Integration

Standard integrals:

(I) $\int x^n dx = \frac{x^{n+1}}{n+1}$ $(n \neq -1)$

(II) $\int dx = x$

(III) $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$

(IV) $\int \frac{dx}{x} = \log|x|$

(V) $\int e^{mx} dx = \frac{e^{mx}}{m}$

(VI) $\int e^x dx = e^x$

(VII) $\int \sin x dx = -\cos x$

(VIII) $\int \cos x dx = \sin x$

(IX) $\int \sec^2 x dx = \tan x$

(X) $\int \operatorname{cosec}^2 x dx = -\cot x$

(XI) $\int \sec x \tan x dx = \sec x$

(XII) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x$

Examples: ① Integrate $\int \sin^2 x dx$

Soln: $\int \sin^2 x dx$

$= \frac{1}{2}\int 2\sin^2 x dx$

$= \frac{1}{2}\int (1-\cos 2x) dx$ $\left[\because 2\sin^2 x = 1-\cos 2x\right]$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c = \frac{1}{2}x - \frac{\sin 2x}{4} + c$

(2) $\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \int \sec^2 x\,dx - \int dx$

$= \tan x - x + c$

(3) $\int \dfrac{5(x-3)^2}{x\sqrt{x}}\,dx$

$= \int \dfrac{5(x^2 - 2\cdot x\cdot 3 + 3^2)}{x\sqrt{x}}\,dx$

$= \int \dfrac{5(x^2 - 6x + 9)}{x\sqrt{x}}\,dx$

$= \int \dfrac{5x^2 - 30x + 45}{x\sqrt{x}}\,dx$

$= \int \dfrac{5x^2 - 30x + 45}{x^1 . x^{\frac{1}{2}}}\,dx$

$= \int \dfrac{5x^2 - 30x + 45}{x^{\frac{3}{2}}}\,dx$

$= 5\int x^{2-\frac{3}{2}}\,dx - 30\int x^{1-\frac{3}{2}}\,dx + 45\int x^{-\frac{3}{2}}\,dx$

$= 5\int x^{\frac{1}{2}}\,dx - 30\int x^{-\frac{1}{2}}\,dx + 45\int x^{-\frac{3}{2}}\,dx$

$= 5\,\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 30\int \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 45\,\dfrac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$

For…

Ex-3: Integrate $\int \tan^{-1}x\,dx$

$$I = \int \tan^{-1}x\,dx$$

$$= \tan^{-1}x \int dx - \int \left\{ \frac{d}{dx}(\tan^{-1}x) \int dx \right\} dx$$

$$= \tan^{-1}x \cdot x - \int \frac{1}{1+x^2} \cdot x\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2).$$

(Ans.)

Ex-4: Integrate $\int \log(x+\sqrt{x^2+a^2})\,dx$

$$I = \int \log(x+\sqrt{x^2+a^2})\,dx$$

$$= \log(x+\sqrt{x^2+a^2})\int dx - \int \left[\frac{d}{dx}\left\{\log(x+\sqrt{x^2+a^2})\right\} \int dx \right] dx$$

$$= x\log(x+\sqrt{x^2+a^2}) - \int \frac{x\,dx}{\sqrt{x^2+a^2}}$$

$$= x\log(x+\sqrt{x^2+a^2}) - \int \frac{x\,dx}{\sqrt{x^2+a^2}}$$

Fore evaluating $\int \frac{x\,dx}{\sqrt{x^2+a^2}}$.

Put, $x^2+a^2 = z^2$

$$2x\,dx = 2z\,dz$$

$$\Rightarrow x\,dx = z\,dz$$

$$\therefore \int \frac{x\,dx}{\sqrt{x^2+a^2}} = \int \frac{z\,dz}{z}$$

$$= \int dz$$

$$= z$$

$$= \sqrt{x^2+a^2}$$

$$\therefore I = x\log\left(x+\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}.$$

Homework:

(i) $\int x^3 e^x\,dx$.

(ii) $\int x \sin 2x\,dx$.

(iii) $\int x^2 \sin x\,dx$.

(iv) $\int x^3 (1+x^2)^{-3}\,dx$.

(v) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

(vi) $\sqrt{1-\sin 2x}\,dx$

Example-5: $\int_{1}^{2} (2x^2+1)\,dx$.

$$= \left[\frac{2x^3}{3} + x\right]_1^2$$

$$= \left[\frac{2.2^3}{3} + 2\right] - \left[\frac{2.1^3}{3} + 1\right]$$

$$= \left[\frac{2.8}{3} + 2\right] - \left[\frac{2}{3} + 1\right]$$

$$= \frac{16}{3} + 2 - \frac{2}{3} - 1$$

$$= \frac{16}{3} - \frac{2}{3} + 1$$

$$= \frac{16-2+3}{3}$$

$$= \frac{17}{3}.$$

(Ans.)

Integration By parts

Formula:

$$\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\} dx.$$

Example-1: Integrate $\int x e^x dx$.

Solution: $I = \int x e^x dx$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\} dx$$

$$= xe^x - \int 1.e^x dx$$

$$= xe^x - e^x$$

$$= e^x(x-1).$$

Example-2: Integrate $\int \log x\, dx$

Solution: $I = \int \log x\, dx$

$$= \log x \int dx - \int\left\{\frac{d}{dx}(\log x)\int dx\right\} dx$$

$$= \log x . x - \int \frac{1}{x} . x\, dx$$

$$= x\log x - \int dx$$

$$= x\log x - x.$$

$$= 5\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 30\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 45\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$= 5.\frac{2}{3}x^{\frac{3}{2}} - 30.2\,x^{\frac{1}{2}} + 45(-2)x^{-\frac{1}{2}} + c$$

$$= \frac{10}{3}x^{\frac{3}{2}} - 60\sqrt{x} - 90\,x^{-\frac{1}{2}} + c$$

(4) $\int \sin 3x \cos 2x\,dx$

$$= \frac{1}{2}\int 2\sin 3x \cos 2x\,dx$$

$$= \frac{1}{2}\int (\sin 5x + \sin x)\,dx \qquad [2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\int \sin 5x\,dx + \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\cancel{\sin 5x}\left[-\frac{\cos 5x}{5} - \cos x\right] + c$$

(Ans.)

$$= 5\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 30\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 45\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$= 5.\frac{2}{3}x^{\frac{3}{2}} - 30.2\,x^{\frac{1}{2}} + 45(-2)x^{-\frac{1}{2}} + c$$

$$= \frac{10}{3}x^{\frac{3}{2}} - 60\sqrt{x} - 90\,x^{-\frac{1}{2}} + c$$

(4) $\int \sin 3x \cos 2x \, dx$

$$= \frac{1}{2}\int 2\sin 3x \cos 2x \, dx$$

$$= \frac{1}{2}\int (\sin 5x + \sin x)\, dx \qquad [2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\int \sin 5x \, dx + \frac{1}{2}\int \sin x \, dx$$

$$= \frac{1}{2}\cancel{\sin 5x}\left[-\frac{\cos 5x}{5} - \cos x\right] + c$$

(Ans.)