An analysis of the LSTM's ability to learn and reproduce Chaotic Systems.

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***Abstract*— a-**

I. Introduction

The constantly evolving and diversely applicable field of machine learning has grown exponentially as both the magnitude and accessibility of computing power have improved in recent times. Accompanying this growth, great strides have been made in testing the limits of the various machine learning frameworks. The frameworks have also been used to try and understand chaotic systems, while previous methodologies would suggest a knowledge based approach to model the equations governing these systems, machine learning methods aim to predict future states of the system by relying just on past time series data, providing a model-free method of accurately predicting these systems.

The Long-Short Term Memory (universally abbreviated LSTM) cell is a hidden unit belonging to the family of artificial Recurrent Neural Networks (RNNs) that was designed to be capable of remembering long term dependencies[1] which previous RNNs found difficult to address due to the vanishing or exploding gradients problem that occurs while back-propagating in time during the training of said RNNs. Networks utilizing LSTMs have already demonstrated their capacity to solve problems with long term dependencies as they have been used in analysis of audio[2] and video[3] data, speech recognition[4], and predictions of traffic flow[5].

There have been many attempts at simulating and reproducing the dynamics of the 1963 Lorenz system[6]. Using machine learning schemes to reproduce the Lorenz system is not a novel task in the field, as various machine learning frameworks have been used to tackle this problem, attempts using Reservoir Computing schemes[7–10] have been thoroughly explored with varying degrees of success, especially in the short term. Even the use of LSTMs in modelling the Lorenz has been explored to some extent, [11] explores the effects of precision of the training data in predicting future time steps of the Lorenz system compared to the true values. In this paper we explore the effect of the LSTMs hyper-parameters in predicting the dynamics of the system, we demonstrate each designed networks ability to predict the system by evaluating the errors between the predicted and true values of the attractor, as well as how well the dynamics are modelled by comparing the maximal Lyapunov exponents of a predicted system to that of the literature value.

II. Long Short Term Memory (LSTM)

TABLE I: LSTM Equations

Forward Pass Equations
$f_t = \sigma_g(W_f x_t + U_f h_{t-1} + b_f)$
$i_t = \sigma_g(W_i x_t + U_i h_{t-1} + b_i)$
$o_t = \sigma_g(W_o x_t + U_o h_{t-1} + b_o)$
$\widetilde{c_t} = \sigma_c(W_c x_t + U_c h_{t-1} + b_c)$
$c_t = f_t \circ c_{t-1} + i_t \circ \widetilde{c_t}$
$h_t = \sigma_t \circ \sigma_h(c_t)$
Notation
$x_t \in \mathbb{R}^d$: input vector to LSTM cell
$f_t \in \mathbb{R}^h$: forget gate function
$i_t \in \mathbb{R}^h$: input gate function
$o_t \in \mathbb{R}^h$: output gate function
$h_t \in \mathbb{R}^h$: hidden state of LSTM cell
$\widetilde{c_t} \in \mathbb{R}^h$: cell input function
$c_t \in \mathbb{R}^h$: cell state
$W \in \mathbb{R}^{h \times d}$: weight matrix for input vector$*^0$
$U \in \mathbb{R}^{h \times h}$: weight matrix for hidden state vector$*^0$
$b \in \mathbb{R}^h$: bias vectors
Activation Functions
σ_g : sigmoid function
σ_c : hyperbolic tangent function

* the values d and h refer to the number of input features (in our case 3 (x,y,z) component

III. Data

A. Data Generation

We generate data from the Lorenz attractor defined by the equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{1}$$

Where $\sigma, \rho,$ and β are constants.To generate the data, we make use of the Forward Euler Integration Scheme which works recursively by solving

$$y'(t) = f(t, y(t)) \tag{2}$$
$$y(t_0) = y_0$$
$$t_n = t_0 + nh$$
$$\therefore y_{n+1} = y_n + hf(t_n, y_n) \tag{3}$$

Where h is the **step size** and t_i is the i^{th} **time step**. Identical equations exist for both x and z as well. We begin with the initial value $(x_0, y_0, z_0) = (0,0,0)$ and solve recursively as defined above to obtain $\mathbf{X} = X_0, X_1, ..., X_n$, where $X_i = (x_i, y_i, z_i)$ for $n = 16000$.

B. Training and Testing

Let us define the target vector, $\mathbf{Y} = Y_0, Y_1, ..Y_{n-1}$, where $Y_i = (x_{i+1} - x_i, y_{i+1} - y_i, z_{i+1} - z_i) = X_{i+1} - X_i$. The target vector $\mathbf{Y}$ is the vector we will **train the model to predict**. As such the model will be fed the true $\mathbf{X}$ and true $\mathbf{Y}$ and be asked to predict a $\hat{Y}_i$ for each time step. That is, each X_i will be mapped to a $\hat{Y}_i$ as predicted by the model; $X_i \longmapsto \hat{Y}_i$. The $\hat{Y}_i$ determined by the model are predicted based on a set of weights $\mathbf{W}, \mathbf{U}$ and a set of bias terms $\mathbf{B}$ which are determined by the model while training to minimize the loss function, $\mathbf{L} = (\mathbf{Y} - \hat{\mathbf{Y}})^2$, this loss function is simply the **mean squared error**, where $\hat{\mathbf{Y}} = \hat{Y}_0, \hat{Y}_1, ..., \hat{Y}_{n-1}$. The specific method through which the weights, $\mathbf{W}, \mathbf{U}$ and bias terms, $\mathbf{B}$ are determined is not imperative to understand, and these will change depending on the size and architecture of the model, however **the model itself** will be discussed to some extent in the following section.

IV. Model

A. Network Structure

We have analyzed the Lorenz attractor through many different neural networks. However, the structure of the neural networks remains the same throughout and they all recieved the same data as described in **2.1**. Each neural network is first composed of an LSTM layer composed of a certain number of LSTM units, this is then fed into a dense layer composed of three units, respectively the (x, y, z) components of the attractor, that is then returned to the user by the network. We evaluated networks LSTM units ranging between 8 and 256 units, while the number of epochs were also varied.

V. Results

Each network was evaluated according to how well it was able to reproduce the literature Lyapunov exponent for it's given attractor. The networks were trained with data pertaining to a Lorenz attractor with $\beta = \mathbf{8/2}, \rho = \mathbf{28}, \sigma = \mathbf{10}$, the literature value of the Lyapunov exponent corresponding to these parameters is $\mathbf{0.9056}$.

A. Lyapunov Exponents

The Lyapunov exponents for the various LSTM's were as follows.

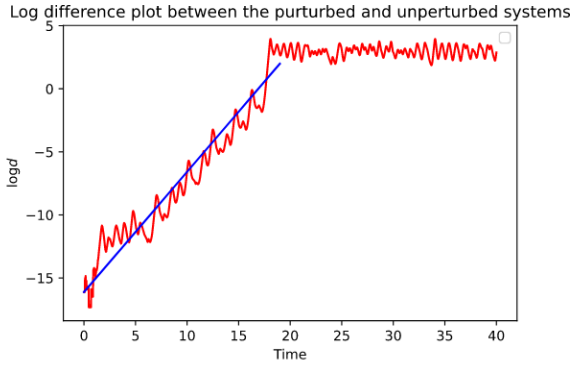


Fig. 1: Lyapunov Plot for Network with 256 LSTM units

1) 256 LSTM Units: The network with 256 LSTM units was only tested under 25 epochs and it performed well. As can be seen the Lyapunov exponent, **blue line**, shares a similar gradient to the plot, thus we can evaluate this network to be succesful.

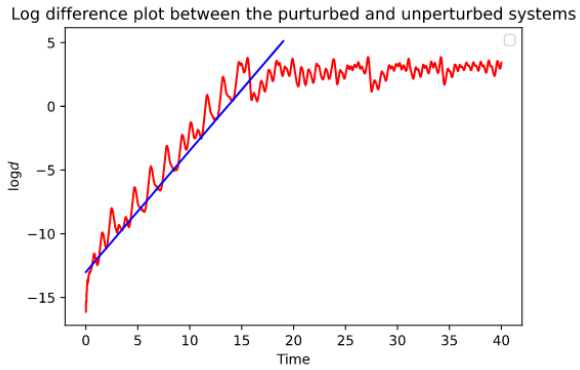


Fig. 2: Lyapunov Plot for Network with 128 LSTM units

2) 128 LSTM units: The network with 128 LSTM units performs similarly to the network with 256 units.

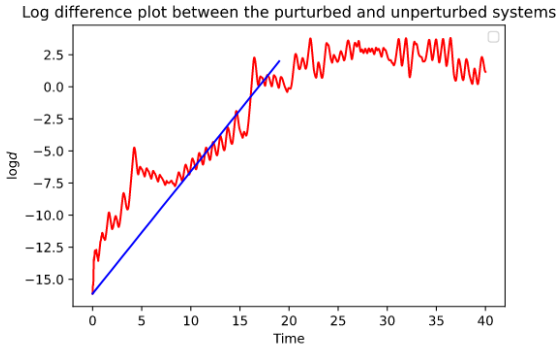
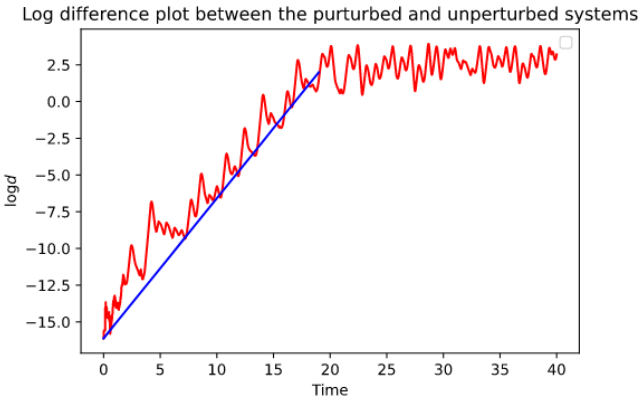
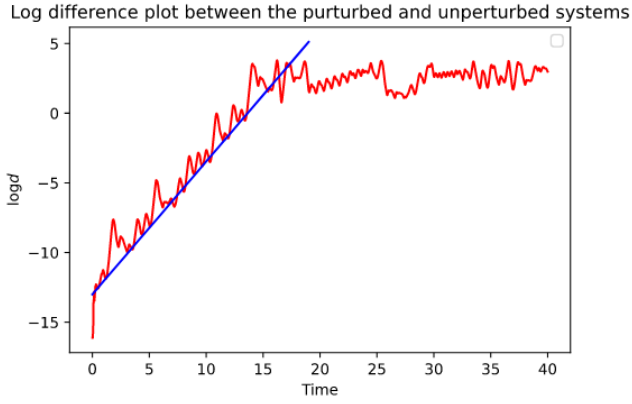


Fig. 3: Lyapunov Plot for Network with 64 LSTM units

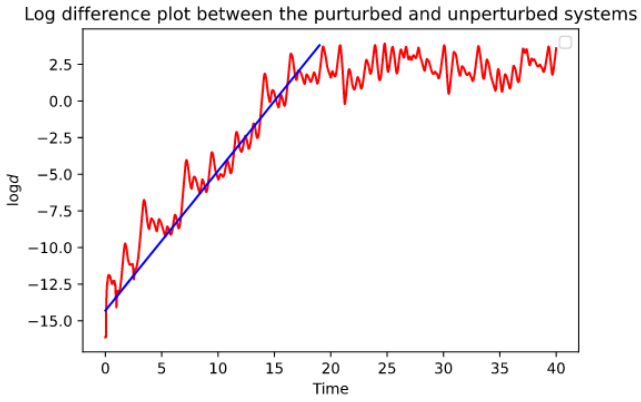
3) 64 LSTM units: We are surprised by the results from the network with 64 LSTM units, as it does not appear to predict the Lyapunov exponent as well as the previous networks or the networks with even few LSTM units as can be seen below.



(a) 25 Epochs



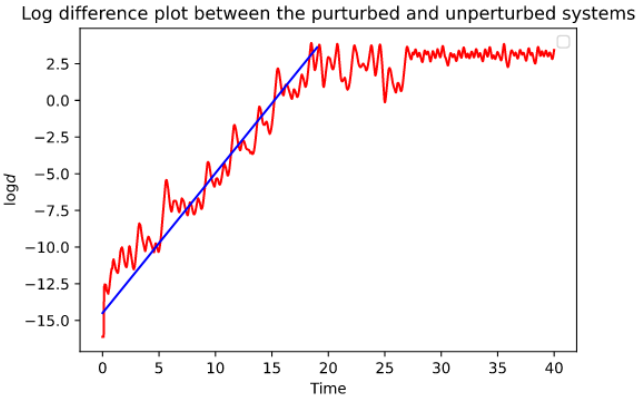
(b) 20 Epochs



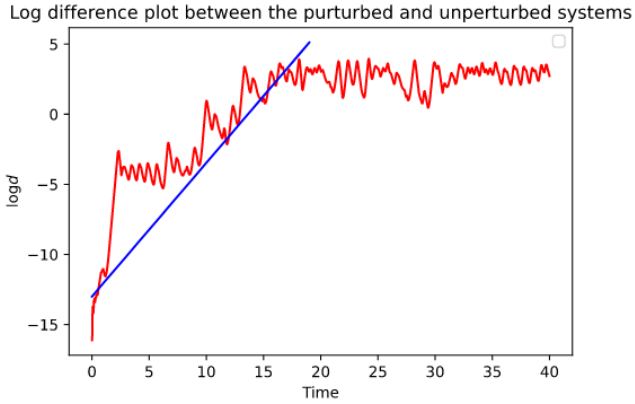
(c) 15 Epochs

Fig. 4: Lyapunov Plots for the Networks with 32 LSTM units

4) 32 LSTM units: The network with 32 LSTM units were tested for 25 epochs as with the previous networks, but the effects of training for 20 and 15 networks was also evaluated. All the networks using 32 LSTM units seem to perform quite well, as they reproduce the literature value for the Lyapunov exponent quite well.



(a) 25 Epochs



(b) 20 Epochs

Fig. 5: Lyapunov Plots for the Networks with 16 LSTM units

5) 16 LSTM units: The networks with 16 LSTM units finally show breakdown, while the network trained for 25 epochs still reproduces the literature Lyapunov exponent, the network trained for 20 epochs breaks down.

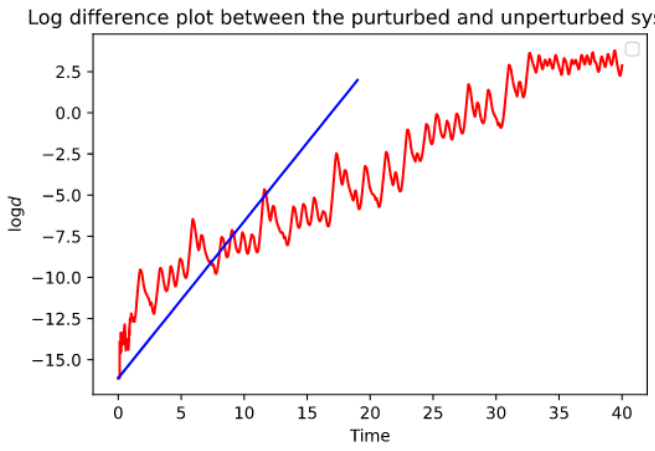


Fig. 6: Lyapunov Plot for Network with 8 LSTM units

6) 8 LSTM units: The system with 8 LSTM units finally breaks down completely when trained for 25 epochs

References

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