CS 33

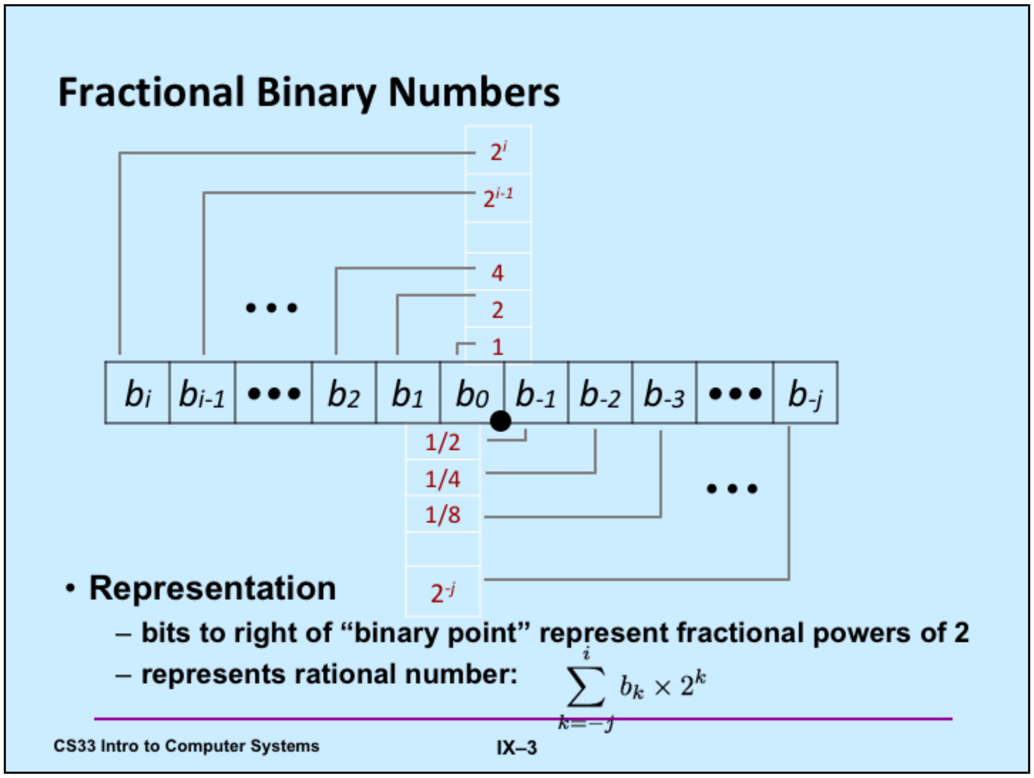
Data Representation (Part 3)

Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.

Fractional binary numbers

- What is 1011.101_2?

Supplied by CMU.



Supplied by CMU.

Representable Numbers

- **Limitation #1**
  - can exactly represent only numbers of the form $n/2^k$
    - other rational numbers have repeating bit representations
  - value — representation
    - 1/3 — `0.0101010101[01]…`$_2$
    - 1/5 — `0.001100110011[0011]…`$_2$
    - 1/10 — `0.0001100110011[0011]…`$_2$

- **Limitation #2**
  - just one setting of decimal point within the *w* bits
    - limited range of numbers (very small values? very large?)

Supplied by CMU.

IEEE Floating Point

- IEEE Standard 754
  - established in 1985 as uniform standard for floating point arithmetic
    - before that, many idiosyncratic formats
  - supported by all major CPUs
- Driven by numerical concerns
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - numerical analysts predominated over hardware designers in defining standard

Supplied by CMU.

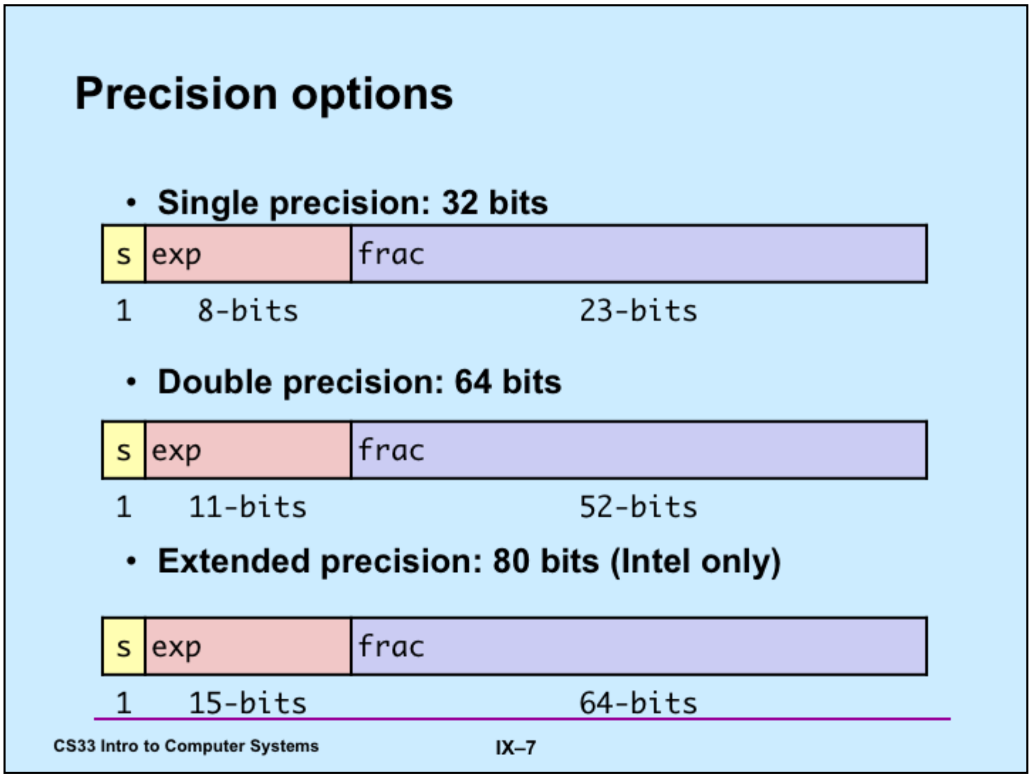
Floating-Point Representation

- **Numerical Form:**

$$(-1)^s\ M\ \ 2^E$$

  - sign bit s determines whether number is negative or positive
  - significand M normally a fractional value in range [1.0,2.0)
  - exponent E weights value by power of two
- **Encoding**
  - MSB `s` is sign bit s
  - `exp` field encodes E (but is not equal to E)
  - `frac` field encodes M (but is not equal to M)

Supplied by CMU.



Supplied by CMU.

On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.

“Normalized” Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as *biased* value: $E = Exp - Bias$
  - *exp*: unsigned value exp
  - *bias* = $2^{k-1} - 1$, where *k* is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: $M = 1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 ($M = 2.0 - \varepsilon$)
  - get extra leading bit for “free”

Supplied by CMU.

Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.`$\underline{1101101101101}_2$

  `frac` = $\underline{1101101101101}0000000000_2$

- **Exponent**

  E = **13**

  bias = **127**

  exp = **140** = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Supplied by CMU.

Denormalized Values

- Condition: exp = `000…0`
- Exponent value: $E = -Bias + 1$ (instead of $E = 0 - Bias$)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = `000…0`, `frac` = `000…0`
    - » represents zero value
    - » note distinct values: +0 and –0 (why?)
  - `exp` = `000…0`, `frac` ≠ `000…0`
    - » numbers closest to 0.0
    - » equispaced

Supplied by CMU.

Special Values

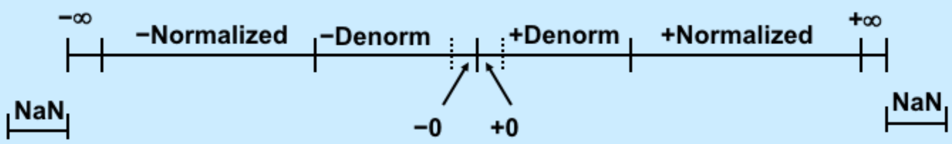
- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

Supplied by CMU.

Visualization: Floating-Point Encodings



Supplied by CMU.

Tiny Floating-Point Example

s	exp	frac
1	4-bits	3-bits

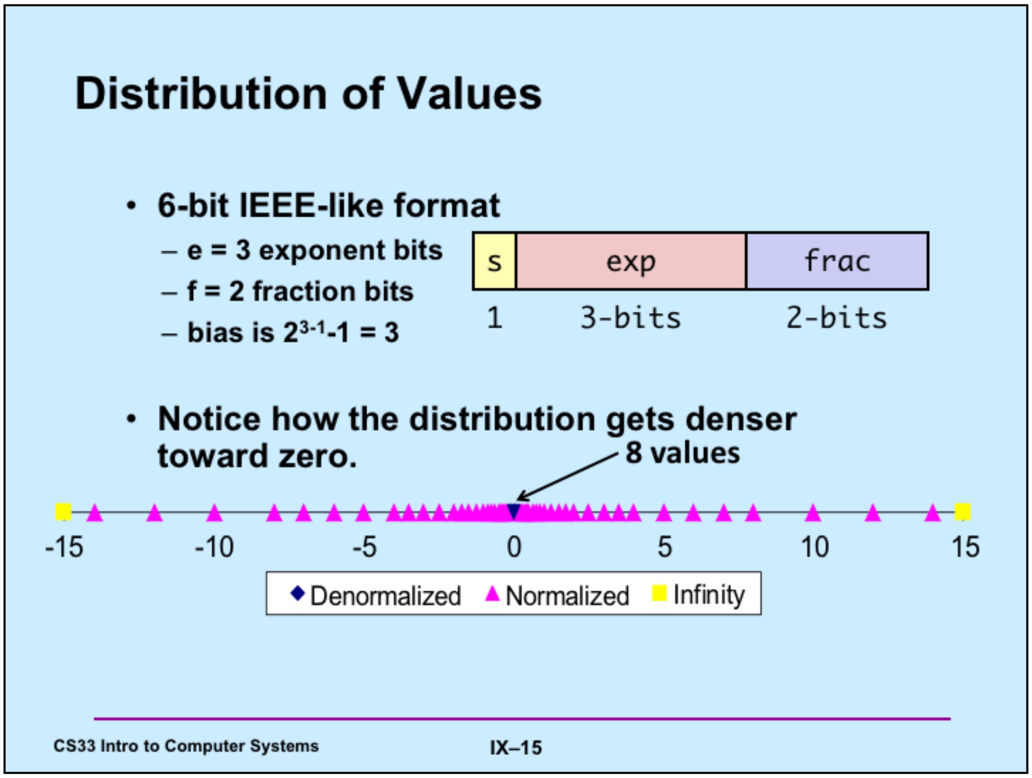
- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Supplied by CMU.

Dynamic Range (Positive Only)

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Supplied by CMU.

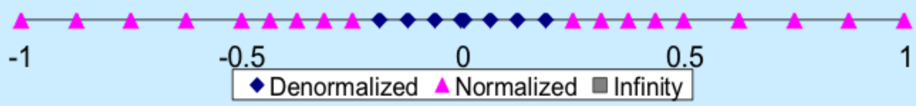


Supplied by CMU.

Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

s	exp	frac
1	3-bits	2-bits



Supplied by CMU.

Quiz 1

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

s	exp	frac
1	3-bits	2-bits

What number is represented by 0 011 10?

a) 12
b) 1.5
c) .5
d) none of the above

Floating-Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x ×f y = Round(x × y)`
- Basic idea
  - first **compute exact result**
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly **round to fit into** `frac`

Supplied by CMU.

Rounding

- **Rounding modes (illustrated with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest integer	$1	$2	?	?	?
nearest even (default)	$1	$2	$2	$2	–$2

Supplied by CMU.

Creating a Floating Point Number

s	exp	frac
1	4-bits	3-bits

- **Steps**
  - normalize to have leading 1
  - round to fit within fraction
  - postnormalize to deal with effects of rounding

- **Case study**
  - convert 8-bit unsigned numbers to tiny floating-point format

example numbers

128	10000000
13	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Supplied by CMU.

Normalize

s	exp	frac
1	4-bits	3-bits

- **Requirement**
  - set binary point so that numbers of form 1.xxxxx
  - adjust all to have leading one
    - » decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
13	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Supplied by CMU.

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round-up conditions**
  - round = 1, sticky = 1 ⇒ > 0.5
  - guard = 1, round = 1, sticky = 0 ⇒ round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
13	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Supplied by CMU.

Postnormalize

- **Issue**
  - rounding may have caused overflow
  - handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
13	1.101	3		13
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	$1.000*2^6$	64

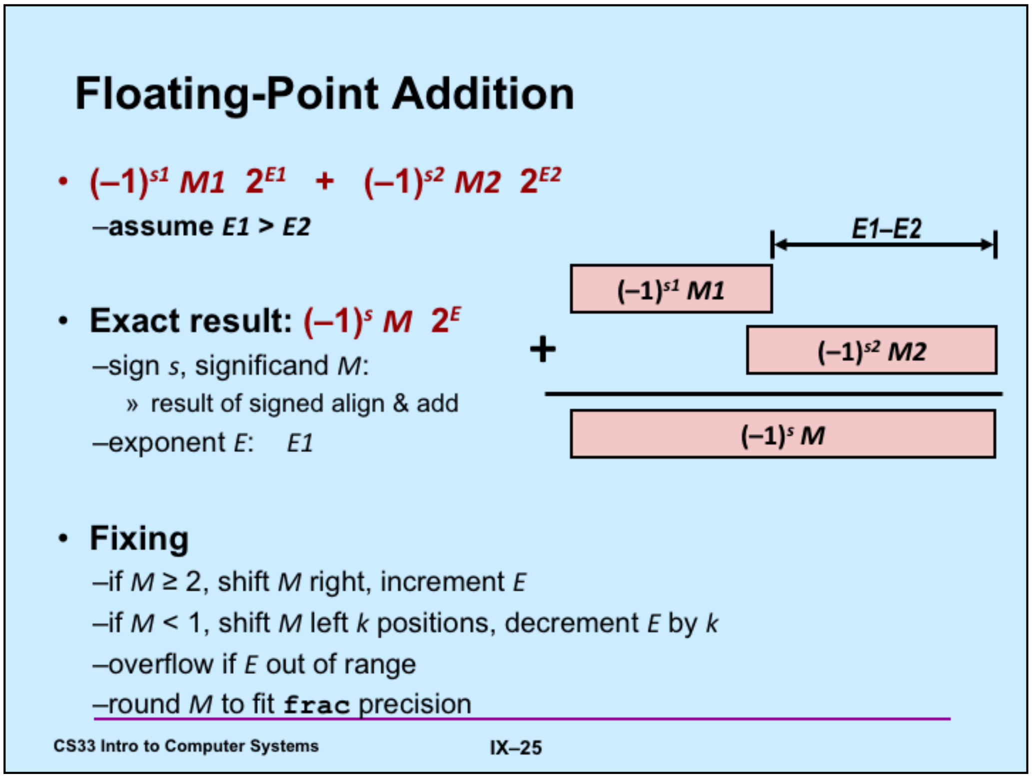
Supplied by CMU.

Floating-Point Multiplication

- $(-1)^{s1}\, M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\, M2\ \ 2^{E2}$
- **Exact result:** $(-1)^{s}\, M\ \ 2^{E}$
  - sign s: $s1$ ^ $s2$
  - significand M: $M1 \times M2$
  - exponent E: $E1 + E2$
- **Fixing**
  - if $M \geq 2$, shift M right, increment E
  - if E out of range, overflow (or underflow)
  - round M to fit `frac` precision
- **Implementation**
  - biggest chore is multiplying significands

Supplied by CMU.

Note that to compute E, one must first convert exp_1 and exp_2 to E_1 and E_2, then add them them together and check for underflow or overflow (corresponding to -∞ and +∞), and then convert to exp.



Supplied by CMU.

Note that, by default, overflow results in either +∞ or -∞.

Floating Point in C

- C guarantees two levels
  - `float` single precision
  - `double` double precision
- Conversions/casting
  - casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - » truncates fractional part
    - » like rounding toward zero
    - » not defined when out of range or NaN: generally sets to TMin
  - `int` → `double`
    - » exact conversion, as long as `int` has ≤ 53-bit word size
  - `int` → `float`
    - » will round according to rounding mode

Supplied by CMU.

Quiz 2

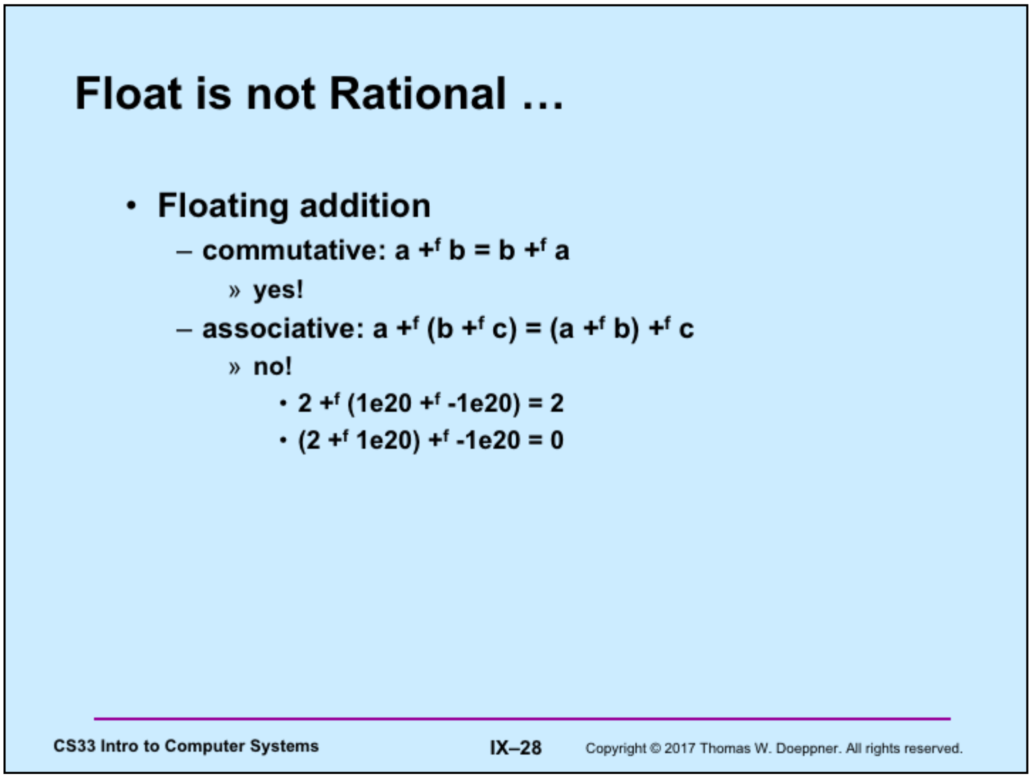
Suppose *f*, declared to be a `float`, is assigned the largest possible floating-point positive value (other than +∞). What is the value of *g = f+1.0*?

a) f

b) +∞

c) NAN

d) 0



Note that the floating-point numbers in this and the next two slides are expressed in base 10, not base 2.

Float is not Rational …

- **Multiplication**
  - commutative: a $*^f$ b = b $*^f$ a
    - » yes!
  - associative: a $*^f$ (b $*^f$ c) = (a $*^f$ b) $*^f$ c
    - » no!
      - 1e20 $*^f$ (1e20 $*^f$ 1e-20) = 1e20
      - (1e20 $*^f$ 1e20) $*^f$ 1e-20 = +∞

Float is not Rational …

- More …
  - multiplication distributes over addition:
    $a *^f (b +^f c) = (a *^f b) +^f (a *^f c)$
    - » no!
    - » $1e20 *^f (1e20 +^f -1e20) = 0$
    - » $(1e20 *^f 1e20) +^f (1e20 *^f -1e20) = NaN$
  - loss of significance:
    x=y+1
    z=2/(x-y)
    z==2?
    - » not necessarily!
      - consider y = 1e20