SOLUTIONS TO PROBLEMS

Problem 1: Solution: $\dfrac{24\sqrt{13}}{13}$.

Pick any point on one line and plug it into the distance formula for a point and a line. So let's pick (6, 0) from the first line, $2x - 3y = 12$.

The distance between the point (6,0) and line $2x - 3y = 36$ is

$$d = \frac{|2(6) - 3(0) - 36|}{\sqrt{2^2 + 3^2}} = \frac{24}{\sqrt{13}} = \frac{24\sqrt{13}}{13}.$$

Problem 2: Solution: $\dfrac{23}{5}$.

First notice that the closest the line gets to the origin is $\dfrac{|3(0) + 4(0) - 48|}{\sqrt{3^2 + 4^2}} = \dfrac{48}{5}$, so it is more than 5 units from the origin. Now subtract the radius of the circle, yielding the shortest distance as $\dfrac{48}{5} - 5 = \dfrac{23}{5}$.

Problem 3: Solution:

Solve for x in the system of equations: $x = \dfrac{156}{13 + 11m} = \dfrac{2^2 \cdot 3 \cdot 13}{13 + 11m}$.

We know that x is an integer, so $13 + 11m$ must be a divisor of 156. The divisors for 156 are 1, 2, 3, 4, 6, 12, 13, 26, 39, 52, 78, and 156. We want to find a positive integer m such that $13 + 11m = d \quad \Rightarrow \quad m = \dfrac{d - 13}{11}$. d is one of these divisors.

Since m is a positive integer, the smallest value for d is 24.

So
(1) if $d = 26$, $d - 13 = 13$ is not divisible by 11.
(2) if $d = 39$, $d - 13 = 26$ is not divisible by 11
(3) if $d = 52$, $d - 13 = 39$ is not divisible by 11.
(4) if $d = 78$, $d - 13 = 65$ is not divisible by 11.
(5) if $d = 156$, $d - 13 = 143$ and $m = 13$.

We can conclude that $m = 13$ is the only positive integer.

Problem 4: Solution: 15.
The points P, Q, R, and S are respectively (3, 2), (3, – 2), (2, – 3) and (– 2, 3).

The rectangle with vertices(– 2, 3), (3, 3), (3, – 3) and (– 2, – 3) has the area of 30. When the three corner regions, with the areas of $\frac{1}{2}$, $\frac{5}{2}$ and 12, respectively, are subtracted, the remaining quadrilateral has the area of 15.

Problem 5: Solution: $3x - y + 9 = 0$.
Let the slope of l_2 be k and l_1 be α.
We have $\tan\alpha = \frac{1}{2}$ and $k = \tan(45° + \alpha) = \frac{1+\tan\alpha}{1-\tan\alpha} = 3$.
Since the intersecting point is $A(-3, 0)$, the equation of l_2 is then $y = 3(x + 3)$ or $3x - y + 9 = 0$.

Problem 6: Solution: $7x - y - 5 = 0$.
Method 1:
Let the equation be $y - 2 = k(x - 1)$. The points of intersection with two lines are A and B, respectively.

Solve the system of equations: $\begin{cases} 4x + 3y + 1 = 0 \\ y - 2 = k(x - 1) \end{cases}$

and $\begin{cases} 4x + 3y + 6 = 0 \\ 4x + 3y + 1 = 0 \end{cases}$

The coordinates of A and B are:

$$\left(\frac{3k-7}{3k+4}, \frac{-5k+8}{3k+4}\right) \text{ and } \left(\frac{3k+12}{3k+4}, \frac{8-10k}{3k+4}\right).$$

Since $|AB| = \sqrt{\left(\frac{5}{3k+4}\right)^2 + \left(\frac{5k}{3k+4}\right)^2} = \sqrt{2}$, $\frac{5\sqrt{1+k^2}}{|3k+4|} = \sqrt{2}$.

The above equation has the form of:

$$(7k + 1)(k - 7) = 0 \quad \Rightarrow \quad k = -\frac{1}{7} \text{ or } 7.$$

So the equation is $x + 3y - 15 = 0$ or $7x - y - 5 = 0$.

Method 2:

The distance between two parallel lines is $d = \dfrac{|1-6|}{\sqrt{4^2+3^2}} = 1$.

Since the length of the segment cut by these two lines is $\sqrt{2}$, the angle formed by the line and two parallel lines is 45°.

Let the equation be $y-2 = k(x-1)$.

$$\tan\theta = \left|\frac{k_2-k_1}{1+k_1k_2}\right| \qquad \Rightarrow \qquad \left|\frac{-\frac{4}{3}-k}{1-\frac{4}{3}k}\right| = 1$$

Solving for k: $k_1 = -\dfrac{1}{7}$, $k_2 = 7$.

So the equation is $x+3y-15=0$ or $7x-y-5=0$.

Problem 7: Solution: $7x+y+22=0$

Method 1:

Solving $\begin{cases} x-y-2=0 \\ 3x-y+3=0 \end{cases}$ gives the point of intersection: $(-\dfrac{5}{2}, -\dfrac{9}{2})$

Let the slope of the line in question be *k:*

$$\left|\frac{3-1}{1+3\times 1}\right| = \left|\frac{k-3}{1+3k}\right|$$

Solve for k: $k = -7$ or $k = 1$(extraneous).

So the equation of the line is $y+\dfrac{9}{2} = -7\ (x+\dfrac{5}{2})$ or $7x+y+22=0$.

Method 2:

Since $P_0(0,-2)$ is a point on $x-y-2=0$, its image is $P(x,y)$.

$$\begin{cases} \dfrac{y+2}{x-0}\times 3 = -1 \\ 3\cdot\dfrac{x}{2} - \dfrac{y-2}{2} + 3 = 0 \end{cases}$$

Solve for x: $\begin{cases} x=-3 \\ y=-1 \end{cases}$ or $\begin{cases} x=0 \\ y=-2 \end{cases}$ (extraneous).

So $k=\dfrac{-\frac{9}{2}+1}{-\frac{5}{2}+3}=-7$.

The equation is $y+\frac{9}{2}=-7\ (x+\frac{5}{2})$.

Problem 8: Solution: $\sqrt{2}$.

g(x) can be re-written as

$$g(x)=\left|\sqrt{(x-1)^2+(0-2)^2}-\sqrt{(x-2)^2+(0-3)^2}\right|.$$

The greatest possible value of *g(x)* is equal to finding the greatest possible difference $|PA|-|PB|$ of the distances from *P*(*x,* 0) to *A* (1, 2) and *P* to *B*(2, 3), where point *P* is the point of intersection of the line segment connecting *A* and *B* and *x*-axis.

$|AB|=\sqrt{(1-2)^2+(2-3)^2}=\sqrt{2}$

Problem 9: Solution: $x+4y-4=0$

Let the equation of the line containing point (0,1) be $y = kx + 1$. The points of intersection of line $y = kx + 1$ with $x-3y+10=0$ and $2x+y-8=0$ are $A(\frac{7}{3k-1},\frac{4k-1}{3k-1})$ and $B(\frac{7}{k+2},\frac{8k+2}{k+2})$, respectively.

By the midpoint formula (4), we get $k=-\frac{1}{4}$.

The equation is $y=-\frac{1}{4}x+1$ or $x+4y-4=0$.

Problem 10: Solution: $a = -7$ or -17.

We know that the distance between l_1: $2x + 3y - 6 = 0$ and l_2: $2x + 3y + \frac{a}{2} = 0$ is $\frac{5\sqrt{13}}{26}$.

By the formula (8) $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$, we have: $\frac{|\frac{a}{2} + 6|}{\sqrt{4+9}} = \frac{5\sqrt{13}}{26}$.

Therefore $|\frac{a}{2} + 6| = \frac{5}{2}$. Solving for a gives $a = -7$ or -17.

Problem 11: Solution: 12.

The midpoints of sides *AB, BC, CD, DA* are *M*, *N*, *P*, and *Q*, respectively. $M = (3, 6)$ and $N = (4, 3)$.

We know that *MN* has slope –3. So the slope of perpendicular segment *MQ* is $\frac{1}{3}$. $MQ = MN = \sqrt{10}$.

The equation of the line containing *MQ* can be written as $y - 6 = \frac{(x-3)}{3}$, or $y = \frac{(x+15)}{3}$.

So let the coordinates of *Q* be $(a, \frac{(a+15)}{3})$.

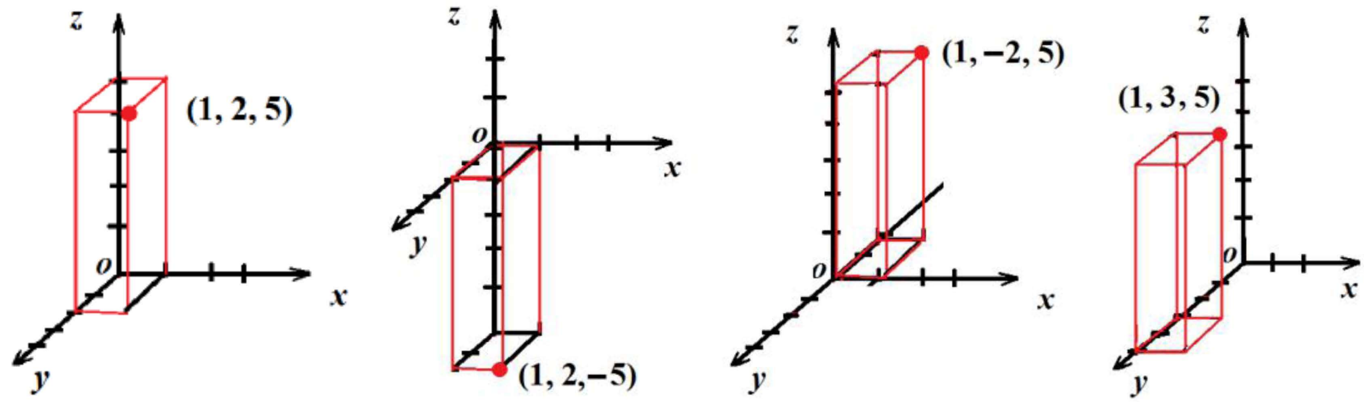
Since $MQ = \sqrt{10}$, using the distance formula to segment *MQ:*

$$(a-3)^2 + (\frac{(a+15)}{3} - 6)^2 = 10 \Rightarrow (a-3)^2 + (\frac{a-3}{3})^2 = 10 \Rightarrow \frac{10}{9}(a-3)^2 = 10 \Rightarrow$$
$$(a-3)^2 = 9 \Rightarrow a - 3 = \pm 3.$$

Since *Q* is in the first quadrant, $a = 6$ and $Q = (6, 7)$. *Q* is the midpoint of *AD* and $A = (4, 10)$. By the midpoint formula, we get the coordinates of *D:* (8, 4). The answer is 8 + 4 = 12.

Problem 12: Solution: (E).

The point *Q* (1, 2, 5) is the result of reflecting point *P* in the *xy*-plane. A 180° rotation about the *x*-axis, or a half turn about the *x*-axis, produces point *R* (1,–2, 5). Translating *R* by 5 units in the positive *y* direction gives us *S* (1, 3, 5). Take a look at the figures below for further explanation.



Problem 13: Solution: 89.

Method 1:

Let the angle bisector be PT and the coordinates of T be (x, y).

From the distance formula, we can get

$$|PQ| = \sqrt{7^2 + 24^2} = 25, \quad |PR| = \sqrt{9^2 + 12^2} = 15.$$

The angle bisector formula gives us

$$\left|\frac{QT}{TR}\right| = \left|\frac{PQ}{PR}\right| = \frac{25}{15} = \frac{5}{3}.$$

By formula (3), we have $x = -5$, $y = \dfrac{-23}{2}$.

The equation for PT is: $y - 5 = -\dfrac{11}{2}(x+8)$ or $11x + 2y + 78 = 0$.

Therefore $a + c = 11 + 78 = 89$.

Method 2:

Let the slope of the angle bisector be k. The slopes of PR and PQ are k_{PR} and k_{PQ}, respectively.

$$k_{PR} = \frac{-7-5}{1+8} = -\frac{4}{3} \qquad k_{PQ} = \frac{-19-5}{-15+8} = \frac{24}{7}.$$

By the formula (1), we have:

$$\frac{-\frac{4}{3}-k}{1-\frac{4}{3}k}=\frac{k-\frac{24}{7}}{1+\frac{24}{7}k} \quad\Rightarrow\quad 22k^2+117k-22=0 \;\Rightarrow\; \therefore\quad k=-\frac{11}{2},\qquad k=\frac{2}{11}\,.$$

Since the angle bisector is the interior angle bisector, $k=-\frac{11}{2}$.

$y-5=\frac{-11}{2}(x+8)$ or $11x+2y+78=0$.

Therefore $a+c=11+78=89$.