ACTUARIAL MATHEMATICS

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Preface for the Society of Actuaries

The first edition of *Actuarial Mathematics*, published in 1986, has served the profession well by presenting the modern mathematical foundations underlying actuarial science and developing powerful tools for the actuary of the future. The text not only introduced a stochastic approach to complement the deterministic approach of earlier texts but also integrated contingency theory with risk theory and population theory. As a result, actuaries now have the methods and skills to quantify the variance of random events as well as the expected value.

Since the publication of the first edition, the technology available to actuaries has changed rapidly and the practice has advanced. To reflect these changes, three of the original five authors agreed to update and revise the text.

The Education and Examination Committee of the Society has directed the project. Richard S. Mattison helped to define the specifications and establish the needed funding. Robert C. Campbell, Warren Luckner, Robert A. Conover, and Sandra L. Rosen have all made major contributions to the project.

Richard Lambert organized the peer review process for the text. He was ably assisted by Bonnie Averback, Jeffrey Beckley, Keith Chun, Janis Cole, Nancy Davis, Roy Goldman, Jeff Groves, Curtis Huntington, Andre L'Esperance, Graham Lord, Esther Portnoy, David Promislow, Elias Shiu, McKenzie Smith, and Judy Strachan.

The Education and Examination Committee, the Peer Review Team, and all others involved in the revision of this text are most appreciative of the efforts of Professors Bowers, Hickman and Jones toward the education of actuaries.

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*Professors Gerber and Nesbitt were involved as consultants with the revisions incorporated in the second edition.

AUTHORS' INTRODUCTIONS AND GUIDE TO STUDY

Introduction to First Edition*

This text represents a first step in communicating the revolution in the actuarial profession that is taking place in this age of high-speed computers. During the short period of time since the invention of the microchip, actuaries have been freed from numerous constraints of primitive computing devices in designing and managing insurance systems. They are now able to focus more of their attention on creative solutions to society's demands for financial security.

To provide an educational basis for this focus, the major objectives of this work are to integrate life contingencies into a full risk theory framework and to demonstrate the wide variety of constructs that are then possible to build from basic models at the foundation of actuarial science. Actuarial science is ever evolving, and the procedures for model building in risk theory are at its forefront. Therefore, we examine the nature of models before proceeding with a more detailed discussion of the text.

Intellectual and physical models are constructed either to organize observations into a comprehensive and coherent theory or to enable us to simulate, in a laboratory or a computer system, the operation of the corresponding full-scale entity. Models are absolutely essential in science, engineering, and the management of large organizations. One must, however, always keep in mind the sharp distinction between a model and the reality it represents. A satisfactory model captures enough of reality to give insights into the successful operation of the system it represents.

The insurance models developed in this text have proved useful and have deepened our insights about insurance systems. Nevertheless, we need to always keep

*Chapter references and nomenclature have been changed to be in accord with the second edition. These changes are indicated by italics.

before us the idea that real insurance systems operate in an environment that is more complex and dynamic than the models studied here. Because models are only approximations of reality, the work of model building is never done; approximations can be improved and reality may shift. It is a continuing endeavor of any scientific discipline to revise and update its basic models. Actuarial science is no exception.

Actuarial science developed at a time when mathematical tools (probability and calculus, in particular), the necessary data (especially mortality data in the form of life tables), and the socially perceived need (to protect families and businesses from the financial consequences of untimely death) coexisted. The models constructed at the genesis of actuarial science are still useful. However, the general environment in which actuarial science exists continues to change, and it is necessary to periodically restate the fundamentals of actuarial science in response to these changes.

We illustrate this with three examples:

1. The insurance needs of modern societies are evolving, and, in response, new systems of employee benefits and social insurance have developed. New models for these systems have been needed and constructed.
2. Mathematics has also evolved, and some concepts that were not available for use in building the original foundations of actuarial science are now part of a general mathematics education. If actuarial science is to remain in the mainstream of the applied sciences, it is necessary to recast basic models in the language of contemporary mathematics.
3. Finally, as previously stated, the development of high-speed computing equipment has greatly increased the ability to manipulate complex models. This has far-reaching consequences for the degree of completeness that can be incorporated into actuarial models.

This work features models that are fundamental to the current practice of actuarial science. They are explored with tools acquired in the study of mathematics, in particular, undergraduate level calculus and probability. The proposition guiding Chapters *1–14* is that there is a set of basic models at the heart of actuarial science that should be studied by all students aspiring to practice within any of the various actuarial specialities. These models are constructed using only a limited number of ideas. We will find many relationships among those models that lead to a unity in the foundations of actuarial science. These basic models are followed, in Chapters *15–21*, by some more elaborate models particularly appropriate to life insurance and pensions.

While this book is intended to be comprehensive, it is not meant to be exhaustive. In order to avoid any misunderstanding, we will indicate the limitations of the text:

- Mathematical ideas that could unify and, in some cases, simplify the ideas presented, but which are not included in typical undergraduate courses, are not used. For example, moment generating functions, but not characteristic functions, are used in developments regarding probability distributions. Stieltjes integrals, which could be used in some cases to unify the presentation of discrete and continuous cases, are not used because of this basic decision on mathematical prerequisites.

- The chapters devoted to life insurance stress the randomness of the time at which a claim payment must be made. In the same chapters, the interest rates used to convert future payments to a present value are considered deterministic and are usually taken as constants. In view of the high volatility possible in interest rates, it is natural to ask why probability models for interest rates were not incorporated. Our answer is that the mathematics of life contingencies on a probabilistic foundation (except for interest) does not involve ideas beyond those covered in an undergraduate program. On the other hand, the modeling of interest rates requires ideas from economics and statistics that are not included in the prerequisites of this volume. In addition, there are some technical problems in building models to combine random interest and random time of claim that are in the process of being solved.
- Methods for estimating the parameters of basic actuarial models from observations are not covered. For example, the construction of life tables is not discussed.
- This is not a text on computing. The issues involved in optimizing the organization of input data and computation in actuarial models are not discussed. This is a rapidly changing area, seemingly best left for readers to resolve as they choose in light of their own resources.
- Many important actuarial problems created by long-term practice and insurance regulation are not discussed. This is true in sections treating topics such as premiums actually charged for life insurance policies, costs reported for pensions, restrictions on benefit provisions, and financial reporting as required by regulators.
- Ideas that lead to interesting puzzles, but which do not appear in basic actuarial models, are avoided. Average age at death problems for a stationary population do not appear for this reason.

This text has a number of features that distinguish it from previous fine textbooks on life contingencies. A number of these features represent decisions by the authors on material to be included and will be discussed under headings suggestive of the topics involved.

Probability Approach

As indicated earlier, the sharpest break between the approach taken here and that taken in earlier English language textbooks on actuarial mathematics is the much fuller use of a probabilistic approach in the treatment of the mathematics of life contingencies. Actuaries have usually written and spoken of applying probabilities in their models, but their results could be, and often were, obtained by a deterministic rate approach. In this work, the treatment of life contingencies is based on the assumption that time-until-death is a continuous-type random variable. This admits a rich field of random variable concepts such as distribution function, probability density function, expected value, variance, and moment generating function. This approach is timely, based on the availability of high-speed computers, and is called for, based on the observation that the economic role of life insurance and pensions can be best seen when the random value of time-until-death is stressed. Also, these probability ideas are now part of general education in mathematics, and a fuller realization thereof relates life contingencies to other fields of applied probability, for example, reliability theory in engineering.

Additionally, the deterministic rate approach is described for completeness and is a tool in some developments. However, the results obtained from using a deterministic model usually can be obtained as expected values in a probabilistic model.

Integration with Risk Theory

Risk theory is defined as the study of deviations of financial results from those expected and methods of avoiding inconvenient consequences from such deviations. The probabilistic approach to life contingencies makes it easy to incorporate long-term contracts into risk theory models and, in fact, makes life contingencies only a part, but a very important one, of risk theory. Ruin theory, another important part of risk theory, is included as it provides insight into one source, the insurance claims, of adverse long-term financial deviations. This source is the most unique aspect of models for insurance enterprises.

Utility Theory

This text contains topics on the economics of insurance. The goal is to provide a motivation, based on a normative theory of individual behavior in the face of uncertainty, for the study of insurance models. Although the models used are highly simplified, they lead to insights into the economic role of insurance, and to an appreciation of some of the issues that arise in making insurance decisions.

Consistent Assumptions

The assumption of a uniform distribution of deaths in each year of age is consistently used to evaluate actuarial functions at nonintegral ages. This eliminates some of the anomalies that have been observed when inconsistent assumptions are applied in situations involving high interest rates.

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Introduction to Second Edition

Actuarial science is not static. In the time since the publication of the first edition of *Actuarial Mathematics,* actuarial science has absorbed additional ideas from economics and the mathematical sciences. At the same time, computing and communications have become cheaper and faster, and this has helped to make feasible more complex actuarial models. During this period the financial risks that modern societies seek to manage have also altered as a result of the globalization of business, technological advances, and political shifts that have changed public policies.

It would be impossible to capture the full effect of all these changes in the revision of a basic textbook. Our objective is more modest, but we hope that it is realistic. This edition is a step in an ongoing process of adaptation designed to keep the fundamentals of actuarial science current with changing realities.

In the second edition, changes in notation and nomenclature appear in almost every section. There are also basic changes from the first edition that should be listed.

1. Commutation functions, a classic tool in actuarial calculations, are not used. This is in response to the declining advantages of these functions in an age when interest rates are often viewed as random variables, or as varying deterministically, and the probability distribution of time until decrement may depend on variables other than attained age. Starting in Chapter 3, exercises that illustrate actuarial calculations using recursion formulas that can be implemented with current software are introduced. It is logically necessary that the challenge of implementing tomorrow's software is left to the reader.
2. Utility theory is no longer confined to the first chapter. Examples are given that illustrate how utility theory can be employed to construct consistent models for premiums and reserves that differ from the conventional model that implicitly depends on linear utility of wealth.
3. In the first edition readers were seldom asked to consider more than the first and second moments of loss random variables. In this edition, following the intellectual path used earlier in physics and statistics, the distribution functions and probability density functions of loss variables are illustrated.
4. The basic material on reserves is now presented in two chapters. This facilitates a more complete development of the theory of reserves for general life insurances with varying premiums and benefits.
5. In recent years considerable actuarial research has been done on joint distributions for several future lifetime random variables where mutual independence is not assumed. This work influences the chapters on multiple life actuarial functions and multiple decrement theory.
6. There are potentially serious estimation and interpretation problems in multiple decrement theory when the random times until decrement for competing causes of decrement are not independent. Those problems are illustrated in the second edition.

7. The applications of multiple decrement theory have been consolidated. No attempt is made to illustrate in this basic textbook the variations in benefit formulas driven by rapid changes in pension practice and regulation.
8. The confluence of new research and computing capabilities has increased the use of recursive formulas in calculating the distribution of total losses derived from risk theory models. This development has influenced Chapter 12.
9. The material on pricing life insurance with death and withdrawal benefits and accounting for life insurance operations has been reorganized. Business and regulatory considerations have been concentrated in one chapter, and the foundations of accounting and provisions for expenses in an earlier chapter. The discussion of regulation has been limited to general issues and options for addressing these issues. No attempt has been made to present a definitive interpretation of regulation for any nation, province, or state.
10. The models for some insurance products that are no longer important in the market have been deleted. Models for new products, such as accelerated benefits for terminal illness or long-term care, are introduced.
11. The final chapter contains a brief introduction to simple models in which interest rates are random variables. In addition, ideas for managing interest rate risk are discussed. It is hoped that this chapter will provide a bridge to recent developments within the intersection of actuarial mathematics and financial economics.

As the project of writing this second edition ends, it is clear that a significant new development is under way. This new endeavor is centered on the creation of general models for managing the risks to individuals and organizations created by uncertain future cash flows when the uncertainty derives from any source. This blending of the actuarial/statistical approach to building models for financial security systems with the approach taken in financial economics is a worthy assignment for the next cohort of actuarial students.

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Guide to Study

The reader can consider this text as covering the two branches of risk theory. Individual risk theory views each policy as a unit and allows construction of a model for a group of policies by adding the financial results for the separate policies in the group. Collective risk theory uses a probabilistic model for total claims that avoids the step of adding the results for individual policies. This distinction is sometimes difficult to maintain in practice. The chapters, however, can be classified as illustrated below.

Chapters Classified by Branch of Risk Theory

Individual Risk Theory	Collective Risk Theory
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	12, 13, 14, 19, 20

It is also possible to divide insurance models into those appropriate for short-term insurance, where investment income is not a significant factor, and long-term insurance, where investment income is important. The following classification scheme provides this division of chapters along with an additional division of long-term models between those for life insurance and those for pensions.

Chapters Classified by Term of Insurance and Field of Application

	Long-Term Insurances	
Short-Term Insurances	Life Insurance	Pensions
1, 2, 12, 13, 14	3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	9, 10, 11, 19, 20, 21

The selection of topics and their organization do not follow a traditional pattern. As stated previously, the new organization arose from the goal to first cover material considered basic for all actuarial students (Chapters 1–14) and then to include a more in-depth treatment of selected topics for students specializing in life insurance and pensions (Chapters 15–21).

The discussion in Chapter 1 is devoted to two ideas: that random events can disrupt the plans of decision makers and that insurance systems are designed to reduce the adverse financial effects of these events. To illustrate the latter, single insurance policies are discussed and convenient, if not necessarily realistic, distributions of the loss random variable are used. In subsequent chapters, more detailed models are constructed for use with insurance systems.

In Chapter 2, the individual risk model is developed, first in regard to single policies, then in regard to a portfolio of policies. In this model, a random variable, S, the total claims in a single period, is the sum of a fixed number of independent random variables, each of which is associated with a single policy. Each component of the sum S can take either the value 0 or a random claim amount in the course of a single period.

From the viewpoint of risk theory, the ideas developed in Chapters 3 through 11 can be seen as extending the ideas of Chapter 2. Instead of considering the potential claims in a short period from an individual policy, we consider loss variables that take into account the financial results of several periods. Since such random variables are no longer restricted to a short time period, they reflect the time value of money. For groups of individuals, we can then proceed, as in

Chapter 2, to use an approximation, such as the normal approximation, to make probability statements about the sum of the random variables that are associated with the individual members.

In Chapter 3, time-of-death is treated as a continuous random variable, and, after defining the probability density function, several features of the probability distribution are introduced and explored. In Chapters 4 and 5, life insurances and annuities are introduced, and the present values of the benefits are expressed as functions of the time-of-death. Several characteristics of the distributions of the present value of future benefits are examined. In Chapter 6, the equivalence principle is introduced and used to define and evaluate periodic benefit premiums. In Chapters 7 and 8, the prospective future loss on a contract already in force is investigated. The distribution of future loss is examined, and the benefit reserve is defined as the expected value of this loss. In Chapter 9, annuity and insurance contracts involving two lives are studied. (Discussion of more advanced multiple life theory is deferred until Chapter 18.) The discussion in Chapters 10 and 11 investigates a more realistic model in which several causes of decrement are possible. In Chapter 10, basic theory is examined, whereas in Chapter 11 the theory is applied to calculating actuarial present values for a variety of insurance and pension benefits.

In Chapter 12, the collective risk model is developed with respect to single-period considerations of a portfolio of policies. The distribution of total claims for the period is developed by postulating the characteristics of the portfolio in the aggregate rather than as a sum of individual policies. In Chapter 13, these ideas are extended to a continuous-time model that can be used to study solvency requirements over a long time period. Applications of risk theory to insurance models are given an overview in Chapter 14.

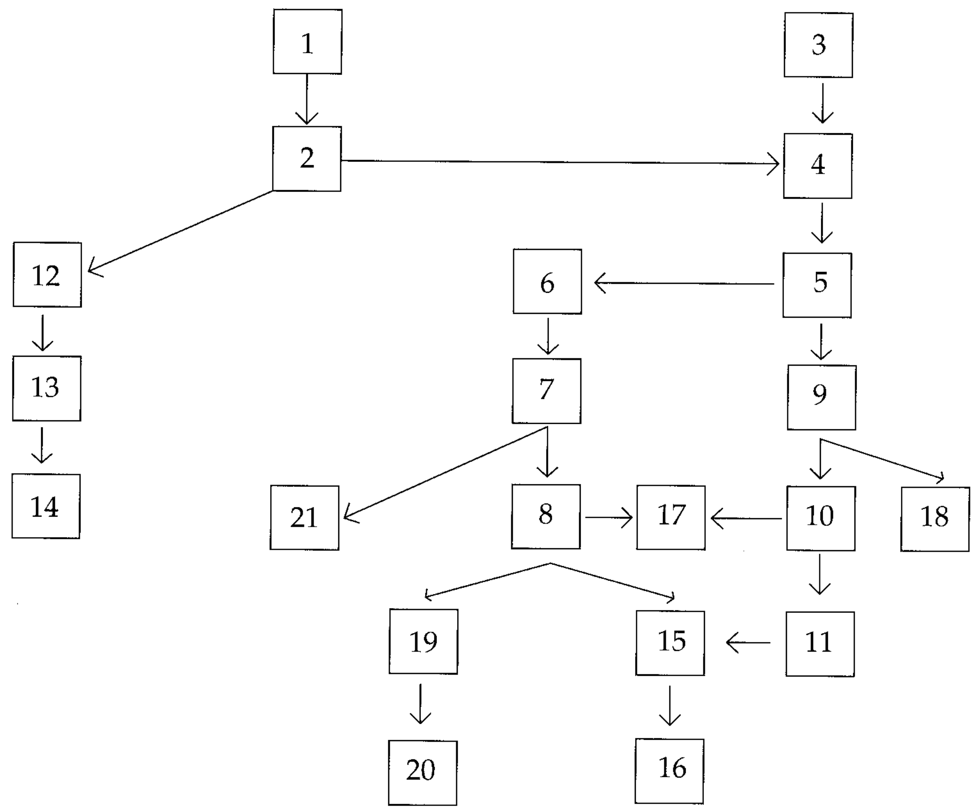
Elaboration of the individual model to incorporate operational constraints such as acquisition and administrative expenses, accounting requirements, and the effects of contract terminations is treated in Chapters 15 and 16. In Chapter 17, individual risk theory models are used to obtain actuarial present values, benefit and contract premiums, and benefit reserves for selected special plans including life annuities with certain periods that depend on the contract premium, variable and flexible products, and accelerated benefits. In Chapter 18, the elementary models for plans involving two lives are extended to incorporate contingencies based on a larger number of lives and more complicated benefits.

In Chapter 19, concepts of population theory are introduced. These concepts are then applied to tracing the progress of life insurance benefits provided on a group, or population, basis. The tools from population theory are applied to tracing the progress of retirement income benefits provided on a group basis in Chapter 20.

Chapter 21 is a step into the future. Interest rates are assumed to be random variables. Several stochastic models are introduced and then integrated into models for basic insurance and annuity contracts.

The following diagram illustrates the prerequisite structure of the chapters. The arrows indicate the direction of the flow. For any chapter, the chapters that are upstream are prerequisite. For example, Chapter 6 has as prerequisites Chapters 1, 2, 3, 4, and 5.

Interrelationships of Chapters



We have a couple of hints for the reader, particularly for one for whom the material is new. The exercises are an important part of the text and include material not covered in the main discussion. In some cases, hints will be offered to aid in the solution. Answers to all exercises are provided except where the answer is given in the formulation of the problem. Writing computer programs and using electronic spreadsheets or mathematical software for the evaluation of basic formulas are excellent ways of enhancing the level of understanding of the material. The student is encouraged to use these tools to work through the computing exercises.

We conclude these introductory comments with some miscellaneous information on the format of the text. First, each chapter concludes with a reference section that provides guidance to those who wish to pursue further study of the topics covered in the chapter. These sections also contain comments that relate the ideas used in insurance models to those used in other areas.

Second, Chapters 1, 12, 13, 14, and 18 contain some theorems with their proofs included as chapter appendices. These proofs are included for completeness, but are not essential to an understanding of the material. They may be excluded from study at the reader's discretion. Exercises associated with these appendices should also be considered optional.

Third, general appendices appear at the end of the text. Included here are numerical tables for computations for examples and exercises, an index to notation, a discussion of general rules for writing actuarial symbols, reference citations, answers to exercises, a subject index, and supplemental mathematical formulas that are not assumed to be a part of the mathematical prerequisites.

Fourth, we observe two notational conventions. A referenced random variable, X, for example, is designated with a capital letter. This notational convention is not used in older texts on probability theory. It will be our practice, in order to indicate the correspondence, to use the appropriate random variable symbol as a subscript on functions and operators that depend on the random variable. We will use the general abbreviation ***log*** to refer to natural (base e) logarithms, because a distinction between natural and common logarithms is unnecessary in the examples and exercises. We assume the natural logarithm in our computations.

Fifth, currencies such as dollar, pound, lira, or yen are not specified in the examples and exercises due to the international character of the required computations.

Finally, since we have discussed prerequisites to this work, some major theorems from undergraduate calculus and probability theory will be used without review or restatement in the discussions and exercises.

1

THE ECONOMICS OF INSURANCE

1.1 Introduction

Each of us makes plans and has expectations about the path his or her life will follow. However, experience teaches that plans will not unfold with certainty and sometimes expectations will not be realized. Occasionally plans are frustrated because they are built on unrealistic assumptions. In other situations, fortuitous circumstances interfere. Insurance is designed to protect against serious financial reversals that result from random events intruding on the plans of individuals.

We should understand certain basic limitations on insurance protection. First, it is restricted to reducing those consequences of random events that can be measured in monetary terms. Other types of losses may be important, but not amenable to reduction through insurance.

For example, pain and suffering may be caused by a random event. However, insurance coverages designed to compensate for pain and suffering often have been troubled by the difficulty of measuring the loss in monetary units. On the other hand, economic losses can be caused by events such as property set on fire by its owner. Whereas the monetary terms of such losses may be easy to define, the events are not insurable because of the nonrandom nature of creating the losses.

A second basic limitation is that insurance does not directly reduce the probability of loss. The existence of windstorm insurance will not alter the probability of a destructive storm. However, a well-designed insurance system often provides financial incentives for loss prevention activities. An insurance product that encouraged the destruction of property or the withdrawal of a productive person from the labor force would affect the probability of these economically adverse events. Such insurance would not be in the public interest.

Several examples of situations where random events may cause financial losses are the following:

- The destruction of property by fire or storm is usually considered a random event in which the loss can be measured in monetary terms.

- A damage award imposed by a court as a result of a negligent act is often considered a random event with resulting monetary loss.
- Prolonged illness may strike at an unexpected time and result in financial losses. These losses will be due to extra health care expenses and reduced earned income.
- The death of a young adult may occur while long-term commitments to family or business remain unfulfilled. Or, if the individual survives to an advanced age, resources for meeting the costs of living may be depleted.

These examples are designed to illustrate the definition:

An ***insurance system*** is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations.

It is helpful to make certain distinctions between insurance and related systems. Banking institutions were developed for the purpose of receiving, investing, and dispensing the savings of individuals and corporations. The cash flows in and out of a savings institution do not follow deterministic paths. However, unlike insurance systems, savings institutions do not make payments based on the size of a financial loss occurring from an event outside the control of the person suffering the loss.

Another system that does make payments based on the occurrence of random events is gambling. Gambling or wagering, however, stands in contrast to an insurance system in that an insurance system is designed to protect against the economic impact of risks that exist independently of, and are largely beyond the control of, the insured. The typical gambling arrangement is established by defining payoff rules about the occurrence of a contrived event, and the risk is voluntarily sought by the participants. Like insurance, a gambling arrangement typically redistributes wealth, but it is there that the similarity ends.

Our definition of an insurance system is purposefully broad. It encompasses systems that cover losses in both property and human-life values. It is intended to cover insurance systems based on individual decisions to participate as well as systems where participation is a condition of employment or residence. These ideas are discussed in Section 1.4.

The economic justification for an insurance system is that it contributes to general welfare by improving the prospect that plans will not be frustrated by random events. Such systems may also increase total production by encouraging individuals and corporations to embark on ventures where the possibility of large losses would inhibit such projects in the absence of insurance. The development of marine insurance, for reducing the financial impact of the perils of the sea, is an example of this point. Foreign trade permitted specialization and more efficient production, yet mutually advantageous trading activity might be too hazardous for some potential trading partners without an insurance system to cover possible losses at sea.

1.2 Utility Theory

If people could foretell the consequences of their decisions, their lives would be simpler but less interesting. We would all make decisions on the basis of preferences for certain consequences. However, we do not possess perfect foresight. At best, we can select an action that will lead to one set of uncertainties rather than another. An elaborate theory has been developed that provides insights into decision making in the face of uncertainty. This body of knowledge is called ***utility theory.*** Because of its relevance to insurance systems, its main points will be outlined here.

One solution to the problem of decision making in the face of uncertainty is to define the value of an economic project with a random outcome to be its expected value. By this ***expected value principle*** the distribution of possible outcomes may be replaced for decision purposes by a single number, the expected value of the random monetary outcomes. By this principle, a decision maker would be indifferent between assuming the random loss X and paying amount $E[X]$ in order to be relieved of the possible loss. Similarly, a decision maker would be willing to pay up to $E[Y]$ to participate in a gamble with random payoff Y. In economics the expected value of random prospects with monetary payments is frequently called the ***fair*** or ***actuarial value*** of the prospect.

Many decision makers do not adopt the expected value principle. For them, their wealth level and other aspects of the distribution of outcomes influence their decisions.

Below is an illustration designed to show the inadequacy of the expected value principle for a decision maker considering the value of accident insurance. In all cases, it is assumed that the probability of an accident is 0.01 and the probability of no accident is 0.99. Three cases are considered according to the amount of loss arising from an accident; the expected loss is tabulated for each.

Case	Possible Losses		Expected Loss
1	0	1	0.01
2	0	1 000	10.00
3	0	1 000 000	10 000.00

A loss of 1 might be of little concern to the decision maker who then might be unwilling to pay more than the expected loss to obtain insurance. However, the loss of 1,000,000, which may exceed his net worth, could be catastrophic. In this case, the decision maker might well be willing to pay more than the expected loss of 10,000 in order to obtain insurance. The fact that the amount a decision maker would pay for protection against a random loss may differ from the expected value suggests that the expected value principle is inadequate to model behavior.

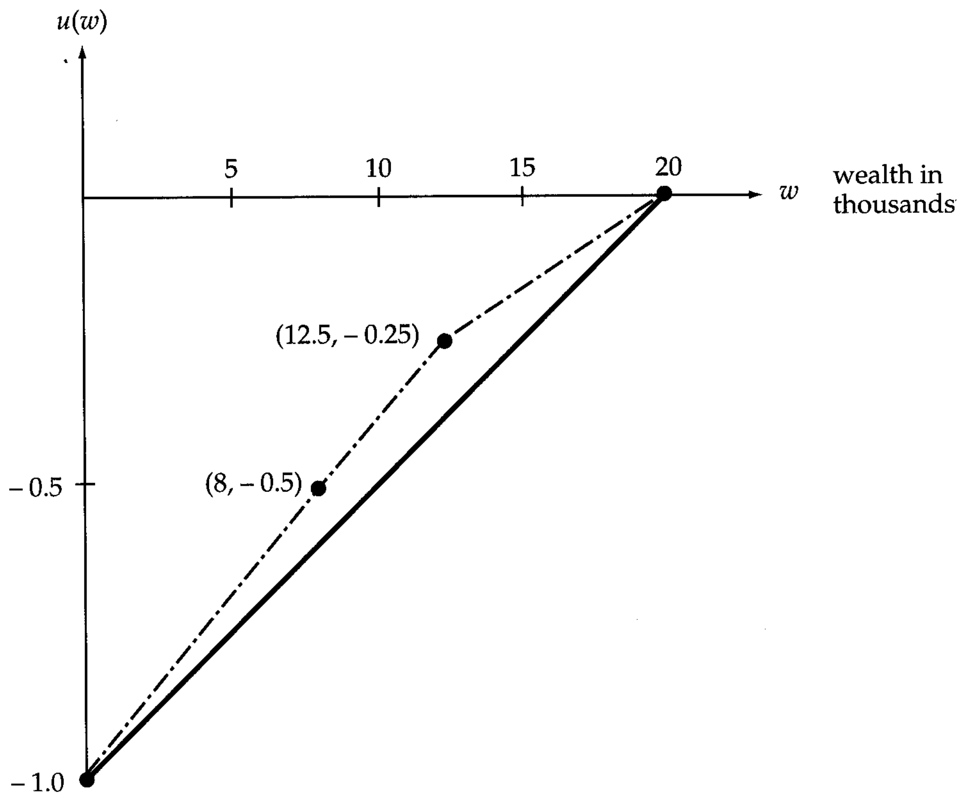
We now study another approach to explain why a decision maker may be willing to pay more than the expected value. At first we simply assume that the value or utility that a particular decision maker attaches to wealth of amount w, measured in monetary units, can be specified in the form of a function $u(w)$, called a ***utility function.*** We demonstrate a procedure by which a few values of such a function can be determined. For this we assume that our decision maker has wealth equal to 20,000. A linear transformation,

$$u^*(w) = a\, u(w) + b \qquad a > 0,$$

yields a function $u^*(w)$, which is essentially equivalent to $u(w)$. It then follows by choice of a and b that we can determine arbitrarily the 0 point and one additional point of an individual's utility function. Therefore, we fix $u(0) = -1$ and $u(20{,}000) = 0$. These values are plotted on the solid line in Figure 1.2.1.

FIGURE 1.2.1

Determination of a Utility Function



We now ask a question of our decision maker: Suppose you face a loss of 20,000 with probability 0.5, and will remain at your current level of wealth with probability 0.5. What is the maximum amount* G you would be willing to pay for

*Premium quantities, by convention in insurance literature, are capitalized although they are not random variables.

complete insurance protection against this random loss? We can express this question in the following way: For what value of G does

$$u(20{,}000 - G) = 0.5\,u(20{,}000) + 0.5\,u(0)$$

$$= (0.5)(0) + (0.5)(-1) = -0.5?$$

If he pays amount G, his wealth will certainly remain at $20{,}000 - G$. The equal sign indicates that the decision maker is indifferent between paying G with certainty and accepting the expected utility of wealth expressed on the right-hand side.

Suppose the decision maker's answer is $G = 12{,}000$. Therefore,

$$u(20{,}000 - 12{,}000) = u(8{,}000) = -0.5.$$

This result is plotted on the dashed line in Figure 1.2.1. Perhaps the most important aspect of the decision maker's response is that he is willing to pay an amount for insurance that is greater than

$$(0.5)(0) + (0.5)(20{,}000) = 10{,}000,$$

the expected value of the loss.

This procedure can be used to add as many points $[w, u(w)]$, for $0 \le w \le 20{,}000$, as needed to obtain a satisfactory approximation to the decision maker's utility of wealth function. Once a utility value has been assigned to wealth levels w_1 and w_2, where $0 \le w_1 < w_2 \le 20{,}000$, we can determine an additional point by asking the decision maker the following question: What is the maximum amount you would pay for complete insurance against a situation that could leave you with wealth w_2 with specified probability p, or at reduced wealth level w_1 with probability $1 - p$? We are asking the decision maker to fix a value G such that

$$u(w_2 - G) = (1 - p)u(w_1) + p\,u(w_2). \tag{1.2.1}$$

Once the value $w_2 - G = w_3$ is available, the point $[w_3, (1 - p)u(w_1) + p\,u(w_2)]$ is determined as another point of the utility function. Such a process has been used to assign a fourth point $(12{,}500, -0.25)$ in Figure 1.2.1. Such solicitation of preferences leads to a set of points on the decision maker's utility function. A smooth function with a second derivative may be fitted to these points to provide for a utility function everywhere.

After a decision maker has determined his utility of wealth function by the method outlined, the function can be used to compare two random economic prospects. The prospects are denoted by the random variables X and Y. We seek a decision rule that is consistent with the preferences already elicited in the determination of the utility of wealth function. Thus, if the decision maker has wealth w, and must compare the random prospects X and Y, the decision maker selects X if

$$\mathrm{E}[u(w + X)] > \mathrm{E}[u(w + Y)],$$

and the decision maker is indifferent between X and Y if

$$E[u(w + X)] = E[u(w + Y)].$$

Although the method of eliciting and using a utility function may seem plausible, it is clear that our informal development must be augmented by a more rigorous chain of reasoning if utility theory is to provide a coherent and comprehensive framework for decision making in the face of uncertainty. If we are to understand the economic role of insurance, such a framework is needed. An outline of this more rigorous theory follows.

The theory starts with the assumption that a rational decision maker, when faced with two distributions of outcomes affecting wealth, is able to express a preference for one of the distributions or indifference between them. Furthermore, the preferences must satisfy certain consistency requirements. The theory culminates in a theorem stating that if preferences satisfy the consistency requirements, there is a utility function $u(w)$ such that if the distribution of X is preferred to the distribution of Y, $E[u(X)] > E[u(Y)]$, and if the decision maker is indifferent between the two distributions, $E[u(X)] = E[u(Y)]$. That is, the qualitative preference or indifference relation may be replaced by a consistent numerical comparison. In Section 1.6, references are given for the detailed development of this theory.

Before turning to applications of utility theory for insights into insurance, we record some observations about utility.

Observations:

1. Utility theory is built on the assumed existence and consistency of preferences for probability distributions of outcomes. A utility function should reveal no surprises. It is a numerical description of existing preferences.
2. A utility function need not, in fact, cannot, be determined uniquely. For example, if

$$u^*(w) = a\,u(w) + b \qquad a > 0,$$

   then

$$E[u(X)] > E[u(Y)]$$

   is equivalent to

$$E[u^*(X)] > E[u^*(Y)].$$

   That is, preferences are preserved when the utility function is an increasing linear transformation of the original form. This fact was used in the Figure 1.2.1 illustration where two points were chosen arbitrarily.
3. Suppose the utility function is linear with a positive slope; that is,

$$u(w) = a\,w + b \qquad a > 0.$$

   Then, if $E[X] = \mu_X$ and $E[Y] = \mu_Y$, we have

$$E[u(X)] = a\,\mu_X + b > E[u(Y)] = a\,\mu_Y + b$$

   if and only if $\mu_X > \mu_Y$. That is, for increasing linear utility functions, preferences for distributions of outcomes are in the same order as the expected values of the distributions being compared. Therefore, the expected value principle for rational economic behavior in the face of uncertainty is consistent with the expected utility rule when the utility function is an increasing linear one.

1.3 Insurance and Utility

In Section 1.2 we outlined utility theory for the purpose of gaining insights into the economic role of insurance. To examine this role we start with an illustration. Suppose a decision maker owns a property that may be damaged or destroyed in the next accounting period. The amount of the loss, which may be 0, is a random variable denoted by X. We assume that the distribution of X is known. Then $E[X]$, the expected loss in the next period, may be interpreted as the long-term average loss if the experiment of exposing the property to damage may be observed under identical conditions a great many times. It is clear that this long-term set of trials could not be performed by an individual decision maker.

Suppose that an insurance organization (***insurer***) was established to help reduce the financial consequences of the damage or destruction of property. The insurer would issue contracts (***policies***) that would promise to pay the owner of a property a defined amount equal to or less than the financial loss if the property were damaged or destroyed during the period of the policy. The contingent payment linked to the amount of the loss is called a ***claim*** payment. In return for the promise contained in the policy, the owner of the property (***insured***) pays a consideration (***premium***).

The amount of the premium payment is determined after an economic decision principle has been adopted by each of the insurer and insured. An opportunity exists for a mutually advantageous insurance policy when the premium for the policy set by the insurer is less than the maximum amount that the property owner is willing to pay for insurance.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this case, the insurer would adopt the expected value principle in setting its premium, as indicated in Section 1.2, Observation 3; that is, the insurer would set its basic price for full insurance coverage as the expected loss, $E[X] = \mu$. In this context μ is called the ***pure*** or ***net premium*** for the 1-period insurance policy. To provide for expenses, taxes, and profit and for some security against adverse loss experience, the insurance system would decide to set the premium for the policy by ***loading,*** adding to, the pure premium. For instance, the loaded premium, denoted by H, might be given by

$$H = (1 + a)\mu + c \qquad a > 0, \qquad c > 0.$$

In this expression the quantity $a\mu$ can be viewed as being associated with expenses that vary with expected losses and with the risk that claims experience will deviate from expected. The constant c provides for expected expenses that do not vary with losses. Later, we will illustrate other economic principles for determining premiums that might be adopted by the insurer.

We now apply utility theory to the decision problems faced by the owner of the property subject to loss. The property owner has a utility of wealth function $u(w)$ where wealth w is measured in monetary terms. The owner faces a possible loss due to random events that may damage the property. The distribution of the random loss X is assumed to be known. Much as in (1.2.1), the owner will be indifferent between paying an amount G to the insurer, who will assume the random financial loss, and assuming the risk himself. This situation can be stated as

$$u(w - G) = \mathrm{E}[u(w - X)]. \tag{1.3.1}$$

The right-hand side of (1.3.1) represents the expected utility of not buying insurance when the owner's current wealth is w. The left-hand side of (1.3.1) represents the expected utility of paying G for complete financial protection.

If the owner has an increasing linear utility function, that is, $u(w) = bw + d$ with $b > 0$, the owner will be adopting the expected value principle. In this case the owner prefers, or is indifferent to, the insurance when

$$u(w - G) = b(w - G) + d \geq \mathrm{E}[u(w - X)] = \mathrm{E}[b(w - X) + d],$$

$$b(w - G) + d \geq b(w - \mu) + d,$$

$$G \leq \mu.$$

That is, if the owner has an increasing linear utility function, the premium payments that will make the owner prefer, or be indifferent to, complete insurance are less than or equal to the expected loss. In the absence of a subsidy, an insurer, over the long term, must charge more than its expected losses. Therefore, in this case, there seems to be little opportunity for a mutually advantageous insurance contract. If an insurance contract is to result, the insurer must charge a premium in excess of expected losses and expenses to avoid a bias toward insufficient income. The property owner then cannot use a linear utility function.

In Section 1.2 we mention that the preferences of a decision maker must satisfy certain consistency requirements to ensure the existence of a utility function. Although these requirements were not listed, they do not include any specifications that would force a utility function to be linear, quadratic, exponential, logarithmic, or any other particular form. In fact, each of these named functions might serve as a utility function for some decision maker or they might be spliced together to reflect some other decision maker's preferences.

Nevertheless, it seems natural to assume that $u(w)$ is an increasing function, "more is better." In addition, it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is the idea of decreasing marginal utility in economics.

The approximate utility function of Figure 1.2.1 consists of straight line segments with positive slopes. It is such that $\Delta^2 u(w) \leq 0$. If these ideas are extended to smoother functions, the two properties suggested by observation are $u'(w) > 0$ and $u''(w) < 0$. The second inequality indicates that $u(w)$ is a strictly concave downward function.

In discussing insurance decisions using strictly concave downward utility functions, we will make use of one form of ***Jensen's inequalities.*** These inequalities state that for a random variable X and function $u(w)$,

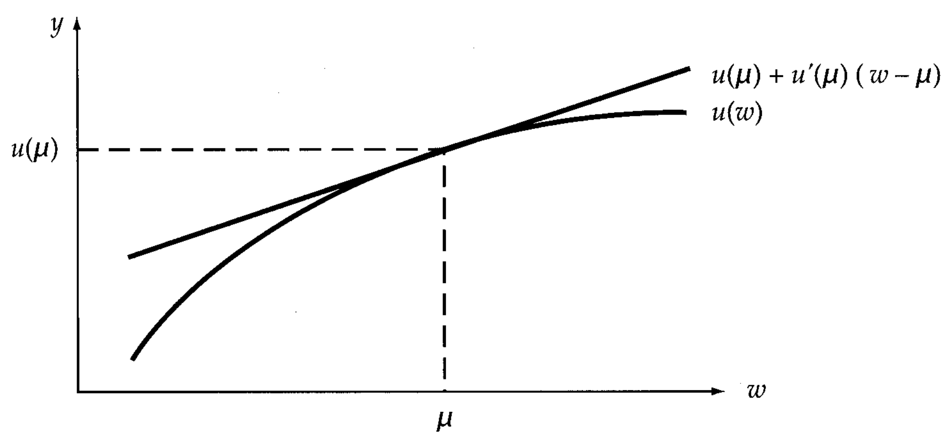
$$\text{if } u''(w) < 0, \text{ then } \mathrm{E}[u(X)] \leq u(\mathrm{E}[X]), \tag{1.3.2}$$

$$\text{if } u''(w) > 0, \text{ then } \mathrm{E}[u(X)] \geq u(\mathrm{E}[X]). \tag{1.3.3}$$

Jensen's inequalities require the existence of the two expected values. Proofs of the inequalities are required by Exercise 1.3. A second proof of (1.3.2) is almost immediate from consideration of Figure 1.3.1 as follows.

FIGURE 1.3.1

Proof of Jensen's Inequalities for the Case $u'(w) > 0$ and $u''(w) < 0$



If $\mathrm{E}[X] = \mu$ exists, one considers the line tangent to $u(w)$,

$$y = u(\mu) + u'(\mu)(w - \mu),$$

at the point $(\mu, u(\mu))$. Because of the strictly concave characteristic of $u(w)$, the graph of $u(w)$ will be below the tangent line; that is,

$$u(w) \leq u(\mu) + u'(\mu)(w - \mu) \tag{1.3.4}$$

for all values of w. If w is replaced by the random variable X, and the expectation is taken on each side of the inequality (1.3.4), we have $\mathrm{E}[u(X)] \leq u(\mu)$.

This basic inequality has several applications in actuarial mathematics. Let us apply Jensen's inequality (1.3.2) to the decision maker's insurance problem as formulated in (1.3.1). We will assume that the decision maker's preferences are such that $u'(w) > 0$ and $u''(w) < 0$. Applying Jensen's inequality to (1.3.1), we have

$$u(w - G) = \mathrm{E}[u(w - X)] \leq u(w - \mu). \tag{1.3.5}$$

Because $u'(w) > 0$, $u(w)$ is an increasing function. Therefore, (1.3.5) implies that $w - G \le w - \mu$, or $G \ge \mu$ with $G > \mu$ unless X is a constant. In economic terms, we have found that if $u'(w) > 0$ and $u''(w) < 0$, the decision maker will pay an amount greater than the expected loss for insurance. If G is at least equal to the premium set by the insurer, there is an opportunity for a mutually advantageous insurance policy.

Formally we say a decision maker with utility function $u(w)$ is ***risk averse*** if, and only if, $u''(w) < 0$.

We now employ a general utility function for the insurer. We let $u_I(w)$ denote the utility of wealth function of the insurer and w_I denote the current wealth of the insurer measured in monetary terms. Then the minimum acceptable premium H for assuming random loss X, from the viewpoint of the insurer, may be determined from (1.3.6):

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)]. \tag{1.3.6}$$

The left-hand side of (1.3.6) is the utility attached to the insurer's current position. The right-hand side is the expected utility associated with collecting premium H and paying random loss X. In other words, the insurer is indifferent between the current position and providing insurance for X at premium H. If the insurer's utility function is such that $u_I'(w) > 0$, $u_I''(w) < 0$, we can use Jensen's inequality (1.3.2) along with (1.3.6) to obtain

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)] \le u_I(w_I + H - \mu).$$

Following the same line of reasoning displayed in connection with (1.3.5), we can conclude that $H \ge \mu$. If G, as determined by the decision maker by solving (1.3.5), is such that $G \ge H \ge \mu$, an insurance contract is possible. That is, the expected utility of neither party to the contract is decreased.

A utility function is based on the decision maker's preferences for various distributions of outcomes. An insurer need not be an individual. It may be a partnership, corporation, or government agency. In this situation the determination of $u_I(w)$, the insurer's utility function, may be a rather complicated matter. For example, if the insurer is a corporation, one of management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflicting attitudes toward risk among the groups of stockholders.

Several elementary functions are used to illustrate properties of utility functions. Here we examine exponential, fractional power, and quadratic functions. Exercises 1.6, 1.8, 1.9, 1.10(b), and 1.13 cover the logarithmic utility function.

An ***exponential utility function*** is of the form

$$u(w) = -e^{-\alpha w} \qquad \text{for all } w \text{ and for a fixed } \alpha > 0$$

and has several attractive features. First,

$$u'(w) = \alpha e^{-\alpha w} > 0.$$

Second,

$$u''(w) = -\alpha^2 e^{-\alpha w} < 0.$$

Therefore, $u(w)$ may serve as the utility function of a risk-averse individual. Third, finding

$$\mathrm{E}[-e^{-\alpha X}] = -\mathrm{E}[e^{-\alpha X}] = -\mathrm{M}_X(-\alpha)$$

is essentially the same as finding the moment generating function (m.g.f.) of X. In this expression,

$$\mathrm{M}_X(t) = \mathrm{E}[e^{tX}]$$

denotes the m.g.f. of X. Fourth, insurance premiums do not depend on the wealth of the decision maker. This statement is verified for the insured by substituting the exponential utility function into (1.3.1). That is,

$$\begin{aligned} -e^{-\alpha(w-G)} &= \mathrm{E}[-e^{-\alpha(w-X)}], \\ e^{\alpha G} &= \mathrm{M}_X(\alpha), \\ G &= \frac{\log \mathrm{M}_X(\alpha)}{\alpha} \end{aligned}$$

and G does not depend on w.

The verification for the insurer is done by substituting the exponential utility function with parameter α_I into (1.3.6):

$$\begin{aligned} -e^{-\alpha_I w_I} &= \mathrm{E}[-e^{-\alpha_I(w_I+H-X)}], \\ -e^{-\alpha_I w_I} &= -e^{-\alpha_I(w_I+H)}\,\mathrm{M}_X(\alpha_I), \\ H &= \frac{\log \mathrm{M}_X(\alpha_I)}{\alpha_I}. \end{aligned}$$

Example 1.3.1

A decision maker's utility function is given by $u(w) = -e^{-5w}$. The decision maker has two random economic prospects (gains) available. The outcome of the first, denoted by X, has a normal distribution with mean 5 and variance 2. Henceforth, a statement about a normal distribution with mean μ and variance σ^2 will be abbreviated as $N(\mu, \sigma^2)$. The second prospect, denoted by Y, is distributed as $N(6, 2.5)$. Which prospect will be preferred?

Solution:

We have

$$\begin{aligned} \mathrm{E}[u(X)] &= \mathrm{E}[-e^{-5X}] \\ &= -\mathrm{M}_X(-5) = -e^{[-5(5)+(5^2)(2)/2]} \\ &= -1, \end{aligned}$$

and

$$\begin{aligned} E[u(Y)] &= E[-e^{-5Y}] \\ &= -M_Y(-5) = -e^{[-5(6)+(5^2)(2.5)/2]} \\ &= -e^{1.25}. \end{aligned}$$

Therefore,

$$E[u(X)] = -1 > E[u(Y)] = -e^{1.25},$$

and the distribution of X is preferred to the distribution of Y. ▼

In Example 1.3.1 prospect X is preferred to Y despite the fact that $\mu_X = 5 < \mu_Y = 6$. Since the decision maker is risk averse, the fact that the distribution of Y is more diffuse than the distribution of X is weighted heavily against the distribution of Y in assessing its desirability. If Y had a $N(6, 2.4)$ distribution, $E[u(Y)] = -1$ and the decision maker would be indifferent between the distributions of X and Y.

The family of ***fractional power utility*** functions is given by

$$u(w) = w^{\gamma} \qquad w > 0,\ 0 < \gamma < 1.$$

A member of this family might represent the preferences of a risk-averse decision maker since

$$u'(w) = \gamma w^{\gamma-1} > 0$$

and

$$u''(w) = \gamma(\gamma - 1)w^{\gamma-2} < 0.$$

In this family, premiums depend on the wealth of the decision maker in a manner that may be sufficiently realistic in many situations.

Example 1.3.2

A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of $w = 10$ and faces a random loss X with a uniform distribution on $(0, 10)$. What is the maximum amount this decision maker will pay for complete insurance against the random loss?

Solution:

Substituting into (1.3.1) we have

$$\begin{aligned} \sqrt{10 - G} &= E[\sqrt{10 - X}] \\ &= \int_0^{10} \sqrt{10 - x}\ 10^{-1}\,dx \\ &= \left.\frac{-2(10 - x)^{3/2}}{3(10)}\right|_0^{10} \\ &= \frac{2}{3}\sqrt{10}, \\ G &= 5.5556. \end{aligned}$$

The decision maker is risk averse and has $u'(w) > 0$. Following the discussion of (1.3.5), we would expect $G > \mathrm{E}[X]$, and in this example $G = 5.5556 > \mathrm{E}[X] = 5$.▼

The family of ***quadratic utility*** functions is given by

$$u(w) = w - \alpha w^2 \qquad w < (2\alpha)^{-1}, \qquad \alpha > 0.$$

A member of this family might represent the preferences of a risk-averse decision maker since $u''(w) = -2\alpha$. While a quadratic utility function is convenient because decisions depend only on the first two moments of the distributions of outcomes under consideration, there are certain consequences of its use that strike some people as being unreasonable. Example 1.3.3 illustrates one of these consequences.

Example 1.3.3

A decision maker's utility of wealth function is given by

$$u(w) = w - 0.01w^2 \qquad w < 50.$$

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount c with probability $1 - p$. For the values of w, c, and p exhibited in the table below, find the maximum insurance premium that the decision maker will pay for complete insurance. Assume $c \le w < 50$.

Solution:

For the facts stated, (1.3.1) becomes

$$\begin{aligned} u(w - G) &= pu(w) + (1 - p)u(w - c), \\ (w - G) - 0.01(w - G)^2 &= p(w - 0.01w^2) \\ &\quad + (1 - p)[(w - c) - 0.01(w - c)^2]. \end{aligned}$$

For given values of w, p, and c this expression becomes a quadratic equation. Two solutions are shown.

Wealth w	**Loss** c	**Probability** p	**Insurance Premium** G
10	10	0.5	5.28
20	10	0.5	5.37

▼

In Example 1.3.3, as anticipated, G is greater than the expected loss of 5. However, the maximum insurance premium for exactly the same loss distribution increases with the wealth of the decision maker. This result seems unreasonable to some who anticipate that more typical behavior would be a decrease in the amount a decision maker would pay for insurance when an increase in wealth would permit the decision maker to absorb more of a random loss. Unfortunately, a maximum insurance premium that increases with wealth is a property of quadratic utility functions. Consequently, these utility functions should not be selected by a decision maker who perceives that his ability to absorb random losses goes up with increases in wealth.

If we rework Example 1.3.3 using an exponential utility function, we know that the premium G will not depend on w, the amount of wealth. In fact, if $u(w) = -e^{-0.01w}$, it can be shown that $G = 5.12$ for both $w = 10$ and $w = 20$.

Example 1.3.4

The probability that a property will not be damaged in the next period is 0.75. The probability density function (p.d.f.) of a positive loss is given by

$$f(x) = 0.25(0.01e^{-0.01x}) \qquad x > 0.$$

The owner of the property has a utility function given by

$$u(w) = -e^{-0.005w}.$$

Calculate the expected loss and the maximum insurance premium the property owner will pay for complete insurance.

Solution:

The expected loss is given by

$$\begin{aligned} \mathrm{E}[X] &= 0.75(0) + 0.25 \int_0^\infty x(0.01e^{-0.01x})dx \\ &= 25. \end{aligned}$$

We apply (1.3.1) to determine the maximum premium that the owner will pay for complete insurance. This premium will be consistent with the property owner's preferences as summarized in the utility function:

$$\begin{aligned} u(w - G) &= 0.75u(w) + \int_0^\infty u(w - x)f(x)dx, \\ -e^{-0.005(w-G)} &= -0.75e^{-0.005w} - 0.25 \int_0^\infty e^{-0.005(w-x)}(0.01e^{-0.01x})dx, \\ e^{0.005G} &= 0.75 + (0.25)(2) \\ &= 1.25, \\ G &= 200 \log 1.25 \\ &= 44.63. \end{aligned}$$

Therefore, in accord with the property owner's preferences, he will pay up to $44.63 - 25 = 19.63$ in excess of the expected loss to purchase insurance covering all losses in the next period. ▼

In Example 1.3.5 the notion of insurance that covers something less than the complete loss is introduced. A modification is made in (1.3.1) to accommodate the fact that losses are shared by the decision maker and the insurance system.

Example 1.3.5

The property owner in Example 1.3.4 is offered an insurance policy that will pay 1/2 of any loss during the next period. The expected value of the partial loss payment is $E[X/2] = 12.50$. Calculate the maximum premium that the property owner will pay for this insurance.

Solution:

Consistent with his attitude toward risk, as summarized in his utility function, the premium is determined from

$$0.75u(w - G) + \int_0^\infty u\left(w - G - \frac{x}{2}\right) f(x)dx$$

$$= 0.75u(w) + \int_0^\infty u(w - x)f(x)dx.$$

The left-hand side of this equation represents the expected utility with the partial insurance coverage. The right-hand side represents the expected utility with no insurance. For the exponential utility function and p.d.f. of losses specified in Example 1.3.4, it can be shown that $G = 28.62$. The property owner is willing to pay up to $G - \mu = 28.62 - 12.50 = 16.12$ more than the expected partial loss for the partial insurance coverage. ▼

1.4 Elements of Insurance

Individuals and organizations face the threat of financial loss due to random events. In Section 1.3 we saw how insurance can increase the expected utility of a decision maker facing such random losses. Insurance systems are unique in that the alleviation of financial losses in which the number, size, or time of occurrence is random is the primary reason for their existence. In this section we review some of the factors influencing the organization and management of an insurance system.

An insurance system can be organized only after the identification of a class of situations where random losses may occur. The word random is taken to mean, along with other attributes, that the frequency, size, or time of loss is not under the control of the prospective insured. If such control exists, or if a claim payment exceeds the actual financial loss, an incentive to incur a loss will exist. In such a situation, the assumptions under which the insurance system was organized will become invalid. The actual conditions under which premiums are collected and claims paid will be different from those assumed in organizing the system. The system will not achieve its intended objective of not decreasing the expected utilities of both the insured and the insurer.

Once a class of insurable situations is identified, information on the expected utilities and the loss-generating process can be obtained. Market research in insurance can be viewed as an effort to learn about the utility functions, that is, the risk preferences of consumers.

The processes generating size and time of loss may be sufficiently stable over time so that past information can be used to plan the system. When a new insurance system is organized, directly relevant statistics are not often available. However, enough ancillary information from similar risk situations may be obtained to identify the risks and to provide preliminary estimates of the probability distributions needed to determine premiums. Because most insurance systems operate under dynamic conditions, it is important that a plan exist for collecting and analyzing insurance operating data so that the insurance system can adapt. Adaptation in this case may mean changing premiums, paying an experience-based dividend or premium refund, or modifying future policies.

In a competitive economy, market forces encourage insurers to price short-term policies so that deviations of experience from expected value behave as independent random variables. Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains. Such consistent deviations would indicate inefficiencies in the insurance market.

As a result, the classification of risks into homogeneous groups is an important function within a market-based insurance system. Experience deviations that are random indicate efficiency or equity in classification. In a competitive insurance market, the continual interaction of numerous buyers and sellers forces experimentation with classification systems as the market participants attempt to take advantage of perceived patterns of deviations. Because insurance losses may be relatively rare events, it is often difficult to identify nonrandom patterns. The cost of classification information for a refined classification system also places a bound on experimentation in this area.

For insurance systems organized to serve groups rather than individuals, the issue is no longer whether deviations in insurance experience are random for each individual. Instead, the question is whether deviations in group experience are random. Consistent deviations in experience from that expected would indicate the need for a revision in the system.

Group insurance decisions do not rest on individual expected utility comparisons. Instead, group insurance plans are based on a collective decision on whether the system increases the total welfare of the group. Group health insurance providing benefits for the employees of a firm is an example.

1.5 Optimal Insurance

The ideas outlined in Sections 1.2, 1.3, and 1.4 have been used as the foundation of an elaborate theory for guiding insurance decision makers to actions consistent with their preferences. In this section we present one of the main results from this theory and review many of the ideas introduced so far.

A decision maker has wealth of amount w and faces a loss in the next period. This loss is a random variable X. The decision maker can buy an insurance contract that will pay $I(x)$ of the loss x. To avoid an incentive to incur the loss, we assume that all ***feasible*** insurance contracts are such that $0 \le I(x) \le x$. We make the simplifying assumption that all feasible insurance contracts with $E[I(X)] = \beta$ can be purchased for the same amount P.

The decision maker has formulated a utility function $u(w)$ that is consistent with his preferences for distributions of outcomes. We assume that the decision maker is risk averse, $u''(w) < 0$. We further assume that the decision maker has decided on the amount, denoted by P, to be paid for insurance. The question is: which of the insurance contracts from the class of feasible contracts with expected claims, β, and premium, P, should be purchased to maximize the expected utility of the decision maker?

One subclass of the class of feasible insurance contracts is defined as follows:

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases} \tag{1.5.1}$$

This class of contracts is characterized by the fact that claim payments do not start until the loss exceeds the ***deductible*** amount d. For losses above the deductible amount, the excess is paid under the terms of the contract. This type of contract is sometimes called ***stop-loss*** or ***excess-of-loss insurance,*** the choice depending on the application.

In the problem discussed in this section the expected claims are denoted by β. In (1.5.2) the symbol $f(x)$ denotes the p.d.f. and the symbol $F(x)$ denotes the distribution function (d.f.) associated with the random loss X:

$$\beta = \int_d^\infty (x - d)f(x)dx \tag{1.5.2A}$$

or

$$\beta = \int_d^\infty [1 - F(x)]dx. \tag{1.5.2B}$$

Equation (1.5.2B) is obtained from (1.5.2A) by integration by parts. When β is given, then (1.5.2) provides explicit equations for the corresponding deductible, denoted by d^*. In Exercise 1.17, it is shown that d^* exists and is unique.

The main result of this section can be stated as a theorem.

Theorem 1.5.1

If a decision maker
- has wealth of amount w
- is risk averse, in other words, has utility of wealth function $u(w)$ such that $u''(w) < 0$
- faces a random loss X
- will spend amount P on insurance

and the insurance market offers for a payment of P all feasible insurance contracts of the form $I(x)$, $0 \le I(x) \le x$, with $E[I(X)] = \beta$, then the decision maker's expected utility will be maximized by purchasing an insurance policy

$$I_{d^*}(x) = \begin{cases} 0 & x < d^* \\ x - d^* & x \ge d^* \end{cases}$$

where d^* is the solution of

$$\beta - \int_d^{\infty} (x - d) f(x) dx = 0.$$

The theorem is proved in the Appendix to this chapter.

Theorem 1.5.1 is an important result and illustrates many of the ideas developed in this chapter. However, it is instructive to consider certain limitations on its applicability. First, the ratio of premium to expected claims is the same for all available contracts. In fact, the distributions of the random variables $I(X)$ can be very different, and the provision for risk in the premium usually depends on the characteristics of the distribution of $I(X)$. Second, in Theorem 1.5.1, it is assumed that the premium P is fixed by a budget constraint. Alternatives to amount P are not considered. In Exercise 1.22, relaxation of the budget constraint is considered. Third, while the theorem indicates the form of insurance, it does not help to determine the amount P to spend. In the theorem, P is fixed.

1.6 Notes and References

Definitions and principles of actuarial science can be found in "Principles of Actuarial Science" (SOA Committee on Actuarial Principles 1992).

The role of risk in business was developed in a pioneering thesis by Willett (1951). Borch (1974) has published a series of papers applying utility theory to insurance questions. DeGroot (1970) gives a complete development of utility theory starting from basic axioms for consistency among preferences for various distributions of outcomes. DeGroot and Borch both discuss the historically important St. Petersburg paradox, outlined in Exercise 1.2. A paper by Friedman and Savage (1948) provides many insights into utility theory and human behavior.

Pratt (1964) has studied (1.3.1) and derived several theorems about premiums and utility functions. Exercise 1.10, which uses two rough approximations, is related to one of Pratt's results.

Theorem 1.5.1 on optimal insurance was proved by Arrow (1963) in the context of health insurance. The theorem in Exercise 1.21, in which the goal of insurance is to minimize the variance of retained losses, was the subject of papers by Borch (1960) and Kahn (1961). The use of the variance of losses as a measure of stability is discussed by Beard, Pentikäinen, and Pesonen (1984). Exercise 1.23 is based on their discussion.

Appendix

Lemma:
If $u''(w) < 0$ for all w in $[a, b]$, then for w and z in $[a, b]$,

$$u(w) - u(z) \leq (w - z)u'(z). \tag{1.A.1}$$

Proof:
The lemma may be established with the aid of Figure 1.3.1. Using the point slope form, a line tangent to $u(w)$ at the point $(z, u(z))$ has the equation $y - u(z) = u'(z)(w - z)$ and is above the graph of the function $u(w)$ except at the point $(z, u(z))$. Therefore,

$$u(w) - u(z) \leq u'(z)(w - z). \qquad \blacksquare$$

Figure 1.3.1 shows the case $u'(w) > 0$. The same argument holds for $u'(w) < 0$.

In Exercise 1.20 an alternative proof is required.

Proof of Theorem 1.5.1:
Let $I(x)$ be associated with an insurance policy satisfying the hypothesis of the theorem. Then from the lemma,

$$\begin{aligned} &u(w - x + I(x) - P) - u(w - x + I_{d^*}(x) - P) \\ &\quad \leq [I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P). \end{aligned} \tag{1.A.2}$$

In addition, we claim

$$\begin{aligned} &[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \\ &\quad \leq [I(x) - I_{d^*}(x)]u'(w - d^* - P). \end{aligned} \tag{1.A.3}$$

To establish inequality (1.A.3), we must consider three cases:

Case I. $I_{d^*}(x) = I(x)$
In this case equality holds, (1.A.3) is 0 on both sides.

Case II. $I_{d^*}(x) > I(x)$
In this case $I_{d^*}(x) > 0$ and from (1.5.1), $x - I_{d^*}(x) = d^*$. Therefore, equality holds with each side of (1.A.3) equal to $[I(x) - I_{d^*}(x)]u'(w - d^* - P)$.

Case III. $I_{d^*}(x) < I(x)$
In this case $I(x) - I_{d^*}(x) > 0$. From (1.5.1) we obtain $I_{d^*}(x) - x \geq -d^*$ and $I_{d^*}(x) - x - P \geq -d^* - P$. Therefore,

$$u'(w - x + I_{d^*}(x) - P) \leq u'(w - d^* - P)$$

since the second derivative of $u(x)$ is negative and $u'(x)$ is a decreasing function.

Therefore, in each case

$$[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \leq [I(x) - I_{d^*}(x)]u'(w - P - d^*),$$

establishing inequality (1.A.3).

Now, combining inequalities (1.A.2) and (1.A.3) and taking expectations, we have

$$\mathrm{E}[u(w - X + I(X) - P)] - \mathrm{E}[u(w - X + I_{d^*}(X) - P)]$$
$$\leq \mathrm{E}[I(X) - I_{d^*}(X)]u'(w - d^* - P) = (\beta - \beta)u'(w - d^* - P) = 0.$$

Therefore,

$$\mathrm{E}[u(w - X + I(X) - P)] \leq \mathrm{E}[u(w - X + I_{d^*}(X) - P)]$$

and the expected utility will be maximized by selecting $I_{d^*}(x)$, the stop-loss policy. ■

Exercises

Section 1.2

1.1. Assume that a decision maker's current wealth is 10,000. Assign $u(0) = -1$ and $u(10{,}000) = 0$.

a. When facing a loss of X with probability 0.5 and remaining at current wealth with probability 0.5, the decision maker would be willing to pay up to G for complete insurance. The values for X and G in three situations are given below.

X	G
10 000	6 000
6 000	3 300
3 300	1 700

Determine three values on the decision maker's utility of wealth function u.

b. Calculate the slopes of the four line segments joining the five points determined on the graph $u(w)$. Determine the rates of change of the slopes from segment to segment.

c. Put yourself in the role of a decision maker with wealth 10,000. In addition to the given values of $u(0)$ and $u(10{,}000)$, elicit three additional values on your utility of wealth function u.

d. On the basis of the five values of your utility function, calculate the slopes and the rates of change of the slopes as done in part (b).

1.2. **St. Petersburg paradox:** Consider a game of chance that consists of tossing a coin until a head appears. The probability of a head is 0.5 and the repeated trials are independent. Let the random variable N be the number of the trial on which the first head occurs.

a. Show that the probability function (p.f.) of N is given by

$$f(n) = \left(\frac{1}{2}\right)^n \qquad n = 1, 2, 3, \ldots.$$

b. Find $E[N]$ and $\mathrm{Var}(N)$.

c. If a reward of $X = 2^N$ is paid, prove that the expectation of the reward does not exist.

d. If this reward has utility $u(w) = \log w$, find $E[u(X)]$.

Section 1.3

1.3. **Jensen's inequalities:**

a. Assume $u''(w) < 0$, $E[X] = \mu$, and $E[u(X)]$ exist; prove that $E[u(X)] \le u(\mu)$. [Hint: Express $u(w)$ as a series around the point $w = \mu$ and terminate the expansion with an error term involving the second derivative. Note that Jensen's inequalities do not require that $u'(w) > 0$.]

b. If $u''(w) > 0$, prove that $E[u(X)] \ge u(\mu)$.

c. Discuss Jensen's inequalities for the special case $u(w) = w^2$. What is

$$E[u(X)] - u(E[X])?$$

1.4. If a utility function is such that $u'(w) > 0$ and $u''(w) > 0$, use (1.3.1) to show $G \le \mu$. A decision maker with preferences consistent with $u''(w) > 0$ is a ***risk lover.***

1.5. Construct a geometric argument, based on a graph like that displayed in Figure 1.3.1, that if $u'(w) < 0$ and $u''(w) < 0$, then (1.3.4) follows.

1.6. Confirm that the utility function $u(w) = \log w$, $w > 0$, is the utility function of a decision maker who is risk averse for $w > 0$.

1.7. A utility function is given by

$$u(w) = \begin{cases} e^{-(w-100)^2/200} & w < 100 \\ 2 - e^{-(w-100)^2/200} & w \ge 100. \end{cases}$$

a. Is $u'(w) \ge 0$?

b. For what range of w is $u''(w) < 0$?

1.8. If one assumes, as did D. Bernoulli in his comments on the St. Petersburg paradox, that utility of wealth satisfies the differential equation

$$\frac{du(w)}{dw} = \frac{k}{w} \qquad w > 0, k > 0,$$

confirm that $u(w) = k \log w + c$.

1.9. A decision maker has utility function $u(w) = k \log w$. The decision maker has wealth w, $w > 1$, and faces a random loss X, which has a uniform distribution on the interval $(0, 1)$. Use (1.3.1) to show that the maximum insurance premium that the decision maker will pay for complete insurance is

$$G = w - \frac{w^w}{e(w - 1)^{w-1}}.$$

1.10. a. In (1.3.1) use the approximations

$$u(w - G) \cong u(w - \mu) + (\mu - G)u'(w - \mu),$$

$$u(w - x) \cong u(w - \mu) + (\mu - x)u'(w - \mu) + \frac{1}{2}(\mu - x)^2 u''(w - \mu)$$

and derive the following approximation for G:

$$G \cong \mu - \frac{1}{2}\frac{u''(w - \mu)}{u'(w - \mu)}\sigma^2.$$

b. If $u(w) = k \log w$, use the approximation developed in part (a) to obtain

$$G \cong \mu + \frac{1}{2}\frac{\sigma^2}{(w - \mu)}.$$

1.11. The decision maker has a utility function $u(w) = -e^{-\alpha w}$ and is faced with a random loss that has a chi-square distribution with n degrees of freedom. If $0 < \alpha < 1/2$, use (1.3.1) to obtain an expression for G, the maximum insurance premium the decision maker will pay, and prove that $G > n = \mu$.

1.12. Rework Example 1.3.4 for
a. $u(w) = -e^{-w/400}$
b. $u(w) = -e^{-w/150}$.

1.13. a. An insurer with net worth 100 has accepted (and collected the premium for) a risk X with the following probability distribution:

$$\Pr(X = 0) = \Pr(X = 51) = \frac{1}{2}.$$

What is the maximum amount G it should pay another insurer to accept 100% of this loss? Assume the first insurer's utility function of wealth is $u(w) = \log w$.

b. An insurer, with wealth 650 and the same utility function, $u(w) = \log w$, is considering accepting the above risk. What is the minimum amount H this insurer would accept as a premium to cover 100% of the loss?

1.14. If the complete insurance of Example 1.3.4 can be purchased for 40 and the 50% coinsurance of Example 1.3.5 can be purchased for 25, the purchase of which insurance maximizes the property owner's expected utility?

Section 1.4

1.15. A hospital expense policy is issued to a group consisting of n individuals. The policy pays B dollars each time a member of the group enters a hospital.

The group is not homogeneous with respect to the expected number of hospital admissions each year. The group may be divided into r subgroups. There are n_i individuals in subgroup i and $\Sigma_1^r n_i = n$. For subgroup i the number of annual hospital admissions for each member has a Poisson distribution with parameter λ_i, $i = 1, 2, \ldots, r$. The number of annual hospital admissions for members of the group are mutually independent.

a. Show that the expected claims payment in one year is

$$B \sum_1^r n_i \lambda_i = Bn\bar{\lambda}$$

where

$$\bar{\lambda} = \frac{\sum_1^r n_i \lambda_i}{n}.$$

b. Show that the number of hospital admissions in 1 year for the group has a Poisson distribution with parameter $n\,\bar{\lambda}$.

Section 1.5

1.16. Perform the integration by parts indicated in (1.5.2). Use the fact that if E[X] exists, if and only if, $\lim_{x\to\infty} x[1 - F(x)] = 0$.

1.17. a. Differentiate the right-hand side of (1.5.2B) with respect to d.

b. Let β be a number such that $0 < \beta < \text{E}[X]$. Show that (1.5.2) has a unique solution d^*.

1.18. Let the loss random variable X have a p.d.f. given by

$$f(x) = 0.1e^{-0.1x} \qquad x > 0.$$

a. Calculate E[X] and Var(X).

b. If $P = 5$ is to be spent for insurance to be purchased by the payment of the pure premium, show that

$$I(x) = \frac{x}{2}$$

and

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d, \text{ where } d = 10 \log 2, \end{cases}$$

both represent feasible insurance policies with pure premium $P = 5$. $I(x)$ is called ***proportional insurance.***

1.19. The loss random variable X has a p.d.f. given by

$$f(x) = \frac{1}{100} \qquad 0 < x < 100.$$

a. Calculate E[X] and Var(X).
b. Consider a proportional policy where

$$I(x) = kx \qquad 0 < k < 1,$$

and a stop-loss policy where

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases}$$

Determine k and d such that the pure premium in each case is $P = 12.5$.
c. Show that $\mathrm{Var}[X - I(X)] > \mathrm{Var}[X - I_d(x)]$.

Appendix

1.20. Establish the lemma by using an analytic rather than a geometric argument. [Hint: Expand $u(w)$ in a series as far as a second derivative remainder around the point z and subtract $u(z)$.]

1.21. Adopt the hypotheses of Theorem 1.5.1 with respect to β and insurance contracts $I(x)$ and assume $E[X] = \mu$. Prove that

$$\mathrm{Var}[X - I(X)] = E[(X - I(X) - \mu + \beta)^2]$$

is a minimum when $I(x) = I_{d^*}(x)$. You will be proving that for a fixed pure premium, a stop-loss insurance contract will minimize the variance of retained claims. [Hint: we may follow the proof of Theorem 1.5.1 by first proving that $x^2 - z^2 \geq (x - z)(2z)$ and then establishing that

$$\begin{aligned}[x - I(x)]^2 - [x - I_{d^*}(x)]^2 &\geq [I_{d^*}(x) - I(x)][2x - 2I_{d^*}(x)] \\ &\geq 2[I_{d^*}(x) - I(x)]d^*.\end{aligned}$$

The final inequality may be established by breaking the proof into three cases. Alternatively, by proper choice of wealth level and utility function, the result of this exercise is a special case of Theorem 1.5.1.]

1.22. Adopt the hypotheses of Theorem 1.5.1, except remove the budget constraint; that is, assume that the decision maker will pay premium P, $0 < P \leq E[X] = \mu$, that will maximize expected utility. In addition, assume that any feasible insurance can be purchased for its expected value. Prove that the optimal insurance is $I_0(x)$. This result can be summarized by stating that full coverage is optimal in the absence of a budget constraint if insurance can be purchased for its pure premium. [Hint: Use the lemma with the role of w played by $w - x + I(x) - P$ and that of z played by $w - x + I_0(x) - E[X] = w - \mu$. Take expectations and establish that $E[u(w - X) + I(X) - P] \leq u(w - \mu)$.]

1.23. Optimality properties of stop-loss insurance were established in Theorem 1.5.1 and Exercise 1.21. These results depended on the decision criteria, the constraints, and the insurance alternatives available. In each of these developments, there was a budget constraint. Consider the situation where there is a risk constraint and the price of insurance depends on the insurance risk as measured by the variance.

(i) The insurance premium is $E[I(X)] + f(\text{Var}[I(X)])$, where $f(w)$ is an increasing function. The amount of $f(\text{Var}[I(X)])$ can be interpreted as a ***security loading.***

(ii) The decision maker elects to retain loss $X - I(X)$ such that $\text{Var}[X - I(X)] = V \geq 0$. This requirement imposes a risk rather than a budget constraint. The constant is determined by the degree of risk aversion of the decision maker. Fixing the accepted variance, and then optimizing expected results, is a decision criterion in investment portfolio theory.

(iii) The decision maker selects $I(x)$ to minimize $f(\text{Var}[I(X)])$. The objective is to minimize the security loading, the premium paid less the expected insurance payments. Confirm the following steps:

a. $\text{Var}[I(X)] = V + \text{Var}(X) - 2\,\text{Cov}[X, X - I(X)]$.

b. The $I(x)$ that minimizes $\text{Var}[I(X)]$ and thereby $f(\text{Var}[I(X)])$ is such that the correlation coefficient between X and $X - I(x)$ is 1.

c. It is known that if two random variables W and Z have correlation coefficient 1, then $\Pr\{W = aZ + b, \text{ where } a > 0\} = 1$. In words, the probability of their joint distribution is concentrated on a line of positive slope. In part (b), the correlation coefficient of X and $X - I(X)$ was found to be 1. Thus, $X - I(X) = aX + b$, which implies that $I(X) = (1 - a)X - b$. To be a feasible insurance, $0 \leq I(x) \leq x$ or $0 \leq (1 - a)x - b \leq x$. These inequalities imply that $b = 0$ and $0 \leq 1 - a \leq 1$ and $0 \leq a \leq 1$.

d. To determine a, set the correlation coefficient of X and $X - I(X)$ equal to 1, or equivalently, their covariance equal to the product of their standard deviations. Thus, show that $a = \sqrt{V/\text{Var}(X)}$ and thus that the insurance that minimizes $f(\text{Var}[X])$ is $I(X) = [1 - \sqrt{V/\text{Var}(X)}]X$.

2

INDIVIDUAL RISK MODELS FOR A SHORT TERM

2.1 Introduction

In Chapter 1 we examined how a decision maker can use insurance to reduce the adverse financial impact of some types of random events. That examination was quite general. The decision maker could have been an individual seeking protection against the loss of property, savings, or income. The decision maker could have been an organization seeking protection against those same types of losses. In fact, the organization could have been an insurance company seeking protection against the loss of funds due to excess claims either by an individual or by its portfolio of insureds. Such protection is called ***reinsurance*** and is introduced in this chapter.

The theory in Chapter 1 requires a probabilistic model for the potential losses. Here we examine one of two models commonly used in insurance pricing, reserving, and reinsurance applications.

For an insuring organization, let the random loss of a segment of its risks be denoted by S. Then S is the random variable for which we seek a probability distribution. Historically, there have been two sets of postulates for distributions of S. The ***individual risk model*** defines

$$S = X_1 + X_2 + \cdots + X_n \tag{2.1.1}$$

where X_i is the loss on insured unit i and n is the number of risk units insured. Usually the X_i's are postulated to be independent random variables, because the mathematics is easier and no historical data on the dependence relationship are needed. The other model is the collective risk model described in Chapter 12.

The individual risk model in this chapter does not recognize the time value of money. This is for simplicity and is why the title refers to short terms. Chapters 4–11 cover models for long terms.

In this chapter we discuss only ***closed models;*** that is, the number of insured units n in (2.1.1) is known and fixed at the beginning of the period. If we postulate about migration in and out of the insurance system, we have an ***open model.***

2.2 Models for Individual Claim Random Variables

First, we review basic concepts with a life insurance product. In a ***one-year term life insurance*** the insurer agrees to pay an amount b if the insured dies within a year of policy issue and to pay nothing if the insured survives the year. The probability of a claim during the year is denoted by q. The claim random variable, X, has a distribution that can be described by either its probability function, p.f., or its distribution function, d.f. The p.f. is

$$f_X(x) = \Pr(X = x) = \begin{cases} 1 - q & x = 0 \\ q & x = b \\ 0 & \text{elsewhere,} \end{cases} \tag{2.2.1}$$

and the d.f. is

$$F_X(x) = \Pr(X \le x) = \begin{cases} 0 & x < 0 \\ 1 - q & 0 \le x < b \\ 1 & x \ge b. \end{cases} \tag{2.2.2}$$

From the p.f. and the definition of moments,

$$\begin{aligned} \mathrm{E}[X] &= bq, \\ \mathrm{E}[X^2] &= b^2q, \end{aligned} \tag{2.2.3}$$

and

$$\mathrm{Var}(X) = b^2q(1 - q). \tag{2.2.4}$$

These formulas can also be obtained by writing

$$X = Ib \tag{2.2.5}$$

where b is the constant amount payable in the event of death and I is the random variable that is 1 for the event of death and 0 otherwise. Thus, $\Pr(I = 0) = 1 - q$ and $\Pr(I = 1) = q$, the mean and variance of I are q and $q(1 - q)$, respectively, and the mean and variance of X are bq and $b^2q(1 - q)$ as above.

The random variable I with its $\{0, 1\}$ range is widely applicable in actuarial models. In probability textbooks it is called an ***indicator, Bernoulli random variable,*** or ***binomial random variable*** for a single trial. We refer to it as an indicator for the sake of brevity and because it indicates the occurrence, $I = 1$, or nonoccurrence, $I = 0$, of a given event.

We now seek more general models in which the amount of claim is also a random variable and several claims can occur in a period. Health, automobile, and other property and liability coverages provide immediate examples. Extending (2.2.5), we postulate that

$$X = IB \tag{2.2.6}$$

where X is the claim random variable for the period, B gives the total claim amount incurred during the period, and I is the indicator for the event that at least one claim has occurred. As the indicator for this event, I reports the occurrence ($I = 1$) or nonoccurrence ($I = 0$) of claims in this period and not the number of claims in the period. $\Pr(I = 1)$ is still denoted by q.

Let us look at several situations and determine the distributions of I and B for a model. First, consider a 1-year term life insurance paying an extra benefit in case of accidental death. To be specific, if death is accidental, the benefit amount is 50,000. For other causes of death, the benefit amount is 25,000. Assume that for the age, health, and occupation of a specific individual, the probability of an accidental death within the year is 0.0005, while the probability of a nonaccidental death is 0.0020. More succinctly,

$$\Pr(I = 1 \text{ and } B = 50{,}000) = 0.0005$$

and

$$\Pr(I = 1 \text{ and } B = 25{,}000) = 0.0020.$$

Summing over the possible values of B, we have

$$\Pr(I = 1) = 0.0025,$$

and then

$$\Pr(I = 0) = 1 - \Pr(I = 1) = 0.9975.$$

The conditional distribution of B, given $I = 1$, is

$$\Pr(B = 25{,}000|I = 1) = \frac{\Pr(B = 25{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0020}{0.0025} = 0.8,$$

$$\Pr(B = 50{,}000|I = 1) = \frac{\Pr(B = 50{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0005}{0.0025} = 0.2.$$

Let us now consider an automobile insurance providing collision coverage (this indemnifies the owner for collision damage to his car) above a 250 deductible up to a maximum claim of 2,000. For illustrative purposes, assume that for a particular individual the probability of one claim in a period is 0.15 and the chance of more than one claim is 0:

$$\Pr(I = 0) = 0.85,$$

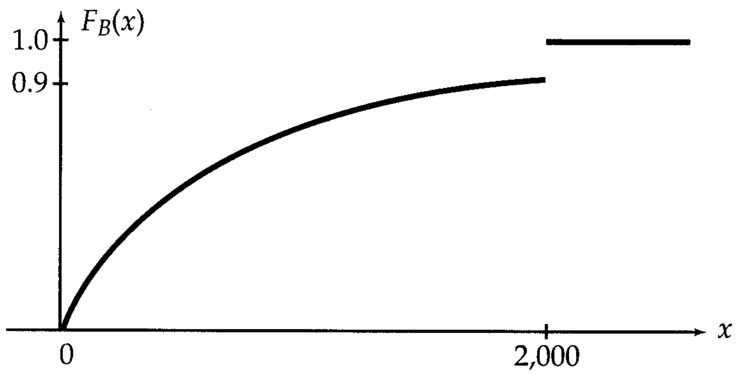
$$\Pr(I = 1) = 0.15.$$

This unrealistic assumption of no more than one claim per period is made to simplify the distribution of B. We remove that assumption in a later section after we discuss the distribution of the sum of a number of claims. Since B is the claim incurred by the insurer, rather than the amount of damage to the car, we can infer two characteristics of I and B. First, the event $I = 0$ includes those collisions in which the damage is less than the 250 deductible. The other inference is that B's distribution has a probability mass at the maximum claim size of 2,000. Assume this probability mass is 0.1. Furthermore, assume that claim amounts between 0 and 2,000 can be modeled by a continuous distribution with a p.d.f. proportional to $1 - x/2{,}000$ for $0 < x < 2{,}000$. (In practice the continuous curve chosen to represent the distribution of claims is the result of a study of claims by size over a recent period.) Summarizing these assumptions about the conditional distribution of B, given $I = 1$, we have a mixed distribution with positive density from 0 to 2,000 and a mass at 2,000. This is illustrated in Figure 2.2.1. The d.f. of this conditional distribution is

$$\Pr(B \le x | I = 1) = \begin{cases} 0 & x \le 0 \\ 0.9 \left[1 - \left(1 - \dfrac{x}{2{,}000} \right)^2 \right] & 0 < x < 2{,}000 \\ 1 & x \ge 2{,}000. \end{cases}$$

FIGURE 2.2.1

Distribution Function for B, given $I = 1$



We see in Section 2.4 that the moments of the claim random variable, X, in particular the mean and variance, are extensively used. For this automobile insurance, we shall calculate the mean and the variance by two methods. First, we derive the distribution of X and use it to calculate $E[X]$ and $\mathrm{Var}(X)$. Letting $F_X(x)$ be the d.f. of X, we have

$$\begin{aligned} F_X(x) &= \Pr(X \le x) = \Pr(IB \le x) \\ &= \Pr(IB \le x | I = 0) \Pr(I = 0) \\ &\quad + \Pr(IB \le x | I = 1) \Pr(I = 1). \end{aligned} \tag{2.2.7}$$

For $x < 0$,

$$F_X(x) = 0(0.85) + 0(0.15) = 0.$$

For $0 \le x < 2{,}000$,

$$F_X(x) = 1(0.85) + 0.9 \left[1 - \left(1 - \frac{x}{2{,}000} \right)^2 \right] (0.15).$$

For $x \geq 2{,}000$,

$$F_X(x) = 1(0.85) + 1(0.15) = 1.$$

This is a mixed distribution. It has both probability masses and a continuous part as can be seen in its graph in Figure 2.2.2.

FIGURE 2.2.2
Distribution Function of $X = IB$

Corresponding to this d.f. is a combination p.f. and p.d.f. given by

$$\Pr(X = 0) = 0.85,$$
$$\Pr(X = 2{,}000) = 0.015 \tag{2.2.8}$$

with p.d.f.

$$f_X(x) = \begin{cases} F'_X(x) = 0.000135\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere.} \end{cases}$$

Moments of X can then be calculated by

$$E[X^k] = 0 \times \Pr(X = 0) + (2{,}000)^k \times \Pr(X = 2{,}000) + \int_0^{2{,}000} x^k f_X(x)\, dx, \tag{2.2.9}$$

specifically,

$$E[X] = 120$$

and

$$E[X^2] = 150{,}000.$$

Thus,

$$\mathrm{Var}(X) = 135{,}600.$$

There are some formulas relating the moments of random variables to certain conditional expectations. General versions of these formulas for the mean and variance are

$$E[W] = E[E[W|V]] \tag{2.2.10}$$

and

$$\mathrm{Var}(W) = \mathrm{Var}(\mathrm{E}[W|V]) + \mathrm{E}[\mathrm{Var}(W|V)]. \tag{2.2.11}$$

In these equations we think of calculating the terms of the left-hand sides by direct use of W's distribution. In the terms on the right-hand sides, $\mathrm{E}[W|V]$ and $\mathrm{Var}(W|V)$ are calculated by use of W's conditional distribution for a given value of V. These components are then functions of the random variable V, and we can calculate their moments by use of V's distribution.

In many actuarial models conditional distributions are used. This makes the formulas above directly applicable. In our model, $X = IB$, we can substitute X for W and I for V to obtain

$$\mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|I]] \tag{2.2.12}$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}[X|I]) + \mathrm{E}[\mathrm{Var}(X|I)]. \tag{2.2.13}$$

Now let us write

$$\mu = \mathrm{E}[B|I = 1], \tag{2.2.14}$$

$$\sigma^2 = \mathrm{Var}(B|I = 1), \tag{2.2.15}$$

and look at the conditional means

$$\mathrm{E}[X|I = 0] = 0 \tag{2.2.16}$$

and

$$\mathrm{E}[X|I = 1] = \mathrm{E}[B|I = 1] = \mu. \tag{2.2.17}$$

Formulas (2.2.16) and (2.2.17) define $\mathrm{E}[X|I]$ as a function of I, which can be written by the formula

$$\mathrm{E}[X|I] = \mu I. \tag{2.2.18}$$

Hence,

$$\mathrm{E}[\mathrm{E}[X|I]] = \mu \mathrm{E}[I] = \mu q \tag{2.2.19}$$

and

$$\mathrm{Var}(\mathrm{E}[X|I]) = \mu^2 \,\mathrm{Var}(I) = \mu^2 q(1 - q). \tag{2.2.20}$$

Since $X = 0$ for $I = 0$, we have

$$\mathrm{Var}(X|I = 0) = 0. \tag{2.2.21}$$

For $I = 1$, we have $X = B$ and

$$\mathrm{Var}(X|I = 1) = \mathrm{Var}(B|I = 1) = \sigma^2. \tag{2.2.22}$$

Formulas (2.2.21) and (2.2.22) can be combined as

$$\mathrm{Var}(X|I) = \sigma^2 I. \tag{2.2.23}$$

Then

$$\mathrm{E}[\mathrm{Var}(X|I)] = \sigma^2\mathrm{E}[I] = \sigma^2 q. \tag{2.2.24}$$

Substituting (2.2.19), (2.2.20), and (2.2.24) into (2.2.12) and (2.2.13), we have

$$\mathrm{E}[X] = \mu q \tag{2.2.25}$$

and

$$\mathrm{Var}(X) = \mu^2 q(1 - q) + \sigma^2 q. \tag{2.2.26}$$

Let us now apply these formulas to calculate E[X] and Var(X) for the automobile insurance in Figure 2.2.2. Since the p.d.f. for B, given $I = 1$, is

$$f_{B|I}(x|1) = \begin{cases} 0.0009\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere,} \end{cases}$$

with $\Pr(B = 2{,}000|I = 1) = 0.1$, we have

$$\mu = \int_0^{2{,}000} 0.0009\, x\left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000) = 800,$$

$$\mathrm{E}[B^2|I = 1] = \int_0^{2{,}000} 0.0009\, x^2\left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000)^2 = 1{,}000{,}000,$$

and

$$\sigma^2 = 1{,}000{,}000 - (800)^2 = 360{,}000.$$

Finally, with $q = 0.15$ we obtain the following from (2.2.25) and (2.2.26):

$$\mathrm{E}[X] = 800(0.15) = 120$$

and

$$\begin{aligned} \mathrm{Var}(X) &= (800)^2(0.15)(0.85) + (360{,}000)(0.15) \\ &= 135{,}600. \end{aligned}$$

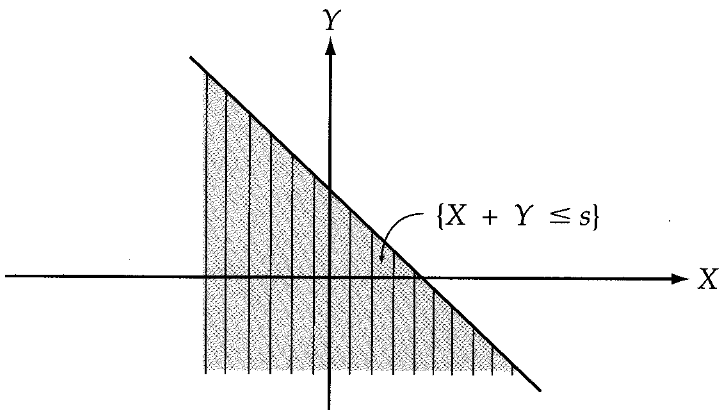
There are other possible models for B in different insurance situations. As an example, let us consider a model for the number of deaths due to crashes during an airline's year of operation. We can start with a random variable for the number of deaths, X, on a single flight and then add up a set of such random variables over the set of flights for the year. For a single flight, the event $I = 1$ will be the event of an accident during the flight. The number of deaths in the accident, B, will be modeled as the product of two random variables, L and Q, where L is the load factor, the number of persons on board at the time of the crash, and Q is the fraction of deaths among persons on board. The number of deaths B is modeled in this way since separate statistical data for the distributions of L and Q may be more readily available than are total data for B. We have $X = ILQ$. While the fraction of passengers killed in a crash and the fraction of seats occupied are probably related, L and Q might be assumed to be independent as a first approximation.

2.3 Sums of Independent Random Variables

In the individual risk model, claims of an insuring organization are modeled as the sum of the claims of many insured individuals.

The claims for the individuals are assumed to be independent in most applications. In this section we review two methods for determining the distribution of the sum of independent random variables. First, let us consider the sum of two random variables, $S = X + Y$, with the sample space shown in Figure 2.3.1.

FIGURE 2.3.1
Event $\{X + Y \leq s\}$



The line $X + Y = s$ and the region below the line represent the event

$$[S = X + Y \leq s].$$

Hence the d.f. of S is

$$F_S(s) = \Pr(S \leq s) = \Pr(X + Y \leq s). \tag{2.3.1}$$

For two discrete, non-negative random variables, we can use the law of total probability to write (2.3.1) as

$$\begin{aligned} F_S(s) &= \sum_{\text{all } y \leq s} \Pr(X + Y \leq s | Y = y) \Pr(Y = y) \\ &= \sum_{\text{all } y \leq s} \Pr(X \leq s - y | Y = y) \Pr(Y = y). \end{aligned} \tag{2.3.2}$$

When X and Y are independent, this last sum can be written

$$F_S(s) = \sum_{\text{all } y \leq s} F_X(s - y) f_Y(y). \tag{2.3.3}$$

The p.f. corresponding to this d.f. can be calculated by

$$f_S(s) = \sum_{\text{all } y \leq s} f_X(s - y) f_Y(y). \tag{2.3.4}$$

For continuous, non-negative random variables the formulas corresponding to (2.3.2), (2.3.3), and (2.3.4) are

$$F_S(s) = \int_0^s \Pr(X \le s - y | Y = y) f_Y(y)\, dy, \tag{2.3.5}$$

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy, \tag{2.3.6}$$

$$f_S(s) = \int_0^s f_X(s - y) f_Y(y)\, dy. \tag{2.3.7}$$

When either one, or both, of X and Y have a mixed-type distribution (typical in individual risk model applications), the formulas are analogous but more complex. For random variables that may also take on negative values, the sums and integrals in the formulas above are over all y values from $-\infty$ to $+\infty$.

In probability, the operation in (2.3.3) and (2.3.6) is called the ***convolution*** of the pair of distribution functions $F_X(x)$ and $F_Y(y)$ and is denoted by $F_X * F_Y$. Convolutions can also be defined for a pair of probability functions or probability density functions as in (2.3.4) and (2.3.7).

To determine the distribution of the sum of more than two random variables, we can use the convolution process iteratively. For $S = X_1 + X_2 + \cdots + X_n$ where the X_i's are independent random variables, F_i is the d.f. of X_i, and $F^{(k)}$ is the d.f. of $X_1 + X_2 + \cdots + X_k$, we proceed thus:

$$\begin{aligned} F^{(2)} &= F_2 * F^{(1)} = F_2 * F_1 \\ F^{(3)} &= F_3 * F^{(2)} \\ F^{(4)} &= F_4 * F^{(3)} \\ &\vdots \\ F^{(n)} &= F_n * F^{(n-1)}. \end{aligned}$$

Example 2.3.1 illustrates the procedure using probability functions for three discrete random variables.

Example 2.3.1

The random variables X_1, X_2, and X_3 are independent with distributions defined by columns (1), (2), and (3) of the table below. Derive the p.f. and d.f. of

$$S = X_1 + X_2 + X_3.$$

Solution:

The notation of the previous paragraph is used in the table:

- Columns (1)–(3) are given information.
- Column (4) is derived from columns (1) and (2) by use of (2.3.4).
- Column (5) is derived from columns (3) and (4) by use of (2.3.4).

The determination of column (5) completes the determination of the distribution of S. Its d.f. in column (8) is the set of partial sums of column (5) from the top.

x	(1) $f_1(x)$	(2) $f_2(x)$	(3) $f_3(x)$	(4) $f^{(2)}(x)$	(5) $f^{(3)}(x)$	(6) $F_1(x)$	(7) $F^{(2)}(x)$	(8) $F^{(3)}(x)$
0	0.4	0.5	0.6	0.20	0.120	0.4	0.20	0.120
1	0.3	0.2	0.0	0.23	0.138	0.7	0.43	0.258
2	0.2	0.1	0.1	0.20	0.140	0.9	0.63	0.398
3	0.1	0.1	0.1	0.16	0.139	1.0	0.79	0.537
4		0.1	0.1	0.11	0.129	1.0	0.90	0.666
5			0.1	0.06	0.115	1.0	0.96	0.781
6				0.03	0.088	1.0	0.99	0.869
7				0.01	0.059	1.0	1.00	0.928
8					0.036	1.0	1.00	0.964
9					0.021	1.0	1.00	0.985
10					0.010	1.0	1.00	0.995
11					0.004	1.0	1.00	0.999
12					0.001	1.0	1.00	1.000

For illustrative purposes we include column (6), the d.f. for column (1), column (7) which can be derived directly from columns (2) and (6) by use of (2.3.3), and column (8), derived similarly from columns (3) and (7). Column (5) can then be obtained by differencing column (8). ▼

We follow with two examples involving continuous random variables.

Example 2.3.2

Let X have a uniform distribution on (0, 2) and let Y be independent of X with a uniform distribution over (0, 3). Determine the d.f. of $S = X + Y$.

Solution:

Since X and Y are continuous, we use (2.3.6):

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{3} & 0 < y < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

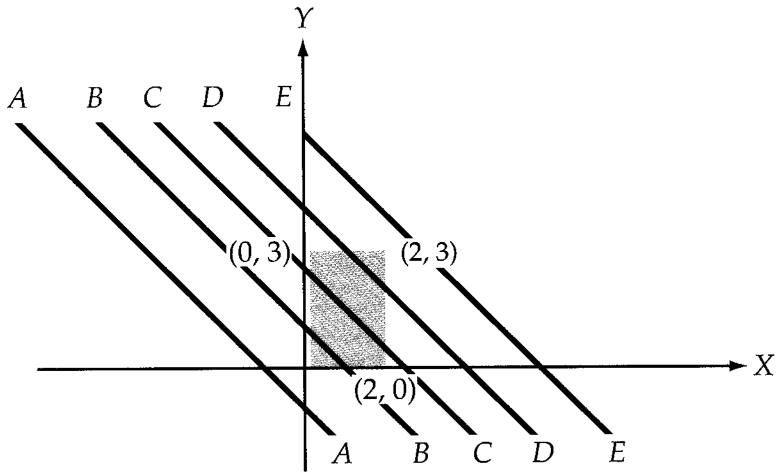
Then

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy.$$

The X, Y sample space is illustrated in Figure 2.3.2. The rectangular region contains all of the probability for X and Y. The event of interest, $X + Y \le s$, has been

FIGURE 2.3.2

Convolution of Two Uniform Distributions



illustrated in the figure for five values of s. For each value, the line intersects the y-axis at s and the line $x = 2$ at $s - 2$. The values of F_S for these five cases are

$$F_S(s) = \begin{cases} 0 & s < 0 \quad \text{line } A \\ \int_0^s \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s^2}{12} & 0 \le s < 2 \quad \text{line } B \\ \int_0^{s-2} 1\frac{1}{3}\,dy + \int_{s-2}^{s} \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s-1}{3} & 2 \le s < 3 \quad \text{line } C \\ \int_0^{s-2} 1\frac{1}{3}\,dy + \int_{s-2}^{3} \frac{s-y}{2}\frac{1}{3}\,dy = 1 - \frac{(5-s)^2}{12} & 3 \le s < 5 \quad \text{line } D \\ 1 & s \ge 5 \quad \text{line } E. \end{cases}$$

▼

Example 2.3.3

Consider three independent random variables X_1, X_2, X_3. For $i = 1, 2, 3$, X_i has an exponential distribution and $E[X_i] = 1/i$. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by the convolution process.

Solution:

$$f_1(x) = e^{-x} \qquad x > 0,$$

$$f_2(x) = 2e^{-2x} \qquad x > 0,$$

$$f_3(x) = 3e^{-3x} \qquad x > 0.$$

Using (2.3.7) twice, we have

$$
\begin{aligned}
f^{(2)}(x) &= \int_0^x f_1(x-y)\,f_2(y)\,dy = \int_0^x e^{-(x-y)}\,2e^{-2y}\,dy \\
&= 2e^{-x}\int_0^x e^{-y}\,dy \\
&= 2e^{-x} - 2e^{-2x} \qquad x > 0, \\
f_s(x) = f^{(3)}(x) &= \int_0^x f^{(2)}(x-y) f_3(y)\,dy \\
&= \int_0^x (2e^{-(x-y)} - 2e^{-2(x-y)})3e^{-3y}\,dy \\
&= 6e^{-x}\int_0^x e^{-2y}\,dy - 6e^{-2x}\int_0^x e^{-y}\,dy \\
&= (3e^{-x} - 3e^{-3x}) - (6e^{-2x} - 6e^{-3x}) \\
&= 3e^{-x} - 6e^{-2x} + 3e^{-3x} \qquad x > 0.
\end{aligned}
$$

▼

Another method to determine the distribution of the sum of random variables is based on the uniqueness of the ***moment generating function*** (m.g.f.), which, for the random variable X, is defined by $M_X(t) = \mathrm{E}[e^{tX}]$. If this expectation is finite for all t in an open interval about the origin, then $M_X(t)$ is the only m.g.f. of the distribution of X, and it is not the m.g.f. of any other distribution. This uniqueness can be used as follows. For the sum $S = X_1 + X_2 + \cdots + X_n$,

$$
\begin{aligned}
M_S(t) &= \mathrm{E}[e^{tS}] = \mathrm{E}[e^{t(X_1+X_2+\cdots+X_n)}] \\
&= \mathrm{E}[e^{tX_1}e^{tX_2}\cdots e^{tX_n}]. \qquad (2.3.8)
\end{aligned}
$$

If $X_1, X_2, \ldots, X_n$ are independent, then the expectation of the product in (2.3.8) is equal to

$$\mathrm{E}[e^{tX_1}]\,\mathrm{E}[e^{tX_2}]\cdots\mathrm{E}[e^{tX_n}]$$

so that

$$M_S(t) = M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t). \qquad (2.3.9)$$

Recognition of the unique distribution corresponding to (2.3.9) would complete the determination of S's distribution. If inversion by recognition is not possible, then inversion by numerical methods may be used. (See Section 2.6.)

Example 2.3.4

Consider the random variables of Example 2.3.3. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by recognition of the m.g.f. of S.

Solution:

By (2.3.9), $M_S(t) = \left(\frac{1}{1-t}\right)\left(\frac{2}{2-t}\right)\left(\frac{3}{3-t}\right)$, which we write, by the method of partial fractions, as

$$M_s(t) = \frac{A}{1-t} + \frac{2B}{2-t} + \frac{3C}{3-t}.$$

The solution for this is $A = 3$, $B = -3$, $C = 1$. But $\beta/(\beta - t)$ is the moment generating function of an exponential distribution with parameter β, so the p.d.f. for S is

$$f_S(x) = 3(e^{-x}) - 3(2e^{-2x}) + (3e^{-3x}).$$ ▼

Example 2.3.5

The ***inverse Gaussian distribution*** was developed in the study of stochastic processes. Here it is used as the distribution of B, the claim amount. It will have a similar role in risk theory in Chapters 12–14. The p.d.f. and m.g.f. associated with the inverse Gaussian distribution are given by

$$f_X(x) = \frac{\alpha}{\sqrt{2\pi\beta}}\, x^{-3/2} \exp\left[-\frac{(\beta x - \alpha)^2}{2\beta x}\right] \qquad x > 0,$$

$$M_X(t) = \exp\left[\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

Find the distribution of $S = X_1 + X_2 + X_3 + \cdots + X_n$ where the random variables $X_1, X_2, \ldots, X_n$ are independent and have identical inverse Gaussian distributions.

Solution:

Using (2.3.9), the m.g.f. of S is given by

$$M_S(t) = [M_X(t)]^n = \exp\left[n\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

The m.g.f. $M_S(t)$ can be recognized and shows that S has an inverse Gaussian distribution with parameters $n\alpha$ and β. ▼

2.4 Approximations for the Distribution of the Sum

The ***central limit theorem*** suggests a method to obtain numerical values for the distribution of the sum of independent random variables. The usual statement of the theorem is for a sequence of independent and identically distributed random variables, $X_1, X_2, \ldots$, with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. For each n, the distribution of $\sqrt{n}\,(\bar{X}_n - \mu)/\sigma$, where $\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n$, has mean 0 and variance 1. The sequence of distributions ($n = 1, 2, \ldots$) is known to approach the standard normal distribution. When n is large the theorem is applied to approximate the distribution of $\bar{X}_n$ by a normal distribution with mean μ and variance σ^2/n. Equivalently, the distribution of the sum of the n random variables is approximated by a normal distribution with mean $n\mu$ and variance $n\sigma^2$. The effectiveness of these approximations depends not only on the number of variables but also on the departure of the distribution of the summands from normality. Many elementary statistics textbooks recommend that n be at least 30 for the approximations to be reasonable. One routine used to generate normally distributed random variables for simulation is based on the average of only 12 independent random variables uniformly distributed over (0, 1).

In many individual risk models the random variables in the sum are not identically distributed. This is illustrated by examples in the next section. The central limit theorem extends to sequences of nonidentically distributed random variables.

To illustrate some applications of the individual risk model, we use a normal approximation to the distribution of the sum of independent random variables to obtain numerical answers. If

$$S = X_1 + X_2 + \cdots + X_n,$$

then

$$\mathrm{E}[S] = \sum_{k=1}^{n} \mathrm{E}[X_k],$$

and, further, under the assumption of independence,

$$\mathrm{Var}(S) = \sum_{k=1}^{n} \mathrm{Var}(X_k).$$

For an application we need only

- Evaluate the means and variances of the individual loss random variables
- Sum them to obtain the mean and variance for the loss of the insuring organization as a whole
- Apply the normal approximation.

Illustrations of this process follow.

2.5 Applications to Insurance

In this section four examples illustrate the results of Section 2.2 and use of the normal approximation.

Example 2.5.1

A life insurance company issues 1-year term life contracts for benefit amounts of 1 and 2 units to individuals with probabilities of death of 0.02 or 0.10. The following table gives the number of individuals n_k in each of the four classes created by a benefit amount b_k and a probability of claim q_k.

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.10	1	300
4	0.10	2	500

The company wants to collect, from this population of 1,800 individuals, an amount equal to the 95th percentile of the distribution of total claims. Moreover, it wants each individual's share of this amount to be proportional to that individual's expected claim. The share for individual j with mean $E[X_j]$ would be $(1 + \theta)E[X_j]$. The 95th percentile requirement suggests that $\theta > 0$. This extra amount, $\theta E[X_j]$, is the ***security loading*** and θ is the ***relative security loading.*** Calculate θ.

Solution:

The criterion for θ is $\Pr(S \le (1 + \theta)E[S]) = 0.95$ where $S = X_1 + X_2 + \cdots + X_{1,800}$. This probability statement is equivalent to

$$\Pr\left[\frac{S - E[S]}{\sqrt{\mathrm{Var}(S)}} \le \frac{\theta E[S]}{\sqrt{\mathrm{Var}(S)}}\right] = 0.95.$$

Following the discussion of the central limit theorem in Section 2.4, we approximate the distribution of $(S - E[S])/\sqrt{\mathrm{Var}(S)}$ by the standard normal distribution and use its 95th percentile to obtain

$$\frac{\theta\, E[S]}{\sqrt{\mathrm{Var}(S)}} = 1.645.$$

It remains to calculate the mean and variance of S and to calculate θ by this equation.

For the four classes of insured individuals, we have the results given below.

k	q_k	b_k	Mean $b_k q_k$	Variance $b_k^2 q_k(1 - q_k)$	n_k
1	0.02	1	0.02	0.0196	500
2	0.02	2	0.04	0.0784	500
3	0.10	1	0.10	0.0900	300
4	0.10	2	0.20	0.3600	500

Then

$$E[S] = \sum_{j=1}^{1,800} E[X_j] = \sum_{k=1}^{4} n_k b_k q_k = 160$$

and

$$\mathrm{Var}(S) = \sum_{j=1}^{1,800} \mathrm{Var}(X_j) = \sum_{k=1}^{4} n_k b_k^2 q_k(1 - q_k) = 256.$$

Thus, the relative security loading is

$$\theta = 1.645 \frac{\sqrt{\mathrm{Var}(S)}}{E[S]} = 1.645 \frac{16}{160} = 0.1645. \quad ▼$$

Example 2.5.2

The policyholders of an automobile insurance company fall into two classes.

Class k	Number in Class n_k	Claim Probability q_k	Distribution of Claim Amount, B_k, Parameters of Truncated Exponential λ	L
1	500	0.10	1	2.5
2	2 000	0.05	2	5.0

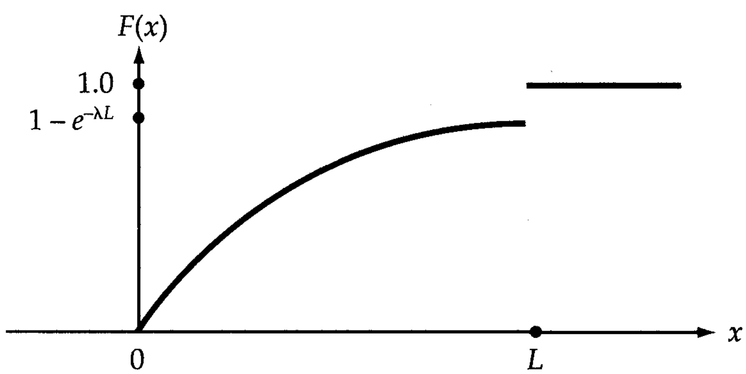
A truncated exponential distribution is defined by the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & 0 \le x < L \\ 1 & x \ge L. \end{cases}$$

This is a mixed distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $0 < x < L$, and a probability mass $e^{-\lambda L}$ at L. A graph of the d.f. appears in Figure 2.5.1.

FIGURE 2.5.1

Truncated Exponential Distribution



Again, the probability that total claims exceed the amount collected from policyholders is 0.05. We assume that the relative security loading, θ, is the same for the two classes. Calculate θ.

Solution:

This example is much like the previous one. It differs in that the claim amounts are random variables. First, we obtain formulas for the moments of the truncated exponential distribution in preparation for applying (2.2.25) and (2.2.26):

$$\mu = \mathrm{E}[B|I = 1] = \int_0^L x\lambda e^{-\lambda x}\, dx + Le^{-\lambda L} = \frac{1 - e^{-\lambda L}}{\lambda},$$

$$\mathrm{E}[B^2|I = 1] = \int_0^L x^2\lambda e^{-\lambda x}\, dx + L^2 e^{-\lambda L} = \frac{2}{\lambda^2}(1 - e^{-\lambda L}) - \frac{2L}{\lambda} e^{-\lambda L},$$

$$\sigma^2 = \mathrm{E}[B^2|I = 1] - (\mathrm{E}[B|I = 1])^2 = \frac{1 - 2\lambda L e^{-\lambda L} - e^{-2\lambda L}}{\lambda^2}.$$

Using the parameter values given and applying (2.2.25) and (2.2.26), we obtain the following results.

k	q_k	μ_k	σ_k^2	Mean $q_k\mu_k$	Variance $\mu_k^2 q_k(1 - q_k) + \sigma_k^2 q_k$	n_k
1	0.10	0.9179	0.5828	0.09179	0.13411	500
2	0.05	0.5000	0.2498	0.02500	0.02436	2 000

Then S, the sum of the claims, has moments

$$\mathrm{E}[S] = 500\ (0.09179) + 2{,}000\ (0.02500) = 95.89,$$

$$\mathrm{Var}(S) = 500\ (0.13411) + 2{,}000\ (0.02436) = 115.78.$$

The criterion for θ is the same as in Example 2.5.1,

$$\Pr(S \le (1 + \theta)\mathrm{E}[S]) = 0.95.$$

Again by the normal approximation,

$$\frac{\theta\mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} = 1.645$$

and

$$\theta = \frac{1.645\sqrt{115.78}}{95.89} = 0.1846.$$

▼

Example 2.5.3

A life insurance company covers 16,000 lives for 1-year term life insurance in amounts shown below.

Benefit Amount	Number Covered
10 000	8 000
20 000	3 500
30 000	2 500
50 000	1 500
100 000	500

The probability of a claim q for each of the 16,000 lives, assumed to be mutually independent, is 0.02. The company wants to set a ***retention limit.*** For each life, the retention limit is the amount below which this (the ***ceding***) company will retain the insurance and above which it will purchase ***reinsurance*** coverage from another (the ***reinsuring***) company. For example, assume the retention limit is 20,000. The company will retain up to 20,000 on each life and purchase reinsurance for the difference between the benefit amount and 20,000 for each of the 4,500 individuals with benefit amounts in excess of 20,000. As a decision criterion, the company wants to minimize the probability that retained claims plus the amount that it pays for reinsurance will exceed 8,250,000. Reinsurance is available at a cost of 0.025 per unit of coverage (i.e., at 125% of the expected claim amount per unit, 0.02). We will consider the block of business as closed. New policies sold during the year are not to enter this decision process. Calculate the retention limit that minimizes the probability that the company's retained claims plus cost of reinsurance exceeds 8,250,000.

Partial Solution:

First, do all calculations in benefit units of 10,000. As an illustrative step, let S be the amount of retained claims paid when the retention limit is 2 (20,000). Our portfolio of retained business is given by

k	Retained Amount b_k	Number Covered n_k
1	1	8 000
2	2	8 000

and

$$E[S] = \sum_{k=1}^{2} n_k b_k q_k = 8{,}000\ (1)(0.02) + 8{,}000\ (2)(0.02) = 480$$

and

$$\begin{aligned}\operatorname{Var}(S) &= \sum_{k=1}^{2} n_k b_k^2 q_k(1 - q_k) \\ &= 8{,}000\ (1)(0.02)(0.98) + 8{,}000\ (4)(0.02)(0.98) = 784.\end{aligned}$$

In addition to the retained claims, S, there is the cost of reinsurance premiums. The total coverage in the plan is

$$8{,}000\ (1) + 3{,}500\ (2) + 2{,}500\ (3) + 1{,}500\ (5) + 500\ (10) = 35{,}000.$$

The retained coverage for the plan is

$$8{,}000\ (1) + 8{,}000\ (2) = 24{,}000.$$

Therefore, the total amount reinsured is $35{,}000 - 24{,}000 = 11{,}000$ and the reinsurance cost is $11{,}000(0.025) = 275$. Thus, at retention limit 2, the retained claims plus reinsurance cost is $S + 275$. The decision criterion is based on the probability that this total cost will exceed 825,

$$
\begin{aligned}
\Pr(S + 275 > 825) &= \Pr(S > 550) \\
&= \Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} > \frac{550 - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}}\right] \\
&= \Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} > 2.5\right].
\end{aligned}
$$

Using the normal distribution, this is approximately 0.0062. The solution is completed in Exercises 2.13 and 2.14. ▼

In Section 1.5 stop-loss insurance, which is available as a reinsurance coverage, was discussed. The expected value of the claims paid under the stop-loss reinsurance coverage can be approximated by using the normal distribution as the distribution of total claims.

Let total claims, X, have a normal distribution with mean μ and variance σ^2 and let d be the deductible of the stop-loss insurance. Then, by (1.5.2A), the expected reinsurance claims equal

$$
\mathrm{E}[I_d(X)] = \frac{1}{\sqrt{2\pi}\sigma}\int_d^{\infty}(x - d)\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]dx. \tag{2.5.1}
$$

Changing the variable of integration to $z = (x - \mu)/\sigma$ and defining β by $d = \mu + \beta\sigma$, we obtain the following general expression for the expected value of stop-loss claims under a normal distribution assumption:

$$
\mathrm{E}[I_d(X)] = \sigma\left\{\frac{\exp(-\beta^2/2)}{(2\pi)^{0.5}} - \beta[1 - \Phi(\beta)]\right\} \tag{2.5.2}
$$

where $\Phi(x)$ is the distribution function for the standard normal distribution.

Example 2.5.4

Consider the portfolio of insurance contracts in Example 2.5.3. Calculate the expected value of the claims provided by a stop-loss reinsurance coverage where

a. There is no individual reinsurance and the deductible amount is 7,500,000
b. There is a retention amount of 20,000 on individual policies and the deductible amount on the business retained is 5,300,000.

Solution:

a. With no individual reinsurance and the use of 10,000 as the unit,

$$
\mathrm{E}[S] = 0.02[8{,}000(1) + 3{,}500(2) + 2{,}500(3) + 1{,}500(5) + 500(10)] = 700
$$

and

$$
\begin{aligned}
\mathrm{Var}(S) &= (0.02)(0.98)[8{,}000(1) + 3{,}500(4) + 2{,}500(9) + 1{,}500(25) + 500(100)] \\
&= 2{,}587.2
\end{aligned}
$$

so

$$\sigma(S) = 50.86.$$

Then, with

$$\beta = \frac{(d - \mu)}{\sigma} = \frac{(750 - 700)}{50.86} = 0.983$$

the application of (2.5.2) gives us

$$P = 50.86[0.24608 - (0.983)(0.16280)] = 4.377.$$

This is equivalent to 43,770 in the example as posed.

b. In Example 2.5.3 we determined the mean and the variance of the aggregate claims, after imposing a 20,000 retention limit per individual, to be 480 and 784, respectively, in units of 10,000. Thus $\sigma(S) = 28$.

Then, with

$$\beta = \frac{d - \mu}{\sigma} = \frac{530 - 480}{28} = 1.786$$

the application of (2.5.2) gives us

$$P = 28[0.08100 - (1.786)(0.03707)] = 0.414.$$

This is equivalent to 4,140 in the example as posed. ▼

2.6 Notes and References

The basis of the material in Sections 2.2, 2.3, and 2.4 can be found in a number of post-calculus probability and statistics texts. Mood et al. (1974) prove the theorems given in (2.2.10) and (2.2.11). They also provide an extensive discussion of properties of the moment generating function. For a discussion of the advanced mathematical methods for deriving the distribution function that corresponds to a given moment generating function, see Bellman et al. (1966). Methods are also available to obtain the p.f. of a discrete distribution from its ***probability generating function;*** see Kornya (1983).

DeGroot (1986) provides a discussion of several conditions under which the central limit theorem holds. Kendall and Stuart (1977) give material on ***normal power expansions*** that may be viewed as modifications of the normal approximation to improve numerical results. Bowers (1967) also describes the use of normal power expansions and gives an application to approximate the distribution of present values for an annuity portfolio.

Exercises

Section 2.2

2.1. Use (2.2.3) and (2.2.4) to obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount is fixed at 10.

2.2. Obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount random variable B is uniformly distributed between 0 and 20.

2.3. Let X be the number of heads observed in five tosses of a true coin. Then, X true dice are thrown. Let Y be the sum of the numbers showing on the dice. Determine the mean and variance of Y. [Hint: Apply (2.2.10) and (2.2.11).]

2.4. Let X be the number showing when one true die is thrown. Let Y be the number of heads obtained when X true coins are then tossed. Calculate $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$.

2.5. Let X be the number obtained when one true die is tossed. Let Y be the sum of the numbers obtained when X true dice are then thrown. Calculate $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$.

2.6. The probability of a fire in a certain structure in a given time period is 0.02. If a fire occurs, the damage to the structure is uniformly distributed over the interval $(0, a)$ where a is its total value. Calculate the mean and variance of fire damage to the structure within the time period.

Section 2.3

2.7. Independent random variables X_k for four lives have the discrete probability functions given below.

x	$\Pr(X_1 = x)$	$\Pr(X_2 = x)$	$\Pr(X_3 = x)$	$\Pr(X_4 = x)$
0	0.6	0.7	0.6	0.9
1	0.0	0.2	0.0	0.0
2	0.3	0.1	0.0	0.0
3	0.0	0.0	0.4	0.0
4	0.1	0.0	0.0	0.1

Use a convolution process on the non-negative integer values of x to obtain $F_S(x)$ for $x = 0, 1, 2, \ldots, 13$ where $S = X_1 + X_2 + X_3 + X_4$.

2.8. Let X_i for $i = 1, 2, 3$ be independent and identically distributed with the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

Let $S = X_1 + X_2 + X_3$.

a. Show that $F_S(x)$ is given by

$$F_S(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^3}{6} & 0 \le x < 1 \\ \dfrac{x^3 - 3(x-1)^3}{6} & 1 \le x < 2 \\ \dfrac{x^3 - 3(x-1)^3 + 3(x-2)^3}{6} & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

b. Show that $E[S] = 1.5$ and $Var(S) = 0.25$.

c. Evaluate the following probabilities using the d.f. of part (a):

(i) $Pr(S \le 0.5)$
(ii) $Pr(S \le 1.0)$
(iii) $Pr(S \le 1.5)$.

2.9. Find the mean and variance of the inverse Gaussian distribution by using its m.g.f. as given in Example 2.3.5.

Section 2.4

2.10. Calculate the mean and variance of X and Y in Example 2.3.2. Use a normal distribution to approximate $Pr(X + Y > 4)$. Compare this with the exact answer.

2.11. a. Use the central limit theorem to calculate b, c, and d, for given a, in the statement

$$Pr\left(\sum_1^n X_i \ge n\mu + a\sqrt{n}\sigma\right) \cong c + b\Phi(d)$$

where the X_i's are independent and identically distributed with mean μ and variance σ^2 and $\Phi(z)$ is the d.f. of the standard normal distribution.

b. Evaluate the probabilities in Exercise 2.8(c) by use of the normal approximation developed in part (a).

2.12. A random variable U has m.g.f.

$$M_U(t) = (1 - 2t)^{-9} \qquad t < \frac{1}{2}.$$

a. Use the m.g.f. to calculate the mean and variance of U.

b. Use a normal approximation to calculate points $y_{0.05}$ and $y_{0.01}$ such that $Pr(U > y_\epsilon) = \epsilon$.

Note the random variable U has a gamma distribution with parameters $\alpha = 9$ and $\beta = 1/2$. Gamma distributions with $\alpha = n/2$ and $\beta = 1/2$ are chi-square distributions with n degrees of freedom. Thus U has a chi-square distribution with 18 degrees of freedom. From tables of d.f.'s of chi-square distributions, we obtain $y_{0.05} = 28.869$ and $y_{0.01} = 34.805$.

Section 2.5

2.13. Calculate the probability that the total cost in Example 2.5.3 will exceed 8,250,000 if the retention limit is
a. 30,000 b. 50,000.

2.14. Calculate the retention limit that minimizes the probability of the total cost in Example 2.5.3 exceeding 8,250,000. Assume that the limit is between 30,000 and 50,000.

2.15. A fire insurance company covers 160 structures against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Contract Amount	Number of Contracts
10 000	80
20 000	35
30 000	25
50 000	15
100 000	5

Assume that for each of the structures, the probability of one claim within a year is 0.04, and the probability of more than one claim is 0. Assume that fires in the structures are mutually independent events. Furthermore, assume that the conditional distribution of the claim size, given that a claim has occurred, is uniformly distributed over the interval from 0 to the contract amount. Let N be the number of claims and let S be the amount of claims in a 1-year period.

a. Calculate the mean and variance of N.
b. Calculate the mean and variance of S.
c. What relative security loading, θ, should be used so the company can collect an amount equal to the 99th percentile of the distribution of total claims? (Use a normal approximation.)

2.16. Consider a portfolio of 32 policies. For each policy, the probability q of a claim is $1/6$ and B, the benefit amount given that there is a claim, has p.d.f.

$$f(y) = \begin{cases} 2(1-y) & 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let S be the total claims for the portfolio. Using a normal approximation, estimate $\Pr(S > 4)$.

3

SURVIVAL DISTRIBUTIONS AND LIFE TABLES

3.1 Introduction

Chapter 1 was dedicated to showing how insurance can increase the expected utility of individuals facing random losses. In Chapter 2 simple models for single-period insurance policies were developed. The foundations of these models were Bernoulli random variables associated with the occurrence or nonoccurrence of a loss. The occurrence of a loss, in some examples, resulted in a second random process generating the amount of the loss. Chapters 4 through 8 deal primarily with models for insurance systems designed to manage random losses where the randomness is related to how long an individual will survive. In these chapters the ***time-until-death*** random variable, $T(x)$, is the basic building block. This chapter develops a set of ideas for describing and using the distribution of time-until-death and the distribution of the corresponding ***age-at-death***, X.

We show how a distribution of the age-at-death random variable can be summarized by a ***life table***. Such tables are useful in many fields of science. Consequently a profusion of notation and nomenclature has developed among the various professions using life tables. For example, engineers use life tables to study the reliability of complex mechanical and electronic systems. Biostatisticians use life tables to compare the effectiveness of alternative treatments of serious diseases. Demographers use life tables as tools in population projections. In this text, life tables are used to build models for insurance systems designed to assist individuals facing uncertainty about the times of their deaths. This application determines the viewpoint adopted. However, when it provides a bridge to other disciplines, notes relating the discussion to alternative applications of life tables are added.

A life table is an indispensable component of many models in actuarial science. In fact, some scholars fix the date of the beginning of actuarial science as 1693. In that year, Edmund Halley published "An Estimate of the Degrees of the Mortality of Mankind, Drawn from Various Tables of Births and Funerals at the City of

Breslau." The life table, called the Breslau Table, contained in Halley's paper remains of interest because of its surprisingly modern notation and ideas.

3.2 Probability for the Age-at-Death

In this section we formulate the uncertainty of age-at-death in probability concepts.

3.2.1 The Survival Function

Let us consider a newborn child. This newborn's age-at-death, X, is a continuous-type random variable. Let $F_X(x)$ denote the distribution function (d.f.) of X,

$$F_X(x) = \Pr(X \le x) \qquad x \ge 0, \tag{3.2.1}$$

and set

$$s(x) = 1 - F_X(x) = \Pr(X > x) \qquad x \ge 0. \tag{3.2.2}$$

We always assume that $F_X(0) = 0$, which implies $s(0) = 1$. The function $s(x)$ is called the ***survival function*** (s.f.). For any positive x, $s(x)$ is the probability a newborn will attain age x. The distribution of X can be defined by specifying either the function $F_X(x)$ or the function $s(x)$. Within actuarial science and demography, the survival function has traditionally been used as a starting point for further developments. Within probability and statistics, the d.f. usually plays this role. However, from the properties of the d.f., we can deduce corresponding properties of the survival function.

Using the laws of probability, we can make probability statements about the age-at-death in terms of either the survival function or the distribution function. For example, the probability that a newborn dies between ages x and z $(x < z)$ is

$$\begin{aligned}\Pr(x < X \le z) &= F_X(z) - F_X(x)\\ &= s(x) - s(z).\end{aligned}$$

3.2.2 Time-until-Death for a Person Age *x*

The conditional probability that a newborn will die between the ages x and z, given survival to age x, is

$$\begin{aligned}\Pr(x < X \le z | X > x) &= \frac{F_X(z) - F_X(x)}{1 - F_X(x)}\\ &= \frac{s(x) - s(z)}{s(x)}.\end{aligned} \tag{3.2.3}$$

The symbol (x) is used to denote a ***life-age-x***. The future lifetime of (x), $X - x$, is denoted by $T(x)$.

Within actuarial science, it is frequently necessary to make probability statements about $T(x)$. For this purpose, and to promote research and communication, a set of symbols, part of the International Actuarial Notation, was originally adopted by the 1898 International Actuarial Congress. Symbols for common actuarial functions and principles to guide the adoption of new symbols were established. This system has been subject to constant review and is revised or extended as necessary by the International Actuarial Association's Permanent Committee on Notation. These notational conventions are followed in this book whenever possible.

These symbols differ from those used for probability notation, and the reader may be unfamiliar with them. For example, a single-variate function that would be written $q(x)$ in probability notation is written q_x in this system. Likewise, a multivariate function is written in actuarial notation using combinations of subscripts, superscripts, and other symbols. The general rules for defining a function in actuarial notation are given in Appendix 4. The reader may want to study these forms before continuing the discussion of the future-lifetime random variable.

To make probability statements about $T(x)$, we use the notations

$$ {}_tq_x = \Pr[T(x) \le t] \qquad t \ge 0, \tag{3.2.4}$$

$$ {}_tp_x = 1 - {}_tq_x = \Pr[T(x) > t] \qquad t \ge 0. \tag{3.2.5}$$

The symbol ${}_tq_x$ can be interpreted as the probability that (x) will die within t years; that is, ${}_tq_x$ is the d.f. of $T(x)$. On the other hand, ${}_tp_x$ can be interpreted as the probability that (x) will attain age $x + t$; that is, ${}_tp_x$ is the s.f. for (x). In the special case of a life-age-0, we have $T(0) = X$ and

$$ {}_xp_0 = s(x) \qquad x \ge 0. \tag{3.2.6}$$

If $t = 1$, convention permits us to omit the prefix in the symbols defined in (3.2.4) and (3.2.5), and we have

$$q_x = \Pr[(x) \text{ will die within 1 year}],$$

$$p_x = \Pr[(x) \text{ will attain age } x + 1].$$

There is a special symbol for the more general event that (x) will survive t years and die within the following u years; that is, (x) will die between ages $x + t$ and $x + t + u$. This special symbol is given by

$$\begin{aligned} {}_{t|u}q_x &= \Pr[t < T(x) \le t + u] \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x. \end{aligned} \tag{3.2.7}$$

As before, if $u = 1$, the prefix is deleted in ${}_{t|u}q_x$, and we have ${}_{t|}q_x$.

At this point it appears there are two expressions for the probability that (x) will die between ages x and $x + u$. Formula (3.2.7) with $t = 0$ is one such expression; (3.2.3) with $z = x + u$ is a second expression. Are these two probabilities different? Formula (3.2.3) can be interpreted as the conditional probability that a newborn will die between ages x and $z = x + u$, given survival to age x. The only information on the newborn, now at age x, is its survival to that age. Hence, the probability statement is based on a conditional distribution of survival for newborns.

On the other hand, (3.2.7) with $t = 0$ defines a probability that a life observed at age x will die between ages x and $x + u$. The observation on the life at age x might include information other than simply survival. Such information might be that the life has just passed a physical examination for insurance, or it might be that the life has commenced treatment for a serious illness. Life tables for situations where the observation of a life at age x implies more than simply survival of a newborn to age x are discussed in Section 3.8, where additional notation for those life tables is introduced. We will continue development of the theory without further reference to the distinction between (3.2.3) and (3.2.7), and we assume that until that section, observation of survival at age x will yield the same conditional distribution of survival as the hypothesis that a newborn has survived to age x; that is,

$$ {}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)}, \tag{3.2.8} $$

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)}. \tag{3.2.9} $$

Under this approach, (3.2.7), and its many special cases, can be expressed as

$$ \begin{aligned} {}_{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \left[\frac{s(x+t)}{s(x)}\right]\left[\frac{s(x+t) - s(x+t+u)}{s(x+t)}\right] \\ &= {}_tp_x \; {}_uq_{x+t}\,. \end{aligned} \tag{3.2.10} $$

3.2.3 Curtate-Future-Lifetime

A discrete random variable associated with the future lifetime is the number of future years completed by (x) prior to death. It is called the ***curtate-future-lifetime*** of (x) and is denoted by $K(x)$. Because $K(x)$ is the greatest integer in $T(x)$, its p.f. is

$$ \begin{aligned} \Pr[K(x) = k] &= \Pr[k \le T(x) < k+1] \\ &= \Pr[k < T(x) \le k+1] \\ &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x \; q_{x+k} = {}_{k|}q_x \qquad k = 0, 1, 2, \ldots . \end{aligned} \tag{3.2.11} $$

The switching of inequalities is possible since, under our assumption that $T(x)$ is a continuous-type random variable, $\Pr[T(x) = k] = \Pr[T(x) = k + 1] = 0$. Expression (3.2.11) is a special case of (3.2.7) where $u = 1$ and k is a non-negative integer. From (3.2.11) we can see that the d.f. of $K(x)$ is the step function

$$ F_{K(x)}(y) = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x, \qquad y \ge 0 \text{ and } k \text{ is the greatest integer in } y. $$

It often follows from the context that $T(x)$ is the future lifetime of (x), in which case we may write T instead of $T(x)$. Likewise, we may write K instead of $K(x)$.

3.2.4 Force of Mortality

Formula (3.2.3) expresses, in terms of the d.f. and in terms of the survival function, the conditional probability that (0) will die between ages x and z, given survival to x. With $z - x$ held constant, say, at c, then considered as a function of x, this conditional probability describes the distribution of the probability of death in the near future (between time 0 and c) for a life of attained age x. An analogue of this function for instantaneous death can be obtained by using the density of probability of death at attained age x, that is, using (3.2.3) with $z = x + \Delta x$,

$$\Pr(x < X \leq x + \Delta x | X > x) = \frac{F_X(x + \Delta x) - F_X(x)}{1 - F_X(x)}$$

$$\cong \frac{f_X(x)\Delta x}{1 - F_X(x)}. \tag{3.2.12}$$

In this expression $F'_X(x) = f_X(x)$ is the p.d.f. of the continuous age-at-death random variable. The function

$$\frac{f_X(x)}{1 - F_X(x)}$$

in (3.2.12) has a conditional probability density interpretation. For each age x, it gives the value of the conditional p.d.f. of X at exact age x, given survival to that age, and is denoted by $\mu(x)$.

We have

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)}$$

$$= \frac{-s'(x)}{s(x)}. \tag{3.2.13}$$

The properties of $f_X(x)$ and $1 - F_X(x)$ imply that $\mu(x) \geq 0$.

In actuarial science and demography $\mu(x)$ is called the ***force of mortality.*** In reliability theory, the study of the survival probabilities of manufactured parts and systems, $\mu(x)$ is called the ***failure rate*** or ***hazard rate*** or, more fully, the ***hazard rate function.***

As is true for the s.f., the force of mortality can be used to specify the distribution of X. To obtain this result, we start with (3.2.13), change x to y, and rearrange to obtain

$$-\mu(y)\, dy = d \log s(y).$$

Integrating this expression from x to $x + n$, we have

$$-\int_x^{x+n} \mu(y)\, dy = \log\left[\frac{s(x+n)}{s(x)}\right]$$
$$= \log {}_np_x,$$

and on taking exponentials obtain

$${}_np_x = \exp[-\textstyle\int_x^{x+n} \mu(y)\, dy]. \tag{3.2.14}$$

Sometimes it is convenient to rewrite (3.2.14), with $s = y - x$, as

$${}_np_x = \exp[-\textstyle\int_0^n \mu(x+s)\, ds]. \tag{3.2.15}$$

In particular, we will change the notation to conform with that used in (3.2.6) by setting the age already lived to 0 and denoting the time of survival by x. We then have

$${}_xp_0 = s(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]. \tag{3.2.16}$$

In addition,

$$F_X(x) = 1 - s(x) = 1 - \exp[-\textstyle\int_0^x \mu(s)\, ds] \tag{3.2.17}$$

and

$$F'_X(x) = f_X(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]\, \mu(x)$$
$$= {}_xp_0\, \mu(x). \tag{3.2.18}$$

Let $F_{T(x)}(t)$ and $f_{T(x)}(t)$ denote, respectively, the d.f. and p.d.f. of $T(x)$, the future lifetime of (x). From (3.2.4) we note that $F_{T(x)}(t) = {}_tq_x$; therefore,

$$\begin{aligned} f_{T(x)}(t) &= \frac{d}{dt}\, {}_tq_x \\ &= \frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right] \\ &= \frac{s(x+t)}{s(x)}\left[-\frac{s'(x+t)}{s(x+t)}\right] \\ &= {}_tp_x\, \mu(x+t) \qquad t \ge 0. \end{aligned} \tag{3.2.19}$$

Thus ${}_tp_x\, \mu(x+t)\, dt$ is the probability that (x) dies between t and $t + dt$, and

$$\int_0^\infty {}_tp_x\, \mu(x+t)\, dt = 1$$

where the upper limit on the integral is written as positive infinity (an abbreviation for integrating over all positive probability density).

It follows from (3.2.19) that

$$\frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}\, {}_tp_x = {}_tp_x\, \mu(x+t). \tag{3.2.20}$$

This equivalent form is useful in several developments in actuarial mathematics.

Since

$$\lim_{n\to\infty} {}_np_x = 0,$$

we have

$$\lim_{n\to\infty} (-\log {}_np_x) = \infty;$$

that is,

$$\lim_{n\to\infty} \int_x^{x+n} \mu(y)\, dy = \infty.$$

The developments of this section are summarized in Table 3.2.1.

The lower half of Table 3.2.1 summarizes some of the relationships among functions of general probability theory and those specific to age-at-death applications. There are many other examples where age-at-death questions can be formed in the more general probability setting. The following will illustrate this point.

TABLE 3.2.1

Probability Theory Functions for Age-at-Death, X

	d.f. $F_X(x)$	**Survival Function $s(x)$**		**p.d.f. $f_X(x)$**	**Force of Mortality $\mu(x)$**
For	Requirements		For	Requirements	
$x < 0$	$F_X(x) = 0$	$s(x) = 1$	$x < 0$	$f_X(x) = 0$	$\mu(x) = 0$
$x = 0$	$F_X(0) = 0$	$s(0) = 1$	$x = 0$	undefined	undefined
$x \ge 0$	nondecreasing	nonincreasing	$x > 0$	$f_X(x) \ge 0$	$\mu(x) \ge 0$
$\lim_{x\to\infty}$	$F_X(\infty) = 1$	$s(\infty) = 0$	$\lim_{x\to\infty}$	$\int_0^\infty f_X(x)\, dt = 1$	$\int_0^\infty \mu(x)dx = \infty$
Functions in Terms of	Relationships				
$F_X(x)$	$F_X(x)$	$1 - F_X(x)$		$F'_X(x)$	$F'_X(x) / [1 - F_X(x)]$
$s(x)$	$1 - s(x)$	$s(x)$		$-s'(x)$	$-s'(x) / s(x)$
$f_X(x)$	$\int_0^x f_X(u)du$	$\int_x^\infty f_X(u)du$		$f_X(x)$	$f_X(x) / \int_x^\infty f_X(u)du$
$\mu(x)$	$1 - \exp[-\int_0^x \mu(t)\, dt]$	$\exp[-\int_0^x \mu(t)\, dt]$		$\mu(x)\exp[-\int_0^x \mu(t)\, dt]$	$\mu(x)$

Example 3.2.1

If $\bar{A}$ refers to the complement of the event A within the sample space and $\Pr(\bar{A}) \neq 0$, the following expresses an identity in probability theory:

$$\Pr(A \cup B) = \Pr(A) + \Pr(\bar{A})\, \Pr(B|\bar{A}).$$

Rewrite this identity in actuarial notation for the events $A = [T(x) \le t]$ and $B = [t < T(x) \le 1]$, $0 < t < 1$.

Solution:

$\Pr(A \cup B)$ becomes $\Pr[T(x) \le 1] = q_x$, $\Pr(A)$ is ${}_tq_x$, and $\Pr(B|\bar{A})$ is ${}_{1-t}q_{x+t}$; hence

$$q_x = {}_tq_x + {}_tp_x\ {}_{1-t}q_{x+t}. \qquad \blacktriangledown$$

3.3 Life Tables

A published life table usually contains tabulations, by individual ages, of the basic functions q_x, l_x, d_x, and, possibly, additional derived functions. Before presenting such a table, we consider an interpretation of these functions that is directly related to the probability functions discussed in the preceding section.

3.3.1 Relation of Life Table Functions to the Survival Function

In (3.2.9) we expressed the conditional probability that (x) will die within t years by

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)}, $$

and, in particular, we have

$$ q_x = 1 - \frac{s(x+1)}{s(x)}. $$

We now consider a group of l_0 newborns, $l_0 = 100{,}000$, for instance. Each newborn's age-at-death has a distribution specified by s.f. $s(x)$. In addition, we let $\mathcal{L}(x)$ denote the group's number of survivors to age x. We index these lives by $j = 1, 2, \ldots, l_0$ and observe that

$$ \mathcal{L}(x) = \sum_{j=1}^{l_0} I_j $$

where I_j is an indicator for the survival of life j; that is,

$$ I_j = \begin{cases} 1 & \text{if life } j \text{ survives to age } x \\ 0 & \text{otherwise.} \end{cases} $$

Since $\mathrm{E}[I_j] = s(x)$,

$$ \mathrm{E}[\mathcal{L}(x)] = \sum_{j=1}^{l_0} \mathrm{E}[I_j] = l_0\, s(x). $$

We denote $\mathrm{E}[\mathcal{L}(x)]$ by l_x; that is, l_x represents the expected number of survivors to age x from the l_0 newborns, and we have

$$ l_x = l_0\, s(x). \tag{3.3.1} $$

Moreover, under the assumption that the indicators I_j are mutually independent, $\mathcal{L}(x)$ has a binomial distribution with parameters $n = l_0$ and $p = s(x)$. Note, however, that (3.3.1) does not require the independence assumption.

In a similar fashion, ${}_n\mathcal{D}_x$ denotes the number of deaths between ages x and $x + n$ from among the initial l_0 lives. We denote $\mathrm{E}[{}_n\mathcal{D}_x]$ by ${}_nd_x$. Since a newborn has probability $s(x) - s(x + n)$ of death between ages x and $x + n$ we can, by an argument similar to that for l_x, express

$$
\begin{aligned}
{}_nd_x = \mathrm{E}[{}_n\mathcal{D}_x] &= l_0[s(x) - s(x + n)] \\
&= l_x - l_{x+n}.
\end{aligned} \tag{3.3.2}
$$

When $n = 1$, we omit the prefixes on ${}_n\mathcal{D}_x$ and ${}_nd_x$.

From (3.3.1), we see that

$$
-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{1}{s(x)}\frac{ds(x)}{dx} = \mu(x) \tag{3.3.3}
$$

and

$$
-dl_x = l_x\,\mu(x)\,dx. \tag{3.3.4}
$$

Since

$$
l_x\,\mu(x) = l_0\,{}_xp_0\,\mu(x) = l_0\,f_X(x),
$$

the factor $l_x\,\mu(x)$ in (3.3.4) can be interpreted as the expected density of deaths in the age interval $(x, x + dx)$. We note further that

$$
l_x = l_0 \exp[-\textstyle\int_0^x \mu(y)\,dy], \tag{3.3.5}
$$

$$
l_{x+n} = l_x \exp[-\textstyle\int_x^{x+n} \mu(y)\,dy], \tag{3.3.6}
$$

$$
l_x - l_{x+n} = \textstyle\int_x^{x+n} l_y\,\mu(y)\,dy. \tag{3.3.7}
$$

For convenience of reference, we call this concept of l_0 newborns, each with survival function $s(x)$, a ***random survivorship group.***

3.3.2 Life Table Example

In "Life Table for the Total Population: United States, 1979–81" (Table 3.3.1), the functions ${}_tq_x$, l_x, and ${}_td_x$ are presented with $l_0 = 100{,}000$. Except for the first year of life, the value of t in the tabulated functions ${}_tq_x$ and ${}_td_x$ is 1. The other functions appearing in the table are discussed in Section 3.5.

The 1979–81 U.S. Life Table was not constructed by observing 100,000 newborns until the last survivor died. Instead, it was based on estimates of probabilities of death, given survival to various ages, derived from the experience of the entire U.S. population in the years around the 1980 census. In using the random survivorship group concept with this table, we must make the assumption that the probabilities derived from the table will be appropriate for the lifetimes of those who belong to the survivorship group.

TABLE 3.3.1

Life Table for the Total Population: United States, 1979–81

(1) Age Interval — Period of Life between Two Ages, x to $x+t$	(2) Proportion Dying — Proportion of Persons Alive at Beginning of Age Interval Dying during Interval, ${}_tq_x$	(3) Of 100,000 Born Alive — Number Living at Beginning of Age Interval, l_x	(4) Of 100,000 Born Alive — Number Dying during Age Interval, ${}_td_x$	(5) Stationary Population* — Years Lived in the Age Interval, ${}_tL_x$	(6) Stationary Population* — Years Lived in This and All Subsequent Age Intervals, T_x	(7) Average Remaining Lifetime — Average Number of Years of Life Remaining at Beginning of Age Interval, $\mathring{e}_x$
Days						
0–1	0.00463	100 000	463	273	7 387 758	73.88
1–7	0.00246	99 537	245	1 635	7 387 485	74.22
7–28	0.00139	99 292	138	5 708	7 385 850	74.38
28–365	0.00418	99 154	414	91 357	7 380 142	74.43
Years						
0–1	0.01260	100 000	1 260	98 973	7 387 758	73.88
1–2	0.00093	98 740	92	98 694	7 288 785	73.82
2–3	0.00065	98 648	64	98 617	7 190 091	72.89
3–4	0.00050	98 584	49	98 560	7 091 474	71.93
4–5	0.00040	98 535	40	98 515	6 992 914	70.97
5–6	0.00037	98 495	36	98 477	6 894 399	70.00
6–7	0.00033	98 459	33	98 442	6 795 922	69.02
7–8	0.00030	98 426	30	98 412	6 697 480	68.05
8–9	0.00027	98 396	26	98 383	6 599 068	67.07
9–10	0.00023	98 370	23	98 358	6 500 685	66.08
10–11	0.00020	98 347	19	98 338	6 402 327	65.10
11–12	0.00019	98 328	19	98 319	6 303 989	64.11
12–13	0.00025	98 309	24	98 297	6 205 670	63.12
13–14	0.00037	98 285	37	98 266	6 107 373	62.14
14–15	0.00053	98 248	52	98 222	6 009 107	61.16
15–16	0.00069	98 196	67	98 163	5 910 885	60.19
16–17	0.00083	98 129	82	98 087	5 812 722	59.24
17–18	0.00095	98 047	94	98 000	5 714 635	58.28
18–19	0.00105	97 953	102	97 902	5 616 635	57.34
19–20	0.00112	97 851	110	97 796	5 518 733	56.40
20–21	0.00120	97 741	118	97 682	5 420 937	55.46
21–22	0.00127	97 623	124	97 561	5 323 255	54.53
22–23	0.00132	97 499	129	97 435	5 225 694	53.60
23–24	0.00134	97 370	130	97 306	5 128 259	52.67
24–25	0.00133	97 240	130	97 175	5 030 953	51.74
25–26	0.00132	97 110	128	97 046	4 933 778	50.81
26–27	0.00131	96 982	126	96 919	4 836 732	49.87
27–28	0.00130	96 856	126	96 793	4 739 813	48.94
28–29	0.00130	96 730	126	96 667	4 643 020	48.00
29–30	0.00131	96 604	127	96 541	4 546 353	47.06

*Stationary population is a demographic concept treated in Chapter 19.

TABLE 3.3.1—Continued

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages x to $x+t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
30–31	0.00133	96 477	127	96 414	4 449 812	46.12
31–32	0.00134	96 350	130	96 284	4 353 398	45.18
32–33	0.00137	96 220	132	96 155	4 257 114	44.24
33–34	0.00142	96 088	137	96 019	4 160 959	43.30
34–35	0.00150	95 951	143	95 880	4 064 940	42.36
35–36	0.00159	95 808	153	95 731	3 969 060	41.43
36–37	0.00170	95 655	163	95 574	3 873 329	40.49
37–38	0.00183	95 492	175	95 404	3 777 755	39.56
38–39	0.00197	95 317	188	95 224	3 682 351	38.63
39–40	0.00213	95 129	203	95 027	3 587 127	37.71
40–41	0.00232	94 926	220	94 817	3 492 100	36.79
41–42	0.00254	94 706	241	94 585	3 397 283	35.87
42–43	0.00279	94 465	264	94 334	3 302 698	34.96
43–44	0.00306	94 201	288	94 057	3 208 364	34.06
44–45	0.00335	93 913	314	93 756	3 114 307	33.16
45–46	0.00366	93 599	343	93 427	3 020 551	32.27
46–47	0.00401	93 256	374	93 069	2 927 124	31.39
47–48	0.00442	92 882	410	92 677	2 834 055	30.51
48–49	0.00488	92 472	451	92 246	2 741 378	29.65
49–50	0.00538	92 021	495	91 773	2 649 132	28.79
50–51	0.00589	91 526	540	91 256	2 557 359	27.94
51–52	0.00642	90 986	584	90 695	2 466 103	27.10
52–53	0.00699	90 402	631	90 086	2 375 408	26.28
53–54	0.00761	89 771	684	89 430	2 285 322	25.46
54–55	0.00830	89 087	739	88 717	2 195 892	24.65
55–56	0.00902	88 348	797	87 950	2 107 175	23.85
56–57	0.00978	87 551	856	87 122	2 019 225	23.06
57–58	0.01059	86 695	919	86 236	1 932 103	22.29
58–59	0.01151	85 776	987	85 283	1 845 867	21.52
59–60	0.01254	84 789	1 063	84 258	1 760 584	20.76
60–61	0.01368	83 726	1 145	83 153	1 676 326	20.02
61–62	0.01493	82 581	1 233	81 965	1 593 173	19.29
62–63	0.01628	81 348	1 324	80 686	1 511 208	18.58
63–64	0.01767	80 024	1 415	79 316	1 430 522	17.88
64–65	0.01911	78 609	1 502	77 859	1 351 206	17.19

*Stationary population is a demographic concept treated in Chapter 19.

TABLE 3.1—Continued

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages, x to $x+t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval, ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval, l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval, ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval, ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals, T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval, $\mathring{e}_x$
Years						
65–66	0.02059	77 107	1 587	76 314	1 273 347	16.51
66–67	0.02216	75 520	1 674	74 683	1 197 033	15.85
67–68	0.02389	73 846	1 764	72 964	1 122 350	15.20
68–69	0.02585	72 082	1 864	71 150	1 049 386	14.56
69–70	0.02806	70 218	1 970	69 233	978 236	13.93
70–71	0.03052	68 248	2 083	67 206	909 003	13.32
71–72	0.03315	66 165	2 193	65 069	841 797	12.72
72–73	0.03593	63 972	2 299	62 823	776 728	12.14
73–74	0.03882	61 673	2 394	60 476	713 905	11.58
74–75	0.04184	59 279	2 480	58 039	653 429	11.02
75–76	0.04507	56 799	2 560	55 520	595 390	10.48
76–77	0.04867	54 239	2 640	52 919	539 870	9.95
77–78	0.05274	51 599	2 721	50 238	486 951	9.44
78–79	0.05742	48 878	2 807	47 475	436 713	8.93
79–80	0.06277	46 071	2 891	44 626	389 238	8.45
80–81	0.06882	43 180	2 972	41 694	344 612	7.98
81–82	0.07552	40 208	3 036	38 689	302 918	7.53
82–83	0.08278	37 172	3 077	35 634	264 229	7.11
83–84	0.09041	34 095	3 083	32 553	228 595	6.70
84–85	0.09842	31 012	3 052	29 486	196 042	6.32
85–86	0.10725	27 960	2 999	26 461	166 556	5.96
86–87	0.11712	24 961	2 923	23 500	140 095	5.61
87–88	0.12717	22 038	2 803	20 636	116 595	5.29
88–89	0.13708	19 235	2 637	17 917	95 959	4.99
89–90	0.14728	16 598	2 444	15 376	78 042	4.70
90–91	0.15868	14 154	2 246	13 031	62 666	4.43
91–92	0.17169	11 908	2 045	10 886	49 635	4.17
92–93	0.18570	9 863	1 831	8 948	38 749	3.93
93–94	0.20023	8 032	1 608	7 228	29 801	3.71
94–95	0.21495	6 424	1 381	5 733	22 573	3.51
95–96	0.22976	5 043	1 159	4 463	16 840	3.34
96–97	0.24338	3 884	945	3 412	12 377	3.19
97–98	0.25637	2 939	754	2 562	8 965	3.05
98–99	0.26868	2 185	587	1 892	6 403	2.93
99–100	0.28030	1 598	448	1 374	4 511	2.82

*Stationary population is a demographic concept treated in Chapter 19.

TABLE 3.3.1—Continued

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages x to $x+t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
100–101	0.29120	1 150	335	983	3 137	2.73
101–102	0.30139	815	245	692	2 154	2.64
102–103	0.31089	570	177	481	1 462	2.57
103–104	0.31970	393	126	330	981	2.50
104–105	0.32786	267	88	223	651	2.44
105–106	0.33539	179	60	150	428	2.38
106–107	0.34233	119	41	99	278	2.33
107–108	0.34870	78	27	64	179	2.29
108–109	0.35453	51	18	42	115	2.24
109–110	0.35988	33	12	27	73	2.20

*Stationary population is a demographic concept treated in Chapter 19.

Several observations about the 1979–81 U.S. Life Table are instructive.

Observations:

1. Approximately 1% of a survivorship group of newborns would be expected to die in the first year of life.
2. It would be expected that about 77% of a group of newborns would survive to age 65.
3. The maximum number of deaths within a group would be expected to occur between ages 83 and 84.
4. For human lives, there have been few observations of age-at-death beyond 110. Consequently, it is often assumed that there is an age ω such that $s(x) > 0$ for $x < \omega$, and $s(x) = 0$ for $x \geq \omega$. The age ω, if assumed, is called the ***limiting age.*** The limiting age for this table is not defined. It is clear that there is a positive probability of survival to age 110, but the table does not indicate the age ω.
5. Local minimums in the expected number of deaths occur around ages 11 and 27 and a local maximum around age 24.
6. Although the values of l_x have been rounded to integers, there is no compelling reason, according to (3.3.1), to do so.

A display such as Table 3.3.1 is the conventional method for describing the distribution of age-at-death. Alternatively, an s.f. can be described in analytic form such as $s(x) = e^{-cx}$, $c > 0$, $x \geq 0$. However, most studies of human mortality for insurance purposes use the representation $s(x) = l_x / l_0$, as illustrated in Table 3.3.1. Since 100,000 $s(x)$ is displayed for only integer values of x, there is a need to interpolate in evaluating $s(x)$ for noninteger values. This is the subject of Section 3.6.

Example 3.3.1

On the basis of Table 3.3.1, evaluate the probability that (20) will

a. Live to 100
b. Die before 70
c. Die in the tenth decade of life.

Solution:

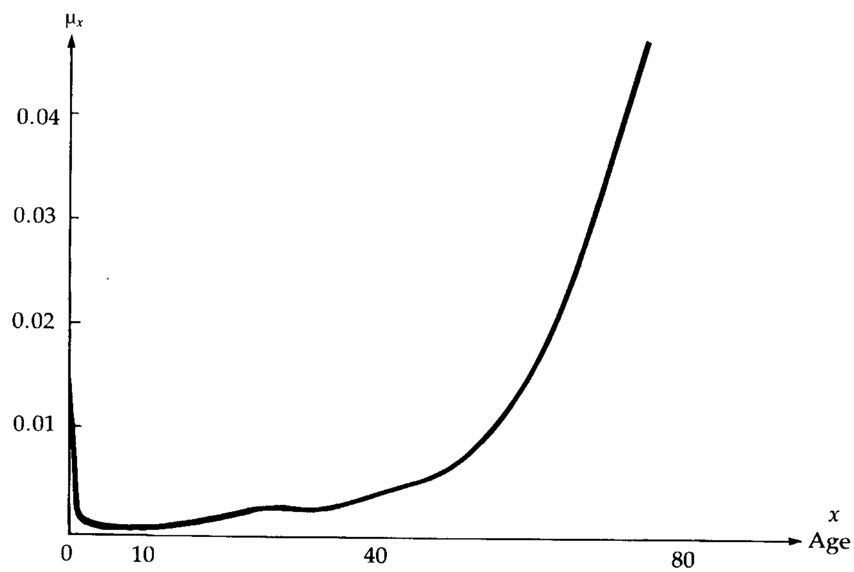
a. $\dfrac{s(100)}{s(20)} = \dfrac{l_{100}}{l_{20}} = \dfrac{1,150}{97,741} = 0.0118$

b. $\dfrac{[s(20) - s(70)]}{s(20)} = 1 - \dfrac{l_{70}}{l_{20}} = 1 - \dfrac{68,248}{97,741} = 0.3017$

c. $\dfrac{[s(90) - s(100)]}{s(20)} = \dfrac{(l_{90} - l_{100})}{l_{20}} = \dfrac{(14,154 - 1,150)}{97,741} = 0.1330.$ ▼

Insight into life table functions can be obtained by studying Figures 3.3.1, 3.3.2, and 3.3.3. These are drawn to be representative of current human mortality and are not taken directly from Table 3.3.1.

FIGURE 3.3.1
Graph of $\mu(x)$



In Figure 3.3.1 note two features:

- The force of mortality is positive and the requirement

$$\int_0^\infty \mu(x)\,dx = \infty$$

  appears satisfied. (See Table 3.2.1.)
- The force of mortality starts out rather large and then drops to a minimum around age 10.

FIGURE 3.3.2

Graph of $l_x\,\mu(x)$

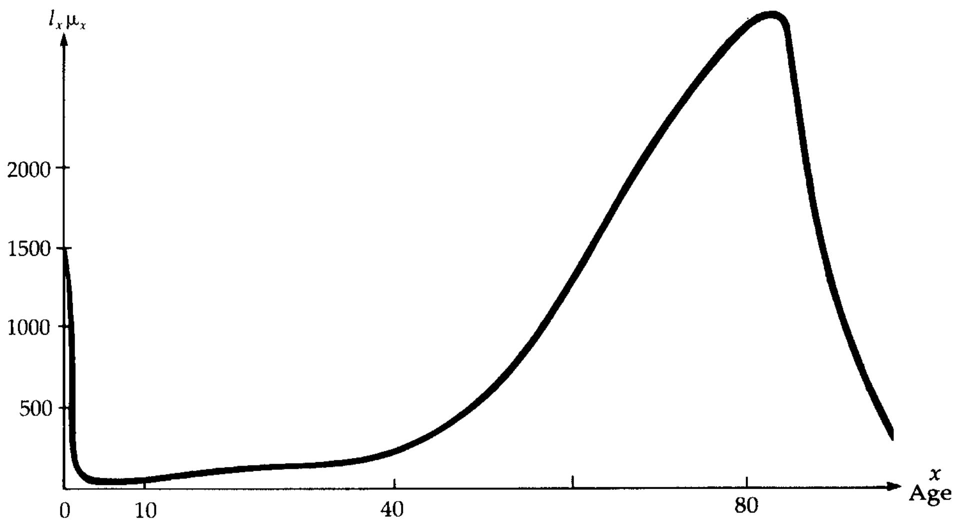
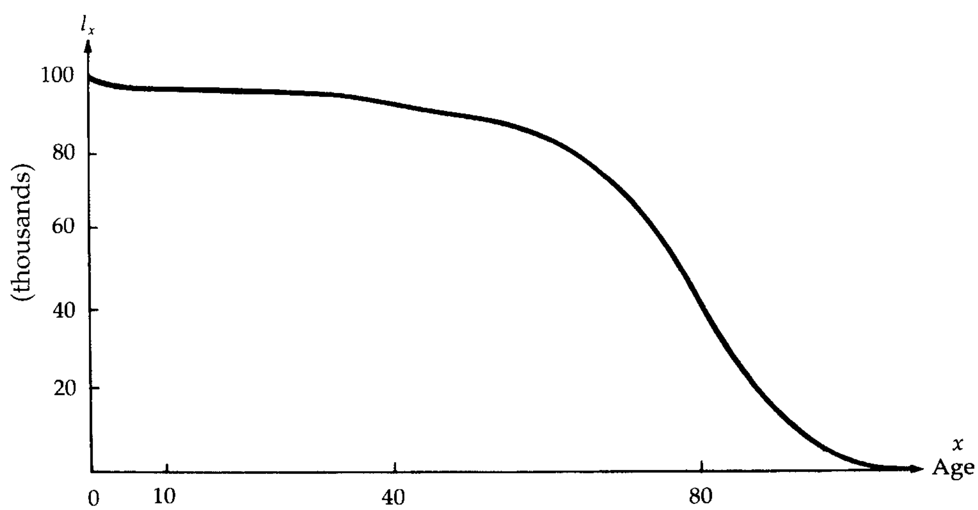


FIGURE 3.3.3

Graph of l_x



In Figures 3.3.2 and 3.3.3 note the following:

- The function $l_x\ \mu(x)$ is proportional to the p.d.f. of the age-at-death of a newborn. Since $l_x\ \mu(x)$ is the expected density of deaths at age x, under the random survivorship group idea, the graph of $l_x\ \mu(x)$ is called ***the curve of deaths.***
- There is a local minimum of $l_x\ \mu(x)$ at about age 10. The mode of the distribution of deaths—the age at which the maximum of the curve of deaths occurs—is around age 80.
- The function l_x is proportional to the survival function $s(x)$. It can also be interpreted as the expected number living at age x out of an initial group of size l_0.
- Local extreme points of $l_x\ \mu(x)$ correspond to points of inflection of l_x since

$$\frac{d}{dx}\, l_x\ \mu(x) = \frac{d}{dx}\left(-\frac{d}{dx}\, l_x\right) = -\frac{d^2}{dx^2}\, l_x .$$

3.4 The Deterministic Survivorship Group

We proceed now to a second, and nonprobabilistic, interpretation of the life table. This is rooted mathematically in the concept of decrement (negative growth) rates. As such, it is related to growth-rate applications in biology and economics. It is deterministic in nature and leads to the concept of a ***deterministic survivorship group*** or ***cohort.***

A deterministic survivorship group, as represented by a life table, has the following characteristics:

- The group initially consists of l_0 lives age 0.
- The members of the group are subject, at each age of their lives, to effective annual rates of mortality (decrement) specified by the values of q_x in the life table.
- The group is closed. No further entrants are allowed beyond the initial l_0. The only decreases come as a result of the effective annual rates of mortality (decrement).

From these characteristics it follows that the progress of the group is determined by

$$\begin{aligned}
l_1 &= l_0(1 - q_0) = l_0 - d_0, \\
l_2 &= l_1(1 - q_1) = l_1 - d_1 = l_0 - (d_0 + d_1), \\
&\ \ \vdots \qquad\quad \vdots \qquad\ \ \vdots \\
l_x &= l_{x-1}(1 - q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - \sum_{y=0}^{x-1} d_y \\
&= l_0\left(1 - \frac{\sum_{y=0}^{x-1} d_y}{l_0}\right) = l_0(1 - {}_xq_0)
\end{aligned} \tag{3.4.1}$$

where l_x is the number of lives attaining age x in the survivorship group. This chain of equalities, generated by a value l_0 called the ***radix*** and a set of q_x values, can be rewritten as

$$
\begin{aligned}
l_1 &= l_0\, p_0, \\
l_2 &= l_1\, p_1 = (l_0\, p_0)\, p_1, \\
&\vdots \qquad \vdots \\
l_x &= l_{x-1}\, p_{x-1} = l_0 \left(\prod_{y=0}^{x-1} p_y \right) = l_0\, {}_xp_0 .
\end{aligned}
\tag{3.4.2}
$$

There is an analogy between the deterministic survivorship group and the model for compound interest. Table 3.4.1 is designed to summarize some of this parallelism.

TABLE 3.4.1

Related Concepts of the Mathematics of Compound Interest and of Deterministic Survivorship Groups

Compound Interest	Survivorship Group
$A(t)$ = Size of fund at time t, time measured in years	l_x = Size of group at age x, age measured in years
Effective annual rate of interest (increment) $$i_t = \frac{A(t+1) - A(t)}{A(t)}$$	Effective annual rate of mortality (decrement) $$q_x = \frac{l_x - l_{x+1}}{l_x}$$
Effective n-year rate of interest, starting at time t $${}_ni_t^* = \frac{A(t+n) - A(t)}{A(t)}$$	Effective n-year rate of mortality, starting at age x $${}_nq_x = \frac{l_x - l_{x+n}}{l_x}$$
Force of interest at time t $$\delta_t = \lim_{\Delta t \to 0} \left[\frac{A(t+\Delta t) - A(t)}{A(t)\, \Delta t} \right] = \frac{1}{A(t)} \frac{dA(t)}{dt}$$	Force of mortality at age x $$\mu(x) = \lim_{\Delta x \to 0} \left(\frac{l_x - l_{x+\Delta x}}{l_x\, \Delta x} \right) = -\frac{1}{l_x} \frac{dl_x}{dx}$$

*There is no universally accepted symbol for an effective n-year rate of interest.

The headings of the ${}_tq_x$, l_x, and ${}_td_x$ columns in Table 3.3.1 refer to the deterministic survivorship group interpretation. Although the mathematical foundations of the random survivorship group and the deterministic survivorship group are different, the resulting functions q_x, l_x, and d_x have the same mathematical properties and subsequent analysis. The random survivorship group concept has the advantage of allowing for the full use of probability theory. The deterministic survivorship group is conceptually simple and easy to apply but does not take account of random variation in the number of survivors.

3.5 Other Life Table Characteristics

In this section we derive expressions for some commonly used characteristics of the distributions of $T(x)$ and $K(x)$ and introduce a general method for computing several of these characteristics.

3.5.1 Characteristics

The expected value of $T(x)$, denoted by $\mathring{e}_x$, is called the ***complete-expectation-of-life.*** By definition and an integration by parts, we have

$$\begin{aligned}\mathring{e}_x = \mathrm{E}[T(x)] &= \int_0^\infty t\ {}_tp_x\ \mu(x+t)\ dt \\ &= \int_0^\infty t\ d_t(-{}_tp_x) \\ &= t(-{}_tp_x)\Big|_0^\infty + \int_0^\infty {}_tp_x\ dt. \end{aligned} \tag{3.5.1}$$

The existence of $\mathrm{E}[T(x)]$ implies the $\lim_{t\to\infty} t(-{}_tp_x) = 0$. Thus

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt. \tag{3.5.2}$$

The complete-expectation-of-life at various ages is often used to compare levels of public health among different populations.

A similar integration by parts yields equivalent expressions for $\mathrm{E}[T(x)^2]$:

$$\begin{aligned}\mathrm{E}[T(x)^2] &= \int_0^\infty t^2\ {}_tp_x\ \mu(x+t)\ dt \\ &= 2\int_0^\infty t\ {}_tp_x\ dt. \end{aligned} \tag{3.5.3}$$

This result is useful in the calculation of Var $[T(x)]$ by

$$\begin{aligned}\mathrm{Var}[T(x)] &= \mathrm{E}[T(x)^2] - \mathrm{E}[T(x)]^2 \\ &= 2\int_0^\infty t\ {}_tp_x\ dt - \mathring{e}_x^2. \end{aligned} \tag{3.5.4}$$

In these developments, we assume that $\mathrm{E}[T(x)]$ and $\mathrm{E}[T(x)^2]$ exist. One can construct s.f.'s such as $s(x) = (1+x)^{-1}$ where this would not be true.

Other characteristics of the distribution of $T(x)$ can be determined. The ***median future lifetime*** of (x), to be denoted by $m(x)$, can be found by solving

$$\Pr[T(x) > m(x)] = \frac{1}{2}$$

or

$$\frac{s[x + m(x)]}{s(x)} = \frac{1}{2} \tag{3.5.5}$$

for $m(x)$. In particular, $m(0)$ is given by solving $s[m(0)] = 1/2$. We can also find the ***mode*** of the distribution of $T(x)$ by locating the value of t that yields a maximum value of ${}_tp_x\ \mu(x + t)$.

The expected value of $K(x)$ is denoted by e_x and is called the ***curtate-expectation-of-life.*** By definition and use of summation by parts as described in Appendix 5, we have

$$\begin{aligned} e_x = \mathrm{E}[K] &= \sum_{k=0}^{\infty} k\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k\ \Delta(-{}_kp_x) \\ &= k(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} {}_{k+1}p_x\,. \end{aligned} \tag{3.5.6}$$

Again, the existence of $\mathrm{E}[K(x)]$ implies the $\lim_{k\to\infty} k(-{}_kp_x) = 0$. Thus, with a change of the summation variable,

$$e_x = \sum_{k=1}^{\infty} {}_kp_x\,. \tag{3.5.7}$$

Following the outline used for the continuous model and using summation by parts, we have

$$\begin{aligned} \mathrm{E}[K(x)^2] &= \sum_{k=0}^{\infty} k^2\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k^2\ \Delta(-{}_kp_x) \\ &= k^2(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} (\Delta k^2)({}_{k+1}p_x). \end{aligned} \tag{3.5.8}$$

The existence of $\mathrm{E}[K(x)^2]$ implies $\lim_{k\to\infty} k^2(-{}_kp_x) = 0$. With a change of the summation variable,

$$\mathrm{E}[K(x)^2] = \sum_{k=0}^{\infty} (2k + 1)\ {}_{k+1}p_x = \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x \tag{3.5.9}$$

Now,

$$\begin{aligned} \mathrm{Var}(K) &= \mathrm{E}[K^2] - \mathrm{E}[K]^2 \\ &= \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x - e_x^2\,. \end{aligned} \tag{3.5.10}$$

To complete the discussion of some of the entries in Table 3.3.1, we must define additional functions. The symbol L_x denotes the total expected number of years lived between ages x and $x + 1$ by survivors of the initial group of l_0 lives. We have

$$L_x = \int_0^1 t\, l_{x+t}\, \mu(x + t)\, dt + l_{x+1} \tag{3.5.11}$$

where the integral counts the years lived of those who die between ages x and $x + 1$, and the term l_{x+1} counts the years lived between ages x and $x + 1$ by those who survive to age $x + 1$. Integration by parts yields

$$\begin{aligned} L_x &= -\int_0^1 t\, dl_{x+t} + l_{x+1} \\ &= -t\, l_{x+t}\Big|_0^1 + \int_0^1 l_{x+t}\, dt + l_{x+1} \\ &= \int_0^1 l_{x+t}\, dt. \end{aligned} \tag{3.5.12}$$

The function L_x is also used in defining the ***central-death-rate*** over the interval from x to $x + 1$, denoted by m_x where

$$m_x = \frac{\int_0^1 l_{x+t}\, \mu(x + t)\, dt}{\int_0^1 l_{x+t}\, dt} = \frac{l_x - l_{x+1}}{L_x}. \tag{3.5.13}$$

An application of this function is found in Chapter 10.

The definitions for m_x and L_x can be extended to age intervals of length other than one:

$$\begin{aligned} {}_nL_x &= \int_0^n t\, l_{x+t}\, \mu(x + t)\, dt + n l_{x+n} \\ &= \int_0^n l_{x+t}\, dt, \end{aligned} \tag{3.5.14}$$

$${}_nm_x = \frac{\int_0^n l_{x+t}\, \mu(x + t)\, dt}{\int_0^n l_{x+t}\, dt} = \frac{l_x - l_{x+n}}{{}_nL_x}. \tag{3.5.15}$$

For the random survivorship group, ${}_nL_x$ is the total expected number of years lived between ages x and $x + n$ by the survivors of the initial group of l_0 lives and ${}_nm_x$ is the average death rate experienced by this group over the interval $(x, x + n)$.

The symbol T_x denotes the total number of years lived beyond age x by the survivorship group with l_0 initial members. We have

$$\begin{aligned} T_x &= \int_0^\infty t\, l_{x+t}\, \mu(x+t)\, dt \\ &= -\int_0^\infty t\, d_t\, l_{x+t} \\ &= \int_0^\infty l_{x+t}\, dt. \end{aligned} \tag{3.5.16}$$

The final expression can be interpreted as the integral of the total time lived between ages $x + t$ and $x + t + dt$ by the l_{x+t} lives who survive to that age interval. We also recognize T_x as the limit of ${}_nL_x$ as n goes to infinity.

The average number of years of future lifetime of the l_x survivors of the group at age x is given by

$$\begin{aligned} \frac{T_x}{l_x} &= \frac{\int_0^\infty l_{x+t}\, dt}{l_x} \\ &= \int_0^\infty {}_tp_x\, dt \\ &= \mathring{e}_x, \end{aligned}$$

as determined in (3.5.1) and (3.5.2).

We can express the average number of years lived between x and $x + n$ by the l_x survivors at age x as

$$\begin{aligned} \frac{{}_nL_x}{l_x} &= \frac{\int_0^n l_{x+t}\, dt}{l_x} \\ &= \frac{T_x - T_{x+n}}{l_x} \\ &= \int_0^n {}_tp_x\, dt. \end{aligned} \tag{3.5.17}$$

This function is the ***n-year temporary complete life expectancy*** of (x) and is denoted by $\mathring{e}_{x:\overline{n}|}$. (See Exercise 3.16.)

A final function, related to the interpretation of the life table developed in this section, is the average number of years lived between ages x and $x + 1$ by those of the survivorship group who die between those ages. This function is denoted by $a(x)$ and is defined by

$$a(x) = \frac{\int_0^1 t\, l_{x+t}\, \mu(x+t)\, dt}{\int_0^1 l_{x+t}\, \mu(x+t)\, dt}. \tag{3.5.18}$$

For the probabilistic view of the life table, we would have

$$a(x) = \frac{\int_0^1 t\, {}_tp_x\, \mu(x+t)\, dt}{\int_0^1 {}_tp_x\, \mu(x+t)\, dt} = \mathrm{E}[T|T < 1].$$

If we assume that

$$l_{x+t}\, \mu(x+t)\, dt = d_x\, dt \qquad 0 \le t \le 1,$$

that is, if deaths are uniformly distributed in the year of age, we have

$$a(x) = \int_0^1 t\, dt = \frac{1}{2}.$$

This is the usual approximation for $a(x)$, except for young and old years of age where Figure 3.3.2 shows that the assumption may be inappropriate.

Example 3.5.1

Show that

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}$$

and

$$L_x \cong \frac{l_x + l_{x+1}}{2}.$$

Solution:

From (3.5.11), (3.5.12), and (3.5.18), we have

$$a(x) = \frac{L_x - l_{x+1}}{l_x - l_{x+1}}$$

or

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}.$$

The formula

$$L_x \cong \frac{l_x + l_{x+1}}{2}$$

can be justified by using the trapezoidal rule for approximate integration on (3.5.12). ▼

Key life table terminology, defined in Sections 3.3–3.5, is summarized as part of Table 3.9.1 in Section 3.9.

3.5.2 Recursion Formulas

Example 3.5.1 illustrates the use of a numerical analysis technique to evaluate a life table characteristic. The trapezoidal rule for approximate integration is used. The calculation of complete and curtate expectations-of-life can be used to illustrate another computational tool called ***recursion formulas.*** The application of recursion formulas in this book typically involves one of two forms:

Backward Recursion Formula

$$u(x) = c(x) + d(x)\, u(x + 1) \tag{3.5.19}$$

or

Forward Recursion Formula

$$u(x + 1) = -\frac{c(x)}{d(x)} + \frac{1}{d(x)}\, u(x). \tag{3.5.20}$$

The variable x is usually a non-negative integer.

To evaluate a function $u(x)$, for a domain of non-negative integer values of x, we need to have available values of $c(x)$ and $d(x)$ and a starting value of $u(x)$. This procedure is used in subsequent chapters and is illustrated in Table 3.5.1 where backward recursion formulas are developed to compute e_x and $\mathring{e}_x$.

TABLE 3.5.1

Backward Recursion Formulas for e_x and $\mathring{e}_x$

Step	e_x	$\mathring{e}_x$
1. Basic equation	$e_x = \sum_{k=1}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^{\infty} {}_sp_x\, ds$
2. Separate the operation	$e_x = p_x + \sum_{k=2}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^1 {}_sp_x\, ds + \int_1^{\infty} {}_sp_x\, ds$
3. Factor p_x and change variable in the operation	$e_x = p_x + p_x \sum_{k=1}^{\infty} {}_kp_{x+1}$ $= p_x + p_x\, e_{x+1}$	$\mathring{e}_x = \int_0^1 {}_sp_x\, ds + p_x \int_0^{\infty} {}_tp_{x+1}\, dt$ $= \int_0^1 {}_sp_x\, ds + p_x\, \mathring{e}_{x+1}$
4. Recursion formula[a]	$u(x) = e_x,\ c(x) = p_x$ $d(x) = p_x$	$u(x) = \mathring{e}_x,\ c(x) = \int_0^1 {}_sp_x ds$ $d(x) = p_x$
5. Starting value[b]	$e_\omega = u(\omega) = 0$	$\mathring{e}_\omega = u(x) = 0$

[a]The integral $c(x) = \int_0^1 {}_sp_x ds$ can be evaluated using the trapezoidal rule as $c(x) = (1 + p_x)/2$.
[b]From Section 3.3.1 we have $s(x) = 0,\ x \geq \omega$, and $s(x) > 0,\ x < \omega$. In this development we will assume that ω is an integer.

3.6 Assumptions for Fractional Ages

In this chapter we have discussed the continuous random variable remaining lifetime, T, and the discrete random variable curtate-future-lifetime, K. The life table developed in Section 3.3 specifies the probability distribution of K completely. To specify the distribution of T, we must postulate an analytic form or adopt a life table and an assumption about the distribution between integers.

We will examine three assumptions that are widely used in actuarial science. These will be stated in terms of the s.f. and in a form to show the nature of interpolation over the interval $(x, x + 1)$ implied by each assumption. In each statement, x is an integer and $0 \le t \le 1$. The assumptions are the following:

- ***Linear interpolation:*** $s(x + t) = (1 - t)\, s(x) + t\, s(x + 1)$. This is known as the ***uniform distribution*** or, perhaps more properly, a uniform distribution of deaths assumption within each year of age. Under this assumption ${}_tp_x$ is a linear function.
- ***Exponential interpolation,*** or linear interpolation on $\log s(x + t)$: $\log s(x + t) = (1 - t) \log s(x) + t \log s(x + 1)$. This is consistent with the assumption of a ***constant force*** of mortality within each year of age. Under this assumption ${}_tp_x$ is exponential.
- ***Harmonic interpolation:*** $1/s(x + t) = (1 - t)/s(x) + t/s(x + 1)$. This is what is known as the ***hyperbolic*** (historically ***Balducci****) assumption, for under it ${}_tp_x$ is a hyperbolic curve.

With these basic definitions, formulas can be derived for other standard probability functions in terms of life table probabilities. These results are presented in Table 3.6.1. Note that we just as well could have elected to propose equivalent definitions in terms of the p.d.f., the d.f., or the force of mortality.

The derivations of the entries in Table 3.6.1 are exercises in substituting the stated assumption about $s(x + t)$ into the appropriate formulas of Sections 3.2 and 3.3. We will illustrate the process for the uniform distribution of deaths, an assumption that is used extensively throughout this text.

To derive the first entry in the uniform distribution column, we start with

$${}_tq_x = \frac{s(x) - s(x + t)}{s(x)} \qquad 0 \le t \le 1,$$

then substitute for $s(x + t)$,

$${}_tq_x = \frac{s(x) - [(1 - t)\, s(x) + t\, s(x + 1)]}{s(x)} = \frac{t\,[s(x) - s(x + 1)]}{s(x)} = t q_x .$$

For the second entry, we use (3.2.13) and

$$\mu(x + t) = -\frac{s'(x + t)}{s(x + t)};$$

*This assumption is named after G. Balducci, an Italian actuary, who pointed out its role in the traditional actuarial method of constructing life tables.

TABLE 3.6.1

Probability Theory Functions for Fractional Ages

Function	Uniform Distribution	Constant Force	Hyperbolic
	Assumption		
${}_tq_x$	tq_x	$1 - p_x^t$	$\dfrac{tq_x}{1-(1-t)q_x}$
$\mu(x+t)$	$\dfrac{q_x}{1-tq_x}$	$-\log p_x$	$\dfrac{q_x}{1-(1-t)q_x}$
${}_{1-t}q_{x+t}$	$\dfrac{(1-t)\,q_x}{1-tq_x}$	$1 - p_x^{1-t}$	$(1-t)q_x$
${}_yq_{x+t}$	$\dfrac{yq_x}{1-tq_x}$	$1 - p_x^y$	$\dfrac{yq_x}{1-(1-y-t)q_x}$
${}_tp_x$	$1 - tq_x$	p_x^t	$\dfrac{p_x}{1-(1-t)q_x}$
${}_tp_x\mu(x+t)$	q_x	$-p_x^t \log p_x$	$\dfrac{q_x\,p_x}{[1-(1-t)q_x]^2}$

Note that, in this table, x is an integer, $0 < t < 1$, $0 \le y \le 1$, $y + t \le 1$. For rows one, three, four, and five, the relationships also hold for $t = 0$ and $t = 1$.

then, substituting for $s(x+t)$, we have

$$\mu(x+t) = \frac{[s(x) - s(x+1)]}{[(1-t)\,s(x) + t\,s(x+1)]}.$$

Dividing both numerator and denominator of the right-hand side by $s(x)$ yields

$$\mu(x+t) = \frac{q_x}{(1-tq_x)}.$$

The third entry is the special case of the fourth entry with $y = 1 - t$.

For the fourth entry we start with

$${}_yq_{x+t} = \frac{s(x+t) - s(x+t+y)}{s(x+t)},$$

then substitute for $s(x+t)$ and $s(x+t+y)$ to obtain

$$\begin{aligned}
{}_yq_{x+t} &= \frac{[(1-t)\,s(x) + t\,s(x+1)] - [(1-t-y)\,s(x) + (t+y)\,s(x+1)]}{(1-t)\,s(x) + t\,s(x+1)} \\
&= \frac{y[s(x) - s(x+1)]/s(x)}{\{s(x) - t[s(x) - s(x+1)]\}/s(x)} \\
&= \frac{yq_x}{1-tq_x}.
\end{aligned}$$

The fifth entry is the complement of the first, and the final entry for the uniform distribution column is the product of the second and fifth entries.

If, as before, x is an integer, insight can be obtained by defining a random variable $S = S(x)$ by

$$T = K + S \tag{3.6.1}$$

where T is time-until-death, K is the curtate-future-lifetime, and S is the random variable representing the fractional part of a year lived in the year of death. Since K is a non-negative integer random variable and S is a continuous-type random variable with all of its probability mass on the interval (0, 1), we can examine their joint distribution by writing

$$\begin{aligned}\Pr\,[(K = k) \cap (S \le s)] &= \Pr\,(k < T \le k + s) \\ &= {}_kp_x\ {}_sq_{x+k}.\end{aligned}$$

Now, using the expression for ${}_sq_{x+k}$ under the uniform distribution assumption as shown in Table 3.6.1, we have

$$\begin{aligned}\Pr\,[(K = k) \cap (S \le s)] &= {}_kp_x\ s\ q_{x+k} \\ &= {}_{k|}q_x\ s \\ &= \Pr(K = k)\Pr(S \le s).\end{aligned} \tag{3.6.2}$$

Therefore, the joint probability involving K and S can be factored into separate probabilities of K and S. It follows that, under the uniform distribution of deaths assumption, the random variables K and S are independent. Since $\Pr\,(S \le s) = s$ is the d.f. of a uniform distribution on (0, 1), S has such a uniform distribution.

Example 3.6.1

Under the constant force of mortality assumption, are the random variables K and S independent?

Solution:

Using entries from Table 3.6.1 for the constant force assumption, we obtain

$$\begin{aligned}\Pr\,[(K = k) \cap (S \le s)] &= {}_kp_x\ {}_sq_{x+k} \\ &= {}_kp_x\,[1 - (p_{x+k})^s].\end{aligned}$$

To discuss this result, we distinguish two cases:

- If p_{x+k} is not independent of k, we cannot factor the joint probability of K and S into separate probabilities. We conclude that K and S are not independent.
- In the special case where $p_{x+k} = p_x$, a constant,

$$\begin{aligned}\Pr\,[(K = k) \cap (S \le s)] = p_x^k(1 - p_x^s) &= \frac{(1 - p_x)p_x^k(1 - p_x^s)}{(1 - p_x)} \\ &= \Pr\,(K = k)\Pr\,(S \le s).\end{aligned}$$

For this special case we conclude that K and S are independent under the constant force assumption. ▼

Example 3.6.2

Under the assumption of uniform distribution of deaths, show that

a. $\mathring{e}_x = e_x + \frac{1}{2}$

b. $\mathrm{Var}(T) = \mathrm{Var}(K) + \frac{1}{12}$.

Solution:

a. $$\begin{aligned}\mathring{e}_x &= \mathrm{E}[T] = \mathrm{E}[K + S] \\ &= \mathrm{E}[K] + \mathrm{E}[S] \\ &= e_x + \frac{1}{2}.\end{aligned}$$

b. $\mathrm{Var}(T) = \mathrm{Var}(K + S)$.

From the independence of K and S, under the uniform distribution assumption, it follows that

$$\mathrm{Var}(T) = \mathrm{Var}(K) + \mathrm{Var}(S).$$

Further, since S is uniformly distributed over (0, 1),

$$\mathrm{Var}(T) = \mathrm{Var}(K) + \frac{1}{12}.$$ ▼

3.7 Some Analytical Laws of Mortality

There are three principal justifications for postulating an analytic form for mortality or survival functions. The first is philosophical. Many phenomena studied in physics can be explained efficiently by simple formulas. Therefore, using biological arguments, some authors have suggested that human survival is governed by an equally simple law. The second justification is practical. It is easier to communicate a function with a few parameters than it is to communicate a life table with perhaps 100 parameters or mortality probabilities. In addition, some of the analytic forms have elegant properties that are convenient in evaluating probability statements that involve more than one life. The third justification for a simple analytic survival function is the ease of estimating a few parameters of the function from mortality data.

The support for simple analytic survival functions has declined in recent years. Many feel that the belief in universal laws of mortality is naive. With the increasing speed and storage capacity of computers, the advantages of some analytic forms in computations involving more than one life are no longer of great importance. Nevertheless, some interesting research has recently reiterated the biological arguments for analytic laws of mortality.

In Table 3.7.1, several families of simple analytic mortality and survival functions, corresponding to various postulated laws, are displayed. The names of the originators of the laws and the dates of publication are included for identification purposes.

TABLE 3.7.1

Mortality and Survival Functions under Various Laws

Originator	$\mu(x)$	$s(x)$	Restrictions
De Moivre (1729)	$(\omega - x)^{-1}$	$1 - \dfrac{x}{\omega}$	$0 \le x < \omega$
Gompertz (1825)	Bc^x	$\exp[-m(c^x - 1)]$	$B > 0,\ c > 1,\ x \ge 0$
Makeham (1860)	$A + Bc^x$	$\exp[-Ax - m(c^x - 1)]$	$B > 0,\ A \ge -B,\ c > 1,\ x \ge 0$
Weibull (1939)	kx^n	$\exp(-ux^{n+1})$	$k > 0,\ n > 0,\ x \ge 0$

Note:

- The special symbols are defined as

$$m = \frac{B}{\log c}, \qquad u = \frac{k}{(n + 1)}.$$

- Gompertz's law is a special case of Makeham's law with $A = 0$.
- If $c = 1$ in Gompertz's and Makeham's laws, the exponential (constant force) distribution results.
- In connection with Makeham's law, the constant A has been interpreted as capturing the accident hazard, and the term Bc^x as capturing the hazard of aging.

The entries in the $s(x)$ column of Table 3.7.1 were obtained by substituting into (3.2.16). For example, for Makeham's law, we have

$$\begin{aligned} s(x) &= \exp[-\textstyle\int_0^x (A + Bc^s)ds] \\ &= \exp\left[-Ax - B\,\frac{(c^x - 1)}{\log c}\right] \\ &= \exp[-Ax - m\,(c^x - 1)] \end{aligned}$$

where $m = B/\log c$.

Two objectives governed the development of a mortality table for computational purposes in the examples and exercises of this book. One objective was to have mortality rates in the middle of the range of variation for groups, such variation caused by factors such as residence, gender, insured status, annuity status, marital status, and occupation. The second objective was to have a Makeham law at most ages to illustrate how calculations for multiple lives can be performed.

The Illustrative Life Table in Appendix 2A is based on the Makeham law for ages 13 and greater,

$$1{,}000\ \mu(x) = 0.7 + 0.05\ (10^{0.04})^x. \tag{3.7.1}$$

The calculations of the basic functions q_x, l_x, and d_x from (3.7.1) were all done directly from (3.7.1) instead of calculating l_x and d_x from the truncated values of q_x.

It was found that the latter choice would make little difference in the applications. It should be kept in mind that the Illustrative Life Table, as its name implies, is for illustrative purposes only.

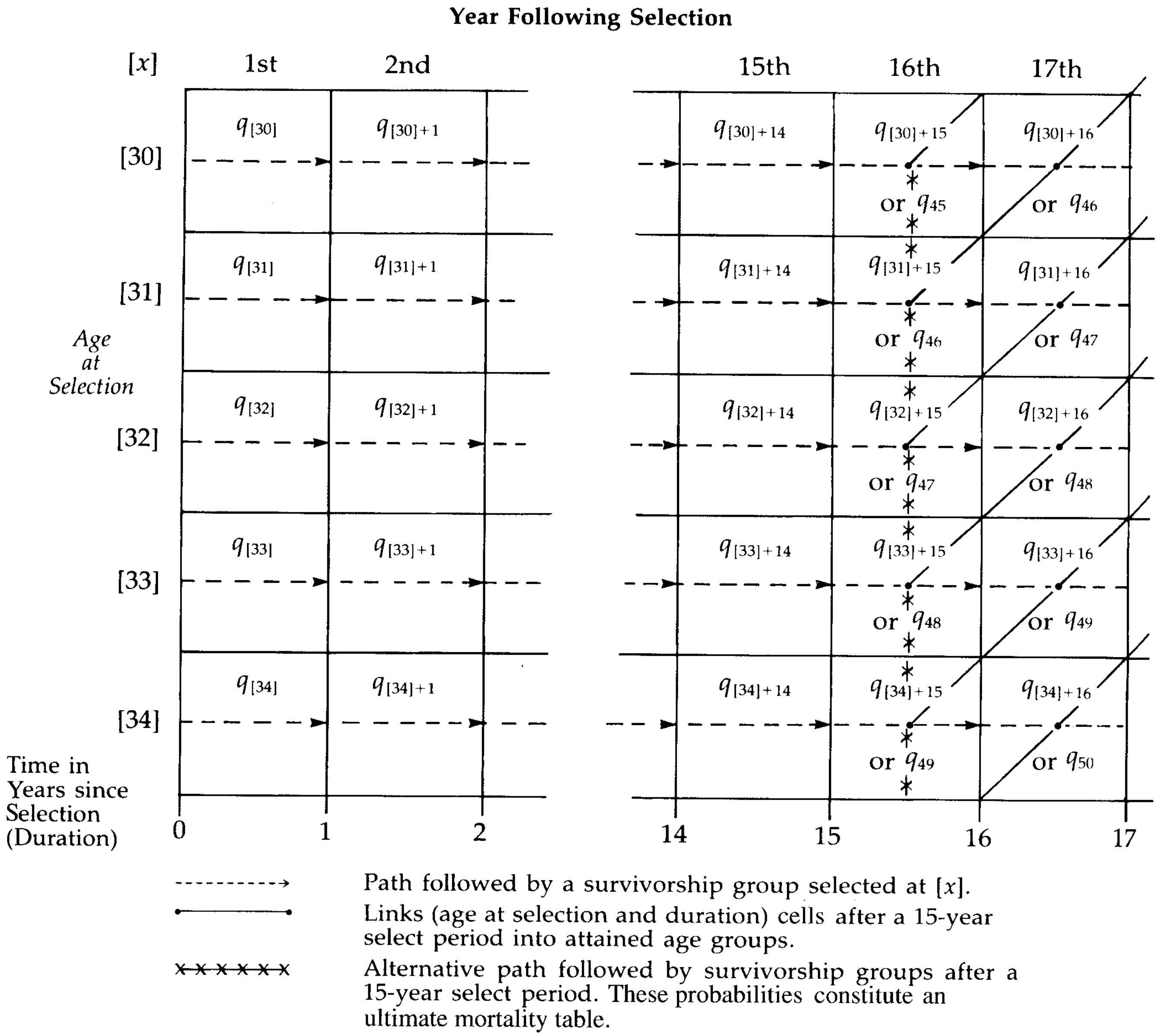
3.8 Select and Ultimate Tables

In Section 3.2 we discussed how ${}_tp_x$ [the probability that (x) will survive to age $x + t$] might be interpreted in two ways. The first interpretation was that the probability can be evaluated by a survival function appropriate for newborns, under the single hypothesis that the newborn has survived to age x. This interpretation has been the basis of the notation and development of the formulas. The second interpretation was that additional knowledge available about the life at age x might make the original survival function inappropriate for evaluating probability statements about the future lifetime of (x). For example, the life might have been underwritten and accepted for life insurance at age x. This information would lead us to believe that (x)'s future-lifetime distribution is different from what we might otherwise assume for lives age x. As a second example, the life might have become disabled at age x. This information would lead us to believe that the future-lifetime distribution for (x) is different from that of those not disabled at age x. In these two illustrations, a special force of mortality that incorporates the particular information available at age x would be preferred. Without this particular information for (x), the form of mortality at duration t would be a function of only the attained age $x + t$, denoted in the previous sections by $\mu(x + t)$. Given the additional information at x, the force of mortality at $x + t$ is a function of this information at x and duration t. Its notation will be $\mu_x(t)$, showing separately the age, x, at which the additional information was available, and the duration, t. The additional information is usually not explicit in the notation but is conveyed by the context. In other words, the complete model for such lives is a set of survival functions including one for each age at which information is available on issue of insurance, disability, and so on. This set of survival functions can be thought of as a function of two variables. One variable is the age at ***selection*** (e.g., at policy issue or the onset of disability), $[x]$, and the second variable is the duration since policy issue or duration since selection, t. Then each of the usual life table functions associated with this bivariate survival function is a two-dimensional array on $[x]$ and t. Note the bracket notation to indicate which variable identifies the age of selection. When the select status can be inferred from the force of mortality, the bracket notation will be suppressed to reduce the clutter of the symbols.

The schematic diagram in Figure 3.8.1 illustrates these ideas. For instance, suppose some special information is available about a group of lives age 30. Perhaps they have been accepted for life insurance or perhaps they have become disabled. A special life table can be built for these lives. The conditional probability of death in each year of duration would be denoted by $q_{[30]+i}$, $i = 0, 1, 2, \ldots$, and would be entered on the first row of Figure 3.8.1. The subscript reflects the bivariate nature of this function with the bracketed thirty, [30], denoting that the survival function in the first row is conditional on special information available at age 30. The second row of Figure 3.8.1 would contain the probabilities of death for lives on which the

FIGURE 3.8.1

Select, Ultimate, and Aggregate Mortality, 15-Year Select Period



Notes:

1. In biostatistics the select table index [x] may not be age. For example, in cancer research, [x] could be a classification index that depends on the size and location of the tumor, and time following selection would be measured from the time of diagnosis.
2. Ultimate mortality, following a 15-year select period, for age [x] + 15, would be estimated by using observations from all cells identified by $[x-j] + 15 + j$, for $j = 0, 1, 2, \ldots$. Therefore, $q_{[x]+15} = q_{x+15}$ is estimated by a weighted average of mortality estimates from several different selection groups. If the effect of selection is not small, the resulting estimate will be influenced by the amount of data from the various cells.

special information became available at age 31. In actuarial science such a two-dimensional life table is called a ***select life*** table.

The impact of selection on the distribution of time-until-death, T, may diminish following selection. Beyond this time period the q's at equal attained ages would be essentially equal regardless of the ages at selection. More precisely, if there is a smallest integer r such that $|q_{[x]+r} - q_{[x-j]+r+j}|$ is less than some small positive constant for all ages of selection $[x]$ and for all $j > 0$, it would be economical to construct a set of ***select-and-ultimate*** tables by truncation of the two-dimensional array after the $(r + 1)$ column. For durations beyond r we would use

$$q_{[x-j]+r+j} \cong q_{[x]+r} \qquad j > 0.$$

The first r years of duration comprise the ***select period.***

The resulting array remains a set of life tables, one for each age at selection. For a single age at selection, the life table entries are horizontal during the select period and then vertical during the ultimate period. This is shown in Figure 3.8.1 by the arrows.

The Society of Actuaries mortality studies of lives who were issued individual life insurance on a standard basis use a 15-year select period as illustrated in Figure 3.8.1; that is, it is accepted that

$$q_{[x-j]+15+j} \cong q_{[x]+15} \qquad j > 0.$$

Beyond the select period, the probabilities of death are subscripted by attained age only; that is, $q_{[x-j]+r+j}$ is written as q_{x+r}. For instance, with $r = 15$, $q_{[30]+15}$ and $q_{[25]+20}$ would both be written as q_{45}.

A life table in which the functions are given only for attained ages is called an ***aggregate table,*** Table 3.3.1, for instance. The last column in a select-and-ultimate table is a special aggregate table that is usually referred to as an ***ultimate table,*** to reflect the select table setting.

Table 3.8.1 contains mortality probabilities and corresponding values of the $l_{[x]+k}$ function, as given in the Permanent Assurances, Females, 1979–82, Table, published by the Institute of Actuaries and the Faculty of Actuaries; it is denoted as the

TABLE 3.8.1

Excerpt from the AF80 Select-and-Ultimate Table

$[x]$	(1) $1{,}000\, q_{[x]}$	(2) $1{,}000\, q_{[x]+1}$	(3) $1{,}000\, q_{x+2}$	(4) $l_{[x]}$	(5) $l_{[x]+1}$	(6) l_{x+2}	(7) $x + 2$
30	0.222	0.330	0.422	9 906.7380	9 904.5387	9 901.2702	32
31	0.234	0.352	0.459	9 902.8941	9 900.5769	9 897.0919	33
32	0.250	0.377	0.500	9 898.7547	9 896.2800	9 892.5491	34
33	0.269	0.407	0.545	9 894.2903	9 891.6287	9 887.6028	35
34	0.291	0.441	0.596	9 889.4519	9 886.5741	9 882.2141	36

AF80 Table. This table has a 2-year select period and is easier to use for illustrative purposes than tables with a 15-year select period such as the Basic Tables, published by the Society of Actuaries.

In Table 3.8.1 we observe three mortality probabilities for age 32, namely,

$$q_{[32]} = 0.000250 < q_{[31]+1} = 0.000352 < q_{32} = 0.000422.$$

The order among these probabilities is plausible since mortality should be lower for lives immediately after acceptance for life insurance. Column (3) can be viewed as providing ultimate mortality probabilities.

Given the 1-year mortality rates of a select-and-ultimate table, the construction of the corresponding select-and-ultimate life table (survival functions) is started with the ultimate portion. Formulas such as (3.4.1) can be used, which would yield a set of values of $l_{x+r} = l_{[x]+r}$ where r is the length of the select period. We would then complete the select segments by using the relation

$$l_{[x]+r-k-1} = \frac{l_{[x]+r-k}}{p_{[x]+r-k-1}} \qquad k = 0, 1, 2, \ldots, r-1,$$

working from duration $r - 1$ down to 0.

Example 3.8.1

Use Table 3.8.1 to evaluate

a. ${}_2p_{[30]}$ b. ${}_5p_{[30]}$

c. ${}_{1|}q_{[31]}$ d. ${}_3q_{[31]+1}$.

Solution:

Formulas developed earlier in this chapter can be adapted to select-and-ultimate tables yielding

a. $${}_2p_{[30]} = \frac{l_{[30]+2}}{l_{[30]}} = \frac{l_{32}}{l_{[30]}} = \frac{9,901.2702}{9,906.7380} = 0.99945$$

b. $${}_5p_{[30]} = \frac{l_{35}}{l_{[30]}} = \frac{9,887.6028}{9,906.7380} = 0.99807$$

c. $${}_{1|}q_{[31]} = \frac{l_{[31]+1} - l_{33}}{l_{[31]}} = \frac{9,900.5769 - 9,897.0919}{9,902.8941} = 0.00035$$

d. $${}_3q_{[31]+1} = \frac{l_{[31]+1} - l_{35}}{l_{[31]+1}} = \frac{9,900.5769 - 9,887.6028}{9,900.5769} = 0.00131.$$ ▼

TABLE 3.9.1

Chapter 3 Concepts

Symbol	Name or Description of the Concept
(x)	Notation for a life age x
$[x]$	Age, or other status, at selection
X	Age at death, a random variable
$T(x)$	Future lifetime of (x), equals $X - x$
$K(x)$	Curtate-future-lifetime of (x), equals the integer part of $T(x)$
$S(x)$	Future lifetime of (x) within the year of death, equals $T(x) - K(x)$
$s(x)$	Survival function, equal to the probability that a newborn will live to at least x
$\mu(x)$	Force of mortality at age x in an aggregate life table
$\mu_x(t)$	Force of mortality at attained age $x + t$ given selection at age x
${}_tq_x$	Probability that (x) dies within t years
${}_tp_x$	Probability that (x) survives at least t years
${}_{t\|u}q_x$	Probability that (x) dies between t and $t + u$ years
$\mathring{e}_x$	Complete expectation of life for (x), equals $\mathrm{E}[T(x)]$
e_x	Curtate expectation of life for (x), equals $\mathrm{E}[K(x)]$
$\mathcal{L}(x)$	Cohort's number of survivors to age x, a random variable
${}_n\mathcal{D}_x$	Cohort's number of deaths between ages x and $x + n$
l_x	Expected number of survivors at age x, equals $\mathrm{E}[\mathcal{L}(x)]$
${}_nd_x$	Expected number of deaths between ages x and $x + n$, equals $\mathrm{E}[{}_n\mathcal{D}_x]$
${}_nL_x$	Expected number of years lived between ages x and $x + n$ by survivors to age x of the initial group of l_0 lives
T_x	Expected number of years lived beyond age x by the survivors to age x of the initial group of l_0 lives
m_x	Central death rate over the interval $(x, x + 1)$
ω	Omega, the limiting age of a life table

3.9 Notes and References

Table 3.9.1 summarizes this chapter's new concepts with their names, symbols, and descriptions. Life tables are a cornerstone of actuarial science. Consequently they are extensively discussed in several English-language textbooks on life contingencies:

- King (1902)
- Spurgeon (1932)
- Jordan (1967)
- Hooker and Longley-Cook (1953)
- Neill (1977).

These have been used in actuarial education. In addition, life tables are used by biostatisticians. An exposition of this latter approach is given by Chiang (1968) and Elandt-Johnson and Johnson (1980). The deterministic rate function interpretation is discussed by Allen (1907). London (1988) summarizes several methods for estimating life tables from data.

The historically important analytic forms for survival functions are referred to in Table 3.6. Brillinger (1961) provides an argument for certain analytic forms from the viewpoint of statistical life testing. Tenenbein and Vanderhoof (1980) restate the case for analytic laws of mortality and develop formulas for select mortality. Balducci's (1921) contribution was preceded by a remarkable set of papers by Wittstein (1873). Wittstein's papers were published first in German and translated into English by T. B. Sprague. Some of the methods for evaluating probabilities for fractional ages are reviewed by Mereu (1961) and in Batten's textbook on mortality estimation (1978) (see also Seal's 1977 historical review). Discussions of the length of the select period for various types of insurance selection procedures have a long history, for example, Williamson (1942), Thompson (1934), and Jenkins (1943). The Society of Actuaries 1975–80 Basic Tables use a 15-year select period and are published in *TSA Reports 1982*. International Actuarial Notation is outlined in *TASA 48* (1947).

We planned to use the 1989–91 U.S. Life Table for illustrative purposes in Table 3.2.1, but this plan was not realized because the life tables based on the 1990 U.S. Census were not completed when this chapter was revised.

Exercises

Section 3.2

3.1. Using the ideas summarized in Table 3.2.1, complete the entries below.

$s(x)$	$F_X(x)$	$f_X(x)$	$\mu(x)$
			$\tan x,\ 0 \le x \le \frac{\pi}{2}$
$e^{-x},\ x \ge 0$			
	$1 - \frac{1}{1+x},\ x \ge 0$		

3.2. Confirm that each of the following functions can serve as a force of mortality. Show the corresponding survival function. In each case $x \ge 0$.

a. $B\,c^x$	$B > 0$	$c > 1$	(Gompertz)
b. $k\,x^n$	$n > 0$	$k > 0$	(Weibull)
c. $a\,(b + x)^{-1}$	$a > 0$	$b > 0$	(Pareto)

3.3. Confirm that the following can serve as a survival function. Show the corresponding $\mu(x)$, $f_X(x)$, and $F_X(x)$.

$$s(x) = e^{-x^3/12} \qquad x \ge 0.$$

3.4. State why each of the following functions cannot serve in the role indicated by the symbol:

a. $\mu(x) = (1 + x)^{-3} \qquad x \ge 0$

b. $s(x) = 1 - \dfrac{22x}{12} + \dfrac{11x^2}{8} - \dfrac{7x^3}{24} \qquad 0 \le x \le 3$

c. $f_X(x) = x^{n-1} e^{-x/2} \qquad x \ge 0, n \ge 1.$

3.5. If $s(x) = 1 - x/100$, $0 \le x \le 100$, calculate

a. $\mu(x)$ b. $F_X(x)$

c. $f_X(x)$ d. $\Pr(10 < X < 40)$.

3.6. Given the survival function of Exercise 3.5, determine the survival function, force of mortality, and p.d.f. of the future lifetime of (40).

3.7. If $s(x) = [1 - (x/100)]^{1/2}$, $0 \le x \le 100$, evaluate

a. ${}_{17}p_{19}$ b. ${}_{15}q_{36}$ c. ${}_{15|13}q_{36}$

d. $\mu(36)$ e. $E[T(36)]$.

3.8. Confirm that ${}_{k|}q_0 = -\Delta s(k)$, and that $\sum_{k=0}^{\infty} {}_{k|}q_0 = 1$.

3.9. If $\mu(x) = 0.001$ for $20 \le x \le 25$, evaluate ${}_{2|2}q_{20}$.

Sections 3.3, 3.4

3.10. If the survival times of 10 lives in a survivorship group are independent with survival defined in Table 3.3.1, exhibit the p.f. of $\mathcal{L}(65)$ and the mean and variance of $\mathcal{L}(65)$.

3.11. If $s(x) = 1 - x/12$, $0 \le x \le 12$, $l_0 = 9$, and the survival times are independent, then $({}_3\mathcal{D}_0, {}_3\mathcal{D}_3, {}_3\mathcal{D}_6, {}_3\mathcal{D}_9)$ is known to have a multinomial distribution. Calculate

a. The expected value of each random variable

b. The variance of each random variable

c. The coefficient of correlation between each pair of random variables.

3.12. On the basis of Table 3.3.1,

a. Compare the values of ${}_5q_0$ and ${}_5q_5$

b. Evaluate the probability that (25) will die between ages 80 and 85.

3.13. Given that l_{x+t} is strictly decreasing in the interval $0 \le t \le 1$, show that

a. If l_{x+t} is concave down, then $q_x > \mu(x)$

b. If l_{x+t} is concave up, then $q_x < \mu(x)$.

3.14. Show that

a. $\dfrac{d}{dx} l_x \, \mu(x) < 0 \qquad$ when $\dfrac{d}{dx} \mu(x) < \mu^2(x)$

b. $\dfrac{d}{dx} l_x \, \mu(x) = 0 \qquad$ when $\dfrac{d}{dx} \mu(x) = \mu^2(x)$

c. $\dfrac{d}{dx} l_x \, \mu(x) > 0 \qquad$ when $\dfrac{d}{dx} \mu(x) > \mu^2(x)$.

3.15. Consider a random survivorship group consisting of two subgroups: (1) the survivors of 1,600 persons joining at birth; (2) the survivors of 540 persons joining at age 10. An excerpt from the appropriate mortality table for both subgroups follows:

x	l_x
0	40
10	39
70	26

If Y_1 and Y_2 are the numbers of survivors to age 70 out of subgroups (1) and (2), respectively, estimate a number c such that $\Pr(Y_1 + Y_2 > c) = 0.05$. Assume the lives are independent and ignore half-unit corrections.

Section 3.5

3.16. Let the random variable

$$\begin{aligned} T^*(x) &= T(x) \qquad & 0 < T(x) \le n \\ &= n & n < T(x) \end{aligned}$$

and denote $\mathrm{E}[T^*(x)]$ by $\mathring{e}_{x:\overline{n}|}$. This expectation is called a ***temporary complete life expectancy.*** It is used in public health planning; the same expectation, under the name ***limited expected value function,*** is used in the analysis of loss amount distributions. Show that

a. $$\begin{aligned} \mathring{e}_{x:\overline{n}|} &= \int_0^n t\ {}_tp_x\ \mu(x+t)\,dt + n\ {}_np_x \\ &= \int_0^n {}_tp_x\,dt = \frac{T_x - T_{x+n}}{l_x} \end{aligned}$$

b. $$\begin{aligned} \mathrm{Var}[T^*(x)] &= \int_0^n t^2\ {}_tp_x\ \mu(x+t)\,dt + n^2\ {}_np_x - (\mathring{e}_{x:\overline{n}|})^2 \\ &= 2\int_0^n t\ {}_tp_x\,dt - \mathring{e}_{x:\overline{n}|}{}^2. \end{aligned}$$

3.17. Let the random variable

$$\begin{aligned} K^*(x) &= K(x) \qquad & K(x) = 0, 1, 2, \ldots, n-1 \\ &= n & K(x) = n, n+1, \ldots \end{aligned}$$

and denote $\mathrm{E}[K^*(x)]$ by $e_{x:\overline{n}|}$. This expectation is called a ***temporary curtate life expectancy.*** Show that

a. $$\begin{aligned} e_{x:\overline{n}|} &= \sum_0^{n-1} k\ {}_{k|}q_x + n\ {}_np_x \\ &= \sum_1^n {}_kp_x \end{aligned}$$

b. $\mathrm{Var}[K^*(x)] = \sum_{0}^{n-1} k^2 \, {}_{k|}q_x + n^2 \, {}_n p_x - (e_{x:\overline{n}|})^2$

$$= \sum_{1}^{n} (2k + 1) \, {}_k p_x - (e_{x:\overline{n}|})^2.$$

3.18. If the random variable T has p.d.f. given by $f_T(t) = ce^{-ct}$ for $t \geq 0$, $c > 0$, calculate

a. $\mathring{e}_x = \mathrm{E}[T]$ b. $\mathrm{Var}(T)$ c. median (T)

d. The mode of the distribution of T.

3.19. If $\mu(x + t) = t$, $t \geq 0$, calculate

a. ${}_t p_x \, \mu(x + t)$ b. $\mathring{e}_x$.

[Hint: Recall, from the study of probability, that $(1/\sqrt{2\pi}) \, e^{-t^2/2}$ is the p.d.f. for the standard normal distribution.]

3.20. If the random variable $T(x)$ has d.f. given by

$$F_{T(x)}(t) = \begin{cases} \dfrac{t}{(100 - x)} & 0 \leq t < 100 - x \\ 1 & t \geq 100 - x, \end{cases}$$

calculate

a. $\mathring{e}_x$ b. $\mathrm{Var}[T(x)]$ c. median $[T(x)]$.

3.21. Show that

a. $\dfrac{\partial}{\partial x} \, {}_t p_x = {}_t p_x \, [\mu(x) - \mu(x + t)]$

b. $\dfrac{d}{dx} \, \mathring{e}_x = \mathring{e}_x \, \mu(x) - 1$

c. $\Delta e_x = q_x \, e_{x+1} - p_x$.

3.22. Confirm the following statements:

a. $a(x) \, d_x = L_x - l_{x+1}$

b. The approximation developed in Example 3.5.1 was not used to calculate L_0 in Table 3.3.1, but was used to calculate L_1

c. $T_x = \sum_{k=0}^{\infty} L_{x+k}$.

3.23. The survival function is given by

$$\begin{aligned} s(x) &= 1 - \frac{x}{10} && 0 \leq x \leq 10 \\ &= 0 && \text{elsewhere.} \end{aligned}$$

Calculate values of $\mathring{e}_x$ and e_x, $x = 0, 1, 2, \ldots, 9$

a. Using formulas (3.5.2) and (3.5.7)

b. Using the formulas developed in Table 3.5.1.

3.24. Find $u(0)$, $-c(x)/d(x)$, and $d(x)$ if $u(x) = \Pr[X = x]$ where (3.5.20) is to be used to produce a table of the p.f. of the random variable X when it has a
a. A Poisson distribution with parameter λ
b. A binomial distribution with parameters n and p.

3.25. Formula (3.5.20) is to be used to produce tables of compound interest functions. Find $u(1)$, $-c(x)/d(x)$, and $1/d(x)$ when
a. $u(x) = \ddot{a}_{\overline{x}|}$
b. $u(x) = \ddot{s}_{\overline{x}|}$.

Section 3.6

3.26. Verify the entries for the constant force of mortality and the hyperbolic assumption in Table 3.6.1. Note that the entry for ${}_sp_x$ in the hyperbolic column provides justification for the hyperbolic name.

3.27. Graph $\mu(x + t)$, $0 < t < 1$, for each of the three assumptions in Table 3.6.1. Also graph the survival function for each assumption.

3.28. Using the l_x column of Table 3.3.1, compute ${}_{1/2}p_{65}$ for each of the three assumptions in Table 3.6.1.

3.29. Use Table 3.3.1 and an assumption of uniform distribution of deaths in each year of age to find the median of the future lifetime of a person
a. Age 0 b. Age 50.

3.30. If $q_{70} = 0.04$ and $q_{71} = 0.05$, calculate the probability that (70) will die between ages 70½ and 71½ under
a. The assumption that deaths are uniformly distributed within each year of age
b. The hyperbolic assumption for each year of age.

3.31. Using the l_x column in Table 3.3.1 and each of the assumptions in Table 3.6.1, compute
a. $\lim_{h \to 0^-} \mu(60 + h)$
b. $\lim_{h \to 0^+} \mu(60 + h)$
c. $\mu(60 + \frac{1}{2})$.

3.32. If the constant force assumption is adopted, show that
a. $a(x) = \dfrac{[(1 - e^{-\mu})/\mu] - e^{-\mu}}{1 - e^{-\mu}}$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{12}$.

3.33. If the hyperbolic assumption is adopted, show
a. $a(x) = -\dfrac{p_x}{q_x^2}(q_x + \log p_x)$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{6}$.

Section 3.7

3.34. Verify the entries in Table 3.7.1 for De Moivre's law and Weibull's law.

3.35. Consider a modification of De Moivre's law given by

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \qquad 0 \le x < \omega, \qquad \alpha > 0.$$

Calculate

a. $\mu(x)$ b. $\overset{\circ}{e}_x$.

Section 3.8

3.36. Using Table 3.8.1, calculate

a. ${}_2q_{[32]+1}$ b. ${}_2p_{[31]+1}$.

3.37. The quantity

$$1 - \frac{q_{[x]+k}}{q_{x+k}} = I(x, k)$$

has been called the ***index of selection.*** When it is close to 0, the indication is that selection has worn off. From Table 3.8.1, calculate the index for $x = 32$, $k = 0, 1$.

3.38. The force of mortality for a life selected at age (x) is given by $\mu_{\mathbf{x}}(t) = \Psi(\mathbf{x})\mu(t)$, $t > 0$. In this formula $\mu(t)$ is the standard force of mortality. The symbol $\mathbf{x}$ denotes a vector of numerical information about the life at the time of selection. This information would include the age and other classification information. It is required that $\Psi(\mathbf{x}) > 0$ and $\Psi(\mathbf{x}_0) = 1$, where $\mathbf{x}_0$ denotes standard information. Show that the select survival function is

$${}_tp_{[\mathbf{x}]} = ({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})}$$

and the p.d.f. of $T(\mathbf{x})$, the random variable time-until-death given the information $\mathbf{x}$, is $-\Psi(\mathbf{x})\ {}_tp'_{[\mathbf{x}_0]}({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})-1}$, where ${}_tp'_{[\mathbf{x}_0]}$ is the derivative with respect to t of ${}_tp_{[\mathbf{x}_0]}$. This is called a ***proportional hazard model.***

Miscellaneous

3.39. A life at age 50 is subject to an extra hazard during the year of age 50 to 51. If the standard probability of death from age 50 to 51 is 0.006, and if the extra risk may be expressed by an addition to the standard force of mortality that decreases uniformly from 0.03 at the beginning of year to 0 at the end of the year, calculate the probability that the life will survive to age 51.

3.40. If the force of mortality $\mu_x(t)$, $0 \le t \le 1$, changes to $\mu_x(t) - c$ where c is a positive constant, find the value of c for which the probability that (x) will die within a year will be halved. Express the answer in terms of $q_{[x]}$.

3.41. From a standard mortality table, a second table is prepared by doubling the force of mortality of the standard table. Is the rate of mortality, q'_x, at any given age under the new table, more than double, exactly double, or less than double the mortality rate, q_x, of the standard table?

3.42. If $\mu(x) = B\,c^x$, $c > 1$, show that the function $l_x\,\mu(x)$ has its maximum at age x_0 where $\mu(x_0) = \log c$. [Hint: This exercise makes use of Exercise 3.14.]

3.43. Assume $\mu(x) = \dfrac{A\,c^x}{1 + B\,c^x}$ for $x > 0$.

a. Calculate the survival function, $s(x)$.

b. Verify that the mode of the distribution of X, the age-at-death, is given by

$$x_0 = \frac{\log(\log c) - \log A}{\log c}.$$

3.44. If $\mu(x) = \dfrac{3}{100 - x} - \dfrac{10}{250 - x}$ for $40 < x < 100$, calculate

a. ${}_{40}p_{50}$

b. The mode of the distribution of X, the age-at-death.

3.45. a. Show that, under the uniform distribution of deaths assumption,

$$m_x = \frac{q_x}{1 - (1/2)q_x} \quad \text{and} \quad q_x = \frac{m_x}{1 + (1/2)m_x}.$$

b. Calculate m_x in terms of q_x under the constant force assumption.

c. Calculate m_x in terms of q_x under the hyperbolic assumption.

d. If $l_x = 100 - x$ for $0 \leq x \leq 100$, calculate ${}_{10}m_{50}$ where

$${}_nm_x = \frac{\displaystyle\int_0^n l_{x+t}\,\mu(x+t)\,dt}{\displaystyle\int_0^n l_{x+t}\,dt}.$$

3.46. Show that K and S are independent if and only if the expression

$$\frac{{}_sq_{[x]+k}}{q_{[x]+k}}$$

does not depend on k for $0 \leq s \leq 1$.

Computing Exercises:

These are the first in a series of exercises that involve sufficient computation to make it worthwhile to use a computer. The series will continue in the following chapters, and in each exercise it is assumed that the results of previous exercises are available. For example, in Exercise 3.47, you are asked to set up a life table that will then be used in risk analysis in Chapters 4 and 5.

3.47. Using spreadsheet or other mathematical software, set up an object that will accept input values for the Makeham law parameters and then calculate and display the values of p_x and q_x for ages 0 to 140. As a check on your output, input the parameter values given in (3.7.1) and compare your q_x values with those for $x = 13, 14, \ldots$ in the Illustrative Life Table in Appendix 2A. We will refer to this computing object as your Illustrative Life Table. When the Makeham parameter values are not stated, those of (3.7.1) are implied. [Remark: With a Makeham Table, $s(x) > 0$ for all $x > 0$, so ω does not exist as defined in Section 3.3.1. For the parameter values of the Illustrative Life Table, q_{140} is zero to eight decimal places; thus we choose $\omega = 140$ for our Illustrative Life Table, that is, Table 2A.]

3.48. In your Illustrative Life Table use the forward recursion formula $l_{x+1} = (p_x)(l_x)$ and initial value $l_{13} = 96{,}807.88$ to calculate the l_x values of Table 2A. [Remark: The Makeham law was not realistic for ages less than 13, so the Illustrative Life Table is a blend of some ad hoc values from 0 through 12 and the Makeham law table from age 13 up.]

3.49. Illustrate the result of Exercise 3.41 by doubling the A and B parameter values in your Illustrative Life Table.

3.50. Use the backward recursion formula of Table 3.5.1 to calculate values of e_x in your Illustrative Life Table for ages 13 to 140.

3.51. Compare the values of e_x at $x = 20, 40, 60, 80$, and 100 in your Illustrative Life Table with those found when the force of mortality is doubled.

3.52. Use the backward recursion formula of Table 3.5.1 and the trapezoidal rule to calculate values of $\mathring{e}_x$ in your Illustrative Life Table for ages 13 to 110.

3.53. Verify the following backward recursion formula for the temporary curtate life expectancy to age y:

$$e_{x:\overline{y-x}|} = p_x + p_x\, e_{x+1:\overline{y-(x+1)}|} \quad \text{for } x = 0, 1, \ldots, y - 1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the curtate temporary life expectancy up to age 45 for ages 13 to 44.

3.54. Verify the following backward recursion formula for the n-year temporary curtate life expectancy:

$$e_{x:\overline{n}|} = p_x\,(1 - {}_np_{x+1}) + p_x\, e_{x+1:\overline{n}|} \quad \text{for } x = 0, 1, \ldots, \omega - 1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the 10-year temporary curtate life expectancy for ages 13 to 139.

3.55. "Look up" $e_{15:\overline{25}|}$ in your Illustrative Life Table. [Hint: Since the $c(x)$ term in the relation in Exercise 3.53 does not depend on n, it may be more efficient to view $e_{15:\overline{25}|}$ as a curtate temporary life expectancy to age 40 for (15).]

4

LIFE INSURANCE

4.1 Introduction

We have stated that insurance systems are established to reduce the adverse financial impact of some types of random events. Within these systems individuals and organizations adopt utility models to represent preferences, stochastic models to represent uncertain financial impact, and economic principles to guide pricing. Agreements are reached after analyses of these models.

In Chapter 2 we developed an elementary model for the financial impact of random events in which the occurrence and the size of impact are both uncertain. In that model, the policy term is assumed to be sufficiently short so the uncertainty of investment income from a random payment time could be ignored.

In this chapter we develop models for life insurances designed to reduce the financial impact of the random event of untimely death. Due to the long-term nature of these insurances, the amount of investment earnings, up to the time of payment, provides a significant element of uncertainty. This uncertainty has two causes: the unknown rate of earnings over, and the unknown length of, the investment period. A probability distribution is used to model the uncertainty in regards to the investment period throughout this book. In this chapter a deterministic model is used for the unknown investment earnings, and in Chapter 21 stochastic models for this uncertainty are discussed. In other words, our model will be built in terms of functions of T, the insured's future-lifetime random variable.

While everything in this chapter will be stated as insurances on human lives, the ideas would be the same for other objects such as equipment, machines, loans, and business ventures. In fact, the general model is useful in any situation where the size and time of a financial impact can be expressed solely in terms of the time of the random event.

4.2 Insurances Payable at the Moment of Death

In this chapter, the amount and the time of payment of a life insurance benefit depend only on the length of the interval from the issue of the insurance to the death of the insured. Our model will be developed with a ***benefit function,*** b_t, and a ***discount function,*** v_t. In our model, v_t is the interest discount factor from the time of payment back to the time of policy issue, and t is the length of the interval from issue to death. In the case of endowments, covered in this section, t can be greater than or equal to the length of the interval from issue to payment.

For the discount function we assume that the underlying force of interest is deterministic; that is, the model does not include a probability distribution for the force of interest. Moreover, we usually show the simple formulas resulting from the assumption of a constant, as well as a deterministic, force of interest.

We define the ***present-value function,*** z_t, by

$$z_t = b_t v_t. \tag{4.2.1}$$

Thus, z_t is the present value, at policy issue, of the benefit payment. The elapsed time from policy issue to the death of the insured is the insured's future-lifetime random variable, $T = T(x)$, defined in Section 3.2.2. Thus, the present value, at policy issue, of the benefit payment is the random variable z_T. Unless the context requires a more elaborate symbol, we denote this random variable by Z and base the model for the insurance on the equation

$$Z = b_T v_T. \tag{4.2.2}$$

The random variable Z is an example of a claim random variable and, as such, of an X_i term in the sum of the individual risk model, as defined by (2.1.1). This model is used in later sections when we consider applications involving portfolios. We now turn to the development of the probability model for Z.

The first step in our analysis of a life insurance will be to define b_t and v_t. The next step is to determine some characteristics of the probability distribution of Z that are consequences of an assumed distribution for T, and we work through these steps for several conventional insurances. A summary is provided in Table 4.2.1 on page 109.

4.2.1 Level Benefit Insurance

An ***n-year term life insurance*** provides for a payment only if the insured dies within the n-year term of an insurance commencing at issue. If a unit is payable at the moment of death of (x), then

$$b_t = \begin{cases} 1 & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t \ge 0,$$

$$Z = \begin{cases} v^T & T \le n \\ 0 & T > n. \end{cases}$$

These definitions use three conventions. First, since the future lifetime is a non-negative variable, we define b_t, v_t, and Z only on non-negative values. Second, for a t value where b_t is 0, the value of v_t is irrelevant. At these values of t, we adopt definitions of v_t by convenience. Third, unless stated otherwise, the force of interest is assumed to be constant.

The expectation of the present-value random variable, Z, is called the ***actuarial present value*** of the insurance. The reader will find that the expectation of the present value of a set of payments contingent on the occurrence of a set of events is referred to by different terms in different actuarial contexts. In Chapter 1, the expected loss was called the pure premium. This vocabulary is commonly used in property-liability insurance. A more exact term, but more cumbersome, would be ***expectation of the present value of the payments.*** We denote actuarial present values by their symbols according to the International Actuarial Notation (see Appendix 4).

The principal symbol for the actuarial present value of an insurance paying a unit benefit is A. The subscript includes the age of the insured life at the time of the calculation. How this age is displayed depends upon the form of the mortality assumption. For the actuarial present value of an insurance on (40), the age might be displayed as [40], 40, or [20] + 20, for example. As in Section 3.8, the bracket indicates selection at that age and hence the use of a select table commencing at that age. The unbracketed age indicates the use of an aggregate or ultimate table. Thus [20] + 20 indicates the calculation for a 40-year-old on the basis of a select table commencing at age 20.

The actuarial present value for the n-year term insurance with a unit payable at the moment of death of (x), E[Z], is denoted by $\bar{A}^{\,1}_{x:\overline{n|}}$. This can be calculated by recognizing Z as a function of T so that $E[Z] = E[z_T]$. Then we use the p.d.f. of T to obtain

$$\bar{A}^{\,1}_{x:\overline{n|}} = E[Z] = E[z_T] = \int_0^\infty z_t\, f_T(t)\, dt = \int_0^n v^t\, {}_tp_x\, \mu_x(t)\, dt. \tag{4.2.3}$$

The j-th moment of the distribution of Z can be found by

$$\begin{aligned} E[Z^j] &= \int_0^n (v^t)^j\, {}_tp_x\, \mu_x(t)\, dt \\ &= \int_0^n e^{-(\delta j)t}\, {}_tp_x\, \mu_x(t)\, dt. \end{aligned}$$

The second integral shows that the j-th moment of Z is equal to the actuarial present value for an n-year term insurance for a unit amount payable at the moment of death of (x), calculated at a force of interest equal to j times the given force of interest, or $j\,\delta$.

This property, which we call the ***rule of moments,*** holds generally for insurances paying only a unit amount when the force of interest is deterministic, constant or not. More precisely,

$$\mathrm{E}[Z^j] \ @\ \delta_t = \mathrm{E}[Z] \ @\ j\,\delta_t. \tag{4.2.4}$$

In addition to the existence of the moments, the sufficient condition for the rule of moments is $b_t^j = b_t$ for all $t \geq 0$, that is, for each t the benefit amount is 0 or 1. Demonstration that this is sufficient is left to Exercise 4.30.

It follows from the rule of moments that

$$\mathrm{Var}(Z) = {}^2\bar{A}^{\,1}_{x:\overline{n}|} - (\bar{A}^{\,1}_{x:\overline{n}|})^2 \tag{4.2.5}$$

where ${}^2\bar{A}^{\,1}_{x:\overline{n}|}$ is the actuarial present value for an n-year term insurance for a unit amount calculated at force of interest $2\,\delta$.

Whole life insurance provides for a payment following the death of the insured at any time in the future. If the payment is to be a unit amount at the moment of death of (x), then

$$\begin{aligned} b_t &= 1 & t &\geq 0, \\ v_t &= v^t & t &\geq 0, \\ Z &= v^T & T &\geq 0. \end{aligned}$$

The actuarial present value is

$$\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t\, {}_tp_x\, \mu_x(t)\, dt. \tag{4.2.6}$$

For a life selected at x and now age $x + h$, the expression would be

$$\bar{A}_{[x]+h} = \int_0^\infty v^t\, {}_tp_{[x]+h}\, \mu_x(h + t)\, dt.$$

Whole life insurance is the limiting case of n-year term insurance as $n \to \infty$.

Example 4.2.1

The p.d.f. of the future lifetime, T, for (x) is assumed to be

$$f_T(t) = \begin{cases} 1/80 & 0 \leq t \leq 80 \\ 0 & \text{elsewhere.} \end{cases}$$

At a force of interest, δ, calculate for Z, the present-value random variable for a whole life insurance of unit amount issued to (x):

a. The actuarial present value

b. The variance

c. The 90th percentile, $\xi_Z^{0.9}$.

Solution:

a. $\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t\, f_T(t)\, dt = \int_0^{80} e^{-\delta t}\, \frac{1}{80}\, dt = \frac{1 - e^{-80\delta}}{80\,\delta} \qquad \delta \neq 0.$

b. By the rule of moments,

$$\mathrm{Var}(Z) = \frac{1 - e^{-160\delta}}{160\,\delta} - \left(\frac{1 - e^{-80\delta}}{80\,\delta}\right)^2 \qquad \delta \neq 0.$$

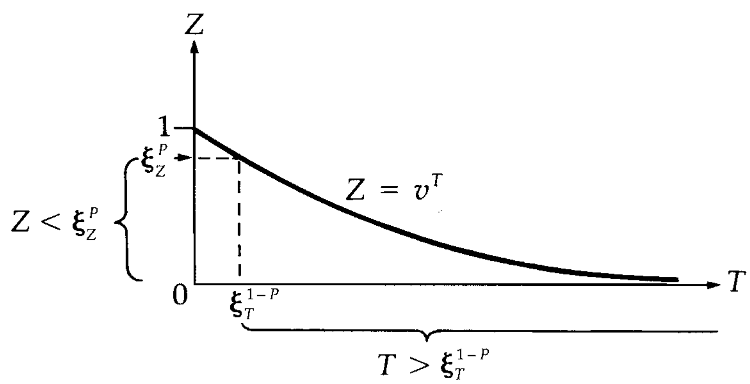
c. For the continuous random variable, Z, we have $\Pr(Z \leq \xi_Z^{0.9}) = 0.9$.

Since we have the p.d.f. for T and not for Z, we proceed by finding the event for T which corresponds to $Z \leq \xi_Z^{0.9}$. From Figure 4.2.1, which shows the general relationship between the sample space of T (on the horizontal axis) and the sample space of Z (on the vertical axis), we see that $\xi_Z^{0.9} = v^{\xi_T^{0.1}}$. Because Z is a strictly decreasing function of T for whole life insurance, the percentile from T's distribution that is related to 90th percentile of Z's distribution is at the complementary probability level, 0.1. In this example T is uniformly distributed over the interval (0, 80), so $\xi_T^{0.1} = 8.0$ and thus $\xi_Z^{0.9} = v^{8.0}$. ▼

The graph in Figure 4.2.1 can be used to establish relationships between the d.f. and p.d.f. of Z and those of T:

FIGURE 4.2.1

Relationship of *Z* to *T* for Whole Life Insurance



For $z \le 0$, $\{Z \le z\}$ is the null event

For $0 < z < 1$, $\{Z \le z\} = \{T \ge \log z/\log v\}$, and

For $z \ge 1$, $\{Z \le z\}$ is the certain event.

Therefore,

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1 - F_T(\log z/\log v) & 0 < z < 1 \\ 1 & 1 \le z. \end{cases} \tag{4.2.7}$$

By differentiation of (4.2.7),

$$f_Z(z) = \begin{cases} f_T[(\log z)/(\log v)][1/(\delta z)] & 0 < z < 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{4.2.8}$$

Example 4.2.2

For the assumptions in Example 4.2.1, determine

a. Z's d.f.

b. Z's p.d.f.

Solution:

a.

$$\text{From } F_T(t) = \begin{cases} t/80 & 0 \le t \le 80 \\ 1 & t \ge 80, \end{cases}$$

we see that $\Pr\{T > 80\} = 0.0$, so $\Pr\{0 < Z < v^{80}\} = 0.0$. Therefore, from (4.2.7)

$$F_Z(z) = \begin{cases} 0 & z < v^{80} \\ 1 - [(\log z)/(\log v)]/80 & v^{80} < z < 1 \\ 1 & z \ge 1. \end{cases}$$

b. By differentiation of the d.f. in part (a),

$$f_Z(z) = \begin{cases} (1/80)(1/\delta z) & v^{80} < z < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

▼

We now turn our attention to a common application involving portfolios of risk: determining an initial investment fund for a segment of insurances in the total portfolio. The individual risk model and the normal approximation (as discussed in Section 2.4) are used.

Example 4.2.3

Assume that each of 100 independent lives

- Is age x
- Is subject to a constant force of mortality, $\mu = 0.04$, and
- Is insured for a death benefit amount of 10 units, payable at the moment of death.

The benefit payments are to be withdrawn from an investment fund earning $\delta = 0.06$. Calculate the minimum amount at $t = 0$ so that the probability is approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.

Solution:

For each life,

$$b_t = 10 \qquad t \geq 0,$$
$$v_t = v^t \qquad t \geq 0,$$
$$Z = 10v^T \qquad T \geq 0.$$

If we think of the lives as numbered, perhaps by the order of issuing policies, then at $t = 0$ the present value of all payments to be made is

$$S = \sum_1^{100} Z_j$$

where Z_j is the present value at $t = 0$ for the payment to be made at the death of the life numbered j.

We can use the fact that Z is 10 times the present-value random variable for the unit amount whole life insurance to calculate the mean and variance. For constant forces of interest, δ, and mortality, μ, the actuarial present value for the unit amount whole life insurance is

$$\bar{A}_x = \int_0^\infty e^{-\delta t}\, e^{-\mu t}\, \mu\, dt = \frac{\mu}{\mu + \delta}.$$

Then, for this example,

$$\mathrm{E}[Z] = 10\bar{A}_x = 10\,\frac{0.04}{0.1} = 4,$$

$$\mathrm{E}[Z^2] = 10^2\, {}^2\bar{A}_x = 100\,\frac{0.04}{0.04 + 2(0.06)} = 25$$

and $\mathrm{Var}(Z) = 9$.

Using these values for the mean and the variance of each term in the sum for S, we have

$$\mathrm{E}[S] = 100(4) = 400,$$
$$\mathrm{Var}(S) = 100(9) = 900.$$

Analytically, the required minimum amount is a number, h, such that

$$\Pr(S \leq h) = 0.95,$$

or equivalently

$$\Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} \leq \frac{h - 400}{30}\right] \doteq 0.95.$$

By use of a normal approximation, we obtain

$$\frac{h - 400}{30} = 1.645,$$

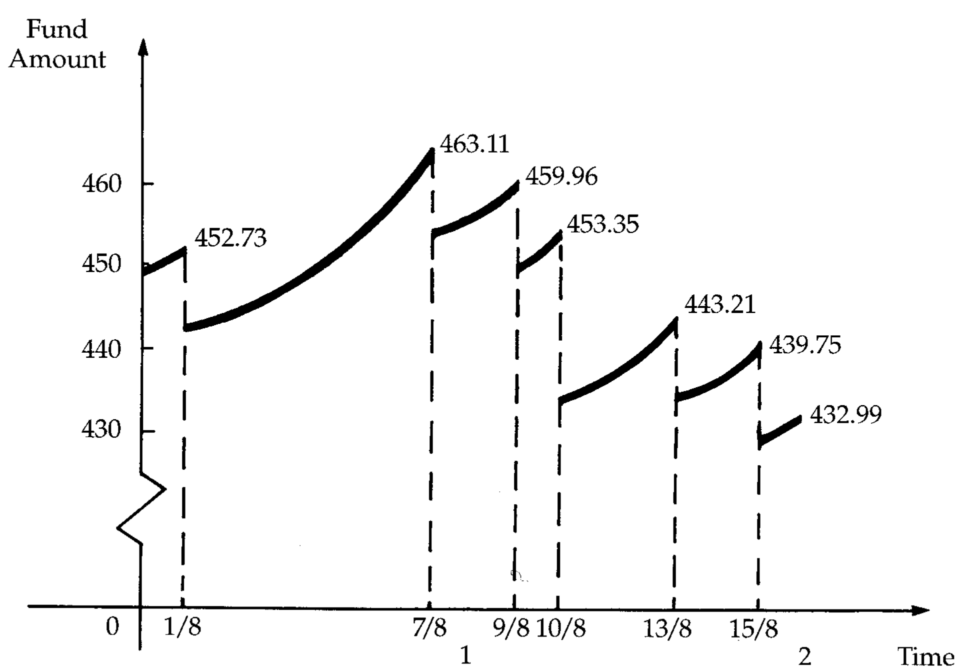
$$h = 449.35. \qquad \blacktriangledown$$

Observations:

1. The 49.35 difference between this initial fund of 449.35 and the expectation of the present value of all payments, 400, is the risk loading of Chapter 1. The loading is 0.4935 per life, or 4.935% per unit payment, or 12.34% of the actuarial present value.
2. This example, like Examples 2.5.2 and 2.5.3, used the individual risk model and a normal approximation to the probability distribution of S. In the short-period examples, the collected income, equal to expected claims plus a risk loading, was determined to have a high probability of being in excess of claims. In this long-period life insurance example, the collected income plus interest income on it at the assumed interest rate is determined to be sufficient to cover the benefit payments. The initial fund of 449.35 will cover less than 45% of the eventual certain payout of 1,000.
3. A graph of the amount in the fund during the first 2 years for a payout pattern when one death occurs at each of times 1/8, 7/8, 9/8, 13/8, and 15/8, and two deaths occur at time 10/8 is shown in Figure 4.2.2. Between the benefit payments, represented by the discontinuities, are exponential arcs representing the growth of the fund at $\delta = 0.06$.

FIGURE 4.2.2

Graph of an Outcome for the Fund



4. There are infinitely many payout patterns, each with its own graph. Both the number of claims and the times of those claims affect the fund. For example, had the seven claims all occurred within the first instant, instead of the payout pattern of Figure 4.2.2, the fund would have dropped immediately to 379.35 and then grown to 427.72 by the end of the second year.

These examples illustrate the different roles of the three random elements in risk model building, that is, whether or not a claim will occur, the size, and the time of payment if one occurs. In Example 2.5.2 there was uncertainty about only the occurrence of the claim. In Example 4.2.2 there was uncertainty about only the time of claim payment. Other uncertainties have been ignored in these models. In Examples 4.2.1, 4.2.2, and 4.2.3 we have ignored the possibility of the fund earning interest at rates different from the deterministic rates assumed.

4.2.2 Endowment Insurance

An ***n*-year pure endowment** provides for a payment at the end of the n years if and only if the insured survives at least n years from the time of policy issue. If the amount payable is a unit, then

$$b_t = \begin{cases} 0 & t \le n \\ 1 & t > n, \end{cases}$$

$$v_t = v^n \qquad t \ge 0,$$

$$Z = \begin{cases} 0 & T \le n \\ v^n & T > n. \end{cases}$$

The only element of uncertainty in the pure endowment is whether or not a claim will occur. The size and time of payment, if a claim occurs, are predetermined. In the expression $Z = v^n Y$, Y is the indicator of the event of survival to age $x + n$. This Y has the value 1 if the insured survives to age $x + n$ and has the value 0 otherwise. The n-year pure endowment's actuarial present value has two symbols. In an insurance context it is $A_{x:\overset{1}{\overline{n|}}}$. We see in the next chapter that it is denoted by ${}_nE_x$ in an annuity context. This distinction is not strict; the reader will have to be ready for either:

$$A_{x:\overset{1}{\overline{n|}}} = \mathrm{E}[Z] = v^n \,\mathrm{E}[Y] = v^n \, {}_np_x,$$

and

$$\begin{aligned} \mathrm{Var}(Z) &= v^{2n} \,\mathrm{Var}(Y) = v^{2n} \, {}_np_x \, {}_nq_x \\ &= {}^2A_{x:\overset{1}{\overline{n|}}} - (A_{x:\overset{1}{\overline{n|}}})^2. \end{aligned} \tag{4.2.9}$$

An ***n*-year endowment insurance** provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first. If the insurance is for a unit amount and the death benefit is payable at the moment of death, then

$$b_t = 1 \qquad t \ge 0,$$

$$v_t = \begin{cases} v^t & t \le n \\ v^n & t > n, \end{cases}$$

$$Z = \begin{cases} v^T & T \le n \\ v^n & T > n. \end{cases}$$

The actuarial present value is denoted by $\bar{A}_{x:\overline{n}|}$. Since $b_t = 1$ for the endowment insurance, we have by the rule of moments

$$\mathrm{E}[Z^j] \text{ @ } \delta = \mathrm{E}[Z] \text{ @ } j\delta.$$

Moreover,

$$\mathrm{Var}(Z) = {}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2. \tag{4.2.10}$$

This insurance can be viewed as the combination of an n-year term insurance and an n-year pure endowment—each for a unit amount. Let Z_1, Z_2, and Z_3 denote the present-value random variables of the term, the pure endowment, and the endowment insurances, respectively, with death benefits payable at the moment of death of (x). From the preceding definitions we have

$$Z_1 = \begin{cases} v^T & T \le n \\ 0 & T > n, \end{cases}$$

$$Z_2 = \begin{cases} 0 & T \le n \\ v^n & T > n, \end{cases}$$

$$Z_3 = \begin{cases} v^T & T \le n \\ v^n & T > n. \end{cases}$$

It follows that

$$Z_3 = Z_1 + Z_2, \tag{4.2.11}$$

and by taking expectations of both sides

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}. \tag{4.2.12}$$

We can also find the $\mathrm{Var}(Z_3)$ by using (4.2.11),

$$\mathrm{Var}(Z_3) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) + 2\,\mathrm{Cov}(Z_1, Z_2). \tag{4.2.13}$$

By use of the formula

$$\mathrm{Cov}(Z_1, Z_2) = \mathrm{E}[Z_1 Z_2] - \mathrm{E}[Z_1]\,\mathrm{E}[Z_2] \tag{4.2.14}$$

and the observation that

$$Z_1 Z_2 = 0$$

for all T, we have

$$\mathrm{Cov}(Z_1, Z_2) = -\mathrm{E}[Z_1]\,\mathrm{E}[Z_2] = -\bar{A}^{1}_{x:\overline{n}|} A_{x:\overset{1}{\overline{n}|}}. \tag{4.2.15}$$

Substituting (4.2.5), (4.2.9), and (4.2.15) into (4.2.13) produces a formula for $\mathrm{Var}(Z_3)$ in terms of actuarial present values for an n-year term insurance and a pure endowment.

Since the actuarial present values are positive, the $\mathrm{Cov}(Z_1, Z_2)$ is negative. This is to be anticipated since, of the pair Z_1 and Z_2, one is always zero and the other positive. On the other hand, the correlation coefficient of Z_1 and Z_2 is not -1 since they are not linear functions of each other; recall Exercise 1.23(c).

4.2.3 Deferred Insurance

An ***m*-year deferred insurance** provides for a benefit following the death of the insured only if the insured dies at least m years following policy issue. The benefit payable and the term of the insurance may be any of those discussed above. For example, an m-year deferred whole life insurance with a unit amount payable at the moment of death has

$$b_t = \begin{cases} 1 & t > m \\ 0 & t \le m, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T & T > m \\ 0 & T \le m. \end{cases}$$

The actuarial present value is denoted by ${}_{m|}\bar{A}_x$ and is equal to

$$\int_m^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.16}$$

Example 4.2.4

Consider a 5-year deferred whole life insurance payable at the moment of the death of (x). The individual is subject to a constant force of mortality $\mu = 0.04$. For the distribution of the present value of the benefit payment, at $\delta = 0.10$:

a. Calculate the expectation
b. Calculate the variance
c. Display the distribution function
d. Calculate the median $\xi_Z^{0.5}$.

Solution:

a. For arbitrary forces μ and δ,

$$_{5|}\bar{A}_x = \int_5^\infty e^{-\delta t} \, e^{-\mu t} \, \mu \, dt = \frac{\mu}{\mu + \delta} \, e^{-5(\mu+\delta)};$$

thus for $\mu = 0.04$ and $\delta = 0.10$,

$$_{5|}\bar{A}_x = \frac{2}{7} \, e^{-0.7} = 0.1419.$$

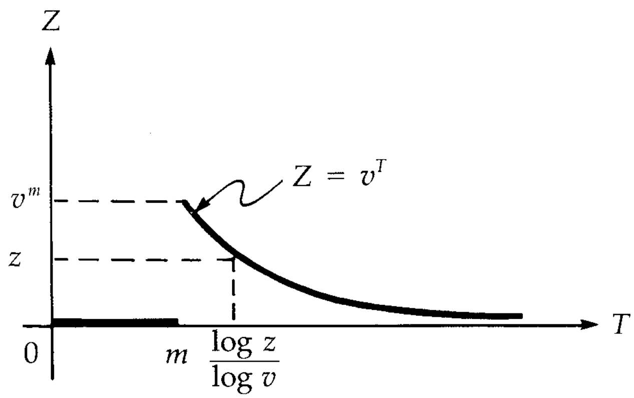
b. By the rule of moments,

$$\mathrm{Var}(Z) = \frac{0.04}{0.04 + 0.20} \, e^{-5(0.04+0.20)} - \frac{4}{49} \, e^{-1.4} = 0.0301.$$

c. As for the case of whole life insurance, a graph of the relation between Z and T provides an outline for the solution. For the general m-year deferred whole life insurance, the graph is given in Figure 4.2.3.

FIGURE 4.2.3

Relationship of Z to T for Deferred Whole Life Insurance



Although T is a continuous random variable, Z is mixed with a probability mass at 0 because $Z = 0$ corresponds to $T \le m$.

For general mortality assumptions and a constant force of interest, we have for $Z = 0$,

$$F_Z(0) = \Pr(T \le m) = F_T(m); \tag{4.2.17}$$

for $0 < z < v^m$,

$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) = \Pr(Z = 0) + \Pr(0 < Z \le z) \\ &= \Pr(T \le m) + \Pr(0 < v^T \le z) \\ &= \Pr(T \le m) + \Pr\left(T > \frac{\log z}{\log v}\right) \\ &= F_T(m) + 1 - F_T\left(\frac{\log z}{\log v}\right); \end{aligned} \tag{4.2.18}$$

for $z > v^m$,

$$F_Z(z) = 1. \tag{4.2.19}$$

In this example of 5-year deferred whole life insurance where $\mu = 0.04$ and $\delta = 0.10$, we have

from (4.2.17),

$$F_Z(0) = F_T(5) = 1 - e^{-0.2} = 0.1813;$$

from (4.2.18) for $0 < z < v^5$,

$$F_Z(z) = F_T(5) + 1 - F_T \frac{\log z}{-0.1}$$

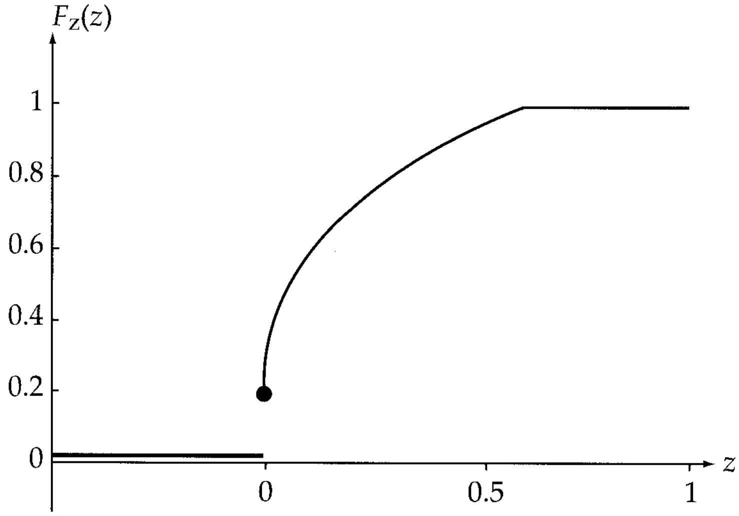
$$= 1 - e^{-0.2} + z^{0.04/0.10} = 0.1813 + z^{0.4}; \quad (4.2.20)$$

from (4.2.19) for $z > v^5$,

$$F_Z(z) = 1.$$

The graph of this d.f. is shown in Figure 4.2.4.

FIGURE 4.2.4
Distribution Function of *Z*



d. From Figure 4.2.4 or (4.2.20), we see that the median is the solution of

$$0.5 = 0.1813 + z^{0.4}.$$

Thus, $\xi_Z^{0.5} = 0.0573$. ▼

Observations:

1. The largest value of Z with nonzero probability density in this example is $e^{-0.1(5)} = 0.6065$, corresponding to $T = 5$.
2. The distribution of Z in this example is highly skewed to the right. While its total mass is in the interval [0, 0.6065] and its mean is 0.1419, its median is only 0.0573. This skewness in the direction of large positive values is characteristic of many claim distributions in all fields of insurance.

4.2.4 Varying Benefit Insurance

The general model given by (4.2.1) can be used for analysis in most applications. We have used it with level benefit life insurances. It can also be applied to insurances where the level of the death benefit either increases or decreases in arithmetic progression over all or a part of the term of the insurance. Such insurances are often sold as an additional benefit when a basic insurance provides for the return of periodic premiums at death or when an annuity contract contains a guarantee of sufficient payments to match its initial premium.

An ***annually increasing whole life insurance*** providing 1 at the moment of death during the first year, 2 at the moment of death in the second year, and so on, is characterized by the following functions:

$$\begin{aligned} b_t &= \lfloor t + 1 \rfloor & t &\geq 0, \\ v_t &= v^t & t &\geq 0, \\ Z &= \lfloor T + 1 \rfloor v^T & T &\geq 0, \end{aligned}$$

where the $\lfloor \ \rfloor$ denote the greatest integer function.

The actuarial present value for such an insurance is

$$(I\bar{A})_x = \mathrm{E}[Z] = \int_0^\infty \lfloor t + 1 \rfloor\, v^t \,{}_tp_x\, \mu_x(t)\, dt.$$

The higher order moments are not equal to the actuarial present value at an adjusted force of interest as was the case for insurances with benefit payments equal to 0 or 1. These moments can be calculated directly from their definitions.

The increases in the benefit of the insurance can occur more, or less, frequently than once per year. For an m-thly increasing whole life insurance the benefit would be $1/m$ at the moment of death during the first m-th of a year of the term of the insurance, $2/m$ at the moment of death during the second m-th of a year during the term of the insurance, and so on, increasing by $1/m$ at m-thly intervals throughout the term of the insurance. For such a whole life insurance the functions are

$$\begin{aligned} b_t &= \frac{\lfloor tm + 1 \rfloor}{m} & t &\geq 0, \\ v_t &= v^t & t &\geq 0, \\ Z &= \frac{v^T \lfloor Tm + 1 \rfloor}{m} & T &\geq 0. \end{aligned}$$

The actuarial present value is

$$(I^{(m)}\,\bar{A})_x = \mathrm{E}[Z].$$

The limiting case, as $m \rightarrow \infty$ in the m-thly increasing whole life insurance, is an insurance paying t at the time of death, t. Its functions are

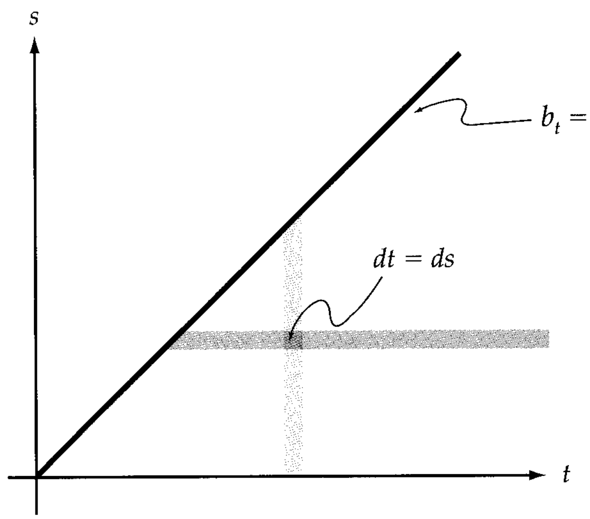
$$\begin{aligned} b_t &= t & t &\geq 0, \\ v_t &= v^t & t &\geq 0, \\ Z &= Tv^T & T &\geq 0. \end{aligned}$$

Its actuarial present-value symbol is $(\bar{I}\bar{A})_x$.

This continuously increasing whole life insurance is equivalent to a set of deferred level whole life insurances. This equivalence is shown graphically in Figure 4.2.5 where the region between the line $b_t = t$ and the t-axis represents the insurance over the future lifetime. If the infinitesimal regions are joined in the vertical direction for a fixed t, the total benefit payable at t is obtained. If they are joined in the horizontal direction for a fixed s, an s-year deferred whole life insurance for the level amount ds is obtained.

FIGURE 4.2.5

Continuously Increasing Insurance



This equivalence implies that the actuarial present values for the coverages are equal. The equality can be established as follows.

By definition,

$$(\bar{I}\bar{A})_x = \int_0^\infty t\, v^t\, {}_tp_x\, \mu_x(t)\, dt,$$

and interpreting t in the integrand as the integral from zero to t in Figure 4.2.5 we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \left(\int_0^t ds \right) v^t\, {}_tp_x\, \mu_x(t)\, dt.$$

If we interchange the order of integration and, for each s value, integrate on t from s to x, we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \int_s^\infty v^t\, {}_tp_x\, \mu_x(t)\, dt\, ds$$

$$= \int_0^\infty {}_{s|}\bar{A}_x\, ds$$

by (4.2.16).

If, for any of these m-thly increasing life insurances, the benefit is payable only if death occurs within a term of n years, the insurance is an ***m-thly increasing n-year term life insurance.***

Complementary to the anually increasing n-year term life insurance is the ***annually decreasing n-year term life insurance*** providing n at the moment of death during the first year, $n - 1$ at the moment of death during the second year, and so on, with coverage terminating at the end of the n-th year. Such an insurance has the following functions:

$$b_t = \begin{cases} n - \lfloor t \rfloor & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T(n - \lfloor T \rfloor) & T \le n \\ 0 & T > n. \end{cases}$$

The actuarial present value for this insurance is

$$(D\bar{A})^1_{x:\overline{n}|} = \int_0^n v^t \, (n - \lfloor t \rfloor) \, {}_tp_x \, \mu_x(t) \, dt.$$

This insurance is complementary to the anually increasing n-year term insurance in the sense that the sum of their benefit functions is the constant $n + 1$ for the n-year term.

Table 4.2.1 is a summary of the models in this section. The insurance plan name appears in the first column followed by the benefit and discount functions that define it in terms of the future lifetime of the insured at policy issue. The present-value function, which is always derived as the product of the previous two functions, is shown next. In the fifth column the International Actuarial Notation for the actuarial present value is shown. In the last column, a reference is given to a footnote stating whether or not the rule of moments can be used in the calculation of higher order moments.

4.3 Insurances Payable at the End of the Year of Death

In the previous section we developed models for life insurances with death benefits payable at the moment of death. In practice, most benefits are considered payable at the moment of death and then earn interest until the payment is actually made. The models were built in terms of T, the future lifetime of the insured at policy issue. In most life insurance applications, the best information available on the probability distribution of T is in the form of a discrete life table. This is the probability distribution of K, the curtate-future-lifetime of the insured at policy issue, a function of T. In this and the following section we bridge this gap by building models for life insurances in which the size and time of payment of the death benefits depend only on the number of complete years lived by the insured from policy issue up to the time of death. We refer to these insurances simply as ***payable at the end of the year of death.***

TABLE 4.2.1

Summary of Insurances Payable Immediately on Death

(1) Insurance Name	(2) Benefit Function b_t	(3) Discount Function v_t	(4) Present-Value Function z_t	(5) Actuarial Present Value	(6) Higher Moments
Whole life	1	v^t	v^t	$\bar{A}_x$	*
n-Year term	$1 \quad t \le n$ $0 \quad t > n$	v^t	$v^t \quad t \le n$ $0 \quad t > n$	$\bar{A}^{1}_{x:\overline{n}\vert}$	*
n-Year pure endowment	$0 \quad t \le n$ $1 \quad t > n$	v^n	$0 \quad t \le n$ $v^n \quad t > n$	$A_{x:\overline{n}\vert}^{\;\;1}$, ${}_nE_x$	*
n-Year endowment	1	$v^t \quad t \le n$ $v^n \quad t > n$	$v^t \quad t \le n$ $v^n \quad t > n$	$\bar{A}_{x:\overline{n}\vert}$	*
m-Year deferred n-Year term	$1 \quad m < t \le n + m$ $0 \quad t \le m, t > n + m$	v^t	$v^t \quad m < t \le n + m$ $0 \quad t \le m, t > n + m$	${}_{m\vert n}\bar{A}_x$	*
n-Year term increasing annually	$\lfloor t + 1 \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$\lfloor t + 1 \rfloor v^t \quad t \le n$ $0 \quad t > n$	$(I\bar{A})^{1}_{x:\overline{n}\vert}$	†
n-Year term decreasing annually	$n - \lfloor t \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$(n - \lfloor t \rfloor)v^t \quad t \le n$ $0 \quad t > n$	$(D\bar{A})^{1}_{x:\overline{n}\vert}$	†
Whole life increasing m-thly	$\lfloor tm + 1 \rfloor / m$	v^t	$v^t \lfloor tm + 1 \rfloor / m$	$(I^{(m)}\bar{A})_x$	†

Note: b_t, v_t, and z_t are defined only for $t \ge 0$.
*The j-th moment is equal to the actuarial present value at j times the given force of interest, denoted by ${}^jA$ for $j > 1$. Then the variance is ${}^2A - A^2$, symbolically.
†Calculated directly from the definition, $E[Z^j]$.

Our model is in terms of functions of the curtate-future-lifetime of the insured. The benefit function, b_{k+1}, and the discount function, v_{k+1}, are, respectively, the benefit amount payable and the discount factor required for the period from the time of payment back to the time of policy issue when the insured's curtate-future-lifetime is k, that is, when the insured dies in year $k + 1$ of insurance. The present value, at policy issue, of this benefit payment, denoted by z_{k+1}, is

$$z_{k+1} = b_{k+1} v_{k+1}. \tag{4.3.1}$$

Measured from the time of policy issue, the insurance year of death is 1 plus the curtate-future-lifetime random variable, K, defined in Section 3.2.3. As in the previous section, we denote the present-value random variable z_{K+1}, by Z.

For an n-year term insurance providing a unit amount at the end of the year of death, we have

$$b_{k+1} = \begin{cases} 1 & k = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere.} \end{cases}$$

The actuarial present value for this insurance is given by

$$A^{1}_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}. \tag{4.3.2}$$

Note that the International Actuarial Notation symbol for the actuarial present value of an insurance payable at the end of the year of death is the symbol for the corresponding insurance payable at the moment of death with the bar removed.

The rule of moments, with the appropriate changes in notation, also holds for insurances payable at the end of the year of death. For example, for the n-year term insurance above,

$$\mathrm{Var}(Z) = {}^2A^{1}_{x:\overline{n}|} - (A^{1}_{x:\overline{n}|})^2$$

where

$${}^2A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} e^{-2\delta(k+1)} \, {}_kp_x \, q_{x+k}.$$

In Section 3.5 recursion relations for life expectancies are derived and used to determine their values. Recursion relations for the term insurance actuarial present values can be derived algebraically from (4.3.2):

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k} = vq_x + \sum_{k=1}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}$$

$$= vq_x + vp_x \sum_{k=1}^{n-1} v^k \, {}_{k-1}p_{x+1} \, q_{x+k}$$

$$= vq_x + vp_x \sum_{j=0}^{n-2} v^{j+1} \, {}_jp_{x+1} \, q_{x+1+j} = vq_x + vp_x \, A^{1}_{x+1:\overline{n-1}|}. \tag{4.3.3}$$

For (4.3.3) to be true at $n = 1$, we define $A^{1}_{x:\overline{0}|} = 0.0$ for all x.

Note: On a select table basis, all x's in the subscripts in (4.3.3) would be enclosed in brackets.

Example 4.3.1

On the basis of the Illustrative Life Table and $i = 0.04$, determine the mean and variance of the present-value random variable for a 10-year term insurance with a unit benefit payable at the end of the year of death issued on (30).

Solution:

Starting with the initial value $A^{1}_{40:\overline{0}|} = 0.0$ and using (4.3.3) adapted to this insurance,

$$A^{1}_{30+k:\overline{10-k}|} = vq_{30+k} + vp_{30+k} \, A^{1}_{30+k+1:\overline{10-(k+1)}|} \quad k = 0, 1, \ldots, 8, 9,$$

we have by working from age 40 to age 30,

$$A^{1}_{30:\overline{10}|} = 0.01577285$$

and

$$\text{Var}(Z) = 0.01271978 - (0.01577285)^2 = 0.1247099.$$

These values were determined by the spreadsheet constructed in the Computing Exercises. ▼

For a whole life insurance issued to (x), the model may be obtained by letting $n \to \infty$ in the n-year term insurance model. For the actuarial present value we have

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \, {}_kp_x \, q_{x+k}. \tag{4.3.4}$$

Multiplication of both sides of (4.3.4) by l_x yields

$$l_x A_x = \sum_{k=0}^{\infty} v^{k+1} \, d_{x+k}. \tag{4.3.5}$$

Formula (4.3.5) shows the balance, at the time of policy issue, between the aggregate fund of actuarial present values for l_x lives insured at age x and the outflow of funds in accordance with the expected deaths of the l_x lives. It is a compound interest equation of value that is stated on an expected value basis.

The expression

$$\sum_{k=r}^{\infty} v^{k+1}\, d_{x+k} \tag{4.3.6}$$

is that part of the fund at issue that, together with interest at the assumed rate, will provide the payments for the expected deaths after the r-th insurance year.

Accumulation of (4.3.6) at the assumed interest rate for r years yields

$$\sum_{k=r}^{\infty} v^{k-r+1}\, d_{x+k}, \tag{4.3.7}$$

the expected amount in the fund after r insurance years. A comparison of expression (4.3.7) with (4.3.5) shows it to be $l_{x+r}\, A_{x+r}$. The difference between this amount and an actual fund is due to deviations of the actual deaths from the expected deaths (according to the life table adopted), and deviations of the actual interest income from the interest income at the assumed rate.

Example 4.3.2

A group of 100 lives age 30 set up a fund to pay 1,000 at the end of the year of death of each member to a designated survivor. Their mutual agreement is to pay into the fund an amount equal to the whole life insurance actuarial present value calculated on the basis of the Illustrative Life Table at 6% interest. The members, not selected by an insurance company, decided to use this population table as the basis of their plan. The actual experience of the fund is one death in each of the second and fifth years; interest income is at 6% in the first year, 6-1/2% in the second and third years, 7% in the fourth and fifth years. What is the difference, at the end of the first 5 years, between the expected size of the fund as determined at the inception of the plan and the actual fund?

Solution:

On the agreed bases, 1,000 $A_{30} = 102.4835$, so, for the 100 lives, the fund starts at 10,248.35. Also, $A_{35} = 0.1287194$ and $l_{35}/l_{30} = 0.9915040$.

For 100 lives age 30, the expected size of the fund after 5 years will be

$$(1{,}000)(100)\,\frac{l_{35}}{l_{30}}\,A_{35} = 12{,}762.58.$$

The development of the actual fund would be as follows, where F_k denotes its size at the end of insurance year k:

$$F_0 = 10{,}248.35$$

$$F_1 = (10{,}248.35)(1.06) = 10{,}863.25$$

$$F_2 = (10{,}863.25)(1.065) - 1{,}000 = 10{,}569.36$$

$$F_3 = (10{,}569.36)(1.065) = 11{,}256.37$$

$$F_4 = (11{,}256.37)(1.07) = 12{,}044.32$$

$$F_5 = (12{,}044.32)(1.07) - 1{,}000 = 11{,}887.42.$$

Thus the required difference is 12,762.58 − 11,887.42 = 875.16. This result combines the investment experience and the mortality experience for the 5-year period. There were gains from the investment earnings in excess of the assumed rate of 6%. On the other hand, there were losses on the mortality experience of two deaths as compared to the expected number of 0.8496. The interpretation of such results in terms of the various sources such as investment earnings and mortality is an actuarial responsibility. ▼

We derived the recursion relations for n-year term insurance actuarial present values (4.3.3) algebraically. Whereas the relationship will hold for whole life insurance actuarial present values as the limiting case of n-year term insurance, as n goes to ∞, we will establish the whole life insurance relationship independently to illustrate a probabilistic derivation.

Consider A_x from its definition $\mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1}]$. For emphasis we now write this as

$$A_x = \mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1} | K(x) \geq 0],$$

which is redundant since all of $K(x)$'s probability is on the non-negative integers.

E[Z] can be calculated by considering the event that (x) dies in the first year, that is, $K(x) = 0$, and its complement, that (x) survives the first year, that is, $K(x) \geq 1$. We can write

$$\mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1} | K(x) = 0] \Pr[K(x) = 0] + \mathrm{E}[v^{K(x)+1} | K(x) \geq 1] \Pr[K(x) \geq 1]. \tag{4.3.8}$$

In this expression we can readily substitute

$$\mathrm{E}[v^{K(x)+1} | K(x) = 0] = v,$$

$$\Pr[K(x) = 0] = q_x,$$

and

$$\Pr[K(x) \geq 1] = p_x.$$

To find an expression for the remaining factor, we rewrite it as

$$\mathrm{E}[v^{K(x)+1} | K(x) \geq 1] = v\,\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0].$$

Since $K(x)$ is the curtate-future-lifetime of (x), given $K(x) \geq 1$, $K(x) - 1$ must be the curtate-future-lifetime of $(x + 1)$.

If we are willing to use the same probabilities for the conditional distribution of $K(x) - 1$ given $K(x) \geq 1$, as we would for a newly considered life age $x + 1$, then we may write

$$\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0] = A_{x+1} \tag{4.3.9}$$

and substitute it into (4.3.8) to obtain

$$A_x = vq_x + vA_{x+1}p_x. \tag{4.3.10}$$

The assumed equality,

(the distribution of the future lifetime
of a newly insured life aged $x + 1$)

= (the distribution of the future lifetime of a life
now age $x + 1$ who was insured 1 year ago),

was discussed in Section 3.8. In terms of select tables, the right-hand side of (4.3.9) would be $A_{[x]+1}$. In (4.3.10) every x would be $[x]$.

Note that (4.3.10) is the same backward recursion formula as (4.3.3). That is,

$$u(x) = vq_x + vp_x \, u(x + 1).$$

It is the starting value that makes the solution the actuarial present value of whole life insurance or of n-year term insurance. We see this same recursion formula for the actuarial present values of n-year endowment insurance where the starting values are the endowment maturity value.

Analysis of relationship (4.3.10) can give more insight into the nature of A_x. After replacement of p_x by $1 - q_x$ and multiplication of both sides by $(1 + i)l_x$, (4.3.10) can be rearranged as

$$l_x (1 + i)A_x = l_x A_{x+1} + d_x(1 - A_{x+1}). \tag{4.3.11}$$

For the random survivorship group, this equation has the following interpretation. Together with 1 year's interest, A_x will provide A_{x+1} for all l_x lives and an additional $1 - A_{x+1}$ for those expected to die within the year. This latter amount for each expected death, that is, $q_x(1 - A_{x+1})$, is considered the ***annual cost of insurance.*** The A_{x+1} is set aside for survivors and deaths, the $1 - A_{x+1}$ is required only for a death.

Dividing by l_x and then subtracting $A_x + q_x(1 - A_{x+1})$ from both sides of (4.3.11), we have

$$A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1}). \tag{4.3.12}$$

In words, the difference between the actuarial present values at age x and one later at age $x + 1$ is equal to the interest on the actuarial present value at x less the annual cost of insurance for the year.

Another expression for A_x can be obtained from (4.3.10) by replacing p_x by $1 - q_x$, multiplying both sides by v^x, and rearranging the terms to get

$$v^{x+1}A_{x+1} - v^x A_x = -v^{x+1}q_x(1 - A_{x+1}),$$

or

$$\Delta v^x A_x = -v^{x+1}q_x(1 - A_{x+1}).$$

Summing from $x = y$ to ∞ (see Appendix 5), we obtain

$$-v^y A_y = -\sum_{x=y}^{\infty} v^{x+1}q_x(1 - A_{x+1})$$

and thus

$$A_y = \sum_{x=y}^{\infty} v^{x+1-y}q_x(1 - A_{x+1}).$$

This expression shows that the actuarial present value at y is the present value at y of the annual costs of insurance over the remaining lifetime of the insured.

The n-year endowment insurance with a unit amount payable at the end of the year of death is a combination of the n-year term insurance of this section and the n-year pure endowment for a unit amount that was discussed in the previous section. Thus the functions for it are

$$b_{k+1} = 1 \qquad k = 0, 1, \ldots,$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, \ldots, n-1 \\ v^n & k = n, n+1, \ldots, \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots. \end{cases}$$

The actuarial present value is

$$A_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} + v^n\, {}_np_x. \tag{4.3.13}$$

The annually increasing whole life insurance, paying $k + 1$ units at the end of insurance year $k + 1$ provided the insured dies in that insurance year, has the benefit and discount functions and present-value random variable as follows:

$$b_{k+1} = k + 1 \qquad k = 0, 1, 2, \ldots,$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, 2, \ldots,$$

$$Z = (K + 1)\, v^{K+1} \qquad K = 0, 1, 2, \ldots.$$

The actuarial present value is denoted by $(IA)_x$.

The annually decreasing n-year term insurance, during the n-year period, provides a benefit at the end of the year of death in an amount equal to $n - k$ where

k is the number of complete years lived by the insured since issue. Its functions are

$$b_{k+1} = \begin{cases} n - k & k = 0, 1, \ldots, n-1 \\ 0 & k = n, n+1, \ldots, \end{cases}$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, \ldots,$$

$$Z = \begin{cases} (n - K)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots. \end{cases}$$

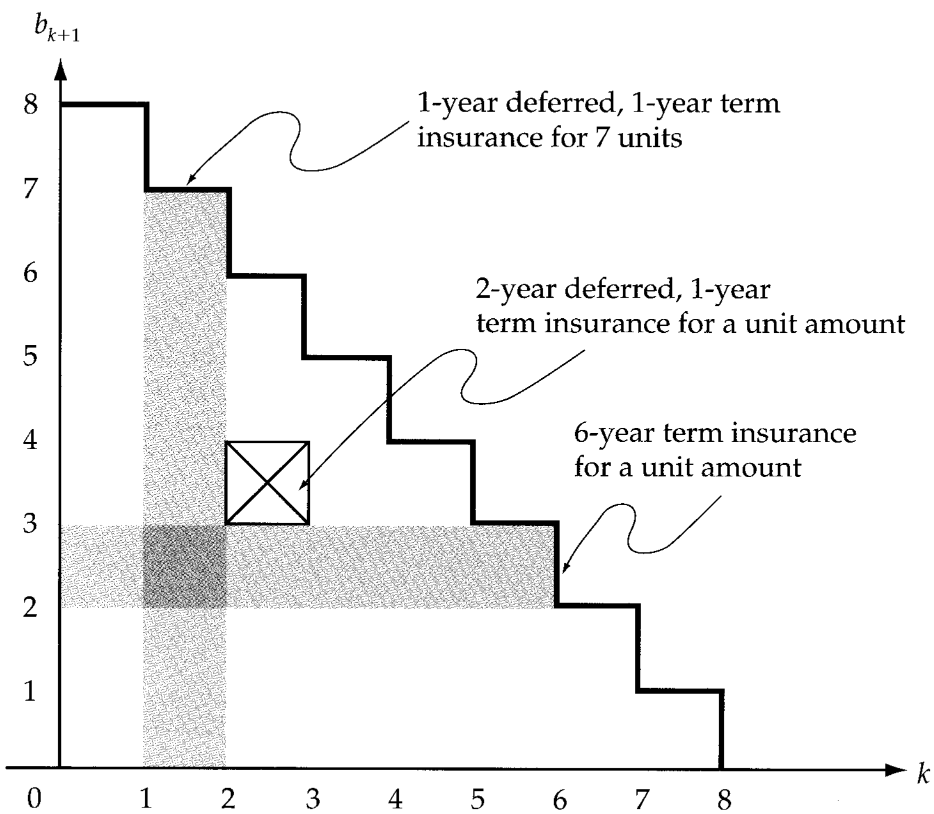
The actuarial present-value symbol for this insurance is $(DA)^{1}_{x:\overline{n}|}$.

As illustrated by Figure 4.2.5 for insurances payable at the moment of death, annually increasing insurances payable at the end of the year of death are equivalent to a combination of deferred level insurances each for a unit amount. Similarly, annually decreasing term insurances are equivalent to a combination of level term insurances of various term lengths. Figure 4.3.1 illustrates this for an annually decreasing 8-year term insurance.

Figure 4.3.1 shows the graph of the benefit function b_{k+1}. Each unit square region between the horizontal steps and the k-axis represents a deferred 1-year term insurance. When these are summed vertically, the deferred 1-year term insurances

FIGURE 4.3.1

Annually Decreasing 8-Year Term Insurance



for the decreasing amounts are obtained. When the squares are summed horizontally, the level amount term insurances of varying duration are obtained. These vertical and horizontal sums are also indicated in Figure 4.3.1.

The equality of the actuarial present values for the combination of level term insurances and the combination of deferred term insurances can be demonstrated analytically. Thus, by definition

$$
\begin{aligned}
(DA)^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} (n - k)\, v^{k+1}\, {}_kp_x\, q_{x+k} \\
&= \sum_{k=0}^{n-1} (n - k)\, (v^k\, {}_kp_x)\, (vq_{x+k}) \\
&= \sum_{k=0}^{n-1} (n - k)\, {}_{k|1}A_x, \qquad (4.3.14)
\end{aligned}
$$

the total of the column sums.

In (4.3.14) we can substitute

$$n - k = \sum_{j=0}^{n-k-1} (1)$$

to obtain

$$(DA)^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k}.$$

By interchanging the order of summation we obtain

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k},$$

and then by comparing the inner summation to (4.3.2), we can write

$$(DA)^{1}_{x:\overline{n}|} = \sum_{j=0}^{n-1} A^{1}_{x:\overline{n-j}|}.$$

Table 4.3.1 provides a summary of functions and symbols for the elementary insurances payable at the end of the year of death.

We close this section with a summary of the recursion relations for the actuarial present values of the insurances payable at the end of the year of death. Consider the list on page 119 arranged in the order of the entries of Table 4.3.1. Each entry is arranged across its line as the recursion relation, the domain for the relation, and the initial condition. Values for the actuarial present value would be generated from the lowest age of the mortality table to age y or ω.

TABLE 4.3.1

Summary of Insurances Payable at End of Year of Death

(1) Insurance Name	(2) Benefit Function b_{k+1}	(3) Discount Function v_{k+1}	(4) Present-Value Function z_{k+1}	(5) Actuarial Present Value	(6) Higher Moments
(a) Whole life	1	v^{k+1}	v^{k+1}	A_x	*
(b) n-Year term	1 $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	v^{k+1}	v^{k+1} $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	$A^{1}_{x:\overline{n}\vert}$	*
(c) n-Year endowment	1	v^{k+1} $k = 0, 1, \ldots, n-1$ v^n $k = n, n+1, \ldots$	v^{k+1} $k = 0, 1, \ldots, n-1$ v^n $k = n, n+1, \ldots$	$A_{x:\overline{n}\vert}$	*
(d) m-Year deferred n-year term	1 $k = m, m+1, \ldots, m+n-1$ 0 $k = 0, \ldots, m-1$ $k = m+n, \ldots$	v^{k+1}	v^{k+1} $k = m, m+1, \ldots, m+n-1$ 0 $k = 0, \ldots, m-1$ $k = m+n, \ldots$	${}_{m\vert n}A_x$	*
(e) n-Year term increasing annually	$k+1$ $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	v^{k+1}	$(k+1)v^{k+1}$ $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	$(IA)^{1}_{x:\overline{n}\vert}$	†
(f) n-Year term decreasing annually	$n-k$ $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	v^{k+1}	$(n-k)v^{k+1}$ $k = 0, 1, \ldots, n-1$ 0 $k = n, n+1, \ldots$	$(DA)^{1}_{x:\overline{n}\vert}$	†
(g) Whole life increasing annually	$k+1$ $k = 0, 1, \ldots$	v^{k+1}	$(k+1)v^{k+1}$ $k = 0, 1, \ldots$	$(IA)_x$	†

b_{k+1}, v_{k+1}, and z_{k+1} are defined only for non-negative integral values of k.
*Rule of moments holds, thus $\text{Var}(Z) = {}^2A - A^2$ symbolically.
†Rule of moments does not hold.

(a) $\qquad A_x = vq_x + vp_x\, A_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$

$\qquad$ and $A_\omega = 0$.

(b) $\qquad A^{1}_{x:\overline{y-x}|} = vq_x + vp_x\, A^{1}_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$

$\qquad$ and $A^{1}_{y:\overline{0}|} = 0$.

(c) $\qquad A_{x:\overline{y-x}|} = vq_x + vp_x\, A_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$

$\qquad$ and $A_{y:\overline{0}|} = 1$.

(d) $\qquad {}_{y-x|n}A_x = 0 + vp_x\, {}_{y-(x+1)|n}A_{x+1} \qquad x = 0, 1, \ldots, y - 1,$

$\qquad$ and ${}_{0|n}A_y = A^{1}_{y:\overline{n}|}$.

(e) $\quad (IA)^{1}_{x:\overline{y-x}|} = [vq_x + vp_x\, A^{1}_{x+1:\overline{y-(x+1)}|}] + vp_x\, (IA)^{1}_{x+1:\overline{y-(x+1)}|}$

$\qquad x = 0, 1, \ldots, y - 1,$ and $(IA)^{1}_{y:\overline{0}|} = 0$.

(f) $\quad (DA)^{1}_{x:\overline{y-x}|} = (y - x)vq_x + vp_x\, (DA)^{1}_{x+1:\overline{y-(x+1)}|}$

$\qquad x = 0, 1, \ldots, y - 1,$ and $(DA)^{1}_{y:\overline{0}|} = 0$.

(g) $\qquad (IA)_x = [vq_x + vp_x\, A_{x+1}] + vp_x\, (IA)_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$

$\qquad$ and $(IA)_\omega = 0$.

Observations:

1. Only (a) and (b) have been justified in this section. Arguments for (c) through (g) are similar to those for (a) and (b).
2. All seven equations are of the form

$$u(x) = c(x) + vp_x\, u(x + 1),$$

where $c(x)$ is a given function defined for the domain of the relation. In the language of difference equations, all seven equations have the same corresponding homogeneous equation, $u(x) = vp_x\, u(x + 1)$. It is linear but does not have constant coefficients.

3. Since $c(x) = vq_x$ for (a), (b), and (c), those actuarial present values are all solutions of the same recursion formula and are distinguished only by their starting values.

4.4 Relationships between Insurances Payable at the Moment of Death and the End of the Year of Death

We begin the study of these relationships with an analysis of the actuarial present value for whole life insurance paying a unit benefit at the moment of death. From (4.2.6) we have

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + \int_1^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt.$$

The change of variables $s = t - 1$ in the second integral gives

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + v \int_0^\infty v^s \, {}_{s+1}p_x \, \mu_x(s + 1) \, ds. \tag{4.4.1}$$

On an aggregate mortality basis

$${}_{s+1}p_x \, \mu_x(s + 1) = p_x \, {}_sp_{x+1} \, \mu(x + s + 1)$$

so the second term of (4.4.1) would be $vp_x\bar{A}_{x+1}$. On a select mortality basis the second term would be $vp_{[x]}\bar{A}_{[x]+1}$. Returning to (4.4.1) and using aggregate notation, we have

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + vp_x \, \bar{A}_{x+1} = \bar{A}^{1}_{x:\overline{1}|} + vp_x \, \bar{A}_{x+1}. \tag{4.4.2}$$

The integral in (4.4.2) can be expressed in discrete life table functions by adopting one of the assumptions about the form of the mortality function between integers as discussed in Section 3.6.

Under the assumption of a uniform distribution of deaths over each year of age,

$${}_tp_y \, \mu_y(t) = q_y \qquad 0 \le t \le 1, \text{ and } y = 0, 1, \ldots$$

which can be placed in (4.4.2) to obtain

$$\bar{A}_x = q_x \int_0^1 v^t \, dt + vp_x\bar{A}_{x+1},$$

$$= \frac{i}{\delta} \, vq_x + vp_x \, \bar{A}_{x+1}. \tag{4.4.3}$$

The domain for this relationship is $x = 0, 1, \ldots, \omega - 1$, and the starting value is $\bar{A}_\omega = 0$.

If we multiply both sides of recursion formula (a) by i/δ, we have

$$\frac{i}{\delta} \, A_x = \frac{i}{\delta} \, vq_x + vp_x \left(\frac{i}{\delta} \, A_{x+1} \right).$$

Since (a) and (4.4.3) embody the same recursion formula and have the same domain and the same initial value of 0 at ω, $(i/\delta)A_x$ is the solution for (4.4.3), and

$$\bar{A}_x = \frac{i}{\delta} \, A_x. \tag{4.4.4}$$

Formula (4.4.4) might have been anticipated under the assumption of a uniform distribution of deaths between integral ages. The effect of the assumption is to make the unit payable at the moment of death equivalent to a unit payable continuously throughout the year of death. With respect to interest, a unit payable continuously over the year is equivalent to i/δ at the end of the year.

The identity in (4.4.4) can be reached using the properties of the future-lifetime random variable under the assumption of a uniform distribution of deaths in each year of age as developed in Section 3.6. From (3.6.1) we write $T = K + S$. We observed there that, under the assumption of a uniform distribution of deaths in each year of age, K and S are independent and S has a uniform distribution over the unit interval. As corollaries to these observations, $K + 1$ and $1 - S$ are also independent, and $1 - S$ has a uniform distribution over the unit interval. In the identity

$$\bar{A}_x = \mathrm{E}[v^T] = \mathrm{E}[v^{K+1}(1 + i)^{1-S}],$$

we can use the independence of $K + 1$ and $1 - S$ to calculate the expectation of the product as the product of the expectations,

$$\mathrm{E}[v^{K+1}(1 + i)^{1-S}] = \mathrm{E}[v^{K+1}]\,\mathrm{E}[(1 + i)^{1-S}]. \tag{4.4.5}$$

The first factor on the right-hand side is A_x. Since $1 - S$ has the uniform distribution over the unit interval, the second factor is

$$\mathrm{E}[(1 + i)^{1-S}] = \int_0^1 (1 + i)^t \, 1 \, dt = \frac{i}{\delta}.$$

Hence, again we have $\bar{A}_x = (i/\delta)A_x$ under the assumption of uniform distribution of deaths in each year of age.

A similar argument, again based on the assumption of a uniform distribution of deaths in each year of age, can be used to show that the actuarial present value of a whole life insurance which pays a unit at the end of the m-th of a year of death is equal to

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x. \tag{4.4.6}$$

This argument is outlined in Exercise 4.19.

In Section 3.6 we also discussed the assumption that the force of mortality is constant between integral ages. The relationship between the actuarial present values for whole life insurances payable at the moment of death and at the end of the year of death under this assumption is developed in Exercise 4.19. Since the hyperbolic assumption implies that the force of mortality decreases over the year of age (see Exercise 3.27), it is seldom realistic for human lives. Moreover, it leads to more complicated relationships that we will not develop here.

Next we turn to an analysis of the annually increasing n-year term insurance payable at the moment of death. For this insurance, the present-value random variable is

$$Z = \begin{cases} \lfloor T + 1 \rfloor v^T & T < n \\ 0 & T \geq n. \end{cases}$$

Since $\lfloor T + 1 \rfloor = K + 1$, we can use the relation $T = K + S$ to obtain

$$Z = \begin{cases} (K+1)v^{K+1}v^{S-1} & T < n \\ 0 & T \ge n. \end{cases}$$

If we let W be the present-value random variable for the annually increasing n-year term insurance payable at the end of the year of death,

$$W = \begin{cases} (K+1)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots. \end{cases}$$

Then

$$Z = W(1+i)^{1-S}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[W(1+i)^{1-S}].$$

Since W is a function of $K+1$ alone and $K+1$ and $1-S$ are independent,

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}[W]\,\mathrm{E}[(1+i)^{1-S}] \\ &= (IA)^{1}_{x:\overline{n}|}\,\frac{i}{\delta}. \end{aligned}$$

These results for the whole life and the increasing term insurances payable at the moment of death, under the assumption of a uniform distribution of deaths over each year of age, are very similar,

$$\bar{A}_x = \frac{i}{\delta} A_x$$

and

$$(I\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta} (IA)^{1}_{x:\overline{n}|}.$$

Let us look at the general model to find the basis of the similarities. From (4.2.2),

$$Z = b_T v_T. \tag{4.4.7}$$

For the two continuous insurances above, the conditions used were

- $v_T = v^T$, and
- b_T was a function of only the integral part of T, the curtate-future-lifetime, K.

Writing this latter property as $b_T = b^*_{K+1}$ we can write (4.4.7) for these insurances as

$$\begin{aligned} Z &= b^*_{K+1} v^T \\ &= b^*_{K+1} v^{K+1} (1+i)^{1-S} \end{aligned}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[b^*_{K+1} v^{K+1} (1+i)^{1-S}]. \tag{4.4.8}$$

Under the assumption of a uniform distribution of deaths over each year of age, we can infer the independence of K and S and that $1-S$ also has a uniform distribution. Then we can write (4.4.8) as

$$\begin{aligned} E[Z] &= E[b^*_{K+1}v^{K+1}]\, E[(1+i)^{1-S}] \\ &= E[b^*_{K+1}v^{K+1}]\, \frac{i}{\delta}. \end{aligned} \tag{4.4.9}$$

Example 4.4.1

Calculate the actuarial present value and the variance for a 10,000 benefit, 30-year endowment insurance providing the death benefit at the moment of death of a male age 35 at issue of the policy. Use the Illustrative Life Table, the uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

For endowment insurance, $v_T \neq v^T$. Therefore, we cannot apply (4.4.9) directly. Recalling (4.2.11), which showed the endowment insurance as the sum of a term insurance and pure endowment, we can apply (4.4.9) to the term insurance component and then calculate the pure endowment insurance part. Thus, using (4.2.12) and (4.2.10), we can calculate the actuarial present value as follows:

$$\begin{aligned} \bar{A}_{35:\overline{30}|} &= \frac{i}{\delta} A^{1}_{35:\overline{30}|} + A_{35:\overset{1}{\overline{30}|}} \\ &= (1.0297087)[0.06748179] + 0.1392408 \\ &= 0.208727, \end{aligned}$$

and the variance as

$$\begin{aligned} \mathrm{Var}(Z) &= {}^2\bar{A}_{35:\overline{30}|} - (\bar{A}_{35:\overline{30}|})^2 \\ &= 0.0309294 + 0.0242432 - (0.208727)^2 \\ &= 0.011606. \end{aligned}$$

For the 10,000 sum insured, 10,000 $\bar{A}_{35:\overline{30}|} = 2{,}087.27$ and $(10{,}000)^2\ \mathrm{Var}(Z) = 1{,}160{,}600$. ▼

Example 4.4.2

Calculate, for a life age 50, the actuarial present value for an annually decreasing 5-year term insurance paying 5,000 at the moment of death in the first year, 4,000 in the second year, and so on. Use the Illustrative Life Table, uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

Referring to Table 4.2.1, we see that

$$b_t = \begin{cases} 5 - \lfloor t \rfloor & t \le 5 \\ 0 & t > 5 \end{cases}$$

is a function of only k, the integral part of t, and hence we may write it as

$$b_t = \begin{cases} 5 - k & k = 0, 1, 2, 3, 4 \\ 0 & k > 4. \end{cases}$$

The discount function is v^t, so we have

$$\begin{aligned}
(D\bar{A})^1_{50:\overline{5}|} &= \frac{i}{\delta} (DA)^1_{50:\overline{5}|} \\
&= (1.0297087) \sum_{k=0}^{4} (5 - k)\, v^{k+1}\, {}_kp_{50}\, q_{50+k} \\
&= (1.0297087) \frac{\sum_{k=0}^{4} (5 - k)\, v^{k+1}\, d_{50+k}}{l_{50}} \\
&= 0.088307.
\end{aligned}$$

Then, 1,000 $(D\bar{A})^1_{50:\overline{5}|} = 88.307$. ▼

For an insurance providing a death benefit at the moment of death that is not a function of K, further analysis is required to express its values in terms of those for an insurance payable at the end of the year of death. Consider the continuously increasing whole life insurance payable at the moment of death. This insurance is discussed extensively in Section 4.2, and its benefit function analyzed in Figure 4.2.5. Its functions are

$$\begin{aligned}
b_t &= t & t &> 0, \\
v_t &= v^t & t &> 0, \\
z_t &= tv^t & t &> 0.
\end{aligned}$$

To find $(\bar{I}\bar{A})_x$ we rewrite

$$\begin{aligned}
Z &= (K + S)v^{K+S} \\
&= (K + 1)v^{K+S} - (1 - S)v^{K+1}(1 + i)^{1-S} \\
&= (K + 1)v^{K+1}(1 + i)^{1-S} - v^{K+1}(1 - S)(1 + i)^{1-S}.
\end{aligned}$$

Now taking expectations, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\begin{aligned}
\mathrm{E}[Z] &= \mathrm{E}[(K + 1)v^{K+1}]\, \mathrm{E}[(1 + i)^{1-S}] - \mathrm{E}[v^{K+1}]\, \mathrm{E}[(1 - S)(1 + i)^{1-S}] \\
&= (IA)_x \frac{i}{\delta} - A_x\, \mathrm{E}[(1 - S)(1 + i)^{1-S}].
\end{aligned}$$

We can simplify the last factor directly since $1 - S$ has a uniform distribution,

$$\mathrm{E}[(1 - S)(1 + i)^{1-S}] = \int_0^1 u(1 + i)^u\, du = (\bar{D}\bar{s})_{\overline{1}|} = \frac{1 + i}{\delta} - \frac{i}{\delta^2}.$$

Thus, we can write

$$(\bar{I}\bar{A})_x = \frac{i}{\delta} \left[(IA)_x - \left(\frac{1}{d} - \frac{1}{\delta} \right) A_x \right].$$

4.5 Differential Equations for Insurances Payable at the Moment of Death

Recursive-type expressions can be established for insurances payable at the moment of death. These are developed using calculus and lead to differential equations.

For a whole life insurance on (x),

$$\frac{d}{dx}\bar{A}_x = -\mu(x) + \bar{A}_x[\delta + \mu(x)] = \delta\bar{A}_x - \mu(x)(1 - \bar{A}_x), \tag{4.5.1}$$

which are the continuous analogues of (4.3.12). The notation used here is for an aggregate mortality basis. Verification of these expressions has been left to Exercise 4.21.

On the other hand, (4.5.1) can be developed from the definition of $\bar{A}_x$ by using conditional expectation as we did for A_x:

$$\begin{aligned}\bar{A}_x &= \mathrm{E}[v^T] \\ &= \mathrm{E}[v^T|T \le h]\ \Pr(T \le h) + \mathrm{E}[v^T|T > h]\ \Pr(T > h).\end{aligned} \tag{4.5.2}$$

Now,

$$\Pr(T \le h) = {}_hq_x \quad \text{and} \quad \Pr(T > h) = {}_hp_x, \tag{4.5.3}$$

and the conditional p.d.f. of T given $T \le h$ is

$$f_T(t|T \le h) = \begin{cases} \dfrac{f_T(t)}{F_T(h)} = \dfrac{{}_tp_x\ \mu(x+t)}{{}_hq_x} & 0 \le t \le h \\ 0 & \text{elsewhere.} \end{cases}$$

Thus,

$$\mathrm{E}[v^T|T \le h] = \int_0^h v^t\ \frac{{}_tp_x\ \mu(x+t)}{{}_hq_x}\ dt. \tag{4.5.4}$$

As we did in the expression for A_x, we will write

$$\begin{aligned}\mathrm{E}[v^T|T > h] &= v^h\mathrm{E}[v^{T-h}|(T-h) > 0] \\ &= v^h\bar{A}_{x+h}.\end{aligned} \tag{4.5.5}$$

Substitution of (4.5.3), (4.5.4), and (4.5.5) into (4.5.2) yields

$$\bar{A}_x = \int_0^h v^t\ \frac{{}_tp_x\ \mu(x+t)}{{}_hq_x}\ dt\ {}_hq_x + v^h\ \bar{A}_{x+h}\ {}_hp_x. \tag{4.5.6}$$

Then, on both sides of (4.5.6), we multiply by -1, add $\bar{A}_{x+h}$, and divide by h to obtain

$$\frac{\bar{A}_{x+h} - \bar{A}_x}{h} = \frac{-1}{h} \int_0^h v^t \, {}_tp_x \, \mu(x + t) \, dt + \bar{A}_{x+h} \frac{1 - v^h \, {}_hp_x}{h}. \tag{4.5.7}$$

Next,

$$\lim_{h \to 0} \frac{1}{h} \int_0^h v^t \, {}_tp_x \, \mu(x + t) \, dt = \frac{d}{ds} \int_0^s v^t \, {}_tp_x \, \mu(x + t) \, dt\Big|_{s=0} = \mu(x)$$

and

$$\lim_{h \to 0} \frac{1 - v^h \, {}_hp_x}{h} = -\frac{d}{dt} (v^t \, {}_tp_x)\Big|_{t=0} = \mu(x) + \delta.$$

Using these two limits as $h \to 0$ in (4.5.7) we obtain (4.5.1),

$$\frac{d}{dx} \bar{A}_x = -\mu(x) + \bar{A}_x[\mu(x) + \delta].$$

Solutions of this differential equation are outlined in Exercise 4.22.

4.6 Notes and References

The life contingencies textbooks listed in Appendix 6 give other developments of formulas for life insurance actuarial present values. Commutation functions, which were fundamental to actuarial calculations until this quarter century, are employed extensively in Jordan (1967).

There is little material in these textbooks on the concept of the time-until-death of an insured as a random variable. Until recently, research and exposition on this concept has been called ***individual risk theory.*** Cramér (1930) gives a detailed exposition of the ideas up to that time. Kahn (1962) and Seal (1969) give concise bibliographical information on both research and expository papers over a 100-year span.

Since 1970 there has been interest in actuarial models that consider both the time-until-death and the investment-rate-of-return as random variables. This combination is discussed in Chapter 21.

Exercises

Assume, unless otherwise stated, that insurances are payable at the moment of death, and that the force of interest is a constant δ with i and d as the equivalent rates of interest and discount.

Section 4.2

4.1. If $\mu(x) = \mu$, a positive constant, for all $x > 0$, show that $\bar{A}_x = \mu/(\mu + \delta)$.

4.2. Let $\mu(x) = 1/(1 + x)$, for all $x > 0$.

a. Integrate by parts to show that

$$\bar{A}_x = 1 - \delta \int_0^\infty e^{-\delta t} \frac{1 + x}{1 + x + t} \, dt.$$

b. Use the expression in (a) to show that $d\bar{A}_x/dx < 0$ for all $x > 0$.

4.3. Show that $d\bar{A}_x/di = -v(\bar{I}\bar{A})_x$.

4.4. Show that the expressions for the variance of the present value of an n-year endowment insurance paying a unit benefit, as given by (4.2.10) and (4.2.13), are identical.

4.5. Let Z_1 and Z_2 be as defined for equation (4.2.11).
a. Show that $\lim_{n\to 0} \text{Cov}(Z_1, Z_2) = \lim_{n\to\infty} \text{Cov}(Z_1, Z_2) = 0$.
b. Develop an implicit equation for the term of the endowment for which $\text{Cov}(Z_1, Z_2)$ is minimized.
c. Develop a formula for the minimum in (b).
d. Simplify the formulas in (b) and (c) for the case when the force of mortality is a constant μ.

4.6. Assume mortality is described by $l_x = 100 - x$ for $0 \le x \le 100$ and that the force of interest is $\delta = 0.05$.
a. Calculate $\bar{A}^{1}_{40:\overline{25}|}$.
b. Determine the actuarial present value for a 25-year term insurance with benefit amount for death at time t, $b_t = e^{0.05t}$, for a person age 40 at policy issue.

4.7. Assuming De Moivre's survival function with $\omega = 100$ and $i = 0.10$, calculate
a. $\bar{A}^{1}_{30:\overline{10}|}$
b. The variance of the present value, at policy issue, of the benefit of the insurance in (a).

4.8. If $\delta_t = 0.2/(1 + 0.05t)$ and $l_x = 100 - x$ for $0 \le x \le 100$, calculate
a. For a whole life insurance issued at age x, the actuarial present value and the variance of the present value of the benefits
b. $(\bar{I}\bar{A})_x$.

4.9. a. Show that $\bar{A}_x$ is the moment generating function of T, the future lifetime of (x), evaluated at $-\delta$.
b. Show that if T has a gamma distribution with parameters α and β, then $\bar{A}_x = (1 + \delta/\beta)^{-\alpha}$.

4.10. Given $b_t = t$, $\mu_x(t) = \mu$, and $\delta_t = \delta$ for all $t > 0$, derive expressions for
a. $(\bar{I}\bar{A})_x = \text{E}[b_T v^T]$ b. $\text{Var}(b_T v^T)$.

4.11. The random variable Z is the present-value random variable for a whole life insurance of unit amount payable at the moment of death and issued to (x). If $\delta = 0.05$ and $\mu_x(t) = 0.01$:

a. Display the formula for the p.d.f. of Z.
b. Graph the p.d.f. of Z.
c. Calculate $\bar{A}_x = \mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

4.12. The random variable Z is the present-value random variable for an n-year endowment insurance as defined in Section 4.2.2. Exhibit the d.f. of Z in terms of the d.f. of T.

4.13. The random variable Z is defined as in Exercise 4.12. If $\delta = 0.05$, $\mu_x(t) = 0.01$, and $n = 20$:

a. Display the d.f. of Z.
b. Graph the d.f. of Z.
c. Calculate $\bar{A}_{x:\overline{n}|} = \mathrm{E}[Z]$ using the distribution of Z. [Hint: Consider using the complement of the d.f.]

Section 4.3

4.14. If $l_x = 100 - x$ for $0 \le x \le 100$ and $i = 0.05$, evaluate

a. $A_{40:\overline{25}|}$ b. $(IA)_{40}$.

4.15. Show that $A_{x:\overline{n}|} = A^{1}_{x:\overline{m}|} + v^m\, {}_mp_x\, A_{x+m:\overline{n-m}|}$ for $m < n$ and interpret the result in words.

4.16. If $A_x = 0.25$, $A_{x+20} = 0.40$, and $A_{x:\overline{20}|} = 0.55$, calculate

a. $A_{x:\overset{1}{\overline{20}|}}$ b. $A^{1}_{x:\overline{20}|}$.

4.17. a. Describe the benefits of the insurance with actuarial present value given by the symbol $(IA)_{x:\overline{m}|}$.

b. Express the actuarial present value of (a) in terms of the symbols given in Tables 4.2.1 and 4.3.1.

4.18. In Example 4.3.2, let the expected size of the fund k years after the agreement, and immediately after the payment of death claims, be denoted by E_k, where $E_0 = (100)(1{,}000\, A_{30}) = 10{,}248.35$.

a. Start with (4.3.10) and develop the forward recursion formula

$$E_k = 1.06E_{k-1} - 100{,}000_{k-1|}q_{30}.$$

b. Use the recursion formula developed in (a) to confirm that $E_5 = 12{,}762.58$.

Section 4.4

4.19. Consider the timescale measured in intervals of length $1/m$ where the unit is a year. Let a whole life insurance for a unit amount be payable at the end of the m-thly interval in which death occurs. Let k be the number of complete insurance years lived prior to death and let j be the number of complete m-ths of a year lived in the year of death.

a. What is the present-value function for this insurance?
b. Set up a formula analogous to (4.4.2) for the actuarial present value, $A_x^{(m)}$, for this insurance.
c. Show algebraically that, under the assumption of a uniform distribution of deaths over the insurance year of age,

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

4.20. Show, under the assumption of a constant force of mortality between integral ages, that

$$\bar{A}_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x \, \mu_x(k) \frac{i + q_{x+k}}{\delta + \mu_x(k)}$$

where $\mu_x(k) = -\log p_{x+k}$.

Section 4.5

4.21. a. Show that (4.2.6), for an aggregate mortality basis, can be rewritten as

$$\bar{A}_x = \frac{1}{{}_xp_0 \, v^x} \int_x^{\infty} v^y \, {}_yp_0 \, \mu(y) \, dy \qquad x \ge 0.$$

b. Differentiate the formula of (a) to establish (4.5.1),

$$\frac{d\bar{A}_x}{dx} = [\mu(x) + \delta]\bar{A}_x - \mu(x) \qquad x \ge 0.$$

c. Use the same technique to show

$$\frac{d\bar{A}^{1}_{x:\overline{n}|}}{dx} = [\mu(x) + \delta]\bar{A}^{1}_{x:\overline{n}|} + \mu(x + n)A_{x:\overline{n}|}^{\;\;\,1} - \mu(x) \qquad x \ge 0.$$

4.22. Solve the differential equation (4.5.1) as follows:
a. Use the integrating factor

$$\exp\left\{-\int_y^x [\delta + \mu(z)] \, dz\right\}$$

to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \exp\left\{-\int_y^x [\delta + \mu(z)] \, dz\right\} dx.$$

b. Use the integrating factor $e^{-\delta x}$ to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \, v^{x-y}(1 - \bar{A}_x) \, dx.$$

Miscellaneous

4.23. Display the actuarial present value of a Double Protection to Age 65 policy that provides a benefit of 2 in the event of death prior to age 65 and a benefit of 1 after age 65 in symbols of Table 4.3.1. Assume benefits are paid at the end of the year of death.

4.24. A policy is issued at age 20 with the following graded scale of death benefits payable at the moment of death.

Age	Death Benefits
20	1 000
21	2 000
22	4 000
23	6 000
24	8 000
25–40	10 000
41 and over	50 000

Calculate the actuarial present value on the basis of the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$. [Hint: A backward recursion formula for the actuarial present value will include a function $c(x)$ that incorporates the death benefit scale.]

4.25. a. Determine whether or not a constant increase in the force of mortality has the same effect on A_x as the same increase in the force interest.

b. Show that if the single probability of death q_{x+n} is increased to $q_{x+n} + c$, then A_x will be increased by

$$c\,v^{n+1}\ {}_np_x\,(1 - A_{x+n+1}).$$

4.26. The actuarial present value for a modified pure endowment of 1,000 issued at age x for n years is 700 with a death benefit equal to the actuarial present value in event of death during the n-year period and is 650 with no death benefit.

a. Calculate the actuarial present value for a modified pure endowment of 1,000 issued at age x for n years if $100k\%$ of the actuarial present value is to be paid at death during the period.

b. For the modified pure endowment in (a), express the variance of the present value at policy issue in terms of actuarial present values for pure endowments and term insurances.

4.27. An appliance manufacturer sells his product with a 5-year warranty promising the return of cash equal to the pro rata share of the initial purchase price for failure within 5 years. For example, if failure is reported 3-3/4 years following purchase, 25% of the purchase price will be returned. From statistical studies, the probability of failure of a new product during the first year is estimated to be 0.2, in each of the second, third, and fourth years 0.1, and in the fifth year 0.2.

a. Assuming that failures are reported uniformly within each year since purchase, determine the fraction of the purchase price that equals the actuarial present value for this warranty. Assume $i = 0.10$.

b. If the warranted return is the reduction on the purchase price of a new product with a 5-year warranty, would the answer to (a) change?

4.28. Assume that $T(x)$ has a p.d.f. given by

$$f_{T(x)}(t) = \frac{2}{10\sqrt{2\pi}} e^{-t^2/200} \qquad t > 0$$

and $\delta = 0.05$.

Show:

a. $\bar{A}_x = 2e^{0.125}[1 - \Phi(0.5)] = 0.6992$

b. ${}^2\bar{A}_x = 2e^{-0.5}[1 - \Phi(1)] = 0.5232$

c. $\mathrm{Var}(Z) = 0.0343$, where $Z = v^T$

d. $\xi_Z^{0.5} = 0.7076$ [Hint: Use Figure 4.2.1]

e. $v^{\mathring{e}_x} = 0.6710 < 0.6992 = \bar{A}_x$.

4.29. Generalize Exercise 4.28(e) by showing that if $\delta > 0$, then

$$v^{\mathring{e}_x} \le \bar{A}_x.$$

[Hint: Use Jensen's Inequality in Section 1.3 when $u''(x) > 0$.]

4.30. For a whole life insurance the benefit amount, b_t, is 0 or 1 for each $t \ge 0$. For calculations at force of interest δ_t:

a. Express the discount function in terms of δ_t.

b. Express the present-value random variable, Z, in terms of T.

c. Show that Z^j@force of interest δ_t equals Z@force of interest $j\delta_t$.

Computing Exercises

4.31. Augment your Illustrative Life Table to include input of a constant interest rate, i, and a payment frequency, m. It will be helpful for the equivalent rates δ, $d^{(m)}$, $i^{(m)}$, d, and v to be shown in your Illustrative Life Table output.

4.32. Develop a set of values of $\ddot{a}_{\overline{n}|}$, $n = 1, 2, \ldots, 100$ at $i = 0.06$ by using the forward recursion formula (3.5.20). [Hint: Review Exercise 3.25 or use $\ddot{a}_{\overline{n+1}|} = 1 + v\ddot{a}_{\overline{n}|}$.]

4.33. a. Using the backward recursion formula of (4.3.10) in your Illustrative Life Table and the appropriate starting value, calculate $1{,}000A_x$ for $x = 13$ to 140 for an interest rate of 0.06.

b. Compare your values to those in Appendix 2A.

4.34. a. Use (4.3.3) and your Illustrative Life to determine $A^{1}_{20:\overline{20}|}$ and ${}^2A^{1}_{20:\overline{20}|}$ at $i = 0.05$.

b. What is the variance for the present-value random variable for a 20-year term insurance with benefit amount 100,000 issued to (20)?

4.35. a. By an algebraic or probabilistic argument, verify the following backward recursion formula of an n-year term insurance with a unit benefit:

$$A^{1}_{x:\overline{n}|} = vq_x - v^{n+1}{}_{n}p_x\, q_{x+n} + vp_x\, A^{\ 1}_{x+1:\overline{n}|}.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with $i = 0.06$ to calculate the actuarial present value of a 10-year term insurance issued at ages $x = 13, \ldots, 130$.

4.36. a. Use recursion relation (g) at the end of Section 4.3 and your Illustrative Life Table to calculate $(IA)_{28}$ at $i = 0.06$.

b. Modify the recursion relation of part (a) to obtain one for $(I\bar{A})_x$ and determine a starting value for it.

c. Modify the recursion relation of part (b) to obtain one for $(\bar{I}\bar{A})_x$ and determine a starting value for it.

d. Make the recursion relations of parts (b) and (c) specific to the assumption of a uniform distribution of deaths over each year of age.

4.37. Use your Illustrative Life Table to verify the numerical solutions to parts (a) and (b) of Example 4.2.4. [Hint: Set $B = 0.00$ and $A = 0.04$ in your Makeham law parameters, $i = e^{0.10} - 1$, and use recursion formula (d) at the end of Section 4.3. Remember that the insurance in Example 4.2.4 is payable at the moment of death.]

4.38. a. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit endowment insurance to age y with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{y-x}|} = \bar{A}^{1}_{x:\overline{1}|} + vp_x\, \bar{A}_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ to calculate the actuarial present value of a unit benefit endowment insurance to age 65 with the death benefit payable at the moment of death for issue ages $x = 13, \ldots,$ 64.

d. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit n-year endowment insurance with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{1}|} + v^n\, {}_{n}p_x\, (1 - vp_{x+n} - \bar{A}^{\ 1}_{x+1:\overline{n}|}) + vp_x\, \bar{A}_{x+1:\overline{n}|}$$

$$x = 0, 1, \ldots, w - 1.$$

4.39. Let Z be the present-value random variable for a 100,000 unit 20-year endowment insurance with the death benefit payable at the moment of death. Use your Illustrative Life Table to calculate the mean and the variance of Z on the basis of a Makeham law with $A = 0.001$, $B = 0.00001$, $c = 1.10$, and $\delta = 0.05$.

5

LIFE ANNUITIES

5.1 Introduction

In the preceding chapter we studied payments contingent on death, as provided by various forms of life insurances. In this chapter we study payments contingent on survival, as provided by various forms of life annuities. A ***life annuity*** is a series of payments made continuously or at equal intervals (such as months, quarters, years) while a given life survives. It may be temporary, that is, limited to a given term of years, or it may be payable for the whole of life. The payment intervals may commence immediately or, alternatively, the annuity may be deferred. Payments may be due at the beginnings of the payment intervals (***annuities-due***) or at the ends of such intervals (***annuities-immediate***).

Through the study of ***annuities-certain*** in the theory of interest, the reader already has a knowledge of annuity terminology, notation, and theory. Life annuity theory is analogous but brings in survival as a condition for payment. This condition has been encountered in Chapter 4 in connection with pure endowments and the maturity payments under endowment insurances.

Life annuities play a major role in life insurance operations. As we see in the next chapter, life insurances are usually purchased by a life annuity of premiums rather than by a single premium. The amount payable at the time of claim may be converted through a settlement option into some form of life annuity for the beneficiary. Some types of life insurance carry this concept even further and, instead of featuring a lump sum payable on death, provide stated forms of income benefits. Thus, for example, there may be a monthly income payable to a surviving spouse or to a retired insured.

Annuities are even more central in pension systems. In fact, a retirement plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) by some form of temporary annuity of contributions during active service. The temporary annuity may consist of varying contributions, and valuation of it may take into account not only interest and mortality, but other factors such as salary increases and the termination of participation for reasons other than death.

Life annuities also have a role in disability and workers' compensation insurances. In the case of disability insurance, termination of the annuity benefit by reason of recovery of the disabled insured may need to be considered. For surviving spouse benefits under workers' compensation, remarriage may terminate the annuity.

We proceed in this chapter as we did in Chapter 4 and express the present value of benefits to be received by the annuitant as a function of T, the annuitant's future-lifetime random variable. It then will be possible to study properties of the distribution of this financial value random variable. The expectation, still called the actuarial present value, can be evaluated in an alternative way using either integration by parts or summation by parts depending, respectively, on whether a continuous or discrete set of payments is being evaluated. The results of this process have a useful interpretation and lead to an alternative method of obtaining actuarial present values called the ***current payment technique.***

As in the preceding chapter on life insurance, unless otherwise stated we assume a constant effective annual rate of interest i (or the equivalent constant force of interest δ). We also assume aggregate mortality for most of the development in this chapter and indicate those situations where a select mortality assumption makes a major difference.

In most applications of the theory developed in this chapter, annuity payments continue while a human life remains in a particular status. However, the possible applications of the theory are much wider. It may be applied to any set of periodic payments where the payments are not made with certainty. Examples of some of these applications are seen in later chapters dealing with multiple lives or multiple causes of decrement.

5.2 Continuous Life Annuities

We start with annuities payable continuously at the rate of 1 per year. This is of course an abstraction but makes use of familiar mathematical tools and as a practical matter closely approximates annuities payable on a monthly basis. A ***whole life annuity*** provides for payments until death. Hence, the present value of payments to be made is $Y = \bar{a}_{\overline{T}|}$ for all $T \geq 0$ where T is the future lifetime of (x). The distribution function of Y can be obtained from that for T as follows:

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr(\bar{a}_{\overline{T}|} \leq y) = \Pr(1 - v^T \leq \delta y) \\ &= \Pr(v^T \geq 1 - \delta y) = \Pr\left[T \leq \frac{-\log(1 - \delta y)}{\delta}\right] \\ &= F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right) \quad \text{for } 0 < y < \frac{1}{\delta}. \end{aligned} \tag{5.2.1}$$

From this we obtain the probability density function for Y as

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right)$$

$$= \frac{f_T([-\log(1 - \delta y)]/\delta)}{1 - \delta y} \quad \text{for } 0 < y < \frac{1}{\delta}. \tag{5.2.2}$$

The distribution function for Y depends on the distribution of T but would resemble that shown at the end of this section in Figure 5.2.1(a).

The actuarial present value for a continuous whole life annuity is denoted by $\bar{a}_x$ where the post fixed subscript, x, indicates that the annuity ceases at the death of (x) and that the distribution of $T(x)$ may depend on information available at age x. Under aggregate mortality the p.d.f. of T is ${}_tp_x\, \mu(x + t)$, and the actuarial present value can be calculated by

$$\bar{a}_x = \mathrm{E}[Y] = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x + t)\, dt. \tag{5.2.3}$$

Using integration-by-parts with $f(t) = \bar{a}_{\overline{t}|}$, $dg(t) = {}_tp_x\, \mu(x + t)\, dt$, $g(t) = -{}_tp_x$, and $df(t) = v^t\, dt$, we obtain

$$\bar{a}_x = \int_0^\infty v^t\, {}_tp_x\, dt = \int_0^\infty {}_tE_x\, dt. \tag{5.2.4}$$

This integral can be considered as involving a momentary payment of 1 dt made at time t, discounted at interest back to time zero by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t. This is the current payment form of the actuarial present value for the whole life annuity. In general, the current payment technique for determining an actuarial present value for an annuity gives

$$\mathrm{APV} = \int_0^\infty v^t\, \Pr[\text{payments are being made at time } t] \times [\text{Payment rate at time } t]\, dt. \tag{5.2.5}$$

Let us rewrite (5.2.4) by splitting off that portion of the integral involving t values from 0 to 1. Thus

$$\begin{aligned}\bar{a}_x &= \int_0^1 v^t\, {}_tp_x\, dt + \int_1^\infty v^t\, {}_tp_x\, dt \\ &= \bar{a}_{x:\overline{1}|} + v\, p_x \int_0^\infty v^s\, {}_sp_{x+1}\, ds \\ &= \bar{a}_{x:\overline{1}|} + v\, p_x\, \bar{a}_{x+1}.\end{aligned} \tag{5.2.6}$$

The actuarial present-value symbol, $\bar{a}_{x:\overline{1}|}$, used above is introduced below in (5.2.11). Expression (5.2.6) is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Section 4.3. Here $u(x) = \bar{a}_x$, $c(x) = \bar{a}_{x:\overline{1}|}$, and $d(x) = v\, p_x$. The initial value to use for the whole life annuity is $\bar{a}_\omega = 0$. There are several ways to evaluate the $c(x)$ term. A simple approach is to use a trapezoid approximation for the integral

$$\bar{a}_{x:\overline{1|}} = \int_0^1 v^t \, {}_tp_x \, dt = \frac{1 + v \, p_x}{2}.$$

Another approach, based on the assumption of a uniform distribution of deaths within each year of age, is examined in Section 5.4.

A relationship, familiar from compound interest theory, is that

$$1 = \delta \, \bar{a}_{\overline{t|}} + v^t.$$

This can be interpreted as indicating that a unit invested now will produce annual interest of δ payable continuously for t years at which point interest ceases and the investment is repaid. This relationship holds for all values of t and thus is true for the random variable T:

$$1 = \delta \, \bar{a}_{\overline{T|}} + v^T. \tag{5.2.7}$$

Then, taking expectations, we obtain

$$1 = \delta \, \bar{a}_x + \bar{A}_x. \tag{5.2.8}$$

This is subject to the same kind of interpretation as above. A unit invested now will produce annual interest of δ payable continuously for as long as (x) survives, and, at the time of death, interest ceases and the investment of 1 is repaid.

To measure, on the basis of the assumptions in our model, the mortality risk in a continuous life annuity, we are interested in $\mathrm{Var}(\bar{a}_{\overline{T|}})$. We determine

$$\begin{aligned}
\mathrm{Var}(\bar{a}_{\overline{T|}}) &= \mathrm{Var}\left(\frac{1 - v^T}{\delta}\right) \\
&= \frac{\mathrm{Var}(v^T)}{\delta^2} \\
&= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}.
\end{aligned} \tag{5.2.9}$$

Further, we can observe that since $1 = \delta \, \bar{a}_{\overline{T|}} + v^T$, $\mathrm{Var}(\delta \, \bar{a}_{\overline{T|}} + v^T) = 0$. Thus there is no mortality risk for the combination of a continuous life annuity of δ per year and a life insurance of 1 payable at the moment of death.

Example 5.2.1

Under the assumptions of a constant force of mortality, μ, and of a constant force of interest, δ, evaluate

a. $\bar{a}_x = \mathrm{E}[\bar{a}_{\overline{T|}}]$
b. $\mathrm{Var}(\bar{a}_{\overline{T|}})$
c. The probability that $\bar{a}_{\overline{T|}}$ will exceed $\bar{a}_x$.

Solution:

a. $\bar{a}_x = \displaystyle\int_0^\infty v^t \, {}_tp_x \, dt = \int_0^\infty e^{-\delta t} \, e^{-\mu t} \, dt = \frac{1}{\delta + \mu}.$

b. $\bar{A}_x = 1 - \delta\, \bar{a}_x = \dfrac{\mu}{\delta + \mu}$

$$^2\bar{A}_x = \frac{\mu}{2\delta + \mu} \quad \text{by the rule of moments}$$

$$\mathrm{Var}(\bar{a}_{\overline{T}|}) = \frac{1}{\delta^2}\left[\frac{\mu}{2\delta + \mu} - \left(\frac{\mu}{\delta + \mu}\right)^2\right] = \frac{\mu}{(2\delta + \mu)(\delta + \mu)^2}.$$

c. $$\Pr(\bar{a}_{\overline{T}|} > \bar{a}_x) = \Pr\left(\frac{1 - v^T}{\delta} > \bar{a}_x\right) = \Pr\left[T > -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)\right]$$
$$= {}_{t_0}p_x \qquad \text{where } t_0 = -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)$$
$$= \left(\frac{\mu}{\delta + \mu}\right)^{\mu/\delta}.$$

▼

We now turn to temporary and deferred life annuities. The present value of a benefits random variable for an ***n*-year temporary life annuity** of 1 per year, payable continuously while (x) survives during the next n years, is

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.10}$$

The distribution of Y in this case is a mixed distribution. In particular, the maximum value of Y is limited to $\bar{a}_{\overline{n}|}$, and there is a positive probability associated with $\bar{a}_{\overline{n}|}$ of $\Pr(T \ge n) = {}_np_x$. A typical distribution function for this random variable is illustrated in Figure 5.2.1(b).

The actuarial present value of an n-year temporary life annuity is denoted by $\bar{a}_{x:\overline{n}|}$ and equals

$$\bar{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x + t)\, dt + \bar{a}_{\overline{n}|}\, {}_np_x. \tag{5.2.11}$$

Integration by parts gives

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t\, {}_tp_x\, dt. \tag{5.2.12}$$

This is the current payment integral for the actuarial present value for the n-year temporary annuity. It can be considered as involving a momentary payment $1\, dt$ made at time t, discounted at interest back to time 0 by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t for times up to time n. No payments are to be made after time n so the probability of such payments is 0.

The same recursion formula as indicated for (5.2.6) applies here with $u(x) = \bar{a}_{x:\overline{y-x}|}$ and the same $c(x)$ function which we now recognize as $\bar{a}_{x:\overline{1}|}$. We use here $n = y - x$. The only thing that needs to be changed is the initial value, for which we use $u(y) = \bar{a}_{y:\overline{0}|} = 0$. Another form of a recursion formula for a temporary annuity with the n-year period fixed is examined in Exercise 5.7.

Returning to (5.2.10) we note that

$$Y = \begin{cases} \bar{a}_{\overline{T}|} = \dfrac{1 - Z}{\delta} & 0 \le T < n \\ \bar{a}_{\overline{n}|} = \dfrac{1 - Z}{\delta} & T \ge n \end{cases} \tag{5.2.13}$$

where

$$Z = \begin{cases} v^T & 0 \le T < n \\ v^n & T \ge n. \end{cases} \tag{5.2.14}$$

In (5.2.14), Z is the present-value random variable for an n-year endowment insurance. Hence

$$\mathrm{E}[Y] = \bar{a}_{x:\overline{n}|} = \mathrm{E}\left[\frac{1 - Z}{\delta}\right] = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta} \tag{5.2.15}$$

and

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{\delta^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}_{x:\overline{n}|}^{\,2}}{\delta^2}. \tag{5.2.16}$$

In terms of annuity values, (5.2.16) becomes

$$\mathrm{Var}(Y) = \frac{1 - 2\delta\, {}^2\bar{a}_{x:\overline{n}|} - (1 - \delta\, \bar{a}_{x:\overline{n}|})^2}{\delta^2}$$

$$= \frac{2}{\delta}\,(\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}) - (\bar{a}_{x:\overline{n}|})^2.$$

The analysis for an **n-*year deferred whole life annuity*** is similar. The present-value random variable Y is defined as

$$Y = \begin{cases} 0 \qquad\quad\; = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{T}|} & 0 \le T < n \\ v^n\, \bar{a}_{\overline{T-n}|} = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.17}$$

Here the random variable Y can take on a value no larger than $(1/\delta) - \bar{a}_{\overline{n}|} = v^n/\delta$, and the probability that it takes on a zero value is $\Pr(T \le n) = {}_nq_x$. A typical distribution function is illustrated in Figure 5.2.1(c).

Then,

$${}_{n|}\bar{a}_x = \mathrm{E}[Y] = \int_n^\infty v^n\, \bar{a}_{\overline{t-n}|}\, {}_tp_x\, \mu(x + t)\, dt$$

$$= \int_0^\infty v^n\, \bar{a}_{\overline{s}|}\, {}_{n+s}p_x\, \mu(x + n + s)\, ds$$

$$= v^n\, {}_np_x \int_0^\infty \bar{a}_{\overline{s}|}\, {}_sp_{x+n}\, \mu(x + n + s)\, ds$$

which shows that

$${}_{n|}\bar{a}_x = {}_nE_x\, \bar{a}_{x+n}. \tag{5.2.18}$$

An alternative development would be to note that, from the definitions of Y,

$$\begin{aligned}(Y \text{ for an } n\text{-year deferred whole life annuity}) &= (Y \text{ for a whole life annuity}) \\ &\quad - (Y \text{ for an } n\text{-year temporary life annuity}).\end{aligned}$$

Taking expectations gives

$$_{n|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|}. \tag{5.2.19}$$

Integration by parts can be employed to verify the result given by the current payment technique. Since the annuity will be paying after time n if x survives, the actuarial present value can be written as

$$_{n|}\bar{a}_x = \int_n^\infty v^t \, _tp_x \, dt = \int_n^\infty \, _tE_x \, dt. \tag{5.2.20}$$

To develop the backward recursion formula for deferred annuities with $n = y - x > 1$, we note that we have no term corresponding to the integral for t values between 0 and 1. Thus, for $u(x) = \, _{y-x|}\bar{a}_x$ at ages less than y, $c(x) = 0$, and $d(x) = v \, p_x$. For a starting value we would use $u(y) = \bar{a}_y$.

One way to calculate the variance of Y for the deferred annuity is the following:

$$\begin{aligned}\text{Var}(Y) &= \int_n^\infty v^{2n} \, (\bar{a}_{\overline{t-n}|})^2 \, _tp_x \, \mu(x + t) \, dt - (_{n|}\bar{a}_x)^2 \\ &= v^{2n} \, _np_x \int_0^\infty (\bar{a}_{\overline{s}|})^2 \, _sp_{x+n} \, \mu(x + n + s) \, ds - (_{n|}\bar{a}_x)^2,\end{aligned}$$

and using integration by parts,

$$\begin{aligned}&= v^{2n} \, _np_x \int_0^\infty 2\bar{a}_{\overline{s}|} \, v^s \, _sp_{x+n} \, ds - (_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} \, v^{2n} \, _np_x \int_0^\infty (v^s - v^{2s}) \, _sp_{x+n} \, ds - (_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} \, v^{2n} \, _np_x \, (\bar{a}_{x+n} - {}^2\bar{a}_{x+n}) - (_{n|}\bar{a}_x)^2.\end{aligned} \tag{5.2.21}$$

For an alternative development of this formula, see Exercise 5.37.

We now turn to analysis of an **n-*year certain and life annuity.*** This is a whole life annuity with a guarantee of payments for the first n years. The present value of annuity payments is

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & T \le n \\ \bar{a}_{\overline{T}|} & T > n. \end{cases} \tag{5.2.22}$$

A typical distribution function is shown in Figure 5.2.1(d), which reflects the mixed nature of the distribution and the minimum value and upper bound of Y, which are $\bar{a}_{\overline{n}|}$ and $1/\delta$, respectively.

The actuarial present value is denoted by $\bar{a}_{\overline{x:\overline{n}|}}$. This symbol is adopted to indicate that payments continue until max$[T(x), n]$:

$$\bar{a}_{\overline{x:\overline{n}|}} = \mathrm{E}[Y] = \int_0^n \bar{a}_{\overline{n}|}\ {}_tp_x\ \mu(x+t)\ dt$$

$$+ \int_n^\infty \bar{a}_{\overline{t}|}\ {}_tp_x\ \mu(x+t)\ dt$$

$$= {}_nq_x\ \bar{a}_{\overline{n}|} + \int_n^\infty \bar{a}_{\overline{t}|}\ {}_tp_x\ \mu(x+t)\ dt. \tag{5.2.23}$$

Integration by parts can be used to obtain

$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + \int_n^\infty v^t\ {}_tp_x\ dt. \tag{5.2.24}$$

This is the current payment form for the actuarial present value, since at times 0 to n payment is certain, whereas for times greater than n payment is made if (x) survives.

Further insight can be obtained by rewriting Y as

$$Y = \begin{cases} \bar{a}_{\overline{n}|} + 0 & T \le n \\ \bar{a}_{\overline{n}|} + (\bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}) & T > n. \end{cases}$$

Here Y is the sum of a constant $\bar{a}_{\overline{n}|}$ and the random variable for the n-year deferred annuity. Thus,

$$\begin{aligned} \bar{a}_{\overline{x:\overline{n}|}} &= \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x \\ &= \bar{a}_{\overline{n}|} + {}_nE_x\ \bar{a}_{x+n} && \text{by (5.2.18)} \\ &= \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|}) && \text{by (5.2.19).} \end{aligned} \tag{5.2.25}$$

Furthermore, since $\mathrm{Var}(Y - \bar{a}_{\overline{n}|}) = \mathrm{Var}(Y)$, the variance for the n-year certain and life annuity is the same as that of the n-year deferred annuity given by (5.2.21).

A backward recursion for $\bar{a}_{\overline{x:\overline{n}|}}$ with a fixed n-year certain period is examined in Exercise 5.9.

Analogous to the function

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t}\ dt$$

in the theory of interest, we have for life annuities

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x} = \int_0^n \frac{1}{{}_{n-t}E_{x+t}}\ dt, \tag{5.2.26}$$

representing the actuarial accumulated value at the end of the term of an n-year temporary life annuity of 1 per year payable continuously while (x) survives.

Such accumulated value, which is often said to have been accumulated under (or with the benefit of) interest and survivorship, is available at age $x + n$ if (x) survives.

We obtain an expression for $d\bar{a}_x/dx$ by differentiating the integral in (5.2.4), assuming that the probabilities are derived from an aggregate table:

$$\frac{d}{dx}\bar{a}_x = \int_0^\infty v^t \left(\frac{\partial}{\partial x}\,{}_tp_x\right) dt = \int_0^\infty v^t\,{}_tp_x\,[\mu(x) - \mu(x+t)]\,dt$$

$$= \mu(x)\,\bar{a}_x - \bar{A}_x = \mu(x)\,\bar{a}_x - (1 - \delta\,\bar{a}_x).$$

Therefore,

$$\frac{d}{dx}\bar{a}_x = [\mu(x) + \delta]\,\bar{a}_x - 1. \tag{5.2.27}$$

The interpretation of (5.2.27) is that the actuarial present value changes at a rate that is the sum of the rate of interest income $\delta\,\bar{a}_x$ and the rate of survivorship benefit $\mu(x)\,\bar{a}_x$, less the rate of payment outgo.

Example 5.2.2

Assuming that probabilities come from an aggregate table, obtain formulas for

$$\text{a. } \frac{\partial}{\partial x}\bar{a}_{x:\overline{n}|} \qquad \text{b. } \frac{\partial}{\partial n}\,{}_{n|}\bar{a}_x.$$

Solution:

a. Proceeding as in the development of (5.2.27), we obtain

$$\frac{\partial}{\partial x}\bar{a}_{x:\overline{n}|} = \mu(x)\,\bar{a}_{x:\overline{n}|} - \bar{A}^{1}_{x:\overline{n}|}$$

$$= \mu(x)\,\bar{a}_{x:\overline{n}|} - (1 - \delta\,\bar{a}_{x:\overline{n}|} - {}_nE_x)$$

$$= [\mu(x) + \delta]\,\bar{a}_{x:\overline{n}|} - (1 - {}_nE_x).$$

b. $\dfrac{\partial}{\partial n}\,{}_{n|}\bar{a}_x = \dfrac{\partial}{\partial n}\displaystyle\int_n^\infty v^t\,{}_tp_x\,dt = -v^n\,{}_np_x.$ ▼

Table 5.2.1 summarizes concepts for continuous life annuities.

Figure 5.2.1 shows typical distribution functions for the several types of continuous life annuities studied in this section. Limiting values and points of discontinuities are indicated on one or both axes.

When $F_Y(0) = 0$, $E[Y] = \int_0^\infty [1 - F_Y(y)]\,dy$, the actuarial present value of Y can be visualized as the area above the graph of $z = F_Y(y)$, below $z = 1$, and to the right of the line $y = 0$. This interpretation can provide a bridge between the actuarial

TABLE 5.2.1

Summary of Continuous Life Annuities (Annuity of 1 per Annum Payable Continuously)

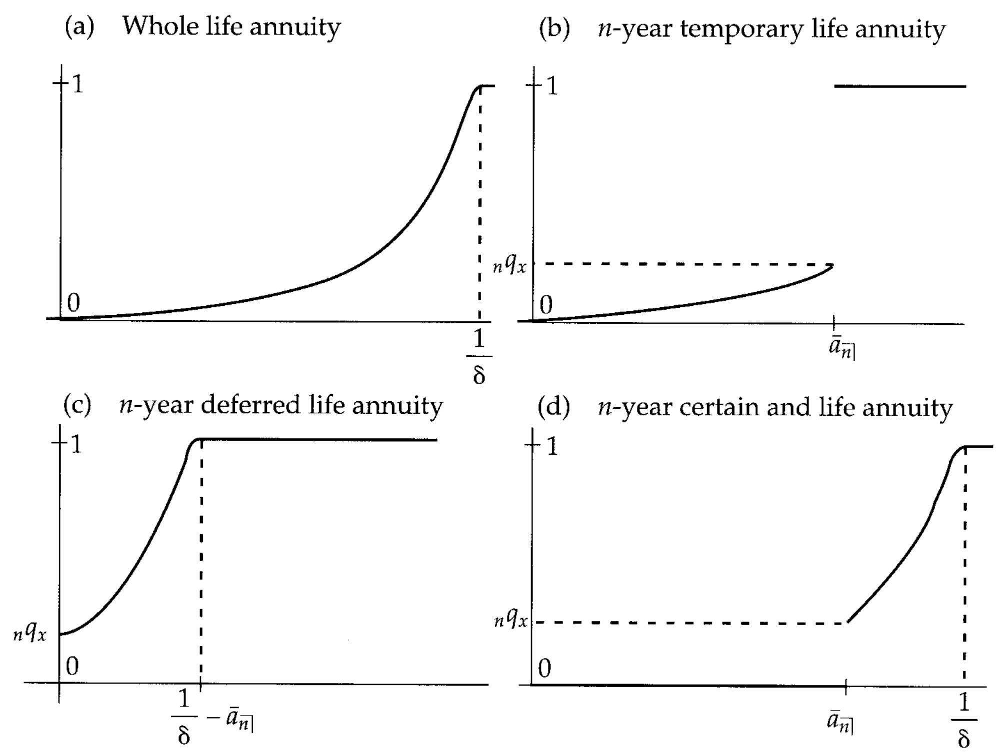
Annuity Name	Present-Value Random Variable Y	Actuarial Present Value E[Y] Equal to
Whole Life Annuity	$\bar{a}_{\overline{T}\rvert} \quad T \ge 0$	$\bar{a}_x = \int_0^\infty v^t\,{}_tp_x\,dt$
n-Year Temporary Life Annuity	$\begin{cases} \bar{a}_{\overline{T}\rvert} & 0 \le T < n \\ \bar{a}_{\overline{n}\rvert} & T \ge n \end{cases}$	$\bar{a}_{x:\overline{n}\rvert} = \int_0^n v^t\,{}_tp_x\,dt$
n-Year Deferred Whole Life Annuity	$\begin{cases} 0 & 0 \le T < n \\ \bar{a}_{\overline{T}\rvert} - \bar{a}_{\overline{n}\rvert} & T \ge n \end{cases}$	${}_{n\rvert}\bar{a}_x = \int_n^\infty v^t\,{}_tp_x\,dt$
n-Year Certain and Life Annuity	$\begin{cases} \bar{a}_{\overline{n}\rvert} & 0 \le T < n \\ \bar{a}_{\overline{T}\rvert} & T \ge n \end{cases}$	$\bar{a}_{\overline{x:\overline{n}\rvert}} = \bar{a}_{\overline{n}\rvert} + \int_n^\infty v^t\,{}_tp_x\,dt$

Additional relations are

- $1 = \delta\,\bar{a}_x + \bar{A}_x$
- $1 = \delta\,\bar{a}_{x:\overline{n}\rvert} + \bar{A}_{x:\overline{n}\rvert}$
- ${}_{n\rvert}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}\rvert}$
- $\bar{s}_{x:\overline{n}\rvert} = \dfrac{\bar{a}_{x:\overline{n}\rvert}}{{}_nE_x} = \int_0^n (1+i)^{n-t}\,\dfrac{l_{x+t}}{l_{x+n}}\,dt.$

FIGURE 5.2.1

Typical Distribution Functions for the Present-Value Random Variables, *Y*



present value as evaluated from the definition of the random variable and the current payment form for the actuarial present value.

5.3 Discrete Life Annuities

The theory of discrete life annuities is analogous, step-by-step, to the theory of continuous life annuities, with integrals replaced by sums, integrands by summands, and differentials by differences. For continuous annuities there was no distinction between payments at the beginning of payment intervals or at the ends, that is, between annuities-due and annuities-immediate. For discrete annuities, the distinction is meaningful, and we start with annuities-due as they have a more prominent role in actuarial applications. For example, most individual life insurances are purchased by an annuity-due of periodic premiums.

We consider an annuity that pays a unit amount at the beginning of each year that the annuitant (x) survives. In the nomenclature this is called a ***whole life annuity-due.*** The present-value random variable, Y, for such an annuity, is given by $Y = \ddot{a}_{\overline{K+1}|}$ where the random variable K is the curtate-future-lifetime of (x). The possible values of this random variable are a discrete set of values ranging from $\ddot{a}_{\overline{1}|} = 1$ to $\ddot{a}_{\overline{\omega-x}|}$, a value which is less than $1/d$. The probability associated with the value $\ddot{a}_{\overline{k+1}|}$ is $\Pr(K = k) = {}_kp_x\, q_{x+k}$.

Let us now consider $\ddot{a}_x$, the actuarial present value of the annuity:

$$\ddot{a}_x = \mathrm{E}[Y] = \mathrm{E}[\ddot{a}_{\overline{K+1}|}]$$

$$= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_kp_x\, q_{x+k}, \tag{5.3.1}$$

since $\Pr(K = k) = {}_kp_x\, q_{x+k}$. By summation-by-parts (see Appendix 5) with $\Delta f(k) = {}_kp_x\, q_{x+k} = {}_kp_x - {}_{k+1}p_x$ and $g(k) = \ddot{a}_{\overline{k+1}|}$ and use of the relations

$$\Delta\, g(k) = \Delta\, \ddot{a}_{\overline{k+1}|} = v^{k+1} \text{ and } f(k) = -{}_kp_x,$$

(5.3.1) converts to

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1}\, {}_{k+1}p_x \tag{5.3.2}$$

$$= \sum_{k=0}^{\infty} v^k\, {}_kp_x. \tag{5.3.3}$$

The expression (5.3.3) is the current payment form of the actuarial present value for a whole life annuity-due where the ${}_kp_x$ term is the probability of a payment of size 1 being made at time k.

Starting with the sum in (5.3.2) above we have

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1} \; {}_{k+1}p_x = 1 + v \; p_x \sum_{k=0}^{\infty} v^k \; {}_kp_{x+1}$$

$$= 1 + v \; p_x \, \ddot{a}_{x+1}. \tag{5.3.4}$$

This expression is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Sections 4.3 and 5.2. Here $u(x) = \ddot{a}_x$, $c(x) = 1$, and $d(x) = v \; p_x$. The initial value to use for the whole life annuity is $\ddot{a}_\omega = 0$.

From (5.3.1) we obtain in succession

$$\ddot{a}_x = \mathrm{E}\left[\frac{1 - v^{K+1}}{d}\right]$$

$$= \frac{1 - A_x}{d}, \tag{5.3.5}$$

and

$$\ddot{a}_x = \ddot{a}_{\overline{\infty}|} - \ddot{a}_{\overline{\infty}|} \, A_x, \tag{5.3.6}$$

$$1 = d\ddot{a}_x + A_x. \tag{5.3.7}$$

This should be compared with its continuous counterpart (5.2.8). Formula (5.3.7) indicates that a unit invested now will produce interest-in-advance of d per year while (x) survives plus repayment of the unit at the end of the year of death of (x).

The variance formula is

$$\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \mathrm{Var}\left(\frac{1 - v^{K+1}}{d}\right) = \frac{\mathrm{Var}(v^{K+1})}{d^2}$$

$$= \frac{{}^2A_x - (A_x)^2}{d^2}; \tag{5.3.8}$$

see (5.2.9).

The present-value random variable of an **n-*year temporary life annuity-due*** of 1 per year is

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \le K < n \\ \ddot{a}_{\overline{n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\ddot{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \; {}_kp_x \; q_{x+k} + \ddot{a}_{\overline{n}|} \; {}_np_x. \tag{5.3.9}$$

Summation by parts can be used to transform (5.3.9) into

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \; {}_kp_x, \tag{5.3.10}$$

which is the actuarial present value in the current payment form.

Again separating out the first term and factoring out vp_x, we obtain the backward recursion formula for a temporary annuity-due payable to age $y = x + n$:

$$\ddot{a}_{x:\overline{y-x}|} = 1 + vp_x\, \ddot{a}_{x+1:\overline{y-(x+1)}|}. \tag{5.3.11}$$

This recursive formula for the actuarial present values is the same as (5.3.4) but differs in that here we use an initial value of $\ddot{a}_{y:\overline{0}|} = 0$.

Since $Y = (1 - Z)/d$, where

$$Z = \begin{cases} v^{K+1} & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

is the present-value random variable for a unit of endowment insurance, payable at the end of the year of death or at maturity, we have

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - \mathrm{E}[Z]}{d} = \frac{1 - A_{x:\overline{n}|}}{d}; \tag{5.3.12}$$

see (5.2.15).

Rearrangement of (5.3.12) yields

$$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}. \tag{5.3.13}$$

To calculate the variance, we can use

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{d^2} = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}. \tag{5.3.14}$$

For an **n-*year deferred whole life annuity-due*** of 1 payable at the beginning of each year while (x) survives from age $x + n$ onward, the present-value random variable is

$$Y = \begin{cases} 0 & 0 \le K < n \\ {}_{n|}\ddot{a}_{\overline{K+1-n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\mathrm{E}[Y] = {}_{n|}\ddot{a}_x = {}_nE_x\, \ddot{a}_{x+n} \tag{5.3.15}$$

$$= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \tag{5.3.16}$$

$$= \sum_{k=n}^{\infty} v^k\, {}_kp_x; \tag{5.3.17}$$

see (5.2.18)–(5.2.20).

The backward recursion formula for a deferred annuity-due with $n = y - x > 1$ is identical to that for the continuous version in that it uses $c(x) = 0$ and $d(x) = vp_x$. The change is that we use the actuarial present value for an annuity-due, $u(y) = \ddot{a}_y$, for the starting value.

The variance of Y can be developed in a manner completely analogous to that used in formula (5.2.21) and leads to the expression

$$\mathrm{Var}(Y) = \frac{2}{d} v^{2n} {}_np_x (\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n}) + {}_{n|}{}^2\ddot{a}_x - ({}_{n|}\ddot{a}_x)^2. \tag{5.3.18}$$

We turn now to analysis of an **n-*year certain and life annuity-due.*** This is a life annuity with a guarantee of payments for at least n years. The present value of the annuity payments is

$$Y = \begin{matrix} \ddot{a}_{\overline{n}|} & 0 \le K < n \\ \ddot{a}_{\overline{K+1}|} & K \ge n. \end{matrix} \tag{5.3.19}$$

Then

$$\ddot{a}_{\overline{\overline{x:n}}|} = \mathrm{E}[Y] = \ddot{a}_{\overline{n}|}\, {}_nq_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_kp_x\, q_{x+k}, \tag{5.3.20}$$

and this can be transformed by summation by parts into the current payment version of the actuarial present value

$$\ddot{a}_{\overline{\overline{x:n}}|} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k\, {}_kp_x. \tag{5.3.21}$$

This can be written as

$$\ddot{a}_{\overline{\overline{x:n}}|} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}.$$

The actuarial accumulated value at the end of the term of an n-year temporary life annuity-due of 1 per year, payable while (x) survives, is denoted by $\ddot{s}_{x:\overline{n}|}$. Formulas for this function are

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}, \tag{5.3.22}$$

which are analogous to formulas for $\ddot{s}_{\overline{n}|}$ in the theory of interest.

The procedures used above for annuities-due can be adapted for annuities-immediate where payments are made at the ends of the payment periods. For instance, for a ***whole life annuity-immediate,*** the present-value random variable is $Y = a_{\overline{K}|}$. Then,

$$a_x = \mathrm{E}[Y] = \sum_{k=0}^{\infty} {}_kp_x\, q_{x+k}\, a_{\overline{k}|}, \tag{5.3.23}$$

and a summation by parts will give the current payment form of the actuarial present value as

$$a_x = \sum_{k=1}^{\infty} v^k\, {}_kp_x. \tag{5.3.24}$$

Since Y equals $(1 - v^K)/i = [1 - (1 + i)\, v^{K+1}]/i$, we have, taking expectations,

$$a_x = \mathrm{E}[Y] = \frac{1 - (1 + i)\, A_x}{i}. \tag{5.3.25}$$

This formula can be rewritten as $1 = ia_x + (1 + i)\, A_x$. A comparison of this formula with (5.3.7) shows that an interest payment of i is made at the end of each year while (x) remains alive and that at the end of the year of death an interest payment of i along with the principal amount of 1 must be paid. This formula has significance for estate taxation. For each unit of an estate, define ia_x as the ***life estate*** and $(1 + i)\,A_x = 1 - ia_x$ as the ***remainder,*** which, if designated for a qualified charitable organization, is exempt from estate taxation.

The analysis for the other forms of the annuity-immediate is similar. The present-value random variable can be formed in a manner analogous to that for the annuity-due. Formulas for the actuarial present value from the definition and by summation by parts can be obtained. Formulas for the variances of the annuities-immediate in this section can also be obtained.

Example 5.3.1

Find formulas for the expectation and variance of the present-value random variable for the temporary life annuity-immediate.

Solution:

We start with the present-value random variable for an n-year temporary annuity-immediate:

$$Y = \begin{cases} a_{\overline{K}|} = \dfrac{1 - v^K}{i} & 0 \le K < n \\[2ex] a_{\overline{n}|} = \dfrac{1 - v^n}{i} & K \ge n. \end{cases}$$

We now introduce two new random variables

$$Z_1 = \begin{cases} (1 + i)v^{K+1} & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

and

$$Z_2 = \begin{cases} 0 & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

so that $Y = (1 - Z_1 - Z_2)/i$ for all K. Now taking expectations, we have

$$\mathrm{E}[Y] = a_{x:\overline{n}|} = \frac{1 - (1 + i)\,A^{1}_{x:\overline{n}|} - A_{x:\overset{\,1}{\overline{n}|}}}{i}.$$

This can be rewritten, following (5.3.13),

$$1 = i\,a_{x:\overline{n}|} + i\,A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}.$$

The variance calculation is as follows:

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z_1 + Z_2)}{i^2} = \frac{\mathrm{Var}(Z_1) + 2\,\mathrm{Cov}(Z_1,Z_2) + \mathrm{Var}(Z_2)}{i^2}.$$

Recall $\mathrm{Var}(Z_1) = (1 + i)^2 \, [{}^2A^{\,1}_{x:\overline{n}|} - (A^{\,1}_{x:\overline{n}|})^2]$ and $\mathrm{Var}(Z_2) = v^{2n} \, {}_np_x \, (1 - {}_np_x)$. Since $Z_1 \, Z_2 = 0$ for all K, we have $\mathrm{Cov}(Z_1, Z_2) = -(1 + i) \, A^{\,1}_{x:\overline{n}|} \, v^n \, {}_np_x$. Combining, we obtain

$$\mathrm{Var}(Y) = \frac{(1 + i)^2 \, [{}^2A^{\,1}_{x:\overline{n}|} - (A^{\,1}_{x:\overline{n}|})^2] - 2(1 + i) \, A^{\,1}_{x:\overline{n}|} \, v^n {}_np_x + v^{2n} {}_np_x \, (1 - {}_np_x)}{i^2}.$$

▼

TABLE 5.3.1

Summary of Discrete Life Annuities [Annuity of 1 Per Annum Payable at the Beginning of Each Year (Annuity-Due) or at the End of Each Year (Annuity-Immediate)]

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value E[Y] Equal to
Whole Life Annuity		
• Due	$\ddot{a}_{\overline{K+1}|} \quad K \ge 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$
• Immediate	$a_{\overline{K}|} \quad K \ge 0$	$a_x = \sum_{k=1}^{\infty} v^k \, {}_kp_x$
n-Year Temporary Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \le K < n \\ \ddot{a}_{\overline{n}|} & K \ge n \end{cases}$	$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{K}|} & 0 \le K < n \\ a_{\overline{n}|} & K \ge n \end{cases}$	$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_x$
n-Year Deferred Whole Life Annuity		
• Due	$\begin{cases} 0 & 0 \le K < n \\ \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|} & K \ge n \end{cases}$	${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} 0 & 0 \le K < n \\ a_{\overline{K}|} - a_{\overline{n}|} & K \ge n \end{cases}$	${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$
n-Year Certain and Whole Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{n}|} & 0 \le K < n \\ \ddot{a}_{\overline{K+1}|} & K \ge n \end{cases}$	$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{n}|} & 0 \le K < n \\ a_{\overline{K}|} & K \ge n \end{cases}$	$a_{\overline{x:\overline{n}|}} = a_{\overline{n}|} + \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$

Additional relations are

- $1 = d \, \ddot{a}_x + A_x$
- $A_x = v \, \ddot{a}_x - a_x$
- $1 = d \, \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$
- $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$
- $A^{\,1}_{x:\overline{n}|} = v \, \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$
- $A_{x:\overline{n}|} = v \, \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$
- ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{s}_{x:\overline{n}|} = \dfrac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \sum_{k=0}^{n-1} (1 + i)^{n-k} \dfrac{l_{x+k}}{l_{x+n}}.$

5.4 Life Annuities with m-thly Payments

In practice, life annuities are often payable on a monthly, quarterly, or semiannual basis. In International Actuarial Notation, the actuarial present value of a life annuity of 1 per year, payable in installments of 1 / m at the beginning of each m-th of the year while (x) survives, is denoted by $\ddot{a}_x^{(m)}$.

We start the analysis of the distribution of Y, the present value of the life annuity-due, with payments made on an m-thly basis, by expressing Y in terms of the interest rate and the random variables, K and $J = \lfloor (T - K)m \rfloor$. The "$\lfloor\ \rfloor$" in the expression for J denote the greatest integer function so that J is the number of complete m-ths of a year lived in the year of death. For an annuity-due there would be m payments for each of the K complete years and then $J + 1$ payments of 1 / m in the year of death; thus,

$$Y = \sum_{j=0}^{mK+J} \frac{1}{m} v^{j/m} = \ddot{a}^{(m)}_{\overline{K+(J+1)/m}|} = \frac{1 - v^{K+(J+1)/m}}{d^{(m)}}. \tag{5.4.1}$$

The actuarial present value, E[Y], which can be determined using Exercise 4.19, is

$$\mathrm{E}[Y] = \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}. \tag{5.4.2}$$

The current payment form, which is the sum of the actuarial present value of the set of payments, is

$$\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{h=0}^{\infty} v^{h/m}\ {}_{h/m}p_x. \tag{5.4.3}$$

Again using (5.4.1), we obtain

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(v^{K+(J+1)/m})}{(d^{(m)})^2} = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}. \tag{5.4.4}$$

It is convenient to use (5.3.7) and (5.4.2) to obtain various relationships between the actuarial present values for m-thly annuities and those with annual payments:

$$1 = d\, \ddot{a}_x + A_x = d^{(m)}\, \ddot{a}_x^{(m)} + A_x^{(m)}. \tag{5.4.5}$$

These show that an investment of 1 will produce interest-in-advance at the beginning of each interest period and repayment of the unit at the end of the period in which death occurs.

From the right two members of (5.4.5) we obtain

$$\begin{aligned}\ddot{a}_x^{(m)} &= \frac{d}{d^{(m)}}\, \ddot{a}_x - \frac{1}{d^{(m)}}\, (A_x^{(m)} - A_x) \\ &= \ddot{a}^{(m)}_{\overline{1}|}\, \ddot{a}_x - \ddot{a}^{(m)}_{\overline{\infty}|}\, (A_x^{(m)} - A_x).\end{aligned} \tag{5.4.6}$$

This can be interpreted as follows: The m-thly payment life annuity is equivalent to a series of 1-year annuities-certain in each year that (x) begins, with cancellation in the year of death of installments payable beyond the m-th (month, quarter, half-year) of death. The cancellation is accomplished by an m-thly payment perpetuity beginning at the end of the m-th of death less a similar perpetuity beginning at the end of the year of death. Alternatively, from (5.4.2), we can write

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} A_x^{(m)}, \tag{5.4.7}$$

which is left to the reader to interpret.

Remark:

In the study of interest theory the calculation of the present value of an annuity with payment periods and effective interest periods of different lengths was reduced to the calculation of the present value of an annuity with payment periods and interest periods of equal length in one of two ways. The first was to replace the payments corresponding to an interest period by a single equivalent payment (at the given interest rate) at one end or the other of the interest period. The expression for the m-thly whole life annuity in (5.4.6) is an extension of this method to a set of contingent payments. When calculators with exponentiation keys replaced interest tables, the second method, using the equivalent effective interest rate per payment period, became the preferred way to match payment period and interest period lengths. The extension of this second method to m-thly whole life annuities would be to use an m-thly mortality table along with the equivalent effective interest rate per m-th of a year. Using this, the recursion relations of Section 5.3 could be used to obtain the actuarial present value of the m-thly whole life annuity. An advantage of this second approach is that it frees us to use other assumptions about the distribution of deaths within each year of age, like constant force or Makeham's, in place of the uniform distribution that is central in the discussion below.

Now let us assume that deaths have a uniform distribution in each year of age. This means that S has a uniform distribution on (0, 1) so that J is uniformly distributed on the integers $\{0, 1, \ldots, m-1\}$. Exercise 4.19 shows that this implies

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x = s_{\overline{1}|}^{(m)} A_x,$$

and from (5.4.6) we have

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} A_x. \tag{5.4.8}$$

Formula (5.4.8) shows that the value of the m-thly life annuity-due is the difference between

a. The value of an annual life annuity-due with each annual payment sufficient to pay a 1-year annuity certain of $1/m$ at the beginning of each m-th; and

b. The value of an insurance payable at the end of the year of death with the benefit equal to the coefficient of A_x. It can be shown that, under the assumption of uniform distribution of deaths in each year of age, this coefficient is the actuarial accumulated value of those payments of $1/m$ for the m-ths after death. See the bracketed expression in Exercise 5.15.

By substituting $1 - d\,\ddot{a}_x$ for A_x, in (5.4.8) and noting that $d^{(m)}\,\ddot{a}^{(m)}_{\overline{1}|} = d$, we obtain a formula involving only annuity functions, namely,

$$\ddot{a}^{(m)}_x = \frac{1 - s^{(m)}_{\overline{1}|}\,(1 - d\,\ddot{a}_x)}{d^{(m)}}$$

$$= s^{(m)}_{\overline{1}|}\,\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \frac{(s^{(m)}_{\overline{1}|} - 1)}{d^{(m)}}. \tag{5.4.9}$$

An alternative, widely used, formula is

$$\ddot{a}^{(m)}_x = \ddot{a}_x - \frac{m-1}{2m}. \tag{5.4.10}$$

This result can be obtained by assuming that the function $v^{k+(j/m)}\,{}_{k+(j/m)}p_x$ is linear in j for $j = 0, 1, 2, \ldots, m-1$. In that case,

$$\sum_{j=0}^{m-1} \frac{1}{m}\,v^{k+(j/m)}\,{}_{k+(j/m)}p_x = \sum_{j=0}^{m-1} \frac{1}{m}\left[\left(1 - \frac{j}{m}\right) v^k\,{}_kp_x + \frac{j}{m}\,v^{k+1}\,{}_{k+1}p_x\right]$$

$$= v^k\,{}_kp_x - (v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x) \sum_{j=0}^{m-1} \frac{j}{m^2}$$

$$= v^k\,{}_kp_x - \frac{m-1}{2m}\,(v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x).$$

Thus

$$\ddot{a}^{(m)}_x = \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} \frac{1}{m}\,v^{k+(j/m)}\,{}_{k+(j/m)}p_x$$

$$= \sum_{k=0}^{\infty} v^k\,{}_kp_x - \frac{m-1}{2m} \sum_{k=0}^{\infty} (v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x)$$

$$= \ddot{a}_x - \frac{m-1}{2m}.$$

Note that this is not the same assumption as that of linearity of ${}_tp_x$, which would follow if a uniform distribution of deaths within each year of age is assumed. Consistent use of the assumption of a uniform distribution of deaths in each year of age assures that relations such as (5.4.5) are satisfied exactly. It has also been observed that formulas derived from (5.4.10) can, for high rates of interest and low rates of mortality, produce distorted annuity values, such as $a^{(m)}_{x:\overline{1}|} > a^{(m)}_{\overline{1}|}$; see Exercise 5.50. For these reasons, (5.4.8) and equivalently (5.4.9) are presented as replacements for the widely used formula (5.4.10).

It is convenient for writing purposes to express (5.4.9) in the form

$$\ddot{a}^{(m)}_x = \alpha(m)\,\ddot{a}_x - \beta(m), \tag{5.4.11}$$

where

$$\alpha(m) = s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} = \frac{id}{i^{(m)}\, d^{(m)}}, \tag{5.4.12}$$

and

$$\beta(m) = \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)}\, d^{(m)}}. \tag{5.4.13}$$

We note that $\alpha(m)$ and $\beta(m)$ depend only on m and the rate of interest and are independent of the year of age. Further, for $m = 1$, (5.4.11) is an identity where $\alpha(1) = 1$ and $\beta(1) = 0$. Also, $\beta(m)$ is the coefficient of the cancellation term in (5.4.8); that is, (5.4.8) can be written as

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_x - \beta(m) A_x. \tag{5.4.14}$$

For series expansions of $\alpha(m)$ and $\beta(m)$, see Exercise 5.41.

Example 5.4.1

On the basis of the Illustrative Life Table, with interest at the effective annual rate of 6%, calculate the actuarial present value of a whole life annuity-due of 1,000 per month for a retiree age 65 by (5.4.9) and (5.4.10) and its standard deviation by (5.4.4).

Solution:

Here

$$\alpha(12) = s_{\overline{1}|}^{(12)}\, \ddot{a}_{\overline{1}|}^{(12)} = (1.02721070)(0.97378368) = 1.0002810,$$

$$\beta(12) = \frac{s_{\overline{1}|}^{(12)} - 1}{d^{(12)}} = 0.46811951,$$

$$\frac{11}{24} = 0.45833333.$$

Observe that $\alpha(12) \cong 1$, and $\beta(12)$ is fairly close to the $11/24$ that appears in the traditional approximation.

By the Illustrative Life Table, as defined by (3.7.1), with interest at 6%,

$$\ddot{a}_{65} = 9.89693,$$

$$A_{65} = 1 - d\, \ddot{a}_{65} = 0.4397965.$$

Then, $12{,}000\ddot{a}_{65}^{(12)}$ can be calculated as follows:

by (5.4.11) $\quad 12{,}000[\alpha(12)\ddot{a}_{65} - \beta(12)]$

$$= 12{,}000[(1.0002810)(9.89693) - 0.46811951]$$

$$= 113{,}179 \text{ and}$$

by (5.4.10) $\quad 12{,}000\left(\ddot{a}_{65} - \dfrac{11}{24}\right) = 113{,}263.$

The variance of $12{,}000Y = 12{,}000(1 - v^{K+(J+1)/12})/d^{(12)}$ is equal to

$$\left(\frac{12{,}000}{d^{(12)}}\right)^2 \mathrm{Var}[v^{K+1}(1+i)^{1-(J+1)/12}]$$

$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \{\mathrm{E}[v^{2(K+1)}(1+i)^{2(1-(J+1)/12)}] - (\mathrm{E}[v^{(K+1)}(1+i)^{(1-(J+1)/12)}])^2\}$$

$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \left\{{}^2A_{65}\,\mathrm{E}[(1+i)^{2(1-(J+1)/12)}] - \left(\frac{A_{65}\,i}{i^{(12)}}\right)^2\right\}$$

$$= (206{,}442.14)^2\,[(0.2360299 \times 1.055458268) - (0.4397965 \times 1.02721069)^2]$$

$$= 1{,}919{,}074{,}762.$$

This means that the standard deviation for the present value for the payments to an individual is 43,807 as compared to the actuarial present value of 113,179. ▼

The development starting with random variables can be followed for m-thly temporary and deferred annuities-due. However, if all we seek is formulas for their actuarial present values, we can proceed by starting with (5.4.14). Thus

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_{x} - {}_nE_x\,\ddot{a}^{(m)}_{x+n}\\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \beta(m)A_x - {}_nE_x[\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x+n} - \beta(m)A_{x+n}]\\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x:\overline{n}|} - \beta(m)A^{1}_{x:\overline{n}|};\end{aligned} \tag{5.4.15}$$

and similarly,

$${}_{n|}\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{\overline{1}|}\;{}_{n|}\ddot{a}_x - \beta(m)\;{}_{n|}A_x. \tag{5.4.16}$$

Alternately from (5.4.11),

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x), \tag{5.4.17}$$

$${}_{n|}\ddot{a}^{(m)}_x = \alpha(m)\;{}_{n|}\ddot{a}_x - \beta(m)\;{}_nE_x. \tag{5.4.18}$$

Backward recursion formulas can be developed directly for m-thly life annuities, and the reader is asked to do this in Exercise 5.16 for an m-thly annuity-due. A more direct approach, however, would be to use the recursions of Section 5.3 and then adjust from annual to m-thly life annuities by means of (5.4.11), (5.4.17), and (5.4.18) or the equivalent formulas (5.4.14), (5.4.15), and (5.4.16).

The distribution of the present value of the payments of a life annuity-immediate with m-thly payments can be explored in steps analogous with those for the annuity-due. For example, the present-value random variable, Y, would be

$$Y = a^{(m)}_{\overline{K+(J/m)}|} = \frac{1 - v^{K+(J/m)}}{i^{(m)}} \tag{5.4.19}$$

for the whole life annuity-immediate with m-thly payments. This exploration leads to the following formula analogous to (5.4.5):

$$1 = i\,a_x + (1+i)\,A_x = i^{(m)}\,a^{(m)}_x + \left(1 + \frac{i^{(m)}}{m}\right)A^{(m)}_x. \tag{5.4.20}$$

The meaning here is that an investment of 1 will produce interest at the end of each interest period plus the repayment of the unit together with interest then due at the end of the interest period in which death occurs.

The actuarial present values for the annuities-immediate can also be obtained by adjusting the actuarial present values of the corresponding life annuities-due. For instance,

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m},$$

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}(1 - {}_nE_x). \tag{5.4.21}$$

5.5 Apportionable Annuities-Due and Complete Annuities-Immediate

With discrete annuities each payment is made either for the following period (an annuity-due) or for the preceding period (an annuity-immediate). A question may arise about having an adjustment for the payment period of death. For instance, suppose that a life insurance contract is purchased by annual payments payable at the beginning of each contract year. If the insured dies 1 month after making an annual payment, a refund of premium for the 11 months the insured did not complete in the policy year of death might seem appropriate. As another example, if a retirement income life annuity-immediate provides annual payments and the annuitant dies 1 month before the due date of the next payment, there might be a final payment for the 11-month period that the annuitant has survived since the last full payment. Consider first the appropriate size for the adjustment payment in such cases.

Let us examine the first case above. The insured dies at time T after paying a full yearly premium of 1 at time K. Assume that the premium is earned or accrued at a uniform rate over the year following the payment. In this case the rate of accrual, c, is given by $c\,\bar{a}_{\overline{1}|} = 1$. If accrual ceases at death, then $c\,\bar{s}_{\overline{T-K}|}$ has been earned to that date, while

$$(1+i)^{T-K} - c\,\bar{s}_{\overline{T-K}|} = 1 \times (1+i)^{T-K} - \frac{\bar{s}_{\overline{T-K}|}}{\bar{a}_{\overline{1}|}} = \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

is unearned and is the amount to be refunded. The present-value random variable, at time 0, of all the payments less the refund is

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+1}|} - v^T \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}} \\ &= \ddot{a}_{\overline{K+1}|} - \frac{v^T - v^{K+1}}{d} \\ &= \frac{1 - v^T}{d} = \ddot{a}_{\overline{T}|}. \end{aligned} \tag{5.5.1}$$

When the annual rate of payments is 1, the actuarial present value at time 0 of the payments is denoted by $\ddot{a}_x^{\{1\}}$:

$$\ddot{a}_x^{\{1\}} = \mathrm{E}[\ddot{a}_{\overline{T}|}] = \mathrm{E}\left[\frac{\delta}{d}\,\bar{a}_{\overline{T}|}\right] = \frac{\delta}{d}\,\bar{a}_x. \tag{5.5.2}$$

This type of life annuity-due, one with a refund for the period between the time of death and the end of the period represented by the last full regular payment, is called an ***apportionable annuity-due.***

We can extend this idea to annuities that are paid more frequently than once a year. As in Section 5.4, we define $J = \lfloor (T - K)m \rfloor$ to be the number of m-ths of a year completed in the year of death, so $K + (J + 1)/m - T$ is the length of the period to be compensated for by the refund. The accrual rate is given by $c\,\bar{a}_{\overline{1/m}|} = 1/m$. Then, proceeding as above,

$$\begin{aligned} Y &= \ddot{a}^{(m)}_{\overline{K+(J+1)/m}|} - v^T \left(\frac{\bar{a}_{\overline{K+(J+1)/m-T}|}}{m\bar{a}_{\overline{1/m}|}}\right) \\ &= \ddot{a}^{(m)}_{\overline{K+(J+1)/m}|} - \frac{v^T - v^{K+(J+1)/m}}{d^{(m)}} \\ &= \frac{1 - v^T}{d^{(m)}} = \ddot{a}^{(m)}_{\overline{T}|}. \end{aligned} \tag{5.5.3}$$

When the annual rate of payments is 1, the actuarial present value of payments, less the refund, is $\ddot{a}_x^{\{m\}}$:

$$\ddot{a}_x^{\{m\}} = \mathrm{E}\left[\frac{1 - v^T}{d^{(m)}}\right] = \frac{\delta}{d^{(m)}}\,\bar{a}_x. \tag{5.5.4}$$

Alternatively, using the second line of (5.5.3),

$$\begin{aligned} \ddot{a}_x^{\{m\}} &= \mathrm{E}\left[\frac{1 - v^{K+(J+1)/m}}{d^{(m)}}\right] - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] \\ &= \ddot{a}_x^{(m)} - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right]. \end{aligned} \tag{5.5.5}$$

The second term on the right-hand side of (5.5.5) is the actuarial present value of the refund. Using the ideas developed in Exercise 4.19 we have

$$\mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] = \frac{\bar{A}_x - A_x^{(m)}}{d^{(m)}}. \tag{5.5.6}$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$\frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x$$

and

$$\ddot{a}_x^{\{m\}} = \ddot{a}_x^{(m)} - \frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x. \tag{5.5.7}$$

Let us now develop a parallel theory for annuities-immediate. Assume the annuitant dies at time T after receiving a last regular payment of size $1/m$ at time $K + J/m$, where $J = \lfloor (T - K)m \rfloor$. Now $T - K - (J/m)$ is the length of the period to be compensated for by an additional payment. Assume that each payment is accrued at a uniform rate over the m-th of the year preceding its payment. In this case the rate of accrual, c, is given by $c\,\bar{s}_{\overline{1/m}|} = 1/m$. If accrual ceases at death, an appropriate payment at death is that portion of the next payment that has been accrued to date and is given by $c\,\bar{s}_{\overline{T-K-(J/m)}|} = \bar{s}_{\overline{T-K-(J/m)}|}/(m\bar{s}_{\overline{1/m}|})$. The present value, at time 0, of all of the payments is

$$
\begin{aligned}
Y &= a^{(m)}_{\overline{K+J/m}|} + v^T \left(\frac{\bar{s}_{\overline{T-K-(J/m)}|}}{m\bar{s}_{\overline{1/m}|}} \right) \\
&= a^{(m)}_{\overline{K+J/m}|} + \frac{v^{K+J/m} - v^T}{i^{(m)}} \\
&= \frac{1 - v^T}{i^{(m)}} = a^{(m)}_{\overline{T}|}.
\end{aligned}
\tag{5.5.8}
$$

When the annual rate of payments is 1, the actuarial present value at 0 of the payments is denoted by $\mathring{a}^{(m)}_x$. When $m = 1$, the (1) in the notation is omitted for this annuity:

$$
\mathring{a}^{(m)}_x = \mathrm{E}\left[\frac{1 - v^T}{i^{(m)}}\right] = \frac{\delta}{i^{(m)}}\,\bar{a}_x. \tag{5.5.9}
$$

Alternatively, using the second line of (5.5.8),

$$
\begin{aligned}
\mathring{a}^{(m)}_x &= \mathrm{E}[a^{(m)}_{\overline{K+J/m}|}] + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right] \\
&= a^{(m)}_x + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right].
\end{aligned}
\tag{5.5.10}
$$

The second term on the right-hand side of (5.5.10) is the actuarial present value of the final partial payment. Using the ideas developed in Exercise 4.19, we have

$$
\mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right] = \frac{(1+i)^{1/m} A^{(m)}_x - \bar{A}_x}{i^{(m)}}. \tag{5.5.11}
$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$
\frac{i}{i^{(m)}}\left(\frac{1}{d^{(m)}} - \frac{1}{\delta}\right) A_x
$$

and

$$
\mathring{a}^{(m)}_x = a^{(m)}_x + \frac{i}{i^{(m)}}\left(\frac{1}{d^{(m)}} - \frac{1}{\delta}\right) A_x. \tag{5.5.12}
$$

This type of life annuity-immediate, one with a partial payment for the period between the last full payment and the time of death, is called a ***complete annuity-immediate.***

For use in subsequent material, (5.5.3) and (5.5.8) seem to be most useful. We illustrate this in the following example.

Example 5.5.1

Compare the variances of the present-value random variables for the complete annuity-immediate and apportionable annuity-due.

Solution:

For the apportionable annuity-due, we have

$$\begin{aligned}\mathrm{Var}(\ddot{a}^{\{m\}}_{\overline{T}|}) &= \mathrm{Var}\left(\frac{1-v^T}{d^{(m)}}\right) && \text{from (5.5.3)}\\ &= \frac{\mathrm{Var}(v^T)}{(d^{(m)})^2}\\ &= \frac{{}^2\bar{A}_x-(\bar{A}_x)^2}{(d^{(m)})^2}.\end{aligned}$$

For the complete annuity-immediate, we have

$$\begin{aligned}\mathrm{Var}(\mathring{a}^{(m)}_{\overline{T}|}) &= \mathrm{Var}\left(\frac{1-v^T}{i^{(m)}}\right) && \text{from (5.5.8)}\\ &= \frac{\mathrm{Var}(v^T)}{(i^{(m)})^2}\\ &= \frac{{}^2\bar{A}_x-(\bar{A}_x)^2}{(i^{(m)})^2}.\end{aligned}$$

Since $i^{(m)}$ is larger than $d^{(m)}$, and in fact $i^{(m)} = d^{(m)}\,(1+i)^{1/m}$, the variance of the complete annuity-immediate is the smaller. ▼

5.6 Notes and References

Taylor (1952) presents new formulas analogous to (5.4.6). Various inquiries into the probability distribution of annuity costs are made by Boermeester (1956), Fretwell and Hickman (1964), and Bowers (1967). This work is summarized by McCrory (1984). Mereu (1962) gives a means of calculating annuity values directly from Makeham constants. The use of the floor function, $\lfloor t \rfloor$, in actuarial science, in particular with respect to actuarial present values of life annuities, is found in Shiu (1982) and in the discussions to that paper. Complete and apportionable annuities are involved, explicitly or implicitly, in papers by Rasor and Greville (1952), Lauer (1967), and Scher (1974) and in the discussions thereto.

Exercises

Section 5.2

5.1. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\bar{a}_{20}$, $\bar{a}_{50}$, $\bar{a}_{80}$

b. $\text{Var}(\bar{a}_{\overline{T}|})$ for $x = 20, 50, 80$.
[Hint: Use (5.2.8), (5.2.9), and (4.4.4).]

5.2. Using the values obtained in Exercise 5.1, calculate the standard deviation and the coefficient of variation, σ / μ, of the following present-value random variables.

a. Individual annuities issued at ages 20, 50, 80 with life incomes of 1,000 per year payable continuously.

b. A group of 100 annuities, each issued at age 50, with life income of 1,000 per year payable continuously.

5.3. Show that $\text{Var}(\bar{a}_{\overline{T}|})$ can be expressed as

$$\frac{2}{\delta}(\bar{a}_x - {}^2\bar{a}_x) - \bar{a}_x^2,$$

where ${}^2\bar{a}_x$ is based on the force of interest 2δ.

5.4. Calculate $\text{Cov}(\delta\, \bar{a}_{\overline{T}|}, v^T)$.

5.5. If a deterministic (rate function) approach is adopted, (5.2.27) could be taken as the starting point for the development of a theory of continuous life annuities. For this, we would begin with

$$\frac{d\bar{a}_y}{dy} = [\mu(y) + \delta]\bar{a}_y - 1 \qquad x \le y < \omega,$$

$$\bar{a}_y = 0 \qquad \omega \le y.$$

a. Use the integrating factor $\exp\{-\int_0^y [\mu(z) + \delta]dz\}$ to solve the differential equation to obtain (5.2.3).

b. Use the integrating factor $e^{-\delta y}$ to obtain

$$\bar{a}_x = \bar{a}_{\overline{\omega - x}|} - \int_x^{\omega} e^{-\delta(y-x)}\, \bar{a}_y\, \mu(y)\, dy$$

and give an interpretation of it in words.

5.6. Assume that $\mu(x + t) = \mu$ and the force of interest is δ for all $t \ge 0$.

a. If $Y = \bar{a}_{\overline{T}|}$, $0 \le T$, display the formula for the distribution function of Y.

b. If

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

c. If

$$Y = \begin{cases} 0 & 0 \le T < n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

d. If

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & 0 \le T < n \\ \bar{a}_{\overline{T}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

5.7. By considering the integral $\int_0^{n+1} v^t \, {}_tp_x \, dt$ and breaking it in two different ways into subintervals (first, from 0 to 1 and 1 to $n + 1$ and then 0 to n and n to $n + 1$), establish a backward recursion formula for the n-year temporary life annuity based on a fixed n-year temporary period. What starting value is appropriate for this recursion formula?

5.8. By considering the integral $\int_n^{\infty} v^t \, {}_tp_x \, dt$ and breaking it into subintervals from n to $n + 1$ and $n + 1$ to ∞, establish a backward recursion formula for the n-year deferred whole life annuity based on a fixed n-year deferral period. What starting value is appropriate for this recursion formula?

5.9. Combine the result from Exercise 5.8 with the first line of (5.2.25) to establish a backward recursion formula for $u(x) = \bar{a}_{\overline{\overline{x:n}|}}$. What starting value is appropriate for this recursion formula?

5.10. If the probabilities come from an aggregate table, establish (5.2.6) by a probabilistic derivation starting with a rewrite of (5.2.3) as

$$\bar{a}_x = \mathrm{E}[\bar{a}_{\overline{T}|}] = \mathrm{E}[\bar{a}_{\overline{T}|} \, |0 \le T < 1] \Pr(0 \le T < 1) + \mathrm{E}[\bar{a}_{\overline{T}|} \, |1 \le T] \Pr(1 \le T).$$

Section 5.3

5.11. Show that

$$\mathrm{Var}(a_{\overline{K}|}) = \mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \frac{\mathrm{Var}(v^{K+1})}{d^2}.$$

5.12. Prove and interpret the given relations:

a. $a_{x:\overline{n}|} = {}_1E_x \, \ddot{a}_{x+1:\overline{n}|}$

b. ${}_{n|}a_x = \dfrac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x.$

5.13. Using (5.3.13), prove and interpret the following relation in words:

$$A_{x:\overline{n}|} = v\, \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}\,.$$

5.14. Obtain an alternative expression for the variance given in Example 5.3.1 by starting with

$$Y^2 = \begin{cases} \dfrac{1 - 2v^K + v^{2K}}{i^2} = \dfrac{2(1 - v^K) - (1 - v^{2K})}{i^2} & K = 0, 1, n - 1 \\ (a_{\overline{n}|})^2 & K = n, n + 1 \ldots . \end{cases}$$

Section 5.4

5.15. Assume a uniform distribution of deaths over each year of age. Simplify

$$\sum_{k=0}^{\infty} {}_kp_x\, v^{k+1} \left[\sum_{j=0}^{m-1} ({}_{j/m}p_{x+k}\ {}_{1/m}q_{x+k+j/m})\ \ddot{s}^{(m)}_{\overline{1-(j+1)/m}|} \right]$$

for use in interpretation of (5.4.8).

5.16. Consider an m-thly temporary life annuity-due that pays 1 per annum to an annuitant age x for $y - x$ years.

a. Express the current payment form of the actuarial present value of the above annuity as a sum of that for payments in the first year and for the remaining $y - x - 1$ years.

b. Express the actuarial present value of payments in the first year in terms of $\alpha(m)$ and $\beta(m)$ under an assumption of the uniform distribution of deaths within each year of age.

c. Find the form of the $c(x)$ and $d(x)$ expressions for a recursion relation for such an annuity and indicate a starting value.

5.17. Using (5.4.10), derive alternative formulas to (5.4.17) and (5.4.18).

5.18. Show that the annuity-immediate analogue for (5.4.6) is

$$a_x^{(m)} = s_{\overline{1}|}^{(m)}\, a_x + \frac{1}{i^{(m)}} \left[(1 + i)A_x - \left(1 + \frac{i^{(m)}}{m}\right) A_x^{(m)} \right],$$

and that under the assumption of a uniform distribution of deaths in each year of age, this becomes

$$a_x^{(m)} = s_{\overline{1}|}^{(m)}\, a_x + (1 + i)\, \frac{1 - \ddot{a}_{\overline{1}|}^{(m)}}{i^{(m)}}\, A_x\,.$$

5.19. Show that the annuity-immediate analogues for (5.4.7) are

$$a_x^{(m)} = \frac{1 - (1 + i^{(m)}/m)A_x^{(m)}}{i^{(m)}} = a_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)}\, A_x^{(m)}$$

and that under the assumption of a uniform distribution of deaths in each year of age these become

$$a_x^{(m)} = \alpha(m)a_x + \frac{1 - \ddot{a}_{\overline{1}|}^{(m)}}{i^{(m)}}.$$

5.20. a. Use (5.4.3) as a starting point to verify that

$$\lim_{m\to\infty} \ddot{a}_x^{(m)} = \bar{a}_x.$$

b. Use (5.4.10) and the result in (a) to show

$$\bar{a}_x \cong a_x + \frac{1}{2}.$$

5.21. Using the traditional approximation given in (5.4.10), establish the following:

a. $a_x^{(m)} \cong a_x + \dfrac{m-1}{2m}$

b. $a_{x:\overline{n}|}^{(m)} \cong a_{x:\overline{n}|} + \dfrac{m-1}{2m}(1 - {}_nE_x)$

c. ${}_{n|}a_x^{(m)} \cong {}_{n|}a_x + \dfrac{m-1}{2m}\, {}_nE_x$.

5.22. a. Develop a formula for $\ddot{s}_{25:\overline{40}|}^{(m)}$ in terms of $\ddot{s}_{25:\overline{40}|}$.

b. On the basis of the Illustrative Life Table with interest at the effective annual rate of 6%, calculate the values of

(i) $\ddot{a}_{25:\overline{40}|}^{(12)}$ (ii) $\ddot{s}_{25:\overline{40}|}^{(12)}$.

5.23. The actuarial present value of a standard increasing temporary life annuity with respect to (x) with

- Yearly income of 1 in the first year, 2 in the second year, and so on, ending with n in the n-th year,
- Payments made m-thly on a due basis

is denoted by $(I\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_{x:\overline{n-k}|}^{(m)}.$$

5.24. The actuarial present value of a standard decreasing temporary life annuity with respect to (x) with

- Yearly income of n in the first year, $n-1$ in the second year, and so on, ending with 1 in the n-th year,
- Payments, made m-thly on a due basis

is denoted by $(D\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=1}^{n} \ddot{a}_{x:\overline{k}|}^{(m)}.$$

5.25. If in Exercise 5.23 the yearly income does not cease at age $x + n$ but continues at the level n while (x) survives therafter, the actuarial present value is denoted by $(I_{\overline{n}|}\, \ddot{a})_x^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_x^{(m)}.$$

5.26. Verify the formula

$$\delta(\bar{I}\bar{a})_{\overline{T}|} + T\, v^T = \bar{a}_{\overline{T}|},$$

where T represents the future lifetime of (x). Use it to prove that

$$\delta(\bar{I}\bar{a})_x + (\bar{I}\bar{A})_x = \bar{a}_x,$$

where $(\bar{I}\bar{a})_x$ is the actuarial present value of a life annuity to (x) under which payments are being made continuously at the rate of t per annum at time t.

5.27. From $a_{\overline{x:n}|}^{(m)} = a_x^{(m)} + a_{\overline{n}|}^{(m)} - a_{x:\overline{n}|}^{(m)}$, show that the assumption of uniform distribution of deaths in each year of age leads to

$$a_{\overline{x:n}|}^{(m)} = \frac{i}{i^{(m)}}\left[a_{\overline{n}|} + v^n\, {}_np_x\, a_{x+n} + \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) v^n\, {}_np_x\, A_{x+n}\right].$$

Section 5.5

5.28. Establish and interpret the following formulas:

a. $1 = i^{(m)}\, \mathring{a}_x^{(m)} + \bar{A}_x$

b. $1 = d^{(m)}\, \ddot{a}_x^{\{m\}} + \bar{A}_x$

c. $\mathring{a}_{x:\overline{n}|}^{(m)} = (\delta / i^{(m)})\, \bar{a}_{x:\overline{n}|}$

d. $\ddot{a}_{x:\overline{n}|}^{\{m\}} = (\delta / d^{(m)})\, \bar{a}_{x:\overline{n}|}$

e. $\ddot{a}_{x:\overline{n}|}^{\{m\}} = (1 + i)^{1/m}\, \mathring{a}_{x:\overline{n}|}^{(m)}$.

5.29. Let $H(m) = \ddot{a}_x^{\{m\}} - \mathring{a}_x^{(m)}$. Prove that $H(m) \geq 0$ and $\lim_{m\to\infty} H(m) = 0$.

Miscellaneous

5.30. For $0 \leq t \leq 1$ and the assumption of a uniform distribution of deaths in each year of age, show that

a. $\ddot{a}_{x+t} = \dfrac{(1 + it)\ddot{a}_x - t(1 + i)}{1 - t\, q_x}$

b. ${}_{t|}\ddot{a}_x = v^t[(1 + it)\ddot{a}_x - t(1 + i)]$

c. ${}_{1-t|}\ddot{a}_{x+t} = \dfrac{(1+i)^t}{1-tq_x}(\ddot{a}_x - 1)$

d. $A_{x+t} = \dfrac{1+it}{1-tq_x}A_x - \dfrac{t\,q_x}{1-t\,q_x}$.

5.31. Obtain formulas for the evaluation of a life annuity-due to (x) with an initial payment of 1 and with annual payments increasing thereafter by

a. 3% of the initial annual payment

b. 3% of the previous year's annual payment.

5.32. Express $(\bar{D}\bar{a})_{x:\overline{n}|}$ as an integral and prove the formula

$$\frac{\partial}{\partial n}(\bar{D}\bar{a})_{x:\overline{n}|} = \bar{a}_{x:\overline{n}|}.$$

5.33. Give an expression for the actuarial accumulated value at age 70 of an annuity with the following monthly payments:

- 100 at the end of each month from age 30 to 40
- 200 at the end of each month from age 40 to 50
- 500 at the end of each month from age 50 to 60
- 1,000 at the end of each month from age 60 to 70.

5.34. Derive a simplified expression for the actuarial present value for a 25-year term insurance payable immediately on the death of (35), under which the death benefit in case of death at age $35 + t$ is $\bar{s}_{\overline{t}|}$, $0 \le t \le 25$. Interpret your result.

5.35. Derive a simplified expression for the actuarial present value for an n-year term insurance payable at the end of the year of death of (x), under which the death benefit in case of death in year $k+1$ is $\ddot{s}_{\overline{k+1}|}$, $0 \le k < n$. Interpret your result.

5.36. Obtain a simplified expression for

$$(I\ddot{a})^{(12)}_{x:\overline{25}|} - (Ia)^{(12)}_{x:\overline{25}|}.$$

5.37. Consider an n-year deferred continuous life annuity of 1 per year as an insurance with probability of claim, ${}_np_x$, and random amount of claim, $v^n\,\bar{a}_{\overline{T}|}$. Here T has p.d.f., ${}_tp_{x+n}\,\mu_x(n+t)$. Apply (2.2.13) to show that the variance of the insurance equals

$$v^{2n}\,{}_np_x\,(1 - {}_np_x)\,\bar{a}^2_{x+n} + v^{2n}\,{}_np_x\,\frac{{}^2\bar{A}_{x+n} - (\bar{A}_{x+n})^2}{\delta^2}$$

and verify that this reduces to (5.2.21).

5.38. Write the discrete analogue of the variance formula in Exercise 5.37.

5.39. Consider the indicator random variable, I_k, defined by

$$I_k = \begin{cases} 1 & T(x) \geq k \\ 0 & T(x) < k. \end{cases}$$

Show the following:

a. The present value of a life annuity to (x), with annual payment b_k on survival to age $x + k$, $k = 0, 1, 2, \ldots$, can be written as

$$\sum_{k=0}^{\infty} v^k \, b_k \, I_k.$$

b. $E[I_j I_k] = {}_kp_x \qquad j \leq k$

$\mathrm{Cov}(I_j, I_k) = {}_kp_x \, {}_jq_x \qquad j \leq k.$

c. $\mathrm{Var}\left(\sum_{k=0}^{\infty} v^k \, b_k \, I_k\right) = \sum_{k=0}^{\infty} v^{2k} \, b_k^2 \, {}_kp_x \, {}_kq_x + 2 \sum_{k=0}^{\infty} \sum_{j<k} v^{j+k} \, b_j \, b_k \, {}_kp_x \, {}_jq_x.$

5.40. If a left superscript 2 indicates that interest is at force 2δ, show that

a. ${}^2A_x = 1 - (2\,d - d^2)\,{}^2\ddot{a}_x$

b. $\mathrm{Var}\,(v^{K+1}) = 2\,d(\ddot{a}_x - {}^2\ddot{a}_x) - d^2\,(\ddot{a}_x^2 - {}^2\ddot{a}_x)$

c. $\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \dfrac{2}{d}\,(\ddot{a}_x - {}^2\ddot{a}_x) - (\ddot{a}_x^2 - {}^2\ddot{a}_x).$

5.41. a. Expand, in terms of powers of δ, the annuity coefficients $\alpha(m)$ and $\beta(m)$.

b. What do the expansions in (a) become for $m = \infty$?

5.42. Use Jensen's inequality to show, for $\delta > 0$, that

a. $\bar{a}_x < \bar{a}_{\overline{\mathring{e}_x}|}$ b. $a_x < a_{\overline{e_x}|} \qquad x < \omega - 1.$

5.43. If $g(x)$ is a non-negative function and X is a random variable with p.d.f. $f(x)$, justify the inequality

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx \geq k\,\Pr[g(X) \geq k] \qquad k > 0$$

and use it to show that

$$\bar{a}_x \geq \bar{a}_{\overline{\mathring{e}_x}|}\,\Pr(\bar{a}_{\overline{T}|} \geq \bar{a}_{\overline{\mathring{e}_x}|}) = \bar{a}_{\overline{\mathring{e}_x}|}\,\Pr(T \geq \mathring{e}_x).$$

5.44. A unit is to be used to purchase a combination benefit consisting of a life income of I per year payable continuously while (x) survives and an insurance of J payable immediately on the death of (x). Write the present-value random variable for this combination and give its mean and variance.

5.45. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\ddot{a}_{40}^{(12)}$ b. $\ddot{a}_{40:\overline{30}|}^{(12)}$ c. ${}_{30|}\ddot{a}_{40}^{(12)}.$

5.46. If $A''_{x:\overline{m}|}$ and $\ddot{a}''_{x:\overline{m}|}$ are actuarial present values calculated using

- An interest rate of i for the first n years, $n < m$, and
- An interest rate i' for the remaining $m - n$ years,

show algebraically and interpret

a. $A''_{x:\overline{m}|} = 1 - d\,\ddot{a}_{x:\overline{n}|} - v^n\,{}_np_x\,d'\,\ddot{a}'_{x+n:\overline{m-n}|}$

b. $A''_{x:\overline{m}|} = 1 - d'\,\ddot{a}''_{x:\overline{m}|} + (d' - d)\,\ddot{a}_{x:\overline{n}|}$.

5.47. Show that

$$\frac{d\ddot{a}_x}{di} = -v(Ia)_x,$$

where

$$(Ia)_x = \sum_{t=1}^{\infty} t\, v^t\, {}_tp_x,$$

and interpret the relation.

5.48. Show that a constant increase in the force of mortality has the same effect on $\ddot{a}_x$ as a constant increase in the force of interest, but that this is not the case for $\ddot{a}_x^{(m)}$ evaluated by $\alpha(m)\,\ddot{a}_x - \beta(m)$.

5.49. Show that

$$\alpha(m) - \beta(m)d = \ddot{a}_{\overline{1}|}^{(m)}.$$

5.50. Show that, if $q_x < (i^{(2)}/2)^2$, the approximation

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x),$$

in the special case with $n = 1$, $m = 2$, leads to

$$\ddot{a}_{x:\overline{1}|}^{(2)} > \ddot{a}_{\overline{1}|}^{(2)}.$$

5.51. Consider the following portfolio of annuities-due currently being paid from the assets of a pension fund.

Age	Number of Annuitants
65	30
75	20
85	10

Each annuity has an annual payment of 1 as long as the annuitant survives. Assume an earned interest rate of 6% and a mortality as given in the Illustrative Life Table. For the present value of these obligations of the pension fund, calculate

a. The expectation

b. The variance

c. The 95th percentile of its distribution.
For parts (b) and (c), assume the lives are mutually independent.

Computing Exercises

5.52. a. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a life annuity-due of 1 per annum for ages 13 to 140.
b. Compare your values to those given in Table 2A.

5.53. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a temporary life annuity-due of 1 per annum payable to age 65 for ages 13 to 64.

5.54. Using your Illustrative Life Table with the assumption of a uniform distribution of deaths within each year of age and $i = 0.06$, calculate the actuarial present value of a 10-year temporary life annuity of 1 per annum payable continuously and issued at ages 13 to 99 using the results of Exercise 5.7.

5.55. a. Add $\alpha(m)$ and $\beta(m)$ to the interest functions calculated and stored in your Illustrative Life Table. Refer to Exercise 4.31.
b. Determine $\alpha(12)$ and $\beta(12)$ at $i = 0.06$ and compare your results to those given in Example 5.4.1.
[Remark: We suggest using the series derived in Exercise 5.41 for accurate results with small interest rates.]

5.56. Using your Illustrative Life Table with $i = 0.06$ and the assumption of uniform distribution of deaths over each year of age, calculate the actuarial present value of a temporary life annuity of 1 per annum payable continuously to age 65 for ages 13 to 64.

5.57. Let Y be the present-value random variable for a continuous 10-year temporary life annuity of 1 per annum commencing at age 60. On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.08$, calculate the mean and variance of Y.

5.58. Let Y be the present-value random variable for a life annuity-due of 1 per annum, payable monthly to (65). On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$, calculate the mean and variance of Y.

5.59. Use the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.07$ to determine $\mathring{a}_{30:\overline{20}|}$.

6

BENEFIT PREMIUMS

6.1 Introduction

In Chapters 4 and 5 we discussed actuarial present values of the payments of various life insurances and annuities. These ideas are combined in this chapter to determine the level of life annuity payments necessary to buy, or fund, the benefits of an insurance or annuity contract. In practice individual life insurance is usually purchased by a life annuity of ***contract premiums***—the insurance contract specifies the premium to be paid. Contract premiums provide for benefits, expenses of initiating and maintaining the insurance, and margins for profit and for offsetting possible unfavorable experience. The premiums studied in this chapter are determined only by the pattern of benefits and premiums and do not consider expenses, profit, or contingency margins.

In Chapter 1 we discussed the idea that determination of the insurance premium requires the adoption of a ***premium principle.*** Example 6.1.1 illustrates the application of three such premium principles. All three principles are based on the impact of the insurance on the wealth of the insuring organization. The random variable that gives the present value at issue of the insurer's loss, if the insurance is contracted at a certain premium level, is the key in the model for the principles. Principle I requires that the loss random variable be positive with no more than a specified probability. Principles II and III are based on the expected utility of the insurer's wealth as discussed in Section 1.3. We will see that Principle II, which uses a linear utility function, could also be characterized as requiring that the loss random variable have zero expected value.

Example 6.1.1

An insurer is planning to issue a policy to a life age 0 whose curtate-future-lifetime, K, is governed by the p.f.

$$_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4.$$

The policy will pay 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Find the annual premium, P, as determined by:

a. Principle I: P will be the least annual premium such that the insurer has probability of a positive financial loss of at most 0.25.
b. Principle II: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = x$, will be indifferent between accepting and not accepting the risk.
c. Principle III: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between accepting and not accepting the risk.

For all three parts assume the insurer will use an annual effective interest rate of $i = 0.06$.

Solution:

For $K = k$ and an arbitrary premium, P, the present value of the financial loss at policy issue is $l(k) = v^{k+1} - P\ddot{a}_{\overline{k+1|}} = (1 + P/d)\, v^{k+1} - P/d,\ k = 0, 1, 2, 3, 4$. The corresponding loss random variable is $L = v^{K+1} - P\ddot{a}_{\overline{K+1|}}$.

a. Since $l(k)$ decreases as k increases, the requirement of Principle I will hold if P is such that $v^2 - P\,\ddot{a}_{\overline{2|}} = 0$. Then the financial loss is positive for only $K = 0$, which has probability $0.2 < 0.25$. Thus, for this principle, $P = 1/\ddot{s}_{\overline{2|}} = 0.45796$.
b. By an extension of (1.3.6), we seek the premium P such that $u(w) = \mathrm{E}[u(w - L)]$. By Principle II $u(x) = x$ so we have

$$w = \mathrm{E}[w - L] = w - \mathrm{E}[L].$$

Thus, Principle II is equivalent to requiring that P be chosen so that $\mathrm{E}[L] = 0$. For this example, we require

$$\sum_{k=0}^{4} (v^{k+1} - P\,\ddot{a}_{\overline{k+1|}}) \Pr(K = k) = 0 \tag{6.1.1}$$

which gives $P = 0.30272$.

c. Again by (1.3.6) and now using the utility function in Principle III, we have

$$-e^{-0.1w} = \mathrm{E}[-e^{-0.1(w-L)}] = -e^{-0.1w}\,\mathrm{E}[e^{0.1L}].$$

Thus, Principle III is equivalent to requiring that P be chosen so that $\mathrm{E}[e^{0.1L}] = 1$. Here, we require

$$\sum_{k=0}^{4} \exp[0.1\,(v^{k+1} - P\,\ddot{a}_{\overline{k+1|}})] \Pr(K = k) = 1, \tag{6.1.2}$$

which gives $P = 0.30628$.

These three results are summarized below.

Outcome k	Probability ${}_{k\mid}q_0$	General Formula	Present Value of Financial Loss When Premium Is by Principle I	II	III	$\exp(0.1L)$ III
0	0.2	$v^1 - P\,\ddot{a}_{\overline{1}\vert}$	0.48544	0.64067	0.63712	1.06579
1	0.2	$v^2 - P\,\ddot{a}_{\overline{2}\vert}$	0	0.30169	0.29477	1.02992
2	0.2	$v^3 - P\,\ddot{a}_{\overline{3}\vert}$	−0.45796	−0.01811	−0.02819	0.99718
3	0.2	$v^4 - P\,\ddot{a}_{\overline{4}\vert}$	−0.89000	−0.31981	−0.33287	0.96726
4	0.2	$v^5 - P\,\ddot{a}_{\overline{5}\vert}$	−1.29758	−0.60443	−0.62031	0.93985
	Premium		0.45796	0.30272	0.30628	
	Expected Value		−0.43202	0.00000	−0.00990	1.00000

The table shows that for this example the decision makers adopting principles I and III reduce their risk in the sense that they are demanding an expected present value of loss to be negative. ▼

Premiums defined by Principle I are known as ***percentile premiums.*** Although the principle is attractive on the surface, it is easy to show that it can lead to quite unsatisfactory premiums. Such cases are examined in Example 6.2.3.

Principle II has many applications in practice. To formalize its concepts, we define the insurer's loss, L, as the random variable of the present value of benefits to be paid by the insurer less the annuity of premiums to be paid by the insured. Principle II is called the ***equivalence principle*** and has the requirement that

$$\mathrm{E}[L] = 0. \tag{6.1.3}$$

We will speak of ***benefit premiums*** as those satisfying (6.1.3). Equivalently, benefit premiums will be such that

$$\mathrm{E}[\text{present value of benefits}] = \mathrm{E}[\text{present value of benefit premiums}].$$

Methods developed in Chapters 4 and 5 for calculating these actuarial present values can be used to reduce this equality to a form that can be solved for the premiums. For instance, in Example 6.1.1, which has constant benefit premiums and constant benefits of 1, equation (6.1.1) can be rewritten as $A_0 = P\ddot{a}_0$, and $\ddot{a}_0$ can be calculated as

$$\sum_{k=0}^{4} v^k \, {}_kp_0.$$

When the equivalence principle is used to determine a single premium at policy issue for a life insurance or a life annuity, the premium is equal to the actuarial present value of benefit payments and is called the ***single benefit premium.***

Premiums based on Principle III, using an exponential utility function, are known as ***exponential premiums.*** Exponential premiums are nonproportional in the sense that the premium for the policy with a benefit level of 10 is more than 10 times the premium for a policy with a benefit level of 1 (see Exercise 6.2). This is consistent for a risk averse utility function.

6.2 Fully Continuous Premiums

The basic concepts involved in the determination of annual benefit premiums using the equivalence principle will be illustrated first for the case of the fully continuous level annual benefit premium for a unit whole life insurance payable immediately on the death of (x). For any continuously paid premium, $\bar{P}$, consider

$$l(t) = v^t - \bar{P}\,\bar{a}_{\overline{t}|}, \tag{6.2.1}$$

the present value of the loss to the insurer if death occurs at time t.

We note that $l(t)$ is a decreasing function of t with $l(0) = 1$ and $l(t)$ approaching $-\bar{P}/\delta$ as $t \to \infty$. If t_0 is the time when $l(t_0) = 0$, death before t_0 results in a positive loss, whereas death after t_0 produces a negative loss, that is, a gain. Figure 6.2.1 later in this section illustrates these ideas.

We now consider the loss random variable,

$$L = l(T) = v^T - \bar{P}\,\bar{a}_{\overline{T}|}, \tag{6.2.2}$$

corresponding to the loss function $l(t)$. If the insurer determines his premium by the equivalence principle, the premium is denoted by $\bar{P}(\bar{A}_x)$ and is such that

$$\mathrm{E}[L] = 0. \tag{6.2.3}$$

It follows from (4.2.6) and (5.2.3) that

$$\bar{A}_x - \bar{P}(\bar{A}_x)\bar{a}_x = 0,$$

or

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}. \tag{6.2.4}$$

Remark:
In this chapter we continue to suppress the select notation except in situations in which it is necessary or helpful to eliminate ambiguity.

The variance of L can be used as a measure of the variability of losses on an individual whole life insurance due to the random nature of time-until-death. When $\mathrm{E}[L] = 0$,

$$\mathrm{Var}(L) = \mathrm{E}[L^2]. \tag{6.2.5}$$

For the loss in (6.2.2), we have

$$\begin{aligned}\mathrm{Var}(v^T - \bar{P}\,\bar{a}_{\overline{T}|}) &= \mathrm{Var}\left[v^T - \frac{\bar{P}(1 - v^T)}{\delta}\right] \\ &= \mathrm{Var}\left[v^T\left(1 + \frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]\end{aligned}$$

$$= \text{Var}\left[v^T\left(1+\frac{\bar{P}}{\delta}\right)\right]$$

$$= \text{Var}\,(v^T)\left(1+\frac{\bar{P}}{\delta}\right)^2$$

$$= [{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1+\frac{\bar{P}}{\delta}\right)^2. \tag{6.2.6}$$

For the premium determined by the equivalence principle, we can use (6.2.4) and (5.2.8), $\delta\bar{a}_x + \bar{A}_x = 1$, to rewrite (6.2.6) as

$$\text{Var}(L) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2}. \tag{6.2.7}$$

Example 6.2.1

Calculate $\bar{P}(\bar{A}_x)$ and $\text{Var}(L)$ with the assumptions that the force of mortality is a constant $\mu = 0.04$ and the force of interest $\delta = 0.06$.

Solution:

These assumptions yield $\bar{a}_x = 10$, $\bar{A}_x = 0.4$, and ${}^2\bar{A}_x = 0.25$. Using (6.2.4), we obtain

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = 0.04,$$

and from (6.2.7)

$$\text{Var}(L) = \frac{0.25 - 0.16}{(0.6)^2} = 0.25.$$

▼

By reference to (6.2.7) we can see that the numerator of this last expression can be interpreted as the variance of the loss, $v^T - \bar{A}_x$, associated with a single-premium whole life insurance. This latter variance is 0.09, and hence the standard deviation of the loss associated with this annual premium insurance is $\sqrt{0.25/0.09} = 5/3$ times the standard deviation of the loss in the single-premium case. Additional uncertainty about the present value of the premium income increases the variability of losses due to the random nature of time-until-death.

In Example 6.2.1, $\bar{P}(\bar{A}_x) = 0.04$, the constant force of mortality. We can confirm this as a general result by using parts of Examples 4.2.3 and 5.2.1. Under the constant force of mortality assumption,

$$\bar{A}_x = \frac{\mu}{\mu + \delta}$$

and

$$\bar{a}_x = \frac{1}{\mu + \delta},$$

thus

$$\bar{P}(\bar{A}_x) = \frac{\mu(\mu + \delta)^{-1}}{(\mu + \delta)^{-1}} = \mu,$$

which does not depend on the force of interest or the age at issue.

Using the equivalence principle, as in (6.1.3), we can determine formulas for annual premiums of a variety of fully continuous life insurances. Our general loss is

$$b_T v_T - \bar{P}\,Y = Z - \bar{P}\,Y \tag{6.2.8}$$

where

- b_t and v_t are, respectively, the benefit amount and discount factor defined in connection with (4.2.1)
- $\bar{P}$ is a general symbol for a fully continuous net annual premium
- Y is a continuous annuity random variable as defined, for example, in (5.2.13), and
- Z is defined by (4.2.2).

Application of the equivalence principle yields

$$\mathrm{E}[b_T v_T - \bar{P}\;Y] = 0$$

or

$$\bar{P} = \frac{\mathrm{E}[b_T v_T]}{\mathrm{E}[Y]}.$$

These ideas are used to display annual premium formulas in Table 6.2.1.

It is of interest to note how these steps can be used for an n-year deferred whole life annuity of 1 per year payable continuously. In this case $b_T v_T = 0$, $T \le n$ and $b_T v_T = \bar{a}_{\overline{T-n|}} v^n$, $T > n$. Then,

$$\begin{aligned}
\mathrm{E}[b_T v_T] &= {}_n p_x \; \mathrm{E}[\bar{a}_{\overline{T-n|}} v^n | T > n] \\
&= v^n \; {}_n p_x \; \bar{a}_{x+n} = A_{x:\overset{\;1}{\overline{n|}}} \; \bar{a}_{x+n}.
\end{aligned}$$

In practice, however, deferred life annuities usually provide some type of death benefit during the period of deferment. One contract of this type is examined in Example 6.6.2.

Example 6.2.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.2.1).

TABLE 6.2.1

Fully Continuous Benefit Premiums

Plan	Loss Components: $b_T v_T$	Loss Components: $\bar{P}\ Y$ Where Y Is	Premium Formula $\bar{P} = \dfrac{E[b_T v_T]}{E[Y]}$
Whole life insurance	$1\ v^T$	$\bar{a}_{\overline{T}\rvert}$	$\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_x}$
n-Year term insurance	$1\ v^T$ 0	$\bar{a}_{\overline{T}\rvert},\ T \le n$ $\bar{a}_{\overline{n}\rvert},\ T > n$	$\bar{P}(\bar{A}^{1}_{x:\overline{n}\rvert}) = \dfrac{\bar{A}^{1}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{n}\rvert}}$
n-Year endowment insurance	$1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rvert},\ T \le n$ $\bar{a}_{\overline{n}\rvert},\ T > n$	$\bar{P}(\bar{A}_{x:\overline{n}\rvert}) = \dfrac{\bar{A}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{n}\rvert}}$
h-Payment* whole life insurance	$1\ v^T$ $1\ v^T$	$\bar{a}_{\overline{T}\rvert},\ T \le h$ $\bar{a}_{\overline{h}\rvert},\ T > h$	${}_h\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_{x:\overline{h}\rvert}}$
h-Payment,* n-year endowment insurance	$1\ v^T$ $1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rvert},\ T \le h$ $\bar{a}_{\overline{h}\rvert},\ h < T \le n$ $\bar{a}_{\overline{h}\rvert},\ T > n$	${}_h\bar{P}(\bar{A}_{x:\overline{n}\rvert}) = \dfrac{\bar{A}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{h}\rvert}}$
n-Year pure endowment	0 $1\ v^n$	$\bar{a}_{\overline{T}\rvert},\ T \le n$ $\bar{a}_{\overline{n}\rvert},\ T > n$	$\bar{P}(A_{x:\overline{n}\rvert}^{\ \ 1}) = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}}{\bar{a}_{x:\overline{n}\rvert}}$
n-Year† deferred whole life annuity	0 $\bar{a}_{\overline{T-n}\rvert}\ v^n$	$\bar{a}_{\overline{T}\rvert},\ T \le n$ $\bar{a}_{\overline{n}\rvert},\ T > n$	$\bar{P}({}_{n\rvert}\bar{a}_x) = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}\ \bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\rvert}}$

*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.

†The annuity product described above provides no death benefits and has a level premium with premiums payable for n years. A different, perhaps more realistic, design for an n-year level premium-deferred annuity is given in Example 6.6.2.

Solution:

Using the notation of (4.2.11), we have

$$\text{Var}(L) = \text{Var}\left\{Z_3\left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rvert})}{\delta}\right] - \frac{\bar{P}(\bar{A}_{x:\overline{n}\rvert})}{\delta}\right\}.$$

We now use (4.2.10) to obtain

$$\text{Var}(L) = \left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rvert})}{\delta}\right]^2 [{}^2\bar{A}_{x:\overline{n}\rvert} - (\bar{A}_{x:\overline{n}\rvert})^2].$$

From the second additional relation given in Table 5.2.1, we have

$$(\delta\,\bar{a}_{x:\overline{n}\rvert})^{-1} = 1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rvert})}{\delta},$$

which implies that

$$\mathrm{Var}(L) = \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{(\delta\,\bar{a}_{x:\overline{n}|})^2}.$$

▼

The two identities, (5.2.8) and (5.2.15), can be used to derive relationships among continuous benefit premiums. For example, starting with (5.2.8),

$$\begin{aligned}
\delta\,\bar{a}_x + \bar{A}_x &= 1,\\
\delta + \bar{P}(\bar{A}_x) &= \frac{1}{\bar{a}_x},\\
\bar{P}(\bar{A}_x) &= \frac{1}{\bar{a}_x} - \delta\\
&= \frac{1 - \delta\,\bar{a}_x}{\bar{a}_x}\\
&= \frac{\delta\,\bar{A}_x}{1 - \bar{A}_x}. \qquad (6.2.9)
\end{aligned}$$

Starting with (5.2.14) we obtain

$$\begin{aligned}
\delta\,\bar{a}_{x:\overline{n}|} + \bar{A}_{x:\overline{n}|} &= 1,\\
\delta + \bar{P}(\bar{A}_{x:\overline{n}|}) &= \frac{1}{\bar{a}_{x:\overline{n}|}},\\
\bar{P}(\bar{A}_{x:\overline{n}|}) &= \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta\\
&= \frac{1 - \delta\,\bar{a}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}\\
&= \frac{\delta\,\bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}}. \qquad (6.2.10)
\end{aligned}$$

Verbal interpretations of the discrete analogues of (6.2.9) and (6.2.10) are given in Example 6.3.4.

The premiums discussed so far in this section are benefit premiums, those derived from the equivalence principle. We now turn to an example that describes two ways that percentile premiums give unsatisfactory results.

Example 6.2.3

Find the 25th percentile premium for an insured age 55 for the following plans of insurance:

a. 20-year endowment
b. 20-year term
c. 10-year term.

Assume a fully continuous basis with a force of interest, $\delta = 0.06$ and mortality following the Illustrative Life Table.

Solution:

a. The loss function for 20-year endowment insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20$$
$$= v^{20} - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20$$

and is a nonincreasing function of T. Thus the values of T for which the loss L is to be positive, which are to have probability of 0.25, are those values below $\xi_T^{0.25}$. Since $l_{55} = 86{,}408.60$ and $l_{70.617} = 64{,}806.45$ (by linear interpolation), $\Pr(T < 15.617) = 0.25$. Thus, the premium required by the 25th percentile principle is that which sets the loss at $T = 15.617$ equal to zero and is $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$.

b. The loss function for 20-year term insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20$$
$$= \quad - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20.$$

This is still a nonincreasing function of T, and since $\Pr(T < 15.617) = 0.25$, the premium required by the 25th percentile principle is again $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$. It is, of course, unsatisfactory that the same premium is generated for two different plans of insurance, particularly since the benefit premium at this age for 20-year endowment is almost two times that for 20-year term.

c. The loss function for a 10-year term insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 10$$
$$= - \bar{P}\bar{a}_{\overline{10}|} \qquad T \geq 10.$$

If the premium is set at zero, then $\Pr(L > 0) = \Pr(T < 10)$, and this probability equals, by the Illustrative Life Table,

$$\frac{(l_{55} - l_{65})}{l_{55}} = 0.1281.$$

Thus, zero is the least non-negative annual premium such that the insurer's probability of financial loss is at most 0.25. In this case $\Pr(L > 0) = 0.1281$, and $\bar{P} = 0$ the 25th percentile premium. The benefit premium in this case is 70% of that for 20-year term insurance. ▼

The conclusion from this example is that the percentile premium principle yields conflicting results for insurances on a single individual. Its use will be minimal in what follows.

For a whole life insurance, as defined in the first row of Table 6.2.1,

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T \geq 0.$$

The d.f. of L can be developed as follows:

$$\begin{aligned}
F_L(u) &= \Pr(L \le u) \\
&= \Pr\left[v^T - \bar{P}\left(\frac{1-v^T}{\delta}\right) \le u\right] \\
&= \Pr\left(v^T \le \frac{\delta u + \bar{P}}{\delta + \bar{P}}\right) \\
&= \Pr\left[T \ge -\frac{1}{\delta}\log\left(\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right)\right] \\
&= 1 - F_T\left(-\frac{1}{\delta}\log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right]\right) \qquad -\frac{\bar{P}}{\delta} < u.
\end{aligned} \tag{6.2.11}$$

The p.d.f. of L is

$$\frac{d}{du}F_L(u) = f_L(u) = f_T\left(-\frac{1}{\delta}\log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right]\right)\left(\frac{1}{\delta u + \bar{P}}\right) \qquad -\frac{\bar{P}}{\delta} < u. \tag{6.2.12}$$

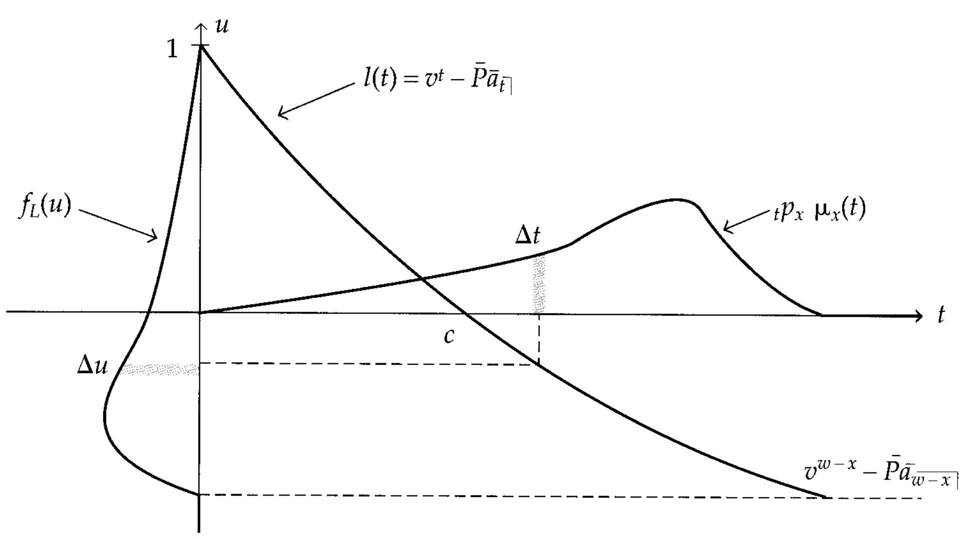
Using the language of decision analysis, we can say that the determination of the premium $\bar{P}$ is equivalent to selecting the distribution of L, given by (6.2.11), that is optimal from the viewpoint of the premium principle adopted by the decision maker. This principle reflects the preferences of the decision maker.

Schematic diagrams of $l(t)$, the p.d.f. of $T(x)$, and the induced p.d.f. of L are combined in Figure 6.2.1.

The set of d.f.'s of L is indexed by the parameter $\bar{P}$. The value of $\bar{P}$ is selected by the premium principle adopted. For illustration, use Figure 6.2.1 where $\Pr(T \le c) = \Pr(L > 0)$ and this probability is taken as 0.25. We assume that the

FIGURE 6.2.1

Schematic Diagrams of $l(t)$ and the p.d.f.'s of $T(x)$ and L



value of $\bar{P}$ will be obtained by solving $F_L(0) = 1 - 0.25 = 0.75$. This illustration uses a percentile premium principle with the probability for a positive value of L set at 0.25.

It is evident from Figure 6.2.1 that the events $(T \le c)$ and $(L > 0)$ are equivalent in the sense that the occurrence of one of the two events implies the occurrence of the other. To continue our illustration, if the decision maker has adopted the percentile premium principle with $\Pr(L > 0) = p$, then $\Pr(T \le c) = p$, where $c = \xi_T^p$, the $100p$-th percentile of the distribution of T. Furthermore, because of the equivalence of these two events, the premium can be determined from an equation involving the loss function, that is, from

$$v^{\xi_T^p} - \bar{P}\bar{a}_{\overline{\xi_T^p}|} = 0,$$

or

$$\bar{P} = \frac{v^{\xi_T^p}}{\bar{a}_{\overline{\xi_T^p}|}} = \frac{1}{\bar{s}_{\overline{\xi_T^p}|}}. \tag{6.2.13}$$

Because $\bar{P}$ is the rate of payment into a fund that will provide a unit payment at time ξ_T^p, there is intuitive support for the result. The accumulation $\bar{s}_{\overline{T}|}/\bar{s}_{\overline{\xi_T^p}|}$ will be less than 1 with probability p and greater than 1 with probability $1 - p$.

Example 6.2.4

This example builds on Example 6.2.3, except that $T(55)$ has a De Moivre distribution, with p.d.f.

$${}_tp_{55}\ \mu_{55}(t) = 1/45 \qquad 0 < t < 45.$$

For the three loss variables, display the d.f. of L and determine the parameter $\bar{P}$ as the smallest non-negative number such that $\Pr(L > 0) \le 0.25$.

Solution:

a. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = v^{20} - \bar{P}\bar{a}_{\overline{20}|}$ induced by the constraint on L if $T \ge 20$, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 && u < v^{20} - \bar{P}\bar{a}_{\overline{20}|} \\ &= 1 + \frac{1}{0.06}\,\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} && v^{20} - \bar{P}\bar{a}_{\overline{20}|} \le u \le 1 \\ &= 1 && 1 < u. \end{aligned}$$

Figure 6.2.2a is a diagram of the d.f. associated with a 20-year endowment insurance. This figure provides a graphical way of thinking of premium determination using the percentile principle. The d.f. within the set of d.f.'s indexed by $\bar{P}$ that crosses the vertical axis at 0.75 is sought. Analytically this means that the premium is determined by solving, for $\bar{P}$,

$$F_L(0) = 0.75$$

$$= 1 + \frac{1}{0.06}\frac{\log[\bar{P}/(0.06 + \bar{P})]}{45} = 0.75,$$

or

$$\log\left(\frac{\bar{P}}{0.06 + \bar{P}}\right) = -0.675,$$

and

$$\bar{P} = \frac{0.06e^{-0.675}}{(1 - e^{-0.675})}$$

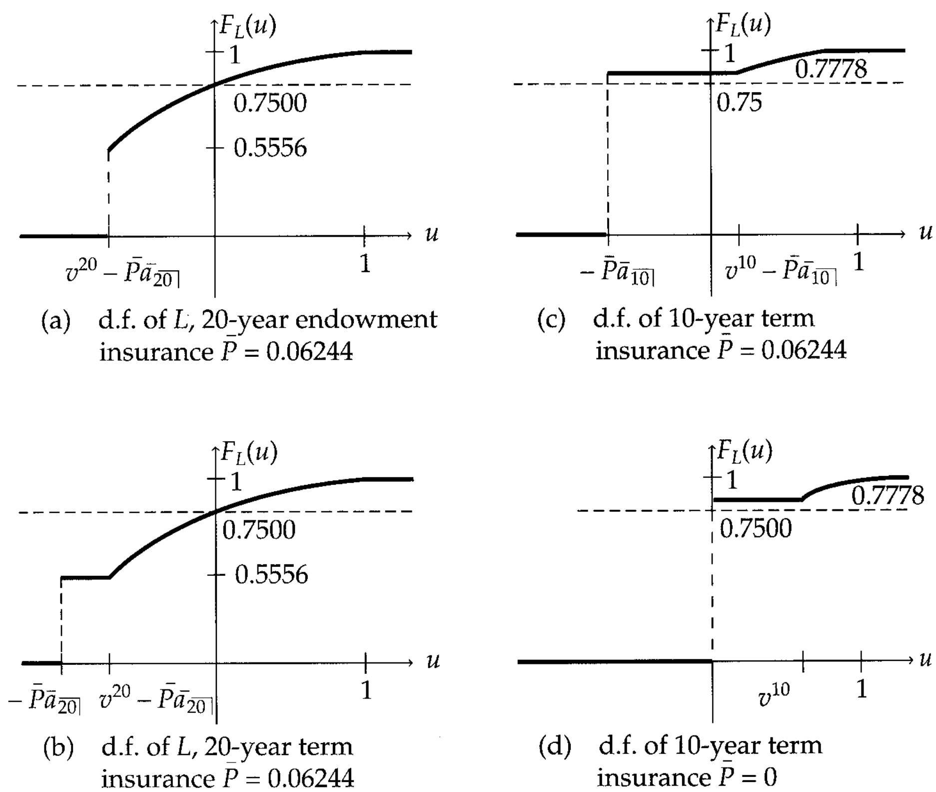
$$= \frac{1}{\bar{s}_{\overline{11.25}|}}$$

$$= 0.06224.$$

In view of the discussion about (6.2.11), this is not surprising. The 25th percentile of the De Moivre distribution of T in this example is $\xi_T^{0.25} = 11.25$.

FIGURE 6.2.2

Distribution Functions of *L* Developed in Example 6.2.4



For comparison, the benefit, or equivalence principle, premium is

$$\bar{P}(\bar{A}_{55:\overline{20}|}) = \frac{\int_0^{20} (v^t/45)\, dt + (25/45)\, v^{20}}{\int_0^{20} v^t\, [1 - (t/45)]\, dt} = 0.04456.$$

b. Adapting (6.2.11) with recognition of the jump in the d.f. of L at $u = -\bar{P}\bar{a}_{\overline{20}|}$, induced by the constraint on the 20-year term insurance loss variable, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 & u &< -\bar{P}\bar{a}_{\overline{20}|} \\ &= \frac{25}{45} & -\bar{P}\bar{a}_{\overline{20}|} &\le u \le v^{20} - \bar{P}\bar{a}_{\overline{20}|} \\ &= 1 + \frac{1}{0.06}\,\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} & v^{20} - \bar{P}\bar{a}_{\overline{20}|} &< u \le 1 \\ &= 1 & 1 &< u. \end{aligned}$$

A diagram of the d.f. associated with a 20-year term insurance is shown in Figure 6.2.2b. The premium is determined by solving $F_L(0) = 0.75$ for $\bar{P}$. Using part (a) we find, once more, that $\bar{P} = 0.06224$.

c. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = -\bar{P}\bar{a}_{\overline{10}|}$ induced by the constraint on L for 10-year term insurance, we have the family of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 & u &< -\bar{P}\bar{a}_{\overline{10}|} \\ &= \frac{35}{45} & -\bar{P}\bar{a}_{\overline{10}|} &\le u \le v^{10} - \bar{P}\bar{a}_{\overline{10}|} \\ &= 1 + \frac{1}{0.06}\,\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} & v^{10} - \bar{P}\bar{a}_{\overline{10}|} &< u \le 1 \\ &= 1 & 1 &< u. \end{aligned}$$

It is tempting to conjecture that $\bar{P} = 0.06224$, as it was in parts (a) and (b), when we observe that the only nonconstant values of the d.f. have the same formula as in the earlier parts of this example. When we observe that for any u in the interval $(-\bar{P}\bar{a}_{\overline{10}|}, v^{10} - \bar{P}\bar{a}_{\overline{10}|})$, $F_L(u) = 35/45 > 0.75$, it appears that the conjecture is wrong. Figure 6.2.2c displays the d.f. of L when $\bar{P} = 0.06224$ and confirms this judgment. As in Example 6.2.3, try $\bar{P} = 0$. The corresponding d.f. of L is

$$\begin{aligned} F_L(u) &= 0 & u &< 0 \\ &= \frac{35}{45} & 0 &\le u \le v^{10} \\ &= 1 + \frac{1}{0.06}\,\frac{\log u}{45} & v^{10} &< u \le 1 \\ &= 1 & 1 &< u \end{aligned}$$

and the probability of a positive value of L is

$$\Pr(L > 0) = \frac{10}{45} < 0.25.$$

This is illustrated in Figure 6.2.2d.

As in Example 6.2.3c, the specifications for applying the percentile premium principle leads to $\bar{P} = 0$, an anomalous result from a business perspective. ▼

6.3 Fully Discrete Premiums

In Section 6.2 we have discussed the theory of fully continuous benefit premiums. In this section we consider annual premium insurances like the one that appeared in Example 6.1.1; that is, the sum insured is payable at the end of the policy year in which death occurs, and the first premium is payable when the insurance is issued. Subsequent premiums are payable on anniversaries of the policy issue date while the insured survives during the contractual premium payment period. The set of annual premiums form a life annuity-due. This model does not conform to practice but is of historic importance in the development of actuarial theory.

Under these circumstances, the level annual benefit premium for a unit whole life insurance is denoted by P_x, where the absence of $(\bar{A}_x)$ means that the *insurance* is payable at the end of the policy year of death. The loss for this insurance is

$$L = v^{K+1} - P_x\, \ddot{a}_{\overline{K+1}|} \qquad K = 0, 1, 2, \ldots. \tag{6.3.1}$$

The equivalence principle requires that $\mathrm{E}[L] = 0$, or

$$\mathrm{E}[v^{K+1}] - P_x\, \mathrm{E}[\ddot{a}_{\overline{K+1}|}] = 0,$$

which yields

$$P_x = \frac{A_x}{\ddot{a}_x}. \tag{6.3.2}$$

This is the discrete analogue of (6.2.4).

Using (5.3.7) in place of (5.2.8) in steps parallel to those taken in obtaining (6.2.7), we obtain

$$\mathrm{Var}(L) = \frac{{}^2A_x - (A_x)^2}{(d\, \ddot{a}_x)^2}. \tag{6.3.3}$$

Example 6.3.1

If

$$_{k|}q_x = c(0.96)^{k+1} \qquad k = 0, 1, 2, \ldots$$

where $c = 0.04/0.96$ and $i = 0.06$, calculate P_x and $\mathrm{Var}(L)$.

Solution:

First we exhibit the components of (6.3.2),

$$A_x = c \sum_{k=0}^{\infty} (1.06)^{-k-1} (0.96)^{k+1} = 0.40,$$

$$\ddot{a}_x = \frac{1 - A_x}{d} = 10.60.$$

Then using (6.3.2) we obtain

$$P_x = \frac{A_x}{\ddot{a}_x} = 0.0377.$$

For $\mathrm{Var}(L)$, we calculate

$$^2A_x = c \sum_{k=0}^{\infty} [(1.06)^2]^{-k-1} (0.96)^{k+1} = 0.2445.$$

Therefore,

$$\begin{aligned}\mathrm{Var}(L) &= \frac{0.2445 - 0.1600}{[(0.06)(10.60)/(1.06)]^2} \\ &= 0.2347.\end{aligned}$$

▼

There is a connection between Examples 6.2.1 and 6.3.1. Since

$$_{k|}q_x = \int_k^{k+1} {}_tp_x\, \mu_x(t)\, dt \qquad k = 0, 1, 2, \ldots \tag{6.3.4}$$

for the situation described in Example 6.3.1, we have

$$\frac{0.04}{0.96} (0.96)^{k+1} = \int_k^{k+1} {}_tp_x\, \mu_x(t)\, dt.$$

If the force of mortality is a constant, μ, it follows that

$$\frac{0.04}{0.96} (0.96)^{k+1} = e^{-(k+1)\mu} (e^{\mu} - 1),$$

and then $e^{-\mu} = 0.96$ and $\mu = 0.0408$. The geometric distribution, with p.f.

$$_{k|}q_x = \frac{0.04}{0.96} (0.96)^{k+1},$$

is a discrete version of the exponential distribution with $\mu = 0.0408$. Formula (6.3.4) provides the bridge between the discrete and continuous versions. The fully continuous annual benefit premium corresponding to $P_x = 0.0377$ in Example 6.3.1 would be $\bar{P}(\bar{A}_x) = \mu = 0.0408$.

Continuing to use the equivalence principle, we can determine formulas for annual benefit premiums for a variety of fully discrete life insurances. Our general loss will be

$$b_{K+1} v_{K+1} - P Y$$

where

- b_{k+1} and v_{k+1} are, respectively, the benefit and discount functions defined in (4.3.1)
- P is a general symbol for an annual premium paid at the beginning of each policy year during the premium paying period while the insured survives and
- Y is a discrete annuity random variable as defined, for example, in connection with (5.3.9).

Application of the equivalence principle yields

$$\mathrm{E}[b_{K+1} v_{K+1} - P Y] = 0,$$

or

$$P = \frac{\mathrm{E}[b_{K+1} v_{K+1}]}{\mathrm{E}[Y]}.$$

These ideas are used in Table 6.3.1 to display premium formulas for fully discrete insurances.

Example 6.3.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.3.1).

Solution:

We start with the notation of Table 6.3.1. Let

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots. \end{cases}$$

Then we can write, by reference to the third row of Table 6.3.1,

$$L = Z - P_{x:\overline{n|}} \frac{1 - Z}{d};$$

therefore we have

$$\mathrm{Var}(L) = \mathrm{Var}\left[Z \left(1 + \frac{P_{x:\overline{n|}}}{d}\right) - \frac{P_{x:\overline{n|}}}{d} \right].$$

We can use the rule of moments to find Var(Z), as indicated in Table 4.3.1, and then obtain