RBF-WHITE Kernel

Covariance functions:

This Kernel is the result of having a white Gaussian noise latent process, with an auto-covariance function defined by

$$\boxed{K_{f_r f_r}(t,t') = \sigma_r^2\, \delta(t-t'),} \qquad (1)$$

for the r-th latent function, convolved with a Gaussian smoothing Kernel,

$$\boxed{\begin{aligned} h_q(t,t') = h_q(t-t') &= \frac{1}{\sqrt{2\pi l_q^2}} \exp\left(-\frac{(t-t')^2}{2 l_q^2}\right) \qquad (2.a) \\ &= \frac{\sqrt{v_q}}{\sqrt{2\pi}} \exp\left(-\frac{v_q}{2}(t-t')^2\right) \qquad (2.b) \end{aligned}},$$

where l_q is the length (or time) scale of the q-th smoothing Kernel, and $\boxed{v_q = 1/l_q^2}$ (i.e. $\boxed{l_q = 1/v_q}$) is its inverse width. The resulting output is given by

$$\boxed{y_{rq}(t) = \int_0^t f_r(\tau)\, h_q(t-\tau)\, d\tau.} \qquad (3)$$

Given the linearity of the convolution and the fact that $f_r(t)$ is a GP, then $y_{rq}(t)$ is also a GP defined by its mean function,

$$\boxed{E\{y_{rq}(t)\} = \mu_{rq}(t) = \int_0^t h_q(t-\tau)\, E\{f_r(\tau)\}\, d\tau = 0}, \qquad (4)$$

and the cross-covariance between $y_{rp}(t)$ and $y_{rq}(t')$, since $y_{mp}(t)$ is uncorrelated with $y_{rq}(t')$ whenever $m \neq r$ for any p, q:

$$\begin{aligned} \boxed{K^{(r)}_{y_p y_q}(t,t')} &= E\{y_{rp}(t)\, y_{rq}(t')\} \\ &= \int_0^t \int_0^{t'} h_p(t-\tau)\, h_q(t'-\tau')\, K_{f_r f_r}(\tau,\tau')\, d\tau\, d\tau' \\ &= \boxed{\sigma_r^2 \int_0^{t \wedge t'} h_p(t-\tau)\, h_q(t'-\tau)\, d\tau}, \end{aligned} \qquad (5)$$

where the upper limit of integration is due to the fact for $t > t \wedge t'$ or $t' > t \wedge t'$, $K_{frfr}(\tau, \tau') = 0$. Replacing $h_p(t-\tau)$ and $h_q(t'-\tau)$ for their values we obtain

$$K^{(r)}_{Y_p Y_q}(t,t') = \sigma_r^2 \int_0^{t\wedge t'} \sqrt{\frac{\vartheta_p}{2\pi}} \exp\left(-\frac{\vartheta_p (t-\tau)^2}{2}\right) \cdot \sqrt{\frac{\vartheta_q}{2\pi}} \exp\left(-\frac{\vartheta_q (t'-\tau)^2}{2}\right) d\tau$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \int_0^{t\wedge t'} \exp\left(-\frac{\vartheta_p (t^2 - 2t\tau + \tau^2) + \vartheta_q ((t')^2 - 2t'\tau + \tau^2)}{2}\right) \times d\tau$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \int_0^{t\wedge t'} \exp\left(-\frac{(\vartheta_p + \vartheta_q)\tau^2 - 2(\vartheta_p t + \vartheta_q t')\tau + \vartheta_p t^2 + \vartheta_q (t')^2}{2}\right) \times d\tau$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \int_0^{t\wedge t'} \exp\left(-\frac{\tau^2 - 2\frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\tau + \frac{\vartheta_p t^2 + \vartheta_q (t')^2}{\vartheta_p + \vartheta_q}}{2/(\vartheta_p + \vartheta_q)}\right) d\tau$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \int_0^{t\wedge t'} \exp\left(-\frac{\left(\tau - (\vartheta_p t + \vartheta_q t')/(\vartheta_p + \vartheta_q)\right)^2}{2/(\vartheta_p + \vartheta_q)}\right) d\tau$$

$$\times \exp\left(-\frac{\vartheta_p t^2 + \vartheta_q (t')^2}{\frac{2\cancel{(\vartheta_p + \vartheta_q)}}{\cancel{(\vartheta_p + \vartheta_q)}}} + \frac{(\vartheta_p t + \vartheta_q t')^2}{\frac{2(\vartheta_p + \vartheta_q)^{\cancel{2}}}{\cancel{(\vartheta_p + \vartheta_q)}}}\right)$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \int_0^{t\wedge t'} \exp\left(-\frac{\vartheta_p + \vartheta_q}{2}\left(\tau - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\right)^2\right) d\tau$$

$$\times \exp\left(-\frac{\cancel{\vartheta_p^2 t^2} + \vartheta_p \vartheta_q t^2 + \vartheta_p \vartheta_q (t')^2 + \cancel{\vartheta_q^2 (t')^2}}{2(\vartheta_p + \vartheta_q)}\right)$$

$$\times \exp\left(\frac{\cancel{\vartheta_p^2 t^2} + 2\vartheta_p \vartheta_q t t' + \cancel{\vartheta_q^2 (t')^2}}{2(\vartheta_p + \vartheta_q)}\right) \Rightarrow$$

$$\boxed{\begin{aligned} K^{(r)}_{Y_p Y_q}(t,t') &= \frac{\sigma_r^2 \sqrt{\vartheta_p \vartheta_q}}{2\pi} \exp\left(-\frac{\vartheta_p \vartheta_q}{2(\vartheta_p+\vartheta_q)} (t-t')^2\right) \\ &\quad \times \underbrace{\int_0^{t\wedge t'} \exp\left(-\frac{\vartheta_p \vartheta_q}{2}\left(\tau - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q}\right)^2\right) d\tau}_{= I} \end{aligned}} \qquad (6)$$

Now, to obtain the final expression of $K^{(r)}_{Y_p Y_q}(t,t')$ we just need to solve I, which is easy, since, making the change of variable

$$Z = \sqrt{\frac{\vartheta_p \vartheta_q}{2}} \left(\tau - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\right) \Rightarrow$$

$$dZ = \sqrt{\frac{\vartheta_p \vartheta_q}{2}}\, d\tau,$$

we obtain the familiar expression of the error function:

$$I = \frac{\sqrt{\pi}}{2} \cdot \sqrt{\frac{2}{\vartheta_p \vartheta_q}} \cdot \frac{2}{\sqrt{\pi}} \int_{-\sqrt{\frac{\vartheta_p\vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q}}^{\sqrt{\frac{\vartheta_p\vartheta_q}{2}}\left(t\wedge t' - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q}\right)} \exp(-Z^2)\, dZ$$

$$= \sqrt{\frac{\pi}{2\vartheta_p \vartheta_q}} \left[\operatorname{erf}\left(\sqrt{\frac{\vartheta_p\vartheta_q}{2}}\left(t\wedge t' - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q}\right)\right) + \operatorname{erf}\left(\sqrt{\frac{\vartheta_p\vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q}\right)\right].$$

The first term can be rewritten to emphasize that it corresponds to the stationary part, since

$$t\wedge t' - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p+\vartheta_q} = -\frac{\vartheta_p (t - t\wedge t') + \vartheta_q (t' - t\wedge t')}{\vartheta_p + \vartheta_q}$$

$$= \begin{cases} -\dfrac{\vartheta_q (t'-t)}{\vartheta_p+\vartheta_q}, & t < t' \ [\text{i.e. } t\wedge t' = t]; \\ -\dfrac{\vartheta_p (t-t')}{\vartheta_p+\vartheta_q}, & t > t' \ [\text{i.e. } t\wedge t' = t']. \end{cases} \Rightarrow$$

$$\boxed{t \wedge t' - \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q} = -\vartheta_v \, |t - t'| / (\vartheta_p + \vartheta_q),} \tag{7}$$

where

$$\vartheta_v = \begin{cases} \vartheta_q, & t < t'; \\ \vartheta_p, & t > t'. \end{cases} \quad (8.a)$$

$$= \vartheta_q \left(1 - u(t - t')\right) + \vartheta_p \, u(t - t') \quad (8.b)$$

$$= \vartheta_q + (\vartheta_p - \vartheta_q) \, u(t - t'). \quad (8.c)$$

Therefore, I can be re-expressed as

$$I = \sqrt{\frac{\pi}{2\vartheta_p \vartheta_q}} \left[\operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q} \right) - \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_v |t - t'|}{\vartheta_p + \vartheta_q} \right) \right],$$

and the kernel becomes

$$\boxed{\begin{aligned} K^{(r)}_{y_p y_q}(t, t') = {} & \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(- \frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)} (t - t')^2 \right) \\ & \times \left[\operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q} \right) - \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_v |t - t'|}{\vartheta_p + \vartheta_q} \right) \right] \end{aligned}} \tag{9}$$

To complete the description of the kernel it is interesting to separate the stationary,

$$\boxed{K^{st\,(r)}_{y_p y_q}(t, t') = \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(- \frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)} (\overbrace{t - t'}^{\Delta t})^2 \right) \left[1 - \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_v \overbrace{|t - t'|}^{|\Delta t|}}{\vartheta_p + \vartheta_q} \right) \right]} \tag{10}$$

and non-stationary versions of the kernel,

$$\boxed{K^{non\text{-}st\,(r)}_{y_p y_q}(t, t') = \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(- \frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)} (t - t')^2 \right) \left[\operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q} \right) - 1 \right]} \tag{11}$$

It is also interesting to observe the behaviour of the main diagonal of the kernel matrix (i.e. $t = t'$):

$$\boxed{K^{(r)}_{y_p y_q}(t, t) = \frac{\sigma_r^2}{\sqrt{8\pi}} \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \, t \right)} \tag{12}$$

And the auto-covariance function (i.e. $p=q$):

$$\boxed{K^{(r)}_{y_p y_p}(t,t') = \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(\frac{\vartheta_p (t-t')^2}{4}\right)\left[\operatorname{erf}\left(\frac{\vartheta_p (t+t')}{2\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\vartheta_p |t-t'|}{2\sqrt{2}}\right)\right]} \quad (13)$$

We also need to obtain the cross-covariance matrix between $y_{rp}(t)$ and $f_r(t')$:

$$K_{y_p f_r}(t,t') = \int_0^t h_p(t-\tau)\, K_{f_r f_r}(t',\tau)\, d\tau$$

$$= \sigma_r^2\, h_p(t-t')\, u(t-t')$$

$$= \frac{\sigma_r^2 \sqrt{\vartheta_p}}{\sqrt{2\pi}} \exp\left(-\frac{\vartheta_p}{2}(t-t')^2\right) u(t-t'), \quad (14)$$

which means that $f_r(t')$ only has influence on $y_p(t)$ for $t > t'$, as expected, since it is a white noise process.

Gradients:

$$\nabla_{\sigma_r^2} K_{y_p f_r}(t,t') = \sqrt{\frac{\vartheta_p}{2\pi}} \exp\left(-\frac{\vartheta_p}{2}(t-t')^2\right) u(t-t') \quad (14.a)$$

$$= \frac{K_{y_p f_r}(t,t')}{\sigma_r^2} \quad (14.b)$$

$\frac{1}{2}\vartheta_p^{-1/2}$

$\parallel$

$$\nabla_{\vartheta_p} K_{y_p f_r}(t,t') = \frac{\sigma_r^2}{\sqrt{2\pi}} \exp\left(-\frac{\vartheta_p}{2}(t-t')^2\right) u(t-t')\, \nabla_{\vartheta_p} \vartheta_p^{1/2} \cdot \frac{\sqrt{\vartheta_p}}{\sqrt{\vartheta_p}}$$

$K_{y_p f_r}(t,t') =$

$$+ \frac{\sigma_r^2 \sqrt{\vartheta_p}}{\sqrt{2\pi}} \exp\left(-\frac{\vartheta_p}{2}(t-t')^2\right)\left(-\frac{1}{2}(t-t')^2\right) u(t-t') \quad (15.a)$$

$$= \left[\frac{1}{\sqrt{\vartheta_p}} - \sqrt{\vartheta_p}\,(t-t')^2\right] \frac{\sigma_r^2}{2\sqrt{2\pi}} \exp\left(-\frac{\vartheta_p}{2}(t-t')^2\right) u(t-t')$$

$$= \frac{1}{2}\left(\frac{1}{\vartheta_p} - (t-t')^2\right) K_{y_p f_r}(t,t') \quad (15.b)$$

Now, before obtaining the gradients of $K_{y_p y_q}^{(r)}(t,t')$ we define

$$E_{pq}^{(r)}(t,t') = \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\right) - \operatorname{erf}\left(\sqrt{\frac{\vartheta_p \vartheta_q}{2}} \cdot \frac{\vartheta_v |t-t'|}{\vartheta_p + \vartheta_q}\right), \tag{16}$$

So that

$$K_{y_p y_q}^{(r)}(t,t') = \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(-\frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)}(t-t')^2\right) E_{pq}^{(r)}(t,t'). \tag{17}$$

And hence,

$$\nabla_{\sigma_r^2} K_{y_p y_q}(t,t') = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)}(t-t')^2\right) E_{pq}^{(r)}(t,t') \tag{18.a}$$

$$= \frac{K_{y_p y_q}^{(r)}(t,t')}{\sigma_r^2} \tag{18.b}$$

$$\nabla_{\vartheta_p} K_{y_p y_q}^{(r)}(t,t') = \underbrace{\frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(-\frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)}(t-t')^2\right) E_{pq}^{(r)}(t,t')}_{K_{y_p y_q}^{(r)}(t,t')}$$
$$\times \left(-\frac{(t-t')^2}{2}\right) \cdot \nabla_{\vartheta_p} \frac{\vartheta_p \vartheta_q}{\vartheta_p + \vartheta_q}$$
$$+ \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(-\frac{\vartheta_p \vartheta_q}{2(\vartheta_p + \vartheta_q)}(t-t')^2\right) \nabla_{\vartheta_p} E_{pq}^{(r)}(t,t'), \tag{19}$$

where

$$\nabla_{\vartheta_p} \frac{\vartheta_p \vartheta_q}{\vartheta_p + \vartheta_q} = \frac{\vartheta_q(\vartheta_p + \vartheta_q) - \vartheta_p \vartheta_q}{(\vartheta_p + \vartheta_q)^2} = \frac{\vartheta_q^2}{(\vartheta_p + \vartheta_q)^2} \tag{20}$$

$$\nabla_{\vartheta_p} E_{pq}^{(r)}(t,t') = \frac{2}{\sqrt{\pi}} \Bigg\{ \exp\left(-\frac{\vartheta_p \vartheta_q}{2} \cdot \left(\frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\right)^2\right) \cdot \frac{\sqrt{\vartheta_q}}{\sqrt{2}}$$
$$\times \nabla_{\vartheta_p}\left(\frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q} \sqrt{\vartheta_p}\right)$$
$$- \exp\left(-\frac{\vartheta_p \vartheta_q}{2} \cdot \left(\frac{\vartheta_v |t-t'|}{\vartheta_p + \vartheta_q}\right)^2\right) \cdot \sqrt{\frac{\vartheta_q}{2}} \cdot \nabla_{\vartheta_p}\left(\frac{\overset{|t-t'|}{\vartheta_v} \sqrt{\vartheta_p}}{\vartheta_p + \vartheta_q}\right) \tag{21}$$

and

$$\boxed{\nabla_{v_p}\left(\sqrt{v_p}\cdot\frac{v_p t + v_q t'}{v_p + v_q}\right)} = \frac{\left[\frac{1}{2}(v_p)^{-1/2}(v_p t + v_q t') + v_p^{1/2} t\right](v_p + v_q)}{(v_p + v_q)^2}$$

$$- \frac{v_p^{1/2}(v_p t + v_q t')}{(v_p + v_q)^2}$$

$$= \big[\ \cdot (v_p t + v_q t')(v_p + v_q)/2$$

$$+ v_p t (v_p + v_q) - v_p (v_p t + v_q t')\big]$$

$$\times (v_p + v_q)^{-2}\, v_p^{-1/2}$$

$$= \big[v_p^2 t/2 + v_p v_q t/2 + v_p v_q t'/2 + v_q^2 t'/2$$

$$+ v_p v_q t - v_p v_q t'\big] / \big[(v_p + v_q)^2 \sqrt{v_p}\big]$$

$$\boxed{= \frac{v_p^2 t/2 + v_p v_q (3t - t')/2 + v_q^2 t'/2}{(v_p + v_q)^2 \sqrt{v_p}}} \qquad (22)$$

$$\nabla_{v_p}\left(\frac{v_v \sqrt{v_p}}{v_p + v_q}\right) = \frac{\frac{1}{2} v_p^{-1/2} v_v (v_p + v_q) + v_p^{1/2}(v_p + v_q)\nabla_{v_p} v_v}{(v_p + v_q)^2}$$

$$- \frac{v_v (v_p)^{1/2}}{(v_p + v_q)^2},$$

where, remembering that

$$v_v = v_q (1 - u(t - t')) + v_p\, u(t - t'),$$

we get

$$\boxed{\nabla_{v_p} v_v = u(t - t')} \qquad (23)$$

so that

$$\boxed{\nabla_{\vartheta_p}\left(\frac{\vartheta_v\sqrt{\vartheta_p}}{\vartheta_p+\vartheta_q}\right)} = \Big[\boxed{\vartheta_p\vartheta_q\,(1-u(t-t'))/2} + \boxed{\vartheta_p^2\,u(t-t')/2} + \cancel{\vartheta_p^2\,u(t-t}$$

$$+\boxed{\vartheta_p\vartheta_q\,u(t-t')} - \vartheta_p\vartheta_q\,(1-u(t-t')) - \cancel{\vartheta_p^2\,u(t-t'}$$

$$\times(\vartheta_p+\vartheta_q)^{-2}\,\vartheta_p^{-1/2}$$

Missing terms:
$\vartheta_q^2(1-u(t-t'))/2$
$+\vartheta_p\vartheta_q\,u(t-t')/2$

$$= \Big[\vartheta_p\vartheta_q/2 - \vartheta_p\vartheta_q\,u(t-t')/2 + \vartheta_p^2\,u(t-t')/2$$

$$+\vartheta_p\vartheta_q\,u(t-t') - \vartheta_p\vartheta_q + \vartheta_p\vartheta_q\,u(t-t')\Big]$$

$$\times(\vartheta_p+\vartheta_q)^{-2}\,\vartheta_p^{-1/2}$$

Missing terms:
$\vartheta_q^2/2 - \vartheta_q^2\,u(t-t')/2$
$+\vartheta_p\,u(t-t')\,\vartheta_q/2$

See p. 8'

$$= \frac{\vartheta_p\vartheta_q/2 + [\vartheta_p^2 - 3\vartheta_p\vartheta_q]\,u(t-t')/2}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}} \tag{24}$$

And finally, the gradient w.r.t. ϑ_q:

$$\nabla_{\vartheta_q} K^{(r)}_{y_p y_q}(t,t') = \boxed{\frac{\sigma_r^2}{\sqrt{8\pi}}}\left(-\frac{(t-t')^2\,\vartheta_p}{2}\right)\boxed{\exp\left(-\frac{\vartheta_p\vartheta_q}{2(\vartheta_p+\vartheta_q)}(t-t')^2\right)}$$

$$K^{(r)}_{y_p y_q}(t,t') = \quad \times \boxed{E^{(r)}_{pq}}(t,t')\;\nabla_{\vartheta_q}\frac{\vartheta_q}{\vartheta_p+\vartheta_q}$$

$$+\frac{\sigma_r^2}{\sqrt{8\pi}}\exp\left(-\frac{\vartheta_p\vartheta_q}{2(\vartheta_p+\vartheta_q)}(t-t')^2\right)\nabla_{\vartheta_q}E^{(r)}_{pq}(t,t') \tag{25}$$

Where

$$\boxed{\nabla_{\vartheta_q}\frac{\vartheta_q}{\vartheta_p+\vartheta_q} = \frac{\vartheta_p+\cancel{\vartheta_q}-\cancel{\vartheta_q}}{(\vartheta_p+\vartheta_q)^2} = \frac{\vartheta_p}{(\vartheta_p+\vartheta_q)^2}}, \tag{26}$$

and

$$\boxed{\nabla_{\vartheta_p}\left(\frac{\vartheta_v\sqrt{\vartheta_p}}{\vartheta_p+\vartheta_q}\right)} = \Big\{\Big[\vartheta_p^{1/2}u(t-t') + \big(\vartheta_q + (\vartheta_p-\vartheta_q)u(t-t')\big)\tfrac{1}{2}\vartheta_p^{-1/2}\Big](\vartheta_p+\vartheta_q)$$

$$- [\vartheta_q+(\vartheta_p-\vartheta_q)u(t-t')]\vartheta_p^{1/2}\Big\}\Big\{(\vartheta_p+\vartheta_q)^2\Big\}^{-1}$$

$$= \Big\{\vartheta_p^2 u(t-t') + \vartheta_p\vartheta_q u(t-t') + \frac{\vartheta_p\vartheta_q}{2} + \frac{\vartheta_q^2}{2}$$

$$+ \frac{\vartheta_p^2 u(t-t')}{2} - \frac{\vartheta_q^2 u(t-t')}{2} - \vartheta_p\vartheta_q - \vartheta_p^2 u(t-t')$$

$$+ \vartheta_p\vartheta_q u(t-t')\Big\}\Big\{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}\Big\}^{-1}$$

$$\boxed{= \frac{\vartheta_q(\vartheta_q-\vartheta_p)/2 + (\vartheta_p^2+4\vartheta_p\vartheta_q-\vartheta_q^2)u(t-t')/2}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}}}$$

Alternatively this could be written as

$$\boxed{\nabla_{\vartheta_p}\left(\frac{\vartheta_v\sqrt{\vartheta_p}}{\vartheta_p+\vartheta_q}\right)} = \Big\{\Big[\vartheta_p^{1/2}u(t-t') + \big(\vartheta_q(1-u(t-t')) + \vartheta_p u(t-t')\big)\tfrac{1}{2}\vartheta_p^{-1/2}\Big]$$

$$\times(\vartheta_p+\vartheta_q) - \big(\vartheta_q(1-u(t-t')) + \vartheta_p u(t-t')\big)\vartheta_p^{1/2}\Big\}$$

$$\times\Big\{(\vartheta_p+\vartheta_q)^2\Big\}^{-1}$$

$$= \Big\{\cancel{\vartheta_p^2 u(t-t')} + \vartheta_p\vartheta_q u(t-t') + \tfrac{1}{2}\vartheta_p\vartheta_q(1-u(t-t'))$$

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$$+ \tfrac{1}{2}\vartheta_p^2 u(t-t') \boxed{+ \tfrac{1}{2}\vartheta_q^2(1-u(t-t')) + \tfrac{1}{2}\vartheta_p\vartheta_q u(t-t')}$$

$$- \vartheta_p\vartheta_q(1-u(t-t')) - \cancel{\vartheta_p^2 u(t-t')}\Big\}\Big\{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}\Big\}^{-1}$$

$$= \Big\{\vartheta_p\vartheta_q u(t-t') + \tfrac{1}{2}\vartheta_p\vartheta_q - \cancel{\tfrac{1}{2}\vartheta_p\vartheta_q u(t-t')} + \tfrac{1}{2}\vartheta_p^2 u(t-t')$$

missing in p. 8

$$\boxed{+ \tfrac{1}{2}\vartheta_q^2 - \tfrac{1}{2}\vartheta_q^2 u(t-t') + \cancel{\tfrac{1}{2}\vartheta_p\vartheta_q u(t-t')}} - \vartheta_p\vartheta_q$$

$$+ \vartheta_p\vartheta_q u(t-t')\Big\}\Big\{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}\Big\}^{-1}$$

$$\boxed{= \Big\{\tfrac{1}{2}(\vartheta_q^2-\vartheta_p\vartheta_q) + \tfrac{1}{2}(4\vartheta_p\vartheta_q + \vartheta_p^2 - \vartheta_q^2)u(t-t')\Big\} \times\Big\{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p}\Big\}^{-1}}$$

$$\nabla_{\vartheta_q} E^{(r)}_{pq}(t,t') = \frac{2}{\sqrt{\pi}} \Bigg\{ \exp\left(-\frac{\vartheta_p \vartheta_q}{2}\left(\frac{\vartheta_p t + \vartheta_q t'}{\vartheta_p + \vartheta_q}\right)^2\right)$$
$$\times \sqrt{\frac{\vartheta_p}{2}} \times \nabla_{\vartheta_q} \frac{(\vartheta_p t + \vartheta_q t')\sqrt{\vartheta_q}}{\vartheta_p + \vartheta_q}$$
$$- \exp\left(-\frac{\vartheta_p \vartheta_q}{2}\left(\frac{\vartheta_v |t-t'|}{\vartheta_p + \vartheta_q}\right)^2\right)$$
$$\times \frac{|t-t'|\sqrt{\vartheta_p}}{\sqrt{2}} \nabla_{\vartheta_q} \frac{\vartheta_v \sqrt{\vartheta_q}}{\vartheta_p + \vartheta_q} \Bigg\}, \qquad (27)$$

with

$$\nabla_{\vartheta_q} \frac{(\vartheta_p t + \vartheta_q t')\sqrt{\vartheta_q}}{\vartheta_p + \vartheta_q} = \frac{\frac{1}{2}\vartheta_q^{-1/2}(\vartheta_p t + \vartheta_q t')(\vartheta_p + \vartheta_q) + \vartheta_q^{1/2} t' (\vartheta_p + \vartheta_q)}{(\vartheta_p + \vartheta_q)^2}$$
$$- \frac{\vartheta_q^{1/2}(\vartheta_p t + \vartheta_q t')}{(\vartheta_p + \vartheta_q)^2}$$
$$= \Big[\vartheta_p^2 t/2 + \vartheta_p \vartheta_q t/2 + \vartheta_p \vartheta_q t'/2 + \vartheta_q^2 t'/2$$
$$+ \vartheta_p \vartheta_q t' + \cancel{\vartheta_q^2 t'} - \vartheta_p \vartheta_q t - \cancel{\vartheta_q^2 t'}\Big]$$
$$\times (\vartheta_p + \vartheta_q)^{-2} (\vartheta_q)^{-1/2}$$
$$= \frac{\vartheta_p^2 t/2 + \vartheta_p \vartheta_q (3t' - t)/2 + \vartheta_q^2 t'/2}{(\vartheta_p + \vartheta_q)^2 \sqrt{\vartheta_q}} \qquad (28)$$

and

$$\boxed{\nabla_{\vartheta_q} \frac{\vartheta_v \sqrt{\vartheta_q}}{\vartheta_p+\vartheta_q}} = \frac{\frac{1}{2}\vartheta_q^{-1/2}\vartheta_v(\vartheta_p+\vartheta_q) + \vartheta_q^{1/2}(\vartheta_p+\vartheta_q)\nabla_{\vartheta_q}\vartheta_v}{(\vartheta_p+\vartheta_q)^2}$$

$$- \frac{\vartheta_v \vartheta_q^{1/2}}{(\vartheta_p+\vartheta_q)^2}$$

$$= \Big[\vartheta_p\vartheta_v/2 + \vartheta_q\vartheta_v/2 + (\vartheta_p\vartheta_q + \vartheta_q^2)(1-u(t-t'))$$

$$- \vartheta_q\vartheta_v\Big] / \Big[(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_q}\Big]$$

$$= \Big[\vartheta_p\vartheta_q\widehat{(1-u(t-t'))}/2 + \vartheta_p^2 u(t-t')/2$$

$$+ \vartheta_q^2\widehat{(1-u(t-t'))}/2 + \vartheta_p\vartheta_q u(t-t')/2$$

$$+ \vartheta_p\vartheta_q\widehat{(1-u(t-t'))} + \vartheta_q^2\widehat{(1-u(t-t'))}$$

$$- \vartheta_q^2\widehat{(1-u(t-t'))} - \vartheta_p\vartheta_q u(t-t')\Big]$$

$$\times (\vartheta_p+\vartheta_q)^{-2}\vartheta_q^{-1/2}$$

$$= \Big[\Big(\vartheta_p\vartheta_q/2 + \vartheta_q^2/2 + \vartheta_p\vartheta_q + \cancel{\vartheta_q^2} - \cancel{\vartheta_q^2}\Big)$$

$$+ \Big(\cancel{\vartheta_p^2/2} - \cancel{\vartheta_p\vartheta_q/2} - \cancel{\vartheta_q^2/2} + \cancel{\vartheta_p\vartheta_q/2}\ u(t-t')$$

$$- \vartheta_p\vartheta_q - \cancel{\vartheta_q^2} + \cancel{\vartheta_q^2} - \vartheta_p\vartheta_q\Big)\Big] / \Big[(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_q}\Big]$$

$$\boxed{= \frac{[3\vartheta_p\vartheta_q + \vartheta_q^2]/2 + (\vartheta_p^2/2 - 2\vartheta_p\vartheta_q - \vartheta_q^2/2)\,u(t-t')}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_q}}} \quad (29)$$

where we have used the fact that

$$\boxed{\nabla_{\vartheta_q}\vartheta_v = 1 - u(t-t')} \quad (30)$$

$$\boxed{\nabla_{\upsilon_\ell} K^{(r)}_{y_p y_q}(t,t')} = -\frac{1}{2}\cdot \boxed{\frac{\sigma_r^2}{\sqrt{8\pi}}}\,(t-t')^2 \boxed{\exp\left(-\frac{\upsilon_p \upsilon_q}{2(\upsilon_p+\upsilon_q)}(t-t')^2\right) \times E^{(r)}_{pq}(t,t')} \times \nabla_{\upsilon_\ell} \frac{\upsilon_p \upsilon_q}{\upsilon_p+\upsilon_q}$$

$K^{(r)}_{y_p y_q}(t,t')$

$$+ \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(-\frac{\upsilon_p \upsilon_q}{2(\upsilon_p+\upsilon_q)}(t-t')^2\right) \nabla_{\upsilon_\ell} E^{(r)}_{pq}(t,t')$$

$$= -\frac{1}{2}(t-t')^2 K^{(r)}_{y_p y_q}(t,t')\, \nabla_{\upsilon_\ell} \frac{\upsilon_p \upsilon_q}{\upsilon_p+\upsilon_q} + \frac{\sigma_r^2}{\sqrt{8\pi}} \exp\left(-\frac{\upsilon_p \upsilon_q}{2(\upsilon_p+\upsilon_q)}(t-t')^2\right) \nabla_{\upsilon_\ell} E^{(r)}_{pq}(t,t')$$

$$\nabla_{\upsilon_\ell} \frac{\upsilon_p \upsilon_q}{\upsilon_p+\upsilon_q} = \frac{(\upsilon_q \nabla_{\upsilon_\ell} \upsilon_p + \upsilon_p \nabla_{\upsilon_\ell} \upsilon_q)(\upsilon_p+\upsilon_q) - \upsilon_p \upsilon_q}{(\upsilon_p+\upsilon_q)^2} \Rightarrow$$

$$\boxed{\nabla_{\upsilon_p} \frac{\upsilon_p \upsilon_q}{\upsilon_p+\upsilon_q} = \frac{\upsilon_q(\cancel{\upsilon_p} + \upsilon_q) - \cancel{\upsilon_p}\upsilon_q}{(\upsilon_p+\upsilon_q)^2} = \frac{\upsilon_q^2}{(\upsilon_p+\upsilon_q)^2}}$$

$$\boxed{\nabla_{\upsilon_q} \frac{\upsilon_p \upsilon_q}{\upsilon_p+\upsilon_q} = \frac{\upsilon_p(\upsilon_p + \cancel{\upsilon_q}) - \upsilon_p\upsilon_q}{(\upsilon_p+\upsilon_q)^2} = \frac{\upsilon_p^2}{(\upsilon_p+\upsilon_q)^2}}$$

$$\nabla_{\upsilon_\ell} E^{(r)}_{pq}(t,t') = \frac{\frac{\sqrt{2}}{2}}{\sqrt{\pi}} \Bigg\{ \cancel{\frac{1}{\sqrt{2}}} \exp\left(-\frac{\upsilon_p \upsilon_q}{2}\left(\frac{\upsilon_p t + \upsilon_q t'}{\upsilon_p+\upsilon_q}\right)^2\right)$$

$\sqrt{\frac{2}{\pi}} =$

$$\times \nabla_{\upsilon_\ell}\left[\sqrt{\upsilon_p \upsilon_q} \cdot \frac{\upsilon_p t + \upsilon_q t'}{\upsilon_p+\upsilon_q}\right]$$

$$- \cancel{\frac{1}{\sqrt{2}}} \exp\left(-\frac{\upsilon_p \upsilon_q}{2}\left(\frac{\upsilon_v |t-t'|}{\upsilon_p+\upsilon_q}\right)^2\right)$$

$$\times \nabla_{\upsilon_\ell}\left[\sqrt{\upsilon_p \upsilon_q} \cdot \frac{\upsilon_v |t-t'|}{\upsilon_p+\upsilon_q}\right] \Bigg\}$$

$$\boxed{\nabla_{\vartheta_l} \frac{(\vartheta_p t + \vartheta_q t')\,\vartheta_p^{1/2}\vartheta_q^{1/2}}{\vartheta_p+\vartheta_q}} = \Big\{\Big[(t\,\nabla_{\vartheta_l}\vartheta_p + t'\,\nabla_{\vartheta_l}\vartheta_q)\,\vartheta_p^{1/2}\vartheta_q^{1/2}$$

$$+ (\vartheta_p t+\vartheta_q t')\,\vartheta_q^{1/2}\,\nabla_{\vartheta_l}\vartheta_p^{1/2}$$

$$+ (\vartheta_p t + \vartheta_q t')\,\vartheta_p^{1/2}\,\nabla_{\vartheta_l}\vartheta_q^{1/2}\Big]$$

$$\times(\vartheta_p+\vartheta_q)$$

$$- (\vartheta_p t+\vartheta_q t')\,\vartheta_p^{1/2}\vartheta_q^{1/2}\Big\}(\vartheta_p+\vartheta_q)^{-2}$$

$$\boxed{\begin{aligned} = \Big\{\Big[&(t\,\delta_{lp} + t'\,\delta_{lq})\,\vartheta_p\vartheta_q \\ &+ \tfrac{1}{2}(\vartheta_p t+\vartheta_q t')(\vartheta_q\,\delta_{lp} + \vartheta_p\,\delta_{lq})\Big] \\ &\times(\vartheta_p+\vartheta_q) - (\vartheta_p t+\vartheta_q t')\,\vartheta_p\vartheta_q\Big\} \\ &\times\Big[(\vartheta_p+\vartheta_q)^2\,\vartheta_p^{1/2}\vartheta_q^{1/2}\Big]^{-1}\end{aligned}}$$

Therefore,

$$\nabla_{\vartheta_p} \frac{(\vartheta_p t + \vartheta_q t')\,\vartheta_p^{1/2}\vartheta_q^{1/2}}{\vartheta_p+\vartheta_q} = \Big\{\Big[\vartheta_p\vartheta_q t + \tfrac{1}{2}\vartheta_q(\vartheta_p t+\vartheta_q t')\Big]$$

$$\times(\vartheta_p+\vartheta_q) - (\vartheta_p t+\vartheta_q t')\,\vartheta_p\vartheta_q\Big\}$$

$$\times\Big[(\vartheta_p+\vartheta_q)^2\,\vartheta_p^{1/2}\vartheta_q^{1/2}\Big]^{-1}$$

$$= \frac{\cancel{\vartheta_p^2\vartheta_q t} + \vartheta_p\vartheta_q^2 t + \tfrac{1}{2}\vartheta_p^2\vartheta_q t + \tfrac{1}{2}\vartheta_p\vartheta_q^2 t + \tfrac{1}{2}\vartheta_p\vartheta_q^2 t' + \tfrac{1}{2}\vartheta_q^3 t'}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}}$$

$$- \frac{\cancel{\vartheta_p^2\vartheta_q t} + \vartheta_p\vartheta_q^2 t'}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}} \quad \Rightarrow$$

$$\boxed{\nabla_{\vartheta_p} \frac{(\vartheta_p t + \vartheta_q t')\vartheta_p^{1/2}\vartheta_q^{1/2}}{\vartheta_p + \vartheta_q} = \frac{(3\vartheta_p\vartheta_q^2 + \vartheta_p^2\vartheta_q)t + (\vartheta_q^3 - \vartheta_p\vartheta_q^2)t'}{2(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}}}$$

$$\nabla_{\vartheta_q} \frac{(\vartheta_p t + \vartheta_q t')\vartheta_p^{1/2}\vartheta_q^{1/2}}{\vartheta_p + \vartheta_q}$$

$$= \left\{\left[\vartheta_p\vartheta_q t' + \frac{1}{2}(\vartheta_p t + \vartheta_q t')\vartheta_p\right](\vartheta_p + \vartheta_q) - (\vartheta_p t + \vartheta_q t')\vartheta_p\vartheta_q\right\} \times \left[(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}\right]^{-1}$$

$$= \frac{\vartheta_p^2\vartheta_q t' + \cancel{\vartheta_p\vartheta_q^2 t'} + \frac{1}{2}\vartheta_p^3 t + \frac{1}{2}\vartheta_p^2\vartheta_q t + \frac{1}{2}\vartheta_p^2\vartheta_q t' + \frac{1}{2}\vartheta_p\vartheta_q^2 t'}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}}$$

$$- \frac{\vartheta_p^2\vartheta_q t + \cancel{\vartheta_p\vartheta_q^2 t'}}{(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}} \Rightarrow$$

$$\boxed{\nabla_{\vartheta_q} \frac{(\vartheta_p t + \vartheta_q t')\sqrt{\vartheta_p\vartheta_q}}{\vartheta_p + \vartheta_q} = \frac{(\vartheta_p^3 - \vartheta_p^2\vartheta_q)t + (3\vartheta_p^2\vartheta_q + \vartheta_p\vartheta_q^2)t'}{2(\vartheta_p+\vartheta_q)^2\sqrt{\vartheta_p\vartheta_q}}}$$

And finally,

$$\nabla_{\vartheta_l} \frac{\vartheta_v\sqrt{\vartheta_p\vartheta_q}\,|t-t'|}{\vartheta_p+\vartheta_q} = \nabla_{\vartheta_l} \frac{[\vartheta_q + (\vartheta_p - \vartheta_q)u(t-t')]\sqrt{\vartheta_p\vartheta_q}\,|t-t'|}{\vartheta_p+\vartheta_q}$$

$$= \Big\{\Big(\left[\nabla_{\vartheta_l}\vartheta_q + (\nabla_{\vartheta_l}\vartheta_p - \nabla_{\vartheta_l}\vartheta_q)u(t-t')\right]\sqrt{\vartheta_p\vartheta_q}\,|t-t'|$$

$$+ [\vartheta_q + (\vartheta_p-\vartheta_q)u(t-t')]\,|t-t'|\Big(\vartheta_q^{1/2}\cdot\frac{1}{2}\vartheta_p^{-1/2}\nabla_{\vartheta_l}\vartheta_p$$

$$+ \vartheta_p^{1/2}\cdot\frac{1}{2}\vartheta_q^{-1/2}\nabla_{\vartheta_l}\vartheta_q\Big)\Big)(\vartheta_p+\vartheta_q)$$

$$- [\vartheta_q + (\vartheta_p-\vartheta_q)u(t-t')]\sqrt{\vartheta_p\vartheta_q}\,|t-t'|\Big\} \times (\vartheta_p+\vartheta_q)^{-2} \Rightarrow$$

$$
\nabla_{v_l} \frac{v_v \sqrt{v_p v_q}\,|t-t'|}{v_p+v_q} = \Big\{ \big[(1-u(t-t'))\,\delta_{lq} + u(t-t')\,\delta_{lp} \big] v_p v_q (v_p+v_q)\,|t-t'|
$$

$$
+\frac{1}{2}\big[v_q(1-u(t-t')) + v_p\, u(t-t') \big] (v_q \delta_{lp} + v_p \delta_{lq})
$$

$$
\times (v_p+v_q)\,|t-t'|
$$

$$
-\big[v_q(1-u(t-t')) + v_p\, u(t-t') \big] v_p v_q\, |t-t'| \Big\}
$$

$$
\times \Big\{ (v_p+v_q)^2 \sqrt{v_p v_q} \Big\}^{-1}
$$

Now, let's focus on the gradient w.r.t. v_p:

$$
\nabla_{v_p} \frac{v_v \sqrt{v_p v_q}\,|t-t'|}{v_p+v_q} = \Big\{ \Big[v_p v_q (\cancel{v_p}+v_q)\, u(t-t') + \frac{1}{2} v_q^2 (v_p+v_q)(1-u(t-t'))
$$

$$
+\frac{1}{2} v_p v_q (v_p+v_q)\, u(t-t')
$$

$$
- v_p v_q^2 (1-u(t-t')) - \cancel{v_p^2 v_q\, u(t-t')} \Big] |t-t'| \Big\}
$$

$$
\times \Big\{ (v_p+v_q)^2 \sqrt{v_p v_q} \Big\}^{-1}
$$

$$
= \Big\{ \Big[\frac{v_q^2}{2}(v_q - v_p) + \frac{v_q}{2}(4 v_p v_q + v_p^2 - v_q^2)\, u(t-t') \Big]
$$

$$
\times |t-t'| \Big\} \Big\{ (v_p+v_q)^2 \sqrt{v_p v_q} \Big\}^{-1}
$$

And w.r.t. v_q:

$$
\nabla_{v_q} \frac{v_v \sqrt{v_p v_q}\,|t-t'|}{v_p+v_q} = \Big\{ \Big[v_p v_q (v_p+v_q)(1-u(t-t')) + \frac{1}{2} v_p v_q (v_p+v_q)(1-u(t-t'))
$$

$$
+\frac{1}{2} v_p^2 (v_p+v_q)\, u(t-t') - v_p v_q^2 (1-u(t-t'))
$$

$$
- v_p^2 v_q\, u(t-t') \Big] |t-t'| \Big\} \Big\{ (v_p+v_q)^2 \sqrt{v_p v_q} \Big\}^{-1}
$$

$$
= \Big\{ \Big[\frac{v_p v_q}{2}(3 v_p + v_q) + \frac{v_p}{2}(v_p^2 - 4 v_p v_q - v_q^2)\, u(t-t') \Big]
$$

$$
\times |t-t'| \Big\} \Big\{ (v_p+v_q)^2 \sqrt{v_p v_q} \Big\}^{-1}
$$