White Kernel

The white kernel is probably the simplest and most widely used possible Gaussian process. It can be considered as the limit of the squared exponential kernel,

$$\boxed{K^{SE}_{f_r f_r}(t,t') = \tilde{\sigma}_r^2 \exp\left(-\frac{(t-t')^2}{\ell_r^2}\right)},$$

when the inverse width, ℓ_r, tends to zero and $\boxed{\tilde{\sigma}_r^2 = \frac{\sigma_r^2}{\ell_r\sqrt{\pi}}}$. Under this circumstances, it can be shown that

$$\boxed{\lim_{\ell_r \to 0} \frac{1}{\ell_r\sqrt{\pi}} \exp\left(-\frac{(t-t')^2}{\ell_r^2}\right) = \delta(t-t')},$$

since

$$\frac{1}{\ell_r\sqrt{\pi}} \exp\left(-\frac{(t-t')^2}{\ell_r^2}\right)\Bigg|_{t=t'} = \frac{1}{\ell_r\sqrt{\pi}} \longrightarrow \infty \quad (\ell_r \to 0),$$

and

$$I = \int_{-\infty}^{\infty} \frac{1}{\ell_r\sqrt{\pi}} \exp\left(-\frac{(t-t')^2}{\ell_r^2}\right) d(t-t')$$

$$= \frac{1}{\ell_r\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2\pi(\ell_r/\sqrt{2})^2}}{\sqrt{2\pi(\ell_r/\sqrt{2})^2}} \cdot \exp\left(-\frac{(t-t')^2}{2(\ell_r/\sqrt{2})^2}\right) d(t-t')$$

$$= \frac{1}{\ell_r\sqrt{\pi}} \cdot \sqrt{2\pi} \cdot \frac{\ell_r}{\sqrt{2}} = 1.$$

Hence, $\exp(-(t-t')^2/\ell_r^2)/[\ell_r\sqrt{\pi}]$ fulfills the two conditions required to consider it a delta function in the limit, and

$$\boxed{\lim_{\ell_r \to 0} K_{f_r f_r}(t,t') = \sigma_r^2\, \delta(t-t') = K^{W}_{f_r f_r}(t,t')},$$

which is the general expression of the white kernel, widely used in communications, signal processing and machine learning to represent noise processes due to the central limit theorem.

LFOS + White Kernel   SIM-WHITE

The output of the linear first-order system (LFOS) is given by

$$\boxed{\begin{aligned} y_q(t) &= \frac{B_q}{D_q} + \sum_{r=1}^{R} L_{qr}[f_r](t), \\ L_{qr}[f_r](t) &= S_{qr} \exp(-D_q t) \int_0^t f_r(\tau) \exp(D_q \tau) \, d\tau . \end{aligned}}$$

Hence, when we have the white kernel as the input of the system, we can easily obtain the kernel that models the outputs:

$$\boxed{E\{y_q(t)\} = \frac{B_q}{D_q}} \qquad [E\{f_r(t)\} = 0]$$

$$\boxed{K_{y_q f_r}(t,t')} = S_{qr} \exp(-D_q t) \int_0^t k_{f_r f_r}(\tau, t') \exp(D_q \tau) \, d\tau$$

$$= \begin{cases} \sigma_r^2 S_{qr} \exp(-D_q (t-t')), & t \geq t'; \\ 0 & , \quad t < t'. \end{cases}$$

$$\boxed{= \sigma_r^2 S_{qr} \exp(-D_q (t-t'))\, u(t-t'),}$$

where $\boxed{u(t-t') = \begin{cases} 1, & t \geq t'; \\ 0, & t < t'. \end{cases}}$ is Heaviside's step function.

Note that the shape of the cross-covariance function is due to the fact that $k_{f_r f_r}(\tau, t') = \sigma_r^2 \, \delta(\tau - t')$, and hence the integral is only different from zero when t' belongs to the interval of integration (i.e. $0 \leq t' \leq t$).

Following the same reasoning we can obtain the cross-covariance function between $y_p(t)$ and $y_q(t')$:

$$\boxed{K_{y_p y_q}(t,t')} = \sum_{r=1}^{R} S_{pr} S_{qr} \exp\left(-(D_p t + D_q t')\right)$$

$$\times \int_0^t \int_0^{t'} K_{f_r f_r}(\tau,\tau') \exp(D_p \tau) \exp(D_q \tau') \, d\tau \, d\tau'$$

$$= \sum_{r=1}^{R} S_{pr} S_{qr} \exp\left(-(D_p t + D_q t')\right)$$

$$\times \int_0^t \exp(D_p \tau) \left[\int_0^{t'} K_{f_r f_r}(\tau,\tau') \exp(D_q \tau') \, d\tau' \right] d\tau$$

$$= \sum_{r=1}^{R} S_{pr} S_{qr} \exp\left(-(D_p t + D_q t')\right)$$

$$\times \int_0^{t \wedge t'} \sigma_r^2 \exp\left((D_p + D_q)\tau\right) d\tau$$

$$= \sum_{r=1}^{R} S_{pr} S_{qr} \sigma_r^2 \exp\left(-(D_p t + D_q t')\right) \cdot \frac{\exp\left((D_p + D_q)(t \wedge t')\right) - 1}{D_p + D_q}$$

$$\boxed{= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{D_p + D_q} \left[\exp\left(-D_v |t - t'|\right) - \exp\left(-(D_p t + D_q t')\right) \right]}$$

where this expression comes from the fact that, once more, the inner integral is only different from zero when $0 \leq \tau' \leq \tau \leq t'$ and since $0 \leq \tau \leq t$ for the second integral also, we have to choose $\min(t, t') = t \wedge t'$ as the upper limit of the integration interval. Also, note that

$$\exp\left(-(D_p t + D_q t')\right) \exp\left((D_p + D_q)(t \wedge t')\right)$$

$$= \exp\left(-D_p t - D_q t' + D_p (t \wedge t') + D_q (t \wedge t')\right)$$

$$= \begin{cases} \exp\left(-D_q (t' - t)\right), & t < t'; \\ \exp\left(-D_p (t - t')\right), & t \geq t'. \end{cases}$$

$$= \exp\left(-D_v \, |t-t'|\right),$$

with

$$\boxed{\begin{aligned} v &= \begin{cases} q, & t < t'; \\ p, & t' \leq t. \end{cases} \\ &= q \cdot \chi_{(-\infty,0)}(t-t') + p \cdot \chi_{[0,\infty)}(t-t'), \end{aligned}}$$

wher $\chi_E(x)$ is the characteristic or indicator function of x for the region E, which determines whether x belongs to E or not:

$$\boxed{\chi_E(x) = \begin{cases} 1, & x \in E; \\ 0, & x \notin E. \end{cases}}$$

Finally, note that, although the input Kernel is clearly stationary, the resulting Kernel of the output is non-stationary, due to the transients caused by the initialization of the system at $t=0$. However, the Kernel is asymptotically stationary. This means that, calling $t' = t - \Delta t$ and taking the limit of the cross-covariance functions as $t \to \infty$ (and meanwhile keeping Δt finite), we recover the stationary version of the Kernel, i.e. the one which should be used when we assume that the system is very far away from its initialization and all the transients have faded away:

$$\boxed{\begin{aligned} K^{st}_{y_q f_r}(t, t-\Delta t) &= \lim_{t \to \infty} K_{y_q f_r}(t, t-\Delta t) \\ &= \sigma_r^2 S_{qr} \exp\left(-D_q \cdot \Delta t\right) u(\Delta t) = k_{y_q f_r}(t, t-\Delta t) \end{aligned}}$$

which is the same as $K_{y_q f_r}(t, t-\Delta t)$, since in this case we do not have non-stationary terms, and

$$\boxed{\begin{aligned} K^{st}_{y_p y_q}(t, t-\Delta t) &= \lim_{t \to \infty} K_{y_p y_q}(t, t-\Delta t) \\ &= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{D_p + D_q} \exp\left(-D_r |\Delta t|\right). \end{aligned}}$$

In some cases it will be advantageous to use the non-stationary case (to model transients of the output), whereas in some others it will be preferable to use the stationary version. That decision will have to be taken case by case depending on whether we can consider that the system has already settled down or not.

LFOS + White Kernel : Gradients

The parameters of the system are now σ_r^2, S_{qr} (S_{pr}) and D_q (D_p). So, the gradients for the cross-covariance between $y_q(t)$ and $f_r(t')$ are:

$$\boxed{\nabla_{\sigma_r^2} K_{y_q f_r}(t,t') = S_{qr} \exp\left(-D_q (t-t')\right) u(t-t')}$$

$$\boxed{\nabla_{S_{qr}} K_{y_q f_r}(t,t') = \sigma_r^2 \exp\left(-D_q (t-t')\right) u(t-t')}$$

$$\boxed{\begin{aligned}\nabla_{D_q} K_{y_q f_r}(t,t') &= -\sigma_r^2 S_{qr} (t-t') \exp\left(-D_q(t-t')\right) u(t-t') \\ &= -(t-t') K_{y_q f_r}(t,t')\end{aligned}}$$

W.r.t. to the cross-covariance between $y_p(t)$ and $y_q(t')$, the gradient for σ_r^2 is quite straightforward:

$$\boxed{\nabla_{\sigma_r^2} K_{y_p y_q}(t,t') = \frac{S_{pr} S_{qr}}{D_p + D_q} \left[\exp\left(-D_v |t-t'|\right) - \exp\left(-(D_p t + D_q t')\right)\right]}$$

For the gradient w.r.t. the S's and D's we introduce, as usual, the generic parameters D_ℓ and $S_{\ell r}$, which can take values D_p or D_q, and S_{pr} or S_{qr} respectively. Now, the gradients become:

$$\boxed{\begin{aligned}\nabla_{S_{\ell r}} K_{y_p y_q}(t,t') = \; & \frac{\sigma_r^2 \left[S_{qr} \nabla_{S_{\ell r}} S_{pr} + S_{pr} \nabla_{S_{\ell r}} S_{qr}\right]}{D_p + D_q} \\ & \times \Big[\overbrace{\exp\left(-D_v |t-t'|\right)}^{\text{st.}} - \overbrace{\exp\left(-(D_p t + D_q t')\right)}^{\text{non-st.}}\Big]\end{aligned}},$$

$$\boxed{\begin{aligned}\nabla_{D_\ell} K_{y_p y_q}(t,t') = \; & \sum_{r=1}^{R} \sigma_r^2 S_{pr} S_{qr} \\ & \times \Bigg\{ \nabla_{D_\ell}\left(\frac{1}{D_p + D_q}\right) \cdot \Big[\overbrace{\exp\left(-D_v |t-t'|\right)}^{\text{st.}} - \overbrace{\exp\left(-(D_p t + D_q t')\right)}^{\text{non-st.}}\Big] \\ & + \frac{1}{D_p + D_q} \Big[\overbrace{-|t-t'| \exp\left(-D_v |t-t'|\right) \nabla_{D_\ell} D_v}^{\text{st.}} \\ & + \overbrace{\left(t \nabla_{D_\ell} D_p + t' \nabla_{D_\ell} D_q\right) \exp\left(-(D_p t + D_q t')\right)}^{\text{non-st.}}\Big]\Bigg\}\end{aligned}},$$

where

$$\boxed{\nabla_{D_\ell}\left(\frac{1}{D_p+D_q}\right) = -\frac{\nabla_{D_\ell}D_p + \nabla_{D_\ell}D_q}{(D_p+D_q)^2}}$$

$$\boxed{\begin{aligned}\nabla_{D_\ell}D_v &= \nabla_{D_\ell}D_q \cdot \chi_{(-\infty,0)}(t-t') + \nabla_{D_\ell}D_p \cdot \chi_{[0,\infty)}(t-t') \\ &= \begin{cases}\nabla_{D_\ell}D_q, & t<t' \\ \nabla_{D_\ell}D_p, & t\geq t'\end{cases}\end{aligned}}$$

Hence, we have three possible situations:

1) $\ell = p \neq q$:

$$\nabla_{S_{pr}} K_{y_p y_q}(t,t') = \frac{\sigma_r^2 S_{qr}}{D_p+D_q}\left[\exp(-D_v|t-t'|) - \exp(-(D_p t + D_q t'))\right]$$

$$\begin{aligned}\boxed{\nabla_{D_p} K_{y_p y_q}(t,t')} = \sum_{r=1}^{R} \sigma_r^2 S_{pr} S_{qr} \\ \times \Big\{ -\frac{1}{(D_p+D_q)^2}\left[\exp(-D_v|t-t'|) - \exp(-(D_p t + D_q t'))\right] \\ + \frac{1}{D_p+D_q}\Big[-|t-t'|\exp(-D_v|t-t'|)\chi_{[0,\infty)}(t-t') \\ + t\exp(-(D_p t + D_q t'))\Big]\Big\}\end{aligned}$$

2) $\ell = q \neq p$:

$$\nabla_{S_{qr}} K_{y_p y_q}(t,t') = \frac{\sigma_r^2 S_{pr}}{D_p+D_q}\left[\exp(-D_v|t-t'|) - \exp(-(D_p t + D_q t'))\right]$$

$$\begin{aligned}\nabla_{D_q} K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \sigma_r^2 S_{pr} S_{qr} \\ \times \Big\{ -\frac{1}{(D_p+D_q)^2}\left[\exp(-D_v|t-t'|) - \exp(-(D_p t + D_q t'))\right] \\ + \frac{1}{D_p+D_q}\Big[-|t-t'|\exp(-D_v|t-t'|)\chi_{(-\infty,0)}(t-t') \\ + t'\exp(-(D_p t + D_q t'))\Big]\Big\}\end{aligned}$$

3) $l = p = q$:

$$\nabla_{S_{pr}} K_{y_p y_p}(t,t') = \frac{2\sigma_r^2 S_{pr}}{D_p + D_q}\left[\exp\left(-D_r|t-t'|\right) - \exp\left(-(D_p t + D_q t')\right)\right]$$

$$\nabla_{D_p} K_{y_p y_p}(t,t') = \sum_{r=1}^{R} \sigma_r^2 S_{pr}^2$$

$$\times \left\{ -\frac{1}{2D_p^2}\left[\exp\left(-D_p|t-t'|\right) - \exp\left(-(D_p t + D_q t')\right)\right]\right.$$

$$+ \frac{1}{2D_p}\left[-|t-t'|\exp\left(-D_p|t-t'|\right)\right.$$

$$\left.\left. + (t+t')\exp\left(-D_p(t+t')\right)\right]\right\}$$

$\boxed{\text{LSOS + White Kernel}}$ $\quad \omega_q = \frac{\sqrt{4m_q D_q - C_q^2}}{2m_q}$ $\quad \boxed{\text{LFM-WHITE}}$

$\alpha_q = C_q/(2m_q)$

The output of the linear second-order system (LSOS) is given by

$$\boxed{\begin{aligned} y_q(t) &= \frac{B_q}{D_q} + \sum_{r=1}^{R} L_{qr}[f_r](t), \\ L_{qr}[f_r](t) &= \frac{S_{qr}}{\omega'_q} \exp(-\alpha_q t) \int_0^t f_r(\tau) \exp(\alpha_q \tau) \sin(\omega'_q (t-\tau))\, d\tau, \end{aligned}}$$

$m_q \omega_q$

where $\omega'_q = \sqrt{4D_q - C_q^2}/2$ and $\alpha_q = C_q/2$. Hence, we have again $\boxed{E\{y_q(t)\} = C}$ and we can compute easily the cross-covariance functions:

$$\boxed{K_{y_q f_r}(t,t')} = \frac{S_{qr}}{\omega'_q} \exp(-\alpha_q t) \int_0^t \underbrace{K_{f_r f_r}(\tau, t')}_{= \sigma_r^2 \delta(\tau - t')} \exp(\alpha_q \tau) \sin(\omega'_q (t-\tau))\, d\tau$$

$$= \begin{cases} 0 & , \ t < t'; \\ \dfrac{\sigma_r^2 S_{qr}}{\omega'_q} \exp(-\alpha_q (t-t')) \sin(\omega'_q (t-t')), & t \geq t'; \end{cases}$$

$$= \boxed{\frac{\sigma_r^2 S_{qr}}{\omega'_q} \exp(-\alpha_q (t-t')) \sin(\omega'_q (t-t'))\, u(t-t').}$$

$m_q \omega_q$

And again, taking $t' = t - \Delta t$:

$$\boxed{K_{y_q f_r}(t, t-\Delta t) = \frac{\sigma_r^2 S_{qr}}{\omega'_q} \exp(-\alpha_q \cdot \Delta t) \sin(\omega'_q \cdot \Delta t)\, u(\Delta t),}$$

$m_q \omega_q$

which means that $\boxed{K_{y_q f_r}(t, t-\Delta t) = K^{st}_{y_q f_r}(t, t-\Delta t)}$, as occurs for the LFOS. Hence, again the non-stationarities in the joint GP formed by the set of inputs, $\{f_r\}_{r=1}^{R}$, and outputs, $\{y_q\}_{q=1}^{Q}$, are located exclusively in $K_{y_p y_q}(t,t')$. Note also that, once more, $K_{y_q f_r}(t,t') = 0$ when $t < t'$ (i.e. when $\Delta t < 0$ in $K_{y_q f_r}(t, t-\Delta t)$), which is an expected feature of the system, since it is causal and the correlation between the inputs is limited to $t = t'$.

The cross-covariance between two outputs now becomes

$$K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{S_{pr} S_{qr}}{\omega_p' \omega_q'} \exp\left(-(\alpha_p t + \alpha_q t')\right)$$

$$\times \int_0^t \exp(\alpha_p \tau) \sin\left(\omega_p'(t-\tau)\right) \left[\int_0^{t'} K_{f_r f_r}(\tau,\tau') \exp(\alpha_q \tau') \sin\left(\omega_q'(t'-\tau')\right) d\tau' \right] d\tau$$

$$= \sum_{r=1}^{R} \frac{S_{pr} S_{qr}}{\omega_p' \omega_q'} \exp\left(-(\alpha_p t + \alpha_q t')\right) \cdot \sigma_r^2$$

$$\times \int_0^{t \wedge t'} \exp\left((\alpha_p + \alpha_q)\tau\right) \sin\left(\omega_p'(t-\tau)\right) \sin\left(\omega_q'(t'-\tau)\right) d\tau .$$

Now, notice that

$\sin a \cdot \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$\sin\left(\omega_p'(t-\tau)\right) \sin\left(\omega_q'(t'-\tau)\right) = \frac{1}{2}\Big[\cos\left((\omega_p' t - \omega_q' t') - (\omega_p' - \omega_q')\tau\right)$$

$$- \cos\left((\omega_p' t + \omega_q' t') - (\omega_p' + \omega_q')\tau\right)\Big],$$

so that, decomposing the cosine into complex exponentials the integral in $K_{y_p y_q}(t,t')$ becomes

$$I_1 = \int_0^{t \wedge t'} \exp\left((\alpha_p + \alpha_q)\tau\right) \sin\left(\omega_p'(t-\tau)\right) \sin\left(\omega_q'(t'-\tau)\right) d\tau$$

$$= \frac{1}{4} \Big\{ \exp\left(j(\omega_p' t - \omega_q' t')\right) \int_0^{t \wedge t'} \exp\Big(\big(\overbrace{\tilde{\gamma}_p' + \gamma_q'}^{\alpha_p' - j\omega_p' + \alpha_q' + j\omega_q'}\big)\tau\Big) d\tau$$

$$+ \exp\Big(\overbrace{j(\omega_q' t' - \omega_p' t)}^{-j(\omega_p' t - \omega_q' t')}\Big) \int_0^{t \wedge t'} \exp\Big(\big(\overbrace{\gamma_p' + \tilde{\gamma}_q'}^{\alpha_p' + j\omega_p' + \alpha_q' - j\omega_q'}\big)\tau\Big) d\tau$$

$$+ \exp\left(j(\omega_p' t + \omega_q' t')\right) \int_0^{t \wedge t'} \exp\Big(\big(\overbrace{\tilde{\gamma}_p' + \tilde{\gamma}_q'}^{\alpha_p' - j\omega_p' + \alpha_q' - j\omega_q'}\big)\tau\Big) d\tau$$

$$- \exp\left(-j(\omega_p' t + \omega_q' t')\right) \int_0^{t \wedge t'} \exp\Big(\big(\overbrace{\gamma_p' + \gamma_q'}^{\alpha_p' + j\omega_p' + \alpha_q' + j\omega_q'}\big)\tau\Big) d\tau \Big\}$$

Now, we are going to denote

$$\boxed{I(r, t_1, t_2) = \int_{t_1}^{t_2} \exp(r\tau)\, d\tau = \frac{\exp(rt_2) - \exp(rt_1)}{r}},$$

so that I_1 finally can be expressed as

$$I_1 = \frac{1}{4}\Big\{ \exp\big(j(\omega_p' t - \omega_q' t')\big)\, I(\tilde{r}_p' + r_q', 0, t \wedge t')$$
$$+ \exp\big(j(\omega_q' t' - \omega_p' t)\big)\, I(r_p' + \tilde{r}_q', 0, t \wedge t')$$
$$- \exp\big(j(\omega_p' t + \omega_q' t')\big)\, I(\tilde{r}_p' + \tilde{r}_q', 0, t \wedge t')$$
$$- \exp\big(-j(\omega_p' t + \omega_q' t')\big)\, I(r_p' + r_q', 0, t \wedge t') \Big\},$$

and the Kernel finally is:

$$K_{y_p y_q}(t, t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4\, \omega_p' \omega_q'} \leftarrow m_p m_q \omega_p \omega_q$$
$$\times \Big[\exp\big(-(\tilde{r}_p' t + r_q' t')\big)\, I(\tilde{r}_p' + r_q', 0, t \wedge t')$$
$$+ \exp\big(-(r_p' t + \tilde{r}_q' t')\big)\, I(r_p' + \tilde{r}_q', 0, t \wedge t')$$
$$- \exp\big(-(\tilde{r}_p' t + \tilde{r}_q' t')\big)\, I(\tilde{r}_p' + \tilde{r}_q', 0, t \wedge t')$$
$$- \exp\big(-(r_p' t + r_q' t')\big)\, I(r_p' + r_q', 0, t \wedge t') \Big]$$

Now, note that the term which appears in each exponential is always the same as the one appearing in the integral I. Hence, $K_{y_p y_q}(t, t')$ can be expressed in a similar way to the LFOS: as the sum of a stationary and a non-stationary part (in this case with 4 exponentials for each part). Calling once more $t' = t - \Delta t$, we have:

$$K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4\omega_p' \omega_q'}$$

$$K^{(r)}_{y_p y_q}(t,t') \quad \times\Bigg[\frac{\exp(-\Psi(\gamma_q', \tilde{\gamma}_p', t-t')\cdot|t-t'|) - \exp(-(\tilde{\gamma}_p' t + \gamma_q' t'))}{\tilde{\gamma}_p' + \gamma_q'}$$

$$+ \frac{\exp(-\Psi(\tilde{\gamma}_q', \gamma_p', t-t')\cdot|t-t'|) - \exp(-(\gamma_p' t + \tilde{\gamma}_q' t'))}{\gamma_p' + \tilde{\gamma}_q'}$$

$$- \frac{\exp(-\Psi(\tilde{\gamma}_q', \tilde{\gamma}_p', t-t')\cdot|t-t'|) - \exp(-(\tilde{\gamma}_p' t + \tilde{\gamma}_q' t'))}{\tilde{\gamma}_p' + \tilde{\gamma}_q'}$$

$$- \frac{\exp(-\Psi(\gamma_q', \gamma_p', t-t')\cdot|t-t'|) - \exp(-(\gamma_p' t + \gamma_q' t'))}{\gamma_p' + \gamma_q'}\Bigg],$$

which shows clearly the stationary part (the first term in each fraction) and the non-stationary part (second term). Finally, taking the limit as $t \to \infty$:

$$\boxed{\begin{aligned} K^{st}_{y_p y_q}(t,t-\Delta t) &= \lim_{t\to\infty} K_{y_p y_q}(t,t-\Delta t) \\ &= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4\omega_p' \omega_q'} \Bigg[\frac{\exp(-\Psi(\gamma_q', \tilde{\gamma}_p', \Delta t)\cdot|\Delta t|)}{\tilde{\gamma}_p' + \gamma_q'} \\ &+ \frac{\exp(-\Psi(\tilde{\gamma}_q', \gamma_p', \Delta t)\cdot|\Delta t|)}{\gamma_p' + \tilde{\gamma}_q'} \\ &- \frac{\exp(-\Psi(\tilde{\gamma}_q', \tilde{\gamma}_p', \Delta t)\cdot|\Delta t|)}{\tilde{\gamma}_p' + \tilde{\gamma}_q'} \\ &- \frac{\exp(-\Psi(\gamma_q', \gamma_p', \Delta t)\cdot|\Delta t|)}{\gamma_p' + \gamma_q'} \Bigg] \end{aligned}}$$

$m_p m_q \omega_p \omega_q$

where the function Ψ in these expressions is defined as

$$\boxed{\begin{aligned}\varphi(r_1, r_2, \Delta t) &= \begin{cases} r_1, & \Delta t < 0; \\ r_2, & \Delta t \geq 0 \end{cases} \\ &= r_1 \cdot \chi_{(-\infty,0)}(\Delta t) + r_2 \cdot \chi_{[0,\infty)}(\Delta t)\end{aligned}}$$

In order to finish with the LSOS, we need to show that it is equivalent to the LFOS in the limit when $m_q \to 0$ and $C_q = 1$. Note that m_q does not appear in the expressions since a normalization has been performed so that $C_q' = C_q/m_q$, $D_q' = D_q/m_q$, $B_q' = B_q/m_q$ and $S_{qr}' = S_{qr}/m_q$, and so we have

$$\alpha_q' = \frac{C_q'}{2} = \frac{C_q}{2m_q} = \alpha_q$$

$$\frac{B_q'}{D_q'} = \frac{B_q}{D_q}$$

$$\omega_q' = \frac{\sqrt{4D_q' - (C_q')^2}}{2} = \frac{\sqrt{4(D_q/m_q) - (C_q/m_q)^2}}{2} = \frac{\sqrt{4m_q D_q - C_q^2}}{2m_q} = \omega_q$$

In the general case we can assume that $m_q = 1$ and so we are working with the real parameters (if we know that $m_q \neq 1$ we can easily recover the true parameters and take that into account), but here we need to consider m_q. So, when $m_q \to 0$ we have:

$$\lim_{m_q \to 0} K_{y_q f_r}(t, t-\Delta t) = \sigma_r^2 S_{qr} \cdot \lim_{m_q \to 0} \frac{1}{m_q \omega_q} \cdot \lim_{m_q \to 0} \exp(-\alpha_q \Delta t) \sin(\omega_q \Delta t) \cdot u(\Delta t)$$

Let us focus on the first limit:

$$\lim_{m_q \to 0} \frac{1}{m_q \omega_q} = \lim_{m_q \to 0} \frac{1}{m_q \cdot \frac{\sqrt{4m_q D_q - C_q^2}}{2m_q}} = \frac{2}{\sqrt{-C_q^2}} = \frac{2}{jC_q} \underset{C_q = 1}{=} -j2$$

And now, the second limit can be expressed as

$$\lim_{m_q \to 0} \exp(-\alpha_q \Delta t) \cdot \sin(\omega_q \Delta t) = \frac{1}{j2}\left[\lim_{m_q \to 0} \exp\left(-(\alpha_q - j\omega_q)\Delta t\right) - \lim_{m_q \to 0} \exp\left(-(\alpha_q + j\omega_q)\right)\Delta t\right]$$

$$= \frac{1}{j2}\left[\exp\left(-\Delta t \cdot \lim_{m_q \to 0} (\alpha_q - j\omega_q)\right) - \exp\left(-\Delta t \cdot \lim_{m_q \to 0} (\alpha_q + j\omega_q)\right)\right]$$

and these two limits are

$$\lim_{m_q \to 0} (\alpha_q - j\omega_q) = \lim_{m_q \to 0} \frac{C_q - j\sqrt{4 m_q D_q - C_q^2}}{2 m_q}$$

$$= \lim_{m_q \to 0} \frac{C_q - j^2 C_q}{2 m_q} = \lim_{m_q \to 0} \frac{2 C_q}{2 m_q} = \infty,$$

$$\lim_{m_q \to 0} \alpha_q + j\omega_q = \lim_{m_q \to 0} \frac{C_q + j\sqrt{4 m_q D_q - C_q^2}}{2 m_q} \quad \left[\text{indeterminacy } \frac{0}{0}\right]$$

$$\underset{\substack{\uparrow \\ \text{L'Hôpital's} \\ \text{rule}}}{=} \lim_{m_q \to 0} \frac{j 4 D_q \cdot 1/2 \left[4 m_q D_q - C_q^2\right]^{-1/2}}{2}$$

$$= \lim_{m_q \to 0} \frac{j \cancel{4} D_q}{\cancel{4}\sqrt{4 m_q D_q - C_q^2}} = \frac{j D_q}{j C_q} \underset{\substack{\uparrow \\ C_q = 1}}{=} D_q$$

So, finally :

$$\lim_{m_q \to 0} \exp(-\alpha_q \Delta t) \sin(\omega_q \Delta t) = \frac{1}{j2}\left[0 - \exp(-D_q \Delta t)\right],$$

and

$$\boxed{\lim_{\substack{m_q \to 0 \\ C_q = 1}} K_{Y_q F_r}^{LSOS}(t, t-\Delta t) = \sigma_r^2 S_{qr} \exp(-D_q \Delta t)\, u(\Delta t) = k_{Y_q F_r}^{LFOS}(t, t-\Delta t)}$$

which is exactly the expression for the LFOS.

And the expression for the cross-covariance between two outputs can be obtained in a similar way noting that

$$\lim_{m_v \to 0} \exp(-\gamma_v t) = \exp(-D_v t),$$

$$\lim_{m_v \to 0} \exp(-\tilde{\gamma}_v t) = 0,$$

So in the limit,

$$\boxed{\begin{aligned}\lim_{\substack{m_p, m_q \to 0\\ C_p = C_q = 1}} k^{LSOS}_{y_p y_q}(t,t') &= \sum_{r=1}^{R} \sigma_r^2 S_{pr} S_{qr} \\ &\times \frac{\exp(-D_p|t-t'|)\,u(t-t') + \exp(-D_q|t-t'|)\,u(t'-t) - \exp(-(D_p t + D_q t'))}{D_p + D_q} \\ &= k^{LFOS}_{y_p y_q}(t,t'),\end{aligned}}$$

where we must note that $\tilde{\gamma}_p' + \gamma_q'$, $\gamma_p' + \tilde{\gamma}_q'$ and $\tilde{\gamma}_p' + \tilde{\gamma}_q' \to \infty$ as $m_q \to 0$, so the first three terms of $K^{LFOS}_{y_p y_q}(t,t')$ vanish, and we have also used

$$\lim_{\substack{m_p, m_q \to 0\\ C_p = C_q = 1}} \frac{1}{m_p \omega_p m_q \omega_q} = \lim_{\substack{m_p \to 0\\ C_p = 1}} \frac{1}{m_p \omega_p} \cdot \lim_{\substack{m_q \to 0\\ C_q = 1}} \frac{1}{m_q \omega_q} = (j2)^2 = -4,$$

$$\lim_{\substack{m_p, m_q \to 0\\ C_p = C_q = 1}} \gamma_p' + \gamma_q' = D_p + D_q$$

LSOS + White Kernel: Gradients

Now, the parameters appearing in the cross-covariance functions are σ_r^2 S_{pr}, S_{qr}, ω_p', ω_q', α_p and α_q. These last four depend on turn on C_p, C_q, D_p and D_q, which will be the ones used for the gradients. First, the gradients of $K_{y_q f_r}(t,t')$ are given by

$$\boxed{\nabla_{S_{qr}} K_{y_q f_r}(t,t') = \frac{\sigma_r^2}{\omega_q'} \exp(-\alpha_q(t-t')) \sin(\omega_q'(t-t'))\, u(t-t')}$$

mq ωq

$$\boxed{\nabla_{\sigma_r^2} K_{y_q f_r}(t,t') = \frac{S_{qr}}{\omega_q'} \exp(-\alpha_q(t-t')) \sin(\omega_q'(t-t'))\, u(t-t')}$$

$$\nabla_{C_q} K_{y_q f_r}(t,t') = \sigma_r^2 S_{qr} \times \Big\{ \nabla_{C_q}(1/\omega_q') \cdot \exp(-\alpha_q(t-t')) \sin(\omega_q'(t-t'))$$

mq ωq

$$+ \frac{1}{\omega_q'} \Big[-(t-t') \exp(-\alpha_q(t-t')) \sin(\omega_q'(t-t'))\, \nabla_{C_q}\alpha_q$$

mq ωq

$$+ (t-t') \exp(-\alpha_q(t-t')) \cos(\omega_q'(t-t'))\, \nabla_{C_q}\omega_q' \Big] \Big\},$$

$$\times u(t-t')$$

where

$$\nabla_{C_q}\omega_q' = -\frac{C_q}{2\sqrt{4D_q - C_q^2}} = -\frac{\alpha_q}{2\omega_q'},$$

$$\nabla_{C_q}(1/\omega_q') = -\frac{1}{(\omega_q')^2} \cdot \nabla_{C_q}\omega_q' = \frac{\alpha_q}{2(\omega_q')^3},$$

$$\nabla_{C_q}\alpha_q = 1/2.$$

Finally,

$$\nabla_{D_q} K_{y_q f_r}(t,t') = \sigma_r^2 S_{qr} \times \Big\{ \nabla_{D_q}(1/\omega_q') \exp(-\alpha_q(t-t')) \sin(\omega_q'(t-t'))$$

$$+ \frac{(t-t')}{\omega_q'} \exp(-\alpha_q(t-t')) \cos(\omega_q'(t-t'))\, \nabla_{D_q}\omega_q' \Big\}\, u(t-t'),$$

with

$$\nabla_{D_q}\omega_q' = 1/\sqrt{4D_q - C_q^2} = 1/(2\omega_q'),$$

$$\nabla_{D_q}(1/\omega_q') = -1/(\omega_q')^2 \cdot \nabla_{D_q}\omega_q' = -1/[2(\omega_q')^3].$$

The gradients of $k_{y_p y_q}(t,t')$ can be obtained in a similar way:

$$\boxed{\nabla_{\sigma_r^2} k_{y_p y_q}(t,t') = \frac{S_{pr} S_{qr}}{4\omega_p' \omega_q'} \, k^{(r)}_{y_p y_q}(t,t')}$$

$$\boxed{\nabla_{S_{\ell r}} k_{y_p y_q}(t,t') = \frac{\sigma_r^2 \left[S_{pr} \nabla_{S_{\ell r}} S_{qr} + S_{qr} \nabla_{S_{\ell r}} S_{pr} \right]}{4\omega_p' \omega_q'} \, k^{(r)}_{y_p y_q}(t,t')}$$

Use $\theta_\ell \in \{C_\ell, D_\ell\}$, $\ell \in \{p,q\}$

$$\boxed{\begin{aligned}\nabla_{C_\ell} k_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4} & \\ \times \Bigg\{ & \nabla_{C_\ell}\left(\frac{1}{\omega_p' \omega_q'}\right) \cdot k^{(r)}_{y_p y_q}(t,t') \\ & + \frac{1}{\omega_p' \omega_q'} \Big[\nabla_{C_\ell} h(r_q', \tilde{r}_p', t, t') \\ & + \nabla_{C_\ell} h(\tilde{r}_q', r_p', t, t') - \nabla_{C_\ell} h(\tilde{r}_q', \tilde{r}_p', t, t') \\ & - \nabla_{C_\ell} h(r_q', r_p', t, t') \Big] \Bigg\},\end{aligned}}$$

where we have defined

$$\boxed{h(r_q', r_p', t, t') = \frac{\exp\left(-\psi(r_q', r_p', t-t')|t-t'|\right) - \exp\left(-(r_p' t + r_q' t')\right)}{r_p' + r_q'},}$$

so that

$$\begin{aligned} k^{(r)}_{y_p y_q}(t,t') = {} & h(r_q', \tilde{r}_p', t, t') + h(\tilde{r}_q', r_p', t, t') \\ & - h(\tilde{r}_q', \tilde{r}_p', t, t') - h(r_q', r_p', t, t'), \end{aligned}$$

and

$$k_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4\omega_p' \omega_q'} \, k^{(r)}_{y_p y_q}(t,t').$$

Now, we only have to obtain the gradient of $h(r_q', r_p', t, t')$ w.r.t. C_ℓ and D_ℓ:

$$\nabla_{C_\ell} h(r_q', r_p', t, t') = \nabla_{C_\ell}\left(\frac{1}{r_p'+r_q'}\right) \times \Big[\exp\left(-\psi(r_q', r_p', t, t')|t-t'|\right) - \exp\left(-(r_p't + r_q't')\right)\Big]$$
$$+ \frac{1}{r_p'+r_q'} \times \Big[-|t-t'| \exp\left(-\psi(r_q', r_p', t, t')|t-t'|\right) \times \nabla_{C_\ell}\psi(r_q', r_p', t, t')$$
$$+ \left(t\nabla_{C_\ell} r_p' + t'\nabla_{C_\ell} r_q'\right)\exp\left(-(r_p't + r_q't')\right)\Big],$$

and similarly

$$\nabla_{D_\ell} h(r_q', r_p', t, t') = \nabla_{D_\ell}\left(\frac{1}{r_p'+r_q'}\right) \times \Big[\exp\left(-\psi(r_q', r_p', t, t')|t-t'|\right) - \exp\left(-(r_p't + r_q't')\right)\Big]$$
$$+ \frac{1}{r_p'+r_q'} \times \Big[-|t-t'| \exp\left(-\psi(r_q', r_p', t, t')|t-t'|\right) \nabla_{D_\ell}\psi(r_q', r_p', t, t')$$
$$+ \left(t\nabla_{D_\ell} r_p' + t'\nabla_{D_\ell} r_q'\right)\exp\left(-(r_p't + r_q't')\right)\Big],$$

so that

$$\nabla_{D_\ell} K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{4} \times \Big\{ \nabla_{D_\ell}\left(\frac{1}{\omega_p'\omega_q'}\right) K^{(r)}_{y_p y_q}(t,t')$$
$$+ \frac{1}{\omega_p'\omega_q'}\Big[\nabla_{D_\ell} h(r_q', \tilde{r}_p', t, t') + \nabla_{D_\ell} h(\tilde{r}_q', r_p', t, t')$$
$$- \nabla_{D_\ell} h(\tilde{r}_q', \tilde{r}_p', t, t') - \nabla_{D_\ell} h(r_q', r_p', t, t')\Big].$$

And finally, we only need to define the last few gradients:

$$\nabla_{C_\ell}\left(\frac{1}{\omega_p'\omega_q'}\right) = -\left[\frac{\nabla_{C_\ell}\omega_p'}{(\omega_p')^2\omega_q'} + \frac{\nabla_{C_\ell}\omega_q'}{\omega_p'(\omega_q')^2}\right]$$

$$\nabla_{D_\ell}\left(\frac{1}{\omega_p'\omega_q'}\right) = -\left[\frac{\nabla_{D_\ell}\omega_p'}{(\omega_p')^2\omega_q'} + \frac{\nabla_{D_\ell}\omega_q'}{\omega_p'(\omega_q')^2}\right]$$

$$\nabla_{C_\ell}\left(\frac{1}{r_p'+r_q'}\right) = -\frac{\nabla_{C_\ell} r_p' + \nabla_{C_\ell} r_q'}{(r_p'+r_q')^2}$$

$$\nabla_{D_\ell}\left(\frac{1}{r_p'+r_q'}\right) = -\frac{\nabla_{D_\ell} r_p' + \nabla_{D_\ell} r_q'}{(r_p'+r_q')}$$

$$\nabla_{C_\ell} \gamma_p' = \nabla_{C_\ell} \alpha_p' + j \nabla_{C_\ell} \gamma_p'$$

$$\nabla_{C_\ell} \alpha_p' = \nabla_{C_\ell} C_p / 2$$

$$\nabla_{C_\ell} \gamma_p' = -\frac{C_p}{2\sqrt{4D_p - C_p^2}} \nabla_{C_\ell} C_p = -\frac{\alpha_p}{2\omega_p'} \nabla_{C_\ell} C_p$$

$$\nabla_{D_\ell} \gamma_p' = \nabla_{D_\ell} \alpha_p' + j \nabla_{D_\ell} \gamma_p' = j \nabla_{D_\ell} \gamma_p'$$

$$\nabla_{D_\ell} \alpha_p' = 0$$

$$\nabla_{D_\ell} \gamma_p' = \frac{\nabla_{D_\ell} D_p}{\sqrt{4D_q - C_q^2}} = \frac{\nabla_{D_\ell} D_p}{2\omega_p'}$$

$$\nabla_{C_\ell} \Psi (\gamma_q', \gamma_p', t, t') = \nabla_{C_\ell} \gamma_q' \cdot \chi_{(-\infty,0)}(t-t') + \nabla_{C_\ell} \gamma_p' \cdot \chi_{[0,\infty)}(t-t')$$

$$\nabla_{D_\ell} \Psi (\gamma_q', \gamma_p', t, t') = \nabla_{D_\ell} \gamma_q' \cdot \chi_{(-\infty,0)}(t-t') + \nabla_{D_\ell} \gamma_p' \cdot \chi_{[0,\infty)}(t-t')$$

More generally:

$$\nabla_{\theta_\ell} \gamma_m' = \nabla_{\theta_\ell} \alpha_m' + j \nabla_{\theta_\ell} \gamma_m'$$

$\theta_\ell \in \{C_\ell, D_\ell\}$

$\ell, m \in \{p, q\}$