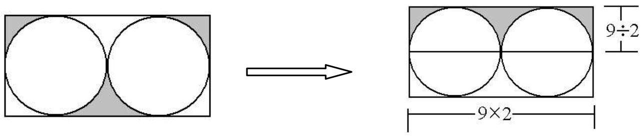
SOLUTIONS:

Problem 1. Solution: (A).



The area of the shaded region for one tile is (9 × 2)(9 ÷ 2) – π × (9 ÷ 2)^2 =

$81-\frac{81}{4}\pi$.

Since there are 36 × 72 ÷(9 × 18) = 16 tiles in the entire floor, the area of the total shaded

region in square feet is $16(81-\frac{81\pi}{4})=1296-324\pi$.

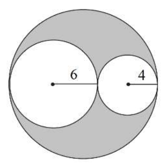
Problem 2. Solution: (B).

The radius of the large circle is (12 + 8)/2 = 10, so its area is 100π.

The area of the shaded region is (100π − 36π − 16π)/2 = 24π.

The area of the unshaded region is 100π −24π = 76π.

The ratio is 24π / 76π = 6/19.



Problem 3. Solution: (E).

Let A_A be the area of sector A, and let A_B be the area of sector B.

Let C_A be the circumference of circle A. Let C_B be the circumference of circle B.

By the formula $A=\pi r^2\times\frac{\theta}{360}$, with $A_A=A_B \Rightarrow \pi r_A^{\ 2}\times\frac{25°}{360}=\pi r_B^{\ 2}\times\frac{100°}{360}$

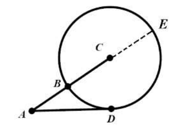
$\Rightarrow \quad (\frac{r_A}{r_B})^2=\frac{100°}{25°}=4 \Rightarrow \quad \frac{r_A}{r_B}=\frac{2}{1}$

Thus the ratio of the circumferences is $\frac{C_A}{C_B}=\frac{2\pi r_A}{2\pi r_B}=\frac{r_A}{r_B}=\frac{2}{1}$.

Problem 4. Solution: (A).

Extend AC to meet the circle at E.

By the Power of Points formula:



$AD^2=AE\times AB \qquad \Rightarrow 10^2=(7+BE)\times 7 \Rightarrow BE=\frac{51}{7}$

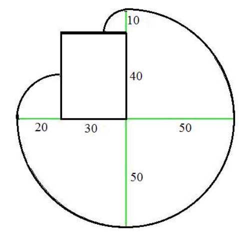
The radius is 51/14, which is half of the diameter BE.

So the area is $\pi\times(\frac{51}{14})^2=\frac{2601}{196}\pi$.

Problem 5. Solution: (D).

The grazing area consists of

1. three quarters of a big 50ft-radius circle, which has area $\frac{3}{4}\pi\times 50^2=1875\pi$.

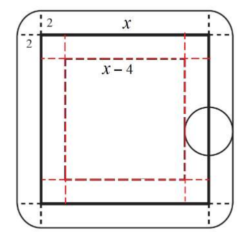


2. one quarter of a 20ft-radius circle on the left, which has area $\frac{1}{4}\pi\times 20^2=100\pi$

3. one quarter of a small 10ft-radius circle on the top, which has area $\frac{1}{4}\pi\times 10^2=25\pi$.

The answer is (1875+ 100 + 25) π = 2000π square feet of grazing area.

Problem 6. Solution: (D).

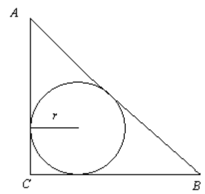


Let x be the length of the side of the square.

At any point on Charles' walk, he can see all the points inside a circle of radius 2 km. The portion of the viewable region inside the square consists of the interior of the square except for a smaller square with side length x – 4 km. This portion of the viewable region has area $x^2-(x-4)^2=8x-16$ km^2. The portion of the viewable region outside the square consists of four rectangles, each x km by 2 km, and four

quarter-circles (which make one whole circle), each with a radius of 2 km. This portion of the viewable region has area $4 \times 2x + \pi \times 2^2 = (8x + 4\pi)$ km^2. The area of the entire
viewable region is $8x - 16 + (8x + 4\pi) = 144 + 4\pi \quad \Rightarrow 16x - 160 = 0 \Rightarrow \quad x = 10$.

Problem 7. Solution: (B).



Since $AC = 6$ and $BC = 8$, $AB = 10$.

By formula (16.11), $r = \dfrac{AC + BC - AB}{2} = \dfrac{6 + 8 - 10}{2} = 2$

The area of the circle is $\pi r^2 = 4\pi$

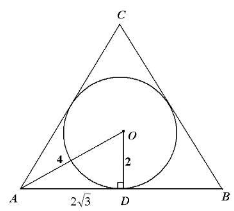
The area of the triangle is $\dfrac{AC \times BC}{2} = \dfrac{6 \times 8}{2} = 24$

The answer is $24 - 4\pi$.

Problem 8. Solution: (B).

Since the area of the circle is 4π, the radius is 2.

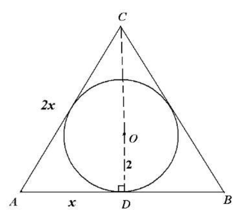
Method 1:



As shown in the figure, $OD = 2$, $\angle DAO = 30°$, $AO = 4$, $AD = 2\sqrt{3}$. The area of ΔADO is $\dfrac{AD \times DO}{2} = \dfrac{2\sqrt{3} \times 2}{2} = 2\sqrt{3}$.

The area of triangle ABC is 6 times of the area of ΔADO, or $6 \times 2\sqrt{3} = 12\sqrt{3}$.

Method 2:



CD is the median of triangle ABC, so since $DO = 2$, $CO = 4$ and $CD = 6$.

Applying Pythagorean Theorem to the right triangle ACD gives: $x^2 + 6^2 = (2x)^2 \Rightarrow \quad x = 2\sqrt{3}$

The area of triangle ABC is

$$\frac{AB \times CD}{2} = \frac{2x \times 6}{2} = 6x = 6 \times 2\sqrt{3} = 12\sqrt{3}.$$

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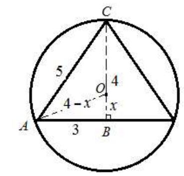
Problem 9. Solution: (C).

Triangle ABC is a $3 - 4 - 5$ right triangle with $AB = 3$ and $BC = 4$. Let $BO = x$. We know that $AO = CO = 4 - x$.



Applying Pythagorean Theorem to triangle ABO gives:

$AB^2 + BO^2 = AO^2$

or $3^2 + x^2 = (4 - x)^2 \quad \Rightarrow \quad 3^2 + x^2 = 16 - 8x + x^2 \quad \Rightarrow \quad 8x = 7 \Rightarrow$

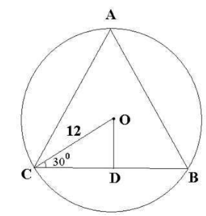
$x = \dfrac{7}{8}$

The radius is $4 - x = 4 - \dfrac{7}{8} = \dfrac{32 - 7}{8} = \dfrac{25}{8}$.

As a consequence, the circle has area $\pi(\dfrac{25}{8})^2 = \dfrac{625}{64}\pi$.

Problem 10. Solution: (A).

Method 1:

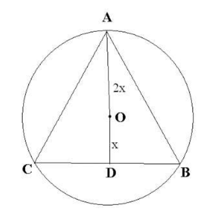


Triangle ODC is a $30 - 60 - 90$ triangle, so $OD = 12/2 = 6$.

From the Pythagorean Theorem, $CD = \sqrt{12^2 - 6^2} = 6\sqrt{3}$ and $BC = 12\sqrt{3}$ follows.

The area of the equilateral triangle with side $12\sqrt{3}$ equals

$$A = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(12\sqrt{3})^2 = 108\sqrt{3}.$$



Method 2: Let AO be $2x$ and OD be x. The diameter of the circle is 12, so $2x = 12$ and $x = 6$. $AD = 3x = 18$. D is the midpoint of CB, so $AC = 2CD$. By the Pythagorean Theorem,

$$AC^2 - CD^2 = 18^2 \Rightarrow AC^2 - (\frac{1}{2}AC)^2 = 18^2 \Rightarrow \; AC = 12\sqrt{3}.$$

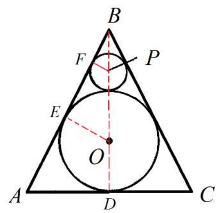
The area of the triangle is equal to $\dfrac{1}{2} \times 12\sqrt{3} \times 18 = 108\sqrt{3}$.

Problem 11 Solution: (D).

Draw $PF \perp AB$ and $OE \perp AB$.
Connect BD.
Triangle BEO is a 30°-60°-90° triangle and $BP = 2PF = 2$.
Let $OE = r$.
We know that $BO = 2OE \quad \Rightarrow \quad 2r = r + 3 \quad \Rightarrow r = 3$

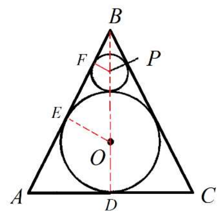
We know that $r = \frac{1}{6}a\sqrt{3}$, where a is the length of the side of triangle ABC. So $3 = \frac{1}{6}a\sqrt{3} \quad \Rightarrow \quad a = 6\sqrt{3}$

We know that $S_{\Delta ABC} = \frac{1}{4}a^2\sqrt{3}$. So $S_{\Delta ABC} = \frac{1}{4}\times(6\sqrt{3})^2\sqrt{3} = 27\sqrt{3}$



Problem 12. Solution: (D).

Draw $PF \perp AB$ and $OE \perp AB$.
Connect BD.
Triangle BEO is a 30°-60°-90° triangle and $BO = 2OE$.
$OE = 45$ and $BO = 2OE = 90$, and $BD = 90 + 45 = 135$.
Let $PF = r$.
Triangle BFP is a 30°-60°-90° triangle and $BP = 2r$.
In other word, $BP = BO - 45 - r$.
So $2r = BO - 45 - r \quad \Rightarrow 3r = 90 - 45 = 45$.
So $r = 15$.
The area is $\pi(15)^2 = 225\pi$.



Problem 13. Solution: (A).

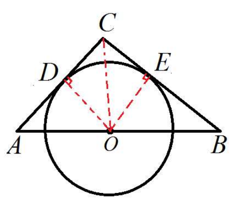
By Heron's formula,

$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(21-15)(21-14)(21-13)} = 84$,

where $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(15 + 14 + 13) = 21$.

Connect OD, OE, and OC.



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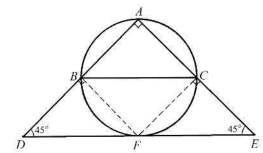
$S_{\Delta ABC} = \frac{1}{2}AC \times OD + \frac{1}{2}BC \times OE$

$= \frac{1}{2}\times(14+13)\times OD = 84$.

So $27OD = 168 \quad \Rightarrow \quad OD = 56/9$.

Problem 14. Solution: (D).

Since $\overline{BC}$ is the diameter, and ABC is the isosceles triangle, $\angle A = 90°$, $\angle D = \angle E = 45°$. When we connect BF and CF and we see that four triangles are congruent. So the ratio is 1:3.



Problem 15. Solution: 4.

Method 1:

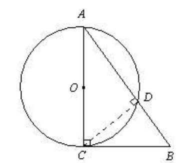
Applying Pythagorean Theorem to right triangle ABC:

$AC = \sqrt{10^2 - (2\sqrt{5})^2} = \sqrt{100-20} = \sqrt{80} = 4\sqrt{5}$.

Connect CD. We get $\angle ADC = 90°$.

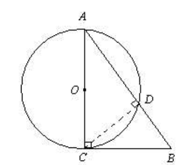
$S_{\Delta ABC} = \frac{1}{2}BC \times AC = \frac{1}{2}AB \times CD$.

So $\frac{1}{2}\times 2\sqrt{5}\times 4\sqrt{5} = \frac{1}{2}\times 10\times CD \quad \Rightarrow \quad 40 = 10CD \Rightarrow \quad CD = 4$.



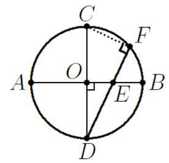
Method 2:

Connect CD. We get $\angle ADC = 90°$.
$\angle ACB = 90°$ (given).
Let $AD = x$. By the power of points formula, we get
$BC^2 = AB \times DB$ or $(2\sqrt{5})^2 = 10 \times DB$, then $DB = 2$.
$AD = 8$, so $CD^2 = AD \times DB = 2\times 8 = 16$, then $CD = 4$.

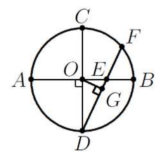


Problem 16. Solution: (C).

Method 1
Draw segment FC. Angle CFD is a right angle since arc CFD is a semicircle. Then right triangles DOE and DFC are similar, so $DO/DF = DE/DC$.
Let $DO = r$ and $DC = 2r$. Substituting, we have $1/8 = 6/2r$, $2r^2 = 48$, $r^2 = 24$.
Then the area of the circle is $\pi r^2 = 24\pi$.

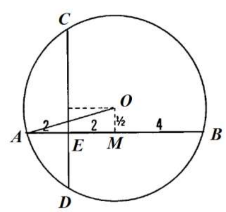


Method 2:
Let $OA = OB = r$ and $OE = x$. Substituting into $AE \cdot EB = DE \cdot EF$ gives $(r + x)(r - x) = 6 \cdot 2$ so $r^2 - x^2 = 12$. In right triangle EOD, $r^2 + x^2 = 36$. Add to find $2r^2 = 48$. Thus, the area of the circle is $\pi r^2 = 24\pi$.



Method 3:
Construct $OG \perp DF$ with G on DF. Then $DG = (1/2)DF = 4$. Since OG is an
altitude to the hypotenuse of right triangle EOD, we have $DE/DO = DO/DG$. Let DO = r. Then $6/r = r/4$, so $r^2 = 24$, and the area of the circle is $\pi r^2 = 24\pi$.

Problem 17. Solution: (B).
Method 1 (official solution)
Since point E divides every chord through E into two segments whose product is constant, $CE \cdot ED = AE \cdot EB$ or $CE \cdot 3 = 2 \cdot 6$, so $CE = 4$.
Thus chords AB and CD have lengths 8 and 7, respectively. Now the center O of the circle lies at the intersection of the perpendicular bisectors of chords CD and AB which is $\frac{1}{2}$ unit above and 4 units to the right of point A. The radius OA is the hypotenuse of a right triangle with legs $AM = 4$ and $OM = \frac{1}{2}$;



$$OA^2 = AM^2 + OM^2 = 4^2 + (\frac{1}{2})^2 = 65/4.$$

Thus the length of the diameter is $2OA = 2\sqrt{65/4} = \sqrt{65}$.

Method 2 (our solution):

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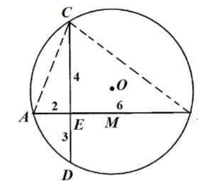
By the Power of points Theorem, we have $CE \cdot ED = AE \cdot EB$ or $CE \cdot 3 = 2 \cdot 6$, so $CE = 4$.
Connect AC, BC. Applying Pythagorean Theorem to ΔAEC: $AC = \sqrt{4^2 + 2^2} = 2\sqrt{5}$. Applying Pythagorean Theorem to ΔBEC: $CB = \sqrt{4^2 + 6^2} = 2\sqrt{13}$. We also have $AB = 8$.
By Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)} = r \times s = 1$,



where $s = \frac{1}{2}(a+b+c)$

Then we use the formula to calculate $R = \frac{abc}{4A} = \frac{2\sqrt{5} \times 2\sqrt{13} \times 8}{4} = \frac{\sqrt{65}}{2}$.

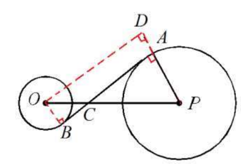
Thus $d = \sqrt{65}$.

Problem 18. Solution: (D).
Method 1:
Connect OB, PA. Extend PA to D with $DA = OB$. Connect OD. So $OD \perp PD$.
Applying Pythagorean Theorem to triangle ODP:

$OD^2 + DP^2 = OP^2 \quad \Rightarrow \quad OD^2 + (5+3)^2 = 16^2 \quad \Rightarrow$

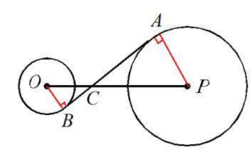
$OD = 8\sqrt{3}$.

$OBAD$ is a rectangle so $AB = 8\sqrt{3}$.



Method 2:
Triangle OBC is similar to triangle PAC.

$\frac{OB}{AP} = \frac{OC}{CP} \quad \Rightarrow \quad \frac{3}{5} = \frac{16-CP}{CP} \Rightarrow \quad CP = 10$.



So $OC = 16 - 10 = 6$.
Applying Pythagorean Theorem to triangle OBC:

$OB^2 + BC^2 = OC^2 \quad \Rightarrow \quad 6^2 - 3^2 = BC^2 \Rightarrow BC = 3\sqrt{3}$.

Applying Pythagorean Theorem to triangle ACP:

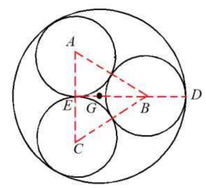
$AP^2 + AC^2 = CP^2 \quad \Rightarrow \quad 5^2 + AC^2 = 10^2 \quad \Rightarrow AC = 5\sqrt{3}$.

$AB = 3\sqrt{3} + 5\sqrt{3} = 8\sqrt{3}$.

Problem 19. Solution: (E).

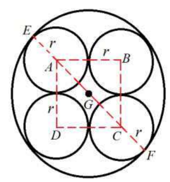
We connect the centers of three small circles and we know that triangle ABC is an equilateral triangle with the side of 8. So the height BE is

$\frac{8}{2} \times \sqrt{3} = 4\sqrt{3}$.



The centroid is G and $GB = \frac{2}{3} \times BE = \frac{2}{3} \times 4\sqrt{3} = \frac{8}{3}\sqrt{3}$. The radius is $GB + BD = \frac{8}{3}\sqrt{3} + 4$.

Problem 20. Solution: (C).



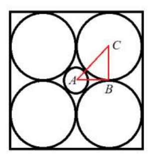
We draw the diameter EF as shown in the figure.

Applying Pythagorean Theorem to right triangles ADC:

$(2r)^2 + (2r)^2 = (2 - 2r)^2 \quad \Rightarrow \quad r = \sqrt{2} - 1$.

Problem 21. Solution: (A).

Method 1:



Annotate the figure as shown. The Pythagorean theory tells us that: $AB^2 + BC^2 = AC^2 \Rightarrow 4^2 + 4^2 = AC^2 \Rightarrow AC = 4\sqrt{2}$.

The square has the side length 8 + 8 = 16.

The area of the smaller circle with a radius of $4\sqrt{2} - 4$ is $\pi(4\sqrt{2} - 4)^2$.

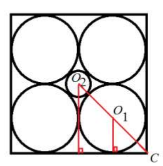
The area of the larger circle with a radius of 4 is $\pi \times 4^2 = 16\pi$.

The area of the shaded region is $16^2 - 4 \times 16\pi - \pi(4\sqrt{2} - 4)^2$

$= 256 + (32\sqrt{2} - 112)\pi$.

Method 2:

Annotate the figure as shown to the right.



$OC^2 = 8^2 + 8^2 = 2 \times 8^2 \Rightarrow OC = 8\sqrt{2}$

$\Rightarrow OO_1 = \frac{1}{2} OC = 4\sqrt{2}$.

The area of the smaller circle with a radius of $4\sqrt{2} - 4$ is $\pi(4\sqrt{2} - 4)^2$.

The square has the side length 8 + 8 = 16.

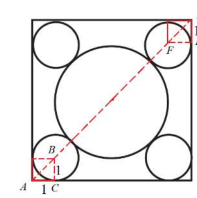
The area of the larger circle with a radius of 4 is $\pi \times 4^2 = 16\pi$.

The area of the shaded region is $16^2 - 4 \times 16\pi - \pi(4\sqrt{2} - 4)^2$

$= 256 + (32\sqrt{2} - 112)\pi$.

Problem 22. Solution: (B).

Connect two opposite vertices of the square and label the figure as shown.



We know that $AC = BC = 1$. So $AB = DF = \sqrt{2}$.

$AD = AB + BF + FD = \sqrt{2} + 1 + 2r + 1 + \sqrt{2}$

$= 2(\sqrt{2} + 2 + r)$.

Since the side of the square is 6, we have

$6\sqrt{2} = 2(\sqrt{2} + 2 + r) \quad \Rightarrow \quad 3\sqrt{2} = (\sqrt{2} + 2 + r)$

$\Rightarrow \quad 2\sqrt{2} - 2 = r$.

The area of the shaded region is $6^2 - \pi \times (1)^2 \times 4 - \pi \times (2\sqrt{2} - 2)^2$

$= 36 + (8\sqrt{2} - 16)\pi$.