CSCB63 Assignment 1 – AVL Trees

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Due: February 9th, 2024

Question 1

a)
Let f, g, $h : \mathbb{N} \to \mathbb{R}^+$. Suppose $f \in \mathcal{O}(g)$, $g \in \mathcal{O}(h)$.

$\because$ f(n) $\leq$ $c_1 \cdot g$(n), $g(n) \leq c_2 \cdot h(n)$, for some c_1, $c_2 \in \mathbb{R}^+$. [def. of $\mathcal{O}$]
$\therefore$ f(n) $\leq c_1 \cdot g(n) \leq c_1 \cdot c_2 \cdot h(n)$. [combining inequalities]

Let c $= c_1 \cdot c_2$. $\implies$ f(n) $\leq$ c $\cdot$ h(n).
Hence, $f \in \mathcal{O}(h)$, as wanted. $\square$

b)
Let f_1, f_2, g_1, $g_2 : \mathbb{N} \to \mathbb{R}^+$. Suppose $f_1 \in \mathcal{O}(g_1)$, $f_2 \in \mathcal{O}(g_2)$.

$\because$ $f_1(n) \leq c_1 \cdot g(n)$, $f_2(n) \leq c_2 \cdot g(n)$, for some c_1, $c_2 \in \mathbb{R}^+$. [def. of $\mathcal{O}$]
$\therefore$ $f(n) = f_1(n) \cdot f_2(n) \leq c_1 \cdot c_2 \cdot g_1(n) \cdot g_2(n) = c_1 \cdot c_2 \cdot g(n)$. [combining inequalities]

Let c $= c_1 \cdot c_2$. $\implies$ f(n) $\leq$ c $\cdot$ g(n).
Hence, $f \in \mathcal{O}(g)$, as wanted. $\square$

c)
BWOC, suppose $2^{2n} \in \mathcal{O}(2^n)$.
Let n $= \lceil log_2 c \rceil + 1 \in \mathbb{N}$.

$\therefore$ c $\cdot$ $2^n \geq 2^{2n} \Leftrightarrow \log_2 c$ + n $\geq$ 2n $\Leftrightarrow \log_2 c \geq$ n $= \lceil log_2 c \rceil + 1$. [Contradiction!]

Hence, $2^{2n} \notin \mathcal{O}(2^n)$, as wanted. $\square$

d)
Let $f_1,\ f_2,\ g:\mathbb{N}\to\mathbb{R}^{+}$. Suppose $f_1,\ f_2\in\mathcal{O}(g)$.

$\because f_1(n)\leq \mathrm{c}_1\cdot g(n)$, $f_2(n)\leq \mathrm{c}_2\cdot g(n)$, for some $\mathrm{c}_1,\ \mathrm{c}_2\in\mathbb{R}^{+}$. [def. of $\mathcal{O}$]

WLOG, suppose $f_{max}=\max(f_1,\ f_2)=f_1$.

$\therefore f_{max}(n)=\mathrm{f}_1(n)\leq c_1\cdot g(n)$. [by assumption]

Let c = c_1. $\implies f_{max}(\mathrm{n})\leq \mathrm{c}\cdot g(\mathrm{n})$.
Hence, $f\in\mathcal{O}(g)$, as wanted. $\square$

Question 2

Inserting (9)

Inserting (10)