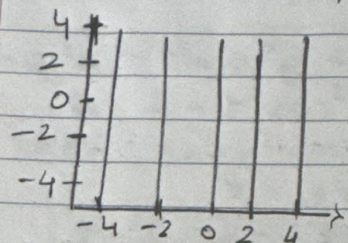
10 $\Gamma_t = (x+at, y+bt)$

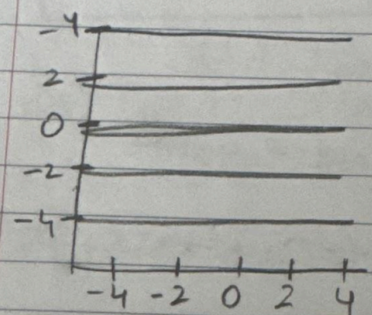
1. $(a,b) = (0,1)$

$\Gamma_t(x,y) = (x, y+t)$ vertical graph



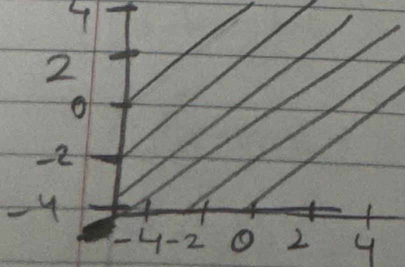
2. $(a,b) : (1,0)$

$\Gamma_t(x,y) = (x+t, y)$



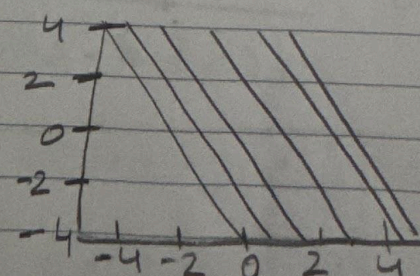
3. $(a,b) = (1,1)$

$\Gamma_t(x,y) = (x+t, y+t)$



4. $(a,b) = (1,-2)$

$\Gamma_t(x,y) = (x+t, y-2t)$



11 $\Gamma_t(x,y) = (e^t x, e^{-t} y)$

① $\Gamma_0(x,y) = (e^0 x, e^{-0} y) = (x,y)$

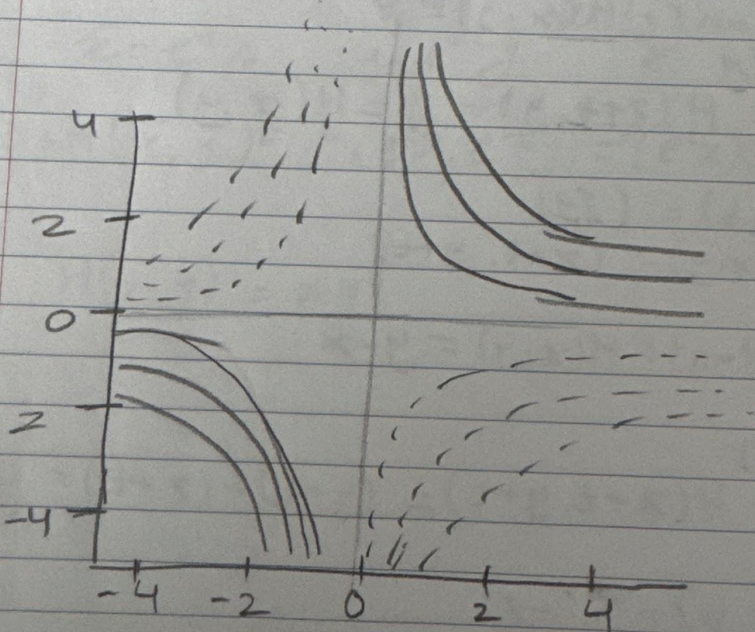
so the identity property is satisfied

② $\Gamma_{t_1} \circ \Gamma_{t_2} = \Gamma_{t_1+t_2}$

$\Gamma_{t_1}(e^{t_2} x, e^{-t_2} y) = (e^{t_1} e^{t_2} x, e^{-t_1} e^{-t_2} y)$

$= (e^{t_1+t_2} x, e^{-(t_1+t_2)} y) = \Gamma_{t_1+t_2}(x,y)$

③ The function is also differentiable.



12

① $(a,b) = (0,1)$:
$\Gamma_t(x,y) = (x, y+t)$

invariant $H(x,y) = x$

Verification:

$$H(x, y+t) = x = H(x,y)$$

② $(a,b) = (1,0)$
$\Gamma_t(x,y) = (x+t, y)$

invariant: $H(x,y) = y$
Verify
$$H(x+t, y) = y = H(x,y)$$

③ $(a,b) = (1,1)$
$\Gamma_t(x,y) = (x+t, y+t)$

invariant: $H(x,y) = y - x$

Verify:
$$H(x+t, y+t) = (y+t) - (x+t) = y - x$$

④ $(a,b) = (1,-2)$
$\Gamma_t(x,y) = (x+t, y-2t)$
invariant: $y + 2x = H(x,y)$
Verification:
$$H(x+t, y-2t) = (y-2t) + 2(x+t) = y + 2x$$
$$= H(x,y)$$

13

$\Gamma_t(x,y) = (e^t x, e^{-t} y)$

x is scaled by e^t & y is scaled by $e^{-t}y$

$H(e^t x, e^{-t} y) = H(x,y)$

Observing

$xy \to (e^t x, e^{-t} y) = xy$

So $H(x,y) = xy$

$\tilde{x} = e^t x \quad \tilde{y} = e^{-t} y$ Verify

$H(\tilde{x}, \tilde{y}) = H(e^t x, e^{-t} y) = (e^t x)(e^{-t} y) = xy$

$H(x,y) = xy$

15

$\Gamma_t(x,y) = (e^{t}x, e^{-t}y)$ $\quad y = 1$

$e^{-t}y = 1$

$t = \ln|y|$

Substitue $t = \ln|y|$

$e^{t}x = e^{\ln y}x = xy$

Thus,

$H(x,y) = xy \Rightarrow$ Invariant

16

$$\frac{dy}{dx} = \frac{2}{x^2} - y^2$$

$$\Gamma_t(x,y) = (e^t x, e^{-t} y)$$

$$e^{-2t}\frac{dy}{dx} = \frac{2}{(e^t x)^2} - e^{-2t} y^2$$

Since $\frac{dy}{dx} = \frac{2}{x^2} - y^2$, we substitue

$$e^{-2t}\left(\frac{2}{x^2} - y^2\right) = \frac{2}{(e^t x)^2} - e^{-2t} y^2$$

Simplify & rearrange

$$e^{-2t}\frac{2}{x^2} - e^{-2t} y^2 = \frac{2}{e^{2t}x^2} - e^{-2t} y^2$$ [This preserves the equation]

Shows symmetry

$$\Gamma_t(x,y) = (e^t x, e^{-t} y) = (x,y)$$

$$e^t x = x \quad \& \quad e^{-t} y = y$$

For these to hold for all t we must have $x=0$ or $y=0$

Since $x=0$ leads to singularities, $y=0$

Zero solution $y(x)=0$ only solution remain unchanged by transformation

17 $\Gamma_t(x,y) = (x+yt, y)$

$\tilde{x} = x + ty \qquad \tilde{y} = y$

differentiate both sides with x

$$d\tilde{x} = \frac{d}{dx}(x+ty)\cdot dx = \left(1 + t\frac{dy}{dx}\right)dx$$

$$d\tilde{y} = \frac{d}{dx}(y)\cdot dx = \frac{dy}{dx}dx$$

Transformed derivative is

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{dy/dx}{1+t(dy/dx)}$$

$$\Gamma_t(x, y, dy/dx) = \left(x+ty, y, \frac{dy/dx}{1+t(dy/dx)}\right)$$

18 $\Gamma_t(x,y) = (e^t x, e^{-t} y)$

① we differentiate

$d\tilde{x} = d(e^t x) = e^t dx$

$d\tilde{y} = d(e^t y) = e^t dy$

$\frac{d\tilde{y}}{d\tilde{x}} = \frac{e^t dy}{e^t dx} = \frac{dy}{dx}$

So diff equation

$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, substituting transformed

$\frac{d\tilde{y}}{d\tilde{x}} = F\left(\frac{e^t y}{e^t x}\right) = F\left(\frac{y}{x}\right)$

② Show $\Gamma_t(x,y) = (e^t x, e^{-t} y)$ is not symmetrical

differentiate

$d\hat{x} = d(e^t x) = e^t dx$

$d\tilde{y} = d(e^{-t} y) \equiv e^{-t} dy - e^{-t} y dt$

$\frac{d\hat{y}}{dx} = \frac{e^{-t} dy - e^{-t} y dt}{e^t dx}$ dividing by dx

$\frac{d\tilde{y}}{dx} = F\left(\frac{y}{x}\right) = F\left(\frac{e^{-t} y}{e^t x}\right) = F\left(e^{-2t} \frac{y}{x}\right)$

$e^{-2t} \frac{dy}{dx} = F\left(e^{-2t} \frac{y}{x}\right)$

This would mean $F(z)$ would have to satisfy

$$F(z) = e^{-2t} F(e^{-2t} z)$$

This is only true if $F(z) = 0$, which would trivialize equation.

So $\Gamma_t(x,y) = (e^{t}x, e^{-t}y)$ is not a symmetry unless $F(z) = 0$

19

$$y' = \sin^2(y-x)$$

(1) Check symmetry

$d\tilde{x} = d(x+t) = dx$, $d\tilde{y} = d(y+t) = dy$

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{dy}{dx}$$

Since it was given

$$\frac{dy}{dx} = \sin^2(y-x)$$ Substitue the transformed Variables

$$\frac{d\tilde{y}}{d\tilde{x}} = \sin^2((y+t)-(x+t)) = \sin^2(y-x)$$

Proving symmetry

(2) Find canonical coordinates

$r = y-x$

$s = x$

$r = y-x$, $s = x$ are new variables

③ Transform the differential equation

$r = y - x$ with respect to x:

$$\frac{dr}{dx} = \frac{dy}{dx} - \frac{dx}{dx} = \frac{dy}{dx} - 1$$

Using original

$$\frac{dy}{dx} = \sin^2(y-x)$$, substitute r

$$\frac{dr}{dx} = \sin^2(r) - 1$$

$$\frac{dr}{dx} = -\cos^2(r)$$

④ Solve simplified ODE

$$\frac{dr}{dx} = -\cos^2(r)$$

$$\int \frac{dr}{\cos^2(r)} = \int -dx$$

$$\int \sec^2(r)\,dr = \int -dx$$

$$\tan(r) = -x + c$$

Thus, solving for r

$r = \tan^{-1}(x+c)$

replace $r = y - x$

$y - x = \tan^{-1}(-x+c)$

Thus,
$y = x + \tan^{-1}(-x+c)$

(5) special cases

if $dr/dx = 0$
$-\cos^2(r) = 0 \Rightarrow \cos^2(r) = 0$ when $r = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

So,
Symmetric solution are:

$$y - x = \frac{\pi}{2} + k\pi \Rightarrow y = x + \frac{\pi}{2} + k\pi$$