Introduction to statistics: Contrast coding

Shravan Vasishth

Universität Potsdam vasishth@uni-potsdam.de http://www.ling.uni-potsdam.de/~vasishth

April 12, 2020

The goals of this lecture

Goals of this lecture:

- Understand contrast coding for two condition experiments.
- ▶ Learn different (custom) contrast codings for more complex designs, such as 2×2 factorial designs.

Typical experiment designs

- ▶ Two conditions: 1×2 "factorial design"
- ▶ Three conditions: 1×3 "factorial design"
- Four conditions: typically 2×2 factorial design, sometimes 1×4
- ▶ More complex designs like 2×3 , $2 \times 2 \times 2$, etc.

Typical experiment designs

- ▶ We will not go beyond 2×2 .
- My advice for your own work is to keep it simple: four conditions maximum.
- ➤ Sometimes one has to go beyond such simple designs, but do that only if you really understand that you are introducing possibly intractable complexity in the design (this will be discussed later, lectures on simulating data).

Typical experiment designs

- A classic beginner mistake (I have done this) is to design an experiment without any clear hypotheses in mind.
- After the data come in, we start speculating and trying out different pairwise comparisons.
- ► A better way: develop an analysis plan in advance using simulated data. This requires some mental discipline.
- ▶ Define *a priori* comparisons.
- Pre-registration is a useful tool (to be discussed later).

We can generate between-subjects data using a function called mixedDesign.

```
library(dplyr)
# load mixedDesign function for simulating data
source("functions/mixedDesign.v0.6.3.R")
M <- matrix(c(0.8, 0.4), nrow=2, ncol=1, byrow=FALSE)
set.seed(1)
# set seed of random number generator for replicability</pre>
```

Generate between-subjects data:

```
simdat <- mixedDesign(B=2, W=NULL, n=5,</pre>
                     M=M, SD=.20, long = TRUE)
##
## Attaching package: 'MASS'
## The following object is masked from 'package:dplyr':
##
      select
##
t(xtabs(~id+B_A,simdat))
## id
## B_A 1 2 3 4 5 6 7 8 9 10
## A1 1 1 1 1 1 0 0 0 0
## A2 0 0 0 0 0 1 1 1 1 1
```

We can generate within-subjects data as well, but some changes to the function are needed:

Using between-subjects simulated data here, rename the columns:

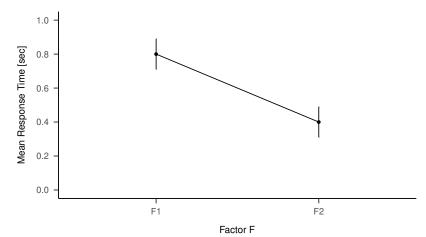
```
names(simdat)[1] \leftarrow "F" # Rename B_A to F(actor)
levels(simdat$F) <- c("F1", "F2")</pre>
head(simdat)
## Fid DV
## 1 F1 1 0.99664
## 2 F1 2 0.84693
## 3 F1 3 0.71202
## 4 F1 4 0.49940
## 5 F1 5 0.94502
## 6 F2 6 0.18293
```

```
str(simdat)
## 'data.frame': 10 obs. of 3 variables:
## $ F : Factor w/ 2 levels "F1", "F2": 1 1 1 1 1 2 2 2 2 2
## $ id: Factor w/ 10 levels "1", "2", "3", "4", ...: 1 2 3 4 9
## $ DV: num 0.997 0.847 0.712 0.499 0.945 ...
```

```
table1 <- simdat %>% group_by(F) %>%
 # Table for main effect F
  summarize(N=n(), M=mean(DV),
           SD=sd(DV), SE=SD/sqrt(N))
table1
## # A tibble: 2 x 5
## F N M SD SE
## <fct> <int> <dbl> <dbl> <dbl>
## 1 F1 5 0.8 0.20 0.0894
## 2 F2 5 0.4 0.2 0.0894
```

[1] 0.6 11/6

(GM <- mean(table1\$M)) # Grand Mean



Intercept = $\hat{\mu}_1$ = estimated mean for F1 Slope (FF2) = $\hat{\mu}_2 - \hat{\mu}_1$ = estim. mean for F2 - estim. mean for F1 (1)

```
contrasts(simdat$F)

## F2
## F1 0
## F2 1
```

The TREATMENT CONTRAST expresses the null hypothesis that the difference in means between the two levels of the factor F is 0; formally, the null hypothesis H_0 is that H_0 : $\beta_1 = 0$:

$$H_0: -1 \cdot \mu_{F1} + 1 \cdot \mu_{F2} = 0 \tag{2}$$

or equivalently:

$$H_0: \mu_{F2} - \mu_{F1} = 0$$
 (3)

The intercept in the TREATMENT CONTRAST expresses a null hypothesis that is usually of no interest: that the mean in condition F1 of the factor F is 0. Formally, the null hypothesis is $H_0: \beta_0 = 0$:

$$H_0: 1 \cdot \mu_{F1} + 0 \cdot \mu_{F2} = 0 \tag{4}$$

or equivalently:

$$H_0: \mu_{F1} = 0.$$
 (5)

Level-ordering is alphabetical and can be changed:

```
simdat$Fb <- factor(simdat$F, levels = c("F2","F1"))
contrasts(simdat$Fb)

## F1
## F2 0
## F1 1</pre>
```

```
(contrasts(simdat$F) <- c(-0.5,+0.5))

## [1] -0.5 0.5

m1_mr <- lm(DV ~ F, simdat)
```

F1

##

```
summary(m1_mr)
##
## Call:
## lm(formula = DV ~ F, data = simdat)
##
## Residuals:
## Min 1Q Median 3Q Max
## -0.3006 -0.1755 0.0524 0.1530 0.2084
##
## Coefficients:
##
             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.6000 0.0632 9.49 1.3e-05
```

-0.4000 0.1265 -3.16 0.013 $^{18/6}$ 18/61

Intercept =
$$(\hat{\mu}_1 + \hat{\mu}_2)/2$$
 = estimated mean of F1 and F2 Slope (F1) = $\hat{\mu}_2 - \hat{\mu}_1$ = est. mean for F2 – estim. mean for F1 (6)

Sum contrasts express the null hypothesis that the difference in means between the two levels of factor F is 0; formally, the null hypothesis H_0 is that

$$H_0: -1 \cdot \mu_{F1} + 1 \cdot \mu_{F2} = 0 \tag{7}$$

This is the same hypothesis that was also tested by the slope in the treatment contrast.

The intercept, however, now expresses a different hypothesis about the data: it expresses the null hypothesis that the average of the two conditions F1 and F2 is 0:

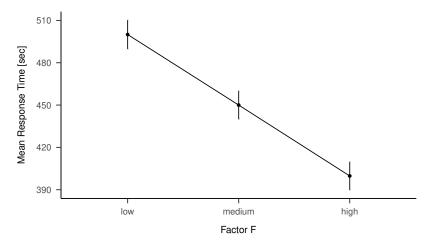
$$H_0: 1/2 \cdot \mu_{F1} + 1/2 \cdot \mu_{F2} = \frac{\mu_{F1} + \mu_{F2}}{2} = 0$$
 (8)

Summary: Treatment and sum contrasts

- ► To summarize, TREATMENT CONTRASTS and SUM CONTRASTS are two possible ways to parameterize the difference between two groups; they test different hypotheses.
- ► TREATMENT CONTRASTS compare one or more means against a baseline condition, whereas SUM CONTRASTS allow us to determine whether we can reject the null hypothesis that a condition's mean is the same as the GM (in the two-group case, this also implies a hypothesis test that the two group means are the same).

```
head(simdat2)
##
      Fid DV
## 1 low 1 497
## 2 low 2 474
## 3 low 3 523
## 4 low 4 506
## 5 medium 5 422
## 6 medium 6 467
table.word <- simdat2 %>% group_by(F) %>%
 summarise(N = length(DV), M = mean(DV),
          SD = sd(DV), SE = sd(DV)/sqrt(N))
```

```
table.word1 <- table.word
names(table.word1) <- c("Factor F","N data points","Estimated means",</pre>
                      "Standard deviations". "Standard errors")
table.word1
## # A tibble: 3 x 5
## `Factor F` `N data points` `Estimated mean... `Standard devia...
## <fct>
                <int>
                                    <dbl>
                                                     <dbl>
## 1 low
                                       500
                                                      20.4
## 2 medium
                                       450
                                                      19.9
## 3 high
                                        400.
                                                      19.9
## # ... with 1 more variable: `Standard errors` <dbl>
```



The estimated means reflect our assumptions about the true means in the data simulation: Response times decrease with increasing word frequency.

Alternative syntax:

- ▶ Notice that the ANOVA only tells that there is some difference, not which conditions are different from which.
- People often do an ANOVA and only if they find some difference do they do pairwise comparisons.
- But this two-step procedure is unnecessary; you can simply specify your hypotheses directly through contrast coding.

Define three hypotheses to test:

$$H_{0_0}: \frac{\mu_1 + \mu_2 + \mu_3}{3} = 0$$

$$H_{0_1}: \mu_1 = \frac{\mu_1 + \mu_2 + \mu_3}{3} = GM$$
(10)

and

$$H_{0_2}: \mu_2 = \frac{\mu_1 + \mu_2 + \mu_3}{3} = GM \tag{11}$$

 H_{00} can be written as:

$$H_{0_0}: \frac{1}{3}\mu_1 + \frac{1}{3}\mu_2 + \frac{1}{3}\mu_3 = 0 \tag{12}$$

Example: simulated data with a three-level factor

 H_{0_1} can be written as:

$$\mu_{1} = \frac{\mu_{1} + \mu_{2} + \mu_{3}}{3}$$

$$\Leftrightarrow \mu_{1} - \frac{\mu_{1} + \mu_{2} + \mu_{3}}{3} = 0$$

$$\Leftrightarrow \frac{2}{3}\mu_{1} - \frac{1}{3}\mu_{2} - \frac{1}{3}\mu_{3} = 0$$
(13)
$$\Leftrightarrow \frac{2}{3}\mu_{1} - \frac{1}{3}\mu_{2} - \frac{1}{3}\mu_{3} = 0$$
(15)

Here, the weights 2/3, -1/3, -1/3 are informative about how to combine the condition means to define the null hypothesis.

Sum contrasts

 H_{0_2} can be rewritten as:

$$\mu_2 = \frac{\mu_1 + \mu_2 + \mu_3}{3}$$

$$\Leftrightarrow \mu_2 - \frac{\mu_1 + \mu_2 + \mu_3}{3} = 0$$

$$\Leftrightarrow -\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 - \frac{1}{3}\mu_3 = 0$$
(16)
$$(17)$$

Here, the weights are -1/3,2/3,-1/3, and they again indicate how to combine the condition means for defining the null hypothesis.

The hypothesis matrix

We can write the weights for the hypotheses in two columns of a matrix:

The hypothesis matrix

The generalized inverse (details require matrix algebra, see earlier lecture):

The hypothesis matrix

Take the generalized inverse of the hypothesis matrix:

```
(XcSum <- ginv2(HcSum))

## cH00 cH01 cH02

## low 1 1 0

## med 1 0 1

## hi 1 -1 -1
```

Compare the second and third columns of the inverse of the hypothesis matrix with the sum contrast matrix:

The hypothesis matrix

```
contrasts(simdat2$F) <- XcSum[,2:3]
## same as:
#contrasts(simdat2£F) <- contr.sum(3)
m1_mr <- lm(DV ~ F, data=simdat2)</pre>
```

The hypothesis matrix

Recall that in matrix form, the linear model is:

$$DV = X\beta + \varepsilon$$
, where X is:

You can see that the model or design matrix X is closely related to the generalized inverse of the hypothesis matrix.

Recall that $\hat{\beta} = (X^T X)^{-1} X^T Y$, and $(X^T X)^{-1} X^T = X^+$ is the generalized inverse of the design matrix.

The hypothesis matrix

```
summary(m1_mr)
##
## Call:
## lm(formula = DV ~ F, data = simdat2)
##
## Residuals:
  Min 10 Median 30 Max
## -28.75 -8.75 4.12 12.75 23.00
## Coefficients:
            Estimate Std. Error t value Pr(>|t|)
## (Intercept) 449.9167 5.7965 77.62 4.9e-14
## FcH01 50.0833 8.1975 6.11 0.00018
## FcH02 0.0833 8.1975 0.01 0.99211
## Residual standard error: 20.1 on 9 degrees of freedom
## Multiple R-squared: 0.847, Adjusted R-squared: 0.813
## F-statistic: 24.9 on 2 and 9 DF, p-value: 0.000214
```

The procedure

From the hypothesis matrix to the contrast matrix:

- Write down the hypotheses
- Extract the weights and write them into a hypothesis matrix as shown above
- ► Apply the **generalized matrix inverse** to the hypothesis matrix to create the contrast matrix
- Assign the contrast matrix (here, contr.sum(3)) to the factor and run the linear model

Repeated difference contrasts

```
fractions(contr.sdif(3))

## 2-1 3-2

## 1 -2/3 -1/3

## 2 1/3 -1/3

## 3 1/3 2/3
```

Repeated difference contrasts

Compares med to low, and high to low frequency.

```
## med vs low
table.word$M[2]-table.word$M[1]

## [1] -50

## high vs med
table.word$M[3]-table.word$M[2]

## [1] -50.25
```

Repeated difference contrasts

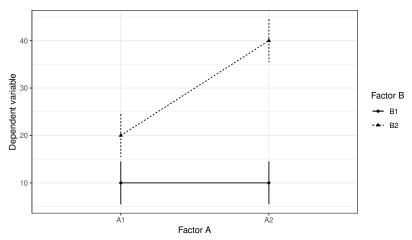
```
contrasts(simdat2$F) <- contr.sdif(3)
m1_mr <- lm(DV ~ F, data=simdat2)
summary(m1_mr)$coefficients

## Estimate Std. Error t value Pr(>|t|)
## (Intercept) 449.92 5.7965 77.6185 4.9491e-14
## F2-1 -50.00 14.1985 -3.5215 6.5001e-03
## F3-2 -50.25 14.1985 -3.5391 6.3229e-03
```

"N data", "Means",

"Std. dev.". "Std. errors")

```
table4a
## # A tibble: 4 x 6
## # Groups: A [2]
## `Factor A` `Factor B` `N data` Means `Std. dev.` `Std. errors`
  <fct> <fct>
                    <int> <dbl>
                                <dbl>
                                           <dbl>
## 1 A1
          B1
                         5 10
                               10
                                           4.47
## 2 A1
      B2
                         5 20
                               10
                                           4.47
                         5 10
## 3 A2
           B1
                               10.0
                                          4.47
## 4 A2
                                    10
                                           4.47
```



```
# ANOVA: B_A(2) times B_B(2)
m2_aov <- aov(DV ~ A*B + Error(id), data=simdat4)
summary(m2_aov)

##
## Error: id
## Df Sum Sq Mean Sq F value Pr(>F)
## A 1 500 500 5 0.03994
## B 1 2000 2000 20 0.00039
## A:B 1 500 500 5 0.03994
## Residuals 16 1600 100
```

```
# MR: B_A(2) times B_B(2)
m2_mr <- lm(DV ~ A*B, data=simdat4)
summary(m2_mr)$coefficients

## Estimate Std. Error t value Pr(>|t|)
## (Intercept) 1.0000e+01 4.4721 2.2361e+00 0.039945
## AA2 5.7595e-15 6.3246 9.1065e-16 1.000000
## BB2 1.0000e+01 6.3246 1.5811e+00 0.133410
## AA2:BB2 2.0000e+01 8.9443 2.2361e+00 0.039945
```

The discrepancy between ANOVA and Im arises from the default treatment contrast coding in R, which Im uses but aov does not:

```
contrasts(simdat4$A)
      A2.
##
## A1 O
## A2 1
contrasts(simdat4$B)
##
      B2.
## B1
## B2 1
```

ANOVA uses sum contrasts, and we can force Im to do this too:

```
# define sum contrasts:
contrasts(simdat4$A) <- contr.sum(2)</pre>
contrasts(simdat4$B) <- contr.sum(2)</pre>
m2_mr.sum <- lm(DV ~ A*B, data=simdat4)
summary(m2_mr.sum)$coefficients
               Estimate Std. Error t value Pr(>|t|)
##
## (Intercept)
                     2.0
                           2.2361 8.9443 1.2670e-07
                     -5
                            2.2361 -2.2361 3.9945e-02
## A1
                    -10 2.2361 -4.4721 3.8511e-04
## B1
## A1:B1
                      5 2.2361 2.2361 3.9945e-02
```

Read the hypr package vignette for examples on how to use it:

```
library(hypr)
vignette("hypr-intro", package = "hypr")
vignette("hypr-regression", package = "hypr")
```

This package allows you to easy go from the hypothesis to contrast matrix and back.

- In my own work, I like experiment designs to be kept simple. $2 \times 2 \times 2$ is too complex for my taste.
- However, sometimes complex designs are unavoidable. In that case, I generally set up sum contrasts for ANOVA-style analyses, or nested contrasts (or both).
- ► Example of 2 × 2 × 2 design: Lena A. Jäger, Daniela Mertzen, Julie A. Van Dyke, and Shravan Vasishth. Interference patterns in subject-verb agreement and reflexives revisited: A large-sample study. Journal of Memory and Language, 111, 2020. https://osf.io/reavs/

- ▶ I never use the contr.sum() etc. functions. I define my own contrast coding columns in the design matrix.
- This becomes relevant in linear mixed models.

For example:

```
persian<-read.table("data/Persiane1crit.txt")</pre>
xtabs(~dist+distance,persian)
##
      distance
## dist long short
## -1 0 756
## 1 756
xtabs(~pred+predability,persian)
      predability
##
## pred predictable unpredictable
##
    -1
               756
##
                             756
```

[Ignore the singularity warning for now]

```
Correlations should not have been computed:
summary(m)
## Linear mixed model fit by REML ['lmerMod']
## Formula:
## rt ~ distance * predability + ((1 | subj) + (0 + distance | subj) +
      (0 + predability | subj) + (0 + distance:predability | subj)) +
##
      ((1 | item) + (0 + distance | item) + (0 + predability |
##
          item) + (0 + distance:predability | item))
##
     Data: persian
##
## REML criterion at convergence: 21880
##
## Scaled residuals:
         10 Median 30
##
     Min
                                Max
## -2.104 -0.454 -0.175 0.160 11.210
##
## Random effects:
## Groups Name
                                                  Variance Std.Dev. Corr
           (Intercept)
   subj
                                                  4.67e+00 2.16
   subi.1
            distancelong
                                                  1.01e+04 100.29
##
            distanceshort
                                                  6.49e+03 80.59
                                                                   1.00
##
   subj.2
            predabilitypredictable
                                                  1.24e+04 111.47
##
            predabilitvunpredictable
                                                  2.17e+04 147.45
                                                                   1.00
   subi.3
            distancelong:predabilitypredictable
                                                 2.43e+03 49.34
##
            distanceshort:predabilitypredictable
                                                  2.48e+03 49.81
##
                                                                   0.83
            distancelong:predabilityunpredictable
                                                  5.55e+02 23.56
                                                                   1.00
##
            distanceshort:predabilityunpredictable 1.58e+04 125.57
                                                                   0.50
            0 00 00 0 00
```

Using my own contrast coding vectors:

```
head(persian)
##
      subj item rt dist distance pred predability
                                       predictable
##
  60
             6
               568
                      -1
                           short -1
## 94
           17 517
                            long
                                    1 unpredictable
            22 675
                           short
## 146
                   -1
                                   -1
                                       predictable
      4 5 575
## 185
                            long
                                    1 unpredictable
             3 581 1
## 215
                            long
                                  -1 predictable
## 285
             7 1171
                            long
                                   -1 predictable
m2<-lmer(rt~dist*pred+(1+dist*pred||subj)+
          (1+dist*pred||item),persian)
```

boundary (singular) fit: see ?isSingular

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Correlations are not computed, as expected:

```
summary(m2)
## Linear mixed model fit by REML ['lmerMod']
## Formula: rt ~ dist * pred + ((1 | subj) + (0 + dist | subj) + (0 + pred |
      subj) + (0 + dist:pred | subj)) + ((1 | item) + (0 + dist |
##
     item) + (0 + pred | item) + (0 + dist:pred | item))
## Data: persian
##
## REML criterion at convergence: 21897
##
## Scaled residuals:
     Min 1Q Median 3Q
## -1 884 -0 452 -0 175 0 159 11 018
##
## Random effects:
## Groups Name
                      Variance Std.Dev.
## subj (Intercept) 2.79e+04 167.15
## subj.1 dist 7.81e+02 27.94
## subj.2 pred 5.51e+02 23.48
## subi.3 dist:pred 2.50e-03 0.05
## item
          (Intercept) 1.97e+03 44.41
## item.1
           dist 0.00e+00 0.00
## item.2
           pred 6.20e+02 24.90
## item.3
           dist:pred 0.00e+00 0.00
## Residual
                      1.06e+05 325.40
## Number of obs: 1512, groups: subj, 42; item, 36
## TI: 1 CC .
```

- ▶ I prefer to define my own contrasts by hand because then I don't get spurious correlations in the random effects, as illustrated above.
- As far as I know, it's not harmful to have those correlations. So maybe this is just a minor technical point.

To learn more

Work through:

Daniel J. Schad, Shravan Vasishth, Sven Hohenstein, and Reinhold Kliegl. How to capitalize on a priori contrasts in linear (mixed) models: A tutorial. Journal of Memory and Language, 110, 2020.

- ► This is a long and detailed paper, and should be read patiently, working through the examples.
- ➤ To fully understand contrasts, you need to become familiar with the matrix formulation of the linear model (some passive matrix algebra knowledge is needed).