Ornstein - Uhlenbeck Process

Definition and Covariance Function:

The Ornstein-Uhlenbeck (OU) process is defined by the following stochastic differential equation [Wikipedia: Ornstein-Uhlenbeck process]:

$$df(t) = -\theta\,(f(t)-\mu)\,dt + \sigma\,dw(t), \qquad (1)$$

where $w(t)$ is the Wiener process (i.e. Brownian motion), and θ, μ and σ are parameters of the system. Dividing by dt in (1), we get

$$\frac{df(t)}{dt} = -\theta\,(f(t)-\mu) + \sigma\,\frac{dw(t)}{dt}, \qquad (2)$$

where the derivatives have no meaning per se (note that $w(t)$ is nowhere differentiable), but the definition of $f(t)$ from an integral equation:

$$f(t) = -\theta\int_0^t (f(t)-\mu)\,dt + \sigma\,w(t). \qquad (3)$$

In any case, (2) can be solved easily using Laplace transforms:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0) = \mathcal{L}\left\{-\theta\,(f(t)-\mu) + \sigma\,\frac{dw(t)}{dt}\right\}$$

$$= -\theta\,(F(s)-\mu/s) + \sigma\,(sW(s)-W(0)) \Rightarrow$$

$$(s+\theta)\,F(s) = f(0) - \sigma W(0) + \frac{\theta\mu}{s} + \sigma s W(s) \Rightarrow$$

$$F(s) = \frac{f(0)-\sigma W(0)}{s+\theta} + \frac{\theta\mu}{s(s+\theta)} + \frac{\sigma s W(s)}{s+\theta}$$

$$= \frac{f(0)}{s+\theta} + \frac{\theta\mu}{s(s+\theta)} + \frac{\sigma(sW(s)-W(0))}{s+\theta} \qquad (4)$$

And $f(t)$ can be obtained as the inverse Laplace transform of $F(s)$ using the following transformed pairs and properties:

(5) $\mathcal{L}\{1\} = 1/s$ [Abramowitz, 29.3.1]

(6) $\mathcal{L}\{\exp(-at)\} = 1/(s+a)$ [Abramowitz, 29.3.8]

(7) $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$ [Abramowitz, 29.2.4]

(8) $$\mathcal{L}\{f_1(t) * f_2(t)\} = \mathcal{L}\left\{\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right\}$$

$$= \mathcal{L}\left\{\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right\}$$

$$= F_1(s) \cdot F_2(s)$$ [Abramowitz, 29.2.8]

So, we have:

(9) $\mathcal{L}^{-1}\left\{\frac{f(0)}{s+\theta}\right\} = f(0) \cdot \exp(-\theta t)$ [using (6)]

(10) $\mathcal{L}^{-1}\left\{\frac{\theta\mu}{s(s+\theta)}\right\} = \mathcal{L}^{-1}\left\{\frac{\theta\mu}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+\theta}\right\}$ [using (8)]

$= \theta\mu * \exp(-\theta t)$ [using (5) and (6)]

$$= \theta\mu \int_0^t \exp(-\theta\tau)\, d\tau$$

$$= \frac{\theta\mu}{-\theta} \cdot \exp(-\theta\tau)\Big|_{\tau=0}^{\tau=t} = \mu\left(1 - \exp(-\theta t)\right)$$

(11) $\mathcal{L}^{-1}\left\{\frac{\sigma(sW(s) - w(0))}{s+\theta}\right\} = \mathcal{L}^{-1}\left\{\sigma(sW(s) - w(0))\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+\theta}\right\}$ [using (8)]

$= \sigma\frac{dW(t)}{dt} * \exp(-\theta t)$ [using (8) and (7)]

$$= \sigma\int_0^t \frac{dw(\tau)}{d\tau} \exp(-\theta(t-\tau))\, d\tau$$

$$= \sigma\exp(-\theta t)\int_0^t \exp(\theta\tau)\, dw(\tau),$$

which is a stochastic integral. Hence, putting it all together we have

$$\boxed{\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \underbrace{f(0)\exp(-\theta t) + \mu\left(1-\exp(-\theta t)\right)}_{=\ \mu + (f(0)-\mu)\exp(-\theta t)} \\ &\quad + \sigma \exp(-\theta t) \int_0^t \exp(\theta \tau)\, dw(\tau), \end{aligned}} \qquad (12)$$

$\rightsquigarrow$ same expression as for GP-SIM

where the first two terms represent the deterministic part of $f(t)$ and the last one the stochastic part. The expected value of $f(t)$ is

$$\begin{aligned} \boxed{E\{f(t)\}} &= f(0)\exp(-\theta t) + \mu\left(1-\exp(-\theta t)\right) \\ &\quad + \sigma \exp(-\theta t) \cdot E\left\{\int_0^t \exp(\theta\tau)\, dw(\tau)\right\}, \\ &\boxed{= f(0)\exp(-\theta t) + \mu\left(1-\exp(-\theta t)\right)} \end{aligned} \qquad (13)$$

where

$$\begin{aligned} E\left\{\int_0^t \exp(\theta\tau)\, dw(\tau)\right\} &= E\left\{\int_0^t \exp(\theta\tau) \cdot \frac{dw(\tau)}{d\tau}\, d\tau\right\} \\ &= \int_0^t \exp(\theta\tau)\, E\left\{\frac{dw(\tau)}{d\tau}\right\} d\tau = 0, \end{aligned} \qquad (14)$$

since $dw(\tau)/d\tau$ is the increment (or decrement) of $w(\tau)$ in a given amount of time, which is zero on average by definition as the increment in time goes to zero, as it happens in the derivative. Now, let's obtain the covariance function,

$$\begin{aligned} \operatorname{cov}\left(f(t), f(t')\right) &= \sigma^2 \exp\left(-\theta(t+t')\right) E\left\{\int_0^t \int_0^{t'} \exp\left(\theta(\tau+\tau')\right) dw(\tau)\, dw(\tau')\right\} \\ &= \sigma^2 \exp\left(-\theta(t+t')\right) \int_0^t \int_0^{t'} \exp\left(\theta(\tau+\tau')\right) E\left\{\frac{dw(\tau)}{d\tau} \cdot \frac{dw(\tau')}{d\tau'}\right\} d\tau\, d\tau' \end{aligned}$$

Now, again by the definition of Brownian motion, $w(\tau)$ is independent of $w(\tau')$. Thus, for $\tau \neq \tau'$:

$$E\left\{\frac{dw(\tau)}{d\tau} \cdot \frac{dw(\tau')}{d\tau'}\right\} = E\left\{\frac{dw(\tau)}{d\tau}\right\} \cdot E\left\{\frac{dw(\tau')}{d\tau'}\right\} = 0. \qquad (15)$$

And for $\tau = \tau'$:

$$E\left\{\frac{dw(\tau)}{d\tau}\cdot\frac{dw(\tau')}{d\tau'}\right\} = E\left\{\left(\frac{dw(\tau)}{d\tau}\right)^2\right\} = 1. \qquad (16)$$

Putting together (15) and (16), we get

$$E\left\{\frac{dw(\tau)}{d\tau}\cdot\frac{dw(\tau')}{d\tau'}\right\} = \delta(\tau-\tau'), \qquad (17)$$

and

$$\boxed{\operatorname{cov}(f(t), f(t'))} = \sigma^2 \exp(-\theta(t+t'))\int_0^{t\wedge t'} \exp(\theta(\tau+\tau))\,d\tau$$

$$= \frac{\sigma^2}{2\theta}\exp(-\theta(t+t'))\cdot \exp(2\theta\tau)\Big|_{\tau=0}^{\tau=t\wedge t'}$$

$$\boxed{= \frac{\sigma^2}{2\theta}\exp(-\theta(t+t'))\left(\exp(2\theta(t\wedge t'))-1\right)}, \qquad (18)$$

where $\boxed{t\wedge t' = \min(t, t')}$ and is the upper limit of integration, since for $\tau > t\wedge t'$ there is no way that $\tau=\tau'$ and the double integral becomes zero. Note that

$$\operatorname{cov}(f(t), f(t')) = \frac{\sigma^2}{2\theta}\left[\exp\left(\theta[2(t\wedge t')-(t+t')]\right) - \exp(-\theta(t+t'))\right],$$

and that

$$t\wedge t' = \begin{cases} t, & t\le t'; \\ t', & t>t'; \end{cases} \quad \Rightarrow$$

$$2(t\wedge t')-(t+t') = \begin{cases} 2t-t-t' = t-t', & t\le t' \\ 2t'-t-t' = t'-t, & t>t' \end{cases} = -|t-t'|,$$

So we can rewrite (18) as

$$\boxed{\operatorname{cov}(f(t), f(t')) = \frac{\sigma^2}{2\theta}\left(\exp(-\theta|t-t'|) - \exp(-\theta(t+t'))\right)} \qquad (19)$$

This expression clearly indicates that the covariance function of $f(t)$ has a stationary part, $\sigma^2/2\theta \cdot \exp(-\theta|t-t'|)$, and a non-stationary part, $-\sigma^2/2\theta \cdot \exp(-\theta(t+t'))$, which vanishes as $t, t' \to \infty$ while keeping $\Delta t = t - t'$ finite, so the process becomes asymptotically stationary, since $E\{f(t)\} \to 0$ as $t \to \infty$. This stationary part is the only one considered in some texts, such as [Rasmussen and Williams, 2006].

An alternative proof of this covariance function can be obtained noting that

$$E\left\{ \int_0^t \int_0^{t'} \exp(\theta(\tau+\tau'))\, dw(\tau)\, dw(\tau') \right\}$$

$$= E\left\{ \int_0^t \exp(\theta\tau)\, dw(\tau) \cdot \int_0^{t'} \exp(\theta\tau')\, dw(\tau') \right\}$$

Now, for $\tau \neq \tau'$, $w(\tau)$ and $w(\tau')$ are independent, and we have

$$E\left\{ \int_0^t \exp(\theta\tau)\, dw(\tau) \right\} \cdot E\left\{ \int_0^{t'} \exp(\theta\tau')\, dw(\tau') \right\} = 0,$$

and for $\tau = \tau'$ we can apply Itô's isometry:

$$E\left\{ \int_0^{t\wedge t'} \exp(\theta\tau)\, dw(\tau) \cdot \int_0^{t\wedge t'} \exp(\theta\tau)\, dw(\tau) \right\}$$

$$= E\left\{ \left(\int_0^{t\wedge t'} \exp(\theta\tau)\, dw(\tau) \right)^2 \right\}$$

$$= E\left\{ \int_0^{t\wedge t'} \left(\exp(\theta\tau)\right)^2 d\tau \right\}$$

$$= \int_0^{t\wedge t'} \exp(2\theta\tau)\, d\tau = \frac{1}{2\theta}\left[\exp(2\theta(t\wedge t')) - 1 \right],$$

where the limit now is changed from t and t' to $t \wedge t'$, since this is the only range where τ can be equal to τ'.

Gradients:

We only have to compute the gradient of $k_{ff}(t,t')$ w.r.t. σ^2 and θ, since μ does not appear in the covariance function. The first one is straightforward,

$$\boxed{\nabla_{\sigma^2} k_{ff}(t,t') = \frac{1}{2\theta}\Big[\underbrace{\exp\left(-\theta|t-t'|\right)}_{\text{stationary}} - \underbrace{\exp\left(-\theta(t+t')\right)}_{\text{non-stationary}}\Big].} \quad (20)$$

The second one is,

$$\nabla_{\theta} k_{ff}(t,t') = -\frac{\sigma^2}{2\theta^2}\Big[\exp\left(-\theta|t-t'|\right) - \exp\left(-\theta(t+t')\right)\Big]$$

$$+ \frac{\sigma^2}{2\theta}\Big[-\theta|t-t'|\exp\left(-\theta|t-t'|\right) + \theta(t+t')\exp\left(-\theta(t+t')\right)\Big] \quad (21)$$

which can be divided into a stationary

$$\nabla_{\theta}^{st} k_{ff}(t,t') = -\frac{\sigma^2}{2\theta}\left[\frac{1}{\theta} + |t-t'|\right]\exp\left(-\theta|t-t'|\right)$$

$$= \boxed{-\frac{\sigma^2}{2\theta}\left(\frac{1}{\theta} + |t-t'|\right)\exp\left(-\theta|t-t'|\right),} \quad (22)$$

and a non-stationary part

$$\boxed{\nabla_{\theta}^{non\text{-}st} k_{ff}(t,t') = \frac{\sigma^2}{2\theta}\left(\frac{1}{\theta} + (t+t')\right)\exp\left(-\theta(t+t')\right).} \quad (23)$$

Finally, we can also evaluate the main diagonal of the covariance function,

$$\boxed{k_{ff}(t,t) = \frac{\sigma^2}{2\theta}\left(1 - \exp\left(-2\theta t\right)\right),} \quad (24)$$

and its gradient w.r.t. t:

$$\boxed{\nabla_{t} k_{ff}(t,t) = -\frac{\sigma^2}{2\theta}\left(-2\theta\right)\exp\left(-2\theta t\right) = \sigma^2 \exp\left(-2\theta t\right),} \quad (25)$$

which is zero if we are evaluating only the stationary part of the kernel.