CS 33

Data Representation (Part 3)

Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook “Computer Systems: A Programmer’s Perspective.” 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O’Hallaron in Fall 2010. These slides are indicated “Supplied by CMU” in the notes section of the slides.

Fractional binary numbers

- What is 1011.101_2?

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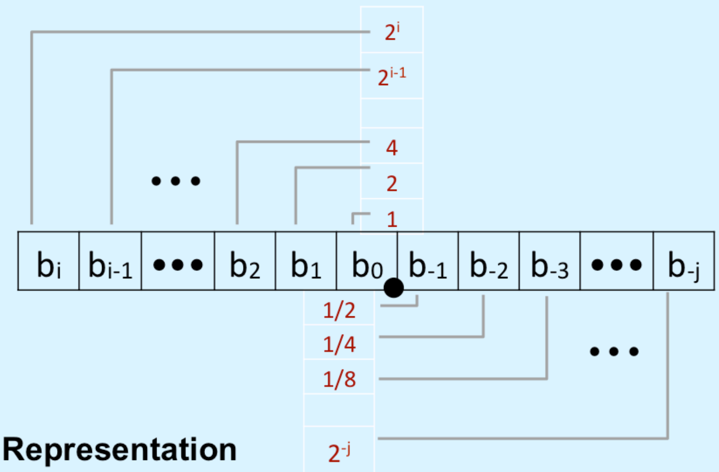
Fractional Binary Numbers



- **Representation**
  - **bits to right of "binary point" represent fractional powers of 2**
  - **represents rational number:** $\sum_{k=-j}^{i} b_k \times 2^k$

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Representable Numbers

- **Limitation #1**
  - **can exactly represent only numbers of the form $n/2^k$**
    - » **other rational numbers have repeating bit representations**
  - **value** | **representation**
    - » **1/3** `0.0101010101[01]…`$_2$
    - » **1/5** `0.001100110011[0011]…`$_2$
    - » **1/10** `0.0001100110011[0011]…`$_2$

- **Limitation #2**
  - **just one setting of decimal point within the *w* bits**
    - » **limited range of numbers (very small values? very large?)**

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IEEE Floating Point

- IEEE Standard 754
  - established in 1985 as uniform standard for floating point arithmetic
    - » before that, many idiosyncratic formats
  - supported by all major CPUs
- Driven by numerical concerns
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - » numerical analysts predominated over hardware designers in defining standard

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Floating-Point Representation

- **Numerical Form:**

$$(-1)^s \; M \;\; 2^E$$

  - sign bit **s** determines whether number is negative or positive
  - significand **M** normally a fractional value in range [1.0,2.0)
  - exponent **E** weights value by power of two
- **Encoding**
  - MSB `s` is sign bit **s**
  - `exp` field encodes **E** (but is not equal to E)
  - `frac` field encodes **M** (but is not equal to M)

`s`	`exp`	`frac`

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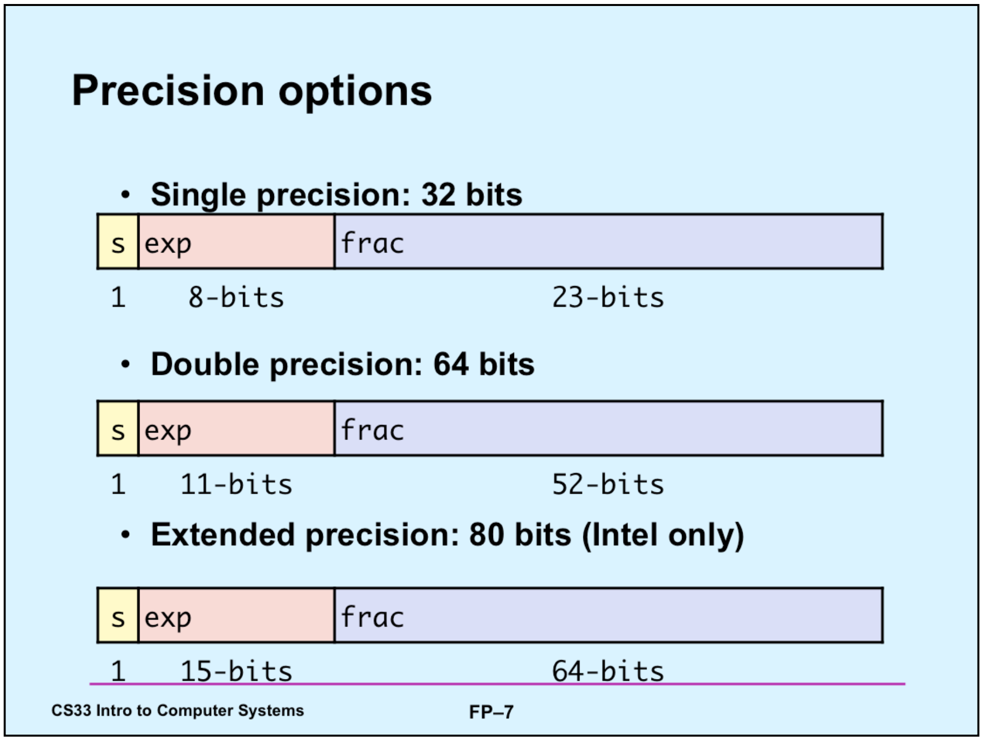

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On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.

"Normalized" Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx\ldots x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for "free"

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Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.1101101101101`$_2$

  `frac` = `11011011011010000000000`$_2$

- **Exponent**

  E = **13**

  bias = **127**

  exp = **140** = `10001100`$_2$

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

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Denormalized Values

- Condition: exp = 000…0
- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0:
  $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = 000…0, `frac` = 000…0
    - » represents zero value
    - » note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - » numbers closest to 0.0
    - » equispaced

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Special Values

- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

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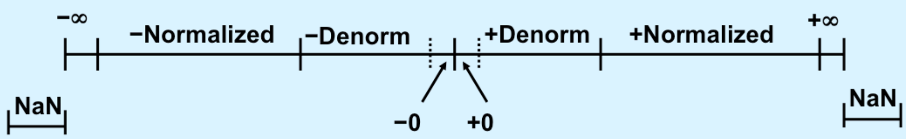
Visualization: Floating-Point Encodings



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Tiny Floating-Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

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Dynamic Range (Positive Only)

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

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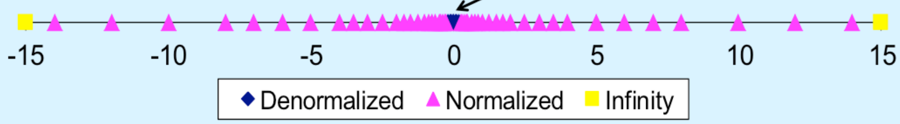
Distribution of Values

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is $2^{3-1}-1 = 3$**

s	exp	frac
1	3-bits	2-bits

- **Notice how the distribution gets denser toward zero.**

8 values



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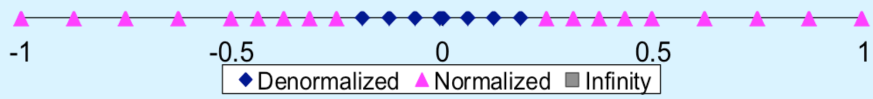
Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**



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Quiz 1

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

s	exp	frac
1	3-bits	2-bits

What number is represented by 0 011 10?

a) 12
b) 1.5
c) .5
d) none of the above

Floating-Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$
- $x \times_f y = \text{Round}(x \times y)$
- **Basic idea**
  - first **compute exact result**
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly **round to fit into** `frac`

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Rounding

- **Rounding modes (illustrated with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest even (default)	$1	$2	$2	$2	–$2

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Floating-Point Multiplication

- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- **Exact result:** $(-1)^{s}$ M 2^{E}
  - sign s: s1 ^ s2
  - significand M: M1 x M2
  - exponent E: E1 + E2
- **Fixing**
  - if M ≥ 2, shift M right, increment E
  - if E out of range, overflow (or underflow)
  - round M to fit `frac` precision
- **Implementation**
  - biggest chore is multiplying significands

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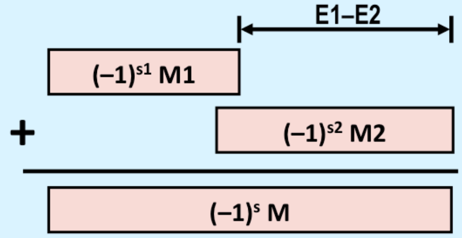
Note that to compute E, one must first convert exp_1 and exp_2 to E_1 and E_2, then add them them together and check for underflow or overflow (corresponding to -∞ and +∞), and then convert to exp.

Floating-Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - –**assume E1 > E2**
- **Exact result:** $(-1)^{s}\ M\ \ 2^{E}$
  - –sign s, significand M:
    - » result of signed align & add
  - –exponent E: E1



- **Fixing**
  - –if M ≥ 2, shift M right, increment E
  - –if M < 1, shift M left k positions, decrement E by k
  - –overflow if E out of range
  - –round M to fit `frac` precision

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Floating Point in C

- **C guarantees two levels**
  - `float` **single precision**
  - `double` **double precision**
- **Conversions/casting**
  - **casting between `int`, `float`, and `double` changes bit representation**
  - `double`/`float` → `int`
    - » **truncates fractional part**
    - » **like rounding toward zero**
    - » **not defined when out of range or NaN: generally sets to TMin**
  - `int` → `double`
    - » **exact conversion, as long as `int` has ≤ 53-bit word size**
  - `int` → `float`
    - » **will round according to rounding mode**

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Quiz 2

Suppose *f*, declared to be a `float`, is assigned the largest possible floating-point positive value (other than +∞). What is the value of *g = f+1.0*?

a) f

b) +∞

c) NAN

d) 0

Float is not Rational …

- **Floating addition**
  - commutative: a $+^f$ b = b $+^f$ a
    - » yes!
  - associative: a $+^f$ (b $+^f$ c) = (a $+^f$ b) $+^f$ c
    - » no!
      - 2 $+^f$ (1e10 $+^f$ -1e10) = 2
      - (2 $+^f$ 1e10) $+^f$ -1e10 = 0

Note that the floating-point numbers in this and the next two slides are expressed in base 10, not base 2.

Float is not Rational …

- Multiplication
  - commutative: $a *^f b = b *^f a$
    - » yes!
  - associative: $a *^f (b *^f c) = (a *^f b) *^f c$
    - » no!
      - $1e20 *^f (1e20 *^f 1e\text{-}20) = 1e20$
      - $(1e20 *^f 1e20) *^f 1e\text{-}20 = +\infty$

Float is not Rational …

- More …
  - multiplication distributes over addition:
    $a *^f (b +^f c) = (a *^f b) +^f (a *^f c)$
    - » no!
    - » $1e20 *^f (1e20 +^f -1e20) = 0$
    - » $(1e20 *^f 1e20) +^f (1e20 *^f -1e20) = NaN$
  - loss of significance:
    x=y+1
    z=2/(x-y)
    z==2?
    - » not necessarily!
      - consider y = 1e20