University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATB41 Fall 2023

Review Assignment

You may wish to work on this assignment prior to your tutorial during the week of Sept. 11th. You may ask questions about this assignment in that tutorial.

This assignment is based on prerequisite A30/A36 or A31/A37 and A23/A22 material. It is **not** being submitted. This assignment contains questions that provide a sample of what you are expected to know from the prerequisite courses. You may seek help with this material from any TA or instructor office hours should you need help refreshing your memory!

STUDY: MATA30/A36 or A31/A37 and A23/A22 (see corresponding portions of your pre-requisite textbook) where appropriate. Our B41 textbook Sections 12.1-12.4 also provide a partial linear algebra review.

EXERCISES:

1. Prove the following vector properties :
   (a) $\forall\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^{\mathbf{3}}, \forall \mathbf{r} \in \mathbb{R},\ \mathbf{r}(\mathbf{u}+\mathbf{v}) = \mathbf{ru}+\mathbf{rv}$
   (b) $\forall\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^{\mathbf{3}}$, $\forall \theta \in [\mathbf{0}, \pi]$ such that θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v} = \parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel \cos\theta$ *Hint : the law of cosines from trigonometry may be helpful.*
   (c) Triangle Inequality $\forall\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^{\mathbf{3}}, \parallel \mathbf{u}+\mathbf{v} \parallel \leq \parallel \mathbf{u} \parallel + \parallel \mathbf{v} \parallel$
2. Let $\mathbf{v} = \langle \mathbf{1}, -\mathbf{2}, \mathbf{1} \rangle$ and $\mathbf{w} = \langle \mathbf{0}, \mathbf{1}, -\mathbf{3} \rangle$ be vectors in $\mathbb{R}^3$.
   (a) Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for $\mathbf{v}$ and $\mathbf{w}$.

(c) Find all unit vectors in $\mathbb{R}^3$ which are orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

(d) Find the projection of

- $\mathbf{v}$ onto $\mathbf{w}$
- $\mathbf{w}$ onto $\mathbf{v}$

3. Let $\mathbf{u},\ \mathbf{v}, \mathbf{w} \in \mathbb{R}^{\mathbf{3}}$.

   (a) Suppose that $\mathbf{u} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v}$ for all $\mathbf{v}$. Prove that $\mathbf{u} = \mathbf{v}$.

   (b) Suppose that $\mathbf{u} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Is it necessarily true that $\mathbf{u} = \mathbf{v}$? Justify your answer.

4. Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 4 \end{pmatrix}$, and $C = \begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$.

   (a) Compute $det(A),\ det(B), \det(C),\ det(AB)$, and $det(A+B)$.

   (b) Verify that A and C are inverse matrices of each other and use this fact to:

   - Solve the simultaneous system of equations
   $$\begin{aligned} 2x + y - z &= 1 \\ 2y + z &= 0 \\ 5x + 2y - 3z &= 1 \end{aligned}$$
   - Show that the only $\mathbf{v} \in \mathbb{R}^{\mathbf{3}}$ such that $A\mathbf{v} = \mathbf{0}$ is the zero vector.

5. Let f be a real-valued function defined on $\mathbb{R}$. Let $a \in \mathbb{R}$. Provide a complete and accurate $\delta - \epsilon$ definition of what it means for $\lim_{x \to a} f(x)$ to exist.

6. Evaluate the following limits or explain why they fail to exist.

   (a) $\displaystyle \lim_{x \to 4} \frac{2x\sqrt{x} + x - 8\sqrt{x} - 4}{x + \sqrt{x} - 6}$

   (b) $\displaystyle \lim_{x \to 0} \frac{|3x - 1| - |3x + 1|}{x}$

   (c) $\displaystyle \lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

(d) $\lim_{x \to \infty} \left(\sqrt{x^2 + 2x} - \sqrt{x^2 - 2x} \right)$

7. Let $a \in \mathbb{R}$. Use only the definition of the derivative to compute $f'(a)$ where $f(x) = x^2 - 3x + 1$.

8. Use only the Riemann definition of the definite integral to compute $\int_0^2 f(x)dx$ where $f(x) = x^2 - 3x + 1$.

9. Let $F(x) = \int_{2^x}^{3+x^3} \sin(t^2)\ dt$. Find $F'(x)$. Make sure to fully and appropriately justify you work.

10. Evaluate each of the following :

    (a) $\int \frac{\sqrt{\ln(x)}}{x}\ dx$

    (b) $\int \sin^3(x) \cos^2(x)\ dx$

    (c) $\int \frac{du}{(u-3)(u+2)}$

    (d) $\int xe^x\ dx$

    (e) $\int \frac{3}{\sqrt{2x - x^2}}\ dx$

11. Do the following improper integrals converge or diverge? Make sure to fully justify your answer.

    (a) $\int_e^\infty \frac{\ln(x)}{x}\ dx$

    (b) $\int_{-3}^{3} \frac{y\ dy}{\sqrt{9 - y^2}}$

    (c) $\int_{-2}^{-1} \frac{\sqrt{x^2 - 1}}{x}\ dx$

It always seems impossible until it's done. — *Nelson Mandela*