)$

Basic Multiple-Server Model Example (cont.)

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\left(\frac{s\mu}{s\mu - \lambda}\right)}$$

$$= \frac{1}{\left[\frac{0}{0!}\left(\frac{10}{4}\right)^0 + \frac{1}{1!}\left(\frac{10}{4}\right)^1 + \frac{1}{2!}\left(\frac{10}{4}\right)^2\right] + \frac{1}{3!}\left(\frac{10}{4}\right)^3 \frac{3(4)}{3(4) - 10}}$$

$= 0.045$ probability that no customers are in the health service.

Basic Multiple-Server Model Example (cont.)

$$L = \frac{\lambda\mu(\lambda/\mu)^s}{(s-1)!(s\mu-\lambda)^2}P_0 + \frac{\lambda}{\mu}$$
$$= \frac{(10)(4)(10/4)^3}{(3-1)![3(4)-10]^2}(0.045) + \frac{10}{4}$$
$$= 6 \text{ students in the health service}$$
$$W = \frac{L}{\lambda}$$
$$= \frac{6}{10}$$
$$= 0.60 \text{ hour or 36 minutes in the health service}$$

Basic Multiple-Server Model Example (cont.)

$$L_q = L - \frac{\lambda}{\mu}$$
$$= 6 - \frac{10}{4}$$
$$= 3.5 \text{ students waiting to be served}$$
$$W_q = \frac{L_q}{\lambda}$$
$$= \frac{3.5}{10}$$
$$= 0.35 \text{ hour or 21 minutes waiting in line}$$

Basic Multiple-Server Model Example (cont.)

$$P_w = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}P_0$$
$$= \frac{1}{3!}\left(\frac{10}{4}\right)^3 \frac{3(4)}{3(4)-(10)}(0.045)$$
$$= 0.703 \text{ probability that a student must wait for service}$$
(i.e., that there are three or more students in the system)

Basic Multiple-Server Model Example (cont.)

- To cut wait time, add another service representative
  - now, $s = 4$
- Therefore:

P_0 = 0.073 probability that no students are in the health service
L = 3.0 students in the health service
W = 0.30 hour, or 18 minutes, in the health service
L_q = 0.5 students waiting to be served
W_q = 0.05 hour, or 3 minutes, waiting in line
P_w = 0.31 probability that a student must wait for service