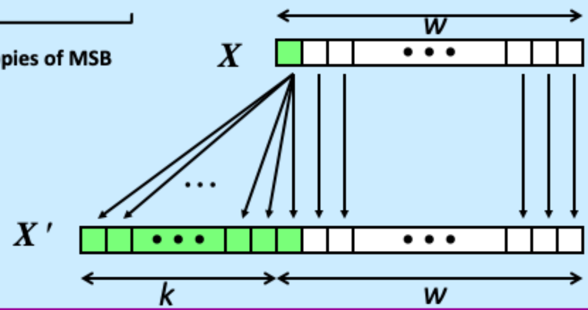
CS 33

Data Representation (Part 2)

Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.

Sign Extension

- **Task:**
  - given ***w***-bit signed integer ***x***
  - convert it to ***w+k***-bit integer with same value
- **Rule:**
  - make ***k*** copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



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Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
  - **C automatically performs sign extension**

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Does it Work?

$$val_w = -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$val_{w+1} = -2^{w} + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$val_{w+2} = -2^{w+1} + 2^{w} + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

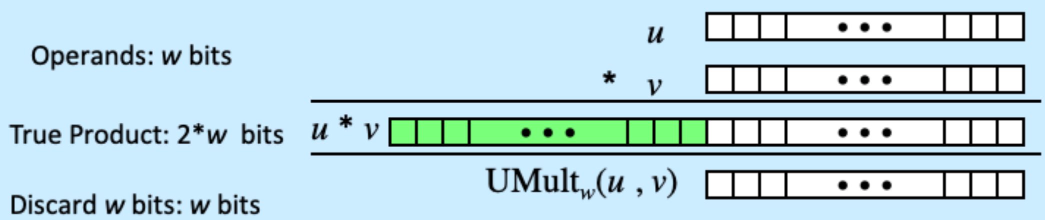
$$= -2^{w} + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

Sign extension clearly works for positive and zero values (where the sign bit is zero). But does it work for negative values? The first line of the slide shows the computation of the value of a w-bit item with a sign bit of one (i.e., it's negative). The next two lines show what happens if we extend this to a w+1-bit item, extending the sign bit. What had been the sign bit becomes one of the value bits, and its contribution to the value is now positive rather than negative. But this is compensated by the new sign bit, whose contribution is a negative value, twice as large as the original sign bit. Thus the net effect is for there to be no change in the value.

We do this again, extending to a w+2-bit item, and again, the resulting value is the same as what we started with.

Unsigned Multiplication



- **Standard multiplication function**
  - **ignores high order *w* bits**
- **Implements modular arithmetic**

  $\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

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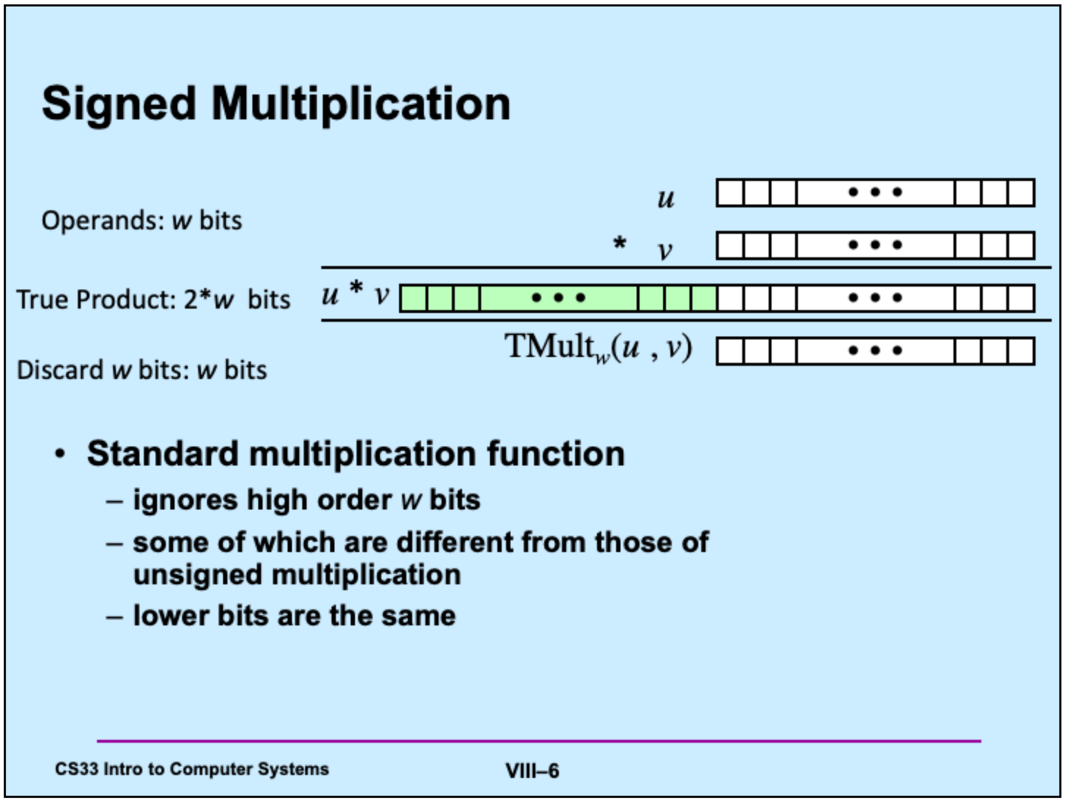

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Why is it that the "true product" is different from that of unsigned multiplication? Consider what the true product should be if the multiplier is -1 and the multiplicand is 1. Thus the multiplier is a w-bit word of all ones and the multiplicand is a w-bit word of all zeroes except for the least-significant bit, which is 1. The high-order w bits of the true product should be all ones (since it's negative), but with unsigned multiplication they'd be all zeroes. However, since we're ignoring the high-order w bits, this doesn't matter.

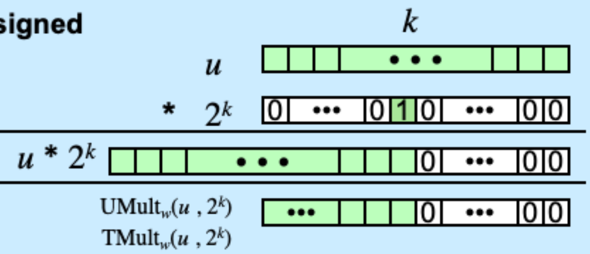
Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` **gives** `u * ` $\boldsymbol{2^k}$
  - **both signed and unsigned**

operands: *w* bits

true product: *w+k* bits

discard *k* bits: *w* bits



- **Examples**

```
u << 3 ==      u * 8
u << 5 - u << 3 == u * 24
```

  - **most machines shift and add faster than multiply**
    - » **compiler generates this code automatically**

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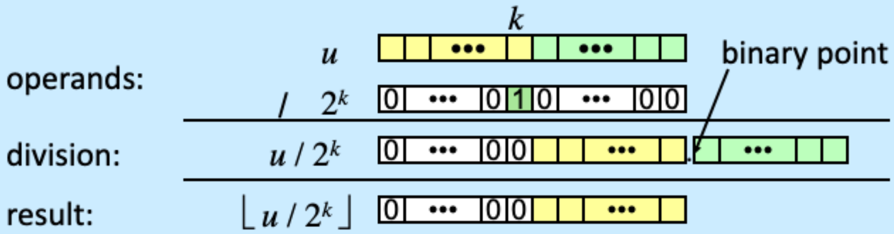
Unsigned Power-of-2 Divide with Shift

- **Quotient of unsigned by power of 2**
  - `u >> k` gives $\lfloor u / 2^k \rfloor$
  - uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

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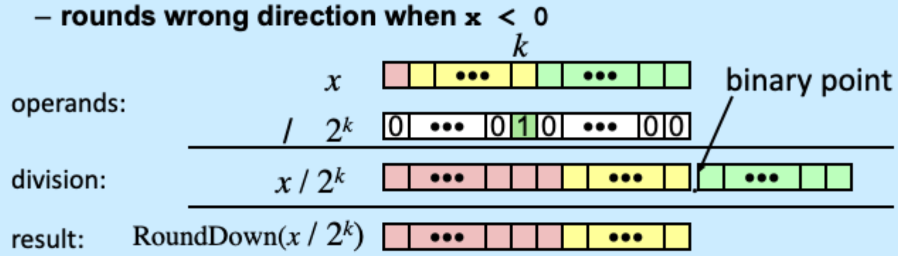
Signed Power-of-2 Divide with Shift

- **Quotient of signed by power of 2**
  - `x >> k` gives $\lfloor x \; / \; 2^k \rfloor$
  - uses arithmetic shift
  - rounds wrong direction when `x < 0`



	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

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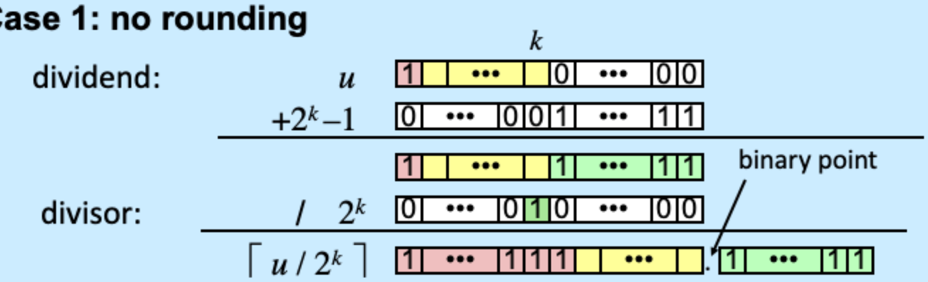
Correct Power-of-2 Divide

- **Quotient of negative number by power of 2**
  - want $\lceil$ `x / 2`$^k \rceil$ (round toward 0)
  - compute as $\lfloor$ `(x+2`^{k}`-1) / 2`$^k \rfloor$
    - » in C: `(x + (1<<k)-1) >> k`
    - » biases dividend toward 0

Case 1: no rounding



Biasing has no effect

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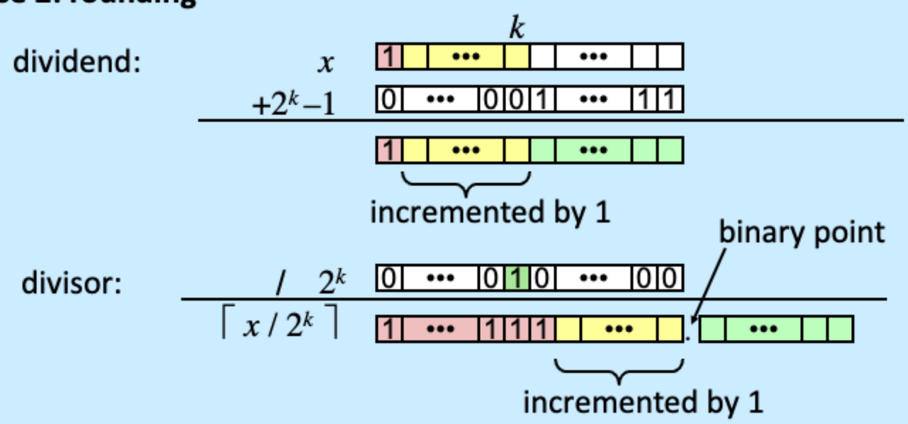
Correct Power-of-2 Divide (Cont.)

Case 2: rounding



Biasing adds 1 to final result

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Why Should I Use Unsigned?

- ***Don't*** **use just because number nonnegative**
  - **easy to make mistakes**

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - **can be very subtle**

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- ***Do*** **use when performing modular arithmetic**
  - **multiprecision arithmetic**
- ***Do*** **use when using bits to represent sets**
  - **logical right shift, no sign extension**

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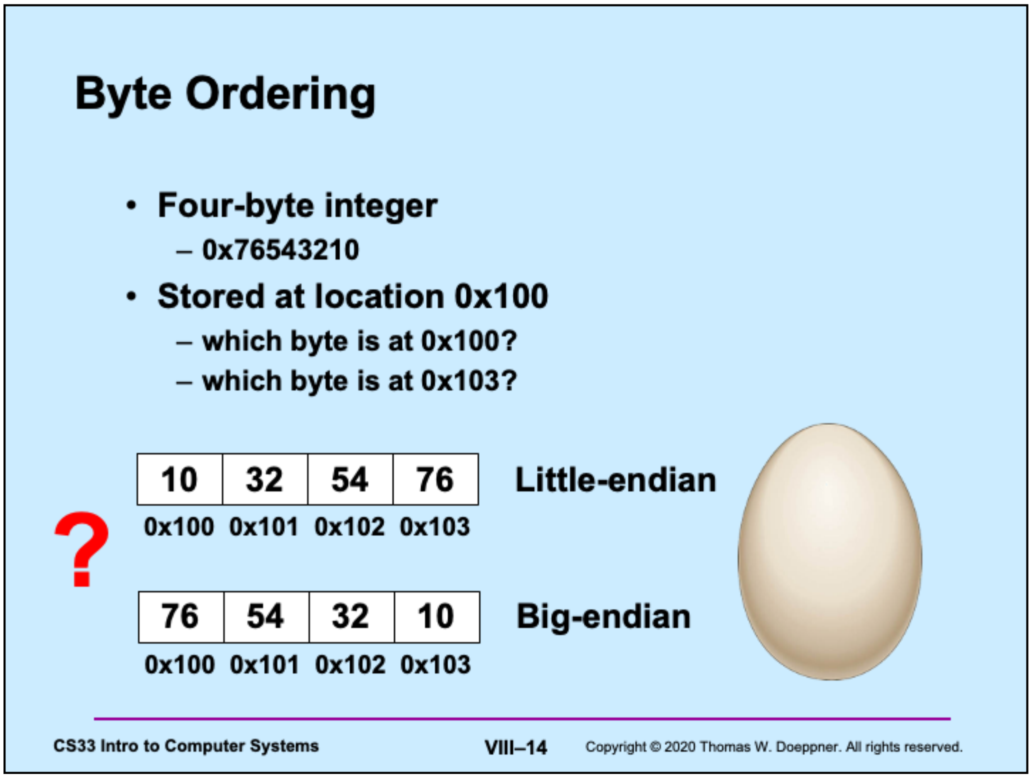
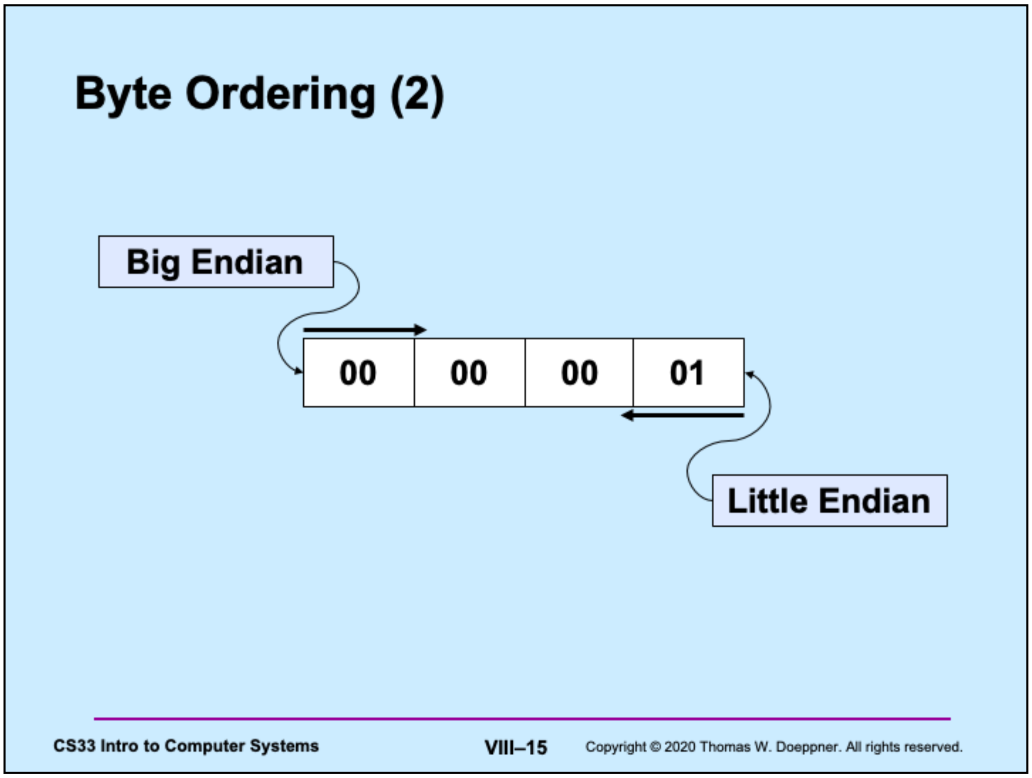
Note that "sizeof" returns an unsigned value. (Recall that, when mixing signed and unsigned items in an expression, the result will be unsigned.)

Word Size

- (Mostly) obsolete term
  - old computers had items of one size: the word size
- Now used to express the number of bits necessary to hold an address
  - 16 bits (really old computers)
  - 32 bits (old computers)
  - 64 bits (most current computers)



Read “Gulliver’s Travels” by Jonathan Swift for an explanation of the egg.



Here we have a four-byte integer one. In the big-endian representation, the address of the integer is the address of the byte containing its most-significant bits (the big end), while in the little-endian representation, the address of the integer is the address of the byte containing its least-significant bits (the little end). Suppose we pass a pointer to this integer to some procedure. However, in a type-mismatch, the procedure assumes that what is passed it is a two-byte integer. On a big-endian system, it would think it was passed a zero, but on a little-endian system, it would think it was passed a one.

This is not an argument in favor of either approach, but simply an observation that behaviors could be different.

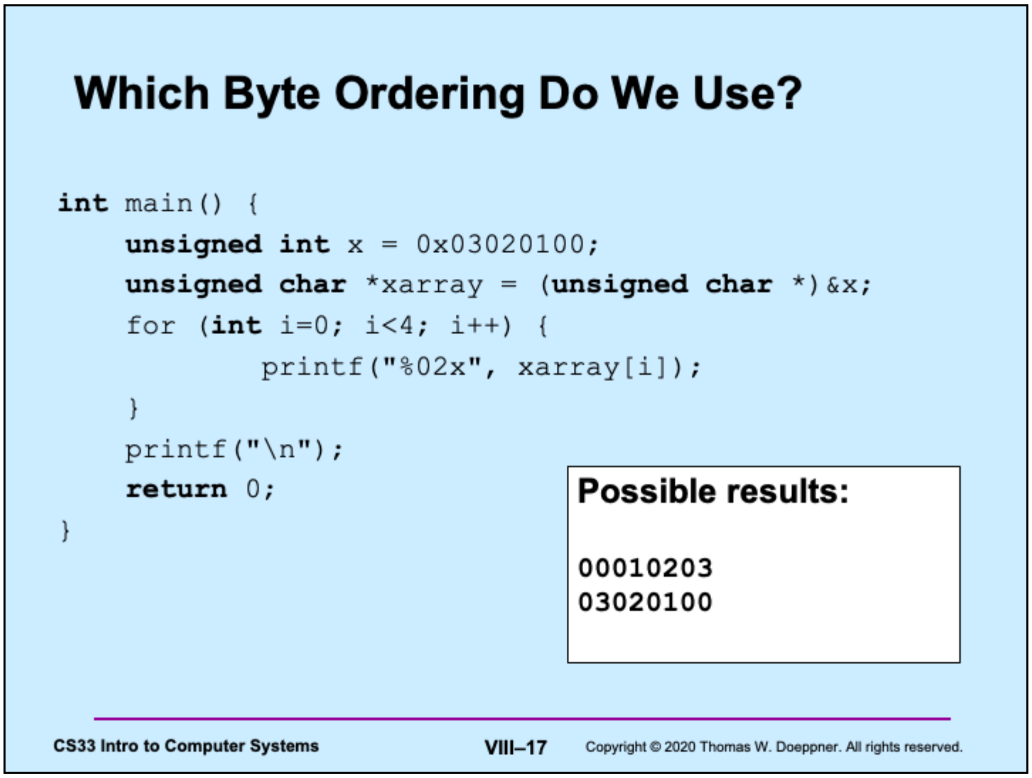
Quiz 1

```
int main() {
  long x=1;
  func((int *)&x);
  return 0;
}

void func(int *arg) {
  printf("%d\n", *arg);
}
```

What value is printed on a big-endian 64-bit computer?

a) 0
b) 1
c) 2^{32}
d) $2^{32}-1$



This code prints out the value of x, one byte at a time, starting with the byte at the lowest address (little end). On x86-based computers, it will print:

00010203

which means that the address of an int is the address of the byte containing its least significant digits (little endian).

Fractional binary numbers

- What is 1011.101_2?

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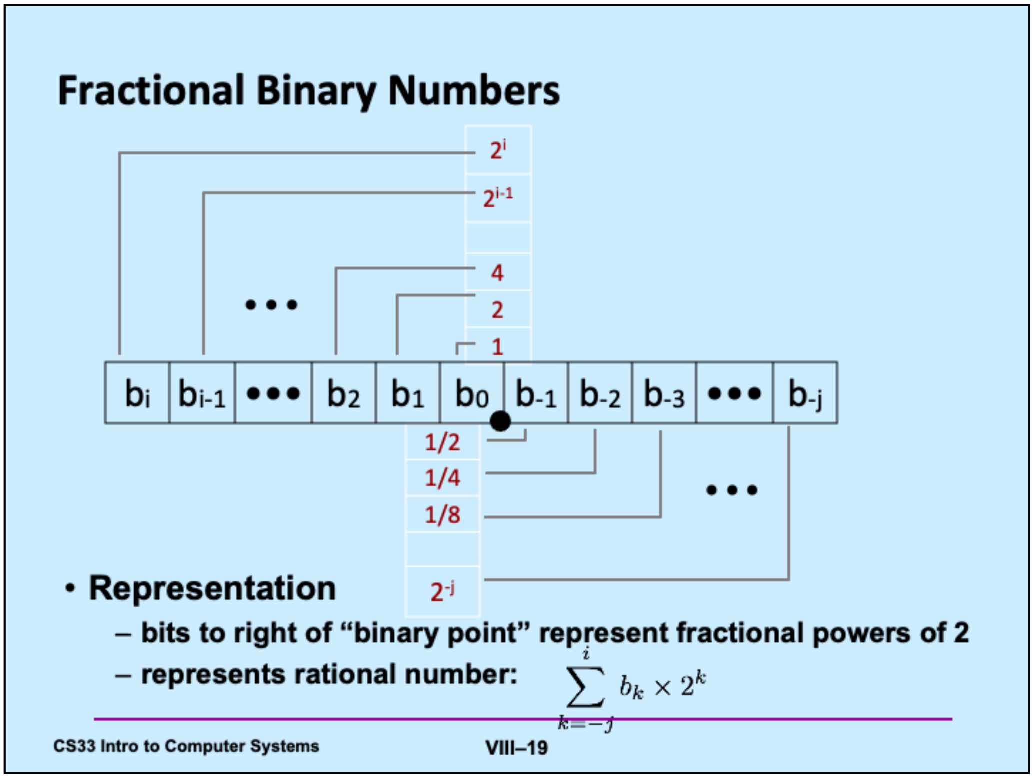

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Representable Numbers

- **Limitation #1**
  - can exactly represent only numbers of the form $n/2^k$
    - » other rational numbers have repeating bit representations
  - value — representation
    - » 1/3 `0.0101010101[01]…2`
    - » 1/5 `0.001100110011[0011]…2`
    - » 1/10 `0.0001100110011[0011]…2`

- **Limitation #2**
  - just one setting of decimal point within the *w* bits
    - » limited range of numbers (very small values? very large?)

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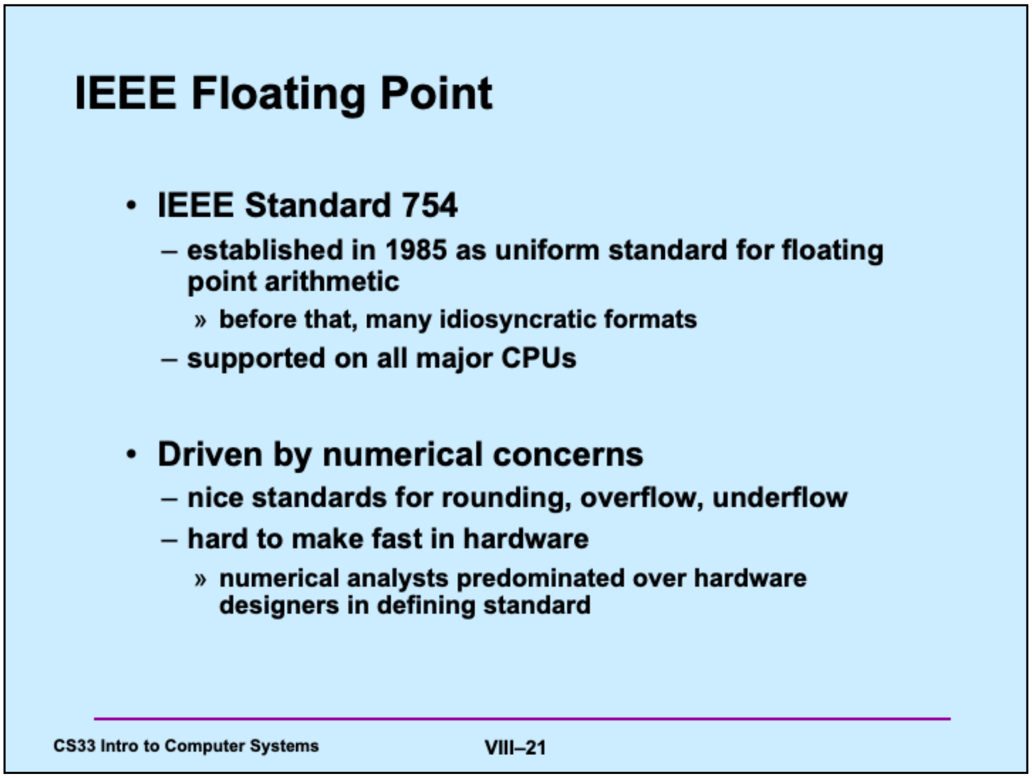

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IEEE is the Institute for Electrical and Electronics Engineers (pronounced "eye triple e").

Floating-Point Representation

- **Numerical Form:**

  $$(-1)^s \; M \; 2^E$$

  - sign bit **s** determines whether number is negative or positive
  - significand **M** normally a fractional value in range [1.0,2.0)
  - exponent **E** weights value by power of two
- **Encoding**
  - MSB `s` is sign bit **s**
  - `exp` field encodes **E** (but is not equal to E)
  - `frac` field encodes **M** (but is not equal to M)

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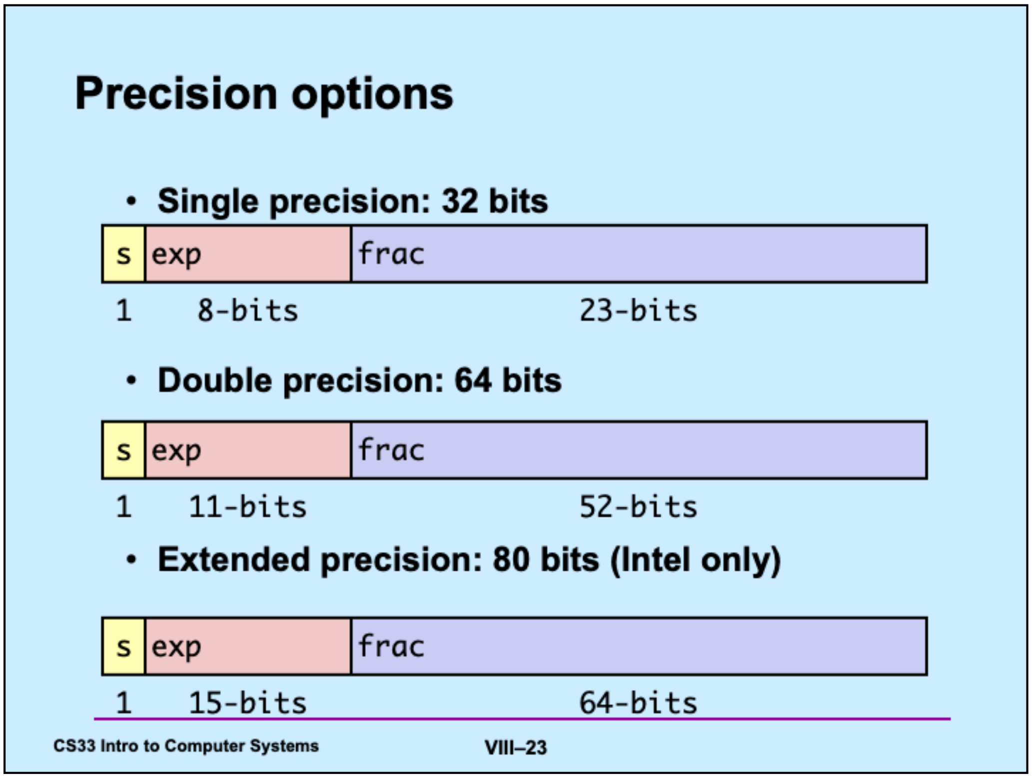

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On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.

"Normalized" Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for "free"

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Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = $1.\underline{1101101101101}_2$

  `frac` = $\underline{1101101101101}0000000000_2$

- **Exponent**

  E = 13

  bias = 127

  exp = 140 = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

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Denormalized Values

- Condition: exp = `000…0`
- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = `000…0`, `frac` = `000…0`
    - » represents zero value
    - » note distinct values: +0 and –0 (why?)
  - `exp` = `000…0`, `frac` ≠ `000…0`
    - » numbers closest to 0.0
    - » equispaced

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Special Values

- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

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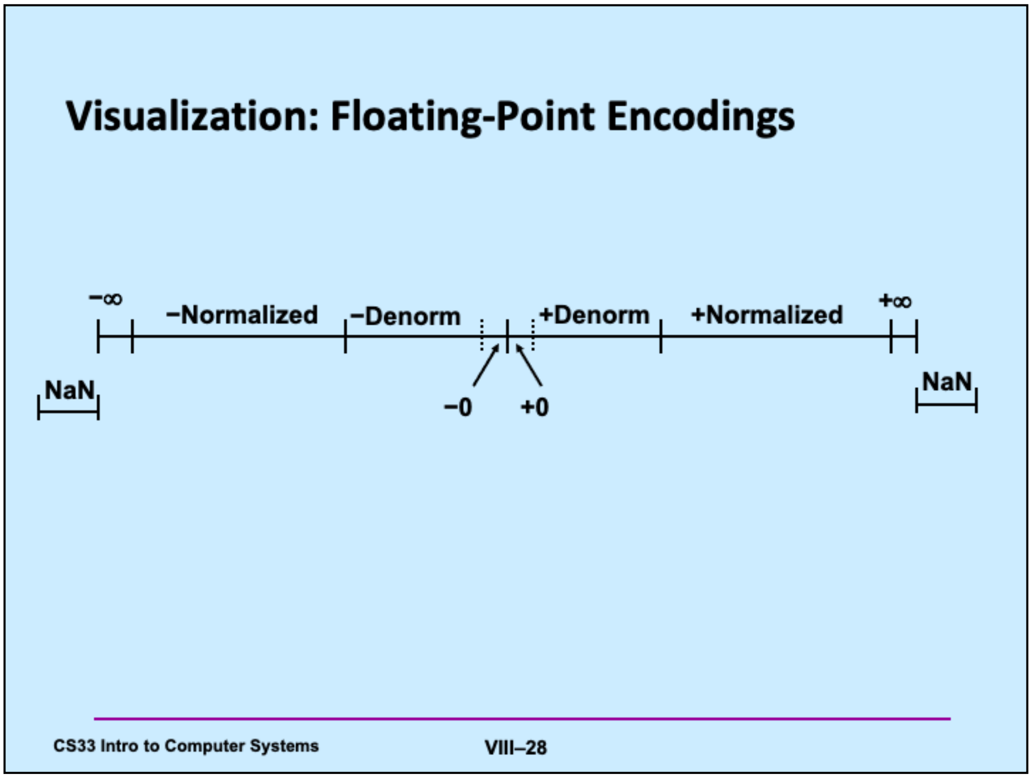

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Tiny Floating-Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

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Dynamic Range (Positive Only)

	s exp frac	E	Value	
	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

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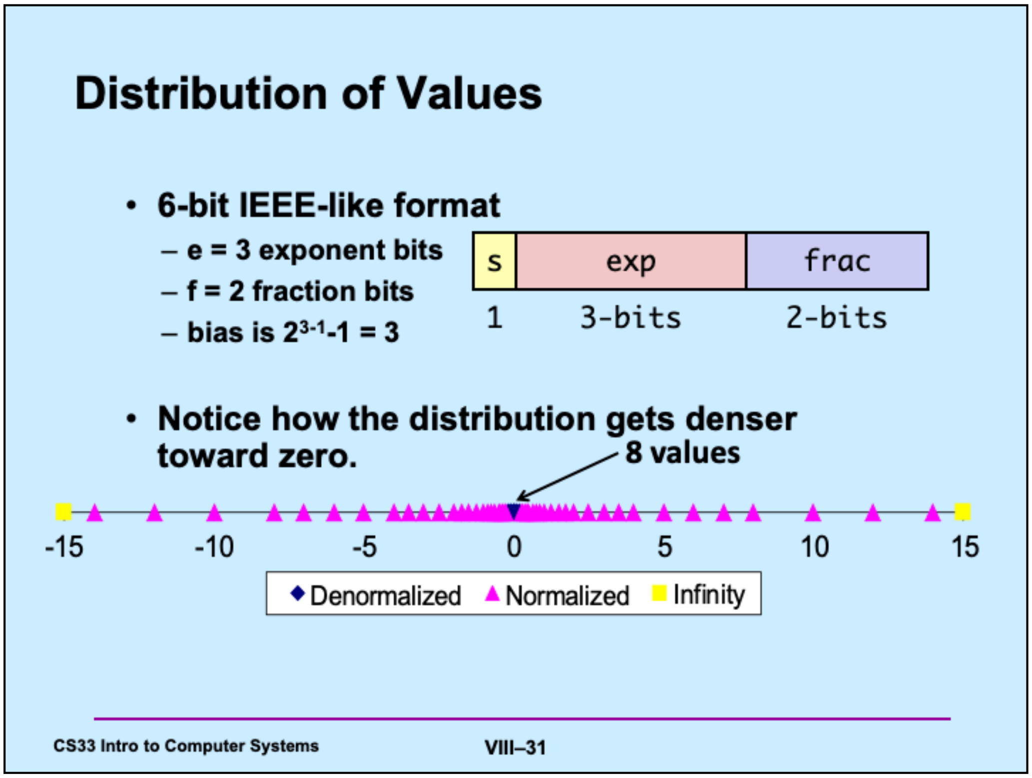

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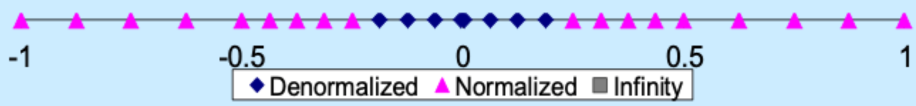
Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**



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Quiz 2

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

`s`	`exp`	`frac`
`1`	`3-bits`	`2-bits`

What number is represented by 0 011 10?

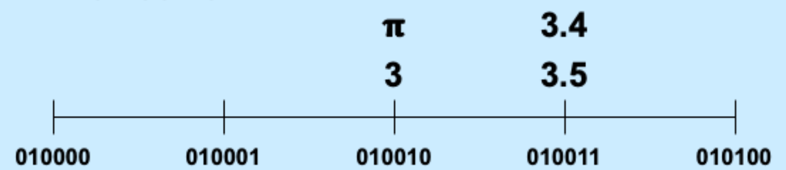
a) 12
b) 1.5
c) .5
d) none of the above

Mapping Real Numbers to Float

- The real number 3 is represented as
  0 100 10
- The real number 3.5 is represented as
  0 100 11
- How is the real number 3.4 represented?

  0 100 11
- How is the real number π represented?

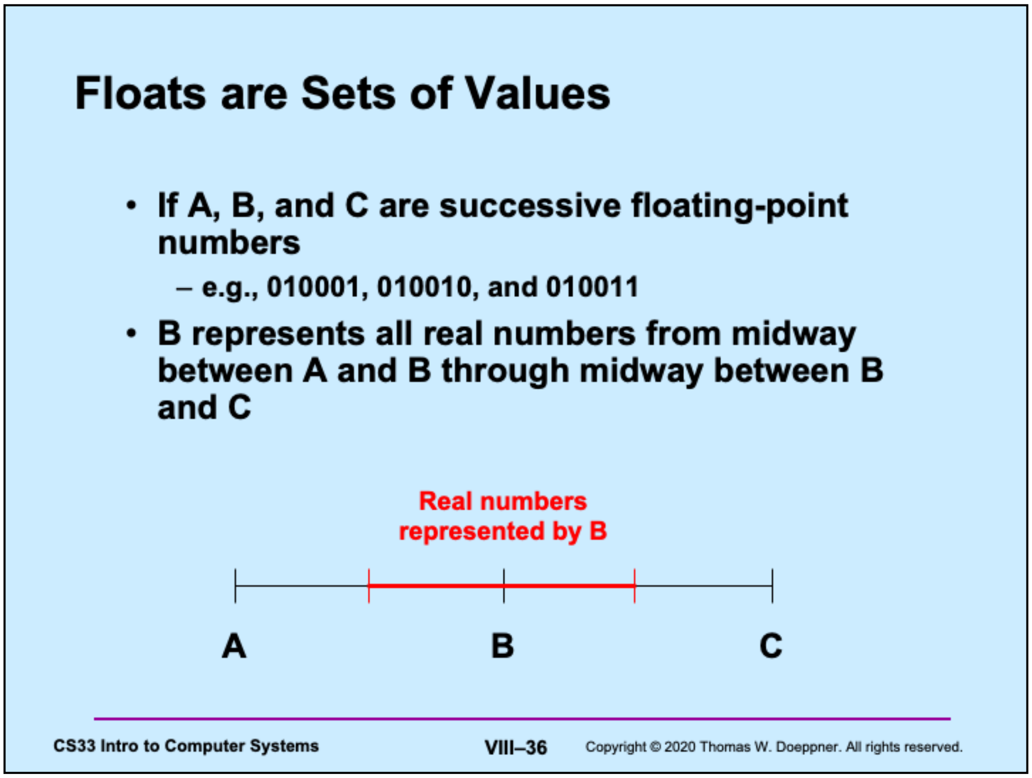
  0 100 10



We're assuming here the six-bit floating-point format.

Mapping Real Numbers to Float

- If R is a real number, it's mapped to the floating-point number whose value is closest to R
- What if it's midway between two values?
  - rounding rules coming up soon!



Note that we still have to discuss rounding so as to accommodate values that are equidistant from A and B or from B and C.

Significance

- Normalized numbers
  - for a particular exponent value E and an S-bit significand, the range from 2^E up to 2^{E+1} is divided into 2^S equi-spaced floating-point values
    - » thus each floating-point value represents $1/2^S$ of the range of values with that exponent
    - » all bits of the signifcand are important
    - » we say that there are S significant bits – for reasonably large S, each floating-point value covers a rather small part of the range
      - high accuracy
      - for S=23 (32-bit float), accurate to one in 2^{23} (.0000119% accuracy)

Significance

- **Unnormalized numbers**
  - high-order zero bits of the significand aren't important
  - in 8-bit floating point, 0 0000 001 represents 2^{-9}
    - » it is the only value with that exponent: 1 significant bit (either 2^{-9} or 0)
  - 0 0000 010 represents 2^{-8}
    0 0000 011 represents $1.5*2^{-8}$
    - » only two values with exponent -8: 2 significant bits (encoding those two values, as well as 2^{-9} and 0)
  - fewer significant bits mean less accuracy
  - 0 0000 01 represents a range of values from $.5*2^{-9}$ to $1.5*2^{-9}$
  - 50% accuracy

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Recall that the bias for the exponent of 8-bit IEEE FP is 7, thus for unnormalized numbers the actual exponent is -6 (-bias+1). The significand has an implied leading 0, thus 0 0000 001 represents $2^{-6} * 2^{-3}$.

With 8-bit IEEE FP. the value 0 0000 01 is interpreted as 2-9, But the number represented could be 50% or 50% more.

Floating-Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x ×f y = Round(x × y)`
- **Basic idea**
  - first compute exact result
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly round to fit into `frac`

Supplied by CMU.

Rounding

- Rounding modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest integer	$1	$2	?	?	?
nearest even (default)	$1	$2	$2	$2	–$2

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Floating-Point Multiplication

- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- **Exact result:** $(-1)^{s}$ M 2^{E}
  - sign s: s1 ^ s2
  - significand M: M1 x M2
  - exponent E: E1 + E2
- **Fixing**
  - if M ≥ 2, shift M right, increment E
  - if E out of range, overflow (or underflow)
  - round M to fit `frac` precision
- **Implementation**
  - biggest chore is multiplying significands

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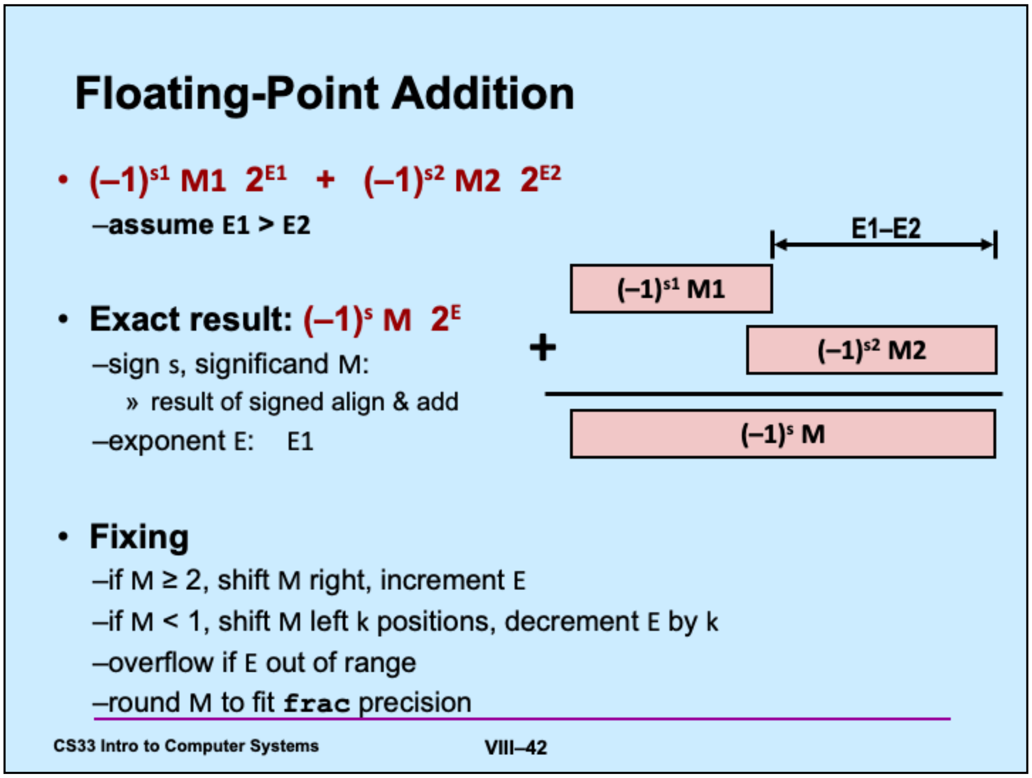
Note that to compute E, one must first convert exp_1 and exp_2 to E_1 and E_2, then add them them together and check for underflow or overflow (corresponding to -∞ and +∞), and then convert to exp.



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Note that, by default, overflow results in either +∞ or -∞.