SIM-WHITE X RBF-WHITE KERNEL

In order to apply the new convolutional variational approach it will be necessary to obtain the cross-covariance matrix between the output of the GP-SIM kernel with white latent force,

$$\boxed{y_p(t) = \frac{B_p}{D_p} + \sum_{r=1}^{R} S_{pr} \exp(-D_p t) \int_0^t f_r(\tau) \exp(D_p \tau)\, d\tau,} \quad (1)$$

and the output of the RBF-WHITE kernel,

$$\boxed{\tilde{y}_q(t) = \sum_{r=1}^{R} \int_0^t f_r(\tau) h_q(t-\tau)\, d\tau.} \quad (2)$$

Concentrating on the r-th force we have

$$\boxed{K^{(r)}_{y_p \tilde{y}_q}(t,t')} = S_{pr} \exp(-D_p t) \int_0^t \int_0^{t'} E\{f_r(\tau) f_r(\tau')\} \exp(D_p \tau) h_q(t'-\tau')\, d\tau d\tau'$$

$$= S_{pr} \exp(-D_p t) \int_0^t \sigma_r^2 \exp(D_p \tau) h_q(t'-\tau)\, d\tau$$

$$\boxed{= \frac{\sigma_r^2 S_{pr}}{\sqrt{2\pi \ell_q^2}} \exp(-D_p t) \int_0^t \exp(D_p \tau) \exp\left(-\frac{(t'-\tau)^2}{2\ell_q^2}\right) d\tau,} \quad (3)$$

where the integral is given by

$$\int_0^t \exp(D_p \tau) \exp\left(-\frac{(t'-\tau)^2}{2\ell_q^2}\right) d\tau = \int_0^t \exp\left(-\frac{\tau^2 - 2t'\tau + (t')^2 - 2D_p \ell_q^2 \tau}{2\ell_q^2}\right) d\tau$$

$$= \int_0^t \exp\left(-\frac{(\tau - (t' + D_p \ell_q^2))^2}{2\ell_q^2}\right) d\tau$$

$$\times \exp\left(-\frac{(t')^2 - (t' + D_p \ell_q^2)^2}{2\ell_q^2}\right)$$

The integral can again be expressed as the sum of two error functions,

$$\int_0^t \exp\left(-\frac{\left(\tau-(t'+D_p l_q^2)\right)^2}{2l_q^2}\right)d\tau = \int_{-(t'+D_p l_q^2)/\sqrt{2l_q^2}}^{(t-t'-D_p l_q^2)/\sqrt{2l_q^2}} \exp(-x^2)\cdot\sqrt{2l_q^2}\,dx$$

$$= \frac{\sqrt{2\pi l_q^2}}{2}\left\{\mathrm{erf}\left(\frac{t-(t'+D_p l_q^2)}{\sqrt{2l_q^2}}\right) + \mathrm{erf}\left(\frac{t'+D_p l_q^2}{\sqrt{2l_q^2}}\right)\right\}$$

And the second term becomes

$$\exp\left(-\frac{(t')^2-(t'+D_p l_q^2)^2}{2l_q^2}\right) = \exp\left(-\frac{\cancel{(t')^2}-\cancel{(t')^2}-2D_p\cancel{l_q^2}\,t'+D_p^2 l_q^{\cancel{4}\,2}}{2\cancel{l_q^2}}\right)$$

$$= \exp\left(-\frac{D_p^2 l_q^2-2D_p t'}{2}\right).$$

Hence, putting it all together

$$\boxed{K^{(r)}_{y_p\tilde{y}_q}(t,t')} = \frac{\sigma_r^2 S_{pr}}{\cancel{\sqrt{2\pi l_q^2}}}\cdot\frac{\cancel{\sqrt{2\pi l_q^2}}}{2}\exp(-D_p t)\,\exp\left(-\frac{D_p^2 l_q^2-2D_p t'}{2}\right)$$

$$\times\left\{\mathrm{erf}\left(\frac{t-t'-D_p l_q^2}{\sqrt{2l_q^2}}\right)+\mathrm{erf}\left(\frac{t'+D_p l_q^2}{\sqrt{2l_q^2}}\right)\right\}$$

$$\boxed{= \frac{\sigma_r^2 S_{pr}}{2}\exp\left(-\frac{D_p^2 l_q^2}{2}\right)\exp\left(-D_p(t-t')\right)\times\left\{\mathrm{erf}\left(\frac{t-t'-D_p l_q^2}{\sqrt{2l_q^2}}\right)+\mathrm{erf}\left(\frac{t'+D_p l_q^2}{\sqrt{2l_q^2}}\right)\right\}}, \quad (4)$$

Where, as usual, there is a stationary part and a non-stationary one.

Now, for the optimisation phase we also need the gradients of this Kernel w.r.t. its hyperparameters; σ_r^2, S_{pr}, D_p, l_q.

$$\boxed{\nabla_{\sigma_r^2} K^{(r)}_{y_p \tilde{y}_q}(t,t') = \frac{1}{\sigma_r^2} K^{(r)}_{y_p \tilde{y}_q}(t,t')} \qquad (5)$$

$$\boxed{\nabla_{S_{pr}} K^{(r)}_{y_p \tilde{y}_q}(t,t') = \frac{1}{S_{pr}} K^{(r)}_{y_p \tilde{y}_q}(t,t')} \qquad (6)$$

$$\begin{aligned}
\boxed{\nabla_{D_p} K^{(r)}_{y_p \tilde{y}_q}(t,t')} &= \frac{\sigma_r^2 S_{pr}}{2} \cdot \frac{l_q^2}{2} \cdot (-2D_p)\, h_{pq}(t,t') \\
&+ \frac{\sigma_r^2 S_{pr}}{2} \cdot \left(-(t-t')\right) h_{pq}(t,t') \\
&+ \frac{\sigma_r^2 S_{pr}}{2} \exp\left(-\frac{D_p^2 l_q^2}{2}\right) \exp\left(-D_p(t-t')\right) \\
&\times \left\{ \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t-t'-D_p l_q^2)^2}{2 l_q^2}\right) \times \left(-\frac{l_q^{\cancel{2}}}{\cancel{l_q}\sqrt{2}}\right) \right. \\
&\quad \left. + \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(t'+D_p l_q^2)^2}{2 l_q^2}\right) \times \left(\frac{l_q^{\cancel{2}}}{\cancel{l_q}\sqrt{2}}\right) \right\}
\end{aligned}$$

$$\boxed{\begin{aligned}
&= -\left[l_q^2 D_p + (t-t')\right] K^{(r)}_{y_p \tilde{y}_q}(t,t') \\
&+ \frac{\sigma_r^2 l_q S_{pr}}{\sqrt{2\pi}} \exp\left(-\frac{D_p^2 l_q^2}{2}\right) \exp\left(-D_p(t-t')\right) \\
&\times \left\{ -\exp\left(-\frac{(t-t'-D_p l_q^2)^2}{2 l_q^2}\right) + \exp\left(-\frac{(t'+D_p l_q^2)^2}{2 l_q^2}\right) \right\}
\end{aligned}} \qquad (7)$$

Where we have used

$$\boxed{\begin{aligned}
h_{pq}(t,t') &= \exp\left(-\frac{D_p^2 l_q^2}{2}\right) \exp\left(-D_p(t-t')\right) \\
&\times \left\{ \operatorname{erf}\left(\frac{t-t'-D_p l_q^2}{\sqrt{2 l_q^2}}\right) + \operatorname{erf}\left(\frac{t'+D_p l_q^2}{\sqrt{2 l_q^2}}\right) \right\}.
\end{aligned}} \qquad (8)$$

Finally,

$$\nabla_{l_q} k^{(r)}_{y_p\tilde{y}_q}(t,t') = \frac{\sigma_r^2 S_{pr}}{2}\cdot\left(-\frac{D_p^2}{\cancel{2}}\right)\cdot\cancel{2}l_q\; h_{pq}(t,t')$$

$$+\frac{\sigma_r^2 S_{pr}}{2}\exp\left(-\frac{D_p^2 l_q^2}{2}\right)\exp\left(-D_p(t-t')\right)$$

$$\times\left\{\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(t-t'-D_p l_q^2)^2}{2l_q^2}\right)\times\nabla_{l_q}\left(-\frac{D_p l_q^2}{\sqrt{2l_q^2}}\right)\right.$$

$$\left.+\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(t'+D_p l_q^2)^2}{2l_q^2}\right)\times\nabla_{l_q}\left(\frac{D_p l_q^2}{\sqrt{2l_q^2}}\right)\right\},$$

where

$$\nabla_{l_q}\frac{D_p l_q^2}{l_q\sqrt{2}} = \frac{D_p}{\sqrt{2}}\cdot\nabla_{l_q}\frac{l_q^{\cancel{2}}}{\cancel{l_q}} = \frac{D_p}{\sqrt{2}},$$

and thus

$$\boxed{\begin{aligned}\nabla_{l_q} K^{(r)}_{y_p\tilde{y}_q}(t,t') &= -D_p^2 l_q K^{(r)}_{y_p\tilde{y}_q}(t,t') + \frac{\sigma_r^2 D_p S_{pr}}{\sqrt{2\pi}}\exp\left(-\frac{D_p^2 l_q^2}{2}\right)\\ &\quad\times\exp\left(-D_p(t-t')\right)\left\{-\exp\left(-\frac{(t-t'-D_p l_q^2)^2}{2l_q^2}\right)\right.\\ &\quad\left.+\exp\left(-\frac{(t'+D_p l_q^2)^2}{2l_q^2}\right)\right\}\end{aligned}} \quad (9)$$