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Tutorial

INTRODUCTION TO ALGEBRAIC GEOMETRY

Abstract. Tutorials for the course "Introduction to algebraic geometry" given at Weizmann institute, fall semester 2019. The lecturer for this course is Nir Avni, these are tutorial note/sketches.

Administrations

The purpose of these sessions are to complement the lectures, we will not learn "new" material that will be used in class, rather we will give a more general context or dive into examples. The Goal of these sessions are:

(1) Algebraic completion (commutative algebra): The theory of algebraic geometry is formulated in algebraic terms, this is a good place to lean these impotent theorems form commutative algebra and to see them is action.
(2) Concrete examples: Analyzing examples in detail!
(3) Exercise: In order to truly understand some math we need to work out the details our-self. In the tutorials there will be exercise sessions and one should do at least a few the exercise each week.
(4) Questions- If you were shy in class or needed some time to think about a question, even from weeks before. Please ask.

1. Week 1

1.1. Preliminaries on Rings.

Definition 1.1.1. A triple $(R,, \cdot)$ is a ring if $(R, +)$ is an abelian group and the multiplication $\cdot$ is distributive on both sides, i.e.

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

and $\cdot$ is associative.

Example 1.1.1.
- $\mathbb{Z}$
- $2\mathbb{Z}$ (but not the "odds")
- $\mathbb{R}[x]$ - the ring of polynomials
- Any field
- $K[x_1, ..., x_n]$

- $K[k_1, ...]$ ring with infinitely many variables, still each element in this ring is a finite collection of symbols
- $Mat_{n\times n}(K)$
- $R[x_1, ...x_n]$ letting R be a ring and not a field is still ok!

Definition 1.1.2. We ay R is a ring with a unit if there is $1 \in R$ s.t. $a \cdot 1 = a = 1 \cdot a$

Exercise 1.1.1. In a ring- the units 0,1 are unique

Definition 1.1.3. A ring R is called commutative if $(R, \cdot)$ is commutative.

Exercise 1.1.2. Which ring above are commutative

From now on we will only deal with commutative rings with unit!

Definition 1.1.4. A ring R is called an integral domain if $a \cdot b = 0 \to a = 0 \ or \ b = 0$

Definition 1.1.5. The set$I$ in a ring R is called an ideal, and denoted $I \lhd R$ if $(I, +)$ is an abelian subgroup of $(R, +)$ and for any $r \in R\, a \in I$, $a \cdot r \in I$.
Note: for non commutative rings we need to separate right and left ideals.

Exercise 1.1.3. Let $S \subset R$, TFAE:

- $I = \bigcap_{S \subset J \lhd R} J$
- $I \lhd R$ (unique) minimal ideal containing S
- $I = \{s_1 r_1 + ... + s_n r_n \ : \ n \in \mathbb{N}, s_i \in S, r_i \in R\}$

Definition 1.1.6. Let S and R as above. We call this ideal "The ideal generated by S and denote it by $< S >$.
If S is a singleton then $< S >$ is called a principle ideal.

Definition 1.1.7. Let R be a (commutative unital..) ring, if any ideal I in R is principle we say that R is a principle ideal domain - PID.

Exercise 1.1.4. Show that $\mathbb{Z}$ is PID.

Exercise 1.1.5. Show that $K[x]$ is PID. (Hint- look at the degree of polynomial)

Exercise 1.1.6. Show that $\mathbb{C}[x, y]$ is NOT PID. ($< x, y >$). Same for $\mathbb{R}, \mathbb{Z}$

1.2. **Some more on ideals.**

Definition 1.2.1. Let I be an ideal in R, the radical of I denoted by $\sqrt{I}$ is the set $\sqrt{I} := \{r \in R \ : \ r^n \in I \quad for\ some\ n\}$
An ideal is called radical if $I = \sqrt{I}$.

Exercise 1.2.1. Show that the radical is indeed an ideal.

Exercise 1.2.2. What is $\sqrt{n\mathbb{Z}}$ in the ring $\mathbb{Z}$.

Lemma 1.2.1. Let R be an integral domain, and I an ideal

- R/I is a field iff I is maximal

1.3. **Fields.**

Definition 1.3.1. We say that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ if there exist $p \in \mathbb{Q}[x]$ (non trivial) s.t. $p(\alpha) = 0$.
If no such polynomial exist then we call α transcendental.

Recall notation, $\mathbb{Q}[\alpha]$ is polynomials in α i.e. formal sums $\sum_i q_i \alpha^i$, if α is algebraic then this ring is actually a field, this condition is iff.

Definition 1.3.2. For $\alpha \in \mathbb{C}$ the field $\mathbb{Q}(\alpha)$ is the minimal sub-field of $\mathbb{Q}$ containing α.

Let $K \subset L$ be a field extension, define $Gal(L/K)$ (The Galois group of L over K to be all field automorphism of L that fix K).
If the extension if finite (as dim vector space viewpoint) then the field fixed by elements of $Gal(L/K)$ is exactly K.
Let $f \in K[x]$ be a polynomial if $f(\alpha) = 0$ then $K(\alpha)/K$ is a finite extension.
Set $G := Gal(K(\alpha)/K)$ we get that $p_\alpha := \prod_{\sigma \in G}(x - \sigma(\alpha))$ is in $K[x]$ since all coefficients are fixed by G.
Also $f(\sigma(\alpha)) = 0$ for any element in the Galois group. Hence $p_\alpha \mid f$. (This is how we went down a degree for our induction claim in class.)

1.4. **Curves.** .
In class we saw a definition of an algebraic curve, it is the "zero set" in $\mathbb{C}^2$ of a polynomial in two variables $f \in \mathbb{C}[x, y]$ we denote the zero set as $\mathcal{Z}(f) := \{(x, y) \ : \ f(x, y) = 0\}$(z for zeros/Zariski).

By corollary of theorem we saw in class ($|\mathcal{Z}(f) \cap \mathcal{Z}(g)| < \infty$) we will often refer to the polynomial itself as the curve, notice any constant times a curve it the same curve.

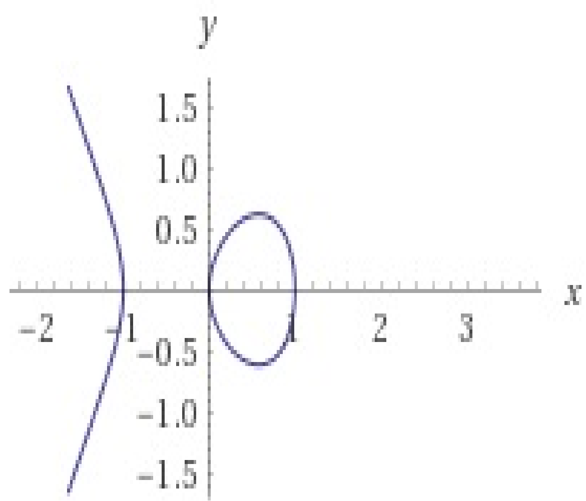
Example 1.4.1. The curve $f(x, y) = x + y$ is the anti diagonal line, the curve $f(x, y) = xy$ is the two axis.

Can we say how these curves are different?
One is irreducible and one is reducible.

Definition 1.4.1. A curve $C = \mathcal{Z}(f)$ is called irreducible if it is reducible as a polynomial, and reducible otherwise i.e. if $f = g \cdot h$ (g, h non units)

We can look at the drawing of the curve and see the difference between the reducible and irreducible ones- but caution!

Exercise 1.4.1 (For home)**.** Describe the curve $f(x, y) = y^2 - x + x^3$ prove that is it irreducible.



$y^2 - x + x^3$

Notice that this is just the "real" picture of the curve, for example the curve $x^2 + y^2 + 1$ is empty in "Real life" but actually is non empty over the complex field.
Some fact are true also over non algebraically closed fields.

1.5. **generic points.** A generic point over field is a transcendental number, for example π over $\mathbb{Q}$.
The fact about these point is that given a field k, and π generic over k, then $k(\pi) \cong k(x)$.
For example given a curve $f(x, y)$ with coefficients in$\mathbb{Q}$ we may regard $f(\pi, y) \in \mathbb{Q}(\pi)[y]$ (a polynomial in one variable defined over a sub-field of $\mathbb{C}$) we know by fundamental theorem of algebra that this polynomial has at most deg(f) solutions.
(The idea behind the generic point is the if we look at the field K which is the field generated by coefficients, and look at the topology in $\mathbb{C}^n$ generated by closed subsets defined as zeros of polynomials in $K[X]$ we get that the generic point is a dense set. i.e. any algebraic statement true for a generic point is true in general.)

Exercise 1.5.1 (Saw in lecture)**.** If $f(x, y)$ irreducible then so is $f(\pi, y)$

Example 1.5.1. The polynomial $x^2 - 2$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$ so we should be careful when just saying irreducible in general.

1.6. **projection to axis.** Given a curve $f(x, y)$ we may project the curve to the x-axis.

$$Prj_X(f) = \{x \in \mathbb{C} \ : \ \exists y \in \mathbb{C} \ f(x, y) = 0\}$$

Exercise 1.6.1 (In class)**.** Show that besides a finite number of points the fiber over each point is finite.
Write $f(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2 + ... + a_n(x)y^n$. If fiber if x_0 is infinite then $|\{y \in \mathbb{C} \ : \ f(x_0,y) = \sum_i a_i(x_0)y^i = 0\}| = \infty$ thus $a_i(x_0) = 0$ there are only finite many x_0 that satisfy this condition.

Exercise 1.6.2 (In class)**.** Let F_q be a finite field with q elements. Let $f(x,y) \in F_q[x,y]$ of degree d, then $|\mathcal{Z}(f)| \leq q \cdot d$.
Since:
$|\mathcal{Z}(f)| = \sum_{x \in F_q} |\{y \in F_q \ : \ f(x,y) = 0\}|$ if $f(x_0,y) \in F_q[x]$ is the zero polynomial by writing $f(x,y) = a_0(x)+a_1(x)y+a_2(x)y^2+...+a_n(x)y^n$. We have that $a_i(x_0) = 0$ for any i. Let $h \coloneqq gcd(\{a_i\}) \in F_q[x,y]$ then of course $h \mid f$ so there exist $g \in F_q[x,y]$ s.t. $f = h \cdot g$. Notice that $f(x',y') = 0$ if ether $h(x',y') = 0$ or $g(x',y') = 0$.
Assume $f(x',y') = 0$, If $h(x',y') = 0$ then $h(x',y) = 0$ for all y. This happens when x' is a root of h thus the amount of such pair is bounded by $\deg(h) \cdot q$.
If $h(x',y') \neq 0$ then $g(x',y') = 0$ and for any such x' there are at most $deg(g)$ such y that give zero thus the amount of such pair is bounded by $deg(g) \cdot q$ we get that

$$|\mathcal{Z}(f)| \leq deg(g) \cdot q + deg(h) \cdot q = deg(f) \cdot q$$

as needed!

Exercise 1.6.3 (For home or in class if time premits)**.** Let $f(x,y) \in \mathbb{C}[x,y]$ non constant, then $|\mathcal{Z}(f)| = \aleph$.

1.7. **Affine Algebraic Varieties.** In the previous part of the course we regarded a single polynomial in 2 variables. In this part we consider any collection of polynomials in many variables.

Definition 1.7.1. Given a sub collection of polynomials $S \subset \mathbb{C}[x_1, ...x_n]$, we define $\mathcal{Z}(S) \coloneqq \{x \in \mathbb{C}^n \ : \ s(x) = 0 \ \forall s \in S\}$.

Exercise 1.7.1. Notice the following

- If $S_1 \subset S_2$ then $\mathcal{Z}(S_2) \subset \mathcal{Z}(S_1)$
- $\mathcal{Z}(S_1 \cup S_2) = \mathcal{Z}(S_1) \cap \mathcal{Z}(S_2)$, $\mathcal{Z}(S_1 \cdot S_2) = \mathcal{Z}(S_1) \cup \mathcal{Z}(S_2)$
- $\mathcal{Z}(S) = \mathcal{Z}(< S >) = \mathcal{Z}(\sqrt{< S >})$

So an "algebraic set" or a "Zariski closed set" is the zero-locus of a radical ideal.
Q: What is a closure of a set with respect to this topology?

Definition 1.7.2. Given any set $X \subset \mathbb{C}^n$ we define $\mathcal{I}(X) \coloneqq \{f \in \mathbb{C}[x_1, ..., x_n] \ : \ f(x) = 0 \forall x \in X\}$.
Notice this is radical ideal.

Exercise 1.7.2. We have the following

- If $X_1 \subset X_2$ then $\mathcal{I}(X_2) \subset \mathcal{I}(X_1)$.
- $\mathcal{I}(X_1 \cap X_2) = \sqrt{\mathcal{I}(X_1) + \mathcal{I}(X_2)}$ (Notice that for radical ideal $I_1 + I_2$ does no need to be radical (take $x, x - y^2$))
- $\mathcal{I}(X_1 \cap X_2) = \mathcal{I}(X_1) \cdot \mathcal{I}(X_2)$
- $I \lhd \mathbb{C}[x_1, ..., x_n]$, $I \subset \mathcal{I}(\mathcal{Z}(I))$
- $X \subset \mathcal{Z}(\mathcal{I}(X))$

We may this of the NSS in the following formulation (seen in class)

Theorem 1.7.1 (NSS).

$$\mathcal{I}(\mathcal{Z}(S)) = \sqrt{< S >}$$

And notice that the closure of a set Y is $\overline{Y} = \mathcal{Z}(\mathcal{I}(Y))$

Examples of affine varietys.

Example 1.7.1. Examples of algebraic sets:

- Unit ball in $\mathbb{C}^n$.
- matrix SL_n
- the setGl_n is open (Next time ee will fallow Rabinowitz trick to show how it can be thought of as a closed set)

Definition 1.7.3. Let G be a group, a representation of G is an n dimensional a homomorphism $\pi\colon G \to GL_n(\mathbb{C})$. Using HBT we can show the following:

Theorem 1.7.2. Let Γ be a finitely generated group (not necessarily finitely presented) then there exist a finitely represented group Δ with a surjection $\Delta \twoheadrightarrow \Gamma$ such that Δ and Γ have same representations for every dimension n.

Proof. What we will show is that the set of n-dimensional representations of Γ is an algebraic set.
Assume $\Gamma =< \gamma_1, ..., \gamma_k >$ (maybe take $k = 2$ for convenience).
Define the following n^2 tuples in $\mathbb{C}^{(2k)n^2}$.

$$\begin{bmatrix} a_{11} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n.1} & \dots & a_{n,n} \end{bmatrix}, \begin{bmatrix} b_{11} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n.1} & \dots & b_{n,n} \end{bmatrix}, \begin{bmatrix} A_{11} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n.1} & \dots & A_{n,n} \end{bmatrix}, \begin{bmatrix} B_{11} & \dots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{n.1} & \dots & B_{n,n} \end{bmatrix}$$

One lower case and one upper case (matrix) for each generators.
And define the polynomials that state $A_{i,j}a_{i,j} = Id_{i,j}$ and all relations etcetera... Since Γ is not necessarily finitely presented there can be infinite amount of polynomials defining this set. From HBT there are just finitely many polynomials defining this set. Take all relations that include such a polynomial from the defining set. And define $\Delta =< \gamma_1, ..., \gamma_k >$ to be the group generated by $\{\gamma_i\}$ but only with the selected relations.
What we have show is that if the representation satisfies the relations of Δ then it satisfies the relations for Γ. $\square$

Note that there are unaccountably many finitely generated groups but just countably many finitely presented ones.

2. Week 2

2.1. Gauss lemma.
Did not get here in tutorial 1
If we know some algebra we can show the theorem proved in class:

Lemma 2.1.1 (Gauss)**.** If $f \in \mathbb{Z}[x]$ is irreducible and *gcd* of coefficient is 1 then f is irreducible over $\mathbb{Q}$.
In general this applies to UFD and ring of fractions.

Corollary 2.1.1. If $p(\pi, y) \in \mathbb{Q}[\pi][y]$ is irreducable then it is irreducable over $\mathbb{Q}(\pi)[y]$. Since irreducibility in this case implies primitive.

2.2. Complex analysis.
First we will define what is a holomorphic and meromorphic function, and then sate important definitions/results.

Definition 2.2.1. Let U be a open subset in $\mathbb{C}$ A function $f\colon U \to \mathbb{C}$ is holomorpic if its derivative exist for any point $w \in U$.
A function on the open disk of radius r around a: $B_r(a)$ is holomorphic if $f(x) = \sum_n c_n(x-a)^n$ (this is given by the Taylor series)

Definition 2.2.2. Let U be a open subset in $\mathbb{C}$ A function $f\colon U \to \mathbb{C} \cup \infty$ is moromorpic if $f|_{U \setminus f^{-1}(\infty)}$ is holomorphic and each $a \in f^{-1}(\infty)$ is a pole.
i.e. in some neighborhood of a, $f(x) = \frac{g(x)}{(x-a)^m}$ for $g(a) \neq 0$ holomorphic and $m \in \mathbb{N}^+$
So in a neighborhood of a we get that

$$f(x) = \sum_{n \geq -m} c_n(x-a)^n = \frac{1}{(x-a)^m} \underbrace{\sum_k c_{k-m}(x-a)^k}_{g(x)}$$

We obtain that $g(a) = c_{-m}$ and define the **residue** of f to be $resf(x); a := c_{-1}$

Example 2.2.1. The function $\sqrt{z}$ is not holomorophic on $\mathbb{C}$, first of all, it is not well defined
But even choosing some sign$\pm\sqrt{z} : re^{i\theta} \mapsto \sqrt{r}e^{\frac{i\theta}{2}}$ is not holomorphic. Since for points on the non negative real axis $r \in [0, \infty]$ we get that for $z_n = r \cdot e^{ia_n}$, $w_n = r \cdot e^{2\pi i - ia_n}$ with $a_n \to 0$ that

$$\lim_{n \to \infty} z_n, w_n \to r,$$

but $\sqrt{z_n} \to \sqrt{r}$ and $\sqrt{w_n} \to -r$, so is not even contiuous.

Exercise 2.2.1. $\pm\sqrt{z}$ is holomorphic on $\mathbb{C} \setminus [0, \infty)$

Theorem 2.2.1 (The implicit function theorem)**.** Let $A(x, y) \in \mathbb{C}[x, y]$ and let $x_0, y_0 \in \mathbb{C}$ such that

$$A(x_0, y_0) = 0, \quad \frac{\partial A}{\partial y}(x_0, y_0) \neq 0$$

Then there are open sets $x_0 \in U,\ y_0 \in V$ and a **holomorphic** function $f\colon U \to V$, such that $f(x_0) = y_0$.
And if $(x, y) \in U \times V$ with $f(x) = y$ then $A(x, y) = 0$ (write$A(x, f(x)) = 0$).

Theorem 2.2.2 (Inverse function theorem).
(1) Let $d\colon U \to V$ be a holomorphic bijection between open sets. Then $\forall u \in U f'(u) \neq 0$ and f^{-1} is holomorphic
(2) Let $d\colon U \to \mathbb{C}$ be holomorphic and $u \in U$ with $f'(u) \neq 0$, then there exist an open set $U' \subset U$ containing a and open V containing $f(u)$ such that f is bijective holomorphic between U' and V.

2.3. **Surfaces.**

Definition 2.3.1 (surface)**.** A surface is a Housdorf topological space locally homoemorphic to $\mathbb{C}$ (or $\mathbb{R}^2$), i.e. for every $x \in S$ there is a open $U \subset S$ containing S such that U is homeomorphic to $\mathbb{C}$.
Such a homoeomorphisim is called a chart

Definition 2.3.2 (atlas)**.** An atlat Φ on a surface S is a collection of charts $\phi_\alpha : U_\alpha \to V_\alpha$ such that $S = \bigcup_\alpha U_\alpha$, together with that property that the image under any chart of $U\alpha \cap U_\beta$ is open.

Definition 2.3.3 (translation function)**.** Given an atlas S, a translation function $\phi_{\alpha\beta} : \phi_\beta(U\alpha \cap U_\beta) \to \phi_\alpha(U\alpha \cap U_\beta)$ defined by $\phi_\alpha \circ \phi_\beta^{-1}$. is a map between open subsets on $\mathbb{C}$.
The atlas is called **Holomorphic** if translation functions are holomorphic.

2.4. **Picturing the curve.**

Recall that we had $\sqrt{z}$ a non holomorphic function.
What about the function $\sqrt{z(z-1)(z+1)}$
We can thik as a curve $y^2 = x(x+1)(x-1)$ - what does this curve look like? (maybe 2 copys of C with 3 glues points).
The each copy for a y point - the plane itself is x.
what does it really look like? since if we go around a point - we get to its minus - recall the sqrt(Z) and draw from there

2.5. **Resultant of polynomials.**

Definition 2.5.1. Let $P(x), Q(x) \in \mathbb{C}[x]$ write $P(x) = \sum_{i=1}^{n} a_i x^i$ $Q(x) = \sum_{i=1}^{m} b_i x^i$ with $a_n, b_m \neq 0$.
The *Resultant* on P and Q, denoted by $\mathcal{R}(P, Q)$ is the determinant of

the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ \vdots & & & & \cdots & & & \vdots \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_0 & b_1 & & \cdots & b_m \end{bmatrix}$$

Lemma 2.5.1. Let $P(x), Q(x) \in \mathbb{C}[x]$ then P, Q have non constant common factor (hence a common zero) iff $\mathcal{R}(P, Q) = 0$

Lemma 2.5.2. If $P(x) = \prod_{i=1}^{n}(x - \lambda_i)$ and $Q(x) = \prod_{i=1}^{m}(x - \mu_i)$ then $\mathcal{R}(P, Q) = \prod_{i,j}(\mu_j - \lambda_i)$.
In particular for polynomails $P, Q, A \in \mathbb{C}[x]$ we get that

$$\mathcal{R}(P, QA) = \mathcal{R}(P, Q)\mathcal{R}(P, A)$$

2.6. **Examples of Zariski closed sets. Recall the Zariski correspondence and NSS**

Exercise 2.6.1. Show that any set in $\mathbb{C}^n$ containing $\{(t, \sin(t)\}$ is Zariski dense.

HInt: Plug in $f(\pi/4, 1/\sqrt{2})$ and play with polynomial...

Exercise 2.6.2. Let $a, b \in K$ what is the Zariski closer of the set $\{(a^n, b^n)\ n \in \mathbb{N}\}$.
Show that the set is dense iff $a^{n_0} \neq b^{m_0}$ for any n_0, m_0.
Hint: If $a^{n_0} = b^{m_0}$ then we have for large enough n that (a^n, b^n) is satisfied by $x^{m_0} = y^{n_0}$.

Exercise 2.6.3. Show that set $\{t, t^2, t^3\}$ is Zariski closed, what is its Ideal?

3. Week 3

3.1. **Bezout thm.**

Theorem 3.1.1 (Bezout)**.** If $f, g \in \mathbb{C}[x, y]$ are relatively prime, then

$$|Z(f) \cap Z(g)| \leq (\deg f)(\deg g)$$

What we saw in class is that actually that the projection to the $x-$axis of $Z(f) \cap Z(g)$ has size at most $n \cdot m$.
We fix this by the following exercise:

Exercise 3.1.1. There exist a change of coordinates such that no two solutions lie over the same x point.

3.2. **Curve minus singular points is a holomorphic surface.**

Lemma 3.2.1. Any complex curve is a surface and actually minus the singular points, it is has a holomorphic atlas.

Proof. Let C by a curve efined by polynomial P ($C = \mathcal{Z}(P)$) and $(a, b) \in \mathbb{C}$ such that $\frac{\partial P}{\partial y}(a, b) \neq 0$
By the imlicit function theorem there is open set $z \in V$, $b \in W$ and hlomorphic function $g \colon V \to W$ with $g(a) = b$ and for $(x, y) \in V \times W$.

$$P(x, y) = 0 \iff g(x) = y$$

We may choose V, W small enough such that $U := C \cap V \times W$ has no singular points.
We get that the projection $(x, y) \mapsto x$ from U is into V and that $(x) \mapsto (x, g(x))$ is a holomorphic inverse.
Same for points with $\frac{\partial P}{\partial x}(a, b) \neq 0$ we get a open set U' in which $(x, y) \mapsto y$ has inverse $(y) \mapsto (h(y), y)$.
If we compute the transition maps we gell get that they may be Id, g, h which are all holomorphic. $\square$

3.3. **Curve next to non singular points.**

Definition 3.3.1. A point $(a, b) \in \mathbb{C}^2$ is singular point of $f \in \mathbb{C}[x, y]$ if $f(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$.

Exercise 3.3.1. What are the singular points of the curve $y^2 - x^2 - x^3$. (notice that $(0, -2/3)$ is not on the curve)

We want to get a feel of how curves look at singular points. First we notice :

Lemma 3.3.1. Let p be a singular point of the curve C, the projection map π_x from $C \cap B_\epsilon(p) \to \mathbb{C}$ is a covering map (for small enough ϵ).

Proof. Let $f(x, y) = \sum_{i=0}^{n} a_i(x) y^i$ then for all but finitely many points $|\pi_X^{-1}(x)| = n$, the points where this does not happen are when $a_n(x_0) = 0$ since then by fundamental theorem pf algebra there are at most $n-1$

roots, and at points where $Res(f(x_0, y), \frac{\partial f}{\partial y}(x_0, y)) = 0$ which account for multiple roots.
The point p is a special point as above and thus there is a small enough neighborhood around it that is a cover. $\square$

But how does the curve look near the singularity p? , we look at the curve intersected with the sphere of radios ϵ in $\mathbb{C}^2$.

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x - p_x|^2 + |y - p_y|^2 = \epsilon^2\}$$

For simplicity and w.l.o.g from now on assume that $p = (0, 0)$.

Example 3.3.1. Let $f(x, y) = xy$, we get that

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x|^2 + |y|^2 = \epsilon^2\} = \{(0, \epsilon e^{it})\} \cup \{(\epsilon e^{it}, 0)\}$$

Which is 2 distinct loops is the 3 dimensional sphere

Example 3.3.2. Let $f(x, y) = x^2 - y^3$, we get that

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x|^2 + |y|^2 = \epsilon^2\} = \{(\delta^3 e^{3it}, \delta^2 e^{2it}) : \ t \in (0, 2\pi)\} = \bigcup_{k=0}^{2} \{(\delta^3 e^{it}, \delta^2 e^{i\frac{2t}{3} + \frac{i2\pi \cdot k}{3}})\}$$

and δ is the unique positive solution for $\delta^4 + \delta^6 = \epsilon^2$.
This is loop circle parameterized by t.
For example take $\epsilon = 1$ and look at the pre-image of the circle.
Starting from the point $(1, 1)$ and going with t from 0 to 2π we get that the second coordinate travels from one solution to the next. This makes one loop.

Notice that the curve $xy = 0$ is the same (after change of basis $(x - y, x + y)$) as $x^2 = y^2$ which has 2 loops on the sphere.

Exercise 3.3.2. Find a curve with 7 loops on the sphere

Proof. Take $f = x^7 - y^7$, the points on the sphere are

$$\{(x, y) \in C \ : \ |x|^2 + |y|^2 = \epsilon^2\} = \bigcup_{k=0}^{7} \{(\delta e^{it}, \delta e^{it + \frac{i2\pi \cdot k}{7}})\}$$

and δ is positive solution for $2\delta^7 = (\epsilon^2 - \delta^2)^{\frac{7}{2}}$.
The loop starting at $(1, 1)$ goes back to $(1, 1)$ after 2π and does not hit another element in the pre-image of 1. $\square$

Lemma 3.3.2. Let $n \leq m$ the amount of loops in the sphere around $(0, 0)$ at the curve $x^n - y^m = 0$ is $gcd(m, n)$.

Recall that this intersection is in a 3 dimensional sphere.
We may do a stereo-graphic projection to $\mathbb{R}^3 + \infty$ and try to imagine the curve.

FACT: The stereo-graphic projection from the ϵ sphere to $\mathbb{R}^3$ can be given by

$$(Re(x), Im(x), Re(y), Im(y)) \mapsto \begin{cases} \frac{\epsilon}{\epsilon - Re(y)}(Re(x), Im(x), Im(y)) & Re(y) \neq \epsilon \\ \infty & Re(y) = \epsilon \end{cases}$$

Lets project some curves and see what we get!

Example 3.3.3. Let $f = xy$ (same example as before) we had that $C \cap S = \{(\cos(t), \sin(t), 0, 0) \,:\, t \in [2\pi]\} \cup \{(0, 0, \cos(s), \sin(s)) \; s \in [2\pi]\}$ under the projection we get the union

$$\{(\cos(t), \sin(t), 0)\} \cup \{\frac{\epsilon}{\epsilon - \sin(s)}(0, 0, \cos(s))\}$$

Which is a circle and a line going throw the circle! we get that the two circles we got above are linked in the sphere

Example 3.3.4. Let $f(x, y) = y^2 - x^3$, (same as before switch x,y) we got that

$C \cap S = \{(\delta^2 e^{2it}, \delta^3 e^{3it}) \,:\, t \in (0, 2\pi)\} = \{(\delta^2 \cos(2t), \delta^2 \sin(2t), \delta^3 \cos(3t), \delta^3 \sin(3t),)\}$

and δ is the unique positive solution for $\delta^4 + \delta^6 = \epsilon^2$.
This is a subset of $\{(x, y) \,:\, |x| = \delta^2, |y| = \delta^3\}$ and this maps under the projection to a points $(a, b, c) \in \mathbb{R}^3$ satisfying

$$2\epsilon^2 \sqrt{a^2 + b^2} = \delta^2(a^2 + b^2 + c^2 + \epsilon^2)$$

which is

$$(\sqrt{a^2 + b^2} - \epsilon^2 \delta^{-2})^2 + c^2 = \epsilon^2 \delta^2$$

which is a surface of revolution of a circle around the c axis, i.e. a torus!
Thus $C \cap S$ maps to a knot on the surface of the torus!

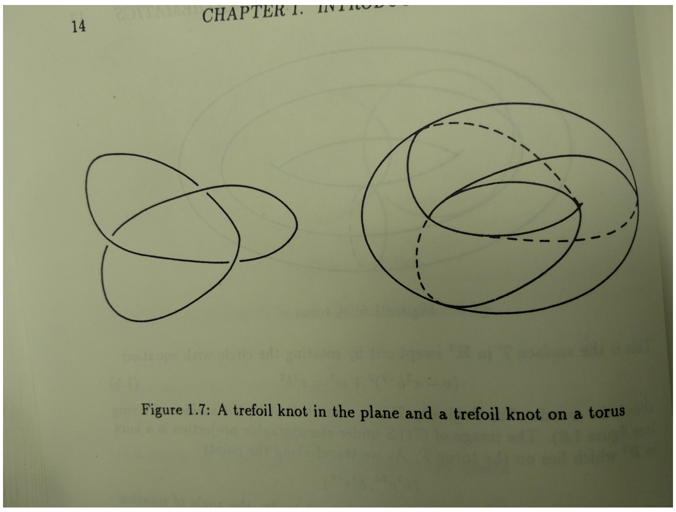


FIGURE 1

3.4. **constructable sets.** The notion of constructability is purely topological.

Definition 3.4.1. Let X, τ be a topological space, a set $Z \subset X$ in called locally-closed if $Z = U \cap B$ where U is open and B is closed.

Definition 3.4.2. Let X, τ be a topological space, a set $C \subset X$ in called construcable if $C = \bigcup_{i=0}^{n} Z_i$ where Z_i is constructable.

Notice that this is equivalent to the definition that was given in class when we look at the Zariski-topology of the space.

Exercise 3.4.1. Let $M_{n,m}(\mathbb{C})$ denote the vector space of $n \times m$ matrices. Show that the matrices of rank r is a constructable set.
Hint: Look at condition on the $r+1$ an r minor of the matrix.