Exercises

November 20, 2019

Questions marked with (*) can be solved using finite fields.

1. Let $X = \{(t, t^2, t^3) \mid t \in \mathbb{C}\} \subset \mathbb{C}^3$, and let $I(X)$ be the ideal of all polynomials vanishing on X. Find generators for X.

2. Show that the plane curve $y^2 = x^3 - x$ is smooth.

3. Show that any algebraic set in $\mathbb{C}^n$ is either finite or has the cardinality of the continuum (Hint: use Chevalley's theorem).

4. Let $\ell_1, \ell_2, \ell_3 \subset \mathbb{C}^3$ be three (affine, complex) lines. Show that there is a quadratic polynomial f such that $\ell_1, \ell_2, \ell_3 \subset Z(f)$. (Hint: prove first that every nine points are contained in $Z(f)$, for some quadratic f).

5. (a) Find an algebraic set $X \subset \mathbb{C}^2$ whose projection on the x-axis is $\mathbb{C} \setminus \{0, 1, 2\}$.
   (b) Find an algebraic set $X \subset \mathbb{C}^n$ whose projection to the x, y-plane is $(\mathbb{C}^2 \setminus (0 \times \mathbb{C})) \cup \{(0, 0)\}$ (i.e., the plane, minus the y-axis, plus the origin).

6. Let $X = Z(x^n - y^m) \subset \mathbb{C}^2$. Show that $X \cap B(0, 1/2) \times B(0, 1/2)$ is homeomorphic to a disk if and only if n, m are relatively prime.

7. Let $X \subset Mat_{m,n}(\mathbb{C})$ be the set of all matrices of rank at most d. Show that X is an algebraic set in $Mat_{m,n}(\mathbb{C}) = \mathbb{C}^{mn}$.

8. Let f be an irreducible polynomial over $\mathbb{Q}$. We say that a point $p = (a, b) \in Z(f)$ is generic if a is a transcendental number (equivalently, if b is transcendental). Show that, for any two generic points $p, q \in Z(f)$, there is an element $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$ such that $\sigma(p) = q$.

9. (*) Let $A \subset M_n(\mathbb{C})$ be an algebraic set of matrices, and suppose that

(a) Every element of A is invertible.

(b) If $g, h \in A$, then $gh \in A$.

Show that A is a group (i.e., $g \in A$ implies $g^{-1} \in A$).

10. (*) Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $f^3 = Id$.

    (a) Show that f must have a fixed point.

    (b) Show that either f has infinitely many fixed points, or the number of fixed points of f is congruent to 1 modulo 3.

11. (a) Let $V \subset \mathbb{C}^2$ be a one-dimensional subspace, and suppose that V is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Show that V contains a non-zero vector with rational coefficients.

    (b) Let $V \subset \mathbb{C}^n$ be a vector subspace, and assume that V is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Show that V has a basis consisting of vectors in $\mathbb{Q}^n$. (Hint: induction on n).

    (c) Let $X \subset \mathbb{C}^n$ be an algebraic set which is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Show that there are polynomials $f_1, \ldots, f_m$ with rational coefficients such that $X = V(I)$.