LFM - WHITE × RBF - WHITE KERNEL

The cross-covariance between the output of the LFM-WHITE kernel and the RBF-WHITE kernel, for the r-th latent process, is given by

$$\boxed{K^{(r)}_{y_p\tilde{y}_q}(t,t') = \frac{S_{pr}\,\sigma_r^2}{m_p\,\omega_p}\exp(-\alpha_p t)\int_0^t \exp(\alpha_p \tau)\sin(\omega_p(t-\tau))\,h_q(t'-\tau)\,d\tau,} \quad (1)$$

where $h_q(t'-\tau)$ is the functional form of the smoothing kernel used to obtain information from the white latent process, which, in this case, is an RBF,

$$\boxed{h_q(t'-\tau) = \frac{1}{\sqrt{2\pi l_q^2}}\exp\left(-\frac{(t'-\tau)^2}{2l_q^2}\right).} \quad (2)$$

Hence, the kernel becomes

$$\boxed{\begin{aligned} K^{(r)}_{y_p\tilde{y}_q}(t,t') &= \frac{S_{pr}\,\sigma_r^2}{j2m_p\omega_p\sqrt{2\pi l_q^2}}\exp(-\alpha_p t) \\ &\quad \times \left\{\exp(j\omega_p t)\,I(\tilde{\gamma}_p) - \exp(-j\omega_p t)\,I(\gamma_p)\right\} \end{aligned}} \quad (3)$$

,

where

$$\begin{aligned}
I(\gamma) &= \int_0^t \exp(\gamma\tau)\exp\left(-\frac{(t'-\tau)^2}{2l_q^2}\right)d\tau \\
&= \int_0^t \exp\left(-\frac{\tau^2 - 2t'\tau + (t')^2 - 2l_q^2\gamma\tau}{2l_q^2}\right)d\tau \\
&= \int_0^t \exp\left(-\frac{(\tau - [t' + l_q^2\gamma])^2}{2l_q^2}\right)d\tau \times \exp\left(-\frac{(t')^2 - (t' + l_q^2\gamma)^2}{2l_q^2}\right) \\
&= \exp\left(-\frac{\cancel{(t')^2} - \cancel{(t')^2} - 2\cancel{l_q^2}\gamma t' - l_q^{2}\gamma^2}{2\cancel{l_q^2}}\right) \times \frac{\sqrt{\pi}}{2} \times \sqrt{2l_q^2} \\
&\quad \times \int_{-(t'+l_q^2\gamma)/\sqrt{2l_q^2}}^{(t-t'-l_q^2\gamma)/\sqrt{2l_q^2}} \exp(-x^2)\,dx \times \frac{\sqrt{2}}{\pi}
\end{aligned}$$

Recognizing in the last integral the familiar expression of the error function, the integral, I, becomes

$$\boxed{\begin{aligned} I(r) &= \frac{\sqrt{2\pi l_q^2}}{2} \exp\left(\frac{l_q^2 r^2 + 2rt'}{2}\right) \\ &\times \left\{ \operatorname{erf}\left(\frac{t-t'-l_q^2 r}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2 r}{\sqrt{2l_q^2}}\right) \right\}. \end{aligned}} \quad (4)$$

And inserting this equation in the expression of the kernel,

$$\boxed{K^{(r)}_{y_p \tilde{y}_q}(t,t')} = \frac{S_{pr}\sigma_r^2}{j4\omega_p \omega_p} \Bigg\{ \exp(-\tilde{\gamma}_p t) \exp\left(\frac{l_q^2 \tilde{\gamma}_p^2 + 2\tilde{\gamma}_p t'}{2}\right)$$

$$\times \left[\operatorname{erf}\left(\frac{t-t'-\tilde{\gamma}_p l_q^2}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2\tilde{\gamma}_p}{\sqrt{2l_q^2}}\right) \right]$$

$$- \exp(-\gamma_p t) \exp\left(\frac{l_q^2 \gamma_p^2 + 2\gamma_p t'}{2}\right)$$

$$\times \left[\operatorname{erf}\left(\frac{t-t'-l_q^2\gamma_p}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2\gamma_p}{\sqrt{2l_q^2}}\right) \right] \Bigg\}$$

$$\boxed{\begin{aligned} &= \frac{S_{pr}\sigma_r^2}{j4\omega_p \omega_p} \Bigg\{ \exp(-\tilde{\gamma}_p (t-t')) \exp\left(\frac{l_q^2 \tilde{\gamma}_p^2}{2}\right) \\ &\times \left[\operatorname{erf}\left(\frac{t-t'-l_q^2\tilde{\gamma}_p}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2\tilde{\gamma}_p}{\sqrt{2l_q^2}}\right) \right] \\ &- \exp(-\gamma_p (t-t')) \exp\left(\frac{l_q^2 \gamma_p^2}{2}\right) \\ &\times \left[\operatorname{erf}\left(\frac{t-t'-l_q^2\gamma_p}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2\gamma_p}{\sqrt{2l_q^2}}\right) \right] \end{aligned}} \quad (5)$$

$$= \frac{S_{pr}\sigma_r^2}{j4\omega_p \omega_p} h_{pq}(t,t')$$

an we further define

$$\boxed{E_q(t,t',\gamma_p) = \operatorname{erf}\left(\frac{t-t'-l_q^2\gamma_p}{\sqrt{2l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+l_q^2\gamma_p}{\sqrt{2l_q^2}}\right)}, \quad (6)$$

so that

$$\boxed{\begin{aligned} h_{pq}(t,t') = {} & \exp(-\tilde{\gamma}_p(t-t'))\exp\left(\frac{l_q^2\tilde{\gamma}_p^2}{2}\right)E_q(t,t',\tilde{\gamma}_p) \\ & - \exp(-\gamma_p(t-t'))\exp\left(\frac{l_q^2\gamma_p^2}{2}\right)E_q(t,t',\gamma_p)\end{aligned}} \quad (7)$$

Now, we have to obtain the gradients of $K^{(r)}_{y_p\tilde{y}_q}(t,t')$ w.r.t. its hyperparameters: S_{pr}, σ_r^2, $\theta_p \in \{m_p, C_p, D_p\}$ and l_q^2.

$$\boxed{\nabla_{S_{pr}} K^{(r)}_{y_p\tilde{y}_q}(t,t') = \frac{1}{\sigma_r^2} K^{(r)}_{y_p\tilde{y}_q}(t,t')} \quad (8)$$

$$\boxed{\nabla_{\sigma_r^2} K^{(r)}_{y_p\tilde{y}_q}(t,t') = \frac{1}{S_{pr}} K^{(r)}_{y_p\tilde{y}_q}(t,t')} \quad (9)$$

$$\boxed{\begin{aligned} \nabla_{\theta_p} K^{(r)}_{y_p\tilde{y}_q}(t,t') = {} & -\left(\frac{\nabla_{\theta_p} m_p}{m_p} + \frac{\nabla_{\theta_p}\omega_p}{\omega_p}\right) K^{(r)}_{y_p\tilde{y}_q}(t,t') \\ & + \frac{S_{pr}\sigma_r^2}{j4m_p\omega_p}\Big\{-(t-t')\nabla_{\theta_p}\tilde{\gamma}_p\cdot\exp(-\tilde{\gamma}_p(t-t')) \\ & \times \exp\left(\frac{l_q^2\tilde{\gamma}_q^2}{2}\right)E_q(t,t',\tilde{\gamma}_p) + \exp(-\tilde{\gamma}_p(t-t')) \\ & \times \exp\left(\frac{l_q^2\tilde{\gamma}_q^2}{2}\right)\nabla_{\theta_p}E_q(t,t',\tilde{\gamma}_p) + (t-t')\nabla_{\theta_p}\gamma_p \\ & \times \exp(-\gamma_p(t-t'))\exp\left(\frac{l_q^2\gamma_q^2}{2}\right)E_q(t,t',\gamma_p) \\ & - \exp(-\gamma_p(t-t'))\exp\left(\frac{l_q^2\gamma_q^2}{2}\right)\nabla_{\theta_p}E_q(t,t',\gamma_p)\Big\}\end{aligned}} \quad (10)$$

where the gradients w.r.t. the parameters of the system are the same as for the LFM-RBF kernel:

$$\nabla_{\theta_p} m_p = \begin{cases} 1, & \theta_p = m_p; \\ 0, & \theta_p \neq m_p. \end{cases}$$

$$\nabla_{\theta_p} \omega_p = \begin{cases} [C_p^2 - 2 m_p D_p] / [2 m_p^2 \sqrt{4 m_p D_p - C_p}\,], & \theta_p = m_p\,; \\ -C_p / [2 m_p \sqrt{4 m_p D_p - C_p^2}\,], & \theta_p = C_p\,; \\ 1 / \sqrt{4 m_p D_p - C_p^2}, & \theta_p = D_p\,. \end{cases}$$

$$\nabla_{\theta_p} \gamma_p = \nabla_{\theta_p} \alpha_p + j \nabla_{\theta_p} \omega_p$$

$$\nabla_{\theta_p} \tilde{\gamma}_p = \nabla_{\theta_p} \alpha_p - j \nabla_{\theta_p} \omega_p$$

$$\nabla_{\theta_p} \alpha_p = \begin{cases} -C_p / (2 m_p^2), & \theta_p = m_p\,; \\ 1/(2 m_p), & \theta_p = C_p\,; \\ 0, & \theta_p = D_p\,. \end{cases}$$

And

$$\begin{aligned} \boxed{\nabla_{\theta_p} E_q(t, t', \gamma_p)} &= \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t - t' - l_q^2 \gamma_p)^2}{2 l_q^2}\right) (-l_q^2 \nabla_{\theta_p} \gamma_p) \\ &\quad + \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t' + l_q^2 \gamma_p)^2}{2 l_q^2}\right) (l_q^2 \nabla_{\theta_p} \gamma_p) \\ &= \boxed{\frac{2 l_q^2}{\sqrt{\pi}} \left\{ -\exp\left(-\frac{(t - t' - l_q^2 \gamma_p)^2}{2 l_q^2}\right) + \exp\left(-\frac{(t' + l_q^2 \gamma_p)^2}{2 l_q^2}\right) \right\} \nabla_{\theta_p} \gamma_p} \end{aligned} \tag{11}$$

Finally,

$$\begin{aligned} \nabla_{l_q^2} k^{(r)}_{y_p \tilde{y}_q}(t, t') = \frac{S_{pr} \sigma_r^2}{j 4 m_p \omega_p} \Bigg\{ & \frac{\tilde{\gamma}_p^2}{2} \exp(-\tilde{\gamma}_p (t - t')) \exp\left(\frac{l_q^2 \tilde{\gamma}_p^2}{2}\right) E_q(t, t', \tilde{\gamma}_p) \\ & + \exp(-\tilde{\gamma}_p (t - t')) \exp\left(\frac{l_q^2 \tilde{\gamma}_p^2}{2}\right) \nabla_{l_q^2} E_q(t, t', \tilde{\gamma}_p) \\ & - \frac{\gamma_p^2}{2} \exp(-\gamma_p (t - t')) \exp\left(\frac{l_q^2 \gamma_p^2}{2}\right) E_q(t, t', \gamma_p) \\ & - \exp(-\gamma_p (t - t')) \exp\left(\frac{l_q^2 \gamma_p^2}{2}\right) \nabla_{l_q^2} E_q(t, t', \gamma_p) \Bigg\} \end{aligned} \tag{12}$$

with

$$\nabla_{l_q^2} E_q(t,t',r_p) = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t-t'-l_q^2 r_p)^2}{2l_q^2}\right) \cdot \nabla_{l_q^2} \frac{t-t'-l_q^2 r_p}{\sqrt{2l_q^2}}$$

$$+ \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t'+l_q^2 r_p)^2}{2l_q^2}\right) \cdot \nabla_{l_q^2} \frac{t'+l_q^2 r_p}{\sqrt{2l_q^2}} ,$$

where

$$\nabla_{l_q^2} \frac{t-t'-l_q^2 r_p}{\sqrt{2l_q^2}} = \frac{-r_p (2l_q^2)^{1/2} - (t-t'-l_q^2 r_p)\frac{1}{\cancel{2}}(2l_q^2)^{-1/2}\cdot \cancel{2}}{2l_q^2}$$

$$= \frac{-2l_q^2 r_p - t + t' + l_q^2 r_p}{(2l_q^2)^{3/2}}$$

$$= -\frac{t-t'+l_q^2 r_p}{(2l_q^2)^{3/2}}$$

$$\nabla_{l_q^2} \frac{t'+l_q^2 r_p}{\sqrt{2l_q^2}} = \frac{r_p (2l_q^2)^{1/2} - (t'+l_q^2 r_p)\frac{1}{\cancel{2}}(2l_q^2)^{-1/2}\cdot \cancel{2}}{2l_q^2}$$

$$= \frac{2l_q^2 r_p - t' - l_q^2 r_p}{(2l_q^2)^{3/2}}$$

$$= -\frac{t'-l_q^2 r_p}{(2l_q^2)^{3/2}}$$

so finally

$$\boxed{\begin{aligned}\nabla_{l_q^2} E_q(t,t',r_p) = -\frac{1}{\sqrt{2\pi l_q^6}} \Bigg\{ &(t-t'+l_q^2 r_p) \exp\left(-\frac{(t-t'-l_q^2 r_p)^2}{2l_q^2}\right) \\ &+ (t'-l_q^2 r_p) \exp\left(-\frac{(t'+l_q^2 r_p)^2}{2l_q^2}\right) \Bigg\}\end{aligned}}$$

(13)