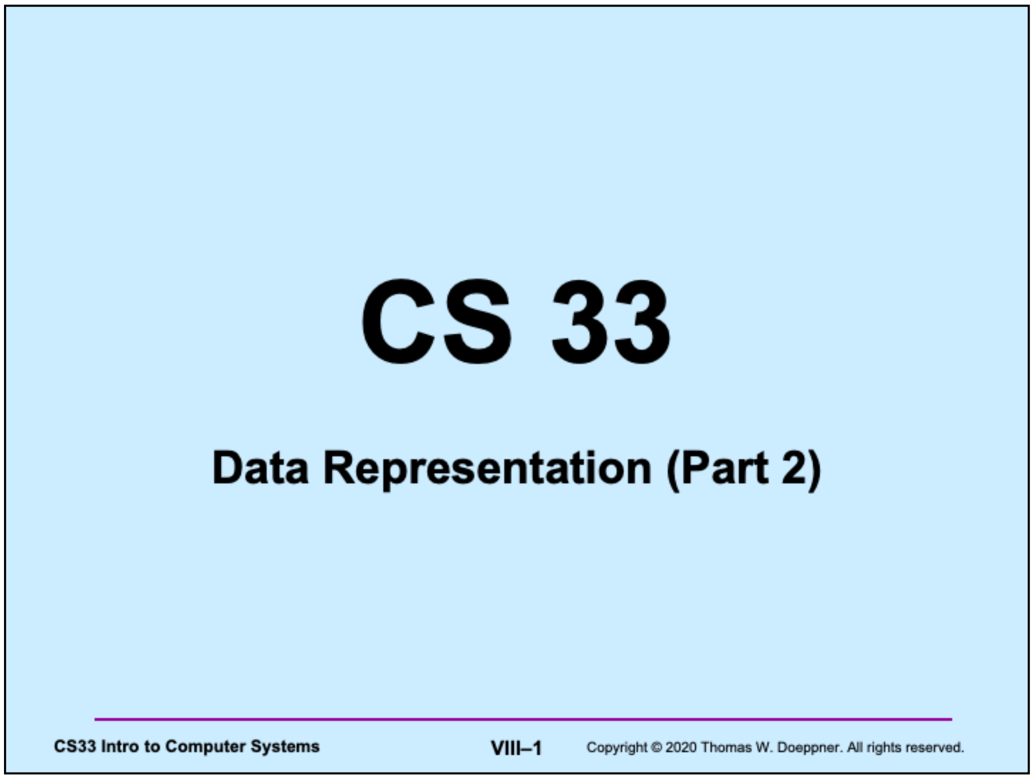
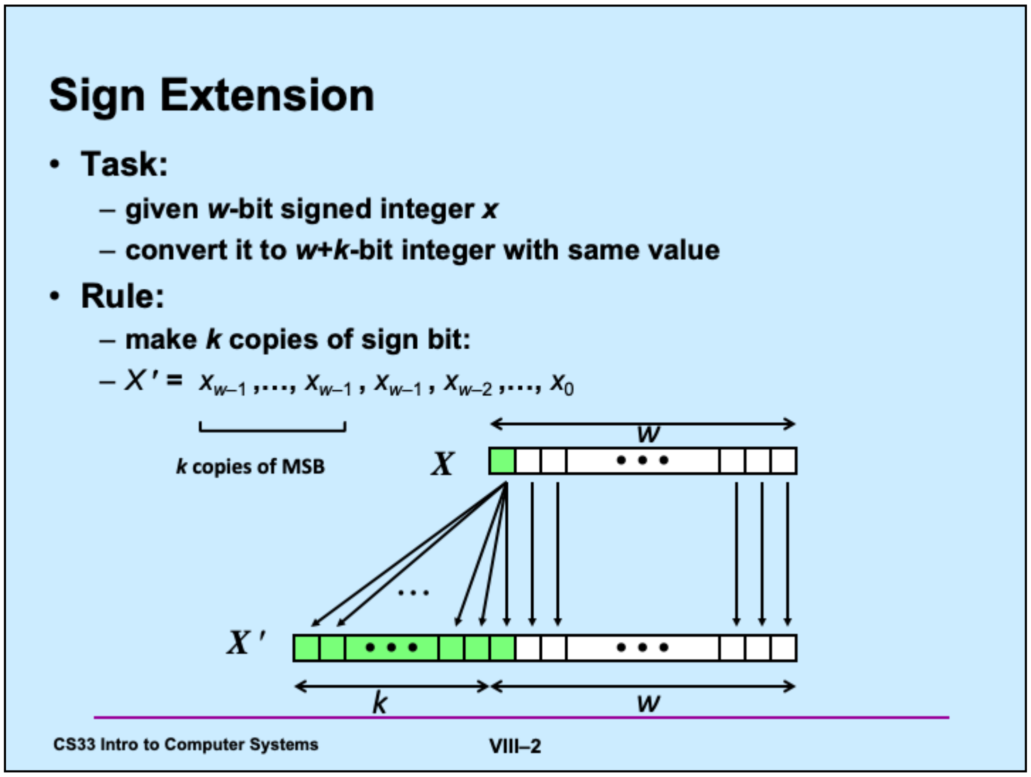


CS 33

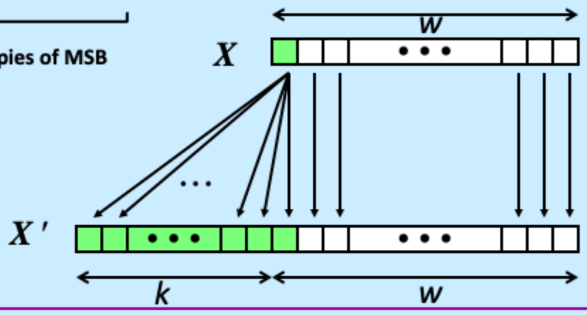
Data Representation (Part 2)

Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.



Sign Extension

- **Task:**
  - **given *w*-bit signed integer *x***
  - **convert it to *w*+*k*-bit integer with same value**
- **Rule:**
  - **make *k* copies of sign bit:**
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



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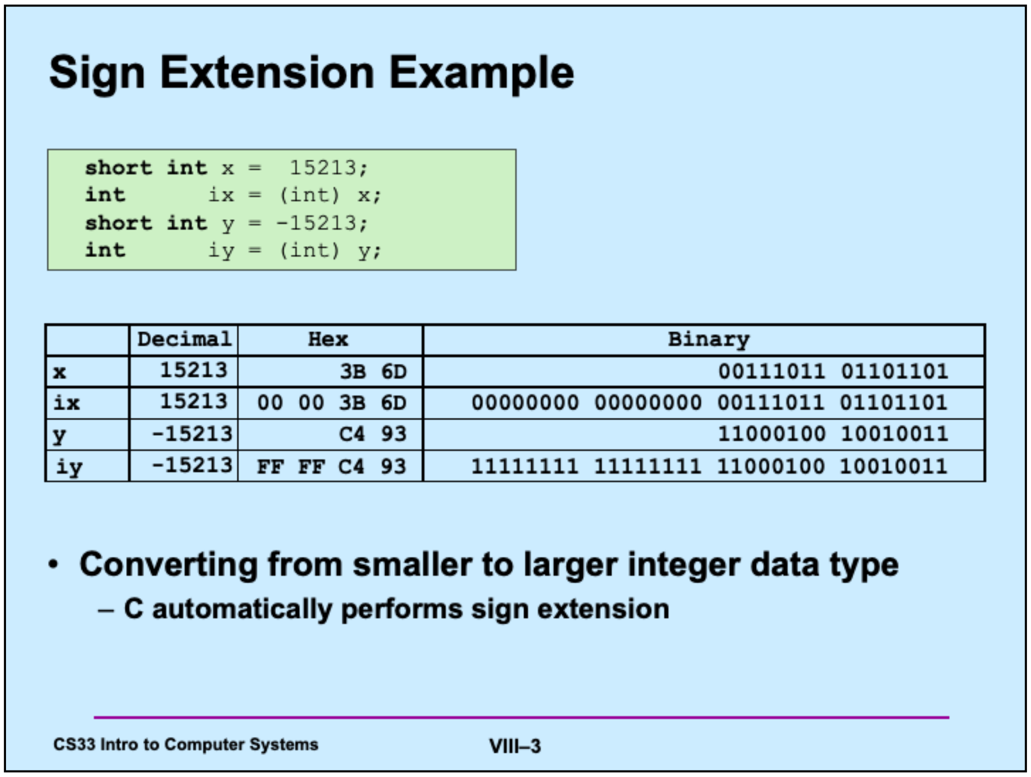


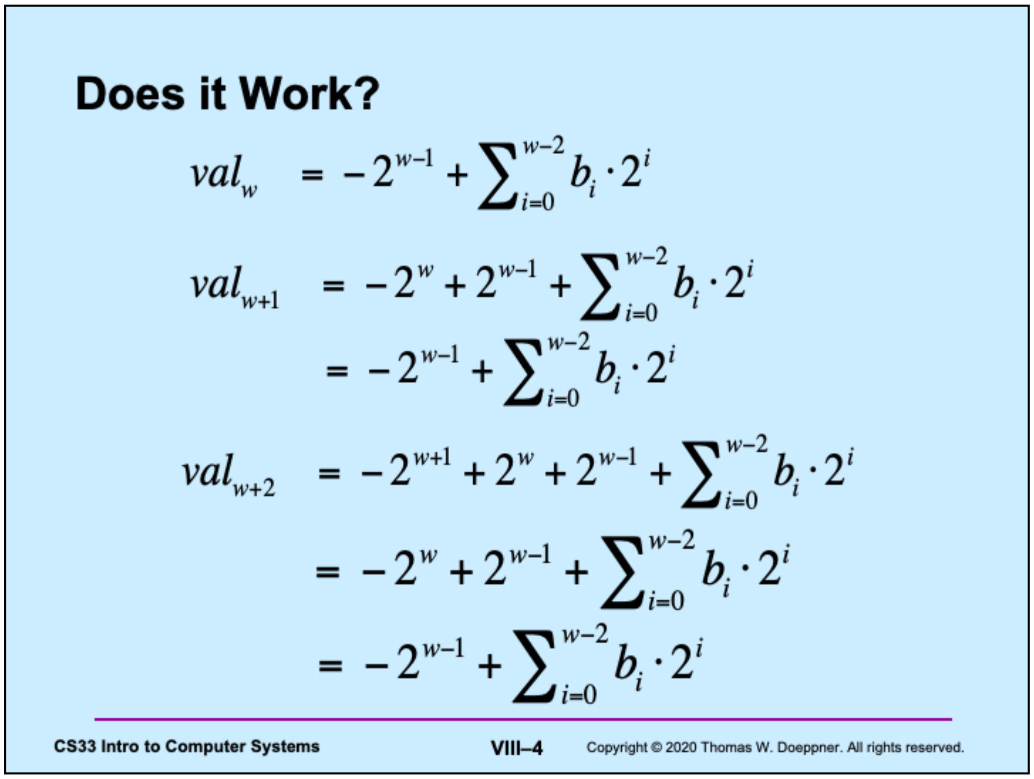
Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
  - C automatically performs sign extension

Supplied by CMU.



Does it Work?

$$val_w = -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$val_{w+1} = -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

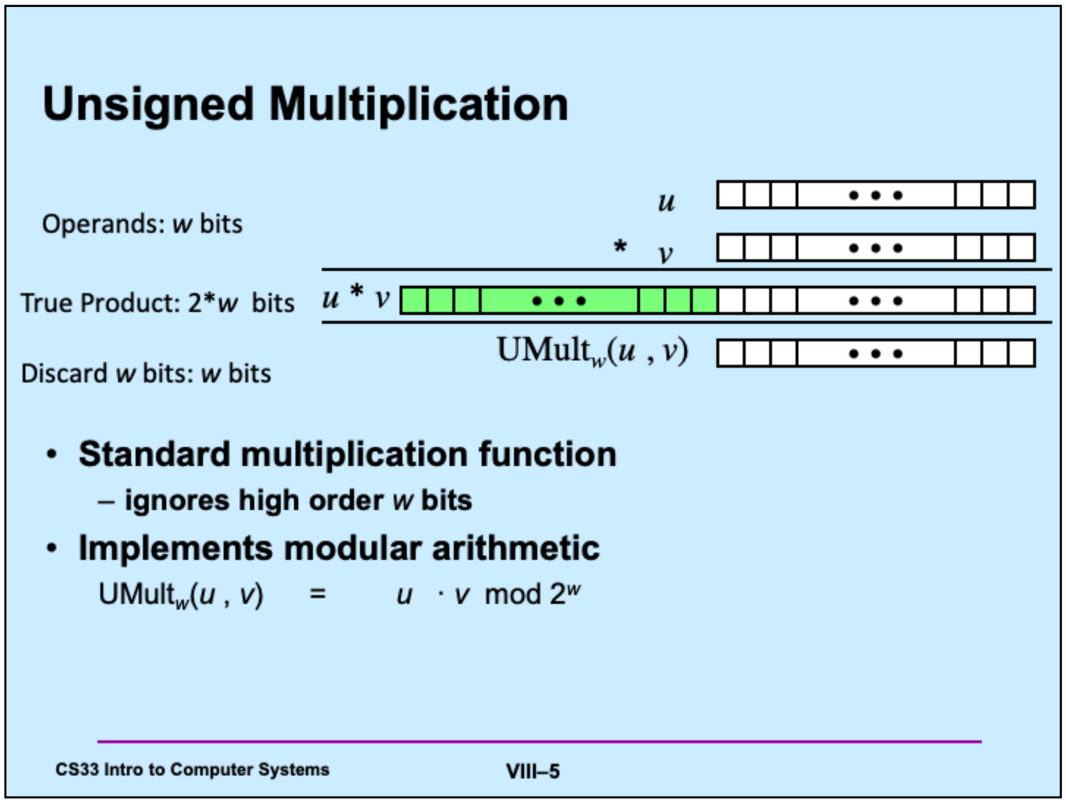
$$val_{w+2} = -2^{w+1} + 2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

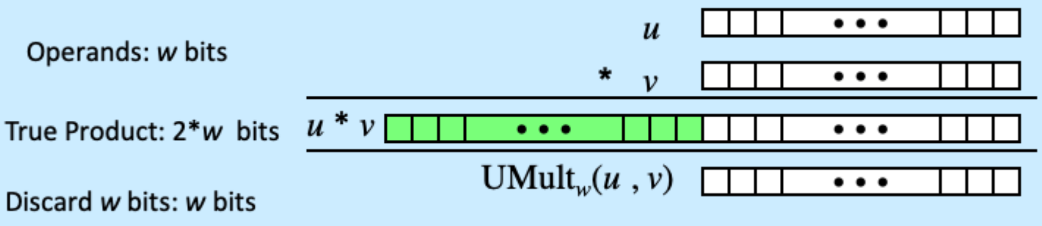
$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

Sign extension clearly works for positive and zero values (where the sign bit is zero). But does it work for negative values? The first line of the slide shows the computation of the value of a w-bit item with a sign bit of one (i.e., it's negative). The next two lines show what happens if we extend this to a w+1-bit item, extending the sign bit. What had been the sign bit becomes one of the value bits, and its contribution to the value is now positive rather than negative. But this is compensated by the new sign bit, whose contribution is a negative value, twice as large as the original sign bit. Thus the net effect is for there to be no change in the value.

We do this again, extending to a w+2-bit item, and again, the resulting value is the same as what we started with.



Unsigned Multiplication



- **Standard multiplication function**
  - **ignores high order *w* bits**
- **Implements modular arithmetic**

  $\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

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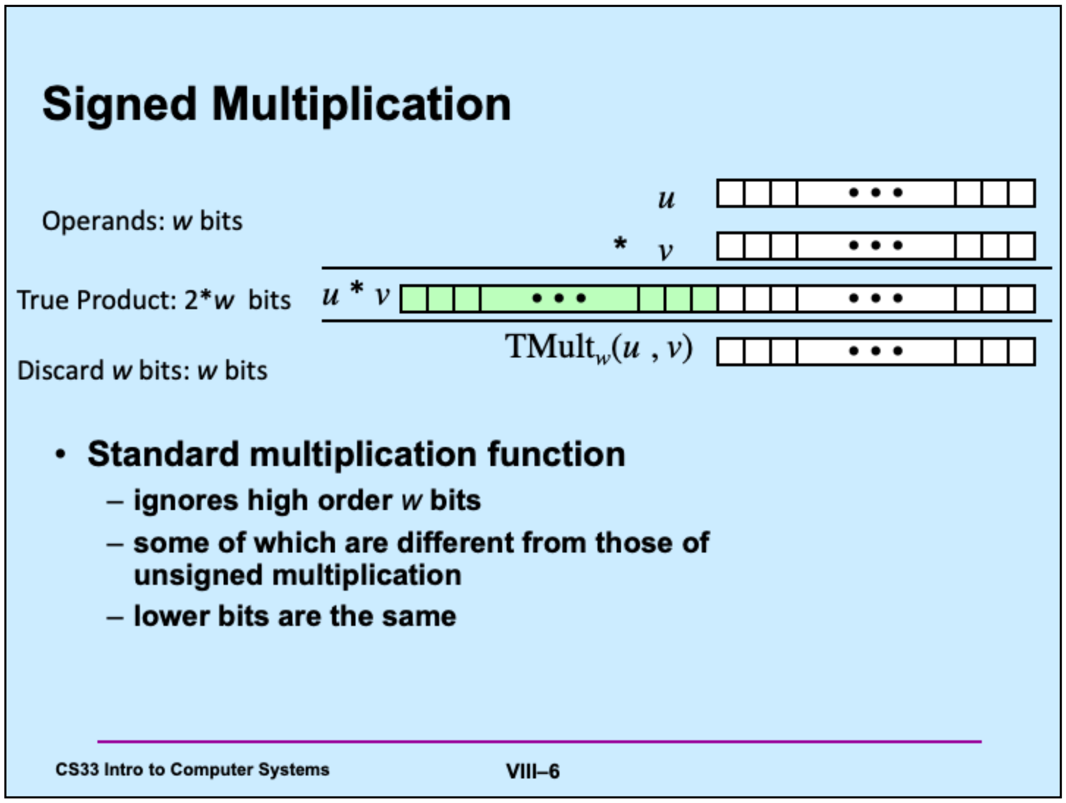
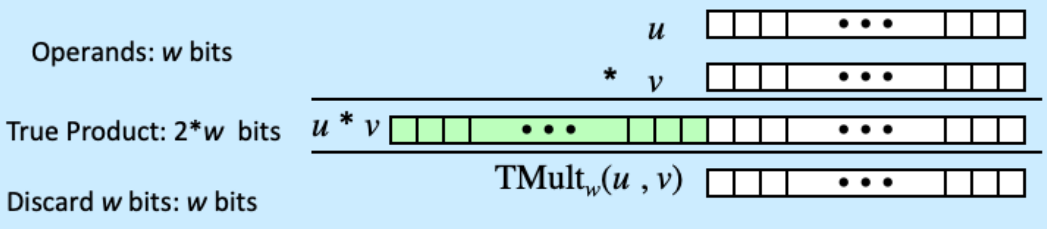


Signed Multiplication



- **Standard multiplication function**
  - **ignores high order *w* bits**
  - **some of which are different from those of unsigned multiplication**
  - **lower bits are the same**

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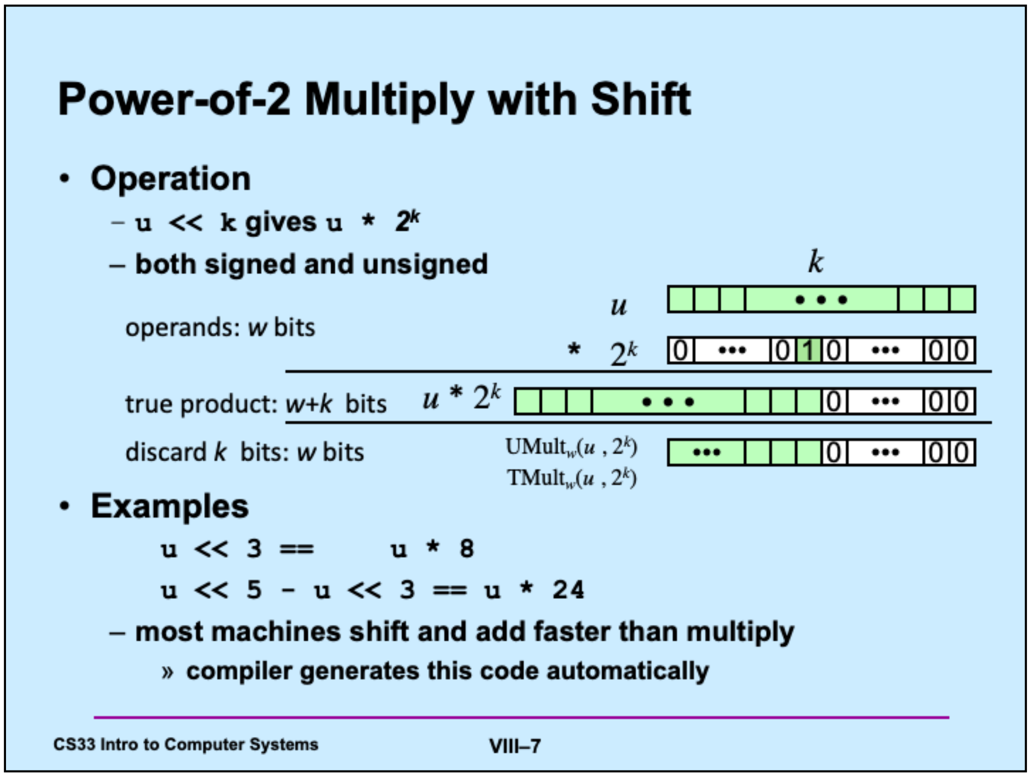
Why is it that the "true product" is different from that of unsigned multiplication? Consider what the true product should be if the multiplier is -1 and the multiplicand is 1. Thus the multiplier is a w-bit word of all ones and the multiplicand is a w-bit word of all zeroes except for the least-significant bit, which is 1. The high-order w bits of the true product should be all ones (since it's negative), but with unsigned multiplication they'd be all zeroes. However, since we're ignoring the high-order w bits, this doesn't matter.



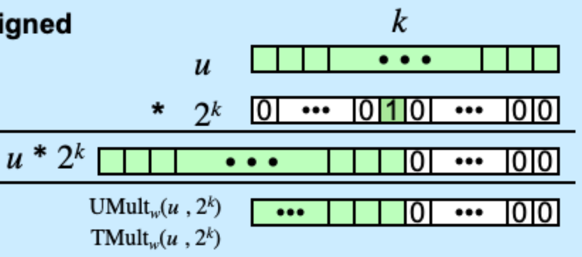
Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` **gives** `u * ` ***2^k***
  - **both signed and unsigned**

operands: *w* bits

true product: *w+k* bits

discard *k* bits: *w* bits

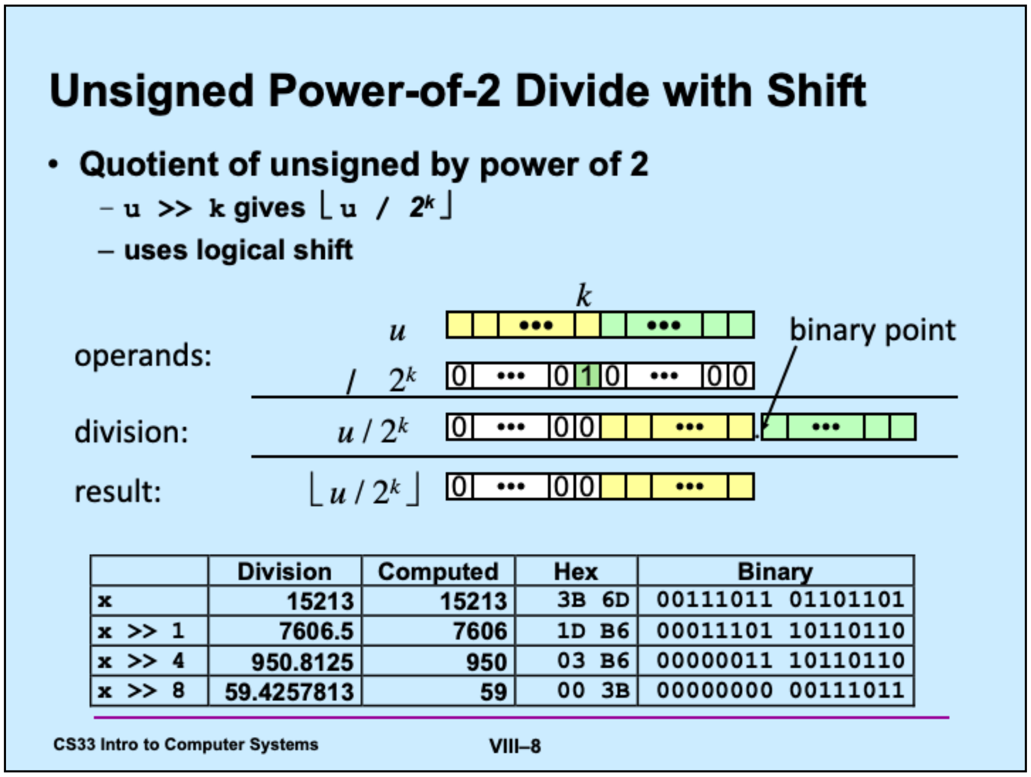


- **Examples**

```
u << 3 ==       u * 8
u << 5 - u << 3 == u * 24
```

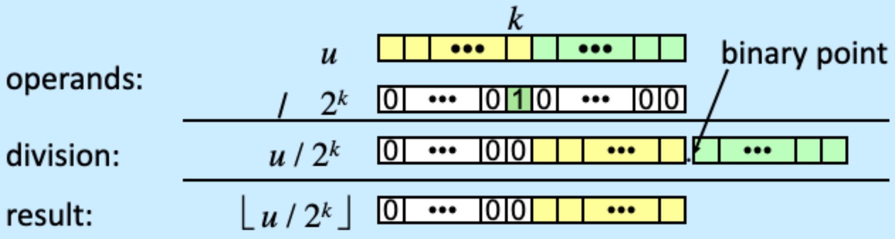
  - **most machines shift and add faster than multiply**
    - » **compiler generates this code automatically**

Supplied by CMU.



Unsigned Power-of-2 Divide with Shift

- **Quotient of unsigned by power of 2**
  - **`u >> k` gives $\lfloor u\ /\ 2^k \rfloor$**
  - **uses logical shift**



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

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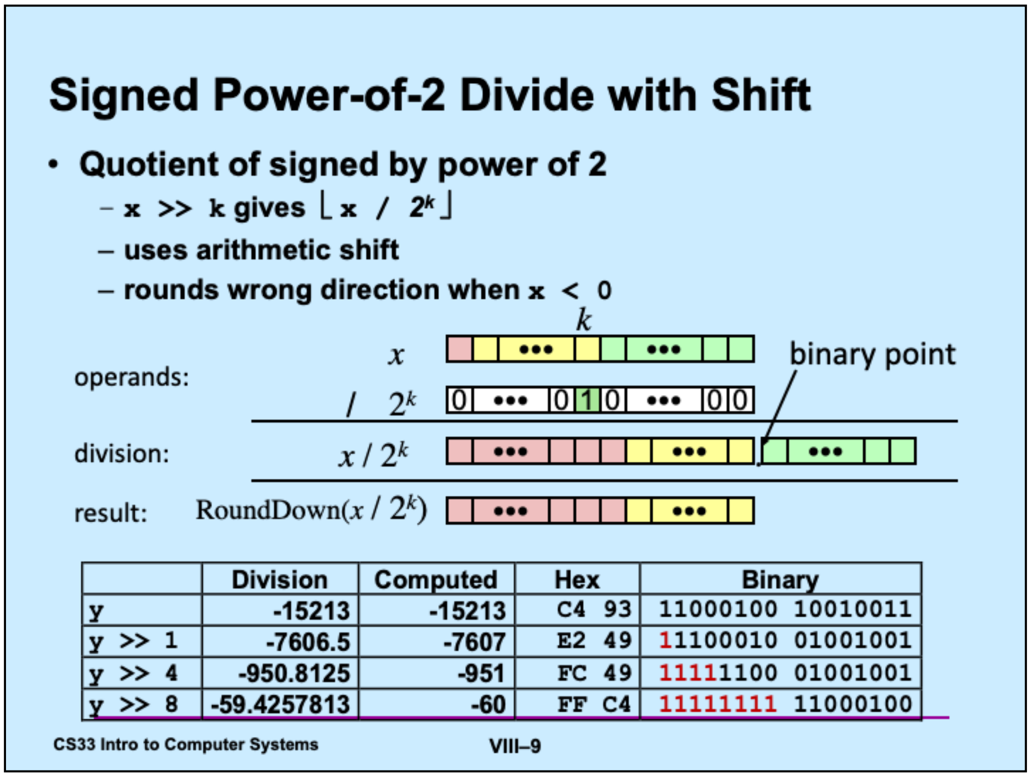
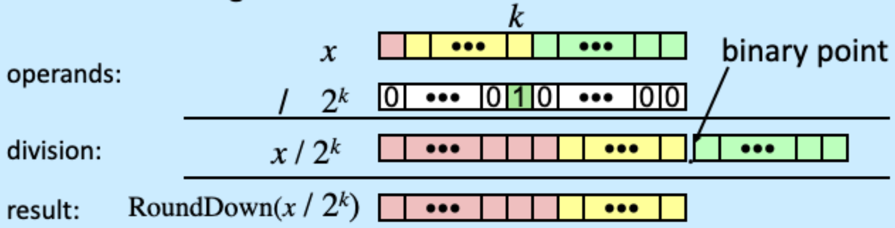


Signed Power-of-2 Divide with Shift

- **Quotient of signed by power of 2**
  - **`x >> k` gives ⌊ `x / 2`^{k} ⌋**
  - **uses arithmetic shift**
  - **rounds wrong direction when `x < 0`**



	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

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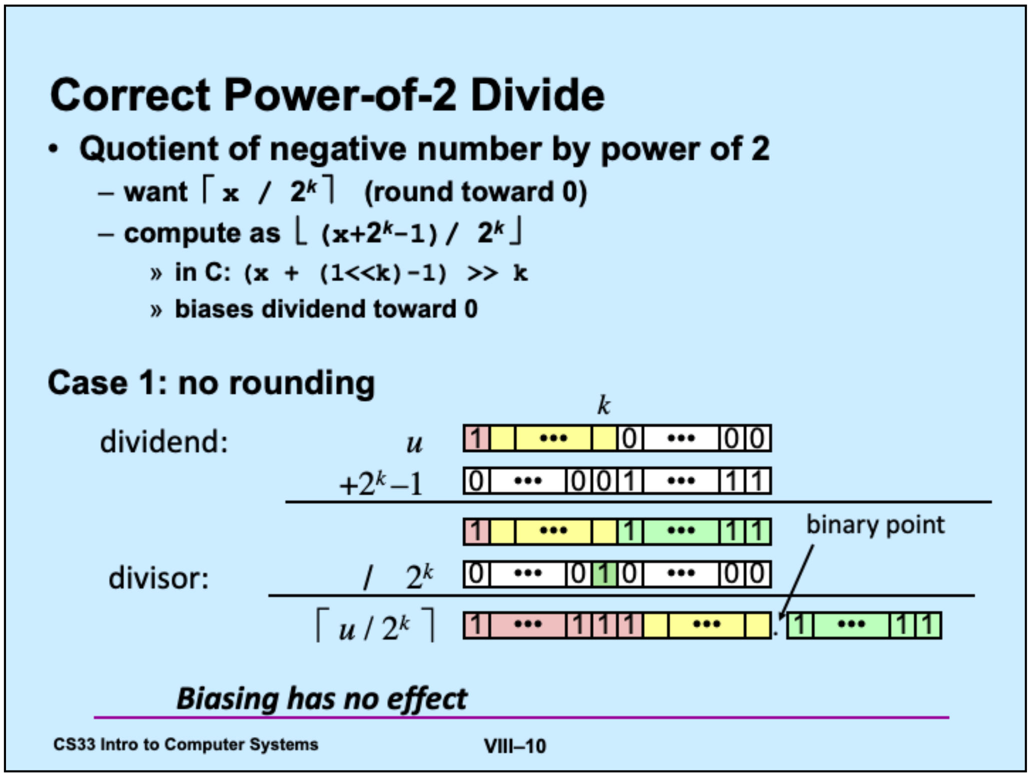
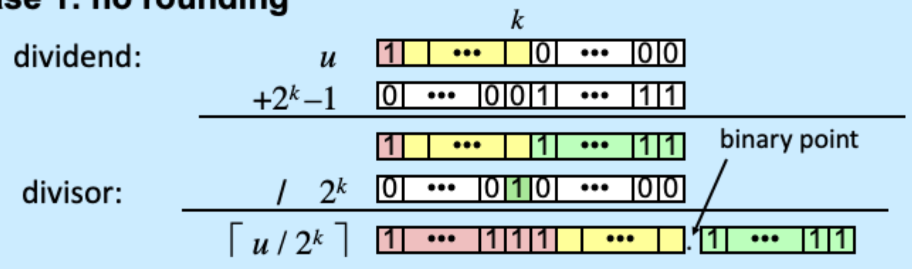


Correct Power-of-2 Divide

- Quotient of negative number by power of 2
  - want $\lceil$ `x / 2`^{k} $\rceil$ (round toward 0)
  - compute as $\lfloor$ `(x+2`^{k}`-1)/ 2`^{k} $\rfloor$
    - » in C: `(x + (1<<k)-1) >> k`
    - » biases dividend toward 0

Case 1: no rounding



Biasing has no effect

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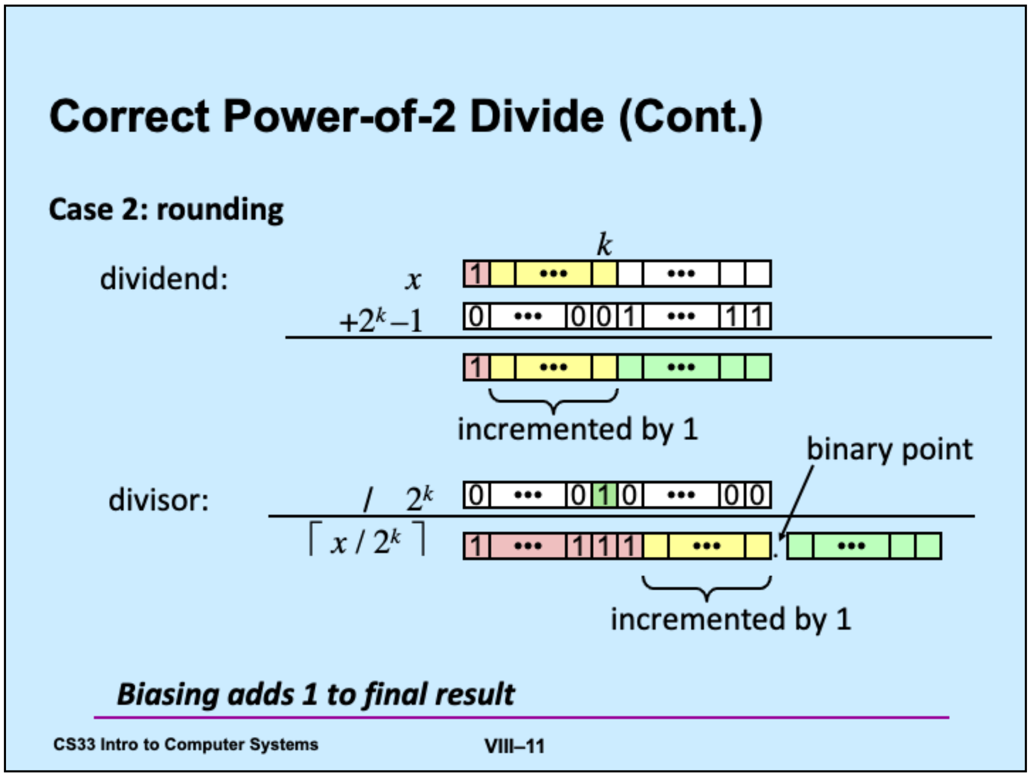
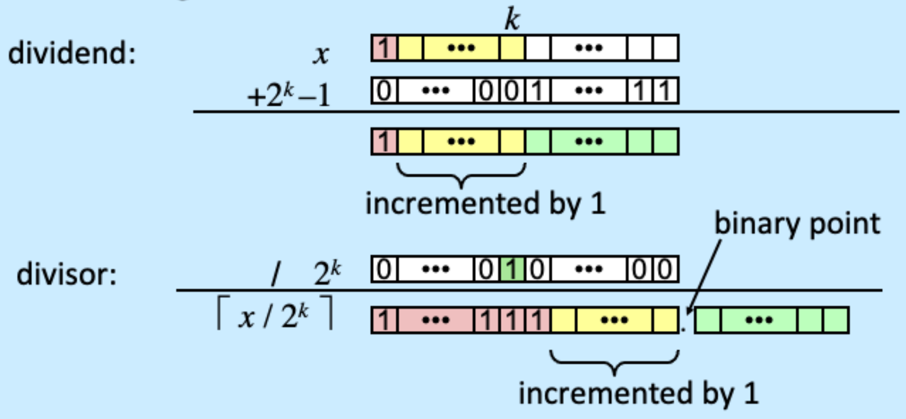


Correct Power-of-2 Divide (Cont.)

Case 2: rounding



Biasing adds 1 to final result

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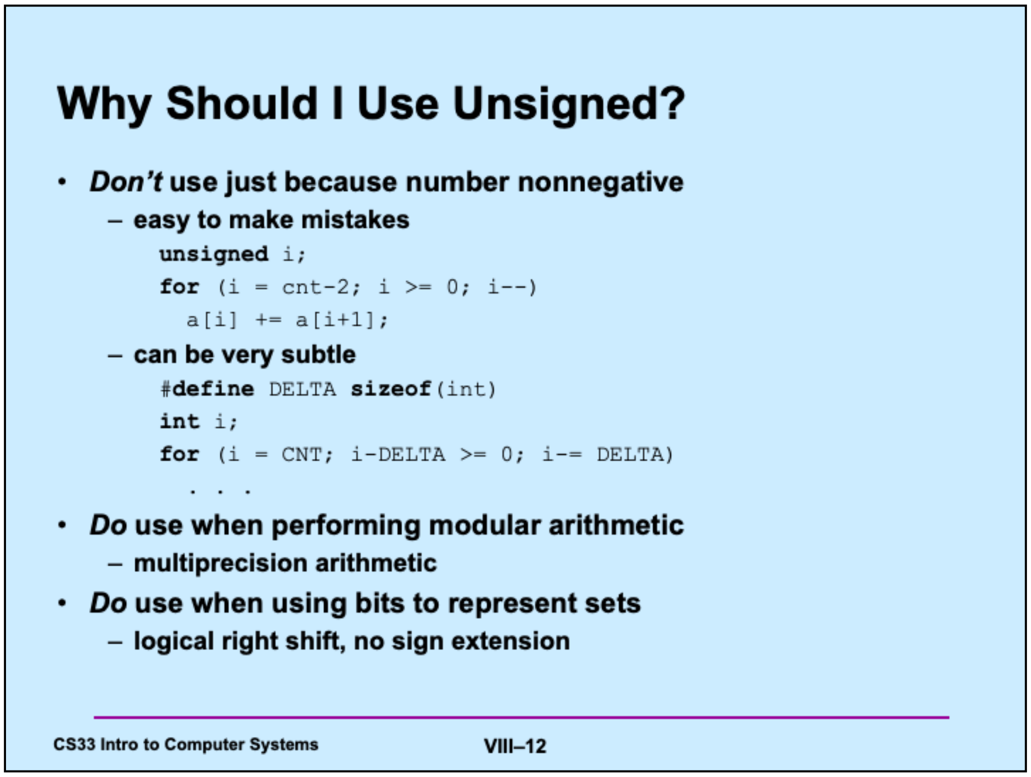

Why Should I Use Unsigned?

- ***Don't*** **use just because number nonnegative**
  - **easy to make mistakes**

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - **can be very subtle**

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- ***Do*** **use when performing modular arithmetic**
  - **multiprecision arithmetic**
- ***Do*** **use when using bits to represent sets**
  - **logical right shift, no sign extension**

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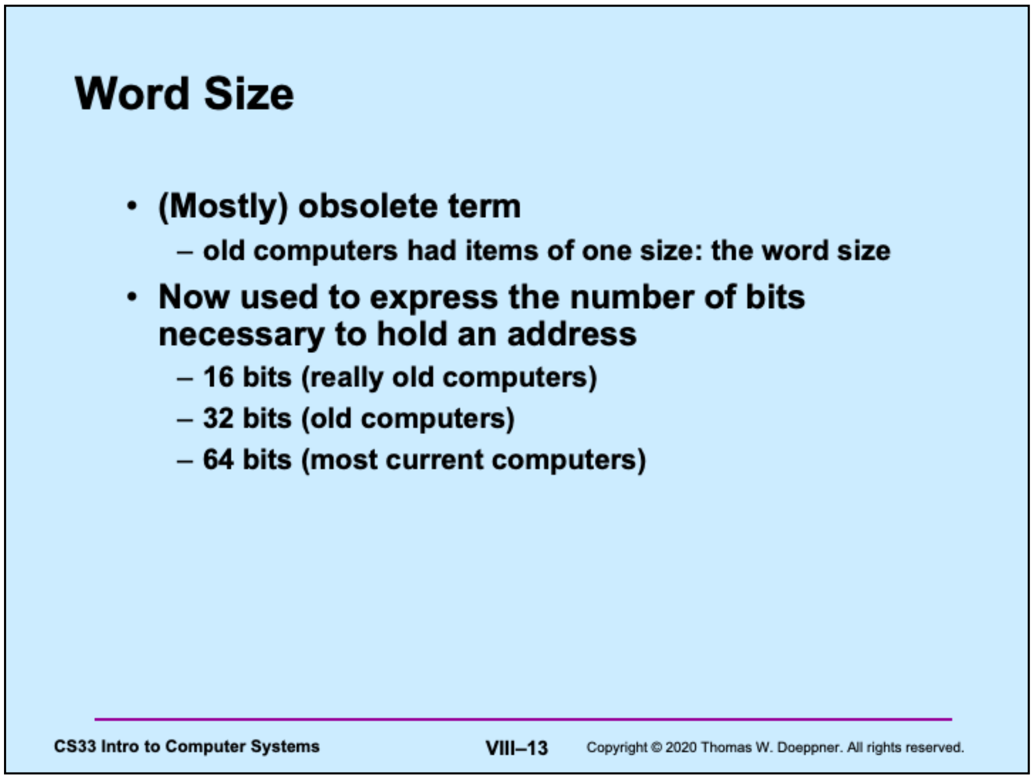
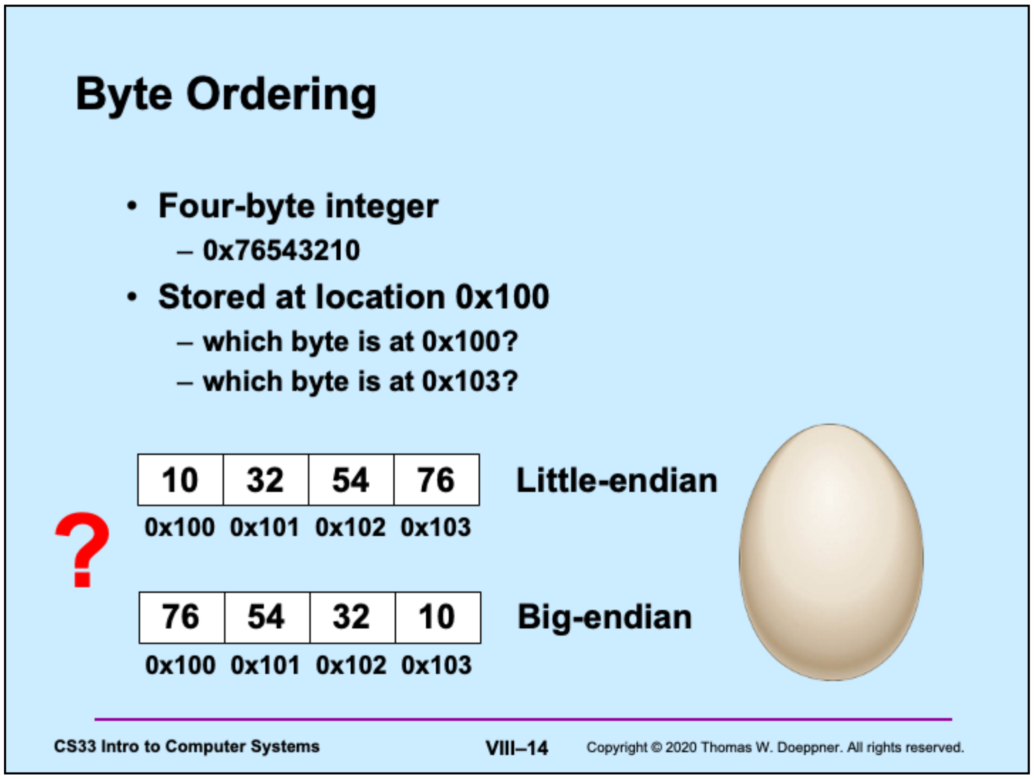
Note that "sizeof" returns an unsigned value. (Recall that, when mixing signed and unsigned items in an expression, the result will be unsigned.)



Word Size

- (Mostly) obsolete term
  - old computers had items of one size: the word size
- Now used to express the number of bits necessary to hold an address
  - 16 bits (really old computers)
  - 32 bits (old computers)
  - 64 bits (most current computers)

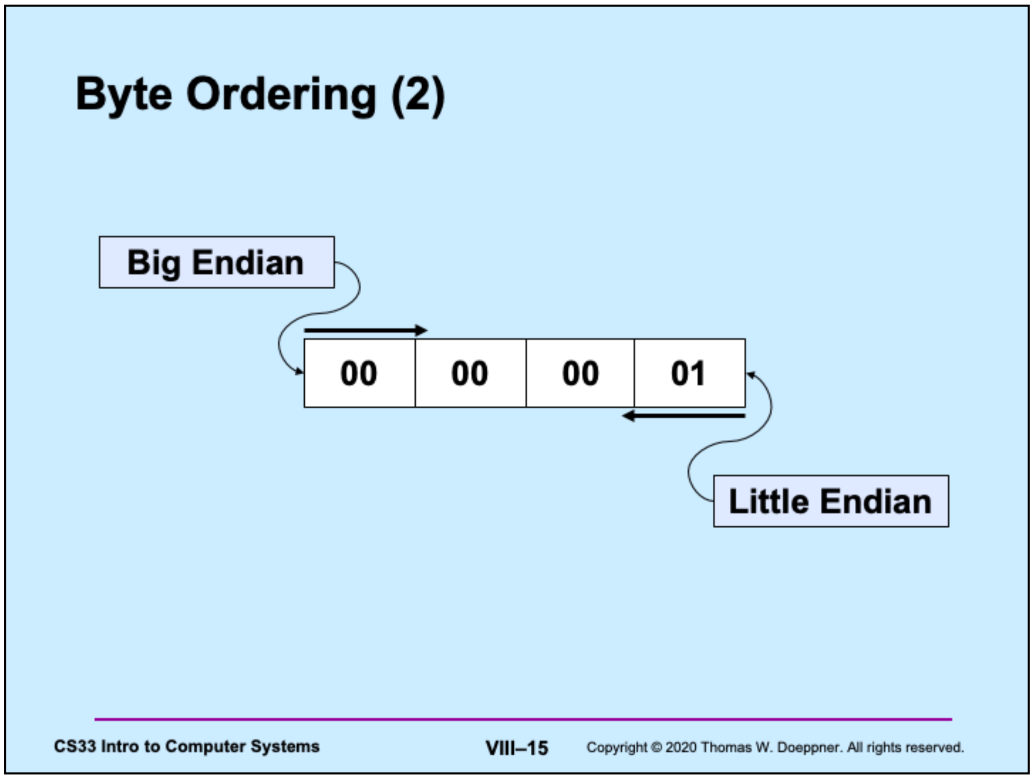


Byte Ordering

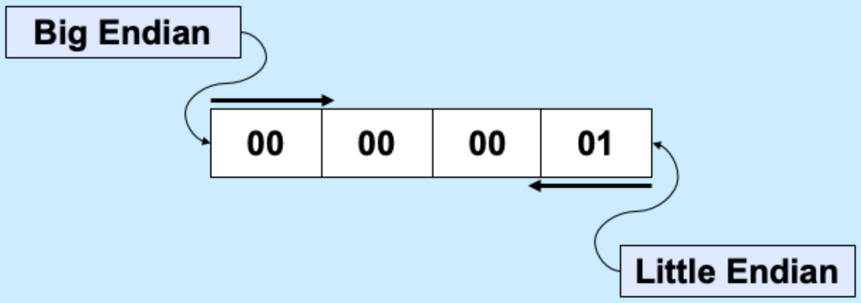
- Four-byte integer
  - 0x76543210
- Stored at location 0x100
  - which byte is at 0x100?
  - which byte is at 0x103?

?
10 32 54 76
0x100 0x101 0x102 0x103
Little-endian
76 54 32 10
0x100 0x101 0x102 0x103
Big-endian

Read “Gulliver’s Travels” by Jonathan Swift for an explanation of the egg.

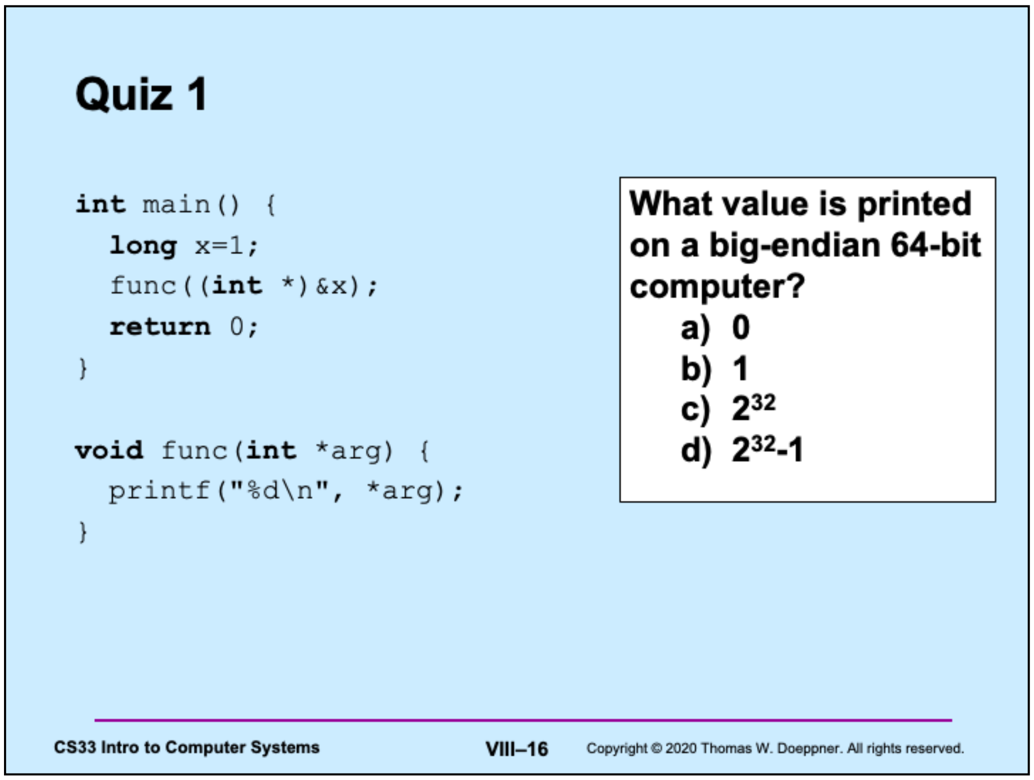


Byte Ordering (2)



Here we have a four-byte integer one. In the big-endian representation, the address of the integer is the address of the byte containing its most-significant bits (the big end), while in the little-endian representation, the address of the integer is the address of the byte containing its least-significant bits (the little end). Suppose we pass a pointer to this integer to some procedure. However, in a type-mismatch, the procedure assumes that what is passed it is a two-byte integer. On a big-endian system, it would think it was passed a zero, but on a little-endian system, it would think it was passed a one.

This is not an argument in favor of either approach, but simply an observation that behaviors could be different.



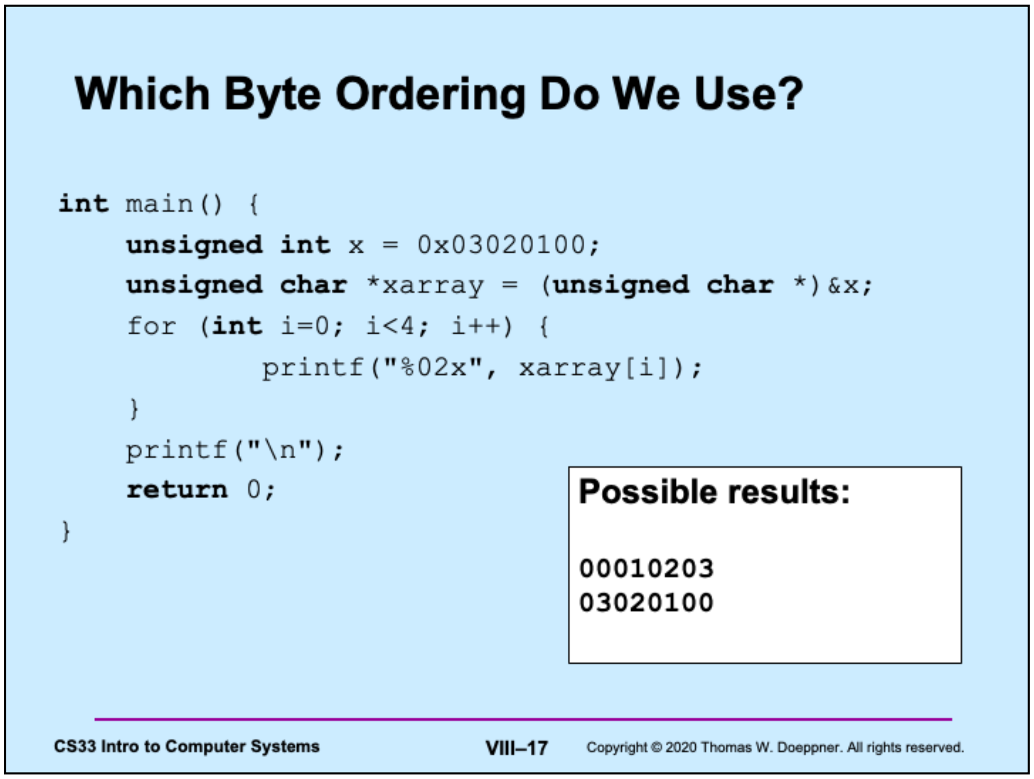
Quiz 1

```
int main() {
  long x=1;
  func((int *)&x);
  return 0;
}

void func(int *arg) {
  printf("%d\n", *arg);
}
```

What value is printed on a big-endian 64-bit computer?

a) 0
b) 1
c) 2^{32}
d) $2^{32}-1$



Which Byte Ordering Do We Use?

```
int main() {
    unsigned int x = 0x03020100;
    unsigned char *xarray = (unsigned char *)&x;
    for (int i=0; i<4; i++) {
            printf("%02x", xarray[i]);
    }
    printf("\n");
    return 0;
}
```

Possible results:

```
00010203
03020100
```

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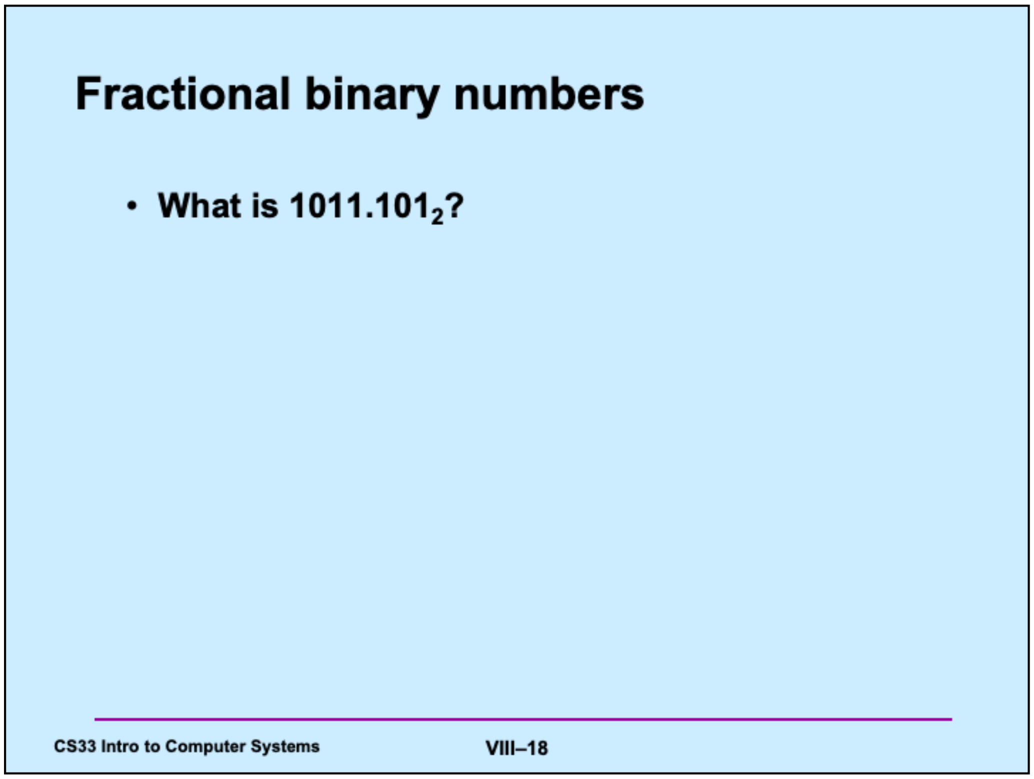
This code prints out the value of x, one byte at a time, starting with the byte at the lowest address (little end). On x86-based computers, it will print:

00010203

which means that the address of an int is the address of the byte containing its least significant digits (little endian).



Fractional binary numbers

- What is 1011.101_2?

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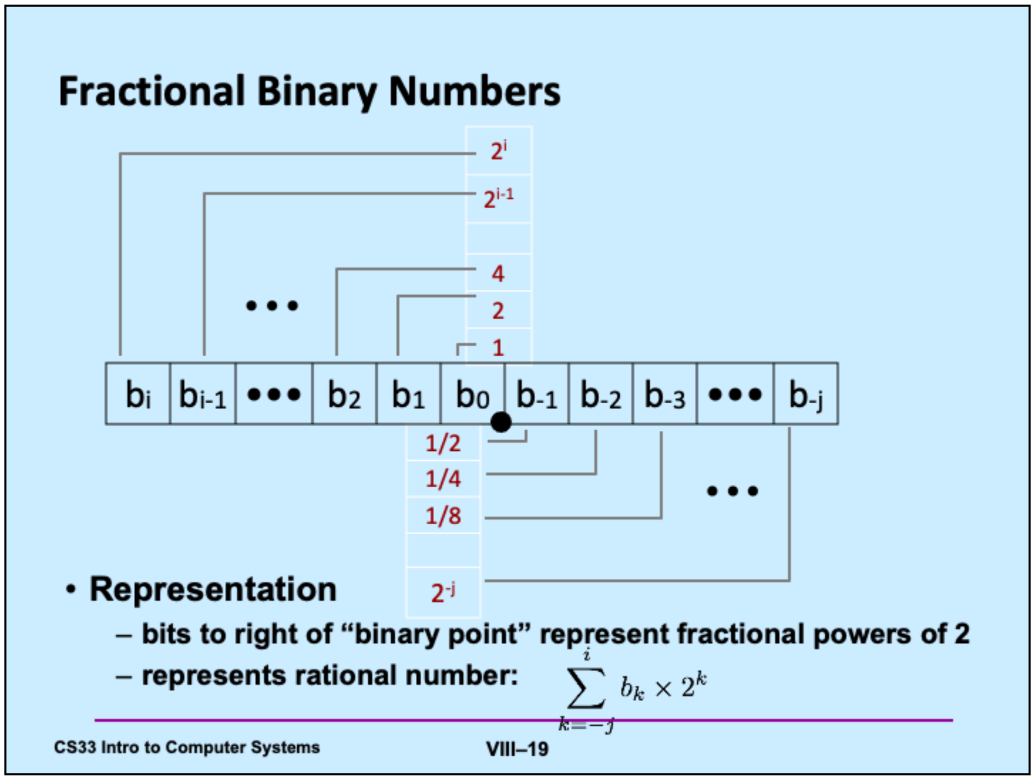
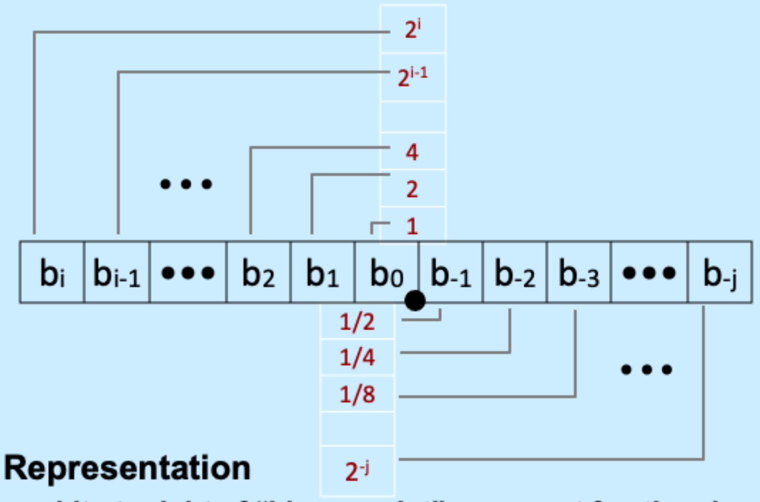


Fractional Binary Numbers



- **Representation**
  - **bits to right of "binary point" represent fractional powers of 2**
  - **represents rational number:** $\sum_{k=-j}^{i} b_k \times 2^k$

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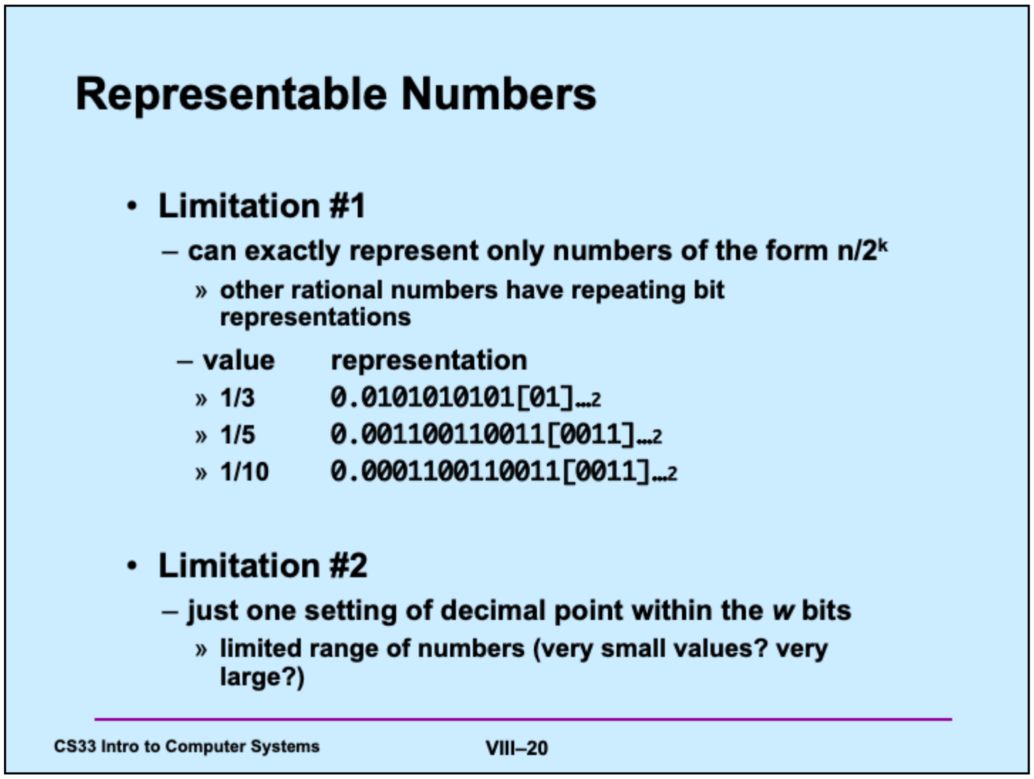

Representable Numbers

- **Limitation #1**
  - can exactly represent only numbers of the form $n/2^k$
    - » other rational numbers have repeating bit representations
  - value — representation

value	representation
» 1/3	`0.0101010101[01]...`$_2$
» 1/5	`0.001100110011[0011]...`$_2$
» 1/10	`0.0001100110011[0011]...`$_2$

- **Limitation #2**
  - just one setting of decimal point within the *w* bits
    - » limited range of numbers (very small values? very large?)

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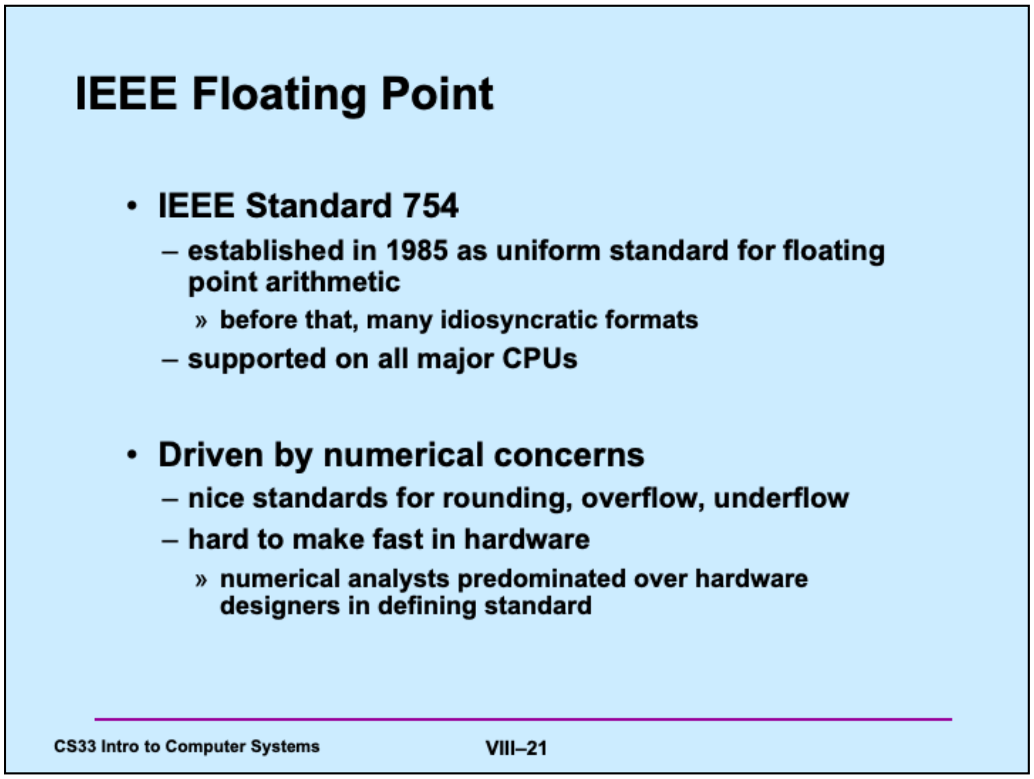

IEEE Floating Point

- IEEE Standard 754
  - established in 1985 as uniform standard for floating point arithmetic
    - » before that, many idiosyncratic formats
  - supported on all major CPUs
- Driven by numerical concerns
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - » numerical analysts predominated over hardware designers in defining standard

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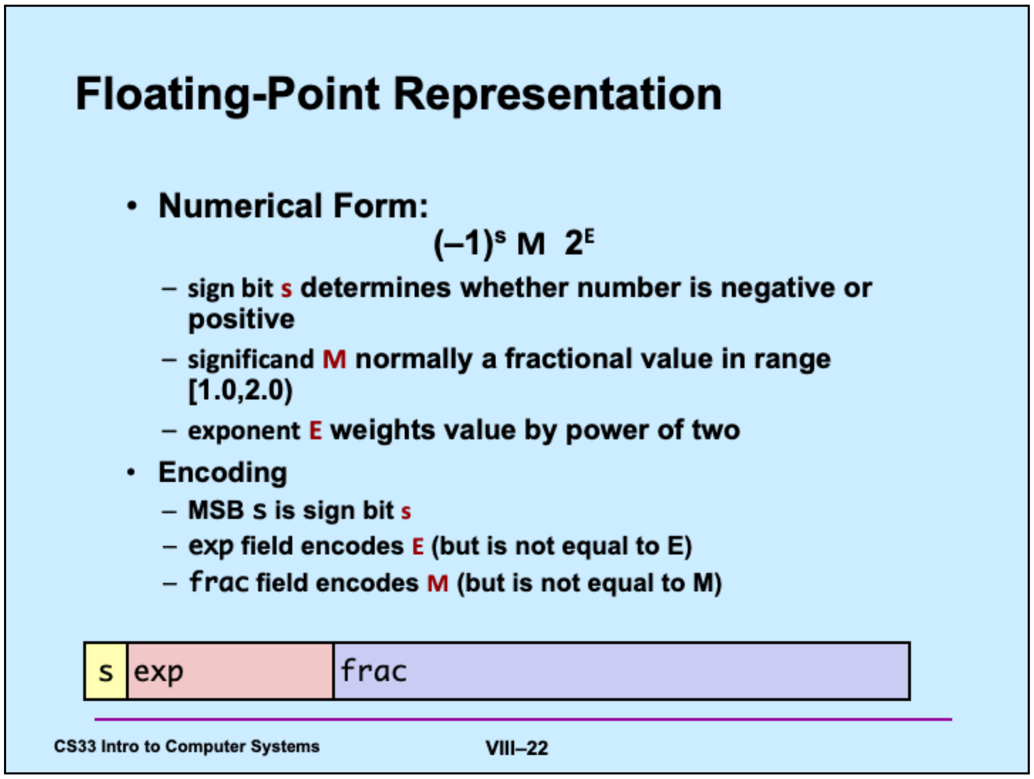
IEEE is the Institute for Electrical and Electronics Engineers (pronounced "eye triple e").



Floating-Point Representation

- Numerical Form:

$$(-1)^s\ M\ \ 2^E$$

  - sign bit s determines whether number is negative or positive
  - significand M normally a fractional value in range [1.0,2.0)
  - exponent E weights value by power of two
- Encoding
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

s	exp	frac

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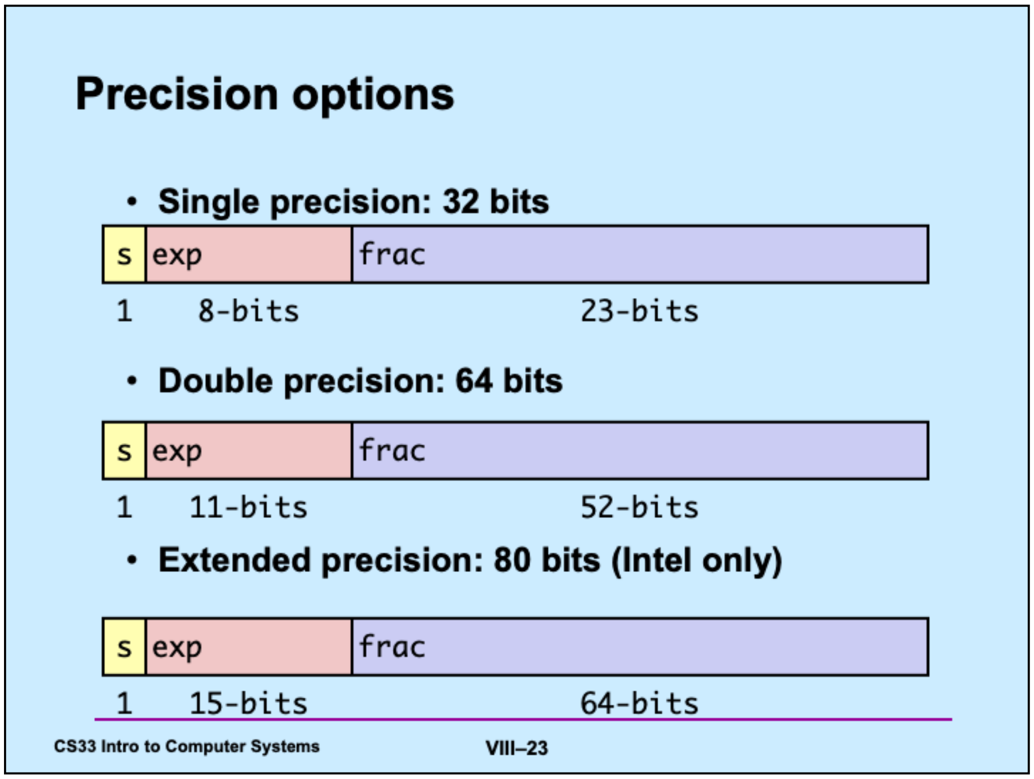
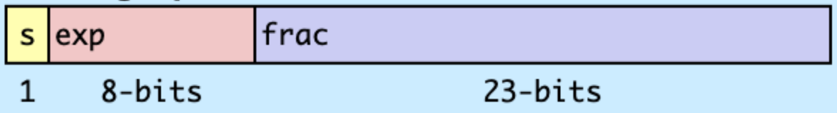
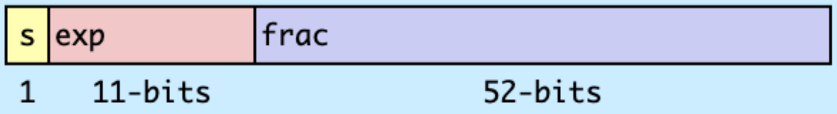


Precision options

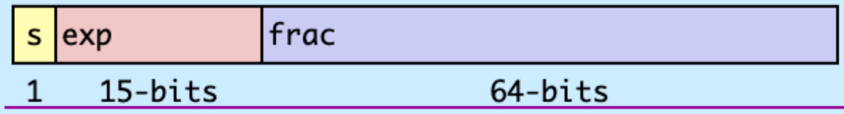
- **Single precision: 32 bits**



- **Double precision: 64 bits**



- **Extended precision: 80 bits (Intel only)**



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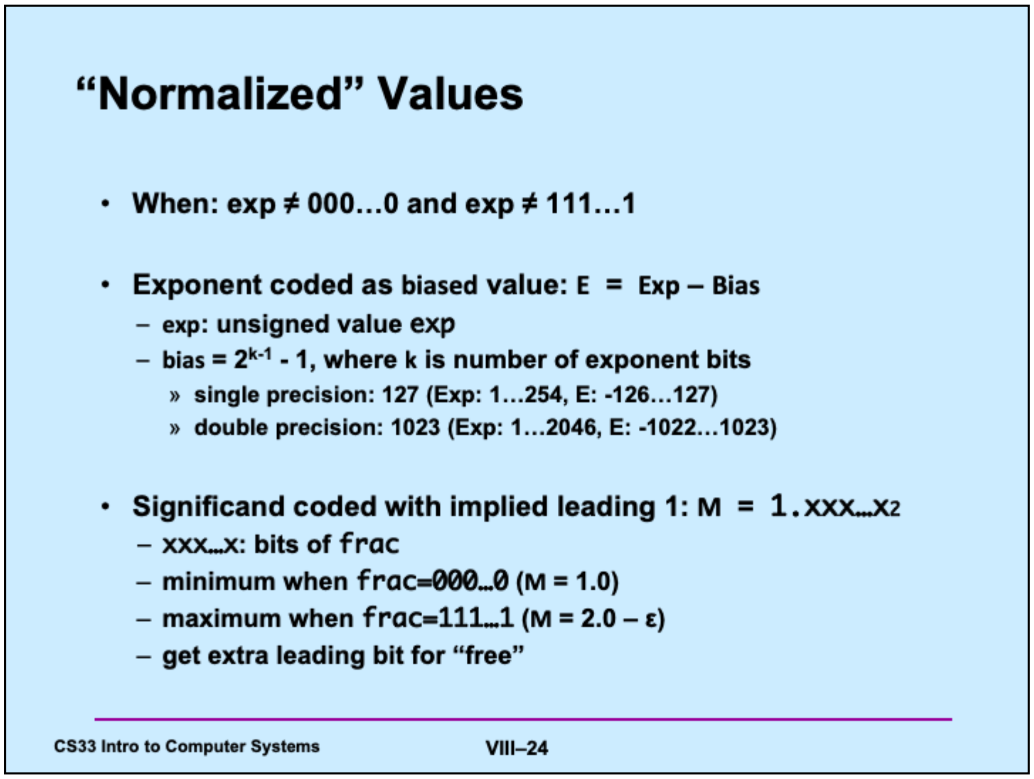
On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.



"Normalized" Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for "free"

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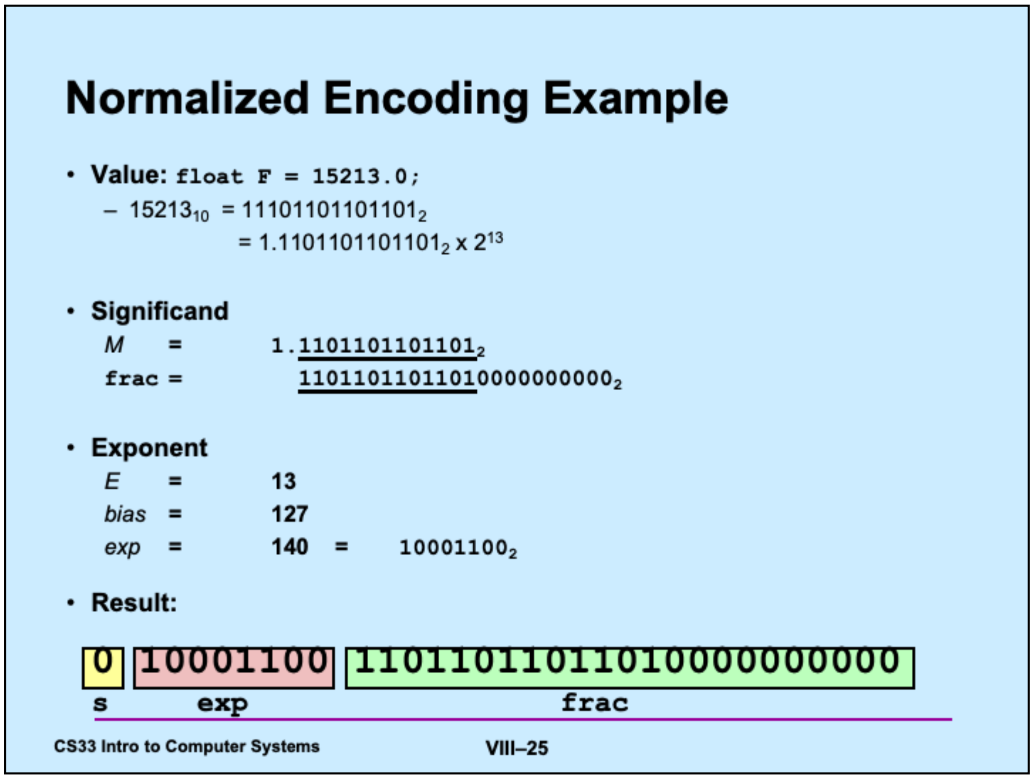

Normalized Encoding Example

- **Value:** **`float F = 15213.0;`**
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**
  - *M* = $1.\underline{1101101101101}_2$
  - **`frac`** = $\underline{1101101101101}0000000000_2$

- **Exponent**
  - *E* = **13**
  - *bias* = **127**
  - *exp* = **140** = 10001100_2

- **Result:**

$$\underset{\texttt{s}}{\boxed{0}}\ \underset{\texttt{exp}}{\boxed{10001100}}\ \underset{\texttt{frac}}{\boxed{11011011011010000000000}}$$

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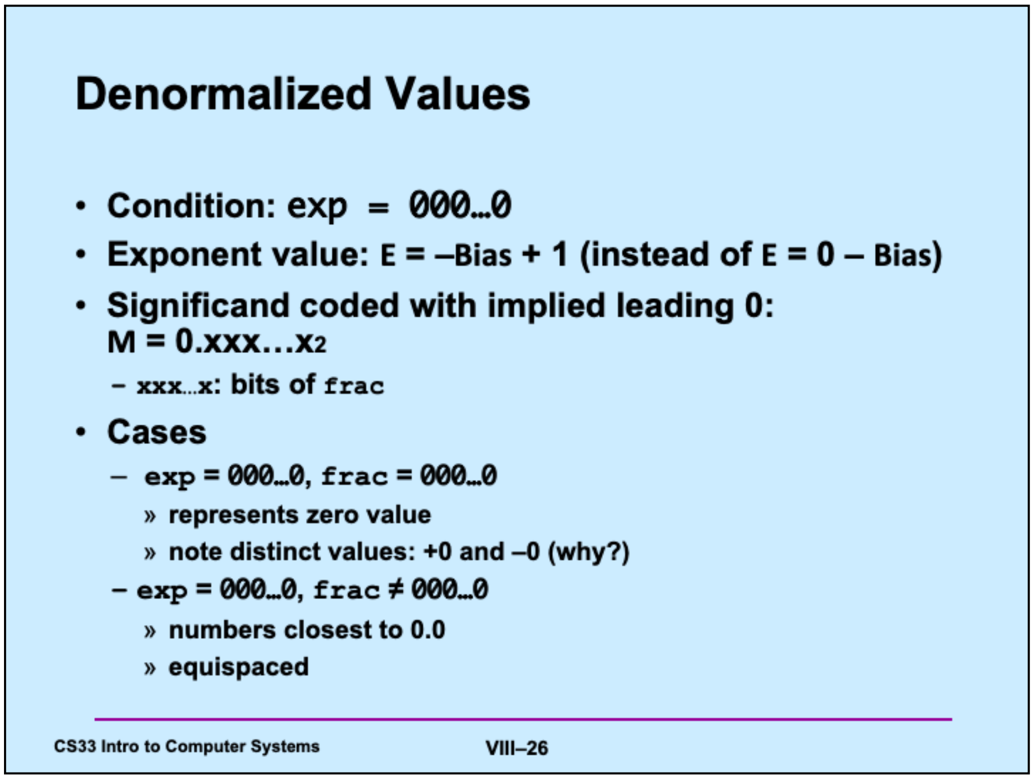

Denormalized Values

- Condition: exp = `000…0`
- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0: M = $0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = `000…0`, `frac` = `000…0`
    - » represents zero value
    - » note distinct values: +0 and –0 (why?)
  - `exp` = `000…0`, `frac` ≠ `000…0`
    - » numbers closest to 0.0
    - » equispaced

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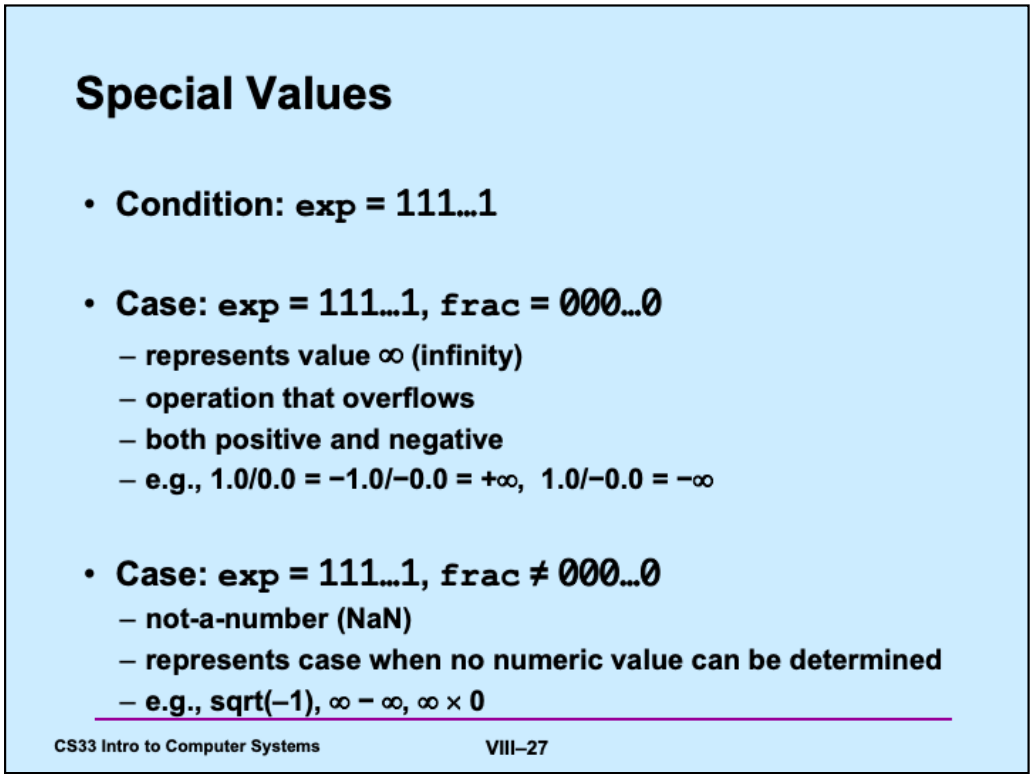
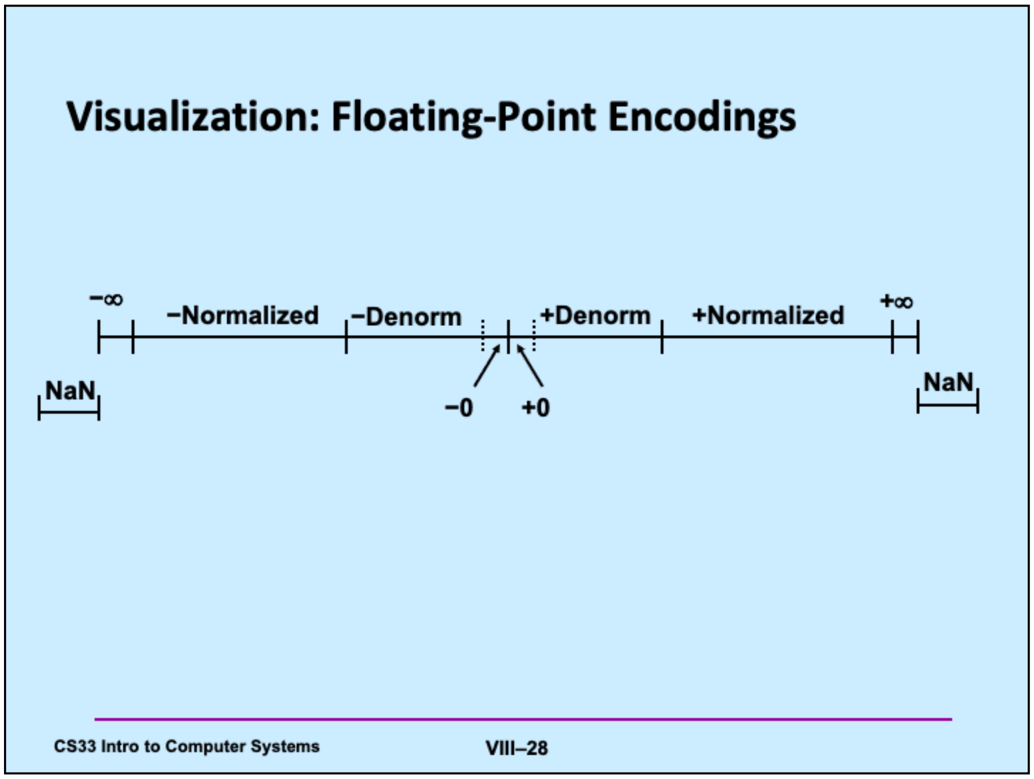


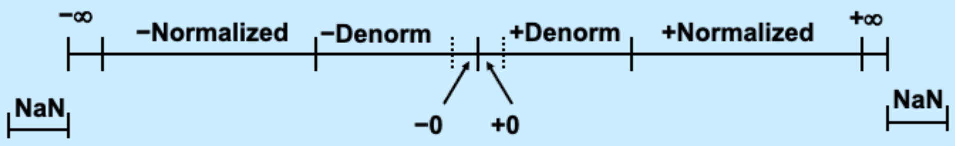
Special Values

- Condition: **`exp`** = 111…1
- Case: **`exp`** = 111…1, **`frac`** = 000…0
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- Case: **`exp`** = 111…1, **`frac`** ≠ 000…0
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

Supplied by CMU.



Visualization: Floating-Point Encodings



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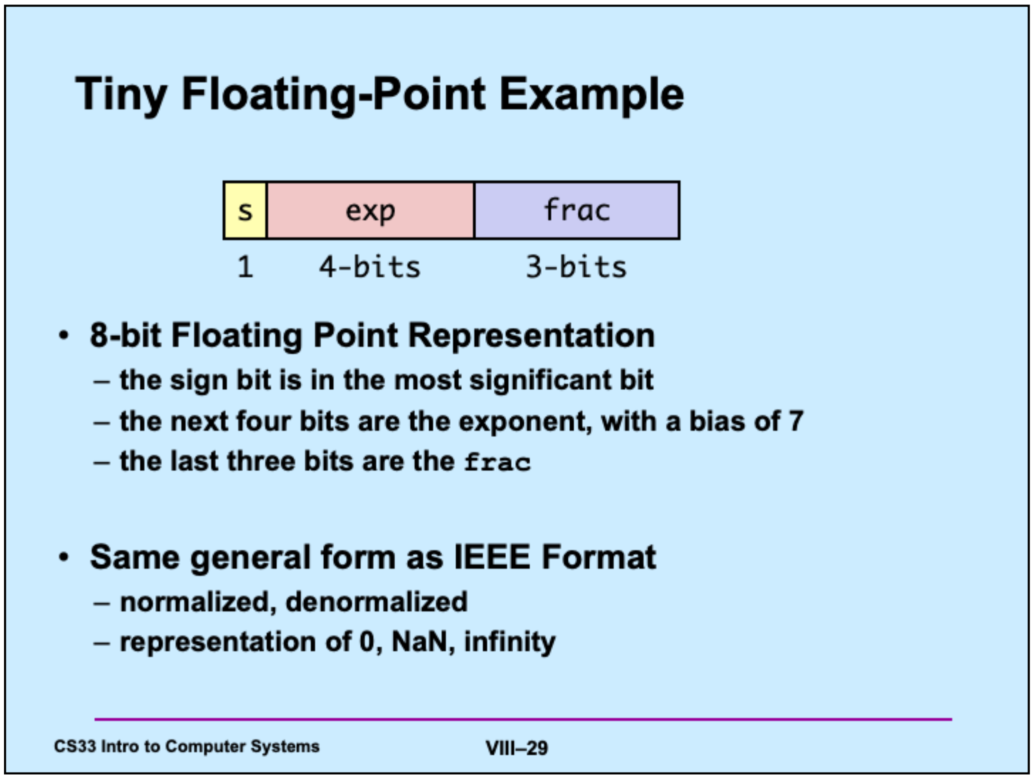
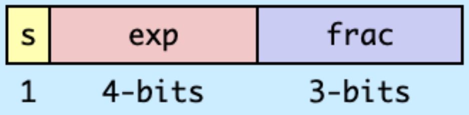
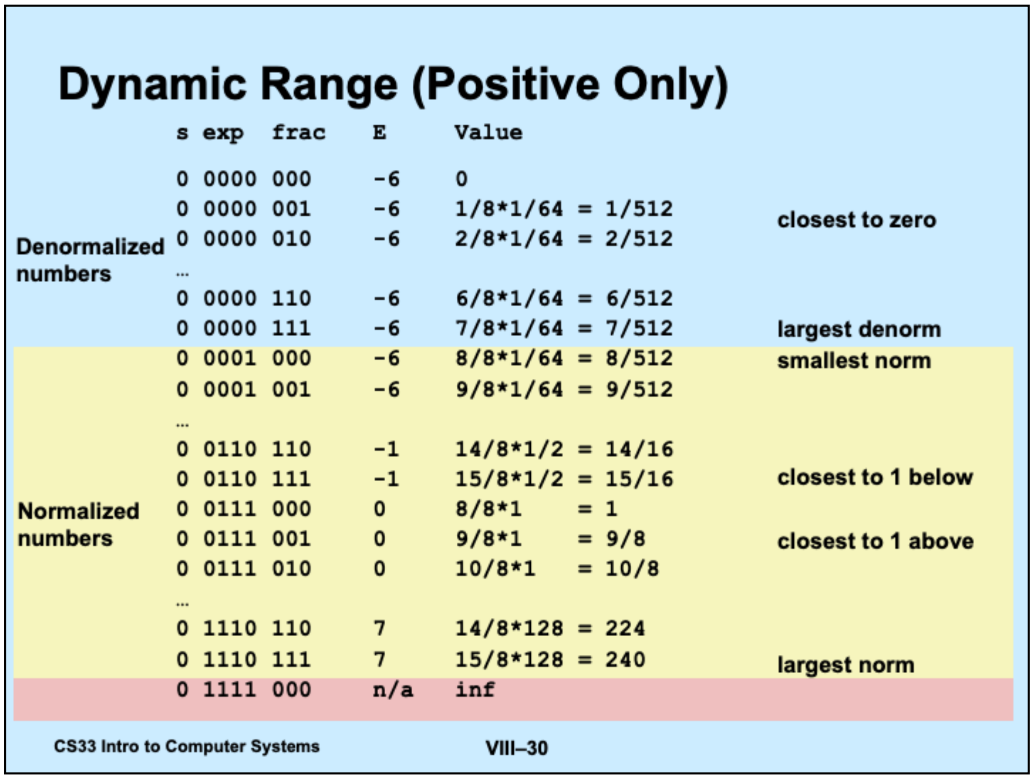


Tiny Floating-Point Example



- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Supplied by CMU.



Dynamic Range (Positive Only)

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

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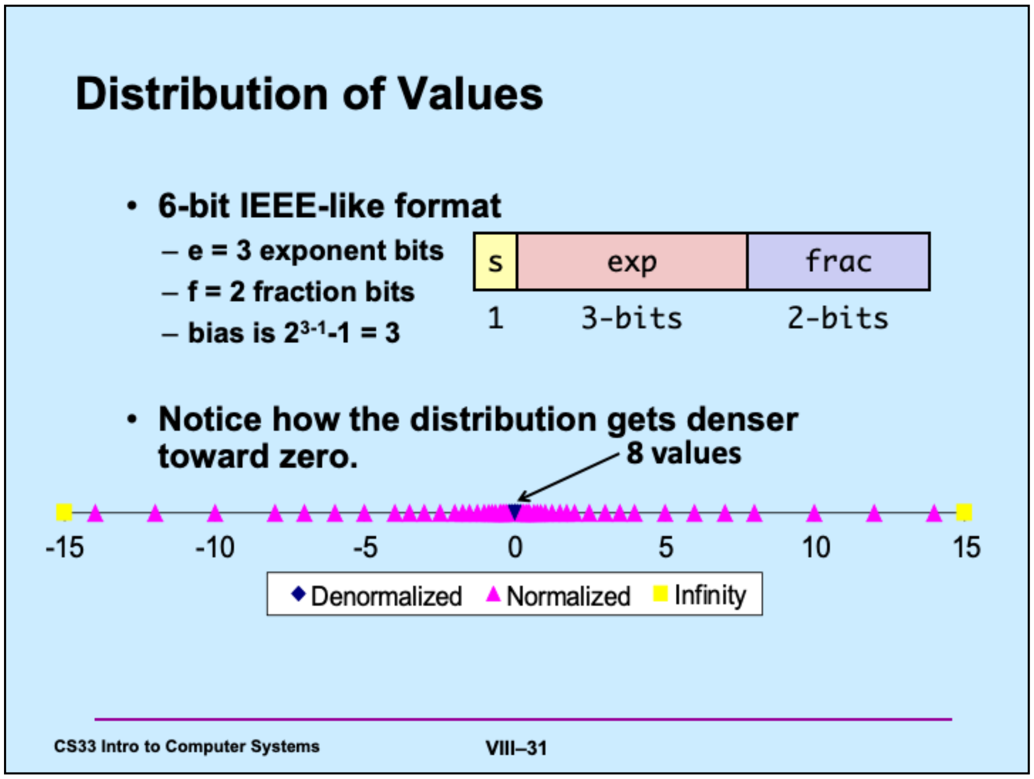
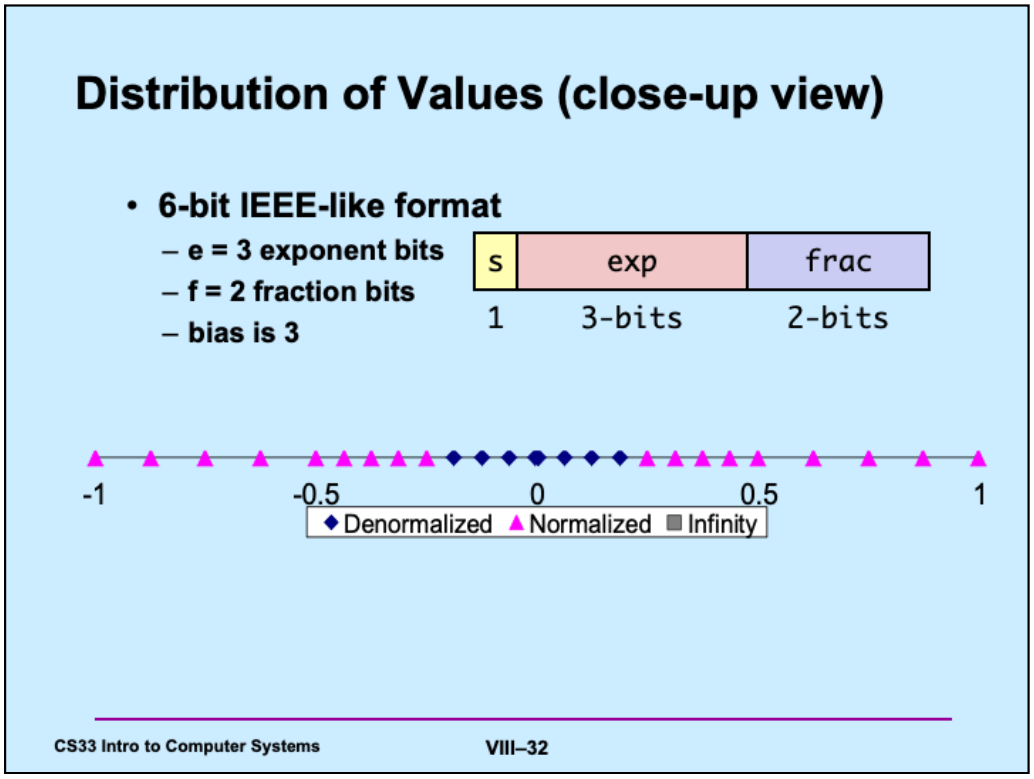


Distribution of Values

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is $2^{3-1}-1 = 3$**

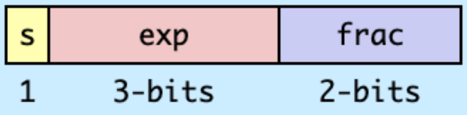
- **Notice how the distribution gets denser toward zero.**

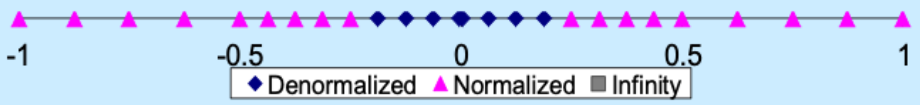
Supplied by CMU.



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**





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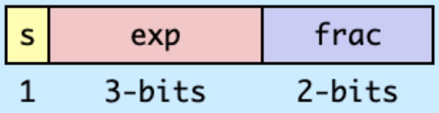
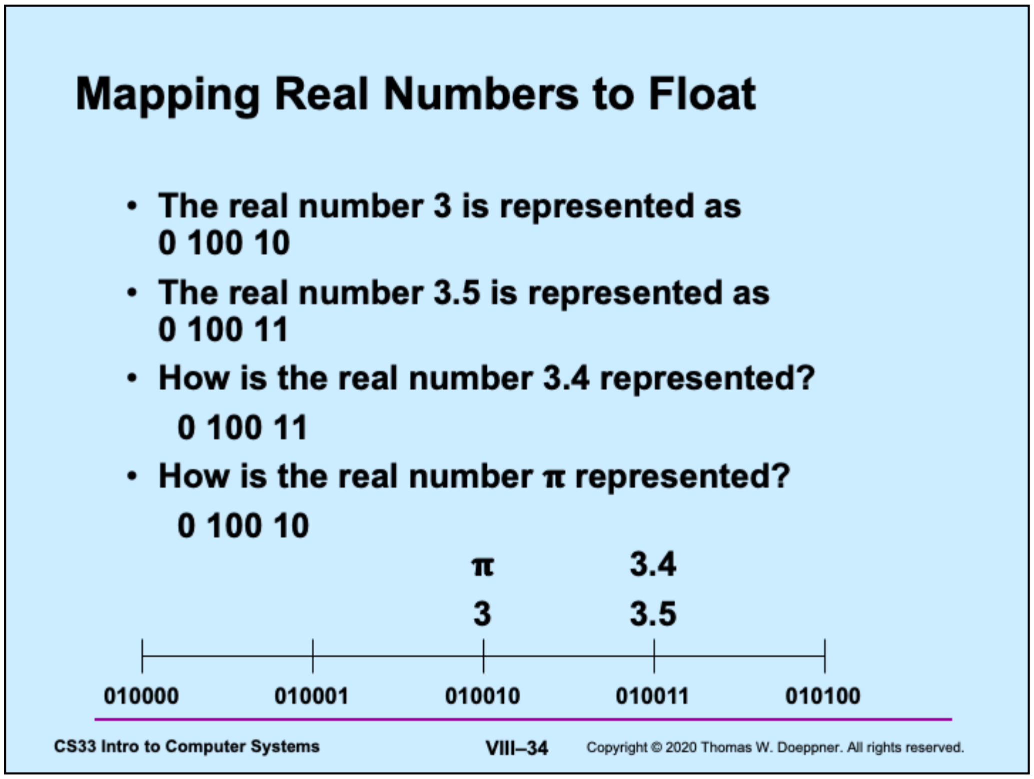
Quiz 2

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3



What number is represented by 0 011 10?

a) 12
b) 1.5
c) .5
d) none of the above

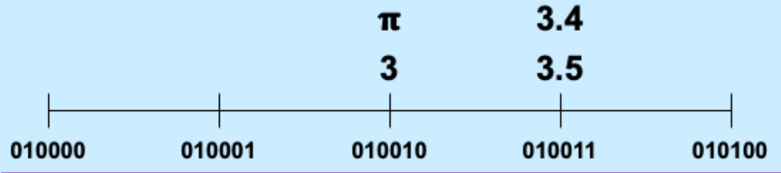


Mapping Real Numbers to Float

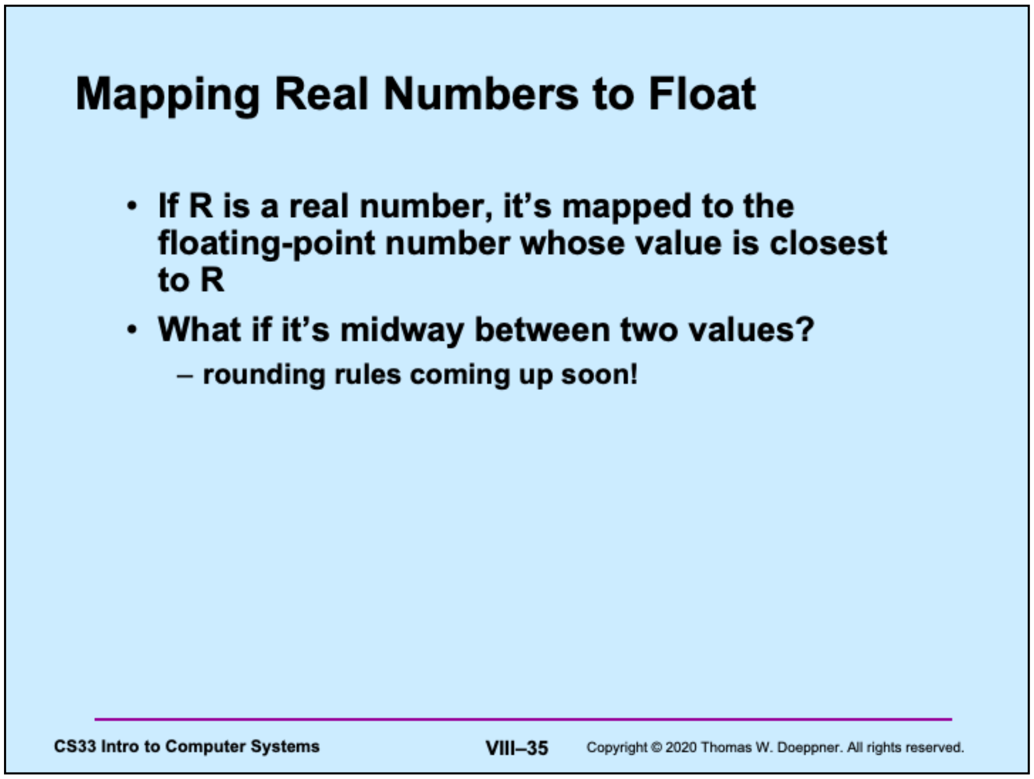
- The real number 3 is represented as 0 100 10
- The real number 3.5 is represented as 0 100 11
- How is the real number 3.4 represented?

  0 100 11
- How is the real number π represented?

  0 100 10

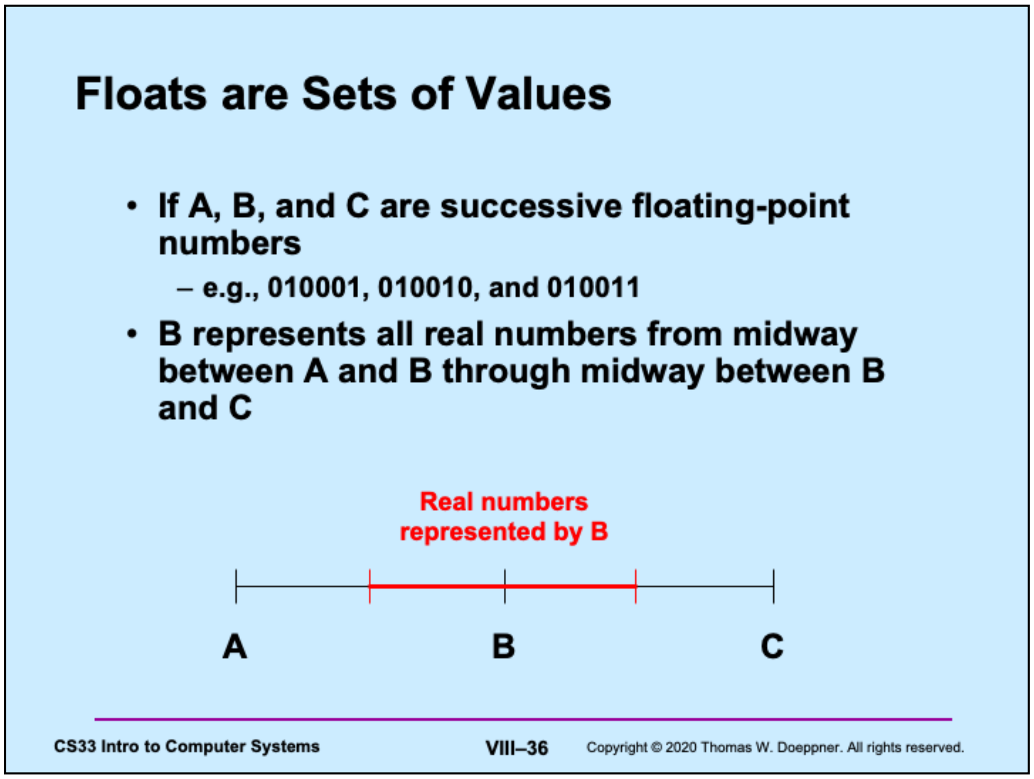


We're assuming here the six-bit floating-point format.



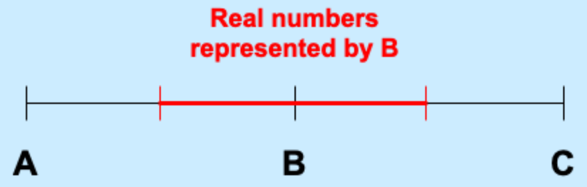
Mapping Real Numbers to Float

- If R is a real number, it's mapped to the floating-point number whose value is closest to R
- What if it's midway between two values?
  - rounding rules coming up soon!

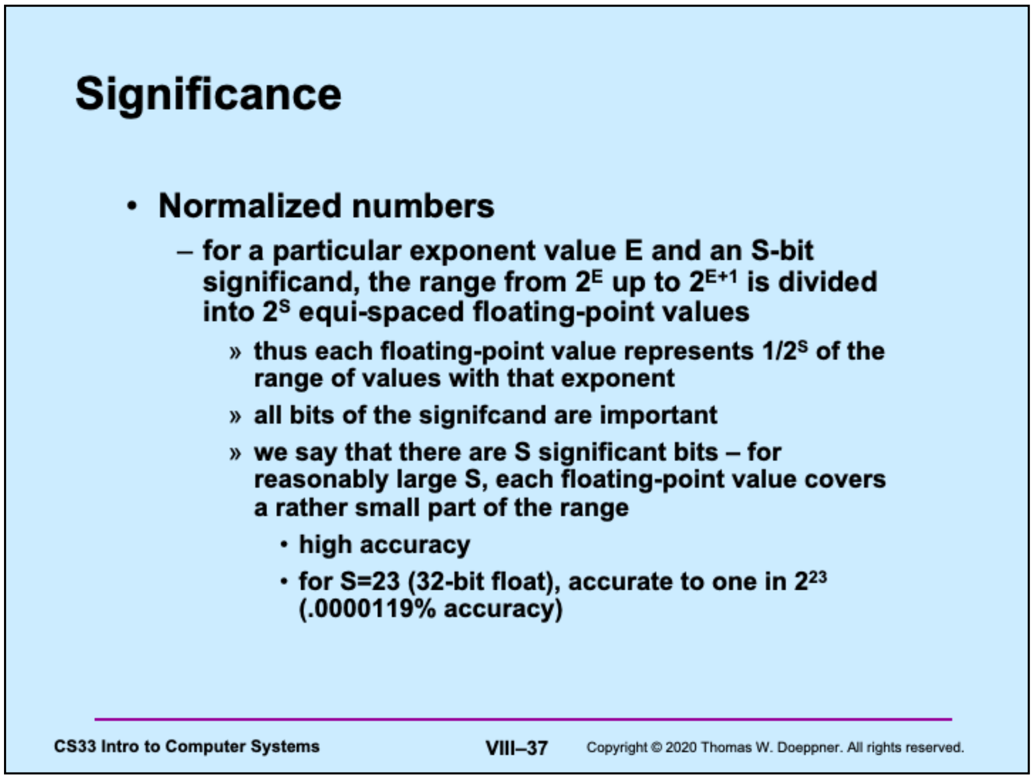


Floats are Sets of Values

- If A, B, and C are successive floating-point numbers
  - e.g., 010001, 010010, and 010011
- B represents all real numbers from midway between A and B through midway between B and C

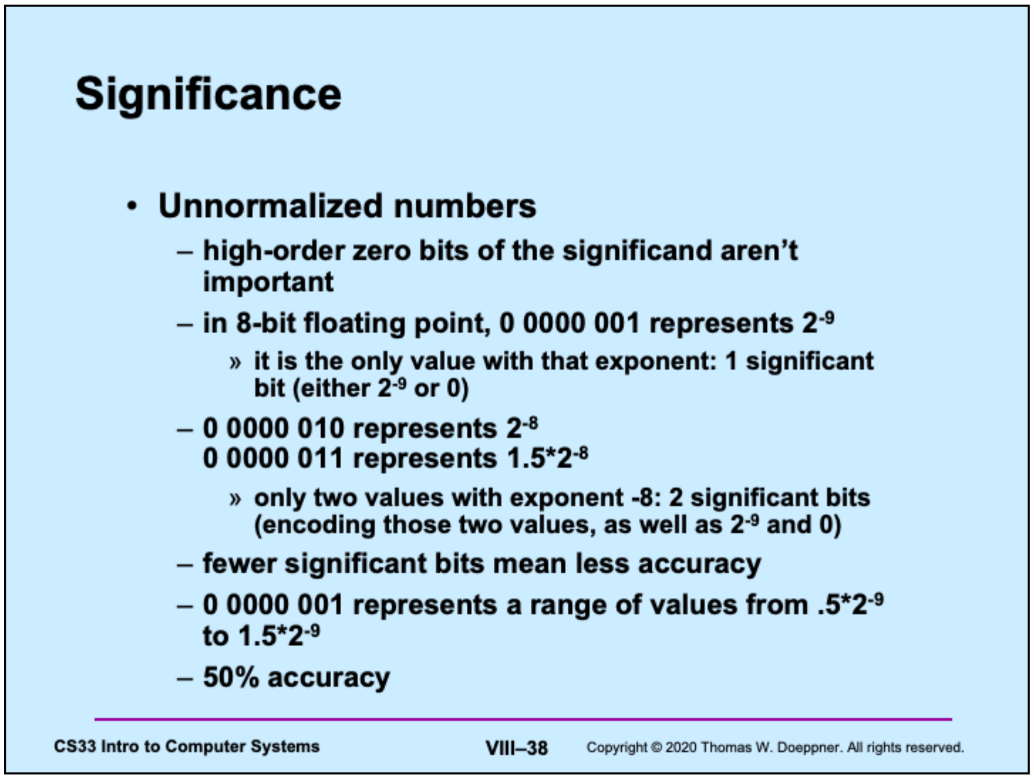


Note that we still have to discuss rounding so as to accommodate values that are equidistant from A and B or from B and C.



Significance

- **Normalized numbers**
  - for a particular exponent value E and an S-bit significand, the range from 2^E up to 2^{E+1} is divided into 2^S equi-spaced floating-point values
    - » thus each floating-point value represents $1/2^S$ of the range of values with that exponent
    - » all bits of the signifcand are important
    - » we say that there are S significant bits – for reasonably large S, each floating-point value covers a rather small part of the range
      - high accuracy
      - for S=23 (32-bit float), accurate to one in 2^{23} (.0000119% accuracy)



Significance

- **Unnormalized numbers**
  - high-order zero bits of the significand aren't important
  - in 8-bit floating point, 0 0000 001 represents 2^{-9}
    - » it is the only value with that exponent: 1 significant bit (either 2^{-9} or 0)
  - 0 0000 010 represents 2^{-8}
    0 0000 011 represents $1.5*2^{-8}$
    - » only two values with exponent -8: 2 significant bits (encoding those two values, as well as 2^{-9} and 0)
  - fewer significant bits mean less accuracy
  - 0 0000 001 represents a range of values from $.5*2^{-9}$ to $1.5*2^{-9}$
  - 50% accuracy

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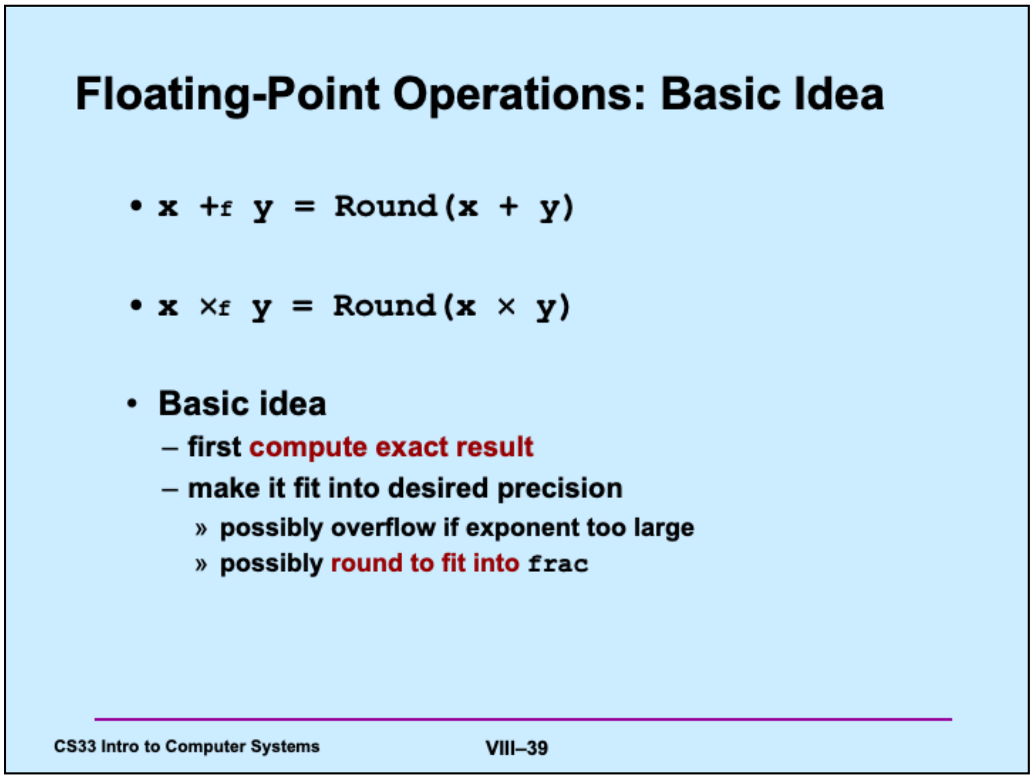
Recall that the bias for the exponent of 8-bit IEEE FP is 7, thus for unnormalized numbers the actual exponent is -6 (-bias+1). The significand has an implied leading 0, thus 0 0000 001 represents $2^{-6} * 2^{-3}$.

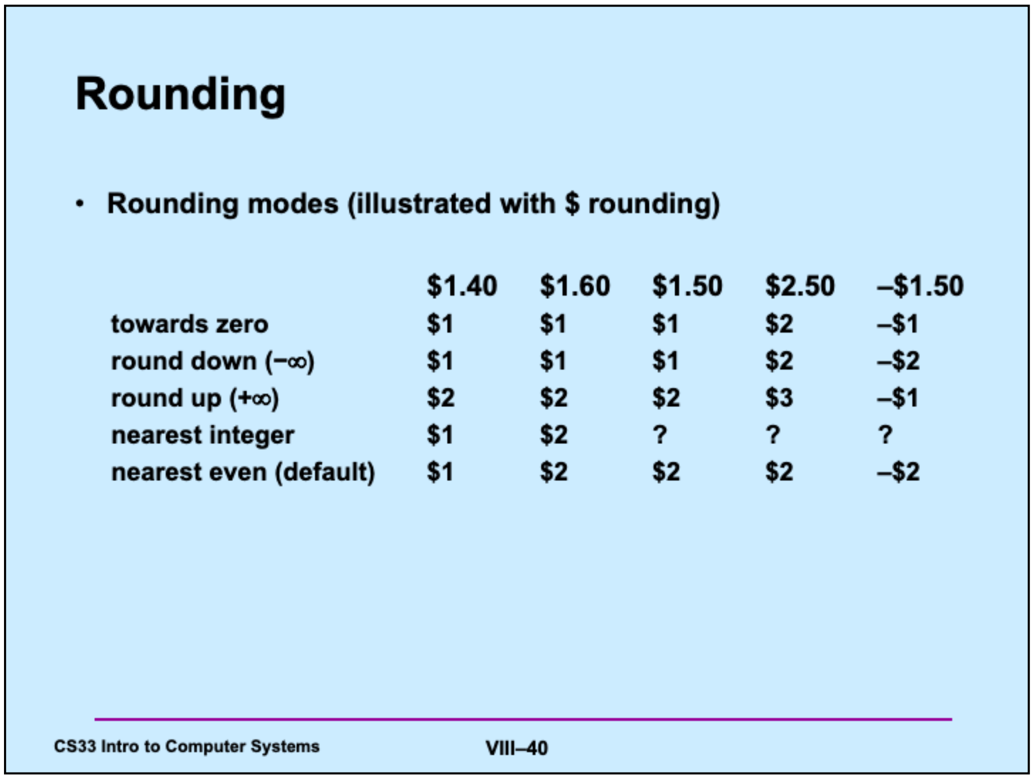
With 8-bit IEEE FP. the value 0 0000 01 is interpreted as 2-9, But the number represented could be 50% or 50% more.



Floating-Point Operations: Basic Idea

- `x` $+_f$ `y = Round(x + y)`
- `x` $\times_f$ `y = Round(x` $\times$ `y)`
- Basic idea
  - first compute exact result
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly round to fit into `frac`

Supplied by CMU.



Rounding

- Rounding modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest integer	$1	$2	?	?	?
nearest even (default)	$1	$2	$2	$2	–$2

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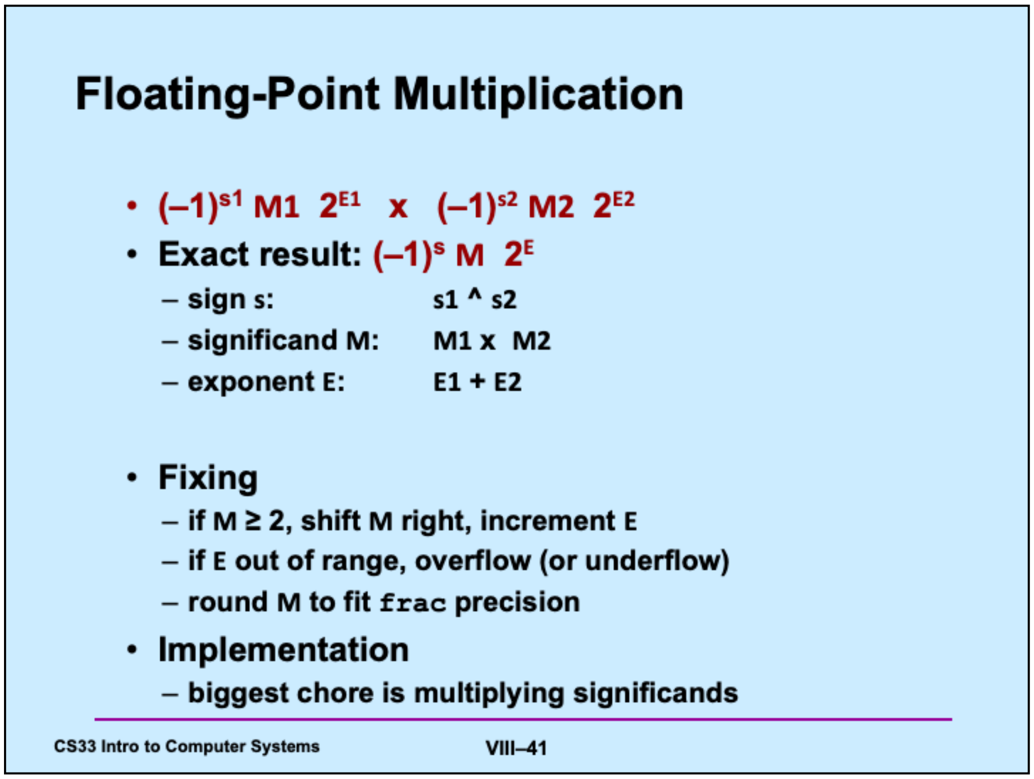
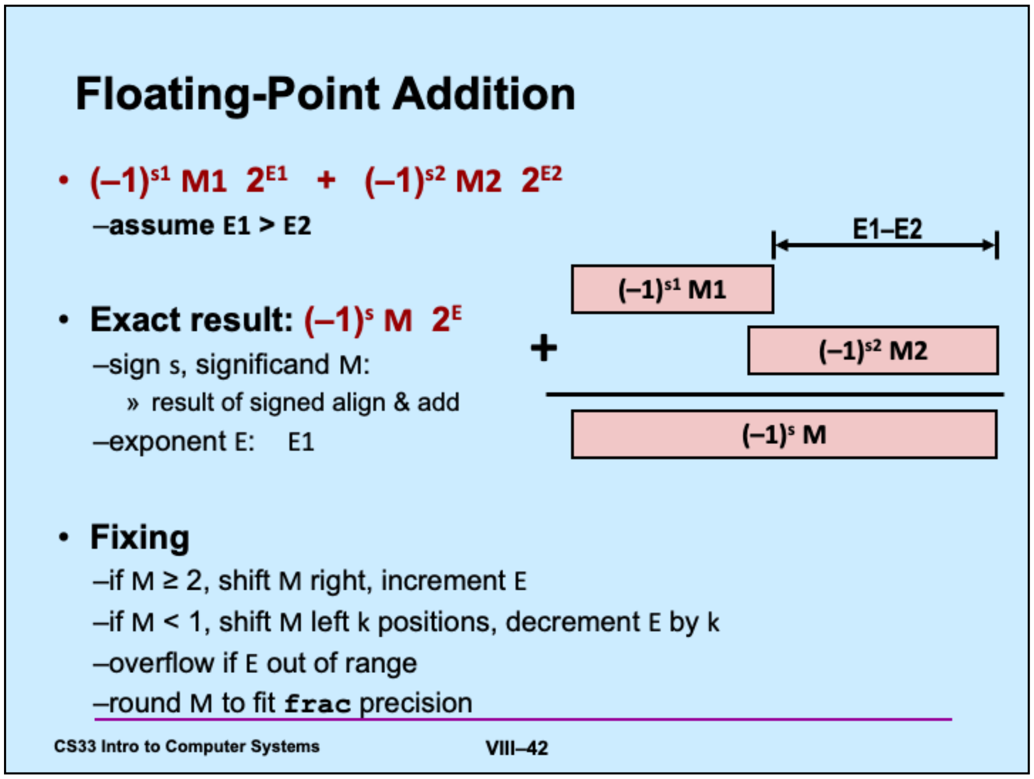


Floating-Point Multiplication

- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- Exact result: $(-1)^{s}$ M 2^{E}
  - sign s: s1 ^ s2
  - significand M: M1 x M2
  - exponent E: E1 + E2
- Fixing
  - if M ≥ 2, shift M right, increment E
  - if E out of range, overflow (or underflow)
  - round M to fit `frac` precision
- Implementation
  - biggest chore is multiplying significands

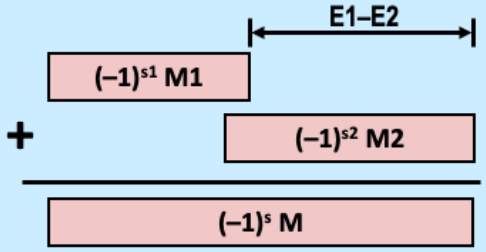
Supplied by CMU.

Note that to compute E, one must first convert $\exp_1$ and $\exp_2$ to E_1 and E_2, then add them them together and check for underflow or overflow (corresponding to -∞ and +∞), and then convert to exp.



Floating-Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - –assume E1 > E2
- **Exact result:** $(-1)^{s}\ M\ \ 2^{E}$
  - –sign s, significand M:
    - » result of signed align & add
  - –exponent E: E1



- **Fixing**
  - –if M ≥ 2, shift M right, increment E
  - –if M < 1, shift M left k positions, decrement E by k
  - –overflow if E out of range
  - –round M to fit `frac` precision

Supplied by CMU.

Note that, by default, overflow results in either +∞ or -∞.