Q1. Write the correct name of the following statement.

Solution:

(i) Cauchy Integral Theorem

(ii) Cauchy Integral Formula

(iii) (a) Cauchy Residue Theorem for single pole

(b) Cauchy Residue Theorem for double pole

(2) Evaluate the following integration:

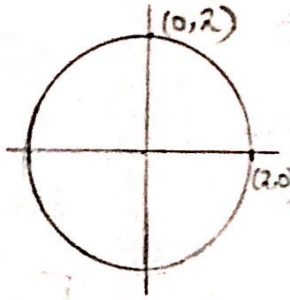
(i) Evaluate $\oint_C (z^2+3z)\,dz$ along (a) the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction, (b) the straight line from $(2,0)$ to $(0,2)$

Solution:

(a) We have, $|z| = 2$

$$\Rightarrow z = 2e^{i\theta} \quad ; \quad 0 \leq \theta < 2\pi$$

$$\therefore dz = 2ie^{i\theta}\,d\theta$$



Now, $$\oint_C (z^2+3z)\,dz = \int_0^{\pi/2} (4e^{2i\theta} + 6e^{i\theta})\,2ie^{i\theta}\,d\theta$$

$$= 4i\int_0^{\pi/2} (2e^{3i\theta} + 3e^{2i\theta})\,d\theta$$

$$= 4i\left[\frac{2e^{3i\theta}}{3i} + \frac{3e^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= 4i\left[-\frac{2}{3} - \frac{3}{2i} - \frac{2}{3i} - \frac{3}{2i}\right]$$

$$= -\frac{8i}{3} - \frac{88}{6}$$

$$= -\frac{44}{3} - \frac{8}{3}i$$

(i)(b) The equation of straight line from (2,0) to (0,2) is:

$$\frac{x-2}{2-0} = \frac{y-0}{0-2}$$

$$\Rightarrow \frac{x-2}{2} = \frac{y}{-2}$$

$$\Rightarrow y = -x+2$$

$$\therefore dy = -dx$$

$$\therefore \oint_c (z^2+3z)dz$$

$$= \oint_c \{(x+iy)^2 + 3(x+iy)\}(dx+idy)$$

$$= \int_2^0 \{x+i(-x+2)\}^2 + 3\{x+i(-x+2)\}(dx-idx)$$

$$= \int_2^0 \{x^2+2xi(2-x)-(2-x)^2+3x-3xi+6i\}(dx-idx)$$

$$= \int_2^0 (x^2+4xi-2x^2i-4+4x-x^2+3x-3xi+6i)(dx-idx)$$

$$= \int_2^0 (7x+xi-2x^2i-4+6i)(dx-idx)$$

$$= \int_2^0 (7x+xi-2x^2i-4+6i-7xi+x-2x^2+4i+6)dx$$

$$= \int_2^0 (8x-6xi-2x^2i-2x^2+10i+2)dx$$

$$= \left[4x^2-3x^2i-\frac{2x^3i}{3}-\frac{2x^3}{3}+10xi+2x\right]_2^0$$

$$= 0-(16-12i-\frac{16}{3}i-\frac{16}{3}+20i+4) = -\frac{44}{3}-\frac{8}{3}i$$

(ii) Let C be any simple closed curve bounding a region having area A. Prove that,

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Solution: According to Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \cdots (1)$$

Putting $P=0$ and $Q=x$ ~~then~~ in (1) then we get,

$$\oint_C x\,dy = \iint_R \frac{\partial x}{\partial x}\,dA = \iint_R dA = A$$

$$\therefore \oint_C x\,dy = A \quad \cdots (2)$$

Putting $Q=0$ and $P=-y$ in (1), then we get,

$$\oint_C -y\,dx = \iint_R -\frac{\partial(-y)}{\partial y}\,dA = \iint_R dA = A$$

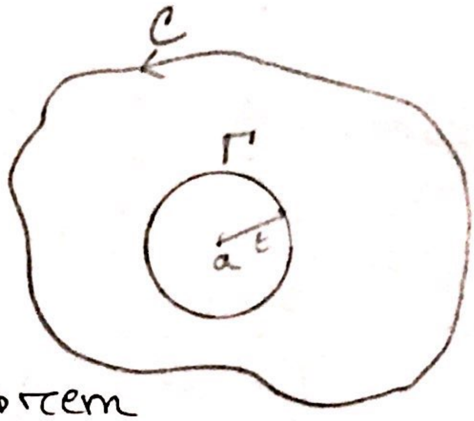
$$\therefore \oint_C -y\,dx = A \quad \cdots (3)$$

Adding (2) and (3), we get,

$$2A = \oint_C x\,dy - y\,dx$$

$$\therefore A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

(iii) Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z=a$ is (1) outside C and (2) ~~outside~~ inside C



Solution:

(1) If $z=a$ is outside C,
$f(z) = \frac{1}{z-a}$ is analytic everywhere inside and on C.
Hence, by cauchy integral theorem

$$\oint_C \frac{dz}{z-a} = 0$$

(2) suppose a is inside C and let Γ be a circle of radius ϵ with center $z=a$ so that Γ is inside C.

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \quad \cdots \text{(i)}$$

Now on Γ, $|z-a| = \epsilon$ or, $z-a = \epsilon e^{i\theta}$

or, $z = a + \epsilon e^{i\theta}$; $0 \le \theta < 2\pi$

$\Rightarrow dz = i\epsilon e^{i\theta} d\theta$

The right side of (i) becomes

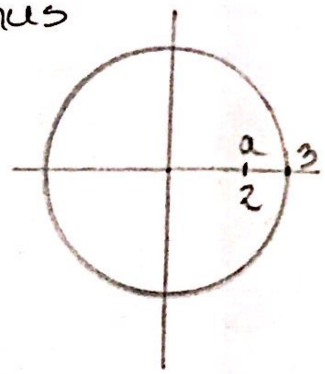
$$\int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

5.30. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ if C is

(a) The circle $|z|=3$ (b) The circle $|z|=1$

Solution:

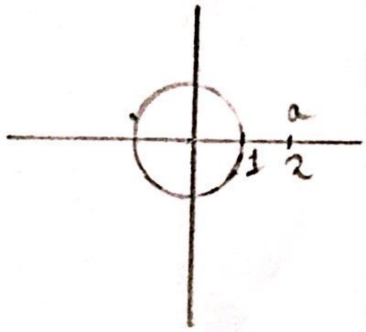
(a) Here $f(z)=e^z$ and $a=2$, radius of circle (region) $=3$, so the point $a=2$ inside C. So it follows cauchy's integral formula thus



$$f(a)=f(z)=e^2=\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$$

$$\therefore \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = e^2$$

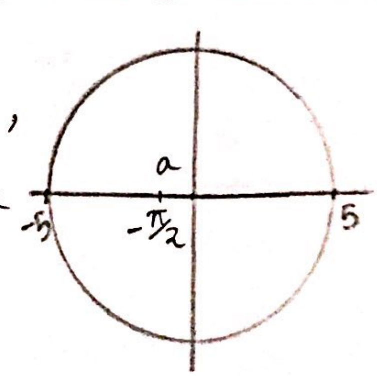
(b) Here, $f(z)=e^z$, $a=2$, radius of circle (region) $=1$; so the point $a=2$ outside C. so it follows cauchy integral theorem,



$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz=0$$

5.31. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$

Solution: Here $f(z)=\sin 3z$, $a=-\frac{\pi}{2}$, radius of circle (region) $=5$, so the point $a=-\frac{\pi}{2}$ inside C. So it



follows cauchy integral formula thus

$$f(a) = \sin(-3 \times \tfrac{\pi}{2}) = 1 = \frac{1}{2\pi i} \oint_C \frac{\sin 3z}{z+\pi/2}\, dz$$

$$\therefore \oint_C \frac{\sin 3z}{z+\pi/2} = 2\pi i$$

5.32. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\, dz$ if C is

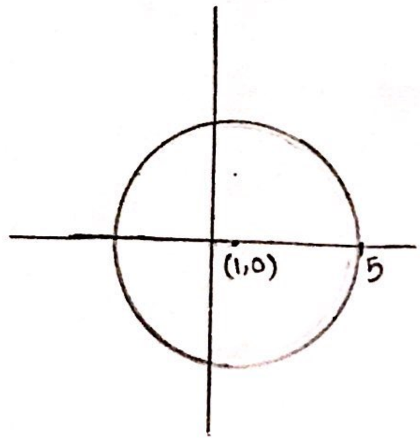
(a) The circle $|z-1| = 4$ (b) The ellipse $|z-2|+|z+2| = 6$

<u>Solution:</u>

(a) here, $f(z) = e^{3z}$; $a = \pi i$

Since $z = \pi i$ is inside the circle $|z-1| = 4$. so it follows cauchy Integral formula

$$f(a) = f(\pi i) = \frac{1}{2\pi i} \oint_C \frac{e^{3z}}{z-\pi i}\, dz$$



$$\therefore \oint_C \frac{e^{3z}}{z-\pi i}\, dz = 2\pi i \times e^{3\pi i}$$

$$= 2\pi i \times (-1)$$

$$= -2\pi i$$

(b) here, $f(z) = e^{3z}$, $a = \pi i$ and C is ellipse

$$|z-2| + |z+2| = 6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$$

$$\Rightarrow \cancel{x^2-4x+4+x^2+4x+4+2y^2=36}$$

$$\Rightarrow \cancel{2x^2+2y^2=28}$$

$$\Rightarrow x^2-4x+4+y^2 = \left\{6-\sqrt{(x+2)^2+y^2}\right\}^2$$

$$\Rightarrow x^2-4x+4+y^2 = 36 - 12\sqrt{(x+2)^2+y^2} + x^2+4x+4+y^2$$

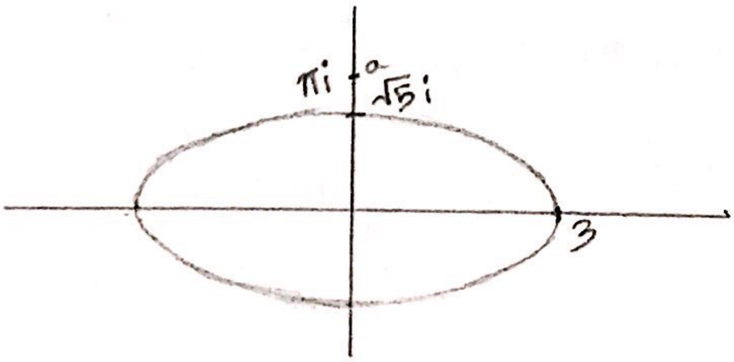
$$\Rightarrow -8x = 36 - 12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow 3\sqrt{(x+2)^2+y^2} = 9+2x$$

$$\Rightarrow 9(x^2+4x+4+y^2) = 81+36x+4x^2$$

$$\Rightarrow 5x^2+9y^2 = 45$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$$



$\therefore$ $z = \pi i$ lies outside C. so it follows Cauchy integral theorem.

$$\therefore \oint_C \frac{e^{3z}}{z-\pi i} = 0$$

Evaluate the following by contour integration.

(i) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

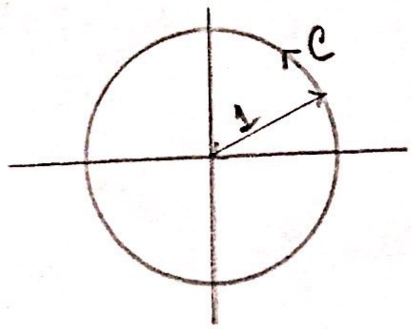
<u>Solution:</u> Let, $z = e^{i\theta}$. Then, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

$$= (z - z^{-1})/2i$$

$\therefore\ dz = ie^{\theta i} d\theta = iz d\theta$; so that

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_c \frac{dz/iz}{a+b(z-z^{-1})/2i} = \oint_c \frac{2dz}{bz^2+2aiz-b}$$

where c is the circle of unit radius with center at the origin.



The poles of $2/(bz^2+2aiz-b)$ are obtained by solving $bz^2+2aiz-b=0$ and are given by,

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2-b^2}\,i}{b}$$

$$= \left\{\frac{-a+\sqrt{a^2-b^2}}{b}\right\}i \text{ , } \left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}i$$

Only $\{(-a+\sqrt{a^2-b^2})/b\}i$ lies inside C. since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b} \cdot \frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{\sqrt{a^2-b^2}+a}\right| < 1$$

when $a > |b|$

Residue at, $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}i$,

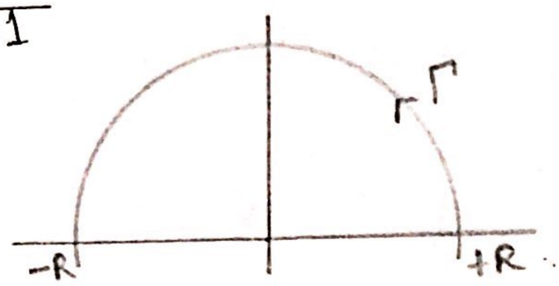
$$= \lim_{z \to z_1} \frac{d}{dz}\left\{(z-z_1)\frac{2}{bz^2+2aiz-b}\right\}$$

$$= \lim_{z \to z_1} \frac{2}{2bz+2ai}$$

$$= \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}\,i}$$

$$\therefore \oint \frac{2dz}{bz^2+2aiz-b} = 2\pi i\left(\frac{1}{\sqrt{a^2-b^2}\,i}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(ii) $\int_0^\infty \frac{dx}{x^4+1}$



<u>Solution:</u> Consider the integral $\oint_C \frac{dz}{z^4+1}$ where C is a closed contour containing the line from $-R$ to $+R$ and semicircle Γ.

The function $F(z) = \frac{1}{z^4+1}$ has poles $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$ but only 1st two poles are in the upper half.

residue at $z = e^{\frac{\pi i}{4}}$, $\lim_{z \to e^{\pi i/4}} \frac{d}{dz}\left\{(z - e^{\pi i/4})\frac{1}{z^4+1}\right\}$

$$= \frac{1}{4} e^{-3\pi i/4}$$

residue at $z = e^{3\pi i/4}$, $\lim_{z \to e^{3\pi i/4}} \frac{d}{dz}\left\{(z - e^{3\pi i/4})\frac{1}{z^4+1}\right\}$

$$= \frac{1}{4} e^{-9\pi i/4}$$

Thus, $\oint_C \frac{dz}{z^4+1} = 2\pi i\{\text{sum of residue}\}$

$$= \frac{2\pi i}{4}\left\{e^{-3\pi i/4} + e^{-9\pi i/4}\right\}$$

$$= \frac{\pi i}{2} \cdot (-\sqrt{2} i) = \frac{\pi}{\sqrt{2}}$$

$$\therefore \oint_C \frac{dz}{z^4+1} = \int_{-R}^{+R} \frac{dx}{x^4+1} + \int_\Gamma \frac{dz}{z^4+1} = \frac{\pi}{\sqrt{2}}$$

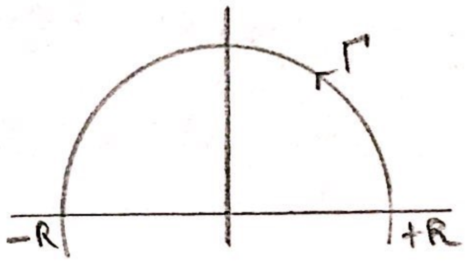
$$= \int_{-\infty}^{+\infty} \frac{dx}{x^4+1} + 0 = \frac{\pi}{\sqrt{2}}$$

$$= 2\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

(iii) $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2\pm1)^2}\,dx$

$$= \oint_C \frac{e^{mzi}}{(z^2\pm1)^2}\,dz$$



Consider the integral $\oint_C \frac{e^{imz}}{(z^2\pm1)^2}\,dz$ where C is a closed contour consists of line from $-R$ to $+R$ and a semicircle Γ traversed in anti-clockwise.

for $\oint_C \frac{e^{imz}}{(z^2 \pm 1)^2} dz$,

the function $\frac{e^{imz}}{(z^2 \pm 1)^2}$ has poles $z = \pm i$ of order 2 but only $z = +i$ lies inside C.

residue at $z = +i$, $\lim_{z \to i} \frac{d}{dz} \left\{ (z-i)^2 \frac{e^{imz}}{(z^2+1)^2} \right\}$

$$= \lim_{z \to i} \frac{d}{dz} \left\{ \frac{e^{imz}}{(z+i)^2} \right\}$$

$$= \lim_{z \to i} \frac{(z+i)^2 im e^{imz} - e^{imz} 2(z+i)}{(z+i)^4}$$

$$= \lim_{z \to i} \frac{e^{imz}(z+i)\{(z+i)im - 2\}}{(z+i)^4}$$

$$= \lim_{z \to i} \frac{e^{imz}(imz - m - 2)}{(z+i)^3}$$

$$= \frac{e^{-m}(-2m-2)}{-8i}$$

$$= \frac{e^{-m}(m+1)}{4i}$$

$$\therefore \oint_C \frac{e^{imz}}{(z^2+1)^2} dz = 2\pi i \times \frac{e^{-m}(m+1)}{4i}$$

$$= \frac{\pi e^{-m}(m+1)}{2}$$

$$\therefore \oint_C \frac{e^{imz}}{(z^2+1)^2}\,dz = \int_{-\infty}^{+\infty} \frac{e^{imx}}{(x^2+1)^2}\,dx + \int_\Gamma \frac{e^{imz}}{(z^2+1)^2}\,dz = \frac{\pi e^{-m}(m+1)}{2}$$

$$\Rightarrow 2\int_0^\infty \frac{e^{imx}}{(x^2+1)^2}\,dx + 0 = \frac{\pi e^{-m}(m+1)}{2}$$

$$\Rightarrow \int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx + i\int_0^\infty \frac{\sin mx}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(m+1)}{4}$$

equating real parts,

$$\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(m+1)}{4}$$