The n-th derivatives of function.

$$y = x^n$$
$$y_1 = nx^{n-1}$$
$$y_2 = n(n-1)x^{n-2}$$
$$y_3 = n(n-1)(n-2)x^{n-3}$$
$$\cdots\cdots$$
$$y_n = n(n-1)(n-2)\cdots 3.2.1 = n!$$

Ex-1 If $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$ find y_n.

Soln: Given,
$$y = \frac{x^2+x-1}{x^3+x^2-6x}$$

Here, $x^3+x^2-6x = x(x^2+x-6)$
$$= x(x^2+3x-2x-6)$$
$$= x(x+3)(x-2)$$

Let, $\dfrac{x^2+x-1}{x^3+x^2-6x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$.

Multiplying both sides by $x(x+3)(x-2)$, we get,
$$x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$$

Putting $x = 0, -3, 2$ successively on both sides,
we get, $A = \frac{1}{6}$, $B = \frac{1}{3}$ and $C = \frac{1}{2}$.

$$\therefore y = \frac{1}{6}\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\cdot\frac{1}{x-2}$$

$$\Rightarrow y_1 = -\frac{1}{6}\frac{1}{x^2} - \frac{1}{3}\frac{1}{(x+3)^2} - \frac{1}{2}\frac{1}{(x-2)^2}$$

$$\Rightarrow y_2 = (-1)^2\frac{1}{6}\frac{2}{x^3} + (-1)^2\frac{1}{3}\frac{2}{(x+3)^3} + (-1)^2\frac{1}{2}\frac{2}{(x-2)^3}$$

$$\Rightarrow y_3 = (-1)^3\frac{1}{6}\frac{1.2.3}{x^4} + (-1)^3\frac{1}{3}\frac{2.3}{(x+3)^4} + (-1)^2\frac{1}{2}\frac{2.3}{(x-2)^4}$$

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$$\therefore y_n = (-1)^n\frac{1}{6}\frac{n!}{x^{n+1}} + (-1)^n\frac{1}{3}\frac{n!}{(x+3)^{n+1}} + (-1)^n\frac{1}{2}\frac{n!}{(x-2)^{n+1}}$$

$$= (-1)^n n!\left\{\frac{1}{6}\frac{1}{x^{n+1}} + \frac{1}{3}\frac{1}{(x+3)^{n+1}} + \frac{1}{2}\frac{1}{(x-2)^{n+1}}\right\}.$$

(Ans.)

Find $\frac{d^2y}{dx^2}$ in the following cases:

(i) If $x = a\cos\theta$, $y = b\sin\theta$

(ii) If $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$

(i) Soln: Given, $x = a\cos\theta$

$$\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta)$$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta \quad \dots (i)$$

Again, $y = b\sin\theta$

$$\Rightarrow \frac{dy}{d\theta} = b\cos\theta \quad \text{—— (ii)}$$

Dividing (ii) by (i), we get,

$$\frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2\theta\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2\theta\left(-\frac{1}{a\sin\theta}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

(Ans.)

(ii) Given, $x = a(\theta+\sin\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(1+\cos\theta) \quad \text{--- (i)}$$

Again, $y = a(1-\cos\theta)$

$$\Rightarrow \frac{dy}{d\theta} = a\sin\theta. \quad \text{--- (ii)}$$

Dividing equation (ii) by equation (i), we get,

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1+\cos\theta)\cos\theta + \sin^2\theta}{(1+\cos\theta)^2} \cdot \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(\cos\theta+1)\cos\theta + (1-\cos^2\theta)}{(1+\cos\theta)^2} \cdot \frac{d\theta}{dx}$$

$$= \frac{(\cos\theta+1)\cos\theta + (1+\cos\theta)(1-\cos\theta)}{(1+\cos\theta)^2} \cdot \frac{d\theta}{dx}$$

$$= \frac{(1+\cos\theta)(\cos\theta+1-\cos\theta)}{(1+\cos\theta)^2} \frac{d\theta}{dx}$$

$$= \frac{1}{1+\cos\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2\cos^2\frac{\theta}{2}} \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2a} \sec^2\frac{\theta}{2} \frac{1}{2\cos^2\frac{\theta}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{4a} \sec^4\frac{\theta}{2}$$

[Ans]

Leibnitz's theorem

Statement: If u and v are two functions of x, each possessing derivatives upto nth order, then the nth derivative of their product, i.e

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + u v_n .$$

where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

Example: If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$, show that

(i) $(1-x^2)y_2 - 3xy_1 - y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$.

Proof: (i) Given, $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$$\Rightarrow y^2 = \frac{(\sin^{-1}x)^2}{(1-x^2)}$$

$$\Rightarrow (1-x^2)y^2 = (\sin^{-1}x)^2$$

$$\Rightarrow (1-x^2)\cdot 2yy_1 + y^2(-2x) = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\,2yy_1 - 2xy^2 = 2y$$

$$\Rightarrow (1-x^2)y_1 - xy = 1 \quad \text{[Dividing both sides by } 2y\text{]}$$

$$\Rightarrow (1-x^2)y_2 - 2xy_1 - xy_1 - y = 0$$

$$\Rightarrow (1-x^2)y_2 - 3xy_1 - y = 0.$$

[showed].

From ①, we get,

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_2 - 3xy_1 - y = 0$$

$$\Rightarrow (1-x^2)y_3 - 2xy_2 - 3xy_2 - 3y_1 - y_1 = 0$$

$$\Rightarrow (1-x^2)y_3 - 5xy_2 - 4y_1 = 0$$

$$\Rightarrow (1-x^2)y_4 - 2xy_3 - 5xy_3 - 5y_2 - 4y_2 = 0$$

$$\Rightarrow (1-x^2)y_4 - 7xy_3 - 9y_2 = 0$$

$$\Rightarrow (1-x^2)y_5 - 2xy_4 - 7xy_4 - 7y_3 - 9y_3 = 0$$

$$\Rightarrow (1-x^2)y_5 - 9xy_4 - 16y_3 = 0$$

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$$\Rightarrow (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$$

[Proved].

Exercise:

① Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$.

② If $y = \sin^3 x$, find y_n.