RBF-White Kernel

This kernel is the result of having a white noise latent process, with an auto-covariance function defined by

$$\boxed{K_{f_r f_r}(t,t') = \sigma_r^2\, \delta(t-t'),} \quad (1)$$

convolved with a Gaussian smoothing kernel,

$$\boxed{h_q(t,t') = h_q(t-t') = \frac{1}{\sqrt{2\pi l_q^2}} \exp\left(-\frac{(t-t')^2}{2l_q^2}\right)} \quad (2)$$

The resulting output is

$$\boxed{y_q(t) = \int_0^t f_r(\tau)\, h_q(t-\tau)\, d\tau,} \quad (3)$$

and this GP is perfectly defined by its mean function,

$$\boxed{E\{y_q(t)\} = \int_0^t E\{f_r(\tau)\}\, h_q(t-\tau)\, d\tau = 0}, \quad (4)$$

and the cross-covariance between $y_p(t)$ and $y_q(t')$,

$$E\{y_p(t)\,y_q(t')\} = \int_0^t \int_0^{t'} E\{f_r(\tau)\, f_r(\tau')\}\, h_p(t-\tau)\, h_q(t'-\tau')\, d\tau\, d\tau'$$

$$= \sigma_r^2 \int_0^t h_p(t-\tau)\, h_q(t'-\tau)\, d\tau$$

$$= \frac{\sigma_r^2}{2\pi l_p l_q} \int_0^t \exp\left(-\frac{(t-\tau)^2}{2l_p^2} - \frac{(t'-\tau)^2}{2l_q^2}\right) d\tau$$

$$= \frac{\sigma_r^2}{2\pi l_p l_q} \int_0^t \exp\left(-\frac{(\tau-\mu_\tau)^2}{2\sigma_\tau^2}\right) \exp\left(-\frac{\theta_{pq}(t,t')}{2\sigma_\tau^2}\right) d\tau$$

$$= \frac{\sigma_r^2}{2\pi l_p l_q} \exp\left(-\frac{\theta_{pq}(t,t')}{2\sigma_\tau^2}\right) \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{-\mu_\tau/\sqrt{2\sigma_\tau^2}}^{(t-\mu_\tau)/\sqrt{2\sigma_\tau^2}} \exp(-x^2)\, dx$$

$x\sqrt{2\sigma_\tau^2}$

Inside the integral we recognize the familier expression of the error function:

$$\frac{2}{\sqrt{\pi}}\int_{-\mu_\tau/\sqrt{2\sigma_\tau^2}}^{(t-\mu_\tau)/\sqrt{2\sigma_\tau^2}} \exp(-x^2)\,dx = \frac{2}{\sqrt{\pi}}\int_{-\mu_\tau/\sqrt{2\sigma_\tau^2}}^{0} \exp(-x^2)\,dx$$

$$+\frac{2}{\sqrt{\pi}}\int_{0}^{(t-\mu_\tau)/\sqrt{2\sigma_\tau^2}} \exp(-x^2)\,dx$$

$$= \operatorname{erf}\left(\frac{t-\mu_\tau}{\sqrt{2\sigma_\tau^2}}\right) + \operatorname{erf}\left(\frac{\mu_\tau}{\sqrt{2\sigma_\tau^2}}\right)$$

Using this expression we finally have

$$\boxed{K_{y_p y_q}(t,t') = \frac{\sigma_r^2}{2\pi l_p l_q}\cdot\sqrt{\frac{\pi\sigma_\tau^2}{2}}\exp\left(-\frac{\theta_{pq}(t,t')}{2\sigma_\tau^2}\right) \times\left\{\operatorname{erf}\left(\frac{t-\mu_\tau}{\sqrt{2\sigma_\tau^2}}\right)+\operatorname{erf}\left(\frac{\mu_\tau}{\sqrt{2\sigma_\tau^2}}\right)\right\}} \quad (5)$$

Now we only have to find μ_τ and σ_τ^2. We concentrate on the exponent of the original integral,

$$\frac{(t-\tau)^2}{2l_p^2}+\frac{(t'-\tau)^2}{2l_q^2} = \frac{l_q^2\left[t^2-2t\tau+\tau^2\right]+l_p^2\left[(t')^2-2t'\tau+\tau^2\right]}{2l_p^2 l_q^2}$$

$$= \frac{(l_p^2+l_q^2)\tau^2-2\left[l_q^2 t+l_p^2 t'\right]\tau+\left(l_q^2t^2+l_p^2(t')^2\right)}{2l_p^2l_q^2}$$

$$= \frac{\tau^2-2\left(l_q^2t+l_p^2t'\right)/(l_p^2+l_q^2)\,\tau+\left(l_q^2t^2+l_p^2(t')^2\right)/(l_p^2+l_q^2)}{2\,l_p^2l_q^2/(l_p^2+l_q^2)}$$

$$= \frac{\left[\tau-\left(l_q^2t+l_p^2t'\right)/(l_p^2+l_q^2)\right]^2-\left(l_q^2t+l_p^2t'\right)^2/(l_p^2+l_q^2)^2}{2l_p^2l_q^2/(l_p^2+l_q^2)}$$

$$+\frac{\left(l_q^2t^2+l_p^2(t')^2\right)/(l_p^2+l_q^2)}{2l_p^2l_q^2/(l_p^2+l_q^2)}$$

Identifying terms,

$$\boxed{\mu_\tau = \frac{l_q^2 t + l_p^2 t'}{l_p^2 + l_q^2}} \qquad (6)$$

$$\boxed{\sigma_\tau^2 = \frac{l_p^2 l_q^2}{l_p^2 + l_q^2}} = \left(\frac{1}{l_p^2} + \frac{1}{l_q^2} \right)^{-1} \qquad (7)$$

$$\boxed{\theta_{pq}(t,t')} = \frac{\left(l_q^2 t^2 + l_p^2 (t')^2\right)\left(l_p^2 + l_q^2\right) - \left(l_q^2 t + l_p^2 t'\right)^2}{\left(l_p^2 + l_q^2\right)^2}$$

$$= \frac{l_p^2 l_q^2 t^2 + \cancel{l_q^4 t^2} + \cancel{l_p^4 (t')^2} + l_p^2 l_q^2 (t')^2 - \cancel{l_q^4 t^2} - 2 l_p^2 l_q^2 t t' - \cancel{l_p^4 (t')^2}}{\left(l_p^2 + l_q^2\right)^2}$$

$$\boxed{= \frac{l_p^2 l_q^2 (t-t')^2}{\left(l_p^2 + l_q^2\right)^2} = \frac{\sigma_\tau^2 (t-t')^2}{l_p^2 + l_q^2}} \qquad (8)$$

From these three equations we can obtain all the expressions that appear in the new kernel:

$$\frac{\theta_{pq}(t,t')}{2\sigma_\tau^2} = \frac{(t-t')^2}{2\left(l_p^2 + l_q^2\right)}$$

$$\frac{\mu_\tau}{\sqrt{2\sigma_\tau^2}} = \frac{l_q^2 t + l_p^2 t'}{l_p^2 + l_q^2} \cdot \frac{\sqrt{l_p^2 + l_q^2}}{\sqrt{2}\, l_p l_q} = \frac{l_q^2 t + l_p^2 t'}{l_p l_q \sqrt{2\left(l_p^2 + l_q^2\right)}}$$

$$\frac{t - \mu_\tau}{\sqrt{2\sigma_\tau^2}} = \frac{\left(l_p^2 + \cancel{l_q^2}\right)t - \cancel{l_q^2 t} - l_p^2 t'}{l_p^2 + l_q^2} \cdot \frac{\sqrt{l_p^2 + l_q^2}}{\sqrt{2}\, l_p l_q} = \frac{l_p (t-t')}{l_q \sqrt{2\left(l_p^2 + l_q^2\right)}}$$

$$\frac{\sigma_r^2}{2\pi l_p l_q} \cdot \sqrt{\frac{\pi \sigma_\tau^2}{2}} = \frac{\sigma_r^2 \sqrt{\pi}\, \cancel{l_p l_q}}{2\pi\, \cancel{l_p^2}\, \cancel{l_q^2} \sqrt{2\left(l_p^2 + l_q^2\right)}} = \frac{\sigma_r^2}{\sqrt{8\pi}\left(l_p^2 + l_q^2\right)}$$

And inserting these expressions into (5) we obtain the final functional form of the kernel:

$$\boxed{K_{y_p y_q}(t,t') = \frac{\sigma_r^2}{\sqrt{8\pi(\ell_p^2+\ell_q^2)}} \cdot \exp\left(-\frac{(t-t')^2}{2(\ell_p^2+\ell_q^2)}\right) \times \left\{ \operatorname{erf}\left(\frac{\ell_p(t-t')}{\ell_q\sqrt{2(\ell_p^2+\ell_q^2)}}\right) + \operatorname{erf}\left(\frac{\ell_q^2 t + \ell_p^2 t'}{\ell_p \ell_q \sqrt{2(\ell_p^2+\ell_q^2)}}\right) \right\}} \quad (9)$$

We will also need the cross-covarience between $y_p(t)$ and $f_r(t')$, which is given by

$$\boxed{K_{y_p f_r}(t,t')} = \int_0^t E\{f_r(\tau) f_r(t')\} h_p(t-\tau)\, d\tau$$

$$= h_p(t-t')\, u(t-t')\, \sigma_r^2$$

$$\boxed{= \frac{\sigma_r^2}{\sqrt{2\pi\ell_p^2}} \exp\left(-\frac{(t-t')^2}{2\ell_p^2}\right) u(t-t')} \quad (10)$$

Finally, to complete the characterization of the kernel we require the gradients of $K_{y_p f_r}(t,t')$ w.r.t. σ_r^2 and ℓ_p^2, and the gradients of $K_{y_p y_q}(t,t')$ w.r.t. σ_r^2, ℓ_p^2 and ℓ_q^2:

$$\boxed{\nabla_{\sigma_r^2} K_{y_p f_r}(t,t') = \frac{K_{y_p f_r}(t,t')}{\sigma_r^2}} \quad (11)$$

$$\boxed{\nabla_{\ell_p^2} K_{y_p f_r}(t,t')} = -\frac{1}{2\ell_p^2} \cdot \frac{\sigma_r^2}{\sqrt{2\pi\ell_p^2}} \exp\left(-\frac{(t-t')^2}{2\ell_p^2}\right) u(t-t')$$

$$+ \frac{(t-t')^2}{2\ell_p^4} \cdot \frac{\sigma_r^2}{\sqrt{2\pi\ell_p^2}} \exp\left(-\frac{(t-t')^2}{2\ell_p^2}\right) u(t-t')$$

$$= \frac{1}{2\ell_p^2}\left(\frac{(t-t')^2}{\ell_p^2} - 1\right) K_{y_p f_r}(t,t')$$

$$\boxed{\nabla_{\ell_p} K_{y_p f_r}(t,t') = \nabla_{\ell_p^2} K_{y_p f_r}(t,t') \times \nabla_{\ell_p} \ell_p^2}$$

$$\boxed{= \frac{(t-t')^2 - \ell_p^2}{2\ell_p^4} K_{y_p f_r}(t,t')} \quad (12)$$

$$\boxed{\nabla_{\ell_p} K_{y_p f_r}(t,t') = \frac{(t-t')^2 - \ell_p^2}{\ell_p^3} K_{y_p f_r}(t,t')} \quad (13)$$

$$\boxed{\nabla_{\sigma_r^2} K_{y_p y_q}(t,t') = \frac{1}{\sigma_r^2} K_{y_p y_q}(t,t')} \qquad (14)$$

$$\nabla_{l_p} K_{y_p y_q}(t,t') = \left[\frac{-l_p}{(l_p^2+l_q^2)^{1/2}} + \frac{l_p\,(t-t')^2}{(l_p^2+l_q^2)^2}\right] K_{y_p y_q}(t,t')$$

$$+ \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)}} \exp\left(-\frac{(t-t')^2}{2(l_p^2+l_q^2)}\right)$$

$$\times \left\{ \frac{2 l_q (t-t')}{\sqrt{2\pi(l_p^2+l_q^2)^3}} \exp\left(-\frac{l_p^2\,(t-t')^2}{2 l_q^2 (l_p^2+l_q^2)}\right)\right.$$

$$\left. - \frac{2 l_q\left[2 l_p^2 (t-t') + l_q^2 t + l_p^2 t'\right]}{l_p^2 \sqrt{2\pi(l_p^2+l_q^2)^3}} \exp\left(-\frac{(l_q^2 t + l_p^2 t')^2}{2 l_p^2 l_q^2 (l_p^2+l_q^2)}\right)\right\} \qquad (15)$$

where we have made use of the following gradients:

$$\nabla_{l_p} \frac{1}{(l_p^2+l_q^2)^{1/2}} \times \frac{\sigma_r^2}{\sqrt{8\pi}} = \cancel{\frac{\sigma_r^2}{\sqrt{8\pi}} \cdot \frac{-(l_p^2+l_q^2)^{1/2} - \frac{1}{\cancel{2}}(l_p^2+l_q^2)^{-1/2} \cdot \cancel{2} l_p^2}{l_p^2 (l_p^2+l_q^2)}}$$

$$= \boxed{-\frac{l_p \sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)^3}}} \qquad (16)$$

$$\cancel{= -\frac{\sigma_r^2}{\sqrt{2\pi}} \cdot \frac{l_p^2 + l_q^2 + l_p^2}{l_p^2 (l_p^2+l_q^2)^{3/2}}} \quad \text{WRONG !!!}$$

$$\cancel{= -\frac{\sigma_r^2 (2 l_p^2 + l_q^2)}{l_p^2 \sqrt{2\pi(l_p^2+l_q^2)^3}}} \quad \cancel{(16)}$$

$$\nabla_{l_p} \exp\left(-\frac{(t-t')^2}{2(l_p^2+l_q^2)}\right) = + \frac{(t-t')^2}{\cancel{2}(l_p^2+l_q^2)^2} (+\cancel{2} l_p) \exp\left(-\frac{(t-t')^2}{2(l_p^2+l_q^2)}\right)$$

$$= \frac{l_p\,(t-t')^2}{(l_p^2+l_q^2)^2} \exp\left(-\frac{(t-t')^2}{2(l_p^2+l_q^2)}\right) \qquad (17)$$

$$(18) \quad \nabla_{l_p} \operatorname{erf}\left(\frac{l_p (t-t')}{l_q \sqrt{2(l_p^2+l_q^2)}}\right) = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{l_p^2 (t-t')^2}{2 l_q^2 (l_p^2+l_q^2)}\right) \cdot \frac{(t-t')}{l_q \sqrt{2}} \nabla_{l_p} \frac{l_p}{\sqrt{l_p^2+l_q^2}}$$

$$(19) \quad \nabla_{l_p} \operatorname{erf}\left(\frac{l_q^2 t + l_p^2 t'}{l_p l_q \sqrt{2(l_p^2+l_q^2)}}\right) = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(l_q^2 t + l_p^2 t')^2}{2 l_p^2 l_q^2 (l_p^2+l_q^2)}\right) \cdot \frac{1}{l_q \sqrt{2}} \nabla_{l_p} \frac{l_q^2 t + l_p^2 t'}{l_p \sqrt{l_p^2+l_q^2}}$$

$$\boxed{\nabla_{\ell_p} \frac{\ell_p}{(\ell_p^2+\ell_q^2)^{1/2}}} = \frac{(\ell_p^2+\ell_q^2)^{1/2} - \ell_p \cancel{\tfrac{1}{2}} (\ell_p^2+\ell_q^2)^{-1/2} \cdot \cancel{2}\, \ell_p}{\ell_p^2+\ell_q^2}$$

$$= \frac{\cancel{\ell_p^2} + \ell_q^2 - \cancel{\ell_p^2}}{(\ell_p^2+\ell_q^2)^{3/2}} = \boxed{\frac{\ell_q^2}{(\ell_p^2+\ell_q^2)^{3/2}}} \qquad (20)$$

$$\boxed{\nabla_{\ell_p} \frac{\ell_q^2 t + \ell_p^2 t'}{\ell_p \sqrt{(\ell_p^2+\ell_q^2)}}} = \frac{2\ell_p^2 t' \sqrt{\ell_p^2+\ell_q^2} - (\ell_q^2 t + \ell_p^2 t')\left[\sqrt{\ell_p^2+\ell_q^2} + \ell_p (\ell_p^2+\ell_q^2)^{-1/2} \cdot \cancel{\tfrac{1}{2}}\cancel{2}\, \ell_p\right]}{\ell_p^2 (\ell_p^2+\ell_q^2)}$$

$$= \frac{2\ell_p^2 (\ell_p^2+\ell_q^2) t' - (\ell_q^2 t + \ell_p^2 t')(\ell_p^2+\ell_q^2+\ell_p^2)}{\ell_p^2 (\ell_p^2+\ell_q^2)^{3/2}}$$

$$= \frac{2\ell_p^2 (\cancel{\ell_p^2}+\ell_q^2) t' - \ell_q^2 (2\ell_p^2+\ell_q^2) t - \ell_p^2 (\cancel{2\ell_p^2}+\ell_q^2) t'}{\ell_p^2 (\ell_p^2+\ell_q^2)^{3/2}}$$

$$= \boxed{\frac{-\ell_q^2 \left[+2\ell_p^2 (t-t') + \ell_q^2 t + \ell_p^2 t'\right]}{\ell_p^2 (\ell_p^2+\ell_q^2)^{3/2}}} \qquad (21)$$

Finally, it will also be useful to compute the diagonal of $\underbrace{K_{y_p y_q}(t,t')}_{\Downarrow\ t=t'}$

$$\boxed{K_{y_p y_q}(t,t) = \frac{\sigma_r^2}{\sqrt{8\pi (\ell_p^2+\ell_q^2)}} \operatorname{erf}\left(\frac{(\ell_p^2+\ell_q^2)\, t}{\ell_p \ell_q \sqrt{2(\ell_p^2+\ell_q^2)}}\right)} \qquad (22)$$

and $K_{y_p y_p}(t,t')$:

$$\boxed{K_{y_p y_p}(t,t') =} \frac{\sigma_r^2}{\sqrt{8\pi \cdot 2\ell_p^2}} \exp\left(-\frac{(t-t')^2}{2 \cdot 2\ell_p^2}\right)$$

$$\times \left\{ \operatorname{erf}\left(\frac{\ell_p (t-t')}{\ell_p \sqrt{2 \cdot 2\ell_p^2}}\right) + \operatorname{erf}\left(\frac{\ell_p^2 (t+t')}{\ell_p^2 \sqrt{2 \cdot 2\ell_p^2}}\right) \right\} \qquad (23)$$

$$= \boxed{\frac{\sigma_r^2}{4\ell_p \sqrt{\pi}} \exp\left(-\frac{(t-t')^2}{4\ell_p^2}\right) \left[\operatorname{erf}\left(\frac{t-t'}{2\ell_p}\right) + \operatorname{erf}\left(\frac{t+t'}{2\ell_p}\right)\right]}$$

Note that setting $t' = t + \Delta t$ and making $t \to \infty$ we recover the stationary versions of all these kernels which, in this case, become

$$\boxed{K^{st}_{y_p y_q}(\Delta t)} = \lim_{t\to\infty} k_{y_p y_q}(t, t+\Delta t)$$

$$\boxed{= \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)}} \exp\left(-\frac{(\Delta t)^2}{2(l_p^2+l_q^2)}\right)\left[1 - \mathrm{erf}\left(\frac{l_p \Delta t}{l_q\sqrt{2(l_p^2+l_q^2)}}\right)\right]} \quad (24)$$

$$\boxed{K^{st}_{y_p y_q}(0) = \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)}}} \quad (25)$$

$$\boxed{K^{st}_{y_p y_p}(\Delta t)} = \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_p^2)}} \exp\left(-\frac{(\Delta t)^2}{2(l_p^2+l_p^2)}\right)\left[1 - \mathrm{erf}\left(\frac{\cancel{l_p}\Delta t}{\cancel{l_p}\sqrt{2(l_p^2+l_p^2)}}\right)\right]$$

$$\boxed{= \frac{\sigma_r^2}{4 l_p\sqrt{\pi}} \exp\left(-\frac{(\Delta t)^2}{4 l_p^2}\right)\left[1 - \mathrm{erf}\left(\frac{\Delta t}{2 l_p}\right)\right]} \quad (26)$$

Note that, in the previous pages the gradient of $K_{y_p y_q}(t,t')$ w.r.t. l_q was missing, so we obtain it now:

$$\nabla_{l_q} K_{y_p y_q}(t,t') = \left[(l_p^2+l_q^2)\nabla_{l_q}\frac{1}{(l_p^2+l_q^2)^{1/2}} - \frac{(t-t')}{2}\nabla_{l_q}\frac{1}{(l_p^2+l_q^2)}\right]$$
$$\times K_{y_p y_q}(t,t') + \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)}}\exp\left(-\frac{(t-t')^2}{2(l_p^2+l_q^2)}\right)\frac{2}{\sqrt{\pi}}$$
$$\times\left\{\exp\left(-\frac{l_p^2(t-t')^2}{l_q^2\, 2(l_p^2+l_q^2)}\right)\times\frac{l_p(t-t')}{\sqrt{2}}\nabla_{l_q}\frac{1}{l_q(l_p^2+l_q^2)^{1/2}}\right.$$
$$\left.+\exp\left(-\frac{(l_q^2 t + l_p^2 t')^2}{l_p^2 l_q^2\, 2(l_p^2+l_q^2)}\right)\times\frac{1}{l_p\sqrt{2}}\nabla_{l_q}\frac{l_q^2 t + l_p^2 t'}{l_q(l_p^2+l_q^2)^{1/2}}\right\} \quad (27)$$

$$\nabla_{l_q}\frac{1}{(l_p^2+l_q^2)^{1/2}} = -\frac{1}{\cancel{2}}\cdot\frac{1}{(l_p^2+l_q^2)^{3/2}}\cdot\cancel{2}l_q = -\frac{l_q}{(l_p^2+l_q^2)^{3/2}}$$

$$\nabla_{l_q}\frac{1}{(l_p^2+l_q^2)} = -\frac{1}{(l_p^2+l_q^2)^2}\cdot 2 l_q = -\frac{2 l_q}{(l_p^2+l_q^2)^2}$$

$$\nabla_{l_q} \frac{1}{l_q (l_p^2+l_q^2)^{1/2}} = - \frac{(l_p^2+l_q^2)^{1/2} + l_q \frac{1}{\cancel{2}}(l_p^2+l_q^2)^{-1/2} \cdot \cancel{2} l_q}{l_q^2 (l_p^2+l_q^2)}$$

$$= - \frac{l_p^2+l_q^2+l_q^2}{l_q^2 (l_p^2+l_q^2)^{3/2}} = - \frac{l_p^2+2l_q^2}{l_q^2 (l_p^2+l_q^2)^{3/2}}$$

$$\nabla_{l_q} \frac{l_q^2 t + l_p^2 t'}{l_q \sqrt{l_p^2+l_q^2}} = \frac{2 l_q t\, l_q (l_p^2+l_q^2)^{1/2} - (l_q^2 t + l_p^2 t')\left[(l_p^2+l_q^2)^{1/2} + l_q \frac{1}{\cancel{2}}(l_p^2+l_q^2)^{-1/2} \times \cancel{2} l_q\right]}{l_q^2 (l_p^2+l_q^2)}$$

$$= \frac{2 l_q^2 (l_p^2+l_q^2) t - (l_q^2 t + l_p^2 t') \overbrace{(l_p^2+l_q^2+l_q^2)}^{= l_p^2+2l_q^2}}{l_q^2 (l_p^2+l_q^2)^{3/2}}$$

$$= \frac{2 l_p^2 l_q^2 t + \cancel{2 l_q^4 t} - l_p^2 l_q^2 t - \cancel{2 l_q^4 t} - l_p^4 t' - 2 l_p^2 l_q^2 t'}{l_q^2 (l_p^2+l_q^2)^{3/2}}$$

$$= \frac{l_p^2 \left[2 l_q^2 (t-t') - l_q^2 t - l_p^2 t'\right]}{l_q^2 (l_p^2+l_q^2)^{3/2}}$$

Introducing all these expressions in (27) we get

$$\boxed{\begin{aligned}
\nabla_{l_q} K_{y_p y_q}(t,t') &= \left[- \frac{l_q}{(l_p^2+l_q^2)^{1/2}} + \frac{l_q (t-t')^2}{(l_p^2+l_q^2)^2} \right] K_{y_p y_q}(t,t') \\
&+ \frac{\sigma_r^2}{\sqrt{8\pi (l_p^2+l_q^2)}} \exp\left(- \frac{(t-t')^2}{2(l_p^2+l_q^2)} \right) \\
&\times \left\{ - \frac{2 l_p (l_p^2+2l_q^2)(t-t')}{l_q^2 \sqrt{2\pi (l_p^2+l_q^2)^3}} \exp\left(- \frac{l_p^2 (t-t')^2}{2 l_q^2 (l_p^2+l_q^2)} \right) \right. \\
&\left. + \frac{2 l_p \left[2 l_q^2 (t-t') - l_q^2 t - l_p^2 t'\right]}{l_q^2 \sqrt{2\pi (l_p^2+l_q^2)^3}} \exp\left(- \frac{(l_q^2 t + l_p^2 t')^2}{2 l_p^2 l_q^2 (l_p^2+l_q^2)} \right) \right\}
\end{aligned}} \quad (28)$$

General gradient of $K_{y_p y_q}$ w.r.t. l_p from which (15) is obtained

$$\boxed{\begin{aligned}\nabla_{l_p} K_{y_p y_q}(t,t') = & \left[(l_p^2+l_q^2)\, \nabla_{l_p} \frac{1}{(l_p^2+l_q^2)^{1/2}} - \frac{(t-t')^2}{2}\, \nabla_{l_p} \frac{1}{(l_p^2+l_q^2)} \right] \\ & \times K_{y_p y_q}(t,t') + \frac{\sigma_r^2}{\sqrt{8\pi(l_p^2+l_q^2)}} \exp\left(- \frac{(t-t')^2}{2(l_p^2+l_q^2)} \right) \\ & * \frac{2}{\sqrt{\pi}} \Bigg\{ \exp\left(- \frac{l_p^2\,(t-t')^2}{l_q^2\, 2\,(l_p^2+l_q^2)} \right) \times \frac{(t-t')}{l_q\sqrt{2}}\, \nabla_{l_p} \frac{l_p}{(l_p^2+l_q^2)^{1/2}} \\ & + \exp\left(- \frac{(l_q^2 t + l_p^2 t')^2}{l_p^2 l_q^2\, 2\,(l_p^2+l_q^2)} \right) \times \frac{1}{l_q\sqrt{2}}\, \nabla_{l_p} \frac{l_q^2 t + l_p^2 t'}{l_p\,(l_p^2+l_q^2)^{1/2}} \Bigg\}\end{aligned}} \qquad (29)$$

$$\nabla_{l_p} \frac{1}{(l_p^2+l_q^2)^{1/2}} = -\frac{1}{2}\, \frac{1}{(l_p^2+l_q^2)^{3/2}} \cdot 2 l_p = - \frac{l_p}{(l_p^2+l_q^2)^{3/2}}$$

$$\nabla_{l_p} \frac{1}{l_p^2+l_q^2} = - \frac{1}{(l_p^2+l_q^2)^2} \cdot 2 l_p = - \frac{2 l_p}{(l_p^2+l_q^2)^2}$$

$$\nabla_{l_p} \frac{l_p}{(l_p^2+l_q^2)^{1/2}} = \frac{(l_p^2+l_q^2)^{1/2} - l_p \cdot \cancel{\tfrac{1}{2}}\,(l_p^2+l_q^2)^{-1/2} \cdot \cancel{2}\, l_p}{l_p^2+l_q^2}$$

$$= \frac{\cancel{l_p^2} + l_q^2 - \cancel{l_p^2}}{(l_p^2+l_q^2)^{3/2}} = \frac{l_q^2}{(l_p^2+l_q^2)^{3/2}}$$

$$\nabla_{l_p} \frac{l_q^2 t + l_p^2 t'}{l_p\,(l_p^2+l_q^2)^{1/2}} = \frac{2 l_p t' l_p\,(l_p^2+l_q^2)^{1/2} - (l_q^2 t + l_p^2 t')\left[(l_p^2+l_q^2)^{1/2} + l_p\, \cancel{\tfrac{1}{2}}\,(l_p^2+l_q^2)^{-1/2}\, \cancel{2}\, l_p \right]}{l_p^2\,(l_p^2+l_q^2)}$$

$$= \frac{2 l_p^2 (l_p^2+l_q^2) t' - (l_q^2 t + l_p^2 t')\,(\overbrace{l_p^2 + l_q^2 + l_p^2}^{2l_p^2+l_q^2})}{l_p^2\,(l_p^2+l_q^2)^{3/2}}$$

$$= \frac{\cancel{2 l_p^4 t'} + 2 l_p^2 l_q^2 t' - 2 l_p^2 l_q^2 t - l_q^4 t - \cancel{2 l_p^4 t'} - l_p^2 l_q^2 t'}{l_p^2\,(l_p^2+l_q^2)^{3/2}}$$

$$= \frac{l_q^2 \left[2 l_p^2 (t'-t) - l_q^2 t - l_p^2 t' \right]}{l_p^2\,(l_p^2+l_q^2)^{3/2}}$$

$$= - \frac{l_q^2 \left[+2 l_p^2 (t-t') + l_q^2 t + l_p^2 t' \right]}{l_p^2\,(l_p^2+l_q^2)^{3/2}}$$