Math 325, Module 3 (Probability and Black Scholes Formula)
Homework
Prof. Min Kang

Instructions: Please submit your **homework** by **11 pm on Sunday, March 16, 2025** on the **Course Moodle Assignment under Module 3** electronically. Your work should be uploaded as one pdf file before the deadline. Any problem with uploading, legibility or format may result in losing some partial credits.

Problem 1. Suppose that a stock price starts at \$800 at time 0. At time 1 (one day later), the stock will either be worth \$850 or \$650. If the stock is worth \$850 at time 1, then the stock will either be worth \$1000 or \$800 at time 2. And If the stock is worth \$650 at time 1, then the stock will either be worth \$750 or \$200 at time 2. Consider a European call option with strike price \$400 and expiry time 2.

Suppose the stock price at time 1 is \$650 and you plan on offering the above European call option to the people in financial market. What is the fair price that you should charge based on "Fair Market (No-Arbitrage)" principle?

Problem 2. Prove that the symmetric simple random walk, under the scaling (the space scale by $\frac{1}{n}$ and the time scale by $\frac{1}{n^2}$), converges to the Brownian motion at time t in distribution sense, by first finding the characteristic functions of both, then taking the limit as n goes to infinity. In other words, prove that $\lim_{n\to\infty} \phi_{\frac{1}{n}S_{n^2t}}(\theta) = \phi_{B_t}(\theta)$. Recall that the $S_m = \sum_{k=1}^{m} X_k$ where X_k are independent and identically distributed random variables taking the values 1 and -1 only, with equal probability. Also recall that the characteristic function for a random variable X is defined as $\phi_X(\theta) = E\left(e^{i\theta X}\right)$.

Problem 3. Consider the symmetric simple random walk described above.
(a) If we take a different scaling for the simple random walk, namely, the space scale by $\frac{1}{n}$ and the time scale by $\frac{1}{n}$, then what happens to the characteristic function of the scaled symmetric simple random walk, $\phi_{\frac{1}{n}S_{nt}}(\theta)$ as n goes to infinity? What does it say about the convergence of $\frac{1}{n}S_{nt}$ in distribution sense?
(b) Secondly, what happens if you keep the same space scale $\frac{1}{n}$ but different time scale $\frac{1}{n^3}$, in other words, what happens to $\phi_{\frac{1}{n}S_{n^3t}}(\theta)$ as n goes to infinity? What does it say about the convergence of $\frac{1}{n}S_{n^3t}$ in distribution sense?

Problem 4. Suppose that the stock price X_t is the solution to $dX_t = \mu X_t dt + \sigma X_t dB_t$ where X_0 is the initial price of the stock. Suppose that you have an European call option (associated with this stock) with a strike price K and a maturity (expiry) time T. Derive the Black-Scholes formula computing the fair price of this European call option. (We did this in class. I want to make sure that you can derive it on your own. The formula should be expressed in terms of the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$)

Problem 5. Suppose that on February 23, 1998, *Wall Street Journal* listed the following prices for European call options maturing on July 23, 1998 on Apple stock.

strike price	75	80	85
option price	11	8.125	5.5

On February 23, 1998, Apple stock was trading at \$81.625 with the annual interest rate of 4%. Adopting the conventional rule-of-thumb in the financial market of taking volatility constant $\sigma = 30\%$ and using Black-Scholes formula, calculate the fair price of the European call option with strike price \$80 maturing on July 23, 1998. Then decide whether you should buy this option or not. (For the function values of Φ, look at the file, 'normal-table.pdf' provided in the same assignment site. Also use 'year' as the time unit, which means the expiry time of your option is $T = 5/12$.)