TABLE 6.3.1

Fully Discrete Annual Benefit Premiums

Plan	Loss Components: $b_{K+1}v_{K+1}$	Loss Components: $P\,Y$ Where Y Is	Premium Formula $P = \dfrac{E[b_{K+1}v_{K+1}]}{E[Y]}$
Whole life insurance	$1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, 2, \ldots$	$P_x = \dfrac{A_x}{\ddot{a}_x}$
n-Year term insurance	$1\ v^{K+1}$ 0	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\mid}$, $K = n, n+1, \ldots$	$P^1_{x:\overline{n}\mid} = \dfrac{A^1_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{n}\mid}}$
n-Year endowment insurance	$1\ v^{K+1}$ $1\ v^n$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\mid}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\mid} = \dfrac{A_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{n}\mid}}$
h-Payment whole life insurance	$1\ v^{K+1}$ $1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\mid}$, $K = h, h+1, \ldots$	${}_hP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\mid}}$
h-Payment, n-year endowment insurance	$1\ v^{K+1}$ $1\ v^{K+1}$ $1\ v^n$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\mid}$, $K = h, \ldots, n-1$ $\ddot{a}_{\overline{h}\mid}$, $K = n, n+1, \ldots$	${}_hP_{x:\overline{n}\mid} = \dfrac{A_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{h}\mid}}$
n-Year pure endowment	0 $1\ v^n$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\mid}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\mid}^{\ \ 1} = \dfrac{A_{x:\overline{n}\mid}^{\ \ 1}}{\ddot{a}_{x:\overline{n}\mid}}$
n-Year deferred whole life annuity	0 $\ddot{a}_{\overline{K+1-n}\mid}v^n$	$\ddot{a}_{\overline{K+1}\mid}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\mid}$, $K = n, n+1, \ldots$	$P({}_{n\mid}\ddot{a}_x) = \dfrac{A_{x:\overline{n}\mid}^{\ \ 1}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\mid}}$

$$\mathrm{Var}(L) = \left(1 + \frac{P_{x:\overline{n}\mid}}{d}\right)^2 [{}^2A_{x:\overline{n}\mid} - (A_{x:\overline{n}\mid})^2].$$

Formula (5.3.13) and the entry from the third row of Table 6.3.1 can be combined as follows:

$$d\,\ddot{a}_{x:\overline{n}\mid} + A_{x:\overline{n}\mid} = 1,$$

$$1 + \frac{P_{x:\overline{n}\mid}}{d} = \frac{1}{d\,\ddot{a}_{x:\overline{n}\mid}}.$$

Therefore, the variance we seek is

$$\frac{{}^2A_{x:\overline{n}\mid} - (A_{x:\overline{n}\mid})^2}{(d\,\ddot{a}_{x:\overline{n}\mid})^2}. \tag{6.3.5}$$

▼

Example 6.3.3

Consider a 10,000 fully discrete whole life insurance. Let π denote an annual premium for this policy and $L(\pi)$ denote the loss-at-issue random variable for one such policy on the basis of the Illustrative Life Table, 6% interest and issue age 35.

a. Determine the premium, π_a, such that the distribution of $L(\pi_a)$ has mean 0. Calculate the variance of $L(\pi_a)$.
b. Approximate the smallest non-negative premium, π_b, such that the probability is less than 0.5 that the loss $L(\pi_b)$ is positive. Find the variance of $L(\pi_b)$.
c. Determine the premium, π_c, such that the probability of a positive total loss on 100 such independent policies is 0.05 by the normal approximation.

Solution:

a. By the equivalence principle, (6.1.3),

$$\pi_a = 10{,}000\, P_{35} = 10{,}000 \frac{A_{35}}{\ddot{a}_{35}} = \frac{1287.194}{15.39262} = 83.62.$$

From (6.3.3)

$$\mathrm{Var}[L(\pi_a)] = (10{,}000)^2 \frac{{}^2A_{35} - (A_{35})^2}{(d\,\ddot{a}_{35})^2} = 10^8 \frac{0.0348843 - (0.1287194)^2}{[(0.06/1.06)(15.39262)]^2} = \frac{1{,}831{,}562}{0.7591295} = 2{,}412{,}713.$$

b. We want π_b such that

$$\Pr[L(\pi_b) > 0] < 0.5,$$

or in terms of curtate-future-lifetime, K,

$$\Pr(10{,}000 v^{K+1} - \pi_b\, \ddot{a}_{\overline{K+1}|} > 0) < 0.5.$$

From the Illustrative Life Table, ${}_{42}p_{35} = 0.5125101$ and ${}_{43}p_{35} = 0.4808964$. Therefore, if π_b is chosen so that

$$10{,}000 v^{43} - \pi_b\, \ddot{a}_{\overline{43}|} = 0,$$

then $\Pr[L(\pi_b) > 0)] = \Pr(K < 42) < 0.5$. Thus,

$$\pi_b = \frac{10{,}000}{\ddot{s}_{\overline{43}|}} = 50.31.$$

Using the fully discrete analogue of (6.2.6) we can write

$$\begin{aligned}\mathrm{Var}[L(\pi_b)] &= (10{,}000)^2[{}^2A_{35} - (A_{35})^2]\left(1 + \frac{\pi_b}{10{,}000}\frac{1}{d}\right)^2\\ &= (1{,}831{,}562)(1.18567)\\ &= 2{,}171{,}630.\end{aligned}$$

c. With a premium π_c, the loss on one policy is

$$L(\pi_c) = 10{,}000v^{K+1} - \pi_c\ddot{a}_{\overline{K+1}|} = \left(10{,}000 + \frac{\pi_c}{d}\right)v^{K+1} - \frac{\pi_c}{d},$$

and its expectation and variance are as follows:

$$\begin{aligned}\mathrm{E}[L(\pi_c)] &= \left(10{,}000 + \frac{\pi_c}{d}\right)A_{35} - \frac{\pi_c}{d}\\ &= (0.1287194)\left(10{,}000 + \frac{\pi_c}{d}\right) - \frac{\pi_c}{d}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}[L(\pi_c)] &= \left(10{,}000 + \frac{\pi_c}{d}\right)^2[{}^2A_{35} - (A_{35})^2]\\ &= \left(10{,}000 + \frac{\pi_c}{d}\right)^2(0.01831562).\end{aligned}$$

For each of 100 such policies each loss $L_i(\pi_c)$ is distributed like $L(\pi_c)$, $i = 1, 2, \ldots, 100$ and

$$S = \sum_{i=1}^{100} L_i(\pi_c)$$

for the total loss on the portfolio. Then

$$\mathrm{E}[S] = 100\ \mathrm{E}[L(\pi_c)],$$

and, using the assumption of independent policies,

$$\mathrm{Var}(S) = 100\ \mathrm{Var}[L(\pi_c)].$$

To determine π_c so that $\Pr(S > 0) = 0.05$ by the normal approximation, we want

$$\frac{0 - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} = 1.645,$$

$$10\left\{\frac{-\mathrm{E}[L(\pi_c)]}{\sqrt{\mathrm{Var}[L(\pi_c)]}}\right\} = 1.645,$$

$$10\left[\frac{-A_{35}[10{,}000 + (\pi_c/d)] + (\pi_c/d)}{[10{,}000 + (\pi_c/d)]\sqrt{{}^2A_{35} - (A_{35})^2}}\right] = 1.645.$$

Thus

$$\pi_c = 10{,}000\,d\left[\frac{(0.1645)\sqrt{{}^2A_{35} - (A_{35})^2} + A_{35}}{1 - (A_{35} + 0.1645\sqrt{{}^2A_{35} - (A_{35})^2})}\right]$$

$$= 100.66.$$

▼

The two identities, (5.3.7) and (5.3.13), can be used to derive relationships among discrete premiums. For example, starting with (5.3.7), we have for whole life insurances

$$d\ddot{a}_x + A_x = 1,$$

$$d + P_x = \frac{1}{\ddot{a}_x},$$

$$P_x = \frac{1}{\ddot{a}_x} - d$$

$$= \frac{1 - d\ddot{a}_x}{\ddot{a}_x}$$

$$= \frac{dA_x}{1 - A_x}. \tag{6.3.6}$$

Starting with (5.3.13) we obtain a similar chain of equalities for n-year endowment insurances:

$$d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|} = 1,$$

$$d + P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}},$$

$$P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d$$

$$= \frac{1 - d\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

$$= \frac{dA_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}. \tag{6.3.7}$$

Example 6.3.4

Give interpretations in words of the following equations from the (6.3.6) set:

$$\frac{1}{\ddot{a}_x} = P_x + d \tag{6.3.8}$$

and

$$P_x = \frac{dA_x}{1 - A_x}. \tag{6.3.9}$$

Solution:

We will use the word equivalent to mean equal in terms of actuarial present value. For (6.3.8), first note that a unit now is equivalent to a life annuity of $\ddot{a}_x^{-1}$ payable at the beginning of each year while (x) survives. A unit now is also equivalent to interest-in-advance of d at the beginning of each year while (x) survives with the repayment of the unit at the end of the year of (x)'s death; that is, $1 = \ddot{a}_x/\ddot{a}_x = d\ddot{a}_x + A_x$. The repayment of the unit at the end of the year of death is, in turn, equivalent to a life annuity-due of P_x while (x) survives. Therefore, the unit now is equivalent to $P_x + d$ at the beginning of each year during the lifetime of (x). Then $\ddot{a}_x^{-1} = P_x + d$, for each side of the equality, represents the annual payment of a life annuity produced by a unit available now.

For (6.3.9), we consider an insured (x) who borrows the single benefit premium A_x for the purchase of a single-premium unit whole life insurance. The insured agrees to pay interest-in-advance in the amount of dA_x on the loan at the beginning of each year during survival and to repay the A_x from the unit death benefit at the end of the year of death. In essence, the insured is paying an annual benefit premium of dA_x for an insurance of amount $1 - A_x$. Then for a full unit of insurance, the annual benefit premium must be $dA_x/(1 - A_x)$. ▼

Similar interpretations exist for corresponding relationships involving endowment insurances as given in the second and fifth equalities in the (6.3.7) set. There is an analogy between (6.3.8), the corresponding formula involving endowment insurances,

$$\ddot{a}_{x:\overline{n}|}^{-1} = P_{x:\overline{n}|} + d,$$

and the interest-only formula

$$\ddot{a}_{\overline{n}|}^{-1} = \ddot{s}_{\overline{n}|}^{-1} + d.$$

Example 6.3.5

Prove and interpret the formula

$$P_{x:\overline{n}|} = {}_nP_x + P_{x:\overline{n}|}^{\;\;1}(1 - A_{x+n}). \tag{6.3.10}$$

Solution:

The proof is completed using entries from Table 6.3.1:

$$P_{x:\overline{n}|}\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|} = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{\;\;1},$$

$${}_nP_x\ddot{a}_{x:\overline{n}|} = A_x = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{\;\;1}A_{x+n}.$$

By subtraction,

$$(P_{x:\overline{n}|} - {}_nP_x)\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|}^{\;\;1}(1 - A_{x+n}),$$

from which (6.3.10) follows.

The interpretation is that both $P_{x:\overline{n}|}$ and ${}_nP_x$ are payable during the survival of (x) to a maximum of n years. During these years, both insurances provide a death benefit of 1 payable at the end of the year of the death of (x). If (x) survives the n years, $P_{x:\overline{n}|}$ provides a maturity benefit of 1, while ${}_nP_x$ provides whole life insurance without further premiums, that is, an insurance with an actuarial present value of A_{x+n}. Hence, the difference $P_{x:\overline{n}|} - {}_nP_x$ is the level annual premium for a pure endowment of $1 - A_{x+n}$. ▼

In practice, life insurances are payable soon after death rather than at the end of the policy year of death, so there is a need for annual payment, semicontinuous benefit premiums. Such premiums, following the same order used in Tables 6.2.1 and 6.3.1, are denoted by $P(\bar{A}_x)$, $P(\bar{A}^{\,1}_{x:\overline{n}|})$, $P(\bar{A}_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$. There is no need for a semicontinuous annual premium n-year pure endowment since no death benefit is involved. The equivalence principle can be applied to produce formulas like those in Table 6.3.1, but with the general symbol A replaced by $\bar{A}$. For example,

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{6.3.11}$$

We observe that the notation for this premium is not $\bar{P}_x$, the annual premium payable continuously for a unit whole life insurance benefit payable at the end of the year of death and equal to $A_x/\bar{a}_x$. If a uniform distribution of deaths is assumed over each year of age, we can use the notations of Section 4.4 to write

$$P(\bar{A}_x) = \frac{i}{\delta}\frac{A_x}{\ddot{a}_x} = \frac{i}{\delta}P_x,$$

$$P(\bar{A}^{\,1}_{x:\overline{n}|}) = \frac{i}{\delta}P^{\,1}_{x:\overline{n}|},$$

and

$$P(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}P^{\,1}_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ \,1}. \tag{6.3.12}$$

6.4 True *m*-thly Payment Premiums

If premiums are payable m times a policy year, rather than annually, with no adjustment in the death benefit, the resulting premiums are called ***true fractional premiums.*** Thus $P_x^{(m)}$ denotes the ***true level annual benefit premium,*** payable in m-thly installments at the beginning of each m-thly period, for a unit whole life insurance payable at the end of the year of death. The symbol $P^{(m)}(\bar{A}_x)$ would have the same interpretation except that the insurance is payable at the moment of death. Typically, m is 2, 4, or 12.

The development in this section stresses the payment of insurance benefits at the end of the policy year of death. Table 6.4.1 specifies the symbols and formulas for true fractional premiums for common life insurances. The premium formulas can be obtained by applying the equivalence principle.

TABLE 6.4.1

True Fractional Benefit Premiums*

Plan	Payment of Proceeds: At End of Policy Year	Payment of Proceeds: At Moment of Death
Whole life insurance	$P_x^{(m)} = \dfrac{A_x}{\ddot{a}_x^{(m)}}$	$P^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_x^{(m)}}$
n-Year term insurance	$P_{x:\overline{n}\vert}^{1\,(m)} = \dfrac{A_{x:\overline{n}\vert}^{1}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$	$P^{(m)}(\bar{A}_{x:\overline{n}\vert}^{1}) = \dfrac{\bar{A}_{x:\overline{n}\vert}^{1}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$
n-Year endowment insurance	$P_{x:\overline{n}\vert}^{(m)} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$	$P^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$
h-Payment years, whole life insurance	${}_hP_x^{(m)} = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$	${}_hP^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$
h-Payment years, n-year endowment insurance	${}_hP_{x:\overline{n}\vert}^{(m)} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$	${}_hP^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$

*The actual amount of each fractional premium, payable m times each policy year, during the premium paying period and the survival of (x), is $P^{(m)}/m$. Note that here h refers to the number of payment years, not to the number of payments.

In some applications it is useful to write the m-thly payment premium as a multiple of the annual premium. This will be illustrated for ${}_hP_{x:\overline{n}\vert}^{(m)}$, the premium for a rather general insurance. The resulting formula can be modified to produce premium formulas for other common insurances. From the last row of Table 6.4.1 we have

$$ {}_hP_{x:\overline{n}\vert}^{(m)} = \frac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}. \tag{6.4.1} $$

Since

$$ A_{x:\overline{n}\vert} = {}_hP_{x:\overline{n}\vert}\, \ddot{a}_{x:\overline{h}\vert}, $$

(6.4.1) can be rearranged as

$$ {}_hP_{x:\overline{n}\vert}^{(m)} = \frac{{}_hP_{x:\overline{n}\vert}\, \ddot{a}_{x:\overline{h}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}. \tag{6.4.2} $$

Formula (6.4.2) is used in the next chapter; it expresses the m-thly payment premium as equal to the corresponding annual payment premium times a ratio of annuity values. This ratio can be arranged in various ways each corresponding to a different formula used to express the relationship between $\ddot{a}_{x:\overline{h}\vert}^{(m)}$ and $\ddot{a}_{x:\overline{h}\vert}$ (see Exercise 6.14).

Example 6.4.1

a. Calculate the level annual benefit premium payable in semiannual installments for a 10,000, 20-year endowment insurance with proceeds paid at the end of the policy year of death (discrete) issued to (50), on the basis of the Illustrative Life Table with interest at the effective annual rate of 6%.
b. Determine the corresponding premium with proceeds paid at the moment of death (semicontinuous).

For both parts, assume a uniform distribution of deaths in each year of age.

Solution:

a. We require 10,000 $P^{(2)}_{50:\overline{20}|}$. As preliminary steps we calculate

$$d = 0.056603774,$$

$$i^{(2)} = 0.059126028,$$

$$d^{(2)} = 0.057428275,$$

$$\ddot{a}^{(2)}_{\overline{1}|} = 0.98564294,$$

$$s^{(2)}_{\overline{1}|} = 1.01478151,$$

$$\alpha(2) = s^{(2)}_{\overline{1}|}\ddot{a}^{(2)}_{\overline{1}|} = 1.0002122,$$

$$\beta(2) = \frac{s^{(2)}_{\overline{1}|} - 1}{d^{(2)}} = 0.25739081,$$

and the following actuarial present values:

$$\ddot{a}_{50:\overline{20}|} = 11.291832,$$

$$A^{\;1}_{50:\overline{20}|} = 0.13036536,$$

$$P^{\;1}_{50:\overline{20}|} = 0.01154510,$$

$${}_{20}E_{50} = 0.23047353,$$

$$A_{50:\overline{20}|} = 0.36083889,$$

$$P_{50:\overline{20}|} = 0.03195574.$$

Then, under the assumption of a uniform distribution of deaths for each year of age, the required premium can be calculated by use of (6.4.1), with $x = 50$, $n = 20$, $h = 20$, and $m = 2$. For this purpose, we calculate

$$\ddot{a}^{(2)}_{50:\overline{20}|} = \alpha(2)\ddot{a}_{50:\overline{20}|} - \beta(2)(1 - {}_{20}E_{50}) = 11.096159,$$

and then

$$10{,}000\,P^{(2)}_{50:\overline{20}|} = 325.19.$$

b. The corresponding semicontinuous premium can be obtained by multiplying the values in (a) by the ratio

$$\frac{P(\bar{A}_{50:\overline{20}|})}{P_{50:\overline{20}|}} = \frac{\bar{A}_{50:\overline{20}|}}{A_{50:\overline{20}|}}.$$

Under the uniform distribution of deaths assumption this ratio is

$$\frac{(i/\delta)\,P^{\;1}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\quad\;\;1}}{P_{50:\overline{20}|}}, \tag{6.4.3}$$

and the result is

$$10{,}000\,P^{(2)}(\bar{A}_{50:\overline{20}|}) = 328.68.$$

▼

6.5 Apportionable Premiums

A second type of fractional premium is the ***apportionable premium.*** Here, at death, a refund is made of a portion of the premium related to the length of time between the time of death and the time of the next scheduled premium payment. In practice this may be on a pro rata basis without interest. In this section we consider interest and view the sequence of m-thly premiums as an apportionable life annuity-due in the sense of Section 5.5. The symbols used to denote these level apportionable annual benefit premiums payable m-thly are like the symbols for true fractional premiums on the semicontinuous basis. They differ in that the superscript m is enclosed in braces rather than parentheses, for example, $P^{\{m\}}(\bar{A}_x)$. In view of the premium refund feature, it is natural to assume that the death benefit is payable at the moment of death.

Again, we use an h-payment years, n-year endowment insurance to illustrate the development of formulas for apportionable premiums paid m-thly. The equivalence principle leads to the formulas

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\,\ddot{a}^{\{m\}}_{x:\overline{h}|} = \bar{A}_{x:\overline{n}|}$$

and

$${}_hP^{\{m\}}\,(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{\{m\}}_{x:\overline{h}|}}. \tag{6.5.1}$$

Utilizing the temporary annuity version of (5.5.4), we obtain

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{(\delta/d^{(m)})\bar{a}_{x:\overline{h}|}} = \frac{d^{(m)}}{\delta}\,{}_h\bar{P}(\bar{A}_{x:\overline{n}|}). \tag{6.5.2}$$

This implies that the m-thly installment is

$$\frac{1}{m}\,{}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\frac{1 - v^{1/m}}{\delta} = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{1/m}|}, \tag{6.5.3}$$

and in particular, for $m = 1$,

$${}_hP^{\{1\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{1}|}. \tag{6.5.4}$$

Formulas (6.5.3) and (6.5.4) demonstrate that these apportionable premiums are equivalent to fully continuous premiums, discounted for interest to the start of each payment period. Similar formulas exist for other types of insurance. For example, by letting h and $n \to \infty$, (6.5.4) becomes

$$P^{\{1\}}(\bar{A}_x) = \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{1}|}. \tag{6.5.5}$$

The apportionable benefit premium $P^{\{1\}}(\bar{A}_x)$ and the semicontinuous benefit premium $P(\bar{A}_x)$ are both payable annually at the beginning of each year while (x) survives. Each insurance provides a unit at the death of (x). The two insurances differ only in respect to the refund provided by $P^{\{1\}}(\bar{A}_x)$. Thus, the difference

$$P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x) \tag{6.5.6}$$

is a level annual benefit premium paid at the beginning of each year for the refund-of-premium feature. We verify this assertion about the expression in (6.5.6) in the following analysis.

From (5.5.1), we note that the random variable for the present value of the refund-of-premium feature is

$$\frac{P^{\{1\}}(\bar{A}_x)\, v^T\, \bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

where K and T are defined as in Chapter 3. The actuarial present value for this feature is

$$\bar{A}_x^{PR} = P^{\{1\}}(\bar{A}_x)\, \mathrm{E}\left[v^T \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}} \right].$$

Using (6.5.5) we obtain

$$\begin{aligned} \bar{A}_x^{PR} &= \bar{P}(\bar{A}_x)\, \mathrm{E}\left[\frac{v^T - v^{K+1}}{\delta} \right] \\ &= \bar{P}(\bar{A}_x) \left(\frac{\bar{A}_x - A_x}{\delta} \right). \end{aligned} \tag{6.5.7}$$

The level annual benefit premium is then, by the equivalence principle,

$$P(\bar{A}_x^{PR}) = \frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta \ddot{a}_x}. \tag{6.5.8}$$

Formula (6.5.7) has the following interpretation: The actuarial present value of the refund feature is the difference between the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ per year beginning at the death of (x), and the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ payable from the end of the year of death of (x).

We return now to (6.5.6) where, by (6.5.5), we have

$$\begin{aligned}
P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x) &= \bar{P}(\bar{A}_x)\frac{d}{\delta} - \frac{\bar{A}_x}{\ddot{a}_x} \\
&= \bar{P}(\bar{A}_x)\left(\frac{d}{\delta} - \frac{\bar{a}_x}{\ddot{a}_x}\right) \\
&= \bar{P}(\bar{A}_x)\frac{d\ddot{a}_x - \delta\bar{a}_x}{\delta\ddot{a}_x} \\
&= \bar{P}(\bar{A}_x)\frac{\bar{A}_x - A_x}{\delta\ddot{a}_x} \\
&= P(\bar{A}_x^{PR}),
\end{aligned} \tag{6.5.9}$$

as obtained in (6.5.8). This confirms our assertion about (6.5.6).

This analysis can be extended to m-thly payment premiums and to other life insurance in addition to whole life. In general,

$$P^{\{m\}}(\bar{A}) - P^{(m)}(\bar{A})$$

is an m-thly payment premium for the refund feature.

Example 6.5.1

If the policy of Example 6.4.1(b) is to have apportionable premiums, what increase occurs in the annual benefit premium?

Solution:

The apportionable annual benefit premium per unit of insurance is given by (6.5.2),

$$P^{\{2\}}(\bar{A}_{50:\overline{20}|}) = \bar{P}(\bar{A}_{50:\overline{20}|})\frac{d^{(2)}}{\delta} = \frac{\bar{A}_{50:\overline{20}|}}{\bar{a}_{50:\overline{20}|}}\frac{d^{(2)}}{\delta}.$$

Under the assumption of a uniform distribution of deaths in each age interval, this becomes

$$= \frac{(i/\delta)A^{1}_{50:\overline{20}|} + A_{50:\overset{\ 1}{\overline{20}|}}}{\alpha(\infty)\ddot{a}_{50:\overline{20}|} - \beta(\infty)(1 - {}_{20}E_{50})}\frac{d^{(2)}}{\delta}$$

$$= \frac{(i/\delta)\,P^{1}_{50:\overline{20}|} + P_{50:\overset{\ 1}{\overline{20}|}}}{\alpha(\infty) - \beta(\infty)(P^{1}_{50:\overline{20}|} + d)}\frac{d^{(2)}}{\delta}.$$

Here $\alpha(\infty) = \bar{s}_{\overline{1}|}\ddot{a}_{\overline{1}|} = id/\delta^2 = 1.00028$, $\beta(\infty) = (\bar{s}_{\overline{1}|} - 1)/\delta = 0.50985$. Using other values available in Example 6.4.1 we find

$$10{,}000\ P^{\{2\}}(\bar{A}_{50:\overline{20}|}) = 329.69.$$

Then the increase in annual premium is

$$10{,}000[P^{\{2\}}(\bar{A}_{50:\overline{20}|}) - P^{(2)}(\bar{A}_{50:\overline{20}|})] = 1.01,$$

which is the annual benefit premium payable semiannually for the refund feature. ▼

6.6 Accumulation-Type Benefits

The analysis in this section is in terms of annual premiums for insurances payable at the end of the year of death. An analogous development is possible for fully continuous premiums and, with some adjustment, for semicontinuous premiums. We first seek the actuarial present value for an n-year term insurance on (x) for which the sum insured, in case death occurs in year $k+1$, is $\ddot{s}_{\overline{k+1}|j}$. The present-value random variable of this benefit, at policy issue, is

$$W = \begin{cases} v^{K+1}\ddot{s}_{\overline{K+1}|j} = \dfrac{1}{d_{(j)}}\,[v^{K+1}(1+j)^{K+1} - v^{K+1}] & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

where the insurer's present values are computed at interest rate i and $d_{(j)}$ is the discount rate equivalent to interest rate j. The actuarial present value is

$$\mathrm{E}[W] = \frac{A'^{\,1}_{x:\overline{n}|} - A^{\,1}_{x:\overline{n}|}}{d_{(j)}} \tag{6.6.1}$$

where $A'^{\,1}_{x:\overline{n}|}$ is calculated at the rate of interest $i' = (i-j)/(1+j)$.

If $i = j$, then $i' = 0$, and the actuarial present value is

$$\begin{aligned} \frac{{}_nq_x - A^{\,1}_{x:\overline{n}|}}{d} &= \frac{1 - {}_np_x - A_{x:\overline{n}|} + v^n\,{}_np_x}{d} \\ &= \ddot{a}_{x:\overline{n}|} - {}_np_x\,\ddot{a}_{\overline{n}|} \\ &= \ddot{a}_{x:\overline{n}|} - {}_nE_x\,\ddot{s}_{\overline{n}|}. \end{aligned} \tag{6.6.2}$$

Formula (6.6.2) indicates that, when $j = i$, this special term insurance is equivalent to an n-year life annuity-due except for the event that (x) survives the n years. Then the term insurance would provide a benefit of zero, whereas the life annuity payments, given survival for n years, would have value $\ddot{s}_{\overline{n}|}$ at time n.

Now let us consider the situation where (x) has the choice of purchasing an n-year unit endowment insurance with an annual premium of $P_{x:\overline{n}|}$ or of establishing a savings fund with deposits of $1/\ddot{s}_{\overline{n}|}$ at the beginning of each of n years and purchasing a special decreasing term insurance. The special insurance will provide, in the event of death in year $k+1$, the difference,

$$1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} \qquad k = 0, 1, 2, \ldots, n-1,$$

between the unit benefit under the endowment insurance and the accumulation in the savings fund. We suppose further that the same interest rate i is applicable in valuing all these transactions. The same benefits are provided by the endowment insurance and by the combination of the special term insurance and the savings fund. Therefore one would anticipate that

(the annual benefit premium $P_{x:\overline{n|}}$ for the endowment insurance) = (the annual benefit premium for the special term insurance) + (the annual savings fund deposit $1/\ddot{s}_{\overline{n|}}$).

To verify this conjecture, we consider the present-value random variable for the special decreasing term insurance,

$$\tilde{W} = \begin{cases} v^{K+1}\left(1 - \dfrac{\ddot{s}_{\overline{K+1|}}}{\ddot{s}_{\overline{n|}}}\right) = v^{K+1} - \dfrac{\ddot{a}_{\overline{K+1|}}}{\ddot{s}_{\overline{n|}}} & 0 \le K < n \\ 0 & K \ge n. \end{cases} \tag{6.6.3}$$

The actuarial present value of $\tilde{W}$ is denoted by $\tilde{A}^{\,1}_{x:\overline{n|}}$ and given by

$$\begin{aligned} \tilde{A}^{\,1}_{x:\overline{n|}} &= \mathrm{E}[\tilde{W}] \\ &= A^{\,1}_{x:\overline{n|}} - \frac{\ddot{a}_{x:\overline{n|}} - {}_np_x\, \ddot{a}_{\overline{n|}}}{\ddot{s}_{\overline{n|}}} \\ &= A^{\,1}_{x:\overline{n|}} - \frac{\ddot{a}_{x:\overline{n|}} - {}_nE_x\, \ddot{s}_{\overline{n|}}}{\ddot{s}_{\overline{n|}}} \end{aligned}$$

[see (6.6.2)].

The annual benefit premium for the special term insurance is therefore

$$\begin{aligned} \tilde{P}^{\,1}_{x:\overline{n|}} = \frac{\tilde{A}^{\,1}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} &= P^{\,1}_{x:\overline{n|}} - \frac{1}{\ddot{s}_{\overline{n|}}} + P_{x:\overline{n|}}^{\;\;\,1} \\ &= P_{x:\overline{n|}} - \frac{1}{\ddot{s}_{\overline{n|}}}, \end{aligned}$$

and then

$$P_{x:\overline{n|}} = \tilde{P}^{\,1}_{x:\overline{n|}} + \frac{1}{\ddot{s}_{\overline{n|}}}. \tag{6.6.4}$$

We have already seen that

$$P_{x:\overline{n|}} = P^{\,1}_{x:\overline{n|}} + P_{x:\overline{n|}}^{\;\;\,1},$$

and now (6.6.4) provides an alternative decomposition of $P_{x:\overline{n|}}$. The components are the annual premium for the special term insurance and the annual savings fund deposits, $1/\ddot{s}_{\overline{n|}}$, which accumulate to one at the end of n years.

Example 6.6.1

Derive formulas for the annual benefit premium for a 5,000, 20-year term insurance on (x) providing, in case of death within the 20 years, the return of the annual benefit premiums paid:

a. Without interest
b. Accumulated at the interest rate used in the determination of premiums.
   In each case, the return of premiums is in addition to the 5,000 sum insured and benefit payments are made at the end of the year of death.

Solution:

a. Let π_a be the benefit premium. Then

$$\pi_a\, \ddot{a}_{x:\overline{20}|} = 5{,}000\, A^{\ 1}_{x:\overline{20}|} + \pi_a (IA)^{\ 1}_{x:\overline{20}|}$$

and

$$\pi_a = 5{,}000\, \frac{A^{\ 1}_{x:\overline{20}|}}{\ddot{a}_{x:\overline{20}|} - (IA)^{\ 1}_{x:\overline{20}|}}.$$

b. Let π_b be the benefit premium. We use (6.6.2) to obtain

$$\pi_b\, \ddot{a}_{x:\overline{20}|} = 5{,}000\, A^{\ 1}_{x:\overline{20}|} + \pi_b(\ddot{a}_{x:\overline{20}|} - {}_{20}E_x\, \ddot{s}_{\overline{20}|}),$$

$$\pi_b\, {}_{20}E_x \ddot{s}_{\overline{20}|} = 5{,}000\, A^{\ 1}_{x:\overline{20}|},$$

$$\pi_b = 5{,}000\, \frac{A^{\ 1}_{x:\overline{20}|}}{{}_{20}E_x\, \ddot{s}_{\overline{20}|}}$$

$$= 5{,}000\, \frac{A^{\ 1}_{x:\overline{20}|}}{{}_{20}p_x\, \ddot{a}_{\overline{20}|}}.$$

In practice, annual contract premiums would be refunded, and the formulas would take this into account. ▼

Example 6.6.2

A deferred annuity issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death during the premium paying period is the return of annual benefit premiums accumulated with interest at the rate used for the premium. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium.

Solution:

Equating the actuarial present value of the annual benefit premiums, π, to the actuarial present value of the benefits, we have

$$\pi \ddot{a}_{x:\overline{n}|} = {}_nE_x\, \ddot{a}_{x+n} + \pi(\ddot{a}_{x:\overline{n}|} - {}_nE_x\, \ddot{s}_{\overline{n}|})$$

where the second term on the right-hand side comes from (6.6.2). Solving for π yields

$$\pi = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

▼

6.7 Notes and References

Lukacs (1948) provides a survey of the development of the equivalence principle. Premiums derived by an application of the equivalence principle are often called actuarial premiums in the literature of the economics of uncertainty. Gerber (1976, 1979) discussed exponential premiums and reserves; these were illustrated in Example 6.1.1 under Principle III. Fractional premiums of various types are important in practice. Scher (1974) has discussed developments in this area, namely, the relations among fully continuous, apportionable, and semicontinuous premiums. The decomposition of an endowment insurance premium appeared in a paper by Linton (1919).

Exercises

Section 6.1

6.1. Calculate the expectation and the variance of the present value of the financial loss for the insurance in Example 6.1.1, when the premium is determined by Principle I.

6.2. Verify that the exponential premium (with $\alpha = 0.1$) for the insurance in Example 6.1.1, modified so that the benefit amount is 10, is 3.45917. (Note that this is roughly 11.3 times as large as the exponential premium for a benefit amount of 1 found in Example 6.1.1.)

6.3. Using the assumptions of Example 6.1.1, determine the annual premium that maximizes the expected utility of an insurer with initial wealth $w = 10$ and utility function $u(x) = x - 0.01x^2$, $x < 50$. [Hint: Use (1.3.6), $w - 0.01w^2 = \mathrm{E}[(w - L) - 0.01(w - L)^2]$.]

Section 6.2

6.4. A fully continuous whole life insurance with unit benefit has a level premium. The time-until-death random variable, $T(x)$, has an exponential distribution with $\mathrm{E}[T(x)] = 50$ and the force of interest is $\delta = 0.06$.
a. If the principle of equivalence is used, find the benefit premium rate.
b. Find the premium rate if it is to be such that $\Pr(L > 0) = 0.50$.
c. Repeat part (b) if the force of interest, δ, equals 0.

6.5. If the force of mortality strictly increases with age, show that $\bar{P}(\bar{A}_x) > \mu_x(0)$. [Hint: Show that $\bar{P}(\bar{A}_x)$ is a weighted average of $\mu_x(t)$, $t > 0$.]

6.6. Following Example 6.2.1, derive a general expression for

$$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta \bar{a}_x)^2}$$

where $\mu_x(t) = \mu$ and δ is the force of interest for $t > 0$.

6.7. If $\delta = 0$, show that

$$\bar{P}(\bar{A}_x) = \frac{1}{\mathring{e}_x}.$$

6.8. Prove that the variance of the loss associated with a single premium whole life insurance is less than the variance of the loss associated with an annual premium whole life insurance. Assume immediate payment of claims on death and continuous payment of benefit premiums.

6.9. Show that

$$\left(1 + \frac{d\bar{a}_x}{dx}\right) \bar{P}(\bar{A}_x) - \frac{d\bar{A}_x}{dx} = \mu(x).$$

Section 6.3

6.10. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate values for the annual premiums in the following table. Note any patterns of inequalities that appear in the matrix of results.

Fully Continuous	**Semicontinuous**	**Fully Discrete**
$\bar{P}(\bar{A}_{35:\overline{10}\vert})$	$P(\bar{A}_{35:\overline{10}\vert})$	$P_{35:\overline{10}\vert}$
$\bar{P}(\bar{A}_{35:\overline{30}\vert})$	$P(\bar{A}_{35:\overline{30}\vert})$	$P_{35:\overline{30}\vert}$
$\bar{P}(\bar{A}_{35:\overline{60}\vert})$	$P(\bar{A}_{35:\overline{60}\vert})$	$P_{35:\overline{60}\vert}$
$\bar{P}(\bar{A}_{35})$	$P(\bar{A}_{35})$	P_{35}
$\bar{P}(\bar{A}^{1}_{35:\overline{30}\vert})$	$P(\bar{A}^{1}_{35:\overline{30}\vert})$	$P^{1}_{35:\overline{30}\vert}$
$\bar{P}(\bar{A}^{1}_{35:\overline{10}\vert})$	$P(\bar{A}^{1}_{35:\overline{10}\vert})$	$P^{1}_{35:\overline{10}\vert}$

6.11. Show that

$${}_{20}P^{1}_{x:\overline{30}\vert} - P^{1}_{x:\overline{20}\vert} = {}_{20}P({}_{20|10}A_x).$$

6.12. Generalize Example 6.3.1 where

$${}_{k|}q_x = (1 - r)r^k \qquad k = 0, 1, 2, \ldots;$$

that is, derive expressions in terms of r and i for A_x, $\ddot{a}_x$, P_x, and $[{}^2A_x - (A_x)^2]/(d\,\ddot{a}_x)^2$.

Section 6.4

6.13. Using the information given in Example 6.4.1, calculate the value $P^{(2)}_{50}$.

6.14. Using various formulas for $\ddot{a}^{(m)}_{x:\overline{n}\vert}$, first under the assumption of a uniform distribution of deaths in each year of age, show that the ratio

$$\frac{\ddot{a}_{x:\overline{h}\vert}}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$$

in (6.4.2) can be expressed as the reciprocal of each of (a) and (b). As an alternative, if the development of (5.4.10) is followed show that the ratio is the reciprocal of (c).

a. $\ddot{a}^{(m)}_{\overline{1}|} - \beta(m)P^{\;1}_{x:\overline{h}|}$

b. $\alpha(m) - \beta(m)(P^{\;1}_{x:\overline{h}|} + d)$

c. $1 - \dfrac{m-1}{2m}(P^{\;1}_{x:\overline{h}|} + d)$.

6.15. Refer to Example 6.4.1(b) and directly calculate

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}}$$

using the Illustrative Life Table for the actuarial present values in the numerator and the denominator.

6.16. If

$$\frac{P^{1\,(12)}_{x:\overline{20}|}}{P^{\;1}_{x:\overline{20}|}} = 1.032$$

and $P_{x:\overline{20}|} = 0.040$, what is the value of $P^{(12)}_{x:\overline{20}|}$?

Section 6.5

6.17. Arrange in order of magnitude and indicate your reasoning:
$P^{(2)}(\bar{A}_{40:\overline{25}|})$, $\bar{P}(\bar{A}_{40:\overline{25}|})$, $P^{\{4\}}(\bar{A}_{40:\overline{25}|})$, $P(\bar{A}_{40:\overline{25}|})$, $P^{\{12\}}(\bar{A}_{40:\overline{25}|})$.

6.18. Given that

$$\frac{d}{d^{(12)}} = \frac{99}{100},$$

evaluate

$$\frac{P^{\{12\}}(\bar{A}_x)}{P^{\{1\}}(\bar{A}_x)}.$$

6.19. If $\bar{P}(\bar{A}_x) = 0.03$, and if interest is at the effective annual rate of 5%, calculate the semiannual benefit premium for a 50,000 whole life insurance on (x) where premiums are apportionable.

6.20. Show that

$$\begin{aligned} P^{\{m\}}(\bar{A}_{x:\overline{n}|}) - P^{(m)}(\bar{A}_{x:\overline{n}|}) &= \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}_{x:\overline{n}|} - A^{(m)}_{x:\overline{n}|}}{\delta\, \ddot{a}^{(m)}_{x:\overline{n}|}}\right) \\ &= \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}^{\;1}_{x:\overline{n}|} - A^{1(m)}_{x:\overline{n}|}}{\delta\, \ddot{a}^{(m)}_{x:\overline{n}|}}\right). \end{aligned}$$

Section 6.6

6.21. Express

$$1 - \frac{\ddot{s}_{\overline{20}|}}{\ddot{s}_{45:\overline{20}|}}$$

as an annual premium. Interpret your result.

6.22. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate the components of the two decompositions

a. $1{,}000\ P_{50:\overline{20}|} = 1{,}000(P^{1}_{50:\overline{20}|} + P_{50:\overset{\ 1}{\overline{20}|}})$

b. $1{,}000\ P_{50:\overline{20}|} = 1{,}000\left(\tilde{P}^{1}_{50:\overline{20}|} + \dfrac{1}{\ddot{s}_{\overline{20}|}}\right).$

6.23. Consider the continuous random variable analogue of (6.6.3),

$$\tilde{\bar{W}} = \begin{cases} v^T\left(1 - \dfrac{\bar{s}_{\overline{T}|}}{\bar{s}_{\overline{n}|}}\right) & 0 \le T < n \\ 0 & T \ge n. \end{cases}$$

The loss,

$$L = \tilde{\bar{W}} - \tilde{\bar{A}}^{1}_{x:\overline{n}|},$$

can be used with the equivalence principle to determine $\tilde{\bar{A}}^{1}_{x:\overline{n}|}$, the single benefit premium for this special policy. Show that

a. $\tilde{\bar{A}}^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} - \dfrac{\bar{a}_{x:\overline{n}|} - {}_np_x\ \bar{a}_{\overline{n}|}}{\bar{s}_{\overline{n}|}}$

b. $\mathrm{E}[\tilde{\bar{W}}^2] = \dfrac{(1+i)^{2n}\ {}^2\bar{A}^{1}_{x:\overline{n}|} - 2(1+i)^n\ \bar{A}^{1}_{x:\overline{n}|} + (1 - {}_np_x)}{[(1+i)^n - 1]^2}.$

Miscellaneous

6.24. Express

$$A_{40}\ P_{40:\overline{25}|} + (1 - A_{40})P_{40}$$

as an annual benefit premium.

6.25. a. Show that

$$\frac{1}{\ddot{a}_{65:\overline{10}|}} - \frac{1}{\ddot{s}_{65:\overline{10}|}} = P^{1}_{65:\overline{10}|} + d.$$

b. What is the corresponding formula for

$$\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}?$$

c. Show that the amount of annual income provided by a single benefit premium of 100,000 where

- The income is payable at the beginning of each month while (65) survives during the next 10 years, and

- The single premium is returned at the end of 10 years if (65) reaches age 75,

is given by

$$100{,}000\left(\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}\right) = 100{,}000(\beta)$$

where (β) denotes the answer to part (b) of this exercise.

6.26. An insurance issued to (35) with level premiums to age 65 provides
- 100,000 in case the insured survives to age 65, and
- The return of the annual contract premiums with interest at the valuation rate to the end of the year of death if the insured dies before age 65.

If the annual contract premium G is 1.1π where π is the annual benefit premium, write an expression for π.

6.27. If ${}_{15}P_{45} = 0.038$, $P_{45:\overline{15}|} = 0.056$, and $A_{60} = 0.625$, calculate $P^{1}_{45:\overline{15}|}$.

6.28. A 20-payment life policy is designed to return, in the event of death, 10,000 plus all contract premiums without interest. The return-of-premium feature applies both during the premium paying period and after. Premiums are annual and death claims are paid at the end of the year of death. For a policy issued to (x), the annual contract premium is to be 110% of the benefit premium plus 25. Express in terms of actuarial present-value symbols the annual contract premium.

6.29. Express in terms of actuarial present-value symbols the initial annual benefit premium for a whole life insurance issued to (25), subject to the following provisions:
- The face amount is to be one for the first 10 years and two thereafter.
- Each premium during the first 10 years is 1/2 of each premium payable thereafter.
- Premiums are payable annually to age 65.
- Claims are paid at the end of the year of death.

6.30. Let L_1 be the insurer's loss on a unit of whole life insurance, issued to (x) on a fully continuous basis. Let L_2 be the loss to (x) on a continuous life annuity purchased for a single premium of one. Show that $L_1 \equiv L_2$ and give an explanation in words.

6.31. An ordinary life contract for a unit amount on a fully discrete basis is issued to a person age x with an annual premium of 0.048. Assume $d = 0.06$, $A_x = 0.4$, and ${}^2A_x = 0.2$. Let L be the insurer's loss function at issue of this policy.

a. Calculate E[L].

b. Calculate Var(L).

c. Consider a portfolio of 100 policies of this type with face amounts given below.

Face Amount	Number of Policies
1	80
4	20

Assume the losses are independent and use a normal approximation to calculate the probability that the present value of gains for the portfolio will exceed 20.

6.32. Express, in terms of actuarial present-value symbols, the initial annual benefit premium for a unit of whole life insurance for (x) if after 5 years the annual benefit premium is double that payable during the first 5 years. Assume that death claims are made at the moment of death.

6.33. Repeat Exercise 6.20 for h-payment whole life insurance.

6.34. The function $l(t)$ is given by (6.2.1).
a. Establish that $l''(t) \geq 0$.
b. Adapt Jensen's inequality from Section 1.3 to establish that if $\bar{P} = \bar{P}(\bar{A}_x)$, then $\bar{P}(\bar{A}_x) \geq v^{\mathring{e}_x}/\bar{a}_{\overline{\mathring{e}_x}|}$.

6.35. If $T(x)$ has an exponential distribution with parameter μ,
a. Exhibit the p.d.f. of L as shown in (6.2.12)
b. Show that $\mathrm{E}[L] = (\mu - \bar{P})/(\mu + \delta)$.
c. Use part (b) to confirm that $\mathrm{E}[L] = 0$ when $\bar{P} = \bar{P}(\bar{A}_x)$.

6.36. Use the assumptions of Exercise 6.35, with $\mu = 0.03$ and $\delta = 0.06$.
a. Evaluate $\Pr(L \leq 0)$ when $\bar{P} = \bar{P}(\bar{A}_x)$.
b. Determine $\bar{P}$ so that $\Pr(L > 0) = 0.5$.

7

BENEFIT RESERVES

7.1 Introduction

In Chapter 6 we introduced several principles that can be used for the determination of benefit premiums. The equivalence principle was used extensively in our discussion in Chapter 6. By it, an equivalence relation is established on the date a long-term contract is entered into by two parties who agree to exchange a set of payments. For example, under an amortized loan, a borrower may pay a series of equal monthly payments equivalent to the single payment by a lender at the date of the loan. An insured may pay a series of benefit premiums to an insurer equivalent, at the date of policy issue, to the sum to be paid on the death of the insured, or on survival of the insured to the maturity date. An individual may purchase a deferred life annuity by means of level premiums payable to an annuity organization equivalent, at the date of contract agreement, to monthly payments by the annuity organization to the individual when that person survives beyond a specified age. Equivalence in the loan example is in terms of present value, whereas in the insurance and annuity examples it is an equivalence between two actuarial present values.

After a period of time, however, there will no longer be an equivalence between the future financial obligations of the two parties. A borrower may have payments remaining to be made, whereas the lending organization has already performed its responsibilities. In other settings both parties may still have obligations. The insured may still be required to pay further benefit premiums, whereas the insurer has the duty to pay the face amount on maturity or the death of the insured. In our deferred annuity example, the individual may have completed the payments, whereas the annuity organization still has monthly remunerations to make.

In this chapter we study payments in time periods beyond the date of initiation. For this, a balancing item is required, and this item is a liability for one of the parties and an asset for the other. In the loan case, the balancing item is the outstanding principal, an asset for the lender and a liability for the borrower. In the other two cases, if the individual continues to survive, the balancing item is called a reserve. This is typically a liability that should be recognized in any financial statement of an insurer or annuity organization, as the case may be. It is also typically an asset for the insured or individual purchasing the annuity.

We illustrate the determination of the balancing item spoken of above by continuation of Example 6.1.1 in the two cases where a utility function was used to define the premium principle.

Example 7.1.1

An insurer has issued a policy paying 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Assume that the insured is still alive 1 year after entering into the contract. Further, assume that the insurer continues to use $i = 0.06$ and the following mortality assumption for K:

$$_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4.$$

Find the reserve, $_1V$, as determined by the following:

a. Principle II: The insurer, using a utility of wealth function $u(x) = x$, will be indifferent between continuing with the risk while receiving premiums of 0.30272 (from Example 6.1.1) and paying the amount $_1V$ to a reinsurer to assume the risk.
b. Principle III: The insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between continuing with the risk while receiving premiums of 0.30628 (from Example 6.1.1) and paying the amount $_1V$ to a reinsurer to assume the risk.

Solution:

The conditional probability function for K, the curtate-future-lifetime, given that $K \geq 1$, is

$$\Pr(K = k|K \geq 1) = \frac{\Pr(K = k)}{\Pr(K \geq 1)} = \frac{0.2}{0.8} = 0.25 \qquad k = 1, 2, 3, 4.$$

The present value at duration 1 of the insurer's future financial loss is $_1L = v^{(K-1)+1} - P\, \ddot{a}_{\overline{(K-1)+1|}}$, where P is the premium determined in Example 6.1.1.

a. According to (1.3.1), we seek the amount $_1V$ such that $u(w - {_1V}) = \mathrm{E}[u(w - {_1L})|K \geq 1]$. By Principle II $u(x) = x$, so we have

$$w - {_1V} = \mathrm{E}[w - {_1L}|K \geq 1] = w - \mathrm{E}[{_1L}|K \geq 1].$$

Thus, Principle II is equivalent to requiring that $_1V$ be chosen so that $_1V = \mathrm{E}[{_1L}|K \geq 1]$. For this example, this requirement is

$$\begin{aligned} {_1V} &= \sum_{k=1}^{4} (v^{(k-1)+1} - 0.30272\, \ddot{a}_{\overline{(k-1)+1|}}) \times \Pr(K = k|K \geq 1) \\ &= \sum_{k=1}^{4} v^{(k-1)+1} \Pr(K = k|K \geq 1) - 0.30272 \sum_{k=1}^{4} \ddot{a}_{\overline{(k-1)+1|}} \times \Pr(K = k|K \geq 1), \end{aligned} \tag{7.1.1}$$

which gives ${}_1V = 0.15111$ as shown in the following calculation.

Outcome k	Conditional Probability	Present Value (1 Year after Issue) of Future Obligations of: Insurer	Insured	Insurer's Prospective Loss
1	0.25	$v = 0.94340$	$P\,\ddot{a}_{\overline{1}\rceil} = 0.30272$	0.64067
2	0.25	$v^2 = 0.89000$	$P\,\ddot{a}_{\overline{2}\rceil} = 0.58831$	0.30169
3	0.25	$v^3 = 0.83962$	$P\,\ddot{a}_{\overline{3}\rceil} = 0.85773$	−0.01811
4	0.25	$v^4 = 0.79209$	$P\,\ddot{a}_{\overline{4}\rceil} = 1.11191$	−0.31981
Expected Value		0.86628	0.71517	0.15111

The actuarial present value of the insurer's prospective loss is

$$0.86628 - 0.71517 = 0.15111.$$

b. Again by (1.3.1) and now using the utility function in Principle III, we have

$$-e^{-0.1(w-{}_1V)} = \mathrm{E}[-e^{-0.1(w-{}_1L)}|K \ge 1] = -e^{-0.1w}\,\mathrm{E}[e^{0.1\,{}_1L}|K \ge 1].$$

Thus, Principle III is equivalent to requiring that ${}_1V$ be chosen so that $e^{0.1\,{}_1V} = \mathrm{E}[e^{0.1\,{}_1L}|K \ge 1]$ or that ${}_1V = 10 \log \mathrm{E}[e^{0.1\,{}_1L}|K \ge 1]$.

The following table summarizes the calculation of ${}_1V$ using the premium (0.30628) from part (c) of Example 6.1.1.

Outcome k	Conditional Probability	Insurer's Prospective Loss, ${}_1L$	$e^{(0.1\,{}_1L)}$
1	0.25	0.63712	1.06579
2	0.25	0.29477	1.02992
3	0.25	−0.02819	0.99718
4	0.25	−0.33287	0.96726

Thus, $\mathrm{E}[e^{0.1\,{}_1L}|K \ge 1] = (1.06579 + 1.02992 + 0.99718 + 0.96726)(0.25) = 1.01504$ and ${}_1V = (\log 1.01504)/0.1 = 0.14925$. ▼

Henceforth, benefit reserves will be based on benefit premiums as determined by the equivalence principle in part (a) of Example 7.1.1. Thus, the ***benefit reserve at time t*** is the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums, the conditioning event being survivorship of the insured to time t. The type of reserve found in part (b) of Example 7.1.1 is called an ***exponential reserve.***

The sections in Chapter 7 parallel sections of Chapter 6 on benefit premiums. We assume, as we do in Example 7.1.1, that the mortality and interest rates adopted at policy issue for the determination of benefit premiums continue to be appropriate and are used in the determination of benefit reserves.

7.2 Fully Continuous Benefit Reserves

We now develop the benefit reserves related to the fully continuous benefit premiums developed in Section 6.2 by application of the equivalence principle.

Let us consider reserves for a whole life insurance of 1 issued to (x) on a fully continuous basis with an annual benefit premium rate of $\bar{P}(\bar{A}_{[x]})$. The corresponding reserve for an insured surviving t years later is defined under the equivalence principle as the conditional expected value of the prospective loss at time t, given that (x) has survived to t. More formally, for $T(x) > t$ the prospective loss is

$$ {}_tL = v^{T(x)-t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{\overline{T(x)-t|}}. \tag{7.2.1}$$

The reserve, as a conditional expectation, is calculated using the conditional distribution of the future lifetime at t for a life selected at x, given it has survived to t. In International Actuarial Notation symbols this is

$$\begin{aligned} {}_t\bar{V}(\bar{A}_{[x]}) &= \mathrm{E}[{}_tL|T(x) > t] \\ &= \mathrm{E}[v^{T(x)-t}|T(x) > t] - \bar{P}(\bar{A}_{[x]})\, \mathrm{E}[\bar{a}_{\overline{T(x)-t|}}|T(x) > t] \\ &= \bar{A}_{[x]+t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{[x]+t}. \end{aligned} \tag{7.2.2}$$

If the attained age was the only given information at issue of the insurance at age x, or for some other reason an aggregate mortality table is used for the distribution of the future lifetime, then the conditional distribution of $T(x) - t$ is the same as the distribution of $T(x + t)$, and (7.2.2) in symbols is

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}. \tag{7.2.3}$$

Formulas (7.2.2) and (7.2.3) state that

(the benefit reserve) = (the actuarial present value for the whole life insurance from age $x + t$)
− (the actuarial present value of future benefit premiums payable after age $x + t$ at an annual rate of $\bar{P}(\bar{A}_x)$).

The formulations of $\bar{P}(\bar{A}_x)$ and ${}_t\bar{V}(\bar{A}_x)$ are related. When $t = 0$, (7.2.3) yields ${}_0\bar{V}(\bar{A}_x) = 0$. This is a consequence of applying the equivalence principle as of the time the contract was established.

Remark on Notation (Restated):

In this book, we will simplify the appearance of the formulas by suppressing the select notation unless its use is necessary or helpful in the particular situation. The symbol $\mu_x(t)$ will be used for the force of mortality in the development of benefit reserves to reinforce the idea that the conditional distributions used in reserve calculations are derived from the distribution of $T(x)$.

Benefit reserves are defined in Section 7.1 as the conditional expectation of loss variables. In evaluating these conditional expected values in this section, the distribution of $T(x) - t$, given $T(x) > t$, was used. The interest rate and the distribution of $T(x)$ used with the equivalence principle at time $t = 0$ to determine the benefit premium were used again. The survival of the life (x) to time t was the only new information incorporated into the expected value calculation. A comprehensive reserve principle would require that all new information relevant to the loss variables and their distributions be incorporated into the reserve calculation. The objective of this requirement would be to estimate liabilities appropriate for the time that the valuation is made. In Chapters 7 and 8 the process of learning from experience to modify the assumptions under which benefit reserves are estimated are not studied.

By steps analogous to those used to obtain (6.2.6), we can determine the variance of ${}_tL$ as follows:

$$ {}_tL = v^{T(x)-t}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}, \tag{7.2.4}$$

thus

$$\begin{aligned}\mathrm{Var}[{}_tL|T(x) > t] &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 \mathrm{Var}[v^{T(x)-t}|T(x) > t] \\ &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2]. \end{aligned}\tag{7.2.5}$$

Note the relation to (6.2.6) and that the development holds for all premium levels. It is not dependent on the equivalence principle.

Example 7.2.1

Follow up Example 6.2.1 by calculating ${}_t\bar{V}(\bar{A}_x)$ and Var $[{}_tL|T(x) > t]$.

Solution:

Since $\bar{A}_x$, $\bar{a}_x$, and $\bar{P}(\bar{A}_x)$ are independent of age x, (7.2.3) becomes

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_x - \bar{P}(\bar{A}_x)\,\bar{a}_x = 0 \qquad t \geq 0.$$

In this case, future premiums are always equivalent to future benefits, and no reserve is needed.

Also, in this case, (7.2.5) reduces to

$$\mathrm{Var}[{}_tL|T(x) > t] = \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_x - (\bar{A}_x)^2] = \mathrm{Var}(L) = 0.25,$$

as in Example 6.2.1. Here the variance of ${}_tL$ depends on neither the age x nor the duration t. ▼

Example 7.2.2

On the basis of De Moivre's law with $l_x = 100 - x$ and the interest rate of 6%, calculate

a. $\bar{P}(\bar{A}_{35})$

b. ${}_t\bar{V}(\bar{A}_{35})$ and $\text{Var}[{}_tL|T(x) > t]$ for $t = 0, 10, 20, \ldots, 60$.

Solution:

a. From $l_x = 100 - x$, we obtain ${}_tp_{35} = 1 - t/65$ and ${}_tp_{35}\,\mu(35+t) = 1/65$ for $0 \le t < 65$. It follows that

$$\bar{A}_{35} = \int_0^{65} v^t \frac{1}{65}\, dt = \frac{\bar{a}_{\overline{65}|0.06}}{65} = 0.258047.$$

Then

$$\bar{P}(\bar{A}_{35}) = \frac{\delta\, \bar{A}_{35}}{1 - \bar{A}_{35}} = 0.020266.$$

b. At age $35 + t$, we have $\bar{A}_{35+t} = \bar{a}_{\overline{65-t}|}/(65 - t)$ and

$${}_t\bar{V}(\bar{A}_{35}) = \bar{A}_{35+t} - 0.020266\, \frac{1 - \bar{A}_{35+t}}{\log(1.06)}.$$

Further,

$${}^2\bar{A}_{35+t} = \int_0^{65-t} v^{2u} \frac{1}{65 - t}\, du = \frac{{}^2\bar{a}_{\overline{65-t}|}}{65 - t}$$

and, from (7.2.5),

$$\text{Var}[{}_tL|T(x) > t] = \left[1 + \frac{0.020266}{\log(1.06)}\right]^2 [{}^2\bar{A}_{35+t} - (\bar{A}_{35+t})^2].$$

Applying these formulas, we obtain the following results.

t	${}_t\bar{V}(\bar{A}_{35})$	$\text{Var}[{}_tL\|T(35) > t]$
0	0.0000	0.1187
10	0.0557	0.1201
20	0.1289	0.1173
30	0.2271	0.1073
40	0.3619	0.0861
50	0.5508	0.0508
60	0.8214	0.0097

Note the convergence of $\text{Var}[{}_tL|T(35) > t]$ toward zero. There is a rendezvous with certainty. ▼

The table in Example 7.2.2 provides the mean and the variance of the conditional distributions of ${}_tL$ for selected values of t. To gain more insight into the nature of reserves, let us explore these distributions of ${}_tL$ in more depth. We previously studied the case for $t = 0$ at (6.2.11) following Example 6.2.3. From (7.2.1),

$$\begin{aligned} {}_tL &= v^{T(x)-t} - \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{T(x)-t}|} \\ &= v^{T(x)-t} \left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}. \end{aligned} \tag{7.2.6}$$

If $\delta > 0$, ${}_tL$ is a decreasing function of $T(x) - t$ and lies in the interval

$$-\frac{\bar{P}(\bar{A}_x)}{\delta} < {}_tL \le 1. \tag{7.2.7}$$

Using Figure 6.2.1 as a guide we can repeat the steps of (6.2.11) to establish the following relationship between $F_{T(x)}(u)$ and the d.f. of the conditional distribution of ${}_tL$, given $T(x) > t$, which we denote by $F_{{}_tL}(y)$. For a y in the interval given by (7.2.7),

$$\begin{aligned} F_{{}_tL}(y) &= \Pr[{}_tL \le y | T(x) > t] \\ &= \Pr\left[v^{T(x)-t} \left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta} \le y \,\middle|\, T(x) > t \right] \\ &= \Pr\left[v^{T(x)-t} \le \frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \,\middle|\, T(x) > t \right] \\ &= \Pr\left[T(x) \ge t - \frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right] \,\middle|\, T(x) > t \right] \\ &= \frac{\Pr[T(x) \ge t - (1/\delta)\{\log[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\}]}{\Pr[T(x) > t]} \qquad (7.2.8) \\ &= \frac{1 - F_{T(x)}(t - (1/\delta)\log\{[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\})}{1 - F_{T(x)}(t)}. \qquad (7.2.9) \end{aligned}$$

Differentiation of (7.2.9) with respect to y derives the p.d.f. for the conditional distribution of ${}_tL$, given $T(x) > t$:

$$f_{{}_tL}(y) = \left\{\frac{1}{[\delta y + \bar{P}(\bar{A}_x)][1 - F_{T(x)}(t)]}\right\} f_{T(x)}\left(t - \frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right). \tag{7.2.10}$$

For an aggregate mortality law the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$, so (7.2.8), (7.2.9), and (7.2.10) reduce to

$$\begin{aligned} F_{{}_tL}(y) &= \Pr\left[T(x+t) \ge -\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right] \qquad (7.2.11) \\ &= 1 - F_{T(x+t)}\left(-\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right), \qquad (7.2.12) \end{aligned}$$

$$f_{{}_tL}(y) = \frac{1}{[\delta y + \bar{P}(\bar{A}_x)]} f_{T(x+t)}\left(-\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right). \tag{7.2.13}$$

To illustrate these concepts we will extend Example 7.2.2.

Example 7.2.3

For the insurance contract and assumptions in Example 7.2.2:

a. Exhibit the formulas for the d.f. and the p.d.f. of the conditional distribution for ${}_tL$, given $T(x) > t$.
b. Display graphs of these conditional p.d.f.'s for $t = 0$, 20, 40, and 60.

Solution:

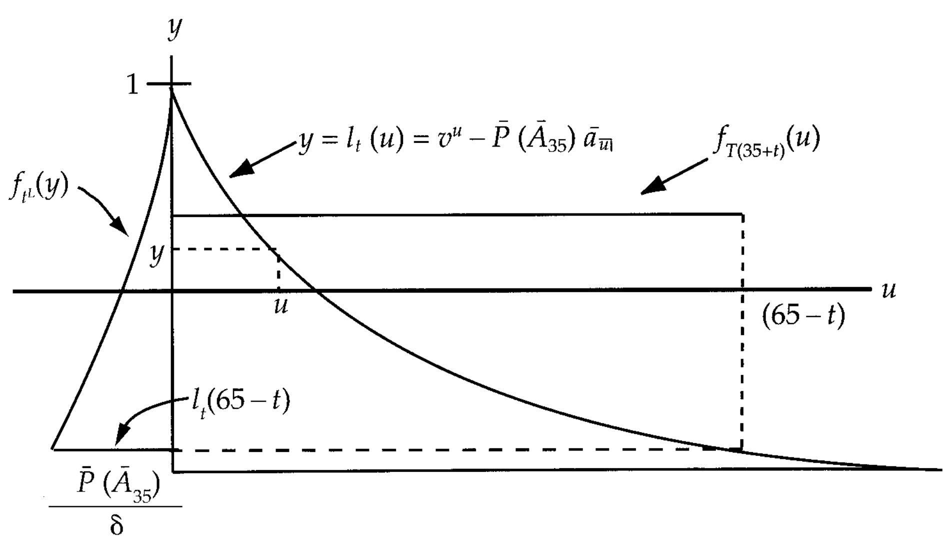
a. Since Example 7.2.2 specifies an aggregate mortality law, we use formulas (7.2.12) and (7.2.13). In Example 7.2.2,

$$\begin{aligned} F_{T(35+t)}(u) &= \frac{u}{65-t} && \text{for } 0 \le u \le 65-t \\ &= 1 && \text{for } 65-t < u, \\ f_{T(35+t)}(u) &= \frac{1}{65-t} && \text{for } 0 \le u \le 65-t \\ &= 0 && \text{elsewhere.} \end{aligned}$$

Figure 7.2.1 shows Figure 6.2.1 as it applies to this example. In this figure the outcome space of $T(35 + t)$ is on the u-axis, and the outcome space of the loss random variable, ${}_tL$, is on the y-axis. The relationship between the outcomes of $T(35 + t)$ and the outcomes of ${}_tL$ is given by the loss function $y = l_t(u)$ and is indicated by the dashed line connecting u and y in the figure. The p.d.f. $f_{T(35+t)}(u)$ has its domain on the u-axis and its range on the y-axis. The domain of the p.d.f. $f_{{}_tL}(y)$ is on the y-axis, and its range is to be imagined on an axis perpendicular to the u-y plane, but for the sketch it has been laid perpendicular to the y-axis in the u-y plane.

FIGURE 7.2.1

Schematic Diagrams of $l_t(u)$, $f_{T(35+t)}(u)$, and $f_{{}_tL}(y)$



To determine the d.f. by ${}_tL$ we start with a value of y corresponding to a value of u in the interval $(0, 65 - t)$. For such a y we have, by (7.2.12),

$$F_{{}_tL}(y) = 1 - \frac{(-1/\delta) \log \{[\delta y + \bar{P}(\bar{A}_{35})]/[\delta + \bar{P}(\bar{A}_{35})]\}}{65 - t} \qquad 0 \le y \le 1.$$

For a $y > 1$, $F_{{}_tL}(y) = 1$.

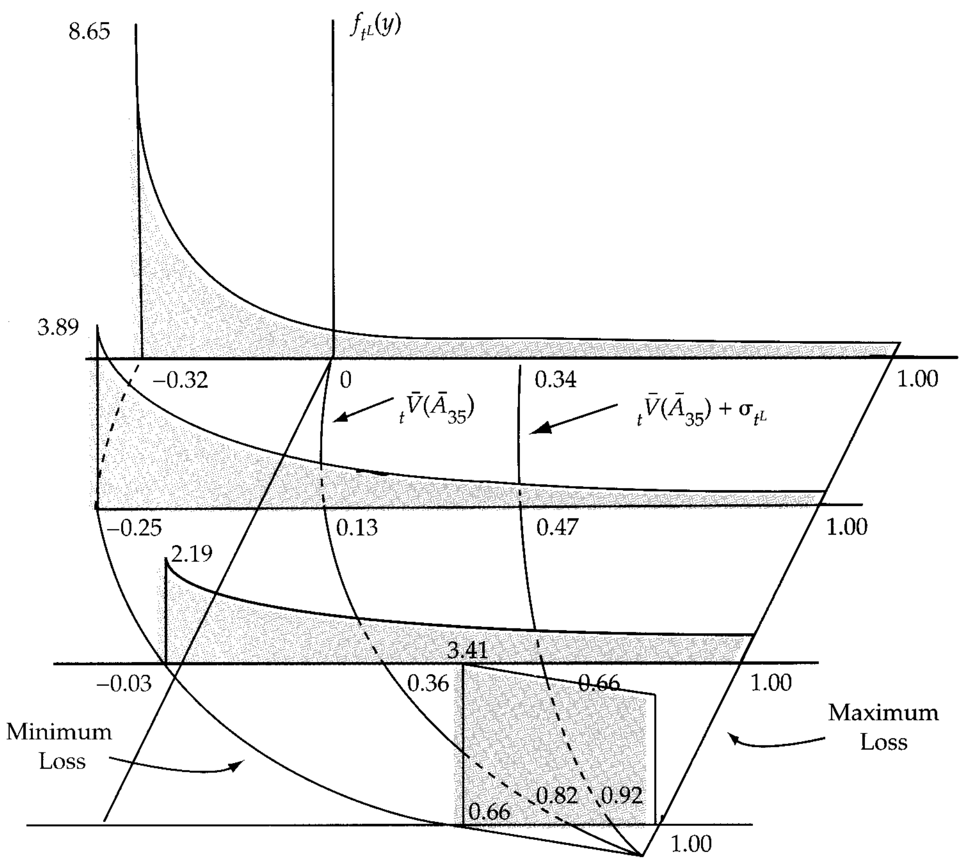
Again, for a value of y corresponding to a value of u in the interval $(0, 65 - t)$, and using (7.2.13), we have

$$f_{{}_tL}(y) = \begin{cases} \left(\dfrac{1}{65 - t}\right)\left[\dfrac{1}{\delta y + \bar{P}(\bar{A}_{35})}\right] & -\dfrac{\bar{P}(\bar{A}_{35})}{\delta} \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

b. Figure 7.2.2 is the composite of the required graphs of the p.d.f.'s $f_{{}_tL}(y)$, for $t = 0, 20, 40$, and 60. Figure 7.2.1 is a graph for one value of t. Compare Figures 7.2.1 and 7.2.2 as follows: The vertical y-axis of Figure 7.2.1 is the horizontal axis in Figure 7.2.2. The axis that was imagined to be perpendicular to the u-y plane in Figure 7.2.1 is the vertical axis in Figure 7.2.2. Note the curves in the y-t plane that indicate the minimum and maximum possible losses, the expected loss (benefit reserve), and the expected loss plus one standard deviation of the loss. ▼

FIGURE 7.2.2

$f_{{}_tL}(y)$ for t = 0, 20, 40, and 60



Corresponding to Table 6.2.1, Table 7.2.1 for benefit reserves is presented. We have not tabulated details of the prospective loss, ${}_tL$, and explicit formulas for $\text{Var}[{}_tL|T(x) > t]$, corresponding to the several benefit reserves, are not displayed.

TABLE 7.2.1

Fully Continuous Benefit Reserves; Age at Issue x, Duration t, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula	
Whole life insurance	${}_t\bar{V}(\bar{A}_x)$	$\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\,\bar{a}_{x+t}$	
n-Year term insurance	${}_t\bar{V}(\bar{A}^{\,1}_{x:\overline{n}\vert})$	$\bar{A}^{\,1}_{x+t:\overline{n-t}\vert} - \bar{P}(\bar{A}^{\,1}_{x:\overline{n}\vert})\,\bar{a}_{x+t:\overline{n-t}\vert}$ 0	$t < n$ $t = n$
n-Year endowment insurance	${}_t\bar{V}(\bar{A}_{x:\overline{n}\vert})$	$\bar{A}_{x+t:\overline{n-t}\vert} - \bar{P}(\bar{A}_{x:\overline{n}\vert})\,\bar{a}_{x+t:\overline{n-t}\vert}$ 1	$t < n$ $t = n$
h-Payment years, whole life insurance	${}^h_t\bar{V}(\bar{A}_x)$	$\bar{A}_{x+t} - {}_h\bar{P}(\bar{A}_x)\,\bar{a}_{x+t:\overline{h-t}\vert}$ $\bar{A}_{x+t}$	$t \le h$ $t > h$
h-Payment years, n-year endowment insurance	${}^h_t\bar{V}(\bar{A}_{x:\overline{n}\vert})$	$\bar{A}_{x+t:\overline{n-t}\vert} - {}_h\bar{P}(\bar{A}_{x:\overline{n}\vert})\,\bar{a}_{x+t:\overline{h-t}\vert}$ $\bar{A}_{x+t:\overline{n-t}\vert}$ 1	$t \le h < n$ $h < t < n$ $t = n$
n-Year pure endowment	${}_t\bar{V}(A_{x:\overline{n}\vert}^{\;\;1})$	$A_{x+t:\overline{n-t}\vert}^{\;\;\;\;1} - \bar{P}(A_{x:\overline{n}\vert}^{\;\;1})\,\bar{a}_{x+t:\overline{n-t}\vert}$ 1	$t < n$ $t = n$
Whole life annuity	${}_t\bar{V}({}_{n\vert}\bar{a}_x)$	${}_{n-t\vert}\bar{a}_{x+t} - \bar{P}({}_{n\vert}\bar{a}_x)\,\bar{a}_{x+t:\overline{n-t}\vert}$ $\bar{a}_{x+t}$	$t \le n$ $t > n$

7.3 Other Formulas for Fully Continuous Benefit Reserves

So far we have defined the benefit reserve as the conditional expectation of the prospective loss random variable and developed only one method to write formulas for fully continuous benefit reserves, namely, the ***prospective method,*** stating that the benefit reserve is the difference between the actuarial present values of future benefits and of future benefit premiums. From the prospective method, we can easily develop three other general formulas for policies with level benefits and level benefit premium rates. We illustrate these for the case of n-year endowment insurances.

The ***premium-difference formula*** for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\vert})$ is obtained by factoring $\bar{a}_{x+t:\overline{n-t}\vert}$ out of the prospective formula for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\vert})$:

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}\vert}) &= \left[\frac{\bar{A}_{x+t:\overline{n-t}\vert}}{\bar{a}_{x+t:\overline{n-t}\vert}} - \bar{P}(\bar{A}_{x:\overline{n}\vert})\right]\bar{a}_{x+t:\overline{n-t}\vert} \\
&= [\bar{P}(\bar{A}_{x+t:\overline{n-t}\vert}) - \bar{P}(\bar{A}_{x:\overline{n}\vert})]\,\bar{a}_{x+t:\overline{n-t}\vert}. \qquad (7.3.1)
\end{aligned}
$$

Formula (7.3.1) exhibits the benefit reserve as the actuarial present value of a premium difference payable over the remaining premium-payment term. The premium difference is obtained by subtracting the original annual benefit premium from the benefit premium for an insurance issued at the attained age $x + t$ for the remaining benefits.

A second formula is obtained by factoring the actuarial present value of future benefits out of the prospective formula. Thus, for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) &= \left[1 - \bar{P}(\bar{A}_{x:\overline{n}|})\,\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{A}_{x+t:\overline{n-t}|}}\right]\bar{A}_{x+t:\overline{n-t}|} \\
&= \left[1 - \frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\bar{P}(\bar{A}_{x+t:\overline{n-t}|})}\right]\bar{A}_{x+t:\overline{n-t}|}. \qquad (7.3.2)
\end{aligned}
$$

This exhibits the benefit reserve as the actuarial present value of a portion of the remaining future benefits, that portion which is not funded by the future benefit premiums still to be collected. Note that $\bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the benefit premium required if the future benefits were to be funded from only the future benefit premiums, but $\bar{P}(\bar{A}_{x:\overline{n}|})$ is the benefit premium actually payable. Thus, $\bar{P}(\bar{A}_{x:\overline{n}|}) / \bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the portion of future benefits funded by future benefit premiums. This is called a ***paid-up insurance formula,*** named from the paid-up insurance nonforfeiture benefit to be discussed in Chapter 16. Formulas analogous to (7.3.1) and (7.3.2) exist for a wide variety of benefit reserves.

A third expression is the ***retrospective formula.*** We develop this from a more general relationship. We have, from Exercise 4.12 and from formulas (5.2.18) and (5.2.19), for $t < n - s$,

$$\bar{A}_{x+s:\overline{n-s}|} = \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\,\bar{A}_{x+s+t:\overline{n-s-t}|}$$

and

$$\bar{a}_{x+s:\overline{n-s}|} = \bar{a}_{x+s:\overline{t}|} + {}_tE_{x+s}\,\bar{a}_{x+s+t:\overline{n-s-t}|}.$$

Substituting these expressions into the prospective formula for ${}_s\bar{V}(\bar{A}_{x:\overline{n}|})$, we obtain

$$
\begin{aligned}
{}_s\bar{V}(\bar{A}_{x:\overline{n}|}) &= \bar{A}^{\,1}_{x+s:\overline{t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{t}|} \\
&\quad + {}_tE_{x+s}\,[\bar{A}_{x+s+t:\overline{n-t-s}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s+t:\overline{n-s-t}|}] \\
&= \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\;{}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}) - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{t}|}. \qquad (7.3.3)
\end{aligned}
$$

Thus the benefit reserves at the beginning and end of a t-year interval are connected by the following argument:

(the benefit reserve at the beginning of the interval) = (the actuarial present value at the beginning of the interval of benefits payable during the interval)

+ (the actuarial present value at the beginning of the interval of a pure endowment for the amount of the benefit reserve at the end of the interval)

− (the actuarial present value of benefit premiums payable during the interval).

The rearranged symbolic form,

$$
\begin{aligned}
{}_s\bar{V}(\bar{A}_{x:\overline{n}|}) &+ \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+s:\overline{n-t}|} \\
&= \bar{A}_{x+s:\overline{t}|} + {}_tE_{x+s}\ {}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}),
\end{aligned} \tag{7.3.4}
$$

shows that the actuarial present values of the insurer's resources and obligations are equal.

The retrospective formula is obtained from (7.3.4) by setting $s = 0$, noting that ${}_0\bar{V}(\bar{A}_{x:\overline{n}|}) = 0$ by the equivalence principle, and solving for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$. Thus,

$$
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \frac{1}{{}_tE_x} [\bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x:\overline{t}|} - \bar{A}^{\,1}_{x:\overline{t}|}].
$$

Further, $\bar{s}_{x:\overline{t}|} = \bar{a}_{x:\overline{t}|}/{}_tE_x$ so the formula reduces to

$$
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x. \tag{7.3.5}
$$

Here

$$
{}_t\bar{k}_x = \frac{\bar{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} \tag{7.3.6}
$$

is called the ***accumulated cost of insurance.*** One notes that

$$
\begin{aligned}
{}_t\bar{k}_x &= \int_0^t \frac{v^s\ {}_sp_x\ \mu_x(s)\, ds}{v^t\ {}_tp_x} \\
&= \int_0^t \frac{(1+i)^{t-s}\, l_{x+s}\ \mu_x(s)\, ds}{l_{x+t}}.
\end{aligned} \tag{7.3.7}
$$

This can be interpreted as the assessment against each of the l_{x+t} survivors to provide for the accumulated value of the death claims in the survivorship group between ages x and $x + t$. Thus, the reserve can be viewed as the difference between the benefit premiums, accumulated with interest and shared among only the survivors at age $x + t$, and the accumulated cost of insurance.

We conclude this section with some special formulas that express the whole life insurance benefit reserves in terms of a single actuarial function. Analogous formulas hold for n-year endowment insurance benefit reserves when benefit premiums are payable continuously for the n years. Because we used (5.2.8) to express $\bar{P}(\bar{A}_x)$ in terms of δ and either $\bar{a}_x$ or $\bar{A}_x$, we can use those ideas here to express ${}_t\bar{V}(\bar{A}_x)$ in terms of one of the actuarial functions $\bar{a}_x$, $\bar{A}_x$, or $\bar{P}(\bar{A}_x)$ and δ.

For an annuity function formula, substitute (6.2.9) and (5.2.8) into the prospective formula (7.2.3) to obtain

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_x) &= 1 - \delta\bar{a}_{x+t} - \left(\frac{1}{\bar{a}_x} - \delta\right) \bar{a}_{x+t} \\
&= 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}.
\end{aligned} \tag{7.3.8}
$$

Further, substituting (6.2.9) into the premium-difference formula, we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_x) &= [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\,\bar{a}_{x+t} \\
&= \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}.
\end{aligned} \tag{7.3.9}
$$

Finally, we can rewrite (7.3.8) using $\bar{A}_{x+t} = 1 - \delta\bar{a}_{x+t}$ to obtain

$$
{}_t\bar{V}(\bar{A}_x) = 1 - \frac{1 - \bar{A}_{x+t}}{1 - \bar{A}_x} = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}. \tag{7.3.10}
$$

These last three formulas depend on relationship (5.2.8) between the annuity for the premium paying period and the insurance for the benefit period. Thus they are available only for whole life and endowment insurances where the premium-paying period and the benefit period are the same. Moreover, the frequency of premium payment must be the same as the "frequency" of benefit payment. We will see that apportionable premiums satisfy these requirements in their own way.

Remark:

Although benefit reserves are non-negative in most applications, there is no mathematical theorem that guarantees this property. Indeed, the reader can combine Exercise 4.2 and formula (7.3.10) for a quick verification that negative benefit reserves are a real possibility.

7.4 Fully Discrete Benefit Reserves

The benefit reserves of this section are for the insurances of Section 6.3 which have annual premium payments and payment of the benefit at the end of the year of death. As in Section 7.2 the underlying mortality assumption can be on a select or aggregate basis. We will display the formulas for the aggregate case, which has simpler notation. Let us consider a whole life insurance with benefit issued to (x) with benefit premium P_x. Following the development in Section 7.2, for an insured surviving k years later, we define the benefit reserve, denoted by ${}_kV_x$, as the conditional expectation of the prospective loss, ${}_kL$, at duration k. More precisely,

$$
{}_kL = v^{(K(x)-k)+1} - P_x\,\ddot{a}_{\overline{(K(x)-k)+1}|} \tag{7.4.1}
$$

$$
{}_kV_x = \mathrm{E}[{}_kL | K(x) = k, k+1, \ldots]. \tag{7.4.2}
$$

The prospective formula for the benefit reserve is

$$
{}_kV_x = A_{x+k} - P_x\,\ddot{a}_{x+k}. \tag{7.4.3}
$$

As in Section 7.2 this formula is the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

Analogous to (7.2.4), we have

$$\begin{aligned}
\mathrm{Var}[{}_kL|K(x) &= k, k+1, \ldots] \\
&= \mathrm{Var}\left[v^{[K(x)-k]+1}\left(1+\frac{P_x}{d}\right)\middle| K(x) = k, k+1, \ldots\right] \\
&= \left(1+\frac{P_x}{d}\right)^2 \mathrm{Var}[v^{[K(x)-k]+1}|K(x) = k, k+1, \ldots] \\
&= \left(1+\frac{P_x}{d}\right)^2 [{}^2A_{x+k} - (A_{x+k})^2].
\end{aligned} \tag{7.4.4}$$

Example 7.4.1

Follow up Example 6.3.1 by calculating ${}_kV_x$ and $\mathrm{Var}[{}_kL|K(x) = k, k+1, \ldots]$.

Solution:

Here A_x, $\ddot{a}_x$, and P_x are independent of age x so that $A_{x+k} = A_x$ and

$${}_kV_x = A_x - P_x\,\ddot{a}_x = 0 \qquad k = 0, 1, 2, \ldots.$$

Also, from (7.4.4), $\mathrm{Var}[{}_kL|K(x) = k, k+1, \ldots] = \mathrm{Var}(L) = 0.2347$. ▼

The benefit reserve formulas tabulated in Table 7.4.1 correspond to the benefit premiums in Table 6.3.1 and are analogous to the benefit reserves in Table 7.2.1.

TABLE 7.4.1

Fully Discrete Benefit Reserves; Age at Issue x, Duration k, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula	
Whole life insurance	${}_kV_x$	$A_{x+k} - P_x\,\ddot{a}_{x+k}$	
n-Year term insurance	${}_kV^{1}_{x:\overline{n}\vert}$	$A^{1}_{x+k:\overline{n-k}\vert} - P^{1}_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{n-k}\vert}$	$k < n$
		0	$k = n$
n-Year endowment insurance	${}_kV_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert} - P_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{n-k}\vert}$	$k < n$
		1	$k = n$
h-Payment years, whole life insurance	${}^{h}_{k}V_x$	$A_{x+k} - {}_hP_x\,\ddot{a}_{x+k:\overline{h-k}\vert}$	$k < h$
		A_{x+k}	$k \ge h$
h-Payment years, n-year endowment insurance	${}^{h}_{k}V_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert} - {}_hP_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{h-k}\vert}$	$k < h < n$
		$A_{x+k:\overline{n-k}\vert}$	$h \le k < n$
		1	$k = n$
n-Year pure endowment	${}_kV_{x:\overline{n}\vert}^{\ \ 1}$	$A_{x+k:\overline{n-k}\vert}^{\ \ \ \ 1} - P_{x:\overline{n}\vert}^{\ \ 1}\,\ddot{a}_{x+k:\overline{n-k}\vert}$	$k < n$
		1	$k = n$
Whole life annuity	${}_kV({}_{n\vert}\ddot{a}_x)$	${}_{n-k\vert}\ddot{a}_{x+k} - P({}_{n\vert}\ddot{a}_x)\,\ddot{a}_{x+k:\overline{n-k}\vert}$	$k < n$
		$\ddot{a}_{x+k}$	$k \ge n$

Example 7.4.2

Determine an expression in actuarial present values and benefit premiums for the $\mathrm{Var}[{}_kL|K(x) = k, k+1, \ldots]$ for a fully discrete n-year endowment insurance with a unit benefit.

Solution:

$$
\begin{aligned}
{}_kL &= v^{K(x)-k+1}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = k, k+1, \ldots, n-1 \\
&= v^{n-k}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = n, n+1, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[{}_kL|K(x) &= k, k+1, \ldots] \\
&= \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 [{}^2A_{x+k:\overline{n-k}|} - (A_{x+k:\overline{n-k}|})^2].
\end{aligned}
$$

▼

In cases other than whole life or endowment insurances with premiums payable throughout the insurance term, the expressions for the variance of the loss may be difficult to summarize in convenient notation.

Formulas similar to those of Section 7.3 can be developed for fully discrete benefit reserves. We illustrate these by writing the formulas for ${}_kV_{\overline{x:n}|}$ with a minimum of discussion. Verbal interpretations and algebraic developments closely parallel those for fully continuous benefit reserves.

The premium difference formula is

$$
{}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{n-k}|}. \tag{7.4.5}
$$

The paid-up insurance formula is

$$
{}_kV_{x:\overline{n}|} = \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right) A_{x+k:\overline{n-k}|}. \tag{7.4.6}
$$

For the retrospective formula, we first establish a result analogous to (7.3.3), namely, for $h < n - j$,

$$
{}_jV_{x:\overline{n}|} = A^{\;1}_{x+j:\overline{h}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+j:\overline{h}|} + {}_hE_{x+j}\; {}_{j+h}V_{x:\overline{n}|}. \tag{7.4.7}
$$

Then, if $j = 0$, we have, since ${}_0V_{x:\overline{n}|} = 0$,

$$
\begin{aligned}
{}_hV_{x:\overline{n}|} &= \frac{1}{{}_hE_x}(P_{x:\overline{n}|}\, \ddot{a}_{x:\overline{h}|} - A^{\;1}_{x:\overline{h}|}) \\
&= P_{x:\overline{n}|}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x.
\end{aligned} \tag{7.4.8}
$$

Here the accumulated cost of insurance is ${}_hk_x = A^{\;1}_{x:\overline{h}|} / {}_hE_x$, and a survivorship group interpretation is possible.

An interesting observation follows from the retrospective formula for the benefit reserve. Let us consider two different policies issued to (x), each for a unit of

Example 7.4.2

Determine an expression in actuarial present values and benefit premiums for the Var[${}_kL|K(x) = k, k + 1, \ldots$] for a fully discrete n-year endowment insurance with a unit benefit.

Solution:

$$\begin{aligned} {}_kL &= v^{K(x)-k+1}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = k, k+1, \ldots, n-1 \\ &= v^{n-k}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = n, n+1, \ldots, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}[{}_kL|K(x) &= k, k+1, \ldots] \\ &= \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 [{}^2A_{x+k:\overline{n-k}|} - (A_{x+k:\overline{n-k}|})^2]. \end{aligned}$$

▼

In cases other than whole life or endowment insurances with premiums payable throughout the insurance term, the expressions for the variance of the loss may be difficult to summarize in convenient notation.

Formulas similar to those of Section 7.3 can be developed for fully discrete benefit reserves. We illustrate these by writing the formulas for ${}_kV_{\overline{x:n}|}$ with a minimum of discussion. Verbal interpretations and algebraic developments closely parallel those for fully continuous benefit reserves.

The premium difference formula is

$$ {}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{n-k}|}. \tag{7.4.5}$$

The paid-up insurance formula is

$$ {}_kV_{x:\overline{n}|} = \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right) A_{x+k:\overline{n-k}|}. \tag{7.4.6}$$

For the retrospective formula, we first establish a result analogous to (7.3.3), namely, for $h < n - j$,

$$ {}_jV_{x:\overline{n}|} = A^{\,1}_{x+j:\overline{h}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+j:\overline{h}|} + {}_hE_{x+j}\; {}_{j+h}V_{x:\overline{n}|}. \tag{7.4.7}$$

Then, if $j = 0$, we have, since ${}_0V_{x:\overline{n}|} = 0$,

$$\begin{aligned} {}_hV_{x:\overline{n}|} &= \frac{1}{{}_hE_x}\,(P_{x:\overline{n}|}\, \ddot{a}_{x:\overline{h}|} - A^{\,1}_{x:\overline{h}|}) \\ &= P_{x:\overline{n}|}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x. \end{aligned} \tag{7.4.8}$$

Here the accumulated cost of insurance is ${}_hk_x = A^{\,1}_{x:\overline{h}|} / {}_hE_x$, and a survivorship group interpretation is possible.

An interesting observation follows from the retrospective formula for the benefit reserve. Let us consider two different policies issued to (x), each for a unit of insurance during the first h years. Here, h is less than or equal to the shorter of the two premium-payment periods. The retrospective formula for the benefit reserve on policy one is

$$ {}_hV_{(1)} = P_{(1)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x $$

and that for the benefit reserve on policy two is

$$ {}_hV_{(2)} = P_{(2)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x. $$

It follows that

$$ {}_hV_{(1)} - {}_hV_{(2)} = (P_{(1)} - P_{(2)})\, \ddot{s}_{x:\overline{h}|}, \tag{7.4.9} $$

which shows that the difference in the two benefit reserves equals the actuarial accumulated value of the difference in the benefit premiums $P_{(1)} - P_{(2)}$. Since $1/\ddot{s}_{x:\overline{h}|} = P_{x:\overline{h}|}^{\;\;1}$, formula (7.4.9) can be rearranged as

$$ P_{(1)} - P_{(2)} = P_{x:\overline{h}|}^{\;\;1}\, ({}_hV_{(1)} - {}_hV_{(2)}). \tag{7.4.10} $$

The difference in the benefit premiums is expressed as the benefit premium for an h-year pure endowment of the difference in the benefit reserves at the end of h years. Formula (6.3.10) is a special case of (7.4.10) with ${}_nV_{x:\overline{n}|} = 1$ and ${}_n^nV_x = A_{x+n}$. Another illustration of (7.4.10) is

$$ P_x = P_{x:\overline{n}|}^1 + P_{x:\overline{n}|}^{\;\;1}\, {}_nV_x \tag{7.4.11} $$

since ${}_nV_{x:\overline{n}|}^1 = 0$.

As in the fully continuous case, there are special formulas for whole life and endowment insurance benefit reserves in the fully discrete case. Parallel to (7.3.8)–(7.3.10), we have, by use of the relations $A_y = 1 - d\,\ddot{a}_y$ and $1/\ddot{a}_y = P_y + d$,

$$ {}_kV_x = 1 - d\,\ddot{a}_{x+k} - \left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+k} $$

$$ = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}, \tag{7.4.12} $$

$$ {}_kV_x = 1 - \frac{P_x + d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}, \tag{7.4.13} $$

and

$$ {}_kV_x = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}. \tag{7.4.14} $$

Similar special formulas also hold for fully discrete n-year endowment insurance benefit reserves, but not for insurance benefit reserves in general.

Fully discrete insurances provide instructive examples for the deterministic, or expected cash flow, model of the operations of benefit reserves. This is displayed in Examples 7.4.3 and 7.4.4.

Example 7.4.3

Assume that a 5-year term life insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and, as a by-product, obtain the benefit reserves.

Solution:

We first calculate the annual benefit premium $\pi = 1{,}000\ P^{1}_{50:\overline{5}|} = 6.55692$. Then the expected accumulation of funds for the group through the collection of benefit premiums, the crediting of interest, and the payment of claims is as stated in the following:

(1) Year h	(2) Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\ \pi$	(3) Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	(4) Expected Interest $(0.06)\ (3)_h$	(5) Expected Death Claims $1{,}000\ d_{50+h-1}$	(6) Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	(7) Expected Number of Survivors at End of Year l_{50+h}	(8) $1{,}000\ {}_hV^{1}_{50:\overline{5}|}$ $(6)_h/(7)_h$
1	586 903	586 903	35 214	529 884	92 233	88 979.11	1.04
2	583 429	675 662	40 540	571 432	144 770	88 407.68	1.64
3	579 682	724 452	43 467	616 416	151 503	87 791.26	1.73
4	575 640	727 143	43 629	665 065	105 707	87 126.20	1.21
5	571 280	676 987	40 619	717 606	0	86 408.60	0.00

Note that the benefit reserves derived in the table match those calculated by formula. For example, at duration 2 we have

$$1{,}000\ A^{1}_{52:\overline{3}|} = 20.09 \text{ and } \ddot{a}_{52:\overline{3}|} = 2.81391.$$

Then

$$1{,}000\ {}_2V^{1}_{50:\overline{5}|} = 20.09 - (6.55692)(2.81391) = 1.54.$$ ▼

Example 7.4.4

Assume that a 5-year endowment insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and as a by-product obtain the benefit reserves.

Solution:

Here the annual benefit premium is $\pi = 1{,}000\ P_{50:\overline{5}|} = 170.083$. The expected cash flow is displayed in the following table:

(1) Year h	(2) Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\,\pi$	(3) Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	(4) Expected Interest $(0.06)\,(3)_h$	(5) Expected Death Claims $1{,}000\,d_{50+h-1}$	(6) Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	(7) Expected Number of Survivors at End of Year l_{50+h}	(8) $1{,}000\ {}_hV^{\,1}_{50:\overline{5}\vert}$ $(6)_h \div (7)_h$
1	15 223 954	15 223 954	913 437	529 884	15 607 507	88 979.11	175.14
2	15 133 829	30 741 336	1 844 480	571 432	32 014 384	88 407.68	362.12
3	15 036 638	47 051 022	2 823 061	616 416	49 257 667	87 791.26	561.08
4	14 931 796	64 189 463	3 851 368	665 065	67 375 766	87 126.20	773.31
5	14 818 680	82 194 446	4 931 667	717 606	86 408 507	86 408.60	1 000.00

▼

Figures 7.4.1 and 7.4.2 display the expected benefit premiums and expected death claims for the preceding two examples. In Example 7.4.3, expected benefit premiums exceed expected death claims for 2 years, but thereafter are less than expected claims. The excess benefit premiums accumulate a fund in the early years to be drawn on in the later years when expected claims are higher. At the end of 5 years, the fund is expected to be exhausted.

FIGURE 7.4.1

Expected Benefit Premiums and Expected Death Claims for Example 7.4.3

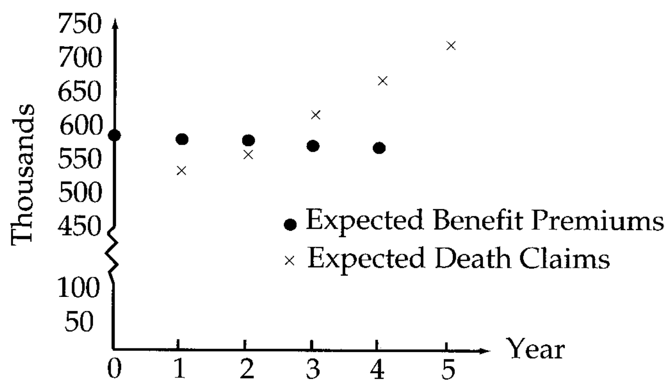
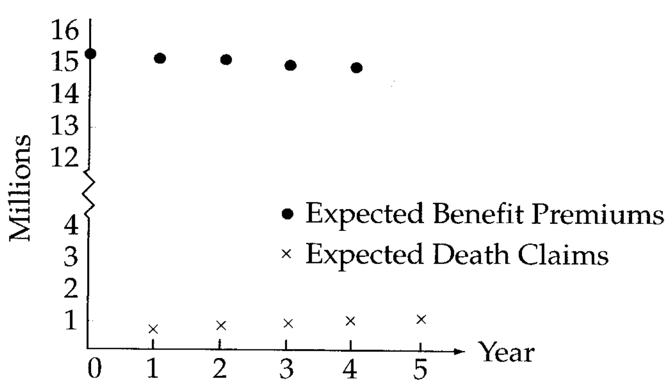


FIGURE 7.4.2

Expected Benefit Premiums and Expected Death Claims for Example 7.4.4



For the 5-year endowment case of Example 7.4.4, the picture is much different. As shown in Figure 7.4.2, the expected benefit premiums remain far in excess of expected death claims throughout. The expected fund at the end of 5 years is sufficient to provide 1,000 in maturity payments to each of the expected survivors.

The 5-year term insurance exemplifies a low-premium, low-accumulation life insurance, whereas the 5-year endowment insurance exemplifies a high-premium, high-accumulation form. Most life insurances would fall between these two extremes.

7.5 Benefit Reserves on a Semicontinuous Basis

We noted at the end of Section 6.3 that, in practice, there is a need for semicontinuous annual benefit premiums $P(\bar{A}_x)$, $P(\bar{A}_{x:\overline{n}|})$, $P(\bar{A}^{\,1}_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$ to take account of immediate payment of death claims. In such cases, the benefit reserve formulas in Table 7.4.1 need to be revised by replacement of A by $\bar{A}$ and of P by $P(\bar{A})$. Moreover, the principal symbol for the benefit reserve is now $V(\bar{A})$ with a subscript on the $\bar{A}$ to indicate the type of insurance as in the benefit premium symbol. For example, for an h-payment years, n-year endowment insurance

$$ {}^h_kV(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}_{x+k:\overline{n-k}|} & h \le k < n \\ 1 & k = n. \end{cases} \tag{7.5.1} $$

If a uniform distribution of deaths over each year of age is assumed, we have, from (4.4.2) and (6.3.12),

$$ {}^h_kV(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}\,{}^h_kV^{1}_{x:\overline{n}|} + {}^h_kV_{x:\overline{n}|}^{\;\;\,1}. \tag{7.5.2} $$

Under this circumstance benefit reserves on a semicontinuous basis are easily calculated from the corresponding fully discrete benefit reserves.

7.6 Benefit Reserves Based on True *m*-thly Benefit Premiums

In this section we examine the benefit reserve formulas corresponding to the formulas for true m-thly benefit premiums discussed in Section 6.4. By the prospective method, one can write a direct formula for ${}^h_kV^{(m)}_{x:\overline{n}|}$, namely,

$$ {}^h_kV^{(m)}_{x:\overline{n}|} = A_{x+k:\overline{n-k}|} - {}_hP^{(m)}_{x:\overline{n}|}\,\ddot{a}^{(m)}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.6.1} $$

This can be evaluated after obtaining ${}_hP^{(m)}_{x:\overline{n}|}$ by means of (6.4.1) or (6.4.2), and $\ddot{a}^{(m)}_{x+k:\overline{h-k}|}$ by means of (5.4.15) or (5.4.17).

We now consider the difference between ${}^h_kV^{(m)}_{x:\overline{n}|}$ and ${}^h_kV_{x:\overline{n}|}$ in the general case of a limited payment endowment insurance. We have, for $k < h$,

$$
\begin{aligned}
{}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} &= {}_hP_{x:\overline{n}|}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_hP_{x:\overline{n}|}^{(m)}\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)} \\
&= {}_hP_{x:\overline{n}|}^{(m)} \frac{\ddot{a}_{x:\overline{h}|}^{(m)}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_hP_{x:\overline{n}|}^{(m)}\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)}.
\end{aligned} \tag{7.6.2}
$$

Under the assumption of a uniform distribution of deaths in each year of age, (7.6.2) becomes

$$
\begin{aligned}
{}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} = {}_hP_{x:\overline{n}|}^{(m)} \Bigg\{ &\frac{\ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_{x:\overline{h}|} - \beta(m)\, A_{x:\overline{h}|}^{1}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} \\
&- \left[\ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_{x+k:\overline{h-k}|} - \beta(m)\, A_{x+k:\overline{h-k}|}^{1} \right] \Bigg\}.
\end{aligned}
$$

The terms involving $\ddot{a}_{\overline{1}|}^{(m)}$ cancel to yield

$$
{}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} = \beta(m)\, {}_hP_{x:\overline{n}|}^{(m)}\, {}_kV_{x:\overline{h}|}^{1}. \tag{7.6.3}
$$

Thus,

(the benefit reserve for an insurance with true m-thly benefit premiums) = (the corresponding fully discrete benefit reserve)

$$
+ \begin{pmatrix} \text{a fully discrete benefit reserve for term insurance over} \\ \text{the premium paying period for a fraction, } \beta(m) \text{, of the} \\ \text{true } m\text{-thly benefit premium for the plan of insurance} \end{pmatrix}.
$$

A similar result holds for benefit reserves on a semicontinuous basis with true m-thly benefit premiums under the assumption of uniform distribution of deaths in each year of age. By the prospective method, we have for $k < h$,

$$
{}_k^hV^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_hP^{(m)}(\bar{A}_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)}. \tag{7.6.4}
$$

By steps analogous to those connecting (7.6.1) and (7.6.3), we obtain

$$
{}_k^hV^{(m)}(\bar{A}_{x:\overline{n}|}) = {}_k^hV(\bar{A}_{x:\overline{n}|}) + \beta(m)\, {}_hP^{(m)}(\bar{A}_{x:\overline{n}|})\, {}_kV_{x:\overline{h}|}^{1}. \tag{7.6.5}
$$

Further, by letting $m \to \infty$ above, we obtain for a fully continuous basis

$$
{}_k^h\bar{V}(\bar{A}_{x:\overline{n}|}) = {}_k^hV(\bar{A}_{x:\overline{n}|}) + \beta(\infty)\, {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\, {}_kV_{x:\overline{h}|}^{1}. \tag{7.6.6}
$$

Note again that the term insurance benefit reserve is on a fully discrete basis.

Example 7.6.1

On the basis of the Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ calculate the following for a 20-year endowment insurance issued to (50) with a unit benefit and true semiannual benefit premiums:

a. The benefit reserve at the end of the tenth year if the benefit is payable at the end of the year of death.
b. The benefit reserve at the end of the tenth year if the benefit is payable at the moment of death.

Also verify (7.6.5) in relation to the benefit reserve in part (b).

Solution:

a. In addition to the values calculated in Example 6.4.1, we require

$$A^{\ \,1}_{60:\overline{10}|} = 0.13678852$$

$$A_{60:\overline{10}|} = 0.58798425$$

$$\ddot{a}_{60:\overline{10}|} = 7.2789425$$

$${}_{10}V^{\ \,1}_{50:\overline{20}|} = A^{\ \,1}_{60:\overline{10}|} - P^{\ \,1}_{50:\overline{20}|}\ \ddot{a}_{60:\overline{10}|} = 0.052752$$

$${}_{10}V_{50:\overline{20}|} = A_{60:\overline{10}|} - P_{50:\overline{20}|}\ \ddot{a}_{60:\overline{10}|} = 0.355380.$$

Then, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\ddot{a}^{(2)}_{60:\overline{10}|} = \alpha(2)\ \ddot{a}_{60:\overline{10}|} - \beta(2)\ (1 - {}_{10}E_{60}) = 7.1392299.$$

The benefit reserve, ${}_{10}V^{(2)}_{50:\overline{20}|}$, can be calculated using either

(7.6.1): $$A_{60:\overline{10}|} - P^{(2)}_{50:\overline{20}|}\ \ddot{a}^{(2)}_{60:\overline{10}|} = 0.355822$$

or

(7.6.3): $${}_{10}V_{50:\overline{20}|} + \beta(2)\ P^{(2)}_{50:\overline{20}|}\ {}_{10}V^{\ \,1}_{50:\overline{20}|} = 0.355822.$$

b. We need additional calculated values:

$$\frac{i}{\delta}\ A^{\ \,1}_{50:\overline{20}|} = 0.13423835$$

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}} = 0.03286830$$

$$\frac{i}{\delta}\ A^{\ \,1}_{60:\overline{10}|} = 0.14085233$$

$$\bar{A}_{50:\overline{20}|} = 0.36471188$$

$$P(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} = 0.03229873$$

$$\bar{A}_{60:\overline{10}|} = 0.59204806$$

$${}_{10}V(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P(\bar{A}_{50:\overline{20}|})\ \ddot{a}_{60:\overline{10}|} = 0.3569475$$

$${}_{10}V^{(2)}(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P^{(2)}(\bar{A}_{50:\overline{20}|})\ \ddot{a}^{(2)}_{60:\overline{10}|} = 0.3573937$$

$$\beta(2)\ P^{(2)}(\bar{A}_{50:\overline{20}|})\ {}_{10}V^{\ \,1}_{50:\overline{20}|} = 0.000446.$$

This last value is the difference between the two directly above it, as shown in (7.6.5). ▼

7.7 Benefit Reserves on an Apportionable or Discounted Continuous Basis

In Section 6.5 we discussed apportionable, or discounted continuous, benefit premiums, and we now consider the corresponding benefit reserves. For integer k, we have by the prospective method

$$ {}_k^hV^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\, \ddot{a}^{\{m\}}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.7.1}$$

But by (6.5.2),

$$ {}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{d^{(m)}}{\delta}\, {}_h\bar{P}(\bar{A}_{x:\overline{n}|}),$$

and by (5.5.4),

$$ \ddot{a}^{\{m\}}_{x+k:\overline{h-k}|} = \frac{\delta}{d^{(m)}}\, \bar{a}_{x+k:\overline{h-k}|}.$$

Substitution into (7.7.1) yields, for an integer k,

$$ {}_k^hV^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+k:\overline{h-k}|} = {}_k^h\bar{V}(\bar{A}_{x:\overline{n}|}). \tag{7.7.2}$$

This means that, on anniversaries of the issue date, fully continuous benefit reserves can be used for all apportionable cases, independent of the premium-paying mode. The condition that k be an integer can be relaxed to being at the end of an m-th for m-thly premiums.

In Section 6.5, it was noted that the apportionable benefit premium could be decomposed as

$$ P^{\{1\}}(\bar{A}_x) = P(\bar{A}_x) + P(\bar{A}_x^{\mathrm{PR}}) \tag{7.7.3}$$

where the superscript PR is used to denote an insurance for the benefit premium refund feature. A similar decomposition for the benefit reserves can be verified by use of the prospective method and (6.5.7). The steps are:

$$\begin{aligned} {}_kV(\bar{A}_x^{\mathrm{PR}}) &= \bar{P}(\bar{A}_x)\,\frac{\bar{A}_{x+k} - A_{x+k}}{\delta} - P(\bar{A}_x^{\mathrm{PR}})\,\ddot{a}_{x+k} \\ &= \bar{P}(\bar{A}_x)\,\frac{d\ddot{a}_{x+k} - \delta\bar{a}_{x+k}}{\delta} - [P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x)]\,\ddot{a}_{x+k}. \end{aligned}$$

Since

$$ \frac{d}{\delta}\,\bar{P}(\bar{A}_x) = P^{\{1\}}(\bar{A}_x),$$

the expression can be reduced to

$$\begin{aligned} {}_kV(\bar{A}_x^{\mathrm{PR}}) &= -\bar{P}(\bar{A}_x)\,\bar{a}_{x+k} + P(\bar{A}_x)\,\ddot{a}_{x+k} \\ &= \bar{A}_{x+k} - \bar{P}(\bar{A}_x)\,\bar{a}_{x+k} - [\bar{A}_{x+k} - P(\bar{A}_x)\,\ddot{a}_{x+k}] \end{aligned}$$

$$= {}_k\bar{V}(\bar{A}_x) - {}_kV(\bar{A}_x)$$
$$= {}_kV^{\{1\}}(\bar{A}_x) - {}_kV(\bar{A}_x).$$

Thus we have

$${}_kV^{\{1\}}(\bar{A}_x) = {}_kV(\bar{A}_k) + {}_kV(\bar{A}_x^{\mathrm{PR}}). \tag{7.7.4}$$

7.8 Notes and References

This chapter has developed the idea of a reserve in parallel to the development of premiums in Chapter 6. Discussion of recursion formulas for reserves is deferred to Chapter 8. Reserve principles based on the utility functions used in Chapter 6 were first applied. Gerber (1976, 1979) develops these reserves in a more abstract setting. Benefit reserves, which followed from a linear utility function, were studied extensively. Scher (1974) explored the apportionable benefit premium reserves as discounted fully continuous benefit reserves.

Exercises

Section 7.1

7.1. Determine the benefit reserve for $t = 2, 3, 4,$ and 5 for the insurance in Example 6.1.1.

7.2. Determine the exponential reserve for $t = 2, 3, 4,$ and 5 for the insurance in Example 6.1.1.

7.3. Determine the exponential reserve for $t = 1, 2, 3, 4,$ and 5 for the insurance in Exercise 6.2.

7.4. Consider the insurance in Example 7.1.1 and the insurer of Exercise 6.3 with utility function $u(x) = x - 0.01\,x^2$, $0 < x < 50$. Determine the reserve, ${}_kV$, for $k = 1, 2, 3,$ and 4 such that the insurer, with wealth 10 at each duration, will be indifferent between continuing the risk while receiving premiums of 0.30360 (from Exercise 6.3) and paying the amount ${}_kV$ to a reinsurer to assume the risk.

7.5. Consider a unit insurance issued to (0) on a fully continuous basis using the following assumptions:
   i. De Moivre's law with $\omega = 5$
   ii. $i = 0.06$
   iii. Principle III of Example 6.1.1 with $\alpha = 0.1$.
   a. Display equations which can be solved for the exponential premium and the exponential reserve at $t = 1$.

b. Solve the equations of (a) for the numerical values for the exponential premium and exponential reserve. Numerical methods must be used to obtain these required solutions.

Section 7.2

7.6. For an n-year unit endowment insurance issued on a fully continuous basis to (x), define ${}_tL$, the prospective loss after duration t. Confirm that

$$\mathrm{Var}({}_tL|T > t) = \frac{{}^2\bar{A}_{x+t:\overline{n-t}|} - (\bar{A}_{x+t:\overline{n-t}|})^2}{(\delta\bar{a}_{x:\overline{n}|})^2}.$$

7.7. The prospective loss, after duration t, for a single benefit premium n-year continuous temporary life annuity of 1 per annum issued to (x) is given by

$${}_tL = \begin{cases} \bar{a}_{\overline{T-t}|} & t \le T < n \\ \bar{a}_{\overline{n-t}|} & T \ge n. \end{cases}$$

Express $\mathrm{E}[{}_tL|T > t]$ and $\mathrm{Var}({}_tL|T > t)$ in symbols of actuarial present values.

7.8. Write prospective formulas for
a. ${}^{20}_{10}\bar{V}(\bar{A}_{35:\overline{30}|})$
b. the benefit reserve at the end of 5 years for a unit benefit 10-year term insurance issued to (45) on a single premium basis.

7.9. a. For the fully continuous whole life insurance with the benefit premium determined by the equivalence principle, determine the outcome $u_0 = T(x) - t$ such that the loss is zero. [Caution: For large values of t, a solution may not exist.]
b. Determine the value of u_0 for $t = 20$ in Example 7.2.3 and compare it to Figure 7.2.1 for reasonableness.

7.10. The assumptions of Example 7.2.3 are repeated. Find the value of t such that the minimum loss is zero. Check your result by examining Figure 7.2.2.

7.11. a. Repeat the development leading to (7.2.9) to obtain the d.f. for the loss variable associated with an n-year fully continuous endowment insurance.
b. Draw the sketch that corresponds to Figure 7.2.1 for this endowment insurance.

7.12. Repeat Exercise 7.11 for an n-year fully continuous term insurance.

7.13. Confirm that (7.2.10) satisfies the conditions for a p.d.f.

Section 7.3

7.14. Write four formulas for ${}^{20}_{10}\bar{V}(\bar{A}_{40})$.

7.15. Write seven formulas for ${}_{10}\bar{V}(\bar{A}_{40:\overline{20}|})$.

7.16. Give the retrospective formula for ${}^{30}_{20}\bar{V}({}_{30|}\bar{a}_{35})$.

7.17. For $0 < t \le m$, show

a. $\bar{P}(\bar{A}_{x:\overline{m+n}|}) = \bar{P}(\bar{A}^{\,1}_{x:\overline{m}|}) + \bar{P}_{x:\overline{m}|}^{\;\;\;\;1}\ {}_{m}\bar{V}(\bar{A}_{x:\overline{m+n}|})$

b. ${}_{t}\bar{V}(\bar{A}_{x:\overline{m+n}|}) = {}_{t}\bar{V}(\bar{A}^{\,1}_{x:\overline{m}|}) + {}_{t}\bar{V}_{x:\overline{m}|}^{\;\;\;\;1}\ {}_{m}\bar{V}(\bar{A}_{x:\overline{m+n}|})$

and give an interpretation in words.

7.18. State what formula in Section 7.3 the following equation is related to, and give an interpretation in words:

$${}^{20}_{10}\bar{V}(\bar{A}_{30}) = \bar{A}^{\,1}_{40:\overline{5}|} + {}_{5}E_{40}\ {}^{20}_{15}\bar{V}(\bar{A}_{30}) - {}_{20}\bar{P}(\bar{A}_{30})\ \bar{a}_{40:\overline{5}|}.$$

Section 7.4

7.19. Write four formulas for ${}^{20}_{10}V_{40}$.

7.20. Write seven formulas for ${}_{10}V_{40:\overline{20}|}$.

7.21. For $0 < k \le m$, show

$${}_{k}V_{x:\overline{m+n}|} = {}_{k}V^{1}_{x:\overline{m}|} + {}_{k}V_{x:\overline{m}|}^{\;\;\;\;1}\ {}_{m}V_{x:\overline{m+n}|}.$$

7.22. If $k < n/2$, ${}_{k}V_{x:\overline{n}|} = 1/6$, and $\ddot{a}_{x:\overline{n}|} + \ddot{a}_{x+2k:\overline{n-2k}|} = 2\ \ddot{a}_{x+k:\overline{n-k}|}$, calculate ${}_{k}V_{x+k:\overline{n-k}|}$.

Section 7.5

7.23. On the basis of the Illustrative Life Table and interest of 6%, calculate values for the benefit reserves in the following table. (See Exercise 6.10.)

Fully Continuous	**Semicontinuous**	**Fully Discrete**
${}_{10}\bar{V}(\bar{A}_{35:\overline{30}|})$	${}_{10}V(\bar{A}_{35:\overline{30}|})$	${}_{10}V_{35:\overline{30}|}$
${}_{10}\bar{V}(\bar{A}_{35})$	${}_{10}V(\bar{A}_{35})$	${}_{10}V_{35}$
${}_{10}\bar{V}(\bar{A}^{\,1}_{35:\overline{30}|})$	${}_{10}V(\bar{A}^{\,1}_{35:\overline{30}|})$	${}_{10}V^{1}_{35:\overline{30}|}$

7.24. Under the assumption of a uniform distribution of deaths in each year of age, which of the following are correct?

a. ${}_{k}V(\bar{A}_{x:\overline{n}|}) = \dfrac{i}{\delta}\ {}_{k}V_{x:\overline{n}|}$

b. ${}_{k}V(\bar{A}_{x}) = \dfrac{i}{\delta}\ {}_{k}V_{x}$

c. ${}_{k}V(\bar{A}^{\,1}_{x:\overline{n}|}) = \dfrac{i}{\delta}\ {}_{k}V^{1}_{x:\overline{n}|}$

Section 7.6

7.25. Show that, under the assumption of a uniform distribution of deaths in each year of age,

$$\frac{{}_{5}V^{(4)}_{30:\overline{20}|} - {}_{5}V_{30:\overline{20}|}}{{}^{20}_{5}V^{(4)}_{30} - {}^{20}_{5}V_{30}} = \frac{A_{30:\overline{20}|}}{A_{30}}.$$

(The assumption is sufficient but not necessary.)

7.26. Which of the following are correct formulas for ${}_{15}V_{40}^{(m)}$?

a. $(P_{55}^{(m)} - P_{40}^{(m)})\,\ddot{a}_{55}^{(m)}$ b. $\left(1 - \dfrac{P_{40}^{(m)}}{P_{55}^{(m)}}\right) A_{55}$

c. $P_{40}^{(m)}\,\ddot{s}_{40:\overline{15}|}^{(m)} - {}_{15}k_{40}$ d. $1 - \dfrac{\ddot{a}_{55}^{(m)}}{\ddot{a}_{40}^{(m)}}$

Section 7.7

7.27. Which of the following are correct formulas for ${}_{15}V^{\{4\}}(\bar{A}_{40})$?

a. ${}_{15}\bar{V}(\bar{A}_{40})$ b. $[P^{\{4\}}(\bar{A}_{55}) - P^{\{4\}}(\bar{A}_{40})]\ddot{a}_{55}^{\{4\}}$

c. $[\bar{P}(\bar{A}_{55}) - \bar{P}(\bar{A}_{40})]\bar{a}_{55}$ d. $\left[1 - \dfrac{\bar{P}(\bar{A}_{40})}{\bar{P}(\bar{A}_{55})}\right] \bar{A}_{55}$

e. $1 - \dfrac{\bar{a}_{55}}{\bar{a}_{40}}$ f. $\bar{P}(\bar{A}_{40})\,\bar{s}_{40:\overline{15}|} - {}_{15}\bar{k}_{40}$

7.28. Show that

a. $P^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_{n}P^{\{m\}}(\bar{A}_{x}) + (1 - \bar{A}_{x+n})P_{x:\overline{n}|}^{\{m\}\;\;1}$

b. ${}_{k}V^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_{k}^{n}V^{\{m\}}(\bar{A}_{x}) + (1 - \bar{A}_{x+n})_{k}V_{x:\overline{n}|}^{\{m\}\;\;1}$.

Give an interpretation in words.

Miscellaneous

7.29. Calculate the value of $P^{1}_{x:\overline{n}|}$ if ${}_{n}V_{x} = 0.080$, $P_{x} = 0.024$, and $P_{x:\overline{n}|}^{\;\;1} = 0.2$.

7.30. If ${}_{10}V_{35} = 0.150$ and ${}_{20}V_{35} = 0.354$, calculate ${}_{10}V_{45}$.

7.31. A whole life insurance issued to (25) pays a unit benefit at the end of the year of death. Premiums are payable annually to age 65. The benefit premium for the first 10 years is P_{25} followed by an increased level annual benefit premium for the next 30 years. Use your Illustrative Life Table and $i = 0.06$ to find the following.

a. The annual benefit premium payable at ages 35 through 64.
b. The tenth-year benefit reserve.
c. At the end of 10 years the policyholder has the option to continue with the benefit premium P_{25} until age 65 in return for reducing the death benefit to B for death after age 35. Calculate B.
d. If the option in (c) is selected, calculate the twentieth-year benefit reserve.

7.32. Assuming $\delta = 0.05$, $q_x = 0.05$, and a uniform distribution of deaths in each year of age, calculate

a. $(\bar{I}\bar{A})^{1}_{x:\overline{1}|}$ b. ${}_{1/2}V((\bar{I}\bar{A})^{1}_{x:\overline{1}|})$.

8

ANALYSIS OF BENEFIT RESERVES

8.1 Introduction

In Chapter 3 probability distributions for future lifetime random variables were developed. Chapters 4 and 5 studied the present-value random variables for insurances and annuities. The funding of insurance and annuities with a system of periodic payments was explored in Chapter 6, and in Chapter 7 the evolution of the liabilities under the periodic payments to fund an insurance or annuity was discussed. In these last two chapters the emphasis was on ***level*** benefits funded by ***level*** periodic payments that are usually determined by an application of the equivalence principle.

Why was the emphasis on level payments? First, traditional insurance products are purchased with level contract premiums. It is natural to think of a constant portion of each premium being for the benefit, hence a level benefit premium. Second, the single equation of the equivalence principle yields a solution for only one parameter. It is natural to think of this parameter as the benefit premium. Third, until the incidence of expenses is discussed in Chapter 15, one of the motivations of nonlevel benefit premiums is not present. Fourth, historically some regulatory standards have been specified in terms of benefit reserves defined by level benefit premiums.

In this chapter we define benefit reserves as we did in Chapter 7, but the definition is applied to general contracts with possibly nonlevel benefits and premiums. Of course, level premiums are a special case of nonlevel premiums, so the ideas here apply to the examples of Chapter 7. However, the reverse is not true. The special technique and relationships of Chapter 7 may not apply to the more general contracts of this chapter.

We start with definitions of general fully continuous and fully discrete insurances, and recursion relations are developed for these general models. The general discrete model is used to obtain formulas for benefit reserves at durations other than a contract anniversary, something that was not obtained in Chapter 7. An allocation of loss and of risk of the contract to the various periods of the contract duration is obtained by use of the general fully discrete model. Again, these ideas apply to the contracts of Chapter 7, and the reader is encouraged to exercise this application.

8.2 Benefit Reserves for General Insurances

Consider a general fully discrete insurance on (x) in which

- The death benefit is payable at the end of the policy year of death
- Premiums are payable annually, at the beginning of the policy year
- The death benefit in the j-th policy year is b_j, $j = 1, 2, \ldots$
- The benefit premium payment in the j-th policy year is π_{j-1}, $j = 1, 2, \ldots$.

Note that the subscripts of b and π are the times of payment.

For a non-negative integer, h, the prospective loss, ${}_hL$, is the present value at h of the future benefits less the present value at h of the future benefit premiums. Expressed as a function of $K(x)$, it is

$$ {}_hL = \begin{cases} 0 & K(x) = 0, 1, \ldots h-1 \\ b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} & K(x) = h, h+1, \ldots. \end{cases} \tag{8.2.1} $$

Note: This definition extends the one given in (7.4.1) by including values (zeros) of ${}_hL$ for $K(x)$ less than h. Of course, this extension will not change the value of the benefit reserve because it is the conditional expectation given $K(x) \geq h$. The extension will be used in the development of recursion relations.

The benefit reserve at h, which we will denote by ${}_hV$, is defined as

$$ \begin{aligned} {}_hV &= \mathrm{E}[{}_hL | K(x) \geq h] \\ &= \mathrm{E}\left[b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} | K(x) \geq h \right] \\ &= \mathrm{E}\left[b_{(K(x)-h)+h+1}\, v^{(K(x)-h)+1} - \sum_{j=0}^{K(x)-h} \pi_{h+j}\, v^{j} | K(x) \geq h \right]. \end{aligned} \tag{8.2.2} $$

Under the assumption that the conditional distribution of $K(x) - h$, given $K(x) = h, h+1, \ldots$, is equal to the distribution of $K(x+h)$, this last expression can be rewritten as

$$ \begin{aligned} {}_hV &= \mathrm{E}\left[b_{K(x+h)+h+1}\, v^{K(x+h)+1} - \sum_{j=0}^{K(x+h)} \pi_{h+j}\, v^{j} \right] \\ &= \sum_{j=0}^{\infty} \left(b_{h+j+1}\, v^{j+1} - \sum_{k=0}^{j} \pi_{h+k}\, v^{k} \right) {}_jp_{x+h}\, q_{x+h+j}. \end{aligned} \tag{8.2.3} $$

Note that if this assumption fails, we are in the select mortality mode. By applying summation by parts (see Appendix 5) or reversing the order of summation, (8.2.3) can be rewritten as

$$ {}_hV = \sum_{j=0}^{\infty} b_{h+j+1}\, v^{j+1}\, {}_jp_{x+h}\, q_{x+h+j} - \sum_{j=0}^{\infty} \pi_{h+j}\, v^j\, {}_jp_{x+h}. \tag{8.2.4} $$

Thus, ${}_hV$ as defined by (8.2.2) converts readily to the prospective formula: the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

In Chapter 7 we discussed four types of formulas for the benefit reserve: prospective, retrospective, premium-difference, and paid-up insurance. These were applicable to benefit reserves for contracts with level benefit premiums and level benefits. Only the prospective and retrospective forms extend naturally to the general fully discrete insurance. The retrospective formula will be developed in the next section.

Example 8.2.1

A fully discrete whole life insurance with a unit benefit issued to (x) has its first year's benefit premium equal to the actuarial present value of the first year's benefit, and the remaining benefit premiums are level and determined by the equivalence principle. Determine formulas for (a) the first year's benefit premium, (b) the level benefit premium after the first year, and (c) the benefit reserve at the first duration.

Solution:

a. From Chapter 4, $\pi_0 = A^{1}_{x:\overline{1|}}$.
b. By the equivalence principle $A^{1}_{x:\overline{1|}} + \pi a_x = A_x$, so $\pi = (A_x - A^{1}_{x:\overline{1|}})/a_x = A_{x+1}/\ddot{a}_{x+1} = P_{x+1}$.
c. By the prospective formula, ${}_1V = A_{x+1} - \pi\ddot{a}_{x+1} = 0$. ▼

Example 8.2.1 illustrates one approach to nonlevel premiums. Another approach would be to set a premium pattern in the loss variable ${}_0L$ as defined in (8.2.1) by a set of weights, w_j, for $j = 0, 1, 2, \ldots$. Applying the equivalence principle, we have in a special case of (8.2.1)

$$ \mathrm{E}[{}_0L] = 0, $$

or

$$ \sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j} = \pi \sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x \tag{8.2.5} $$

and

$$ \pi = \frac{\sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j}}{\sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x}. \tag{8.2.6} $$

By different selections of sequences $\{b_{j+1}; j = 0, 1, 2, \ldots\}$ and $\{w_j; j = 0, 1, 2, \ldots\}$, the various benefit premium formulas can be obtained.

If we consider the sequence $\{b_{j+1}; j = 0, 1, 2, \ldots\}$ to be fixed, there remains great flexibility in selecting the sequence $\{w_j; j = 0, 1, 2, \ldots\}$, which in turn determines the sequence $\{\pi_j; j = 0, 1, 2, \ldots\}$. There may be commercial considerations to require that $w_j \geq 0$ for all j, but the equivalence principle does not impose this condition. In Example 8.2.1, $b_{j+1} = 1$ for all j, but $\pi w_0 = A^{1}_{x:\overline{1|}}$ and $\pi w_j = (A_x - A^{1}_{x:\overline{1|}})/a_x$, $j = 1, 2, \ldots$, and (8.2.6) is satisfied. A different application is found in Example 8.2.2.

Example 8.2.2

The annual benefit premiums for a fully discrete whole life insurance with a unit benefit issued to (x) are $\pi_j = \pi w_j$, where $w_j = (1 + r)^j$. The rate r might be selected to estimate the expected growth rate in the insured's income.

Develop formulas for

a. π

b. ${}_hV$ and

c. ${}_hV$ when $r = i$.

Solution:

a. Using (8.2.5), $\pi = A_x / \ddot{a}^*_x$, where $\ddot{a}^*_x$ is valued at the rate of interest $i^* = (i - r)/(1 + r)$. When $r = i$, $\pi = A_x/(e_x + 1)$.

b. Using (8.2.4),

$$\begin{aligned} {}_hV &= A_{x+h} - \sum_{j=0}^{\infty} \pi_{j+h}\, v^j\, {}_jp_{x+h} \\ &= A_{x+h} - \frac{A_x}{\ddot{a}^*_x}(1 + r)^h \sum_{j=0}^{\infty} \left(\frac{1 + r}{1 + i}\right)^j {}_jp_{x+h} \\ &= A_{x+h} - \frac{A_x}{\ddot{a}^*_x}(1 + r)^h \ddot{a}^*_{x+h}. \end{aligned}$$

c. ${}_hV = A_{x+h} - [A_x/(e_x + 1)](1 + r)^h(e_{x+h} + 1)$. ▼

In Example 8.2.2, negative benefit reserves are possible with higher values of r. See Exercise 8.32 for a variation of this policy.

Now consider a general fully continuous insurance on (x) under which

- The death benefit payable at the moment of death, t, is b_t, and
- Benefit premiums are payable continuously at t at the annual rate, π_t.

The prospective loss for a life insured at x and surviving at t is the present value at t of the future benefits less the present value at t of the future benefit premiums:

$$
{}_tL = \begin{cases} 0 & T(x) \le t \\ b_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} \pi_u\, v^{u-t}\, du & T(x) > t. \end{cases} \tag{8.2.7}
$$

The benefit reserve for this general case, which we will denote by ${}_t\bar{V}$, is then

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}[{}_tL | T(x) > t] \\
&= \mathrm{E}\left[b_{T(x)}\, v^{T(x)-t} - \int_t^{T(x)} \pi_u\, v^{u-t}\, du \Big| T(x) > t \right] \\
&= \mathrm{E}\left[b_{(T(x)-t)+t}\, v^{T(x)-t} - \int_0^{T(x)-t} \pi_{t+r}\, v^r\, dr \Big| T(x) > t \right].
\end{aligned} \tag{8.2.8}
$$

As assumed to obtain (8.2.3) for the fully discrete insurance, we assume here that the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$ and proceed to

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}\left[b_{T(x+t)+t}\, v^{T(x+t)} - \int_0^{T(x+t)} \pi_{t+r}\, v^r\, dr \right] \\
&= \int_0^\infty \left(b_{t+u}\, v^u - \int_0^u \pi_{t+r}\, v^r\, dr \right) {}_up_{x+t}\, \mu_x(t+u)\, du \\
&= \int_0^\infty b_{t+u}\, v^u\, {}_up_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+r}\, v^r\, {}_rp_{x+t}\, dr.
\end{aligned} \tag{8.2.9}
$$

The second integral in (8.2.9) is obtained by integration by parts, or, alternatively, by reversing the order of integration. In other words, ${}_t\bar{V}$ can be expressed as the actuarial present value of future benefits less the actuarial present value of future benefit premiums. If the assumption about the conditional distribution of $T(x) - t$, given $T(x) > t$, does not hold, we are in the select mortality mode.

8.3 Recursion Relations for Fully Discrete Benefit Reserves

One objective of this chapter is to explore recursion relations among the loss random variables, their expected values, and variances. We start with a definition for the insurer's net cash loss (negative cash flow) within each insurance year for the fully discrete model as defined for (8.2.1). Figure 8.3.1 is a time diagram that shows the annual cash income and cash outgo.

FIGURE 8.3.1

Insurer's Cash Income and Outgo for General Fully Discrete Insurance

Outgo		0	0	0		0	0	$b_{K(x)+1}$	0	0	
Income	π_0	π_1	π_2	π_3	…	$\pi_{K(x)-1}$	$\pi_{K(x)}$	0	0	0	etc.
	0	1	2	3		$K(x)-1$	$K(x)$	$K(x)+1$	$K(x)+2$	$K(x)+3$	

Let C_h denote the present value at h of the net cash loss during the year $(h, h+1)$. If $(h, h+1)$ is before the year of death $[h < K(x)]$, then $C_h = -\pi_h$. If $(h, h+1)$ is the year of death $[h = K(x)]$, then $C_h = v\, b_{h+1} - \pi_h$. And if $(h, h+1)$ is after the year of death, of course, $C_h = 0$. Restating this definition as an explicit function of $K(x)$,

$$C_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ v\, b_{h+1} - \pi_h & K(x) = h \\ -\pi_h & K(x) = h+1, h+2, \ldots . \end{cases} \tag{8.3.1}$$

For the conditional distribution of C_h, given $K(x) \geq h$, we observe that

$$C_h = v b_{h+1}\, I - \pi_h,$$

where

$$I = \begin{cases} 1 & \text{with probability } q_{x+h} \\ 0 & \text{with probability } p_{x+h}. \end{cases}$$

Therefore,

$$\mathrm{E}[C_h | K(x) \geq h] = v\, b_{h+1}\, q_{x+h} - \pi_h \tag{8.3.2}$$

and

$$\mathrm{Var}[C_h | K(x) \geq h] = (v\, b_{h+1})^2\, q_{x+h}\, p_{x+h}. \tag{8.3.3}$$

Moreover, using (2.2.10) and (2.2.11) along with (8.3.1)–(8.3.3),

$$\mathrm{E}[C_h] = (v\, b_{h+1}\, q_{x+h} - \pi_h)\, {}_hp_x \tag{8.3.4}$$

and

$$\mathrm{Var}(C_h) = (v\, b_{h+1}\, q_{x+h} - \pi_h)^2\, {}_hp_x\, {}_hq_x + (v\, b_{h+1})^2 q_{x+h}\, p_{x+h}\, {}_hp_x. \tag{8.3.5}$$

Finally, for $j > h$, C_j and C_h are correlated, an assertion that is left for the student to verify in Exercises 8.5 and 8.6.

As previously defined in (8.2.1), ${}_hL$ is the present value at h of the insurer's future cash outflow less the present value at h of the insurer's future cash income. By rearranging the terms in this definition, an equivalent one that states ${}_hL$ as the sum of the present values at h of the insurer's future net annual cash losses is obtained. This is

$${}_hL = \sum_{j=h}^{\infty} v^{j-h}\, C_j. \tag{8.3.6}$$

For $h < K(x)$, we have from (8.3.1),

$$\begin{aligned} {}_hL = \sum_{j=h}^{K(x)} v^{j-h}\, C_j &= v^{K(x)-h}\, (v\, b_{K(x)+1} - \pi_{K(x)}) - \sum_{j=h}^{K(x)-1} v^{j-h} \pi_j \\ &= v^{K(x)+1-h}\, b_{K(x)+1} - \sum_{j=h}^{K(x)} v^{j-h}\, \pi_j \end{aligned}$$

as before. For $h = K(x)$, ${}_hL = C_{K(x)} = (v\, b_{K(x)+1} - \pi_{K(x)})$. And for $h > K(x)$, both sides of (8.3.6) are zero.

A recursion relation for the loss variables follows from (8.3.6):

$$ {}_hL = C_h + v \sum_{j=h+1}^{\infty} v^{j-(h+1)} C_j = C_h + v \; {}_{h+1}L. \tag{8.3.7} $$

A recursion relation for the benefit reserves can be obtained from (8.3.7) by

$$ \begin{aligned} {}_hV &= \mathrm{E}[{}_hL|K(x) \ge h] \\ &= \mathrm{E}[C_h + v \; {}_{h+1}L|K(x) \ge h] \\ &= v \; b_{h+1} \; q_{x+h} - \pi_h + v \; \mathrm{E}[{}_{h+1}L|K(x) \ge h]. \end{aligned} \tag{8.3.8} $$

Since ${}_{h+1}L$ is zero when $K(x)$ is h, we have

$$ \begin{aligned} {}_hV &= v \; b_{h+1} \; q_{x+h} - \pi_h + v \; \mathrm{E}[{}_{h+1}L|K(x) \ge h + 1] \; p_{x+h} \\ &= v \; b_{h+1} \; q_{x+h} - \pi_h + v \; {}_{h+1}V \; p_{x+h}. \end{aligned} \tag{8.3.9} $$

Formula (8.3.9) is a backward recursion formula $[u(h) = c(h) + d(h) \times u(h + 1)]$ for the general fully discrete benefit reserve. Note that $d(h) = vp_{x+h}$ again and $c(h) = vb_{h+1}q_{x+h} - \pi_h$. A forward recursion formula can be obtained by solving (8.3.9) for ${}_{h+1}V$. (See Exercise 8.7.) This forward formula was used in Examples 7.4.3 and 7.4.4 in an aggregate mode; that is, the mortality functions were in life table form.

Further insight to the progress of benefit reserves can be gained by rearrangements of (8.3.9). First, add π_h to both sides, to see

$$ {}_hV + \pi_h = b_{h+1} \; v \; q_{x+h} + {}_{h+1}V \; v \; p_{x+h}. \tag{8.3.10} $$

In words, the resources required at the beginning of insurance year $h + 1$ equal the actuarial present value of the year-end requirements. The sum ${}_hV + \pi_h$ is called the ***initial benefit reserve*** for the policy year $h + 1$. In contrast, ${}_hV$ and ${}_{h+1}V$ are called the ***terminal benefit reserves*** for insurance years h and $h + 1$ to indicate that they are year-end benefit reserves.

Formula (8.3.10) can be rearranged to separate the benefit premium π_h into components for insurance year $h + 1$, namely,

$$ \pi_h = b_{h+1} \; v \; q_{x+h} + ({}_{h+1}V \; v \; p_{x+h} - {}_hV). \tag{8.3.11} $$

The first component on the right-hand side of (8.3.11) is the 1-year term insurance benefit premium for the sum insured b_{h+1}. The second component, ${}_{h+1}V \; v \; p_{x+h} - {}_hV$, represents the amount which, if added to ${}_hV$ at the beginning of the year, would accumulate under interest and survivorship to ${}_{h+1}V$ at the end of the year.

For the purpose of subsequent comparison with formulas for a fully continuous insurance, we multiply both sides of (8.3.11) by $1 + i$ and rearrange the formula to

$$ \pi_h + ({}_hV + \pi_h)i + {}_{h+1}Vq_{x+h} = b_{h+1} \; q_{x+h} + \Delta({}_hV). \tag{8.3.12} $$

The left-hand side of (8.3.12) indicates resources for insurance year $h + 1$, namely, the benefit premium, interest for the year on the initial benefit reserve, and the expected release by death of the terminal benefit reserve. The right-hand side consists of the expected payment of the death benefit at the end of the year and the increment ${}_{h+1}V - {}_hV$ in the benefit reserve.

An analysis different than (8.3.10)–(8.3.12) results if one considers that the benefit reserve ${}_{h+1}V$ is to be available to offset the death benefit b_{h+1}, and that only the ***net amount at risk***, $b_{h+1} - {}_{h+1}V$, needs to be covered by 1-year term insurance. For this analysis we have, on substituting $1 - q_{x+h}$ for p_{x+h} in (8.3.10) and multiplying through by $1 + i$,

$$ {}_{h+1}V = ({}_hV + \pi_h)(1 + i) - (b_{h+1} - {}_{h+1}V)q_{x+h}. \tag{8.3.13} $$

Corresponding to (8.3.11), we now have

$$ \pi_h = (b_{h+1} - {}_{h+1}V)v\, q_{x+h} + (v\, {}_{h+1}V - {}_hV). \tag{8.3.14} $$

The first component on the right-hand side of (8.3.14) is the 1-year term insurance benefit premium for the net amount of risk. The second component, $v\, {}_{h+1}V - {}_hV$, is the amount which, if added to ${}_hV$ at the beginning of the year, would accumulate under interest to ${}_{h+1}V$ at the end of the year. In this formulation ${}_{h+1}V$ is used, in case of death, to offset the death benefit. Consequently, the benefit reserve accumulates as a savings fund. This is shown again by the formula corresponding to (8.3.12), namely,

$$ \pi_h + ({}_hV + \pi_h)i = (b_{h+1} - {}_{h+1}V)q_{x+h} + \Delta({}_hV), \tag{8.3.15} $$

which is left for the reader to interpret.

The analysis by (8.3.11) does not use the benefit reserve to offset the death benefit, and consequently the benefit reserve accumulates under interest and survivorship. Both components of the right-hand side of (8.3.11) involve mortality risk, whereas in (8.3.14) only the first component does. We see in Section 8.5 that (8.3.14) is related to a flexible means for calculating the variance of loss attributable to the random nature of time until death.

Formulas (8.3.10)–(8.3.15) are all recursion relations for the benefit reserve at integral durations. None of the six is written in the form of an explicit backward or forward formula; rather, each is written to give an insight. In Example 8.3.1, recursion relation (8.3.14) is used to obtain an explicit formula for the benefit premium and benefit reserve.

Example 8.3.1

A deferred whole life annuity-due issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death prior to age $x + n$ is the benefit reserve. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium and the benefit reserve at the end of year k for $k \leq n$.

Solution:

Using the fact that $b_{h+1} = {}_{h+1}V$ for $h = 0, 1, 2, \ldots, n-1$, in (8.3.14), we have

$$\pi = v\,{}_{h+1}V - {}_hV.$$

On multiplication by v^h, we have

$$\pi v^h = v^{h+1}\,{}_{h+1}V - v^h\,{}_hV = \Delta(v^h\,{}_hV). \tag{8.3.16}$$

Summing over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\,{}_nV - v^0\,{}_0V = \pi \sum_{h=0}^{n-1} v^h = \pi \ddot{a}_{\overline{n}|},$$

and, since ${}_0V = 0$ and ${}_nV = \ddot{a}_{x+n}$, it follows that

$$\pi = v^n \frac{\ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}} = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

Thus, this annuity is identical to that described in Example 6.6.2. The benefit reserve at the end of k years can be found by summing (8.3.16) over $h = 0, 1, 2, \ldots, k-1$ to give

$$v^k\,{}_kV = \pi \ddot{a}_{\overline{k}|},$$

from which

$${}_kV = \pi \ddot{s}_{\overline{k}|}.$$

▼

Example 8.3.2

A fully discrete n-year endowment insurance on (x) provides, in case of death within n years, a payment of 1 plus the benefit reserve. Obtain formulas for the level benefit premium and the benefit reserve at the end of k years, given that the maturity value is 1.

Solution:

In this case $b_h = 1 + {}_hV$, and the net amount at risk has constant value 1. Denoting the annual benefit premium by π and using (8.3.14), we have

$$v\,{}_{h+1}V - {}_hV = \pi - v\,q_{x+h} \qquad h = 0, 1, \ldots, n-1.$$

On multiplication by v^h, this becomes

$$\Delta(v^h\,{}_hV) = \pi\,v^h - v^{h+1}\,q_{x+h}. \tag{8.3.17}$$

Summing this over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\,{}_nV = \pi\,\ddot{a}_{\overline{n}|} - \sum_{h=0}^{n-1} v^{h+1} q_{x+h}$$

so that, with ${}_nV = 1$ (the maturity value is 1),

$$\pi = \frac{v^n + \sum_{h=0}^{n-1} v^{h+1} q_{x+h}}{\ddot{a}_{\overline{n}|}}.$$

By summing (8.3.17) over $h = 0, 1, 2, \ldots, k - 1$, and solving for ${}_kV$, we have

$$ {}_kV = \pi \ddot{s}_{\overline{k}|} - \sum_{h=0}^{k-1} (1 + i)^{k-h-1} q_{x+h}. $$

▼

Just before Example 8.2.1 we promised to develop a retrospective formula for the benefit reserve of the general fully discrete insurance in this section. We start by rewriting recursion relation (8.3.11) in the form

$$ \pi_h - b_{h+1}\, v\, q_{x+h} = {}_{h+1}V\, v\, p_{x+h} - {}_hV $$

and then multiplying both sides by $v^h\, {}_hp_x$ to obtain

$$ \begin{aligned} \pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h} &= {}_{h+1}V\, v^{h+1}\, {}_{h+1}p_x - {}_hV\, v^h\, {}_hp_x \\ &= \Delta({}_hV\, v^h\, {}_hp_x), \end{aligned} \tag{8.3.18} $$

which holds for $h = 0, 1, 2, \ldots$. When we sum both sides of (8.3.18) over the values from 0 to $k - 1$, we have

$$ \sum_{h=0}^{k-1} (\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}) = {}_kV\, v^k\, {}_kp_x - {}_0V. $$

With equivalence principle premiums, ${}_0V = 0$, so we can rewrite this last equation for the terminal benefit reserve of the general fully discrete insurance as

$$ {}_kV = \sum_{h=0}^{k-1} \frac{\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}}{v^k\, {}_kp_x} $$

and then as

$$ {}_kV = \sum_{h=0}^{k-1} (\pi_h - v b_{h+1}\, q_{x+h}) \frac{(1 + i)^{k-h}}{{}_{k-h}p_{x+h}}. \tag{8.3.19} $$

Formula (8.3.19) shows the benefit reserve at k as the sum over the first k years of each year's premium less its expected death benefit accumulated with respect to interest and mortality to k.

8.4 Benefit Reserves at Fractional Durations

We consider again the general fully discrete insurance of (8.2.1) on (x) for a death benefit of b_{j+1} at the end of insurance year $j + 1$, purchased by annual benefit premiums of π_j, $j = 0, 1, \ldots$, payable at the beginning of the insurance year. We seek a formula for, and an approximation to, the ***interim benefit reserve***, that is, ${}_{h+s}V$ for $h = 0, 1, 2, \ldots$ and $0 < s < 1$. Extending the earlier definition of the benefit reserve as stated in (8.2.1) and (8.2.2), we have for the interim case

$$ {}_{h+s}L = \begin{cases} 0 & K(x) = 0, 1, \ldots, h - 1 \\ v^{1-s}\, b_{K(x)+1} & K(x) = h \\ v^{K(x)+1-(h+s)}\, b_{K(x)+1} - \displaystyle\sum_{j=h+1}^{K(x)} v^{j-(h+s)}\, \pi_j & K(x) = h + 1, h + 2, \ldots \end{cases} \tag{8.4.1} $$

and

$$ {}_{h+s}V = \mathrm{E}[{}_{h+s}L | T(x) > h + s]. \tag{8.4.2} $$

From (8.4.2),

$$ {}_{h+s}V = v^{1-s}\ b_{h+1}\ {}_{1-s}q_{x+h+s} + v^{1-s}\ {}_{h+1}V\ {}_{1-s}p_{x+h+s}. \tag{8.4.3} $$

Now multiply both sides of (8.4.3) by $v^s\ {}_sp_{x+h}$ to obtain

$$ v^s{}_sp_{x+h}\ {}_{h+s}V = v\ b_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V)\ p_{x+h}. \tag{8.4.4} $$

Equation (8.3.9) provides an expression for $vb_{h+1}\ q_{x+h}$, which can be substituted into (8.4.4) to obtain

$$ v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h - {}_{h+1}V\ v\ p_{x+h})\ \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + v\ {}_{h+1}V\ p_{x+h}, $$

which can be rearranged to

$$ v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h)\ \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + ({}_{h+1}V\ v\ p_{x+h})\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.5} $$

This exact expression shows that when the interim benefit reserve at $h + s$ is discounted with respect to interest and mortality to h, the result is equal to an interpolated value between the initial benefit reserve at h and the value of the terminal benefit reserve at $h + 1$ discounted to h.

We emphasize that the interpolation is, in general, not linear; however, under the assumption of uniform distribution of deaths over the age interval the interpolation weights are linear and (8.4.5) is

$$ v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h)(1 - s) + ({}_{h+1}V\ v\ p_{x+h})(s). \tag{8.4.6} $$

By replacing i and q_{x+h} with zeros in (8.4.6), as an approximation, the result is linear interpolation between the initial benefit reserve at h and the terminal benefit reserve. The approximate result is

$$ {}_{h+s}V = (1 - s)({}_hV + \pi_h) + s({}_{h+1}V), \tag{8.4.7} $$

which is often written in the form

$$ {}_{h+s}V = (1 - s)({}_hV) + s({}_{h+1}V) + (1 - s)\pi_h. \tag{8.4.8} $$

Here the interim benefit reserve is the sum of the value obtained by linear interpolation between the terminal benefit reserves,

$$ (1 - s)({}_hV) + s({}_{h+1}V), $$

and the ***unearned benefit premium*** $(1 - s)\pi_h$. In general,

(the unearned benefit premium at a given time during the year) = (the benefit premium for the year)

× (the difference between the time through which the premium has been paid and the given time).

Thus, on an annual premium basis, the benefit premium has been paid to the end of the year so at time s the unearned benefit premium is $(1 - s)\pi_h$. This notion of an unearned benefit premium will be used in discussing approximations to benefit reserves when the premiums are collected by installments more frequent than an annual basis.

We consider now one such case, that of true semiannual premiums with claims paid at the end of the insurance year of death. For $0 < s \leq 1/2$ we could start with the random variable giving the present value of prospective losses as of time $h + s$ and then calculate its conditional expectation given that (x) has survived to $h + s$. This is a bit more complex than it was for (8.4.4), so we start with the equation corresponding to (8.4.3) by noting that it is the prospective formula

$$
\begin{aligned}
{}_{h+s}V^{(2)} = v^{1-s}\, b_{h+1}({}_{1-s}q_{x+h+s}) + v^{1-s}({}_{h+1}V^{(2)})\ {}_{1-s}p_{x+h+s} \\
- \frac{\pi_h}{2}\,(v^{0.5-s})({}_{0.5-s}p_{x+h+s}).
\end{aligned}
\tag{8.4.9}
$$

The first two terms of (8.4.9) can be viewed as the actuarial present value of the death benefit and an endowment benefit of amount equal to the reserve, and the third term is the actuarial present value of the future benefit premium for the $1 - s$ year endowment insurance. Multiplying both sides of (8.4.9) by ${}_sp_{x+h}\, v^s$, we have

$$
\begin{aligned}
{}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} = v b_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V^{(2)})\ p_{x+h} \\
- \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h}).
\end{aligned}
\tag{8.4.10}
$$

For the semiannual premium policy the equation corresponding to (8.3.10) is also a prospective benefit reserve formula:

$$
{}_hV^{(2)} = b_{h+1} v q_{x+h} + {}_{h+1}V^{(2)} v p_{x+h} - \frac{\pi_h}{2}\,(1 + v^{0.5}\,{}_{0.5}p_{x+h}). \tag{8.4.11}
$$

Equation (8.4.11) provides an expression for vb_{h+1} to substitute in (8.4.10), which yields

$$
\begin{aligned}
{}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} \\
+ \left[v({}_{h+1}V^{(2)})\ p_{x+h} - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h})\right] \\
\times \left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right).
\end{aligned}
\tag{8.4.12}
$$

Formula (8.4.12) corresponds to (8.4.5), which showed that the interim benefit reserve at $h + s$, discounted with respect to interest and mortality to h, is equal to a nonlinear interpolated value between the initial benefit reserve at h and the discounted value to h of the terminal benefit reserve at $h + 1$. For the semiannual premium case, this terminal reserve has been reduced by the amount of the discounted value of the midyear benefit premium. Under the assumption of a uniform distribution of deaths over the age interval, we have linear interpolation on the right-hand side:

$$
{}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)(1 - s)
$$

$$
+ \left[v({}_{h+1}V^{(2)})\, p_{x+h} - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h})\right](s). \tag{8.4.13}
$$

Again setting i and q_{x+h} equal to zero, as an approximation, we obtain simple linear interpolation between the initial benefit reserve and the terminal benefit reserve reduced by the benefit premium due at midyear:

$$
{}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)(1 - s) + \left({}_{h+1}V^{(2)} - \frac{\pi_h}{2}\right)(s).
$$

This formula can be rearranged as the interpolated value between the terminal benefit reserves plus the unearned benefit premium $(\pi_h)(1/2 - s)$:

$$
{}_{h+s}V^{(2)} = [(1 - s){}_hV^{(2)} + s\, {}_{h+1}V^{(2)}] + \left(\frac{1}{2} - s\right)\pi_h. \tag{8.4.14}
$$

For the last half of the year when $1/2 < s \le 1$, we can proceed as above to obtain the following exact formula for the benefit reserve at $h + s$ discounted with respect to interest and mortality to h:

$$
{}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = \left[{}_hV^{(2)} + \frac{\pi_h}{2}\,(1 + v^{0.5}{}_{0.5}p_{x+h})\right]\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}
$$

$$
+ [v({}_{h+1}V^{(2)})\, p_{x+h}]\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.15}
$$

Again under uniform distribution of death in the year of age, we have the linear interpolation

$$
{}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = (1 - s)\left[{}_hV^{(2)} + \frac{\pi_h}{2}\,(1 + v^{0.5}{}_{0.5}p_{x+h})\right]
$$

$$
+ s[v({}_{h+1}V^{(2)})\, p_{x+h}], \tag{8.4.16}
$$

and with i and q_{x+h} set equal to zero, as an approximation, we have the simple linear interpolation plus the unearned benefit premium

$$
{}_{h+s}V^{(2)} = {}_hV^{(2)}(1 - s) + {}_{h+1}V^{(2)}(s) + \pi_h(1 - s). \tag{8.4.17}
$$

(For the general result for m-thly reserves see Exercise 8.12.)

8.5 Allocation of the Risk to Insurance Years

In Section 8.3 recursion relations for benefit reserves are developed by an analysis of the insurer's annual cash income and cash outflow. Now we extend this analysis to an accrual or incurred basis and develop allocations of the risk, as measured by the variance of the loss variables, to the insurance years. Figure 8.5.1 shows in a time diagram the insurer's annual cash incomes, cash outflows, and changes in liability for the general fully discrete insurance of (8.2.1). The random variable C_h is related to the cash flows of the policy year $(h, h + 1)$. We now define a random variable related to the total change in liability, cash flow, and reserves.

FIGURE 8.5.1

Insurer's Cash Incomes, Outflows, and Changes in Liability for Fully Discrete General Insurance

Outflow		0	0	0	$b_{K(x)+1}$	0	
Income	π_0	π_1	π_2	$\pi_{K(x)}$	0	0	
ΔLiability		${}_1V$	${}_2V-(1+i)\,{}_1V$	${}_{K(x)}V-(1+i)_{K(x)-1}V$	$-(1+i)_{K(x)}V$	0	etc.
	0	1	2	$K(x)$	$K(x)+1$	$K(x)+2$	

Let Λ_h denote the present value at h (a non-negative integer) of the insurer's cash loss plus change in liability during the year $(h, h + 1)$. If $(h, h + 1)$ is before the year of death $[h < K(x)]$, then

$$\Lambda_h = C_h + v\ \Delta\text{Liability} = -\pi_h + v\ {}_{h+1}V - {}_hV.$$

If $(h, h + 1)$ is the year of death $[h = K(x)]$, then

$$\Lambda_h = C_h + v\Delta\text{Liability} = v\ b_{h+1} - \pi_h - {}_hV.$$

And if $(h, h + 1)$ is after the year of death, of course $\Lambda_h = 0$. Restating this definition as a function of $K(x)$, and rearranging the terms,

$$\Lambda_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ (v\ b_{h+1} - \pi_h) + \quad (-\ {}_hV) & K(x) = h \\ (-\pi_h) \quad + (v\ {}_{h+1}V - {}_hV) & K(x) = h+1, h+2, \ldots. \end{cases} \tag{8.5.1}$$

The definition of Λ_h in (8.5.1) can be rewritten to display Λ_h as the loss variable for a 1-year term insurance with a benefit equal to the amount at risk on the basic policy. See Exercise 8.31.

It follows that

$$E[\Lambda_h | K(x) \geq h] = v\ b_{h+1}\ q_{x+h} + v\ {}_{h+1}V\ p_{x+h} - (\pi_h + {}_hV), \tag{8.5.2}$$

which is zero by (8.3.10).

Since the conditional distribution of Λ_h, given $K(x) = h, h + 1, \ldots$, is a two-point distribution, then

$$\text{Var}[\Lambda_h | K(x) \geq h] = [v(b_{h+1} - {}_{h+1}V)]^2\ p_{x+h}\ q_{x+h}. \tag{8.5.3}$$

With $j \leq h$ we can use (2.2.10) and (2.2.11) to obtain

$$E[\Lambda_h | K(x) \geq j] = 0 \tag{8.5.4}$$

and

$$\mathrm{Var}[\Lambda_h | K(x) \geq j] = \mathrm{Var}[\Lambda_h | K(x) \geq h]\ {}_{h-j}p_{x+j}. \tag{8.5.5}$$

Unlike the C_h's of Section 8.3, the Λ_h's are uncorrelated, an assertion that is proved in the following lemma. This fact conveys some sense of the role of reserves in stabilizing financial reporting of insurance operations.

Lemma 8.5.1:

For non-negative integers satisfying $g \leq h < j$,

$$\mathrm{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = 0. \tag{8.5.6}$$

Proof:

From (8.5.4), $\mathrm{E}[\Lambda_h | K(x) \geq g] = 0$; therefore,

$$\mathrm{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = \mathrm{E}[\Lambda_h \Lambda_j | K(x) \geq g].$$

From (8.5.1) we see that Λ_h is equal to the constant $(v\ {}_{h+1}V - {}_hV - \pi_h)$ where Λ_j is nonzero. Thus,

$$\Lambda_h \Lambda_j = (v\ {}_{h+1}V - {}_hV - \pi_h)\Lambda_j \qquad \text{for all } K(x), \tag{8.5.7}$$

$$\mathrm{E}[\Lambda_h \Lambda_j | K(x) \geq g] = (v\ {}_{h+1}V - {}_hV - \pi_h)\mathrm{E}[\Lambda_j | K(x) \geq g] = 0,$$

and

$$\mathrm{Cov}[\Lambda_k \Lambda_j | K(x) \geq g] = 0. \qquad \blacksquare$$

We now express the loss variables ${}_hL$ in terms of the Λ_h's. From the definition of the Λ_h's and formula (8.3.6),

$$\begin{aligned} \sum_{j=h}^{\infty} v^{j-h}\ \Lambda_j &= \sum_{j=h}^{\infty} v^{j-h}[C_j + v\Delta\text{Liability } (j, j+1)] \\ &= {}_hL + \sum_{j=h}^{\infty} v^{j-h+1}\ \Delta\text{Liability } (j, j+1). \end{aligned} \tag{8.5.8}$$

Conceptually the last term will be the present value of the final liability minus the liability at h, that is, $0 - {}_hV$. Thus we have the relationship

$${}_hL = \begin{cases} 0 & K(x) < h \\ \displaystyle\sum_{j=h}^{\infty} v^{j-h}\ \Lambda_j + {}_hV & K(x) \geq h, \end{cases} \tag{8.5.9}$$

which can be rewritten as

$${}_hL = \begin{cases} 0 & K(x) < h \\ \displaystyle\sum_{j=h}^{h+i-1} v^{j-h}\ \Lambda_j + \sum_{j=h+i}^{\infty} v^{j-h}\ \Lambda_j + {}_hV & K(x) \geq h. \end{cases} \tag{8.5.10}$$

These relationships can be interpreted as stating that the present value of future losses, measured at time h following issue, is equal to the present value of future cash flows, adjusted for changes in reserves, plus the reserve at h.

Using the representation of L_h shown in (8.5.9), we have

$$\operatorname{Var}[{}_hL|K(x) \geq h] = \sum_{j=h}^{\infty} v^{2(j-h)} \operatorname{Var}[\Lambda_j|K(x) \geq h]$$

$$= \sum_{j=h}^{\infty} v^{2(j-h)}\ {}_{j-h}p_{x+h} \operatorname{Var}[\Lambda_j|K(x) \geq j]$$

$$= \sum_{j=h}^{\infty} v^{2(j-h)}\ {}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+i}V)]^2\ p_{x+j}q_{x+j}\}. \quad (8.5.11)$$

In this development the first line makes use of Lemma 8.5.1, the second (8.5.5), and the third (8.5.3).

Starting with (8.5.10) we can follow identical steps to obtain

$$\operatorname{Var}[{}_hL|K(x) \geq h] = \sum_{j=h}^{h+i-1} v^{2(j-h)} \operatorname{Var}[\Lambda_j|K(x) \geq h]$$

$$+ \sum_{j=h+i}^{\infty} v^{2(j-h)} \operatorname{Var}[\Lambda_j|K(x) \geq h]$$

$$= \sum_{j=h}^{h+i-1} v^{2(j-h)}\ {}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\ p_{x+j}q_{x+j}\}$$

$$+ \sum_{j=h+i}^{\infty} v^{2(j-h)}\ {}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\ p_{x+j}q_{x+j}\}. \quad (8.5.12)$$

The second summation can be rewritten by replacing the summation variable j by $l + h$ to obtain

$$\sum_{l=i}^{\infty} v^{2l}\ {}_lp_{x+h}\{[v(b_{h+l+1} - {}_{h+l+1}V)]^2\ p_{x+h+l}q_{x+h+l}\}$$

$$= v^{2i}\ {}_ip_{x+h} \sum_{l=i}^{\infty} v^{2(l-i)}\ {}_{l-i}p_{x+h+i}\ \{[v(b_{h+l+1} - {}_{h+l+1}V)]^2\ p_{x+h+l}q_{x+h+l}\}$$

$$= v^{2i}\ {}_ip_{x+h} \operatorname{Var}[{}_{h+i}L|K(x) \geq h + i]. \quad (8.5.13)$$

The main results of these developments will be summarized as a theorem.

Theorem 8.5.1

$$\operatorname{Var}({}_hL|K(x) \geq h)$$

a. $= \sum_{j=h}^{\infty} v^{2(j-h)} \operatorname{Var}[\Lambda_j|K(x) \geq h]$ (8.5.14)

b. $= \sum_{j=h}^{\infty} v^{2(j-h)}\ {}_{j-h}p_{x+n}\ \{[v(b_{j+1} - {}_{j+1}V)]^2\ p_{x+j}q_{x+j}\}$ (8.5.15)

c. $= \sum_{j=h}^{h+i-1} v^{2(j-h)}\ {}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\ p_{x+j}q_{x+j}\}$
$+ v^{2i}\ {}_ip_{x+h} \operatorname{Var}[{}_{h+i}L|K(x) \geq h + i].$ (8.5.16)

Proof:

(a) follows from (8.5.11), first line
(b) follows from (8.5.11), third line
(c) follows from (8.5.12) and (8.5.13).

We refer to this theorem as the Hattendorf theorem, and we illustrate its application in the following two examples. Items (b) and (c) of the theorem can be used as backward recursion formulas that are useful for understanding the duration allocation of risk and, perhaps, for computing.

Just as the random variables C, introduced in (8.3.1), allocate each loss to insurance years, and the random variables Λ, introduced in (8.5.1), allocate cash loss and liability adjustment to insurance years, the Hattendorf theorem facilitates the allocation of mortality risk, as measured by $\mathrm{Var}[{}_hL|K(x) \geq h]$ to insurance years. This allocation facilitates risk management planning for a limited number of future insurance years rather than for the entire insurance period. This option permits sequential risk management decisions.

The formula $\mathrm{Var}[\Lambda_h|K(x) \geq h] = [v(b_{h+1} - {}_{h+1}V)]^2 p_{x+h} q_{x+h}$ confirms that the amount at risk $(b_{h+1} - {}_{h+1}V)$ is a major determinate of mortality risk, as measured by the variance. In fact if $b_{h+1} = {}_{h+1}V$ for all non-negative integer values of h, mortality risk drops to zero.

Example 8.5.1

Consider an insured from Example 7.4.3 who has survived to the end of the second policy year. For this insured, evaluate
a. $\mathrm{Var}[{}_2L|K(50) \geq 2]$ directly
b. $\mathrm{Var}[{}_2L|K(50) \geq 2]$ by means of the Hattendorf theorem
c. $\mathrm{Var}[{}_3L|K(50) \geq 3]$
d. $\mathrm{Var}[{}_4L|K(50) \geq 4]$.

Solution:

a. For the direct calculation, we need a table of values for ${}_2L$.

Outcome of $K(50) - 2 = j$	${}_2L$	Conditional Probability of Outcome
0	$1{,}000v - 6.55692\, \ddot{a}_{\overline{1}\rceil} = 936.84$	${}_{0\vert}q_{52} = 0.0069724$
1	$1{,}000v^2 - 6.55692\, \ddot{a}_{\overline{2}\rceil} = 877.25$	${}_{1\vert}q_{52} = 0.0075227$
2	$1{,}000v^3 - 6.55692\, \ddot{a}_{\overline{3}\rceil} = 821.04$	${}_{2\vert}q_{52} = 0.0081170$
≥ 3	$0 - 6.55692\, \ddot{a}_{\overline{3}\rceil} = -18.58$	${}_3p_{52} = 0.9773879$

Then $E[{}_2L|K(50) \geq 2] = 1.64$, in agreement with the value shown in Example 7.4.3 and

$$\text{Var}[{}_2L|K(50) \geq 2] = \text{E}[{}_2L^2|K(50) \geq 2] - (\text{E}[{}_2L|K(50) \geq 2])^2$$
$$= 17{,}717.82 - (1.64)^2$$
$$= 17{,}715.1.$$

b. To apply the Hattendorf theorem, we can use the benefit reserves from Example 7.4.3 to calculate the variances of the losses associated with the 1-year term insurances.

j	q_{52+j}	$v^2\,(1{,}000 - 1{,}000\ {}_{2+j+1}V^{1}_{50:\overline{5}|})^2\ p_{52+j}\ q_{52+j}$
0	0.0069724	6 140.842
1	0.0075755	6 674.910
2	0.0082364	7 269.991

Then by (8.5.15),

$$\text{Var}[{}_2L|K(50) \geq 2] = 6{,}140.842 + (1.06)^{-2}(6{,}674.910)p_{52} + (1.06)^{-4}(7{,}269.991)\,{}_2p_{52} = 17{,}715.1,$$

which agrees with the value found by the direct calculation in part (a).

Note that in the direct method it was necessary to consider the gain in the event of survival to age 55; but for the Hattendorf theorem, we need to consider only the losses associated with the 1-year term insurances for the net amounts at risk in the remaining policy years. Thereafter, the net amount at risk is 0, and the corresponding terms in (8.5.15) vanish.

Also note that the standard deviation, $\sqrt{17{,}751.1} = 133.1$, for a single policy is more than 80 times the benefit reserve, $\text{E}[{}_2L|K(50)=2, 3, \ldots] = 1.64$.

Similarly, we use (8.5.15) to calculate

c. $\text{Var}[{}_3L|K(50) \geq 3] = 6{,}674.910 + (1.06)^{-2}(7{,}269.991)\ p_{53} = 13{,}096.2$

d. $\text{Var}[{}_4L|K(50) \geq 4] = 7{,}269.991$, or after rounding, 7,270.0. ▼

Example 8.5.2

Consider a portfolio of 1,500 policies of the type described in Example 7.4.3 and discussed in Example 8.5.1. Assume all policies have annual premiums due immediately. Further, assume 750 policies are at duration 2, 500 are at duration 3, and 250 are at duration 4, and that the policies in each group are evenly divided between those with 1,000 face amount and those with 3,000 face amount.

a. Calculate the aggregate benefit reserve.
b. Calculate the variance of the prospective losses over the remaining periods of coverage of the policies assuming such losses are independent. Also, calculate the amount which, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the future obligations to this block of business.
c. Calculate the variance of the losses associated with the 1-year term insurances for the net amounts at risk under the policies and the amount of supplement to

the aggregate benefit reserve that, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the obligations to this block of business for the 1-year period.

d. Redo (b) and (c) with each set of policies increased 100-fold in number.

Solution:

a. Let Z be the sum of the prospective losses on the 1,500 policies. The symbols E[Z] and Var(Z) used below for the mean and variance of the portfolio of 1,500 policies are abridged, for in both cases the expectations are to be computed with respect to the set of conditions given above for the insureds. Using the results of Example 7.4.3, we have for the aggregate benefit reserve

$$\begin{aligned} E[Z] &= [375(1) + 375(3)](1.64) + [250(1) + 250(3)](1.73) \\ &\quad + [125(1) + 125(3)](1.21) \\ &= 4{,}795. \end{aligned}$$

b. From Example 8.5.1, we have

$$\begin{aligned} \text{Var}(Z) &= [375(1) + 375(9)](17{,}715.1) \\ &\quad + [250(1) + 250(9)](13{,}096.2) \\ &\quad + [125(1) + 125(9)](7{,}270.0) \\ &= (1.0825962) \times 10^8 \end{aligned}$$

and $\sigma_Z = 10{,}404.8$.

Then, if

$$0.05 = \Pr(Z > c) = \Pr\left(\frac{Z - 4{,}795.0}{10{,}404.8} > \frac{c - 4{,}795.0}{10{,}404.8}\right),$$

the normal approximation would imply

$$\frac{c - 4{,}795.0}{10{,}404.8} = 1.645,$$

or

$$c = 21{,}911,$$

which is 4.6 times the aggregate benefit reserve, E[Z].

c. Here we take account of only the next year's risk. For each policy, we consider a variable equal to the loss associated with a 1-year term insurance for the net amount at risk. Let Z_1 be the sum of these loss variables. The expected loss for each of the 1-year term insurances is 0, hence $E[Z_1] = 0$.

From the table in part (b) of Example 8.5.1 we can obtain the variances of the losses in regard to the 1-year term insurances, and hence

$$\text{Var}(Z_1) = [375(1) + 375(9)](6{,}140.8) + [250(1) + 250(9)](6{,}674.9)$$
$$+ [125(1) + 125(9)](7{,}270.0)$$
$$= (4.880275) \times 10^7$$

and $\sigma_{Z_1} = 6985.9$.

If c_1 is the required supplement to the aggregate benefit reserve, then

$$0.05 = \Pr(Z_1 > c_1) = \Pr\left(\frac{Z_1 - 0}{6{,}985.9} > \frac{c_1 - 0}{6{,}985.9}\right),$$

and we determine, again by the normal approximation,

$$c_1 = (1.645)(6{,}985.9) = 11{,}492,$$

which is 2.4 times the aggregate benefit reserve 4,795.

d. In this case, $E[Z] = 479{,}500$ and $\text{Var}(Z) = (1.0825962) \times 10^{10}$. By the normal approximation the amount c required to provide a probability of 0.95 that all future obligations will be met is

$$479{,}500 + 1.645\sqrt{1.0825962} \times 10^5 = 650{,}659,$$

which is 1.36 times the aggregate benefit reserve $E[Z]$.

Also, $\text{Var}(Z_1)$ is now $(4.880275) \times 10^9$. The amount c_1 of supplement to the aggregate benefit reserve required to give a 0.95 probability that the insurer can meet policy obligations for the next year is $1.645\sqrt{4.880275} \times 10^{4.5} = 114{,}918$, or 24% of the aggregate benefit reserve. ▼

8.6 Differential Equations for Fully Continuous Benefit Reserves

In Section 8.2 a general fully discrete insurance and a general fully continuous model is developed. Section 8.3 contains the recursion relations for the fully discrete model. The parallel results for the fully continuous model are developed in this section.

The expression for the benefit reserve at t, ${}_t\bar{V}$, is given in (8.2.9) and is restated here:

$${}_t\bar{V} = \int_0^\infty b_{t+u}\, v^u\, {}_up_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+u}\, v^u\, {}_up_{x+t}\, du.$$

To simplify the calculation of the derivative with respect to t of ${}_t\bar{V}$, we combine the two integrals, replace the variable of integration by the substitution $s = t + u$, and then multiply inside and divide outside by the factor $v^t\, {}_tp_x$ to obtain

$$
{}_t\bar{V} = \frac{\int_t^\infty [b_s\ \mu_x(s) - \pi_s] v^s\ {}_sp_x\ ds}{v^t\ {}_tp_x}. \tag{8.6.1}
$$

Now

$$
\frac{d_t\bar{V}}{dt} = (-1)[b_t\ \mu_x(t) - \pi_t] + \frac{\mu_x(t) + \delta}{v^t\ {}_tp_x} \int_t^\infty [b_s\ \mu_x(s) - \pi_s] v^s\ {}_sp_x ds,
$$

$$
\frac{d_t\bar{V}}{dt} = \pi_t + [\delta + \mu_x(t)]_t\bar{V} - b_t\ \mu_x(t). \tag{8.6.2}
$$

Here the rate of change of the benefit reserve is made up of three components: the benefit premium rate, the rate of increase of the benefit reserve under interest and survivorship, and the rate of benefit outgo. A rearrangement of formula (8.6.2) provides a formula corresponding to (8.3.12):

$$
\pi_t + \delta\ {}_t\bar{V} + {}_t\bar{V}\ \mu_x(t) = b_t\ \mu_x(t) + \frac{d_t\bar{V}}{dt}. \tag{8.6.3}
$$

This balances the sum of income rates to the sum of the rate of benefit outgo and the rate of change in the benefit reserve.

If the benefit reserve is treated as a savings fund available to offset the death benefit, we have

$$
\pi_t + \delta\ {}_t\bar{V} = (b_t - {}_t\bar{V})\ \mu_x(t) + \frac{d_t\bar{V}}{dt}. \tag{8.6.4}
$$

Here the income rates are in respect to benefit premiums and to interest on the benefit reserve, and these balance with the outgo rate, $(b_t - {}_t\bar{V})\ \mu_x(t)$, based on the net amount at risk and the rate of change in the benefit reserve. Formula (8.6.4) corresponds to (8.3.15). Again, the left side represents the resources available, benefit premiums, and investment income, and the right side represents their allocation to benefits and benefit resources.

Example 8.6.1

Use (8.6.2) to develop a retrospective formula for the benefit reserve for the general fully continuous insurance.

Solution:

We start by moving all of the benefit reserve terms of (8.6.2) to the left-hand side and then multiplying both sides by the integrating factor $\exp\{-\int_0^t[\delta + \mu_x(s)]\ ds\}$. Thus,

$$
v^t\ {}_tp_x \left\{\frac{d_t\bar{V}}{dt} - [\delta + \mu_x(t)]\ {}_t\bar{V}\right\} = [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x,
$$

or

$$\frac{d}{dt}(v^t\,{}_tp_x\,{}_t\bar{V}) = [\pi_t - b_t\,\mu_x(t)]\,v^t\,{}_tp_x.$$

Integration of both sides of this last equation over the interval $(0, r)$ yields

$$v^r\,{}_rp_x\,{}_r\bar{V} - {}_0\bar{V} = \int_0^r [\pi_t - b_t\,\mu_x(t)]\,v^t\,{}_tp_x\,dt.$$

For equivalence principle benefit premium rates, ${}_0V = 0$, so

$$_r\bar{V} = \frac{\int_0^r [\pi_t - b_t\,\mu_x(t)]\,v^t\,{}_tp_x\,dt}{v^r\,{}_rp_x}. \tag{8.6.5}$$

▼

8.7 Notes and References

Recursive formulas and differential equations for the loss variables as functions of duration and the expectations and variances of these loss variables provide basic insight into long-term insurance and annuity processes. In particular, one of these recursive formulas is applied to develop Hattendorf's theorem (1868); for references, see Steffensen (1929), Hickman (1964), and Gerber (1976). This formula allocates the variance of the loss to the separate insurance years. This discussion of reserves can be easily extended to more general insurances using martingales of probability theory; see, for example, Gerber (1979). Another application of the recursion relations is to the formulation of the interim reserves at fractional durations, which is discussed for the fully discrete case in Section 8.4.

Exercises

Section 8.2

8.1. Assume that ${}_jp_x = r^j$, $b_{j+1} = 1$, $j = 0, 1, 2, 3, \ldots$, and $0 < r < 1$.
   a. If $w_0 = w_1 = w_2 = \ldots = 1$, use (8.2.6) to calculate π at the interest rate i.
   b. If $w_j = (-1)^j$, $j = 0, 1, 2, \ldots$, use (8.2.6) to calculate π at the interest rate i.

8.2. Develop a continuous analogue of (8.2.6) by applying the equivalence principle to the loss variable of (8.2.7) with $t = 0$ and $\pi_t = \pi w(t)$, where $w(t)$ is given.

8.3. If ${}_0L = T(x)\,v^{T(x)} - \pi\bar{a}_{\overline{T(x)}|}$ and the forces of mortality and interest are constant, express (a) π and (b) ${}_t\bar{V}$ in terms of μ and δ.

8.4. For the general fully discrete insurance of Section 8.2, show that for $j < h$,

$$\text{Cov}(C_j, C_h) = (\pi_h - vb_{h+1}\,q_{x+h})\,{}_hp_x(\pi_j\,{}_jq_x + vb_{j+1}\,{}_jp_x\,q_{x+j}).$$

Section 8.3

8.5. Consider the life insurance policy described in Example 8.2.1. Display, for $0 < j < h$:

a. The covariance of C_0 and C_h.

b. Repeat part (a) for C_j and C_h.

c. Give a rule for determining h such that the covariance of C_j and C_h is negative.

8.6. Consider the deferred annuity described in Example 8.3.1. Find $\mathrm{Cov}(C_j, C_h)$, $j < h \leq n$ and, for a fixed j, determine a condition on h such that $\mathrm{Cov}(C_j, C_h) < 0$. [Note that this condition on h does not depend on j.]

8.7. Show that (8.3.9), with h replaced by $h + 1$, can be rearranged as

$$_{h+1}V = ({}_hV + \pi_h) \frac{1 + i}{p_{x+h}} - b_{h+1} \frac{q_{x+h}}{p_{x+h}}.$$

Give an interpretation in words. (This is called the ***Fackler reserve*** accumulation formula, after the American actuary David Parks Fackler.)

8.8. For a fully discrete whole life insurance of 1 issued to (x), use recursion relations [(8.3.11) in (a) and (8.3.14 in (b)] to prove that

a. $_kV_x = \sum_{h=0}^{k-1} \dfrac{P_x - vq_{x+h}}{_{k-h}E_{x+h}}$

b. $_kV_x = \sum_{h=0}^{k-1} [P_x - vq_{x+h}(1 - {}_{h+1}V_x)](1 + i)^{k-h}$.

Give an interpretation of the formulas in words.

8.9. If $b_{h+1} = {}_{h+1}V$, $_0V = 0$, and $\pi_h = \pi$, for $h = 0, 1, \ldots, k - 1$, prove that $_kV = \pi\ddot{s}_{\overline{k|}}$. [Hint: Use (8.3.14).]

8.10. Show that if π is the level annual benefit premium for an n-year term insurance with $b_h = \ddot{a}_{\overline{n+1-h|}}$, $h = 1, 2, \ldots, n$, $_0V = {}_nV = 0$, then

a. $\pi = \dfrac{a_{\overline{n|}} - a_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$

b. $_kV = a_{\overline{n-k|}} - a_{x+k:\overline{n-k|}} - \pi\ddot{a}_{x+k:\overline{n-k|}}$.

[Hint: This can be shown directly or by use of (8.3.10).]

Section 8.4

8.11. Starting with (8.4.3), establish the equation

$$_sp_{x+h}\ {}_{h+s}V + v^{1-s}\ {}_sq_{x+h}\ b_{h+1} = (1 + i)^s({}_hV + \pi_h) \qquad 0 < s < 1.$$

Explain the result by general reasoning.

8.12. Interpret the formulas

a. $_{k+(h/m)+r}V^{(m)} \cong \left(1 - \frac{h}{m} - r\right) {}_kV^{(m)}$

$$+ \left(\frac{h}{m} + r\right) {}_{k+1}V^{(m)} + \left(\frac{1}{m} - r\right) P^{(m)}$$

b. $_{k+(h/m)+r}V^{\{m\}} \cong \left(1 - \frac{h}{m} - r\right) {}_kV^{\{m\}}$

$$+ \left(\frac{h}{m} + r\right) {}_{k+1}V^{\{m\}} + \left(\frac{1}{m} - r\right) P^{\{m\}},$$

where $0 < r < 1/m$.

8.13. For each of the following benefit reserves, develop formulas similar to one or more of (8.4.8), (8.4.14), and (8.4.18).

a. $_{20^1/_2}V(\bar{A}_{x:\overline{40|}})$
b. $_{20^1/_2}\bar{V}(\bar{A}_{x:\overline{40|}})$
c. $_{20^1/_2}V^{(2)}(\bar{A}_{x:\overline{40|}})$
d. $_{20^2/_3}V^{(2)}(\bar{A}_{x:\overline{40|}})$
e. $_{20^1/_2}V^{\{2\}}(\bar{A}_{x:\overline{40|}})$
f. $_{20^2/_3}V^{\{2\}}(\bar{A}_{x:\overline{40|}})$

8.14. On the basis of the Illustrative Life Table and interest of 6%, approximate $_{10^1/_6}V^{\{4\}}(\bar{A}_{25})$.

Section 8.5

8.15. For a fully discrete whole life insurance of amount 1 issued to (x) with premiums payable for life, show that

a. $\mathrm{Var}[L] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)} {}_hp_x \, p_{x+h} \, q_{x+h}$

b. $\mathrm{Var}[{}_kL|K(x) \geq k] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+k+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)} {}_hp_{x+k} \, p_{x+k+h} \, q_{x+k+h}.$

8.16. For a life annuity-due of 1 per annum payable while (x) survives, consider the whole life loss

$$L = \ddot{a}_{\overline{K+1|}} - \ddot{a}_x \qquad K = 0, 1, 2, \ldots$$

and the loss Λ_h, valued at time h, that is allocated to annuity year h, namely,

$$\Lambda_h = \begin{cases} 0 & & K \leq h - 1 \\ -(\ddot{a}_{x+h} - 1) & = -vp_{x+h} \, \ddot{a}_{x+h+1} & K = h \\ v\ddot{a}_{x+h+1} - (\ddot{a}_{x+h} - 1) & = vq_{x+h} \, \ddot{a}_{x+h+1} & K \geq h + 1. \end{cases}$$

a. Interpret the formulas for Λ_h.
b. Show that

(i) $L = \sum_{h=0}^{\infty} v^h \Lambda_h$

(ii) $\mathrm{E}[\Lambda_h] = 0$

(iii) $\mathrm{Var}(\Lambda_h) = v^2 (\ddot{a}_{x+h+1})^2 {}_hp_x \, p_{x+h} \, q_{x+h}.$

8.17. a. For the insurance of Example 8.3.2, establish that

$$\mathrm{Var}(L) = \sum_{h=0}^{n-1} v^{2(h+1)} \, {}_hp_x \, p_{x+h} \, q_{x+h}.$$

b. If $\delta = 0.05$, $n = 20$, and $\mu_x(t) = 0.01$, $t \ge 0$, calculate $\mathrm{Var}(L)$ for the insurance in (a).

8.18. A 20-payment whole life policy with unit face amount was issued on a fully discrete basis to a person age 25. On the basis of your Illustrative Life Table and interest of 6%, calculate

a. ${}_{20}P_{25}$ b. ${}^{20}_{19}V_{25}$ c. ${}^{20}_{20}V_{25}$

d. $\mathrm{Var}[{}_{20}L|K(25) \ge 20]$ e. $\mathrm{Var}[{}_{18}L|K(25) \ge 18]$, using Theorem 8.5.1.

Section 8.6

8.19. Interpret the differential equations

a. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + [\delta + \mu_x(t)]\,{}_t\bar{V} - b_t\,\mu_x(t)$

b. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + \delta\,{}_t\bar{V} - (b_t - {}_t\bar{V})\,\mu_x(t).$

8.20. If $b_t = {}_t\bar{V}$, ${}_0\bar{V} = 0$, and $\pi_t = \pi$, $t \ge 0$, show that ${}_t\bar{V} = \pi\bar{s}_{\overline{t}|}$.

8.21. Evaluate $(d/dt)\,\{[1 - {}_t\bar{V}(\bar{A}_x)]\,{}_tp_x\}$.

8.22. Use (8.6.2) to write expressions for

a. $\dfrac{d}{dt}\,({}_tp_x\,{}_t\bar{V})$ b. $\dfrac{d}{dt}\,(v^t\,{}_t\bar{V})$ c. $\dfrac{d}{dt}\,(v^t\,{}_tp_x\,{}_t\bar{V})$

and interpret the results.

Miscellaneous

8.23. Show that the formula equivalent to (8.4.6) under the hyperbolic assumption for mortality within the year of age is

$${}_{k+s}V = v^{1-s}[(1 - s)({}_kV + \pi_k)(1 + i) + s\;{}_{k+1}V].$$

8.24. Prove that

$$\int_0^\infty [v^t - \bar{P}(\bar{A}_x)\bar{a}_{\overline{t}|}]^2\,{}_tp_x\,\mu_x(t)\,dt = \int_0^\infty [1 - {}_t\bar{V}(\bar{A}_x)]^2\,v^{2t}\,{}_tp_x\,\mu_x(t)\,dt$$

and interpret the result.

8.25. For a different form of the Hattendorf theorem, consider the following:

$${}_{k,m}L = \begin{cases} b_{K+1}\,v^{(K-k)+1} - {}_kV - \displaystyle\sum_{h=0}^{(K-k)} \pi_{k+h}\,v^h & K(x) = k, k+1, \ldots, k+m-1 \\ {}_{k+m}V\,v^m - {}_kV - \displaystyle\sum_{h=0}^{m-1} \pi_{k+h}\,v^h & K(x) = k+m, k+m+1, \ldots, \end{cases}$$

and, for $h = 0, 1, \ldots, m-1$,

$$\Lambda_{k+h} = \begin{cases} 0 & K(x) = k, k+1, \ldots, k+h-1 \\ vb_{k+h+1} - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h \\ v\ {}_{k+h+1}V - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h+1, k+h+2, \ldots . \end{cases}$$

Show that

a. ${}_{k,m}L = \sum_{h=0}^{m-1} v^h\ \Lambda_{k+h}$

b. $\text{Var}[{}_{k,m}L|K(x) \ge k] = \sum_{h=0}^{m-1} v^{2h}\ \text{Var}[\Lambda_{k+h}|K(x) \ge k]$.

8.26. Repeat Example 8.5.1 in terms of an insured from Example 7.4.4 who has survived to the end of the second policy year.

8.27. Repeat Example 8.5.2 in terms of a portfolio of 1,500 policies of the type described in Example 7.4.4 and discussed in Exercise 8.26.

8.28. In Exercise 8.27 there is no uncertainty about the amount or time of payment for the insureds who have survived to the end of the fourth policy year. Redo Exercise 8.27 for just those insureds at durations 2 and 3.

8.29. Write a formula, in terms of benefit premium and terminal benefit reserve symbols, for the benefit reserve at the middle of the eleventh policy year for a 10,000 whole life insurance with apportionable premiums payable annually issued to (30).

8.30. A 3-year endowment policy for a face amount of 3 has the death benefit payable at the end of the year of death and a benefit premium of 0.94 payable annually. Using an interest rate of 20%, the following benefit reserves are generated:

End of Year	Benefit Reserve
1	0.66
2	1.56
3	3.00

Calculate

a. q_x
b. q_{x+1}
c. The variance of the loss at policy issue, ${}_0L$
d. The conditional variance, given that the insured has survived through the first year, of the loss at the end of the first year, ${}_1L$.

8.31. a. Use (8.3.10) to transform (8.5.1) to

$$\Lambda_h = \begin{cases} 0 & K(x) \le h - 1 \\ (b_{h+1} - {}_{h+1}V)\, v - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) = h \\ 0 \quad\quad - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) \ge h + 1. \end{cases}$$

In this interpretation Λ_h is the loss on the 1-year term insurance for the amount at risk in the year $(h, h + 1)$.

b. Use the display in (a) to verify $E[\Lambda_h] = 0$.

c. Use parts (a) and (b) to obtain $Var(\Lambda_h)$.

Computing Exercises

8.32. Consider a variation of the insurance of Example 8.2.2 which provides a unit benefit whole life insurance to (20) with geometrically increasing benefit premiums payable to age 65. On the basis of your Illustrative Life Table and $i = 0.06$, determine the maximum value of r such that the benefit reserve is non-negative at all durations.

8.33. Use the backward recursion formula (8.3.9) to calculate the benefit reserves of

a. Example 7.4.3 [Hint: ${}_5V = 0.0$.]

b. Example 7.4.4. [Hint ${}_5V = 1.0$.]

8.34. A decreasing term insurance to age 65 with immediate payment of death claims is issued to (30) with the following benefits:

For Death between Ages	**Benefit**
30–50	100,000
50–55	90,000
55–60	80,000
60–65	60,000

On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $i = 0.06$, determine

a. The annual apportionable benefit premium, payable semiannually and

b. The reserve at the end of 30 years, if benefit premiums are as in (a).

8.35. A single premium insurance contract issued to (35) provides 100,000 in case the insured survives to age 65, and it returns (at the end of the year of death) the single benefit premium without interest if the insured dies before age 65. If the single benefit premium is denoted by S, write expressions, in terms of actuarial functions, for

a. S

b. The prospective formula for the benefit reserve at the end of k years

c. The retrospective formula for the benefit reserve at the end of k years

d. On the basis of your Illustrative Life Table and $d = 0.05$, calculate S and the benefit reserve ${}_{20}V$.

8.36. In terms of $P = {}_{20}P^{(12)}(\bar{A}_{30:\overline{35}|})$ and actuarial functions, write prospective and retrospective formulas for the following:

a. ${}^{20}_{10}V^{(12)}(\bar{A}_{30:\overline{35}|})$

b. ${}^{20}_{25}V^{(12)}(\bar{A}_{30:\overline{35}|})$

c. On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $\delta = 0.05$, calculate P and the benefit reserves of parts (a) and (b).

9

MULTIPLE LIFE FUNCTIONS

9.1 Introduction

In Chapters 3 through 8 we developed a theory for the analysis of financial benefits contingent on the time of death of a single life. We can extend this theory to benefits involving several lives. An application of this extension commonly found in pension plans is the joint-and-survivor annuity option. Other applications of multiple life actuarial calculations are common. In estate and gift taxation, for example, the investment income from a trust can be paid to a group of heirs as long as at least one of the group survives. Upon the last death, the principal from the trust is to be donated to a qualified charitable institution. The amount of the charitable deduction allowed for estate tax purposes is determined by an actuarial calculation. There are family policies in which benefits differ due to the order of the deaths of the insured and the spouse, and there are insurance policies with benefits payable on the first or last death providing cash in accordance with an estate plan.

In this chapter we discuss models involving two lives. Actuarial present values for basic benefits are derived by applying the concepts and techniques developed in Chapters 3 through 5. Models built on the assumption that the two future lifetime random variables are independent constitute most of the chapter. Section 9.6 introduces special models in which the two future lifetime random variables are dependent. Annual benefit premiums, reserves, and models involving three or more lives are covered in Chapter 18.

A useful abstraction in the theory of life contingencies, particularly as it is applied to several lives, is that of ***status*** for which there are definitions of survival and failure. Two elements are necessary for a status to be defined. The general term ***entities*** is used in the definition because of the broad range of application of the concept:

- There must be a finite set of entities, and for each member it must be possible to define a future lifetime random variable.
- There must be a rule by which the survival of the status can be determined at any future time.

To compute probabilities or actuarial present values associated with the survival of a status, the joint distribution of the future lifetime random variables must be available. Some of these random variables may have a marginal distribution such that all the probability is at one point.

Several illustrations of the status concept may be helpful. A single life age x defines a status that survives while (x) lives. Thus, the random variable $T(x)$, used in Chapter 3 to denote the future lifetime of (x), can be interpreted as the period of survival of the status and also as the time-until-failure of the status. A term certain, $\overline{n|}$, defines a status surviving for exactly n years and then failing. More complex statuses can be defined in terms of several lives in various ways. Survival can mean that all members survive or, alternatively, that at least one member survives. Still more complicated statuses can be in regard to two men and two women with the status considered to survive only as long as at least one man and at least one woman survive.

After a status and its survival have been defined, we can apply the definition to develop models for annuities and insurances. An annuity is payable as long as the status survives, whereas an insurance is payable upon the failure of the status. Insurances also can be restricted so they are payable only if the individuals die in a specific order.

9.2 Joint Distributions of Future Lifetimes

The time-until-failure of a status is a function of the future lifetimes of the lives involved. In theory these future lifetimes will be dependent random variables. We will explore the consequences of that dependence. For convenience, or because of the lack of data on dependent lives, in practice, an assumption of independence among the future lifetimes has traditionally been made. With the independence assumption numerical values from the marginal distributions (life tables) for single lives can be used.

Example 9.2.1

While the distribution in this example is not realistic, it is offered as a vehicle to explore a joint distribution for two dependent future lifetimes. For two lives (x) and (y), the joint p.d.f. of their future lifetimes, $T(x)$ and $T(y)$, is

$$f_{T(x)T(y)}(s, t) = \begin{cases} 0.0006(t-s)^2 & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the following:

a. The joint d.f. of $T(x)$ and $T(y)$
b. The p.d.f., d.f., ${}_sp_x$, and $\mu(x+s)$ for the marginal distribution of $T(x)$. Note the symmetry of the distribution in s and t, which implies that $T(x)$ and $T(y)$ are identically distributed.
c. The correlation coefficient of $T(x)$ and $T(y)$.

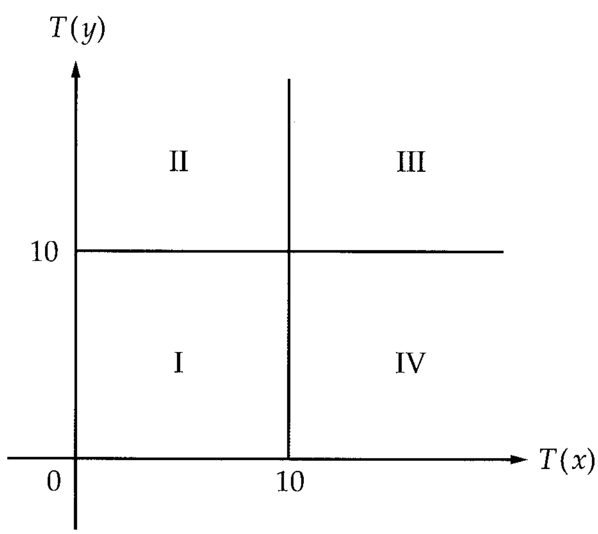
Solution:

a. Before calculating, we look at the sample space of $T(x)$ and $T(y)$ in Figure 9.2.1 and observe the region where the joint p.d.f. is positive. At points outside the first quadrant, the d.f. will be 0. In the first quadrant we start by calculating the d.f. at a point in Region I where both s and t are between 0 and 10:

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= \Pr[T(x) \leq s \text{ and } T(y) \leq t] \\ &= \int_{-\infty}^{s} \int_{-\infty}^{t} f_{T(x)T(y)}(u, v)\, dv\, du \\ &= \int_{0}^{s} \int_{0}^{t} 0.0006(v - u)^2\, dv\, du \\ &= 0.00005[s^4 + t^4 - (t - s)^4] \\ &\quad 0 < s \leq 10,\ 0 < t \leq 10. \end{aligned}$$

Figure 9.2.1

Sample Space of *T*(*x*) and *T*(*y*)



Sample Space of $T(x)$ and $T(y)$

Since the joint p.d.f. is 0 in regions II, III, and IV, we have

$$\left.\begin{aligned} F_{T(x)T(y)}(s, t) &= F_{T(x)T(y)}(s, 10) = F_{T(x)}(s) \\ &= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] \end{aligned}\right\} \quad \text{in Region II}$$

$$\left.\begin{aligned} &= F_{T(x)T(y)}(10, t) = F_{T(y)}(t) \\ &= \frac{1}{2} + 0.00005[t^4 - (10 - t)^4] \end{aligned}\right\} \quad \text{in Region IV}$$

$$= 1 \qquad\qquad \text{in Region III.}$$

b. Using the d.f. obtained in part (a), we have

$$
\begin{aligned}
F_{T(x)T(y)}(s, 10) &= F_{T(x)}(s) \\
&= 0 && s \le 0 \\
&= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] && 0 < s \le 10 \\
&= 1 && s > 10
\end{aligned}
$$

and

$$
f_{T(x)}(s) = F'_{T(x)}(s) = \begin{cases} 0.0002[s^3 + (10 - s)^3] & 0 < s \le 10 \\ 0 & \text{elsewhere.} \end{cases}
$$

The survival probability and force of mortality are given by

$$
\begin{aligned}
{}_sp_x &= 1 - F_{T(x)}(s) \\
&= \frac{1}{2} + 0.00005[(10 - s)^4 - s^4] && 0 < s \le 10 \\
&= 0 && s > 10,
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(x + t) &= \frac{f_{T(x)}(t)}{1 - F_{T(x)}(t)} \\
&= \frac{0.0002[s^3 + (10 - s)^3]}{1/2 + 0.00005[(10 - s)^4 - s^4]} && 0 < s \le 10.
\end{aligned}
$$

c.

$$
\begin{aligned}
\mathrm{E}[T(x)] &= \int_0^{10} s(0.0002)[s^3 + (10 - s)^3]\, ds = 5 = \mathrm{E}[T(y)], \\
\mathrm{E}[T(x)^2] &= \int_0^{10} s^2(0.0002)[s^3 + (10 - s)^3]\, ds = \frac{110}{3} = \mathrm{E}[T(y)^2], \\
\mathrm{Var}[T(x)] &= \frac{35}{3} = \mathrm{Var}[T(y)], \\
\mathrm{E}[T(x)T(y)] &= \int_0^{10}\int_0^{10} st(0.0006)(t - s)^2\, ds\, dt = \frac{50}{3}, \\
\mathrm{Cov}[T(x), T(y)] &= \mathrm{E}[T(x)T(y)] - \mathrm{E}[T(x)]\mathrm{E}[T(y)] = -\frac{25}{3}, \\
\rho_{T(x)T(y)} &= \frac{\mathrm{Cov}[T(x), T(y)]}{\sigma_{T(x)}\sigma_{T(y)}} = \frac{-25/3}{35/3} = -\frac{5}{7}.
\end{aligned}
$$

▼

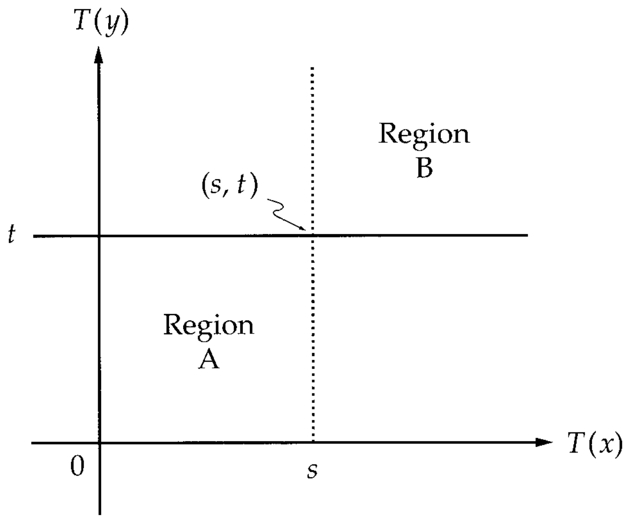
For the joint life distribution, we define the ***joint survival function*** as

$$
s_{T(x)T(y)}(s, t) = \Pr[T(x) > s \text{ and } T(y) > t]. \tag{9.2.1}
$$

Unlike the single life distribution, the d.f. and survival function do not necessarily add up to 1. Their relationship for the joint life distribution can be illustrated by a graph of their joint sample space as shown in Figure 9.2.2. The d.f. $F_{T(x)\,T(y)}\,(s, t)$ gives the probability of Region A, "southwest" of the point (s, t), and $s_{T(x)\,T(y)}(s, t)$ gives the probability of Region B, "northeast" of (s, t).

Figure 9.2.2

Sample Space of Future Lifetime Random Variables *T*(*x*) and *T*(*y*)



Sample Space of Future Lifetime Random Variables $T(x)$ and $T(y)$

Example 9.2.2

For the distribution of $T(x)$ and $T(y)$ in Example 9.2.1 determine the joint survival function.

Solution:

For $0 < s < 10$ and $0 < t < 10$,

$$
\begin{aligned}
s_{T(x)T(y)}(s, t) &= \Pr[T(x) > s \cap T(y) > t] \\
&= \int_s^{\infty} \int_t^{\infty} f_{T(x)T(y)}(u, v)\, dv\, du \\
&= \int_s^{10} \int_t^{10} 0.0006(v - u)^2\, dv\, du \\
&= 0.00005[(10 - t)^4 + (10 - s)^4 - (t - s)^4].
\end{aligned}
$$

For other points in the first quadrant, $s_{T(x)\,T(y)}(s, t)$ will be 0 and for all points in the third quadrant it will be 1. In the second quadrant, where $s < 0$ and $t > 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(y)}(t) = {}_tp_y.$$

In the fourth quadrant, where $s > 0$ and $t < 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(x)}(s) = {}_tp_x.$$

▼

In Example 9.2.1 we were given the joint distribution of two dependent future lifetimes and then determined their marginal distributions and their correlation coefficient, which indicated their degree of dependence. In applications, the dependence of the time-until-death random variables may be difficult to quantify. Consequently, the future lifetimes are usually assumed to be independent, and then their joint distribution is obtained from their marginal single life distributions that we discussed in Chapter 3. This is illustrated in the next example.

Example 9.2.3

The future lifetimes $T(x)$ and $T(y)$ are independent, and each has the distribution defined by the p.d.f.

$$f(t) = \begin{cases} 0.02\,(10 - t) & 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Determine the d.f., survival function, and force of mortality of this distribution.
b. Determine the joint p.d.f., d.f., and survival function for $T(x)$ and $T(y)$.

Solution:

a.

$$F_{T(x)}(t) = \int_{-\infty}^{t} f_{T(x)}(s)\,ds$$

$$= \begin{cases} 0 & t \le 0 \\ 1 - 0.01(10 - t)^2 = 0.2t - 0.01t^2 & 0 < t \le 10 \\ 1 & t > 10, \end{cases}$$

$$s_{T(x)}(t) = 1 - F_{T(x)}(t) = \begin{cases} 1 & t < 0 \\ 0.01(10 - t)^2 & 0 \le t < 10 \\ 0 & t \ge 10, \end{cases}$$

$$\mu(x + t) = \frac{f_{T(x)}(t)}{s_{T(x)}(t)} = \frac{2}{10 - t} \qquad 0 < t < 10.$$

b. $f_{T(x)T(y)}(s, t) = f_{T(x)}(s) f_{T(y)}(t)$

$$= \begin{cases} (0.02)^2(10 - s)(10 - t) & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere,} \end{cases}$$

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= F_{T(x)}(s) F_{T(y)}(t) \\ &= (0.2)^2(t - 0.05t^2)(s - 0.05s^2) && 0 < s \le 10,\ 0 < t \le 10 \\ &= F_{T(x)}(s) = (0.2)(s - 0.05s^2) && 0 < s \le 10,\ t > 10 \\ &= F_{T(y)}(t) = (0.2)(t - 0.05t^2) && s > 10,\ 0 < t \le 10, \end{aligned}$$

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= s_{T(x)}(s) s_{T(y)}(t) \\ &= (0.01)^2(10 - s)^2(10 - t)^2 && 0 \le s < 10,\ 0 \le t < 10 \\ &= s_{T(x)}(s) = (0.01)(10 - s)^2 && 0 \le s < 10,\ t < 0 \\ &= s_{T(y)}(t) = (0.01)(10 - t)^2 && s < 0,\ 0 \le t < 10 \\ &= 0 && s \ge 10,\ t \ge 10. \end{aligned}$$

▼

9.3 The Joint-Life Status

A status that survives as long as all members of a set of lives survive and fails upon the first death is called a ***joint-life status.*** It is denoted by $(x_1, x_2, \ldots, x_m)$, where x_i represents the age of member i of the set and m represents the number of members. Notation introduced in Chapters 3 through 5 is used here with the subscript listing several ages rather than a single age. For example, A_{xy} and ${}_tp_{xy}$ have the same meaning for the joint-life status (xy) as A_x and ${}_tp_x$ have for the single life (x).

A joint-life status is an example of what we call a ***survival status,*** that is, a status for which there is a future lifetime random variable, and, therefore, a survival function can be defined. For the future lifetime of a survival status, the concepts and relationships established in Sections 3.2.2 through 3.5 (excluding the life table example in Section 3.3.2) apply to the distribution of the survival status. These concepts will be used here without new proofs.

We now consider the distribution of the time-until-failure of a joint-life status. For m lives, $T(x_1, x_2, \ldots, x_m) = \min[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time of death of individual i. For the special case of two lives, (x) and (y), we have $T(xy) = \min[T(x), T(y)]$. When clear by context, we denote the future lifetime of the joint-life status by simply T. The student can interpret the time-until-failure of the joint-life status as the ***smallest order statistic*** of the m lifetimes in the set. In previous studies of order statistics, the random variables in the sample have usually been independent and identically distributed. Here the random variables are typically independent by assumption but are rarely identically distributed.

We begin by expressing the distribution function of T, for $t > 0$, in terms of the joint distribution of $T(x)$ and $T(y)$ for the general (dependent) case:

$$\begin{aligned} F_T(t) &= {}_tq_{xy} = \Pr(T \le t) \\ &= \Pr\{\min[T(x), T(y)] \le t\} \\ &= 1 - \Pr\{\min[T(x), T(y)] > t\} \\ &= 1 - \Pr\{T(x) > t \text{ and } T(y) > t\} \\ &= 1 - s_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.1}$$

Another equation can be obtained from the second line by recognizing that the event $\{\min[T(x), T(y)] \le t\}$ is the union of $\{T(x) \le t\}$ and $\{T(y) \le t\}$. Then,

$$F_T(t) = \Pr\{\min[T(x), T(y)] \le t\},$$

and using a basic result in probability, we have

$$\begin{aligned} F_T(t) &= \Pr[T(x) \le t] + \Pr[T(y) \le t] - \Pr[T(x) \le t \cap T(y) \le t] \\ &= {}_tq_x + {}_tq_y - F_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.2}$$

The mixture of IAN and standard probability/statistics notation in (9.3.2) demonstrates that although the IAN system may accommodate survival statuses, it does not provide for the joint distribution of several statuses except in the independent case which can be expressed in single survival status symbols.

When $T(x)$ and $T(y)$ are independent, the two expressions for the d.f. of T can be written in terms of single life functions as:

$$F_T(t) = \Pr\{\min[T(x), T(y)] \le t\}$$
$$= 1 - s_{T(x)\ T(y)}(t, t) = 1 - {}_tp_x\ {}_tp_y, \tag{9.3.3}$$

and

$$F_T(t) = {}_tq_x + {}_tq_y - F_{T(x)\ T(y)}(t, t) = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y. \tag{9.3.4}$$

The survival function for the joint-life status, ${}_tp_{xy}$, is obtained by subtracting the d.f. from 1.

For the general case, ${}_tp_{xy} = s_{T(x)\ T(y)}(t, t)$ using (9.3.1). In the independent case, we have by (9.3.3)

$${}_tp_{xy} = {}_tp_x\ {}_tp_y. \tag{9.3.5}$$

Expression (9.3.5) is the convenient starting point for the independent case since the joint-life status survives to t if, and only if, both (x) and (y) survive to t.

Example 9.3.1

Determine the d.f., survival function, and complete expectation for the joint-life status, $T(xy)$, for the lives of Example 9.2.1.

Solution:

For $t \le 0$ and $t > 10$, the value of $F_{T(xy)}(t)$ would be 0 and 1, respectively. For $0 < t \le 10$, we have by the results of Example 9.2.1(a) and (b) and (9.3.2)

$$F_{T(xy)}(t) = 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - 0.0001\, t^4$$
$$= 1 - 0.0001\,(10 - t)^4$$

for $0 < t \le 10$.

Now,

$${}_tp_{xy} = 1 - F_{T(xy)}(t) = 0.0001(10 - t)^4 \qquad 0 \le t < 10.$$

From (3.5.2),

$$\mathring{e}_{xy} = \mathrm{E}[T(xy)] = \textstyle\int_0^\infty {}_tp_{xy}\, dt = \int_0^{10} 0.0001(10 - t)^4\, dt = 2. \qquad \blacktriangledown$$

Example 9.3.2

Determine the d.f. and survival function and the complete expectation for the joint-life status, $T(xy)$, for the distribution of Example 9.2.3.

Solution:

For independent lives we use (9.3.5) and the results of Example 9.2.3 to obtain

$$ {}_tp_{xy} = [0.01(10 - t)^2]^2 = 0.0001(10 - t)^4 \qquad \text{for } 0 \le t < 10. $$

Then

$$ F_{T(xy)}(t) = 1 - (0.0001)(10 - t)^4 \qquad \text{for } 0 < t \le 10 $$

and

$$ \mathring{e}_{xy} = \int_0^{10} 0.0001(10 - t)^4\, dt = 2. $$

▼

An insight can be gained from the two previous examples. Although the joint distributions of the two underlying future lifetime random variables of the two examples are not the same, the distributions of their joint-life statuses are the same. This is a important point in practice when only the first-to-die can be observed. In such case the underlying joint distribution is not uniquely determined. In statistics this is called ***nonidentifiability*** because of the difficulty in distinguishing among two or more models for the same observed data.

The p.d.f. for T can be obtained by differentiating its d.f. as displayed in either (9.3.1) or (9.3.2). For (9.3.1) we will need the derivative, with respect to t, of

$$ s_{T(x)\ T(y)}(t, t) = \int_t^\infty \int_t^\infty f_{T(x)T(y)}(u, v)\, du\, dv. $$

Using the formula from calculus in Appendix 5, we have

$$ \frac{d}{dt}\, s_{T(x)\ T(y)}(t, t) = -\left[\int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du\right]. $$

Hence,

$$ f_{T(xy)}(t) = \int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du. \tag{9.3.6} $$

Using (9.3.2), the reader can show that the p.d.f. of T can also be written as

$$ f_{T(xy)}(t) = f_{T(x)}(t) + f_{T(y)}(t) - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du\right], $$

or with actuarial notation as

$$ \begin{aligned} f_{T(xy)}(t) &= {}_tp_x\, \mu(x + t) + {}_tp_y\, \mu(y + t) \\ &\quad - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du\right]. \end{aligned} \tag{9.3.7} $$

When $T(x)$ and $T(y)$ are independent, $f_{T(x)T(y)}(u, v) = {}_up_x\, \mu(x + u)\, {}_vp_y\, \mu(y + v)$, and (9.3.6) reduces directly to

$$ = {}_tp_y\, {}_tp_x\, [\mu(x + t) + \mu(y + t)]. \tag{9.3.8} $$

The mixture of IAN and standard probability/statistics notation in (9.3.2) demonstrates that although the IAN system may accommodate survival statuses, it does not provide for the joint distribution of several statuses except in the independent case which can be expressed in single survival status symbols.

When $T(x)$ and $T(y)$ are independent, the two expressions for the d.f. of T can be written in terms of single life functions as:

$$F_T(t) = \Pr\{\min[T(x), T(y)] \leq t\}$$
$$= 1 - s_{T(x)\ T(y)}(t, t) = 1 - {}_tp_x\ {}_tp_y, \tag{9.3.3}$$

and

$$F_T(t) = {}_tq_x + {}_tq_y - F_{T(x)\ T(y)}(t, t) = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y. \tag{9.3.4}$$

The survival function for the joint-life status, ${}_tp_{xy}$, is obtained by subtracting the d.f. from 1.

For the general case, ${}_tp_{xy} = s_{T(x)\ T(y)}(t, t)$ using (9.3.1). In the independent case, we have by (9.3.3)

$${}_tp_{xy} = {}_tp_x\ {}_tp_y. \tag{9.3.5}$$

Expression (9.3.5) is the convenient starting point for the independent case since the joint-life status survives to t if, and only if, both (x) and (y) survive to t.

Example 9.3.1

Determine the d.f., survival function, and complete expectation for the joint-life status, $T(xy)$, for the lives of Example 9.2.1.

Solution:

For $t \leq 0$ and $t > 10$, the value of $F_{T(xy)}(t)$ would be 0 and 1, respectively. For $0 < t \leq 10$, we have by the results of Example 9.2.1(a) and (b) and (9.3.2)

$$F_{T(xy)}(t) = 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - 0.0001\ t^4$$
$$= 1 - 0.0001\ (10 - t)^4$$

for $0 < t \leq 10$.

Now,

$${}_tp_{xy} = 1 - F_{T(xy)}(t) = 0.0001(10 - t)^4 \qquad 0 \leq t < 10.$$

From (3.5.2),

$$\mathring{e}_{xy} = \mathrm{E}[T(xy)] = \textstyle\int_0^\infty {}_tp_{xy}\, dt = \int_0^{10} 0.0001(10 - t)^4\, dt = 2.$$ ▼

Example 9.3.2

Determine the d.f. and survival function and the complete expectation for the joint-life status, $T(xy)$, for the distribution of Example 9.2.3.

Solution:

For independent lives we use (9.3.5) and the results of Example 9.2.3 to obtain

$$ {}_tp_{xy} = [0.01(10 - t)^2]^2 = 0.0001(10 - t)^4 \qquad \text{for } 0 \le t < 10. $$

Then

$$ F_{T(xy)}(t) = 1 - (0.0001)(10 - t)^4 \qquad \text{for } 0 < t \le 10 $$

and

$$ \mathring{e}_{xy} = \int_0^{10} 0.0001(10 - t)^4 \, dt = 2. $$

▼

An insight can be gained from the two previous examples. Although the joint distributions of the two underlying future lifetime random variables of the two examples are not the same, the distributions of their joint-life statuses are the same. This is a important point in practice when only the first-to-die can be observed. In such case the underlying joint distribution is not uniquely determined. In statistics this is called ***nonidentifiability*** because of the difficulty in distinguishing among two or more models for the same observed data.

The p.d.f. for T can be obtained by differentiating its d.f. as displayed in either (9.3.1) or (9.3.2). For (9.3.1) we will need the derivative, with respect to t, of

$$ s_{T(x)\ T(y)}(t, t) = \int_t^\infty \int_t^\infty f_{T(x)T(y)}(u, v)\, du\, dv. $$

Using the formula from calculus in Appendix 5, we have

$$ \frac{d}{dt} s_{T(x)\ T(y)}(t, t) = -\left[\int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du \right]. $$

Hence,

$$ f_{T(xy)}(t) = \int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du. \tag{9.3.6} $$

Using (9.3.2), the reader can show that the p.d.f. of T can also be written as

$$ f_{T(xy)}(t) = f_{T(x)}(t) + f_{T(y)}(t) - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du \right], $$

or with actuarial notation as

$$ \begin{aligned} f_{T(xy)}(t) &= {}_tp_x\, \mu(x + t) + {}_tp_y\, \mu(y + t) \\ &\quad - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du \right]. \end{aligned} \tag{9.3.7} $$

When $T(x)$ and $T(y)$ are independent, $f_{T(x)T(y)}(u, v) = {}_up_x\, \mu(x + u)\, {}_vp_y\, \mu(y + v)$, and (9.3.6) reduces directly to

$$ = {}_tp_y\, {}_tp_x\, [\mu(x + t) + \mu(y + t)]. \tag{9.3.8} $$

Example 9.3.3

By use of (9.3.6) determine the p.d.f. of $T(xy)$ for Example 9.2.1. Verify your result by examination of the d.f. in Example 9.3.1.

Solution:

Using (9.3.6) we obtain

$$f_T(t) = \begin{cases} \displaystyle\int_t^{10} 0.0006\,(t-v)^2\,dv + \int_t^{10} 0.0006\,(u-t)^2\,du \\ \qquad = 0.0004(10-t)^3 & \text{for } 0 < t < 10, \\ 0 & \text{elsewhere.} \end{cases}$$

This is the derivative of the d.f. in Example 9.3.1. ▼

We saw that $T(xy)$ has the same distribution in Examples 9.3.1 and 9.3.2. If we use (9.3.8) to obtain the p.d.f. of $T(xy)$ for Example 9.2.3, we will see this again. This is left as Exercise 9.8.

As explained in Chapter 3, the distribution of $T = T(xy)$ can also be specified by the force of "mortality," or more generally, the force of "failure." First we consider a notation for the force of failure of the status at time t. The traditional notation for this force is $\mu_{x+t:y+t}$ (in analogy with μ_{x+t}), but, in preparation for discussing other statuses where duration must be recognized, and in accordance with the notational convention adopted in Chapter 3, we use the notation $\mu_{xy}(t)$. The notation $\mu_{xy}(t)$ does not necessarily mean that (x) and (y) or the survival status (xy) were subject to a selection process, but the status did come into existence at these ages.

By analogy with the first formula of (3.2.12) and with $f_{T(x)}(x)$ and $F_{T(x)}(x)$ replaced by $f_{T(xy)}(t)$ and $F_{T(xy)}(t)$, we have

$$\mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1 - F_{T(xy)}(t)}. \tag{9.3.9}$$

For dependent $T(x)$, $T(y)$, this expression does not simplify beyond the general form. However, using (9.3.3) and (9.3.8) for the independent case, we have $\mu_{xy}(t) = \mu(x+t) + \mu(y+t)$.

In words, if the future lifetimes are independent, the force of failure for their joint-life status is the sum of the forces of mortality for the individuals. As in Chapter 3 with the single life case, we can characterize the distribution of $T(xy)$ by the p.d.f., the d.f., the survival function, or the force of failure.

Example 9.3.4

Determine the force of failure for the joint-life statuses of (x) and (y) in Examples 9.2.1 and 9.2.3.

Solution:

Since the joint distributions of the two examples produce the same distribution for $T(xy)$, we will use the independent case. From the results of part (a) of Example 9.2.3 and (9.3.9),

$$\mu_{xy}(t) = \frac{4}{10 - t} \qquad 0 < t < 10.$$

▼

We now turn to the curtate future lifetime of the joint-life status.

The probability that the joint-life status fails during the time k to $k + 1$ is determined by

$$\begin{aligned} \Pr(k < T \le k + 1) &= \Pr(T \le k + 1) - \Pr(T \le k) \\ &= {}_kp_{xy} - {}_{k+1}p_{xy} \\ &= {}_kp_{xy}\, q_{x+k:y+k}. \end{aligned} \tag{9.3.10}$$

When the future lifetimes of (x) and (y) are independent, the probability of the joint-life status $(x + k{:}y + k)$ failing within the next year can be written in terms of the probabilities of independent failure of the individual lives as follows:

$$\begin{aligned} q_{x+k:y+k} &= 1 - p_{x+k:y+k} \\ &= 1 - p_{x+k}\, p_{y+k} \\ &= 1 - (1 - q_{x+k})(1 - q_{y+k}) \\ &= q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k} \\ &= q_{x+k} + (1 - q_{x+k})\, q_{y+k}. \end{aligned} \tag{9.3.11}$$

From the discussion of curtate-future-lifetime of (x) in Section 3.2.3, we see that (9.3.10) also provides the p.f. of the random variable K, the number of years completed prior to failure of the joint-life status; that is, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \Pr(K = k) &= \Pr(k \le T < k + 1) \\ &= \Pr(k < T \le k + 1) \\ &= {}_kp_{xy}\, q_{x+k:y+k} \\ &= {}_{k|}q_{xy}. \end{aligned} \tag{9.3.12}$$

Example 9.3.5

Determine the p.f. and curtate expectation of $K(xy)$ using the common d.f. of Examples 9.3.1 and 9.3.2.

Solution:

From Example 9.3.2,

$${}_kp_{xy} = 0.0001(10 - k)^4.$$

Hence

$$\Pr[K(xy) = k] = 0.0001[(10 - k)^4 - (9 - k)^4], \quad k = 0, 1, 2, 3, \ldots, 9.$$

From (3.5.5),

$$e_{xy} = \mathrm{E}[K(xy)] = \sum_{k=0}^{\infty} {}_{k+1}p_{xy} = \sum_{k=0}^{9} 0.0001(9 - k)^4 = 1.5333.$$

▼

9.4 The Last-Survivor Status

In addition to benefits defined in terms of the time of the first death, there are those defined in terms of the time of the last death. In this section we will examine situations in which the random variable is the time of the last death.

A survival status that exists as long as at least one member of a set of lives is alive and fails upon the last death is called the ***last-survivor status***. It is denoted by $(\overline{x_1 \, x_2 \cdots x_m})$, where x_i represents the age of member i and m represents the number of members of the set. We consider the distribution of the time-until-failure of the last-survivor status, the random variable $T = \max[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time-until-death of individual i. The random variable T can be interpreted as the ***largest order statistic*** associated with $[T(x_1), T(x_2), \ldots, T(x_m)]$. Unlike the typical situation in the study of inferential statistics, the m random variables here are not necessarily independent and identically distributed.

For the case of two lives (x) and (y), $T(\overline{xy}) = \max[T(x), T(y)]$. General relationships exist among $T(xy)$, $T(\overline{xy})$, $T(x)$, and $T(y)$. For each outcome, $T(xy)$ equals either $T(x)$ or $T(y)$ and $T(\overline{xy})$ equals the other. Thus, for all joint distributions of $T(x)$ and $T(y)$, the following relationships hold:

$$T(xy) + T(\overline{xy}) = T(x) + T(y), \tag{9.4.1}$$

$$T(xy)\, T(\overline{xy}) = T(x)\, T(y), \tag{9.4.2}$$

$$a^{T(xy)} + a^{T(\overline{xy})} = a^{T(x)} + a^{T(y)} \qquad \text{for } a > 0. \tag{9.4.3}$$

There are also some general relationships among the distributions of these four random variables that come from the method of inclusion-exclusion of probability; that is,

$$\Pr(A \cup B) + \Pr(A \cap B) = \Pr(A) + \Pr(B). \tag{9.4.4}$$

Defining A as $\{T(x) \le t\}$ and B as $\{T(y) \le t\}$, we have $A \cap B = \{T(\overline{xy}) \le t\}$ and $A \cup B = \{T(xy) \le t\}$, which lead to

$$F_{T(xy)}(t) + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t). \tag{9.4.5}$$

From this it follows that

$${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y \tag{9.4.6}$$

and

$$f_{T(xy)}(t) + f_{T(\overline{xy})}(t) = f_{T(x)}(t) + f_{T(y)}(t). \tag{9.4.7}$$

The relationships developed above allow the distribution of the last-survivor status to be explored by use of the distribution of the joint-life status that is developed in the previous section. An illustration of this fact is provided by substituting (9.3.2) into (9.4.5) to obtain

$$F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t) - F_{T(xy)}(t) = F_{T(x)T(y)}(t, t),$$

a relationship that also follows from $F_{T(\overline{xy})}(t) = \Pr[T(x) \le t \cap T(y) \le t]$.

Example 9.4.1

Determine the d.f., survival function and p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.1.

Solution:

From (9.4.5) and the solutions to part (b) of Example 9.2.1 and Example 9.3.1,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - [1 - 0.0001(10 - t)^4] \\ &= 0.0001t^4 = F_{T(x)T(y)}(t, t) \qquad 0 \le t < 10, \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - 0.0001t^4 \qquad 0 < t \le 10. \end{aligned}$$

By differentiation,

$$f_{T(\overline{xy})}(t) = 0.0004\, t^3 \qquad 0 < t < 10. \qquad \blacktriangledown$$

Example 9.4.2

Determine the d.f., survival function and p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.3.

Solution:

From (9.4.5) and the solutions to Examples 9.2.3 and 9.3.2 for $0 < t \le 10$,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2[1 - 0.01(10 - t)^2] - [1 - 0.0001(10 - t)^4] \\ &= [1 - 0.01(10 - t)^2]^2 \\ &= t^2(0.2 - 0.01t)^2 = F_{T(x)T(y)}(t, t), \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - t^2(0.2 - 0.01t)^2, \\ f_{T(\overline{xy})}(t) &= 0.04t(2 - 0.1t)(1 - 0.1t) \qquad 0 < t < 10. \end{aligned} \qquad \blacktriangledown$$

An observation derived by comparing Examples 9.3.1 and 9.3.2 with Examples 9.4.1 and 9.4.2 is that two different joint distributions may produce the same distribution for the joint-life status but different distributions for the last-survivor status. This possibility could have been anticipated by the general nature of (9.4.5).

For applications it is preferable to rearrange and to restate (9.4.5) and (9.4.7) in actuarial notation:

$$ {}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}, \qquad \text{(9.4.5) restated} $$

$$ {}_tp_{\overline{xy}}\ \mu_{\overline{xy}}(t) = {}_tp_x\ \mu(x+t) + {}_tp_y\ \mu(y+t) - {}_tp_{xy}\ \mu_{xy}(t). \qquad \text{(9.4.7) restated} $$

The force of failure for the last-survivor status is implicitly defined in this restatement of (9.4.7) as

$$ \mu_{\overline{xy}}(t) = \frac{f_{T(\overline{xy})}(t)}{1 - F_{T(\overline{xy})}(t)} $$

$$ = \frac{{}_tp_x\ \mu(x+t) + {}_tp_y\ \mu(y+t) - {}_tp_{xy}\ \mu_{xy}(t)}{{}_tp_{\overline{xy}}}. \tag{9.4.8} $$

When $T(x)$ and $T(y)$ are independent $\mu_{xy}(t)$ can be replaced by $\mu_x(t) + \mu_y(t)$ in (9.4.7) and (9.4.8), and they can be rewritten as

$$ {}_tp_{\overline{xy}}\ \mu_{\overline{xy}}(t) = {}_tq_y\ {}_tp_x\ \mu(x+t) + {}_tq_x\ {}_tp_y\ \mu(y+t) \tag{9.4.9} $$

and

$$ \mu_{\overline{xy}}(t) = \frac{{}_tq_y\ {}_tp_x\ \mu(x+t) + {}_tq_x\ {}_tp_y\ \mu(y+t)}{{}_tq_y\ {}_tp_x + {}_tq_x\ {}_tp_y + {}_tp_x\ {}_tp_y}. \tag{9.4.10} $$

In this form the force of failure of the last-survivor status is a weighted average of forces of mortality. Forces of mortality are conditional p.d.f.'s, and the probability density that (x) and (y) will die at t is 0. As a result, the force associated with ${}_tp_x\ {}_tp_y$ in the weighted average is 0.

Example 9.4.3

Determine the force of failure for the last survivor of the two lives in
a. Example 9.2.1, and
b. Example 9.2.3.

Solution:
a. Using the results of Example 9.4.1,

$$ \mu_{\overline{xy}}(t) = \frac{0.0004t^3}{1 - 0.0001t^4} = \frac{4t^3}{10{,}000 - t^4}. $$

b. Using the results of Example 9.4.2,

$$ \mu_{\overline{xy}}(t) = \frac{0.04t(2 - 0.1t)(1 - 0.1t)}{1 - t^2(0.2 - 0.01t)^2} = \frac{4t(20 - t)(10 - t)}{10{,}000 - t^2(20 - t)^2}. $$ ▼

Discrete analogues of relationships (9.4.1)–(9.4.3) and (9.4.5)–(9.4.7) exist for the curtate-future-lifetimes. These are

$$ K(xy) + K(\overline{xy}) = K(x) + K(y), \tag{9.4.11} $$

$$ K(xy)\ K(\overline{xy}) = K(x)\ K(y), \tag{9.4.12} $$

$$a^{K(xy)} + a^{K(\overline{xy})} = a^{K(x)} + a^{K(y)} \qquad \text{for } a > 0, \tag{9.4.13}$$

$$F_{K(xy)}(k) + F_{K(\overline{xy})}(k) = F_{K(x)}(k) + F_{K(y)}(k). \tag{9.4.14}$$

From (9.4.14) it follows that

$$f_{K(xy)}(k) + f_{K(\overline{xy})}(k) = f_{K(x)}(k) + f_{K(y)}(k). \tag{9.4.15}$$

The distribution of $K(\overline{xy})$, the number of years completed prior to failure of the last-survivor status, that is, the number of years completed prior to the last death, can now be determined from these relationships and the results for the curtate-future-lifetime of the joint-life survivor status. From (9.4.15),

$$\Pr[K(\overline{xy}) = k] = f_{K(\overline{xy})}(k) = {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}. \tag{9.4.16}$$

For independent lives, (9.3.5) and (9.3.12) allow us to write (9.4.16) as

$$\begin{aligned}\Pr[K(\overline{xy}) = k] &= {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_x\, {}_kp_y\, (q_{x+k} + q_{y+k} - q_{x+k}q_{y+k}) \\ &= (1 - {}_kp_y)\, {}_kp_x\, q_{x+k} + (1 - {}_kp_x)\, {}_kp_y\, q_{y+k} + {}_kp_x\, {}_kp_y\, q_{x+k}\, q_{y+k}.\end{aligned}$$

In this last form, the first two terms are the probability that only the second death occurs between times k and $k + 1$. The third term is the probability that both deaths occur during that year. This expression for $\Pr[K(\overline{xy}) = k]$ is analogous to (9.4.9) for the p.d.f. of $T(\overline{xy})$ where, since the probability that two deaths occur in the same instant is 0, there are only two terms.

9.5 More Probabilities and Expectations

In Sections 9.3 and 9.4 we express the p.d.f.'s and the d.f.'s of the future lifetimes of the joint-life status and the last-survivor status in terms of the functions of the probability distributions for the single lives. In this section we use these expressions to solve probability problems and to obtain expectations, variances, and, for independent individual future lifetimes, the covariance of the joint-life and last-survivor future lifetimes.

Example 9.5.1

Assuming the future lifetimes of (80) and (85) are independent, obtain an expression, in single life table functions, for the probability that their

a. First death occurs after 5 and before 10 years from now, and
b. Last death occurs after 5 and before 10 years from now.

Solution:

a. With $T = T(80{:}85)$ we obtain

$$\begin{aligned}\Pr(5 < T \le 10) &= \Pr(T > 5) - \Pr(T > 10) \\ &= {}_5p_{80:85} - {}_{10}p_{80:85} \\ &= {}_5p_{80}\, {}_5p_{85} - {}_{10}p_{80}\, {}_{10}p_{85}.\end{aligned}$$

Note that the independence assumption is used only in the last step.

b. With $T = T(\overline{80:85})$

$$\Pr(5 < T \le 10) = \Pr(T > 5) - \Pr(T > 10)$$
$$= {}_5p_{\overline{80:85}} - {}_{10}p_{\overline{80:85}},$$

and from (9.4.6) we obtain

$$= {}_5p_{80} - {}_{10}p_{80} + {}_5p_{85} - {}_{10}p_{85} - ({}_5p_{80:85} - {}_{10}p_{80:85}).$$

Using the independence assumption, we can substitute ${}_5p_{80}\ {}_5p_{85}$ for ${}_5p_{80:85}$ and ${}_{10}p_{80}\ {}_{10}p_{85}$ for ${}_{10}p_{80:85}$. ▼

The results of Section 3.5 concerning expected values of the distribution of T, the time-until-the death of (x), are also valid if $T = T(u)$ the time-until-failure of a survival status (u). We used some of these ideas in the previous two sections; now we will state them explicitly.

From (3.5.2) we have that $\mathring{e}_u = \mathrm{E}[T(u)]$, which for a survival status (u) can be obtained from the formula

$$\mathring{e}_u = \int_0^\infty {}_tp_u\, dt. \tag{9.5.1}$$

If (u) is the joint-life status (xy), then

$$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy}\, dt, \tag{9.5.2}$$

and for the last-survivor status $(\overline{xy})$ we have

$$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}\, dt. \tag{9.5.3}$$

Upon taking expectation of both sides of (9.4.1), we see that

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}. \tag{9.5.4}$$

From (3.5.5), the expected value of $K = K(u)$ is

$$e_u = \sum_1^\infty {}_kp_u$$

for a survival status, (u). Special cases include

$$e_{xy} = \sum_1^\infty {}_kp_{xy}$$

and

$$e_{\overline{xy}} = \sum_1^\infty {}_kp_{\overline{xy}}.$$

It follows from (9.4.11) that

$$e_{\overline{xy}} = e_x + e_y - e_{xy}. \tag{9.5.5}$$

The variance formulas derived in Section 3.5 can be used to calculate the variance of the future lifetime, or curtate-future-lifetime, of any survival status, (u). Thus,

$$\mathrm{Var}[T(xy)] = 2 \int_0^\infty t \; {}_tp_{xy} \, dt - (\mathring{e}_{xy})^2 \tag{9.5.6}$$

and

$$\mathrm{Var}[T(\overline{xy})] = 2 \int_0^\infty t \; {}_tp_{\overline{xy}} \, dt - (\mathring{e}_{\overline{xy}})^2. \tag{9.5.7}$$

In Example 9.2.1 we calculated the covariance of $T(x)$ and $T(y)$ for dependent future lifetimes. For the moment, we return to dependent future lifetimes to explore the covariance of $T(xy)$ and $T(\overline{xy})$ in the general case:

$$\mathrm{Cov}[T(xy), T(\overline{xy})] = \mathrm{E}[T(xy)\; T(\overline{xy})] - \mathrm{E}[T(xy)]\; \mathrm{E}[T(\overline{xy})]. \tag{9.5.8}$$

On the basis of (9.4.2),

$$\mathrm{E}[T(xy)\; T(\overline{xy})] = \mathrm{E}[T(x)\; T(y)].$$

Using this result and (9.5.4), we can write

$$\mathrm{Cov}[T(xy), T(\overline{xy})] = \mathrm{E}[T(x)\; T(y)] - \mathrm{E}[T(xy)]\; \{\mathrm{E}[T(x)] + \mathrm{E}[T(y)] - \mathrm{E}[T(xy)]\},$$

which can be rewritten as

$$= \mathrm{Cov}[T(x), T(y)] + \{\mathrm{E}[T(x)] - \mathrm{E}[T(xy)]\}\; \{\mathrm{E}[T(y)] - \mathrm{E}[T(xy)]\}. \tag{9.5.9}$$

If $T(x)$ and $T(y)$ are uncorrelated, then

$$\mathrm{Cov}[T(xy), T(\overline{xy})] = (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}). \tag{9.5.10}$$

Since both factors of (9.5.10) must be non-negative, we can see that when $T(x)$ and $T(y)$ are uncorrelated, $T(xy)$ and $T(\overline{xy})$ are positively correlated, except in the trivial case where $\mathring{e}_x$ or $\mathring{e}_y$ equals $\mathring{e}_{xy}$.

Example 9.5.2

For $T(x)$ and $T(y)$ of Examples 9.2.1 and 9.2.3 determine (a) $\mathrm{Cov}[T(xy), T(\overline{xy})]$, and (b) the correlation coefficient of $T(xy)$ and $T(\overline{xy})$.

Solution:

Most of the required calculations have been done in previous examples. The remaining calculations can be done readily with the p.d.f.'s that were determined there. We will complete the calculations by displaying intermediate results in tabular form along with the number of the formula being illustrated.

Item	Distribution	
	Examples 9.2.1, 9.3.1, 9.4.1	Examples 9.2.3, 9.3.2, 9.4.2
a. $\overset{\circ}{e}_x = \overset{\circ}{e}_y = \mathrm{E}[T(x)] = \mathrm{E}[T(y)]$	5	10/3
$\overset{\circ}{e}_{xy} = \mathrm{E}[T(xy)]$	2	2
$\mathrm{Cov}[T(x), T(y)]$	−25/3	0
$\mathrm{Cov}[T(xy), T(\overline{xy})]$	2/3	16/9
b. $\mathrm{E}[T(xy)^2]$	20/3	20/3
$\mathrm{Var}[T(xy)]$	8/3	8/3
$\mathrm{E}[T(\overline{xy})]$	8	14/3
$\mathrm{E}[T(\overline{xy})^2]$	200/3	80/3
$\mathrm{Var}[T(\overline{xy})]$	8/3	44/9
$\rho_{T(xy)\ T(\overline{xy})}$	1/4	$\sqrt{8/33}$

▼

9.6 Dependent Lifetime Models

In Section 9.2 the concept of the joint distribution of $[T(x), T(y)]$ was introduced. Two examples, which did not appear plausible as models for the joint distribution of $[T(x), T(y)]$ for human lives, were used to illustrate the ideas. Example 9.2.1 illustrated the ideas when $T(x)$ and $T(y)$ are dependent, and Example 9.2.3 provided similar illustrations for independent random variables.

In this section, two general approaches to specifying the joint distribution of $[T(x), T(y)]$ will be studied. The convenient independent case is included within each model.

9.6.1 Common Shock

Let $T^*(x)$ and $T^*(y)$ denote two future lifetime random variables that, in the absence of the possibility of a common shock, are independent; that is,

$$\begin{aligned} s_{T^*(x)\ T^*(y)}(s, t) &= \Pr[T^*(x) > s \cap T^*(y) > t] \\ &= s_{T^*(x)}(s)\ s_{T^*(y)}(t). \end{aligned} \tag{9.6.1}$$

In addition, there is a ***common shock*** random variable, to be denoted by Z, that can affect the joint distribution of the time-until-death of lives (x) and (y). This common shock random variable is independent of $[T^*(x), T^*(y)]$ and has an exponential distribution; that is,

$$s_Z(z) = e^{-\lambda z} \quad z > 0, \lambda \geq 0.$$

We can picture the random variable Z as being associated with the time of a catastrophe such as an earthquake or aircraft crash. The random variables of interest in building models for life insurance or annuities to (x) and (y) are $T(x) = \min[T^*(x), Z]$ and $T(y) = \min[T^*(y), Z]$. The joint survival function of $[T(x), T(y)]$ is

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= \Pr\{\min[T^*(x), Z] > s \cap \min[T^*(y), Z] > t\} \\ &= \Pr\{[T^*(x) > s \cap Z > s] \cap [T^*(y) > t \cap Z > t]\} \\ &= \Pr[T^*(x) > s \cap T^*(y) > t \cap Z > \max(s, t)] \\ &= s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]}. \end{aligned} \tag{9.6.2}$$

The final line of (9.6.2) follows from the independence of $[T^*(x), T^*(y), Z]$.

Using the joint survival function displayed in (9.6.2), we can determine the joint p.d.f. of $[T(x), T(y)]$. Except when $s = t$, the p.d.f. can be found by partial differentiation. We have

$$\begin{aligned} f_{T(x)T(y)}(s, t) &= \frac{\partial^2}{\partial s \partial t} s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]} \\ &= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s_{T^*(x)}(s)\, s'_{T^*(y)}(t)]\, e^{-\lambda s} \quad 0 < t < s \qquad (9.6.3a) \\ &= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s'_{T^*(x)}(s)\, s_{T^*(y)}(t)]\, e^{-\lambda t} \quad 0 < s < t. \qquad (9.6.3b) \end{aligned}$$

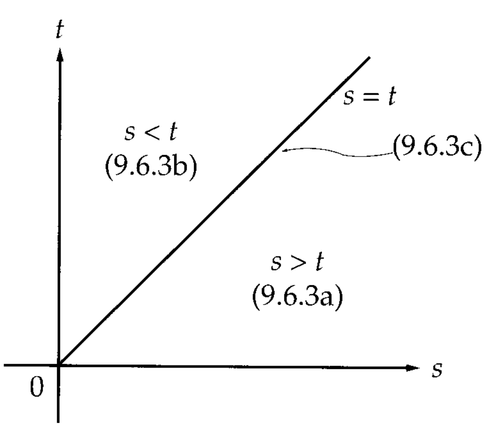
This display does not complete the definition of the p.d.f. When $s = t$, the common shock contribution to the p.d.f. is

$$f_{T(x)T(y)}(t, t) = \lambda e^{-\lambda t}\, s_{T^*(x)}(t)\, s_{T^*(y)}(t) \qquad t \geq 0. \tag{9.6.3c}$$

The domain of this mixed p.d.f., with a ridge of density along the line $s = t$, is shown in Figure 9.6.1.

Figure 9.6.1

Domain of Common Shock p.d.f.



Remark:

The interpretation of the p.d.f. given in (9.6.3.a), (9.6.3.b), and (9.6.3c) requires a careful analysis of the distribution of $[T^*(x), T^*(y)]$ and the derived distribution of $\{T(x) = \min[T^*(x), Z], T(y) = \min[T^*(y), Z]\}$. Table 9.6.1 facilitates this analysis. Because of the intervention of the common shock, realizations of $T^*(x)$ and $T^*(y)$ may not be observed because of the prior realization of Z. The common shock random variable can mask or censor observations of $T^*(x)$ and $T^*(y)$. In addition, events such as $T^*(x) < Z < T^*(y)$ and $T^*(y) < Z < T^*(x)$ can contribute to the p.d.f.

The marginal survival functions are given by

$$\begin{aligned} s_{T(x)}(s) &= \Pr\{[T(x) > s] \cap [T(y) > 0]\} \\ &= s_{T^*(x)}(s)e^{-\lambda s} \end{aligned} \tag{9.6.4a}$$

and

$$\begin{aligned} s_{T(y)}(t) &= \Pr\{[T(x) > 0] \cap [T(y) > t]\} \\ &= s_{T^*(y)}(t)e^{-\lambda t}. \end{aligned} \tag{9.6.4b}$$

Table 9.6.1

Interpretation of p.d.f. of $(T(x), T(y))$, Common Shock Model

Formula	Interpretation	Domain
9.6.3a	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > s)] \cup [(T^*(x) > s) \cap (t < T^*(y) \le t + \Delta t) \cap (s < Z \le s + \Delta s)]\} \approx [f_{T^*(x)}(s)\, f_{T^*(y)}(t)\, s_Z(s) + s_{T^*(x)}(s)\, f_{T^*(y)}(t)\, f_Z(s)]\Delta s\Delta t$	$0 < t < s$
9.6.3b	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > t)] \cup [(s < T^*(x) \le s + \Delta s) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)]\} \approx [f_{T^*(x)}(s)\, f_{T^*(y)}(t)\, s_Z(t) + f_{T^*(x)}(s)\, s_{T^*(y)}(t) f_Z(t)]\Delta s\Delta t$	$0 < s < t$
9.6.3c	$\Pr\{(T^*(x) > t) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)\} \approx s_{T^*(x)}(t)\, s_{T^*(y)}(t)\, f_Z(t)\, \Delta t$	$s = t$

If we are interested in the distribution of $T(xy) = \min[T(x), T(y)]$, the joint-life status in the common shock model, the survival function can be obtained using (9.3.1) and (9.6.3):

$$s_{T(xy)}(t) = s_{T^*(x)}(t)\, s_{T^*(y)}(t)\, e^{-\lambda t} \quad 0 < t. \tag{9.6.5}$$

The distribution of $T(\overline{xy}) = \max[T(x), T(y)]$, the last-survivor status in the common shock model, can be derived by using (9.4.5), (9.6.5), (9.6.4a), and (9.6.4b):

$$s_{T(\overline{xy})}(t) = [s_{T^*(x)}(t) + s_{T^*(y)}(t) - s_{T^*(x)T^*(y)}(t, t)]\, e^{-\lambda t} \quad 0 < t. \tag{9.6.6}$$

If the common shock parameter $\lambda = 0$, formulas (9.6.5) and (9.6.6) revert to the form where $T(x)$ and $T(y)$ are independent. When $\lambda > 0$ the joint-life and last-survivor survival functions are each less than the corresponding survival function when $\lambda = 0$.

Example 9.6.1

Exhibit $\mu_{xy}(t)$ as derived from (9.6.5).

Solution:

$$\begin{aligned}\mu_{xy}(t) &= -\frac{d}{dt}\log\,[s_{T^*(x)}(t)\; s_{T^*(y)}(t)e^{-\lambda t}]\\ &= \mu(x+t) + \mu(y+t) + \lambda.\end{aligned}$$ ▼

Example 9.6.2

The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameter μ_1, μ_2, and λ. These three random variables are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\, ds + f_{T(x)T(y)}(t, t) + \int_t^\infty f_{T(x)T(y)}(s, t)\, ds.$$

b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\, du.$$

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^\infty f_{T(x)T(y)}(t, t)\, dt.$$

Solution:

a. We use the three elements of (9.6.3), adapted for exponential distributions, to obtain

$$\begin{aligned}f_{T(y)}(t) &= \textstyle\int_0^t \mu_1(\mu_2+\lambda)e^{-\mu_1 s-(\mu_2+\lambda)t}\, ds + \lambda e^{-(\mu_1+\mu_2+\lambda)t}\\ &\quad + \textstyle\int_t^\infty \mu_2(\mu_1+\lambda)e^{-(\mu_1+\lambda)s-\mu_2 t}\, ds\\ &= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t}\,(1-e^{-\mu_1 t}) + \lambda e^{-(\mu_1+\mu_2+\lambda)t} + \mu_2 e^{-(\mu_1+\mu_2+\lambda)t}\\ &= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t} \quad 0 < t.\end{aligned}$$

b. $s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\, du = e^{-(\mu_2+\lambda)t} = s_{T^*(y)}(t)e^{-\lambda t}$, which agrees with (9.6.4.b).

c. $\Pr[T(x) = T(y)] = \int_0^\infty \lambda e^{-(\mu_1+\mu_2+\lambda)t}\, dt = \dfrac{\lambda}{\lambda+\mu_1+\mu_2}$. ▼

9.6.2 Copulas

The word ***copula*** means something that connects or joins together. Copula is used in multivariate statistical analysis to define a class of bivariate distributions with specified marginal distributions.

In this section we will illustrate actuarial applications of Frank's copula. The notation will be that used in Section 9.2. It is claimed that

$$F_{T(x)T(y)}(s, t) = \frac{1}{\alpha} \log \left[1 + \frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1} \right] \tag{9.6.7}$$

when $\alpha \neq 0$ is a joint d.f. with marginal distribution $F_{T(x)}(s)$ and $F_{T(y)}(t)$. This claim can be verified by confirming that

$$F_{T(x)T(y)}(0, 0) = 0, \tag{9.6.8}$$

$$F_{T(x)T(y)}(\infty, \infty) = 1, \tag{9.6.9}$$

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} F_{T(x)T(y)}(s, t) &= f_{T(x)T(y)}(s, t) \\ &= \frac{\alpha f_{T(x)}(s)\, f_{T(y)}(t)[e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]}]}{[(e^{\alpha} - 1) + (e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)]^2} (e^{\alpha} - 1) \geq 0, \end{aligned} \tag{9.6.10}$$

and

$$F_{T(x)T(y)}(s, \infty) = F_{T(x)}(s),$$

$$F_{T(x)T(y)}(\infty, t) = F_{T(y)}(t).$$

Statements (9.6.8) and (9.6.9) are necessary for $F_{T(x)T(y)}(s, t)$ to be a d.f. of two future lifetime random variables. Statement (9.6.10) exhibits the joint p.d.f. and shows that it is non-negative.

The parameter α in the d.f. displayed in (9.6.7) and the p.d.f. displayed in (9.6.10) controls the dependence of $T(x)$ and $T(y)$. This can be appreciated by finding

$$\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s)\, f_{T(y)}(t) \left\{ \lim_{\alpha \to 0} [A(\alpha)\ B(\alpha)\ C(\alpha)] \right\},$$

where

$$A(\alpha) = e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]},$$

$$B(\alpha) = \frac{(e^{\alpha} - 1)\alpha}{(e^{\alpha} - 1)^2},$$

and

$$C(\alpha) = \frac{1}{\{1 + [(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)/(e^{\alpha} - 1)]\}^2}.$$

We have

$$\lim_{\alpha \to 0} A(\alpha) = 1,$$

$$\lim_{\alpha \to 0} B(\alpha) = 1,$$

and $\lim_{\alpha \to 0} C(\alpha)$ depends on the term in the denominator

$$\lim_{\alpha \to 0}\left[\frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1}\right]$$

$$= \lim_{\alpha \to 0}\left\{\frac{F_{T(x)}(s)\, e^{\alpha F_{T(x)}(s)}(e^{\alpha F_{T(y)}(t)} - 1) + F_{T(y)}(t) e^{\alpha F_{T(y)}(t)}(e^{\alpha F_{T(x)}(s)} - 1)}{e^{\alpha}}\right\}$$

$$= 0.$$

Therefore, $\lim_{\alpha \to 0} C(\alpha) = 1$, and $\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s)\, f_{T(y)}(t)$. This means that $T(x)$ and $T(y)$ are independent in the limit as $\alpha \to 0$. We interpret the joint p.d.f. in this fashion when $\alpha = 0$.

Example 9.6.3

Let $F_{T(x)}(s) = s$, $0 < s \le 1$, and $F_{T(y)}(t) = t$, $0 < t \le 1$, in (9.6.7) and $T(xy) = \min[T(x), T(y)]$.

a. Find the d.f. of $T(xy)$.

b. Find the p.d.f. of $T(xy)$.

Solution:

a. Using (9.3.2),

$$F_{T(xy)}(t) = F_{T(x)}(t) + F_{T(y)}(t) - F_{T(x)T(y)}(t, t)$$

$$= 2t - \frac{1}{\alpha}\log\left[1 + \frac{(e^{\alpha t} - 1)^2}{e^{\alpha} - 1}\right] \qquad 0 < t < 1.$$

b.

$$f_{T(xy)}(t) = \frac{d}{dt} F_{T(xy)}(t)$$

$$= 2 - \frac{2\alpha(e^{\alpha t} - 1)e^{\alpha t}/(e^{\alpha} - 1)}{\alpha\{1 + [(e^{\alpha t} - 1)^2/(e^{\alpha} - 1)]\}}$$

$$= 2 - \left[\frac{2(e^{\alpha t} - 1)e^{\alpha t}}{(e^{\alpha} - 1) + (e^{\alpha t} - 1)^2}\right] \qquad 0 < t < 1.$$

▼

9.7 Insurance and Annuity Benefits

Insurances and annuities, previously discussed for an individual life, can be defined for the survival status, (u), and are discussed in this section. We also investigate more complicated examples in which an annuity, payable to a survival status, (u), is deferred until another status, (v), has failed.

9.7.1 Survival Statuses

With the single-life status, (x), replaced by the survival status, (u), the models and formulas of Chapters 4 and 5 are applicable here. Expressions for the actuarial present values, variances, and percentiles in terms of the distribution of (u) are immediately available. The relationships of Sections 9.3 and 9.4 can then be used to be these expressions in terms of functions for the individual lives of the survival status.

For an insurance of unit amount payable at the end of the year in which the "survival" status fails, the model and formulas of Section 4.3 apply. Thus if K denotes the curtate-future-lifetime of (u), then the

- Time of payment is $K + 1$,
- Present value at issue of the payment is $Z = v^{K+1}$,
- Actuarial present value, A_u, is $E[Z] = \sum_{0}^{\infty} v^{k+1} \Pr(K = k)$, (9.7.1)
- $\mathrm{Var}(Z) = {}^2A_u - (A_u)^2$. (9.7.2)

As an illustration, consider a unit sum insured payable at the end of the year in which the last survivor of (x) and (y) dies. From (9.4.16) and (9.7.1) we have

$$A_{\overline{xy}} = \sum_{0}^{\infty} v^{k+1} ({}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}),$$

which can be used at forces of interest δ and 2δ to obtain the variance by (9.7.2).

The numerous formulas for discrete annuities in Section 5.3 are valid when the annuity payments are contingent on the survival of a survival status. For example, if we replace x with u to emphasize that K is the curtate-future-lifetime of the survival status, (u), we can restate the following formulas for an n-year temporary life annuity in regard to (u):

$$Y = \begin{cases} \ddot{a}_{\overline{K+1|}} & K = 0, 1, \ldots, n-1 \\ \ddot{a}_{\overline{n|}} & K = n, n+1, \ldots, \end{cases} \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n|}} = E[Y] = \sum_{0}^{n-1} \ddot{a}_{\overline{K+1|}}\ {}_{k|}q_u + \ddot{a}_{\overline{n|}}\ {}_np_u, \qquad (9.7.3)$$

$$\ddot{a}_{u:\overline{n|}} = \sum_{0}^{n-1} {}_kE_u = \sum_{0}^{n-1} v^k\ {}_kp_u, \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n|}} = \frac{1}{d}(1 - A_{u:\overline{n|}}), \qquad \text{(5.3.12) restated}$$

$$\mathrm{Var}(Y) = \frac{1}{d^2}[{}^2A_{u:\overline{n|}} - (A_{u:\overline{n|}})^2]. \qquad \text{(5.3.14) restated}$$

As an illustration, consider an annuity of 1, payable at the beginning of each year to which both (x) and (y) survive during the next n years. This is an annuity to the joint-life status (xy). By substituting ${}_tp_{xy}$, or ${}_tp_x\ {}_tp_y$ if the lifetimes are independent, for ${}_tp_u$ in the above formulas, we can obtain the actuarial present value of the annuity. For the variance as given in (5.3.14), we can use

$$A_{xy:\overline{n|}} = 1 - d\ddot{a}_{xy:\overline{n|}}$$

and

$$^{2}A_{xy:\overline{n}|} = 1 - (2d - d^2)\ {}^{2}\ddot{a}_{xy:\overline{n}|},$$

or we can calculate the actuarial present values directly.

In addition, we can establish relationships among the present-value random variables for annuities and insurances on the last-survivor status and the joint-life status. From relationship (9.4.13), we have

$$v^{K(\overline{xy})+1} + v^{K(xy)+1} = v^{K(x)+1} + v^{K(y)+1}, \tag{9.7.4}$$

$$\ddot{a}_{\overline{K(\overline{xy})+1}|} + \ddot{a}_{\overline{K(xy)+1}|} = \ddot{a}_{\overline{K(x)+1}|} + \ddot{a}_{\overline{K(y)+1}|}. \tag{9.7.5}$$

By taking the expectations of both sides of (9.7.4) and (9.7.5), we have

$$A_{\overline{xy}} + A_{xy} = A_x + A_y$$

and

$$\ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y.$$

These formulas allow us to express the actuarial present values of last-survivor annuities and insurances in terms of those for the individual lives and the joint-life status. Note that these formulas hold for all joint distributions; independence is not required.

We now consider continuous insurances and annuities. If T, the future-lifetime random variable in Sections 4.2 and 5.2, is reinterpreted as $T(u)$, the time-until-failure of the survival status, (u), the formulas of those sections for present values, actuarial present values, percentiles, and variances hold for insurances and annuities for the status (u).

For an insurance paying a unit amount at the moment of failure of (u), the present value at policy issue, the actuarial present value and the variance are given by

$$Z = v^T,$$

$$\bar{A}_u = \int_0^\infty v^t\ {}_tp_u\ \mu_u(t)\ dt, \tag{4.2.6) restated\quad(}$$

$$\mathrm{Var}(Z) = {}^{2}\bar{A}_u - (\bar{A}_u)^2.$$

As an illustration, the restated (4.2.6) for the last survivor of (x) and (y) would be

$$\bar{A}_{\overline{xy}} = \int_0^\infty v^t\ {}_tp_{\overline{xy}}\ \mu_{\overline{xy}}(t)\ dt.$$

From (9.4.7), this is

$$\bar{A}_{\overline{xy}} = \int_0^\infty v^t\ [{}_tp_x\ \mu(x+t) + {}_tp_y\ \mu(y+t) - {}_tp_{xy}\ \mu_{xy}(t)]\ dt.$$

For an annuity payable continuously at the rate of 1 per annum until the time-of-failure of (u), we have

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|}\ {}_tp_u\ \mu_u(t)\ dt \qquad (5.2.3)\text{ restated}$$

$$= \int_0^\infty v^t\ {}_tp_u\ dt, \qquad (5.2.4)\text{ restated}$$

$$\operatorname{Var}(Y) = \frac{{}^2\bar{A}_u - (\bar{A}_u)^2}{\delta^2}. \qquad (5.2.9)\text{ restated}$$

The interest identity,

$$\delta\ \bar{a}_{\overline{T}|} + v^T = 1, \qquad (5.2.7)\text{ restated}$$

is also available for $T = T(u)$ and provides the connection between the models for insurances and annuities.

As an application, consider an annuity payable continuously at the rate of 1 per year as long as at least one of (x) or (y) survives. This is an annuity in respect to $(\overline{xy})$, so we have from the above formulas with $T = T(\overline{xy})$

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_{\overline{xy}} = \int_0^\infty \bar{a}_{\overline{t}|}\ [{}_tp_x\ \mu(x+t) + {}_tp_y\ \mu(y+t) - {}_tp_{xy}\ \mu_{xy}(t)]\ dt$$

$$= \int_0^\infty v^t\ {}_tp_{\overline{xy}}\ dt,$$

$$\operatorname{Var}(Y) = \frac{{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2}{\delta^2}.$$

Formula (9.4.3) implies that

$$v^{T(\overline{xy})} + v^{T(xy)} = v^{T(x)} + v^{T(y)} \qquad (9.7.6)$$

and

$$\bar{a}_{\overline{T(\overline{xy})}|} + \bar{a}_{\overline{T(xy)}|} = \bar{a}_{\overline{T(x)}|} + \bar{a}_{\overline{T(y)}|}, \qquad (9.7.7)$$

and (9.4.1) implies that

$$v^{T(\overline{xy})}\ v^{T(xy)} = v^{T(x)}\ v^{T(y)}. \qquad (9.7.8)$$

These identities can be used to obtain the relations among the actuarial present values, variances, and covariances of insurances and annuities for the various statuses. For example, taking the expectations of both sides of (9.7.6) and (9.7.7), we obtain

$$\bar{A}_{\overline{xy}} + \bar{A}_{xy} = \bar{A}_x + \bar{A}_y, \qquad (9.7.9)$$

$$\bar{a}_{\overline{xy}} + \bar{a}_{xy} = \bar{a}_x + \bar{a}_y. \qquad (9.7.10)$$

In the same way that $\text{Cov}[T(\overline{xy}), T(xy)]$ was expressed as

$$\text{Cov}[T(\overline{xy}), T(xy)] = \text{Cov}[T(x), T(y)] + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}),$$

it can be shown that

$$\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = \text{Cov}(v^{T(x)}, v^{T(y)}) + (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}). \tag{9.7.11}$$

Both factors in the second term of (9.7.11) are nonpositive, so it will be non-negative and will be zero only in the trivial case where either $\bar{A}_x$ or $\bar{A}_y$ equals $\bar{A}_{xy}$.

The actuarial present value of a continuous annuity paid with respect to $(\overline{xy})$ where, because (x) and (y) are subject to common shock, the joint survival function is given by (9.6.5) can be written in current payment form as

$$\bar{a}_{\overline{xy}} = \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(x)}(t)\, dt + \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(y)}(t)\, dt$$
$$- \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(x)}(t) s_{T^*(y)}(t)\, dt. \tag{9.7.12}$$

Formula (9.7.12) illustrates how the common shock parameter λ can be combined with the force of interest in some calculations.

Example 9.7.1

a. Extend (9.7.9) to the actuarial present value of an n-year term insurance paying a death benefit of 1 at the moment of the last death of (x) and (y) if this death occurs before time n. If at least one individual survives to time n, no payment is made.
b. Use the formula to calculate the actuarial present value, on the basis of $\delta = 0.05$, of a 5-year term insurance payable on the death of the last survivor of the two lives in Example 9.2.1.

Solution:

a. By restating (9.7.6) for n-year term insurance random variables and then taking expectation of both sides, we have

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + \bar{A}^1_{y:\overline{n}|} - \bar{A}^{\;1}_{xy:\overline{n}|}.$$

The symbol $\bar{A}^{\;1}_{xy:\overline{n}|}$ represents the actuarial present value of an n-year term insurance payable at the failure of the joint-life status if it occurs prior to n.

b. From Example 9.2.1, $T(x)$ and $T(y)$ each has p.d.f. $f_T(t) = 0.0002\,[t^3 + (10 - t)^3]$, $0 \le t < 10$. Therefore,

$$\bar{A}^1_{x:\overline{5}|} = \bar{A}^1_{y:\overline{5}|} = \int_0^5 e^{-0.05t}\{0.0002[t^3 + (10 - t)^3]\}\, dt$$
$$= 0.4563.$$

From Example 9.3.3, we have that $f_{T(xy)}(t) = 0.0004(10 - t)^3$, $0 < t < 10$, so

$$\bar{A}^{\;1}_{xy:\overline{5}|} = \int_0^5 e^{-0.05t}[0.0004(10 - t)^3]\, dt = 0.8614.$$

Using the result from part (a),

$$\bar{A}_{\overline{xy}:\overline{5|}}^{\,1} = 2(0.4563) - 0.8614 = 0.0512. \qquad \blacktriangledown$$

9.7.2 Special Two-Life Annuities

In this section we illustrate by an example special annuities in which the payment rate depends on the survival of two lives.

Example 9.7.2

An annuity is payable continuously at the rate of

- 1 per year while both (x) and (y) are alive,
- 2/3 per year while one of (x) or (y) is alive and the other is dead.

Derive expressions for

a. The annuity's present-value random variable
b. The annuity's actuarial present value
c. The variance of the random variable in (a), under the assumption that $T(x)$ and $T(y)$ are independent.

Solution:

a. The annuity is a combination of one that is payable at the rate of 2/3 per year while at least one of (x) and (y) is alive—until time $T(\overline{xy})$]—and one that is payable at the rate of 1/3 per year while both individuals are alive—until time $T(xy)$. The present value of the payments is

$$Z = \frac{2}{3}\,\bar{a}_{\overline{T(\overline{xy})|}} + \frac{1}{3}\,\bar{a}_{\overline{T(xy)|}}.$$

b. The actuarial present value is

$$\mathrm{E}[Z] = \frac{2}{3}\,\bar{a}_{\overline{xy}} + \frac{1}{3}\,\bar{a}_{xy}.$$

Using (9.7.10) to substitute for $\bar{a}_{\overline{xy}}$, we have

$$\mathrm{E}[Z] = \frac{2}{3}\,\bar{a}_x + \frac{2}{3}\,\bar{a}_y - \frac{1}{3}\,\bar{a}_{xy}.$$

Alternatively, from (5.3.2B) restated, we have

$$\mathrm{E}[Z] = \frac{2}{3}\int_0^\infty v^t\,{}_tp_{\overline{xy}}\,dt + \frac{1}{3}\int_0^\infty v^t\,{}_tp_{xy}\,dt.$$

Then, by considering the three mutually exclusive cases as to which of the lives may be surviving when $(\overline{xy})$ is surviving at time t, we can write

$${}_tp_{\overline{xy}} = {}_tp_{xy} + ({}_tp_x - {}_tp_{xy}) + ({}_tp_y - {}_tp_{xy}).$$

Substitution of this expression into the first integral and combining the results with the second give

$$E[Z] = \int_0^\infty v^t\, {}_tp_{xy}\, dt + \frac{2}{3}\int_0^\infty v^t\, ({}_tp_x - {}_tp_{xy})\, dt$$
$$+ \frac{2}{3}\int_0^\infty v^t\, ({}_tp_y - {}_tp_{xy})\, dt.$$

This expression of E[Z] can be directly obtained by considering the three cases. The first term is the actuarial present value of the payments at the rate of 1 per year while both (x) and (y) survive. The second term is the actuarial present value of the payments at the rate of 2/3 per year at those times t when (x) is alive (with probability ${}_tp_x$) but not both of (x) and (y) are alive (with probability ${}_tp_{xy}$). The third term has a similar interpretation with x and y interchanged.

c.
$$\text{Var}(Z) = \text{Var}\left(\frac{2}{3}\,\bar{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\,\bar{a}_{\overline{T(xy)}|}\right)$$
$$= \frac{4}{9}\,\text{Var}(\bar{a}_{\overline{T(\overline{xy})}|}) + \frac{1}{9}\,\text{Var}(\bar{a}_{\overline{T(xy)}|}) + \frac{4}{9}\,\text{Cov}(\bar{a}_{\overline{T(\overline{xy})}|}, \bar{a}_{\overline{T(xy)}|}).$$

But, by Exercise 9.23, for independent $T(x)$ and $T(y)$,

$$\text{Cov}(\bar{a}_{\overline{T(\overline{xy})}|}, \bar{a}_{\overline{T(xy)}|}) = \frac{\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)})}{\delta^2}$$
$$= \frac{(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}.$$

Hence

$$\text{Var}(Z) = \left\{\frac{4/9\,[{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2] + 1/9\,[{}^2\bar{A}_{xy} - (\bar{A}_{xy})^2] + 4/9\,(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}\right\}.$$ ▼

9.7.3 Reversionary Annuities

A ***reversionary annuity*** is payable during the existence of a status (u), but only after the failure of a second status (v). Conceptually, this is a deferred life annuity with a random deferment period equal to the time until failure of the second status. In fact, it is a generalization of the deferred life annuity, for if (v) is an n-year term certain, then reversionary annuity reduces to an n-year deferred annuity. If (u) is a term certain, the reversionary annuity reduces to a form of insurance, family income insurance, studied in Chapter 17. The basic notation for the actuarial present value of this annuity is $a_{v|u}$ with adornments to indicate frequency and timing of payments. The concept has been useful to obtain expressions for the more complex annuity arrangements in terms of single and joint status annuities (see Example 9.7.3). Here we will study the present-value random variables for reversionary annuities.

We start with an annuity of 1 per year payable continuously to (y) after the death of (x). The present value at 0, denoted by Z, is

$$Z = \begin{cases} {}_{T(x)|}\bar{a}_{\overline{T(y)-T(x)|}} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases} \tag{9.7.13}$$

This can be written as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(x)|}} & T(x) \le T(y) \\ 0 & T(x) > T(y), \end{cases} \tag{9.7.14}$$

or as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(x)|}} & T(x) \le T(y) \\ \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(y)|}} & T(x) > T(y), \end{cases} \tag{9.7.15}$$

which is the same as

$$Z = \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(xy)|}}. \tag{9.7.16}$$

Thus,

$$\bar{a}_{x|y} = \mathrm{E}[Z] = \mathrm{E}[\bar{a}_{\overline{T(y)|}}] - \mathrm{E}[\bar{a}_{\overline{T(xy)|}}] = \bar{a}_y - \bar{a}_{xy}. \tag{9.7.17}$$

Formulas (9.7.16) and (9.7.17) hold for survival statuses (u) and (v). For example,

$$\bar{a}_{x:\overline{n|}|y} = \bar{a}_y - \bar{a}_{xy:\overline{n|}},$$

$$\bar{a}_{x|y:\overline{n|}} = \bar{a}_{y:\overline{n|}} - \bar{a}_{xy:\overline{n|}}.$$

We note that (9.7.17) holds for dependent future lifetimes.

Example 9.7.3

Calculate the actuarial present value of an annuity payable continuously at the rates shown in the following display.

Case, Rate, and Condition

1. 1 per year with certainty until time n,
2. 1 per year after time n if both (x) and (y) are alive,
3. 3/4 per year after time n if (x) is alive and (y) is dead, and
4. 1/2 per year after time n if (y) is alive and (x) is dead.

Solution:

We use the reversionary annuity idea to write the actuarial present value of this arrangement in terms of single-life and joint-life annuities.

- Case 1: This is an n-year annuity-certain: $\bar{a}_{\overline{n|}}$.
- Case 2: This is n-year deferred joint life annuity to (xy):

$${}_{n|}\bar{a}_{xy} = \bar{a}_{xy} - \bar{a}_{xy:\overline{n|}}.$$

- Case 3: This is a reversionary annuity of 3/4 per annum to x after $\overline{y:\overline{n|}}$:

$$\frac{3}{4}\,\bar{a}_{y:\overline{n|}|x} = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{x:(y:\overline{n|})}) = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{xy} - \bar{a}_{x:\overline{n|}} + \bar{a}_{xy:\overline{n|}}).$$

- Case 4: This is a reversionary annuity of 1/2 per annum to y after $\overline{x{:}\overline{n}|}$:

$$\frac{1}{2}\,\bar{a}_{\overline{x:\overline{n}|}|y} = \frac{1}{2}\,(\bar{a}_y - \bar{a}_{xy} - \bar{a}_{y:\overline{n}|} + \bar{a}_{xy:\overline{n}|}).$$

Adding the results for the four cases together, we obtain the required actuarial present value,

$$\bar{a}_{\overline{n}|} + \frac{3}{4}\,\bar{a}_x + \frac{1}{2}\,\bar{a}_y - \frac{1}{4}\,\bar{a}_{xy} - \frac{3}{4}\,\bar{a}_{x:\overline{n}|} - \frac{1}{2}\,\bar{a}_{y:\overline{n}|} + \frac{1}{4}\,\bar{a}_{xy:\overline{n}|}.$$

▼

9.8 Evaluation—Special Mortality Assumptions

In Section 9.7 the actuarial present values of a variety of insurance and annuity benefits that involve two lifetime random variables were developed. These developments typically culminated in an integral or summation. In this section we study several assumptions about the distribution of $T(u)$ that will simplify the evaluation of these integrals and summations.

9.8.1 Gompertz and Makeham Laws

Here we examine the assumption that mortality follows Makeham's law, or its important special case, Gompertz's law, and the implications for the computations of actuarial present values with respect to multiple life statuses. Independent future lifetime random variables will be assumed.

We begin with the assumption that mortality for each life follows Gompertz's law, $\mu(x) = Bc^x$. We seek to substitute a single-life survival status (w) that has a force of failure equal to the force of failure of the joint-life status (xy) for all $t \ge 0$. Consider

$$\mu_{xy}(s) = \mu(w + s) \qquad s \ge 0; \tag{9.8.1}$$

that is,

$$Bc^{x+s} + Bc^{y+s} = Bc^{w+s},$$

or

$$c^x + c^y = c^w, \tag{9.8.2}$$

which defines the desired w. It follows that for $t > 0$,

$$\begin{aligned} {}_tp_w &= \exp\left[-\int_0^t \mu(w + s)\,ds\right] \\ &= \exp\left[-\int_0^t \mu_{xy}(s)\,ds\right] \\ &= {}_tp_{xy}. \end{aligned} \tag{9.8.3}$$

Thus for w defined in (9.8.2), all probabilities, expected values, and variances for the joint-life status (xy) equal those for the single life (w). For tabled values, the need for a two-dimensional array has been replaced by the need for a one-dimensional array, but typically w is nonintegral, and therefore the determination of its values requires interpolation in the single array.

The assumption that mortality for each life follows Makeham's law (see Table 3.6) makes the search more complex. The force of mortality for the joint-life status is

$$\mu_{xy}(s) = \mu(x + s) + \mu(y + s) = 2A + Bc^s (c^x + c^y). \tag{9.8.4}$$

We cannot substitute a single life for the two lives because of the $2A$. Instead, we replace (xy) with another joint-life status (ww), and then

$$\mu_{ww}(s) = 2\mu(w + s) = 2(A + Bc^s c^w), \tag{9.8.5}$$

and we choose w such that

$$2c^w = c^x + c^y. \tag{9.8.6}$$

Unlike the Gompertz case where the one-dimensional array is based on functions from a single life table, this one-dimensional array is based on functions for a joint-life status (ww) involving equal-age lives.

Example 9.8.1

Use (3.7.1) and the $\ddot{a}_{xx}$ values based on the Illustrative Life Table (Appendix 2A) with interest at 6% to calculate the value of $\ddot{a}_{60:70}$. Compare your result with the values of $\ddot{a}_{60:70}$ in the table of $\ddot{a}_{x:x+10}$.

Solution:

From $c = 10^{0.04}$ and $c^{60} + c^{70} = 2c^w$, we obtain $w = 66.11276$. Then using linear interpolation, $\ddot{a}_{60:70} = 0.88724\ddot{a}_{66:66} + 0.11276\ddot{a}_{67:67} = 7.55637$. The value by the $\ddot{a}_{x:x+10}$ table is 7.55633. ▼

9.8.2 Uniform Distribution

We retain the independence assumption, and in addition we assume a uniform distribution of deaths in each year of age for each individual in the joint-life status. With this additional assumption, we can evaluate the actuarial present values of annuities payable more frequently than once a year and insurance benefits payable at the moment of death.

We recall from Table 3.6 that, under the assumption of a uniform distribution of deaths for each year of age, ${}_tp_x = 1 - tq_x$ and

$${}_tp_x\,\mu(x + t) = \frac{d}{dt}(1 - {}_tp_x) = q_x. \tag{9.8.7}$$

When we apply this assumption to a joint-life status (xy), with independent $T(x)$ and $T(y)$, we obtain, for $0 \le t \le 1$,

$$
\begin{aligned}
{}_tp_{xy}\,\mu_{xy}(t) &= {}_tp_x\,{}_tp_y\,[\mu(x+t)+\mu(y+t)] \\
&= {}_tp_y\,{}_tp_x\,\mu(x+t) + {}_tp_x\,{}_tp_y\,\mu(y+t) \\
&= (1-tq_y)q_x + (1-tq_x)q_y \\
&= q_x + q_y - q_xq_y + (1-2t)q_xq_y \\
&= q_{xy} + (1-2t)q_xq_y. \qquad (9.8.8)
\end{aligned}
$$

On the basis of (4.4.1), the actuarial present value for an insurance benefit in regard to a survival status, (u), can be written as

$$\bar{A}_u = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_u \int_0^1 (1+i)^{1-s}\,\frac{{}_{k+s}p_u}{{}_kp_u}\,\mu_u(k+s)\,ds.$$

Using (9.8.8), we can rewrite this for the joint-life status, (xy), as

$$
\begin{aligned}
\bar{A}_{xy} &= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\left[q_{x+k:y+k}\int_0^1 (1+i)^{1-s}\,ds \right.\\
&\qquad \left. + \, q_{x+k}\,q_{y+k}\int_0^1 (1+i)^{1-s}\,(1-2s)\,ds\right] \\
&= \frac{i}{\delta}\,A_{xy} + \frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \qquad (9.8.9)
\end{aligned}
$$

To interpret the right-hand side of (9.8.9), we see from (4.4.2) that the first term is equal to $\bar{A}_{xy}$ if $T(xy)$, the time-until-failure of (xy), is uniformly distributed in each year of future lifetime. Such is not the case for $T(xy) = \{\min[T(x), T(y)]\}$ when $T(x)$ and $T(y)$ are distributed independently and uniformly over such years. Under this latter assumption, the conditional distribution of $T(xy)$, given that $T(x)$ and $T(y)$ are in different yearly intervals, is also uniform over each year of future lifetime. However, given that $T(x)$ and $T(y)$ are within the same interval, the distribution of their minimum is shifted toward the beginning of the interval (see Exercise 9.38). A consequence of this shift is to require the second term in (9.8.9) to cover the additional expected costs of the earlier claims in those years. The second term, which is the product of an interest term that is close to $i/6$ (see Exercise 9.39) and a actuarial present value for an insurance payable if both individuals die in the same future year, is very small. The actuarial present value $\bar{A}_{xy}$ is often approximated by ignoring the small correction term, thereby simplifying (9.8.9) to

$$\bar{A}_{xy} \cong \frac{i}{\delta}\,A_{xy}, \qquad (9.8.10)$$

which is exact, as noted previously, if $T(xy)$ is uniformly distributed in each year of future lifetime.

To evaluate $\bar{a}_{xy}$, we have from (5.2.8), with survival status (x) replaced by (xy),

$$\bar{a}_{xy} = \frac{1}{\delta}\,(1 - \bar{A}_{xy}),$$

and, on substitution from (9.8.9), obtain

$$\bar{a}_{xy} = \frac{1}{\delta}\left\{1 - \frac{i}{\delta}\left[A_{xy} + \left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}\right]\right\}.$$

Now, on the basis of (5.3.7) for the status (xy), we substitute $1 - d\ddot{a}_{xy}$ for A_{xy} and use (5.4.12) and (5.4.13) to write

$$\bar{a}_{xy} = [\alpha(\infty)\ddot{a}_{xy} - \beta(\infty)] - \frac{i}{\delta^2}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}. \tag{9.8.11}$$

Formula (9.8.11) follows from the assumption that $T(x)$ and $T(y)$ are distributed independently and uniformly over future years. If we assume that $T(xy)$ itself is uniformly distributed over each future year, then from the continuous case of (5.4.11), $m = \infty$, we would have immediately

$$\bar{a}_{xy} = \alpha(\infty)\ddot{a}_{xy} - \beta(\infty). \tag{9.8.12}$$

Formula (9.8.12) differs from (9.8.11) by a small amount, which approximates the product of $i/(6\delta)$ and the actuarial present value for an insurance payable if both individuals die in the same future year.

To use the same approach to evaluate the actuarial present value of an annuity-due payable m-thly, we need an expression for $A_{xy}^{(m)}$ under the assumption of a uniform distribution of deaths for each of the individuals in each year of age. In analogy to the continuous case, we start with

$$A_{xy}^{(m)} = \sum_{k=0}^{\infty} v^k\, {}_kp_{xy} \sum_{j=1}^{m} v^{j/m}\, ({}_{(j-1)/m}p_{x+k:y+k} - {}_{j/m}p_{x+k:y+k}). \tag{9.8.13}$$

In Exercise 9.40 this expression, under the assumption that $T(x)$ and $T(y)$ are independently and uniformly distributed over each year of age, is reduced to

$$A_{xy}^{(m)} = \frac{i}{i^{(m)}} A_{xy} + \frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}. \tag{9.8.14}$$

As $m \to \infty$, the expression in (9.8.14) approach their counterparts in (9.8.9). To interpret the right-hand side of (9.8.14), we see by analogy to (9.8.9) that the first term is the usual approximation for $A_{xy}^{(m)}$ and is exact if $T(xy)$ is uniformly distributed in each year. Then,

$$\frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right) \cong \frac{m^2 - 1}{6m^2}\, i,$$

which is less than $i/6$.

By substituting (9.8.14) into (5.4.4) restated for (xy), and replacing A_{xy} by $1 - d\ddot{a}_{xy}$, we obtain the formula for $\ddot{a}_{xy}^{(m)}$ that is analogous to (9.8.11). If the second term of (9.8.14) is ignored, the formula for $\ddot{a}_{xy}^{(m)}$ reduces to

$$\ddot{a}_{xy}^{(m)} = \alpha(m)\ddot{a}_{xy} - \beta(m). \tag{9.8.15}$$

Again by (5.4.11), this is exact under the assumption that the distribution of $T(xy)$ is uniform over each year of future lifetime.

9.9 Simple Contingent Functions

In this section we study insurances that, in addition to being dependent on the time of failure of the status, are contingent on the order of the deaths of the individuals in the group. In this section we will assume that $T(x)$ and $T(y)$ have a continuous joint d.f. This is done to exclude the common shock model of Section 9.6.1.

We begin with an evaluation of the probability that (x) dies before (y) and before n years from now. In IAN this probability is denoted by ${}_nq^{\,1}_{xy}$, where the 1 over the x indicates the probability is for an event in which (x) dies before (y), and n indicates that the event occurs within n years. Then ${}_nq^{\,1}_{xy}$ equals the double integral of the joint p.d.f. of $T(x)$ and $T(y)$ over the set of outcomes such that $T(x) \le T(y)$ and $T(x) \le n$:

$$\begin{aligned}
{}_nq^{\,1}_{xy} &= \int_0^n \int_s^\infty f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^\infty f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s | T(x) = s]\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s | T(x) = s]\, {}_sp_x\, \mu(x+s)\, ds. \qquad (9.9.1)
\end{aligned}$$

For the independent case, $\Pr[T(y) \ge s|T(x) = s] = {}_sp_y$, so

$${}_nq^{\,1}_{xy} = \int_0^n {}_sp_y\, {}_sp_x\, \mu(x+s)\, ds. \qquad (9.9.2)$$

An interpretation of (9.9.2) involves three elements. First, because s is the time of death of (x), the probability ${}_sp_x\, {}_sp_y$ indicates that both (x) and (y) survive to time s. Second, $\mu(x + s)\, ds$ is the probability that (x), now age $x + s$, will die in the interval $(s, s + ds)$. Third, the probabilities are summed for all times s between 0 and n.

Example 9.9.1

Calculate ${}_5q^{\,1}_{xy}$ for the lives in Example 9.2.1.

Solution:

From (9.9.1),

$$\begin{aligned}
{}_5q^{\,1}_{xy} &= \int_0^5 \int_s^{10} 0.0006(t - s)^2\, dt\, ds \\
&= \int_0^5 0.0002(10 - s)^3\, ds = 0.46875.
\end{aligned}$$

▼

We can also evaluate the probability that (y) dies after (x) and before n years from now. This probability is denoted by ${}_nq_{xy}^{\;2}$, the 2 above the y indicating that (y) dies second and n requiring that this occurs within n years. To evaluate ${}_nq_{xy}^{\;2}$, we integrate the joint p.d.f. of $T(x)$ and $T(y)$ over the event $[0 \le T(x) \le T(y) \le n]$:

$$\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n \int_0^t f_{T(x)T(y)}(s, t)\, ds\, dt \\
&= \int_0^n \int_0^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\; {}_tp_y\, \mu(y + t)\, dt.
\end{aligned} \tag{9.9.3}$$

Again in the independent case, we have

$$\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\
&= {}_nq_y - {}_nq_{xy}^{1}.
\end{aligned} \tag{9.9.4}$$

If the integration of (9.9.3) is set up in the reverse order, we have

$$\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n \int_s^n f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^n f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[s < T(y) \le n | T(x) = s]\; {}_sp_x\, \mu(x + s)\, ds.
\end{aligned}$$

Making the assumption of independence for $T(x)$ and $T(y)$, we can rewrite this as

$$\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n ({}_sp_y - {}_np_y)\; {}_sp_x\, \mu(x + s)\, ds \\
&= {}_nq_{xy}^{1} - {}_np_y\, {}_nq_x.
\end{aligned} \tag{9.9.5}$$

In (9.9.5) the integrand is interpreted as the probability that (x) dies at time s, with $0 < s < n$, and (y) survives to time s but not to time n. Moreover, we now have that

$${}_nq_{xy}^{1} = {}_nq_{xy}^{\;2} + {}_np_y\, {}_nq_x.$$

This implies

$${}_nq_{xy}^{1} \ge {}_nq_{xy}^{\;2}.$$

Similar integrals can be written for the actuarial present values of contingent insurances, but some do not simplify to the same extent. Consider the actuarial present value of an insurance of 1 payable at the moment of death of (x) provided that (y) is still alive. This actuarial present value, denoted by $\bar{A}_{xy}^{1}$, is $E[Z]$ where

$$Z = \begin{cases} v^{T(x)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Since Z is a function of $T(x)$ and $T(y)$, the expectation of Z can be obtained by using the joint p.d.f. of $T(x)$ and $T(y)$:

$$\begin{aligned} \bar{A}^{1}_{xy} &= \int_0^\infty \int_s^\infty v^s f_{T(x)T(y)}(s, t)\, dt\, ds \\ &= \int_0^\infty \int_s^\infty v^s f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\ &= \int_0^\infty \left[\int_s^\infty F_{T(y)|T(x)}(t|s)\, dt \right] v^s\, {}_s p_x\, \mu(x+s)\, ds. \end{aligned} \tag{9.9.6}$$

In the case of independent future lifetimes, $T(x)$ and $T(y)$, we can simplify (9.9.6) and express it in IAN as

$$\begin{aligned} \bar{A}^{1}_{xy} &= \int_0^\infty \left[\int_s^\infty {}_t p_y\, \mu(y+t)\, dt \right] v^s\, {}_s p_x\, \mu(x+s)\, ds \\ &= \int_0^\infty v^s\, {}_s p_y\, {}_s p_x\, \mu(x+s)\, ds. \end{aligned} \tag{9.9.7}$$

The final expression can be interpreted as follows: If (x) dies at any future time s and (y) is still surviving, then a payment of 1, with present value v^s, is made. When $\delta = 0$, $\bar{A}^{1}_{xy} = {}_\infty q^{1}_{xy}$.

Example 9.9.2

Determine the actuarial present value of a payment of 1,000 at the moment of death of (x) providing that (y) is still alive for (x) and (y) in Example 9.2.3 and on the basis of $\delta = 0.04$.

Solution:

Since $T(x)$ and $T(y)$ are independent in Example 9.2.3, we can use the results of that example in (9.9.7) to have

$$\begin{aligned} 1{,}000\, \bar{A}^{1}_{xy} &= 1{,}000 \int_0^\infty e^{-0.04s}\, {}_s p_y\, {}_s p_x\, \mu(x+s)\, ds \\ &= 1{,}000 \int_0^{10} e^{-0.04s}\, 0.01(10-s)^2\, 0.02(10-s)\, ds \\ &= 0.2 \int_0^{10} e^{-0.04s}\, (10-s)^3\, ds = 462.52. \end{aligned}$$

▼

Example 9.9.3

a. Derive the single integral expression for the actuarial present value of an insurance of 1 payable at the time of death of (y) if predeceased by (x).
b. Simplify the integral under the assumption of independent future lifetimes.

c. Obtain a second answer for part (b) by reversing the order of integration in the part (b) double integral.

Solution:

a. The actuarial present value, denoted by $\bar{A}_{xy}^{\;2}$, is E[Z] where

$$Z = \begin{cases} v^{T(y)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Again, Z is a function of $T(x)$ and $T(y)$, so we write an integral for the expectation of Z by using the joint p.d.f. of $T(x)$ and $T(y)$,

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty \int_0^t v^t f_{T(x)T(y)}(s, t)\, ds\, dt \\ &= \int_0^\infty v^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\ &= \int_0^\infty v^t \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt. \end{aligned}$$

b. Invoking the independence assumption and writing in IAN we have

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty v^t\, {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\ &= \int_0^\infty v^t(1 - {}_tp_x)\, {}_tp_y\, \mu(y + t)\, dt \\ &= \bar{A}_y - \bar{A}_{xy}^{\;1}. \end{aligned}$$

We note for the independent case that we can express the actuarial present value for a simple contingent insurance, payable on a death other than the first death, in terms of the actuarial present values for insurances payable on the first death. This is the initial step in the numerical evaluation of simple contingent insurances for independent lives.

c. We have

$$\bar{A}_{xy}^{\;2} = \int_0^\infty \int_s^\infty v^t\, {}_sp_x\, \mu(x + s)\, {}_tp_y\, \mu(y + t)\, dt\, ds.$$

To simplify we replace t with $r + s$ in the inner integral and rewrite the expression

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty \int_0^\infty v^{r+s}\, {}_{r+s}p_y\, \mu(y + r + s)\, {}_sp_x\, \mu(x + s)\, dr\, ds \\ &= \int_0^\infty v^s\, {}_sp_y\, {}_sp_x\, \mu(x + s) \left[\int_0^\infty v^r\, {}_rp_{y+s}\, \mu(y + s + r)\, dr \right] ds \\ &= \int_0^\infty v^s\, \bar{A}_{y+s}\, {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \end{aligned}$$

This last integral is an application of the general result given in (2.2.10), $\mathrm{E}[W] = \mathrm{E}[\mathrm{E}[W|V]]$. Here $V = T(x)$, $W = Z$, and we see that the conditional expectation of Z, given $T(x) = s$, is the actuarial present value, $v^s\ {}_sp_y\ \bar{A}_{y+s}$, of the pure endowment for an amount A_{y+s} sufficient to fund a unit insurance on $(y + s)$. ▼

9.10 Evaluation—Simple Contingent Functions

We now turn to the evaluation of simple contingent probabilities and actuarial present values, noting the effects of assuming Gompertz's law, Makeham's law, and a uniform distribution of deaths. The ubiquitous assumption of independence will be made.

Example 9.10.1

Assuming Gompertz's law for the forces of mortality, calculate

a. The actuarial present value for an n-year term contingent insurance paying a unit amount at the moment of death of (x) only if (x) dies before (y)

b. The probability that (x) dies within n years and predeceases (y).

Solution:

a.
$$\bar{A}^{1}_{xy:\overline{n}|} = \int_0^n v^t\ {}_tp_{xy}\ \mu(x + t)\, dt.$$

Under Gompertz's law,

$$\begin{aligned}
\bar{A}^{1}_{xy:\overline{n}|} &= \int_0^n v^t\ {}_tp_{xy}\ Bc^x\, c^t\, dt \\
&= \frac{c^x}{c^x + c^y} \int_0^n v^t\ {}_tp_{xy}\ B(c^x + c^y)c^t\, dt \\
&= \frac{c^x}{c^x + c^y} \int_0^n v^t\ {}_tp_{xy}\ \mu_{xy}(t)\, dt \\
&= \frac{c^x}{c^x + c^y}\ \bar{A}^{\ \ 1}_{\overline{xy}:\overline{n}|}. \qquad (9.10.1)
\end{aligned}$$

Furthermore, if (9.8.2) holds, then

$$\bar{A}^{\ \ 1}_{\overline{xy}:\overline{n}|} = \bar{A}^{1}_{w:\overline{n}|},$$

and

$$\bar{A}^{1}_{xy:\overline{n}|} = \frac{c^x}{c^w}\ \bar{A}^{1}_{w:\overline{n}|}. \qquad (9.10.2)$$

b. Referring to (9.9.2) we see that ${}_nq^{1}_{xy}$ is $\bar{A}^{1}_{xy:\overline{n}|}$ with $v = 1$. Thus, it follows from (9.10.2) that, under Gompertz's law,

$${}_nq^{1}_{xy} = \frac{c^x}{c^w}\ {}_nq_w, \qquad (9.10.3)$$

where $c^w = c^x + c^y$. ▼

Example 9.10.2

Assuming Makeham's law for the forces of mortality, repeat Example 9.10.1.

Solution:

a.
$$\begin{aligned}\bar{A}^{1}_{xy:\overline{n}|} &= \int_0^n v^t\, {}_tp_{xy}\,(A + Bc^x c^t)\,dt \\ &= A\int_0^n v^t\, {}_tp_{xy}\,dt + \frac{c^x}{c^x + c^y}\int_0^n v^t\, {}_tp_{xy}\,B(c^x + c^y)c^t\,dt \\ &= A\left(1 - \frac{2c^x}{c^x + c^y}\right)\int_0^n v^t\, {}_tp_{xy}\,dt \\ &\quad + \frac{c^x}{c^x + c^y}\int_0^n v^t\, {}_tp_{xy}\,[2A + B(c^x + c^y)c^t]\,dt \\ &= A\left(1 - \frac{2c^x}{c^x + c^y}\right)\bar{a}_{xy:\overline{n}|} + \frac{c^x}{c^x + c^y}\,\bar{A}^{\;1}_{\overline{xy}:\overline{n}|}.\end{aligned}$$

Then by using (9.8.6), we obtain

$$\bar{A}^{1}_{xy:\overline{n}|} = A\left(1 - \frac{c^x}{c^w}\right)\bar{a}_{ww:\overline{n}|} + \frac{c^x}{2c^w}\,\bar{A}^{\;1}_{\overline{ww}:\overline{n}|}. \tag{9.10.4}$$

b. Again, we set $v = 1$ in the result of part (a) to have

$${}_nq^{1}_{xy} = A\left(1 - \frac{c^x}{c^w}\right)\mathring{e}_{ww:\overline{n}|} + \frac{c^x}{2c^w}\,{}_nq_{ww}. \tag{9.10.5}$$

▼

The actuarial present value for a contingent insurance payable at the end of the year of death is

$$A^{1}_{xy} = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q^{\;1}_{x+k:y+k}. \tag{9.10.6}$$

Under the assumption of a uniform distribution of deaths for each individual and independence between the pair of future lifetime random variables, we have

$$\begin{aligned}q^{\;1}_{x+k:y+k} &= \int_0^1 {}_sp_{x+k:y+k}\,\mu(x + k + s)\,ds \\ &= \int_0^1 q_{x+k}(1 - s\,q_{y+k})\,ds \\ &= q_{x+k}\left(1 - \frac{1}{2}\,q_{y+k}\right).\end{aligned} \tag{9.10.7}$$

We can now rewrite ${}_sp_{x+k:y+k}\,\mu(x + k + s)$ in terms of $q^{\;1}_{x+k:y+k}$,

$$
\begin{aligned}
{}_sp_{x+k:y+k}\,\mu(x+k+s) &= q_{x+k}(1 - s q_{y+k}) \\
&= q_{x+k}\left(1 - \frac{1}{2}\,q_{y+k}\right) \\
&\quad + \left(\frac{1}{2} - s\right) q_{x+k}q_{y+k} \\
&= q_{\overset{1}{x+k}:y+k} + \left(\frac{1}{2} - s\right) q_{x+k}q_{y+k}. \qquad (9.10.8)
\end{aligned}
$$

When the benefit is payable at the moment of death, the actuarial present value is

$$
\begin{aligned}
\bar{A}^{1}_{xy} &= \sum_{k=0}^{\infty} v^k\,{}_kp_{xy} \int_0^1 v^s\,{}_sp_{x+k:y+k}\,\mu(x+k+s)\,ds \\
&= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\left[q_{\overset{1}{x+k}:y+k} \int_0^1 (1+i)^{1-s}\,ds\right. \\
&\quad \left. + q_{x+k}\,q_{y+k} \int_0^1 (1+i)^{1-s}\left(\frac{1}{2} - s\right) ds\right] \\
&= \frac{i}{\delta}\,A^{1}_{xy} + \frac{1}{2}\frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right) \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \qquad (9.10.9)
\end{aligned}
$$

The second term (9.10.9) is very small relative to the total amount. It is 1/2 of the second term in (9.8.9).

9.11 Notes and References

The concept of the future-lifetime random variable, developed for a single life in previous chapters, has been extended to a survival status, particular cases of which are statuses defined by several lives. Probability distributions, actuarial present values, variances, and covariances based on these new random variables were obtained for statuses defined by two lives by adaptation of the single life theory. These concepts are developed for more than two lives in Chapter 18.

Discussions of the ideas of this chapter without the use of random variables can be found in Chapters 9–13 of Jordan (1967) and Chapters 7–8 of Neill (1977). A general analysis of laws of mortality, which simplify the formulas for actuarial functions based on more than one life, is given by Greville (1956). Exercise 9.36 illustrates Greville's analysis.

Marshall and Olkin (1967, 1988) contributed to the literature on families of bivariate distributions. In particular, they wrote about the common shock model. Frank's family of bivariate distributions is named for M. J. Frank, who developed it. This family is reviewed by Genest (1987).

Frees, Carriere, and Valdez (1996) used Frank's copula to analyze data from last-survivor annuity experience. They assumed that the marginal distributions belong to the Gompertz family. The estimation process yielded an estimate of α of approximately -3.4. Comparing this estimate with the approximate standard error of the estimate leads to the conclusion that $T(x)$ and $T(y)$ were dependent in that experience.

The parameter α is not a conventional measure of association. The value -3.4 is associated with a positive correlation between $T(x)$ and $T(y)$. This might have been expected because in practice lives receiving payments until the last survivor dies live in the same environment.

Frees et al. also found that the assumption of independence between $T(x)$ and $T(y)$ resulted in higher estimated last-survivor annuity actuarial present values than those estimated using a model that permits dependence. The difference was in the range of 3% to 5%.

Exercises

Unless otherwise indicated, all lives in question are subject to the same table of mortality rates, and their times-until-death are independent random variables.

Section 9.2

9.1. The joint p.d.f. of $T(x)$ and $T(y)$ is given by

$$f_{T(x)T(y)}(s, t) = \frac{(n-1)(n-2)}{(1+s+t)^n} \qquad 0 < s,\ 0 < t,\ n > 2.$$

Find:

a. The joint d.f. of $T(x)$ and $T(y)$.
b. The p.d.f., d.f., and $\mu(x+s)$ for the marginal distribution of $T(x)$. Note the symmetry in s and t which implies that $T(x)$ and $T(y)$ are identically distributed.
c. The covariance and correlation coefficient of $T(x)$ and $T(y)$, given that $n > 4$.

9.2. Display the joint survival function of $[T(x), T(y)]$ where the distribution is defined in Exercise 9.1.

9.3. The future lifetime random variables $T(x)$ and $T(y)$ are independent and identically distributed with p.d.f.

$$f(t) = \frac{n-2}{(1+t)^{n-1}} \qquad n > 3,\ t > 0.$$

Determine the joint d.f. and the joint survival function.

Section 9.3

9.4. In terms of the single life probabilities ${}_np_x$ and ${}_np_y$, express
   a. The probability that (xy) will survive n years
   b. The probability that exactly one of the lives (x) and (y) will survive n years
   c. The probability that at least one of the lives (x) and (y) will survive n years
   d. The probability that (xy) will fail within n years
   e. The probability that at least one of the lives will die within n years
   f. The probability that both lives will die within n years.

9.5. Show that the probability that (x) survives n years and (y) survives $n - 1$ years may be expressed either as

$$\frac{{}_np_{x:y-1}}{p_{y-1}}$$

or as

$$p_x \ {}_{n-1}p_{x+1:y}.$$

9.6. Evaluate

$$\int_0^n {}_tp_{xx} \ \mu_{xx}(t) \ dt.$$

9.7. Using the distribution of $T(x)$ and $T(y)$ shown in Exercise 9.1 display (a) the d.f., (b) the survival function, and (c) $E[T(xy)]$ for $T(xy)$. Assume $n > 3$.

9.8. Use (9.3.8) to obtain the p.d.f. of $T(xy)$ for the joint distribution of $T(x)$ and $T(y)$ given in Example 9.2.3.

Section 9.4

9.9. Show

$$_tp_{\overline{xy}} = {}_tp_{xy} + {}_tp_x (1 - {}_tp_y) + {}_tp_y (1 - {}_tp_x)$$

algebraically and by general reasoning.

9.10. Find the probability that at least one of two lives (x) and (y) will die in the year $(n + 1)$. Is this the same as ${}_{n|}q_{\overline{xy}}$? Explain.

9.11. The random variables $T(x)$ and $T(y)$ have the joint p.d.f. displayed in Exercise 9.1. Find (a) the d.f. and the p.d.f. of $T(\overline{xy})$, (b) $E[T(\overline{xy})]$ for $n > 3$, and (c) $\mu_{\overline{xy}}(t)$.

Section 9.5

9.12. Given that ${}_{25}p_{25:50} = 0.2$ and ${}_{15}p_{25} = 0.9$, calculate the probability that a person age 40 will survive to age 75.

9.13. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate

a. ${}_{10}p_{40:50}$

b. ${}_{10}p_{\overline{40:50}}$

c. $\mathring{e}_{40:50}$

d. $\mathring{e}_{\overline{40:50}}$

e. $\mathrm{Var}[T(40:50)]$

f. $\mathrm{Var}[T(\overline{40:50})]$

g. $\mathrm{Cov}[T(40:50), T(\overline{40:50})]$

h. The correlation between $T(40:50)$ and $T(\overline{40:50})$.

9.14. Evaluate $\dfrac{d\mathring{e}_{xx}}{dx}$.

9.15. Show that the probability of two lives (30) and (40) dying in the same year can be expressed as

$$1 + e_{30:40} - p_{30}(1 + e_{31:40}) - p_{40}(1 + e_{30:41}) + p_{30:40}(1 + e_{31:41}).$$

9.16. Show that the probability of two lives (30) and (40) dying at the same age last birthday can be expressed as

$${}_{10}p_{30}(1 + e_{40:40}) - 2\ {}_{11}p_{30}\ (1 + e_{40:41}) + p_{40}\ {}_{11}p_{30}\ (1 + e_{41:41}).$$

9.17. Assume that the forces of mortality that apply to individuals I and II, respectively, are

$$\mu^{\mathrm{I}}(x) = \log \frac{10}{9} \quad \text{for all } x$$

and

$$\mu^{\mathrm{II}}(x) = (10 - x)^{-1} \quad \text{for } 0 \le x < 10.$$

Evaluate the probability that, if both individuals are of exact age 1, the first death will occur between exact ages 3 and 5.

Section 9.6

9.18. This is a continuation of Example 9.6.3. Exhibit (a) the d.f. and (b) the p.d.f. of $T(\overline{xy})$.

9.19. If ${}_5q_x = 0.05$ and ${}_5q_y = 0.03$, calculate the corresponding value of $F_{T(x)T(y)}(5, 5)$ using (9.6.7). Your answer will depend on the parameter α.

9.20. Use the result of Exercise 9.19 to evaluate ${}_5q_{\overline{xy}}$ for (a) $\alpha = 0$, (b) $\alpha = 3$, and (c) $\alpha = -3$. [Hint: Recall (9.4.5) and (9.3.2).]

Section 9.7

9.21. Show that

$$a_{\overline{\overline{xy}:\overline{n}|}} = a_{\overline{n}|} + {}_{n|}a_{\overline{xy}}.$$

Describe the underlying benefit.

9.22. For an actuarial present value denoted by $\bar{A}_{\overline{x:\overline{n}|}}$, describe the benefit. Show that

$$\bar{A}_{\overline{x:\overline{n}|}} = \bar{A}_x - \bar{A}_{x:\overline{n}|} + v^n.$$

9.23. For independent lifetimes $T(x)$ and $T(y)$, show that

$$\mathrm{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}).$$

9.24. Express, in terms of single- and joint-life annuity values, the actuarial present value of an annuity payable continuously at a rate of 1 per year while at least one of (25) and (30) survives and is below age 50.

9.25. Express, in terms of single- and joint-life annuity values, the actuarial present value of a deferred annuity of 1 payable at the end of any year as long as either (25) or (30) is living after age 50.

9.26. Express, in terms of single- and joint-life annuity values, the actuarial present value of an n-year temporary annuity-due, payable in respect to $(\overline{xy})$, providing annual payments of 1 while both lives survive, reducing to 1/2 on the death of (x) and to 1/3 on the death of (y).

9.27. An annuity-immediate of 1 is payable to (x) as long as he lives jointly with (y) and for n years after the death of (y), except that in no event will payments be made after m years from the present time, $m > n$. Show that the actuarial present value is

$$a_{x:\overline{n}|} + {}_nE_x\, a_{x+n:\overline{y:\overline{m-n}|}}.$$

9.28. Obtain an expression for the actuarial present value of a continuous annuity of 1 per annum payable while at least one of two lives (40) and (55) is living and is over age 60, but not if (40) is alive and under age 55.

9.29. A joint-and-survivor annuity to (x) and (y) is payable at an initial rate per year while (x) lives, and, if (y) survives (x), is continued at the fraction p, $1/2 \le p \le 1$, of the initial rate per year during the lifetime of (y) following the death of (x).

a. Express the actuarial present value of such an annuity-due with an initial rate of 1 per year, payable in m-thly installments, in terms of the actuarial present values of single-life and joint-life annuities.

b. A joint-and-survivor annuity to (x) and (y) and a life annuity to (x) are said to be actuarially equivalent on the basis of stated assumptions if they have equal actuarial present values on such basis. Derive an expression for the ratio of the initial payment of the joint-and-survivor annuity to the payment rate of the actuarially equivalent life annuity to (x).

9.30. Show that

a. $\bar{A}^{\,2}_{xy} = \bar{A}^{\,1}_{xy} - \delta \bar{a}_{y|x}$ [This exercise depends on material in Section 9.9.]

b. $\dfrac{\partial}{\partial x}\, \bar{a}_{y|x} = \mu(x)\, \bar{a}_{y|x} - \bar{A}^{\,2}_{xy}$.

9.31. When, under Makeham's law, the status (xy) is replaced by the status (ww), show that

$$w - y = \frac{\log(c^{\Delta} + 1) - \log 2}{\log c}$$

where $\Delta = x - y \geq 0$. (This indicates that w can be obtained from the younger age y by adding an amount that is a function of $\Delta = x - y$. Such a property is referred to as a ***law of uniform seniority***.)

9.32. On the basis of your Illustrative Life Table with interest of 6% calculate $\ddot{a}_{50:60:\overline{10}|}$. In your solution, use
a. Values interpolated in the $\ddot{a}_{xx}$ table
b. Values from the $\ddot{a}_{x:x+10}$ table.

9.33. Given a mortality table that follows Makeham's law and ages x and y for which (ww) is the equivalent equal-age status, show that
a. ${}_tp_w$ is the geometric mean of ${}_tp_x$ and ${}_tp_y$
b. ${}_tp_x + {}_tp_y > 2, p_w$ for $x \neq y$
c. $a_{\overline{xy}} > a_{\overline{ww}}$ for $x \neq y$.

9.34. Given a mortality table that follows Makeham's law, show that $\bar{a}_{xy}$ is equal to the actuarial present value of an annuity with a single life (w) where $c^w = c^x + c^y$ and force of interest $\delta' = \delta + A$. Further, show that

$$\bar{A}_{xy} = \bar{A}'_w + A\bar{a}'_w$$

where the primed functions are evaluated at force of interest δ'.

9.35. Consider two mortality tables, one for males, M, and one for females, F, with

$$\mu^M(z) = 3a + \frac{3bz}{2} \quad \text{and} \quad \mu^F(z) = a + bz.$$

We wish to use a table of actuarial present values for two lives, one male and one female, each of age w, to evaluate the actuarial present value of a joint-life annuity for a male age x and a female age y. Express w in terms of x and y.

9.36. From Section 9.5 we know that if $T(x)$ and $T(y)$ are independent,

$$\bar{a}_{xy} = \int_0^{\infty} e^{-\int_0^t [\delta + \mu(x+s) + \mu(y+s)]ds}\, dt.$$

If we could find δ', k, and w such that

$$\delta + \mu(x + t) + \mu(y + t) = \delta' + k\mu(w + t) \qquad (*)$$

we would have

$$\bar{a}_{xy} = \int_0^\infty v^t ({}_tp_w)^k \, dt$$
$$= \bar{a}'_{w(k)},$$

where the prime on the discount factor indicates that it is valued at force of interest δ' and $w(k)$ indicates a joint-life status with k "lives" (k is not necessarily an integer).

If $\mu(x + t) = a + b(x + t) + c(x + t)^2$, confirm that (*) is satisfied if

$$k = 2,$$
$$w = \frac{x + y}{2},$$
$$\delta' = \delta + c(x^2 + y^2 - 2w^2).$$

9.37. Find $\mathring{e}_{xy}$ if $q_x = q_y = 1$ and the deaths are uniformly distributed over the year of age for each of (x) and (y).

9.38. Let $T(x)$ and $T(y)$ be independent and uniformly distributed in the next year of age. Given that both (x) and (y) die within the next year, demonstrate that the time-of-failure of (xy) is not uniformly distributed over the year. [Hint: Show that $\Pr[T(xy) \le t | (T(x) \le 1) \cap (T(y) \le 1)] = 2t - t^2$.]

9.39. Show

$$\frac{1}{\delta} = \frac{1}{i[1 - (i/2 - i^2/3 + i^2/4 - i^4/5 + \cdots)]}$$
$$= \frac{1}{i}\left(1 + \frac{i}{2} - \frac{i^2}{12} + \frac{i^3}{24} - \frac{19i^4}{720} + \cdots\right).$$

Hence, show

$$\frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right) \cong \frac{i}{6} - \frac{i^3}{360} + \cdots.$$

9.40. Show that if deaths are uniformly distributed over each year of age, then

$${}_{(j-1)/m}p_{xy} - {}_{j/m}p_{xy} = \frac{1}{m} q_{xy} + \frac{m + 1 - 2j}{m^2} q_x \, q_y$$

for any x and y and $j = 1, 2, 3, \ldots, m$. Hence, verify (9.8.14).

Section 9.9

9.41. Show by general reasoning that

$${}_nq^1_{xy} = {}_nq^{\ 2}_{xy} + {}_nq_x \, {}_np_y.$$

When $n \to \infty$, what does the equation become?

9.42. Show that the actuarial present value for an insurance of 1 payable at the end of the year of death of (x), provided that (y) survives to the time of payment, can be expressed as $vp_y\ddot{a}_{x:y+1} - a_{xy}$.

9.43. Show that $A^{1}_{xy} - A^{\;2}_{xy} = A_{xy} - A_y$.

9.44. Express, in terms of actuarial present values for single life and first death contingent insurances, the net single premium for an insurance of 1 payable at the moment of death of (50), provided that (20), at that time, has died or attained age 40.

9.45. Express, in terms of actuarial present values for pure endowment and first death contingent insurances, the actuarial present value for an insurance of 1 payable at the time of the death of (x) provided (y) dies during the n years preceding the death of (x).

9.46. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate ${}_{25}q_{25:\overset{2}{50}}$.

Section 9.10

9.47. In a mortality table known to follow Makeham's law, you are given that $A = 0.003$ and $c^{10} = 3$.

a. If $\mathring{e}_{40:50} = 17$, calculate ${}_{\infty}q_{\overset{1}{40}:50}$.

b. Express $\bar{A}_{\overset{1}{40}:50}$ in terms of $\bar{A}_{40:50}$ and $\bar{a}_{40:50}$.

9.48. Given that mortality follows Gompertz's law with $\mu(x) = 10^{-4} \times 2^{x/8}$ for $x > 35$ and that by (9.10.2)

$$\bar{A}_{\overset{1}{40}:48:\overline{10|}} = f\bar{A}^{1}_{w:\overline{10|}},$$

calculate f and w.

Miscellaneous

9.49. The survival status $(\overline{n|})$ is one that exists for exactly n years. It has been used in conjunction with life statuses, for example, in $\bar{A}_{x:\overline{n|}}$, $\bar{A}^{1}_{x:\overline{n|}}$, $\bar{A}_{x:\overset{1}{\overline{n|}}}$, $\ddot{a}_{x:\overline{n|}}$, $A_{xy:\overline{n|}}$. Simplify and interpret the following:

a. $\bar{a}_{\overline{x:\overline{n|}}}$

b. $\bar{A}_{x:\overset{2}{\overline{n|}}}$.

9.50. Use the probability rule $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ to obtain (9.4.6).

9.51. Evaluate $\dfrac{\partial}{\partial x}\,\mathring{e}_{xy}$.

9.52. The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameters μ_1, μ_2, and λ. These three random variable are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\, ds + f_{T(x)T(y)}(t, t) + \int_t^\infty f_{T(x)T(y)}(s, t)\, ds.$$

b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\, du.$$

Compare the result with (9.6.4b).

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^\infty f_{T(x)T(y)}(t, t)\, dt.$$

10

MULTIPLE DECREMENT MODELS

10.1 Introduction

In Chapter 9 we extended the theory of Chapters 3 through 8 from an individual life to multiple lives, subject to a single contingency of death. We now return to the case of a single life, but here subject to multiple contingencies. As an application of this extension, we observe that the number of workers for an employer is reduced when an employee withdraws, becomes disabled, dies, or retires. In manpower planning, it might be necessary to estimate only the numbers of those presently at work who will remain active to various years into the future. For this task, the model for survivorship developed in Chapter 3 is adequate, with time-until-termination of employment rather than time-until-death as the interpretation of the basic random variable. Employee benefit plans, however, provide benefits paid on termination of employment that may depend on the cause of termination. For example, the benefits on retirement often differ from those payable on death or disability. Therefore, survivorship models for employee benefit systems include random variables for both time-of-termination and cause of termination. Also, the benefit structure often depends on earnings, which is another and different kind of uncertainty that is discussed in Chapter 11.

As another application, most individual life insurances provide payment of a nonforfeiture benefit if premiums stop before the end of the specified premium payment term. A comprehensive model for such insurances incorporates both time-until-termination and cause of termination as random variables.

Disability income insurance provides periodic payments to insureds who satisfy the definition of disability contained in the policy. In some cases, the amount of the periodic payments may depend on whether the disability was caused by illness or accident. A person may cease to be an active insured by dying, withdrawing, becoming disabled, or reaching the end of the coverage period. A complete model for disability insurance incorporates a random variable for time-until-termination, when the insured ceases to be a member of the active insureds, as well as a random variable for the cause of termination.

Public health planners are interested in the analysis of mortality and survivorship by cause of death. Public health goals may be set by a study of the joint distribution of time-until-death and cause of death. Priorities in cardiovascular and cancer research were established by this type of analysis.

The main purpose of this chapter is to build a theory for studying the distribution of two random variables in regard to a single life: time-until-termination from a given status and cause of the termination. The resulting model is used in each of the applications described in this section. Within actuarial science, the termination from a given status is called ***decrement,*** and the subject of this chapter is called ***multiple decrement theory.*** Within biostatistics it is referred to as the ***theory of competing risks.***

It is also possible to develop multiple decrement theory in terms of deterministic rates and rate functions. There is some recapitulation of the theory from this point of view in Section 10.4.

10.2 Two Random Variables

Chapter 3 was devoted in part to methods for specifying and using the distribution of the continuous random variable $T(x)$, the time-until-death of (x). The same methods can be used to study time-until-termination from a status, such as employment with a particular employer, with only minor changes in vocabulary. In fact, we use the same notation $T(x)$, or T, to denote the time random variable in this new setting.

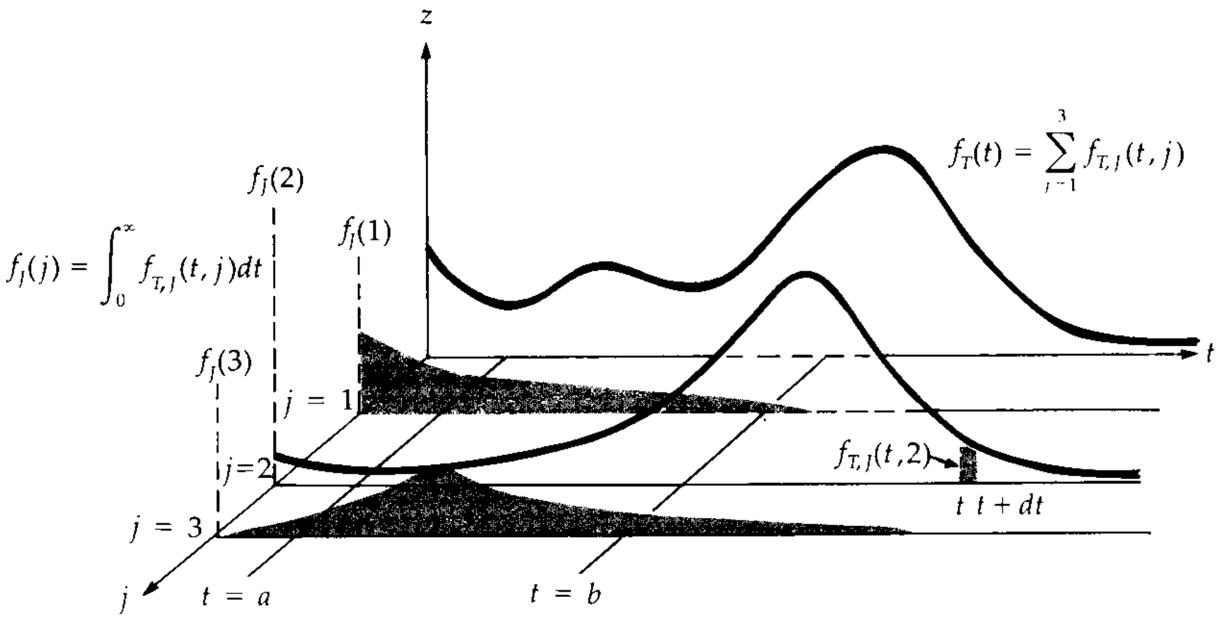
In this section we expand the basic model by introducing a second random variable, cause of decrement, to be denoted by $J(x) = J$. We assume that J is a discrete random variable.

The applications in Section 10.1 provide examples of these random variables. For employee benefit plan applications, the random variable J could be assigned the values 1, 2, 3, or 4 depending on whether termination is due to withdrawal, disability, death, or retirement, respectively. In the life insurance application, J could be assigned the values 1 or 2, depending on whether the insured dies or chooses to terminate payment of premiums. For the disability insurance application, J could be assigned the values 1, 2, 3, or 4 depending on whether the insured dies, withdraws, becomes disabled, or reaches the end of the coverage period. Finally, in the public health application, there are many possibilities for causes of decrement. For example, in a given study, J could be assigned the values 1, 2, 3, or 4 depending on whether death was caused by cardiovascular disease, cancer, accident, or all other causes.

Our purpose is to describe the joint distribution of T and J and the related marginal and conditional distributions. We denote the joint p.d.f. of T and J by $f_{T,J}(t, j)$, the marginal p.f. of J by $f_J(j)$, and the marginal p.d.f. of T by $f_T(t)$. Figure

10.2.1 illustrates these distributions. They may seem strange at first because J is a discrete random variable and T is continuous.

FIGURE 10.2.1
Graph of $f_{T,J}(t, j)$



The joint p.d.f. of T and J, $f_{T,J}(t, j)$, can be pictured as falling on m parallel sheets, as illustrated in Figure 10.2.1 for three causes of decrement ($m = 3$). There is a separate sheet for each of the m causes of decrement recognized in the model. In Figure 10.2.1 the following relations hold:

$$\sum_{j=1}^{3} f_J(j) = 1$$

and

$$\int_0^{\infty} f_T(t)\, dt = 1.$$

The p.d.f. $f_{T,J}(t, j)$ can be used in the usual ways to calculate the probabilities of events defined by T and J. For example,

$$f_{T,J}(t, j)\, dt = \Pr\{(t < T \le t + dt) \cap (J = j)\} \qquad (10.2.1)$$

expresses the probability of decrement due to cause j between t and $t + dt$,

$$\int_0^t f_{T,J}(s, j)\, ds = \Pr\{(0 < T \le t) \cap (J = j)\} \qquad (10.2.2)$$

expresses the probability of decrement due to cause j before time t, and

$$\sum_{j=1}^{m} \int_a^b f_{T,J}(t, j)\, dt = \Pr\{a < T \le b\}$$

expresses the probability of decrement due to all causes between a and b.

The probability of decrement before time t due to cause j given in (10.2.2) has the special symbol

$$ {}_tq_x^{(j)} = \int_0^t f_{T,J}(s, j)\, ds \qquad t \ge 0, j = 1, 2, \ldots, m, \tag{10.2.3}$$

which is illustrative of the use of the superscript to denote the cause of decrement in multiple decrement theory.

The use of information given at age x to select a distribution is similar to the concepts in Chapter 3. If being in the survival status at age x is the only information available, then an aggregate distribution would be used. On the other hand, if there is additional information, then the distribution would be a select distribution and the age of selection would be enclosed in brackets.

By the definition of the marginal distribution for J, appearing as $f_J(j)$ in the (j, z) plane of Figure 10.2.1, we have the probability of decrement due to cause j at any time in the future to be

$$f_J(j) = \int_0^\infty f_{T,J}(s, j)\, ds = {}_\infty q_x^{(j)} \qquad j = 1, 2, \ldots, m. \tag{10.2.4}$$

This is new and without a counterpart in Chapter 3, unlike the marginal p.d.f. for T, $f_T(t)$ in the (t, z) plane of Figure 10.2.1. For $f_T(t)$, and the d.f., $F_T(t)$, we have for $t \ge 0$

$$f_T(t) = \sum_{j=1}^{m} f_{T,J}(t, j) \tag{10.2.5}$$

and

$$F_T(t) = \int_0^t f_T(s)\, ds.$$

The notations introduced in Chapter 3 for the functions of the distribution of the future-lifetime random variable, T, can be extended to accommodate those of the time-until-decrement random variable of the multiple decrement model. Using the superscript (τ) to indicate that a function refers to all causes, or total force, of decrement, we obtain

$${}_tq_x^{(\tau)} = \Pr\{T \le t\} = F_T(t) = \int_0^t f_T(s)\, ds, \tag{10.2.6}$$

$${}_tp_x^{(\tau)} = \Pr\{T > t\} = 1 - {}_tq_x^{(\tau)}, \tag{10.2.7}$$

$$\mu_x^{(\tau)}(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt}\, {}_tq_x^{(\tau)}$$

$$= -\frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt}\, {}_tp_x^{(\tau)}$$

$$= -\frac{d}{dt} \log {}_tp_x^{(\tau)}, \tag{10.2.8}$$

and

$${}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)\, ds}. \tag{10.2.9}$$

Mathematically, these functions for the random variable T of this chapter are identical to those for the T of Chapter 3; the difference is in their interpretation in the applications. The choice of the symbol $\mu_x^{(\tau)}(t)$ for the force of total decrement is influenced by these applications. For example, in pension applications (x) is an age of entry into the pension plan, and although no special selection information may be available, subsequent causes of decrement may depend on this age.

As with the applications in previous chapters, the statement in (10.2.1) can be analyzed by conditioning on survival in the given status to time t. In this way, we have

$$f_{T,J}(t, j)\, dt = \Pr\{T > t\} \Pr\{[(t < T \le t + dt) \cap (J = j)] | T > t\}. \tag{10.2.10}$$

By analogy with (3.2.12) this suggests the definition of the ***force of decrement due to cause*** j as

$$\mu_x^{(j)}(t) = \frac{f_{T,J}(t, j)}{1 - F_T(t)} = \frac{f_{T,J}(t, j)}{{}_tp_x^{(\tau)}}. \tag{10.2.11}$$

The force of decrement at age $x + t$ due to cause j has a conditional probability interpretation. It is the value of the joint conditional p.d.f. of T and J at $x + t$ and j, given survival to $x + t$. Then (10.2.10) can be rewritten as

$$f_{T,J}(t, j)\, dt = {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \qquad j = 1, 2, \ldots, m,\ t \ge 0. \qquad \text{(10.2.10) restated}$$

Restated in words,

(the probability of decrement between t and $t + dt$ due to cause j) = (the probability, ${}_tp_x^{(\tau)}$, that (x) remains in the given status until time t)

$\times$ (the conditional probability, $\mu_x^{(j)}(t)$ that decrement occurs between t and $t + dt$ due to cause j, given that decrement has not occurred before time t).

It follows, from differentiation of (10.2.3) and use of (10.2.11), that

$$\mu_x^{(j)}(t) = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt}\, {}_tq_x^{(j)}. \tag{10.2.12}$$

Now, from (10.2.6), (10.2.5), and (10.2.3),

$$
\begin{aligned}
{}_tq_x^{(\tau)} &= \int_0^t f_T(s)\, ds = \int_0^t \sum_{j=1}^{m} f_{T,J}(s, j)\, ds \\
&= \sum_{j=1}^{m} \int_0^t f_{T,J}(s, j)\, ds = \sum_{j=1}^{m} {}_tq_x^{(j)}.
\end{aligned} \tag{10.2.13}
$$

That the first and last members of (10.2.13) are equal is immediately interpretable. Combining (10.2.8), (10.2.13), and (10.2.12), we have

$$
\mu_x^{(\tau)}(t) = \sum_{j=1}^{m} \mu_x^{(j)}(t); \tag{10.2.14}
$$

that is, the force of total decrement is the sum of the forces of decrement due to the m causes.

We can summarize the definitions here by expressing the joint, marginal, and conditional p.d.f.'s in actuarial notation and repeating the defining equation numbers:

$$f_{T,J}(t, j) = {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t), \qquad \text{(10.2.11) restated}$$

$$f_J(j) = {}_\infty q_x^{(j)}, \qquad \text{(10.2.4) restated}$$

$$f_T(t) = {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t). \qquad \text{(10.2.8) restated}$$

The conditional p.f. of J, given decrement at time t, is

$$
\begin{aligned}
f_{J|T}(j|t) &= \frac{f_{T,J}(t, j)}{f_T(t)} = \frac{{}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)}{{}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)} \\
&= \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)}.
\end{aligned} \tag{10.2.15}
$$

Finally, we note that the probability in (10.2.3) can be rewritten as

$$
{}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\, \mu_x^{(j)}(s)\, ds. \tag{10.2.16}
$$

Example 10.2.1

Consider a multiple decrement model with two causes of decrement; the forces of decrement are given by

$$
\begin{aligned}
\mu_x^{(1)}(t) &= \frac{t}{100} \qquad t \ge 0, \\
\mu_x^{(2)}(t) &= \frac{1}{100} \qquad t \ge 0.
\end{aligned}
$$

For this model, calculate the p.f. (or p.d.f.) for the joint, marginal, and conditional distributions.

Solution:

Since

$$\mu_x^{(\tau)}(s) = \mu_x^{(1)}(s) + \mu_x^{(2)}(s) = \frac{s+1}{100},$$

the survival probability ${}_tp_x^{(\tau)}$ is

$$\begin{aligned} {}_tp_x^{(\tau)} &= \exp\left(-\int_0^t \frac{s+1}{100}\, ds\right) \\ &= \exp\left(\frac{-(t^2+2t)}{200}\right) \qquad t \ge 0, \end{aligned}$$

and the joint p.d.f. of T and J is

$$f_{T,J}(t, j) = \begin{cases} \dfrac{t}{100} \exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \ge 0,\, j = 1 \\[2ex] \dfrac{1}{100} \exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \ge 0,\, j = 2. \end{cases}$$

The marginal p.d.f. of T is

$$f_T(t) = \sum_{j=1}^{2} f_{T,J}(t, j) = \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200}\right] \qquad t \ge 0,$$

and the marginal p.f. of J is

$$f_J(j) = \begin{cases} \displaystyle\int_0^\infty f_{T,J}(t, 1)\, dt & j = 1 \\[2ex] \displaystyle\int_0^\infty f_{T,J}(t, 2)\, dt & j = 2. \end{cases}$$

It is somewhat easier to evaluate $f_J(2)$. In the following development, $\Phi(x)$ is the d.f. for the standard normal distribution $N(0, 1)$. By completing the square we have

$$\begin{aligned} f_J(2) &= \frac{1}{100} e^{0.005} \int_0^\infty \exp\left[\frac{-(t+1)^2}{200}\right] dt \\ &= \frac{1}{100} e^{0.005} \sqrt{2\pi}\, 10 \int_0^\infty \frac{1}{\sqrt{2\pi}\, 10} \exp\left[\frac{-(t+1)^2}{200}\right] dt. \end{aligned}$$

We now make the change of variable $z = (t+1)/10$ and obtain

$$\begin{aligned} f_J(2) &= \frac{1}{10} e^{0.005} \sqrt{2\pi} \int_{0.1}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \\ &= \frac{1}{10} e^{0.005} \sqrt{2\pi}\, [1 - \Phi(0.1)] \\ &= 0.1159. \end{aligned}$$

Therefore $f_J(1) = 0.8841$. Finally, the conditional p.f. of J, given decrement at t, is derived from (10.2.15) as

$$f_{J|T}(1|t) = \frac{t}{t+1}$$

and

$$f_{J|T}(2|t) = \frac{1}{t+1}. \qquad \blacktriangledown$$

Example 10.2.2

For the joint distribution of T and J specified in Example 10.2.1, calculate $\mathrm{E}[T]$ and $\mathrm{E}[T|J = 2]$.

Solution:

Using the marginal p.d.f. $f_T(t)$, we have

$$\mathrm{E}[T] = \int_0^\infty t \left\{ \frac{t+1}{100} \exp\left[\frac{-(t^2 + 2t)}{200} \right] \right\} dt.$$

Integration by parts, as in (3.5.1), yields

$$\begin{aligned}\mathrm{E}[T] &= -t \exp\left[\frac{-(t^2+2t)}{200} \right] \Bigg|_0^\infty \\ &\quad + \int_0^\infty \exp\left[\frac{-(t^2+2t)}{200} \right] dt \\ &= 0 + 100 f_J(2),\end{aligned}$$

hence

$$\mathrm{E}[T] = 11.59.$$

Using the conditional p.d.f. $f_{T,J}(t, 2)/f_J(2)$, we have

$$\mathrm{E}[T|J = 2] = \int_0^\infty t \left\{ 100^{-1} \exp\left[\frac{-(t^2+2t)}{200} \right] \right\} (0.1159)^{-1}\, dt.$$

This integral may be evaluated as follows:

$$\begin{aligned}\mathrm{E}[T|J = 2] &= \mathrm{E}[(T+1) - 1 | J = 2] \\ &= (0.1159)^{-1} \int_0^\infty \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200} \right] dt - 1 \\ &= -(0.1159)^{-1} \exp\left[\frac{-(t^2+2t)}{200} \right] \Bigg|_0^\infty - 1 \\ &= 7.63.\end{aligned}$$

The point of Examples 10.2.1 and 10.2.2 is that once the joint distribution of T and J is specified, marginal and conditional distributions can be derived, and the moments of these distributions determined. ▼

In some instances, a particular application may require a modification of the above model. A continuous distribution for time-until-termination, T, is inadequate in applications where there is a time at which there is a positive probability of decrement. One example of this is a pension plan with a mandatory retirement age, an age at which all remaining active employees must retire. A second example is term life insurance in which there is typically no benefit paid on withdrawal. Thus, after a premium is paid, none of the remaining insureds withdraw until the next premium due date. Here we do not attempt to extend the notation to cover such situations. However, in Section 10.7 we describe extended models for each of these examples.

The random variable K, the curtate-future-years before decrement of (x), is defined as in Chapter 3 to be the greatest integer less than T. Using (10.2.1) and (10.2.11), we can write the joint p.f. K and J as

$$\Pr\{(K = k) \cap (J = j)\} = \Pr\{(k < T \le k + 1) \cap (J = j)\}$$
$$= \int_k^{k+1} {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt. \tag{10.2.17}$$

Rewriting ${}_tp_x^{(\tau)}$ of the integrand in the exponential form of (10.2.9) and factoring it into two factors changes (10.2.17) to

$$= {}_kp_x^{(\tau)} \int_k^{k+1} e^{-\int_k^t \mu_x^{(\tau)}(u)\, du}\, \mu_x^{(j)}(t)\, dt.$$

Changing the variables of the integrations by $r = u - k$ and $s = t - k$ yields

$$= {}_kp_x^{(\tau)} \int_0^1 e^{-\int_0^s \mu_x^{(\tau)}(k+r)\, dr}\, \mu_x^{(j)}(k + s)\, ds.$$

Thus far we have done manipulations that hold in all tables. If we are using an aggregate or ultimate (a nonselect) table where the forces of decrement depend on an initial age and the duration only through their sum, that is, the ***attained age***, then for τ and all j,

$$\mu_x(k + s) = \mu_{x+k}(s) \qquad \text{for all } x,\ k, \text{ and } s \ge 0,$$

and (10.2.17) may be written

$${}_kp_x^{(\tau)} \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(j)}(s)\, ds = {}_kp_x^{(\tau)}\, q_{x+k}^{(j)}. \tag{10.2.18}$$

The probability of decrement from all causes between ages $x + k$ and $x + k + 1$, given survival to age $x + k$, is denoted by $q_{x+k}^{(\tau)}$, and it follows that

$$q_{x+k}^{(\tau)} = \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(\tau)}(s)\, ds$$
$$= \int_0^1 {}_sp_{x+k}^{(\tau)} \sum_{j=1}^m \mu_{x+k}^{(j)}(s)\, ds$$
$$= \sum_{j=1}^m q_{x+k}^{(j)}. \tag{10.2.19}$$

An examination of (10.2.18) and (10.2.19) discloses why multiple decrement theory is also called the theory of competing risks. The probability of decrement between ages $x + k$ and $x + k + 1$ due to cause j depends on ${}_sp_{x+k}^{(\tau)}$, $0 \le s \le 1$, and thus on all the component forces. When the forces for other decrements are increased, ${}_sp_{x+k}^{(\tau)}$ is reduced, and then $q_{x+k}^{(j)}$ is also decreased.

10.3 Random Survivorship Group

Let us consider a group of $l_a^{(\tau)}$ lives age a years. Each life is assumed to have a distribution of time-until-decrement and cause of decrement specified by the p.d.f.

$$f_{T,J}(t, j) = {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t) \qquad t \ge 0,\ j = 1, 2, \ldots, m.$$

We denote by ${}_n\mathcal{D}_x^{(j)}$ the random variable equal to the number of lives who leave the group between ages x and $x + n$, $x \ge a$, from cause j. We denote $\mathrm{E}[{}_n\mathcal{D}_x^{(j)}]$ by ${}_nd_x^{(j)}$ and obtain

$$\begin{aligned} {}_nd_x^{(j)} &= \mathrm{E}[{}_n\mathcal{D}_x^{(j)}] \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt. \end{aligned} \tag{10.3.1a}$$

As usual, if $n = 1$, we delete the prefixes on ${}_n\mathcal{D}_x^{(j)}$ and ${}_nd_x^{(j)}$. We note that

$${}_n\mathcal{D}_x^{(\tau)} = \sum_{j=1}^{m} {}_n\mathcal{D}_x^{(j)}$$

and define

$${}_nd_x^{(\tau)} = \mathrm{E}[{}_n\mathcal{D}_x^{(\tau)}] = \sum_{j=1}^{m} {}_nd_x^{(j)}. \tag{10.3.1b}$$

Then, using (10.3.1a), we have

$$\begin{aligned} {}_nd_x^{(\tau)} &= l_a^{(\tau)} \sum_{j=1}^{m} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(\tau)}(t)\, dt. \end{aligned} \tag{10.3.2}$$

If $\mathcal{L}^{(\tau)}(x)$ is defined as the random variable equal to the number of survivors at age x out of the $l_a^{(\tau)}$ lives in the original group at age a, then by analogy with (3.3.1) we can write

$$\begin{aligned} l_x^{(\tau)} &= \mathrm{E}[\mathcal{L}^{(\tau)}(x)] \\ &= l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}. \end{aligned} \tag{10.3.3}$$

We recognize the integral of (10.3.1a) with $n = 1$ as the integral of (10.2.17) with $x = a$ and $k = x - a$. Thus for a nonselect table, we have from (10.2.18)

$$d_x^{(j)} = l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}\, q_x^{(j)} = l_x^{(\tau)}\, q_x^{(j)}. \tag{10.3.4}$$

This result lets us display a table of $p_x^{(\tau)}$ and $q_x^{(j)}$ values in a corresponding table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values. Either table is called a ***multiple decrement table.***

Example 10.3.1

Construct a table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values corresponding to the probabilities of decrement given below.

x	$q_x^{(1)}$	$q_x^{(2)}$
65	0.02	0.05
66	0.03	0.06
67	0.04	0.07
68	0.05	0.08
69	0.06	0.09
70	0.00	1.00

Although this display is designed for computational ease, it may be roughly suggestive of a double decrement situation with cause 1 related to death and cause 2 to retirement. It appears that, in this case, 70 is the mandatory retirement age.

Solution:

We assume the arbitrary value of $l_{65}^{(\tau)} = 1{,}000$ and use (10.3.4) as indicated below.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$	$l_x^{(\tau)} = l_{x-1}^{(\tau)}\, p_{x-1}^{(\tau)}$	$d_x^{(1)} = l_x^{(\tau)}\, q_x^{(1)}$	$d_x^{(2)} = l_x^{(\tau)}\, q_x^{(2)}$
65	0.02	0.05	0.07	0.93	1 000.00	20.00	50.00
66	0.03	0.06	0.09	0.91	930.00	27.90	55.80
67	0.04	0.07	0.11	0.89	846.30	33.85	59.24
68	0.05	0.08	0.13	0.87	753.21	37.66	60.26
69	0.06	0.09	0.15	0.85	655.29	39.32	58.98
70	0.00	1.00	1.00	0.00	557.00	0.00	557.00

Note, as a check on the calculations, that $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$, except for rounding error.

We continue this example with the evaluation, from first principles, of several probabilities:

$$_2p_{65}^{(\tau)} = p_{65}^{(\tau)}\, p_{66}^{(\tau)} = (0.93)(0.91) = 0.8463,$$

$$_{2|}q_{66}^{(1)} = p_{66}^{(\tau)}\, p_{67}^{(\tau)}\, q_{68}^{(1)} = (0.91)(0.89)(0.05) = 0.0405,$$

$$_2q_{67}^{(2)} = q_{67}^{(2)} + p_{67}^{(\tau)}\, q_{68}^{(2)} = 0.07 + (0.89)(0.08) = 0.1412.$$

The last three columns of the above table may be used to obtain the same probabilities. The answers agree to four decimal places:

$$ {}_2p_{65}^{(\tau)} = \frac{l_{67}^{(\tau)}}{l_{65}^{(\tau)}} = \frac{846.30}{1{,}000.00} = 0.8463, $$

$$ {}_{2|}q_{66}^{(1)} = \frac{d_{68}^{(1)}}{l_{66}^{(\tau)}} = \frac{37.66}{930.00} = 0.0405, $$

$$ {}_2q_{67}^{(2)} = \frac{d_{67}^{(2)} + d_{68}^{(2)}}{l_{67}^{(\tau)}} = \frac{59.24 + 60.26}{846.30} = 0.1412. $$ ▼

10.4 Deterministic Survivorship Group

The total force of decrement can also be viewed as a total (nominal annual) rate of decrement rather than as a conditional probability density. In this view, where we assume a continuous model, a group of $l_a^{(\tau)}$ lives advance through age subject to deterministic forces of decrement $\mu_a^{(\tau)}(y - a)$, $y \geq a$. The number of survivors to age x from the original group of $l_a^{(\tau)}$ lives at age a is given by

$$ l_x^{(\tau)} = l_a^{(\tau)} \exp\left[-\int_a^x \mu_a^{(\tau)}(y - a)\, dy\right], \tag{10.4.1} $$

and the total decrement between ages x and $x + 1$ is

$$ \begin{aligned} d_x^{(\tau)} &= l_x^{(\tau)} - l_{x+1}^{(\tau)} \\ &= l_x^{(\tau)}\left(1 - \frac{l_{x+1}^{(\tau)}}{l_x^{(\tau)}}\right) \\ &= l_x^{(\tau)}\left\{1 - \exp\left[-\int_x^{x+1} \mu_a^{(\tau)}(y - a)\, dy\right]\right\} \\ &= l_x^{(\tau)}(1 - p_x^{(\tau)}) \\ &= l_x^{(\tau)} q_x^{(\tau)}. \end{aligned} \tag{10.4.2} $$

Further, by definition or from differentiating (10.4.1), we have

$$ \mu_a^{(\tau)}(x - a) = -\frac{1}{l_x^{(\tau)}} \frac{dl_x^{(\tau)}}{dx}. \tag{10.4.3} $$

These formulas are analogous to those for life tables in Section 3.4. Here $q_x^{(\tau)}$ is the effective annual total rate of decrement for the year of age x to $x + 1$ equivalent to the forces $\mu_a^{(\tau)}(y - a)$, $x \leq y < x + 1$.

Consider next m causes of decrement and assume that the $l_x^{(\tau)}$ survivors to age x will, at future ages, be fully depleted by these m forms of decrement. Then the $l_x^{(\tau)}$ survivors can be visualized as falling into distinct subgroups $l_x^{(j)}$, $j = 1, 2, \ldots, m$, where $l_x^{(j)}$ denotes the number from the $l_x^{(\tau)}$ survivors who will terminate at future ages due to cause j, so that

$$ l_x^{(\tau)} = \sum_{j=1}^{m} l_x^{(j)}. \tag{10.4.4} $$

We define the force of decrement at age x due to cause j by

$$\mu_a^{(j)}(x-a) = \lim_{h\to 0} \frac{l_x^{(j)} - l_{x+h}^{(j)}}{h l_x^{(\tau)}}$$

where $l_x^{(\tau)}$, not $l_x^{(j)}$, appears in the denominator. This yields

$$\mu_a^{(j)}(x-a) = -\frac{1}{l_x^{(\tau)}} \frac{dl_x^{(j)}}{dx}. \tag{10.4.5}$$

From (10.4.3)–(10.4.5) it follows that

$$\mu_a^{(\tau)}(x-a) = -\frac{1}{l_x^{(\tau)}} \frac{d}{dx} \sum_{j=1}^{m} l_x^{(j)} = \sum_{j=1}^{m} \mu_a^{(j)}(x-a). \tag{10.4.6}$$

Formula (10.4.5), substituting y for x, can be written as

$$-dl_y^{(j)} = l_y^{(\tau)}\, \mu_a^{(j)}(y-a)\, dy,$$

and integration from $y = x$ to $y = x + 1$ gives

$$l_x^{(j)} - l_{x+1}^{(j)} = d_x^{(j)} = \int_x^{x+1} l_y^{(\tau)}\, \mu_a^{(j)}(y-a)\, dy. \tag{10.4.7}$$

Summation over $j = 1, 2, \ldots, m$ yields

$$l_x^{(\tau)} - l_{x+1}^{(\tau)} = d_x^{(\tau)} = \int_x^{x+1} l_y^{(\tau)}\, \mu_a^{(\tau)}(y-a)\, dy. \tag{10.4.8}$$

Further, from division of formula (10.4.7) by $l_x^{(\tau)}$, we have

$$\frac{d_x^{(j)}}{l_x^{(\tau)}} = \int_x^{x+1} {}_{y-x}p_x^{(\tau)}\, \mu_a^{(j)}(y-a)\, dy = q_x^{(j)}. \tag{10.4.9}$$

Here $q_x^{(j)}$ is defined as the proportion of the $l_x^{(\tau)}$ survivors to age x who terminate due to cause j before age $x + 1$ when all m causes of decrement are operating.

As was the case for life tables, the deterministic model provides an alternative language and conceptual framework for multiple decrement theory.

10.5 Associated Single Decrement Tables

For each of the causes of decrement recognized in a multiple decrement model, it is possible to define a single decrement model that depends only on the particular cause of decrement. We define the ***associated single decrement model*** functions as follows:

$$\begin{aligned} {}_tp_x'^{(j)} &= \exp\left[-\int_0^t \mu_x^{(j)}(s)\, ds\right], \\ {}_tq_x'^{(j)} &= 1 - {}_tp_x'^{(j)}. \end{aligned} \tag{10.5.1}$$

Quantities such as ${}_tq_x'^{(j)}$ are called ***net probabilities of decrement*** in biostatistics because they are net of other causes of decrement. However, many other names have been given to the same quantity. One is ***independent rate of decrement,*** chosen because cause j does not compete with other causes in determining ${}_tq_x'^{(j)}$. The term we use for ${}_tq_x'^{(j)}$ is ***absolute rate of decrement.*** The use of the word ***rate*** in describing ${}_tq_x'^{(j)}$ stems from a desire to distinguish between q and q'. The symbol ${}_tq_x^{(j)}$ denotes a probability of decrement for cause j between ages x and $x + t$, and we will show that it differs from ${}_tq_x'^{(j)}$. In addition, ${}_tp_x'^{(j)}$, unlike ${}_tp_x^{(\tau)}$, is not necessarily a survivorship function, because it is not required that $\lim_{t\to\infty} {}_tp_x'^{(j)} = 0$.

While

$$\int_0^\infty \mu_x^{(\tau)}(t)\, dt = \infty,$$

we can conclude from (10.2.14) only that

$$\int_0^\infty \mu_x^{(j)}(t)\, dt = \infty$$

for at least one j. There may be causes of decrement for which this integral is finite.

We seldom have an opportunity to observe the operation of a random survival system in which a single cause of decrement operates. In an employee benefit plan, retirement, disabilities, and voluntary terminations make it impossible to directly observe the operation of a single decrement model for mortality during active service. In biostatistical applications random withdrawals from observation and arbitrary ending of the period of study may prevent the observation of mortality alone operating on a group of lives.

As we see in Section 10.6, a usual first step in constructing a multiple decrement model is to select absolute rates of decrement and to make assumptions concerning the incidence of the decrements within any single year of age to obtain probabilities $q_x^{(j)}$. The converse problem of obtaining absolute rates from the probabilities also involves assumptions about the incidence of the decrements. These assumptions are implicit in statistical methods for estimating absolute rates and will be discussed in Section 10.5.5.

In the next subsection we examine a number of relationships between a multiple decrement table and its associated single decrement tables. Then we examine a number of special assumptions about incidence of decrement over the year of age and note some implied relationships. In Section 10.5.5 some of the statistical issues in estimating a multiple decrement distribution are examined.

10.5.1 Basic Relationships

First, note that since

$${}_tp_x^{(\tau)} = \exp\left\{-\int_0^t [\mu_x^{(1)}(s) + \mu_x^{(2)}(s) + \cdots + \mu_x^{(m)}(s)]\, ds\right\},$$

we have

$$ {}_tp_x^{(\tau)} = \prod_{i=1}^{m} {}_tp_x'^{(i)}. \tag{10.5.2} $$

This result does not involve any approximation. It is based on the definition of an associated single decrement table where the sole force of decrement is equal to the force for that decrement in the multiple decrement model. We require that it hold for any method used to construct a multiple decrement table from a set of absolute rates of decrement.

Now compare the size of the absolute rates and the probabilities. From (10.5.2) we see, if some cause other than j is operating, that

$$ {}_tp_x'^{(j)} \geq {}_tp_x^{(\tau)}. $$

This implies

$$ {}_tp_x'^{(j)} \, \mu_x^{(j)}(t) \geq {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t), $$

and if these functions are integrated with respect to t over the interval (0, 1), we obtain

$$ q_x'^{(j)} = \int_0^1 {}_tp_x'^{(j)} \, \mu_x^{(j)}(t) \, dt \geq \int_0^1 {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt = q_x^{(j)}. \tag{10.5.3} $$

The magnitude of other forces of decrement can cause ${}_tp_x'^{(j)}$ to be considerably greater than ${}_tp_x^{(\tau)}$, and thus there can be corresponding differences between the absolute rates and the probabilities.

10.5.2 Central Rates of Multiple Decrement

There is one function of the multiple decrement model that is quite close to the corresponding function for an associated single decrement model. To introduce this function, we return to a mortality table and recall the central rate of mortality, or central-death-rate at age x, denoted by m_x and defined in (3.5.13) by

$$ m_x = \frac{\int_0^1 {}_tp_x \, \mu_x(t) \, dt}{\int_0^1 {}_tp_x \, dt} = \frac{\int_0^1 l_{x+t} \, \mu_x(t) \, dt}{\int_0^1 l_{x+t} \, dt} = \frac{d_x}{L_x}. \tag{10.5.4} $$

Thus, m_x is a weighted average of the force of mortality between ages x and $x + 1$, and this justifies the term ***central rate.***

Such central rates can be defined in a multiple decrement context. The ***central rate of decrement from all causes*** is defined by

$$ m_x^{(\tau)} = \frac{\int_0^1 {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t) \, dt}{\int_0^1 {}_tp_x^{(\tau)} \, dt} \tag{10.5.5} $$

and is a weighted average of $\mu_x^{(\tau)}(t)$, $0 \le t < 1$. Similarly, the ***central rate of decrement from cause j*** is

$$m_x^{(j)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x^{(\tau)}\, dt} \tag{10.5.6}$$

and is a weighted average of $\mu_x^{(j)}(t)$, $0 \le t < 1$. Clearly,

$$m_x^{(\tau)} = \sum_{j=1}^{m} m_x^{(j)}.$$

The corresponding central rate for the associated single decrement table is given by

$$m_x'^{(j)} = \frac{\int_0^1 {}_tp_x'^{(j)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x'^{(j)}\, dt}. \tag{10.5.7}$$

This is again a weighted average of $\mu_x^{(j)}(t)$ over the same age range, the weight function now ${}_tp_x'^{(j)}$ rather than ${}_tp_x^{(\tau)}$. If the force $\mu_x^{(j)}(t)$ is constant for $0 \le t < 1$, we have $m_x^{(j)} = m_x'^{(j)} = \mu_x^{(j)}(0)$. If $\mu_x^{(j)}(t)$ is an increasing function of t, then ${}_tp_x'^{(j)}$ gives more weight to higher values than does ${}_tp_x^{(\tau)}$, and $m_x'^{(j)} > m_x^{(j)}$. If $\mu_x^{(j)}(t)$ is a decreasing function of t, then $m_x'^{(j)} < m_x^{(j)}$. See Exercise 10.33 for a more formal treatment of these statements.

Central rates provide a convenient but approximate means of proceeding from the $q_x'^{(j)}$ to the $q_x^{(j)}$, $j = 1, 2, \ldots, m$, and vice versa. This is illustrated in Exercise 10.18.

10.5.3 Constant Force Assumption for Multiple Decrements

Let us examine specific assumptions concerning the incidence of decrements. First, let us use an assumption of a constant force for decrement j and for the total decrement over the interval $(x, x + 1)$. This implies

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$$

and

$$\mu_x^{(\tau)}(t) = \mu_x^{(\tau)}(0) \qquad 0 \le t < 1.$$

Then, for $0 \le s \le 1$, we have

$$\begin{aligned} {}_sq_x^{(j)} &= \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \\ &= \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)} \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)\, dt = \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)}\, {}_sq_x^{(\tau)}. \end{aligned} \tag{10.5.8}$$

But also for any r in $(0, 1)$, under the constant force assumption,

$$r\mu_x^{(\tau)}(0) = -\log {}_rp_x^{(\tau)}$$

and

$$r\mu_x^{(j)}(0) = -\log {}_rp_x'^{(j)},$$

so that from (10.5.8),

$${}_sq_x^{(j)} = \frac{\log {}_rp_x'^{(j)}}{\log {}_rp_x^{(\tau)}} \, {}_sq_x^{(\tau)}. \tag{10.5.9}$$

Equation (10.5.9) can be rearranged as

$${}_rp_x'^{(j)} = ({}_rp_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}}$$

and then in the limit as r goes to 1 can be solved for $q_x'^{(j)}$ to give

$$q_x'^{(j)} = 1 - (p_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}} . \tag{10.5.10}$$

If the constant force assumption holds for all decrements (and then automatically for the total decrement), (10.5.9), as r and s approach 1, together with (10.5.2), can be used for calculating $q_x^{(j)}$ from given values of $q_x'^{(j)}$, $j = 1, 2, \ldots, m$. Also, (10.5.10) is useful for obtaining absolute rates from a set of probabilities of decrement. Note that for (10.5.9) and (10.5.10) special treatment is required if $p_x'^{(j)}$ or $p_x^{(\tau)}$ equals 0.

10.5.4 Uniform Distribution Assumption for Multiple Decrements

Formula (10.5.10) holds under alternative assumptions. One of these is that both decrement j and total decrement, in the multiple decrement context, have a uniform distribution of decrement over the interval $(x, x + 1)$. Thus we assume that

$${}_tq_x^{(j)} = t\, q_x^{(j)}$$

and

$${}_tq_x^{(\tau)} = t\, q_x^{(\tau)}.$$

Also under the given assumption, we see from (10.2.12) that

$${}_tp_x^{(\tau)}\, \mu_x^{(j)}(t) = q_x^{(j)} \tag{10.5.11}$$

and

$$\mu_x^{(j)}(t) = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - t\, q_x^{(\tau)}} .$$

Then

$$\begin{aligned}
{}_sp_x'^{(j)} &= \exp\left[-\int_0^s \mu_x^{(j)}(t)\, dt\right] \\
&= \exp\left(-\int_0^s \frac{q_x^{(j)}}{1 - t\, q_x^{(\tau)}}\, dt\right) \\
&= \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \log\,(1 - sq_x^{(\tau)})\right] \\
&= ({}_sp_x^{(\tau)})^{q_x^{(j)}/q_x^{(\tau)}}.
\end{aligned} \tag{10.5.12}$$

At $s = 1$, (10.5.10) and (10.5.12) yield the same equation relating $q_x'^{(j)}$ with $q_x^{(j)}$ and $q_x^{(\tau)}$. As a result, (10.5.9) with $r = 1$ can be used to obtain $q_x^{(j)}$. Exercise 10.22 provides additional insights into the connection between the developments in Sections 10.5.3 and 10.5.4.

Example 10.5.1

Continue Example 10.3.1 evaluating $q_x'^{(1)}$ and $q_x'^{(2)}$ by (10.5.10).

Solution:

By (10.5.10), the following results are obtained.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x'^{(1)}$	$q_x'^{(2)}$
65	0.02	0.05	0.02052	0.05052
66	0.03	0.06	0.03095	0.06094
67	0.04	0.07	0.04149	0.07147
68	0.05	0.08	0.05215	0.08213
69	0.06	0.09	0.06294	0.09291
70	0.00	1.00	—	—

At age 70, the rates depend on mandatory retirement, and there is no particular need for $q_{70}'^{(1)}$, $q_{70}'^{(2)}$ although they could be identified, respectively, using $q_{70}^{(1)}$ and $q_{70}^{(2)}$. ▼

10.5.5 Estimation Issues

The definition of the absolute rate of decrement given in (10.5.1) depends on the force of decrement in multiple decrement theory as defined in (10.2.11). From definition (10.5.1), the developments in this section and their application in constructing multiple decrement distributions from assumed absolute rates of decrement follow. Questions remain, however, about the interpretation and estimation of ${}_tp_x'^{(j)}$.

If the joint p.d.f. $f_{T,J}(t, j)$ is known, then the survival function and forces of decrement are determined using the formulas of Section 10.2. For example, (10.2.13) follows as a consequence of the assumption that decrements occur from m mutually exclusive causes. The issue is, under what conditions can information obtained in a single decrement environment be used to construct the distribution of (T, J)?

We will illustrate this issue by considering two causes of decrement. Each associated single decrement environment has its time-until-decrement random variable $T_j(x)$ and its survival function $s_{T_j}(t) = \Pr\{T_j(x) > t\}$, $j = 1, 2$. The joint survival function of $T_1(x)$ and $T_2(x)$ is given by

$$s_{T_1,T_2}(t_1, t_2) = \Pr\{(T_1(x) > t_1) \cap (T_2(x) > t_2)\}.$$

In this context, the time-until-decrement random variable T equals the minimum

At $s = 1$, (10.5.10) and (10.5.12) yield the same equation relating $q_x'^{(j)}$ with $q_x^{(j)}$ and $q_x^{(\tau)}$. As a result, (10.5.9) with $r = 1$ can be used to obtain $q_x^{(j)}$. Exercise 10.22 provides additional insights into the connection between the developments in Sections 10.5.3 and 10.5.4.

Example 10.5.1

Continue Example 10.3.1 evaluating $q_x'^{(1)}$ and $q_x'^{(2)}$ by (10.5.10).

Solution:

By (10.5.10), the following results are obtained.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x'^{(1)}$	$q_x'^{(2)}$
65	0.02	0.05	0.02052	0.05052
66	0.03	0.06	0.03095	0.06094
67	0.04	0.07	0.04149	0.07147
68	0.05	0.08	0.05215	0.08213
69	0.06	0.09	0.06294	0.09291
70	0.00	1.00	—	—

At age 70, the rates depend on mandatory retirement, and there is no particular need for $q_{70}'^{(1)}$, $q_{70}'^{(2)}$ although they could be identified, respectively, using $q_{70}^{(1)}$ and $q_{70}^{(2)}$. ▼

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We will illustrate this issue by considering two causes of decrement. Each associated single decrement environment has its time-until-decrement random variable $T_j(x)$ and its survival function $s_{T_j}(t) = \Pr\{T_j(x) > t\}$, $j = 1, 2$. The joint survival function of $T_1(x)$ and $T_2(x)$ is given by

$$s_{T_1,T_2}(t_1, t_2) = \Pr\{(T_1(x) > t_1) \cap (T_2(x) > t_2)\}.$$

In this context, the time-until-decrement random variable T equals the minimum of $T_1(x)$ and $T_2(x)$ and, in accordance with Section 9.3, (9.3.1), its survival function is

$$s_T(t) = s_{T_1,T_2}(t, t).$$

If $T_1(x)$ and $T_2(x)$ are independent,

$$s_T(t) = s_{T_1}(t)s_{T_2}(t) = s_{T_1,T_2}(t, 0)s_{T_1,T_2}(0, t),$$

and

$$\mu_x^{(\tau)}(t) = -\frac{d}{dt}\log s_{T_1}(t)s_{T_2}(t)$$
$$= \mu_x^{(1)}(t) + \mu_x^{(2)}(t). \tag{10.5.13}$$

On the other hand, if $T_1(x)$ and $T_2(x)$ are dependent,

$$\mu_x^{(\tau)}(t) = -\frac{d}{dt}\log s_{T_1,T_2}(t, t)$$
$$\neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t). \tag{10.5.14}$$

The two terms on the right-hand side of (10.5.13) are called ***marginal forces of decrement*** associated, in order, with $T_1(x)$ and $T_2(x)$.

If $T_1(x)$ and $T_2(x)$ are independent, then the marginal forces of decrement from a single decrement environment can be used with (10.5.2) to determine ${}_tp_x^{(\tau)}$. If $T_1(x)$ and $T_2(x)$ are dependent, we have no assurance that assuming (10.5.2) yields the survival function of time-until-decrement in a multiple decrement environment.

Example 10.5.2

This example builds on Examples 9.2.1, 9.2.2, and 9.3.1. The dependent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$f_{T_1,T_2}(s, t) = 0.0006(s - t)^2 \qquad 0 < s < 10, 0 < t < 10$$
$$= 0 \qquad \text{elsewhere.}$$

The joint survival function is exhibited in Example 9.2.2, and the survival function of $T = \min[T_1, T_2]$ is exhibited in Example 9.3.1.

Show that

$$-\frac{d}{dt}\log s_T(t) \neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t).$$

Solution:

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{(10 - t)} \qquad 0 < t < 10,$$

and

$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{4{,}000 - 1{,}200t + 120t^2}{20{,}000 - 4{,}000t + 600t^2 - 40t^3}$$
$$= \frac{100 - 30t + 3t^2}{500 - 100t + 15t^2 - t^3}$$
$$= \frac{100 - 30t + 3t^2}{(10 - t)(50 - 5t + t^2)},$$

which by symmetry is also equal to $-\frac{d}{dt}\log s_{T_1,T_2}(0, t)$. Therefore

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10 - t} \neq \frac{1}{10 - t}\left(\frac{200 - 60t + 6t^2}{50 - 5t + t^2}\right).$$ ▼

Example 10.5.3

This example builds on Examples 9.2.3 and 9.3.2. The independent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$f_{T_1,T_2}(s, t) = [0.02(10 - s)][0.02(10 - t)] \qquad \begin{array}{l} 0 < s < 10 \\ 0 < t < 10. \end{array}$$

Show that

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t).$$

Solution:

The survival function of $T = \min[T_1(x), T_2(x)]$ is displayed in Example 9.3.2. Therefore,

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10 - t} \qquad 0 < t < 10,$$
$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{2}{10 - t} \qquad 0 < t < 10,$$

and by symmetry

$$-\frac{d}{dt}\log s_{T_1,T_2}(0, t) = \frac{2}{10 - t} \qquad 0 < t < 10.$$

As a result,

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1 T_2}(t, 0) - \frac{d}{dt}\log s_{T_1 T_2}(0, t).$$ ▼

An interesting but distressing aspect of Examples 10.5.2 and 10.5.3 is that two dependent time-until-decrement random variables and two independent time-until-decrement random variables yield the same distribution of $T = \min[T_1(x), T_2(x)]$. Values of T can be observed, but without additional information it is impossible to select between the two models that may be generating the data. This, as in Section 9.3, is an example of nonidentifiability. Henceforth, in this chapter when constructing multiple decrement distributions from associated single decrement distributions, we assume that the component random variables are independent.

Remark:

The correspondence between the theory for the joint life model and the theory for the multiple decrement model can provide insights, but it is not complete. The difference between the two models centers on two facts that were identified in the discussion of (10.5.2) and Example 10.5.2. Realizations of both $T(x)$ and $T(y)$ can, at least in theory, be observed, while only the minimum of $T_1(x)$ and $T_2(x)$ and which one is the minimum can be observed. The corresponding problem in estimating joint life models was mentioned in Section 9.3. In addition, $\lim_{t\to\infty} {}_tp_x = \lim_{t\to\infty} {}_tp_y = 0$, whereas there is no assurance that

$$\lim_{t\to\infty} {}_tp_x'^{(j)} = 0 \qquad j = 1, 2.$$

10.6 Construction of a Multiple Decrement Table

In building a multiple decrement model it is best if data, including that on age and cause of decrement for the population under study, can be used to estimate directly the probabilities $q_x^{(j)}$. Large, well-established employee benefit plans may have such data. For other plans, such data are frequently not available. An alternative is to construct the model from associated single decrement rates assumed appropriate for the population under study. The adequacy of the model should then be tested by reviewing data as they become available.

Once satisfactory associated single decrement tables are selected, the results of Section 10.5 can be used to complete the construction of the multiple decrement table. The availability of a set of $p_x'^{(j)}$, for $j = 1, 2, \ldots, m$ and all values of x, will permit the computation of $p_x^{(\tau)}$ by (10.5.2) and of $q_x^{(\tau)}$ by $q_x^{(\tau)} = 1 - p_x^{(\tau)}$. The remaining step is to break $q_x^{(\tau)}$ into its components $q_x^{(j)}$ for $j = 1, 2, \ldots, m$. If either the constant force or the uniform distribution of decrement assumption is adopted in the model, (10.5.9) can be used for the calculation of the $q_x^{(j)}$.

Example 10.6.1

Use (10.5.2) and (10.5.9) to obtain the multiple decrement table corresponding to absolute rates of decrement given below. Presumably the actuary has examined the characteristics of the participant group and has decided that associated single decrement tables yielding these rates are appropriate for the group under study. It is also assumed that cause 3 is retirement that can occur between ages 65 and 70 and is mandatory at 70.

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
65	0.020	0.02	0.04
66	0.025	0.02	0.06
67	0.030	0.02	0.08
68	0.035	0.02	0.10
69	0.040	0.02	0.12

Solution:

The table below contains the results of the calculation of the probabilities of decrement. Formula (10.5.2) can be rewritten as

$$q_x^{(\tau)} = 1 - \prod_{j=1}^{3} (1 - q_x'^{(j)}).$$

In this equation the assumed independence among the three causes of decrement is apparent. Formula (10.5.9), and the mandatory retirement condition, yield the multiple decrement probabilities. The multiple decrement table is constructed as in Example 10.3.1.

x	$q_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
65	0.07802	0.01940	0.01940	0.03921	1 000.00	19.40	19.40	39.21
66	0.10183	0.02401	0.01916	0.05867	921.99	22.14	17.67	54.09
67	0.12545	0.02851	0.01891	0.07803	828.09	23.61	15.66	64.62
68	0.14887	0.03290	0.01866	0.09731	724.20	23.83	13.51	70.47
69	0.17210	0.03720	0.01841	0.11649	616.39	22.93	11.35	71.80
70	1.00000	0.00000	0.00000	1.00000	510.31	0.00	0.00	510.31

▼

It has been noted that (10.5.9) and (10.5.10) will not be used if $p_x'^{(j)}$ or $p_x^{(\tau)} = 0$; some alternative device will be necessary. One such method, which handles this indeterminacy and lends itself to special adjustments, is based on assumed distributions of decrement in the associated single decrement tables rather than on assumptions about multiple decrement probabilities as in Section 10.5. We first examine an assumption of uniform distribution of decrement (in each year of age) in the associated single decrement tables. We restrict our attention to situations with three decrements, but the method and formulas easily extend for $m > 3$. Under the stated assumption,

$$ {}_tp_x'^{(j)} = 1 - t\, q_x'^{(j)} \qquad j = 1, 2, 3;\ 0 \le t \le 1 \tag{10.6.1}$$

and

$$ {}_tp_x'^{(j)}\, \mu_x^{(j)}(t) = \frac{d}{dt}\,(-{}_tp_x'^{(j)}) = q_x'^{(j)}. \tag{10.6.2}$$

It follows that

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt$$

$$= \int_0^1 {}_tp_x'^{(1)}\ \mu_x^{(1)}(t)\ {}_tp_x'^{(2)}\ {}_tp_x'^{(3)}\ dt$$

$$= q_x'^{(1)} \int_0^1 (1 - t\ q_x'^{(2)})(1 - t\ q_x'^{(3)})\ dt$$

$$= q_x'^{(1)} \left[1 - \frac{1}{2}(q_x'^{(2)} + q_x'^{(3)}) + \frac{1}{3} q_x'^{(2)} q_x'^{(3)} \right]. \tag{10.6.3}$$

Similar formulas hold for $q_x^{(2)}$, $q_x^{(3)}$, and it can be verified that

$$\begin{aligned} q_x^{(1)} + q_x^{(2)} + q_x^{(3)} &= q_x'^{(1)} + q_x'^{(2)} + q_x'^{(3)} \\ &\quad - (q_x'^{(1)} q_x'^{(2)} + q_x'^{(1)} q_x'^{(3)} + q_x'^{(2)} q_x'^{(3)}) \\ &\quad + q_x'^{(1)} q_x'^{(2)} q_x'^{(3)} \\ &= 1 - (1 - q_x'^{(1)})(1 - q_x'^{(2)})(1 - q_x'^{(3)}) = q_x^{(\tau)}. \end{aligned} \tag{10.6.4}$$

Example 10.6.2

Obtain the probabilities of decrement for ages 65–69 from the data in Example 10.6.1, under the assumption of a uniform distribution of decrement in each year of age in each of the associated single decrement tables.

Solution:

This is an application of (10.6.3).

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
65	0.020	0.02	0.04	0.01941	0.01941	0.03921
66	0.025	0.02	0.06	0.02401	0.01916	0.05866
67	0.030	0.02	0.08	0.02852	0.01892	0.07802
68	0.035	0.02	0.10	0.03292	0.01867	0.09727
69	0.040	0.02	0.12	0.03723	0.01843	0.11643

These probabilities are close to those obtained by (10.5.9), displayed in Example 10.6.1. ▼

We conclude this section with another example illustrating the use of a special distribution for one of the decrements. Special distributions are sometimes required by the facts of the situation being modeled.

Example 10.6.3

Consider a situation with three causes of decrement: mortality, disability, and withdrawal. Assume mortality and disability are uniformly distributed in each year of age in the associated single decrement tables with absolute rates of $q_x'^{(1)}$ and

$q_x'^{(2)}$, respectively. Also assume that withdrawals occur only at the end of the year with an absolute rate of $q_x'^{(3)}$.

a. Give formulas for the probabilities of decrement in the year of age x to $x + 1$ for the three causes.
b. Reformulate the probabilities under the assumptions that
   (i) In the associated single decrement model, withdrawals occur only at the age's midyear or year end, and
   (ii) Equal proportions, namely $(1/2)$, $q_x'^{(3)}$, of those beginning the year withdraw at the midyear and at the year end.

Remark:

Until now our multiple decrement models have been fully continuous, except possibly to recognize a mandatory retirement age. Moreover, our theory began with a multiple decrement model and after defining the forces $\mu_x^{(j)}(t)$, $j = 1, 2, \ldots, m$ proceeded to the associated single decrement tables. In this example we start with the single decrement tables, and in one of these tables the decrement takes place discretely at the ends of stated intervals. We do not attempt to define a force of decrement for this discrete case but proceed by direct methods to build, from the single decrement tables, a multiple decrement model possessing the relationships (10.2.19) and (10.5.2) established in our prior theory.

Solution:

a. Figure 10.6.1 displays survival factors for the given single decrement tables and for a multiple decrement table where

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\ {}_tp_x'^{(2)}\ {}_tp_x'^{(3)} $$

for nonintegral $t \ge 0$. At $t = 1$, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ are discontinuous, so we consider

$$ \lim_{t\to 1-} {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\ {}_tp_x'^{(2)}\ 1 $$

and

$$ p_x^{(\tau)} = p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

We also require that, for our multiple decrement table,

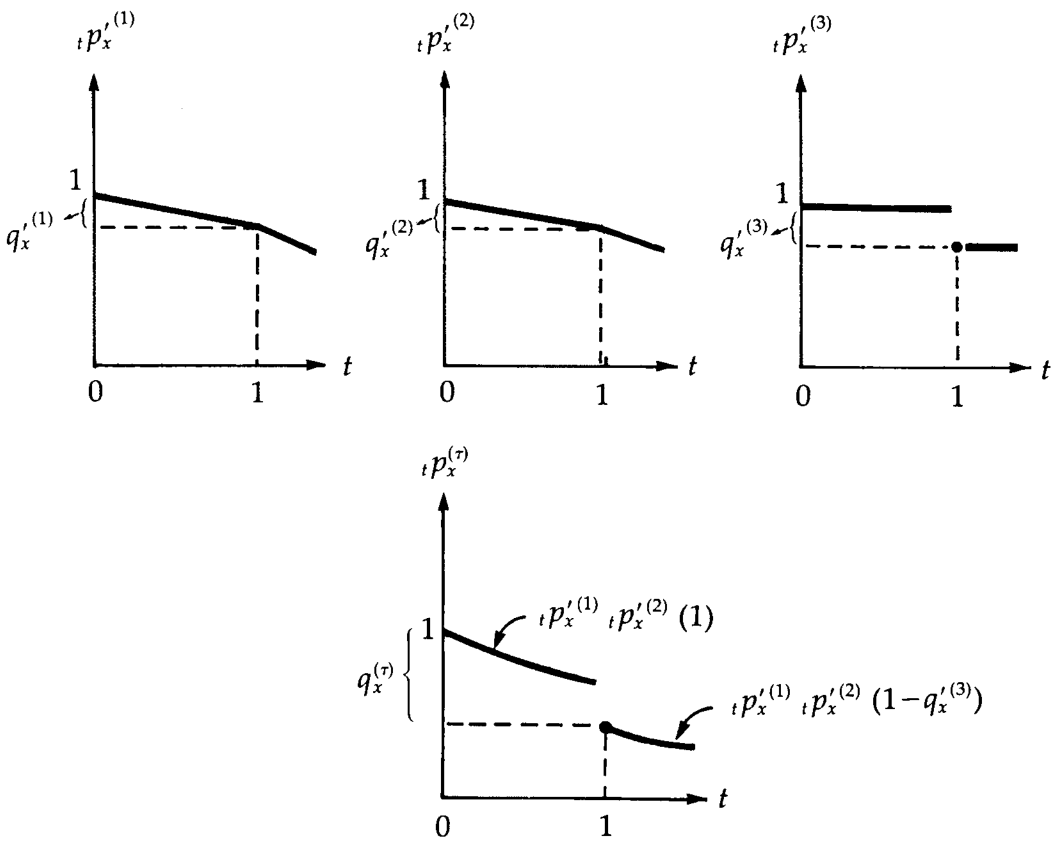
$$ q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = 1 - p_x^{(\tau)} = 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

We set

$$ \begin{aligned} q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt \\ &= \int_0^1 {}_tp_x'^{(1)}\,{}_tp_x'^{(2)}(1)\mu_x^{(1)}(t)\ dt \\ &= q_x'^{(1)} \int_0^1 (1 - t q_x'^{(2)})\ dt \\ &= q_x'^{(1)} \left(1 - \frac{1}{2}\, q_x'^{(2)}\right). \end{aligned} $$

FIGURE 10.6.1

Survival Factors, ${}_tp_x'^{(j)}$, j = 1, 2, 3, and ${}_tp_x^{(\tau)}$



Similarly, we set

$$q_x^{(2)} = q_x'^{(2)}\left(1 - \frac{1}{2}\,q_x'^{(1)}\right).$$

Then

$$\begin{aligned} q_x^{(3)} &= q_x^{(\tau)} - (q_x^{(1)} + q_x^{(2)}) \\ &= 1 - p_x'^{(1)}p_x'^{(2)}(1 - q_x'^{(3)}) - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)}, \end{aligned}$$

and, since

$$1 - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)} = p_x'^{(1)}p_x'^{(2)},$$

$$q_x^{(3)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)}.$$

Note that

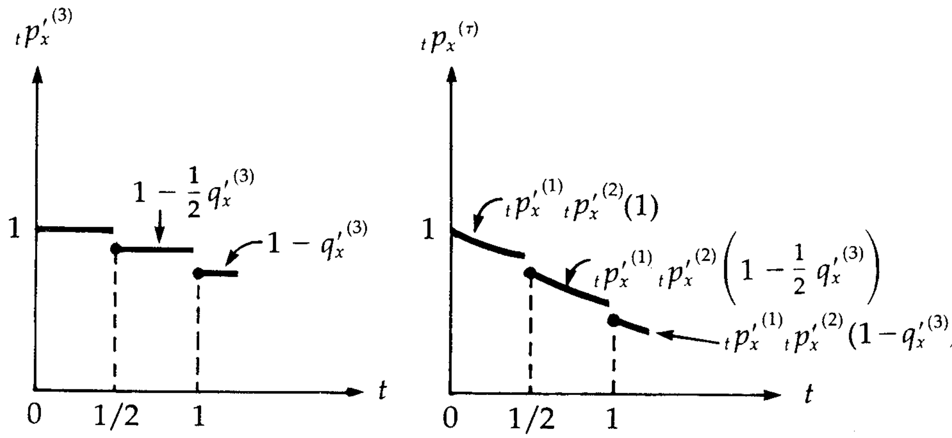
$$\lim_{t\to 1-} {}_tp_x^{(\tau)} - \lim_{t\to 1+} {}_tp_x^{(\tau)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)} = q_x^{(3)}\ ;$$

that is, the discontinuity at $t = 1$ equals $q_x^{(3)}$.

b. Here ${}_tp_x'^{(1)}$ and ${}_tp_x'^{(2)}$ are as in Figure 10.6.1, but ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ now have discontinuities at $t = 1/2$ and $t = 1$, as shown in Figure 10.6.2.

FIGURE 10.6.2

Survival Factors, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$



Proceeding as in (a), but taking account of the intervals [0, 1/2) and [1/2, 1), we set

$$\begin{aligned}
q_x^{(1)} &= q_x'^{(1)} \int_0^{1/2} (1 - t\, q_x'^{(2)})\, dt \\
&\quad + q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(3)}\right) \int_{1/2}^{1} (1 - t\, q_x'^{(2)})\, dt \\
&= q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(2)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(2)} q_x'^{(3)}\right).
\end{aligned}$$

Similarly, we set

$$q_x^{(2)} = q_x'^{(2)} \left(1 - \frac{1}{2} q_x'^{(1)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(1)} q_x'^{(3)}\right).$$

Then,

$$\begin{aligned}
q_x^{(3)} &= 1 - p_x^{(\tau)} - q_x^{(1)} - q_x^{(2)} \\
&= 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}) - q_x^{(1)} - q_x^{(2)},
\end{aligned}$$

which reduces to

$$q_x^{(3)} = q_x'^{(3)} \left(1 - \frac{3}{4} q_x'^{(1)} - \frac{3}{4} q_x'^{(2)} + \frac{5}{8} q_x'^{(1)} q_x'^{(2)}\right).$$ ▼

10.7 Notes and References

The history of multiple decrement theory was reviewed by Seal (1977). Chiang (1968) developed the theory using the language of competing risks. The foundation for the actuarial theory of multiple decrement models was built by Makeham (1874). Menge (1932) and Nesbitt and Van Eenam (1948) provided insight into the deterministic interpretation of forces of decrement and of increment. Bicknell and

Nesbitt (1956) developed a very general theory for individual insurances using a deterministic multiple decrement model. Hickman (1964) redeveloped this theory using the language of the stochastic model, and this redevelopment is the basis for much of this chapter. The analysis of life tables by cause of death is the subject of papers by Greville (1948) and Preston, Keyfitz, and Schoen (1973).

The perplexing estimation issues that arise when the times-until-decrement are not independent are discussed by Elandt-Johnson and Johnson (1980). Promislow (1991b) makes the excellent point that in practice multiple decrement models should be select in the sense of Chapter 3. He developed a theory and associated notation for select multiple decrement models. Exercises 10.3 and 10.24 are built on a discussion by Robinson (1984).

Carriere (1994) applied copulas, discussed in Section 9.6.2, to create multiple decrement distributions that incorporate dependent component random variables. Carriere also discusses the problem of identifiability and reviews the conditions under which it is possible to identify a unique joint survival function $s_{T_1(x)\ T_2(x)}(t, t)$.

Exercises

Section 10.2

10.1. Let $\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$, $j = 1, 2, \ldots, m$, $t \geq 0$. Obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_J(j)$ c. $f_T(t)$.
The functions called for in (a) and (c) are p.d.f.'s, and the function in (b) is a p.f. Show that T and J are independent random variables.

10.2. A multiple decrement model with two causes of decrement has forces of decrement given by

$$\mu_x^{(1)}(t) = \frac{1}{100 - (x + t)}$$

and

$$\mu_x^{(2)}(t) = \frac{2}{100 - (x + t)} \qquad t < 100 - x.$$

If $x = 50$, obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_T(t)$ c. $f_J(j)$ d. $f_{J|T}(j|t)$.

10.3. Given the joint p.d.f.

$$\begin{aligned} f_{T,J}(t, j) &= pu_1\, e^{-(u_1+v_1)t} + (1 - p)u_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t, j = 1 \\ &= pv_1\, e^{-(u_1+v_1)t} + (1 - p)v_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t, j = 2 \end{aligned}$$

where $0 < p < 1$ and $0 < u_1, u_2, v_1, v_2,$

find

a. The marginal p.d.f.s $f_T(t)$ and $f_J(j)$

b. The survival function $s_T(t)$.

Section 10.3

10.4. Using the multiple decrement probabilities given in Example 10.3.1, evaluate the following:

a. ${}_3p_{65}^{(\tau)}$ b. ${}_{3|}q_{65}^{(1)}$ c. ${}_3q_{65}^{(2)}$.

10.5. The following multiple decrement probabilities apply to students entering a 4-year college.

Curtate Duration, at Beginning of Academic Year	**Probability of**		
	Academic Failure, $j = 1$	**Withdrawal for All Other Reasons, $j = 2$**	**Survival through the Academic Year**
0	0.15	0.25	0.60
1	0.10	0.20	0.70
2	0.05	0.15	0.80
3	0.00	0.10	0.90

An entering class has 1,000 members.

a. What is the expectation of the number of graduates? What is the variance?

b. What is the expected number of those who will fail sometime during the 4-year program? What is the variance of the number of students who will fail?

10.6. Construct a multiple decrement table on the basis of the data in Exercise 10.5 and use it to exhibit

a. The marginal distribution of the random variable J (mode of exit), which takes on values for academic failure, withdrawal, and graduation

b. The conditional distribution of the mode of termination, given that a student has terminated in the third year.

Section 10.4

10.7. Given that $\mu^{(1)}(x) = 1/(a - x)$, $0 \le x < a$, and $\mu^{(2)}(x) = 1$, derive expressions for

a. $l_x^{(\tau)}$ b. $d_x^{(1)}$ c. $d_x^{(2)}$.

Assume $l_0^{(\tau)} = a$.

10.8. Given $\mu^{(1)}(x) = 2x/(a - x^2)$, $0 \le x \le \sqrt{a}$, and $\mu^{(2)}(x) = c$, $c > 0$, and $l_0^{(\tau)} = 1{,}000$, derive an expression for $l_x^{(\tau)}$.

10.9. Derive expressions for the following derivatives:

a. $\dfrac{d}{dx}\,{}_tq_x^{(\tau)}$ b. $\dfrac{d}{dx}\,{}_tq_x^{(j)}$ c. $\dfrac{d}{dt}\,{}_tq_x^{(j)}$.

Section 10.5

10.10. Using the data in Exercise 10.5, and assuming a uniform distribution of all decrements in the multiple decrement model, calculate a table of $q_k'^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration).

10.11. If $\mu_x^{(1)}(t)$ is a constant c for $0 \le t \le 1$, derive expressions in terms of c and ${}_tp_x^{(\tau)}$ for
a. $q_x'^{(1)}$ b. $m_x^{(1)}$ c. $q_x^{(1)}$.

10.12. Show that under appropriate assumptions of a uniform distribution of decrements

a. $m_x^{(\tau)} = \dfrac{q_x^{(\tau)}}{1 - (1/2)\, q_x^{(\tau)}}$ b. $m_x^{(j)} = \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}$ c. $m_x'^{(j)} = \dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}}$

and, conversely,

d. $q_x^{(\tau)} = \dfrac{m_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$ e. $q_x^{(j)} = \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}$ f. $q_x'^{(j)} = \dfrac{m_x'^{(j)}}{1 + (1/2)\, m_x'^{(j)}}$.

10.13. Order the following in terms of magnitude and state your reasons:

$$q_x'^{(j)},\quad q_x^{(j)},\quad m_x'^{(j)}.$$

10.14. Given, for a double decrement table, that $q_{40}'^{(1)} = 0.02$ and $q_{40}'^{(2)} = 0.04$, calculate $q_{40}^{(\tau)}$ to four decimal places.

10.15. For a double decrement table you are given that $m_{40}^{(\tau)} = 0.2$ and $q_{40}'^{(1)} = 0.1$. Calculate $q_{40}'^{(2)}$ to four decimal places assuming
a. Uniform distribution of decrements in the multiple decrement model
b. Uniform distribution of decrements in the associated single decrement tables.

10.16. Using the data in Exercise 10.5 and assuming a uniform distribution of decrements in the multiple decrement model, construct a table of $m_k^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration). Calculate each result to five decimal places.

10.17. Given that decrement may be due to death, 1, disability, 2, or retirement, 3, use (10.5.9) to construct a multiple decrement table based on the following absolute rates.

Age x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
62	0.020	0.030	0.200
63	0.022	0.034	0.100
64	0.028	0.040	0.120

10.18. Recalculate the multiple decrement table from the absolute rates of decrement in Exercise 10.17 by means of the ***central rate bridge***. [Hint: To use the central rate bridge, first calculate $m_x'^{(j)}$ by the formula

$$m_x'^{(j)} \cong \frac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \qquad j = 1, 2, 3,$$

which holds if there is a uniform distribution of decrement in the associated single decrement tables. Next, assume $m_x^{(j)} \cong m_x'^{(j)}$, $j = 1, 2, 3$, and proceed to $q_x^{(j)}$ by

$$q_x^{(j)} = \frac{d_x^{(j)}}{l_x^{(\tau)}} = \frac{d_x^{(j)}}{l_x^{(\tau)} - (1/2)\, d_x^{(\tau)} + (1/2)\, d_x^{(\tau)}} = \frac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}.$$

This second relation holds if there is a uniform distribution of total decrement in the multiple decrement table. But then

$$\begin{aligned} {}_tp_x^{(\tau)} = 1 - tq_x^{(\tau)} &\neq {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)} \\ &= (1 - tq_x'^{(1)})(1 - tq_x'^{(2)})(1 - tq_x'^{(3)}) \end{aligned}$$

under the condition of a uniform distribution in the associated single decrement tables. Thus there is an inconsistency in the stated conditions, but the calculations may be accurate enough for this purpose.]

10.19. Indicate arguments for the following relations:

a. $m_x'^{(j)} \cong m_x^{(j)}$

b. $\dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}$.

Show that these lead to

c. $q_x^{(j)} \cong \dfrac{q_x'^{(j)}\, [1 - (1/2)\, q_x^{(\tau)}]}{1 - (1/2)\, q_x'^{(j)}}$

d. $q_x'^{(j)} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\, (q_x^{(\tau)} - q_x^{(j)})}$.

Compare (c) and (d) to (10.5.9) and (10.5.10).

10.20. Use the values of $q_x^{(j)}$, $q_x'^{(j)}$ from Example 10.5.1 to calculate values of $m_x^{(j)}$, $m_x'^{(j)}$, $j = 1, 2$, $x = 65, \ldots, 69$, under appropriate assumptions of uniform distribution of decrements (see Exercise 10.12).

10.21. Which of the following statements would you accept? Revise where necessary.

a. $q_x^{(j)} \cong \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(j)}}$

b. $\displaystyle\int_0^1 l_{x+t}^{(\tau)}\, dt \cong \frac{l_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$

c. $q_x^{(1)} = q_x'^{(1)}[1 - (1/2)q_x'^{(2)}]$ in a double decrement table where there is a uniform distribution of decrement for the year of age x to $x + 1$ in each of the associated single decrement tables.

10.22. a. For a certain age x, particular cause of decrement j, and constant K_j, show that the following conditions are equivalent:

(i) ${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1$

(ii) $\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t \le 1$

(iii) $1 - {}_tq_x'^{(j)} = (1 - {}_tq_x^{(\tau)})^{K_j} \qquad 0 \le t \le 1.$

[Hint: Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).]

b. Verify that, in a multiple decrement table, where either

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0) \qquad 0 \le t \le 1, j = 1, 2, \ldots, m$$

(the constant force assumption for each cause of decrement) or

$${}_tq_x^{(j)} = t \, q_x^{(j)} \qquad 0 \le t \le 1, j = 1, 2, \ldots, m$$

(the uniform distribution for each cause of decrement), then

$${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1, j = 1, 2, \ldots, m.$$

c. Assume that in part (a), condition (ii), $\mu_x^{(\tau)}(t)$, $0 \le t \le 1$, is given by

(i) $kt^n \qquad k > 0, n > 0 \qquad$ (Weibull)

(ii) $Bc^t \qquad B > 0, c > 1 \qquad$ (Gompertz)

and for each example find the corresponding expressions for ${}_tq_x^{(j)}$ and $1 - {}_tq_x'^{(j)}$.

10.23. a. Prove that

$$\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t, j = 1, 2$$

where

$$K_j = \int_0^\infty {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt \qquad j = 1, 2,$$

if and only if the random variables T and J are independent.

b. If $T_1(x)$ and $T_2(x)$ are independent and J and T are independent, show that

$${}_tp_x'^{(j)} = ({}_tp_x^{(\tau)})^{K_j} \qquad j = 1, 2.$$

[Remark: Note that $K_j = f_J(j)$.]

10.24. This exercise is a continuation of Exercise 10.3 and uses the notation of Section 10.5.5. The joint survival function is given by

$$s_{T_1,T_2}(t_1, t_2) = pe^{-u_1t_1-v_1t_2} + (1 - p)e^{-u_2t_1-v_2t_2}$$

$$0 \le t_1, t_2, u_1, u_2, v_1, v_2$$

$$0 < p < 1.$$

Confirm that

$$s_{T_1,T_2}(t, t) \ne s_{T_1,T_2}(t, 0) \, s_{T_1,T_2}(0, t)$$

and

$$-\left.\frac{\partial \log s_{T_1,T_2}(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=t} \neq -\frac{d \log s_{T_1,T_2}(t, 0)}{dt}.$$

Section 10.6

10.25. Redo Exercise 10.10 by use of the formula for $q_x'^{(j)}$ in Exercise 10.19.

10.26. Show that $\mu_x^{(j)}(1/2) = m_x^{(j)}$, under the assumption of a uniform distribution of each decrement in each year of age in a multiple decrement context.

10.27. How would you proceed to construct the multiple decrement table if the given rates were those given below?
a. $q_x'^{(1)}, q_x'^{(2)}, q_x^{(3)}$
b. $q_x'^{(1)}, q_x^{(2)}, q_x^{(3)}$

10.28. In Example 10.6.2 suppose that decrement 3 at age 69 is not uniformly distributed but follows the pattern

$$_tp_{69}'^{(3)} = \begin{cases} 1 - 0.12t & 0 < t < 1 \\ 0 & t = 1. \end{cases}$$

In words, the cause 3 absolute rate is 0.12 during the year. Then, just before age 70, all remaining survivors terminate due to cause 3. This is consistent with an assumption that $q_{69}'^{(3)} = 1$. What then is the value of $q_{69}^{(3)}$?

10.29. In a double decrement table where cause 1 is death and cause 2 is withdrawal, it is assumed that
- Deaths in the year from age h to $h + 1$ are uniformly distributed,
- Withdrawals in the year from age h to age $h + 1$ occur immediately after the attainment of age h.

From this table it is noted that, at age 50, $l_{50}^{(\tau)} = 1{,}000$, $q_{50}^{(2)} = 0.2$, and $d_{50}^{(1)} = 0.06\ d_{50}^{(2)}$. Determine $q_{50}'^{(1)}$.

Miscellaneous

10.30. On the basis of a triple decrement table, display an expression for the probability that (20) will not terminate before age 65 for cause 2.

10.31. a. You are given $q_x'^{(1)}, q_x'^{(2)}, m_x^{(3)}, m_x^{(4)}$. How would you proceed to construct a multiple decrement table where active service of an employee group is subject to decrement from death, 1, withdrawal, 2, disability, 3, and retirement, 4?
b. On the basis of the table in (a), give an expression for the probability that, in the future, an active member age y will not retire but will terminate from service for some other cause.

10.32. Prove and interpret the relation

$$q_x^{(j)} = q_x'^{(j)} - \sum_{k\neq j} \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(k)}(t)\, {}_{1-t}q_{x+t}'^{(j)}\, dt.$$

10.33. Let

$$w^{(\tau)}(t) = \frac{{}_tp_x^{(\tau)}}{\int_0^1 {}_tp_x^{(\tau)}\, dt}$$

and

$$w^{(j)}(t) = \frac{{}_tp_x'^{(j)}}{\int_0^1 {}_tp_x'^{(j)}\, dt} \qquad 0 \le t \le 1.$$

Assume that j and at least one other cause have positive forces of decrement on the interval $0 \le t \le 1$.

a. Show that
   (i) $w^{(\tau)}(0) > w^{(j)}(0)$
   (ii) $w^{(\tau)}(1) < w^{(j)}(1)$
   (iii) There exists a unique number r, $0 < r < 1$, such that $w^{(\tau)}(r) = w^{(j)}(r)$.

b. Let

$$-I = \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

Show that

$$I = \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

c. Assume that $\mu_x^{(j)}(t)$ is an increasing function on the interval $0 \le t \le 1$. Use the mean value theorem for integrals to establish the following inequalities:

$$\begin{aligned} m_x'^{(j)} - m_x^{(j)} &= \int_0^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\, dt \\ &= \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\, dt \\ &\quad + \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\, dt \\ &= -\mu_x^{(j)}(t_0)\, I + \mu_x^{(j)}(t_1)\, I \qquad 0 < t_0 < r < t_1 < 1 \\ &= I\, [\mu_x^{(j)}(t_1) - \mu_x^{(j)}(t_0)] > 0. \end{aligned}$$

10.34. The joint distribution of T and J is specified by

$$\left.\begin{aligned} \mu_x^{(1)}(t) &= \frac{\theta t^{\alpha-1}\, e^{-\beta t}}{\displaystyle\int_t^{\infty} s^{\alpha-1}\, e^{-\beta s} ds} \\ \mu_x^{(2)}(t) &= \frac{(1-\theta) t^{\alpha-1}\, e^{-\beta t}}{\displaystyle\int_t^{\infty} s^{\alpha-1}\, e^{-\beta s}\, ds} \end{aligned}\right\} \begin{array}{c} 0 < \theta < 1 \\ \alpha > 0 \\ \beta > 0 \\ t \geq 0. \end{array}$$

a. Obtain expressions for $f_{T,J}(t, j)$, $f_J(j)$, and $f_T(t)$.
b. Express E[T] and Var(T) in terms of α and β.
c. Confirm that J and T are independent.

11

APPLICATIONS OF MULTIPLE DECREMENT THEORY

11.1 Introduction

The multiple decrement model developed in Chapter 10 provides a framework for studying many financial security systems. For example, life insurance policies frequently provide for special benefits if death occurs by accidental means or if the insured becomes disabled. The single decrement model, the subject of Chapters 3 through 9, does not provide a mathematical model for policies with such multiple benefits. In addition, there may be nonforfeiture benefits that are paid when the insured withdraws from the set of premium-paying policyholders. The determination of the amount of these nonforfeiture benefits, and related public policy issues, are discussed in Chapter 16. The basic models associated with these multiple benefits are developed in this chapter.

Another major application of multiple decrement models is in pension plans. In this chapter we consider basic methods used in calculating the actuarial present values of benefits and contributions for a participant in a pension plan. The participants of a plan may be a group of employees of a single employer, or they may be the employees of a group of employers engaged in similar activities. A plan, upon a participant's retirement, typically provides pensions for age and service or for disability. In case of withdrawal from employment, there can be a return of accumulated participant contributions or a deferred pension. For death occurring before the other contingencies, there can be a lump sum or income payable to a beneficiary. Payments to meet the costs of the benefits are referred to as contributions, not premiums as for insurance, and are payable in various proportions by the participants and the plan sponsor.

A pension plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) and certain ancillary benefits with a temporary annuity of contributions during active service. The balancing of the actuarial present values of benefits and contributions may be on an individual basis, but frequently it is on some aggregate basis for the whole group of participants. Methods to accomplish this balance comprise the theory of pension funding. Here we are concerned with only the separate valuation of the pension plan's actuarial present value of benefits and contributions with respect to a typical participant. Aggregate values can then be obtained by summation over all the participants. The basic tools for valuing the benefits of, and the contributions to, a pension plan are presented here, but their application to the possible funding methods for a plan is deferred to Chapter 20.

In Section 11.6 we study disability benefits commonly found in conjunction with individual life insurance. The benefits include those for waiver of premium and for disability income. There is a discussion of a widely used single decrement approximation for calculating benefit premiums and benefit reserves for these disability coverages.

11.2 Actuarial Present Values and Their Numerical Evaluation

Actuarial applications of multiple decrement models arise when the amount of benefit payment depends on the mode of exit from the group of active lives. We let $B_{x+t}^{(j)}$ denote the value of a benefit at age $x + t$ incurred by a decrement at that age by cause j. Then the actuarial present value of the benefits, denoted in general by $\bar{A}$, will be given by

$$\bar{A} = \sum_{j=1}^{m} \int_0^{\infty} B_{x+t}^{(j)}\, v^t\, {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt. \tag{11.2.1}$$

If $m = 1$ and $B_{x+t}^{(1)} = 1$, $\bar{A}$ reduces to $\bar{A}_x$, the actuarial present value for a unit of whole life insurance with immediate payment of claims.

More appropriate for this chapter is the example of a ***double indemnity provision***, which provides for the death benefit to be doubled when death is caused by accidental means. Let $J = 1$ for death by accidental means and $J = 2$ for death by other means, and take $B_{x+t}^{(1)} = 2$ and $B_{x+t}^{(2)} = 1$. The actuarial present value for an n-year term insurance is given by

$$\bar{A} = 2 \int_0^n v^t\, {}_tp_x^{(\tau)} \mu_x^{(1)}(t)\, dt + \int_0^n v^t\, {}_tp_x^{(\tau)}\, \mu_x^{(2)}(t)\, dt. \tag{11.2.2}$$

For numerical evaluation, the first step is to break the expression into a set of integrals, one for each of the years involved. For the first integral,

$$2 \int_0^n v^t\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt = 2 \sum_{k=0}^{n-1} v^k\, {}_kp_x^{(\tau)} \int_0^1 v^s\, {}_sp_{x+k}^{(\tau)}\, \mu_x^{(1)}(k + s)\, ds.$$

If now we assume, as for (10.5.11), that each decrement in the multiple decrement context has a uniform distribution in each year of age, we have

$$2\int_0^n v^t\,{}_tp_x^{(\tau)}\,\mu_x^{(1)}(t)\,dt = 2\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\int_0^1 (1+i)^{1-s}\,ds$$
$$= \frac{2i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}.$$

Applying a similar argument for the second integral and combining, we get

$$\begin{aligned}\bar{A} &= \frac{i}{\delta}\left[\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)(}\,(2q_{x+k}^{(1)} + q_{x+k}^{(2)})\right]\\ &= \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,(q_{x+k}^{(1)} + q_{x+k}^{(\tau)})\\ &= \bar{A}_{x:\overline{n|}}^{1\,(1)} + \bar{A}_{x:\overline{n|}}^{1},\end{aligned} \tag{11.2.3}$$

where $\bar{A}_{x:\overline{n|}}^{1\,(1)}$ is the actuarial present value of term insurance benefits of 1 covering death from accidental means and $\bar{A}_{x:\overline{n|}}^{1}$ is the actuarial present value for term insurance benefits of 1 covering death from all causes. Here ${}_kp_x^{(\tau)}$ could be taken as the survival function from a mortality table. If values of $q_{x+k}^{(1)}$ are available, it would be unnecessary to develop the full double decrement table in order to calculate (11.2.3) under the assumption that each decrement has a uniform distribution in each year of age.

This example is simple because the benefit amount does not change as a function of age at decrement, and, in particular, it does not change within a year of age. For a contrasting example, we take $B_{x+t}^{(1)} = t$ and $B_{x+t}^{(2)} = 0$ for $t > 0$. In this case,

$$\bar{A} = \int_0^\infty t\,v^t\,{}_tp_x^{(\tau)}\,\mu_x^{(1)}(t)\,dt = \sum_{k=0}^{\infty} v^k\,{}_kp_x^{(\tau)}\int_0^1 (k+s)v^s\,{}_sp_{x+k}^{(\tau)}\,\mu_x^{(1)}(k+s)\,ds.$$

We again make the assumption that each decrement in the multiple decrement context has a uniform distribution in each year of age, and we obtain

$$\begin{aligned}\bar{A} &= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\int_0^1 (k+s)(1+i)^{1-s}\,ds\\ &= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\,\frac{i}{\delta}\left(k + \frac{1}{\delta} - \frac{1}{i}\right).\end{aligned} \tag{11.2.4}$$

In practice, $B_{x+t}^{(j)}$ is often a complicated function, possibly requiring some degree of approximation. For such a case, if we apply the uniform distribution assumption to the j-th integral in (11.2.1), we obtain

$$\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(j)}\int_0^1 B_{x+k+s}^{(j)}\,(1+i)^{1-s}\,ds.$$

Then, use of the midpoint integration rule yields

$$\sum_{k=0}^{\infty} v^{k+1/2}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(j)}\,B_{x+k+1/2}^{(j)} \tag{11.2.5}$$

as a practical formula for the evaluation of the integral.

As an example, we return to (11.2.4) where the quantity

$$k + \frac{1}{\delta} - \frac{1}{i}$$

can be viewed as an effective mean benefit amount for the year $k + 1$, and the familiar i/δ term can be viewed as the correction needed to provide immediate payment of claims. The value given by (11.2.4) is closely approximated by

$$\sum_{k=0}^{\infty} v^{k+1/2} \, {}_kp_x^{(\tau)} \, q_{x+k}^{(j)} \left(k + \frac{1}{2}\right), \tag{11.2.6}$$

which makes use of the midpoint rule for approximate integration to evaluate

$$\int_0^1 (k + s)(1 + i)^{1-s} \, ds.$$

In Section 10.6 we discussed situations where a uniform distribution of decrement assumption was not appropriate. For such situations, special adjustments to the actuarial present value should be made. We reexamine Example 10.6.3 where, in the associated single decrement model for decrement (3), one-half the expected withdrawals occur at midyear and the other half occur at year end. The actuarial present value for withdrawal benefits is given by

$$\begin{aligned} \bar{A} = \sum_{k=0}^{\infty} v^k \, {}_kp_x^{(\tau)} \Bigg[& \frac{1}{2} q_{x+k}^{\prime(3)} \, v^{1/2} \, B_{x+k+1/2}^{(3)} \left(1 - \frac{1}{2} q_{x+k}^{\prime(1)}\right)\left(1 - \frac{1}{2} q_{x+k}^{\prime(2)}\right) \\ & + \frac{1}{2} q_{x+k}^{\prime(3)} \, v \, B_{x+k+1}^{(3)} \, (1 - q_{x+k}^{\prime(1)})(1 - q_{x+k}^{\prime(2)}) \Bigg]. \end{aligned}$$

Here we are dealing with the distribution of decrement in the context of the associated single decrement tables, rather than in the multiple decrement context. A possible approximation would be to use a geometric average value of the interest factor in the year of withdrawal, such as $v^{3/4}$, and the arithmetic average value of the withdrawal benefit, such as

$$\hat{B}_{x+k}^{(3)} = \frac{1}{2}\left(B_{x+k+1/2}^{(3)} + B_{x+k+1}^{(3)}\right).$$

Thus,

$$\begin{aligned} \bar{A} \cong \sum_{k=0}^{\infty} v^{k+3/4} \, {}_kp_x^{(\tau)} \, \hat{B}_{x+k}^{(3)} \Bigg[& \frac{1}{2} q_{x+k}^{\prime(3)} \left(1 - \frac{1}{2} q_{x+k}^{\prime(1)}\right)\left(1 - \frac{1}{2} q_{x+k}^{\prime(2)}\right) \\ & + \frac{1}{2} q_{x+k}^{\prime(3)} \, (1 - q_{x+k}^{\prime(1)})(1 - q_{x+k}^{\prime(2)}) \Bigg] \\ = \sum_{k=0}^{\infty} v^{k+3/4} \, {}_kp_x^{(\tau)} \, & q_{x+k}^{(3)} \, \hat{B}_{x+k}^{(3)}. \end{aligned}$$

Remark:

In this section we have not used the format employed in Chapter 6 to state premium determination problems. This was done to achieve brevity. The premium problems of this section could have been approached by formulating a loss function and invoking the equivalence principle or some other premium principle.

Assume, for example, an insurance to (x) paying

a. $2B$ upon death due to an accident before age r

b. B upon death due to all other causes before age r, and

c. B upon death after age r.

Two causes of decrement are recognized, $J = 1$, the accidental cause, and $J = 2$, the nonaccidental cause. The loss function is

$$L = \begin{cases} 2Bv^T - \pi & J = 1 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 2 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 1, 2 \quad T > r - x. \end{cases}$$

The equivalence principle requires that $\mathrm{E}[L] = 0$, or

$$\pi = B\left[\int_0^{r-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt + \int_0^{\infty} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t) \, dt\right].$$

A measure of the dispersion due to the random natures of time and cause of death is provided by $\mathrm{Var}(L) = \mathrm{E}[L^2]$. One can verify that, for this case,

$$\mathrm{Var}(L) = B^2\left[3\int_0^{r-x} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt + \int_0^{\infty} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t) \, dt\right] - \pi^2.$$

In the general case, with actuarial present value given by (11.2.1), we have

$$\mathrm{Var}(L) = \mathrm{E}[L^2] = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t - \bar{A})^2 \, {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt,$$

which can be reduced to

$$\mathrm{Var}(L) = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t)^2 \, {}_tp_x^{(\tau)} \, \mu_{x(t)}^{(j)} \, dt - (\bar{A})^2. \tag{11.2.7}$$

11.3 Benefit Premiums and Reserves

We examine, in this section, a method of paying for benefits included in a life insurance policy in a multiple decrement setting. Often these extra benefits are included in life insurance contracts on a policy rider basis; that is, a specified extra premium is charged for the extra benefit, and a separate reserve is held for this benefit. The extra premium is payable only for as long as the benefit has value. In the case of double indemnity it is common to pay the extra amount only for death from accidental means before a specific age, such as 65, and thus the specified extra premiums would be payable only until that age.

We henceforth consider the double indemnity benefit as such a benefit. The model is not complete because the possibility of withdrawal, with a corresponding withdrawal benefit, is not included. We consider withdrawal benefits in Section 11.4, but most of this subject is discussed in Chapters 15 and 16.

Consider the fully discrete model for a whole life policy to a person age 30 with a double indemnity rider. The benefit amount is one for nonaccidental death, decrement $J = 1$, and is two for death by accidental means, decrement $J = 2$. The principle of equivalence is now applied twice, once for the premium payable for life for the policy without the rider and once for the premium payable to age 65 for the extra benefit payable on accidental death before age 65.

For the policy without the rider the benefit level is one under either decrement 1 or 2 and the premium is payable for life. Thus

$$P_{30}^{(\tau)} = \frac{\sum_{k=0}^{\infty} v^{k+1} \, {}_kp_{30}^{(\tau)} \, q_{30+k}^{(\tau)}}{\sum_{k=0}^{\infty} v^k \, {}_kp_{30}^{(\tau)}}. \tag{11.3.1}$$

The benefit premium for the rider reflects that the premium is payable through age 64, and its benefit amount, payable under decrement 2 only, is unity. It is given by

$${}_{35}P_{30}^{(2)} = \frac{\sum_{k=0}^{34} v^{k+1} \, {}_kp_{30}^{(\tau)} \, q_{30+k}^{(2)}}{\sum_{k=0}^{34} v^k \, {}_kp_{30}^{(\tau)}}. \tag{11.3.2}$$

We now display the benefit reserve for the policy with the rider for years prior to attaining age 65:

$$\begin{aligned} {}_kV &= \sum_{h=0}^{\infty} v^{h+1} \, {}_hp_{30+k}^{(\tau)} \, q_{30+k+h}^{(\tau)} + \sum_{h=0}^{34-k} v^{h+1} \, {}_hp_{30+k}^{(\tau)} \, q_{30+k+h}^{(2)} \\ &\quad - \left(P_{30}^{(\tau)} \sum_{h=0}^{\infty} v^h \, {}_hp_{30+k}^{(\tau)} + {}_{35}P_{30}^{(2)} \sum_{h=0}^{34-k} v^h \, {}_hp_{30+k}^{(\tau)} \right). \end{aligned}$$

This reserve is the sum of a reserve on the base policy plus a reserve on a policy that pays only on failure through decrement 2. The reserve on the base policy is a Chapter 7 benefit reserve for a fully discrete whole life insurance with $q_x = q_x^{(\tau)} = q_x^{(1)} + p_x^{(2)}$.

11.4 Withdrawal Benefit Patterns That Can Be Ignored in Evaluating Premiums and Reserves

A single decrement model for an individual life insurance benefit with annual premiums and reserves was built in Chapters 6 through 8. In that model the timing and, perhaps, the amount of benefit payments are determined by the time of death of the insured, and premiums are paid until death or the end of the premium period as specified in the policy. In practice, there is no way to prevent the cessation of premium payments by the policyholder before death or the end of the premium period. In this situation an issue arises about how to reconcile the interests of the parties to the policy for which a model derived from multiple decrement theory is appropriate. Public policy considerations that should guide the reconciliation of the interests of the insurance system and the terminating insured have been subject to discussion since the early days of insurance.

Before premiums and reserves can be determined, a guiding principle must be adopted. A guiding principle is required as well in the determination of ***nonforfeiture benefits,*** those benefits that will not be lost because of the premature cessation of premium payments. In this section we adopt a simple operational principle, one that is, in effect, close to that adopted in U.S. insurance regulation. The principle is that the withdrawing insured receives a value such that the benefit, premium, and reserve structure, built using the single decrement model, remains appropriate in the multiple decrement context.

This principle is motivated by a particular concept of equity about the treatment of the two classes of policyholders, those who terminate before the basic insurance contract is fulfilled and those who continue. Clearly several concepts of what constitutes equity are possible, ranging from the view that terminating policyholders have not fulfilled the contract, and are therefore not entitled to nonforfeiture benefits, to the view that a terminating policyholder should be returned to his original position by the return of the accumulated value of all premiums, perhaps less an insurance charge. The concept of equity, which is the foundation of the principle adopted in the United States, is an intermediate one; that is, withdrawing life insurance policyholders are entitled to nonforfeiture benefits, but these benefits should not force a change in the price-benefit structure for continuing policyholders.

To illustrate some of the implications of this principle, we will develop a model for a whole life policy on a fully continuous payment basis with death and withdrawal benefits. The force of withdrawal is denoted by $\mu_x^{(2)}(t)$ with $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t)$. For multiple decrement models, it is required that

$$\int_0^\infty \mu_{x+t}^{(\tau)}\, dt = \infty$$

so that

$$\lim_{t\to\infty} {}_tp_x^{(\tau)} = 0,$$

but it is not necessary for $\mu_x^{(2)}(t)$ and the derived ${}_tp_x'^{(2)}$ to have these properties.

We assume that the introduction of withdrawals into the model does not change the force of mortality, which is labeled for this development $\mu_x^{(1)}(t)$ in both the single and double decrement models. In other words, time-until-death and time-until-withdrawal will be assumed to be independent, but this assumption may not be realized in practice. This issue was discussed in Chapter 10.

We start our model by specializing (8.6.4) to the case of a whole life insurance and single decrement premiums and reserves:

$$\frac{d}{dt}\,{}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)\,[1 - {}_t\bar{V}(\bar{A}_x)]. \tag{11.4.1}$$

Recalling from Section 10.2 that

$$\frac{d}{dt}\,{}_tp_x^{(\tau)} = -{}_tp_x^{(\tau)}[\mu_x^{(1)}(t) + \mu_x^{(2)}(t)],$$

we can express the following derivative as

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)] &= v^t\,{}_tp_x^{(\tau)}\{\bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)]\}\\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)]\\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x) - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)].\end{aligned} \tag{11.4.2}$$

The progress of the reserves for a whole life insurance that includes withdrawal benefit ${}_tV(A_x)$ using premiums and reserves derived from a double decrement model is analogous to (11.4.1) and is shown in (11.4.3). In this expression, the superscript $\underline{2}$ denotes premiums and reserves based on the double decrement model:

$$\begin{aligned}\frac{d}{dt}\,[{}_t\bar{V}(\bar{A}_x)^{\underline{2}}] &= \bar{P}(\bar{A}_x)^{\underline{2}} + \delta\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}}\\ &\quad - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}] - \mu_x^{(2)}(t)[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}].\end{aligned} \tag{11.4.3}$$

The last term in (11.4.3) is the net cost of withdrawal when the reserve ${}_tV(A_x)^{\underline{2}}$ is treated as a savings fund available to offset benefits [see (8.4.5)]. Thus,

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}}] &= v^t\,{}_tp_x^{(\tau)}\,\{\bar{P}(\bar{A}_x)^{\underline{2}} + \delta\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}]\\ &\quad - \mu_{x+t}^{(2)}[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}]\}\\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}}\,[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)]\\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x)^{\underline{2}} - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)].\end{aligned} \tag{11.4.4}$$

Combining (11.4.2) and (11.4.4), we obtain

$$\frac{d}{dt}\,\{v^t\,{}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - {}_t\bar{V}(\bar{A}_x)]\} = v^t\,{}_tp_x^{(\tau)}[\bar{P}(A_x)^{\underline{2}} - \bar{P}(\bar{A}_x)]. \tag{11.4.5}$$

We now integrate (11.4.5) from $t = 0$ to $t = \infty$ to obtain

$$0 = \bar{a}_x^{(\tau)}[\bar{P}(\bar{A}_x)^{\underline{2}} - \bar{P}(\bar{A}_x)], \tag{11.4.6}$$

which implies that

$$\bar{P}(\bar{A}_x)^{\underline{2}} = \bar{P}(\bar{A}_x).$$

Thus (11.4.5) reduces to

$$\frac{d}{dt}\{v^t\ {}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - {}_t\bar{V}(\bar{A}_x)]\} = 0,$$

which, with the initial condition that

$${}_0\bar{V}(\bar{A}_x)^{\underline{2}} = {}_0\bar{V}(\bar{A}_x),$$

implies that

$${}_t\bar{V}(\bar{A}_x)^{\underline{2}} = {}_t\bar{V}(\bar{A}_x) \qquad \text{for all } t \ge 0. \tag{11.4.7}$$

Therefore, if the withdrawal benefit in a double decrement model whole life insurance, fully continuous payment basis, is equal to the reserve under the single decrement model, the premium and reserves under the double decrement model are equal to the premium and reserves under the single decrement model. This result is not directly applied to the practical problem of defining nonforfeiture benefits. However, it does suggest the basic idea of how to minimize the impact of withdrawal or nonforfeiture benefits on premiums and reserves (determined under a single decrement model). These ideas are developed further in Chapter 16.

The ideas of this section are closely related to Example 6.6.2, where it was demonstrated that if the death benefit during the premium-paying period for a deferred life annuity is the accumulated value of the premiums, then the premium does not depend on the mortality assumption during the deferral period. This idea is elaborated in Example 11.4.1.

Example 11.4.1

A continuously paid life annuity issued on (x) provides an income benefit commencing at age $x + n$ at an annual rate of 1. The benefit for death (decrement $J = 1$) or withdrawal (decrement $J = 2$) during the n-year deferral period, paid at the moment of death, will be the accumulated benefit premiums with interest at the rate used in the premium calculation. Premiums are paid continuously from age x to $x + n$ or to the age of decrement, if less than $x + n$. Assume that there are no withdrawals after the commencement of the annuity payments.

a. Formulate a loss variable.
b. Determine the annual benefit premium rate π using the principle of equivalence.
c. Determine the benefit reserve at time t, $0 \le t \le n$.

Solution:

a.
$$L = \begin{cases} \pi\, v^T\, \bar{s}_{\overline{T|}} - \pi\, \bar{a}_{\overline{T|}} & 0 \le T \le n, \quad J = 1, 2 \\ v^n\, \bar{a}_{\overline{T-n|}} - \pi\, \bar{a}_{\overline{n|}} & T > n, \qquad\quad J = 1. \end{cases}$$

b. Applying the principle of equivalence, we obtain

$$\mathrm{E}[L] = \int_n^\infty (v^n\, \bar{a}_{\overline{t-n|}} - \pi\, \bar{a}_{\overline{n|}})\ {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt.$$

This yields

$$v^n\ {}_np_x^{(\tau)}\ \bar{a}_{x+n} = \pi\ \bar{a}_{\overline{n}|}\ {}_np_x^{(\tau)} \quad \text{and} \quad \pi = \frac{v^n\ \bar{a}_{x+n}}{\bar{a}_{\overline{n}|}} = \frac{\bar{a}_{x+n}}{\bar{s}_{\overline{n}|}}.$$

c. The reserve at time t, $t \le n$, viewed prospectively, is given by

$$\int_0^{n-t} (\pi\ v^s\ \bar{s}_{\overline{t+s}|} - \pi\ \bar{a}_{\overline{s}|})\ {}_sp_{x+t}^{(\tau)}\ \mu_x^{(\tau)}(t+s)\ ds$$

$$+ \int_{n-t}^{\infty} (v^{n-t}\ \bar{a}_{\overline{s-(n-t)}|} - \pi\ \bar{a}_{\overline{n-t}|})\ {}_sp_{x+t}^{(\tau)}\ \mu_x^{(1)}(t+s)\ ds$$

$$= \pi\ \bar{s}_{\overline{t}|}(1 - {}_{n-t}p_{x+t}^{(\tau)}) + {}_{n-t|}\bar{a}_{x+t} - \pi\ \bar{a}_{\overline{n-t}|}\ {}_{n-t}p_{x+t}^{(\tau)}$$

$$= \pi\ \bar{s}_{\overline{t}|}\ .$$

The simplification of the last term comes from the definition of π in part (b). The benefit premium and reserve during the deferred period can be viewed as derived from a zero decrement model. ▼

11.5 Valuation of Pension Plans

Two sets of assumptions are needed to determine the actuarial present values of pension plan benefits and of contributions to support these benefits. These sets can be identified as demographic (the service table and survival functions for retired lives, disabled lives, and perhaps lives who have withdrawn) and economic (investment return and salary scale) assumptions.

11.5.1 Demographic Assumptions

A starting point for the valuation of pension plan benefits is a multiple decrement (service) table constructed to represent a survivorship group of participants subject, in the various years of active service, to given probabilities of

- Withdrawal from service
- Death in service
- Retirement for disability, and
- Retirement for age-service.

The notations for these probabilities for the year of age x to $x+1$ are $q_x^{(w)}$, $q_x^{(d)}$, $q_x^{(i)}$, and $q_x^{(r)}$, respectively. These are consistent with the notations developed in Chapter 10. Also, we use the survivorship function $l_x^{(\tau)}$ from Chapter 10, which satisfies

$$l_{x+1}^{(\tau)} = l_x^{(\tau)}[1 - (q_x^{(w)} + q_x^{(d)} + q_x^{(i)} + q_x^{(r)})] = l_x^{(\tau)}\ p_x^{(\tau)}.$$

This function can be used to evaluate such expression as ${}_kp_x^{(\tau)}$, thus,

$${}_kp_x^{(\tau)} = \frac{l_{x+k}^{(\tau)}}{l_x^{(\tau)}}.$$

One can also proceed by direct recursion, namely,

$$_kp_x^{(\tau)} = {}_{k-1}p_x^{(\tau)}\, p_{x+k-1}^{(\tau)}.$$

The forces of decrement related to a service table will be continuous at most ages. They will be denoted by $\mu_x^{(w)}(t)$, $\mu_x^{(d)}(t)$, $\mu_x^{(i)}(t)$, and $\mu_x^{(r)}(t)$. At some ages, discontinuities may occur. This occurs most frequently at age α, the first eligible age for retirement. We generally assume that decrements are spread across each year of age.

In the early years of service, withdrawal rates tend to be high, and the benefit for withdrawal may be only the participant's contributions, if any, possibly accumulated with interest. After a period of time, for example, 5 years, withdrawal rates will be somewhat lower, and the withdrawing participant may be eligible for a deferred pension. If these conditions hold, it may be necessary to use select rates of withdrawal for an appropriate number of years. Conditions for disability retirement may also indicate a need for a select basis. The mathematical modifications to a select basis are relatively easy to make, and the theory is more adaptable if select functions are used. In this chapter we denote the age of entry by x, but we do not otherwise indicate whether an aggregate table, select table, or select-and-ultimate table is intended.

The Illustrative Service Table in Appendix 2B illustrates a service table for entry age 30, earliest age for retirement $\alpha = 60$, and with no probability of active service beyond age 71. Here $l_{71}^{(\tau)} = 0$.

As noted earlier, the principal benefits under a pension plan are annuities to eligible beneficiaries. For the valuation of such annuity benefits, it is necessary to adopt appropriate mortality tables that will differ if retirement is for disability, for age-service, or perhaps withdrawal. The corresponding annuity values will be indicated by post-fixed superscripts. The continuous annuity value is used as a convenient means of approximating the actual form of pension payment that usually is monthly, but may have particular conditions as to initial and final payments.

11.5.2 Projecting Benefit Payment and Contribution Rates

A common form of pension plan is one that defines the rate of retirement income by formula. These plans are called ***defined benefit plans.*** Some pension plans define benefit income rates as a function of the level of compensation at or near retirement. In these cases, it is necessary to estimate future salaries to value the benefits. Sponsor contributions are also often expressed as a percentage of salary, so here too estimation of future salaries is important. To accomplish these estimations, we define the following salary functions:

$(AS)_{x+h}$ is the actual annual salary rate at age $x + h$, for a participant who entered at age x and is now at attained age $x + h$,

$(ES)_{x+h+t}$ is the projected (estimated) annual salary rate at age $x + h + t$.

Further, we assume that we have a salary scale function S_y to use for these projections, such that

$$(ES)_{x+h+t} = (AS)_{x+h} \frac{S_{x+h+t}}{S_{x+h}}. \tag{11.5.1}$$

The salary functions S_y may reflect merit and seniority increases in salary as well as those caused by inflation. For example, in the Illustrative Service Table, $S_y = (1.06)^{y-30} s_y$, where the s_y factor represents the progression of salary due to individual merit and experience increases, and the 6% accumulation factor is to allow for long-term effects of inflation and of increases in productivity of all members of the plan. As was the case of the $l_x^{(\tau)}$ function, one of the values of S_y can be chosen arbitrarily. For instance, in the Illustrative Service Table, S_{30} is taken as unity. The S_y function is usually assumed to be a step function, with constant level throughout any given year of age.

We now move to the problem of estimating the benefit level for a pension plan. For this purpose, we introduce the function $R(x, h, t)$ to denote the projected annual income benefit rate to commence at age $x + h + t$ for a participant, who entered h years ago at age x. Both x and h are assumed to be integers. We assume that the income benefit rate remains level during payout so that when we come to expressing the actuarial present value of the benefit at time of retirement, it will simply be $R(x, h, t)\bar{a}^r_{x+h+t}$. As stated in the previous section, the post-fixed superscript r indicates that a mortality table appropriate for retired lives should be used.

We now consider several common types of income benefit rate functions $R(x, h, t)$. The estimation procedure falls into two groups. First, there are functions that do not depend on salary levels. For other types of benefit formulas, which depend on future salaries, the projected annual income rate must be estimated. There are those that depend on either the final salary rate or on an average salary rate over the last several years prior to retirement. There are also formulas that depend on the average salary over the career with the plan sponsor. The following are examples of the more common types of benefit formula together with their estimation.

a. Consider an income benefit rate that is a fraction d of the final salary rate. Thus $R(x, h, t) = d(ES)_{x+h+t}$. Here we estimate the final salary from the current salary at age $x + h$ by $(ES)_{x+h+t} = (AS)_{x+h}(S_{x+h+t}/S_{x+h})$ so that $R(x, h, t) = d(AS)_{x+h}(S_{x+h+t}/S_{x+h})$.
b. A ***final m-year average salary benefit*** rate is a fraction d of the average salary rate over the last m years prior to retirement. We illustrate this in the common case where $m = 5$. In this case, if $t > 5$, an estimate of the average salary over the last 5 years is given as

$$(AS)_{x+h} \frac{0.5\ S_{x+h+k-5} + S_{x+h+k-4} + S_{x+h+k-3} + S_{x+h+k-2} + S_{x+h+k-1} + 0.5\ S_{x+h+k}}{5\ S_{x+h}}$$

where k is the greatest integer in t.

The thinking behind this expression is that if retirement occurs at midyear, the current year's salary is earned only for the last half year of service. A notation in common usage for the above average is ${}_5Z_{x+h+k}/S_{x+h}$. If the participant is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.

The above formulas do not reflect the amount of service of the participant at retirement. We now look at three formulas where the benefits are proportional to the number of years of service at retirement.

c. Consider an income benefit that is d times the total number of years of service, including any fraction in the final year of employment. In this case $R(x, h, t) = d(h + t)$. If only whole years of service are to be counted, then $R(x, h, t) = d(h + k)$, where k is the greatest integer in t.
d. Consider an income benefit rate that is the product of a fraction d of the final 5-year average salary and the number of years of service at retirement. A typical formula would be, where d is a designated fraction,

$$= d(h + t)(AS)_{x+h} \frac{{}_5Z_{x+h+k}}{S_{x+h}}.$$

Again, if the participant is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.
e. Consider an income benefit rate that is d times the number of years of service times the average salary over the entire career. Such a benefit formula is called a ***career average benefit.*** This formula is equivalent to a benefit rate of a fraction d of the entire career earnings of the retiree.

The analysis of career average retirement benefits breaks naturally into two parts, one for past service for which the salary information is known and one for future service where salaries must be estimated. Here past salaries enter into the valuation of benefits for all participants and not just for those participants very near to retirement age. If the total of past salaries for a participant at age $x + h$ is denoted by $(TPS)_{x+h}$, the benefit rate attributed to past service is $d(TPS)_{x+h}$. The retirement income benefit rate based on future service is given by

$$d(AS)_{x+h} \frac{S_{x+h} + S_{x+h+1} + \cdots + S_{x+h+k-1} + 0.5\, S_{x+h+k}}{S_{x+h}},$$

where k is the greatest integer in t and retirements are assumed to occur at midyear.

Finally, we display one benefit formula where the service component for participants with a large number of years of service at retirement is modified.

f. Consider an income benefit rate that is the product of the 3-year final average salary and 0.02 times the number of years of service at retirement for the first 30 years of service with an additional 0.01 per year of service above 30 years. Following (d),

$$= 0.02\,(h + t)\,(AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} \qquad h + t \le 30$$

$$= [0.30 + 0.01\,(h + t)]\,(AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} \qquad h + t > 30.$$

11.5.3 Defined-Benefit Plans

We now seek to develop formulas for actuarial present values of such benefits and of the contributions expected to be used to fund the promised benefits. We do so first for a general case of a defined-benefit plan and then examine a specific example that includes a typical pattern of defined benefits.

Let us first look at the evaluation of, and approximation to, the actuarial present value for an age-retirement benefit. Assume that the benefit rate function has been found as $R(x, h, t)$ and that the benefit involves life annuities with no certain period. We can then write an integral expression for the actuarial present value of the retirement benefit as

$$\text{APV} = \int_{\alpha-x-h}^{\infty} v^t \, {}_tp_{x+h}^{(\tau)} \, \mu_x^{(r)}(h + t) \, R(x, h, t) \, \bar{a}_{x+h+t}^r \, dt. \tag{11.5.2}$$

As in Section 11.2, we approximate the integral for practical calculation of the actuarial present value. To do so, we write

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \int_0^1 v^s \, {}_sp_{x+h+k}^{(\tau)} \, \mu_x^{(r)}(h + k + s) \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^r \, ds.$$

By assuming a uniform distribution of retirements in each year of age, we can rewrite this as

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \int_0^1 v^s \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^r \, ds.$$

Using the midpoint approximation for the remaining integrals gives

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^{k+1/2} \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \, R(x, h, k + 1/2) \, \bar{a}_{x+h+k+1/2}^r. \tag{11.5.3}$$

Formula (11.5.3) is the general means by which we calculate the actuarial present value of retirement and, by extension, other benefits of a pension plan.

We now present an example that shows the types of calculations that might be used for the valuation of the several benefits of a hypothetical defined-benefit pension plan.

Example 11.5.1

Find the actuarial present values of the following benefits for a participant who was hired 3 years ago at age 30 and who currently has a salary of $45,000.

a. Retirement income for any participant of at least age 65 or whenever the sum of the attained age and the number of years of service exceeds a total of 90. The benefit is in the form of a 10-year certain and life annuity, payable monthly, at an annual rate of 0.02 times the final 5-year average salary times the total number of years of service, including any final fraction.

b. Retirement income for any participant with at least 5 years of service upon withdrawal. The benefit and income benefit rate formula is as for age retirement. However, the initial payment of the annuity is deferred until the earliest possible date of age retirement had the participant continued in the active status.
c. Retirement income for those disabled participants too young for age retirement. The income benefit rate is the larger of 50% or the percentage based on years of service, uses the average salary over the preceding 5 years, and is for a 10-year certain and life annuity.
d. Lump sum benefit for those participants who die while still in active status. The benefit amount is two times the salary rate at the time of death.

Solution:

The participant was hired at age 30 and so is eligible for retirement at age 60 [60 + (60 − 30) = 90]. Assuming midyear retirements, the income benefit rate is given by

$$= 0.02\ (h + t)\ (AS)_{x+h} \frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\ (3 + k + 0.5)\ (45{,}000) \frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

for retirements starting in the year following age 33 + k. The actuarial present value of age-retirement benefits is approximated by

$$\text{APV} = 900 \sum_{k=27}^{\infty} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(r)}$$

$$\times\ (3.5 + k) \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ \bar{a}^{r}_{\overline{33+k+1/2:\overline{10}|}}.$$

Benefits are paid for those withdrawing from active status between ages 35 and 60. After attaining age 60, the withdrawals are classified as retirements. The income benefit rate is again given by

$$= 0.02\ (h + t)\ (AS)_{x+h} \frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\ (3 + k + 0.5)\ (45{,}000) \frac{{}_5Z_{30+3+k}}{S_{30+3}}.$$

For withdrawals at ages 35 through 37, adjustments using actual wage data rather than the Z function could be made. The actuarial present value of the withdrawal benefits is approximated by

$$\text{APV} = 900 \sum_{k=2}^{26} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(w)}$$

$$\times\ (3.5 + k) \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ {}_{27-k-1/2|}\bar{a}^{w}_{\overline{33+k+1/2:\overline{10}|}}.$$

For the disability benefit the income benefit rate function makes a distinction between disabilities starting before and after the participant has worked for 25 years. Thus the income benefit rate is

$$= 0.5\ (AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.5\ (45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}} \qquad \text{for } 0 \le k \le 21$$

$$= 0.02\ (h + t)\ (AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.02\ (3 + k + 0.5)\ (45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

$$\text{for } 22 \le k \le 26.$$

Thus, the actuarial present value of the disability benefits is approximated by

$$\text{APV} = 22{,}500 \sum_{k=0}^{21} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(i)}\ \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ \bar{a}^{\,i}_{33+k+1/2:\overline{10}|}$$

$$+\ 900 \sum_{k=22}^{26} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(i)}\ \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ (3.5 + k)\ \bar{a}^{\,i}_{33+k+1/2:\overline{10}|}.$$

For the death benefit, the projected lump sum benefit amount is

$$= 2\ (AS)_{x+h}\,\frac{S_{x+h+k}}{S_{x+h}}$$

$$= 2\ (45{,}000)\,\frac{S_{30+3+k}}{S_{30+3}}.$$

Thus, the actuarial present value of the death benefits is approximated by

$$\text{APV} = 90{,}000 \sum_{k=0}^{\infty} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(d)}\ \frac{S_{30+3+k}}{S_{30+3}}. \qquad \blacktriangledown$$

There are many funding or budgeting methods available to assure that contributions are made to the plan in an orderly and appropriate manner. An overview of these methods is presented in Chapter 20.

11.5.4 Defined-Contribution Plans

The principal benefit under a pension plan is normally the deferred annuity for age-service retirement. In ***defined-contribution plans,*** the actuarial present value is simply the accumulation under interest of contributions made by or for the participant, and the benefit is an annuity that can be purchased by such accumulation. The accumulated amount is typically available upon death and, under certain conditions, upon withdrawal before retirement. We examine the interplay between the rate of contribution and the rate of income provided to the participant in the following example. The defined-contribution rate can be determined with a retirement income goal. The risk that the goal will not be achieved is held by the participant of the plan, not the sponsor. Budget constraints on the sponsor may, of course, restrict the amount of the contributions.

Example 11.5.2

Find the contribution level for the sponsor to provide for age-retirement at age 65 that has as its objective a 10-year certain and life annuity with an initial benefit rate of 50% of the average salary over the 5 years between ages 60 and 65. The contribution rate, which is to be applied as a proportion of salary, is calculated for a new participant at age 30. Assume that there are no withdrawal benefits for the first 5 years but after that the contributions accumulated with interest are vested, that is, become the property of the withdrawing participant, and will be applied toward an annuity to start no earlier than age 60. (An active participant who becomes disabled is treated as a withdrawal and is covered by a separate disability income coverage for the period between the date of disability and age 65 at which time a regular age retirement commences.) Upon death after the end of the 5-year vesting period but before retirement income has commenced, the accumulated contributions are paid out.

Solution:

We start by calculating the actuarial present value of a contribution rate of c times the annual salary rate, assumed for convenience to be 1, at age 30:

$$\text{APV} = c\left\{\sum_{k=0}^{34} {}_kp_{30}^{(\tau)}\,(q_{30+k}^{(d)} + q_{30+k}^{(w)})\left[\sum_{j=0}^{k} v^{j+1/2}\,\frac{S_{30+j}}{S_{30}} - \frac{v^{k+1/2}S_{30+k}}{2S_{30}}\right] + {}_{35}p_{30}^{(\tau)}\sum_{k=0}^{34} v^{k+1/2}\,\frac{S_{30+k}}{S_{30}}\right\}. \tag{11.5.4}$$

In (11.5.4) contributions are assumed to occur at midyear, and the projected salary rate at age $30 + k$ is (S_{30+k}/S_{30}) times 1, the initial salary at age 30.

We now estimate the desired benefit payment rate at age 65 as the first step in estimating the actuarial present value of the target benefits. The average salary projected to be earned between the ages of 60 and 65 is $(S_{60} + S_{61} + S_{62} + S_{63} + S_{64})/(5\,S_{30})$. The desired benefit rate is one-half of this, and the actuarial present value of the target benefit is given by

$$\text{APV} = {}_{35}p_{30}^{(\tau)}\,v^{35}\,(0.5)\,\frac{S_{60} + S_{61} + S_{62} + S_{63} + S_{64}}{5\,S_{30}}\,\bar{a}^{r}_{\overline{65:\overline{10|}}}. \tag{11.5.5}$$

This expression is the actuarial present value at age 30 of the new participant's target income benefit.

The actuarial present value of the vested benefit is given by

$$\text{APV} = c\left\{\sum_{k=5}^{34} v^{k+1/2}\,{}_kp_{30}^{(\tau)}\,(q_{30+k}^{(d)} + q_{30+k}^{(w)})\left[\sum_{j=0}^{k}(1+i)^{k-j}\,\frac{S_{30+j}}{S_{30}} - \frac{1}{2}\,\frac{S_{30+k}}{S_{30}}\right]\right\}. \tag{11.5.6}$$

The expression for the actuarial present value of contributions, (11.5.4), less the expression for the actuarial present value of vested benefits, (11.5.6), is

$$c\left\{\sum_{k=0}^{4} {}_kp_{30}^{(\tau)}\,(q_{30+k}^{(d)} + q_{30+k}^{(w)})\left[\sum_{j=0}^{k} v^{j+1/2}\frac{S_{30+j}}{S_{30}} - v^{k+1/2}\frac{S_{30+k}}{2S_{30}}\right]\right.$$
$$\left. + {}_{35}p_{30}^{(\tau)}\sum_{k=0}^{34} v^{k+1/2}\frac{S_{30+k}}{S_{30}}\right\}. \qquad (11.5.7)$$

The benefits in this plan, after the 5 years when vesting occurs, are similar to those discussed in Example 11.4.1.

We now equate the two actuarial present values from (11.5.4) and (11.5.7) to solve for c, the sponsor's contribution rate, which will be applied to all future salary payment in accordance with the plan to achieve the stated retirement income goal. ▼

Some plans of this type have both sponsor and participant contributions. It is common here for some kind of matching between the size of the sponsor contribution and the size of the participant contribution.

11.6 Disability Benefits with Individual Life Insurance

In Section 11.5.3 we discuss disability benefits included in pension plans. We now turn to disability benefits commonly found with individual life insurance. Provision can be made for the waiver of life insurance premiums during periods of disability. Alternatively, policies can contain a provision for a monthly income, sometimes related to the face amount, if disability occurs. The multiple decrement model is appropriate for studying these provisions.

The usual disability clause provides a benefit for total disability. Total disability can require a disability severe enough to prevent engaging in any gainful occupation, or it can require only the inability to engage in one's own occupation. Total disability that has been continuous for a period of time specified in the policy, called the ***waiting or elimination period,*** qualifies the policyholder to receive benefit payments. The waiting period can be 1, 3, 6, or 12 months. In policies with waiver of premium, it is common to make the benefits ***retroactive,*** that is, to refund any premiums paid by the insured during the waiting periods. Coverage is only for disabilities that occur prior to a disability benefit expiry age, typically 60 or 65. However, benefits in the form of an annuity, either as disability income or as waiver or premiums, will often continue to a higher age, typically the maturity date or paid-up date of the life insurance policy.

11.6.1 Disability Income Benefits

Let us start by expressing the actuarial present value of a disability income benefit of 1,000 per month issued to (x) under coverage expiring at age y and with income running to age u. We assume that the waiting period is m months. Using notation from Chapter 10 and earlier sections of Chapter 11, we can express the actuarial present value as a definite integral as

$$\bar{A} = \int_0^{y-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(i)}(t) \, v^{m/12} \, {}_{m/12}p^i_{[x+t]} \, (12{,}000 \, \ddot{a}^{(12)i}_{[x+t]+m/12:\overline{u-x-t-m/12}|}) \, dt. \quad (11.6.1)$$

The i superscript on ${}_{m/12}p^i_{[x+t]}$ indicates a survival probability for a disabled life. We now break up the integral into separate integrals for each year. Upon making the assumption of uniform distribution for the disability decrement within each year of age and replacing t by $k + s$, we obtain an expression for the actuarial present value much like (11.2.5):

$$\bar{A} = 12{,}000 \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \, q^{(i)}_{x+k} \, v^{m/12}$$

$$\times \int_0^1 v^s \, {}_{m/12}p^i_{[x+k+s]} \, \ddot{a}^{(12)i}_{[x+k+s]+m/12:\,\overline{u-x-k-s-m/12}|} \, ds. \quad (11.6.2)$$

A simplification of this formula occurs when the decrement i (disability) is defined to occur only if the person who is disabled survives to the end of the waiting period of m months. If death occurs during the waiting period, the decrement is regarded as death. This means that the disabled life survivorship factor, ${}_{m/12}p^i_{[x+k+s]}$, is unnecessary as it has been taken into account in the definition of $q^{(i)}_{x+k}$. We note that it also means that the attained age at entry into the disabled life state is reached at the completion of the waiting period and is so indicated in the select age of the disabled life annuity function.

By the midpoint method the integrals in (11.6.2) are evaluated as

$$\int_0^1 v^s \, \ddot{a}^{(12)i}_{[x+k+s+m/12]:\overline{u-x-k-s-m/12}|} \, ds = v^{1/2} \, \ddot{a}^{(12)i}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|} \, . \quad (11.6.3)$$

With these two changes (11.6.2) can be written as

$$A = 12{,}000 \sum_{k=0}^{y-x-1} v^{k+1/2} \, {}_kp_x^{(\tau)} q^{(i)}_{x+k} v^{m/12} \ddot{a}^{(12)i}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|} \, . \quad (11.6.4)$$

11.6.2 Waiver-of-Premium Benefits

Let us go through the same process for a waiver of premium benefit. We assume that the premium, P, to be waived is payable g times per year for life. The primary difference between this and the disability income benefit is that the waiver benefit is a g-thly payment annuity starting at the first premium due date after the end of the waiting period.

We start with a special case with its actuarial present value written with the definite integrals already broken down to individual years of time of disablement. The case chosen is the waiver of semiannual premiums, payable for life, in the event of a disability occurring prior to age y and continuing through a 4-month waiting period. We further assume that the benefits are retroactive by which we mean that for a premium paid to the insurer on a due date during the waiting period, reimbursement with interest at the valuation rate will be made. This will increase the number of integrals within each year of age because disabilities that start within the first 2 months of each half year do not have premiums due during the waiting period:

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \left[\int_0^{1/6} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1/2-s|}\ddot{a}^{(2)i}_{[x+k+s]} \, ds \right.$$
$$+ \int_{1/6}^{1/2} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{1-s|}\ddot{a}^{(2)i}_{[x+k+s]} + {}_{4/12}p^i_{[x+k+s]} \, v^{1/2-s} \frac{1}{2} \right) ds$$
$$+ \int_{1/2}^{2/3} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1-s|}\ddot{a}^{(2)i}_{[x+k+s]} \, ds$$
$$\left. + \int_{2/3}^{1} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{3/2-s|}\ddot{a}^{(2)i}_{[x+k+s]} + {}_{4/12}p^i_{[x+k+s]} \, v^{1-s} \frac{1}{2} \right) ds \right]. \quad (11.6.5)$$

If we incorporate the assumption of a uniform distribution of disability within each year of age and include only those disabilities that survive the waiting period as disabilities, (11.6.5) becomes

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} q_{x+k}^{(i)} \left[\int_0^{1/6} v^{s+4/12} \, {}_{1/2-s-4/12|}\ddot{a}^{(2)i}_{[x+k+s+4/12]} \, ds \right.$$
$$+ \int_{1/6}^{1/2} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}^{(2)i}_{[x+k+s+4/12]} + v^{1/2} \frac{1}{2} \, ds$$
$$+ \int_{1/2}^{2/3} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}^{(2)i}_{[x+k+s+4/12]} \, ds$$
$$\left. + \int_{2/3}^{1} v^{s+4/12} \, {}_{3/2-s-4/12|}\ddot{a}^{(2)i}_{[x+k+s+4/12]} + v \frac{1}{2} \, ds \right]. \quad (11.6.6)$$

We now use the midpoint approximate integration method for each of the several integrals within each year of age to obtain

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \, q_{x+k}^{(i)} \left[\frac{1}{6} v^{5/12} \, {}_{1/12|}\ddot{a}^{(2)i}_{[x+k+5/12]} \right.$$
$$+ \frac{1}{3} \left(v^{2/3} \, {}_{1/3|}\ddot{a}^{(2)i}_{[x+k+2/3]} + v^{1/2} \frac{1}{2} \right) + \frac{1}{6} v^{11/12} \, {}_{1/12|}\ddot{a}^{(2)i}_{[x+k+11/12]}$$
$$\left. + \frac{1}{3} \left(v^{7/6} \, {}_{1/3|}\ddot{a}^{(2)i}_{[x+k+7/6]} + v \frac{1}{2} \right) \right]. \quad (11.6.7)$$

11.6.3 Benefit Premiums and Reserves

Equivalence principle benefit premiums for disability income and waiver of premium benefits are found by equating the actuarial present value of benefits to the actuarial present value of premiums. For the waiver benefit discussed in Section 11.6.2, the annual benefit premium, ${}_{y-x}\Pi_x$, equals $\bar{A}$ of (11.6.7) divided by $\ddot{a}^{(\tau)}_{x:\overline{y-x}|}$.

Active life benefit reserves, that is, the reserve when premiums are not being waived, are most conveniently expressed by a premium difference formula:

$$_kV = ({}_{y-x-k}\Pi_{x+k} - {}_{y-x}\Pi_x)\, \ddot{a}^{(\tau)}_{x+k:\overline{y-x-k}|}. \tag{11.6.8}$$

The terminal reserve for a disabled life is the actuarial present value of future disability benefits, calculated on the assumption that the insured has incurred a disability. The amount of premium waived, or disability income rate, is multiplied by the actuarial present value of an appropriate disabled life annuity. This value takes into account the age at disablement, the duration since disablement, and the terminal age for benefits.

11.7 Notes and References

We have not defined insurer's losses and studied their variances in this chapter. Formula (11.2.7) gave a means of doing so if we consider the total benefits for all causes of decrements. If we consider only a single benefit, such as the retirement benefit, there is more than one way of defining losses. The usual concept is that premiums and reserves, for a benefit in regard to a particular cause of decrement, apply only to that decrement. Thus, if decrement due to a second cause occurs, then, with respect to the first cause, there is zero benefit and a gain emerges. An insurer's loss based on this concept would lead, for example, to (11.4.3). However, losses defined in this way may have nonzero covariances, so that the loss variance for all benefits is not the sum of the loss variances for the individual benefits.

Alternatively, one may consider that when a particular cause of decrement occurs, the reserves accumulated for the benefits in regard to all the other causes are released to offset the benefit outgo for the given cause. In this case, the loss random variables defined for the benefits for the several causes of decrement have zero-valued covariances, and the loss variance for all benefits is the sum of the loss variances for the individual benefits. However, the premiums and reserves for the individual benefits are more difficult to compute on this second basis and individually differ significantly from those on the usual basis. For insights into these matters, see Hickman (1964).

The result stated in Section 11.4 concerning the neutral impact on premiums and reserves when a withdrawal benefit equals the reserve on the death benefit does not hold for fully discrete insurances. This was pointed out by Nesbitt (1964), who reported on work by Schuette. The problem results from the fact that, in the discrete model, the probability of withdrawal

$$q^{(w)}_{x+k} = \int_0^1 \exp\left[-\int_0^t \mu^{(\tau)}_x(k+s)\,ds\right] \mu^{(w)}_x(k+t)\,dt$$

depends on the force of mortality.

While there are many papers and a number of books dealing with pension fund mathematics, it seems useful for the purposes of this introductory treatment to refer only to other actuarial texts with similar chapters; see, for example, Hooker and Longley-Cooke (1957), Jordan (1967), and Neill (1977). These authors stress the formulation of actuarial present values in terms of pension commutation functions and the use of tables of such functions to carry out computations.

In contrast, a major portion of our presentation has been in terms of integrals and approximating sums, with the integrands or summands expressed in terms of basic functions. These approximating sums can be computed by various processes that may or may not make use of commutation functions. For pension benefits determined by complex eligibility or income conditions, it can be more flexible and efficient to calculate by processes not requiring extensive formulation by commutation functions. An opposing view, indicating the power of commutation functions for expressing actuarial present values and controlling their computation, is given by Chamberlain (1982).

There is an alternative foundation for constructing a model for disability insurance. In Section 11.6 we used a multiple decrement model that did not explicitly provide for recovery from disability. Models with several states of disability, with provision for transition from state to state, have been developed. These models are frequently the foundation of long-term care insurance. Hoem (1988) provides an introduction with valuable references to these ideas.

The multiple decrement model developed in Chapter 10 and applied in this chapter can be viewed as being made up of $m + 1$ states; m are called absorbing states in that it is impossible to return from them to the active state. These m states are associated with the m causes of decrement, and the remaining state is associated with continuing survival. If some of the m decrements are not absorbing, but are such that transition to the active state or one of the other nonabsorbing states is possible, a more complex but possibly more realistic model results. Estimation of the probabilities of transition among the states can be difficult because the probabilities can depend on the path followed to the current state.

Exercises

Section 11.2

11.1. Employees enter a benefit plan at age 30. If an employee remains in service until retirement, the employee receives an annual pension of 300 times years of service. If the employee dies in service before retirement, the beneficiary is paid 20,000 immediately. If the employee withdraws before age 70 for any reason except death, the member receives a deferred (to age 70) life annuity of 300 times years of service. Give an expression, in terms of integrals and continuous annuities, for the actuarial present value of these benefits at age 30.

11.2. Let $J = 1$ represent death by accidental means and $J = 2$ represent death by other means. You are given that

(i) $\delta = 0.05$
(ii) $\mu_x^{(1)}(t) = 0.005$ for $t \geq 0$ where $\mu_x^{(1)}(t)$ is the force of decrement for death by accidental means.
(iii) $\mu_x^{(2)}(t) = 0.020$ for $t \geq 0$.

A 20-year term insurance policy, payable at the moment of death, is issued to a life age (x) providing a benefit of 2 if death is by accidental means and providing a benefit of 1 for other deaths. Find the expectation and variance of the present value of benefits random variable.

Section 11.4

11.3. A double decrement model is defined by $\mu_x^{(1)}(t) = \mu_x^{(2)}(t) = 1/(a - t)$, $0 \leq t < a$.

a. In the single decrement model with decrement (1) only, the prospective loss variable at duration t is given by

$$ {}_tL^1 = v^{T(x)-t} \qquad 0 < t \leq T(x),\ J = 1. $$

Confirm that

$$ {}_tV^1 = \frac{1 - e^{-\delta(a-t)}}{\delta(a - t)} = \bar{A}^{\,1}_{x+t}. $$

b. In the double decrement model, the prospective loss variable at duration t is given by

$$ \begin{aligned} {}_tL^2 &= v^{T(x)-t} && 0 < t \leq T(x),\ J = 1 \\ &= v^{T(x)-t}\ {}_{T(x)}V^1 && 0 < t \leq T(x),\ J = 2. \end{aligned} $$

Confirm that $\mathrm{E}[{}_tL^2 | T \geq t] = {}_tV^1$.

Section 11.5

11.4. A pension plan valuation assumes a linear salary scale function satisfying $S_{20} = 1$. If $(ES)_{45} = 2(AS)_{25}$, find S_x for $x \geq 20$.

11.5. A new pension plan with two participants, (35) and (40), provides annual income at retirement equal to 2% of salary at the final rate times the number of years of service, including any fraction of a year. If
(i) Salary increases occur continuously,
(ii) For (40), $(AS)_{40} = 50{,}000$ and $S_{40+t} = 1 + 0.06t$, and
(iii) For (35), $(AS)_{35} = 35{,}000$ and $S_{35+t} = 1 + 0.10t$,
calculate the maximum value of $[R(40, 0, t) - R(35, 0, t)]$ for $t \geq 0$.

11.6. It is assumed that, for a new participant entering at age 30, there will be annual increases in salary at the rate of 5% per year to take care of the effects of inflation and increases in productivity. In addition, it is assumed that promotion raises of 10% of the existing salary will occur at ages 40, 50, and 60.

a. Construct a salary scale function, S_{30+k}, to express these assumptions.

b. Write an expression for the actuarial present value of contributions of 10% of future salary for a new entrant with annual salary 24,000 and with increases in salary according to the scale constructed in (a).

11.7. Every year, a plan sponsor contributes 10% of that portion of each participant's salary in excess of a certain amount. That amount is 15,000 this year and will increase by 5% annually. Express the actuarial present value of the sponsor's contribution for a participant entering now at age 35 with a salary of 40,000.

11.8. A plan provides for an income benefit rate, payable from retirement to age 65, of 2% of the final 3-year average salary for each year of service. After age 65 the income benefit rate is 1-1/3% of the final 3-year average salary for each year of service.

a. For a participant age 50, who entered service at age 30 and currently has a salary of 48,000, express the actuarial present value of the participant's benefit if the earliest retirement age is 55 and there is no mandatory retirement age.
b. If the maximum number of years to be credited in the plan is 35, express the actuarial present value of the benefit for the above participant.
c. Give an expression for the actuarial present value of the income benefit associated with past service for the above participant.

11.9. A career average plan provides a retirement income of 2% of aggregate salary during a participant's years of service. The earliest age of retirement is 58, and all retirements are completed by age 68. For a participant age 50 who entered service at age 30 and has 450,000 total of past salaries with a current salary of 42,000, write expressions for

a. The participant's total income benefit rate in case of retirement at exact age 65
b. The participant's midyear total income benefit rate in case of retirement between ages 65 and 66
c. The actuarial present value of this participant's retirement benefit for past service
d. The actuarial present value of this participant's retirement benefit for future service.

11.10. A new participant in a pension plan, age 45, has a choice of two benefit options:

(1) A defined-contribution plan with contributions of 20% of salary each year. Contributions are made at the beginning of each year and earn 5% per year. Accumulated contributions are used to purchase a monthly life annuity-due.

(2) A defined-benefit plan with an annual benefit, payable monthly, of 40% of the final 2-year average salary.

You are given that (a) $\ddot{a}_{65}^{(12)} = 10$ and (b) $S_{45+k} = (1.05)^k$ for $k \geq 0$ where S_y is a step function, constant over each year of age. Assuming that retirement occurs at exact age 65 and that the participant survives to retirement, calculate the ratio of the expected monthly payment under the defined-contribution plan to that under the defined-benefit plan.

11.11. Display definite integrals for the actuarial present values of the following possible benefits of an employee benefit package. Assume that the employee is currently age 40 and earning 40,000 annually. This employee was hired at age 25 and has received a total of 320,000 in salary since hire. Retirement benefits are available only after age 55, and withdrawal benefits are only available before age 55 in the form of an annuity with payments deferred until the employee reaches age 55.

a. A retirement benefit at the annual benefit rate of 50% of the final salary.
b. A retirement benefit at the rate of 0.015 times the product of the final salary rate multiplied by the exact number of years (including fractions) of service at the moment of retirement.
c. A withdrawal benefit using the benefit income formula in (b).
d. A retirement benefit at the rate of 0.025 times the total salary paid over the whole career to the employee.
e. A withdrawal benefit using the benefit income formula in (d).

11.12. A retirement benefit consisting of a continuous annuity, payable for life, is part of an employer's benefit package. The annual benefit income rate is 60% of the salary rate applicable at the moment of retirement for retirements between ages 60 and 70. For retirements after attaining age 70, the benefit rate is 60% of the salary rate applicable between ages 69 and 70. Give an approximating sum for the actuarial present value of this benefit for a person age 30 who has just been hired at a salary of 35,000.

Section 11.6

11.13. a. Give an expression for the annual benefit premium, payable to age 60, for a disability income insurance issued to (35) of 2,000 per month payable to age 65 in case (35) becomes disabled before age 60 and survives a waiting period of 6 months.

b. Give an expression for the active life benefit reserve at the end of 10 years for the insurance in (a).

Miscellaneous

11.14. The Hattendorf theorem for the fully continuous model as stated in Exercise 8.24 can be restated in the definitions and notation of this chapter for the fully continuous multiple decrement model:

$$\mathrm{Var}({}_0L^{\underline{m}}) = \sum_{j=1}^{m} \int_0^\infty [v^t\,(B_{x+t}^{(j)} - {}_t\bar{V}^{\underline{m}})]^2\, {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt.$$

Confirm that this result holds for the fully continuous whole life insurance discussed in Section 11.4.

Outline of solution:

a. Confirm that the loss random variable for this insurance is

$$ {}_0L^{\underline{2}} = \begin{cases} v^T - \bar{P}(\bar{A}_x)^{\underline{2}}\, \bar{a}_{\overline{T}|} & 0 \le T,\ J = 1 \\ v^T\, {}_T\bar{V}(\bar{A}_x) - \bar{P}(\bar{A}_x)^{\underline{2}}\, \bar{a}_{\overline{T}|} & 0 \le T,\ J = 2. \end{cases} $$

b. Use (11.4.6) and (11.4.7) to rewrite the differential equation (11.4.3) and then employ the integrating factor $e^{-\delta t}$ to obtain the solution

$$ v^t\, {}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{t}|} - \int_0^t e^{-\delta s}\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds. $$

c. Use the result of part (b) to modify both lines of the definition of ${}_0L^{\underline{2}}$ in part (a) and then show that

$$ \begin{aligned} \mathrm{Var}({}_0L^{\underline{2}}) = \int_0^\infty \Bigg\{ & v^t[1 - {}_t\bar{V}(\bar{A}_x)] \\ & - \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \Bigg\}^2\, {}_tp_x^{(\tau)} \mu_x^{(1)}(t)\, dt \\ & + \int_0^\infty \left\{ \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \right\}^2\, {}_tp_x^{(\tau)}\, \mu_x^{(2)}(t)\, dt. \end{aligned} $$

d. Perform the indicated squaring operation on the factor in the integrand of the first integral in part (c) and combine the two integrals that include $\{\int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds\}^2$ as a component of the integrand. Then use integration by parts to obtain

$$ \mathrm{Var}({}_0L^{\underline{2}}) = \int_0^\infty \{v^t\, [1 - {}_t\bar{V}(\bar{A}_x)]\}^2\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt. $$

This result provides a reduction of variance argument for establishing the withdrawal benefit as ${}_t\bar{V}(\bar{A}_x)$.

12

COLLECTIVE RISK MODELS FOR A SINGLE PERIOD

12.1 Introduction

In Chapters 3 through 11 we considered models for long-term insurances. The inclusion of interest in these models was essential. In this chapter we return to a topic introduced in Chapter 2, namely, short-term insurance policies. Consequently interest will be ignored. The purpose of this chapter is to present an alternative to the individual policy model discussed in Chapter 2.

The individual risk model of Chapter 2 considers individual policies and the claims produced by each policy. Then aggregate claims are obtained by summing over all the policies in the portfolio.

For the ***collective risk model*** we assume a random process that generates claims for a portfolio of policies. This process is characterized in terms of the portfolio as a whole rather than in terms of the individual policies comprising the portfolio. The mathematical formulation is as follows: Let N denote the number of claims produced by a portfolio of policies in a given time period. Let X_1 denote the amount of the first claim, X_2 the amount of the second claim, and so on. Then,

$$S = X_1 + X_2 + \cdots + X_N \tag{12.1.1}$$

represents the aggregate claims generated by the portfolio for the period under study. The number of claims, N, is a random variable and is associated with the frequency of claim. The individual claim amounts $X_1, X_2, \ldots$ are also random variables and are said to measure the severity of claims.

In order to make the model tractable, we usually make two fundamental assumptions:

1. $X_1, X_2, \ldots$ are identically distributed random variables.
2. The random variables $N, X_1, X_2, \ldots$ are mutually independent.

Expression (12.1.1) will be called a random sum, and unless stated otherwise, assumptions (1) and (2) will be made concerning its components.

A principal tool for developing the theory of this chapter is the moment generating function (m.g.f.). These functions provide a simple but powerful means for the reader to gain a working knowledge of the collective theory of risk. A reader who has not worked with them recently would do well to review the m.g.f.'s, means, and variances of the widely used probability distributions summarized in Appendix 5.

12.2 The Distribution of Aggregate Claims

In this section we see how the distribution of aggregate claims in a fixed time period can be obtained from the distribution of the number of claims and the distribution of individual claim amounts.

Let $P(x)$ denote the common d.f. of the independent and identically distributed X_i's. Let X be a random variable with this d.f. Then let

$$p_k = \mathrm{E}[X^k] \tag{12.2.1}$$

denote the k-th moment about the origin, and

$$M_X(t) = \mathrm{E}[e^{tX}] \tag{12.2.2}$$

denote the m.g.f. of X. In addition, let

$$M_N(t) = \mathrm{E}[e^{tN}] \tag{12.2.3}$$

denote the m.g.f. of the number of claims, and let

$$M_S(t) = \mathrm{E}[e^{tS}] \tag{12.2.4}$$

denote the m.g.f. of aggregate claims. The d.f. of aggregate claims will be denoted by $F_S(s)$.

Using (2.2.10) and (2.2.11), in conjunction with assumptions (1) and (2) of Section 12.1, we obtain

$$\mathrm{E}[S] = \mathrm{E}[\mathrm{E}[S|N]] = \mathrm{E}[p_1\, N] = p_1\, \mathrm{E}[N] \tag{12.2.5}$$

and

$$\begin{aligned} \mathrm{Var}(S) &= \mathrm{E}[\mathrm{Var}(S|N)] + \mathrm{Var}(\mathrm{E}[S|N]) \\ &= \mathrm{E}[N\, \mathrm{Var}(X)] + \mathrm{Var}(p_1 N) \\ &= \mathrm{E}[N]\, \mathrm{Var}(X) + p_1^2\, \mathrm{Var}(N) \end{aligned} \tag{12.2.6}$$

where $\mathrm{Var}(X) = p_2 - p_1^2$.

The result stated in (12.2.5), that the expected value of aggregate claims is the product of the expected individual claim amount and the expected number of claims, is not surprising. Expression (12.2.6) for the variance of aggregate claims also has a natural interpretation. The variance of aggregate claims is the sum of two components where the first is attributed to the variability of individual claim amounts and the second to the variability of the number of claims.

In a similar fashion we derive an expression for the m.g.f. of S:

$$
\begin{aligned}
M_S(t) &= \mathrm{E}[e^{tS}] = \mathrm{E}[\mathrm{E}[e^{tS}|N]] \\
&= \mathrm{E}[M_X(t)^N] = \mathrm{E}[e^{N \log M_X(t)}] \\
&= M_N[\log M_X(t)]. \qquad (12.2.7)
\end{aligned}
$$

Example 12.2.1

Assume that N has a geometric distribution; that is, the p.f. of N is given by

$$
\Pr(N = n) = pq^n \qquad n = 0, 1, 2, \ldots \qquad (12.2.8)
$$

where $0 < q < 1$ and $p = 1 - q$. Determine $M_S(t)$ in terms of $M_X(t)$.

Solution:

Since

$$
M_N(t) = \mathrm{E}[e^{tN}] = \sum_{n=0}^{\infty} p(qe^t)^n = \frac{p}{1 - qe^t},
$$

(12.2.7) tells us that

$$
M_S(t) = \frac{p}{1 - qM_X(t)}. \qquad (12.2.9)
$$

▼

To derive the d.f. of S, we distinguish according to how many claims occur and use the law of total probability,

$$
\begin{aligned}
F_S(x) = \Pr(S \le x) &= \sum_{n=0}^{\infty} \Pr(S \le x | N = n) \Pr(N = n) \\
&= \sum_{n=0}^{\infty} \Pr(X_1 + X_2 + \cdots + X_n \le x) \Pr(N = n). \qquad (12.2.10)
\end{aligned}
$$

In terms of the convolution defined in Section 2.3, we can write

$$
\begin{aligned}
\Pr(X_1 + X_2 + \cdots + X_n \le x) &= P{*}P{*}P{*} \cdots {*}P(x) \\
&= P^{*n}(x), \qquad (12.2.11)
\end{aligned}
$$

which is the n-th convolution of P defined in Chapter 2. Recall that

$$
P^{*0}(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}
$$

Thus (12.2.10) becomes

$$
F_S(x) = \sum_{n=0}^{\infty} P^{*n}(x) \Pr(N = n). \qquad (12.2.12)
$$

If the individual claim amount distribution is discrete with p.f. $p(x) = \Pr(X = x)$, the distribution of aggregate claims is also discrete. By analogy with the above derivation, the p.f. of S can be obtained directly as

$$f_S(x) = \sum_{n=0}^{\infty} p^{*n}(x) \Pr(N = n) \qquad (12.2.13)$$

where

$$p^{*n}(x) = p*p* \cdots * p(x) = \Pr(X_1 + X_2 + \cdots + X_n = x) \qquad (12.2.11A)$$

$$\text{and } p^{*0}(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Here the inequality sign in the probability symbol in (12.2.11) has been replaced by the equal sign.

Example 12.2.2

Consider an insurance portfolio that will produce zero, one, two, or three claims in a fixed time period with probabilities 0.1, 0.3, 0.4, and 0.2, respectively. An individual claim will be of amount 1, 2, or 3 with probabilities 0.5, 0.4, and 0.1, respectively. Calculate the p.f. and d.f. of the aggregate claims.

Solution:

The calculations are summarized below. Only nonzero entries are exhibited.

(1) x	(2) $p^{*0}(x)$	(3) $p^{*1}(x) = p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $f_S(x)$	(7) $F_S(x)$
0	1.0	—	—	—	0.1000	0.1000
1	—	0.5	—	—	0.1500	0.2500
2	—	0.4	0.25	—	0.2200	0.4700
3	—	0.1	0.40	0.125	0.2150	0.6850
4	—	—	0.26	0.300	0.1640	0.8490
5	—	—	0.08	0.315	0.0950	0.9440
6	—	—	0.01	0.184	0.0408	0.9848
7	—	—	—	0.063	0.0126	0.9974
8	—	—	—	0.012	0.0024	0.9998
9	—	—	—	0.001	0.0002	1.0000
n	0	1	2	3	—	—
$\Pr(N = n)$	0.1	0.3	0.4	0.2	—	—

Since there are at most three claims and each produces a claim amount of at most 3, we can limit the calculations to $x = 0, 1, 2, \ldots, 9$.

Column (2) lists the p.f. of a degenerate distribution with all the probability mass at 0. Column (3) lists the p.f. of the individual claim amount random variable. Columns (4) and (5) are obtained recursively by applying

$$\begin{aligned} p^{*(n+1)}(x) &= \Pr(X_1 + X_2 + \cdots + X_{n+1} = x) \\ &= \sum_y \Pr(X_{n+1} = y) \Pr(X_1 + X_2 + \cdots + X_n = x - y) \\ &= \sum_y p(y)\, p^{*n}(x - y). \end{aligned} \qquad (12.2.14)$$

Since only three different claim amounts are possible, the evaluation of (12.2.14) will involve a sum of three or fewer terms. Next, (12.2.13) is used to compute the p.f. displayed in column (6). For this step, it is convenient to record the p.f. of N in the last row of the results. Finally, the elements of column (7) are obtained as partial sums of column (6). An alternative approach, not illustrated here, would be to perform the convolutions in terms of the d.f.'s, obtain $F_S(x)$ from (12.2.12), and calculate $f_S(x) = F_S(x) - F_S(x - 1)$. ▼

If the claim amount distribution is continuous, it cannot be concluded that the distribution of S is continuous. If $\Pr(N = 0) > 0$, the distribution of S will be of the mixed type; that is, it will have a mass of probability at 0 and be continuous elsewhere. This idea is illustrated in the following example.

Example 12.2.3

In Example 12.2.1, add the assumption that

$$P(x) = 1 - e^{-x} \qquad x > 0;$$

that is, the individual claim amount distribution is exponential with mean 1. Then show that

$$M_S(t) = p + q\,\frac{p}{p - t} \tag{12.2.15}$$

and interpret the formula.

Solution:

First, we rewrite (12.2.9) as follows:

$$M_S(t) = p + q\,\frac{pM_X(t)}{1 - qM_X(t)}\,.$$

Then we substitute

$$M_X(t) = \int_0^\infty e^{tx}\, e^{-x}\, dx = (1 - t)^{-1}$$

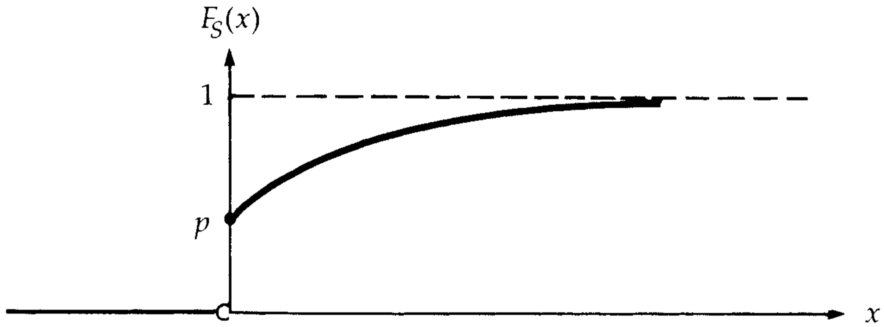
to obtain (12.2.15).

Since 1 is the m.g.f. of the constant 0 and $p/(p - t)$ is the m.g.f. of the exponential distribution with d.f. $1 - e^{-px}$, $x > 0$, (12.2.15) can be interpreted as a weighted average (with weights p and q, respectively). It follows that the d.f. of S is the corresponding weighted average of distributions. Thus, for $x > 0$

$$F_S(x) = p(1) + q(1 - e^{-px}) = 1 - qe^{-px}. \tag{12.2.16}$$

This distribution is of the mixed type. Its d.f. is shown in Figure 12.2.1. ▼

FIGURE 12.2.1
Graph of $F_S(x)$



12.3 Selection of Basic Distributions

In this section we discuss some issues in selecting the distribution of the number of claims N and the common distribution of the X_i's. As different considerations apply to these two selections, a separate subsection will be devoted to each.

12.3.1 The Distribution of *N*

One choice for the distribution of N is the Poisson with p.f. given by

$$\Pr(N = n) = \frac{\lambda^n e^{-\lambda}}{n!} \qquad n = 0, 1, 2, \ldots \tag{12.3.1}$$

where $\lambda > 0$. For the Poisson distribution, $E[N] = \mathrm{Var}(N) = \lambda$. With this choice for the distribution of N, the distribution of S is called a ***compound Poisson distribution.*** Using (12.2.5) and (12.2.6), we have that

$$E[S] = \lambda p_1 \tag{12.3.2}$$

and

$$\mathrm{Var}(S) = \lambda p_2 . \tag{12.3.3}$$

Substituting the m.g.f. of the Poisson distribution

$$M_N(t) = e^{\lambda(e^t - 1)} \tag{12.3.4}$$

into (12.2.7), we obtain the m.g.f. of the compound Poisson distribution,

$$M_S(t) = e^{\lambda[M_X(t) - 1]} . \tag{12.3.5}$$

The compound Poisson distribution has many attractive features, some of which are discussed in Section 12.4.

When the variance of the number of claims exceeds its mean, the Poisson distribution is not appropriate. In this situation, use of the negative binomial distribution has been suggested. The negative binomial distribution has a p.f. given by

$$\Pr(N = n) = \binom{r + n - 1}{n} p^r q^n \qquad n = 0, 1, 2, \ldots. \tag{12.3.6}$$

This distribution has two parameters: $r > 0$ and $0 < p < 1$; $q = 1 - p$. For this distribution, we have

$$M_N(t) = \left(\frac{p}{1 - q e^t}\right)^r, \tag{12.3.7}$$

$$\mathrm{E}[N] = \frac{rq}{p}, \tag{12.3.8}$$

and

$$\mathrm{Var}(N) = \frac{rq}{p^2}. \tag{12.3.9}$$

When a negative binomial distribution is chosen for N, the distribution of S is called a ***compound negative binomial distribution.*** Substituting from (12.3.8) and (12.3.9) into (12.2.5) and (12.2.6), we have

$$\mathrm{E}[S] = \frac{rq}{p} p_1 \tag{12.3.10}$$

and

$$\mathrm{Var}(S) = \frac{rq}{p} p_2 + \frac{rq^2}{p^2} p_1^2. \tag{12.3.11}$$

Substituting from (12.3.7) into (12.2.7), we obtain

$$M_S(t) = \left[\frac{p}{1 - qM_X(t)}\right]^r. \tag{12.3.12}$$

We observe that the family of geometric distributions used in Examples 12.2.1 and 12.2.3 is contained as a special case ($r = 1$) of the two-parameter family of negative binomial distributions.

A family of distributions for the number of claims can be generated by assuming that the Poisson parameter Λ is a random variable with p.d.f. $u(\lambda)$, $\lambda > 0$, and that the conditional distribution of N, given $\Lambda = \lambda$, is Poisson with parameter λ. There are several situations in which this might be a useful way to consider the distribution of N. For example, consider a population of insureds where various classes of insureds within the population generate numbers of claims according to Poisson distributions with different values of λ for the various classes. If the relative frequency of the values of λ is denoted by $u(\lambda)$, we can use the law of total probability to obtain

$$\begin{aligned}\Pr(N = n) &= \int_0^\infty \Pr(N = n \mid \Lambda = \lambda)\, u(\lambda) d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} u(\lambda) d\lambda. \end{aligned} \tag{12.3.13}$$

Furthermore, using (2.2.10) and (2.2.11), we have

$$\mathrm{E}[N] = \mathrm{E}[\mathrm{E}[N|\Lambda]] = \mathrm{E}[\Lambda] \tag{12.3.14}$$

and

$$\begin{aligned} \mathrm{Var}(N) &= \mathrm{E}[\mathrm{Var}(N|\Lambda)] + \mathrm{Var}(\mathrm{E}[N|\Lambda]) \\ &= \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda). \end{aligned} \tag{12.3.15}$$

Also,

$$M_N(t) = \mathrm{E}[e^{tN}] = \mathrm{E}[\mathrm{E}[e^{tN}|\Lambda]] = \mathrm{E}[e^{\Lambda(e^t-1)}] = M_\Lambda(e^t - 1). \tag{12.3.16}$$

The equality,

$$\mathrm{E}[e^{tN}|\Lambda] = e^{\Lambda(e^t-1)},$$

follows from the hypothesis that the conditional distribution of N, given Λ, is Poisson with parameter Λ.

A comparison of (12.3.14) and (12.3.15) shows that, as in the case of the negative binomial distribution, $\mathrm{E}[N] < \mathrm{Var}(N)$. In fact, the negative binomial distribution can be derived in this fashion, which will be shown in the following example.

Example 12.3.1

Assume that $u(\lambda)$ is the gamma p.d.f. with parameters α and β,

$$u(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \qquad \lambda > 0 \tag{12.3.17}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy.$$

a. Show that the marginal distribution of N is negative binomial with parameters

$$r = \alpha, \qquad p = \frac{\beta}{1+\beta}. \tag{12.3.18}$$

b. By substituting $\mathrm{E}[\Lambda] = \alpha/\beta$ and $\mathrm{Var}(\Lambda) = \alpha/\beta^2$ into (12.3.14) and (12.3.15), verify (12.3.8) and (12.3.9).

Solution:

a. Substituting

$$M_\Lambda(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \tag{12.3.19}$$

into (12.3.16), we have

$$\begin{aligned} M_N(t) = M_\Lambda(e^t - 1) &= \left[\frac{\beta}{\beta - (e^t - 1)}\right]^\alpha \\ &= \left\{\frac{\beta/(\beta+1)}{1 - [1 - \beta/(\beta+1)]e^t}\right\}^\alpha. \end{aligned} \tag{12.3.20}$$

Comparison of (12.3.20) with (12.3.7) confirms that this distribution for N is negative binomial with parameters $r = \alpha$,

$$p = \frac{\beta}{1 + \beta}, \tag{12.3.21}$$

$$q = 1 - p = \frac{1}{1 + \beta}.$$

b. The suggested substitutions into (12.3.14) and (12.3.15) yield

$$\mathrm{E}[N] = \frac{\alpha}{\beta} = \frac{rq}{p}$$

as in (12.3.8) and

$$\mathrm{Var}(N) = \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{rq}{p}\left(1 + \frac{q}{p}\right) = \frac{rq}{p^2}$$

as in (12.3.9). ▼

The following is another example of a distribution for N that is obtained by mixing Poisson distributions.

Example 12.3.2

Assume that $u(\lambda)$ is the inverse Gaussian p.d.f. with parameters α and β. Exhibit the moment generating function of N, E[N], and Var(N).

Solution:

Example 2.3.5 contains the basic facts about the inverse Gaussian distribution.

Applying (12.3.16) yields

$$M_N(t) = M_\Lambda(e^t - 1) = e^{\alpha\{1-[1-2(e^t-1)/\beta]^{1/2}\}},$$

and from (12.3.14) and (12.3.15) we obtain

$$\mathrm{E}[N] = \mathrm{E}[\Lambda] = \frac{\alpha}{\beta}$$

and

$$\begin{aligned}\mathrm{Var}(N) &= \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda)\\ &= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{\alpha(\beta + 1)}{\beta^2}.\end{aligned}$$

This distribution is called the ***Poisson inverse Gaussian distribution.*** ▼

Table 12.3.1 summarizes pertinent information on the compound distributions resulting from the selections for N discussed here.

TABLE 12.3.1

Compound Distributions of S

$$S = \sum_{j=1}^{N} X_j$$

$N, X_1, X_2, \ldots$ are independent random variables. Each X_j has d.f. $P(x)$, m.g.f. $M_X(t)$, and $p_k = \mathrm{E}[X^k] \quad k = 1, 2, \ldots$

$$P^{*0}(x) = \begin{bmatrix} 1 & x \geq 0 \\ 0 & \text{elsewhere} \end{bmatrix}$$

$$P^{*n}(x) = \begin{cases} \sum_{j=0}^{x} p(x-j)P^{*(n-1)}(j), \text{ or} \\ \int_0^x p(x-y)P^{*(n-1)}(y)dy \end{cases}$$

Definitions	Distribution Function, $F_S(x)$	Restrictions on Parameters	Moment Generating Function, $M_S(t)$	Mean	Variance
General	$\sum_{n=0}^{\infty} \Pr(N=n)P^{*n}(x)$	—	$M_N[\log M_X(t)]$	$p_1\,\mathrm{E}[N]$	$\mathrm{E}[N](p_2 - p_1^2) + p_1^2\,\mathrm{Var}(N)$
Compound Poisson	$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} P^{*n}(x)$	$\lambda > 0$	$e^{\lambda[M_X(t)-1]}$	$\lambda\, p_1$	$\lambda\, p_2$
Compound Negative Binomial	$\sum_{n=0}^{\infty} \binom{r+n-1}{n} p^r q^n P^{*n}(x)$	$0 < p < 1$, $q = 1 - p$, $r > 0$	$\left[\frac{p}{1 - qM_X(t)}\right]^r \quad qM_X(t) < 1$	$\frac{rqp_1}{p}$	$\frac{rqp_2}{p} + \frac{rq^2p_1^2}{p^2}$
Compound Poisson Inverse Gaussian	no known closed form	$\alpha > 0$, $\beta > 0$	$\exp\left\{\alpha\left[1 - \left\{1 - \frac{2[M_X(t)-1]}{\beta}\right\}^{1/2}\right]\right\}$	$\frac{\alpha}{\beta}p_1$	$\frac{\alpha}{\beta}\left(p_2 + \frac{p_1^2}{\beta}\right)$

12.3.2 The Individual Claim Amount Distribution

On the basis of (12.2.12) we see that convolutions of the individual claim amount distribution may be required. Thus, when possible, it is convenient to select that distribution from a family of distributions for which convolutions can be calculated easily either by formula or numerically. For example, if the claim amount has the normal distribution with mean μ and variance σ^2, then its n-th convolution is the normal distribution with mean $n\mu$ and variance $n\sigma^2$. For many types of insurance, the claim amount random variable is only positive, and its distribution is skewed to the right. For these insurances we might choose a gamma distribution that also has these properties. The n-th convolution of a gamma distribution with parameters α and β is also a gamma distribution but with parameters $n\alpha$ and β. This can be confirmed by noting from (12.3.19) that $M_X(t) = [\beta/(\beta - t)]^\alpha$, and hence the m.g.f. associated with $P^{*n}(x)$ is

$$M_X(t)^n = \left(\frac{\beta}{\beta - t}\right)^{n\alpha} \qquad t < \beta. \tag{12.3.22}$$

If the claim amounts have an exponential distribution with parameter 1, the p.d.f. is given by

$$p(x) = e^{-x} \qquad x > 0.$$

This is a gamma distribution with $\alpha = \beta = 1$. Then, by using (12.3.19), we conclude that the n-th convolution is a gamma distribution with parameters $\alpha = n$, $\beta = 1$; that is,

$$p^{*n}(x) = \frac{x^{n-1}\, e^{-x}}{(n-1)!} \qquad x > 0. \tag{12.3.23}$$

To obtain an expression for $P^{*n}(x)$, we perform integration by parts n times as follows:

$$\begin{aligned}
1 - P^{*n}(x) &= \int_x^\infty \frac{y^{n-1}\, e^{-y}}{(n-1)!}\, dy \\
&= -\frac{y^{n-1}}{(n-1)!}\, e^{-y}\Bigg|_x^\infty + \int_x^\infty \frac{y^{n-2}\, e^{-y}}{(n-2)!}\, dy \\
&= \frac{x^{n-1}}{(n-1)!}\, e^{-x} + [1 - P^{*(n-1)}(x)] \\
&= e^{-x} \sum_{i=0}^{n-1} \frac{x^i}{i!}\, .
\end{aligned} \tag{12.3.24}$$

Then, using (12.2.12), we have

$$1 - F_S(x) = \sum_{n=1}^{\infty} \Pr(N = n) e^{-x} \sum_{i=0}^{n-1} \frac{x^i}{i!} \qquad x > 0. \tag{12.3.25}$$

This exponential distribution case shows that even with simple assumed distributions, the distribution of aggregate claims may not have a simple form. Therefore, it may be more practical to select a discrete claim amount distribution and calculate the required convolutions numerically. For compound Poisson distributions it has been established that the convolution method can be shortened or, alternatively, that it can be bypassed by use of a recursive formula for directly calculating the distribution function of S. These computational shortcuts are discussed in the following section.

12.4 Properties of Certain Compound Distributions

In this section we discuss some mathematical properties of certain compound distributions. Two theorems concerning the compound Poisson are presented.

The first shows that the sum of independent random variables, each having a compound Poisson distribution, also has a compound Poisson distribution.

Theorem 12.4.1

If $S_1, S_2, \ldots, S_m$ are mutually independent random variables such that S_i has a compound Poisson distribution with parameter λ_i and d.f. of claim amount $P_i(x)$, $i = 1, 2, \ldots, m$, then $S = S_1 + S_2 + \cdots + S_m$ has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i \tag{12.4.1}$$

and

$$P(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} P_i(x). \tag{12.4.2}$$

Proof:

We let $M_i(t)$ denote the m.g.f. of $P_i(x)$. According to (12.3.5), the m.g.f. of S_i is

$$M_{S_i}(t) = \exp\{\lambda_i[M_i(t) - 1]\}.$$

By the assumed independence of $S_1, \ldots, S_m$, the m.g.f. of their sum is

$$M_S(t) = \prod_{i=1}^{m} M_{S_i}(t) = \exp\left\{\sum_{i=1}^{m} \lambda_i[M_i(t) - 1]\right\}.$$

Finally, we rewrite the exponent to obtain

$$M_S(t) = \exp\left\{\lambda\left[\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_i(t) - 1\right]\right\}. \tag{12.4.3}$$

Since this is the m.g.f. of the compound Poisson distribution, specified by (12.4.1) and (12.4.2), the theorem follows. ■

This result has two important consequences for building insurance models. First, if we combine m insurance portfolios, where the aggregate claims of each of the portfolios have compound Poisson distributions and are mutually independent, then the aggregate claims for the combined portfolio will also have a compound Poisson distribution. Second, we can consider a single insurance portfolio for a period of m years. Here we assume independence among the annual aggregate claims for the m years and that the aggregate claims for each year have a compound Poisson distribution. It is not necessary that the annual aggregate claims distributions be identical. Then it follows from Theorem 12.4.1 that the total claims for the m-year period will have a compound Poisson distribution.

Example 12.4.1

Let $x_1, x_2, \ldots, x_m$ be m different numbers and suppose that $N_1, N_2, \ldots, N_m$ are mutually independent random variables. Further, suppose that N_i $(i = 1, 2, \ldots, m)$ has a Poisson distribution with parameter λ_i. What is the distribution of

$$x_1 N_1 + x_2 N_2 + \cdots + x_m N_m? \tag{12.4.4}$$

Solution:

By interpreting $x_i N_i$ to have a compound Poisson distribution with Poisson parameter λ_i and a degenerate claim amount distribution at x_i, we can apply Theorem 12.4.1 to establish that the sum in (12.4.4) has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i$$

and p.f. of claim amount $p(x)$ where

$$p(x) = \begin{cases} \dfrac{\lambda_i}{\lambda} & x = x_i, \quad i = 1, 2, \ldots, m \\ 0 & \text{elsewhere.} \end{cases} \tag{12.4.5}$$

▼

We show in Theorem 12.4.2 that the construction in Example 12.4.1 is reversible: that is, every compound Poisson distribution with a discrete claim amount distribution can be represented as a sum of the form (12.4.4). We let $x_1, x_2, \ldots, x_m$ denote the discrete values for individual claim amounts and let

$$\pi_i = p(x_i) \qquad i = 1, 2, \ldots, m \tag{12.4.6}$$

denote their respective probabilities. Let N_i be the number of terms in (12.1.1) that are equal to x_i. Then, by collecting terms, we see that

$$S = x_1 N_1 + x_2 N_2 + \cdots + x_m N_m . \tag{12.4.7}$$

In general, the N_i's of (12.4.7) are dependent random variables. However, in the special case of a compound Poisson distribution for S, they are independent, as is shown in Theorem 12.4.2.

Before stating Theorem 12.4.2, we cite some properties of the ***multinomial distribution*** that are used in the proof. For the multinomial, each of n independent