CSCB63 Assignment 1 – AVL Trees

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Due: February 9th, 2024

Question 1

a)
Let f, g, h : $\mathbb{N} \to \mathbb{R}^+$. Suppose $f \in \mathcal{O}(g)$, $g \in \mathcal{O}(h)$.

$\because$ f(n) $\leq$ c$_1$ $\cdot$ g(n), $g(n) \leq$ c$_2$ $\cdot$ $h(n)$, for some c$_1$, c$_2$ $\in \mathbb{R}^+$. [def. of $\mathcal{O}$]
$\therefore$ f(n) $\leq$ c$_1$ $\cdot$ $g(n) \leq c_1 \cdot$ c$_2$ $\cdot$ $h(n)$. [combining inequalities]

Let c $= c_1 \cdot c_2$. $\implies$ f(n) $\leq$ c $\cdot$ h(n).
Hence, $f \in \mathcal{O}(h)$, as wanted. $\square$

b)
Let f_1, f_2, g_1, g_2 : $\mathbb{N} \to \mathbb{R}^+$. Suppose $f_1 \in \mathcal{O}(g_1)$, $f_2 \in \mathcal{O}(g_2)$.

$\because$ $f_1(n) \leq$ c$_1$ $\cdot$ $g(n)$, $f_2(n) \leq$ c$_2$ $\cdot$ $g(n)$, for some c$_1$, c$_2$ $\in \mathbb{R}^+$. [def. of $\mathcal{O}$]
$\therefore$ $f(n) =$ f$_1(n)$ $\cdot$ f$_2(n) \leq c_1 \cdot$ c$_2$ $\cdot$ $g_1(n) \cdot g_2(n) = c_1 \cdot$ c$_2$ $\cdot$ $g(n)$. [combining inequalities]

Let c $= c_1 \cdot c_2$. $\implies$ f(n) $\leq$ c $\cdot$ g(n).
Hence, $f \in \mathcal{O}(g)$, as wanted. $\square$

c)
BWOC, suppose $2^{2n} \in \mathcal{O}(2^n)$.
Let n = $\lceil log_2 c \rceil$ + 1 $\in \mathbb{N}$.

$\therefore$ c $\cdot$ $2^n \geq 2^{2n} \Leftrightarrow \log_2 c$ + n $\geq$ 2n $\Leftrightarrow \log_2 c \geq$ n = $\lceil log_2 c \rceil$ + 1. [Contradiction!]

Hence, $2^{2n} \notin \mathcal{O}(2^n)$, as wanted. $\square$

d)
Let $f_1,\ f_2,\ g : \mathbb{N} \to \mathbb{R}^+$. Suppose $f_1,\ f_2 \in \mathcal{O}(g)$.

$\because f_1(n) \leq \mathrm{c}_1 \cdot g(n)$, $f_2(n) \leq \mathrm{c}_2 \cdot g(n)$, for some $\mathrm{c}_1,\ \mathrm{c}_2 \in \mathbb{R}^+$. [def. of $\mathcal{O}$]

WLOG, suppose $f_{max} = \max(f_1,\ f_2) = f_1$.

$\therefore f_{max}(n) = \mathrm{f}_1(n) \leq c_1 \cdot g(n)$. [by assumption]

Let c = c_1. $\implies f_{max}(\mathrm{n}) \leq \mathrm{c} \cdot g(\mathrm{n})$.
Hence, $f \in \mathcal{O}(g)$, as wanted. $\square$

Question 2