1.

(a) $p(x)\log_2\left(\frac{1}{p(x)}\right)$

$= p(x)\left(\log_2 1 - \log_2 p(x)\right)$

$= p(x)\left(-\log_2 p(x)\right)$

since $p(x) \in [0,1]$

then $-\log_2 p(x) \in [0, +\infty)$

so the multiplication of $p(x)\cdot(-\log_2 p(x))$ is non-negative

since $H(X) = \sum_x p(x)\log_2\left(\frac{1}{p(x)}\right)$

which is the sum of non-negative numbers.

so $H(X)$ is non-negative.

(b). $KL(p\|q) = \sum_x p(x) \log_2 \frac{p(x)}{q(x)} = -\sum p(x) \log_2 \frac{q(x)}{p(x)}$

$= \sum p(x) \left[-\log_2 \frac{q(x)}{p(x)} \right]$

$= -E(\log \frac{q}{p})$

since $\log_2 x$ is concave.

$-\log_2 x$ is convex

$\geq -\log(E[\frac{q}{p}])$

$= -\log(\sum p(x) \frac{q(x)}{p(x)})$

$= -\log \sum q(x)$

$= -\log 1 = 0$,

so $KL(p\|q)$ is non-negative.

1.(c) $I(Y;X) = H(Y) - H(Y|X)$

$$= \sum_x p(x) \log \frac{1}{p(x)} - \sum_{x,y} p(y) \log H(X|Y=y)$$

$$= \sum_{x,y} p(x,y) \log \frac{1}{p(x)} - \sum_{x,y} p(x,y) \log \frac{1}{p(x|y)}$$

$$= \sum p(x,y) \log \frac{p(x|y)}{p(x)}$$

$$= \sum p(x,y) \log \frac{p(y)\, p(x|y)}{p(x)\, p(y)}$$

$$= \sum p(x,y) \log \frac{p(x,y)}{p(x)\, p(y)}.$$

$$= KL\left(p(x,y) \,\|\, p(x)\, p(y)\right)$$

2. $L(\bar{h}(x), t) = \frac{1}{2}(\bar{h}(x) - t)^2 = \frac{1}{2}(\frac{1}{m}\Sigma h_i(x) - t)^2$

$= \frac{1}{2}(E(h_i(x)) - t)^2$

since $y = x^2$ is convex.

so

$\frac{1}{2}(E(h_i(x)) - t)^2 \leq E(\frac{1}{2}(h_i(x) - t)^2)$

$= \frac{1}{m}\Sigma L(h_i(x), t)$

so $L(\bar{h}(x), t) \leq \frac{1}{m}\sum_{i=1}^{m} L(h_i(x), t)$

3. $err' = \frac{\sum_{i\in E} W_i'}{\sum_{i=1}^{N} W_i'}$

$= \frac{\sum_{i\in E} W_i'}{\sum_{i\in E} W_i' + \sum_{i\in E^c} W_i'}$

$= \frac{1}{1+\frac{\sum_{i\in E^c} W_i'}{\sum_{i\in E} W_i'}}$

$\sum_{i\in E} W_i' = \sum_E W_i \exp\{-\alpha_t t^{(i)} h_t(x^{(i)})\}$

$t^{(i)} h_t(x^{(i)}) = -1$ since these are ~~err~~ in set E.

$= \sum_E W_i \exp(\alpha_t)$

$= \sum_E W_i \exp\{\frac{1}{2}\log\frac{1-err_t}{err_t}\}$

$= \sum_E W_i \left(\frac{1-err}{err}\right)^{\frac{1}{2}}$

$\sum_{i\in E^c} W_i' = \sum_{E^c} W_i \exp\{-\alpha_t t^{(i)} h_t(x^{(i)})\}$

$t^{(i)} h_t(x^{i}) = 1$ since these are in set E^c

$= \sum_{E^c} W_i \exp(-\alpha_t)$

$= \sum_{E^c} W_i \left(\frac{1-err}{err}\right)^{-\frac{1}{2}}$

$\frac{\sum_{i\in E^c} W_i'}{\sum_{i\in E} W_i'} = \frac{\sum_{E^c} W_i \left(\frac{1-err}{err}\right)^{-\frac{1}{2}}}{\sum_{i\in E} W_i \left(\frac{1-err}{err}\right)^{\frac{1}{2}}} = \left(\frac{1-err}{err}\right)^{-1} \cdot \frac{\sum_{E^c} W_i}{\sum_E W_i}$

$= \frac{err}{1-err}\cdot\frac{\sum_{E^c} W_i}{\sum_E W_i} = \frac{\left(\frac{\sum_{i\in E} W_i}{\sum_{i=1}^N W_i}\right)\left(\sum_{i=1}^N W_i\right)}{\left(1-\frac{\sum_{i\in E} W_i}{\sum_{i=1}^N W_i}\right)\left(\sum_{i=1}^N W_i\right)}\cdot\frac{\sum_{E^c} W_i}{\sum_E W_i} = \frac{\sum_E W_i}{\sum_{E^c} W_i}\cdot\frac{\sum_{E^c} W_i}{\sum_E W_i} = 1$

Therefore $err' = \frac{1}{1+1} = \frac{1}{2}$.