LFOS : $\boxed{\text{SIM-OU KERNEL}}$

Once more, recall that the output of the system (assuming the right initial conditions) is

$$y_q(t) = \frac{B_q}{D_q} + \sum_{r=1}^{R} L_{qr}[f_r](t), \tag{1}$$

$$L_{qr}[f_r](t) = S_{qr} \exp(-D_q t) \int_0^t f_r(\tau) \exp(D_q \tau)\, d\tau \tag{2}$$

Hence, if $f_r(t)$ is an OU process,

$$E\{y_q(t)\} = \frac{B_q}{D_q} + \sum_{r=1}^{R} E\{L_{qr}[f_r](t)\} \tag{3}$$

$$
\begin{aligned}
E\{L_{qr}[f_r](t)\} &= S_{qr} \exp(-D_q t) \int_0^t \exp(D_q \tau)\, E\{f_r(\tau)\}\, d\tau \\
&= S_{qr} \exp(-D_q t) \\
&\quad \times \Big\{ \int_0^t f(0) \exp((D_q - \theta_r)\tau)\, d\tau \\
&\qquad + \int_0^t \mu_r \exp(D_q \tau)\left(1 - \exp(-\theta_r \tau)\right) d\tau \Big\} \\
&= S_{qr} \exp(-D_q t) \\
&\quad \times \Big\{ \int_0^t (f(0) - \mu_r) \exp((D_q - \theta_r)\tau)\, d\tau \\
&\qquad + \int_0^t \mu_r \exp(D_q \tau)\, d\tau \Big\} \\
&= S_{qr} \exp(-D_q t) \times \Big\{ \frac{f(0) - \mu_r}{D_q - \theta_r}\left[\exp((D_q - \theta_r)t) - 1\right] \\
&\qquad + \frac{\mu_r}{D_q}\left[\exp(D_q t) - 1\right] \Big\} \\
&= S_{qr} \exp(-D_q t) \left[(f(0) - \mu_r)\, I(D_q - \theta_r, 0, t) + \mu_r I(D_q, 0, t) \right]
\end{aligned} \tag{4}
$$

where we have defined

$$\boxed{I(\gamma, t_1, t_2) = \int_{t_1}^{t_2} \exp(\gamma t)\, dt = \frac{\exp(\gamma t_2) - \exp(\gamma t_1)}{\gamma}} \quad (5)$$

Putting (3) and (4) together:

$$\boxed{\begin{aligned} E\{y_q(t)\} &= \frac{B_q}{D_q} + \sum_{r=1}^{R} S_{qr} \exp(-D_q t) \\ &\times \left[(f(0) - \mu_r)\, I(D_q - \theta_r, 0, t) + \mu_r I(D_q, 0, t) \right] \end{aligned}} \quad (6)$$

Now, the cross-covariance between the q-th output and the r-th input is:

$$k_{y_q f_r}(t, t') = S_{qr} \exp(-D_q t) \int_0^t k_{f_r f_r}(\tau, t') \exp(D_q \tau)\, d\tau,$$

where $k_{f_r f_r}(\tau, t')$ is given by (19) in the OU process. Thus,

$$\begin{aligned} k_{y_q f_r}(t, t') &= S_{qr} \exp(-D_q t) \cdot \frac{\sigma_r^2}{2\theta_r} \\ &\times \left\{ \int_0^t \exp(D_q \tau) \exp(-\theta_r |\tau - t'|)\, d\tau \right. \\ &\left. - \int_0^t \exp(D_q \tau) \exp(-\theta_r (\tau + t'))\, d\tau \right\}. \end{aligned} \quad (7)$$

The second integral can be solved easily using (5), and for the first one we are going to define a new generic integral:

$$\boxed{I_2(\gamma, \theta, t_1, t_2) = \int_0^{t_1} \exp(\gamma \tau) \exp(-\theta |\tau - t_2|)\, d\tau,} \quad (8)$$

which can be solved noting that, when $t_1 \leq t_2 \Rightarrow 0 \leq \tau \leq t_1 \leq t_2 \Rightarrow |\tau - t_2| = t_2 - \tau \Rightarrow$ (8) becomes:

$$I_2(r,\theta,t_1,t_2) = \int_0^{t_1} \exp(r\tau)\exp(-\theta(t_2-\tau))\,d\tau$$

$$= \exp(-\theta t_2)\int_0^{t_1} \exp((r+\theta)\tau)\,d\tau$$

$$= \exp(-\theta t_2)\, I(r+\theta, 0, t_1). \qquad (9)$$

On the other hand, when $t_1 > t_2$ we have to distinguish between two intervals in the integral: $[0,t_2]$ and $[t_2,t_1]$. Now we have

$$I_2(r,\theta,t_1,t_2) = \int_0^{t_2} \exp(r\tau)\exp(-\theta|\tau-t_2|)\,d\tau$$

$$+\int_{t_2}^{t_1} \exp(r\tau)\exp(-\theta|\tau-t_2|)\,d\tau$$

$$\left\{\begin{array}{l} 0\leq\tau\leq t_2 \Rightarrow |\tau-t_2| = t_2-\tau \\ t_2<\tau\leq t_1 \Rightarrow |\tau-t_2| = \tau-t_2 \end{array}\right\}$$

$$= \int_0^{t_2} \exp(r\tau)\exp(-\theta(t_2-\tau))\,d\tau$$

$$+\int_{t_2}^{t_1} \exp(r\tau)\exp(-\theta(\tau-t_2))\,d\tau$$

$$= \exp(-\theta t_2)\int_0^{t_2} \exp((r+\theta)\tau)\,d\tau$$

$$+\exp(\theta t_2)\int_{t_2}^{t_1} \exp((r-\theta)\tau)\,d\tau$$

$$= \exp(-\theta t_2)\, I(r+\theta, 0, t_2) + \exp(\theta t_2)\, I(r-\theta, t_2, t_1). \qquad (10)$$

Observing (9) and (10), it is straightforward to see that (8) is given by

$$\boxed{\begin{aligned} I_2(r,\theta,t_1,t_2) &= \exp(-\theta t_2)\, I(r+\theta, 0, t_1\wedge t_2) \\ &+ \exp(\theta t_2)\, I(r-\theta, t_1\wedge t_2, t_1). \end{aligned}} \qquad (11)$$

Note that, when $t_1 \leq t_2 \Rightarrow t_1\wedge t_2 = t_1$, so that the second integral is zero, and (11) reduces to (9). Similarly, when $t_1 > t_2 \Rightarrow t_1 \wedge t_2 = t_2$, and we obtain (10).

Using (5) and (11), we can express (7) as

$$\boxed{\begin{aligned} K_{y_q f_r}(t,t') &= \frac{\sigma_r^2 S_{qr}}{2\theta_r} \exp(-D_q t) \Big\{ I_2(D_q, \theta_r, t, t') \\ &\quad - \exp(-\theta_r t') I(D_q - \theta_r, 0, t) \Big\} \\ &= \frac{\sigma_r^2 S_{qr}}{2\theta_r} \exp(-D_q t) \Big\{ \exp(-\theta_r t') I(D_q + \theta_r, 0, t \wedge t') \\ &\quad + \exp(\theta_r t') I(D_q - \theta_r, t \wedge t', t) - \exp(-\theta_r t') I(D_q - \theta_r, 0, t) \Big\} \end{aligned}} \tag{12}$$

As usual, this expression can be decomposed into a stationary and a non-stationary part,

$$\boxed{K_{y_q f_r}(t,t') = K^{st}_{y_q f_r}(t,t') + K^{non\text{-}st}_{y_q f_r}(t,t'),} \tag{13}$$

which can be obtained from (12) developing further that expression:

$$\begin{aligned} K_{y_q f_r}(t,t') = \frac{\sigma_r^2 S_{fr}}{2\theta_r} \Bigg\{ & \frac{\exp(-\xi_1 |t-t'|) - \exp(-(D_q t + \theta_r t'))}{D_q + \theta_r} \\ & + \frac{\exp(\theta_r (t'-t)) - \exp(-\xi_2 |t-t'|)}{D_q - \theta_r} \\ & - \frac{\exp(-\theta_r (t+t')) - \exp(-(D_q t + \theta_r t'))}{D_q - \theta_r} \Bigg\}, \end{aligned}$$

where

$$\boxed{\xi_1 = \theta_r \cdot \chi_{(-\infty,0]}(t-t') + D_q \cdot \chi_{(0,\infty)}(t-t') = \begin{cases} \theta_r, & t \le t'; \\ D_q, & t > t'; \end{cases}} \tag{14}$$

$$\boxed{\xi_2 = -\theta_r \cdot \chi_{(-\infty,0]}(t-t') + D_q \cdot \chi_{(0,\infty)}(t-t') = \begin{cases} -\theta_r, & t \le t'; \\ D_q, & t > t'. \end{cases}} \tag{15}$$

Remember that $\chi_E(x) = \begin{cases} 1, & x \in E; \\ 0, & x \notin E. \end{cases}$

Note also that, although $\xi_2 < 0$ for $t \leq t'$, this does not mean that the covariance function increases with $|t-t'|$ in this case, since then $\exp(-\xi_2 |t-t'|) = \exp(\theta_r (t'-t))$, which cancels the previous term in the fraction. Therefore, the stationary part is simply

$$\boxed{k^{st}_{y_q f_r}(t,t')} = \frac{\sigma_r^2 S_{qr}}{2\theta_r}\left[\frac{\exp(\theta_r (t'-t)) - \exp(-\xi_2 |t-t'|)}{D_q - \theta_r} + \frac{\exp(-\xi_1 |t-t'|)}{D_q + \theta_r}\right]$$

$$= \frac{\sigma_r^2 S_{qr}}{2\theta_r}\left[\frac{\exp(-\xi_1 |t-t'|)}{D_q + \theta_r} + \frac{\exp(-\theta_r (t-t')) - \exp(-D_q (t-t'))}{D_q - \theta_r}\,\chi_{(0,\infty)}(t-t')\right]$$

$$\boxed{\equiv \begin{cases} \dfrac{\exp(-\theta_r (t'-t))}{D_q + \theta_r}, & t \leq t'; \\[2ex] \dfrac{\exp(-D_q (t-t'))}{D_q + \theta_r} + \dfrac{\exp(-\theta_r (t-t')) - \exp(-D_q (t-t'))}{D_q - \theta_r}, & t > t'. \end{cases}} \qquad (16)$$

And the non-stationary part:

$$\boxed{k^{non\text{-}st}_{y_q f_r}(t,t') = -\frac{\sigma_r^2 S_{qr}}{2\theta_r}\left[\frac{\exp(-(D_q t + \theta_r t'))}{D_q + \theta_r} + \frac{\exp(-\theta_r (t+t')) - \exp(-(D_q t + \theta_r t'))}{D_q - \theta_r}\right]}, \qquad (17)$$

which again tends to zero as $t, t' \to \infty$.

Note also that, although not done previously, $E\{y_q(t)\}$ can also be decomposed into a stationary and a non-stationary part. Expanding (6), we get

$$\boxed{E\{y_q(t)\}} = \frac{B_q}{D_q} + \sum_{r=1}^{R} \left\{ \frac{S_{qr}(f(0)-\mu_r)}{D_q-\theta_r}\left[\exp(-\theta_r t) - \exp(-D_q t)\right] + \frac{S_{qr}\mu_r}{D_q}\left[1-\exp(-D_q t)\right] \right\}$$

$$\boxed{= \mu_{y_q}^{st}(t) + \mu_{y_q}^{non-st}(t),} \qquad (18)$$

where

$$\boxed{\mu_{y_q}^{st}(t) = \mu_{y_q}^{st} = \frac{B_q}{D_q} + \frac{1}{D_q}\sum_{r=1}^{R} S_{qr}\mu_r}, \qquad (19)$$

is just the bias term introduced by the system, B_q, plus a weighted average of the means of the R inputs, all divided by D_q. And the non-stationary part is:

$$\boxed{\mu_{y_q}^{non-st}(t) = \sum_{r=1}^{R} S_{qr}\left[\frac{f(0)-\mu_r}{D_q-\theta_r}\exp(-\theta_r t) - \left(\frac{f(0)-\mu_r}{D_q-\theta_r} + \frac{\mu_r}{D_q}\right)\exp(-D_q t)\right]}, \qquad (20)$$

which tends to zero once more as $t \to \infty$.

Finally, we will obtain the cross-covariance between two outputs:

$$K_{y_p y_q}(t,t') = \sum_{r=1}^{R} S_{pr} S_{qr} \exp\left(-(D_p t + D_q t')\right) \times \int_0^t \int_0^{t'} K_{f_r f_r}(\tau,\tau') \exp(D_p \tau) \exp(D_q \tau')\, d\tau\, d\tau'$$

$$= \sum_{r=1}^{R} S_{pr} S_{qr} \exp\left(-(D_p t + D_q t')\right) \cdot \frac{\sigma_r^2}{2\theta_r} \times \int_0^t \int_0^{t'} \exp(D_p \tau + D_q \tau')\left[\exp(-\theta_r|\tau-\tau'|) - \exp(-\theta_r(\tau+\tau'))\right] d\tau\, d\tau'$$

$$= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right) \times \int_0^t \exp(D_p \tau)\left[I_2(D_q, \theta_r, t', \tau) - \exp(-\theta_r \tau)\, I(D_q - \theta_r, 0, t')\right] d\tau$$

(21)

The second integral is immediate, since $I(Dq-\theta r, 0, t')$ does not depend on τ, so we only need to worry about

$$\int_0^t \exp(Dp\tau)\, I_2(Dq, \theta r, t', \tau)\, d\tau$$

$$= \int_0^t \exp((Dp-\theta r)\tau)\, I(Dq+\theta r, 0, t'\wedge\tau)\, d\tau$$

$$+ \int_0^t \exp((Dp+\theta r)\tau)\, I(Dq-\theta r, t'\wedge\tau, t')\, d\tau$$

$$= \int_0^t \exp((Dp-\theta r)\tau)\left[\exp((Dq+\theta r)(t'\wedge\tau) - 1\right]\cdot\frac{1}{Dq+\theta r}\, d\tau$$

$$+ \frac{1}{Dq-\theta r}\int_0^t \exp((Dp+\theta r)\tau)\left[\exp((Dq-\theta r)t') - \exp((Dq-\theta r)(t'\wedge\tau))\right] d\tau \qquad (22)$$

Again, instead of solving directly this integral, we are going to define a generic integral:

$$\boxed{I_3(r_1, r_2, \theta, t_1, t_2) = \int_0^{t_1} \exp(r_1\tau)\, I_2(r_2, \theta, t_2, \tau)\, d\tau}$$

$$= \int_0^{t_1} \exp((r_1-\theta)\tau)\, I(r_2+\theta, 0, t_2\wedge\tau)\, d\tau$$

$$+ \int_0^{t_1} \exp((r_1+\theta)\tau)\, I(r_2-\theta, t_2\wedge\tau, t_2)\, d\tau$$

$$= \frac{1}{r_2+\theta}\int_0^{t_1} \exp((r_1-\theta)\tau)\left[\exp((r_2+\theta)(t_2\wedge\tau)) - 1\right] d\tau$$

$$+ \frac{1}{r_2-\theta}\int_0^{t_1} \exp((r_1+\theta)\tau)\left[\exp((r_2-\theta)t_2) - \exp((r_2-\theta)(t_2\wedge\tau))\right] d\tau \qquad (23)$$

And, once more, we have to distinguish between two cases. First, when $t_1 \le t_2$, $0 \le \tau \le t_1 \le t_2$, and I_3 becomes:

$$I_3(r_1, r_2, \theta, t_1, t_2) = \frac{1}{r_2+\theta}\left\{\int_0^{t_1} \exp((r_2+\theta)\tau)\exp((r_1-\theta)\tau)\, d\tau - \int_0^{t_1} \exp((r_1-\theta)\tau)\, d\tau\right\}$$

$$+ \frac{1}{r_2-\theta}\left\{\exp((r_2-\theta)t_2)\int_0^{t_1} \exp((r_1+\theta)\tau)\, d\tau - \int_0^{t_1} \exp((r_1+\theta)\tau)\exp((r_2-\theta)\tau)\, d\tau\right\}$$

And, using (5.), I_3 finally becomes

$$\boxed{\begin{aligned} I_3(r_1,r_2,\theta,t_1,t_2) &= \frac{1}{r_2+\theta}\Big[I(r_1+r_2,0,t_1) - I(r_1-\theta,0,t_1)\Big] \\ &+ \frac{1}{r_2-\theta}\Big[\exp\big((r_2-\theta)t_2\big) I(r_1+\theta,0,t_1) - I(r_1+r_2,0,t_1)\Big] \\ &= \frac{\exp\big((r_2-\theta)t_2\big)}{r_2-\theta} I(r_1+\theta,0,t_1) - \frac{1}{r_2+\theta} I(r_1-\theta,0,t_1) \\ &- \frac{2\theta}{r_2^2-\theta^2} I(r_1+r_2,0,t_1). \end{aligned}} \qquad (24)$$

When $t_1 > t_2$, we have to divide the integrals in two:

$$\begin{aligned} I_3(r_1,r_2,\theta,t_1,t_2) = \frac{1}{r_2+\theta}\Big\{ &\int_0^{t_2} \exp((r_1-\theta)\tau)\exp((r_2+\theta)\tau)\,d\tau \\ &+ \int_{t_2}^{t_1} \exp((r_1-\theta)\tau)\exp((r_2+\theta)t_2)\,d\tau \\ &- \int_0^{t_1} \exp((r_1-\theta)\tau)\,d\tau \Big\} \\ + \frac{1}{r_2-\theta}\Big\{ &\int_0^{t_1} \exp((r_1+\theta)\tau)\exp((r_2-\theta)t_2)\,d\tau \\ &- \int_0^{t_2} \exp((r_1+\theta)\tau)\exp((r_2-\theta)\tau)\,d\tau \\ &- \int_{t_2}^{t_1} \exp((r_1+\theta)\tau)\exp((r_2-\theta)t_2)\,d\tau \Big\} \end{aligned}$$

$$\begin{aligned} = \frac{1}{r_2+\theta}\Big[& I(r_1+r_2,0,t_2) + \exp((r_2+\theta)t_2) I(r_1-\theta,t_2,t_1) \\ & - I(r_1-\theta,0,t_1)\Big] + \frac{1}{r_2-\theta}\Big[\exp((r_2-\theta)t_2) I(r_1+\theta,0,t_1) \\ & - I(r_1+r_2,0,t_2) - \exp((r_2-\theta)t_2) I(r_1+\theta,t_2,t_1)\Big]. \end{aligned} \qquad (25)$$

Which can be simplified, as before, yielding:

$$
\begin{aligned}
\mathbb{I}_3(r_1, r_2, \theta, t_1, t_2) = {} & \frac{\exp((r_2-\theta)t_2)}{r_2-\theta}\Big[\underbrace{I(r_1+\theta, 0, t_1) - I(r_1+\theta, t_2, t_1)}_{= I(r_1+\theta,0,t_2)}\Big] \\
& + \frac{\exp((r_2+\theta)t_2)}{r_2+\theta} I(r_1-\theta, t_2, t_1) \\
& - \frac{1}{r_2+\theta} I(r_1-\theta, 0, t_1) - \frac{2\theta}{r_2^2-\theta^2} I(r_1+r_2, 0, t_2)
\end{aligned}
\qquad (26)
$$

Putting together (24) and (26), we obtain

$$
\boxed{
\begin{aligned}
I_3(r_1, r_2, \theta, t_1, t_2) = {} & \frac{\exp((r_2-\theta)t_2)}{r_2-\theta}\Big[\underbrace{I(r_1+\theta, 0, t_1) - I(r_1+\theta, t_1\wedge t_2, t_1)}_{= I(r_1+\theta,0,t_1\wedge t_2)}\Big] \\
& + \frac{\exp((r_2+\theta)t_2)}{r_2+\theta} I(r_1-\theta, t_1\wedge t_2, t_1) \\
& - \frac{1}{r_2+\theta} I(r_1-\theta, 0, t_1) - \frac{2\theta}{r_2^2-\theta^2} I(r_1+r_2, 0, t_1\wedge t_2)
\end{aligned}
}
\qquad (27)
$$

So, using (27) and (5), (21) becomes:

$$
\boxed{
\begin{aligned}
K_{y_p y_q}(t,t') = {} & \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right) \\
& \times \Bigg\{ \frac{\exp((D_q-\theta_r)t')}{D_q-\theta_r}\Big[\underbrace{I(D_p+\theta_r, 0, t) - I(D_p+\theta_r, t\wedge t', t)}_{= I(D_p+\theta_r, 0, t\wedge t')}\Big] \\
& + \frac{\exp((D_q+\theta_r)t')}{D_q+\theta_r} I(D_p-\theta_r, t\wedge t', t) \\
& - \frac{1}{D_q+\theta_r} I(D_p-\theta_r, 0, t) - \frac{2\theta_r}{D_q^2-\theta_r^2} I(D_p+D_q, 0, t\wedge t') \\
& - \underbrace{\frac{1}{D_q-\theta_r}\Big[\exp((D_q-\theta_r)t') - 1\Big]}_{= I(D_q-\theta_r, 0, t')} I(D_p-\theta_r, 0, t)
\end{aligned}
}
\qquad (28)
$$

In order to check this result, we are now going to obtain this cross-covariance interchanging the order of the integrals:

$$\boxed{K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right)}$$

$$\times \int_0^{t'} \exp(D_q \tau') \left[\int_0^t \exp(D_p \tau) \left[\exp(-\theta_r |\tau-\tau'|) - \exp(-\theta_r(\tau+\tau'))\right] d\tau \right] d\tau'$$

$$= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right)$$

$$\times \int_0^{t'} \exp(D_q \tau') \left[I_2(D_p, \theta_r, t, \tau') - \exp(-\theta_r \tau') I(D_p - \theta_r, 0, t)\right] d\tau'$$

$$\boxed{= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right) \times \left[I_3(D_q, D_p, \theta_r, t', t) - I(D_p - \theta_r, 0, t)\, I(D_q - \theta_r, 0, t')\right]} \quad (29)$$

Note that (28) can be written in a similar way as

$$\boxed{K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right) \times \left[I_3(D_p, D_q, \theta_r, t, t') - I(D_q - \theta_r, 0, t')\, I(D_p - \theta_r, 0, t)\right]} \quad (30)$$

Obviously, the last part, $I(D_q - \theta_r, 0, t')\, I(D_p - \theta_r, 0, t)$ is identical in both cases, so we only have to prove that

$$I_3(D_p, D_q, \theta_r, t, t') = I_3(D_q, D_p, \theta_r, t', t).$$

And, in fact, we can use the generic expression of I_3 to prove

$$\boxed{I_3(\gamma_1, \gamma_2, \theta_r, t_1, t_2) = I_3(\gamma_2, \gamma_1, \theta, t_2, t_1),} \quad (31)$$

in one case (i.e. when $t_1 \le t_2$ or $t_1 > t_2$), since the other one comes straightforward from this. Noting that in any case, $t_1 \le t_2$ in the first expression means that $t_2 \geq t_1$ in the second (and viceversa), what we really need to prove is that

$$
\begin{aligned}
I_3(r_2, r_1, \theta, t_2, t_1) \underset{t_2 \geq t_1}{=} & \frac{1}{r_1+\theta}\Big[I(r_1+r_2, 0, t_1) + \exp((r_1+\theta)t_1) I(r_2-\theta, t_1, t_2) \\
& - I(r_2-\theta, 0, t_2)\Big] + \frac{1}{r_1-\theta}\Big[\exp((r_1-\theta)t_1) I(r_2+\theta, 0, t_2) \\
& - I(r_1+r_2, 0, t_1) - \exp((r_1-\theta)t_1) I(r_2+\theta, t_1, t_2)\Big] \\
= & \frac{1}{r_2+\theta}\Big[I(r_1+r_2, 0, t_1) - I(r_1-\theta, 0, t_1)\Big] \\
& + \frac{1}{r_2-\theta}\Big[\exp((r_2-\theta)t_2) I(r_1+\theta, 0, t_1) - I(r_1+r_2, 0, t_1)\Big] \\
\underset{t_1 \leq t_2}{=} & I_3(r_1, r_2, \theta, t_1, t_2)
\end{aligned}
$$

Let us concentrate on the first part of $I_3(r_2, r_1, \theta, t_2, t_1)$ for $t_2 \geq t_1$:

$$
\begin{aligned}
& \frac{1}{r_1+\theta}\Big[I(r_1+r_2, 0, t_1) + \exp((r_1+\theta)t_1) I(r_2-\theta, t_1, t_2) - I(r_2-\theta, 0, t_2)\Big] \\
& = \frac{1}{r_1+\theta}\Big[I(r_1+r_2, 0, t_1) + \Big\{\exp((r_1+\theta)t_1 + (r_2-\theta)t_2) \\
& \quad - \exp((r_1+\cancel{\theta})t_1 + (r_2-\cancel{\theta})t_1) - \exp((r_2-\theta)t_2) + 1\Big\}(r_2-\theta)^{-1}\Big] \\
& = \frac{1}{(r_1+\theta)(r_2-\theta)}\Big[(r_2-\theta) I(r_1+r_2, 0, t_1) - \Big\{\exp((r_1+r_2)t_1) - 1\Big\} \\
& \quad + \exp((r_2-\theta)t_2)\big[\exp((r_1+\theta)t_1) - 1\big]\Big] \\
& = \frac{1}{(r_1+\theta)(r_2-\theta)}\Big[(\cancel{r_2}-\theta) I(r_1+r_2, 0, t_1) - (r_1+\cancel{r_2}) I(r_1+r_2, 0, t_1) \\
& \quad + (r_1+\theta)\exp((r_2-\theta)t_2) I(r_1+\theta, 0, t_1)\Big]
\end{aligned}
$$

$$= -\frac{\cancel{r_1+\theta}}{\cancel{(r_1+\theta)}(r_2-\theta)} I(r_1+r_2,0,t_1) + \frac{\cancel{r_1+\theta}}{\cancel{(r_1+\theta)}(r_2-\theta)} \exp((r_2-\theta)t_2) I(r_1+\theta,0,t_1)$$

$$= \left[\exp((r_2-\theta)t_2) I(r_1+\theta,0,t_1) - I(r_1+r_2,0,t_1)\right] \cdot (r_2-\theta)^{-1},$$

which is exactly the second term in $I_3(r_1,r_2,\theta,t_1,t_2)$ when $t_1 \le t_2$. Now, let us take a look at the second part:

$$\frac{1}{r_1-\theta}\left\{\exp((r_1-\theta)t_1)\left[I(r_2+\theta,0,t_2) - I(r_2+\theta,0,t_1)\right] - I(r_1+r_2,0,t_1)\right\}$$

$$= \frac{1}{r_1-\theta}\left\{\exp((r_1-\theta)t_1) I(r_2+\theta,0,t_1) - I(r_1+r_2,0,t_1)\right\}$$

$$= \frac{1}{(r_1-\theta)(r_2+\theta)}\Big\{\exp((r_1-\cancel{\theta})t_1 + (r_2+\cancel{\theta})t_1) - \exp((r_1-\theta)t_1)$$
$$- (r_2+\theta) I(r_1+r_2,0,t_1)\Big\}$$

$$= \frac{1}{(r_1-\theta)(r_2+\theta)}\Big\{\left[\exp((r_1+r_2)t_1) - 1\right] - \left[\exp((r_1-\theta)t_1) - 1\right]$$
$$- (r_2+\theta) I(r_1+r_2,0,t_1)\Big\}$$

$$= \frac{1}{(r_1-\theta)(r_2+\theta)}\Big\{(r_1+\cancel{r_2}) I(r_1+r_2,0,t_1) - (r_1-\theta) I(r_1-\theta,0,t_1)$$
$$- (\cancel{r_2}+\theta) I(r_1+r_2,0,t_1)\Big\}$$

$$= \frac{1}{(r_1-\theta)(r_2+\theta)}\Big\{(r_1-\theta) I(r_1+r_2,0,t_1) - (r_1-\theta) I(r_1-\theta,0,t_1)\Big\}$$

$$= \frac{1}{r_2+\theta}\left[I(r_1+r_2,0,t_1) - I(r_1-\theta,0,t_1)\right],$$

which is exactly the first term in $I_3(r_1,r_2,\theta,t_1,t_2)$ for $t_1 \le t_2$.

Now, as usual, we are going to obtain the stationary and non-stationary parts. The cross-covariance can be compactly expressed as

$$K_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r} \exp\left(-(D_p t + D_q t')\right)$$

$$\times \Big\{ \frac{1}{D_q+\theta_r} \Big[I(D_p+D_q, 0, t\wedge t') - I(D_p-\theta_r, 0, t)$$

$$+ \exp((D_q+\theta_r)t')\, I(D_p-\theta_r, t\wedge t', t)\Big]$$

$$+ \frac{1}{D_q-\theta_r}\Big[\exp((D_q-\theta_r)t')\, I(D_p+\theta_r, 0, t\wedge t')$$

$$- I(D_p+D_q, 0, t\wedge t')\Big] - I(D_p-\theta_r, 0, t)\, I(D_q-\theta_r, 0, t')\Big\}$$

$$= \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r}$$

$$\times \Big\{ \frac{1}{D_q+\theta_r}\Big[\frac{\exp(-D_\upsilon|t-t'|) - \exp(-(D_p t + D_q t'))}{D_p+D_q}$$

$$- \frac{\exp(-(D_q t' + \theta_r t)) - \exp(-(D_p t + D_q t'))}{D_p-\theta_r}$$

$$+ \frac{\exp(-\theta_r(t-t')) - \exp(-\xi_2|t-t'|)}{D_p-\theta_r}\Big]$$

$$+ \frac{1}{D_q-\theta_r}\Big[\frac{\exp(-\xi_1|t-t'|) - \exp(-(D_p t + D_q t'))}{D_p+\theta_r}$$

$$- \frac{\exp(-D_\upsilon|t-t'|) - \exp(-(D_p t + D_q t'))}{D_p+D_q}\Big]$$

$$- \frac{\exp(-\theta_r t) - \exp(-D_p t)}{D_p-\theta_r} \cdot \frac{\exp(-\theta_r t') - \exp(-D_q t')}{D_q-\theta_r}\Big\},$$

where $D_v = D_q \cdot \chi_{(-\infty,0)} + D_p \cdot \chi_{[0,\infty)}$, as for the white kernel, and ξ_1 and ξ_2 are given by (14) and (15) respectively. Now, the stationary part is:

$$K^{st}_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r}$$
$$\times \left\{ \frac{1}{D_q+\theta_r} \left[\frac{\exp(-D_v|t-t'|)}{D_p+D_q} + \frac{\exp(-\theta_r(t-t')) - \exp(-\xi_2|t-t'|)}{D_p-\theta_r} \right] \right.$$
$$\left. + \frac{1}{D_q-\theta_r} \left[\frac{\exp(-\xi_1|t-t'|)}{D_p+\theta_r} - \frac{\exp(-D_v|t-t'|)}{D_p+D_q} \right] \right\}, \tag{32}$$

and the non-stationary part:

$$K^{non-st}_{y_p y_q}(t,t') = \sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{2\theta_r}$$
$$\times \left\{ \frac{\exp(-(D_p t + D_q t'))}{D_q-\theta_r} \left[\frac{1}{D_p+D_q} - \frac{1}{D_p+\theta_r} \right] \right.$$
$$- \frac{1}{D_q+\theta_r} \left[\exp(-(D_p t + D_q t')) \left\{ \frac{1}{D_p+D_q} + \frac{1}{D_p-\theta_r} \right\} \right.$$
$$\left. + \frac{1}{D_p-\theta_r} \exp(-(D_q t' + \theta_r t)) \right]$$
$$- \frac{\exp(-\theta_r t) - \exp(-D_p t)}{D_p-\theta_r} \cdot \frac{\exp(-\theta_r t') - \exp(-D_q t')}{D_q-\theta_r} \tag{33}$$

LSOS:

LFM–OU KERNEL

The output of the system (given the proper initial conditions) is

$$\boxed{y_q(t) = \frac{B_q}{D_q} + \sum_{r=1}^{R} L_{qr}[f_r](t)} \qquad (34)$$

$$\boxed{L_{qr}[f_r](t)} = \frac{S_{qr}}{\omega_q'} \exp(-\alpha_q t) \int_0^t f_r(\tau) \exp(\alpha_q \tau) \sin(\omega_q'(t-\tau)) d\tau$$

$$= \frac{S_{qr}}{j2\omega_q'} \Big\{ \exp(-\tilde{r}_q' t) \int_0^t \exp(\tilde{r}_q' \tau) f_r(\tau) d\tau - \exp(-r_q' t) \int_0^t f_r(\tau) \exp(r_q' \tau) d\tau \Big\} \qquad (35)$$

where $r_q' = \alpha_q + j\omega_q'$ and $\tilde{r}_q' = \alpha_q - j\omega_q'$. Comparing (35) to (2), we see that

$$\boxed{L_{qr}^{LSOS}[f_r](t) = \frac{1}{j2\omega_q'} \left[L_{qr}^{LFOS}[f_r, \tilde{r}_q'](t) - L_{qr}^{LFOS}[f_r, r_q'](t) \right].} \qquad (36)$$

Therefore, we can speed up all the calculations by using the results of the LFOS:

$$E\{y_q(t)\} = \frac{B_q}{D_q} + \sum_{r=1}^{R} E\{ L_{qr}^{LSOS}[f_r](t) \} \qquad (37)$$

$$E\{ L_{qr}^{LSOS}[f_r](t) \} = \frac{1}{j2\omega_q'} \left[E\{ L_{qr}^{LFOS}[f_r, \tilde{r}_q'](t) \} - E\{ L_{qr}^{LFOS}[f_r, r_q'](t) \} \right]$$

$$= \frac{S_{qr}}{j2\omega_q'} \Big[\exp(-\tilde{r}_q' t) \big((f(0) - \mu_r) I(\tilde{r}_q' - \theta_r, 0, t) + \mu_r I(\tilde{r}_q', 0, t) \big) - \exp(-r_q' t) \big((f(0) - \mu_r) I(r_q' - \theta_r, 0, t) + \mu_r I(r_q', 0, t) \big) \Big] \qquad (38)$$

Putting together (37) and (38):

$$\boxed{E\{y_q(t)\} = \frac{B_q}{D_q} + \sum_{r=1}^{R} \frac{S_{qr}}{j2\omega_q'} \Big[\exp(-\tilde{r}_q' t) \big[(f(0) - \mu_r) I(\tilde{r}_q' - \theta_r, 0, t) + \mu_r I(\tilde{r}_q', 0, t) \big] - \exp(-r_q' t) \big[(f(0) - \mu_r) I(r_q' - \theta_r, 0, t) + \mu_r I(r_q', 0, t) \big] \Big]} \qquad (39)$$

The cross-covariance between $y_q(t)$ and $f_r(t)$ is:

$$\boxed{\begin{aligned}
K_{y_q f_r}(t,t') &= \frac{S_{qr}}{j2\omega_q'}\Big\{\exp(-\tilde{\gamma}_q' t)\int_0^t K_{f_r f_r}(\tau,t')\exp(\tilde{\gamma}_q'\tau)\,d\tau \\
&\quad - \exp(-\gamma_q' t)\int_0^t K_{f_r f_r}(\tau,t')\exp(\gamma_q'\tau)\,d\tau\Big\} \\
&= \frac{\sigma_r^2 S_{qr}}{j4\omega_q'\theta_r}\Big\{\exp(-\tilde{\gamma}_q' t)\Big[I_2(\tilde{\gamma}_q',\theta_r,t,t') - \exp(-\theta_r t') \\
&\quad \times I(\tilde{\gamma}_q'-\theta_r,0,t)\Big] - \exp(-\gamma_q' t)\Big[I_2(\gamma_q',\theta_r,t,t') \\
&\quad - \exp(-\theta_r t')\, I(\gamma_q'-\theta_r,0,t)\Big]\Big\} \\
&= \frac{\sigma_r^2 S_{qr}}{j4\omega_q'\theta_r}\Big\{\exp(-\tilde{\gamma}_q' t)\Big[\exp(-\theta_r t')\, I(\tilde{\gamma}_q'+\theta_r,0,t\wedge t') \\
&\quad + \exp(\theta_r t')\, I(\tilde{\gamma}_q'-\theta_r,t\wedge t',t) - \exp(-\theta_r t')\, I(\tilde{\gamma}_q'-\theta_r,0,t)\Big] \\
&\quad - \exp(-\gamma_q' t)\Big[\exp(-\theta_r t')\, I(\gamma_q'+\theta_r,0,t\wedge t') \\
&\quad + \exp(\theta_r t')\, I(\gamma_q'-\theta_r,t\wedge t',t) - \exp(-\theta_r t')\, I(\gamma_q'-\theta_r,0,t)\Big]\Big\}
\end{aligned}} \quad (40)$$

And its stationary and non-stationary parts can be obtained also from those of the LFOS:

$$\boxed{\begin{aligned}
K^{st}_{y_q f_r}(t,t') &= \frac{\sigma_r^2 S_{qr}}{j4\omega_q'\theta_r}\Big\{\frac{\exp(-\tilde{\xi}_1'|t-t'|)}{\tilde{\gamma}_q'+\theta_r} - \frac{\exp(-\xi_1'|t-t'|)}{\gamma_q'+\theta_r} \\
&\quad + \frac{\exp(-\theta_r(t-t')) - \exp(-\tilde{\gamma}_q'(t-t'))}{\tilde{\gamma}_q'-\theta_r}\,\chi_{(0,\infty)}(t-t') \\
&\quad - \frac{\exp(-\theta_r(t-t')) - \exp(-\gamma_q'(t-t'))}{\gamma_q'-\theta_r}\,\chi_{(0,\infty)}(t-t')\Big\},
\end{aligned}} \quad (41)$$

where

$$\boxed{\xi_1' = \theta_r\cdot\chi_{(-\infty,0]}(t-t') + \gamma_q'\cdot\chi_{(0,\infty)}(t-t')} \quad (42)$$

$$\boxed{\tilde{\xi}_1' = \theta_r\cdot\chi_{(-\infty,0]}(t-t') + \tilde{\gamma}_q'\cdot\chi_{(0,\infty)}(t-t')} \quad (43)$$

and

$$\boxed{k^{\text{non-st}}_{y_q f_r}(t,t') = -\frac{\sigma_r^2 S_{qr}}{j4\theta_r \omega_q'}\left\{\frac{\exp(-(\tilde{\gamma}_q' t + \theta_r t'))}{\tilde{\gamma}_q' + \theta_r} - \frac{\exp(-(\gamma_q' t + \theta_r t'))}{\gamma_q' + \theta_r} + \frac{\exp(-\theta_r(t+t')) - \exp(-(\tilde{\gamma}_q' t + \theta_r t'))}{\tilde{\gamma}_q' - \theta_r} - \frac{\exp(-\theta_r(t+t')) - \exp(-(\gamma_q' t + \theta_r t'))}{\gamma_q' - \theta_r}\right\}.} \quad (44)$$

Regarding the cross-covariance between two outputs, this becomes

$$\boxed{\begin{aligned} k_{y_p y_q}(t,t') = -\sum_{r=1}^{R} \frac{\sigma_r^2 S_{pr} S_{qr}}{\theta_r \omega_p' \omega_q'} & \times\Big\{\exp(-(\tilde{\gamma}_p' t + \tilde{\gamma}_q' t'))\left[I_3(\tilde{\gamma}_q', \tilde{\gamma}_p', \theta_r, t', t) - I(\tilde{\gamma}_p' - \theta_r, 0, t)\, I(\tilde{\gamma}_q' - \theta_r, 0, t')\right] \\ & - \exp(-(\gamma_p' t + \tilde{\gamma}_q' t'))\left[I_3(\tilde{\gamma}_q', \gamma_p', \theta_r, t', t) - I(\gamma_p' - \theta_r, 0, t)\, I(\tilde{\gamma}_q' - \theta_r, 0, t')\right] \\ & - \exp(-(\tilde{\gamma}_p' t + \gamma_q' t'))\left[I_3(\gamma_q', \tilde{\gamma}_p', \theta_r, t', t) - I(\tilde{\gamma}_p' - \theta_r, 0, t)\, I(\gamma_q' - \theta_r, 0, t')\right] \\ & + \exp(-(\gamma_p' t + \gamma_q' t'))\left[I_3(\gamma_q', \gamma_p', \theta_r, t', t) - I(\gamma_p' - \theta_r, 0, t)\, I(\gamma_q' - \theta_r, 0, t')\right]\Big\} \end{aligned}} \quad (45)$$

And as usual we can obtain the stationary and non-stationary parts from (32) and (33) respectively.