trials result in one of m different outcomes. The probability that a trial ends in outcome i is denoted by π_i. We denote the random variable that counts the number of outcomes i in n trials by N_i. Then

$$1 = \sum_{i=1}^{m} \pi_i, \qquad n = \sum_{i=1}^{m} N_i,$$

and the joint p.f. of $N_1, N_2, \ldots, N_m$ is given by

$$\Pr(N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!}\, \pi_1^{n_1}\, \pi_2^{n_2} \cdots \pi_m^{n_m}. \quad (12.4.8)$$

By using (12.4.8) we obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i\, N_i\right)\right] = (\pi_1\, e^{t_1} + \pi_2\, e^{t_2} + \cdots + \pi_m\, e^{t_m})^n. \quad (12.4.9)$$

The multivariate discrete distribution with p.f. given by (12.4.8) and m.g.f. given by (12.4.9) is called a multinomial distribution with parameters $n, \pi_1, \ldots, \pi_m$.

Theorem 12.4.2

If S, as given in (12.4.7), has a compound Poisson distribution with parameter λ and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6), then

a. $N_1, N_2, \ldots, N_m$ are mutually independent.

b. N_i has a Poisson distribution with parameter $\lambda_i = \lambda\, \pi_i$, $i = 1, 2, \ldots, m$.

Proof:

We start by defining the m.g.f. of the joint distribution of $N_1, N_2, \ldots, N_m$ by use of (2.2.10) for conditional expectations. Note that for a fixed number of independent claims (trials) where each claim results in one of m claim amounts, the numbers of claims of each amount have a multinomial distribution with parameters $n, \pi_1, \pi_2, \ldots, \pi_m$. Hence, given

$$N = \sum_{i=1}^{m} N_i = n,$$

the conditional distribution of $N_1, N_2, \ldots, N_m$ is this multinomial distribution. For this case, we use (12.4.9) to obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i\, N_i\right)\right] = \sum_{n=0}^{\infty} \mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i\, N_i\right) \middle| N = n\right] \Pr(N = n)$$

$$= \sum_{n=0}^{\infty} (\pi_1\, e^{t_1} + \cdots + \pi_m\, e^{t_m})^n\, \frac{e^{-\lambda}\, \lambda^n}{n!}. \quad (12.4.10)$$

We now perform the required summation by recognizing (12.4.10) as a Taylor series expansion of an exponential function. We obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = \exp(-\lambda)\exp\left(\lambda \sum_{i=1}^{m} \pi_i e^{t_i}\right)$$

$$= \prod_{i=1}^{m} \exp[\lambda \pi_i(e^{t_i} - 1)]. \qquad (12.4.11)$$

Since this is the product of m functions each of a single variable t_i, (12.4.11) shows the mutual independence of the N_i's. Furthermore, if we set $t_i = t$, and $t_j = 0$ for $j \neq i$ in (12.4.11), we obtain

$$\mathrm{E}[\exp(tN_i)] = \exp[\lambda \pi_i(e^t - 1)] , \qquad (12.4.12)$$

which is the m.g.f. of the Poisson distribution with parameter $\lambda \pi_i$. This proves statement (b). ■

Formula (12.4.7) and Theorem 12.4.2 provide an alternative method for tabulating a compound Poisson distribution with a discrete claim amount distribution. First, we compute the p.f.'s of $x_1 N_1, x_2 N_2, \ldots, x_m N_m$. Since the nonzero entries for the p.f. of $x_i N_i$ are at multiples of x_i and are Poisson probabilities, this is an easy task. Then the convolution of these m distributions is calculated to obtain the p.f. of S. This method is particularly convenient if m, the number of different claim amounts, is small. Even if a continuous distribution has been selected for the individual claim amounts, a discrete approximation can sometimes be used with this alternative method to produce a satisfactory approximation to the distribution of S. The basic and the alternative methods for tabulating the distribution of S are compared in the following example.

Example 12.4.2

Suppose that S has a compound Poisson distribution with $\lambda = 0.8$ and individual claim amounts that are 1, 2, or 3 with probabilities 0.25, 0.375, and 0.375, respectively. Compute $f_S(x) = \Pr(S = x)$ for $x = 0, 1, \ldots, 6$.

Solution:

For the basic method, the calculations parallel those in Example 12.2.2 and are summarized below.

Basic Method Calculations

(1) x	(2) $p^{*0}(x)$	(3) $p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $p^{*4}(x)$	(7) $p^{*5}(x)$	(8) $p^{*6}(x)$	(9) $f_S(x)$
0	1	—	—	—	—	—	—	0.449329
1	—	0.250000	—	—	—	—	—	0.089866
2	—	0.375000	0.062500	—	—	—	—	0.143785
3	—	0.375000	0.187500	0.015625	—	—	—	0.162358
4	—	—	0.328125	0.070313	0.003906	—	—	0.049905
5	—	—	0.281250	0.175781	0.023438	0.000977	—	0.047360
6	—	—	0.140625	0.263672	0.076172	0.007324	0.000244	0.030923
n	0	1	2	3	4	5	6	
$e^{-0.8}\dfrac{(0.8)^n}{n!}$	0.449329	0.359463	0.143785	0.038343	0.007669	0.001227	0.000164	

For the alternative method outlined in this section, the calculations are displayed below.

Alternative Method Calculations

(1) (x)	(2) $\Pr(1N_1 = x)$	(3) $\Pr(2N_2 = x)$	(4) $\Pr(3N_3 = x)$	(5) $\Pr(N_1 + 2N_2 = x)$ = (2)*(3)	(6) $\Pr(N_1 + 2N_2 + 3N_3 = x)$ = (4)*(5) = $f_S(x)$
0	0.818731	0.740818	0.740818	0.606531	0.449329
1	0.163746	—	—	0.121306	0.089866
2	0.016375	0.222245	—	0.194090	0.143785
3	0.001092	—	0.222245	0.037201	0.162358
4	0.000055	0.033337	—	0.030973	0.049905
5	0.000002	—	—	0.005703	0.047360
6	0.000000	0.003334	0.033337	0.003287	0.030923
i	1	2	3		
λ_i	0.2	0.3	0.3		
	$\dfrac{e^{-0.2}\,(0.2)^x}{x!}$	$\dfrac{e^{-0.3}\,(0.3)^{x/2}}{(x/2)!}$	$\dfrac{e^{-0.3}\,(0.3)^{x/3}}{(x/3)!}$		

For the application of the formulas of this section, we note that $m = 3$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $\lambda_1 = \lambda\, p(1) = 0.2$, $\lambda_2 = \lambda\, p(2) = 0.3$, and $\lambda_3 = \lambda\, p(3) = 0.3$. First, we compute columns (2), (3), and (4). The nonzero entries are Poisson probabilities. Then we obtain the convolution of the p.f.'s in columns (2) and (3) and record the result in column (5). Finally, we convolute the p.f.'s displayed in columns (4) and (5) and record the result in column (6).

Remember that the complete p.f. is not displayed in either set of calculations. The example required probabilities for only $x = 0, 1, \ldots, 6$. However, $\Pr(S \leq 6) = f_S(0) + f_S(1) + \cdots + f_S(6) = 0.973526$. ▼

Formula (12.4.7) and Theorem 12.4.2 have another implication. Instead of defining a compound Poisson distribution of S by specifying the parameter λ and the d.f. $P(x)$ of the discrete individual claim amounts, we can define the distribution in terms of the possible individual claim amounts $x_1, x_2, \ldots, x_m$ and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the associated Poisson distributions described in part (b) of Theorem 12.4.2. Thus for x_i there is an associated Poisson distribution of N_i with parameter λ_i. In terms of this new definition of the distribution of S, we have from $\mathrm{E}[N_i] = \mathrm{Var}(N_i) = \lambda_i$ and the independence of the N_i's that

$$\mathrm{E}[S] = \mathrm{E}\left[\sum_{i=1}^{m} x_i N_i\right] = \sum_{i=1}^{m} x_i \lambda_i \tag{12.4.13}$$

and

$$\mathrm{Var}(S) = \mathrm{Var}\left(\sum_{i=1}^{m} x_i N_i\right) = \sum_{i=1}^{m} x_i^2 \lambda_i . \tag{12.4.14}$$

Formula (12.4.13) could be obtained by starting from (12.3.2) and noting that

$$\lambda p_1 = \lambda \sum_{i=1}^{m} x_i \pi_i = \sum_{i=1}^{m} x_i \lambda_i .$$

Similarly we can obtain (12.4.14) from (12.3.3).

In some cases it is useful, as in Example 12.4.1, to regard S as a sum of mutually independent random variables $x_i N_i$, $i = 1, 2, \ldots, m$, where $x_i N_i$ has a compound Poisson distribution with parameter λ_i and degenerate claim amount distribution at x_i. This interpretation follows from Theorem 12.4.2 and underlies the alternative method illustrated in Example 12.4.2.

There is a third way, the ***recursive method,*** for evaluating certain compound distributions for which the only possible claim amounts are positive integers. It is based on the recursive formula of the following theorem.

Theorem 12.4.3

For compound distributions where the probability distribution for N, the number of claims, satisfies the condition $\Pr(N = n)/\Pr(N = n - 1) = a + (b/n)$ for $n = 1, 2, \ldots$, and where the distribution of claim amounts is restricted to the positive integers,

$$f_S(x) = \sum_{i=1}^{x} \left[a + \left(\frac{bi}{x} \right) \right] p(i) \, f_S(x - i) \qquad x = 1, 2, \ldots, \tag{12.4.15}$$

with the starting value given by $f_S(0) = \Pr(N = 0)$.

We establish the following lemma to be used in the proof of the theorem.

Lemma

For $X_1, X_2, X_3, \ldots, X_n$ which are independent and identically distributed random variables taking on values restricted to the positive integers, we have, for positive integer values of x,

(i) $$p^{*n}(x) = \sum_{i=1}^{x} p(i) \, p^{*(n-1)}(x - i)$$

(ii) $$p^{*n}(x) = \frac{n}{x} \sum_{i=1}^{x} i \, p(i) \, p^{*(n-1)}(x - i).$$

Proof of the Lemma:

For $n = 1$, both (i) and (ii) reduce to $p^{*1}(x) = p(x) \times p^{*(0)}(0)$. For $n > 1$ we establish (i) by using the Law of Total Probability to evaluate $\Pr(X_1 + X_2 + \cdots + X_n = x)$ by conditioning on the value taken by X_1 as

$$\sum_{i=1}^{x} \Pr(X_1 = i) \Pr(X_2 + X_3 + \cdots + X_n = x - i).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x - i)$ and $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n - 1)$-fold and n-fold convolutions, respectively, of $p(i)$; see (2.3.4).

For $n > 1$, we establish (ii) by considering the conditional expectations $E[X_k|X_1+X_2+X_3+\cdots+X_n = x]$ for $k = 1, 2, 3, \ldots, n$. From reasons of symmetry, these quantities are the same for all such k. Since their sum is x, each is equal to x/n. The conditional expectation $E[X_1|X_1+X_2+X_3+\cdots+X_n = x]$ is evaluated as

$$\sum_{i=1}^{x} i \Pr(X_1 = i) \Pr(X_2+X_3+\cdots+X_n = x - i)/\Pr(X_1 + X_2 + X_3+\cdots+X_n = x).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x - i)$ and that $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n - 1)$-fold and n-fold convolutions, respectively, of $p(i)$. Solving for $p^{*n}(x)$ completes the proof. ■

Proof of the Theorem:

First,

$$f_S(x) = \sum_{n=1}^{\infty} \Pr(N = n)\, p^{*n}(x).$$

With $\Pr(N = n) = [a + (b/n)] \Pr(N = n - 1)$, we have

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*n}(x) + \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n - 1)\, p^{*n}(x),$$

and by the two parts of the lemma

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n - 1) \sum_{i=1}^{x} p(i)\, p^{*\,n-1}(x - i)$$
$$+ \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n - 1) \frac{n}{x} \sum_{i=1}^{x} i\, p(i)\, p^{*\,n-1}(x - i).$$

Interchanging the order of summation, we get

$$f_S(x) = a \sum_{i=1}^{x} p(i) \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*\,n-1}(x - i)$$
$$+ \frac{b}{x} \sum_{i=1}^{x} ip(i) \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*\,n-1}(x - i)$$
$$= a \sum_{i=1}^{x} p(i)\, f_S(x - i) + \frac{b}{x} \sum_{i=1}^{x} i\, p(i)\, f_S(x - i)$$
$$= \sum_{i=1}^{x} \left(a + \frac{b\,i}{x}\right) p(i)\, f_S(x - i).$$ ■

We now examine the only three distributions that satisfy the required relationship between successive values of $\Pr(N = n)$.

a. Poisson: $[\Pr(N = n)/\Pr(N = n - 1)] = \lambda/n$. The recursion formula for the compound Poisson is

$$f_S(x) = \frac{\lambda}{x} \sum_{i=1}^{x} i p(i) f_S(x-i) \quad \text{with} \quad f_S(0) = e^{-\lambda}. \tag{12.4.16}$$

b. Negative binomial: $[\Pr(N = n)/\Pr(N = n-1)] = (1-p)[(n+r-1)/n]$ so that $a = (1-p)$ and $b = (1-p)(r-1)$. The recursion formula for the compound negative binomial is

$$f_S(x) = (1-p) \sum_{i=1}^{x} \left[(r-1)\frac{i}{x} + 1 \right] p(i) f_S(x-i) \tag{12.4.17}$$

with $f_S(0) = p^r$.

c. Binomial with parameters m and p:

$$\frac{\Pr(N = n)}{\Pr(N = n-1)} = \frac{m+1-n}{n} \frac{p}{1-p}$$

so that $a = -[p/(1-p)]$ and $b = (m+1)[p/(1-p)]$. The recursion formula in this case is

$$f_S(x) = \left(\frac{p}{1-p}\right) \sum_{i=1}^{x} \left[(m+1)\frac{i}{x} - 1 \right] p(i) f_S(x-i) \tag{12.4.18}$$

with $f_S(0) = (1-p)^m$.

Example 12.4.2 (recomputed)

For the compound Poisson distribution of this example, compute $f_S(x) = \Pr(S = x)$ by the recursive method.

Solution:

Substituting the values used for the alternative method of calculation into (12.4.16) yields

$$f_S(x) = \frac{1}{x} [0.2 f_S(x-1) + 0.6 f_S(x-2) + 0.9 f_S(x-3)] \qquad x = 1, 2, \ldots .$$

Recalling that $f_S(x) = 0$, $x < 0$, and $f_S(0) = e^{-\lambda} = 0.449329$, we readily reproduce the values of $f_S(x)$ given in the basic method calculations. ▼

12.5 Approximations to the Distribution of Aggregate Claims

In Section 2.4 the normal distribution was employed as an approximation to the distribution of aggregate claims in the individual model. The normal approximation is the first developed here for use with the collective model.

For the compound Poisson distribution, the two parameters of the normal approximation are given by (12.3.2) and (12.3.3). For the compound negative binomial distribution the parameters are given by (12.3.10) and (12.3.11). In each of the two cases the approximation is better when the expected number of claims is large, or, in other words, when λ is large for the compound Poisson case and when r is large for the negative binomial. These two results are contained in Theorem 12.5.1, which may be interpreted as a version of the central limit theorem.

Theorem 12.5.1

a. If S has a compound Poisson distribution, specified by λ and $P(x)$, then the distribution of

$$Z = \frac{S - \lambda\, p_1}{\sqrt{\lambda\, p_2}} \tag{12.5.1}$$

converges to the standard normal distribution as $\lambda \to \infty$.

b. If S has a compound negative binomial distribution, specified by r, p, and $P(x)$, then the distribution of

$$Z = \frac{S - r(q/p)p_1}{\sqrt{r(q/p)p_2 + r(q^2/p^2)p_1^2}} \tag{12.5.2}$$

converges to the standard normal distribution as $r \to \infty$.

Proof:

We shall prove statement (a) by showing that

$$\lim_{\lambda\to\infty} M_Z(t) = e^{t^2/2}.$$

Statement (b) can be proved using a similar strategy, but the proof involves additional steps.

From (12.5.1), it follows that

$$M_Z(t) = M_S\left(\frac{t}{\sqrt{\lambda p_2}}\right) \exp\left(-\frac{\lambda p_1\, t}{\sqrt{\lambda p_2}}\right).$$

Now we use (12.3.5) to obtain

$$M_Z(t) = \exp\left\{\lambda\left[M_X\left(\frac{t}{\sqrt{\lambda p_2}}\right) - 1\right] - \frac{\lambda p_1\, t}{\sqrt{\lambda p_2}}\right\}, \tag{12.5.3}$$

and then substitute the expansion

$$M_X(t) = 1 + \frac{p_1 t}{1!} + \frac{p_2 t^2}{2!} + \cdots, \tag{12.5.4}$$

with $t/\sqrt{\lambda p_2}$ in place of t, into (12.5.3) to obtain

$$M_Z(t) = \exp\left(\frac{1}{2}\, t^2 + \frac{1}{6}\frac{1}{\sqrt{\lambda}}\frac{p_3}{p_2^{3/2}}\, t^3 + \cdots\right). \tag{12.5.5}$$

Then as $\lambda \to \infty$, $M_Z(t)$ approaches $e^{t^2/2}$, the m.g.f. of the standard normal distribution. ■

A normal distribution may not be the best approximation to the aggregate claim distribution because the normal distribution is symmetric and the distribution of aggregate claims is often skewed. This skewness is evident in Table 12.5.1, which shows that the third central moment of S under each of the compound Poisson and compound negative binomial distributions is not 0. For positive claim amount distributions, $P(0) = 0$, and the third central moment of S is positive in each case.

TABLE 12.5.1

Calculation of Third Central Moment of S

	Distribution of S	
Step	**Compound Poisson**	**Compound Negative Binomial**
$M_S(t)$	$\exp\{\lambda[M_X(t) - 1]\}$	$\left[\dfrac{p}{1 - qM_X(t)}\right]^r$
$\log M_S(t)$	$\lambda[M_X(t) - 1]$	$r \log p - r \log[1 - qM_X(t)]$
$\dfrac{d^3}{dt^3} \log M_S(t)$	$\lambda\, M'''_X(t)$	$\dfrac{rqM'''_X(t)}{1 - qM_X(t)} + \dfrac{3rq^2M'_X(t)M''_X(t)}{[1 - qM_X(t)]^2} + \dfrac{2rq^3M'_X(t)^3}{[1 - qM_X(t)]^3}$
$\mathrm{E}[(S - \mathrm{E}[S])^3] = \left.\dfrac{d^3}{dt^3} \log M_S(t)\right\rvert_{t=0}$ *	$\lambda\, p_3$	$\dfrac{rqp_3}{p} + \dfrac{3rq^2p_1p_2}{p^2} + \dfrac{2rq^3p_1^3}{p^3}$

*For $k \geq 4$, $\left.\dfrac{d^k}{dt^k} \log M(t)\right\rvert_{t=0}$ is not the k-th central moment.

In completing Table 12.5.1 we use properties of the logarithm of a m.g.f., for example,

$$\left.\frac{d}{dt} \log M_X(t)\right\rvert_{t=0} = \frac{M'_X(0)}{M_X(0)} = \mu$$

and

$$\frac{d^2}{dt^2} \log M_X(t)\big\rvert_{t=0} = \frac{M''_X(0)M_X(0) - M'_X(0)^2}{M_X(0)^2} = \sigma^2.$$

In Exercise 12.23(a), the reader is asked to confirm the relation used in the last row of Table 12.5.1.

Because of this skewness we seek a more general approximation to the distribution of aggregate claims, one that accommodates skewness. For this second approximation, we begin with a gamma distribution. This choice is motivated by the fact that the gamma distribution has a positive third central moment as do the compound Poisson and compound negative binomial distributions with positive claim amounts. We let $G(x{:}\alpha, \beta)$ denote the d.f. of the gamma distribution with parameters α and β; that is,

$$G(x{:}\alpha, \beta) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)}\, t^{\alpha-1}\, e^{-\beta t}\, dt. \tag{12.5.6}$$

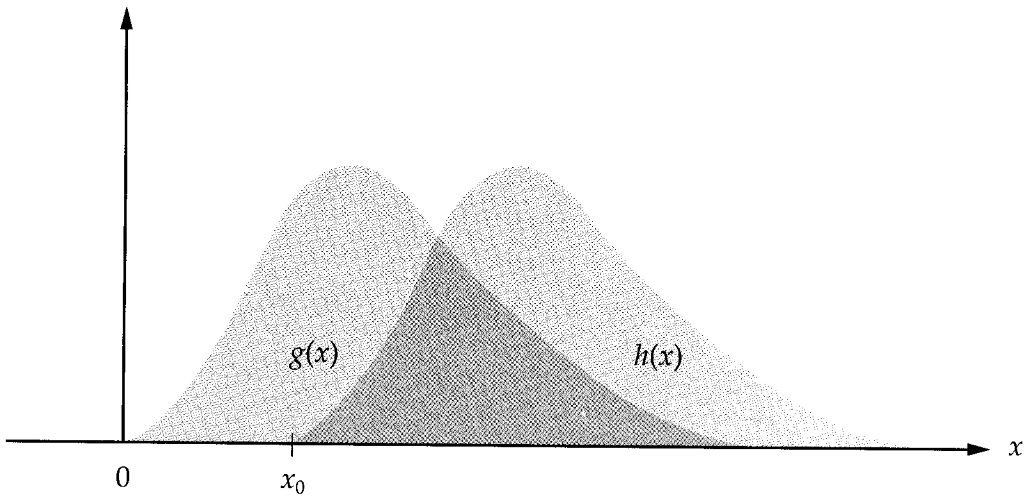
Then for any x_0 we define a new d.f., denoted by $H(x{:}\alpha, \beta, x_0)$, as

$$H(x{:}\alpha, \beta, x_0) = G(x - x_0{:}\alpha, \beta). \tag{12.5.7}$$

This amounts to a translation of the distribution $G(x{:}\alpha, \beta)$ by x_0. Figure 12.5.1 illustrates this for the case $x_0 > 0$ where $g(x)$, $x \geq 0$, and $h(x)$, $x \geq x_0$, denote, respectively, the p.d.f.'s associated with $G(x{:}\alpha, \beta)$ and $H(x{:}\alpha, \beta, x_0)$.

FIGURE 12.5.1

Translated Gamma Distribution



We approximate the distribution of aggregate claims S by a translated gamma distribution where the parameters α, β, and x_0 are selected by equating the first moment and second and third central moments of S with the corresponding characteristics of the translated gamma distribution. Since central moments of the translated gamma are the same as for the basic gamma distribution, this procedure imposes the requirements

$$\mathrm{E}[S] = x_0 + \frac{\alpha}{\beta}, \tag{12.5.8}$$

$$\mathrm{Var}(S) = \frac{\alpha}{\beta^2}, \tag{12.5.9}$$

$$\mathrm{E}[(S - \mathrm{E}[S])^3] = \frac{2\alpha}{\beta^3}. \tag{12.5.10}$$

From these we obtain

$$\beta = 2\frac{\mathrm{Var}(S)}{\mathrm{E}[(S - \mathrm{E}[S])^3]}, \tag{12.5.11}$$

$$\alpha = 4\frac{[\mathrm{Var}(S)]^3}{\mathrm{E}[(S - \mathrm{E}[S])^3]^2}, \tag{12.5.12}$$

$$x_0 = \mathrm{E}[S] - 2\frac{[\mathrm{Var}(S)]^2}{\mathrm{E}[(S - \mathrm{E}[S])^3]}. \tag{12.5.13}$$

For a compound Poisson distribution this procedure leads to

$$\alpha = 4\lambda \frac{p_2^3}{p_3^2}, \tag{12.5.14}$$

$$\beta = 2\frac{p_2}{p_3}, \tag{12.5.15}$$

$$x_0 = \lambda\, p_1 - 2\lambda \frac{p_2^2}{p_3}. \tag{12.5.16}$$

Remark:

We can show that if $\alpha \to \infty$, $\beta \to \infty$, and $x_0 \to -\infty$ such that

$$\begin{aligned} x_0 + \frac{\alpha}{\beta} &= \mu \text{ (constant)}, \\ \frac{\alpha}{\beta^2} &= \sigma^2 \text{ (constant)}, \end{aligned} \tag{12.5.17}$$

the distribution $H(x{:}\alpha, \beta, x_0)$ converges to the $N(\mu, \sigma^2)$ distribution. Therefore, the family of normal distributions is contained, as limiting distributions, within this family of three-parameter gamma distributions. In this sense, this approximation is a generalization of the normal approximation.

Example 12.5.1

Consider the Poisson distribution with parameter $\lambda = 16$. This is the same as the compound Poisson distribution with $\lambda = 16$ and a degenerate claim amount distribution at 1. Compare this distribution with approximations by

a. A translated gamma distribution
b. A normal distribution.

Solution:

a. Here $p_k = 1$, $k = 1, 2, 3$, and from (12.5.14)–(12.5.16), we have $\alpha = 64$, $\beta = 2$, $x_0 = -16$. Note that, unlike the case in Figure 12.5.1, x_0 is negative.
b. For the normal approximation, we use $\mu = 16$ and $\sigma = 4$.

The results given below compare the three distributions. In the approximations, the half-integer discontinuity correction was used to approximate $F_S(x)$ for $x = 5, 10, \ldots, 40$.

	Exact	Approximations	
x	$\sum_{y=0}^{x} \frac{e^{-16}(16)^y}{y!}$	$G(x+16.5:64, 2)$	$\Phi\left(\frac{x+0.5-16}{4}\right)$
5	0.001384	0.001636	0.004332
10	0.077396	0.077739	0.084566
15	0.466745	0.466560	0.450262
20	0.868168	0.868093	0.869705
25	0.986881	0.986604	0.991226
30	0.999433	0.999378	0.999856
35	0.999988	0.999985	0.999999
40	1.000000	1.000000	1.000000

▼

In the case of the compound negative binomial distribution there is an additional argument that supports the use of a gamma approximation. The argument is outlined in the Appendix to this chapter.

12.6 Notes and References

Chapter 2 of Seal (1969) contains an extensive survey of the literature on collective risk models, including the pioneering work of Lundberg on the compound Poisson distribution. Several authors, for example, Dropkin (1959) and Simon (1960), have used the negative binomial distribution to model the number of automobile accidents by a collection of policyholders in a fixed period.

In Example 12.3.1 we derived the negative binomial distribution by assuming that the unknown Poisson parameter has a gamma distribution. This idea goes back at least as far as Greenwood and Yule's work on accident proneness (1920). An alternative derivation in terms of a contagion model is due to Polya and Eggenberger and may be found in Chapter 2 of Bühlmann (1970). In the special case where r is an integer, the negative binomial can be obtained as the distribution of the number of Bernoulli trials that end in failure prior to the r-th success. This development may be found in most probability texts, but has little relevance to the subject of this chapter.

Theorem 12.4.2 has been known for some time and can be studied in Section 2, Chapter 2 of Feller (1968). The alternative method for computing probabilities for a compound Poisson distribution, which is based on Theorem 12.4.2, was suggested by Pesonen (1967) and implemented by Halmstad (1976) in the calculation of stop-loss premiums. Theorem 12.4.2 has a converse, which was not stated in Section 12.4. Renyi (1962) shows that the mutual independence of $N_1, N_2, \ldots, N_m$ implies that N has a Poisson distribution. Hence the alternative method of computing will work only for the compound Poisson distribution.

The alternative method of computation may also be adopted to build a simulation model for aggregate claims. Instead of determining the individual claim amounts, one simulates $N_1, N_2, \ldots, N_m$ and obtains a realization of S directly from (12.4.7). For one determination of S, the expected number of random numbers required is $1 + \lambda$ under the basic method. If the alternative method is used, exactly m random numbers are required for each determination of a value of S.

There are several more elaborate methods of approximating the distribution of aggregate claims. The normal power and Esscher approximations are described in Beard, Pesonen, and Pentikäinen (1984). Several of the approximation methods have been compared by Bohman and Esscher (1963, 1964). Seal (1978a) presents the case for the translated gamma approximation and illustrates its excellent performance. Bowers (1966) approximated the distribution of aggregate claims by a sum of orthogonal functions, the first term of which is the gamma distribution. The result, stated in the Appendix to this chapter, that the gamma distribution can be obtained as a limit from the compound negative binomial is due to Lundberg (1940).

Sometimes the distribution of aggregate claims can be obtained from a numerical inversion of its m.g.f.; this is developed in Chapter 3 of Seal (1978).

A monograph by Panjer and Willmot (1992) develops more completely the ideas of this chapter with particular emphasis on recursive calculation and discrete approximations.

Appendix

Theorem 12.A.1

If the random variables S_k, $k = 0, 1, 2, \ldots$, have compound negative binomial distributions with parameters r and $p(k)$ and claim amount d.f. $P(x)$, and if the parameters of the negative binomial distributions are such that

$$\frac{q(k)}{p(k)} = k\,\frac{q}{p}$$

for $k = 1, 2, 3, \ldots$, where $q = 1 - p$ is a constant, then the distribution of

$$\frac{S_k}{\mathrm{E}[S_k]}$$

approaches $G(x{:}r, r)$ as $k \to \infty$.

Proof:

Using (12.3.12), we find the m.g.f. of $S_k/\mathrm{E}[S_k]$ to be

$$\left[\frac{p(k)}{1 - q(k)M_X(t/\mathrm{E}[S_k])}\right]^r. \tag{12.A.1}$$

We also have

$$M_X\left(\frac{t}{\mathrm{E}[S_k]}\right) = 1 + \frac{p_1}{\mathrm{E}[S_k]}\,t + \frac{p_2}{2\mathrm{E}[S_k]^2}\,t^2 + \cdots. \tag{12.A.2}$$

If (12.A.2) is substituted into (12.A.1), we obtain

$$\left\{\frac{p(k)}{1 - q(k) - [q(k)p_1/\mathrm{E}[S_k]]t - [q(k)p_2/2\mathrm{E}[S_k]^2]t^2 - \cdots}\right\}^r. \tag{12.A.3}$$

Now, since

$$\mathrm{E}[S_k] = r\,\frac{q(k)p_1}{p(k)} = r\,\frac{kqp_1}{p},$$

we see that the m.g.f. of $S_k/\mathrm{E}[S_k]$ is

$$\left[1 - \frac{1}{r}t - \frac{p_2}{2r^2p_1^2(q/p)k}t^2 - \cdots\right]^{-r} = \left[1 - \frac{1}{r}t - R(k)\right]^{-r}$$

where the remainder term $R(k)$ is such that $\lim_{k\to\infty} R(k) = 0$. Therefore,

$$\lim_{k\to\infty} \mathrm{E}\left[\exp\left(t\frac{S_k}{\mathrm{E}[S_k]}\right)\right] = \left(\frac{r}{r-t}\right)^r, \tag{12.A.4}$$

which is the m.g.f. of a $G(x{:}r, r)$ distribution. ■

It follows from (12.A.4) that the m.g.f. of S_k itself is approximately

$$\left(\frac{r}{r - \mathrm{E}[S_k]t}\right)^r = \left\{\frac{r}{r - [rq(k)/p(k)]p_1 t}\right\}^r = \left\{\frac{p(k)/[q(k)p_1]}{p(k)/[q(k)p_1] - t}\right\}^r,$$

which is the m.g.f. of $G\{x{:}r, [p(k)/q(k)p_1]\}$. Thus, when k is large, which under the hypothesis of Theorem 12.A.1 implies that the expected number of claims, $rq(k)/p(k) = rk(q/p)$, is large, the distribution of aggregate claims is approximately a gamma distribution.

Theorem 12.A.1 is presented to provide an argument supporting the use of gamma distributions to approximate the distribution of aggregate claims. Comparison of the main ideas in Theorem 12.5.1(b) and Theorem 12.A.1 leads to insights. Theorem 12.5.1(b) follows closely the pattern of the central limit theorem. If in (12.5.2) one writes

$$Z = \frac{S/r - (q/p)p_1}{\sqrt{(q/p)p_2 + (q^2/p^2)p_1^2}/\sqrt{r}},$$

the correspondence is clear, with the parameter r playing the role of n in the central limit theorem.

In Theorem 12.A.1 the parameter r of the negative binomial distribution remains fixed. The expected number of claims changes in proportion to a size parameter k, by compensating changes in $q(k)$ and $p(k) = 1 - q(k)$. Under the hypothesis of Theorem 12.A.1,

$$\mathrm{Var}(S_k) = \frac{rkq}{p}p_2 + r\frac{k^2q^2}{p^2}p_1^2$$

and

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) = \frac{p}{rkqp_1^2}p_2 + \frac{1}{r}.$$

As the size parameter $k \to \infty$,

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) \to \frac{1}{r},$$

as indicated by Theorem 12.A.1. Thus the gamma approximation may be considered in the negative binomial case when the expected number of claims is large and the claim amount distribution has relatively small dispersion.

Exercises

Section 12.1

12.1. Let S denote the number of people crossing a certain intersection by car in a given hour. How would you model S as a random sum?

12.2. Let S denote the total amount of rain that falls at a weather station in a given month. How would you model S as a random sum?

Section 12.2

12.3. Suppose N has a binomial distribution with parameters n and p. Express each of the following in terms of n, p, p_1, p_2, and $M_X(t)$:
a. $E[S]$ b. $Var(S)$ c. $M_S(t)$.

12.4. For the distribution specified in Example 12.2.2, calculate
a. $E[N]$ b. $Var(N)$ c. $E[X]$
d. $Var(X)$ e. $E[S]$ f. $Var(S)$.

Section 12.3

12.5. Suppose that the claim amount distribution is the same as in Example 12.2.2, but that N has a Poisson distribution with $E[N] = 1.7$. Calculate
a. $E[S]$ b. $Var(S)$.

12.6. Suppose that S has a compound Poisson distribution with $\lambda = 2$ and $p(x) = 0.1x$, $x = 1, 2, 3, 4$. Calculate probabilities that aggregate claims equal 0, 1, 2, 3, and 4.

12.7. Consider the family of negative binomial distributions with parameters r and p. Let $r \to \infty$ and $p \to 1$ such that $r(1 - p) = \lambda$ remains constant. Show that the limit obtained is the Poisson distribution with parameter λ. [Hint: Note that $p^r = [1 - (\lambda / r)]^r \to e^{-\lambda}$ as $r \to \infty$, and consider the convergence of the m.g.f.]

12.8. Suppose that S has a compound Poisson distribution with Poisson parameter λ and claim amount p.f.

$$p(x) = [-\log(1 - c)]^{-1} \frac{c^x}{x} \qquad x = 1, 2, 3, \ldots, \qquad 0 < c < 1.$$

Consider the m.g.f. of S and show that S has a negative binomial distribution with parameters p and r. Express p and r in terms of c and λ.

12.9. Let

$$g(x) = 3^{18} x^{17} \frac{e^{-3x}}{17!}$$

and

$$h(x) = 3^6\, x^5 \frac{e^{-3x}}{5!} \qquad x > 0$$

be two p.d.f.'s. Write the convolution of these two distributions, that is, exhibit $g*h(x)$. [Hint: Proceed directly from the definition of convolution in Section 2.3, or make use of (12.3.19).]

12.10. Suppose that the number of accidents incurred by an insured driver in a single year has a Poisson distribution with parameter λ. If an accident happens, the probability is p that the damage amount will exceed a deductible amount. On the assumption that the number of accidents is independent of the severity of the accidents, derive the distribution of the number of accidents that result in a claim payment.

12.11. The m.g.f. of the Poisson inverse Gaussian distribution is given in the solution to Example 12.3.2. Replace the α parameter by $\lambda\beta$ so that the mean is now λ and the variance is $\lambda + \lambda/\beta$. Show that

$$\lim_{\beta \to \infty} M_N(t) = e^{\lambda(e^t - 1)}.$$

This confirms that the Poisson inverse Gaussian distribution approaches the Poisson distribution as $\beta \to \infty$ and the mean remains constant.

Section 12.4

12.12. Suppose that S_1 has a compound Poisson distribution with Poisson parameter $\lambda = 2$ and claim amounts that are 1, 2, or 3 with probabilities 0.2, 0.6, and 0.2, respectively. In addition, S_2 has a compound Poisson distribution with Poisson parameter $\lambda = 6$ and claim amounts that are either 3 or 4 with probability 0.5 for each. If S_1 and S_2 are independent, what is the distribution of $S_1 + S_2$?

12.13. Suppose that N_1, N_2, N_3 are mutually independent and that N_i has a Poisson distribution with $\mathrm{E}[N_i] = i^2$, $i = 1, 2, 3$. What is the distribution of $S = -2N_1 + N_2 + 3N_3$?

12.14. If N has a Poisson distribution with parameter λ, express $\Pr(N = n + 1)$ in terms of $\Pr(N = n)$.

Note that this recursive formula may be useful in calculations such as those for successive entries in columns (2), (3), and (4) of the alternate method calculations of Example 12.4.2.

12.15. Suppose that S has a compound Poisson distribution with parameter λ and discrete p.f. $p(x)$, $x > 0$. Let $0 < \alpha < 1$.

Consider $\tilde{S}$ with a distribution that is compound Poisson with Poisson parameter $\tilde{\lambda} = \lambda/\alpha$ and claim amount p.f. $\tilde{p}(x)$ where

$$\tilde{p}(x) = \begin{cases} \alpha p(x) & x > 0 \\ 1 - \alpha & x = 0. \end{cases}$$

This means we are allowing for claim amounts of 0 (as could happen if there is a deductible) and are modifying the distributions accordingly. Show that S and $\tilde{S}$ have the same distribution by

a. Comparing the m.g.f.'s of S and $\tilde{S}$

b. Comparing the definition of the distribution of S and $\tilde{S}$ in terms of possible claim amounts and the Poisson parameters of the distributions of their frequencies.

12.16. In Example 12.2.2, let N_1 be the random number of claims of amount 1 and N_2 the random number of claims of amount 2. Compute

a. $\Pr(N_1 = 1)$ b. $\Pr(N_2 = 1)$ c. $\Pr(N_1 = 1, N_2 = 1)$.

Are N_1 and N_2 independent?

12.17. Compute $f_S(x)$ for $x = 0, 1, 2, \ldots, 5$ for the following three compound distributions, each with claim amount distribution given by $p(1) = 0.7$ and $p(2) = 0.3$:

a. Poisson with $\lambda = 4.5$

b. Negative binomial with $r = 4.5$ and $p = 0.5$

c. Binomial with $m = 9$ and $p = 0.5$

d. For each of the distributions of (a), (b), and (c) calculate the mean and variance of the number of claims.

12.18. Let S, as given in (12.4.7), have a compound negative binomial distribution with parameters r and p and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).

a. Show that N_i has a negative binomial distribution with parameters r and $p/(p + q\pi_i)$.

b. Show that, in general, N_1 and N_2 are not independent.
[Hint: Use the joint m.g.f. of $N_1, N_2, \ldots, N_m$ as in the proof of Theorem 12.4.2.]

12.19. Show that the compound distribution of Example 12.2.2 does not satisfy the hypotheses of Theorem 12.4.3.

Section 12.5

12.20. Show that if N has a Poisson distribution with parameter λ, the distribution of

$$Z = \frac{N - \lambda}{\sqrt{\lambda}}$$

approaches a $N(0, 1)$ distribution as $\lambda \rightarrow \infty$.

12.21. Use $\log M_S(t)$ as given in Table 12.5.1 to verify (12.3.3) and (12.3.11).

12.22. Suppose that the d.f. of S is $G(x{:}\alpha, \beta)$. Use the m.g.f. [see (12.3.19)] to show that

$$E[S^h] = \frac{\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + h - 1)}{\beta^h} \qquad h = 1, 2, 3, \ldots .$$

12.23. a. Verify that

$$\frac{d^3}{dt^3} \log M_X(t)\bigg|_{t=0} = E[(X - E[X])^3].$$

b. Use (a) to show that if S has a $G(x{:}\alpha, \beta)$ distribution, then

$$E[(S - E[S])^3] = \frac{2\alpha}{\beta^3}.$$

12.24. a. For a given α, determine β and x_0 so that $H(x{:}\alpha, \beta, x_0)$ has mean 0 and variance 1.

b. What is the limit of $H(x{:}\alpha, \sqrt{\alpha}, -\sqrt{\alpha})$ as $\alpha \to \infty$?

12.25. Suppose that S has a compound Poisson distribution with $\lambda = 12$ and claim amounts that are uniformly distributed between 0 and 1. Approximate $\Pr(S < 10)$ using

a. The normal approximation

b. The translated gamma approximation.

Miscellaneous

12.26. The loss ratio for a collection of insurance policies over a single premium period is defined as $R = S/G$ where S is aggregate claims and G is aggregate premiums. Assume that $G = p_1 E[N](1 + \theta)$, $\theta > 0$.

a. Show that

$$E[R] = (1 + \theta)^{-1}$$

and that

$$\operatorname{Var}(R) = \frac{E[N] \operatorname{Var}(X) + p_1^2 \operatorname{Var}(N)}{[p_1 E[N](1 + \theta)]^2}.$$

b. Develop an expression for $\operatorname{Var}(R)$ if

(i) N has a Poisson distribution

(ii) N has a negative binomial distribution.

12.27. Suppose that the distribution of S_1 is compound Poisson, given by λ and $P_1(x)$, and that the distribution of S_2 is compound negative binomial, given by r, p with $q = 1 - p$, and $P_2(x)$. Show that S_1 and S_2 have the same distribution provided that $\lambda = -r \log p$ and

$$P_1(x) = \frac{\sum_{k=1}^{\infty} (q^k/k)\, P_2^{*k}(x)}{-\log p}.$$

[Hint: Show equality of the m.g.f.'s.]
Note that, in the sense of this exercise, every compound negative binomial distribution can be considered as compound Poisson.

12.28. Let S, as given in (12.4.7), have a compound Poisson inverse Gaussian distribution with parameters α and β and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).

a. Show that N_i has a Poisson inverse Gaussian distribution and determine its parameter values.
b. Show that, in general, N_1 and N_2 are not independent.

12.29. Follow the steps displayed in Table 12.5.1 to show that the third central moment of S, when it has a compound Poisson inverse Gaussian distribution, can be expressed in the parameters of its distribution as

$$\frac{\alpha}{\beta} p_3 + \frac{3\alpha}{\beta^2} p_1 p_2 + \frac{3\alpha}{\beta^3} p_1^3.$$

12.30. a. Verify that the extension of (2.2.10) for the mean and (2.2.11) for the variance to the third central moment, $\mu_3(W) = E[\{W - E[W]\}^3]$, is $E[\mu_3\ W|V\ ] + 3\ \mathrm{Cov}(\mathrm{Var}(W|V), E[W|V]\ \ + \mu_3(E[W|V])$.
[Hint: Write $E[W - E[W]] = E[(W - E[W|V]) + (E[W|V] - E[W])]$, expand the third power, and take expectations termwise.]

b. Apply the result of (a) to S of (12.1.1) to express its third central moment in terms of the parameters of its distribution. [Hint: Refer to (12.2.5) and (12.2.6).]
c. Apply the formula of (b) to the compound distributions of Table 12.5.1 to confirm the third central moments shown there.
d. Apply the formula of (b) to the compound Poisson inverse Gaussian distribution to confirm the formula of the previous exercise.

13

COLLECTIVE RISK MODELS OVER AN EXTENDED PERIOD

13.1 Introduction

The purpose of this chapter is to present mathematical models for the variations in the amount of an insurer's ***surplus*** over an extended period of time. By surplus we mean the excess of the initial fund plus premiums collected over claims paid. This is a convenient mathematical, but not accounting, definition of surplus.

Let $U(t)$ denote the surplus at time t, $c(t)$ denote premiums collected through time t, and $S(t)$ denote aggregate claims paid through time t. If u is the surplus on hand at time 0, perhaps as a result of past operations, then $U(t)$ is given by

$$U(t) = u + c(t) - S(t) \qquad t \geq 0. \tag{13.1.1}$$

Now motivated by the question of exhausting surplus at least once in a period of time, we want to explore probability questions about $U(t)$ for many—indeed, infinitely many—values of t simultaneously. Since the questions involve more than a finite number of random variables, consistent with the language of probability, we call $U(t)$ the ***surplus process*** and $S(t)$ the ***aggregate claims process.*** The premiums collected, $c(t)$, will be deterministic, not a stochastic process.

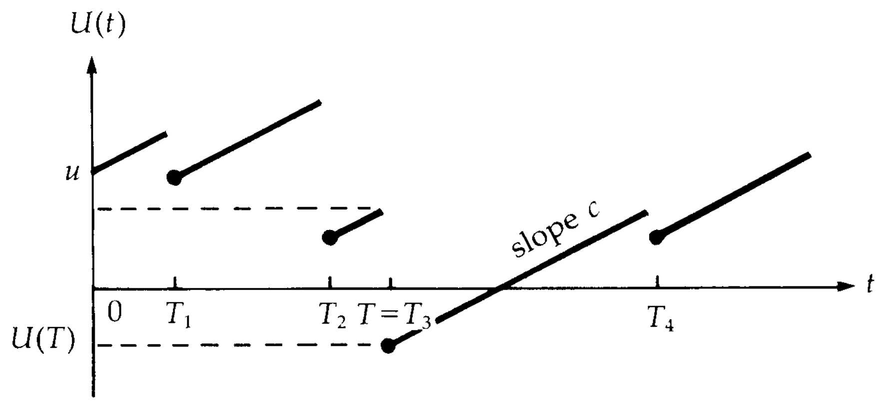
In this setting the models in the previous chapter can be viewed as being concerned with the distribution of the random variable, $U(t)$, for a single value of t. The monitoring for a negative value of the surplus, $U(t)$, can be on a period-to-period basis or on an ongoing basis. In the latter case we would look at the continuous time surplus process

$$\{U(t): t \geq 0\}.$$

A typical outcome of this surplus process, $\{U(t), t \geq 0\}$, is shown in Figure 13.1.1. We assume a constant premium rate, c, $c > 0$, and set $c(t) = ct$ throughout this chapter. Then the surplus increases linearly (with slope c) except at those times when a claim occurs. At that time the surplus drops by the amount of the claim. If the initial surplus u were increased or decreased by the amount h, the graph of

FIGURE 13.1.1

A Typical Outcome of the Continuous Time Surplus Process



$U\ t$ would be raised or lowered h units of height but would be unchanged otherwise.

As illustrated in Figure 13.1.1, the surplus might become negative at certain times. When this first happens, we speak of ***ruin*** having occurred. This term has its roots in the gambler's ruin problem of probability theory. It is not equivalent to ***insolvency*** of an insurer. The typical application of the ideas of this chapter is to a project or line of business of an insurer. However, a useful measure of the financial risk in an insurance organization can be obtained by calculating the probability of ruin as a consequence of variations in the amount and timing of claims.

Let

$$T = \min \{t : t \geq 0 \text{ and } U\ t\ < 0\} \tag{13.1.2}$$

denote the time of ruin with the understanding that $T = \infty$ if $U\ t\ \geq 0$ for all t. Let

$$\psi(u) = \Pr(T < \infty) \tag{13.1.3}$$

denote the probability of ruin considered as a function of the initial surplus u. We are also interested in $U\ T$, the negative surplus at the time of ruin.

Because of changes in the world of applications, the useful lifetime of a model is limited so that a realistic planning horizon might be a long—but finite—period. More precisely, consideration would be limited to

$$\psi(u, t) = \Pr(T < t), \tag{13.1.4}$$

the probability of ruin before time t. We discuss, however, only the probability of ruin over an infinite horizon, $\psi(u)$, which is more tractable mathematically. Of course, $\psi(u)$ is an upper bound for $\psi(u, t)$.

The ideas for this chapter can be used to provide an early warning system for the guidance of an insurance project. A model must necessarily be selected to represent the risk process. The probability of ruin, on the basis of that model, warns management of the project about some of the risks involved. Again, particular models developed in this chapter make simplifying assumptions to keep the mathematics tractable. The effects of interest, expenses, dividends, and experience rating are not included. Nevertheless, these models provide a means of analyzing the risk process. In practice, they would be supplemented by additional analyses.

Before we consider the continuous time model we turn to the ***discrete time surplus process*** defined by considering the values of $U(t)$ at only integer values of t. Traditionally, this sequence of random variables is denoted

$$\{U_n : n = 0, 1, 2, \ldots\}.$$

This can be viewed as examining the amount of surplus on a periodic basis, much as would be done by the managers of insurance enterprises who are required to submit financial reports on the operations on a yearly, semiannual, quarterly, or monthly basis.

13.2 A Discrete Time Model

Let U_n denote the insurer's surplus at time n, $n = 0, 1, 2, \ldots$. We assume that

$$U_n = u + n\,c - S_n \tag{13.2.1}$$

where u is the initial surplus, the amount of premiums received each period is constant and denoted by c, and S_n is the aggregate claims of the first n periods. Further we assume that

$$S_n = W_1 + W_2 + \cdots + W_n \tag{13.2.2}$$

where W_i is the sum of the claims in period i. At first we assume $W_1, W_2, \ldots$ are independent, identically distributed random variables with $\mu = \mathrm{E}[W_i] < c$; later we will relax this constraint.

Thus, we can write U_n as

$$U_n = u + (c - W_1) + (c - W_2) + \cdots + (c - W_n). \tag{13.2.3}$$

Let

$$\tilde{T} = \min\{n : U_n < 0\} \tag{13.2.4}$$

denote the time of ruin (again with the understanding that $\tilde{T} = \infty$ if $U_n \geq 0$ for all n) and let

$$\tilde{\psi}(u) = \Pr(\tilde{T} < \infty) \tag{13.2.5}$$

denote the probability of ruin in this context.

There is an important connection between a quantity that we now define, the ***adjustment coefficient,*** and the probability of ruin. We define the adjustment coefficient $\tilde{R}$ as the positive solution of the equation

$$M_{W-c}(r) = \mathrm{E}[e^{r(W-c)}] = e^{-rc}\,M_W(r) = 1 \tag{13.2.6}$$

or the equivalent equation

$$\log M_W \, r \ = rc \tag{13.2.7}$$

where W denotes a random variable with the distribution of the annual claims, W_i.

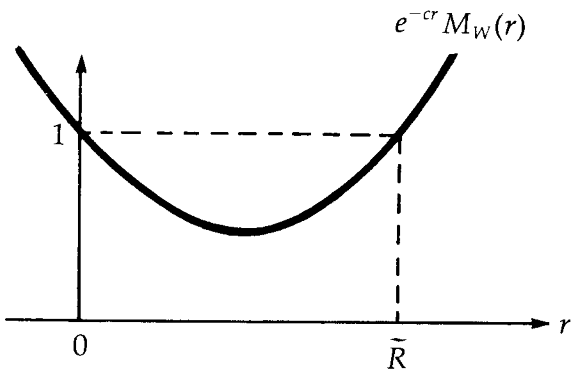
The graph of $e^{-rc} M_W \, r$ (see Figure 13.2.1) can be traced by observing that

$$\frac{d}{dr} \mathrm{E}[e^{r \, W-c}] = \mathrm{E}[(W - c)e^{r \, W-c}]$$

and

$$\frac{d^2}{dr^2} \mathrm{E}[e^{r \, W-c}] = \mathrm{E}[(W - c)^2 e^{r \, W-c}].$$

FIGURE 13.2.1
Definition of $\tilde{R}$



The first of these observations shows that the slope at $r = 0$ is $\mu - c$, a negative quantity, and the second shows that the graph is concave upward. Further, provided W has positive probability over values in excess of c, the first derivative, for some large enough r, becomes positive and remains so. Thus, $\mathrm{E}[e^{r \, W-c}]$ has a minimum as indicated in Figure 13.2.1, and (13.2.6) will typically have a positive root as shown. This positive root is called the ***adjustment coefficient.*** Example 13.4.3 can be used to illustrate that R does not exist for all distributions and values of c.

Example 13.2.1

Derive an expression for R in the special case where the W_i's common distribution is $N(\mu, \sigma^2)$.

Solution:

Here

$$\log[M_W \, r \,] = \mu \, r + \frac{\sigma^2 r^2}{2}.$$

Hence, the positive solution of (13.2.7) is

$$\tilde{R} = \frac{2(c - \mu)}{\sigma^2}$$

where, as assumed above, $\mu < c$. ▼

The connection between the adjustment coefficient and the probability of ruin is given in the following result.

Theorem 13.2.1

Let $U_n = u + nc - \sum_{i=1}^{n} W_i$ for $n = 1, 2, \ldots$, and $W_1, W_2, \ldots$ be mutually independent and identically distributed with $E[W_i] = \mu < c$. For $u > 0$,

$$\tilde{\psi}(u) = \frac{\exp(-\tilde{R}\, u)}{E[\exp(-\tilde{R}\, U_{\tilde{T}}) \mid \tilde{T} < \infty]}. \tag{13.2.8}$$

Theorem 13.2.1 is a special case of Theorem 13.2.2 which is proved in the Appendix to this chapter.

Since $U_{\tilde{T}} < 0$ by definition, it follows from Theorem 13.2.1 that

$$\tilde{\psi}(u) < \exp(-\tilde{R}\, u). \tag{13.2.9}$$

We now derive an approximation for $\tilde{R}$. In the discussion of Table 12.5.1 we saw that for a random variable X

$$\frac{d}{dt} \log M_X(t) \,|_{t=0} = E[X]$$

and

$$\frac{d^2}{dt^2} \log M_X(t) \,|_{t=0} = \mathrm{Var}(X).$$

Hence, using the Maclaurin series expansion, we have

$$\log M_W(r) = \mu\, r + \frac{1}{2}\, \sigma^2\, r^2 + \cdots$$

where $\sigma^2 = \mathrm{Var}(W)$. If we use only the first two terms of this expansion in (13.2.7), we obtain the approximation

$$\tilde{R} \cong \frac{2(c - \mu)}{\sigma^2}. \tag{13.2.10}$$

Comparing this with Example 13.2.1, we observe that (13.2.10) is exact in the case where the W_i's common distribution is normal. Furthermore, if W has a compound distribution and the relative security loading θ is given by $c = (1 + \theta)\, \mu$, then (12.2.5) and (12.2.6) yield

$$R \cong \frac{2\theta\, p_1\, \mathrm{E}[N]}{(p_2 - p_1^2)\mathrm{E}[N] + p_1^2\, \mathrm{Var}(N)} \tag{13.2.11}$$

where N is a random variable distributed as the number of claims in a period.

Example 13.2.2

Approximate R if

a. N has a Poisson distribution with parameter λ

b. N has a negative binomial distribution with parameters r and p.

Solution:

a. Here $\mathrm{E}[N] = \mathrm{Var}(N) = \lambda$, and (13.2.11) reduces to

$$R \cong \frac{2\theta\, p_1}{p_2}. \tag{13.2.12}$$

It follows from Exercise 13.6 that the right-hand side of (13.2.12) is actually an upper bound.

b. In this case,

$$\mathrm{E}[N] = \frac{rq}{p},$$

$$\mathrm{Var}(N) = \frac{rq}{p^2},$$

so it follows from (13.2.11) that

$$R \cong \frac{2\theta\, p_1}{p_2 + p_1^2[(1/p) - 1]}. \tag{13.2.13}$$

Note that for $p \rightarrow 1$, the result in (a) is obtained. ▼

It has been assumed that the total amounts of claims in different periods are independent random variables. In many cases, this assumption may be unrealistic. To investigate a situation of this type, we now consider an autoregressive model for the insurer's claim costs that generalizes the model previously considered and allows for correlation between the claims of successive periods.

We assume that W_i, the sum of claims in period i, is of the form

$$W_i = Y_i + a\, W_{i-1} \qquad i = 1, 2, \ldots. \tag{13.2.14}$$

Here $-1 < a < 1$, and $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $\mathrm{E}[Y_i] < (1 - a)\, c$. The initial value $W_0 = w$ completes the description of this first-order autoregressive model for the W_i's.

The insurer's surplus U_n at time n is defined as in (13.2.1) and T, the time of ruin, as in (13.2.4). Note that the probability of ruin,

$$\psi(u, w) = \Pr(T < \infty), \tag{13.2.15}$$

is now a function of two variables. This generalizes the model considered previously, which corresponds to the special case $a = 0$.

We now use the iterative rule in (13.2.14) to obtain

$$W_i = Y_i + a\, Y_{i-1} + \cdots + a^{i-1}\, Y_1 + a^i\, w. \tag{13.2.16}$$

Thus the total of the claims in the first n periods is

$$\begin{aligned} S_n &= Y_n + (1 + a)\, Y_{n-1} + \cdots + (1 + a + \cdots + a^{n-1})\, Y_1 \\ &\quad + (a + a^2 + \cdots + a^n)\, w \\ &= Y_n + \frac{1 - a^2}{1 - a} Y_{n-1} + \cdots + \frac{1 - a^n}{1 - a} Y_1 + a \frac{1 - a^n}{1 - a} w. \end{aligned} \tag{13.2.17}$$

This shows that Y_1 ultimately contributes $Y_1/(1 - a)$ to the total claims. Hence, we assume that $c > \mathrm{E}[Y_1]/(1 - a)$, and in analogy to (13.2.6), we define the adjustment coefficient as the positive solution of the equation

$$e^{-cr}\, M_{Y/(1-a)}\,(r) = 1. \tag{13.2.18}$$

Thus $\tilde{R}$ is a positive number with the property that

$$\log \mathrm{E}\left[\exp\left(\frac{\tilde{R}Y}{1 - a}\right)\right] - c\tilde{R} = 0. \tag{13.2.19}$$

Note that $\tilde{R}$ depends on the common distribution of the Y_i's and on the values of a and c.

In the Appendix to this chapter, the following result will be derived.

Theorem 13.2.2

$$\tilde{\psi}(u, w) = \frac{\exp(-\tilde{R}\hat{u})}{\mathrm{E}[\exp(-\tilde{R}\hat{U}_{\tilde{T}}) \mid \tilde{T} < \infty]}. \tag{13.2.20}$$

Here we have used the notation

$$\hat{U}_n = U_n - \frac{a}{1 - a} W_n \qquad \hat{u} = \hat{U}_0. \tag{13.2.21}$$

In a sense, $\hat{U}_n$ is a modified surplus. It is the actual surplus U_n adjusted by all future claims that are related to W_n. This interpretation of $\hat{U}_n$ is developed in Exercise 13.2.

If $a \geq 0$, it follows that $\hat{U}_{\tilde{T}} \leq U_{\tilde{T}} < 0$. Thus, in this case, the denominator of (13.2.20) is greater than 1, and we get a simplified upper bound for the probability of ruin.

Corollary 13.2.1

If $0 \leq a < 1$, then

$$\tilde{\psi}(u, w) \leq \exp(-\tilde{R}\hat{u}). \tag{13.2.22}$$

Note that this generalizes (13.2.9).

13.3 A Continuous Time Model

We now formulate the ruin model using two continuous time random processes, the claim number process and the aggregate claim process. We usually model the first by a Poisson process and the second by a compound Poisson process.

For a certain portfolio of insurance, let $N\ t$ denote the number of claims and $S\ t$ the aggregate claims up to time t. We start the count at time 0; thus, $N(0) = 0$. Furthermore, $S\ t\ = 0$ as long as $N\ t\ = 0$. As in Chapter 12, we let X_i denote the amount of the i-th claim. Then

$$S\ t\ = X_1 + X_2 + X_3 + \cdots + X_{N\ t}. \tag{13.3.1}$$

The process $\{N\ t\ , t \geq 0\}$ is called the ***claim number process,*** whereas $\{S\ t\ , t \geq 0\}$ is called the ***aggregate claim process.*** These collections of random variables are called processes, and we are interested in the distributions simultaneously at all times $t \geq 0$. This is in contrast to Chapter 12 where we were interested in the number of claims and aggregate claims for a single period.

Let $t \geq 0$ and $h > 0$. From the definitions it follows that $N\ t + h) - N\ t$ is the number of claims and $S\ t + h) - S\ t$ is the aggregate amount of claims that occur in the time interval between t and $t + h$. Let T_i denote the time when the i-th claim occurs. Thus $T_1, T_2, \ldots$ are random variables such that $T_1 < T_2 < T_3 < \ldots$ to exclude the possibility that two or more claims occur at the same time. The time elapsed between successive claims is denoted by $V_1 = T_1$, and

$$V_i = T_i - T_{i-1} \qquad i > 1. \tag{13.3.2}$$

Typical outcomes of the claim number process and the aggregate claim process are depicted in Figures 13.3.1 and 13.3.2. Note that $N\ t$ and $S\ t$ are increasing step functions. The discontinuities are at times T_i when the claims occur, and the size of the steps at these times is 1 for $N\ t$ and the corresponding claim amount X_i for $S\ t$.

FIGURE 13.3.1

A Typical Outcome of the Claim Number Process

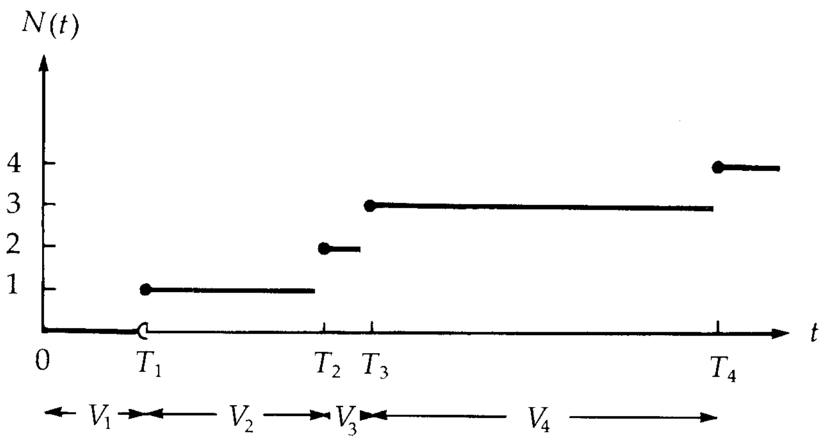
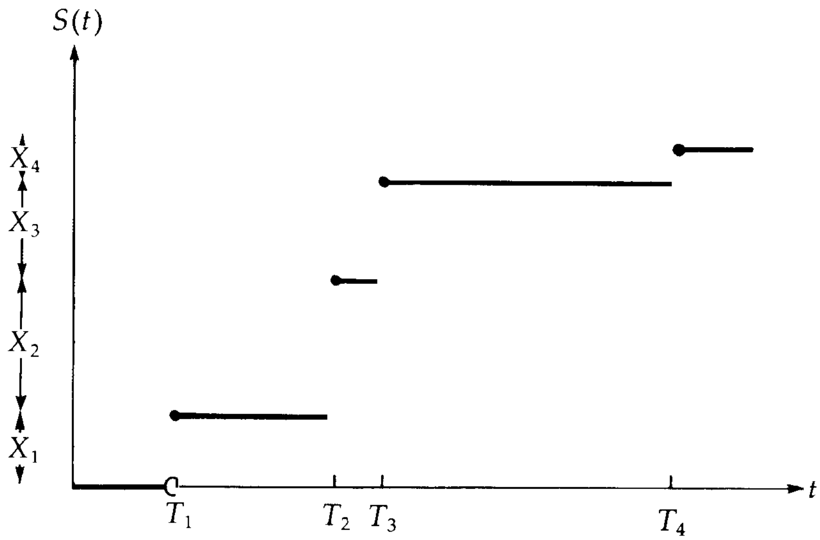


FIGURE 13.3.2

A Typical Outcome of the Aggregate Claim Process



There are several ways to define the distribution of a claim number process. We examine two of them:

a. The ***global method:*** For all $t \geq 0$ and all $h > 0$ we specify the conditional distribution of $N\ t + h) - N\ t$, given the values of $N\ s$ for $s \leq t$.
b. The ***discrete method:*** We specify the joint distribution of $V_1, V_2, V_3, \ldots$ or, equivalently, that of $T_1, T_2, T_3, \ldots$.

Example 13.3.1

Consider n people age x at time 0. Let $N\ t$ denote the number of deaths that have occurred by time t and T_i denote the time when the i-th death occurs ($i = 1, 2, \ldots, n$). We assume independence of the times-until-death. Specify the process $\{N\ t\ , t \geq 0\}$ by each of the two methods above.

Solution:

a. The conditional distribution of $N\ t + h) - N\ t$, given $N\ t\ = i$, is binomial with parameters $n - i$ and ${}_hq_{x+t}$, $i = 0, 1, 2, \ldots, n - 1$. Thus,

$$\Pr[N\ t + h) - N\ t\ = k \mid N\ t\ = i]$$
$$= \binom{n-i}{k}({}_hq_{x+t})^k(1 - {}_hq_{x+t})^{n-i-k} \qquad k = 0, 1, \ldots, n - i.$$

Note here that the conditional distribution depends on t and only on $N\ s$ at $s = t$.

b. We specify the joint distribution of $T_1, T_2, \ldots, T_n$ by iteration. First,

$$\Pr(T_1 > s) = ({}_sp_x)^n,$$

$$f_{T_1}(s) = n\ {}_sp_x{}^n\ \mu_x(s).$$

For $i = 1, 2, \ldots, n - 1$,

$$\Pr(T_{i+1} > t \mid T_i = s) = ({}_{t-s}p_{x+s})^{n-i}$$

and

$$f_{T_{i+1}|T_i}(t \mid s) = (n - i)({}_{t-s}p_{x+s})^{n-i}\ \mu_x(t).$$

We observe that in this example the elapsed times between successive deaths are not mutually independent or identically distributed. ▼

We turn now to the Poisson process where the elapsed times between successive claims are mutually independent and identically distributed.

The ***global method definition*** of a ***Poisson process*** is as follows:

$$\Pr[N(t + h) - N(t) = k \mid N(s) \text{ for all } s \le t] = \frac{e^{-\lambda h}(\lambda h)^k}{k!}$$

$$k = 0, 1, 2, 3, \ldots \text{ for all } t \ge 0 \text{ and } h > 0. \qquad (13.3.3)$$

The following properties come from this definition of the Poisson process:

(i) The ***increments are stationary;*** that is, the distribution of $N(t + h) - N(t)$, which is Poisson with parameter λh, depends on the length of the interval but not on its location, t.

(ii) For any set of disjoint time intervals, the ***increments are independent;*** that is, for $t_1 < t_1 + h_1 < t_2 < t_2 + h_2 < t_3 \cdots < t_n + h_n$ the increments $N(t_1 + h_1) - N(t_1), N(t_2 + h_2) - N(t_2), \ldots, N(t_n + h_n) - N(t_n)$ are mutually independent.

(iii) The ***probability of simultaneous claims is zero:*** that is,

$$\lim_{h\to 0} \frac{\Pr[N(t + h) - N(t) > 1]}{h} = \lim_{h\to 0} \frac{1 - e^{-\lambda h} - \lambda h e^{-\lambda h}}{h} = 0.$$

The ***discrete method definition of the Poisson process*** states that elapsed times $V_1, V_2, V_3, \ldots$ are mutually independent and that each has the exponential distribution with parameter λ.

We now show the equivalence of these two definitions of the Poisson process. To see that the process defined by the global method implies the salient feature of the discrete method, we observe

$$\begin{aligned}\Pr(V_{i+1} > h \mid V_1, V_2, \ldots, V_i) &= \Pr\left[V_{i+1} > h \mid N(s) \text{ for all } s \le t = \sum_{j=1}^{i} V_j\right] \\ &= \Pr[N(t + h) - N(t) = 0 \mid N(s), s \le t] \\ &= e^{-\lambda h}, \qquad (13.3.4)\end{aligned}$$

the survival function of each V_i in the Poisson process.

To see that the process defined by the discrete method implies the salient feature of the global method, apply the definition of the $N(t)$'s, T_j's, and V_j's, the mutual independence and exponential distribution of the V_j's, to show that

$$\Pr[N(t + h) - N(t) = k \mid N(s) \text{ for } s \le t] = \Pr[T_k \le h \text{ and } T_k + V_{k+1} > h]. \quad (13.3.5)$$

Since $T_k = V_1 + V_2 + \cdots + V_k$, and the V_i's are independent and identically distributed with an exponential distribution with parameter λ, T_k has a gamma distribution with parameters k and λ. Then (13.3.5) is equal to

$$\int_0^h \Pr\ T_k + V_{k+1} > h \mid T_k = u\ \ f_{T_k}\ u\ \ du = \int_0^h \Pr\ V_{k+1} > h - u\ \ f_{T_k}\ u\ \ du$$

$$= \int_0^h e^{-\lambda\ h-u}\ \frac{\lambda^k\ u^{k-1}\ e^{-\lambda u}}{(k-1)!}\ du$$

$$= \frac{e^{-\lambda h}\ \lambda^k}{(k-1)!} \int_0^h u^{k-1}\ du = e^{-\lambda h}\ \frac{(\lambda h)^k}{k!},$$

the p.f. of the number of occurrences in a period of length h in a Poisson process.

We now define a compound Poisson process in this context. If for $S\ t$, defined in (13.3.1), the random variables $X_1, X_2, X_3, \ldots$ are independent, identically distributed random variables with common d.f. $P\ x$, and if they are also independent of the process $\{N\ t\ ,\ t \geq 0\}$, assumed to be a Poisson process, the process $\{S\ t\ ,\ t \geq 0\}$ is said to be a ***compound Poisson process.***

If the aggregate claims process is a compound Poisson process given by λ and $P\ x$, the following properties correspond to properties of the underlying claim number process:

a. If $t \geq 0$ and $h > 0$, the distribution of $S\ t + h) - S\ t$ is compound Poisson with specifications λh and $P\ x$, that is,

$$\Pr[S\ t + h) - S\ t\ \leq x \mid S\ s\ \text{ for } s \leq t] = \sum_{k=0}^{\infty} e^{-\lambda h}\ (\lambda h)^k\ \frac{P^{*k}(x)}{k!}$$

where $P^{*k}(x)$ is the k-fold convolution of the d.f. $P\ x$.

b. At any time t, the probability that the next claim occurs between $t + h$ and $t + h + dh$ and that the claim amount is less than or equal to x is $e^{-\lambda dh}\ (\lambda dh)\ P\ x$.

c. The process $\{S\ t\ ,\ t \geq 0\}$ has independent and stationary increments; that is, the aggregate claims of disjoint time intervals are independent random variables, and the distribution of each of these depends on only the length of the corresponding time interval and not on its location.

d. If $S\ t$ denotes a compound Poisson process and the value of t is fixed, $S\ t$ has a compound Poisson distribution with Poisson parameter λt. Formulas (12.3.2) and (12.3.3) give the mean and variance of $S\ t$:

$$\mathrm{E}[S\ t\ ] = \lambda t p_1, \tag{13.3.6}$$

$$\mathrm{Var}[S\ t\ ] = \lambda t p_2. \tag{13.3.7}$$

13.4 Ruin Probabilities and the Claim Amount Distribution

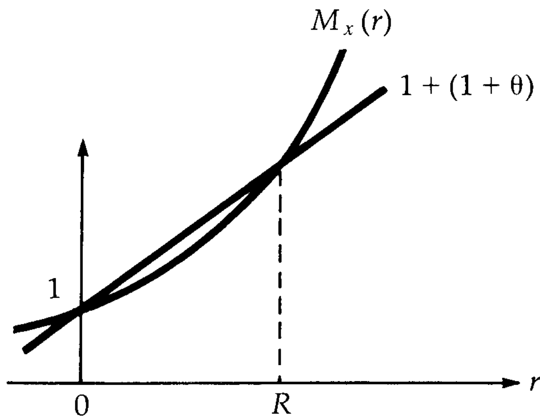
The surplus process $\{U\ t\ ,\ t \geq 0\}$ can be studied by its relation, given in (13.1.1), to the claim process $S\ t$. For the remainder of the chapter, we assume that S(t) is a compound Poisson process. With this assumption we can develop upper and lower bounds on $\psi(u)$. In the particular case of an exponential distribution of individual claims, there is an explicit form for $\psi(u)$.

First, we assume that the rate of premium collection c exceeds the expected claim payments per unit time, which is λp_1. Further, we define the relative security loading θ by the equation $c = (1 + \theta) \lambda p_1$ where θ is positive. We can see that $\theta = 0$ or $\theta < 0$ implies $\psi(u) = 1$; that is, ruin is certain.

Next, let $(-\infty, \gamma)$ denote the largest open interval for which the m.g.f. of $P(x)$ exists. We assume that γ is positive. In the case of the exponential distribution with parameter β, γ is equal to β, while for any bounded claim amount distribution, γ is $+\infty$. Furthermore, we assume that $M_X(r)$ tends to $+\infty$ as r tends to γ. That this assumption does not always hold for finite γ is demonstrated with an inverse Gaussian distribution in Example 13.4.3. A comparison of the figure that accompanies Example 13.4.3 with Figure 13.4.1 shows the importance of this assumption.

FIGURE 13.4.1

The Definition of *R*



We now proceed by analogy with the definition of the adjustment coefficient in (13.2.6). The usefulness of this adjustment coefficient appears in the proof of Theorem 13.4.1.

Consider a period of length $t > 0$ where the amount of premiums collected is ct and the claim distribution is compound Poisson with expected number of claims λt. Then, by analogy with (13.2.6), we choose, for the adjustment coefficient, the smallest positive root of

$$\begin{aligned} M_{S(t)-ct}(r) &= \mathrm{E}[e^{r(S(t)-ct)}] = e^{-rct} M_{S(t)}(r) \\ &= e^{-rct}e^{\lambda t[M_X(r)-1]} = 1. \end{aligned} \tag{13.4.1}$$

Thus,

$$\lambda[M_X(r) - 1] = cr. \tag{13.4.2}$$

Substituting $c = (1 + \theta)\lambda p_1$, we have the equivalent form

$$1 + (1 + \theta)p_1 r = M_X r . \tag{13.4.3}$$

The left-hand side of (13.4.3) is a linear function of r, whereas the right-hand side is a positive increasing function which by assumption tends to $+\infty$ as r tends to γ. Furthermore, the second derivative of the right-hand side is positive, so its graph is concave upward. The assumption that $c > \lambda p_1$ (equivalent to $\theta > 0$) means the slope, $(1 + \theta) p_1$, of the left-hand side of (13.4.3) exceeds the slope, $M_X'(0) = p_1$, of the right-hand side at $r = 0$. From Figure 13.4.1, we see that (13.4.3) has two solutions. Aside from the trivial solution $r = 0$, there is a positive solution $r = R$, which is defined to be the ***adjustment coefficient.***

Example 13.4.1

Determine the adjustment coefficient if the claim amount distribution is exponential with parameter $\beta > 0$.

Solution:

The adjustment coefficient is obtained from (13.4.3) which is, for this example,

$$1 + \frac{(1 + \theta)r}{\beta} = \frac{\beta}{\beta - r}$$

or, as a quadratic equation in r,

$$(1 + \theta)r^2 - \theta\beta r = 0.$$

As expected, $r = 0$ is a solution, and the adjustment coefficient solution, the smallest positive root, is

$$R = \frac{\theta\beta}{1 + \theta}.$$

▼

Example 13.4.2

Calculate the adjustment coefficient if all claims are of size 1.

Solution:

Formula (13.4.3) gives the adjustment coefficient as the positive root of

$$1 + (1 + \theta)r = e^r.$$

The results of numerical evaluation of the above equation for several values of θ are displayed below.

θ	R	θ	R
0.2	0.35420	1.0	1.25643
0.4	0.63903	1.2	1.41318
0.6	0.87640	1.4	1.55368
0.8	1.07941		

▼

In general, the adjustment coefficient is an increasing function of the relative security loading, θ. This can be seen from Figure 13.4.1. As θ is increased, the slope of the straight line through the point (0, 1) is increased so that the point of intersection of the line and the curve moves to the right and upward.

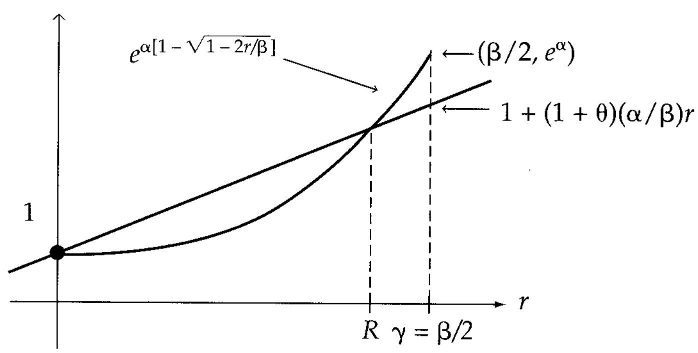
Example 13.4.3

Assume that the claim amount distribution is inverse Gaussian with parameters α and β. For this assumption:

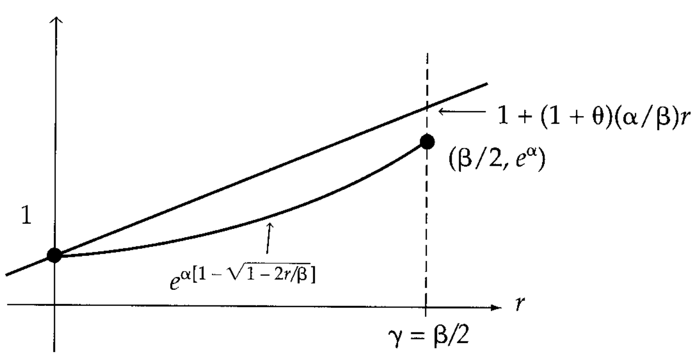
a. Determine the largest open interval $(-\infty, \gamma)$ for which $M_X(t)$ is defined.
b. Determine limit $M_X(t)$ as $t \to \gamma$.
c. Display the equation for the adjustment coefficient.
d. Display the graph corresponding to Figure 13.4.1 for the case when the limit in (b) is greater than $1 + (1 + \theta)(\alpha/\beta)\gamma$.
e. Display the graph corresponding to Figure 13.4.1 for the case when the limit in (b) is less than $1 + (1 + \theta)(\alpha/\beta)\gamma$.

Solution:

a. $M_X(t) = e^{\alpha[1-\sqrt{(1-2t/\beta)}]}$, for $t < \beta/2$. So $\gamma = \beta/2$.
b. $\lim_{t \to \beta/2} M_X(t) = e^{\alpha}$.
c. $1 + (1 + \theta)(\alpha/\beta)r = e^{\alpha[1-\sqrt{(1-2r/\beta)}]}$, for $r < \beta/2$.
d. If $e^{\alpha} > 1 + (1 + \theta)(\alpha/2)$, the graph is



e. If $e^{\alpha} < 1 + (1 + \theta)(\alpha/2)$, the graph is



Observations:

1. We see in the graphs the possible failure of the definition of the adjustment coefficient when the limit of the claim distribution's m.g.f. is finite at the end of the open interval of its definition.
2. For a fixed expected claim size, α / β, the lines of the graphs are fixed for all β. As β increases the claim size's variance, α / β^2, decreases, and the point $(\beta / 2, e^{\alpha})$ moves to the right and up, making the existence of R more likely. As β decreases the opposite is true.
3. An examination of the proof of Theorem 13.4.1 in the Appendix to this chapter reveals the use of the adjustment coefficient to set $-Rc + \lambda[M_X(R) - 1] = 0$ to obtain the equality of Theorem 13.4.1. For the situation in part (e) when the adjustment coefficient does not exist, this expression evaluated at γ is less than zero and the steps of the proof can be followed with R replaced by γ and the equalities replaced by inequalities. The resulting inequality is less useful than the equality obtained when the adjustment coefficient exists. ▼

An analog to Theorem 13.2.1 is the following result, which is proved in the Appendix to this chapter.

Theorem 13.4.1

If $U(t)$ is the surplus process based upon a compound Poisson aggregate claims process, $S(t)$, with $c > \lambda p_1$, that is, with positive relative security loading, then for $u \geq 0$,

$$\psi(u) = \frac{\exp(-Ru)}{\mathrm{E}[\exp(-RU(T)) \mid T < \infty]} \tag{13.4.4}$$

where R is the smallest positive root of (13.4.3).

The denominator is calculated with respect to the conditional distribution of the negative surplus, $U(T)$, given that ruin occurs: that is, $T < \infty$.

From Figure 13.4.1, we see that if $\theta \rightarrow 0$, the secant approaches the tangent to $M_X(r)$ at $r = 0$, which implies $R \rightarrow 0$. But then, from (13.4.4), $\psi(u) = 1$, or ruin is certain. Further, $U(t)$, $t > 0$, for the case where $\theta < 0$, will always be less than the corresponding $U(t)$ for $\theta \rightarrow 0$, and hence since ruin is certain for $\theta = 0$, ruin is also certain for $\theta < 0$. For these reasons, we remain with the assumption that $\theta > 0$.

In general, a closed form evaluation of the denominator of (13.4.4) is not possible. Exceptions are the case $u = 0$ (see Exercise 13.10), and the case where the claim amount distribution is exponential (see Example 13.4.4). However, the theorem can be used to derive inequalities. Since $U(T)$, given $T < \infty$, is necessarily negative, the denominator in (13.4.4) exceeds 1. It follows that

$$\psi(u) < e^{-Ru}. \tag{13.4.5}$$

If the claim amount distribution is bounded so that $P(m) = 1$ for some finite m, it follows, given $T < \infty$, that $U(T) > -m$ since the surplus just before the claim must have been positive.

Thus,

$$\psi(u) > e^{-Ru}\, e^{-Rm} = e^{-R(u+m)}. \tag{13.4.6}$$

Some authors suggest the use of the approximation

$$\psi(u) \cong e^{-Ru}, \tag{13.4.7}$$

which, in view of (13.4.5), overstates the probability of ruin.

We now examine a special case where Theorem 13.4.1 can be applied to obtain an explicit expression for the ruin probability $\psi(u)$.

Example 13.4.4

Calculate the probability of ruin in the case that the claim amount distribution is exponential with parameter $\beta > 0$.

Solution:

Ruin, if it occurs, is assumed to take place at time T. Let $\hat{u}$ be the amount of surplus just prior to T. The event that $-U(T) > y$ can be restated as the event that X, the size of the claim causing ruin, exceeds $\hat{u} + y$, given that it exceeds $\hat{u}$. The conditional probability of this event is given by

$$\frac{\beta \displaystyle\int_{\hat{u}+y}^{\infty} e^{-\beta x}\, dx}{\beta \displaystyle\int_{\hat{u}}^{\infty} e^{-\beta x}\, dx} = e^{-\beta y},$$

so that the p.d.f. of $-U(T)$, given $T < \infty$, is

$$\frac{d}{dy}(1 - e^{-\beta y}) = \beta e^{-\beta y}.$$

Therefore,

$$\begin{aligned} E[\exp(-RU(T)) \mid T < \infty] &= \beta \int_0^{\infty} e^{-\beta y} e^{Ry}\, dy \\ &= \frac{\beta}{\beta - R}. \end{aligned}$$

From Example 13.4.1, we know that the adjustment coefficient in this case is $R = \theta\beta/(1 + \theta)$. Combining this with (13.4.4) gives us

$$\begin{aligned}\psi(u) &= \frac{(\beta - R)e^{-Ru}}{\beta} \\ &= \frac{1}{1+\theta}\exp\left(\frac{-\theta\beta u}{1+\theta}\right) \\ &= \frac{1}{1+\theta}\exp\left[\frac{-\theta u}{(1+\theta)p_1}\right]. \end{aligned} \tag{13.4.8}$$

▼

13.5 The First Surplus below the Initial Level

We continue with the continuous time model where the aggregate claim process, $S(t)$, is compound Poisson. Specifically, we consider the amount of the surplus at the time it first falls below the initial level (this, of course, may never happen). As an application, we find a simple expression for $\psi(0)$, the probability of ruin, if the initial surplus is 0.

The main theorem of this section, proved in the Appendix to this chapter, is the following.

Theorem 13.5.1

For a compound Poisson process, the probability that the surplus will ever fall below its initial level u, and will be between $u - y$ and $u - y - dy$ when it happens for the first time, is

$$\frac{\lambda}{c}[1 - P(y)]\,dy = \frac{1 - P(y)}{(1+\theta)\,p_1}\,dy \qquad y > 0.$$

As an application of Theorem 13.5.1, we note that the probability that the surplus will ever fall below its original level is

$$\frac{1}{(1+\theta)\,p_1}\int_0^\infty [1 - P(y)]\,dy = \frac{1}{1+\theta} \tag{13.5.1}$$

since

$$\int_0^\infty [1 - P(y)]\,dy = p_1.$$

In the special case of $u = 0$, (13.5.1) gives the probability that the surplus will even fall below zero, that is, its original level. Hence

$$\psi(0) = \frac{1}{1+\theta}. \tag{13.5.2}$$

It is remarkable that $\psi(0)$ depends only upon the relative security loading θ and not on the specific form of the claim amount distribution.

We note that

$$\frac{\lambda}{c}\,[1 - P\ y\ ] = \frac{1 - P\ y}{(1 + \theta)\ p_1} \qquad y > 0$$

is not a p.d.f. since it does not integrate to 1. However, there is a related p.d.f. in the proper sense. Let L_1 be a random variable denoting the amount by which the surplus falls below the initial level for the first time, given that this ever happens. The p.d.f. for L_1 is obtained by dividing

$$\frac{1 - P\ y}{(1 + \theta)\ p_1}$$

by

$$\psi(0) = \frac{1}{1 + \theta}$$

and is

$$f_{L_1}(y) = \frac{1}{p_1}\,[1 - P\ y\ ] \qquad y > 0. \tag{13.5.3}$$

The relationship between the m.g.f. of L_1 and that of the distribution of claim size X can be obtained by integration by parts:

$$\begin{aligned} M_{L_1}(r) &= \frac{1}{p_1}\int_0^\infty e^{ry}\,[1 - P\ y\ ]\,dy \\ &= \frac{1}{p_1}\left\{\frac{e^{ry}}{r}\,[1 - P\ y\ ]\Big|_0^\infty + \frac{1}{r}\int_0^\infty e^{ry}\,p\ y\ \ dy\right\} \\ &= \frac{1}{p_1\,r}\,[M_X\ r) - 1]. \end{aligned} \tag{13.5.4}$$

We illustrate further applications of Theorem 13.5.1 by means of the following examples.

Example 13.5.1

Write an expression for the distribution of the surplus level at the first time the surplus falls below the initial level u, given that it does fall below u, if all claims are of size 2.

Solution:

We have

$$1 - P\ y\ = \begin{cases} 1 & 0 \le y < 2 \\ 0 & y \ge 2. \end{cases}$$

Thus the p.d.f. for L_1 is

$$\frac{1}{p_1}\,[1 - P\ y\ ] = \begin{cases} \frac{1}{2} & 0 \le y < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore, L_1 is uniformly distributed between 0 and 2, so the surplus level after the first such drop is uniformly distributed between $u - 2$ and u. ▼

Example 13.5.2

Write an expression for the distribution of L_1 if the size of the individual claims has an exponential distribution with parameter β.

Solution:

Since $1 - P(y) = e^{-\beta y}$ for $y > 0$, the p.d.f. of L_1 is

$$\frac{1}{p_1}[1 - P(y)] = \beta e^{-\beta y} \qquad y > 0.$$

Hence, the distribution of L_1 is also exponential with parameter β. ▼

13.6 The Maximal Aggregate Loss

A new random variable, the ***maximal aggregate loss,*** is defined as

$$L = \max_{t \geq 0} \{S(t) - ct\}, \tag{13.6.1}$$

that is, as the maximal excess of aggregate claims over premiums received. Since $S(t) - ct = 0$ for $t = 0$, it follows that $L \geq 0$.

Theorem 13.5.1 is used in the proof of another theorem that gives an explicit formula for the m.g.f. of L. This can be used to provide information about $\psi(u)$. As an application, $\psi(u)$ is expressed for the case where the individual claim amount distribution is a weighted sum of exponential distributions.

To obtain the d.f. of the random variable L we consider, for $u \geq 0$, that

$$\begin{aligned} 1 - \psi(u) &= \Pr[U(t) \geq 0 \text{ for all } t] \\ &= \Pr[u + ct - S(t) \geq 0 \text{ for all } t] \\ &= \Pr[S(t) - ct \leq u \text{ for all } t]. \end{aligned}$$

But the right-hand side is equivalent to $\Pr(L \leq u)$, so we have

$$1 - \psi(u) = \Pr(L \leq u) \qquad u \geq 0. \tag{13.6.2}$$

It follows that $1 - \psi(u)$, the complement of the probability of ruin, can be interpreted as the d.f. of L. In particular, we have

$$1 - \psi(0) = \Pr(L \leq 0) = \Pr(L = 0) \tag{13.6.3}$$

since $L \geq 0$. In this case, the maximum loss is attained at time $t = 0$. Also, the distribution of L is of mixed type. There is a point mass of $1 - \psi(0)$ at the origin with the remaining probability distributed continuously over positive values of L.

The main result of this section is the following explicit formula for the m.g.f. of L, which, in view of (13.6.2), can be used to obtain information about $\psi(u)$.

Theorem 13.6.1

$$M_L(r) = \frac{\theta\, p_1\, r}{1 + (1+\theta)\, p_1\, r - M_X(r)}. \tag{13.6.4}$$

An equivalent formula is

$$M_L(r) = \frac{\theta}{1+\theta} + \frac{1}{1+\theta}\,\frac{\theta[M_X(r) - 1]}{1 + (1+\theta)\, p_1\, r - M_X(r)}, \tag{13.6.4A}$$

which reflects the point mass at the origin more directly, since the contribution to the m.g.f. of the probability at $L = 0$ is

$$1 - \psi(0) = \frac{\theta}{1+\theta}$$

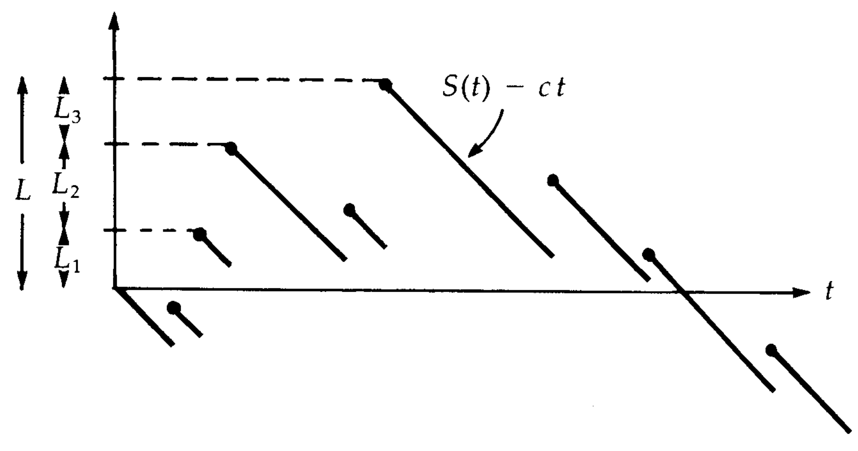
by (13.5.2). Notice that the equation used to define the adjustment coefficient is obtained by equating the denominator of (13.6.4) to 0.

Proof:

The proof of the theorem involves the consideration of the times when the aggregate loss process assumes new record highs. An outcome of the aggregate loss process is shown in Figure 13.6.1. In this outcome a new record high is established three times. After each record high there is a probability of $1 - \psi(0)$ that this record will not be broken and a probability of $\psi(0)$ that it will be broken. In making this statement we are relying on the fact that a compound Poisson process has stationary and independent increments.

FIGURE 13.6.1

A Typical Outcome of the Aggregate Loss Process



If the record is broken, the p.d.f. of the increase is that of L_1, which is given in (13.5.3). Figure 13.6.1 illustrates that we can represent L as a sum of a random number of random variables, thus

$$L = L_1 + L_2 + \cdots + L_N. \tag{13.6.5}$$

Here N is the number of new record highs and has a geometric distribution with

$$\begin{aligned}\Pr(N = n) &= [1 - \psi(0)][\psi(0)]^n \\ &= \theta \left(\frac{1}{1 + \theta}\right)^{n+1} \qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{13.6.6}$$

and m.g.f.

$$M_N(r) = \frac{\theta}{1 + \theta - e^r}. \tag{13.6.7}$$

The random variables $N, L_1, L_2, \ldots$ are mutually independent, and the common p.d.f. of the L_i's is given by (13.5.3). According to (12.2.7), the m.g.f. of L is

$$\begin{aligned}M_L(r) &= M_N [\log M_{L_1}(r)] \\ &= \frac{\theta}{1 + \theta - M_{L_1}(r)}. \end{aligned} \tag{13.6.8}$$

Formula (13.5.4) gives $M_{L_1}(r)$ in terms of $M_X(r)$ and thus

$$\begin{aligned}M_L(r) &= \frac{\theta}{1 + \theta - [1/(p_1 r)][M_X(r) - 1]} \\ &= \frac{\theta\, p_1\, r}{1 + (1 + \theta)\, p_1 r - M_X(r)}, \end{aligned}$$

which was to be shown. The alternative formula (13.6.4A) can be verified by collecting terms over a common denominator. ■

Observe that since $1 - \psi(u)$ is the d.f. of the random variable L—see (13.6.2)—where L has a point mass at the origin and a continuous density for positive values of u, we have

$$\begin{aligned}M_L(r) &= 1 - \psi(0) + \int_0^\infty e^{ur}\, [-\psi'(u)]\, du \\ &= \frac{\theta}{1 + \theta} + \int_0^\infty e^{ur}\, [-\psi'(u)]\, du. \end{aligned}$$

Hence, (13.6.4A) states that

$$\int_0^\infty e^{ur}\, [-\psi'(u)]\, du = \frac{1}{1 + \theta} \frac{\theta[M_X(r) - 1]}{1 + (1 + \theta)\, p_1\, r - M_X(r)}. \tag{13.6.9}$$

This formula can be used to find explicit expressions for $\psi(u)$ for certain families of claim amount distributions. One such family consists of mixtures of exponential distributions of the form

$$p(x) = \sum_{i=1}^{n} A_i\, \beta_i\, e^{-\beta_i x} \qquad x > 0,$$

$$\beta_i > 0,\ A_i > 0,\ A_1 + A_2 + \cdots + A_n = 1. \tag{13.6.10}$$

Then

$$M_X(r) = \sum_{i=1}^{n} A_i \frac{\beta_i}{\beta_i - r}. \tag{13.6.11}$$

Originally, $M_X(r)$ is defined as an expectation and exists only for $r < \gamma = \min\{\beta_1, \ldots, \beta_n\}$. However, this function can be extended in a natural way to all $r \neq \beta_i$. For simplicity, we use the same symbol, $M_X(r)$, for this function.

We substitute (13.6.11) into (13.6.9) and recognize that the right-hand side of the result is a rational function of r, which, by applying the method of partial fractions, we can write in the following form:

$$\int_0^\infty e^{ur}\,[-\psi'(u)]\,du = \sum_{i=1}^{n} \frac{C_i\, r_i}{r_i - r}. \tag{13.6.12}$$

The only function that satisfies this and $\psi(\infty) = 0$ is

$$\psi(u) = \sum_{i=1}^{n} C_i\, e^{-r_i u}. \tag{13.6.13}$$

It follows that the probability of ruin is given by this expression. We illustrate this procedure with two examples.

Example 13.6.1

Derive an expression for $\psi(u)$ if the X_i's have an exponential claim amount distribution; that is, $n = 1$ in (13.6.10).

Solution:

Since

$$M_X(r) = \frac{\beta}{\beta - r} = \frac{1}{1 - p_1 r},$$

the right-hand side of (13.6.9) is

$$\left(\frac{1}{1+\theta}\right)\left\{\frac{\theta[1/(1 - p_1 r) - 1]}{1 + (1+\theta)\,p_1 r - 1/(1 - p_1 r)}\right\} = \left(\frac{1}{1+\theta}\right)\left[\frac{\theta}{\theta - (1+\theta)\,p_1 r}\right]$$

$$= C_1 \frac{r_1}{r_1 - r}$$

where $C_1 = 1/(1+\theta)$ and $r_1 = \theta/[(1+\theta)\,p_1]$. Then

$$\psi(u) = C_1\, e^{-r_1 u}.$$

This formula was established previously in Example 13.4.4, formula (13.4.8). ▼

Example 13.6.2

Given that $\theta = 2/5$ and $p(x)$ is given by

$$p(x) = \frac{3}{2}e^{-3x} + \frac{7}{2}e^{-7x} \qquad x > 0,$$

calculate $\psi(u)$.

Solution:

From $p(x)$ we have

$$M_X(r) = \frac{3/2}{3 - r} + \frac{7/2}{7 - r}$$

and

$$p_1 = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{7}\right) = \frac{5}{21}.$$

We substitute these expressions into the right-hand side of (13.6.9), which, after some simplification, becomes

$$\left(\frac{6}{7}\right)\left(\frac{5 - r}{6 - 7r + r^2}\right).$$

The roots of the denominator are $r_1 = 1$ and $r_2 = 6$; hence this expression, rewritten in the form of (13.6.12), is

$$\frac{C_1}{1 - r} + \frac{6\, C_2}{6 - r}.$$

The coefficients are determined to be

$$C_1 = \frac{24}{35},$$

$$C_2 = \frac{1}{35}.$$

Thus

$$\psi(u) = \frac{24}{35}e^{-u} + \frac{1}{35}e^{-6u} \qquad u \ge 0.$$

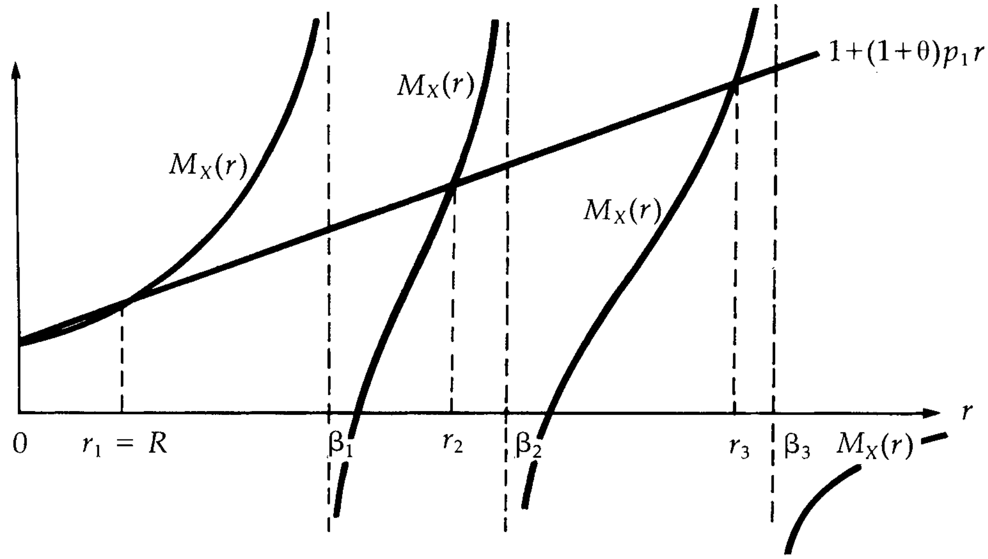
▼

In Figure 13.6.2 a graph illustrates the points r_i satisfying the equation

$$1 + (1 + \theta)\, p_1\, r = M_X(r) \tag{13.6.14}$$

where the distribution of X is a mixture of exponentials. The function $M_X(r)$ is given in (13.6.11) with $n = 3$.

FIGURE 13.6.2
The Solution of (13.6.14) for n = 3



The right-hand side of (13.6.11) has discontinuities at $\beta_1, \beta_2, \ldots,$ and at each of these arguments the value of the function shifts from $+\infty$ to $-\infty$. The figure illustrates that, in general, the r_i's will satisfy a condition of the form

$$r_1 = R < \beta_1 < r_2 < \beta_2 < \cdots < r_n < \beta_n. \tag{13.6.15}$$

Figure 13.4.1 illustrates that part of Figure 13.6.2 to the left of $\beta_1 = \gamma$.

Practical problems will necessitate consideration of claim distributions that are not mixtures of exponential distributions. For some distributions it may even be difficult to calculate the adjustment coefficient so as to be able to approximate ruin probabilities. The following method based on the first two moments of the claim amount distribution is easy to apply and seems to give satisfactory results for moderate values of u.

The first moment for the distribution of L is obtained in Exercise 13.13(b). The result is

$$\mathrm{E}[L] = \frac{p_2}{2\,\theta\,p_1}. \tag{13.6.16}$$

Further, we know from (13.5.2) that $\psi(0) = 1/(1 + \theta)$, and from (13.4.5) that $\psi(u) < e^{-Ru}$. The approximation proposed here is that

$$1 - F_L\,u \;=\; \psi(u) \cong \frac{1}{1+\theta}\,e^{-Ku} \qquad u > 0$$

where K is chosen so that the approximated value of E[L] is equal to that given in (13.6.16). But

$$\mathrm{E}[L] = \int_0^\infty [1 - F_L\ u\ ]\, du \cong \frac{1}{1+\theta}\frac{1}{K},$$

so that

$$K = \frac{2\,\theta\, p_1}{(1+\theta)\, p_2}$$

gives us the required equality. Thus our approximation is

$$\psi(u) \cong \frac{1}{1+\theta}\exp\left[-\frac{2\,\theta\, p_1\, u}{(1+\theta)\, p_2}\right] \qquad u > 0.$$

Note that if the claim distribution is exponential with mean p_1, so that $p_2 = 2\,p_1^2$, the result is exact; see (13.4.8). This method is extended in Exercise 13.19 to give an improved approximation.

13.7 Notes and References

The importance of the joint probability distribution of the time of ruin, T, and the surplus level immediately after ruin, $U(T)$, for a surplus process with aggregate claims following a compound Poisson process is indicated in the denominator of the expression for the probability of ruin in Theorem 13.4.1. The possibility of expressing ruin probabilities for such a surplus process in terms of the distribution of the claim amounts is demonstrated in Theorems 13.5.1 and 13.6.1. Several results in this area have been published since the publication of the first edition of *Actuarial Mathematics*. A recent paper by Gerber and Shiu 1998 provides a readable summary of these results. We quote their abstract:

> This paper studies the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The classical model is generalized by discounting with respect to time of ruin. We show how to calculate an expected discounted penalty, which is due at ruin, and may depend on the deficit at ruin and the surplus immediately before ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation, which has a probabilistic interpretation. Explicit answers are obtained for zero and very large initial surplus, and for arbitrary surplus when the claim amount distribution is exponential. We generalize D. C. M. Dickson's formula, which expresses the joint distribution of the surplus immediately prior to and at ruin in terms of the probability of ultimate ruin. Explicit results are obtained when dividends are paid out to the stockholders according to a constant barrier strategy.

The reader of the proof of Theorem 13.5.1 will recognize the use of the penalty function in their paper.

General references are the texts by

- Beard, Pentikäinen, and Pesonen (1984)
- Beekman (1974)
- Bühlmann (1970)

- Gerber (1979)
- Panjer and Willmot (1992) and
- Seal (1969).

Ruin theory has been developed by the Scandinavian school (F. Lundberg, Cramér), and by the Italian school (DeFinetti); this development is accurately described in the text by Dubourdieu (1952).

We did not discuss the famous asymptotic formula for the probability of ruin,

$$\psi(u) \cong Ce^{-Ru} \qquad u \to \infty, \tag{13.7.1}$$

where C is some constant. In view of Theorem 13.4.1, this formula is quite plausible; it means that the denominator in (13.4.4) has a limit as $u \to \infty$ with the C in (13.7.1) as the reciprocal of this limit. In the case where the claim amount distribution is a mixture of exponential distributions, this asymptotic form is illustrated by (13.6.13).

The formula in Exercise 13.11 is called a defective renewal equation. Feller (1966) discussed the solutions of equations of this type; in particular, he proves (13.7.1). By the same technique, Gerber (1974) finds the limit of the conditional distribution of $-U(T)$, given $T < \infty$, for $u \to \infty$.

If we modify the model by assuming that the surplus earns interest at a constant force $\delta > 0$, we have to replace c by $c + \delta u$ in the integro-differential equation (13.A.12) in the Appendix to this chapter. In the case of exponential claim amounts, the resulting equation has an explicit solution in terms of the gamma function.

If the roles of premiums and claims are interchanged, with premiums representing payments by the insurer and claims representing payments to the insurer, so that

$$U(t) = u - ct + S(t)$$

where it is now assumed that $c < \lambda\, p_1$, then there is an explicit formula,

$$\psi(u) = e^{-Ru}.$$

To prove this, one establishes a result like Theorem 13.4.1 and observes that the surplus at the time of ruin is necessarily 0. It has been suggested that this model could be used for a portfolio of annuities where a death frees the reserve of the policyholder and leads to a negative claim.

Seal (1978b) discusses numerical methods for evaluating the probability of ruin in a finite time interval. Beekman and Bowers (1972) approximate $\psi(u, t)$ by matching moments. Gerber (1974) and DeVylder (1977) give upper bounds for $\psi(u, t)$ by martingale arguments.

Panjer and Willmot (1992, Chapter 11) discuss the ideas of this chapter using more advanced mathematical tools. Sections 11.4 and 11.5 are of particular interest because of the development of recursive methods for calculating the approximate probability of ultimate ruin.

Appendix

Proof of Theorem 13.2.2

The following calculation yields a simple recursion formula for the modified surplus:

$$
\begin{aligned}
U_i &= U_i - \frac{a}{1-a} W_i \\
&= U_{i-1} + c - W_i - \frac{a}{1-a} W_i \\
&= U_{i-1} + c - \frac{1}{1-a} W_i \\
&= U_{i-1} + c - \frac{1}{1-a} (Y_i + a\, W_{i-1}) \\
&= U_{i-1} + c - \frac{Y_i}{1-a} \qquad i = 1, 2, \ldots . \qquad (13.A.1)
\end{aligned}
$$

From (13.A.1), (13.2.18) and the independence of the Y_i's it follows that for any n,

$$E[\exp -R[U_n - U_i])] = 1 \qquad i = 0, 1, \ldots, n. \qquad (13.A.2)$$

Now consider the identity

$$
\begin{aligned}
E[\exp(-RU_n)] = \sum_{i=1}^{n} &E[\exp(-RU_n) \mid T = i] \Pr(T = i) \\
&+ E[\exp(-RU_n) \mid T > n] \Pr(T > n). \qquad (13.A.3)
\end{aligned}
$$

From (13.A.2) for $i = 0$, it follows that the expression on the left-hand side of (13.A.3) is $\exp(-R\hat{u})$. In the summation on the right-hand side we replace U_n by $U_i + (U_n - U_i)$. The difference, $U_n - U_i$, is independent of $U_1, U_2, \ldots, U_i$. This can be confirmed by using (13.A.1) and the independence of the Y's. In particular, $(U_n - U_i)$ is independent of the event $T = i$. It follows from (13.A.2) that

$$E[\exp(-RU_n) \mid T = i] = E[\exp(-RU_i) \mid T = i].$$

Thus (13.A.3) can be written as

$$
\begin{aligned}
\exp(-R\hat{u}) = \sum_{i=1}^{n} &E[\exp(-RU_i) \mid T = i] \Pr(T = i) \\
&+ E[\exp(-RU_n) \mid T > n] \Pr(T > n), \qquad (13.A.4)
\end{aligned}
$$

which is similar to (13.A.8) in the proof below of Theorem 13.4.1. Now we let $n \rightarrow \infty$. Then the first term on the right-hand side of (13.A.4) converges to

$$\sum_{i=1}^{\infty} E[\exp(-RU_i) \mid T = i] \Pr(T = i) = E[\exp(-RU_T) \mid T < \infty] \Pr(T < \infty).$$

Thus to complete the proof of Theorem 13.2.2 we have to show that the second term on the right-hand side of (13.A.4) vanishes for $n \rightarrow \infty$. We do this as follows.

From (13.2.17) it follows that

$$\mathrm{E}[S_n] = n\,\frac{\mathrm{E}[Y]}{1-a} - a\,\frac{1-a^n}{1-a}\left(\frac{\mathrm{E}[Y]}{1-a} - w\right).$$

Since $c > \mathrm{E}[Y]/(1-a)$, there is a positive number α such that $\mathrm{E}[\hat{U}_n] > u + \alpha n$ if n is sufficiently large. Furthermore, it follows from (13.2.17) that there is a number β^2 such that $\mathrm{Var}(\hat{U}_n) < \beta^2 n$. Now the remainder of the proof that the second term on the right-hand side of (13.A.4) converges to 0 as $n \to \infty$ is similar to the proof that is shown in detail for the convergence of the second term on the right-hand side of (13.A.8) in the proof given below for Theorem 13.4.1. See Promislow (1991a) for a complete demonstration of this step. ■

Proof of Theorem 13.4.1

For $t > 0$ and $r > 0$, we consider

$$\begin{aligned}\mathrm{E}[e^{-rU(t)}] &= \mathrm{E}[e^{-rU(t)} \mid T \le t]\,\Pr(T \le t)\\ &\quad + \mathrm{E}[e^{-rU(t)} \mid T > t]\,\Pr(T > t).\end{aligned} \tag{13.A.5}$$

Since $U(t) = u + ct - S(t)$, the term on the left-hand side is

$$\exp\{-ru - rct + \lambda t[M_X(r) - 1]\}. \tag{13.A.6}$$

In the first term on the right-hand side, we write

$$\begin{aligned}U(t) &= U(T) + [U(t) - U(T)]\\ &= U(T) + c(t-T) - [S(t) - S(T)].\end{aligned}$$

For a given T, the term in brackets is independent of $U(T)$ and has a compound Poisson distribution with Poisson parameter $\lambda(t-T)$. Hence the first term on the right-hand side of (13.A.5) can be written as

$$\mathrm{E}[\exp(-r\,U(T))\exp\{-r\,c(t-T) + \lambda(t-T)[M_X(r) - 1]\} \mid T \le t]\,\Pr(T \le t). \tag{13.A.7}$$

Expressions (13.A.6) and (13.A.7) can be greatly simplified if we choose r such that

$$-rc + \lambda\,[M_X(r) - 1] = 0.$$

Two solutions exist (see Figure 13.4.1). The solution $r = 0$ gives a trivial identity when substituted into (13.A.5). The other solution, $r = R$, is the adjustment coefficient. If, with $r = R$, we substitute the simplified expressions into (13.A.5), we obtain

$$\begin{aligned}e^{-Ru} &= \mathrm{E}[e^{-RU(T)} \mid T \le t]\,\Pr(T \le t)\\ &\quad + \mathrm{E}[e^{-RU(t)} \mid T > t]\,\Pr(T > t).\end{aligned} \tag{13.A.8}$$

Now we let $t \to \infty$. The first term on the right-hand side converges to

$$\mathrm{E}[e^{-RU(T)} \mid T < \infty]\,\psi(u).$$

Hence Theorem 13.4.1 follows if we can show that the second term on the right-hand side vanishes for $t \to \infty$. We shall show this as follows:

Let $\alpha = c - \lambda p_1$, $\beta^2 = \lambda p_2$. Thus, from (13.3.6) and (13.3.7),

$$\mathrm{E}[U\ t\ ] = \mathrm{E}[u + ct - S\ t\ ] = u + \alpha t,$$

and

$$\mathrm{Var}[U\ t\ ] = \mathrm{Var}[S\ t\ ] = \beta^2 t.$$

We consider $u + \alpha t - \beta t^{2/3}$, which is positive for t sufficiently large. Now we split the second term on the right-hand side of (13.A.8) by distinguishing whether $U\ t$ is less than or greater than $u + \alpha t - \beta t^{2/3}$. With this splitting, we have

$$\begin{aligned}
&\mathrm{E}[e^{-RU\ t}\ \mid T > t, 0 \le U\ t\ \ \le u + \alpha t - \beta t^{2/3}] \\
&\Pr[T > t, 0 \le U\ t\ \ \le u + \alpha t - \beta t^{2/3}] \\
&\quad + \mathrm{E}[e^{-RU\ t}\ \mid T > t, U\ t\ \ > u + \alpha t - \beta t^{2/3}] \\
&\Pr[T > t, U\ t\ \ > u + \alpha t - \beta t^{2/3}] \\
\le\ &\Pr[U\ t\ \ \le u + \alpha t - \beta t^{2/3}] + \exp[-R\ u + \alpha t - \beta t^{2/3})] \\
\le\ &t^{-1/3} + \exp[-R\ u + \alpha t - \beta t^{2/3})]
\end{aligned}$$

by Chebychev's inequality. But with this upper bound, the second term on the right-hand side of (13.A.8) vanishes for $t \to \infty$. ■

Proof of Theorem 13.5.1:

For this proof we introduce a new concept that is of interest in itself. Let $w\ x\ $, $x \le 0$, be a function with $w\ x\ \ \ge 0$. We define

$$\psi(u; w) = \mathrm{E}[w(U\ T\ ) \mid T < \infty]\ \psi(u), \tag{13.A.9}$$

considered as a function of the initial surplus, u. We may interpret $w\ x\ $ as a ***penalty*** if the surplus at the time of ruin is x. In this case $\psi(u; w)$ is the expected value of the penalty. Examples are

a. If $w\ x\ \ = e^{-Rx}$, then (13.4.4) shows that $\psi(u; w) = e^{-Ru}$

b. If $w\ x\ \ = 1$, then (13.A.9) shows that $\psi(u; w) = \psi(u)$

c. If

$$w_h\ x\ \ = \begin{cases} 1 & x < -h \\ 0 & -h \le x \le 0, \end{cases}$$

then

$$\psi(0; w_h) = \Pr[U\ T\ \ < -h \mid T < \infty]\ \psi(0).$$

We start the proof by showing that, for every bounded function $w\ x\ $, we have

$$\psi(0; w) = \frac{\lambda}{c} \int_0^\infty w\ -y\ [1 - P\ y\ ]\ dy. \tag{13.A.10}$$

Using the law of total probability applied to the definition of the expected penalty (13.A.9) by conditioning on the number claims in the initial interval $(0, b)$, we have

$$\begin{aligned}
\psi(u; w) = &\Pr[N\ b\ \ = 0]\{\text{Conditional Expected Penalty given } N\ b\ \ = 0\} \\
&+ \Pr[N\ b\ \ = 1]\{\text{Conditional Expected Penalty given } N\ b\ \ = 1\} \\
&+ \Pr[N\ b\ \ > 1]\{\text{Conditional Expected Penalty given } N\ b\ \ > 1\}.
\end{aligned} \tag{13.A.11}$$

From the properties of the Poisson counting process,

$$\Pr[N(b) = 0] = e^{-\lambda b},$$

$$\Pr[N(b) = 1] = \lambda b e^{-\lambda b},$$

$$\Pr[N(b) > 1] = b\,A(b), \text{ where the limit } A(b) \text{ is 0 as } b \text{ goes to 0.}$$

By the limitation of $w(x)$ to bounded non-negative functions, $0 \leq w(x) \leq M$ for some M. With this restriction, and the properties of the compound Poisson process, we proceed as follows:

For $N(b) = 0$, the conditional expected penalty is $\psi(u + cb; w)$. Since no claims have occurred and income has been received at the rate c, the surplus has advanced to $u + cb$ and the stationary independent increments place the process at a new starting point.

For $N(b) = 1$, we need to condition further on the amount, x, of the single claim:

- If $x \leq u$, the surplus has remained positive, and as for the case of no claim, the conditional expected penalty is $\psi(u + cb - x; w)$.
- If $u < x \leq u + cb$, it is sufficient to observe that the conditional expected penalty, say, $A(x, b)$, is less than or equal to the bound on $w(x)$, M.
- If $x > u + cb$, then the surplus became negative at the occurrence of the claim, and the conditional expected penalty is $w(u + \xi - x)$, where ξ is the amount of income received up to the time of claim.

Thus for $N(b) = 1$, the conditional expected penalty is

$$\int_0^u \psi(u + cb - x: w)\, p(x)dx + \int_u^{u+cb} A(x, b)p(x)\, dx + \int_{u+cb}^{\infty} w(u + \xi - x)p(x)\, dx.$$

For $N(b) > 1$, it is also sufficient to observe that the conditional expected penalty, say, $D(b)$, is less than or equal to the bound of $w(x)$, M.

Combining these three cases into (13.A.11), we obtain

$$\begin{aligned}\psi(u; w) = e^{-\lambda b}\, \psi(u + cb; w) + \lambda b e^{-\lambda b} &\left[\int_0^u \psi(u + cb - x; w)p(x)\, dx\right. \\ &\left. + \int_u^{u+cb} A(x, b)p(x)\, dx + \int_{u+cb}^{\infty} w(u + \xi - x)p(x)\, dx\right] \\ &+ b\, A(b)\, D(b). \end{aligned} \tag{13.A.12}$$

Now subtract $\psi(u; w)$ from both sides of (13.A.12) and divide the result by cb to get

$$\begin{aligned}&\frac{e^{-\lambda b}\psi(u + cb; w) - \psi(u; w)}{cb} + \frac{\lambda}{c} e^{-\lambda b} \left[\int_0^u \psi(u + cb - x; w)p(x)\, dx\right. \\ &\left. + \int_u^{u+cb} A(x, b)p(x)\, dx + \int_{u+cb}^{\infty} w(u + \xi - x)p(x)\, dx\right] + \frac{A(b)D(b)}{c} = 0. \end{aligned} \tag{13.A.13}$$

As b goes to 0, the limits of the three terms of (13.A.13) are

$$\frac{-\lambda\psi(u; w) + c\psi'(u; w)}{c},$$

$$\frac{\lambda}{c}\left[\int_0^u \psi(u - x; w)p\ x\ \ dx + \int_u^\infty w\ u - x\ p\ x\ \ dx\right],$$

and 0.

Thus,

$$\psi'(u; w) = \frac{\lambda}{c}\psi(u; w) - \frac{\lambda}{c}\int_0^u \psi(u - x; w)\ p\ x\ \ dx$$
$$- \frac{\lambda}{c}\int_u^\infty w\ u - x)\ p\ x\ \ dx. \qquad (13.A.14)$$

We now integrate this equation over u from 0 to z. The resulting double integrals can be reduced to single integrals by a change in variables. For the first double integral we replace x and u by x and $y = u - x$. Then

$$\int_0^z \int_0^u \psi(u - x; w)\ p\ x\ \ dx\, du = \int_0^z \int_0^{z-y} \psi(y; w)\ p\ x\ \ dx\, dy$$
$$= \int_0^z \psi(y; w)\ P\ z - y\ \ dy.$$

In the second double integral we replace x and u by x and $y = x - u$. Then

$$\int_0^z \int_u^\infty w\ u - x\ \ p\ x\ \ dx\, du = \int_0^\infty \int_y^{y+z} w\ -y\ \ p\ x\ \ dx\, dy$$
$$= \int_0^\infty w\ -y\ \ [P\ y + z) - P\ y\ ]\, dy.$$

Thus (13.A.14), integrated from 0 to z, gives

$$\psi(z; w) - \psi(0; w) = \frac{\lambda}{c}\int_0^z \psi(y; w)[1 - P\ z - y)]\, dy$$
$$- \frac{\lambda}{c}\int_0^\infty w\ -y\ \ [P\ y + z) - P\ y\ ]\, dy. \qquad (13.A.15)$$

For $z \to \infty$, the first terms on both sides vanish leaving

$$-\psi(0; w) = -\frac{\lambda}{c}\int_0^\infty w\ -y\ \ [1 - P\ y\ ]\, dy.$$

Now let $w_h\ x$ be defined as in Example (c); that is,

$$w_h\ x\ = \begin{cases} 1 & x < -h \\ 0 & -h \le x \le 0. \end{cases}$$

Then

$$\Pr[U\ T\ < -h \mid T < \infty]\ \psi(0) = \psi(0; w_h) = \frac{\lambda}{c}\int_h^\infty [1 - P\ y\ ]\, dy.$$

Hence, when $u = 0$, the probability that the surplus ever falls below 0, and will be between $-h$ and $-h - dh$ when it happens, is

$$\frac{\lambda}{c} [1 - P(h)] \, dh.$$

If $u > 0$, an event with equal probability is that the surplus will ever fall below u, and will be between $u - h$ and $u - h - dh$ when it happens. This proves Theorem 13.5.1. ■

Exercises

Section 13.2

13.1. Suppose that W_i assumes only the value 0 and +2 and that $\Pr(W = 0) = p$, $\Pr(W = 2) = q$ where $p + q = 1$. Assume that $c = 1$, $p > 1/2$ and that u is an integer. For this case, determine

a. $U(\tilde{T})$ b. $\tilde{\psi}(u)$ in terms of $\tilde{R}$

c. $\tilde{R}$ in terms of p, q d. $\tilde{\psi}(u)$ in terms of p, q.

13.2. Consider the claims in periods $n + 1, n + 2, \ldots, n + m$ and denote their total by $S_{n,m}$; that is,

$$S_{n,m} = W_{n+1} + W_{n+2} + \cdots + W_{n+m}.$$

The claim amount for each period is generated by the stochastic process described in (13.2.14). Verify the following:

a.
$$\begin{aligned} S_{n,m} &= \sum_{i=1}^{m} Y_{n+i} \sum_{j=0}^{m-i} a^j + W_n \sum_{j=1}^{m} a^j \\ &= \sum_{i=1}^{m} \left(\frac{1 - a^{(m-i+1)}}{1 - a} \right) Y_{n+i} + \left(\frac{a - a^{m+1}}{1 - a} \right) W_n. \end{aligned}$$

b. As $m \to \infty$ the final term on the right-hand side of the expression in (a) converges to

$$\frac{a \, W_n}{1 - a}.$$

c. $\mathrm{E}[S_{n,m} \mid W_1 = w_1, W_2 = w_2, \ldots, W_n = w_n]$

$$= \left[\frac{m\,(1 - a) - a + a^{m+1}}{(1 - a)^2} \right] \mu + \left(\frac{a - a^{m+1}}{1 - a} \right) w_n$$

where $\mathrm{E}(Y_{n+i}) = \mu$.

Section 13.3

13.3. Suppose that $V_1, V_2, \ldots$ are independent, identically distributed random variables with common d.f. $F(x)$ and p.d.f. $f(x)$, $x \geq 0$. Given $N(t) = i$ and $T_i = s$ $(s < t)$, what is the probability of the occurrence of a claim between times t and $t + dt$? (This generalization of the Poisson process is called a ***renewal process***.)

13.4. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with parameter λ and $p_n(t) = \Pr[N(t) = n]$.

a. Show that

$$p_0'(t) = -\lambda\, p_0(t),$$

$$p_n'(t) = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t) \qquad n \geq 1.$$

b. Interpret these formulas.

Section 13.4

13.5. Calculate $\lim_{\theta \to 0} R$ and $\lim_{\theta \to \infty} R$.

13.6. Use

$$e^{rx} > 1 + r\,x + \frac{1}{2}\,(rx)^2 \qquad r > 0,\ x > 0$$

to show that

$$R < \frac{2\,\theta\,p_1}{p_2}.$$

13.7. Suppose that $\theta = 2/5$ and

$$p(x) = \frac{3}{2}\,e^{-3x} + \frac{7}{2}\,e^{-7x} \qquad x > 0,$$

calculate

a. γ b. R.

13.8. Suppose that the claim amount distribution is discrete with $p(1) = 1/4$ and $p(2) = 3/4$. If $R = \log 2$, calculate θ.

13.9. Show that the adjustment coefficient can also be obtained as the unique solution of the equation

$$\int_0^\infty e^{rx}\,[1 - P(x)]\,dx = \frac{c}{\lambda} \qquad r < \gamma.$$

Section 13.5

13.10. Use Theorem 13.5.1 to evaluate the denominator in (13.4.4) in the case $u = 0$. Is your result consistent with (13.5.2)?

13.11. a. Show by interpretation that $\psi(u)$, $u \geq 0$, satisfies

$$\psi(u) = \frac{\lambda}{c} \int_0^u [1 - P\ y\ ]\ \psi(u - y\ \ dy + \frac{\lambda}{c} \int_u^\infty [1 - P\ y\ ]\ dy.$$

[Hint: Use Theorem 13.5.1.]

b. The equation shown in part (a) is called a ***renewal equation.*** [See Exercise 13.3.] It is an integral equation because it is a relationship between functions where the relationship involves an integral. One of the methods of seeking a solution for an integral equation is to convert it to an ***integro-differential equation.*** Show that

$$\psi'(u) = \frac{\lambda}{c} \left\{ \psi(u) - \int_0^u \psi(u - y) p(y) dy - [1 - P(u)] \right\}$$

13.12. Substitute

$$M_X(r) = 1 + p_1\, r + p_2 \frac{r^2}{2} + p_3 \frac{r^3}{6} + \cdots$$

into (13.5.4) to derive expressions for

a. $\mathrm{E}[L_1]$ b. $\mathrm{E}[L_1^2]$ c. $\mathrm{Var}\ L_1$.

Section 13.6

13.13. a. For N as in (13.6.5), show that

$$\mathrm{E}[N] = \frac{1}{\theta}$$

and

$$\mathrm{Var}\ N = \frac{1 + \theta}{\theta^2}.$$

b. Use the result of Exercise 13.12 to derive expressions for $\mathrm{E}[L]$ and $\mathrm{Var}\ L$. [Hint: Formulas (13.6.5), (12.2.5), and (12.2.6), with $p_1 = \mathrm{E}[L_1]$, $\mathrm{Var}\ X = \mathrm{Var}\ L_1$, are helpful. For an alternative derivation, expand $M_L(r)$ in (13.6.4) in powers of r.]

13.14. Determine the m.g.f. of L if all claims are of size 2.

13.15. Under certain assumptions, the probability of ruin is

$$\psi(u) = (0.3)\ e^{-2u} + (0.2)\ e^{-4u} + (0.1)\ e^{-7u} \qquad u \geq 0.$$

Calculate

a. θ b. R.

13.16. What is the expected claim size for a distribution of the form (13.6.10)?

13.17. Suppose that $\lambda = 3$, $c = 1$, and

$$p(x) = \frac{1}{3} e^{-3x} + \frac{16}{3} e^{-6x} \qquad x > 0.$$

Calculate or derive

a. p_1
b. θ
c. $M_X(r)$
d. Expressions for the right-hand sides of (13.6.9) and (13.6.12)
e. An explicit formula for $\psi(u)$.

13.18. Suppose that $\lambda = 1$, $c = 10$, and

$$p(x) = \frac{9x}{25} e^{-3x/5} \qquad x > 0.$$

Calculate or derive

a. p_1
b. θ
c. $M_X(r)$
d. Expressions for the right-hand sides of (13.6.9) and (13.6.12)
e. An explicit formula for $\psi(u)$.

13.19. Beekman (1969) and Bowers's discussion thereof suggested the following approximation for the d.f. of L:

$$\Pr(L \le u) \cong \xi I(u) + (1 - \xi)\, G(u\colon \alpha, \beta)$$

where $I(x)$ is the degenerate d.f. of the constant 0 and $G(x\colon \alpha, \beta)$ is the d.f. of the gamma distribution with parameters α and β.

a. Determine ξ, α and β to match the point mass at the origin and the first two moments of L; see Exercise 13.13(b).
b. What is the resulting approximation for $\psi(u)$, $u \ge 0$?

13.20. In a surplus process, (1) aggregate claims follow a compound Poisson process, and (2) the claim amount distribution is a mixture of n exponential distributions with p.d.f. given by (13.6.10).

a. Show that the conditional distribution of L_1, given that the surplus falls below its initial level, is a mixture of the same n exponential distributions.
b. Determine an expression for the n weights of this mixture in terms of the weights and parameters of the claim amount distribution.
c. Determine $E[L_1]$.

13.21. For the surplus process in Example 13.6.2, determine:

a. The p.d.f. of L_1
b. $E[L_1]$
c. $\mathrm{Var}(L_1)$.

Miscellaneous

13.22. In the context of formula (13.2.1) let $G_i = U_i - U_{i-1}$ denote the insurer's gain between times $i - 1$ and i. Suppose that $G_1, G_2, \ldots$ are independent, identically distributed random variables. Suppose further that $u\ x = -e^{-\alpha x}$, $\alpha > 0$, is the insurer's utility function. Show that

$$\mathrm{E}[u\ U_{n+1}) \mid U_n = x] \geq u\ x$$

if and only if $\alpha \leq R$, and interpret the result.

13.23. If we change the time units so that the new units are f times the old units (for some $f > 0$), and let $\tilde{c}, \lambda, \psi(u, t)$ denote parameters of the model in terms of the new units,
a. What are these new parameters in terms of c, λ, and $\psi(u, t)$?
b. For which value of f is $\lambda = 1$? (Some authors refer to these units as ***operational time.***)

13.24. In a surplus process:
1. Aggregate claims have a compound Poisson process
2. Claim amounts are uniformly distributed over (0, 10)
3. $\theta = 0.05$
4. $u = 0.0$
5. $U(T)$ is the negative surplus at ruin
6. $U(T\text{-})$ is the surplus position immediately before ruin [Note: $U(T\text{-}) - X = U(T)$, where X is the claim amount at ruin.]

Determine $\mathrm{E}[U(T\text{-}) | T < \infty]$.
[Hint: $U(T\text{-})$ and $|U(T)|$ are identically distributed in this exercise. To verify this fact, see Gerber and Shiu (1998), formula (3.7).]

Appendix

13.25. Suppose that all claims are of size 1.
a. State the equation for $\psi(u)$ that corresponds to (13.A.14).
b. Solve the equation if $0 \leq u \leq 1$.

14

APPLICATIONS OF RISK THEORY

14.1 Introduction

The collective risk model was developed in Chapters 12 and 13. This model is built on the assumptions that a collection of policies generates a random number of claims in each period and that each claim can be for a random amount. To apply the model, we need information about the distribution of the number of claims and the distribution of individual claim amounts. The selection of these distributions was discussed in Section 12.3, and only definitional remarks are added in this chapter for the distribution of the number of claims. Here, the distribution of individual claim amounts is illustrated in terms of four different lines of insurance: fire, automobile, short-term disability, and hospital.

We discuss two methods of approximating the individual risk theory model for a portfolio of insurances by a collective model. For short-term situations, this provides the means of substituting collective models for individual models.

The concept of stop-loss reinsurance for a portfolio of policies is explored in general. The methods of Chapter 12 for calculating the distribution of aggregate claims provide the means for the calculation of net stop-loss reinsurance premiums. Additionally, we discuss the interpretation of one form of a group insurance dividend formula as a stop-loss insurance.

We give illustrations of analyses of reinsurance agreements using tools from Chapters 12 and 13, as well as a comparison of the adjustment coefficient, which is related to the probability of ruin, with E[L], which is related more to the depth of ruin.

The main purpose of this chapter is to indicate various ways of applying risk theory to insurance problems.

14.2 Claim Amount Distributions

To provide an idea of the broad range of applications or risk theory models, four specific but diverse applications are presented in this section. Here the discussion is of the individual claim amount distribution. This can then be combined with probabilities of individual claims occurring, to provide an individual risk model, or it can be compounded with a distribution for number of claims from a collection of insurances, to provide a collective risk model. The applications suggested in this section might be used by an insurance company in managing a line of business or block of similar policies, or by an industrial firm that uses modeling in its risk management program.

Fire Insurance:

In this line of insurance, the claim event is a fire in an insured structure that creates a loss. Because fires can cause heavy damage, adequate probability should be assigned to the higher claim amounts by the d.f. $P\ x$. In actuarial literature, some standard distributions have been suggested. Three of these distributions are listed in Table 14.2.1.

TABLE 14.2.1

Typical Claim Amount Distributions

Name	$p\ x$	Mean	Variance
Lognormal	$(x\sigma\sqrt{2\pi})^{-1} \exp\left[\frac{-(\log x - m)^2}{2\sigma^2}\right]$ $x > 0,\quad \sigma > 0$	$\exp(m + \sigma^2/2)$	$(e^{\sigma^2} - 1)\exp(2m + \sigma^2)$
Pareto	$\frac{\alpha\, x_0^\alpha}{x^{\alpha+1}}$ $x > x_0 > 0,\quad \alpha > 0$	$\frac{\alpha\, x_0}{\alpha - 1}$ $\alpha > 1$	$\frac{\alpha\, x_0^2}{(\alpha-2)(\alpha-1)^2}$ $\alpha > 2$
Mixture of exponentials	$p\,\alpha\, e^{-\alpha x} + q\,\beta\, e^{-\beta x}$ $x > 0,\quad 0 < p < 1,\quad q = 1 - p,\quad \alpha, \beta > 0$	$\frac{p}{\alpha} + \frac{q}{\beta}$	$\frac{p(1+q)}{\alpha^2} + \frac{q(1+p)}{\beta^2} - \frac{2pq}{\alpha\beta}$

An indication of the wide dispersion of probability in the case of the Pareto distribution is given by the mean's not existing unless $\alpha > 1$ and the variance's not existing unless $\alpha > 2$. Such distributions are said to have ***heavy tails.***

To apply one of these standard distributions, the parameters of the distribution could be estimated from a sample of claim amounts.

Automobile Physical Damage:

In this line of insurance, a claim event is an incident causing damage to an insured automobile. The claim amount will not have the wide variability found in fire insurance. For this reason, the gamma distribution (12.3.17) has given reasonable fit to data and has been used on occasion for the claim amount distribution. Again, the parameters of the distribution could be estimated from a sample of claim amounts.

Disability Insurance:

This insurance provides income benefits to disabled lives. Usually there is a defined ***elimination period,*** seven days for example, following the occurrence of disability until benefits commence. There is also an upper limit on the payment period, as short as 13 weeks or as long as the period until retirement age. When issued to a group, the insurance is called ***group weekly indemnity insurance*** or ***group long-term disability insurance,*** depending on the length of the payment period.

The benefit is a fixed amount per period, and the amount of claim is this fixed amount times the number of periods the disability continues beyond the elimination period. Let the random variable, Y, represent this number of periods. From claims statistics, the distribution of Y can be estimated and tabulated in a form such as Table 14.2.2. Note the analogy between the function represented in the second column of Table 14.2.2 and the survival function discussed for life tables in Chapter 3. As used in this case, the function is referred to as a **continuance function.** It yields probabilities of continuance or survival of a disability claim for the indicated length of time. Similar to the way the survival function can be employed to provide various probabilities of survival and death, the continuance function can be employed to provide various probabilities of continuance and termination of disability.

In applying a collective risk model for a group disability insurance of the type illustrated in Table 14.2.2, $\Pr(N = n)$ should be interpreted as the probability that n disabilities, each of which continues at least seven days, occur during the insurance term among the group of insureds.

If the income benefit is an amount c per day, the claim amount distribution is given by

$$p(x) = \Pr\left(Y = \frac{x}{c}\right) \qquad x = c, 2c, 3c, \ldots, 28c, 31c, \ldots, 87c, 91c.$$

Interest is not considered for this short-term insurance.

For group disability insurance a Poisson distribution is often appropriate for the distribution of the number of disablements that occur and continue through the elimination period. The expected number of disablements for the distribution is often assumed to be proportional to the number of lives in the group. The following example illustrates how a compound Poisson distribution might be used to model the experience of a group disability contract with a medium length benefit period. The number of claims generated by the group within a period of fixed length and the lengths of the benefit periods of the claims that occur are assumed to be mutually independent.

Example 14.2.1

Consider a disability insurance contract covering a group of 200 females all age 32. The benefit is a set of monthly payments of 2,000 each that commence three

TABLE 14.2.2

Illustrative Distribution of Y, the Length of Claim under Group Weekly Disability Income Insurance (13-Week Maximum Benefit, 7-Day Elimination Period)

Length of Claim (in Days) y	$\Pr(Y > y)$	$\Pr(Y = y)$
0	1.00000	—
1	0.96500	0.03500
2	0.93026	0.03474
3	0.89677	0.03349
4	0.86359	0.03318
5	0.83164	0.03195
6	0.80004	0.03160
7	0.76964	0.03040
8	0.73962	0.03002
9	0.71077	0.02885
10	0.68376	0.02701
11	0.65846	0.02530
12	0.63476	0.02370
13	0.61254	0.02222
14	0.59171	0.02083
15	0.57218	0.01953
16	0.55387	0.01831
17	0.53615	0.01772
18	0.51953	0.01662
19	0.50342	0.01611
20	0.48832	0.01510
21	0.47367	0.01465
22	0.45993	0.01374
23	0.44659	0.01334
24	0.43364	0.01295
25	0.42150	0.01214
26	0.40970	0.01180
27	0.39864	0.01106
28	0.38788	0.01076
31	—	0.06361*
35	0.32427	—
38	—	0.04832
42	0.27595	—
45	—	0.03753
49	0.23842	—
52	—	0.02980
56	0.20862	—
59	—	0.02399
63	0.18463	—
66	—	0.01939
70	0.16524	—
73	—	0.01586
77	0.14938	—
80	—	0.01300
84	0.13638	—
87	—	0.01077
91	0.00000	0.12561
		1.00000

*For convenience, claim terminations of a week from here on have been considered as terminations at the end of the third day of the week.

months following the date of disablement and continue as long as the disability up to a maximum of 21 payments. We assume that a compound Poisson distribution is appropriate for S, the total claims for the group.

For the rate of disablement and the claim amount distribution we make use of an excerpt of a continuance table shown below that is appropriate for a group of 1,000 females age 32 as published in the 1987 Commissioners Group Long-Term Disability (GLTD) table. The table was constructed in a deterministic context. For this example we make the reasonable interpretation that the data are appropriate for a compound Poisson distribution, and the column headings have been adapted with Y in Column (3) used as defined above. The constant of proportionality for the rate of disablement per life is denoted by λ_{32}.

Continuance in GLTD Table
Female, Three-Month Elimination Period
Age at Disablement 32

(1) Months from Disablement	(2) Number of Payments (y)	(3) $1{,}000\,\lambda_{32}\,\Pr(Y \geq y)$
3	1	2.6640
4	2	2.4008
5	3	2.1343
6	4	1.9277
7	5	1.7664
8	6	1.6431
9	7	1.5442
10	8	1.4614
11	9	1.3898
12	10	1.3288
13	11	1.2767
14	12	1.2320
15	13	1.1926
16	14	1.1575
17	15	1.1261
18	16	1.0983
19	17	1.0735
20	18	1.0512
21	19	1.0308
22	20	1.0125
23	21	0.9961
24	22	0.9815

This table continues to 25 years following disablement.

Determine the mean and variance of S.

Solution:

The benefit (claim) amount is denoted by $X = 2{,}000\ Y$, $Y = 1, 2, \ldots, 21$, and the p.f. of Y is denoted by $p\ y$. Note that this formulation ignores interest over the 21-month period. The expected number of disablements that continue beyond the elimination period for the group of 200 lives is $200\,\lambda_{32}$. In this setup,

$$E[S] = 200\,\lambda_{32}\,E[X] = (2{,}000)(200)\,\lambda_{32}\,E[Y], \tag{14.2.1}$$

and

$$\mathrm{Var}(S) = 200\,\lambda_{32}\,E[X^2] = (2{,}000^2)(200)\,\lambda_{32}\,E[Y^2]. \tag{14.2.2}$$

To express the moments of S in terms of the values given in the continuance table we proceed as follows:

$$\begin{aligned}
E[Y] &= \sum_{y=1}^{21} y\,p(y) + 21\sum_{y=22}^{\infty} p(y) \\
&= \sum_{y=1}^{21}\left(\sum_{x=1}^{y} 1\right) p(y) + 21\sum_{y=22}^{\infty} p(y) \\
&= \sum_{x=1}^{21}\left[\sum_{y=x}^{21} p(y)\right] + 21\sum_{y=22}^{\infty} p(y) \\
&= \sum_{x=1}^{21}\left[\sum_{y=x}^{\infty} p(y) - \sum_{y=22}^{\infty} p(y)\right] + 21\sum_{y=22}^{\infty} p(y) \\
&= \sum_{x=1}^{21}\sum_{y=x}^{\infty} p(y) = \sum_{x=1}^{21}\Pr(Y \ge x).
\end{aligned}$$

Upon substituting this expression into (14.2.1), we have

$$\begin{aligned}
E[S] &= 200\,\lambda_{32}\,E[X] = (2{,}000)(200)\,\lambda_{32}\,E[Y] \\
&= 400(2.6640 + 2.4008 + \cdots + 0.99610) = 12{,}203.12.
\end{aligned}$$

For the calculation of the variance for the total claims of the group of 200 lives, we need the second moment equivalent of the substitution $m = \sum_{x=1}^{m}(1)$. It can be verified that $m^2 = \sum_{x=1}^{m}(2x - 1)$. Thus

$$\begin{aligned}
E(Y^2) &= \sum_{y=1}^{21} y^2 p(y) + (21)^2\sum_{y=22}^{\infty} p(y) \\
&= \left\{\sum_{y=1}^{21}\left[\sum_{x=1}^{y}(2x-1)\right] p(y) + (21)^2\sum_{y=22}^{\infty} p(y)\right\} \\
&= \left\{\sum_{x=1}^{21}(2x-1)\left[\sum_{y=x}^{\infty} p(y) - \sum_{y=22}^{\infty} p(y)\right] + (21)^2\sum_{y=22}^{\infty} p(y)\right\} \\
&= \sum_{x=1}^{21}(2x-1)\sum_{y=x}^{\infty} p(y) = \sum_{x=1}^{21}(2x-1)\Pr(y \ge x).
\end{aligned}$$

Upon substituting this expression into (14.2.2), we have

$$\begin{aligned}
\mathrm{Var}(S) &= (2{,}000)^2(200)\,\lambda_{32}\,E[Y^2] \\
&= (800{,}000)1{,}000\,\lambda_{32}\sum_{x=1}^{21}(2x-1)\Pr(y \ge x) \\
&= 800{,}000[1(2.6640) + 3(2.4008) + \cdots + 41(0.9961)] \\
&= 425.9808 \times 10^6.
\end{aligned}$$

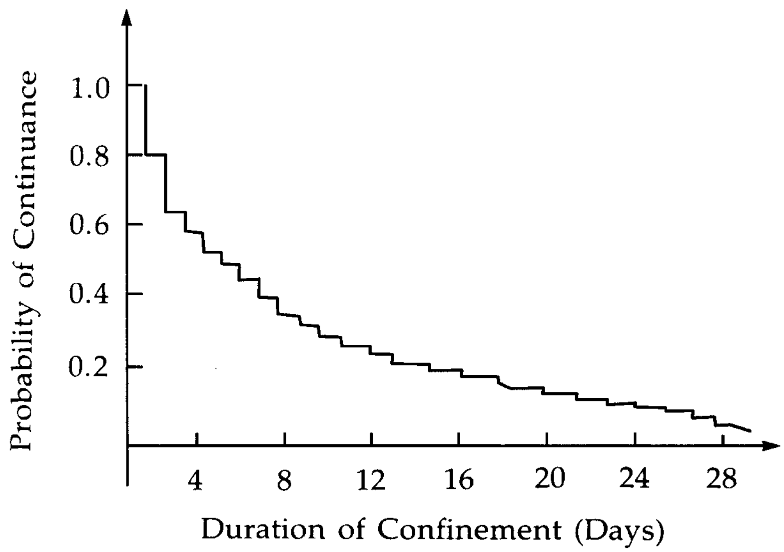
▼

Hospital Insurance:

Here we consider hospital insurance that provides a flat daily benefit during hospitalization. A hospitalization continuance table can be used to produce a p.f. for the length of stay in a hospital for each hospitalization. A graph of a hospitalization continuance function is given in Figure 14.2.1.

FIGURE 14.2.1

Continuance Function for Hospital Insurance



In applying a collective risk model to a hospital insurance of this type issued to a group of lives, $\Pr(N = n)$ should be interpreted as the probability that n hospitalizations, which meet the definition contained in the policy, occur during the period to members of the covered group. If the benefit amount is c per day, the p.f. of the claim amount is given by

$$p\ x = \Pr\left(Y = \frac{x}{c}\right) \qquad x = c, 2c, \ldots, mc$$

where the random variable Y represents the length of hospitalization in days and m is the maximum number of days for which benefits are paid.

The use of risk models in these applications permits us to estimate the required total pure premium. In addition, this estimate may be supplemented with statements about variability of losses.

14.3 Approximating the Individual Model

The individual and collective risk models are alternative constructions designed to capture key aspects of insurance systems. Each model leads to the development of a distribution of total claims for the modeled insurance system. In this section we develop two methods by which the compound Poisson distribution, usually associated with the collective risk model, can be used to approximate the distribution of total claims in the individual model.

We consider the individual model developed in Chapter 2 for application to a group of n policies. The total claims in a policy period for the group is $S = X_1 + X_2 + \cdots + X_n$, where X_j is the claim that results from policy $j (j = 1, 2, \ldots, n)$. We distinguish between the occurrence of a claim and its amount, and we write

$$X_j = I_j B_j. \tag{14.3.1}$$

Here, I_j is 1 if policy j leads to a claim and 0 otherwise; B_j is the amount of such a claim, given that it occurs. On the assumption that $I_j, B_j, j = 1, 2, \ldots, n$, are mutually independent, it follows that

$$\mathrm{E}[S] = \sum_{j=1}^{n} q_j \mu_j \tag{14.3.2}$$

and

$$\mathrm{Var}(S) = \sum_{j=1}^{n} q_j(1 - q_j) \mu_j^2 + \sum_{j=1}^{n} q_j \sigma_j^2 \tag{14.3.3}$$

[see (2.2.25) and (2.2.26)], where q_j denotes the probability that policy j leads to a claim, $\mu_j = \mathrm{E}[B_j]$, and $\sigma_j^2 = \mathrm{Var}(B_j)$.

We denote the d.f. of B_j by $P_j(x)$. If a claim occurs, the probability that it comes from policy j is, by Bayes theorem, approximately $q_j/(q_1 + q_2 + \cdots + q_n)$. Then, by the law of total probability, the d.f. of the amount of a given claim is approximately

$$\sum_{j=1}^{n} \frac{q_j P_j(x)}{q_1 + \cdots + q_n}. \tag{14.3.4}$$

We next consider two methods of approximating the distribution of S by a compound Poisson distribution.

The first method uses the compound Poisson distribution with Poisson parameter

$$\lambda = q_1 + q_2 + \cdots + q_n \tag{14.3.5}$$

and d.f. of individual claim amounts

$$P(x) = \sum_{j=1}^{n} \frac{q_j}{\lambda} P_j(x). \tag{14.3.6}$$

The interpretation of (14.3.5) is that the expected number of claims in the compound Poisson model is the same as in the original individual risk model. Similarly, (14.3.6) means that the distribution of a claim, given that it has occurred, is the same in the two models, as can be seen from (14.3.4).

The compound Poisson distribution specified by (14.3.5) and (14.3.6) can also be explained as follows: In the individual model, the number of claims produced by policy j is a Bernoulli random variable. We approximate its distribution by the Poisson distribution with parameter q_j. Correspondingly, the distribution of X_j is approximated by the compound Poisson distribution given by q_j and $P_j\ x$. Then we use Theorem 12.4.1 to approximate the distribution of S by the compound Poisson distribution given by (14.3.5) and (14.3.6).

From (14.3.6), it follows that

$$p_k = \sum_{j=1}^{n} \frac{q_j}{\lambda} \mathrm{E}[B_j^k] \qquad k = 1, 2, \ldots . \tag{14.3.7}$$

In particular,

$$p_1 = \sum_{j=1}^{n} \frac{q_j}{\lambda} \mu_i$$

and

$$p_2 = \sum_{j=1}^{n} \frac{q_j}{\lambda} (\mu_j^2 + \sigma_j^2).$$

Thus, the mean of the approximating compound Poisson distribution, λp_1, coincides with the mean of the total claims in the original individual model [see (14.3.2)]. On the other hand, the variance of the approximating compound Poisson distribution, λp_2, is

$$\sum_{j=1}^{n} q_j(\mu_j^2 + \sigma_j^2) \tag{14.3.8}$$

and exceeds the variance of total claims in the individual model [see (14.3.3)]. However, if the q_j's are small, the two variances are approximately the same.

Let us consider the special case where the claim amount for each policy is constant, $B_j = b_j$, so that $\mu_j = b_j$ and $\sigma_j = 0$. Then the p.f. of individual claim amounts according to (14.3.6) is

$$p\ x\ = \sum_{b_j=x} \frac{q_j}{\lambda} \tag{14.3.9}$$

where the sum is taken over the policies for which $b_j = x$. Furthermore, the ratio of the variance of total claims in the individual model [see (14.3.3)] to the variance of the approximating compound Poisson distribution [see (14.3.8)] is

$$\frac{\sum_{j=1}^{n} q_j b_j^2(1 - q_j)}{\sum_{j=1}^{n} q_j b_j^2}. \tag{14.3.10}$$

This ratio can be interpreted as a weighted average of the probabilities of no claims, $1 - q_j$.

Example 14.3.1

In Example 2.5.1 we considered a portfolio of 1,800 policies. Approximate the distribution of aggregate claims by a compound Poisson distribution and discuss the resulting approximation for the variance of aggregate claims.

Solution:

According to (14.3.5),

$$\lambda = 500(0.02) + 500(0.02) + 300(0.1) + 500(0.1) = 100.$$

According to (14.3.9),

$$p(1) = \frac{500(0.02) + 300(0.1)}{100} = 0.4$$

$$p(2) = \frac{500(0.02) + 500(0.1)}{100} = 0.6.$$

Then $p_2 = p(1) + 4\,p(2) = 2.8$, and the variance of the compound Poisson approximation is $\lambda p_2 = 280$. As expected, this exceeds the variance of aggregate claims in the individual model, which was found to be 256 in Example 2.5.1. ▼

The second method to approximate the distribution of S uses the compound Poisson distribution with Poisson parameter

$$\tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n \qquad (14.3.11)$$

where $\tilde{\lambda}_j = -\log(1 - q_j)$ and d.f. of individual claim amounts

$$\tilde{P}(x) = \sum_{j=1}^{n} \frac{\tilde{\lambda}_j}{\tilde{\lambda}} P_j(x). \qquad (14.3.12)$$

The motivation for (14.3.11) and (14.3.12) is similar to that for (14.3.5) and (14.3.6). The key difference is that in (14.3.5) the expected numbers of claims in the two models are matched, whereas (14.3.11) implies

$$e^{-\tilde{\lambda}} = \prod_{j=1}^{n} (1 - q_j);$$

that is, the probabilities of no claims are the same in the two models.

Example 14.3.2

For the portfolio of 1,800 policies studied in Examples 2.5.1 and 14.3.1, calculate the compound Poisson approximation to the distribution of aggregate claims by the second method.

Solution:

$$\begin{aligned}\tilde{\lambda} &= -500\log(0.98) - 500\log(0.98) - 300\log(0.9)\\ &\quad -500\log(0.9) = 104.5\end{aligned}$$

$$\tilde{p}(1) = \frac{-500 \log(0.98) - 300 \log(0.9)}{104.5} = 0.399$$

$$\tilde{p}(2) = \frac{-500 \log(0.98) - 500 \log(0.9)}{104.5} = 0.601.$$

▼

In this section we have presented two methods for approximating the distribution of aggregate claims in the individual model by a compound Poisson distribution. If all the q_j's for the individual model are small (which could well be the case in connection with life insurance policies), the two methods produce very similar results since, in that case,

$$\lambda_j = -\log(1 - q_j) = q_j + \frac{1}{2} q_j^2 + \cdots \cong q_j.$$

14.4 Stop-Loss Reinsurance

The concept of an insurance with a deductible is discussed in Section 1.5. A definition is given in (1.5.1), and a property of optimality is established in Theorem 1.5.1. When such a coverage is written for a collection of insurance risks, it is called stop-loss reinsurance, the topic of this section. In a given application, S may denote the total claims in a given period for an insurance company, or for a block of business of a company, or for a life or health group insurance contract.

For a stop-loss contract with deductible d, the amount paid by the reinsurer to the ceding insurer is

$$I_d = \begin{cases} 0 & S \le d \\ S - d & S > d. \end{cases} \tag{14.4.1}$$

Sometimes this is written as $I_d = (S - d)_+$, where the plus subscript denotes the positive part of $S - d$.

Note that I_d, as a function of the aggregate claims S, is also a random variable. The amount of claims retained by the ceding insurer is

$$S - I_d = \begin{cases} S & S \le d \\ d & S > d. \end{cases} \tag{14.4.2}$$

Thus, the amount retained is bounded by d, which explains the name stop-loss contract.

We discuss methods to calculate $\mathrm{E}[I_d]$, the expected claims paid by the reinsurance when the deductible is d. We denote the d.f. of S by $F_S\, x$ and first assume that S has a p.d.f. $f_S\, x$. Then,

$$\mathrm{E}[I_d] = \int_d^\infty (x - d)\, f_S\, x \;\; dx. \tag{14.4.3}$$

Usually S cannot assume any negative values. We can extend the integral to $(0, \infty)$ and subtract the integral over $(0, d)$ to see that

$$E[I_d] = E[S] - d + \int_0^d (d - x) f_S(x)\, dx. \tag{14.4.4}$$

If we set

$$f_S(x) = -\frac{d}{dx}[1 - F_S(x)]$$

in (14.4.3), and integrate by parts, we get

$$E[I_d] = \int_d^\infty [1 - F_S(x)]\, dx. \tag{14.4.5}$$

Similarly, we obtain

$$E[I_d] = E[S] - \int_0^d [1 - F_S(x)]\, dx \tag{14.4.6}$$

from (14.4.4).

Each of these four expressions for $E[I_d]$ has its own merit. If $E[S]$ is available, (14.4.4) and (14.4.6) are preferable where numerical integration is required, since the range of integration is finite. This reduces the possibilities of inaccurate approximation of $f_S(x)$ for large x. Formulas (14.4.5) and (14.4.6) hold for general distributions, including those of discrete or of mixed type. If the distribution of S is given in analytical form, for example, by a normal or gamma distribution, (14.4.3) might be the most tractable formula.

Example 14.4.1

If S has a gamma distribution, show that

$$E[I_d] = \frac{\alpha}{\beta}[1 - G(d{:}\alpha + 1, \beta)] - d\,[1 - G(d{:}\alpha, \beta)].$$

Solution:

From (14.4.3), we obtain

$$E[I_d] = \int_d^\infty x f_S(x)\, dx - d\,[1 - F_S(d)]$$

$$= \int_d^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x}\, dx - d\,[1 - G(d{:}\alpha, \beta)].$$

Since $\alpha\Gamma(\alpha) = \Gamma(\alpha + 1)$, the integrand is α/β times the gamma p.d.f. with parameters $\alpha + 1$ and β. Hence the given formula follows. ▼

Example 14.4.2

Suppose that a, b are numbers with $\Pr(a < S < b) = 0$. Show that, for $a < d < b$, $E[I_d]$ can be obtained from $E[I_a]$ and $E[I_b]$ by linear interpolation.

Solution:

From the assumption, it follows that $F_S(x) = F_S(a)$ for $a \le x < b$. We use this in (14.4.6) to see that

$$\mathrm{E}[I_d] = \mathrm{E}[I_a] - (d - a)[1 - F_S(a)];$$

that is, $\mathrm{E}[I_d]$ is a linear function of d in the interval $[a, b]$. ▼

We now consider the case where the possible values of S are non-negative integers and denote by $f_S(x)$ the p.f. of S ($x = 0, 1, 2, \ldots$). We assume that the deductible d is an integer. According to the preceding example, the expected stop-loss reinsurance claims for noninteger deductibles can be obtained by linear interpolation.

The formulas

$$\mathrm{E}[I_d] = \sum_{x=d+1}^{\infty} (x - d)\, f_S(x) \tag{14.4.7}$$

and

$$\mathrm{E}[I_d] = \mathrm{E}[S] - d + \sum_{x=0}^{d-1} (d - x)\, f_S(x) \tag{14.4.8}$$

are the counterparts of (14.4.3) and (14.4.4). The integrals in (14.4.5) and (14.4.6) can be written as sums, since $F_S(x)$ is piecewise constant. We obtain

$$\mathrm{E}[I_d] = \sum_{x=d}^{\infty} [1 - F_S(x)] \tag{14.4.9}$$

and

$$\mathrm{E}[I_d] = \mathrm{E}[S] - \sum_{x=0}^{d-1} [1 - F_S(x)]. \tag{14.4.10}$$

Example 14.4.3

For the aggregate claims distribution in Example 12.2.2 calculate, by two methods, the expected stop-loss reinsurance claims when the deductible is 7.

Solution:

According to (14.4.7),

$$\mathrm{E}[I_7] = f_S(8) + 2\, f_S(9) = 0.0028.$$

Alternatively, according to (14.4.9),

$$\mathrm{E}[I_7] = [1 - F_S(7)] + [1 - F_S(8)] = 0.0028.$$ ▼

Example 14.4.4

Calculate $E[I_6]$ for the compound Poisson distribution used in Example 12.4.2.

Solution:

Since the compound Poisson distribution has an infinite range, the use of (14.4.8) and (14.4.10) is more practical here. For example, using (14.4.8) we obtain

$$E[I_6] = E[S] - 6 + \sum_{x=0}^{5} (6 - x) f_S\ x$$

$$= 1.7 - 6 + 4.3547 = 0.0547.$$

▼

In general, from (14.4.9), we obtain a recursive formula

$$E[I_{d+1}] = E[I_d] - [1 - F_S\ d\ ] \qquad d = 0, 1, 2, \ldots . \tag{14.4.11}$$

Thus $E[I_d]$ can be obtained recursively with starting value $E[I_0] = E[S]$.

This recursive approach is particularly convenient if S has a compound distribution that satisfies the conditions of Theorem 12.4.3. In this case, $f_S\ x$ also can be calculated recursively [see (12.4.16)–(12.4.18)]. As an example, for the compound Poisson distribution, we start with

$$f_S(0) = F_S(0) = e^{-\lambda}$$

and

$$E[I_0] = \lambda p_1,$$

and use the recursive formulas

$$f_S\ x\ = \frac{\lambda}{x} \sum_{j=1}^{\infty} j p\ j\ f_S\ x - j),$$

$$F_S\ x\ = F_S\ x - 1) + f_S\ x\ ,$$

$$E[I_x] = E[I_{x-1}] - [1 - F_S\ x - 1)]$$

successively for $x = 1, 2, 3, \ldots .$

Example 14.4.5

Assume that S has a compound Poisson distribution with $\lambda = 1.5$, $p(1) = 2/3$, $p(2) = 1/3$. Calculate values of $f_S\ x$, $F_S\ x$, $E[I_x]$ for $x = 0, 1, 2, \ldots, 6$.

Solution:

First,

$$f_S(0) = F_S(0) = e^{-1.5} = 0.223$$

and

$$E[I_0] = \lambda p_1 = 1.5 \frac{4}{3} = 2.$$

Then, since $\lambda j p(j) = 1$ for $j = 1, 2,$

$$f_S(x) = \frac{1}{x} [f_S(x-1) + f_S(x-2)] \qquad x = 1, 2, \ldots, 6.$$

Note that $f_S(1) = f_S(0)$.

The remaining steps and the results are displayed below.

x	$f_S(x) = (1/x)\,[f_S(x-1) + f_S(x-2)]$	$F_S(x) = F_S(x-1) + f_S(x)$	$E[I_x] = E[I_{x-1}] + F_S(x-1) - 1$
0	0.223	0.223	2.000
1	0.223	0.446	1.223
2	0.223	0.669	0.669
3	0.149	0.818	0.338
4	0.093	0.911	0.156
5	0.048	0.959	0.067
6	0.024	0.983	0.026

▼

Our discussion has focused on the calculation of $E[I_d]$, the expected stop-loss reinsurance claims. Typically, this is a lower bound for a stop-loss premium. The actual premium will contain a loading that reflects the variability of the reinsurer's payment, I_d. One measure of this variability is

$$\operatorname{Var}(I_d) = E[I_d^2] - E[I_d]^2.$$

In the discrete case, it is possible to compute $E[I_d^2]$ recursively (see Exercise 14.8).

We now turn to a dividend formula of group insurance because it is identical in concept to a stop-loss reinsurance. Recall that group insurance is the name used when an insurance covering many individuals is purchased in the form of a single contract by a sponsor such as an employer. Examples are given in the short-term disability illustration in Section 14.2 and Example 14.2.1. In this section we discuss one type of dividend formula that can be used in relation to group insurance.

We assume that for a premium of G the insurer will provide full coverage for total claims S in a given period. With the policyholder's knowledge, the premium contains a substantial loading $G - E[S] > 0$. Consequently, the policyholder anticipates a dividend D at the end of the period, which will be a function of S. Specifically, we assume that the dividend is of the form

$$D = \begin{cases} kG - S & S < kG \\ 0 & S \geq kG \end{cases} \qquad (14.4.12)$$

where $0 < k < 1$. Thus the policyholder pays G and in return receives S and D.

We now consider the expected value of D. For notational convenience, we assume that the distribution of S is continuous and denote the p.d.f. of S by $f_S\ x$; the discrete case is very similar, as shown in Example 14.4.6. From (14.4.12), we have

$$\mathrm{E}[D] = \int_0^{kG} (kG - x)\, f_S\ x\ \ dx. \tag{14.4.13}$$

Presumably, the insurer will set k small enough so that $\mathrm{E}[S] + \mathrm{E}[D] < G$.

Example 14.4.6

For a premium of five the insurer covers total claims S, having the compound Poisson distribution considered in Example 14.4.5. The insurer agrees to pay a dividend equal to the excess of 80% of the premium over the claims. Calculate $G - \mathrm{E}[S] - \mathrm{E}[D]$ (this is the expected value of the amount available to cover expenses, security loading, and so on).

Solution:

The dividend is of the form (14.4.12) with $k = 0.8$. Thus

$$\mathrm{E}[D] = 4\,f_S(0) + 3\,f_S(1) + 2\,f_S(2) + f_S(3) = 2.156.$$

Then,

$$G - \mathrm{E}[S] - \mathrm{E}[D] = 5 - 2 - 2.156 = 0.844.$$

▼

We can rewrite the right-hand side of (14.4.13) as

$$\mathrm{E}[D] = \int_0^{\infty} (kG - x)\, f_S\ x\ \ dx + \int_{kG}^{\infty} (x - kG)\, f_S\ x\ \ dx.$$

Thus

$$\mathrm{E}[D] = kG - \mathrm{E}[S] + \mathrm{E}[I_{kG}] \tag{14.4.14}$$

where I_{kG} is the payment under a stop-loss contract with deductible kG. If the expected stop-loss claims $\mathrm{E}[I_d]$ have already been calculated for various deductibles, this is a convenient formula to obtain $\mathrm{E}[D]$.

Example 14.4.7

Use (14.4.14) to obtain $\mathrm{E}[D]$ in Example 14.4.6.

Solution:

Since $\mathrm{E}[I_4] = 0.156$,

$$\mathrm{E}[D] = 4 - 2 + 0.156 = 2.156.$$

▼

There are more facets to the connection between a dividend formula of the type (14.4.12) and a stop-loss contract. We start with the identity

$$S + D = kG + I_{kG}. \tag{14.4.15}$$

This can be verified by distinguishing the cases: $S \leq kG$ where both sides equal kG, and $S > kG$ where both sides equal S. Subtracting G from both sides, we obtain

$$S + D - G = I_{kG} - (1 - k)G. \tag{14.4.16}$$

We have the following interpretation: The balance of the claim payments and dividends received over the premium paid is the same as the corresponding balance for a stop-loss contract with deductible kG and stop-loss premium $(1 - k)G$.

This interpretation suggests that the insurer can regard the premium as split into two components,

$$G = kG + (1 - k)G. \tag{14.4.17}$$

Claims are first paid from kG, and any remaining balance $kG - S$ (for $S < kG$) is paid as a dividend in accordance with (14.4.12). The second component, $(1 - k)G$, is used to provide a stop-loss reinsurance with deductible kG.

Formula (14.4.15) rearranged as

$$D = kG - S + I_{kG}$$

yields (14.4.14) again when expectations are taken on each side.

14.5 Analysis of Reinsurance Using Ruin Theory

Questions about type and amount of reinsurance to purchase can be answered in various ways. One approach is provided by the insurer adopting a utility function. From all available reinsurance arrangements, the insurer selects the one yielding the highest expected utility. This ideal approach, which is very simple in concept, is not commonly used in practice.

In preparation for a second approach, note that in Chapter 13 we considered the insurer's premium rate c to have a relative security loading θ such that $c = (1 + \theta)\,\lambda p_1$ [see (13.4.3)]. Here θ did not include any loading for expenses, and all of c was available for the risk process. For our further discussion of reinsurance, it is useful to define a reinsurance loading ξ by the formula

$$\text{(reinsurance premium rate)} = (1 + \xi)\text{(expected rate of reinsurance payment)}. \tag{14.5.1}$$

The reinsurance premium rate, as determined by the reinsurer, will provide for reinsurance payments, expenses, security, and profit. The insurer can express the reinsurance premium rate in the format of the right-hand side of (14.5.1) to determine ξ. In particular, for expected stop-loss claims $E[I_d]$, given by (14.4.3), the loading ξ is 0.

The second approach recognizes that the purchase of reinsurance is necessarily a compromise between expected gain and security. Because of the loading contained in the reinsurance premium, the purchase of reinsurance will reduce the insurer's expected gain; on the other hand, a reasonable reinsurance arrangement will increase security in some sense. For this approach, a required standard of security is first defined, and then only reinsurance arrangements satisfying this standard are considered. From this set of admissible arrangements, the insurer selects the one that allows for the highest expected gain.

We consider two tools from ruin theory for evaluating reinsurance agreements. The first is the adjustment coefficient because it can be used to obtain information about the probability of ruin. As a second tool we examine the use of E[L], the expected value of the maximal aggregate loss. At this point the name ***adjustment coefficient*** reveals its meaning: If a certain reinsurance arrangement produces a value of R (or $\tilde{R}$) that is not large enough, the arrangement needs to be adjusted.

Example 14.5.1

An insurer has a portfolio producing annual aggregate claims that are independent and identically distributed; their common distribution is compound Poisson with $\lambda = 1.5$, $p(1) = 2/3$, $p(2) = 1/3$ (see Example 14.4.5). The annual premiums received are $c = 2.5$.

a. Calculate the adjustment coefficient that results from this portfolio.
b. Stop-loss coverage can be obtained for a reinsurance loading charge of 100%. Calculate the adjustment coefficient if a stop-loss contract is purchased with a deductible of

(i) 3 (ii) 4 (iii) 5.

Also, compare these alternatives from the point of view of expected gain.

Solution:

a. In this case, $R = \tilde{R}$, and we can obtain R from (13.2.6) or (13.4.2). The latter condition can be written

$$1.5 + 2.5r = e^r + (0.5)\, e^{2r}.$$

We obtain $R = \tilde{R} = 0.28$.

b. We consider case (i), $d = 3$, in detail. In Example 14.4.5 we computed $\mathrm{E}[I_3] = 0.338$. The actual stop-loss premium is twice this amount, or 0.676. Thus the insurer's retained premium in year i is $2.5 - 0.676 = 1.824$, and retained claims are

$$\hat{W}_i = \begin{cases} W_i & W_i = 0, 1, 2, 3 \\ 3 & W_i > 3 \end{cases}$$

where W_i denotes the aggregate claims of year i. According to (13.2.6), $\tilde{R}$ is the positive solution of the equation

$$e^{-1.824r} \left\{ \sum_{x=0}^{2} f_W\, x \;\; e^{xr} + [1 - F_W(2)]\, e^{3r} \right\} = 1$$

(see Example 14.4.5). We calculate $\tilde{R} = 0.25$. The expected gain per year is

(the expected gain in the absence of reinsurance, $c - \mathrm{E}[W_i] = 2.5 - 2 = 0.5$)	$-$	(the expected return of the reinsurer, which, because the reinsurer charges at rate 2 $\mathrm{E}[I_3]$, is $\mathrm{E}[I_3] = 0.338$)	$= 0.162.$

The calculations for cases (ii) and (iii) are similar. The results are displayed below where $d = \infty$ represents the case of no reinsurance.

Stop-Loss Deductible, d	Adjustment Coefficient, $\tilde{R}$	Expected Gain
3	0.25	0.162
4	0.35	0.344
5	0.34	0.433
∞	0.28	0.500

With respect to security (as measured by the adjustment coefficient), a deductible of 4 is better than one of 5, which in turn is better than no reinsurance. With respect to expected gain, this order is reversed. Further, it can be observed that selecting a deductible of 3 would be an irrational decision. It is worse than no reinsurance with respect to both security and expected gain. ▼

We next consider reinsurance arrangements where the reinsurer's payments depend on the individual claim amounts. In general, such a coverage is defined in terms of a function $h(x)$ with $0 \le h(x) \le x$ for all x. The interpretation is that $h(x)$ is the amount payable (by the reinsurer to the insurer) if a claim is of size x. A special case is ***proportional reinsurance*** where

$$h(x) = \alpha x \qquad 0 \le \alpha \le 1, \tag{14.5.2}$$

that is, where the reinsurer reimburses a constant percentage of the claim. A second case is ***excess-of-loss reinsurance*** where

$$h(x) = \begin{cases} 0 & x \le \beta \\ x - \beta & x > \beta \end{cases} \tag{14.5.3}$$

with $\beta \ge 0$ playing the role of a deductible. An excess-of-loss coverage is reminiscent of a stop-loss coverage [see (14.4.1)]. However, the excess-of-loss is applied to individual claims, while the stop-loss is applied to aggregate claims.

We assume the continuous time compound Poisson model of Chapter 13 and its notation. Correspondingly, we assume the reinsurance premiums are payable continuously at a rate c_h. Then the ceding insurer's adjustment coefficient, R_h, is the nontrivial solution of the equation

$$\lambda \, [M_{x-h(x)}(r) - 1] = (c - c_h) \, r. \tag{14.5.4}$$

This follows from (13.3.1) since the ceding insurer now receives income at a net rate of $c - c_h$ and pays $x - h(x)$ for a claim of size x.

Example 14.5.2

Consider a surplus process with (1) the aggregate claim process, $S\ t$, being compound Poisson where the claims have an exponential distribution with mean = 1, (2) the relative security loading is 25%, and (3) proportional reinsurance is available at a price 140% of the expected reinsured claims. Determine the proportion, α, of each claim reinsured that maximizes the adjustment coefficient, R, for the process with this reinsurance.

Solution:

By (14.5.4) R is the smallest positive root of

$$\lambda + (c - c_h)r = \lambda\ \mathrm{E}[\exp\{r[X - h(X)]\}].$$

In this situation with $p_1 = 1$ and $h\ x\ = \alpha x$, we have $c = 1.25\lambda$ and $c_h = 1.4\alpha\lambda$. Further, for X with an exponential distribution,

$$\mathrm{E}[\exp\{r[X - h(X)]\}] = \frac{1}{1 - (1 - \alpha)r}.$$

This leads to the equation for R as

$$1 + (1.25 - 1.4\alpha)r = \frac{1}{1 - (1 - \alpha)r},$$

and the solution for the adjustment coefficient is

$$R = \frac{(0.25 - 0.4\alpha)}{(1 - \alpha)(1.25 - 1.4\alpha)}.$$

The value of α that maximizes the value of the adjustment coefficient is

$$\alpha = \frac{5 - [3\ (35)^{1/2}/7]}{8} = 0.308067,$$

and this results in a value of the adjustment coefficient of

$$R = 0.223787.$$ ▼

This answer depends upon the relationship between the two loading rates in the example as explored in Exercise 14.16.

Example 14.5.3

Consider the same situation as described in Example 14.5.2. This time excess-of-loss reinsurance is available at a price of 140% of expected claims. Find the deductible amount, β, which maximizes the adjustment coefficient, R, for the process with this reinsurance.

Solution:

With excess-of-loss reinsurance, the claim size distribution is the exponential distribution truncated at β. Again we have

$c = 1.25\lambda$ but now $h\ x\ = x - \beta$ for $x > \beta$ and zero elsewhere, so

$$c_h = 1.4\lambda \int_{\beta}^{\infty} (x - \beta)\, e^{-x}\, dx = 1.4\lambda\, e^{-\beta}.$$

Further,

$$\begin{aligned} \mathrm{E}[\exp\{r[X - h\ X\ ]\}] &= \int_0^{\beta} e^{rx}\, e^{-x}\, dx + \int_{\beta}^{\infty} e^{r\beta}\, e^{-x}\, dx \\ &= \frac{1 - r e^{-\beta\ 1-r)}}{(1 - r)}. \end{aligned}$$

Thus, the nonlinear equation for R as a function of β is

$$1 + (1.25 - 1.4\, e^{-\beta})r - \frac{1 - r e^{-\beta(1-r)}}{(1 - r)} = 0.$$

The following table shows the values of R corresponding to several different values of β.

$\mathrm{E}[h\ X\ ]$	β	R
0.00	infinite	0.2000
0.05	2.9957	0.2393
0.10	2.3026	0.2649
0.15	1.8971	0.2871
0.20	1.6094	0.3070
0.25	1.3863	0.3244
0.30	1.2040	0.3384
0.35	1.0498	0.3474
0.40	0.9163	0.3486
0.45	0.7985	0.3371
0.50	0.6931	0.3047
0.55	0.5978	0.2366
0.60	0.5108	0.1051

The value of $\mathrm{E}[h\ X\ ]$ that leads to the largest value of R requires additional work but can be determined to be 0.38167 corresponding to a value of the deductible, β, of 0.9632. This results in a value of R of 0.3493 which is considerably larger than that for the most favorable value available under proportional reinsurance examined in Example 14.5.2. ▼

Example 14.5.4

Compare the values of the adjustment coefficient, R, for the situations described in Examples 14.5.2 and 14.5.3 for pairs of α and β such that the reinsurer's expected payments, $\mathrm{E}[h\ X\ ]$, are the same, that is, if $\alpha = e^{-\beta}$.

Solution:

	Proportional		Excess-of-Loss	
E[$h(X)$]	α	R	β	R
0.00	0.00	0.2000	infinite	0.2000
0.05	0.05	0.2052	2.9957	0.2393
0.10	0.10	0.2102	2.3026	0.2649
0.15	0.15	0.2149	1.8971	0.2871
0.20	0.20	0.2191	1.6094	0.3070
0.25	0.25	0.2222	1.3863	0.3244
0.30	0.30	0.2238	1.2040	0.3384
0.35	0.35	0.2227	1.0498	0.3474
0.40	0.40	0.2174	0.9163	0.3486
0.45	0.45	0.2053	0.7985	0.3371
0.50	0.50	0.1818	0.6931	0.3047
0.55	0.55	0.1389	0.5978	0.2366
0.60	0.60	0.0610	0.5108	0.1051

For a given reinsurance loading, the excess-of-loss coverage consistently leads to a higher adjustment coefficient than that provided by the corresponding proportional coverage. We see below that this is not a coincidence. ▼

Somewhat similar to Theorem 1.5.1 is another theorem giving an optimality feature of excess-of-loss reinsurance. The proof of Theorem 14.5.1 is given in the Appendix to this chapter.

Theorem 14.5.1

Assume the compound Poisson model of Chapter 13. Let an arbitrary reinsurance be defined by $h(x)$, $0 \leq h(x) \leq x$, and c_h, its premium rate. Similarly, let an excess-of-loss reinsurance with deductible β be defined by $h_\beta(x)$ and c_β (simplified notation for c_{h_β}). Furthermore, let R_h and R_β denote the resulting adjustment coefficients, respectively. If $\mathrm{E}[h(X)] = \mathrm{E}[h_\beta(X)]$ and $c_h = c_\beta$, then $R_h \leq R_\beta$.

Since $c_h = (1 + \xi_h)\,\lambda\,\mathrm{E}[h(X)]$ and $c_\beta = (1 + \xi_\beta)\,\lambda\,\mathrm{E}[h_\beta(X)]$ where ξ_h and ξ_β are the loadings for the respective reinsurance coverages, the conditions of the theorem imply $\xi_h = \xi_\beta$. This limits the application of the theorem, as it may not be possible to secure excess-of-loss reinsurance with the same loading as for other reinsurances.

To illustrate this point, reconsider the situations of Examples 14.5.2 and 14.5.3. Assume that excess-of-loss reinsurance can be obtained only with a reinsurance loading of 75%, whereas the proportional reinsurance remains available with reinsurance loading of 40%. Proportional reinsurance with $\alpha = 0.25$, $\xi_h = 0.40$ has a premium rate of $1.4\ \alpha = 0.35$, and from Example 14.5.4 we see that $R = 0.2222$. The excess-of-loss reinsurance with the same expected cost has a deductible of $\beta = 1.3863$ and the premium rate would be $1.75\ e^{-\beta} = 0.4375$. Here, however, going through the process of Example 14.5.3 we obtain an R value of 0.1459. Thus the reinsurance premium rate would be higher than for proportional reinsurance, but the resulting adjustment coefficient would be lower and imply less protection against ruin.

We now shift attention to a second criterion for analyzing reinsurance arrangements. The criterion is to minimize the expected value of the maximal aggregate loss random variable, L. By its definition, $\Pr(L > u) = \psi(u)$. Since L is a non-negative random variable,

$$\mathrm{E}[L] = \int_0^\infty \Pr(L > u)\, du = \int_0^\infty \psi(u)\, du, \tag{14.5.5}$$

and minimizing E[L] is related to the problem of reducing the probability of ruin. The two criteria are quite closely related since, by (13.4.5),

$$\psi(u) < e^{-Ru}$$

so that

$$\mathrm{E}[L] = \int_0^\infty \psi(u)\, du < \frac{1}{R}.$$

Thus maximizing R is closely related to minimizing E[L].

In the next two examples, we again determine a reinsurance arrangement of the proportional type and then of the excess-of-loss type, both of which minimize E[L].

Example 14.5.5

Consider a surplus process where (1) the aggregate claim process, $S\ t$, is compound Poisson with claims having an exponential distribution with mean = 1, (2) the relative security loading is 25%, and (3) proportional reinsurance is available at a price 140% of the expected reinsured claims. Determine the proportion, α, of each claim reinsured that minimizes E[L], the expected value of the maximal aggregate loss random variable.

Solution:

By (13.6.16) $\mathrm{E}[L] = p_2/(2p_1\theta)$. All numbers are in terms of premiums and claims retained by the original insurer, so $p_2 = \mathrm{E}[(1-\alpha)^2X^2] = (1-\alpha)^2\,\mathrm{E}[X^2] = 2(1-\alpha)^2$. The expression θp_1 is the net amount of loading collected and retained by the original insurer and is the difference $1.25 - 1.4\,\alpha - (1-\alpha) = 0.25 - 0.4\,\alpha$. Thus $\mathrm{E}[L] = (1-\alpha)^2/(0.25 - 0.4\,\alpha)$, and this is minimized at $\alpha = 0.25$. At this value of α, $\mathrm{E}[L] = 3.75$. ▼

Example 14.5.6

Consider the same situation as described in Example 14.5.5. Find the deductible amount, β, which minimizes E[L] if excess-of-loss reinsurance is available at a price of

a. 140% of the expected claims reinsured, and
b. 175% of the expected claims reinsured.

Solution:

With excess-of-loss reinsurance, the claim size distribution is the exponential distribution truncated at β. Here,

$$p_1 = \int_0^{\beta} x\, e^{-x}\, dx + \int_{\beta}^{\infty} \beta\, e^{-x}\, dx = 1 - e^{-\beta}$$

and

$$p_2 = \int_0^{\beta} x^2 e^{-x}\, dx + \int_{\beta}^{\infty} \beta^2\, e^{-x}\, dx = 2\,[1 - (\beta + 1)e^{-\beta}].$$

a. The net amount of loading collected and retained by the original insurer is the difference

$$1.25 - 1.4e^{-\beta} - (1 - e^{-\beta}) = 0.25 - 0.4e^{-\beta}.$$

Thus $\mathrm{E}[L] = [1 - (\beta + 1)e^{-\beta}]/(0.25 - 0.4e^{-\beta})$. The value of β that minimizes this expression satisfies the equation

$$0.4 - 0.25\,\beta = 0.4\, e^{-\beta},$$

and the β that satisfies this is 1.02717. At this value of β, $\mathrm{E}[L] = 2.56793$. This value of $\mathrm{E}[L]$ is smaller than the one found in Example 14.5.5, suggesting that excess-of-loss reinsurance is also to be preferred over proportional reinsurance by this criterion.

b. By a similar process the value of β that minimizes $\mathrm{E}[L]$ in part (b) satisfies the equation

$$0.75 - 0.25\,\beta = 0.75\, e^{-\beta},$$

and the β that satisfies this is 2.82143. At this value of β, $\mathrm{E}[L] = 3.76192$. This value of $\mathrm{E}[L]$ is slightly larger than the one found in Example 14.5.5 for proportional reinsurance. This suggests that excess-of-loss reinsurance, with higher loading, is not preferred over proportional reinsurance by the $\mathrm{E}[L]$ criterion. ▼

Example 14.5.7

Compare the results of Examples 14.5.5 and 14.5.6 for pairs of α and β such that the reinsurer's expected payments, $\mathrm{E}[h(X)]$, are the same, that is, if $\alpha = e^{-\beta}$.

Solution:

	Proportional 40% Loading		Excess-of-Loss 40% Loading		Excess-of-Loss 75% Loading	
$E[h\ X]$	α	$E[L]$	β	$E[L]$	β	$E[L]$
0.00	0.00	4.000	infinite	4.000	infinite	4.000
0.05	0.05	3.924	2.9957	3.479	2.9957	3.766
0.10	0.10	3.857	2.3026	3.189	2.3026	3.827
0.15	0.15	3.803	1.8971	2.976	1.8971	4.112
0.20	0.20	3.765	1.6094	2.812	1.6094	4.781
0.25	0.25	3.750	1.3863	2.690	1.3863	6.455
0.30	0.30	3.769	1.2040	2.606	1.2040	13.552
0.35	0.35	3.841	1.0498	2.569		
0.40	0.40	4.000	0.9163	2.594		
0.45	0.45	4.321	0.7985	2.724		
0.50	0.50	5.000	0.6931	3.069		
0.55	0.55	6.750	0.5978	4.040		
0.60	0.60	16.000	0.5108	9.350		

▼

Example 14.5.7 suggests that for the same loading levels, excess-of-loss reinsurance is to be preferred over proportional reinsurance. Exercise 14.24 illustrates the development of a result that formalizes this observation. At higher loading levels for excess-of-loss reinsurance the picture is mixed. For low amounts of reinsurance purchased, that is, low values of $E[h\ X]$, a superiority of excess-of-loss can still be observed. With higher amounts purchased, the proportional reinsurance with its smaller loading will be preferred.

14.6 Notes and References

A monograph by Hogg and Klugman (1984) demonstrates the use of claim statistics for selecting a claim amount distribution and estimating the parameters. Other references for this can be found in Seal (1969). The claim amount distribution for group weekly indemnity insurance was taken from papers by Miller (1951) and Bartlett (1965). The hospitalization continuance curve was derived from data in a paper by Gingery (1952).

The two methods for approximating the individual model by a collective model were suggested by Mereu (1972) and Wooddy (1973).

Calculating stop-loss premiums has been the subject of many papers. Bohman and Esscher (1963–64) reported on an extensive study of alternative methods of approximating the distribution of total claims and expected stop-loss claims. Bartlett (1965) discussed the use of the gamma distribution for the calculation of expected stop-loss claims. Bowers (1969) presented an upper bound, in terms of the mean and variance of aggregate claims, for expected stop-loss claims. This result has been generalized by Taylor (1977) and by Goovaerts and DeVylder (1980). In recent years several papers have developed methods for use with discrete claim distributions. These include Halmstad (1972), Mereu (1972), Gerber and Jones (1977), and Panjer (1980).

A link between the applications of risk theory and financial economics is established in Exercise 14.23. The result was obtained by Black and Scholes (1973), starting with assumptions about the operations of an efficient securities market. Their work is widely regarded as starting a new approach to many issues in financial economics.

The effect of reinsurance on the probability of ruin is discussed by Gerber (1980).

Appendix

Proof of Theorem 14.5.1:

We know that R_h is the positive root of

$$\lambda + (c - c_h)r = \lambda M_{X-h(X)}(r)$$

and R_β is the positive root of

$$\lambda + (c - c_\beta)r = \lambda M_{X-h_\beta(X)}(r).$$

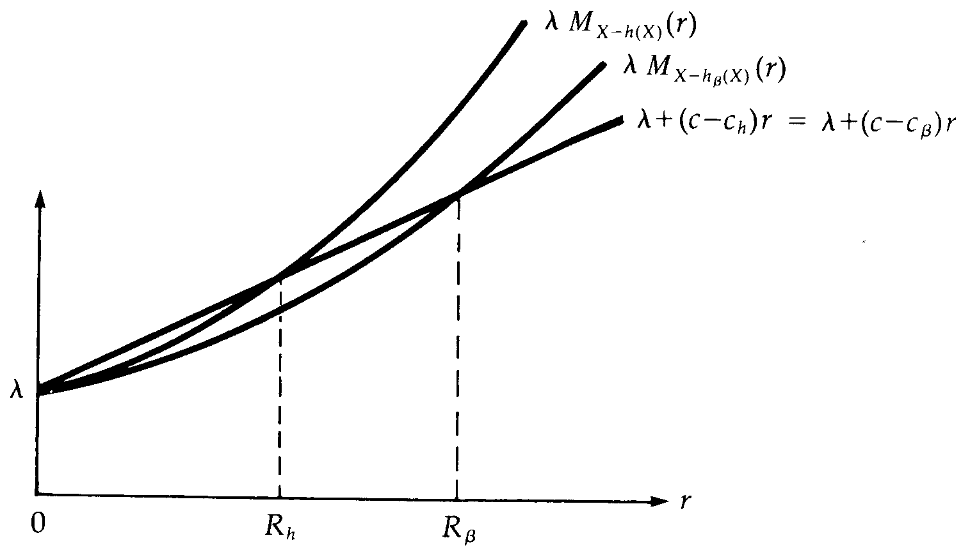
Since $c_h = c_\beta$, we can see from Figure 14.A.1 that

$$M_{X-h(X)}(r) \geq M_{X-h_\beta(X)}(r) \qquad r > 0 \tag{14.A.1}$$

implies $R_h \leq R_\beta$.

FIGURE 14.A.1

Proof of Theorem 14.5.1



To establish (14.A.1), we first use the convexity of the exponential function to show that

$$\exp\{r[x - h\ x\ ]\} \geq \exp\{r[x - h_\beta(x)]\}$$
$$+ \ r\exp\{r[x - h_\beta(x)]\}\ [h_\beta(x) - h\ x\ ].$$

Since $x - h_\beta(x) \leq \beta$ and $x - h_\beta(x) = \beta$ whenever $h_\beta(x) - h\ x\ > 0$, it follows that

$$\exp\{r[x - h\ x\ ]\} \geq \exp\{r[x - h_\beta(x)]\} + r\exp(r\,\beta)[h_\beta(x) - h\ x\ ].$$

Then

$$\mathrm{E}[\exp\{r[X - h\ X\ ]\}] \geq \mathrm{E}[\exp\{r[X - h_\beta(X)]\}]$$
$$+ \ r\exp(r\ \beta)\ \mathrm{E}[h_\beta(X) - h\ X\ ].$$

By the hypothesis of the theorem the last expectation is 0. This yields (14.A.1), from which the theorem follows. ■

Exercises

Section 14.1

14.1. A term insurance provides the amount b if a claim occurs. The probability of a claim occurring is q.

a. Consider the following loss random variable:

$$L = \begin{cases} b - bq & \text{with probability } q \\ 0 - bq & \text{with probability } p = 1 - q. \end{cases}$$

Verify that $\mathrm{E}[L] = 0$.

b. Calculate $\mathrm{Var}(L)$.

c. The security-loaded premium is taken as $bq + s\sqrt{\mathrm{Var}(L)}$. If 100 identical policies of this type are sold and the loss random variables, as defined in part (a), for these policies are mutually independent, calculate the loading factor s such that the probability that the sum of these loss random variables exceeds the total security loading is less than 0.01.

Section 14.2

14.2. Verify the mean and variance entries in Table 14.2.1.

14.3. a. Calculate the mean of the distribution described in Table 14.2.2.

b. By how much would the expected benefit per case of disability be reduced in the short-term disability insurance illustration if the 13-week maximum were replaced by a 10-week maximum?

14.4. Given that a disability has occurred, evaluate, on the basis of Table 14.2.2,

a. $\Pr(3 \leq Y \leq 6)$

b. $\Pr(10 \leq Y \leq 13)$

c. $\Pr(20 \leq Y \leq 23)$.

Section 14.3

14.5. Consider a portfolio of 100 policies, each of which is for a 1-year term life insurance.

One-Year Mortality Rate	Amount Insured	
	1	4
0.01	10	20
0.02	30	40

The matrix entries give the number of policies in the portfolio for the indicated combination of amount insured and mortality rate.

a. If S represents the aggregate claims, calculate $\mathrm{E}[S]$ and $\mathrm{Var}(S)$.
b. What compound Poisson distribution would be used for approximating the individual model by the first method? What would be the resulting approximation for $\mathrm{Var}(S)$?

14.6. Suppose that $B_j = b_j > 0$ for $j = 1, 2, \ldots, n$.

a. Write expressions for the mean and variance of the compound Poisson distribution chosen according to the second method.
b. Show that the values in (a) are higher than the corresponding values obtained by the first method. [Hint: First show that $\tilde{\lambda}_j > q_j$, $j = 1, 2, \ldots, n$.]
c. Compute the mean and variance of the compound Poisson distribution in Example 14.3.1 by the second method.

14.7. Calculate the probability that two claims that occur in the individual model are from policies i and j $(i \neq j)$.

Section 14.4

14.8. Suppose that the possible claims are integers. Show that

$$\mathrm{E}[I_d^2] = \mathrm{E}[I_{d-1}^2] - 2\,\mathrm{E}[I_{d-1}] + 1 - F_S(d-1).$$

14.9. Calculate $\mathrm{E}[I_d]$ if S has the normal distribution with parameters μ and σ.

14.10. Suppose that the possible claims are integers. Express the following in terms of $F_S(x)$ and $f_S(x)$:

a. $\Delta \mathrm{E}[I_x]$ b. $\Delta^2 \mathrm{E}[I_x]$.

14.11. It is known that $\mathrm{E}[I_d] = 1 - d - (1 - d^3)/3$ for $0 \le d \le 1$ and is equal to 0 for $d > 1$. Derive the p.d.f. of the underlying distribution of aggregate claims.

14.12. If S has a compound Poisson distribution given by $\lambda = 3$, $p(1) = 5/6$, $p(2) = 1/6$, calculate $f_S(x)$, $F_S(x)$, $\mathrm{E}[I_x]$ for $x = 0, 1, 2$.

14.13. A dividend of the form (14.4.12) is to be used in Examples 14.4.6 and 14.4.7.

a. Calculate $G - \mathrm{E}[S] - \mathrm{E}[D]$ if $k = 0.9$.
b. Determine k such that $G - \mathrm{E}[S] - \mathrm{E}[D] = 0$.

14.14. A reinsurer will pay 80% of the excess of S over a deductible d, subject to a maximum payment of m. Express the expected claims under this coverage in terms of expected stop-loss claims.

14.15. In Example 14.4.5 determine d such that $\mathrm{E}[I_d] = 0.2$.

Section 14.5

14.16. a. Repeat Example 14.5.2 with $\lambda = 1$, $c = 1 + \theta$, and $c_h = (1 + \xi)\alpha$.
b. Develop a relationship between θ and ξ so that the maximum of R occurs at $\alpha = 0$.

14.17. a. Repeat Example 14.5.5 with $\lambda = 1$, $c = 1 + \theta$, and $c_h = (1 + \xi)\alpha$.
b. Develop a relationship between θ and ξ so that the minimum of $\mathrm{E}[L]$ occurs at $\alpha = 0$.

14.18. Reconsider the situation of Example 14.5.3, now with $\lambda = 1$, insurer's relative security loading θ, and excess-of-loss reinsurance available at a price of $1 + \xi$ times expected claims covered by the reinsurance.
a. Determine an expression for the ceding insurer's relative security loading after purchase of the reinsurance described.
b. Determine the equation from which the ceding insurer's adjustment coefficient can be obtained.

14.19. The annual claims, W_i, $i = 1, 2, \ldots$ for an insurance company are mutually independent and identically distributed, $N(10, 4)$. The company collects a relative security loading of 25%. A reinsurer is willing to accept the risk on any part, α, of the portfolio on a proportional basis for a reinsurance premium equal to 140% of the expected value of the claims reinsured.
a. Express the adjustment coefficient, $\tilde{R}$, for the portfolio with proportional reinsurance as a function of α.
b. Determine the value of α that maximizes the security of the insurance company by giving the largest value of $\tilde{R}$.

Miscellaneous

14.20. A reinsurer with wealth w and utility function $u(w)$ sets the stop-loss premium H_d corresponding to a deductible d so that $u(w) = \mathrm{E}[u(w + H_d - I_d)]$ [see (1.3.6)]. Calculate H_d if $u(w) = -\alpha e^{-\alpha w}$ $(\alpha > 0)$ and if S has the normal distribution with parameters μ and σ.

14.21. It is known that

$$\mathrm{E}[I_d] = \left(\frac{\alpha}{\beta} - d\right)\left[1 - \Phi\left(-\frac{\alpha}{\sqrt{\beta d}} + \sqrt{\beta d}\right)\right] + \left(\frac{\alpha}{\beta} + d\right) e^{2\alpha}\, \Phi\left(-\frac{\alpha}{\sqrt{\beta d}} - \sqrt{\beta d}\right) \quad \text{for } d > 0.$$

Derive the p.d.f. of the underlying distribution of aggregate claims. Identify the distribution.

14.22. Let N have a Poisson distribution with parameter λ, a positive integer.

a. Show that $\mathrm{E}[I_\lambda] = \dfrac{\lambda^{\lambda+1}e^{-\lambda}}{\lambda!}$.

b. Use the result of Exercise 12.20 in which an approximation to the distribution of $(N - \lambda)/\sqrt{\lambda}$, when λ is large, is derived to confirm that

$$\mathrm{E}[I_\lambda] \cong \frac{\sqrt{\lambda}}{\sqrt{2\pi}}.$$

c. Combine the results of parts (a) and (b) to derive the approximation

$$\lambda! \cong \lambda^{\lambda+1/2}e^{-\lambda}\sqrt{2\pi}.$$

(Historical Comment: This is known as Stirling's approximation for $\lambda!$, when λ is large. "In 1730 James Stirling, with help from De Moivre, derived this exponential approximation for factorials. De Moivre then showed in a paper of 1733 that the exponential error function gave a very good approximation of the distribution of possible outcomes for problems like the result of 1000 coin tosses" [*The Rise of Statistical Thinking,* T. M. Porter, Princeton University Press, 1986, p. 93]. In this exercise we have reversed the route followed by De Moivre and Stirling. We used the Central Limit Theorem to derive Stirling's approximation.)

14.23. a. If S has a lognormal distribution with parameters tm and $t\sigma^2$, $t > 0$, derive an expression for

$$\mathrm{E}[e^{-\delta t}I_d], \text{ where } \delta > 0.$$

b. Determine the value of m such that the expectation of the discounted value of S is 1.

c. Rewrite your result of a. with this value of m.
[Remark: Your answer is the Black Scholes formula for a European call option, with exercise price d, on a stock with price 1 at $t = 0$ and price S at time t.]

14.24. For the compound Poisson process model of Chapter 13 with security loading θ, consider the excess-of-loss reinsurance contract $h_\beta(x)$ with loading ξ and the set of reinsurance contracts, $h(x)$, with $\mathrm{E}[h(x)] = \mathrm{E}[h_\beta(x)]$ and loading ξ. Assume that $\theta\mathrm{E}[X] > \xi\,\mathrm{E}[h(x)]$.

a. Verify that $\mathrm{E}[L{:}h] = p_{2:h}/2\theta p_{1:h}$ where the h subscripts indicate the moments of the process of retained claims under reinsurance contract h.

b. Verify that under the hypothesis, the minimization of $\mathrm{E}[L{:}h]$ over the set of possible reinsurance contracts with the specified premium and loading is equivalent to minimizing $p_{2:h}$.

c. Using the method of Exercise 1.21, confirm that $\mathrm{E}[L{:}h_\beta] \leq \mathrm{E}[L{:}h]$, thereby establishing another optimal property of excess-of-loss reinsurance.

15

INSURANCE MODELS INCLUDING EXPENSES

15.1 Introduction

The equivalence principle was introduced in Chapter 6 as a means for determining insurance premiums. In that chapter, the principle imposed the condition that the actuarial present values of benefits and benefit premiums be equal at the time the insurance is issued. In Chapters 7 and 8 this principle was applied to time periods beyond the initial date of contract. Benefit reserves were expressed as the actuarial present value of the difference between future benefit payments and future benefit premium income.

The foregoing chapters were devoted, in large part, to building a comprehensive model for insurance systems based on the equivalence principle. However, this model did not incorporate several aspects of insurance practice and economic reality. For example, an insurer has cash outflows other than claim payments. Expenditures of this general type include those for taxes and licenses as well as those for selling and servicing insurance policies. These expenses must be met from premium and investment income. In this chapter we incorporate expenses into the model for premiums and reserves.

The equivalence principle is extended to incorporate expense payments along with benefits as expenditures, and provisions for expenses are included in premiums and reserves. In this extension, it is assumed that the expenses incurred in connection with each policy are known with certainty. This extension is shown to provide a reasonable foundation for financial reporting for insurance enterprises.

Withdrawal benefits are common in life insurance, and they are required, and their amounts regulated, in many jurisdictions. In Section 15.3 a multiple decrement example is used to illustrate a comprehensive model involving death and withdrawal benefits and expenses. This model is used with the equivalence principle to derive premiums, reserves, and financial reports.

In the history of insurance practice and regulation, it has been convenient to approximate double decrement models incorporating expenses by using single-decrement-death-benefit-only models with suitably modified sets of benefit premiums. Some of these regulatory issues are discussed in Sections 16.2, 16.6, and 16.7.

15.2 Expense Augmented Models

The main ideas needed to incorporate expenses are first examined in an extended illustration. Tables 15.2.1A and 15.2.1B specify the salient features, selected for convenience and ease of calculation rather than for realism.

TABLE 15.2.1A

Specifications of Illustration: Description

1. Plan of insurance	3-year annual premium endowment insurance, issued to (x) with level benefits and premiums
2. Payment basis	Fully discrete
3. Mortality	$q_x = 0.1$, $q_{x+1} = 0.1111$, $q_{x+2} = 0.5$
4. Interest	Annual effective rate of $i = 0.15$
5. Amount of insurance	1,000
6. Expenses	
a. Timing	Paid at the beginning of each policy year
b. Amount	(as given in Table 15.2.1B)

TABLE 15.2.1B

Specifications of Illustration: Amount of Expenses

	First Year		Renewal Years	
Type of Expense	**Percentage of Premium**	**Constant**	**Percentage of Premium**	**Constant**
Sales commission	10%	—	2%	—
General expense	4	3	—	1
Taxes, licenses, and fees	2	—	2	—
Policy maintenance	2	1	2	1
Issue and classification	2	4	—	—
Total	20%	8	6%	2

15.2.1 Premiums and Reserves

Descriptive specifications 1 through 5 from Table 15.2.1A can be used with the equivalence principle to determine the level annual benefit premium for this insurance, 1,000 $P_{x:\overline{3}|} = 288.41$. Table 15.2.2 provides the details of the calculation of the corresponding benefit reserves.

TABLE 15.2.2

Benefit Reserve Calculations

(1) Curtate Future Lifetime	(2) Loss Variable	(3) Conditional Probability of Outcome	(4) (2) × (3)
At Issue ($_0L$)			
$K(x) = 0$	581.16	0.1	58.12
$K(x) = 1$	216.94	0.1	21.69
$K(x) \geq 2$	− 99.76	0.8	−79.81
		$1{,}000\ {}_0V_{x:\overline{3}\vert} = \mathrm{E}[{}_0L] =$	0.00
		$\sigma({}_0L) =$	215.51
One Year after Issue ($_1L$)			
$K(x) = 1$	581.16	0.1111	64.57
$K(x) \geq 2$	216.94	0.8889	192.84
		$1{,}000\ {}_1V_{x:\overline{3}\vert} = \mathrm{E}[{}_1L] =$	257.41
		$\sigma({}_1L) =$	114.46
Two Years after Issue ($_2L$)			
$K(x) \geq 2$	581.16	1.0	581.16
		$1{,}000\ {}_2V_{x:\overline{3}\vert} = \mathrm{E}[{}_2L] =$	581.16
		$\sigma({}_2L) =$	0

As a final confirmation we can verify that ${}_3V_{x:\overline{3}|} = 1.0$:

$$1{,}000({}_2V_{x:\overline{3}|} + P_{x:\overline{3}|})(1 + i) = 1{,}000$$

$$(581.16 + 288.41)(1.15) = 1{,}000$$

The expenses, as provided by Table 15.2.1B, are incorporated by modifying the loss variables. The present value of benefits is increased by the present value of expenses. This new total is then offset by the present value of level expense-loaded premiums, denoted by G. Table 15.2.3 is constructed using information from Table 15.2.1B.

TABLE 15.2.3

Expense Augmented Loss Variable (${}_0L_e$)

Curtate Future Lifetime	Present Values: Benefits + Expenses	Present Values: − Premiums	Probability of Outcome
$K(x) = 0$	$1{,}000v + (0.20G + 8)$	$-G\ddot{a}_{\overline{1}|}$	0.1
$K(x) = 1$	$1{,}000v^2 + (0.20G + 8) + (0.06G + 2)a_{\overline{1}|}$	$-G\ddot{a}_{\overline{2}|}$	0.1
$K(x) \geq 2$	$1{,}000v^3 + (0.20G + 8) + (0.06G + 2)a_{\overline{2}|}$	$-G\ddot{a}_{\overline{3}|}$	0.8
Expected Values	$688.58387 + (0.20G + 8) + (0.06G + 2)(1.3875236)$	$-G(2.3875236)$	

Implicit in the expense augmented loss variable (${}_0L_e$) displayed in Table 15.2.3 is the decision to fund benefits and expenses with a level annual premium G. Other patterns for premiums are possible. In this case, the expense-loaded premium is determined by the equivalence principle; that is, the expected value of the expense augmented loss variable is 0. From Table 15.2.3,

$$688.58387 + (0.20G + 8) + (0.06G + 2)(1.3875236) - G(2.3875236) = 0.0.$$

This yields

$$G = 332.35,$$

which may be written

$$G = 1{,}000P_{x:\overline{3}|} + \text{level expense premium } (e)$$

$$= 288.41 + 43.94 = 332.35.$$

Table 15.2.4 exhibits the calculations of the expected values and standard deviations of the expense augmented loss variables at policy issue and at 1 and 2 years after issue. The total reserve is allocated into benefit and expense components. In each year, expected income from level benefit premium payments does not match expected benefit payments. This mismatching creates a non-negative benefit reserve. Likewise in each year, expected income from level expense loadings does not match expected payments for expenses. This mismatching creates a nonpositive expense reserve.

TABLE 15.2.4

Expected Values of Expense Augmented Loss Variables

Curtate Future Lifetime	[(Present Value of Benefits) − (1,000 × $P_{x:\overline{3}|}$ × $\ddot{a}_{\overline{k+1}|}$)]	+	[(Present Value of Expenses) − $e\ \ddot{a}_{\overline{k+1}|}$]	Conditional Probability of Outcome
At Issue ($_0L_e$)				
$K(x) = 0$	(869.57 − 288.41)	+	(74.47 − 43.94)	0.1
$K(x) = 1$	(756.14 − 539.20)	+	(93.55 − 82.15)	0.1
$K(x) \geq 2$	(657.52 − 757.28)	+	(110.14 − 115.37)	0.8
Expected Values:	Benefit reserve	+	Expense reserve = Total reserve	
	0	+	0 = 0	
			$\sigma(_0L_e)$ = 226.82	
One Year after Issue ($_1L_e$)				
$K(x) = 1$	(869.57 − 288.41)	+	(21.94 − 43.94)	0.1111
$K(x) \geq 2$	(756.14 − 539.20)	+	(41.02 − 82.15)	0.8889
Expected Values:	Benefit reserve	+	Expense reserve = Total reserve	
	257.41	−	39.00 = 218.41	
			$\sigma(_1L_e)$ = 120.47	
Two Years after Issue ($_2L_e$)				
$K(x) \geq 2$	(869.57 − 288.41)	+	(21.94 − 43.94)	1.0
Expected Values:	Benefit reserve	+	Expense reserve = Total reserve	
	581.16	−	22.00 = 559.16	
			$\sigma(_2L_e)$ = 0	

As a confirmation, the terminal total reserve (at the end of 3 years) is:

(total reserve at end of year 2 + loaded premium − expenses)(1 + i)
=(559.16 + 332.35 − 21.94)(1.15) = 1,000

The following observations identify some of the key ideas in the illustration.

Observations:

1. Loss variables, as originally introduced, measure the present value of benefits less the present value of benefit premiums at the various times when benefits might be paid. These variables can be augmented to incorporate expenses and expense-loaded premiums.
2. The equivalence principle can be used to determine expense-loaded premiums and the associated total reserves (benefit reserves plus expense reserves).
3. The expense reserve is often negative in early policy years. This is a consequence of matching a decreasing stream of expense payments with a level stream of expense loadings.
4. Analysis and projection of expenses precede the determination of expense-loaded premiums.
5. The standard deviation of the expense augmented loss variable can be used to determine a contingency fund. This fund guards against inadequate balancing of premium and investment income with benefit and expense payments. Such a situation is possible due to the random nature of the time benefits are paid. Methods for this were illustrated in Chapter 8.

15.2.2 Accounting

In manufacturing enterprises, a product is usually built before it is sold. In most businesses that provide services, the service is performed before payment is received. An insurance operation is unusual in that premium income is received before the service of risk assumption is performed. It is this fact that motivates concerns by regulators and consumers about insurer solvency, and it also creates financial reporting issues.

Accounting is directed, in part, to matching the cost of providing a product or service with the revenue derived from selling it. The objective is to measure the economic gain or loss from engaging in these activities. Accounting in life insurance and annuity operations differs from that in many enterprises because income is received before costs are known. The reserve systems illustrated earlier, level benefit premium and expense-loaded premium, can be used to achieve an improved match between premium income and associated expenditures.

The illustration is continued in Tables 15.2.5 and 15.2.6, in which the following assumptions are made:

- The annual contract premium for each policy is 342.35, the expense-loaded premium plus an arbitrary amount of 10. The remainder, after paying the percent of premium expenses on the additional 10, is for profit and contingencies.

TABLE 15.2.5

Income Statements (10 Initial Insureds)

Income Statements	(a) Reporting Benefit Reserves as Liabilities	(b) Reporting Benefit Plus Expense Reserves as Liabilities
During First Year		
Income		
Premium (10)	3 423.50	3 423.50
Investment (15%)	548.82	548.82
	3 972.32	3 972.32
Charges to Income		
Expenses		
Percentage (20%)	684.70	684.70
Constant (8)	80.00	80.00
Claims (1)	1 000.00	1 000.00
Increase in reserves	2 316.69	1 965.69
	4 081.39	3 730.39
Net Income	−109.07	241.93
During Second Year		
Income		
Premium (9)	3 081.15	3 081.15
Investment (15%)	912.88	912.88
	3 994.03	3 994.03
Charges to Income		
Expenses		
Percentage (6%)	184.87	184.87
Constant (2)	18.00	18.00
Claims (1)	1 000.00	1 000.00
Increase in reserves	2 332.59	2 507.59
	3 535.46	3 710.46
Net Income	458.57	283.57
During Third Year		
Income		
Premium (8)	2 738.80	2 738.80
Investment (15%)	1 283.59	1 283.59
	4 022.39	4 022.39
Charges to Income		
Expenses		
Percentage (6%)	164.33	164.33
Constant (2)	16.00	16.00
Claims and maturities (8)	8 000.00	8 000.00
Increase in reserves	−4 649.28	−4 473.28
	3 531.05	3 707.05
Net income	491.34	315.34

Notes:

1. Investment income = (assets at end of prior year + premium income − expenses) (0.15).
2. Total net income = −109.07 + 458.57 + 491.34 Col. (a)
   = 241.93 + 283.57 + 315.34 Col. (b)
   = 840.84.
3. Alternative calculation (review specifications in Table 15.2.1B):
   total net income = (interest income on initial funds) + (accumulated value of net profit loadings)
   $= 1{,}000[(1.15)^3 - 1] + 10[(10)(0.8)(1.15)^3 + 9(0.94)(1.15)^2 + (8)(0.94)(1.15)]$
   = 840.91.

   The difference in results of these two calculations is attributed to rounding errors.

TABLE 15.2.6

Balance Sheets (10 Initial Insureds)

Balance Sheets	(a) Reporting Benefit Reserves as Liabilities	(b) Reporting Benefit Plus Expense Reserves as Liabilities
At End of First Year		
Assets	3 207.62	3 207.62
Liabilities (Reserves)	2 316.69	1 965.69
Surplus	890.93	1 241.93
	3 207.62	3 207.62
At End of Second Year		
Assets	5 998.78	5 998.78
Liabilities (Reserves)	4 649.28	4 473.28
Surplus	1 349.50	1 525.50
	5 998.78	5 998.78
At End of Third Year		
Assets	1 840.84	1 840.84
Liabilities (Reserves)	0	0
Surplus	1 840.84	1 840.84
	1 840.84	1 840.84

Notes:
1. Increase in surplus = total gains (see note 2 to Table 15.2.5). 1,840.84 − 1,000 = 840.84.
2. Surplus = (surplus at end of previous year + net income).
3. Assets = [assets at end of previous year + (net income + increase in reserves)]
   = [assets at end of previous year + (premiums + investment income − claims − expenses)].

- The accounting statements are derived using a deterministic survival group, initially consisting of 10 insureds. Each accounting entry can be divided by 10 to produce entries that can be interpreted as expected amounts for each initial insured.
- Expenses are paid and investment income is earned exactly as specified in Tables 15.2.1A and B.
- The hypothetical insurance operations start with an initial fund of 1,000.
- In the accounting statements in column (a), benefit reserves alone are reported as liabilities. In the accounting entries in column (b), benefit reserves plus expense reserves are reported as liabilities.

The set of accounting statements that use benefit reserves as liabilities is internally inconsistent in a sense. This is true because future expenses and provisions for these expenses in future premiums are not incorporated into liabilities.

The following observations indicate some additional key points in the accounting illustration.

Observations:

6. The amounts recognized as net income in the accounting statements are less variable when benefit plus expense reserves are reported as liabilities than in

the situation where benefit reserves alone are reported. Also, net income can be related to interest on surplus and the net profit loadings accumulated with interest.

7. Total gain over the 3-year period is not affected by the method selected for reporting liabilities.
8. In actual practice, expected results are not realized with the degree of certainty assumed in the illustration.

The practice and therefore the vocabulary of life insurance accounting are complicated by the fact that several groups of stakeholders, each with its own responsibilities and interests, rely on these statements. The ideas illustrated in column (a) of Tables 15.2.5 and 15.2.6 are related to those historically used in the United States for regulatory purposes. The ideas illustrated in column (b) are related to those used in financial statements intended for use by the capital markets.

15.3 Withdrawal Benefits

In Section 11.4 the idea of withdrawal benefits to be paid to terminating policyholders was introduced. The primary purpose of that section was the establishment of conditions under which premiums and reserves incorporating death and withdrawal benefits would be identical to those determined within a mortality-only model. This required that death and withdrawal be independent, and in Section 11.7 it was indicated that there are barriers to extending the result to the discrete model.

15.3.1 Premiums and Reserves

In this section the extended illustration constructed in Section 15.2 is expanded to a double decrement model with withdrawal benefits. The amounts of the withdrawal benefits are denoted by $b_{x+t}^{(2)}$, and they are determined in Example 15.3.1. This example is based on the principle incorporated into the law regulating minimum withdrawal benefits in the United States. It involves using a mortality-only benefit model, with an arbitrary provision for expenses. The principle is based on an extension of ideas introduced in Section 11.4; namely, that if withdrawal benefits in the double decrement model are approximately equal to the expense augmented reserves, benefit plus expense reserves, determined using an associated mortality-only model, then the effect on premiums and reserves of adding a withdrawal benefit will be small.

Example 15.3.1

The benefit premium determined by the equivalence principle and the assumptions listed in Table 15.2.1A is $1{,}000\, P_{x:\overline{3|}} = 288.41$. An arbitrary expense loading of $40/\ddot{a}_{x:\overline{3|}} = 40/2.3875 = 16.75$ is added to the benefit premium to produce an expense-loaded premium of 305.16. Within regulations this is called an ***adjusted***

premium and is denoted in this example by 1,000 $P^{A}_{x:\overline{3}|}$. Determine withdrawal benefits with the assumptions listed in Table 15.2.1 using the following prospective reserve-type formula:

$$b^{(2)}_{x+t} = 1{,}000(A_{x+t:\overline{3-t}|} - P^{A}_{x:\overline{3}|}\, \ddot{a}_{x+t:\overline{3-t}|}).$$

Solution:

$$\begin{aligned} b^{(2)}_{x+1} &= 1{,}000\ A_{x+1:\overline{2}|} - 305.16\ \ddot{a}_{x+1:\overline{2}|} \\ &= 768.75 \qquad - 305.16(1.7729) = 227.73, \\ b^{(2)}_{x+2} &= 1{,}000\ A_{x+2:\overline{1}|} - 305.16\ \ddot{a}_{x+2:\overline{1}|} \\ &= 869.57 \qquad - 305.16 \qquad = 564.41. \end{aligned}$$

▼

The assumptions used to expand the illustration are shown in Table 15.3.1. Tables 15.3.2 and 15.3.3 are closely related to each other. In Table 15.3.2 the annual expense-loaded premium using the equivalence principle is determined. Table 15.3.3 corresponds to Table 15.2.4 and exhibits the calculation of benefit and expense reserves in the double decrement model. The observations following Table 15.2.4 remain valid.

TABLE 15.3.1

Specifications of Illustration Including Withdrawal Benefits

Withdrawal Benefits	
$b^{(2)}_{x+1} = 227.73$	$b^{(2)}_{x+2} = 564.41$
Multiple Decrement Probabilities	
$q^{(1)}_{x} = 0.1$	$q^{(1)}_{x+1} = 0.1111$
$q^{(2)}_{x} = 0.1$	$q^{(2)}_{x+1} = 0.1111$

Deaths and withdrawals are assumed to be independent as described in Section 11.4. Withdrawals are assumed to occur only at the end of each year of age; that is, ${}_tp'^{(2)}_x$ is a step function as shown in Figure 10.2A, the graph of ${}_tp'^{(3)}_x$. In this example, the value of $q^{(1)}_x = q_x$ and $q^{(1)}_{x+1} = q_{x+1}$.

Table 15.3.2 contains the data necessary to determine the (double decrement model) annual benefit premium, $P^{2}_{x:\overline{3}|}$, annual expense loading, e, and the annual expense-loaded premium, G, by the equivalence principle. We have

$$621.0011 - P^{2}_{x:\overline{3}|}\,(2.1661) = 0,$$

$$P^{2}_{x:\overline{3}|} = 286.69,$$

$$621.0011 + (0.2G + 8) + (0.06G + 2)(1.1661) - G(2.1661) = 0,$$

$$G = 332.96,$$

TABLE 15.3.2

Expense Augmented Loss Variable, Double Decrement, at Issue

Outcome of: Curtate Future Lifetime $K(x)$	Outcome of: Cause of Decrement $J(x)$	(Present Value of Benefits)	+	(Present Value of Expenses)	−	(Present Value of Premiums)	Probability of Outcome
0	1	$1{,}000.00v$	+	$(0.2G + 8.0)$	−	G	0.1
	2	$227.73v$	+	$(0.2G + 8.0)$	−	G	0.1
1	1	$1{,}000.00v^2$	+	$(0.2G + 8.0) + (0.06G + 2.0)a_{\overline{1}\rceil}$	−	$G\ddot{a}_{\overline{2}\rceil}$	0.0889
	2	$564.41v^2$	+	$(0.2G + 8.0) + (0.06G + 2.0)a_{\overline{1}\rceil}$	−	$G\ddot{a}_{\overline{2}\rceil}$	0.0889
≥ 2	1 or 2	$1{,}000.00v^3$	+	$(0.2G + 8.0) + (0.06G + 2.0)a_{\overline{2}\rceil}$	−	$G\ddot{a}_{\overline{3}\rceil}$	0.6222
Expected Values		621.011	+	$(0.2G + 8.0) + (0.06G + 2.0)(1.1661)$	−	$(1{,}000P^{2}_{x:\overline{3}\rceil} + e)(2.1661)$	
$\sigma({}_0L^2_e) = 224.25$							

TABLE 15.5.3

Expense Augmented Loss Variable, Double Decrement

Outcome of: Curtate Future Lifetime $K(x)$	Outcome of: Cause of Decrement $J(x)$	(Present Value of Benefits)	$-$	(Present Value of Benefit Premiums)	$+$	(Present Value of Expenses)		$-$	(Present Value of Loadings)		Probability of Outcome
At Duration 1											
1	1	$1{,}000.00v$	$-$	$1{,}000P^2_{x:\overline{3}\vert}$	$+$	$(0.06G + 2.0)$		$-$	e		0.1111
	2	$564.41v$	$-$	$1{,}000P^2_{x:\overline{3}\vert}$	$+$	$(0.06G + 2.0)$		$-$	e		0.1111
≥ 2	1 or 2	$1{,}000.00v^2$	$-$	$1{,}000P^2_{x:\overline{3}\vert}\ddot{a}_{\overline{2}\vert}$	$+$	$(0.06G + 2.0)\ddot{a}_{\overline{2}\vert}$		$-$	$e\ddot{a}_{\overline{2}\vert}$		0.7778
Expected Values		Benefit reserve			$+$		Expense reserve			= Total reserve	
		258.67			$+$		(-40.73)			= 217.94	
$\sigma({}_2L^2_e) = 120.44$											
At Duration 2											
≥ 2	1 or 2	$1{,}000.00v$	$-$	$1{,}000P^2_{x:\overline{3}\vert}$	$+$	$[0.06G + 2.0]$		$-$	e		1.0000
Expected Values		Benefit reserve			$+$		Expense reserve			= Total reserve	
		582.88			$+$		(-24.29)			= 558.59	
$\sigma({}_2L^2_e) = 0$											

As a confirmation of the terminal total reserve at end of year 3:
(Total reserve + loaded premium − expenses, all at duration 2)$(1 + i) = (558.59 + 332.96 - 21.98)(1.15) = 1{,}000$.

$$(0.2G + 8) + (0.06G + 2)(1.1661) - e(2.1661) = 0$$
$$e = 46.27.$$

As a confirmation, we have

$$G = P^{2}_{\overline{x:3}|} + e,$$
$$332.96 = 286.69 + 46.27.$$

15.3.2 Accounting

As indicated in Section 15.1, the equivalence principle provides a conceptual framework for financial reporting of an insurance enterprise. In this section the financial reporting illustration of Section 15.2.2 will be extended to the double decrement model.

- The annual contract premium will be the annual level expense-loaded premium plus an arbitrary amount of 10 for profits and contingencies (less the percent of premium expenses on the 10).
- The accounting statements are derived using expected values for death and withdrawal benefit payments and for number of survivals. There are 10 initial insureds.
- Expenses are paid and investment income is earned as specified in Tables 15.2.1A and B.
- The hypothetical insurance operation starts with an initial fund of 1,000.
- In the accounting statements in column (a), benefit reserves, mortality-only model, are reported as liabilities, and in column (b) benefit and expenses reserves are reported as liabilities.

The financial statements in column (a) can be viewed as internally inconsistent because of the failure to incorporate future expenses and withdrawal benefits and provisions for these expenditures in future premiums into liabilities. Column (a) is displayed because of its historic role in regulation.

A comparison of Tables 15.2.5 and 15.2.6 with Tables 15.3.4 and 15.3.5 confirms the increased realism of financial statements using a double decrement model that incorporates expenses. The leveling effect on reported net income of the more comprehensive reserve system is also apparent. If the withdrawal benefits had not been selected to reduce their impact on premiums and reserves in the change from the single decrement model to the double decrement model, the difference in financial results between the two models would have been more pronounced.

15.4 Types of Expenses

The accounting system of an insurance enterprise is designed to record, classify, and summarize financial transactions. The same system, though, will furnish data on activity levels: the number and amount of sales, the number of claims paid, the number of premiums billed, and so on. After collecting this information, analysis

TABLE 15.3.4

Income Statements (10 Initial Insureds)

	(a) Reporting Single Decrement Benefit Reserves as Liabilities	(b) Reporting Double Decrement Benefit and Expense Reserves as Liabilities
During First Year		
Income		
Premiums (10)	3 429.60	3 429.60
Investment (15%)	549.55	549.55
	3 979.15	3 979.15
Charges to income		
Expenses		
Percentage (20%)	685.92	685.92
Constant (8)	80.00	80.00
Death benefits (1)	1 000.00	1 000.00
Withdrawal benefits (1)	227.73	227.73
Increase in reserve	2 059.28	1 743.52
	4 052.93	3 737.17
Net income	−73.78	241.98
During Second Year		
Income		
Premium (8)	2 743.68	2 743.68
Investment (15%)	832.28	832.28
	3 575.96	3 575.96
Charges to income		
Expenses		
Percentage (6%)	164.62	164.62
Constant (2)	16.00	16.00
Death benefits (0.8889)	888.90	888.90
Withdrawal benefits (0.8889)	501.70	501.70
Increase in reserves	1 556.83	1 732.15
	3 128.05	3 303.37
Net income	447.91	272.59
During Third Year		
Income		
Premium (6.2222)	2 133.97	2 133.97
Investment (15%)	1 047.56	1 047.56
	3 181.53	3 181.53
Charges to income		
Expenses		
Percentage (6%)	128.04	128.04
Constant (2)	12.44	12.44
Benefits (6.2222)	6 222.22	6 222.22
Increase in reserves	−3 616.11	−3 475.67
	2 746.59	2 887.03
Net income	434.94	294.50

TABLE 15.3.5

Balance Sheets (10 Initial Insureds)

	(a)	(b)
At end of first year		
Assets	2 985.50	2 985.50
Liabilities (Reserves)	2 059.29	1 743.52
Surplus	926.22	1 241.68
	2 985.50	2 985.20
At end of second year		
Assets	4 990.24	4 990.24
Liabilities (Reserves)	3 616.11	3 475.67
Surplus	1 374.13	1 514.57
	4 990.24	4 990.24
At end of third year		
Assets	1 809.07	1 809.07
Liabilities (Reserves)	0	0
Surplus	1 809.07	1 809.07

Notes on Tables 15.3.4 and 15.3.5

1. Total net income $= -73.78 + 447.91 + 434.94$
   $= 809.07$ Col. (a)
   $= 241.98 + 272.59 + 294.50$
   $= 809.07$ Col. (b)
2. Alternative calculation of total income
   (Interest income on initial funds) + (Accumulated value of profit loadings) =
   $1{,}000[(1.15)^3 - 1] + 10[10(0.8)(1.15)^3 + 8(0.94)(1.15)^2 + (6.2222)(0.94)(1.15)] = 809.26$
   The difference in the results of these two calculations is attributed to accumulated rounding errors that started with the use of a contract premium rounded to two decimal places.

can be performed with the goal of relating major expense items to the activities they support. These allocations will guide the determination of expense loading on premiums for insurance policies sold in the future. If the equivalence principle is applied, the actuarial present value of expense loadings will equal the actuarial present value of expenses charged to the policy.

Classification and allocation of the expenses of an insurance organization are perplexing tasks. An example is given in Table 15.4.1. Here a tentative classification system is adopted and the results traced.

In the determination of expense-loaded premiums, attention is concentrated on the insurance expenses. However, investment expenses are typically viewed as an offset to investment income and reflected in premiums through a reduction in the assumed interest rate.

In some instances practice indicates a natural relationship between expense items and activity levels. For example, it is common to compensate sales agents by a commission structure of percentages applied to first-year and renewal premiums. In Section 15.2 the commission paid was 10% of the premium in the first year and

TABLE 15.4.1

Classification Scheme for the Expenses of an Insurance Organization

Expense Classification	Components
Investment	(a) Analysis (b) Costs of buying, selling, and servicing
Insurance	
1. Acquisition	(a) Selling expense, including agents' commissions and advertising (b) Risk classification, including health examinations (c) Preparing new policies and records
2. Maintenance	(a) Premium collection and accounting (b) Beneficiary change and settlement option preparation (c) Policyholder correspondence
3. General	(a) Research (b) Actuarial and general legal services (c) General accounting and administration (d) Taxes, licenses, and fees
4. Settlement	(a) Claim investigation and legal defense (b) Costs of disbursing benefit payments

2% in the second and third years. Taxes on insurance organizations, especially those levied by the states, are often a percentage of the premium collected within the taxing jurisdiction. In Section 15.2, 2% of premiums were allocated to taxes, licenses, and fees.

The allocation of other items of expense is less clear-cut, and a combination of statistical analysis and judgment is often used. It is common practice to allocate acquisition expenses to the first policy year in premium-loading formulas because marketing and risk classification expenses are incurred for the purpose of generating and selling new insurance business. Some of these acquisition expenses vary with the size of the premium, commissions, for instance. Some vary with the amount of insurance, such as risk classification expense. Some expenses, like the creation of records, are incurred for each policy issued, independent of the size of the policy or premium.

The classification and allocation of expenses is an important management tool for controlling the operation of an insurance system. However, in the determination of premiums, the view of expenses is prospective rather than retrospective. The goal is to match future expenses with future premium loadings. Therefore, expense trends with expectations for inflation or deflation and future economies attributable to automation are built into expense loadings.

The provision within expense-loaded premiums for expenses classified in Table 15.4.1 as Insurance, General (a), (b), and (c) and for the expenses incurred in creating an insurance distribution system remain controversial. These expenses do not relate directly to activity associated with an individual policy. Some of these issues are discussed in Section 16.4.

Table 15.4.2 provides an illustration of the classification system in Table 15.4.1 for insurance expenses and associated loading factors.

TABLE 15.4.2

Illustration of the Allocation of Future Insurance Expenses

	First Year			Renewal Years				
						Percent of Premium by Policy Year		
Classification	Per Policy	Per 1,000 Insurance	Percent of Premium	Per Policy	Per 1,000 Insurance	2–9	10–15	16 over
1. Acquisition								
a. Sales expenses								
Commission	—	—	60%	—	—	7.0%	5.0%	3%
Sales offices	—	—	25	—	—	2.5	1.5	1
Other sales related	12.50	4.00	—	—	—	—	—	—
b. Classification	18.00	0.50	—	—	—	—	—	—
c. Issue and records	4.00	—	—	—	—	—	—	—
2. Maintenance	2.00	0.25	—	2.00	0.25	—	—	—
3. General								
a, b, c	4.00	0.25	—	4.00	0.25	—	—	—
d. Taxes	—	—	2	—	—	2.0	2.0	2
Total (1, 2, 3)	40.50	5.00	87%	6.00	0.50	11.5%	8.5%	6%
4. Settlement	18.00 per policy plus 0.10 per 1,000 insurance							

Example 15.4.1

Using the equivalence principle, develop a formula for the expense-loaded annual premium on a whole life policy, semicontinuous basis, issued to (x) for an amount $1{,}000b$. The expenses are those listed in Table 15.4.2. Use a single decrement model, or assume that withdrawal benefits will be determined so as to have no effect on premiums determined using the single decrement, mortality-only, model. A 15-year select life will be used under the assumption that expenditures for risk classification will result in select mortality.

Solution:

Let $G(b)$ denote the expense-loaded premium for a policy with a death benefit of b thousand. Then, using the equivalence principle,

(actuarial present value of expense-loaded premium) = (actuarial present value of claim and claim settlement expense plus other expenses);

$$
\begin{aligned}
G(b)\ddot{a}_{[x]} &= 1{,}000b\,\bar{A}_{[x]} + [40.50 + 5.00b + 0.87G(b)] \\
&\quad + (6.00a_{[x]} + 0.50ba_{[x]}) \\
&\quad + [(0.115a_{[x]:\overline{8}|} + 0.085\,{}_{9|6}\ddot{a}_{[x]} + 0.06\,{}_{15|}\ddot{a}_{[x]})\,G(b)] \\
&\quad + (18.00 + 0.10b)\bar{A}_{[x]}\ ;
\end{aligned}
$$

$$G(b)(\ddot{a}_{[x]} - 0.87 - 0.115a_{[x]:\overline{8}|} - 0.085_{9|6}\ddot{a}_{[x]} - 0.06_{15|}\ddot{a}_{[x]})$$
$$= (1{,}000.1\ \bar{A}_{[x]} + 5.00 + 0.50a_{[x]})b + 40.50 + 6.00a_{[x]} + 18.00\bar{A}_{[x]}\ .$$

The level expense-loaded annual premium rate for death benefit 1,000b is

$$G(b) = \frac{(1{,}000.1\ \bar{A}_{[x]} + 5.00 + 0.50a_{[x]})b + 40.50 + 6.00a_{[x]} + 18.00\bar{A}_{[x]}}{0.94\ddot{a}_{[x]} - 0.755 - 0.03\ddot{a}_{[x]:\overline{9}|} - 0.025\ddot{a}_{[x]:\overline{15}|}}\ .$$

The level expense-loaded annual premium for each death benefit amount of b, measured in units of 1,000, is

$$\frac{G(b)}{b} = \frac{1{,}000.1\ \bar{A}_{[x]} + 5.00 + 0.50a_{[x]} + (40.50 + 6.00a_{[x]} + 18.00\ \bar{A}_{[x]})/b}{0.94\ddot{a}_{[x]} - 0.755 - 0.03\ddot{a}_{[x]:\overline{9}|} - 0.025\ddot{a}_{[x]:\overline{15}|}}\ . \qquad \blacktriangledown$$

In practice, premiums are usually stated as a rate per unit of insurance. For life insurance, these rates have typically been per 1,000 of initial death benefit. For immediate life annuities, the rates have typically been stated per unit of monthly income.

In Example 15.4.1, because of expenses that do not vary directly with b, the expense-loaded premium rate $G(b)$ depends on b. Provision for these per policy expenses can be made by special methods. One method is to replace b with an expected policy amount. A second method would be to separate the per policy expenses from those expense elements that vary directly with policy size and to balance these per policy expenses with a separate policy fee, independent of policy size. In Example 15.4.1 the annual policy fee would be

$$\frac{40.50 + 6.00a_{[x]} + 18.00\bar{A}_{[x]}}{0.94\ddot{a}_{[x]} - 0.755 - 0.03\ddot{a}_{[x]:\overline{9}|} - 0.025\ddot{a}_{[x]:\overline{15}|}}\ .$$

Often the policy fees are averaged over issue age so the policy fee is constant with respect to issue age.

15.5 Algebraic Foundations of Accounting: Single Decrement Model

In this section many of the ideas illustrated in Section 15.2.2 will be made more precise. Frequent reference to Tables 15.2.5 and 15.2.6 can help the reader follow the arguments.

One of the objectives of financial accounting is the determination, at periodic intervals, of the elements of the balance sheet equation

$$A(h) = L(h) + U(h). \qquad (15.5.1)$$

In (15.5.1), $A(h)$ denotes the amount of assets, $L(h)$ the amount of liabilities, and $U(h)$ the amount of owner's equity (surplus in the terminology of insurance accounting) at the end of accounting period h. Changes in surplus can be represented by

$$\Delta U(h) = \Delta A(h) - \Delta L(h) = \text{net income in period } h + 1. \tag{15.5.2}$$

We illustrate this basic model using an algebraic development under idealized conditions as stated in Table 15.5.1.

TABLE 15.5.1

Specifications of Accounting Illustration

	Level Benefit, Level Premium Insurance
1. Plan of insurance	Whole life, unit amount
2. Payment basis	Fully discrete
3. Age and time of issue	Issued to (x) at the beginning of the first accounting period
4. Expenses	No expenses or expense loadings
5. Experience	Investment experience conforms to that assumed
	Accounting entries will be in terms of expected values at policy issue for each initial insured

The mathematics of the illustration build on the reserve recursion formula (8.3.10) with $b_h = 1$, $\pi_{h-1} = P_x$, and ${}_hV = {}_hV_x$. Multiplying by ${}_{h-1}p_x\ (1 + i)$ we have

$${}_{h-1}p_x\ ({}_{h-1}V_x + P_x)(1 + i) - {}_{h-1}p_x\ q_{x+h-1} = {}_hp_x\ {}_hV_x \qquad h = 1, 2, \ldots. \tag{15.5.3}$$

Under the restrictive assumptions that have been made, (15.5.3) can be interpreted as the expected progress of insurance assets and liabilities for each member of an initial group of insureds. The left-hand side can be interpreted as the expected cash flows affecting assets. The right-hand side is the measure of the expected liabilities for each member of an initial group of insureds.

We illustrate by examining the first accounting period. During this period the expected assets per initial insured change as follows:

Increase	Premium income	$= P_x$
	Interest income	$= P_x\, i$
Decrease	Death claims	$= q_x$

If there are no initial funds,

$$A(1) = A(0) + [A(1) - A(0)] = 0 + [P_x(1 + i) - q_x],$$

and using (15.5.3) with $h = 1$, we have

$$A(1) = p_x\ V_x = L(1).$$

In this illustration, $A(1) - L(1) = U(1) = 0$.

Formula (15.5.3) can also be used to study the progress of accounting statements in a recursive fashion for all policy years. Suppose that at the end of accounting period h

$$A(h) = L(h)$$

and that we start the process during accounting period $h + 1$:

$$\Delta A(h) = \begin{Bmatrix} \text{premium income} \\ + \text{ interest income} \\ - \text{ death claims} \end{Bmatrix} = \begin{Bmatrix} {}_hp_x\, P_x \\ + {}_hp_x\, ({}_hV_x + P_x)i \\ - {}_hp_x\, q_{x+h}. \end{Bmatrix}$$

Then,

$$\begin{aligned} A(h+1) &= A(h) + \Delta A(h) \\ &= {}_hp_x\, {}_hV_x + \{{}_hp_x\, [P_x + ({}_hV_x + P_x)i] - {}_hp_x\, q_{x+h}\} \\ &= {}_hp_x\, [(P_x + {}_hV_x)(1 + i)] - {}_hp_x\, q_{x+h} \\ &= {}_{h+1}p_x\, {}_{h+1}V_x = L(h+1). \end{aligned} \tag{15.5.4}$$

In this illustration, with no initial funds, profit, or contingency loadings, tracing expected results yields $A(h) - L(h) = U(h) = 0$, $h = 0, 1, 2, 3, \ldots$.

We now modify the assumptions of Table 15.5.1 by assuming that the benefit premium is loaded by the positive constant c and that the expenses for each surviving policy, paid at the beginning of accounting period h, are e_{h-1}. The loading constant may contain a component for profit; that is, the actuarial present value of the loadings c may be greater than the actuarial present value of the set of e_{h-1}, $h = 1, 2, \ldots$.

The augmented version of (15.5.3), incorporating loaded premiums and expenses, is

$$\begin{aligned} {}_{h-1}p_x\, \{[{}_{h-1}V_x + \underline{u(h-1)}] + (P_x + \underline{c}) - \underline{e_{h-1}}\}\, (1 + i) - {}_{h-1}p_x\, q_{x+h-1} \\ = {}_hp_x\, [{}_hV_x + \underline{u(h)}] \qquad h = 1, 2, 3, \ldots. \end{aligned} \tag{15.5.5}$$

The elements introduced into the augmented version of (15.5.3) are underlined. In (15.5.5), $u(h)$ denotes the anticipated surplus for each surviving insured at the end of accounting period h.

Subtracting the unloaded version, (15.5.3), from (15.5.5) yields

$${}_{h-1}p_x\, [u(h-1) + (c - e_{h-1})]\, (1 + i) = {}_hp_x\, u(h) \qquad h = 1, 2, 3, \ldots. \tag{15.5.6}$$

Multiplying difference equation (15.5.6) by v^h and rearranging terms yields

$$\begin{aligned} v^{h-1}\, {}_{h-1}p_x\, [u(h-1) + (c - e_{h-1})] &= v^h\, {}_hp_x\, u(h), \\ \Delta[v^{h-1}\, {}_{h-1}p_x\, u(h-1)] &= v^{h-1}\, {}_{h-1}p_x\, (c - e_{h-1}). \end{aligned} \tag{15.5.7}$$

Imposing the initial condition $u(0) = 0$, we obtain from (15.5.7)

$$\sum_{j=1}^{h} \Delta[v^{j-1} \, {}_{j-1}p_x \, u(j-1)] = \sum_{j=1}^{h} v^{j-1} \, {}_{j-1}p_x \, (c - e_{j-1}),$$

$$v^h \, {}_hp_x \, u(h) = \sum_{j=1}^{h} v^{j-1} \, {}_{j-1}p_x \, (c - e_{j-1}),$$

$$ {}_hp_x \, u(h) = \sum_{j=1}^{h} (1 + i)^{h-j+1} \, {}_{j-1}p_x \, (c - e_{j-1}). \tag{15.5.8}$$

That is, the expected surplus at the end of h accounting periods for each initial insured is the accumulated value of the expected contributions to surplus in each earlier accounting period. This result should be compared with Table 15.2.5, footnote 3.

If benefit reserves are reported as the measure of liabilities, the expected entries for each initial insured in the accounting statements of our idealized insurance system at the end of the accounting period h are as follows.

Balance Sheet (at end of accounting period h)

$$\begin{aligned} A(h) &= L(h) + U(h) \\ &= {}_hp_x \, {}_hV_x + {}_hp_x \, u(h) \\ &= {}_hp_x \, {}_hV_x + \sum_{1}^{h} (1 + i)^{h-j+1} \, {}_{j-1}p_x \, (c - e_{j-1}) \end{aligned}$$

Income Statement (h-th accounting period)

Income	
Premium income	${}_{h-1}p_x \, (P_x + c)$
Investment income	${}_{h-1}p_x \, [{}_{h-1}V_x + u(h-1) + P_x + c - e_{h-1}] \, i$
Total	${}_{h-1}p_x \, \{(P_x + c)(1 + i) + [{}_{h-1}V_x + u(h-1) - e_{h-1}]i\}$
Charges to Income	
Death claims	${}_{h-1}p_x \, q_{x+h-1}$
Expenses	${}_{h-1}p_x \, e_{h-1}$
Changes in reserve liability	${}_hp_x \, {}_hV_x - {}_{h-1}p_x \, {}_{h-1}V_x$
Total	${}_hp_x \, {}_hV_x - {}_{h-1}p_x \, ({}_{h-1}V_x - e_{h-1}) + {}_{h-1}p_x \, q_{x+h-1}$
Net income (change in surplus)	${}_{h-1}p_x \, [u(h-1)i + (c - e_{h-1})(1 + i)]$ (15.5.9)

In completing the accounting statements we have made use of (15.5.8) and (15.5.3). The left-hand columns of Tables 15.2.5 and 15.2.6 provide numerical illustrations of this display. The tables are in terms of a deterministic survivorship group rather than expected entries for each initial insured. Thus the expected surplus at the end of h accounting periods for each initial insured is

$$ {}_hp_x \, u(h) = {}_{h-1}p_x \, u(h-1) + {}_{h-1}p_x \, [u(h-1)i + (c - e_{h-1})(1 + i)]. \tag{15.5.10}$$

Formula (15.5.10) is identical to (15.5.6); however, it was derived from an accounting viewpoint. Multiplying by v^h and rearranging yields

$$\Delta[v^{h-1}\ {}_{h-1}p_x\ u(h-1)] = v^{h-1}\ {}_{h-1}p_x\ (c - e_{h-1}),$$

which is (15.5.7) rederived with accounting interpretations.

Earlier in this chapter the point was made that, in practice, expenses tend to decrease as duration increases. Thus the expected surplus,

$${}_hp_x\ u(h) = \sum_{j=1}^{h} (1+i)^{h-j+1}\ {}_{j-1}p_x\ (c - e_{j-1}),$$

will typically be negative for small values of h and positive for larger values. This observation is made with respect to an accounting model in which benefit reserves are reported as the measure of liabilities, loadings are level, and expenses decrease with time following policy issue.

To avoid the situation in which assets are less than liabilities, in the early durations, several actions are possible:

- The insurance organization may obtain additional capital for the initial surplus, $u(0)$, to keep

  $$u(0)(1+i)^h + \sum_{j=1}^{h} (1+i)^{h-j+1}\ {}_{j-1}p_x\ (c - e_{j-1})$$

  positive for $h = 0, 1, 2, 3, \ldots.$
- Loadings may depend on duration so that $c_{h-1} - e_{h-1} \ge 0$, $h = 1, 2, 3, \ldots.$
- The liabilities of the insurance organization could be based on a reserve principle that would reduce reported liabilities in early policy years. The reserve principle of reporting benefit plus expense reserves used in column (b) of Tables 15.2.5 and 15.2.6 is an example of such an action. (This last alternative is the subject of Sections 16.6 and 16.7.)

15.6 Asset Shares

To provide an algebraic foundation for accounting within a double decrement model, using the equivalence principle, it is necessary to develop a set of recursion relations. These recursion relations have many applications, and some of these applications are developed in Sections 16.4.2 and 16.5. The basic variable in these recursion equations has been given different names, depending on the application. In this section we call it ***asset share,*** a term laden with a long history. In other sections, depending on the application, different operational meanings are attached.

15.6.1 Recursion Relations

A life insurance policy is a long-term contract involving income to the insurer from premiums and investments and outgo as a result of death and withdrawal benefit payments and expenses. Contract premiums actually charged for a unit of insurance are influenced by competition, and withdrawal values are influenced by law and competition. There is a need for a calculation of the balance, in the sense of actuarial present values, between the various elements of the price-benefit structure. The asset share calculation, outlined in this section, is designed to fill this need. It is not a historic summary of past results, but a prospective calculation of some complexity attempting to capture most elements that influence the expected financial progress of a group of policies.

We will start with an extension of (15.5.5) for a unit of insurance,

$$
\begin{aligned}
{}_hp_x^{(\tau)}\,({}_hAS) = {}_{h-1}p_x^{(\tau)}\,\{&[{}_{h-1}AS + G(1 - c_{h-1}) - e_{h-1}](1 + i) \\
&- q_{x+h-1}^{(1)} - q_{x+h-1}^{(2)}\,{}_hCV\} \qquad h = 1, 2, 3, \ldots . \qquad (15.6.1)
\end{aligned}
$$

Multiplying (15.6.1) by $1/{}_{h-1}p_x^{(\tau)}$ yields

$$
\begin{aligned}
p_{x+h-1}^{(\tau)}\,({}_hAS) = &[{}_{h-1}AS + G(1 - c_{h-1}) - e_{h-1}]\,(1 + i) \\
&- q_{x+h-1}^{(1)} - q_{x+h-1}^{(2)}\,{}_hCV \qquad h = 1, 2, 3, \ldots . \qquad (15.6.2)
\end{aligned}
$$

In (15.6.1) and (15.6.2),

- ${}_hAS$ denotes the expected asset share h years following policy issue, immediately before the start of policy year $h + 1$
- G denotes the level contract premium
- c_h denotes the fraction of the contract premium paid at time h for expenses
- e_h denotes the amount of per policy expenses paid at time h
- $q_{x+h}^{(1)}$ denotes the probability of decrement by death, before the attainment of age $x + h + 1$, for an insured now age $x + h$
- $q_{x+h}^{(2)}$ denotes the probability of decrement by withdrawal, before the attainment of age $x + h + 1$, for an insured now age $x + h$
- ${}_hCV$ denotes the amount of the withdrawal benefit paid at time h. This is also called a ***cash value.***

Formula (15.6.1) is based on the assumptions of a fully discrete payment basis, unit death claims paid at the end of the year of death, and ${}_hCV$ paid at the end of the year of withdrawal.

Formula (15.6.1) is a generalization of the recursion relationship connecting successive terminal reserves. It will be rewritten in several ways that are reminiscent of similar manipulations with reserve equations. Multiplying (15.6.2) by $v^h l_{x+h-1}^{(\tau)}$ and rearranging the terms yields

$$
\begin{aligned}
\Delta(l_{x+h-1}^{(\tau)}\, v^{h-1}\, {}_{h-1}AS) = &[G(1 - c_{h-1}) - e_{h-1}]l_{x+h-1}^{(\tau)}\, v^{h-1} \\
&- (d_{x+h-1}^{(1)} + d_{x+h-1}^{(2)}\, {}_hCV)\, v^h \qquad h = 1, 2, 3, \ldots . \qquad (15.6.3)
\end{aligned}
$$

The sum of the left-hand side over $h = 1, 2, \ldots, n$ telescopes to

$$l_{x+n}^{(\tau)}\, v^n\, {}_nAS - l_x^{(\tau)}\, {}_0AS = \sum_{h=1}^{n} \{[G(1 - c_{h-1}) - e_{h-1}] l_{x+h-1}^{(\tau)}\, v^{h-1}$$
$$- (d_{x+h-1}^{(1)} + d_{x+h-1}^{(2)}\, {}_hCV)\, v^h\}. \qquad (15.6.4)$$

If ${}_0AS = 0$, we have ${}_nAS$ equal to

$$\sum_{h=1}^{n} \frac{\{[G(1 - c_{h-1}) - e_{h-1}] l_{x+h-1}^{(\tau)}(1 + i) - (d_{x+h-1}^{(1)} + d_{x+h-1}^{(2)}\, {}_hCV)\}(1 + i)^{n-h}}{l_{x+n}^{(\tau)}}. \qquad (15.6.5)$$

For a whole life insurance, we set $n = \omega - x$ in (15.6.4). Then, recognizing that ultimately the expected asset share is zero, we rearrange (15.6.5) to obtain

$$G\ddot{a}_x^{(\tau)} = A_x^{(1)} + \sum_{h=1}^{\omega-x} (Gc_{h-1} + e_{h-1})\, v^{h-1}\, {}_{h-1}p_x^{(\tau)} + \sum_{h=1}^{\omega-x} {}_{h-1}p_x^{(\tau)}\, q_{x+h-1}^{(2)}\, v^h\, {}_hCV. \qquad (15.6.6)$$

Formula (15.6.6) can be interpreted as a general formula for an expense-loaded premium using the equivalence principle. Appropriate modifications can alter the formula from a whole life to an endowment or term policy.

Making the substitution

$$p_{x+h-1}^{(\tau)} = 1 - q_{x+h-1}^{(1)} - q_{x+h-1}^{(2)}$$

allows us to rewrite (15.6.2) as

$${}_hAS = [{}_{h-1}AS + G(1 - c_{h-1}) - e_{h-1}]\,(1 + i)$$
$$- q_{x+h-1}^{(1)}\,(1 - {}_hAS) - q_{x+h-1}^{(2)}\,({}_hCV - {}_hAS). \qquad (15.6.7)$$

This form emphasizes the importance of the difference, ${}_hCV - {}_hAS$, on the progression of asset shares.

Asset share calculations can be viewed as tracing the expected progress of the assets, per surviving policy, of a block of similar policies. The calculations for fixed contract premiums, expense commitments, and cash values can be made to check the balance between the various components of the price-benefit structure. The objective of the calculations might be to determine if ${}_kAS \geq {}_kV$, for all but the very early policy years.

15.6.2 Accounting

Let ${}_hAS = {}_hV + u(h)$, where ${}_hV$ is the reserve liability and $u(h)$ the anticipated surplus for each surviving insured at the end of accounting period h. The values for $u(h)$ may be negative, especially during early policy years. Assume that reserve liabilities are generated by the recursion relation

$${}_hV\, p_{x+h-1}^{(\tau)} = ({}_{h-1}V + P)\,(1 + i)$$
$$- q_{x+h-1}^{(1)} - q_{x+h-1}^{(2)}\, {}_hCV \qquad h = 1, 2, 3, \ldots. \qquad (15.6.8)$$

Formula (15.6.8) is a recursion relation that determines benefit reserves within the double decrement model. These correspond to the benefit reserves calculated in Table 15.3.3.

Multiply (15.6.2) and (15.6.8) by ${}_{h-1}p_x^{(\tau)}$ and subtract to obtain

$$u(h)\ {}_hp_x^{(\tau)} = [u(h-1) + G(1 - c_{h-1}) - e_{h-1} - P](1+i)\ {}_{h-1}p_x^{(\tau)}$$
$$h = 1, 2, \ldots. \qquad (15.6.9)$$

Let $G = P + c$ and $Gc_{h-1} + e_{h-1} = E_{h-1}$ (total expenses), and (15.6.9) becomes

$$u(h)\ {}_hp_x^{(\tau)} = [u(h-1) + c - E_{h-1}]\ (1+i)\ {}_{h-1}p_x^{(\tau)} \qquad (15.6.10)$$

which is a double decrement version of (15.5.10). Multiplying recursion relation (15.6.10) by v^h and rearranging terms yields

$$\Delta v^{h-1}\ {}_{h-1}p_x^{(\tau)}\ u(h-1) = v^{h-1}\ {}_{h-1}p_x^{(\tau)}\ (c - E_{h-1}).$$

Following exactly the same steps as in Section 15.5 to obtain (15.5.8), we obtain

$${}_hp_x^{(\tau)}\ u(h) = \sum_{j=1}^{h} (1+i)^{h-j+1}\ {}_{j-1}p_x^{(\tau)}\ (c - E_{j-1}). \qquad (15.6.11)$$

The remaining developments in Section (15.5) follow in identical fashion with the substitution of corresponding multiple decrement for single decrement probabilities and the addition of expected withdrawal benefits ${}_{h-1}p_x^{(\tau)}\ q_{x+h-1}^{(2)}\ {}_{h-1}CV$. The illustration in Tables 15.3.4 and 15.3.5, column (b), provides a worked example of these ideas.

Example 15.6.1

Consider again the illustration that was started in Table 15.2.1 and expanded to include withdrawal benefits in Table 15.3.1. Assume, as in Tables 15.3.4 and 15.3.5, that $G = 342.96$. Calculate a set of asset shares.

Solution:

We use (15.6.2) to guide our calculation.

Period

h $\{[{}_{h-1}AS + G(1 - c_{h-1}) - e_{h-1}](1+i) - 1{,}000\ q_{x+h-1}^{(1)} - {}_hCV\ q_{x+h-1}^{(2)}\} / p_{x+h-1}^{(\tau)} = {}_hAS$

1 $\{[0 + 342.96\ (1 - 0.20) - 8.0](1.15) - 1{,}000(0.1) - 227.73(0.1)\} / 0.8 = 229.44$

2 $\{[229.44 + 342.96(1 - 0.06) - 2.0](1.15) - 1{,}000(0.1111) - 564.41(0.1111)\} / 0.7778 = 589.46$

3 $\{[589.46 + 342.96(1 - 0.06) - 2.0](1.15) - 1{,}000\} / 1.0 = 46.32$

The motivation for calling ${}_hAS$ an asset share can be appreciated by a comparison of the solution to this example and the final assets reported at the end of the third year in Table 15.3.5:

(Asset Share) (Expected number of insureds receiving death, maturity, or withdrawal benefits at the end of policy year three) = Total Expected Assets

$$(46.32)[(10)(0.6222)] = 288.22,$$

(Expected assets at the end of the third policy year) − (Assets assumulated from the initial fund)

$$1{,}809.07 - 1{,}000\ (1.15)^3 = 288.20.$$

The difference is due to rounding in each computation. ▼

15.7 Expenses, Reserves, and General Insurances

A number of new ideas, using two simple illustrations, are developed in Sections 15.2 and 15.3. In Section 8.2 reserves for a general life insurance, single decrement model, ignoring expenses, were displayed. The development started with the following loss variable for $T(x) > t$:

$$_tL = b_{T(x)}\, v^{T(x)-t} - \int_t^{T(x)} \pi_u\, v^{u-t}\, du.$$

Our goal is to extend this model to incorporate features of recent sections. First we will add a benefit associated with withdrawal,

$$_tL^2 = \begin{cases} b^{(1)}_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} \pi^2_u\, v^{u-t}\, du, & \text{decrement by death} \\ b^{(2)}_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} \pi^2_u\, v^{u-t}\, du, & \text{decrement by withdrawal.} \end{cases}$$

The superscript ^{2} has been added to the loss variable and premium rate symbols to denote that a withdrawal benefit has been added to the initial model.

Second, we add expenses at rate E_t at time t, measured from issue. This corresponds to what was done in (15.6.10). The subscript e is joined to the loss variable symbol and the premium rate symbol to denote that expenses have been added to the model. It is assumed that $_e\pi^2_u$ has been determined by the equivalence principle:

$$_tL^2_e = \begin{cases} b^{(1)}_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} (_e\pi^2_u - E_u) v^{u-t}\, du, & \text{decrement by death} \\ b^{(2)}_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} (_e\pi^2_u - E_u) v^{u-t}\, du, & \text{decrement by withdrawal.} \end{cases}$$

Then, changing the integration variable to $s = u - t$, we can write the conditional expectation of $_tL^2_e$, given $T(x) > t$,

$$\mathrm{E}[{}_tL_e^2] = \int_t^\infty \left[b_y^{(1)}\, v^{y-t} - \int_0^{y-t} ({}_e\pi_{t+s}^2 - E_{t+s}) v^s\, ds \right] {}_{y-t}p_{x+t}^{(\tau)}\, \mu_{x+t}^{(1)}(y-t)\, dy$$

$$+ \int_t^\infty \left[b_y^{(2)}\, v^{y-t} - \int_0^{y-t} ({}_e\pi_{t+s}^2 - E_{t+s}) v^s\, ds \right] {}_{y-t}p_{x+t}^{(\tau)}\, \mu_{x+t}^{(2)}(y-t)\, dy. \quad (15.7.1)$$

Now change the outer integration variable to $u = y - t$ and collect terms as

$$\mathrm{E}[{}_tL_e^2] = \int_0^\infty \left[\sum_{j=1}^{2} b_{t+u}^{(j)}\, \mu_{x+t}^{(j)}(u) v^u - \int_0^u ({}_e\pi_{t+s}^2 - E_{t+s}) v^s\, ds\, \mu_{x+t}^{(\tau)}(u) \right] {}_up_{x+t}^{(\tau)}\, du.$$

Using integration by parts on the second term yields

$$\mathrm{E}[{}_tL_e^2] = \int_0^\infty v^u \left\{ \left[\sum_{j=1}^{2} b_{t+u}^{(j)}\, \mu_{x+t}^{(j)}(u) - {}_e\pi_{t+u}^2 + E_{t+u} \right] {}_up_{x+t}^{(\tau)} \right\} du$$

$$= \int_0^\infty v^u [f(u{:}t)]\, du. \quad (15.7.2)$$

Formula (15.7.2) is of interest because the function

$$f(u{:}t) = [E_{t+u} + \sum_{j=1}^{2} b_{t+u}^{(j)}\, \mu_{x+t}^{(j)}(u) - {}_e\pi_{t+u}^2]\, {}_up_{x+t}^{(\tau)} \quad (15.7.3)$$

can be interpreted as the ***expected cash flow*** at time $t + u$ arising from the insurance policy, given survival to time t.

Because positive values of $f(u{:}t)$ are associated with expected cash outflows from the insurance enterprise, some actuaries prefer to consider the function $g(u{:}t) = -f(u{:}t)$ in which expected cash inflow would have positive values. Formula (15.7.3) also provides yet another interpretation of reserves. Reserves become the present values of future expected net cash outflows. Formula (8.6.1) provides the same interpretation in a less comprehensive model.

In Sections 15.2.2 and 15.3.2 it is illustrated that reserves can differ in their degree of comprehensiveness in accordance with the purpose of the valuation. In this section a very general insurance was introduced. For regulatory purposes reserves might be based on formula (8.2.4) under the assumption that to ignore expenses would be conservative in the sense that typically reserves are decreased by including expenses. The withdrawal benefit might also be omitted under the assumption that a properly determined withdrawal benefit will have only a small effect on reserves. For financial accounting to be used by the capital markets, (15.7.2) would usually be the basis of reserves.

In the financial management of life insurance companies, the rate of change in reserve liabilities with respect to changes in the valuation interest rate is of considerable concern. Using (15.7.2) and viewing the premium rate set at time 0 to be independent of future valuation interest rates, and assuming that the valuation interest rate is independent of the probabilities of death and withdrawal, we have

$$\frac{d}{d\delta}\,\mathrm{E}[{}_tL_e^2] = -\int_0^\infty uv^u\, f(u{:}t)du. \tag{15.7.4}$$

Example 15.7.1

Exhibit the expected cash flow function $f(u{:}t)$ for a fully continuous whole life policy, ignoring expenses, and display $d\mathrm{E}[{}_tL_e^1]/d\delta$ in actuarial present-value functions. Remember that the premium was set at time 0 and is not a function of the current valuation interest rate.

Solution:

Modifying formulas (15.7.3) and (15.7.4) we have

$$f(u{:}t) = [\mu_{x+t}(u) - \bar{P}(\bar{A}_x)]_u p_{x+t}$$

and

$$\begin{aligned}\frac{d}{d\delta}\,\mathrm{E}[{}_tL_e^1] &= -\int_0^\infty uv^u\,[\mu_{x+t}(u) - \bar{P}(\bar{A}_x)]_u p_{x+t}\,du \\ &= -(\bar{I}\bar{A})_{x+t} + \bar{P}(\bar{A}_x)(\bar{I}\bar{a})_{x+t}\,,\end{aligned}$$

where $(\bar{I}\bar{A})_{x+t}$ and $(\bar{I}\bar{a})_{x+t}$ are calculated at the valuation interest rate. ▼

Typically, $g(0{:}0) < 0$ and there exists a u_0 such that $g(u{:}o) > 0$ for $u > u_0$. The time u_0 that expected cash flows cross from negative to positive may be useful in financial planning.

Example 15.7.2

Display the equation that would be used to determine the number u_0 for the policy described in Example 15.7.1.

Solution:

$$f(u{:}0) = [\mu_x(u) - \bar{P}(\bar{A}_x)]_u p_x = 0$$

or

$$\mu_x(u) = \bar{P}(\bar{A}_x).$$

▼

15.8 Notes and References

In this chapter we have used the clumsy term "expense-loaded premiums" for what some would call gross premiums. The use of the longer term is intended as a warning that the subject of premiums contains many topics other than the benefit premiums discussed in previous chapters and the expense loadings introduced here. These topics include competitive considerations, profit loadings in nonparticipating insurance, expected dividends in participating insurance, risk factors, and the impact of withdrawal benefits. Guertin (1965) discusses many of these topics. Chalke (1991) has criticized the traditional cost-plus determination of premiums. The criticisms are built on a foundation of classical microeconomics.

Fassel (1956) discusses the issues involved in estimating and allocating per policy expenses. Until the time of Fassel's paper, most per policy expenses were included in the loading for expenses that vary with the amount of insurance (assuming an average size policy). In fact, the use of either policy fees or the band system was viewed for many years as an inequitable allocation of expense charges.

Brenner et al. (1988) is a treatise on life insurance accounting. Horn (1971) studies the impact of various reserve systems on the time incidence of reported net income. Asset share calculations have a long history. Huffman (1978) discusses refinements in asset share calculations.

Exercises

Section 15.2

15.1. a. A gambling enterprise collects 0.55 from each of 1,000 customers on July 1 of year Z. It immediately invests the funds in a savings account earning 3% interest each 6 months. On July 1 of year $Z + 1$, 1,000 coins will be tossed, each assigned to a specific customer. If the customer's coin comes up heads, the customer receives a prize of 1. If the coin comes up tails, the prize is 0. Supply the figures for the balance sheet and income statement for the gambling enterprise on December 31 of year Z. Use actuarial present values for liabilities.

Balance Sheet	
Assets	Liabilities
Savings account	Reserves
	Surplus
Income Statement	
Premium income	
Interest income	
Increase in revenues	

b. The random variable Y is the amount of the payments made on July 1 of year $Z + 1$ and has a binomial distribution. Using a normal approximation, evaluate

$$\Pr[Y(1.03)^{-1} - A > 0]$$

where A represents the assets as of December 31 of year Z.

c. If the enterprise had only one customer, the scale of the operation would be 0.001 of that in part (a). Show that the probability displayed in (b) is equal to $1/2$ for the reduced enterprise.

15.2. An expense augmented loss variable for use with a whole life policy, fully continuous model, is given by

$$L_e = L + X$$

where

$$L = v^T - \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{T}|}$$

and

$$X = c_0 + (g - e)\, \bar{a}_{\overline{T}|}.$$

In these expressions, L is interpreted as the loss variable associated with the benefit portion of the policy and X with the expenses. The symbol c_0 denotes nonrandom initial expenses, g the rate of continuous maintenance expense, and e the expense loading in the premium. The equivalence principle has been adopted and $\mathrm{E}[L] = \mathrm{E}[X] = 0$. Show that

a. $X = c_0 L$

b. $\mathrm{Var}(L_e) = (1 + c_0)^2\, \mathrm{Var}(L)$

c. $\mathrm{Cov}(L, X) = c_0[{}^2\bar{A}_x - (\bar{A}_x)^2] / (\delta \bar{a}_x)^2$.

15.3. A merchant has accounts receivable at the end of each annual accounting period. Experience has indicated that there is a probability of 0.25 that no payment will be received and 0.75 that full payment will be received at time T. The p.d.f. of T is given by $f(t) = 4 - 8t$, $0 < t < 0.5$, where t is measured in years. At the end of a year, the merchant has 100 accounts receivable, each of amount 100. The random variable R_i is the present value at the end of the year of account i.

a. Verify that

$$R_i = 0 \text{ with probability } 0.25,$$
$$= 100\, v^T \text{ with probability } 0.75.$$

b. If R_i, $j = 1, 2, 3, \ldots, 100$ are independent random variables, calculate (i) $\mathrm{E}[\sum_1^{100} R_i]$, (ii) $\mathrm{Var}(\sum_1^{100} R_i)$, when $\delta = 0.06$.

c. Repeat (b) if δ is 0.

d. The merchant elects to report an actuarial present value as the value of the accounts receivable. The results of part (a) are used. What is the reported value of accounts receivable?

e. As an alternative, the merchant might use the results of part (c) as the reported value of accounts receivable. What amount would be reported?

f. Another method for reporting accounts receivable would be to use the results of (c), $\delta = 0.00$, and to report

$$\mathrm{E}\left[\sum_1^{100} R_i\right] - k\sqrt{\mathrm{Var}\left(\sum_1^{100} R_i\right)}.$$

For what value of k is the amount reported in part (f) the same as in part (d)?

Section 15.3

15.4. a. In the illustration in Table 15.3.1 assume that $b_{x+1}^{(2)} = 257.41$ and $b_{x+2}^{(2)} = 581.16$, the benefit reserves for the single decrement model, and determine $P^2_{x:\overline{3}|}$ using the equivalence principle.

b. Assume that $b_{x+1}^{(2)} = 218.41$ and $b_{x+2}^{(2)} = 559.16$, the total reserves for the single decrement model, and determine G, the expense-loaded premium, using the equivalence principle.

Section 15.4

15.5. The expense-loaded annual premium for a 1,000 endowment-at-age-65 life insurance with level annual premiums issued at age 40 is calculated using the following assumptions:

- Selling commission is 40% of the expense-loaded premium in the first year
- Renewal commissions are 5% of the expense-loaded premium for policy years 2 through 10
- Premium tax is 2% of the expense-loaded premium each year
- Maintenance expense is 12.50 per 1,000 of insurance in the first year and 4.00 per 1,000 of insurance thereafter
- The benefit premium is to provide for the immediate payment of death claims with no premium adjustment on death
- A 15-year select-and-ultimate mortality table is to be used.

Write an expression for the expense-loaded premium.

15.6. The expense-loaded premium for a single premium n-year endowment insurance is determined using the following assumptions:

- Taxes are 2-1/2% of the expense-loaded premium
- Commission is 4% of the expense-loaded premium
- Other expenses are 5 in the first year and 2.50 in each renewal year per 1,000 of insurance.

Claims are paid at the moment of death and expenses are incurred at the start of each policy year. Develop a formula for the expense-loaded premium issued to (x) for an insurance of 1,000.

15.7. For a fully discrete whole life policy of amount 1, the level expense-loaded premium is based on the following schedule of expenses:

- An initial expense of e_0
- Each policy year, including the first, an expense of $e_1 + e_2 P_x$
- The cost of claims settlement, paid along with the claim, of e_3 per unit of insurance.

If $G = a\,P_x + c$, determine a and c.

15.8. There are two random variables in this exercise. The random variable $T(x)$ is interpreted as time-until-death of a life age x at issue. The random variable

B is interpreted as the death benefit chosen by a randomly selected applicant for a whole life policy that uses a continuous model. The expense augmented loss variable associated with this policy is

$$
\begin{aligned}
L(T(x), B)_e &= 0 \qquad\qquad T < 0 \\
&= Bv^T + \alpha B\bar{a}_{\overline{T}|} + \theta\bar{a}_{\overline{T}|} \\
&\quad + \rho(B\pi + f)\bar{a}_{\overline{T}|} - (B\pi + f)\bar{a}_{\overline{T}|} \qquad 0 < T.
\end{aligned}
$$

In this loss variable:

αB = rate of expense payments that are proportional to the death benefit, paid continuously during the lifetime of (x)

θ = rate of expense payments that are independent of benefit and premium amounts, paid continuously during the lifetime of (x)

$\rho(B\pi + f)$ = rate of expense payments that are proportional to premium payments, paid continuously during the lifetime of (x)

π = portion of continuously paid premium proportional to the death benefit

f = rate of policy fee paid continuously during the lifetime of (x).

Assume that $T(x)$ and B are independent.

a. Use a conditional equivalence principle, that is,

$$E[L(T(x), B)_e | B = b] = 0,$$

and exhibit a formula for the premium rate per unit of insurance; that is, derive a formula for $\pi + f/b$.

b. Use the unconditional equivalence principle

$$E[L(T(x), B)_e] = 0$$

and exhibit a formula for the constant premium rate per unit of insurance.

15.9. The p.d.f. of the amount of insurance issued on an individual policy for a particular plan of insurance is given by

$$f(b) = kb^{-3} \qquad b > 10$$

where b is measured in thousands. Calculate

a. The normalizing constant k

b. The expected policy amount

c. The median of the distribution of amounts of insurance.

15.10. A type of whole life policy, issued on a semicontinuous basis, has the following expense allocations.

	Percentage of Expense-Loaded Premium	Per 1,000 Insurance	Per Policy
First-year	30%	3.00	10.00
Renewal	5%	0.50	2.50

a. Write formulas for the expense-loaded first-year and renewal premiums assuming that per policy expenses are matched separately by first-year and renewal policy fees.
b. Write the formula for the policy fee to be paid in each year if per policy expenses are not matched separately by first-year and renewal policy fees.

Section 15.5

15.11. The continuous analogue of (15.5.5) is the differential equation

$$\frac{d}{dt}\, {}_tp_x\, [{}_t\bar{V}(\bar{A}_x) + \bar{u}(t)] = {}_tp_x\, [\bar{P}(\bar{A}_x) + \delta_t\, \bar{V}(\bar{A}_x)$$

$$+ \bar{c} - \bar{e}(t) + \delta\bar{u}(t) - \mu_x(t)].$$

Using this equation, and (8.6.4) rearranged, to express

$$\frac{d}{dt}\, [{}_tp_x\, {}_t\bar{V}(\bar{A}_x)],$$

show that

$${}_tp_x\, \bar{u}(t) = \int_0^t e^{\delta(t-y)}\, {}_yp_x\, [\bar{c} - \bar{e}(y)]\, dy.$$

Section 15.6

15.12. If, in relation to (15.6.7), ${}_{10}AS_1$ is the asset share at the end of 10 years based on G_1 and ${}_{10}AS_2$ is the corresponding quantity based on G_2, write a formula for ${}_{10}AS_2 - {}_{10}AS_1$.

15.13. The continuous analogue of the difference equation (recursion relationship) in (15.6.3) is

$$\frac{d}{dt}\, {}_tp_x^{(\tau)}\, v^t\, {}_t\overline{AS} = [\bar{G}(1 - \bar{c}(t) - \bar{e}(t)]\, {}_tp_x^{(\tau)}\, v^t$$

$$- {}_tp_x^{(\tau)}\, [\mu_x^{(1)}(t) + \mu_x^{(2)}(t)\, {}_t\overline{CV}]\, v^t.$$

In this differential equation bars have been added to payment rate symbols to denote continuous payments.

a. Solve this differential equation and use the initial condition ${}_0\overline{AS} = 0$ to obtain the continuous version of (15.6.5):

$${}_t\overline{AS} = \left\{\int_0^t [\bar{G}(1 - \bar{c}(s)) - \bar{e}(s)]\, {}_sp_x^{(\tau)}\, v^s\, ds\right.$$

$$\left. - \int_0^t {}_sp_x^{(\tau)}\, [\mu_x^{(1)}(s) + \mu_x^{(2)}(s)\, {}_s\overline{CV}]\, v^s\, ds\right\} \Big/ (v^t\, {}_tp_x^{(\tau)}).$$

b. If the policy is a continuous model whole life policy, ${}_0\overline{AS} = {}_{\omega-x}\overline{AS} = 0$, show that

$$G\bar{a}_x^{(\tau)} = \bar{A}_x^{(1)} + \int_0^\infty [\bar{G}\bar{c}(s) + \bar{e}(s) + \mu_x^{(2)}(s)\, {}_s\overline{CV}]\, {}_sp_x^{(\tau)}\, v^s\, ds.$$

15.14. a. Start with (8.2.7) and derive the single decrement, without provision for expenses, version of (15.7.2):

$$
\begin{aligned}
{}_t\bar{V} &= \int_0^\infty v^u \left\{ b_{t+u}\, \mu_x(t+u) - \pi_{t+u}\right]\, {}_u p_{x+t} \bigg\}\, du \\
&= \int_0^\infty v^u f(u{:}t)\, du = \int_0^t (1+i)^{t-\mu}\, g(u{:}\ 0)\, \frac{du}{{}_t p_x}.
\end{aligned}
$$

In solving this exercise the assumption should be made that an aggregate mortality table is used as it was in Sections 7.2 and 7.3.

b. At time t the future force of interest changes from δ to δ'. No change in the distribution of time to death is expected because of this change in interest rates, and the insurance contract prohibits a change in π_{t+u}. The symbol ${}_t\bar{V}'$ denotes a reserve valued at rate δ'. Exhibit a formula for

$$
{}_t\bar{V}' - {}_t\bar{V}.
$$

[If the change from δ to δ' is related to observed changes in capital market interest rates, the difference ${}_t\bar{V}' - {}_t\bar{V}$ might be used to estimate the change in the market value of the reserve liability.]

16

BUSINESS AND REGULATORY CONSIDERATIONS

16.1 Introduction

In Chapter 15 a major step is taken toward bringing the models for life insurance developed in earlier chapters into accordance with business reality. Operating expenses and withdrawal benefits are introduced, and the implications of these new elements for premiums, reserves, and financial reports are illustrated.

Several topics are presented in this chapter that extend the theme of Chapter 15. Two basic ideas provide a thread that connects them.

The first idea is the development of single decrement models that approximate the results of more comprehensive multiple decrement models with expenses to a sufficient degree of accuracy for a particular purpose. For example, in many jurisdictions such approximate models are used in defining regulations. The motivation for these approximations comes from the perceived conceptual and computational complexity of multiple decrement models that incorporate expenses. With current computing capabilities, this motivation is reduced.

The second of these basic ideas comes from economics. Those who supply the capital to start and stabilize insurance enterprises expect to be rewarded for their investment. Contract premiums need to have an expected profit component that is related to the risk assumed by the investors. If the insurance organization developed from a mutual or cooperative endeavor, the corresponding issue is how to return favorable financial experience in an equitable fashion to the members. To address these issues, ideas must be drawn from economics.

16.2 Cash Values

Sections 11.4 and 15.3 are devoted to benefits paid on withdrawal and the premium and financial reporting consequences of these benefits. Withdrawal benefits are also called ***nonforfeiture benefits***, because they cannot be lost as a result of the premature cessation of premium payments.

The determination of premiums and reserves has as a prerequisite the adoption of a principle. Likewise, in the determination of nonforfeiture benefits, a guiding principle is required. In this section we adopt a simple operational principle that is close to the one adopted in United States insurance regulation. The principle is that the withdrawing insured receives a value such that the benefit, premium, and reserve structure based on the single decrement model remains appropriate in the multiple decrement context. Adoption of this principle also permits the use of a less complex model for regulatory purposes.

This principle is motivated by a particular concept of equity involving the treatment of the two classes of policyholders, those who terminate before the basic insurance contract is fulfilled and those who continue. Clearly several concepts of what constitutes equity are possible, ranging from the view that terminating policyholders have not fulfilled the contract and are therefore not entitled to nonforfeiture benefits, to the view that a terminating policyholder should be returned to his original position by the return of the accumulated value of all premiums, perhaps less an insurance charge. The concept of equity, illustrated in this section, is an intermediate one; that is, withdrawing life insurance policyholders are entitled to nonforfeiture benefits, but these benefits should not force a change in the price-benefit structure for continuing policyholders.

The development in Section 11.4 does not include consideration of expenses and corresponding premium loadings. Therefore, if the general principle stated in that section is adopted for determining the value, at the time of premium default, of nonforfeiture benefits, some allowance needs to be made for these missing factors. An approximate method for adjusting benefit reserves for initial expenses not yet recovered from premium loadings, and for the risk of withdrawal at a financially inopportune time for the insurer, is to define

$$ {}_kCV = {}_kV - {}_kSC. \tag{16.2.1} $$

In (16.2.1), ${}_kCV$ is the ***cash value*** of nonforfeiture benefits, ${}_kV$ is the terminal reserve, and ${}_kSC$ is the ***surrender charge,*** all at time $k = 1, 2, 3, \ldots$ following policy issue. Because of the difficulty of collecting additional funds from a withdrawing policyholder, ${}_kCV \geq 0$.

The cash values defined in (16.2.1) are the basis of what are called withdrawal benefits in Sections 11.4 and 15.3. In the earlier sections, these were denoted by $B_{x+t}^{(2)}$. These nonforfeiture benefits need not be paid in cash but are the basis for actuarially equivalent insurance benefits as described in Section 16.3.

A continuing theme in the regulation of the cash value of nonforfeiture benefits has been the need for direct recognition of the amount and incidence of expenses. One idea, in accord with this theme, is to define a minimum cash value for a unit insurance as

$$ {}_kCV = A(k) - P^a\, \ddot{a}(k) $$
$$ = {}_kV - (P^a - P)\, \ddot{a}(k). \tag{16.2.2} $$

Here $A(k)$ and $\ddot{a}(k)$ are, respectively, actuarial present-value insurance and annuity symbols appropriate for time k, $k = 1, 2, 3, \ldots$ following policy issue, ${}_kV$ denotes a terminal reserve at the same time, P is an annual benefit premium, and P^a is called an ***adjusted premium.*** The symbols $A(0)$ and $\ddot{a}(0)$ will be abbreviated to A and $\ddot{a}$, respectively. The regulatory problem becomes that of defining adjusted premiums.

The 1975 report of the Society of Actuaries committee studying nonforfeiture benefits and related matters contained consideration of two types of expenses in defining adjusted premiums. First is a level amount per unit of insurance, denoted by E, incurred each year throughout the premium paying period. Second is an additional expense for the first year of amount E_0. The contract premium rate G is assumed to be composed of an adjusted premium and the level annual expense component E. The first-year expense component E_0 is assumed to be provided by the adjusted premium. That is,

$$ G = P^a + E, \tag{16.2.3A} $$

$$ G\ddot{a} = (P^a + E)\ddot{a} = A + E_0 + E\ddot{a}. \tag{16.2.3B} $$

From (16.2.3B) we obtain

$$ P^a = \frac{A + E_0}{\ddot{a}}. \tag{16.2.4} $$

Formula (16.2.4) can be rewritten, by substituting $\ddot{a} = a + 1$, as

$$ P^a - E_0 + P^a\, a = A. \tag{16.2.5} $$

Example 16.2.1

The 1980 National Association of Insurance Commissioners (NAIC) Standard Nonforfeiture Law adopted (16.2.2) and (16.2.4) to define minimum cash values. The law specified that for policies with level benefits and contract premiums $E_0 = 1.25 \min(P, 0.04) + 0.01$, where P denotes the benefit premium rate per unit of death benefit for the policy. Exhibit the provision for first-year expenses E_0 and the corresponding adjusted premium if (a) $P < 0.04$ and (b) $P \geq 0.04$.

Solution:

	E_0	P^a
a. $P < 0.04$	$1.25P + 0.01$	$\dfrac{A + 1.25P + 0.01}{\ddot{a}}$
b. $P \geq 0.04$	0.06	$\dfrac{A + 0.06}{\ddot{a}}.$

Example 15.3.1 can be interpreted as using (16.2.2) and (16.2.4) to determine cash values with $E_0 = 0.04$. ▼

In this section we have discussed the framework for defining minimum cash values used in many jurisdictions. The method is another example of a single decrement model, with somewhat arbitrary assumptions about the amount and time incidence of expenses, being used as an approximation to a more comprehensive double decrement model. In the history of nonforfeiture value regulation, changes in the general framework have been infrequent.

For a jurisdiction to complete the definition of minimum cash values, the interest rate and the life table to be used must be specified. Legislative changes to update the interest and mortality bases of minimum cash values have occurred more frequently than have changes to the general framework. The 1980 NAIC law provides for revisions of the maximum interest rate according to a formula based, in part, on an index of average interest rates prevailing during a period of time before policy issue.

Some of the modifications required when contract premiums or benefits are not level are discussed in Section 16.9. These modifications are covered with the closely related modifications required to adapt regulatory reserve liability standards to nonlevel contract premiums or benefits.

Because the original contract was one of insurance, there is a view that at least one of the nonforfeiture benefits should also be insurance. In this view, the cash value is a device to define the new insurance benefit. These insurance options are discussed in Section 16.3. Also, cash values form the basis of another important policy provision, the policy loan clause. This provision provides that the insurer will grant, on the security of the policy's cash value, a loan not greater than the cash value. At one time the interest rate on such loans was stated in the policy. In response to volatile interest rates, however, there has been a move to link policy loan interest rates to some market interest rate appropriate at the time the loan is made. Upon settlement of the policy on death, maturity, or surrender for cash, the outstanding indebtedness is subtracted from the proceeds.

16.3 Insurance Options

Cash values are available as a lump sum or as an insurance benefit of equal actuarial present value. Three common insurance benefits are discussed here.

16.3.1 Paid-up Insurance

The equivalence principle is used to determine the reduced amount of paid-up insurance according to the benefit provision in the policy. If premium default occurs at time k, as measured from policy issue, the general equation for the amount of paid-up insurance available, denoted by b_k, is

$$ {}_kCV = b_k A(k), $$

$$ b_k = \frac{{}_kCV}{A(k)} \tag{16.3.1} $$

where ${}_kCV$ is the cash value available and $A(k)$ is the actuarial present value for a unit of future benefits under the policy at time k. In practice, various elaborations on the symbol $A(k)$ indicate, if appropriate, that continuous payment of claims and various term and endowment benefits are required.

For a unit of insurance and in the special case when ${}_kCV = {}_kV$, where ${}_kV$ is a benefit reserve, (16.3.1) may be rewritten to provide additional insight. Some of these ideas were developed in connection with (7.3.2) and (7.4.6). In this special case, the symbol ${}_kW = b_k = {}_kV / A(k)$ is used to denote the amount of paid-up insurance. In Table 16.3.1 some of the relationships between ${}_kW$ and other actuarial quantities are listed. Additional relationships of this type are called for in Exercises 16.7 and 16.8.

There is a general reasoning argument that leads to the results in Table 16.3.1. At age $x + k$ the benefit premium for a whole life insurance of 1 is P_{x+k}. Thus an annual premium of P_x, payable commencing at age $x + k$, would be sufficient to provide insurance of only P_x / P_{x+k}. Since P_x is the benefit premium actually paid for a unit of insurance, the difference at time k, $1 - P_x / P_{x+k}$, must be provided by the reserve. This reasoning can be applied to the other insurances.

TABLE 16.3.1

Amounts of Reduced Paid-up Insurance

Special Case $b_k = {}_kW = {}_kV/A(k)$ — Fully Continuous Basis	Fully Discrete
Whole Life	
${}_k\bar{W}(\bar{A}_x) = \dfrac{\bar{A}_{x+k} - \bar{P}(\bar{A}_x)\,\bar{a}_{x+k}}{\bar{A}_{x+k}}$	${}_kW_x = \dfrac{A_{x+k} - P_x\,\ddot{a}_{x+k}}{A_{x+k}}$
$= 1 - \dfrac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+k})}$	$= 1 - \dfrac{P_x}{P_{x+k}}$
n-Payment Life (k < n)	
${}^n_k\bar{W}(\bar{A}_x) = \dfrac{\bar{A}_{x+k} - {}_n\bar{P}(\bar{A}_x)\,\bar{a}_{x+k:\overline{n-k}\rvert}}{\bar{A}_{x+k}}$	${}^n_kW_x = \dfrac{A_{x+k} - {}_nP_x\,\ddot{a}_{x+k:\overline{n-k}\rvert}}{A_{x+k}}$
$= 1 - \dfrac{{}_n\bar{P}(\bar{A}_x)}{{}_{n-k}\bar{P}(\bar{A}_{x+k})}$	$= 1 - \dfrac{{}_nP_x}{{}_{n-k}P_{x+k}}$
n-Year Endowment (k < n)	
${}_k\bar{W}(\bar{A}_{x:\overline{n}\rvert}) = \dfrac{\bar{A}_{x+k:\overline{n-k}\rvert} - \bar{P}(\bar{A}_{x:\overline{n}\rvert})\,\bar{a}_{x+k:\overline{n-k}\rvert}}{\bar{A}_{x+k:\overline{n-k}\rvert}}$	${}_kW_{x:\overline{n}\rvert} = \dfrac{A_{x+k:\overline{n-k}\rvert} - P_{x:\overline{n}\rvert}\,\ddot{a}_{x+k:\overline{n-k}\rvert}}{A_{x+k:\overline{n-k}\rvert}}$
$= 1 - \dfrac{\bar{P}(\bar{A}_{x:\overline{n}\rvert})}{\bar{P}(\bar{A}_{x+k:\overline{n-k}\rvert})}$	$= 1 - \dfrac{P_{x:\overline{n}\rvert}}{P_{x+k:\overline{n-k}\rvert}}$

16.3.2 Extended Term

The equivalence principle is used to determine the length of a paid-up term insurance for the full amount of the policy. The equation to be solved for s, in connection with an insurance with unit amount, is

$$ {}_kCV = \bar{A}^{\,1}_{x+k:\overline{s}|}. \tag{16.3.2}$$

In the case of an endowment insurance, it may be that $s > n - k$, the remaining time to maturity. In that case the amount of cash value not used to purchase paid-up term insurance is used to buy a pure endowment of amount

$$\frac{{}_kCV - \bar{A}^{\,1}_{x+k:\overline{n-k}|}}{A_{x+k:\overline{n-k}|}^{\;\;\;\;\;\;\;\;1}}. \tag{16.3.3}$$

If a policy of amount b is subject to an outstanding policy loan of amount L at the time of premium default, life insurance policies usually provide that the extended-term insurance will be for amount $b - L$. Without this provision a policyholder with a policy loan of amount L and current death benefit of $b - L$ could, by the act of premium default, increase the amount of insurance to b. In the case of a policy loan of amount L, (16.3.2) is modified to

$$b\ {}_kCV - L = (b - L)\ \bar{A}^{\,1}_{x+k:\overline{s}|}.$$

If the terminating insured can elect either a reduced paid-up or an extended-term insurance equivalent, the insurer has granted an option to the terminating insured. Those in good health tend to elect reduced paid-up insurance, and those in impaired health extended term. To compensate for the cost of this option, some insurers use a life table with higher mortality in determining $A^{\,1}_{x+k:\overline{s}|}$ in (16.3.2) than is used in determining $A(k)$ in (16.3.1).

The basic idea is that the transition of a policyholder from one status defined in the policy to another is information that should be used in determining the conditional distribution of time until death following the transition. This idea also appears in the discussion of accelerated benefits in Section 17.7 and of last-survivor insurance in Section 18.9.

Example 16.3.1

A semicontinuous whole life insurance with a 100,000 death benefit is issued to (40). On the basis of the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.06$, determine the following as of duration 10:

a. The minimum nonforfeiture value according to the 1980 NAIC law.
b. If the policy cash value is 8,700, the length of the extended term insurance period.
c. Repeat part (b) with a policy loan of 5,000 outstanding.

Solution:

[The following calculations are done on the Illustrative Life Table spreadsheet constructed for the computing exercises.] Use of the Illustrative Life Table for *both* the original pricing and the determination of the length of the period extended term insurance available at termination may not be realistic, as the future lifetime of an insured electing the extended term option may be governed by higher mortality rates than the future lifetime of a newly insured life.

a. $P(\bar{A}_{40}) = \bar{A}_{40} / \ddot{a}_{40} = (i/\delta)A_{40}/\ddot{a}_{40} = 0.0112\ 11537$
$E_0 = 1.25\ \text{Min}[P(\bar{A}_{40}), 0.04] + 0.01 = 0.0240\ 14421$
$P^a = P(\bar{A}_{40}) + E_0/\ddot{a}_{40} = 0.0128\ 32315$
Min Cash Value @ 10 = 8,620.2247
Benefit Reserve @ 10 = 10,770.4823.

b. From (16.3.2) we seek an s such that

$$8{,}700 = 100{,}000\bar{A}^{1}_{50:\overline{s}|}$$

or

$$0.08700 = \bar{A}^{1}_{50:\overline{s}|}.$$

To accommodate the assumption of a uniform distribution of deaths within each year of age, we start by searching for the integer $\lfloor s \rfloor$ such that

$$\bar{A}^{1}_{50:\overline{\lfloor s\rfloor}|} < 0.08700 < \bar{A}^{1}_{50:\overline{\lfloor s\rfloor+1}|}.$$

From the Illustrative Life Table we have

$$\bar{A}^{1}_{50:\overline{13}|} = 0.083094960$$

and

$$\bar{A}^{1}_{50:\overline{14}|} = 0.090194845.$$

[Hint: See (b), page 119.]

Thus,

$$0.087 = \bar{A}^{1}_{50:\overline{s}|} = 0.083094960 + v^{13}\ {}_{13}p_{50} \int_0^{s-13} q_{63}\, v^t\, dt \text{ and}$$

$$\frac{0.003905040(1.06)^{13}}{{}_{13}p_{50}\ q_{63}} = \frac{1 - v^{s-13}}{\log(1.06)},$$

$$v^{s-13} = 0.968948162 \text{ and}$$

$$s = 13 + 0.541355012.$$

This would probably be rounded up to 13 years and 198 days.

c. We now seek an s such that

$$(8{,}700 - 5{,}000) = (100{,}000 - 5{,}000)\bar{A}^{1}_{50:\overline{s}|}$$

or

$$0.038947368 = \bar{A}^{1}_{50:\overline{s}|}.$$

Proceeding as in part (b), $s = 6.444084519$, which rounds up to 6 years and 163 days. ▼

16.3.3 Automatic Premium Loan

Another policy provision, one that is not classified as a nonforfeiture benefit by all actuaries, is the automatic premium loan. This provision keeps the policy in full force, if premium default occurs, for as long as the cash value is greater than the balance of the policy loan, which will be increasing because of interest and unpaid premiums added to the balance. The option to restore the policy to full force without a loan is available unilaterally to the insured under this option. Because the restoration to full force under the other nonforfeiture options typically requires the payment of accumulated due contract premiums and, often, evidence of insurability, there is lack of agreement on the classification of this benefit.

For default at time k, on a policy with a continuous payment basis for unit amount, the maximum length of the premium loan period would be determined by solving

$$G\,\bar{s}_{\overline{t}|i} = {}_{k+t}CV \tag{16.3.4}$$

for t. In (16.3.4),

- G is the contract premium per unit of insurance
- ${}_{k+t}CV$ is the cash value per unit of insurance and
- i is the policy loan interest rate.

In practice, t is sometimes taken as an integer such that

$$G\ddot{s}_{\overline{t}|i} \leq {}_{k+t}CV$$

and

$$G\ddot{s}_{\overline{t+1}|i} > {}_{k+t+1}CV.$$

The remaining cash value, ${}_{k+t}CV - G\ddot{s}_{\overline{t}|i}$, is used to buy extended-term insurance.

Example 16.3.2

A fully continuous whole life insurance for unit amount issued to (x) is changed to a nonforfeiture benefit at the end of k years.

a. If ${}_kCV = {}_k\bar{V}(\bar{A}_x)$, express the ratio of the variance of the future loss associated with the changed insurance, immediately after the change, to the variance of the future loss at duration k on the original insurance if the nonforfeiture benefit is
   (i) Paid-up insurance
   (ii) Extended-term insurance.

b. If $x = 35$ and $k = 10$, calculate the ratios in parts [a-(i)] and [a-(ii)] on the basis of the Illustrative Life Table with interest at 6%. (See Exercise 6.10 and Exercise 7.23.)

Solution:

a. (i) By (7.2.5), the variance before the change is

$$\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_{x+k} - (\bar{A}_{x+k})^2] = \frac{{}^2\bar{A}_{x+k} - (\bar{A}_{x+k})^2}{(1 - \bar{A}_x)^2}.$$

For the paid-up insurance the loss variable is

$${}_k\bar{W}(\bar{A}_x)\, v^{T(x)-k} - {}_k\bar{V}(\bar{A}_x),$$

which has variance

$$[{}_k\bar{W}(\bar{A}_x)]^2\, [{}^2\bar{A}_{x+k} - (\bar{A}_{x+k})^2].$$

The ratio of this to the variance before the change is

$$[{}_k\bar{W}(\bar{A}_x)]^2\, (1 - \bar{A}_x)^2,$$

which is less than 1.

(ii) In the case of extended-term insurance, we have from (16.3.2) and (4.2.5) that the variance after the change is

$${}^2\bar{A}^{\,1}_{x+k:\overline{s|}} - (\bar{A}^{\,1}_{x+k:\overline{s|}})^2.$$

The ratio of the variances is

$$\frac{[{}^2\bar{A}^{\,1}_{x+k:\overline{s|}} - (\bar{A}^{\,1}_{x+k:\overline{s|}})^2]\,(1 - \bar{A}_x)^2}{{}^2\bar{A}_{x+k} - (\bar{A}_{x+k})^2}.$$

b. (i) ${}_{10}\bar{W}(\bar{A}_{35}) = {}_{10}\bar{V}(\bar{A}_{35}) / \bar{A}_{45} = 0.08604 / 0.20718 = 0.41529$

$\bar{A}_{35} = 0.13254$

$[0.41529\,(1 - 0.13254)]^2 = 0.13.$

The variance of the loss on the paid-up insurance is 13% of the variance of the loss at duration ten on the original insurance.

(ii) ${}_{10}\bar{V}(\bar{A}_{35}) = 0.08604\, A^{1}_{45:\overline{s|}}$

yields a value of s between 19 and 20. With $s = 19$,

$$\frac{{}^2\bar{A}^{1}_{45:\overline{19|}} - (\bar{A}^{1}_{45:\overline{19|}})^2}{{}^2\bar{A}_{45} - (\bar{A}_{45})^2}\,(1 - \bar{A}_{35})^2 = \frac{0.04308}{0.02922}\,(0.86746)^2 = 1.11.$$

Since s is between 19 and 20, the variance has increased to approximately 111% of what it was before the cessation of premium payments. ▼

16.4 Premiums and Economic Considerations

Sections 15.2.1 and 15.3.1 develop ideas about introducing expected expenses into premium formulas by applying the equivalence principle. The resulting premiums are described as expense-loaded premiums, but they also could be called expected-cost premiums. In Sections 15.2.2 and 15.3.2 a provision for contingencies and profits is introduced by adding a flat extra amount to the expense-loaded premium. The formulation of profit objectives and the inclusion of these objectives into the premium determination process are not discussed.

In this section four premium determination methods, which can be stated in terms of the models that incorporate expenses, are reviewed. The four methods are studied in an order related to increasing economic sophistication of the profit objective.

16.4.1 Natural Premiums

In determining the contract premium, a method called ***the natural premium and reserve method*** can be used to fix premiums and cash values in one coordinated process. The method starts with the calculation of a set of expense-loaded premiums and reserves using the single decrement model as is done in Table 15.2.4. The withdrawal benefits are then set equal to the total reserves, benefit plus expense reserves, and a profit component is added to the expense-loaded premium to produce a contract premium. The method relies on the fact that withdrawal benefits determined in this fashion will have a small impact on premiums and reserves determined within a double decrement model recognizing expenses. Of course, the resulting withdrawal benefits must be checked to confirm that they satisfy regulatory minimum values. The terms natural premiums and natural reserves have been used with a different meaning in some actuarial applications. Section 15.3 provides insights into this method.

16.4.2 Fund Objective

Formula (15.6.6) is described there as a general formula for an expense-loaded premium determined by the equivalence principle. The developments of Section 15.6.1 can be extended to provide a method for introducing an expected profit objective component to produce a contract premium.

In this extension, management sets an asset share goal, $K > {}_{20}V$, for instance. If expected expense and cash value obligations are known, (15.6.5) can be used to determine G, the contract premium, using the following development. Let a trial value of the contract premium, denoted by H, be selected arbitrarily, and let ${}_{20}AS$ be the result of using (15.6.5) with this H and $n = 20$:

$$\begin{aligned} {}_{20}AS = \Bigg\{ \sum_{h=1}^{20} & [H(1 - c_{h-1}) - e_{h-1}]\, l_{x+h-1}^{(\tau)} (1+i)^{21-h} \\ & - (d_{x+h-1}^{(1)} + d_{x+h-1}^{(2)}\, {}_hCV)(1+i)^{20-h} \Bigg\} \Bigg/ l_{x+20}^{(\tau)}. \end{aligned} \tag{16.4.1}$$

The fund goal is K, which should be the result of applying (16.4.1) with G replacing H. Then by subtraction,

$$K - {}_{20}AS = \sum_{h=1}^{20} \frac{(G-H)(1 - c_{h-1}) l_{x+h-1}^{(\tau)} (1+i)^{21-h}}{l_{x+20}^{(\tau)}},$$

and the desired contract premium is given by

$$G = H + \frac{(K - {}_{20}AS)\ {}_{20}p_x^{(\tau)}\ v^{20}}{\sum_{h=1}^{20} (1 - c_{h-1})\ {}_{h-1}p_x^{(\tau)}\ v^{h-1}}. \tag{16.4.2}$$

The effect of the second term in (16.4.2) is to produce a correction to the initial premium H that will cause the fund goal K to be reached in an actuarial present-value sense. The denominator of the second term can be interpreted as the actuarial present value of the increase in profit of a unit increase in the contract premiums.

16.4.3 Rate of Return Objective

Profit objectives in business are often stated in terms of a rate of return on the investment in the business. One view is that the principal investment made in creating an insurance enterprise is in developing a distribution system. An expected rate of return on the investment in the distribution system, in this view, can be expressed as a fraction of the actuarial present values of the sales commissions generated by the distribution system. In addition, this view also recognizes that because of the time incidence of expenses, the insurance enterprise must finance the acquisition expenses when a policy is issued, with the expectation that the policy will yield a future stream of returns.

To incorporate these ideas into a pricing method we start with the asset share model developed in Section 15.6. One necessary change is to separate the percentage expense term, $c_h = g_h + t_h$, where g_h is the commission rate paid and t_h combines other percentage expenses such as premium taxes.

In Table 15.3.4 the financial operations of a very small life insurance enterprise are traced assuming no deviations from expected results. The recursion relationship used to generate Tables 15.3.4 and 15.3.5 is closely related to asset share formula (15.6.7). We rewrite (15.6.7) assuming that after each year the difference ${}_hAS' - {}_hV$ is recognized as book profit or loss, to be denoted by ${}_hBP$, and the initial asset share for policy year $h + 1$ will be set as ${}_hV$. The symbol ${}_hV$ denotes a reserve liability fixed in advance. The prime has been added to the asset share symbol as a reminder that ${}_hBP$ is extracted from the expected fund progress as a profit.

Because ${}_hBP$ is measured at the end of the policy year for each insured who entered the year, we have

$$\begin{aligned} {}_hBP = {}_hAS' - {}_hV = \{[{}_{h-1}V + H(1 - g_{h-1} - t_{h-1}) - e_{h-1}](1 + i) \\ - q_{x+h-1}^{(1)}(1 - {}_hV) - q_{x+h-1}^{(2)}({}_hCV - {}_hV)\} - {}_hV. \end{aligned}$$

The book profits (${}_hBP$) are computed using an arbitrary initial premium H, which could be taken as the benefit premium; ${}_hBP$ can be negative. This is likely especially in early policy years.

The actuarial present values of book profits at the time of issue are given by

$$v_j^h\ {}_{h-1}p_x^{(\tau)}\ {}_hBP = v_j\ {}_{h-1}E_x^*\ {}_hBP \quad h = 1, 2, 3, \ldots$$

where v_j is valued at rate of interest j, the rate of return desired on the investment in new business. The asterisk has been added to the pure endowment symbol as a reminder that it is valued using a double decrement model and interest rate j.

For the initial choice of premium H, the actuarial present value at issue of book profits to the investor in the insurance enterprise is

$$Z = \sum_{1}^{\infty} v_{j\ h-1}E_x^*\ {}_hBP.$$

The actuarial present value at issue of future sales commissions based on the initial choice of premiums is

$$X = H \sum_{h=1}^{\infty} g_{h-1\ h-1}E_x^*,$$

and the actuarial present value at issue of increased book profits to the investor for each unit increase in premium H is

$$Y = \sum_{h=1}^{\infty} (1 - g_{h-1} - t_{h-1})(1 + i)v_{j\ h-1}E_x^*.$$

The quantity Y is closely related to the denominator in (16.4.2). The difference is that Y measures the impact of a unit change in premium over the entire policy period, while in (16.4.2), the impact is measured only for 20 years, at the end of which time a fund objective is to be met.

We let G, as before, denote the contract premium that meets the economic goal and assume that $G > H$. Two modified values are introduced. Let

$$Z' = Z + (G - H)Y \tag{16.4.3}$$

denote a modified value of the actuarial present value of book profits to the investor and

$$X' = \frac{G}{H} X \tag{16.4.4}$$

denote a modified value of the actuarial present value of future sales commissions.

The economic goal can be described as

$$Z' = bX' \tag{16.4.5}$$

where b is the required profit rate on the investment in the distribution system. Then, from (16.4.3)

$$(G - H)Y = Z' - Z$$

and using (16.4.4) and (16.4.5), we have

$$Z + (G - H)Y = bX' = b\frac{G}{H}X. \tag{16.4.6}$$

Formula (16.4.6) has an economic interpretation. The right-hand side of the equation states the economic goal. The left-hand side has two terms. The first term (Z)

is the actuarial present value of book profits with the initial choice of contract premium H. The second term, $(G - H)Y$, is the increase in the actuarial present value of book profits from increasing the contract premium by $G - H$.

Solving (16.4.6) for G yields

$$G = \frac{H^2Y - HZ}{HY - bX},$$

which will meet the rate of return requirement j on investment in new business and the profit rate on commissions b, which is related to the investment in the distribution system.

16.4.4 Risk-Based Objectives

Another justification of profit loadings is that the investor should be compensated directly for accepting risks. As in Section 15.3, let ${}_0L_e$ denote the expense augmented loss variable at issue associated with an insurance policy in a double decrement model. Then $\mathrm{Var}({}_0L_e)$ and $\sqrt{\mathrm{Var}({}_0L_e)}$ would be possible measures of the risk associated with the policy. Let the expense-loaded premium determined by the equivalence principle be denoted by P'. The contract premium, with a risk-adjusted profit loading, might be

$$P' + c\,\mathrm{Var}({}_0L_e) \qquad c > 0 \tag{16.4.7}$$

or

$$P' + k\sqrt{\mathrm{Var}({}_0L_e)} \qquad k > 0. \tag{16.4.8}$$

The premium in (16.4.7) involves the ***variance premium principle***, and (16.4.8) involves the ***standard deviation premium principle***.

$\mathrm{Var}({}_0L_e)$ measures the risk resulting from the uncertainty of the time and the cause of decrement. Because of the importance of investment rate risk in long-term insurances, some actuaries would either limit the application of the variance or standard deviation premium principles to short-term policies or incorporate random interest into the loss variables.

If c and k are both constant, then the standard deviation premium principle has the appeal that both the expense-loaded premium and the contract premium are in the same monetary units. If, however, c is a function of the risk in terms of $(\text{monetary unit})^{-1}$, then the expense-loaded premium and the contract premium are both in the same monetary units under the variance premium principle.

An appealing property of the variance premium principle can be demonstrated by considering two independent expense-augmented loss variables, denoted by ${}_0L_e(1)$ and ${}_0L_e(2)$. Then,

$$c\,\mathrm{Var}[{}_0L_e(1) + {}_0L_e(2)] = c\,\mathrm{Var}[{}_0L_e(1)] + c\,\mathrm{Var}[{}_0L_e(2)]. \tag{16.4.9}$$

On the other hand,

$$k\sqrt{\mathrm{Var}[{}_0L_e(1)] + \mathrm{Var}[{}_0L_e(2)]} < k\sqrt{\mathrm{Var}[{}_0L_e(1)]} + k\sqrt{\mathrm{Var}[{}_0L_e(2)]}.$$

The relationship in (16.4.9) is called the ***additive property*** of the variance premium principle. The additive property means that the premium for a sum of independent risks is the sum of their individual premiums. The profit and contingencies loading for a collection of independent policies is the sum of the loadings for the individual policies. Exercise 16.11 shows that a simple proportional profit and contingencies loading also has the additive property.

Example 16.4.1

Use the results in Table 15.3.2 to determine contract premiums by (a) the variance principle with $c = 0.05$ and (b) the standard deviation principle with $k = 1.0$.

Solution:

a. $332.96 + 0.05(224.25) = 344.17$
b. $332.96 + 1.0\sqrt{224.25} = 347.93.$ ▼

16.5 Experience Adjustments

The set of asset shares as computed using (15.6.7), before a block of policies is issued, will almost certainly not equal the assets per surviving insured developed by experience. Nevertheless, the formulas of Section 15.6 can be used to gain insights into sources of financial gain or loss, measured with respect to expected results. Suppose that we trace the progress of ${}_kAS$ to ${}_{k+1}\widehat{AS}$ where the hat (circumflex) indicates that the $(k+1)$-st asset share is derived from the expected k-th asset share using experience cost factors. A cost factor based on experience will have a hat added. In particular, $\hat{i}_{k+1}$ is the experience interest rate earned over the $(k+1)$-st policy year. The experience asset share will be given by

$$\begin{aligned}{}_{k+1}\widehat{AS} &= [{}_kAS + G(1 - \hat{c}_k) - \hat{e}_k](1 + \hat{i}_{k+1}) - \hat{q}^{(1)}_{x+k}\,(1 - {}_{k+1}\widehat{AS}) \\ &\quad - \hat{q}^{(2)}_{x+k}\,({}_{k+1}CV - {}_{k+1}\widehat{AS}).\end{aligned} \tag{16.5.1}$$

Subtracting (15.6.7), which traced the expected progress of the asset share, from (16.5.1) yields

$$\begin{aligned}
&\text{a. } {}_{k+1}\widehat{AS} - {}_{k+1}AS = ({}_kAS + G)(\hat{i}_{k+1} - i) \\
&\text{b. } \qquad + [(G\,c_k + e_k)(1 + i) - (G\,\hat{c}_k + \hat{e}_k)(1 + \hat{i}_{k+1})] \\
&\text{c. } \qquad + [q^{(1)}_{x+k}\,(1 - {}_{k+1}AS) - \hat{q}^{(1)}_{x+k}\,(1 - {}_{k+1}\widehat{AS})] \\
&\text{d. } \qquad + [q^{(2)}_{x+k}\,({}_{k+1}CV - {}_{k+1}AS) - \hat{q}^{(2)}_{x+k}\,({}_{k+1}CV - {}_{k+1}\widehat{AS})].
\end{aligned} \tag{16.5.2}$$

In (16.5.2) the total deviation between the experience asset share and the expected asset share has been allocated into four components. Component (a) is associated with the deviation between the experience and the assumed interest rates. Component (b) is the difference between experience expenses and expected expenses, with an interest adjustment. Component (c) is the difference between assumed and experience mortality costs, and component (d) is the difference between assumed and experience withdrawal costs.

Participating life insurance is based on the principle that premiums are set at a level such that the probability is very low that premiums and associated investment income will be insufficient to fulfill the benefit and expense commitments implicit in the issuance of a block of policies. Under this principle, which can be viewed as an alternative to the equivalence principle, the expected value of a loss variable should be negative to generate financial margins to provide for deviations in experience unfavorable to the insurance system. As the uncertainty about experience is removed by the passage of time, the margins for adverse deviations built into the original price-benefit structure can be released and returned to the policyholders who, through higher premiums, carried the risks. These returns of funds not needed to match future risks are called ***dividends.*** A simplified version of (16.5.2) is used frequently in the analysis that leads to the determination of dividends.

For the simplified version, we start with a modification of (15.6.7). In this development ${}_kF$ will take the place of ${}_kAS$. The new symbol denotes a fund share that is no longer a consequence of the expected operation of the insurance system. Instead, the values of ${}_kF$ are amounts set in advance such that, with future premium and investment income, the block of policies under consideration has a high probability of meeting its benefit and expense obligations. Thus,

$$\begin{aligned} {}_{k+1}F = {} & [{}_kF + G(1 - c_k) - e_k](1 + i) - q_{x+k}^{(1)} (1 - {}_{k+1}F) \\ & - q_{x+k}^{(2)} ({}_{k+1}CV - {}_{k+1}F). \end{aligned} \tag{16.5.3}$$

In (16.5.3), c_k, e_k, $q_{x+k}^{(1)}$, and $q_{x+k}^{(2)}$ are typically set at levels somewhat higher than expected, and i is set at a level somewhat lower than expected in order to provide margins to cover adverse deviations. These margins result in a small probability of a need for outside funds.

Formula (16.5.4) describes the fund share progress for a unit of insurance where experience cost factors are denoted with hats; that is,

$$\begin{aligned} {}_{k+1}F + {}_{k+1}D = {} & [{}_kF + G(1 - \hat{c}_k) - \hat{e}_k](1 + \hat{i}_{k+1}) \\ & - \hat{q}_{x+k}^{(1)} (1 - {}_{k+1}F - {}_{k+1}D) \\ & - \hat{q}_{x+k}^{(2)} ({}_{k+1}CV - {}_{k+1}F - {}_{k+1}D). \end{aligned} \tag{16.5.4}$$

In (16.5.4) the dividend is denoted by ${}_{k+1}D$; it is the difference between the predetermined goal of ${}_{k+1}F$ and the fund share generated by experience. Subtracting (16.5.3) from (16.5.4) yields

$$
\begin{aligned}
&\text{a. } {}_{k+1}D = ({}_kF + G)(\hat{i}_{k+1} - i) \\
&\text{b. } \qquad + [(G\,c_k + e_k)(1 + i) - (G\,\hat{c}_k + \hat{e}_k)(1 + \hat{i}_{k+1})] \\
&\text{c. } \qquad + (1 - {}_{k+1}F)(q_{x+k}^{(1)} - \hat{q}_{x+k}^{(1)}) \\
&\text{d. } \qquad + ({}_{k+1}CV - {}_{k+1}F)(q_{x+k}^{(2)} - \hat{q}_{x+k}^{(2)}) \\
&\text{e. } \qquad + {}_{k+1}D\,(\hat{q}_{x+k}^{(1)} + \hat{q}_{x+k}^{(2)}). \qquad (16.5.5)
\end{aligned}
$$

The components of (16.5.5) may be identified with experience factors that determine their size. Thus, (a) is associated with interest, (b) with expenses, (c) with mortality, (d) with voluntary terminations, and (e) with the payment of dividends only to survivors. If ${}_{k+1}CV = {}_{k+1}F$ and dividends are paid to insureds who die and to insureds who withdraw, and if $G\,c_k + e_k$ is denoted by E_k whereas $G\,\hat{c}_k + \hat{e}_k$ is denoted by $\hat{E}_k$, then (16.5.5) can be written in a three-term form,

$$
\begin{aligned}
{}_{k+1}D = {} & ({}_kF + G)(\hat{i}_{k+1} - i) \\
& + [E_k(1 + i) - \hat{E}_k(1 + \hat{i}_{k+1})] \\
& + (1 - {}_{k+1}F)(q_{x+k}^{(1)} - \hat{q}_{x+k}^{(1)}). \qquad (16.5.6)
\end{aligned}
$$

Example 16.5.1

In Example 15.6.1 a set of asset shares was computed for the 3-year annual level premium endowment insurance. Recall that the example was designed for convenience and ease of calculation rather than realism. Make the following assumptions about experience:

$$
\begin{aligned}
{}_0F &= {}_0AS = 0 \\
{}_1F &= {}_1AS = 229.44 \\
{}_2F &= {}_2AS = 589.47 \\
\hat{i}_1 &= 0.15,\ \hat{i}_2 = 0.16 \\
\hat{q}_x^{(1)} &= 0.085,\ \hat{q}_x^{(2)} = 0.200 \\
\hat{q}_{x+1}^{(1)} &= 0.090,\ \hat{q}_x^{(2)} = 0.100 \\
c_k &= \hat{c}_k, \quad \text{for } k = 0, 1, \text{ and } 2 \\
\hat{e}_0 &= 10,\ \hat{e}_1 = 1.
\end{aligned}
$$

Calculate ${}_1D$ and ${}_2D$ using (16.5.5), assuming that dividends are paid to those who die or withdraw.

Solution:

	${}_1D$	${}_2D$
(a) Interest $({}_kF + G)(\hat{\imath} - i)$	$(0 + 342.96)(0)$	$(299.44 + 342.96)(0.01)$
+	+	+
(b) Expenses $(G\,c_k + e_k)(1 + i) - (G\,\hat{c}_k + \hat{e}_k)(1 + \hat{\imath}_{k+1})$	$[342.96(0.20) + 8](1.15) - [342.96(0.20) + 10](1.15)$	$[342.96(0.06) + 2](1.15) - [342.96(0.06) + 1](1.16)$
+	+	+
(c) Mortality $[1{,}000 - {}_{k+1}F](q^{(1)}_{x+k} - \hat{q}^{(1)}_{x+k})$	$[1{,}000 - 229.44](0.015)$	$[1{,}000 - 589.47](0.0211)$
+	+	+
(d) Withdrawal $[{}_{k+1}CV - {}_{k+1}F](q^{(2)}_{x+k} - \hat{q}^{(2)}_{x+k})$	$[227.73 - 229.44](-0.1)$	$[564.41 - 589.47] \times (0.0111)$
(a)	0.0000	5.7240
(b)	−2.3000	0.9342
(c)	11.5584	8.6623
(d)	0.1708	−0.2781
Total	9.4292	15.0424

▼

16.6 Modified Reserve Methods

Two salient ideas about financial reporting for life insurance enterprises are indicated in the extended illustrations developed in Sections 15.2.2 and 15.3.2. The first of these ideas is that cash flows, and thus economic gains, are not affected by the method selected for reporting liabilities. The second is that using reserves derived from a multiple decrement model and incorporating expenses results in a more level stream of reported annual net income than when single decrement benefit reserves are used.

In Section 16.1 it was pointed out that single decrement approximations to more comprehensive multiple decrement models incorporating expenses are frequently used, especially for regulatory purposes. In this section the use of such an approximation for financial reporting will be developed.

A general fully discrete insurance on (x) is introduced by way of (8.2.1). Later in Chapter 8 it was shown that for a fixed schedule of death benefits, it is not necessary to determine a sequence of constant benefit premiums to define the corresponding benefit reserves by the equivalence principle. Nevertheless, regulatory or economic restrictions, such as ${}_kV \geq 0$, may make some sequences of nonconstant benefit premiums infeasible. Other examples of such restrictions might be the requirement that the benefit premium used for reserve purposes be less than or equal to the contract premium and, except for the first policy year, the benefit premium be a fixed proportion of the contract premium.

For insurance policies that provide for constant benefits and constant benefit premiums, a system of notation and nomenclature has been developed to define modified reserve systems with nonconstant benefit premiums that provide a more realistic matching of premium reserves and related expenses. These modified reserve systems are constructed using a single decrement model under the assumption that nonforfeiture values have been determined so as to minimize the gains and losses due to withdrawals.

A modified reserve method is one that does not use the actuarial present value of a set of level benefit premiums as a deduction from the actuarial present value of future benefits in defining reserves. Instead, a sequence of ***step premiums*** is defined. Usually no more than three different levels are involved, although the theory would permit more steps in the path. The three premium levels are denoted by α, the first-year benefit premium; β, the benefit premium for the next $j-1$ years; and P, the level benefit premium assumed payable beyond the first j policy years. This set of premiums is constrained to have the same actuarial present value as the set of level benefit premiums so that

$$\alpha + \beta\,(\ddot{a}_{x:\overline{j}|} - 1) + P\,(\ddot{a}_{x:\overline{h}|} - \ddot{a}_{x:\overline{j}|}) = P\,\ddot{a}_{x:\overline{h}|},$$

$$\alpha + \beta\, a_{x:\overline{j-1}|} = P\,\ddot{a}_{x:\overline{j}|} \tag{16.6.1}$$

where h is the length of the premium paying period.

In the notation of Section 15.5 for level benefit premium reserves, amount c is expected to be available in the first policy year. This comes from a loaded premium of $P + c$ to offset first-year expenses. If $\alpha < P$, then $P + c - \alpha > c$ will be expected to be available, within the modified reserve accounting method, to match first-year expenses. If $\alpha < P$, a consequence is that $\beta > P$. This can be seen by rewriting (16.6.1) as

$$\alpha + \beta\, a_{x:\overline{j-1}|} = P(a_{x:\overline{j-1}|} + 1),$$

$$\beta = P + \frac{P - \alpha}{a_{x:\overline{j-1}|}}. \tag{16.6.2}$$

A second useful expression can be obtained from (16.6.1) as follows:

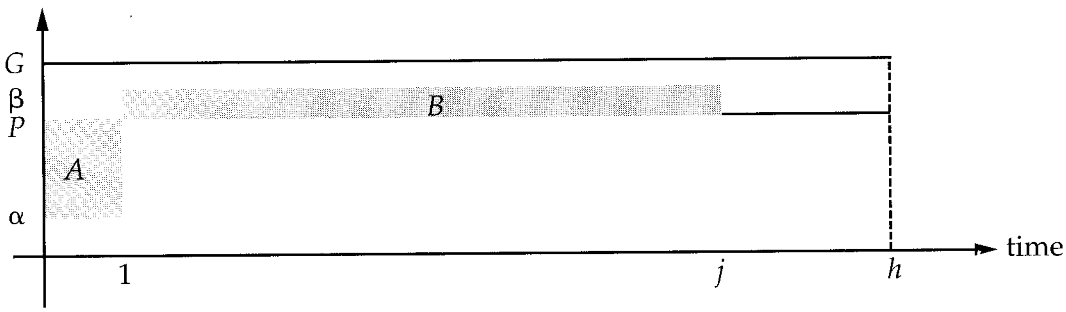
$$\beta(\ddot{a}_{x:\overline{j}|} - 1) = P\,\ddot{a}_{x:\overline{j}|} - \alpha,$$

$$\beta = P + \frac{\beta - \alpha}{\ddot{a}_{x:\overline{j}|}}. \tag{16.6.3}$$

Thus a modified reserve method for policies with constant benefits and premiums can be defined by specifying the length of the modification period j and either the first-year premium α, the renewal premium β, or the difference $\beta - \alpha$. Figure 16.6.1 provides a schematic diagram summarizing the relationship among the premiums making up a modified reserve method. The actuarial present value at issue of the premium components depicted by the shaded areas A and B are equal as can be seen from (16.6.1) rearranged as $P - \alpha = (\beta - P)a_{x:\overline{j-1}|}$.

FIGURE 16.6.1

Premiums in a Modified Reserve Method



In (16.6.1) and (16.6.2) we use the symbols α and β to denote, respectively, the first-year and renewal benefit premiums for a j-year modified reserve method and P to denote the general symbol for the level benefit premium. In a similar fashion we use the symbol V^{Mod} to denote a terminal reserve computed by a modified method.

The following formula is used to calculate terminal reserves in the fully discrete case of an h-payment, n-year endowment insurance under a modified reserve method with a j-year modification period. During the modification period, $k < j$,

$$\begin{aligned} {}^h_kV^{Mod}_{x:\overline{n}|} &= A_{x+k:\overline{n-k}|} - \beta\,\ddot{a}_{x+k:\overline{j-k}|} - {}_hP_{x:\overline{n}|}\ {}_{j-k|}\ddot{a}_{x+k:\overline{h-j}|} \\ &= A_{x+k:\overline{n-k}|} - {}_hP_{x:\overline{n}|}\,\ddot{a}_{x+k:\overline{h-k}|} - (\beta - {}_hP_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{j-k}|} \\ &= {}^h_kV_{x:\overline{n}|} - (\beta - {}_hP_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{j-k}|}\,. \end{aligned}$$

The results of Chapter 8 for a general fully discrete insurance apply to a modified reserve system as described in this section. In (8.2.1), the correspondence is $b_{K(x)+1} = 1$, $\pi_0 = \alpha$, $\pi_h = \beta$, $h = 1, 2, \ldots$. This means that Theorem 8.5.1 holds for modified benefit reserves and can be used in risk management decisions.

After duration j, reserves under the modified reserve method are equal to those under the level benefit premium method.

Example 16.6.1

Whole life insurances, on a fully continuous basis, are to have reserves determined by a modified reserve method. Under this method, the first-year and renewal benefit premium rates are $\bar{\alpha}_x$ and $\bar{\beta}_x$, $\bar{\alpha}_x < \bar{P}(\bar{A}_x)$; the modification period is the entire policy period. Define the future loss variable and write equations that may be used to evaluate reserves.

Solution:

$$
{}_tL^{Mod} = \begin{cases} v^{T(x)-t} - \bar{\alpha}_x\, \bar{a}_{\overline{T(x)-t}|} & 0 \le T(x) - t < 1 - t \\ v^{T(x)-t} - \bar{\alpha}_x\, \bar{a}_{\overline{1-t}|} - \bar{\beta}_x\, {}_{1-t|}\bar{a}_{\overline{(T(x)-t)-(1-t)}|} & T(x) - T \ge 1 - t \end{cases} \Big\}\, 0 \le t < 1 \\ \quad v^{T(x)-t} - \bar{\beta}_x\, \bar{a}_{\overline{T(x)-t}|} \qquad t \ge 1.
$$

By analogy with (7.2.3), the reserve is

$$
{}_t\bar{V}(\bar{A}_x)^{Mod} = \begin{cases} \bar{A}_{x+t} - \bar{\alpha}_x\, \bar{a}_{x+t:\overline{1-t}|} - \bar{\beta}_x\, {}_{1-t|}\bar{a}_{x+t} & 0 \le t < 1 \\ \bar{A}_{x+t} - \bar{\beta}_x\, \bar{a}_{x+t} & t \ge 1. \end{cases}
$$

In addition,

$$
{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{Mod} = [\bar{\beta}_x - \bar{P}(\bar{A}_x)]\, \bar{a}_{x+t} \qquad t \ge 1.
$$

Since we require that

$$
\bar{\alpha}_x\, \bar{a}_{x:\overline{1}|} + \bar{\beta}_x\, {}_{1|}\bar{a}_x = \bar{P}(\bar{A}_x)(\bar{a}_{x:\overline{1}|} + {}_{1|}\bar{a}_x), \tag{16.6.4}
$$

we have, similar to (16.6.2),

$$
\bar{\beta}_x = \bar{P}(\bar{A}_x) + \frac{[\bar{P}(\bar{A}_x) - \bar{\alpha}_x]\, \bar{a}_{x:\overline{1}|}}{{}_{1|}\bar{a}_x}
$$

and

$$
\bar{\beta}_x > \bar{P}(\bar{A}_x).
$$

Therefore,

$$
{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{Mod} \ge 0 \qquad t \ge 1.
$$

▼

Example 16.6.2

Using the information given in Example 16.6.1, derive a retrospective formula for ${}_t\bar{V}(\bar{A}_x)^{Mod}$.

Solution:

Consider the case where $0 \le t < 1$ and recall that

$$
\begin{aligned}
\bar{A}_x &= \bar{\alpha}_x\, \bar{a}_{x:\overline{1}|} + \bar{\beta}_x\, {}_{1|}\bar{a}_x \\
&= \bar{\alpha}_x\, (\bar{a}_{x:\overline{t}|} + \bar{a}_{x+t:\overline{1-t}|}\, {}_tE_x) + \bar{\beta}_x\, ({}_{1-t|}\bar{a}_{x+t}\, {}_tE_x).
\end{aligned}
$$

Then, using notation from Section 7.3, we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_x)^{Mod} &= \bar{A}_{x+t} - \bar{\alpha}_x\, \bar{a}_{x+t:\overline{1-t}|} - \bar{\beta}_x\, {}_{1-t|}\bar{a}_{x+t} \\
&= \bar{A}_{x+t} - \frac{\bar{A}_x - \bar{\alpha}_x\, \bar{a}_{x:\overline{t}|}}{{}_tE_x} \qquad 0 \le t < 1 \\
&= \bar{\alpha}_x\, \bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x.
\end{aligned}
$$

In addition,

$$
{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{Mod} = [\bar{P}(\bar{A}_x) - \bar{\alpha}_x]\, \bar{s}_{x:\overline{t}|} \qquad 0 \le t < 1.
$$

In the case $t \ge 1$, we recall that

$$
\bar{A}_x = \bar{\alpha}_x\, \bar{a}_{x:\overline{1}|} + \bar{\beta}_x\, ({}_{1|}\bar{a}_{x:\overline{t-1}|} + \bar{a}_{x+t}\, {}_tE_x).
$$

Then

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_x)^{Mod} &= \bar{A}_{x+t} - \bar{\beta}_x\, \bar{a}_{x+t} \\
&= \bar{A}_{x+t} - \frac{\bar{A}_x - \bar{\alpha}_x\, \bar{a}_{x:\overline{1}|} - \bar{\beta}_x\, {}_{1|}\bar{a}_{x:\overline{t-1}|}}{{}_tE_x} \\
&= \frac{\bar{\alpha}_x\, \bar{a}_{x:\overline{1}|}}{{}_tE_x} + \bar{\beta}_x\, \bar{s}_{x+1:\overline{t-1}|} - {}_t\bar{k}_x.
\end{aligned}
$$

▼

16.7 Full Preliminary Term

In order to develop a reserve method that increases the effective expense loading, $G - \alpha$, in the first policy year to better match the large first-year expenses, α is usually constrained to be less than P. However, in accordance with certain regulatory principles, there is a practical lower bound on the value of α.

This lower bound is derived from the recognition that negative reserve liabilities are effectively accounting assets. Since the collection of future contract premiums is uncertain, some regulatory agencies have not permitted such negative reserves to be included in the balance sheet in statutory assessment of insurance company solvency. Thus a practical objective of the reserve method is to avoid a negative reserve at the end of the first policy year. This means that, for a level unit benefit policy, the smallest feasible value of α will be $A^{1}_{x:\overline{1}|}$ on the fully discrete basis. This requirement was applied in Example 8.2.1. The result may be seen as follows:

$$
\begin{aligned}
{}_1V &\geq 0, \\
\alpha\, \ddot{s}_{x:\overline{1}|} - {}_1k_x &\geq 0, \qquad (16.7.1)\\
\alpha &\geq A^{1}_{x:\overline{1}|}.
\end{aligned}
$$

If α is set at the minimum level and the modification period, j, is the entire premium paying period, the resulting method is called the ***full preliminary term (FPT)*** method. Under the *FPT* method, the reserve at the end of the first policy year is 0.

For this fully discrete basis, the renewal valuation premium β may be obtained from (16.6.1) by substituting for a general h-payment level premium insurance with actuarial present value denoted by A; that is, let $A(1)$ denote the actuarial present value for an insurance issued at age $x + 1$ for the benefits remaining thereafter. Then

$$
\begin{aligned}
A^{1}_{x:\overline{1}|} + \beta\, {}_{1|}\ddot{a}_{x:\overline{h-1}|} &= P\, \ddot{a}_{x:\overline{h}|} \\
&= A \\
&= A^{1}_{x:\overline{1}|} + {}_1E_x\, A(1) \qquad (16.7.2)
\end{aligned}
$$

or

$$
\beta = \frac{{}_1E_x\, A(1)}{{}_{1|}\ddot{a}_{x:\overline{h-1}|}} = \frac{A(1)}{\ddot{a}_{x+1:\overline{h-1}|}}.
$$

In words, β is the annual benefit premium for a similar insurance issued at an age 1 year older, with premiums paid for 1 less year in the case of a limited premium paying period, and maturing at the same age as the original insurance.

For a fully continuous basis, the smallest feasible value for the first-year premium rate, $\bar{\alpha}$, is $\bar{A}^{1}_{x:\overline{1}|} / \bar{a}_{x:\overline{1}|}$, again based on avoiding a negative terminal reserve at time 1. That is,

$$
\begin{aligned}
{}_1\bar{V}(\bar{A}_x) &\geq 0, \\
\frac{\bar{\alpha}\,\bar{a}_{x:\overline{1}|}}{{}_1E_x} - {}_1\bar{k}_x &\geq 0, \qquad (16.7.3) \\
\bar{\alpha} &\geq \frac{\bar{A}^{1}_{x:\overline{1}|}}{\bar{a}_{x:\overline{1}|}}.
\end{aligned}
$$

The development for the renewal premium rate, $\bar{\beta}$, corresponds closely with the development of (16.7.2) above:

$$
\begin{aligned}
\bar{A}^{1}_{x:\overline{1}|} + \bar{\beta}_{1|}\bar{a}_{x:\overline{h-1}|} &= \bar{P}(\bar{A})\,\bar{a}_{x:\overline{h}|} \\
&= \bar{A} \\
&= \bar{A}^{1}_{x:\overline{1}|} + {}_1E_x\,\bar{A}(1) \qquad (16.7.4)
\end{aligned}
$$

or

$$
\bar{\beta} = \frac{{}_1E_x\,\bar{A}(1)}{{}_{1|}\bar{a}_{x:\overline{h-1}|}} = \frac{\bar{A}(1)}{\bar{a}_{x+1:\overline{h-1}|}}.
$$

The effect of the *FPT* method on accounting statements can be demonstrated by modifying (15.5.5). We note that expense-loaded premiums under *FPT* are given by

$$
P_x + c = A^{1}_{x:\overline{1}|} + c_0 = \beta_x + c_1
$$

where c_0 is the loading in the first year and c_1 is the loading in renewal years. The analogue of (15.5.5) where, as before, $u(k)$ denotes the anticipated surplus for each expected surviving insured at end of accounting period k is, for $u(0) = 0$,

$$
\begin{aligned}
[(A^{1}_{x:\overline{1}|} + c_0) - e_0](1 + i) - q_x &= p_x\,u(1) \\
(c_0 - e_0)(1 + i) &= p_x\,u(1) \qquad k = 0, \qquad (16.7.5\text{A})
\end{aligned}
$$

$$
\begin{aligned}
&{}_kp_x\{[{}_kV^{FPT}_x + u(k)] + (\beta_x + c_1) - e_k\}\,(1 + i) - {}_kp_x\,q_{x+k} \\
&\qquad = {}_{k+1}p_x\,[{}_{k+1}V^{FPT}_x + u(k + 1)] \qquad k = 1, 2, \ldots. \qquad (16.7.5\text{B})
\end{aligned}
$$

Therefore, if

$$
c_0 - e_0 = (P_x + c - A^{1}_{x:\overline{1}|} - e_0) > 0,
$$

the first-year surplus in our idealized accounting illustration will be positive. In realistic situations,

$$
c_0 = P_x + c - A^{1}_{x:\overline{1}|} > c
$$

and $p_x\ u(1)$ will be greater than when liabilities are measured by level benefit premium reserves.

The recursion formulas, analogous to (8.3.10), are

$$A^{1}_{x:\overline{1}|}(1 + i) - q_x = 0 \tag{16.7.6A}$$

and

$${}_kp_x({}_kV_x^{FPT} + \beta_x)(1 + i) - {}_kp_x\ q_{x+k} = {}_{k+1}p_x\ {}_{k+1}V_x^{FPT}. \tag{16.7.6B}$$

Subtracting (16.7.6B) from (16.7.5B) we obtain

$${}_kp_x\ [u(k) + c_1 - e_k](1 + i) = {}_{k+1}p_x\ u(k + 1) \qquad k = 1, 2, 3, \ldots. \tag{16.7.7}$$

Multiply (16.7.5A) and (16.7.7) by v^{k+1}, let $c_k' = c_0$ when $k = 0$ and $c_k' = c_1$ when $k = 1, 2, \ldots$, and we get

$$\Delta[v^k\ {}_kp_x\ u(k)] = v^k\ {}_kp_x\ (c_k' - e_k). \tag{16.7.8}$$

With $u(0) = 0$, difference equation (16.7.8) yields

$$\sum_{j=0}^{k-1} \Delta[v^j\ {}_jp_x\ u(j)] = \sum_{j=0}^{k-1} v^j\ {}_jp_x\ (c_j' - e_j), \tag{16.7.9}$$

$${}_kp_x\ u(k) = \sum_{j=0}^{k-1} (1 + i)^{k-j}\ {}_jp_x\ (c_j' - e_j). \tag{16.7.10}$$

As in (15.5.8), the expected surplus for each initial insured in our idealized model is the accumulated value of the excess of loadings over expenses in each earlier year. The following comparisons of the annual expected contributions to surplus for each surviving insured in the cases of *FPT* reserves and of ***level benefit premium (LBP)*** reserves may be developed using (15.5.7) and (16.7.8).

FPT		LBP		
$c_0 - e_0$	$>$	$c - e_0$		
$c_1 - e_k$	$<$	$c - e_k$	$k = 1, 2, \ldots$	(16.7.11)

The inequalities displayed in (16.7.11) for the expected contributions to surplus are valid when $\alpha < P$ and $\beta > P$.

16.8 Modified Preliminary Term

If we adopt the principle that negative reserve liabilities are inappropriate on the balance sheet of an insurance enterprise, the *FPT* reserve method provides a minimum first-year terminal reserve and minimum first-year premium. According to (16.7.11), the annual surplus contributions in the first year under *FPT* and *LBP* are

$$\begin{array}{c} \qquad\qquad\qquad\qquad FPT \qquad LBP \\ P + c - A^{1}_{x:\overline{1}|} - e_0 = c_0 - e_0 > c - e_0. \end{array}$$

A difficulty arises in that the magnitude of $P - A^{\ 1}_{x:\overline{1}|}$ depends on the plan of insurance. Since $P_{x:\overline{n}|}$ is normally much greater than $P^{\ 1}_{x:\overline{n}|}$, the expense margin to be used to offset first-year expenses will also be much greater for an n-year endowment insurance than for an n-year term insurance. One school of thought holds that if $P - A^{\ 1}_{x:\overline{1}|}$ provides an acceptable expense margin for low-premium policies, it provides an excessive margin for high-premium policies. Under this line of reasoning, low-premium policies may be valued satisfactorily under the *FPT* method, but high-premium policies should use a modified reserve method that will produce a positive first-year terminal reserve.

A ***modified preliminary term reserve standard*** requires: (1) A decision rule by which policies are separated into low- and high-premium classes. (2) That for low-premium policies the *FPT* method, which specifies $\alpha = A^{\ 1}_{x:\overline{1}|}$, is permitted. (3) A definition of a valuation method for high-premium policies by specifying β, $\beta - \alpha$, or an $\alpha > A^{\ 1}_{x:\overline{1}|}$ and the length of the modification period.

An objective of government regulation is to reduce the threat to insureds that an insurance company cannot meet its obligations. In accordance with this objective, in some jurisdictions, an actuary is designated to have the responsibility to determine that reserves make adequate provision for future obligations for an insurance enterprise.

In other jurisdictions laws and regulations limit the choice of the methods and assumptions that may be used to estimate reserve liabilities for regulatory purposes. In some cases these laws and regulations define a modified reserve standard. Only one such standard remains of direct interest to actuaries practicing in the United States. The Standard Valuation Law defines the Commissioners Reserve Valuation Standard for life insurance. The elements of this standard are that

- High-premium policies are defined as those for which $\beta^{FPT} > {}_{19}P_{x+1}$, the *FPT* renewal benefit premium for a 20-payment life
- The *FPT* method is a minimum for low-premium policies
- A specific Commissioners Reserve Valuation Method (CRVM) be used for high-premium policies. Here the premium payment period is the modification period and

$$\beta^{\mathrm{CRVM}} - \alpha^{\mathrm{CRVM}} = {}_{19}P_{x+1} - A^{\ 1}_{x:\overline{1}|}.$$

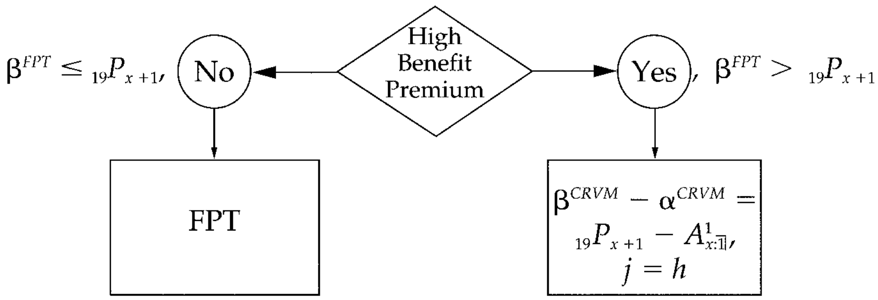
An application of (16.6.3) yields

$$\beta^{\mathrm{CRVM}} = P + \frac{{}_{19}P_{x+1} - A^{\ 1}_{x:\overline{1}|}}{\ddot{a}_{x:\overline{h}|}} \tag{16.8.1}$$

where h is the length of the premium paying period.

The decision flowchart implicit in a modified preliminary term standard, illustrated by the Commissioners Reserve Valuation Standard, is displayed in Figure 16.8.1.

FIGURE 16.8.1
Modified Preliminary Term Decision Flowchart



16.9 Nonlevel Premiums or Benefits

Section 16.2 covers the development of a single decrement model with arbitrary assumptions about the level of expenses and their time incidence for defining minimum cash values. In Section 16.8 a related development is done for the purposes of defining minimum reserves for financial reporting to regulators. In both of these sections the developments were limited to level premiums and benefits. The statutes supporting minimum reserve and minimum cash value regulation have typically contained specific provisions for level premium and benefit policies. Because of the limitless variation that can exist in schedules of premiums and benefits that are not level, the interpretations needed to apply the principles of the statutes have been left to regulators and professional judgment. The job is to take statutory and contractual language and to produce a mathematical model that is consistent with the intent of the language. This type of endeavor is typical of many actuarial assignments.

The main tool in adapting regulations designed for level death benefit and level contract premium policies to policies with nonlevel premiums or benefits is that of averaging. A sequence of nonlevel premiums or benefits will be replaced by a level sequence made up of weighted average premiums or benefits, respectively. The methods of adapting regulations for nonconstant premiums or benefits will differ by the weights used in the averaging.

16.9.1 Valuation

In Section 8.2 a general insurance, using a fully discrete basis, is discussed. This general insurance will be used to illustrate the problem. Recall that, under the insurance, a death benefit b_{j+1} is paid at the end of policy year $j + 1$ if death has occurred in that year. Annual premiums are payable, contingent on survival, at the beginning of each policy year during the premium period. The contract premium G_j is paid at time j, the beginning of policy year $j + 1$.

We describe here the interpretation by Menge (1946) for applying the Commissioners Reserve Valuation Standard to this general insurance. Formulas are given for a term insurance with a description of the modifications for an endowment insurance. These rules are directly applicable to insurance policies with schedules of benefits and premiums. Policies providing for benefits or premiums that are tied to investment performance, or that can be changed at the election of the owner of the policy, require special consideration.

The first task is to determine the criterion for using *FPT*. As a first step, we calculate an ***equivalent level renewal amount (ELRA).*** For this insurance we have

$$ELRA = \frac{\sum_{j=0}^{n-2} b_{j+2}\, v^{j+1}\, {}_jp_{x+1}\, q_{x+1+j}}{A^{\;1}_{x+1:\overline{n-1|}}}. \tag{16.9.1}$$

The *ELRA* for an endowment insurance is calculated on the basis of the death benefits only and is thus also given by (16.9.1). *ELRA* can be interpreted as a weighted average benefit amount with weight

$$v^{j+1}\, \frac{{}_jp_{x+1}\, q_{x+1+j}}{A^{\;1}_{x+1:\overline{n-1|}}}.$$

In addition, the average ratio of renewal benefit premiums to renewal contract premiums is denoted by r_F and is determined by

$$r_F = \frac{\sum_{j=0}^{n-2} b_{j+2}\, v^{j+1}\, {}_jp_{x+1}\, q_{x+1+j}}{\sum_{j=0}^{h-2} G_{j+1}\, v^j\, {}_jp_{x+1}}. \tag{16.9.2}$$

The Menge interpretation depends on the assumption that the valuation premium is a constant fraction of the contract premium.

For endowment insurances, pure endowment benefits are included in the numerator for r_F.

Under the Menge interpretation, *FPT* is allowed if

$$r_F\, G_0 \le ELRA\ {}_{19}P_{x+1}. \tag{16.9.3}$$

If this condition is satisfied, *FPT* is interpreted as involving benefit premiums $\pi_0 = vb_1q_x$ and $\pi_j = r_FG_j$, $j = 1, 2, \ldots, h-1$, where h is the length of the premium paying period. The modified reserve, for $k \ge 1$, is given by

$${}_kV^{Mod} = \sum_{j=0}^{n-k-1} b_{k+j+1}\, v^{j+1}\, {}_jp_{x+k}\, q_{x+k+j} - r_F \sum_{j=0}^{h-k-1} G_{k+j}\, v^j\, {}_jp_{x+k}. \tag{16.9.4}$$

For endowment insurances, pure endowment benefits are added to the first term of (16.9.4).

In cases where (16.9.3) is not satisfied, the excess first-year expense allowance, comparable to $\beta - \alpha$, is given by

$$ELRA\ {}_{19}P_{x+1} - b_1\, A^{\;1}_{x:\overline{1|}}.$$

A modified average ratio of benefit premiums to contract premiums, denoted by r_C, is determined as

$$r_C = \frac{\sum_{j=0}^{n-1} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j} + (ELRA\ {}_{19}P_{x+1} - b_1\, A^{\ 1}_{x:\overline{1}|})}{\sum_{j=0}^{h-1} G_j\, v^j\, {}_jp_x}. \tag{16.9.5}$$

The modified reserve in this high-premium case is given by

$${}_kV^{Mod} = \sum_{j=0}^{n-k-1} b_{k+j+1}\, v^{j+1}\, {}_jp_{x+k}\, q_{x+k+j} - r_C \sum_{j=0}^{h-k-1} G_{k+j}\, v^j\, {}_jp_{x+k}. \tag{16.9.6}$$

For endowment insurances, the numerator of r_C in (16.9.5) and the right-hand side of (16.9.6) are adjusted to include pure endowment benefits.

Example 16.9.1

Calculate the annual benefit premiums under the Commissioners Reserve Valuation Standard for a special 30-year endowment policy issued at age 35. The benefit is 150,000 for the first 20 years and 100,000 thereafter with a maturity benefit of 100,000. The contract premium is a level 2,500 for 10 years and thereafter is 1,250. Use the Illustrative Life Table with $i = 0.06$.

Solution:

The $ELRA$ is based on the death benefit and is calculated as

$$\begin{aligned} ELRA &= 50{,}000 \left(\frac{2A^{\ 1}_{36:\overline{29}|} + A^{\ 1}_{36:\overline{19}|}}{A^{\ 1}_{36:\overline{29}|}} \right) \\ &= 130{,}153.30. \end{aligned}$$

The r_F factor is given by

$$\begin{aligned} r_F &= \frac{50{,}000}{1{,}250} \left(\frac{2A_{36:\overline{29}|} + A^{\ 1}_{36:\overline{19}|}}{\ddot{a}_{36:\overline{29}|} + \ddot{a}_{36:\overline{9}|}} \right) \\ &= 0.91014604 \end{aligned}$$

and

$${}_{19}P_{36} = 0.0116543.$$

For FPT to apply, (16.9.3) must be satisfied. However, since

$$r_F\, G_0 = (0.91014604)(2{,}500) = 2{,}275.37 > 1{,}516.85 = ELRA\ {}_{19}P_{36},$$

FPT is not allowed. The excess first-year expense allowance, corresponding to $\beta - \alpha$, is given by

$$ELRA\ {}_{19}P_{x+1} - b_1\, A^{\ 1}_{x:\overline{1}|} = 1{,}516.8456 - 284.9395 = 1{,}231.9061$$

and

$$r_C = \frac{50{,}000(2A_{36:\overline{29}|} + A^{1}_{36:\overline{19}|}) + 1{,}231.9061}{1{,}250(\ddot{a}_{36:\overline{29}|} + \ddot{a}_{36:\overline{9}|})}$$

$$= 0.88223578.$$

Thus the renewal benefit premiums are

$$r_C(2{,}500) = 2{,}205.59, \quad \text{years } 2, 3, \ldots, 10$$

$$r_C(1{,}250) = 1{,}102.79, \quad \text{years } 11, 12, \ldots, 30,$$

and the first-year benefit premium

$$= r_C(2{,}500) - 1{,}231.9061$$

$$= 973.68.$$

▼

An alternative interpretation of the Commissioners Reserve Valuation Standard as it applies to policies with nonlevel death benefits is contained in Actuarial Guideline XVII developed by the National Association of Insurance Commissioners. These guidelines do not have the force of law in most states but are designed to serve as guides in applying statutes to specific circumstances.

The guideline first requires the calculation of the average of the death benefits at the ends of policy years 2 through 10. Let M denote the average value; that is,

$$M = \frac{\sum_{j=0}^{8} b_{j+2}}{9}.$$

If

$$\frac{\sum_{j=0}^{n-2} b_{j+2}\, v^{j+1}\, {}_{j}p_{x+1}\, q_{x+1+j}}{\ddot{a}_{x+1:\overline{h-1}|}} > M\, {}_{19}P_{x+1},$$

then r_C is determined by (16.9.5) with M replacing *ELRA*. If the alternative inequality holds, then r_F is determined by (16.9.2). Reserve formulas (16.9.4) and (16.9.6) remain valid. The purpose of the guideline is to replace *ELRA* with the simpler M.

16.9.2 Cash Values

The issues that arise with nonforfeiture benefits for such policies are similar, but there are differences. The 1980 NAIC law refers to an ***average amount of insurance (AAI)*** and indicates that this amount is based on benefit amounts at the ends of the first 10 policy years. Thus, in the notation of the general insurance of Section 8.2,

$$AAI = \frac{\sum_{j=0}^{9} b_{j+1}}{10}$$

when $n \geq 10$. A level benefit premium is

$$P = \frac{\sum_{j=0}^{n-1} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j}}{\ddot{a}_{x:\overline{h}|}}. \tag{16.9.7}$$

An additional term is included in the numerator of (16.9.7) if pure endowment benefits are included. The formula for the first-year expense allowance depends on P and AAI and is defined as follows:

$$E_1 = \begin{cases} 1.25P + 0.01AAI & P < 0.04AAI \\ 0.06AAI & P \geq 0.04AAI. \end{cases} \tag{16.9.8}$$

The adjusted premium for any policy year is a multiple, r_N, of the corresponding contract premium for that policy year where

$$r_N = \frac{E_1 + \sum_{j=0}^{n-1} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j}}{\sum_{j=0}^{h-1} G_j\, v^j\, {}_jp_x}. \tag{16.9.9}$$

The minimum cash value is given as

$${}_kCV = \sum_{j=0}^{n-k-1} b_{k+j+1}\, v^{j+1}\, {}_jp_{x+k}\, q_{x+k+j} - r_N \sum_{j=0}^{h-k-1} G_{k+j}\, v^j\, {}_jp_{x+k}. \tag{16.9.10}$$

Again, both (16.9.9) and (16.9.10) must be modified if pure endowment benefits are provided.

Example 16.9.2

Calculate the minimum cash values at durations 1, 2, 10, and 20 for the special 30-year endowment policy described in Example 16.9.1. Use (16.9.8), (16.9.9), and (16.9.10) with the Illustrative Life Table and 6% interest.

Solution:

The death benefit is 150,000 for the first 20 years and 100,000 thereafter. However, since the AAI is based on the first 10 policy years, $AAI = 150{,}000$. The level benefit premium for this policy is, at issue age 35,

$$P = \frac{50{,}000(2A_{35:\overline{30}|} + A^{\;1}_{35:\overline{20}|})}{\ddot{a}_{35:\overline{30}|}}$$

$$= 1622.9358.$$

Since

$$P = 1{,}622.94 \leq 6{,}000 = 0.04AAI,$$

then, according to (16.9.8),

$$E_1 = 1.25P + 0.01AAI$$
$$= 3{,}528.6698.$$

The contract premiums are 2,500 annually for 10 years and 1,250 annually for the remaining 20 years. The adjusted premium multiple, r_N, is

$$r_N = \frac{E_1 + 50{,}000(2A_{35:\overline{30}|} + A^{\;1}_{35:\overline{20}|})}{1{,}250(\ddot{a}_{35:\overline{30}|} + \ddot{a}_{35:\overline{10}|})}$$
$$= 0.9667466.$$

Formula (16.9.10) was used to obtain the following cash values for this contract.

k	${}_kCV$
1	0.00
2	669.73
10	22 519.81
20	48 776.92

Formula (16.9.10) gives $-1{,}483.53$ for ${}_1CV$; however, negative values of CV cannot be collected from withdrawing policyholders, so it is arbitrarily set equal to zero.

▼

16.10 Notes and References

Cummins (1973) outlines the history of nonforfeiture values in the United States with emphasis on the pioneering work of Elizur Wright. Developments in other nations followed different paths. The 1941 NAIC report on nonforfeiture benefits provides an historical summary of United States regulation, a brief review of practices in other countries, discussion of the philosophic considerations regarding equity for withdrawing policyholders, and a development of the adjusted premium approach for defining minimum cash values. The 1975 Society of Actuaries committee report on nonforfeiture requirements is especially interesting for its discussions of problems in connection with policies that provide for variation in benefits after issue. Richardson (1977) supplemented this later report with an expense investigation for the purpose of defining the loading factors in adjusted premiums.

Shepherd (1940) developed the ideas on natural reserves and premiums. In two papers, one covering asset shares and nonforfeiture values (1939) and one covering gross premiums (1929), Hoskins developed many of the ideas in Section 16.4.2. The premium determination method that depends on a rate of return objective is a simplified version of work by Anderson (1959). Anderson also provides extensive background material on margins for contingencies and profits.

Dividends are discussed in detail by Jackson (1959) and by Maclean and Marshall (1937) in a monograph that also traces the history of surplus distribution. Cody (1981) expands earlier models to a more comprehensive model.

Full preliminary term reserves are also called Zillmerized reserves for the European actuary who developed the method. The Commissioners method and related recommendations were developed by two National Association of Insurance Commissioners (NAIC) committees with almost identical membership: the Committee to Study the Need for a New Mortality Table and Related Topics (1939) and the Committee to Study Nonforfeiture Benefits and Related Matters (1941). In recognition of their common chairman, these committees were popularly known as the Guertin Committees. Menge (1946) wrote a comprehensive paper on technical aspects of the Commissioners method. Tullis and Polkinghorn (1992) have written a monograph on the many issues that arise in valuing life insurance liabilities.

Exercises

Section 16.3

16.1. A company is planning to adopt a new life table. Indicate how you would determine the ages at policy issue, and times since issue, for which the reduced paid-up life insurance nonforfeiture benefits under whole life policies will be increased and for which they will be decreased. Use only a table of the annual benefit premiums for whole life policies computed on the bases of both the old and the new mortality tables. Assume that the cash values are calculated by applying the same percentage of the benefit reserve under the new table as was used under the old table.

16.2. An n-year, n-payment endowment insurance on a fully discrete payment basis, with unit amount of insurance, is issued to (x). In the event of default of premium payments, the insured has the option of
- Reduced paid-up whole life insurance or
- An extended term insurance, to the end of the endowment period, with a reduced pure endowment paid at age $x + n$.
- The cash value at time t is ${}_tV_{x:\overline{n}|}$ and is sufficient to purchase paid-up whole life insurance of amount b or to purchase extended-term insurance of one together with a pure endowment at age $x + n$ of amount f. If $A_{x+t:\overline{n-t}|} = 2A_{x+t}$, express f in terms of b, $A^{1}_{x+t:\overline{n-t}|}$, and ${}_{n-t}E_{x+t}$.

16.3. A 20-year endowment insurance of unit amount on a fully continuous payment basis issued to (30) is lapsed at the end of 10 years when there is an indebtedness of amount L outstanding against the cash value ${}_{10}CV$. Express, in terms of actuarial present values,
   a. The amount of pure endowment, E, at the regular maturity date if extended term insurance for the amount of the policy less indebtedness, $1 - L$, can be continued to the maturity date;

b. The amount of the reserve on the extended-term insurance and the pure endowment 5 years after the date of lapse.

16.4. It has been suggested that the amount of reduced paid-up insurance should be in proportion to the number of annual premiums paid to the total number of premiums payable under the terms of the policy. Compare the amount of paid-up insurance under this suggested rule with ${}^{20}_{10}W_{40}$ and ${}_{10}W_{40:\overline{20}|}$ using the Illustrative Life Table and 6% interest.

16.5. Show that

$$\frac{d}{dt}\,[{}_t\bar{W}(\bar{A}_{x:\overline{n}|})] = \frac{\bar{P}(\bar{A}_{x:\overline{n}|}) - \mu_x(t)[1 - {}_t\bar{W}(\bar{A}_{x:\overline{n}|})]}{\bar{A}_{x+t:\overline{n-t}|}}$$

and interpret the equation. [Hint: Use (8.6.2) to write the derivatives of ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ and $\bar{A}_{x+t:\overline{n-t}|}$.]

16.6. In the early years of life insurance, one company defined its cash values as

$${}_kCV = h\,(G_{x+k} - G_x)\,\ddot{a}(k) \qquad k = 1, 2, \ldots$$

where the symbol G denotes a contract premium at the indicated age, and $\ddot{a}(k)$ denotes a life annuity due commencing at age $x + k$, continuing on survival to the end of the premium paying period. In practice h was set at 2/3. If, by the 1980 NAIC law, the contract premiums for a whole life policy are taken as adjusted premiums, and if it is given that P_x and P_{x+k} are each less than 0.04 and $h = 0.9$, show that

$${}_kCV = (0.909 + 1.125P_x)\,{}_kV_x + 1.125(P_{x+k} - P_x).$$

16.7. Let ${}_k\hat{W} = {}_kCV/A(k)$ where ${}_kCV = A(k) - P^a\,\ddot{a}(k)$, as in (16.2.2), and P^a is an adjusted premium. Construct a table for the three policies shown in Table 16.3.1 under a fully discrete model, relating ${}_k\hat{W}$ to adjusted and level benefit premiums.

16.8. If ${}_kW^{Mod} = {}_kV^{Mod}/A(k)$, where ${}_kV^{Mod}$ is the reserve at the end of the k-th policy year under the Commissioners Standard, as discussed in Section 16.8, construct a table for the three policies shown in Table 16.3.1, under a fully discrete model, relating ${}_kW^{Mod}$ to renewal and level benefit premiums. Assume the premium payment period in the limited payment plans is less than 20 years.

16.9. If ${}_{k+t}CV = {}_{k+t}\bar{V}(\bar{A}_x)$,

a. Show that (16.3.4) for the length of the automatic premium loan period can be written as $H(t) = 0$ where $H(t) = \bar{a}_x\,G\,\bar{s}_{\overline{t}|i} + \bar{a}_{x+k+t} - \bar{a}_x$;
b. Confirm that $H(0) < 0$ for survival functions where the force of mortality is increasing and, that as $t \to \infty$, $H(t)$ becomes positive and unbounded;
c. Calculate $H'(t)$.

16.10. If, in relation to (16.4.1), ${}_{10}AS_1$ is the asset share at the end of 10 years based on G_1 and ${}_{10}AS_2$ is the corresponding quantity based on G_2, write a formula for ${}_{10}AS_2 - {}_{10}AS_1$.

16.11. Denote the expense-loaded premium determined by the equivalence principle by P' and the contract premium, loaded for profit and contingencies by G. If $G = P'(1 + \theta)$, $\theta > 0$, the contract premium has been determined by the ***expected value premium principle.*** This principle derives its name because the contract premium is proportional to the equivalence principle premium. Confirm that the expected value premium principle has the additive property. This can be done by working with two independent loss variables, ${}_0L_e^*(1)$ and ${}_0L_e^*(2)$, where the asterisk indicates that the premium included in the loss function is $(1 + \theta)$ times the expense loaded premium determined by the equivalence principle. Confirm that

$$\mathrm{E}[{}_0L_e^*(1) + {}_0L_e^*(2)] = \mathrm{E}[{}_0L_e^*(1)] + \mathrm{E}[{}_0L_e^*(2)].$$

16.12. Suppose that there is an ***experience premium,*** denoted by $\hat{G}$, based on realistic mortality and expense assumptions such that

$$\hat{G} = v\ {}_{k+1}F - {}_kF + \hat{g} + v\,\hat{q}_{x+k}^{(1)}\ (1 - {}_{k+1}F)$$

where $\hat{g} = G\hat{c}_k + \hat{e}_k$, $k = 0, 1, 2, \ldots.$ In addition, assume ${}_{k+1}CV = {}_{k+1}F$ for $k = 0, 1, 2, 3, \ldots.$ Show that under these assumptions

a. ${}_{k+1}F = ({}_kF + \hat{G} - \hat{g})(1 + i) - \hat{q}_{x+k}^{(1)}(1 - {}_{k+1}F);$

b. If dividends are paid to insureds who die or withdraw, (16.5.4) can be written as

$$ {}_{k+1}F + {}_{k+1}D = [{}_kF + G - \hat{g}](1 + \hat{i}_{k+1}) - \hat{q}_{x+k}^{(1)}\ (1 - {}_{k+1}F)$$

and

$$ {}_{k+1}D = (G - \hat{G})(1 + \hat{i}_{k+1}) + ({}_kF + \hat{G} - \hat{g})(\hat{i}_{k+1} - i).$$

This exercise outlines the ***experience premium method*** of dividend calculation.

16.13. The recursion relation between successive life annuity values is given by

$$(\ddot{a}_{x+h} - 1)(1 + i) = p_{x+h}\ \ddot{a}_{x+h+1} \qquad h = 0, 1, 2, \ldots.$$

a. If the actual experience interest rate is $\hat{i}_{h+1}$ and the experience survival probability is $\hat{p}_{x+h}$, the fund progress will be described by

$$(\ddot{a}_{x+h} - 1)(1 + \hat{i}_{h+1}) = \hat{p}_{x+h}(\ddot{a}_{x+h+1} + \Delta_{h+1})$$

where Δ_{h+1} is the survivor's share of deviations. Show that

$$\Delta_{h+1} = \frac{(\hat{i}_{h+1} - i)(\ddot{a}_{x+h} - 1) + (p_{x+h} - \hat{p}_{x+h})\ \ddot{a}_{x+h+1}}{\hat{p}_{x+h}}$$

and interpret the result. [This formula is the basis of the two-factor contribution formula for annuity dividends.]

b. If the annuity income at the end of the year is adjusted to be r_{h+1} times the income as of the beginning of the year where

$$(\ddot{a}_{x+h} - 1)(1 + \hat{i}_{h+1}) = \hat{p}_{x+h}(r_{h+1})\, \ddot{a}_{x+h+1}\,,$$

express r_{h+1} in terms of i, $\hat{i}_{h+1}$, p_{x+h}, and $\hat{p}_{x+h}$.

Section 16.6

16.14. For a modified reserve method the period of modification is equal to the premium paying period. Show that

$${}_kV^{Mod}_{x:\overline{n}|} = 1 - (\beta + d)\, \ddot{a}_{x+k:\overline{n-k}|}.$$

16.15. A modified reserve method for a whole life insurance, fully continuous basis, is defined by

$$\bar{\alpha}(t) = \frac{t}{m}\,\bar{\beta} \qquad 0 \le t < m$$

where $\bar{\beta}$ is the level premium for $t \ge m$.

a. Write a formula for $\bar{\beta}$.

b. Write a prospective formula for ${}_t\bar{V}(\bar{A}_x)^{Mod}$, $t < m$.

16.16. Calculate α^{Mod}_x and β^{Mod}_x in a modified reserve method for a fully discrete whole life insurance where ${}_1V^{Mod}_x = K$. The modification period is the premium paying period.

16.17. Under a modified reserve method for a fully discrete whole life insurance policy, the annual benefit premiums, P_x, are replaced (for reserve purposes) by annual premiums of α^{Mod}_x for the first n years and β^{Mod}_x thereafter. Show that

$$\frac{\beta^{Mod}_x - P_x}{P_x - \alpha^{Mod}_x} = \frac{\ddot{a}_x}{{}_{n|}\ddot{a}_x} - 1.$$

16.18. Show that

$${}_kV - {}_kV^{Mod} = \left(\frac{\beta - \alpha}{\ddot{a}_{x:\overline{j}|}}\right) \ddot{a}_{x+k:\overline{j-k}|}$$

where j is the length of the modification period. Note that this difference can be interpreted as the unamortized portion of the extra first-year expense allowance.

16.19. Use the general symbols of (16.2.2) to define a modified reserve model for a level benefit, level premium policy:

$${}_kV^{Mod} = A(k) - \beta\, \ddot{a}(k) \qquad k = 1, 2, 3, \ldots.$$

Confirm the following steps:

a. $$_kV^{Mod} = vq_k + vp_k\, A(k+1) - \beta - \beta v p_k\, \ddot{a}(k+1)$$
$$= vq_k - \beta + vp_k\; {}_{k+1}V^{Mod}$$
$$({}_kV^{Mod} + \beta)(1+i) = q_k(1 - {}_{k+1}V^{Mod}) + {}_{k+1}V^{Mod} \qquad k = 1, 2, \ldots .$$

b. If ${}_0V^{Mod}$ is defined as $(\alpha - \beta)$ the recursion relation in (a) can be extended to $k = 0$.

c. $\Delta\, v^k\, {}_kV^{Mod} = v^k\, \beta - q_k(1 - {}_{k+1}V^{Mod})\, v^{k+1}$, $k = 0, 1, 2, \ldots .$

d. Summing the expression in part (c) over $k = 0, 1, 2, \ldots, n-1$, confirm that

$$_nV^{Mod} = \beta \ddot{s}_{\overline{n}|} - (\beta - \alpha)(1+i)^n - \sum_{k=0}^{n-1} (1+i)^{n-k-1}\, q_k\, (1 - {}_{k+1}V^{Mod}).$$

Section 16.7

16.20. Show that

$$_kV^{FPT}_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x+1:\overline{n-1}|}} \qquad k = 1, 2, \ldots, n.$$

16.21. A ***two-year preliminary term reserve method*** is defined with three valuation premiums:

First year:	$A^{1}_{x:\overline{1}|}$
Second year:	$A^{1}_{x+1:\overline{1}|}$
Thereafter:	Level benefit premium for age $x+2$, no change in benefits or premium paying period.

Show that, for this method, the reserve on a whole life policy to (x) is given by

$$_1V^{Mod} = {}_2V^{Mod} = 0$$
$$_kV^{Mod} = {}_kV_x - (P_{x+2} - P_x)\, \ddot{a}_{x+k} \qquad k = 3, 4, 5, \ldots .$$

(This type of reserve system is common in health insurance.)

16.22. Consider a fully discrete whole life insurance on (x). Λ_h^{FPT} is defined as in (8.5.1) with $\pi_0 = A^{1}_{x:\overline{1}|}$, $\pi_h = P_{x+1}$, $h = 1, 2, \ldots$, and ${}_hV = {}_hV^{FPT}$. Show that, if ${}_1V_x > 0$,

$$\mathrm{Var}[\Lambda_0^{FPT}|K(x) = 0, 1, \ldots] = v^2\, p_x q_x > \mathrm{Var}[\Lambda_0|K(x) = 0, 1, \ldots].$$

Section 16.8

16.23. A proposed reserve valuation method has a first-year premium α and a renewal premium β applicable in all renewal years. The first-year premium is constrained to be at least equal to $A^{1}_{x:\overline{1}|}$. This provision allows the use of *FPT* for some policies and some ages. The excess of β over α cannot exceed 0.05. Assume that on a particular valuation basis $d = 0.03$, $\ddot{a}_x = 17$, $\ddot{a}_{x:\overline{12}|} = 9$, $A_{x:\overline{12}|}^{\ \ 1} = 2/3$, and $A^{1}_{x:\overline{1}|} = 0.01$.

a. Calculate β for a whole life policy issued to (x).
b. Calculate ${}_{12}V_x^{Mod}$ for this method.
c. Assuming $\alpha = A^{1}_{x:\overline{1}|} = 0.01$, calculate a test value for β for a 12-year endowment insurance on (x). Confirm that this value for β is inadmissible since $\beta - \alpha > 0.05$.
d. Using the result in (c), calculate β for a 12-year endowment insurance on (x).
e. Calculate ${}_1V^{Mod}_{x:\overline{12}|}$.

16.24. A modified preliminary term reserve standard is described as follows:
- Policies are divided into two classes: Class I, if the *FPT* renewal premium is greater than ${}_{19}P_{x+1}$; Class II, if the policy does not belong to Class I.
- For policies in Class I, the first-year premium is the same as that defined by the Commissioners Reserve Valuation Method. The renewal premium is such that the level benefit premium reserve is reached at the end of the premium paying period or 15 years, whichever comes first.
- For policies in Class II, the *FPT* method is prescribed.

For a fully discrete payment basis, 20-payment, 20-year endowment issued to (x), write an expression for α and β under this standard.

16.25. If $\beta^{FPT} > {}_{19}P_{x+1}$, the k-th terminal reserve by the Commissioners method on a fully discrete basis life insurance issued to (x) and paying level benefits may be written as

$$ {}_kV^{\text{CRVM}} = \frac{A^{1}_{x:\overline{1}|}}{{}_kE_x} + {}_{19}P_{x+1}\, \ddot{s}_{x+1:\overline{k-1}|} + T\, \ddot{s}_{x:\overline{k}|} - \frac{A^{1}_{x:\overline{k}|}}{{}_kE_x}. $$

Derive an expression for T.

Section 16.9

16.26. Assume $\beta^{FPT} > {}_{19}P_{x+1}$ and

$$ \begin{aligned} b_{j+1} &= 1 \qquad & j &= 0, 1, \ldots, n-1, \\ G_j &= P(1+\theta) \qquad & j &= 0, 1, \ldots, h-1. \end{aligned} $$

Show that (16.9.7) reduces to the formula for the k-th reserve by the Commissioners Standard.

17

SPECIAL ANNUITIES AND INSURANCES

17.1 Introduction

In this chapter we study a wide variety of policies that provide special annuity and insurance benefits; our aim is to determine actuarial present values, benefit and contract premiums, and benefit premium reserves. In Section 17.2 we examine a number of annuity contracts for a single life where the period during which payments are made may be longer than the future lifetime of the annuitant, or where there are benefits payable upon death. These contracts can arise from a settlement option under a life insurance policy, from the provisions of a pension plan, or from provisions of an individual annuity policy. Section 17.3 covers the closely related matter of the family income policy. Variable products, where benefit levels and reserves depend on investment results, are the subject of Section 17.4. These products become important when price inflation erodes the value of benefits stated in terms of fixed monetary units. In Section 17.5 types of policies that provide wide flexibility for changing benefit amounts and premium levels are examined. Some life insurance policies provide for benefit payments before death when the insured becomes terminally ill or has lost the ability to perform specified activities of daily living. These are called ***accelerated benefits.*** In Section 17.6 a multiple decrement model for such policies will be introduced. The model resembles the disability insurance models introduced in Chapter 11 in that both the time and amount of accelerated benefits payments may be random variables.

17.2 Special Types of Annuity Benefits

In this section we concentrate on calculating the actuarial present values of special forms of annuity benefits. The payment patterns depend on the contract premium collected. We determine the appropriate contract premium using the equivalence principle. We emphasize continuous payment annuities and justify the corresponding results for m-thly payment annuities by analogy.

Section 5.2 contained an analysis of n-year certain and life annuities. That type of life annuity provides for a guarantee of payments for at least n years. The annuities in this section can be viewed as special cases where the number n bears a relationship to the consideration paid for the annuity.

One illustration is the ***installment refund annuity.*** A sufficient number of payments is guaranteed so that the annuitant receives at least as much as the contract premium that was paid. Thus, for such a continuous annuity with contract premium G, the actuarial present value of benefits is

$$\bar{a}_{\overline{G|}} + {}_GE_x\, \bar{a}_{x+G}.$$

If the contract premium is to contain a loading of r times the premium, the equivalence principle requires that G satisfy

$$G(1 - r) = \bar{a}_{\overline{G|}} + {}_GE_x\, \bar{a}_{x+G}. \tag{17.2.1}$$

The difference between the left- and right-hand sides of the above expression could be evaluated for integer values of G. Then, an approximate value of G that equates the two sides is found by linear interpolation. Similar expressions hold for discrete versions, and G could be found for those in a similar manner.

A related annuity that contains some insurance features is the ***cash refund annuity.*** A death benefit is defined as the excess, if any, of the contract premium paid over the annuity payments received. If G is the single contract premium and T is the time of death, the present value of benefits on a continuous basis is

$$Z = \begin{cases} \bar{a}_{\overline{T|}} + (G - T)v^T & T \le G \\ \bar{a}_{\overline{T|}} & T > G. \end{cases} \tag{17.2.2}$$

The actuarial present value of these benefits is given by

$$\begin{aligned} \mathrm{E}(Z) &= \int_0^\infty \bar{a}_{\overline{t|}}\, {}_tp_x\, \mu_x(t)\, dt + \int_0^G (G - t)\, v^t\, {}_tp_x\, \mu_x(t)\, dt \\ &= \bar{a}_x + G\, \bar{A}^{1}_{x:\overline{G|}} - (\bar{I}\bar{A})^{1}_{x:\overline{G|}}. \end{aligned} \tag{17.2.3}$$

As for the installment refund annuity, the principle equivalence is used to determine G. If the loading is r times the contract premium, then

$$G(1 - r) = \bar{a}_x + G\, \bar{A}^{1}_{x:\overline{G|}} - (\bar{I}\bar{A})^{1}_{x:\overline{G|}}. \tag{17.2.4}$$

Linear interpolation can be used to approximate G after the difference between the left- and right-hand sides of (17.2.4) is evaluated for integer values of G.

Example 17.2.1

Consider a ***partial cash refund annuity*** with the present-value random variable defined as

$$Z = \begin{cases} \bar{a}_{\overline{T|}} + (\rho G - T)v^T & T < \rho G,\ 0 < \rho < 1 \\ \bar{a}_{\overline{T|}} & T \ge \rho G. \end{cases}$$

a. Show that (17.2.4) may be rewritten for a partial cash refund annuity as

$$G(1 - r) = \bar{a}_x + \rho G \bar{A}^{1}_{x:\overline{\rho G|}} - (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}}.$$

b. Form $H(G) = Gr + \bar{a}_x + \rho G\, \bar{A}^{1}_{x:\overline{\rho G|}} - (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}} = G$ for the purpose of determining G such that $H(G) = G$, by using an iterative solution method. The method will employ $H(G_i) = G_{i+1}$.
   (i) Display $H'(G)$ and $H''(G)$.
   (ii) Discuss the signs of $H'(G)$ and $H''(G)$ in the neighborhood of a root.

Solution:

a. Apply the equivalence principle to obtain

$$\begin{aligned}
G(1 - r) &= \mathrm{E}[Z^*] \\
&= \int_0^{\rho G} [\bar{a}_{\overline{t|}} + (\rho G - t)v^t]\, {}_tp_x\, \mu_x(t)\, dt \\
&\quad + \int_{\rho G}^{\infty} \bar{a}_{\overline{t|}}\, {}_tp_x\, \mu_x(t)\, dt \\
&= \bar{a}_x + \rho G \int_0^{\rho G} v^t\, {}_tp_x\, \mu_x(t)\, dt \\
&\quad - \int_0^{\rho G} t\, v^t\, {}_tp_x\, \mu_x(t)\, dt \\
&= \bar{a}_x + \rho G\, \bar{A}^{1}_{x:\overline{\rho G|}} - (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}}.
\end{aligned}$$

b. (i) $H'(G) = r + \rho \bar{A}^{1}_{x:\overline{\rho G|}} + \rho[\rho G\, v^{\rho G}\, {}_{\rho G}p_x\, \mu_x(\rho G) - \rho G\, v^{\rho G}\, {}_{\rho G}p_x\, \mu_x(\rho G)]$

$\qquad = r + \rho \bar{A}^{1}_{x:\overline{\rho G|}},$

$H''(G) = \rho^2\, v^{\rho G}\, {}_{\rho G}p_x\, \mu_x(\rho G) \geq 0.$

   (ii) The rate of convergence of iterative methods for solving nonlinear equations depends on the magnitude of the first derivative $[H'(G)]$ in the neighborhood of the solution of the equation. Assume $H(G_i) = G_{i+1}$ is used to solve iteratively for G. We know that $H''(G) > 0$ and $H'(G) = r + \rho \bar{A}^{1}_{x:\overline{\rho G|}} = [G - \bar{a}_x + (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}}]/G$. But $G > \bar{a}_x > \bar{a}_x - (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}} > \bar{a}_x - (\bar{I}\bar{A})_x > 0$, so $0 < G - \bar{a}_x + (\bar{I}\bar{A})^{1}_{x:\overline{\rho G|}} < G$. Thus $0 < H'(G) < 1$ in the neighborhood of the value of G such that $H(G) = G$. The condition that $|H'(G)| < 1$ in the neighborhood of the solution is a sufficient condition for the iterative solution procedure, $H(G_i) = G_{i+1}$, to converge to G. ▼

17.3 Family Income Insurances

An ***n-year family income insurance*** provides an income from the date of death of the insured, continuing until n years have elapsed from the date of issue of the policy. It is typically paid for by premiums over the n-year period, or some period shorter than n years, to keep benefit reserves positive. Again, we start with a continuous annuity. If T is the time of death of the insured, the present value of benefits is

$$Z = \begin{cases} v^T \, \bar{a}_{\overline{n-T}|} & T \le n \\ 0 & t > n. \end{cases} \tag{17.3.1}$$

Usually, the interest rate involved in the annuity factor, $\bar{a}_{\overline{n-T}|}$, is the same as that in the present-value factor, v^T. A variation of this type of contract is the ***mortgage protection policy*** where the annuity factor in the benefit function represents the outstanding balance on a mortgage. The mortgage interest rate used in evaluating $\bar{a}_{\overline{n-T}|}$ may then be different from that used for evaluating v^T.

The actuarial present value for the family income benefit is given by

$$\mathrm{E}[Z] = \int_0^n v^t \, \bar{a}_{\overline{n-t}|} \, {}_tp_x \, \mu_x(t) \, dt. \tag{17.3.2}$$

This integral can be converted to a current payment integral by integration by parts,

$$\bar{a}_{\overline{n}|} - \int_0^n v^t \, {}_tp_x \, dt = \int_0^n v^t \, (1 - {}_tp_x) \, dt = \bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|}. \tag{17.3.3}$$

The interpretation of the middle integral is that the annuity is payable at time t, for $t < n$, only if (x) is dead at that time, the probability of that event being $1 - {}_tp_x$. The first and third expressions in (17.3.3) can be interpreted as requiring a continuous payment, at a constant rate of 1 per year, until time n, but the payments must be returned if (x) is alive.

We conclude this section with an example combining aspects of a family income policy and a retirement annuity with a term certain.

Example 17.3.1

Calculate the actuarial present value of the benefits under a policy issued at age 40 providing the following annuity benefits payable continuously at a rate of 1 per year:

- In the event of death prior to age 65, a family income benefit ceasing at age 65 or 10 years after death, if later, and
- In the event of survival to age 65, a life annuity with 10 years certain.

Solution:

We prepare to write the current payment integral. The following table gives the conditions required for payments at time t and the corresponding probabilities.

Time	Condition	Probability
$0 < t \le 25$	(40) is dead	$1 - {}_tp_{40}$
$25 < t \le 35$	(40) was alive at $t - 10$	${}_{t-10}p_{40}$
$t > 35$	(40) is alive	${}_tp_{40}$

The actuarial present value is

$$\int_0^{25} v^t\,(1 - {}_tp_{40})\,dt + \int_{25}^{35} v^t\,{}_{t-10}p_{40}\,dt + \int_{35}^{\infty} v^t\,{}_tp_{40}\,dt.$$

If we replace $t - 10$ by s in the middle integral, we obtain

$$\int_{15}^{25} v^{s+10}\,{}_sp_{40}\,ds = v^{10}\,(\bar{a}_{40:\overline{25}|} - \bar{a}_{40:\overline{15}|}).$$

Thus the actuarial present value of the benefit can be written as

$$\bar{a}_{\overline{25}|} - \bar{a}_{40:\overline{25}|} + v^{10}\,(\bar{a}_{40:\overline{25}|} - \bar{a}_{40:\overline{15}|}) + \bar{a}_{40} - \bar{a}_{40:\overline{35}|}.$$ ▼

17.4 Variable Products

We consider several products where the benefit levels and reserves depend upon the investment results. The investments associated with a product may be of any type. Typically, the particular investments are selected to be in accord with the announced objective of the insurance or annuity product. The original impetus for these products was to participate in the higher expected total returns (dividends, interest, and capital gains) available in equity investments and thus to provide some measure of protection against inflation. A typical contract provides guarantees concerning mortality and expense charges. Thus, the policyholder is not charged for adverse experience, nor does the policy benefit from favorable experience from these two sources. We examine the mechanisms for change in benefit level on an individual policy basis.

17.4.1 Variable Annuity

We consider here the ***variable annuity.*** During the premium-paying or ***accumulation period,*** a fund is accumulated from a single contribution, or from periodic deposits, at rates of return depending upon the investment performance of the fund. Typically, variable annuities make guarantees on maximum sales, administrative and investment expense charges, and the mortality basis in use. If we ignore gains and losses attributed to withdrawals and to death benefits at guaranteed levels, the expected growth of the fund share is given by

$$[F_k + \pi_k(1 - c_k) - e_k](1 + i'_{k+1}) = F_{k+1} + q_{x+k}(b_{k+1} - F_{k+1}); \qquad (17.4.1)$$

see (16.5.3). Here F_k is the fund share at time k; π_k is the size of the deposit at time k; c_k is the fraction of the premium, π_k, charged for those expenses at time k that are proportional to the premium paid at that time; e_k is the charge at time k for expenses not proportional to the premium; b_{k+1} is the benefit paid at time $k + 1$ for death between times k and $k + 1$; i'_{k+1} is the actual investment return, net of investment expenses, for the year following time k. The second term on the right-hand side of the equation is equal to 0 during the accumulation period, so we have the fund growing with interest only.

At retirement, the existing fund share is used to purchase a paid-up annuity, the purchase rates computed on a predetermined mortality basis and an ***assumed investment return (AIR).*** If the AIR is low, then the initial annuity payment will be low relative to the fund share, but the contract can be expected to provide an increasing payment pattern that offsets some of the effects of inflation. Let the AIR be denoted by i and, again, let the net actual investment return in the year following time k be i'_{k+1}. If the annuity benefit is paid only to those living at the beginning of each year, with the annuity payment level at time k equal to b_k, the reserve just before the payment is $b_k\, \ddot{a}_{x+k}$, with x the retirement age. The equation for the progress of the fund share would be

$$(b_k\, \ddot{a}_{x+k} - b_k)(1 + i'_{k+1}) = b_{k+1} p_{x+k} \ddot{a}_{x+k+1}. \tag{17.4.2}$$

But from (5.3.4), we have

$$(\ddot{a}_{x+k} - 1)(1 + i) = p_{x+k}\, \ddot{a}_{x+k+1}. \tag{17.4.3}$$

Dividing (17.4.2) by (17.4.3) gives

$$b_{k+1} = b_k \frac{1 + i'_{k+1}}{1 + i}. \tag{17.4.4}$$

Thus, if $i'_{k+1} > i$, the benefit level will increase. Note that a high AIR can lead to a situation where the benefit amounts are frequently decreased.

The result, (17.4.4), holds for other payout options. This is indicated for the n-year certain and life annuity in Exercise 17.11. It also holds for m-thly payouts in slightly modified forms. First, let us consider adjusting the payout amount monthly. The formula connecting the annuity values for the first and second months of a contract year is

$$\left(\ddot{a}^{(12)}_{x+k} - \frac{1}{12}\right)\left(1 + \frac{i^{(12)}}{12}\right) = {}_{1/12}p_{x+k}\, \ddot{a}^{(12)}_{x+k+1/12},$$

while the progress of the fund share would be expressed by

$$\left(b_k\, \ddot{a}^{(12)}_{x+k} - \frac{b_k}{12}\right)\left(1 + \frac{i'^{(12)}_{k+1}}{12}\right) = {}_{1/12}p_{x+k}\, b_{k+1/12}\, \ddot{a}^{(12)}_{x+k+1/12}.$$

Division yields

$$b_{k+1/12} = \frac{b_k(1 + i'^{(12)}_{k+1}/12)}{1 + i^{(12)}/12}. \tag{17.4.5}$$

Alternatively, we could adjust the payment size on an annual basis even though the payout is monthly. First, the formula for successive annual reserves for a monthly annuity is

$$(\ddot{a}^{(12)}_{x+k} - \ddot{a}^{(12)}_{x+k:\overline{1|}})(1 + i) = p_{x+k}\, \ddot{a}^{(12)}_{x+k+1}.$$

The equation for the growth of the fund share would be

$$(b_k\, \ddot{a}^{(12)}_{x+k} - b_k\, \ddot{a}^{(12)}_{x+k:\overline{1|}})(1 + i'_{k+1}) = p_{x+k}\, b_{k+1}\, \ddot{a}^{(12)}_{x+k+1}.$$

Thus we make a charge, $b_k\, \ddot{a}^{(12)}_{x+k:\overline{1}|}$, for the present year's annuity payments. Dividing the last two expressions gives (17.4.4) again,

$$b_{k+1} = b_k \frac{1 + i'_{k+1}}{1 + i}.$$

17.4.2 Fully Variable Life Insurance

There are a large number of possible designs for ***variable life insurance.*** We examine three distinct designs, all based on a whole life insurance. Each of these designs can be adapted to limited payment life insurances or to endowment insurances. Each is based on annual premiums with immediate payment of claims. Benefit amounts are changed at the beginning of each year.

The first design is what we call ***fully variable life insurance.*** Benefit amounts change with investment results, and premiums are kept proportional to benefit amounts. We start with a unit benefit amount and benefit premium $P(\bar{A}_x)$. The terminal reserve at time k is equal to the product of ${}_kV(\bar{A}_x)$ and b_k, the benefit amount to be paid in the year following time k. The benefit premium payable at time k is $b_k\, P(\bar{A}_x)$. Upon receipt of the benefit premium, term insurance for the benefit for the year is purchased, the cost being $b_k\, \bar{A}^{\,1}_{x+k:\overline{1}|}$. The equation that connects the expected fund size at the beginning and end of the year and is used to define the benefit for the subsequent year is

$$[b_k\, {}_kV(\bar{A}_x) + b_k\, P(\bar{A}_x) - b_k\, \bar{A}^{\,1}_{x+k:\overline{1}|}](1 + i'_{k+1}) = p_{x+k}\, b_{k+1}\, {}_{k+1}V(\bar{A}_x). \tag{17.4.6}$$

But we know

$$[{}_kV(\bar{A}_x) + P(\bar{A}_x) - \bar{A}^{\,1}_{x+k:\overline{1}|}](1 + i) = p_{x+k}\, {}_{k+1}V(\bar{A}_x). \tag{17.4.7}$$

Dividing (17.4.6) by (17.4.7), we obtain

$$b_{k+1} = b_k \frac{1 + i'_{k+1}}{1 + i}. \tag{17.4.8}$$

This is the same relationship that holds during the payout phase of a variable annuity, namely, (17.4.4).

17.4.3 Fixed Premium Variable Life Insurance

We next examine a ***fixed premium variable life insurance.*** The main difference from the fully variable design, as the name suggests, is that the benefit premium remains constant. Again we start with a unit benefit amount and write the equation connecting expected fund sizes,

$$[b_k\, {}_kV(\bar{A}_x) + P(\bar{A}_x) - b_k\, \bar{A}^{\,1}_{x+k:\overline{1}|}](1 + i'_{k+1}) = p_{x+k}\, b_{k+1}\, {}_{k+1}V(\bar{A}_x). \tag{17.4.9}$$

Combining this with (17.4.7) gives us

$$b_{k+1} = b_k \left[\frac{{}_kV(\bar{A}_x) + P(\bar{A}_x)/b_k - \bar{A}^{\,1}_{x+k:\overline{1}|}}{{}_kV(\bar{A}_x) + P(\bar{A}_x) - \bar{A}^{\,1}_{x+k:\overline{1}|}} \right] \frac{1 + i'_{k+1}}{1 + i}. \tag{17.4.10}$$

The first factor on the left-hand side of (17.4.9) can be written as

$$(b_k - 1)\ {}_kV(\bar{A}_x) + {}_kV(\bar{A}_x) + P(\bar{A}_x) - b_k\ \bar{A}^{\ 1}_{x+k:\overline{1}|}.$$

This shows that the fixed benefit premium supports both the initial face amount of 1 and the additional benefit, $b_k - 1$, generated by the actual investment returns.

17.4.4 Paid-up Insurance Increments

Here we consider an alternative used for the third design. We consider the changes in the benefit amount as paid-up and use the premium to support only the original benefit level. The equation connecting fund shares becomes

$$[(b_k - 1)\ \bar{A}_{x+k} + {}_kV(\bar{A}_x) + P(\bar{A}_x) - b_k\ \bar{A}^{\ 1}_{x+k:\overline{1}|}](1 + i'_{k+1})$$
$$= p_{x+k}\ [(b_{k+1} - 1)\ \bar{A}_{x+k+1} + {}_{k+1}V(\bar{A}_x)]. \quad (17.4.11)$$

The left-hand side of (17.4.11) can be transformed as follows:

$$\{b_k\ (\bar{A}_{x+k} - \bar{A}^{\ 1}_{x+k:\overline{1}|}) - [\bar{A}_{x+k} - {}_kV(\bar{A}_x) - P(\bar{A}_x)]\}(1 + i'_{k+1})$$
$$= [b_k\ {}_1E_{x+k}\ \bar{A}_{x+k+1} - P(\bar{A}_x)(\ddot{a}_{x+k} - 1)](1 + i'_{k+1})$$
$$= [b_k\ {}_1E_{x+k}\ \bar{A}_{x+k+1} - P(\bar{A}_x)({}_1E_{x+k}\ \ddot{a}_{x+k+1})](1 + i'_{k+1})$$
$$= p_{x+k}\ \bar{A}_{x+k+1}\left[b_k - \frac{P(\bar{A}_x)}{P(\bar{A}_{x+k+1})}\right]\frac{1 + i'_{k+1}}{1 + i}.$$

The right-hand side of (17.4.11) is most easily transformed by using the paid-up insurance formula for the reserve. It becomes

$$p_{x+k}\left\{(b_{x+1} - 1)\ \bar{A}_{x+k+1} + \bar{A}_{x+k+1}\left[1 - \frac{P(\bar{A}_x)}{P(\bar{A}_{x+k+1})}\right]\right\}$$
$$= p_{x+k}\ \bar{A}_{x+k+1}\left[b_{k+1} - \frac{P(\bar{A}_x)}{P(\bar{A}_{x+k+1})}\right].$$

Making these substitutions into (17.4.11), we find that the recursion formula for the benefit amount is

$$b_{k+1} - \frac{P(\bar{A}_x)}{P(\bar{A}_{x+k+1})} = \left[b_k - \frac{P(\bar{A}_x)}{P(\bar{A}_{x+k+1})}\right]\frac{1 + i'_{k+1}}{1 + i}. \quad (17.4.12)$$

This third design has the advantage that if, after a period of years with favorable investment returns resulting in $b_k > 1$, the investment returns level off at the AIR, then the benefit amounts will remain fixed. This is not true for the second design that led to (17.4.10).

The paid-up insurance increment design has been used most frequently in commercial practice. Exercise 17.14 explores another design.

17.5 Flexible Plan Products

In the early 1970s insurance companies began to offer several types of policies intended to provide the policyholder a broad range of options for changing benefit amounts, premiums, and plan of insurance. The companies typically allowed small increases in death benefit amounts without new evidence of insurability, but larger increases required such evidence. The insurance plans usually included all types of level premium, level benefit term plans, which as a limiting case include whole life. The more expensive plans offered were either all limited payment life plans or all endowment plans.

In Section 8.2 a model for life insurance policies with nonconstant benefits and premiums was developed. The schedule of premiums and benefits was assumed to have been determined when the policy is issued. In this section the option for changing benefits and premiums within bounds established by the contract is available to the insured. Both participating (with experience-based dividends) and nonparticipating versions were made available. A special dividend option was devised to allow the dividend to be added at net rates directly related to the cash value. This larger cash value was then used to extend the expiry date on term plans or to increase the benefit amounts on permanent plans. We refer to such products as ***flexible plans*** and illustrate a particularly simple version that shows some of the inherent complexities. We conclude the section by describing a second design that has less emphasis on the plan of insurance and some features in common with variable life plans, as described in the previous section.

17.5.1 Flexible Plan Illustration

Basic to the design of the type of flexible plan considered here is a formula used in reserve calculations relating the contract premium to the benefit premium. We use the following very simple relationship applied to both term and limited payment life plans, the latter being our choice for permanent coverage,

$$0.8G = P. \tag{17.5.1}$$

Here G is the contract premium and P is the benefit premium.

Another basic decision is to determine the form of the total expense charge and the related question of the adjustment of the reserves at times of plan change. We use full preliminary term reserves and nonforfeiture values in our illustration. It should be noted that the nonforfeiture and valuation laws may require higher minimum values, particularly for limited payment life plans. We define ${}_0V = -E_0$, where E_0 is the excess first-year expense allowance. This idea also appeared in Section 16.2. Then, from our adoption of full preliminary term reserves, we have ${}_1V = 0$ and, assuming a fully discrete basis,

$${}_0V + P = vq_x b,$$

thus

$$E_0 = -{}_0V = P - vq_x b. \tag{17.5.2}$$

The equation connecting initial benefit amount, b, initial benefit premium, P, and initial plan of insurance with h, the premium payment term, is

$${}_0V + P\ddot{a}_{x:\overline{h|}} = bA^1_{x:\overline{j|}}. \tag{17.5.3}$$

Here j equals either h (in case of a term plan) or $\omega - x$ (in case of limited payment life). Reserves are most easily determined by retrospective formulas since, as we will see, minor adjustments in benefits in the final year of a policy are usually required. Thus

$${}_kV = \frac{{}_0V + P\ddot{a}_{x:\overline{k|}} - bA^1_{x:\overline{k|}}}{{}_kE_x}. \tag{17.5.4}$$

We illustrate the application of these formulas with the following example.

Example 17.5.1

Consider a policy issued at age 35 with an initial gross premium of 1,000 and an initial benefit amount of 120,000. Use the Illustrative Life Table with 6% interest to determine the excess first-year expense allowance, the fifth-year reserve, and the plan of insurance.

Solution:

From (17.5.1), we have $P = 800$. Therefore, the excess first-year expense allowance is

$$-{}_0V = P - 120{,}000vq_{35} = 572.05,$$

and the fifth-year reserve is given by

$$\begin{aligned}{}_5V &= \frac{-572.05 + 800\ddot{a}_{35:\overline{5|}} - 120{,}000A^1_{35:\overline{5|}}}{{}_5E_{35}} \\ &= 2{,}491.24.\end{aligned}$$

The renewal benefit premium for 120,000 of whole life insurance at age 35 on a full preliminary basis is $120{,}000\, P_{36} = 1{,}057.37$. Since our benefit premium is only 800, the plan of insurance is one of the term plans. It can be verified that, by using retrospective formulas,

$${}_{39}V = 3{,}375.72$$

and

$${}_{40}V = -1{,}313.14.$$

Thus the plan of insurance is Term to Age 74. The reserve remaining at time 39 would typically be used to provide term insurance for a fraction of the following year. In our example, the number of days is given by

$$\frac{{}_{39}V}{120{,}000A^1_{74:\overline{1|}}}\, 365 = 230.$$

▼

At the time of change of benefit amount or premium, a new benefit premium is calculated along with any change in the reserve that might result from, for instance, a change in the assumed excess first-year expense allowance. For our simplified plan, the benefit premiums are a constant percentage of the contract premiums, and we assume that the revised reserve at the time of change, ${}_kV'$, is equal to the full preliminary term reserve on hand. The relationship between the revised reserve, new net premium, P', and new benefit amount, b', is of the same form as (17.5.3), namely,

$$ {}_kV' + P'\ddot{a}_{x+k:\overline{h}|} = b' \, A^{\;1}_{x+k:\overline{j}|}. \tag{17.5.5} $$

Here j and h would, in general, change with the new relationship between premium and benefit amounts. Again, j equals either h or $\omega - x - k$, and it is most convenient to evaluate reserves by a retrospective formula. Thus, for $g = 1, 2, 3, \ldots,$ we have

$$ {}_{k+g}V' = \frac{{}_kV' + P'\ddot{a}_{x+k:\overline{g}|} - b' \, A^{\;1}_{x+k:\overline{g}|}}{{}_gE_{x+k}}. \tag{17.5.6} $$

We continue with three examples that are continuations of Example 17.5.1. These illustrate different types of change and show some characteristic calculations.

Example 17.5.2

The policyholder in Example 17.5.1 wishes, 5 years after issue, to change the contract premium of the policy to 2,000 and the benefit amount to 150,000. Determine the reserve 10 years after original issue and the new plan of insurance.

Solution:

$P' = 1{,}600$ and ${}_5V' = 2{,}491.24$ (${}_5V' = {}_5V$ in Example 17.5.1). Thus, by (17.5.6),

$$ {}_{10}V' = \frac{2{,}491.24 + 1{,}600\ddot{a}_{40:\overline{5}|} - 150{,}000A^{\;1}_{40:\overline{5}|}}{{}_5E_{40}} $$

$$ = 10{,}319.89. $$

We know the plan of insurance is one of the limited payment life plans since $2{,}491.24 + 1{,}600\ddot{a}_{40}$ exceeds $150{,}000A_{40}$. It can be determined that the reserve at age 69 is the first one to exceed the actuarial present value of 150,000 of whole life insurance at the same age. Thus,

$$ {}_{34}V' = \frac{2{,}491.24 + 1{,}600\ddot{a}_{40:\overline{29}|} - 150{,}000A^{\;1}_{40:\overline{29}|}}{{}_{29}E_{40}} $$

$$ = 75{,}597.32, $$

while $150{,}000\,A_{69} = 74{,}954.44$. When the policyholder attains age 69, the policy will probably be changed to a paid-up life policy with face amount

$$ \frac{75{,}597.32}{A_{69}} = 151{,}287. $$

▼

Example 17.5.3

The policyholder in Example 17.5.1 wishes to change the policy after 5 years to Life Paid-up at Age 60 with a contract premium of 2,000. Determine the benefit level that results from these changes.

Solution:

$P = 0.8(2{,}000) = 1{,}600$, thus (17.5.5) gives us

$$2{,}491.24 + 1{,}600\ddot{a}_{40:\overline{20}|} = b'\,A_{40}.$$

Solving for b' gives $b' = 132{,}090$. ▼

Example 17.5.4

After 5 years, the policyholder in Example 17.5.1 wishes to change his policy to Term to Age 65 with a coverage of 150,000. Determine the contract premium appropriate after the change.

Solution:

$P = 0.8G$; thus (17.5.5) gives us

$$2{,}491.24 + 0.8G\ddot{a}_{40:\overline{25}|} = 150{,}000A^{1}_{40:\overline{25}|}.$$

The solution of this equation is $G = 895.00$. ▼

17.5.2 An Alternative Design

A second design combines aspects of variable life insurance with the preceding design of a flexible plan policy. The emphasis on the plan of insurance is not as strong in this design as it was in the first. Further, the emphasis is on the ***risk amount*** (previously referred to as the net amount at risk), rather than on the benefit amount. The risk amount can be determined at the beginning of policy year $k + 1$, and a fund growth equation can be written in terms of this factor, which we denote by r_k. Our analysis will be in terms of an annual model but, in practice, a monthly or even more frequent calculation is more common. The basic growth equation for the fund share, the analogue of (16.5.3) without allowance for withdrawals, is

$$({}_kF + G - E - r_k\,\bar{A}^{\,1}_{x+k:\overline{1}|})(1 + i'_{k+1}) = {}_{k+1}F. \tag{17.5.7}$$

Note that accumulation is under interest only, and, in case of death, the policyholder receives both the fund share, that is, the fund at the beginning of the year, ${}_kF + G - E - r_k\,\bar{A}^{\,1}_{x+k:\overline{1}|}$, and the risk amount adjusted for interest to the date of death. The risk amount might be selected to maintain an approximate level total benefit amount. The policyholder is given considerable flexibility in the choice of G, the contract premium, and r_k, the risk amount. The insurer typically makes a number of guarantees. Usually, i'_{k+1} is an investment return that must be at least equal to some minimum rate i. The risk charge is typically guaranteed to be no more than $r_k\,\bar{A}^{\,1}_{x+k:\overline{1}|}$, where the 1-year term insurance actuarial present value is calculated on the basis of interest rate i and a mortality table used in statutory reserve calculations.

The amount r_k typically must be equal to or greater than a lower bound established by tax regulations. The objective of these regulations is to prevent the favorable tax treatment given to life insurance contracts to be extended to contracts that are essentially savings programs.

Plans of insurance based on recursion relationship (17.5.7) have been called ***universal life.*** The encompassing title has been justified by the options granted to the insured to change the relative emphasis on death benefits and savings by changing the premium and death benefits. In some cases, the policy commits the insurer to use accumulation rates i_k that are based on a investments with a particular allocation. For example, i_k might be based on investments in common stocks. Such policies are called ***variable universal life policies.***

Expense charges, E in (17.5.7), currently used are of several forms, including

- A constant percentage charge against all contract premiums
- Surrender charges such as a large but declining (with duration) percentage of the first-year premium or a transaction charge such as 25 for each withdrawal
- A flat amount per policy either in the first year only or a smaller amount for each policy year, and
- A first-year charge expressed as an amount per 1,000 of benefit.

The charges most subject to regulation are the excess first-year expense charges and the risk charges. Expenses are covered by the insurer, in addition to the stated formula charges, by a number of devices. Some of these are

- Reduced interest credits, limited to the guaranteed rate, for an initial corridor of policy cash values, for example, the first 1,000 of cash values
- An interest rate spread of 1 to 1 1/2% between the net investment yield and the rate applied to the cash values, and
- Recognizing that part of the risk charge actually contains some provision for expenses, just as do regular term insurance premiums.

As stated above, the emphasis on plan of insurance is not strong. At any time, a calculation parallel to that used in Examples 17.5.1 and 17.5.2 could be performed to determine the plan that is implicit in any specific pattern of premiums and benefits, current risk charges, expense charges, interest rates, and reserve.

17.6 Accelerated Benefits

Some life insurance contracts provide special benefits if the insured transfers to a state characterized by serious restrictions on daily activities and extraordinary care expenditures. Payment of those benefits reduces the basic death and withdrawal benefits. As a consequence, they are called ***accelerated benefits.***

The word ***state*** is used to describe the two environments that characterize the distributions of time and cause of decrement because the word plays a similar role in the study of stochastic processes. Within insurance, the two states are labeled ***active*** and ***disabled.***

Accelerated benefits can be divided into two classes. In one class a lump sum is paid, usually at the time of a confirmed diagnosis of the existence of a terminal illness. These are frequently called ***dread disease benefits.*** A second class involves income payments that commence when the insured has become unable to perform certain specified necessary activities of daily living. Some policies provide that the assistance needed to perform these activities must be provided in a long-term care institution. Other policies may permit the care to be provided at home or in a long-term care institution as long as that care is made necessary by the incapacity of the insured. This second class is often called ***long-term care benefits.***

In presenting formulas for benefit premiums determined by the equivalence principle for such policies, we will use the multiple decrement model developed in Chapter 10. There is one important addition. As in Chapter 10, the symbol ${}_tp_x^{(\tau)}\,\mu_x^{(j)}(t)$ denotes the joint p.d.f. of the random variables time until decrement and cause of, or type of, decrement. The index $j = 1$ is associated with death, $j = 2$ with withdrawal, and $j = 3$ with transfer to a new state denoted by h. In our applications, h is a state such as being diagnosed as having a terminal illness, or being disabled to an extent that necessary daily services must be provided. For lives in state h, who entered that state at age $x + t$, the p.d.f. of time until decrement and cause, or type, of decrement is denoted by ${}_u(hp)_{x+t}^{(\tau)}\,(h\mu)_{x+t}^{(hj)}(u)$ for $u > 0$, where $(hj) = 1$ denotes decrement as a result of death, and $(hj) = 2$ denotes decrement due to withdrawal. This can be viewed as a conditional p.d.f. given transfer to state h at time t, measured from the time the policy was issued.

Our model does not provide for transfer from state h, the disabled state, back to the active state. Thus the three states to which an active life can transfer can be called ***absorbing states.*** References to models that permit returns to the active state are provided in Section 17.7.

Because of the known health impairment in state h, the distribution of time and cause of decrement in state h is undoubtedly different from that in the active state.

17.6.1 Lump Sum Benefits

This section consists of an extended example that employs the equivalence principle to determine the benefit premium rate for a life insurance policy that pays an accelerated benefit at the moment of transfer to state h. In practice h is typically the state of being diagnosed as having a terminal illness.

Example 17.6.1

The elements of the policy are given in Table 17.6.1.

a. Display the loss variable associated with the policy described in Table 17.6.1 for a life (x) to whom the policy is issued.
b. Use the equivalence principle to derive a formula for the annual benefit premium rate.

TABLE 17.6.1

Description of Immediate Benefit Policy

Death benefit:	1 paid at the moment of death while in the active state 0.75 paid at the moment of death while in state h
Withdrawal benefit:	${}_tCV$ paid at the moment of withdrawal from the active state 0.75 ${}_{t+u}CV$ paid at the moment of withdrawal from state h at time $t + u$, where t is the time of transfer to state h
Accelerate benefit:	0.25 paid at the moment of transfer to state h
Premiums:	The premium is paid at a constant rate until death, withdrawal or transfer to state h

Solution:

a. The loss variable associated with this policy contains a new element. The symbol $B_{x+t}^{(3)}$ was introduced in Section 11.2. In this example it denotes the actuarial present value of benefits paid while the insured is in state h. The actuarial present value is determined as of the moment of transfer to state h.

Loss Variable	Domain	p.d.f.	Formula
$L = v^T - \pi\bar{a}_{\overline{T}\rvert}$	$0 \le T, J = 1$	${}_tp_x^{(\tau)}\mu_x^{(1)}(t)$	17.6.1(a)
$\quad = v^T\,{}_TCV - \pi\bar{a}_{\overline{T}\rvert}$	$0 \le T, J = 2$	${}_tp_x^{(\tau)}\mu_x^{(2)}(t)$	17.6.1(b)
$\quad = v^T\,B_{x+T}^{(3)} - \pi\bar{a}_{\overline{T}\rvert}$	$0 \le T, J = 3$	${}_tp_x^{(\tau)}\mu_x^{(3)}(t)$	17.6.1(c)
$B_{x+t}^{(3)} = \mathrm{E}_{U,HJ\mid T,J=3}[(b(U, HJ)]$ where			
$b(u, hj) = \begin{cases} 0.25 + 0.75\,v^u \\ 0.25 + 0.75\,v^u\,{}_{t+u}CV \end{cases}$	$0 \le u, hj = 1$ $0 \le u, hj = 2$	${}_u(hp)_{x+t}^{(\tau)}\,(h\mu)_{x+t}^{(1)}(u)$ ${}_u(hp)_{x+t}^{(\tau)}(h\mu)_{x+t}^{(2)}(u)$	17.6.1(d)

b. There are three components of $E_{T,J}[L]$. The components relate to (17.6.1[a]), (b), and (c), and they will be denoted, respectively by I_a, I_b, and I_c:

$$I_a = \int_0^\infty (v^t - \pi\bar{a}_{\overline{t}\rvert})\,{}_tp_x^{(\tau)}\,\mu_x^{(1)}(t)dt, \tag{17.6.2(a)}$$

$$I_b = \int_0^\infty (v^t\,{}_tCV - \pi\bar{a}_{\overline{t}\rvert})\,{}_tp_x^{(\tau)}\,\mu_x^{(2)}(t)dt, \tag{17.6.2(b)}$$

and

$$I_c = \int_0^\infty (v^t\,B_{x+t}^{(3)} - \pi\bar{a}_{\overline{t}\rvert})\,{}_tp_x^{(\tau)}\,\mu_x^{(3)}(t)dt. \tag{17.6.2(c)}$$

In turn

$$B_{x+t}^{(3)} = \Big[0.25 + 0.75\int_0^\infty v^u\,{}_u(hp)_{x+t}^{(\tau)}\,(h\mu)_{x+t}^{(1)}(u)\,du$$
$$+ 0.75\int_0^\infty v^u\,{}_{t+u}CV\,{}_u(hp)_{x+t}^{(\tau)}\,(h\mu)_{x+t}^{(2)}(u)\,du\Big]. \tag{17.6.2(d)}$$

Using the equivalence principle and (2.2.10), we have

$$I_a + I_b + I_c = 0.$$

Solving for π, the benefit premium rate, we obtain

$$\pi = \frac{\left[\bar{A}_x^{(1)} + \int_0^\infty v^t \, {}_tCV \, {}_tp_x^{(\tau)} \mu_x^{(2)}(t)dt + \int_0^\infty v^t \, B_{x+t}^{(3)} \, {}_tp_x^{(\tau)} \mu_x^{(3)}(t)dt\right]}{\bar{a}_x^{(\tau)}}. \quad (17.6.3)$$

The methods illustrated in Section 11.2 can be used to evaluate integrals in (17.6.3). ▼

In Table 17.6.1 it was specified that an immediate payment of 0.25 be made at the moment of transfer to state h, and an additional payment of 0.75 is paid as a benefit on subsequent death. The table could have called for an amount b, $0 \le b \le 1$, to be paid on transfer to state h and $1 - b$ at the time of subsequent death.

17.6.2 Income Benefits

This section will consist of an extended example that will use the equivalence principle to determine the benefit premium rate for a life insurance policy that provides an accelerated benefit consisting of income payments that reduce the death benefit. The salient difference between Example 17.6.1 and Example 17.6.2 is in the definition of the conditional actuarial present-value function $B_{x+T}^{(3)}$.

Example 17.6.2

The policy under consideration has many of the feature of the policy described in Table 17.6.1. There is, however, no cash benefit at the moment of transferring to state h. While in state h, the death and withdrawal benefits are not reduced during a short elimination period of length e. For example, e might be 2 months. If the insured does not terminate in the elimination period, an income benefit at annual rate of 0.25 is paid for 2 years with a corresponding reduction in death and withdrawal benefits. Table 17.6.2 displays the definition of the benefit function from which $B_{x+t}^{(3)} = \mathrm{E}_{U,HJ|T,J=3}[b(U, HJ)]$ is found. Apply the equivalence principle to determine the benefit premium rate.

Solution:

The solution follows the path of Example 17.6.1 except that $B_{x+t}^{(3)}$ is the conditional expected value of the function defined in Table 17.6.2 and replaces the corresponding function shown in 17.6.2(d). and (17.6.3). ▼

In Example 17.6.2 the income benefit was paid at an annual rate of 0.25 for 2 years; the annual income rate could have been b for n years where $0 \le bn \le 1$. Changes in b and n would change the relative emphasis on income and death benefits.

TABLE 17.6.2

Definition of Present Value of Benefits While in State h

P. V. Benefits $b(u, hj)$	Domain	p.d.f.
v^u	$0 \le u < e,\ hj = 1$	${}_u(hp)^{(\tau)}_{x+t}\ (h\mu)^{(1)}_{x+t}(u)$
$0.25\ v^e\ \bar{a}_{\overline{u-e}\rvert} + [1 - 0.25(u - e)]\ v^u$	$e \le u < e + 2,\ hj = 1$	
$0.25\ v^e\ \bar{a}_{\overline{2}\rvert} + 0.5\ v^u$	$e + 2 \le u,\ hj = 1$	
$v^u\ {}_{t+u}CV$	$0 \le u < e,\ hj = 2$	${}_u(hp)^{(\tau)}_{x+t}\ (h\mu)^{(2)}_{x+t}(u)$
$0.25\ v^e\ \bar{a}_{\overline{u-e}\rvert} + [1 - 0.25(u - e)]\ v^u\ {}_{t+u}CV$	$e \le u < e + 2,\ hj = 2$	
$0.25\ v^e\ \bar{a}_{\overline{2}\rvert} + 0.5\ v^u\ {}_{t+u}CV$	$e + 2 \le u,\ hj = 2$	

17.7 Notes and References

The foundations for variable annuities were built in a paper by Duncan (1952). Since 1969 there has been a flurry of activity on variable life insurance. The paper by Fraser, Miller, and Sternhell (1969), and its extensive discussion, is the basic reference. Miller (1971) provides a less formal introduction and some numerical illustrations. Papers by Biggs (1969) and Macarchuk (1969), and discussions of those, provide additional information on variable annuities.

It is difficult to discuss flexible plans of insurance in a book devoted to basic actuarial models. Issues related to these plans are primarily regulatory and administrative. The type of policy described in Section 17.5.1 is studied by Chapin (1976). The type of policy in Section 17.5.2 is the subject of a paper by Chalke and Davlin (1983).

Accelerated benefits, when issued as a rider to a basic policy, were discussed by Keller (1990). An appendix to the paper contains data on rates of transfer to a long-term care facility and the expected length of stay.

Jones (1994) provides an introduction to actuarial models in which transfers in both directions between the active and disabled states are possible.

Exercises

Section 17.2

17.1. The present value of a continuous annuity providing payments until n years after the death of an annuitant (x) is

$$Z = \bar{a}_{\overline{T+n}\rvert}.$$

Express the actuarial present value in current payment form.

17.2. Show that Var(Z), where Z is defined in Exercise 17.1, is

$$\frac{v^{2n}[{}^2\bar{A}_x - \bar{A}_x^2]}{\delta^2}.$$

17.3. Assume that $\delta > 0$ and $\mu_x(t) = \mu$, $0 \le t$. Use (17.2.1) to develop the formula

$$G(1 - r) = \frac{1 - e^{-\delta G}}{\delta} + \frac{e^{-(\mu+\delta)G}}{\mu + \delta}$$

to be solved for G.

17.4. Restate the formula of Exercise 17.3 for the case $\delta = 0$, $\mu > 0$, and $0 < r < 1$, and confirm that it does not have a positive solution.

Section 17.3

17.5. If Z is defined as in (17.3.1), show that

$$\text{Var}(Z) = \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{\delta^2}.$$

17.6. Prove that the actuarial present value of an n-year continuous family income insurance with the annuity value calculated at a force of interest δ' is

$$\frac{\bar{A}^{1}_{x:\overline{n}|} - e^{-\delta' n}\, {}''\bar{A}^{1}_{x:\overline{n}|}}{\delta'}$$

where ${}''\bar{A}^{1}_{x:\overline{n}|}$ is evaluated at a force of interest $\delta'' = \delta - \delta'$.

17.7. A policy provides a continuous annuity-certain of 1 per annum beginning at the date of death of (x). If death occurs within 15 years of policy issue, the annuity is payable to the end of 20 years from policy issue. If death occurs between 15 and 20 years from policy issue, the annuity is payable for 5 years certain. Coverage ceases 20 years from policy issue. Write an exact expression for the actuarial present value.

17.8. A contract provides for the payment of 1,000 at the end of 20 years if the insured is then living or an income of 10 a month in the event of death before the 20th anniversary of the policy. The first income payment is due at the end of the policy month of death, but no payments are made after 20 years from the date of policy issue. Write the formula for the annual benefit premium at age x. Premiums are paid at the beginning of each policy year, and at most 20 payments are made.

17.9. Show that

$$\bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|} = \frac{\bar{A}^{1}_{x:\overline{n}|} - v^n\, {}_nq_x}{\delta}.$$

17.10. a. In relation to Example 17.3.1, construct the present value of benefits as a function of the time-until-death.
b. Express the actuarial present value of benefits by the aggregate payment technique.
c. Verify that integration by parts in the answer for (b) yields the expression obtained in Example 17.3.1.

Section 17.4

17.11. a. Verify that

$$(\ddot{a}_{\overline{x:n}|} - 1)(1 + i) = p_x \ddot{a}_{\overline{x+1:n-1}|} + q_x \ddot{a}_{\overline{n-1}|}.$$

b. Verify that (17.4.4) holds for a variable annuity with the payout made on an n-year certain and life basis.

17.12. a. Rearrange (17.4.10) to the following equivalent form:

$$b_{k+1} = \left[b_k - \frac{(b_k - 1)P(\bar{A}_x)}{{}_1E_{x+k}\ {}_{k+1}V(\bar{A}_x)} \right] \frac{1 + i'_{k+1}}{1 + i}.$$

b. If for the formula in part (a), $i'_{k+1} = i$, $k = 0, 1, 2, \ldots$, and $b_0 = 1$, show that $b_{k+1} = 1$, $k = 0, 1, 2, \ldots$.

c. If for the formula in part (a) $i'_{k+h} = i$ for some $k > 0$ and $h = 1, 2, \ldots$, show that the b_{k+h} will be constrained toward 1.

17.13. Rework Exercise 16.13(b), assuming $p'_{x+h} = p_{x+h}$ and show that $r_{h+1} = b_{h+1}/b_h$ as given in (17.4.4).

17.14. A fixed premium variable whole life insurance, discrete model, has death benefit b_{k+1} in year $k + 1$ equal to $F_{k+1} + (1 - {}_{k+1}V_x) = 1 + (F_{k+1} - {}_{k+1}V_x)$, where the fund share F_k satisfies the recursion equation

$$(F_k + P_x)(1 + i'_{k+1}) = q_{x+k} b_{k+1} + p_{x+k} F_{k+1}.$$

Here i'_{k+1} is the interest rate earned on the matching investments in year $k + 1$, and the premium P_x and reserve ${}_kV_x$ are based on the interest rate i.

a. Show that the recursion equation can be rearranged as

$$(F_k + P_x)(1 + i'_{k+1}) - q_{x+k}(1 - {}_{k+1}V_x) = F_{k+1}$$

and interpret this equation.

b. If $i'_{k+1} = i$, $k = 0, 1, 2, \ldots$, show that $F_{k+1} = {}_{k+1}V_x$ so that b_{k+1} is constant at 1.

c. Show that

$$b_{k+1} = b_k + (F_k + P_x)i'_{k+1} - ({}_kV_x + P_x)i.$$

[Note that in this design the death benefit for year $k + 1$ is b_{k+1} rather than b_k, as in (17.4.9). Further, the 1-year term insurance cost, as of the beginning of year $k + 1$, is here $b_{k+1}\, q_{x+k}/(1 + i'_{k+1})$ rather than $b_k\, q_{x+k}/(1 + i)$, as in (17.4.9): The administration of this design depends on the discrete model and the fact that death claims are paid at the end of the period.]

Section 17.5

17.15. The policyholder in Example 17.5.1 wishes to change the policy after 5 years to Endowment Insurance to Age 65 with a annual contract premium of 5,000. Determine the benefit level that results from these changes.

17.16. a. The policyholder in Exercise 17.15 decides to elect only 160,000 of Endowment Insurance to Age 65, but will still pay a annual contract premium of 5,000 until a final, fractional premium is payable 1 year after the date of the last full premium. At what age is this fractional premium payable?

b. For the policy in part (a), what would be the reserve at the end of 10 years after the change to the endowment form?

Section 17.6

17.17. Use the results of Example 17.6.1 and the assumptions that $\mu_x^{(1)}(t) = \mu^{(1)}$, $\mu_x^{(2)}(t) = \mu^{(2)}$, $\mu_x^{(3)}(t) = \mu^{(3)}$, $(h\mu)_{x+t}^{(1)}(u) = (h\mu)^{(1)}$, $(h\mu)_{x+t}^{(2)}(u) = (h\mu)^{(2)}$, and ${}_tCV = 0$ and the force of interest $\delta > 0$ to show that

$$\pi = \mu^{(1)} + \mu^{(3)} \left[\frac{(hu)^{(1)} + (0.25)(hu)^{(2)} + 0.25\delta}{(hu)^{(1)} + (hu)^{(2)} + \delta} \right].$$

17.18. a. Rewrite the result in Exercise 17.17 as

$$\pi - \bar{P}(\bar{A}_x) = \mu^{(3)} \left[\frac{(h\mu)^{(1)} + 0.25(h\mu)^{(2)} + 0.25\delta}{(h\mu)^{(1)} + (h\mu)^{(2)} + \delta} \right].$$

b. This difference can be viewed as the extra benefit premium rate associated with the accelerated benefit. If $(h\mu)^{(2)} = 0$,

$$\pi - \bar{P}(\bar{A}_x) = \mu^{(3)} \left[\frac{(h\mu)^{(1)} + 0.25\delta}{(h\mu)^{(1)} + \delta} \right].$$

c. Observe the behavior of $\pi - \bar{P}(\bar{A}_x)$ as $\delta \to 0$ and as $\delta \to \infty$ in (b).

18

ADVANCED MULTIPLE LIFE THEORY

18.1 Introduction

In Chapter 9 we defined the joint-life and last-survivor statuses, expressing their time-until-failure random variables in terms of those for the individual lives. We extended the concepts of Chapters 3–5 for this to obtain actuarial functions for the statuses of just two lives. On the assumption that the future lifetimes of the two lives were independent, the multiple-life actuarial functions were expressed in terms of the single-life functions, making it possible to calculate them using readily available life tables for single lives. Probabilities, annuities, and insurances, contingent on the order of the deaths of the two lives, were also discussed in Chapter 9.

In this chapter we extend these ideas to more than two lives. In fact, with more than two lives, the idea of a surviving status can be generalized. (See Sections 18.2 and 18.3.) With the ultimate goal of numerical evaluation of these functions, we use Theorem 18.2.1 to express the survival functions of these statuses in terms of only joint-life survival functions. Again, under the independent lifetimes assumptions, we evaluate these joint-life survival functions as products of individual life survival probabilities. Theorem 18.2.1 is a form of a general theorem of probability theory used in the so-called inclusion-exclusion method. A statement and proof of this more general theorem, designated as Theorem 18.2.2, is found in the Appendix to this chapter.

Contingent probabilities and functions and reversionary annuities are also generalized to more than two lives.

The annual benefit premium models of Chapters 6 and 7 are developed for the multiple-life statuses in Section 18.7, with some discussion of practical issues involved with one of the most common of these products, the second-to-die insurance policy.

18.2 More General Statuses

For m lives, $(x_1), (x_2), \ldots, (x_m)$, the ***k-survivor status,*** denoted by

$$\left(\overline{x_1 x_2 \cdots x_m}^{\,k}\right),$$

exists as long as at least k of the m lives survive and fails upon the $m-k+1$-st death among the m lives. This is a survival status as defined in Section 9.3. The previously defined joint-life status and the last-survivor status are, respectively, the m-survivor status and the 1-survivor status. When referring to either of these statuses, we will use its special symbol rather than the general k-survivor form. The future lifetime of the k-survivor status is the k-th largest of the set of m lifetimes $T(x_1), T(x_2), \ldots, T(x_m)$. Like the future lifetimes in Chapters 3 and 9, this one for the k-survivor status is the period of existence from a fixed initial time to a random termination time. With only a change in notation to display the status

$$\left(\overline{x_1 x_2 \cdots x_m}^{\,k}\right),$$

the probability distribution and life table functions of Chapter 3 are applicable to the future lifetime of this status.

Annuity and insurance benefits are defined in terms of the future lifetime of a k-survivor status just as they were in Chapters 4 and 5 for (x). For a continuous annuity of 1 payable annually as long as at least k of the m lives survive, we have from (5.2.4) that the actuarial present value is

$$\bar{a}_{\overline{x_1 x_2 \cdots x_m}^{\,k}} = \int_0^\infty v^t \, {}_tp_{\overline{x_1 x_2 \cdots x_m}^{\,k}} \, dt. \tag{18.2.1}$$

The insurance paying a unit on the $m-k+1$-st death among the m lives would have the actuarial present value given by (4.2.6), that is,

$$\bar{A}_{\overline{x_1 x_2 \cdots x_m}^{\,k}} = \int_0^\infty v^t \, {}_tp_{\overline{x_1 x_2 \cdots x_m}^{\,k}} \, \mu_{\overline{x_1 x_2 \cdots x_m}^{\,k}}(t) \, dt. \tag{18.2.2}$$

For the analysis and evaluation of probabilities and actuarial present values for these benefits (and other combinations of benefits) we will define a new type of status. For the m lives $(x_1), (x_2), \ldots, (x_m)$, the ***[k]-deferred survivor status*** exists while exactly k of the m lives survive; that is, it comes into existence upon the $m-k$-th death and remains in existence until the next death. The notation for this status will be

$$\left(\overline{x_1 x_2 \cdots x_m}^{\,[k]}\right).$$

For $k = m$, the $[m]$-deferred survivor status coincides with the m-survivor status. For $k = 0$, the $[0]$-deferred survivor status exists forever following the m-th death.

The status

$$\left(\overline{x_1 x_2 \cdots x_m}^{\,[k]}\right)$$

is not a survival status as defined in Chapter 9. For instance, for $k < m$, ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[k]}$, which is the probability that exactly k of the m lives are surviving at time t, does not equal 1 at $t = 0$ and thus does not meet the requirements of a survival function as given in Table 3.2.1. In addition, as $t \to \infty$, ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[0]}$ goes to 1, which also violates those requirements. Moreover, for a $[k]$-deferred survivor status, the period of existence does not equal the time to failure. This means that annuity benefits must be carefully defined for this new status. The annuity with actuarial present value $\bar{a}_{\overline{x_1x_2\cdots x_m}}^{[k]}$ is defined to be payable during the future lifetime of the $[k]$-deferred survivor status; hence it is a deferred annuity with a deferral period of random length. Since the time of failure of the $[k]$-deferred survivor status is equal to the time of failure of the k-survivor status, insurance benefits payable upon failure of the deferred status are essentially applications of the k-survivor status.

Example 18.2.1

A continuous annuity is payable as long as any of (w), (x), (y), and (z) are alive. At each death the annual rate of payment is reduced by 50%. Express the actuarial present value of such an annuity in terms of $\bar{a}_{\overline{wxyz}}^{[k]}$, $k = 1, 2, 3, 4$. Assume a unit initial benefit rate.

Solution:

The actuarial present value is

$$\bar{a}_{\overline{wxyz}}^{[4]} + \frac{1}{2}\,\bar{a}_{\overline{wxyz}}^{[3]} + \frac{1}{4}\,\bar{a}_{\overline{wxyz}}^{[2]} + \frac{1}{8}\,\bar{a}_{\overline{wxyz}}^{[1]}.$$

The discussion of this annuity is continued (after Theorem 18.2.1) in Example 18.2.2.

▼

In Chapter 9 we expressed last-survivor probabilities and related actuarial present values in terms of those for single- and joint-life statuses. We shall use the following theorem to aid in obtaining the same results for the k-survivor statuses. A more general statement of the theorem for arbitrary events, as well as its proof, is given in the Appendix to this chapter. The basic symbols and operations of the calculus of finite differences, as used below in Theorem 18.2.1, are reviewed in Appendix 5.

Theorem 18.2.1

Let

$$ {}_tD_j = \sum {}_tp_{x_{(1)}x_{(2)}\cdots x_{(j)}}$$

where the sum is over all combinations of j out of the m lives. Then, for arbitrary numbers, $c_0, c_1, \ldots, c_m$,

$$\sum_{j=0}^{m} c_j\, {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]} = c_0 + \sum_{j=1}^{m} \Delta^j c_0\, {}_tD_j. \tag{18.2.3}$$

Theorem 18.2.1 is applicable for lives with dependent future-lifetime random variables. In most applications, however, we will assume mutually independent future lifetimes and calculate the terms of the ${}_tD_j'$s as products of individual survival probabilities.

An illustration of the algebra implicit in (18.2.3) is provided by

$$\begin{aligned} c_1\, {}_tp_{\overline{xy}}^{[1]} + c_2\, {}_tp_{\overline{xy}}^{[2]} &= (\Delta c_0)({}_tp_x + {}_tp_y) + \Delta^2 c_0({}_tp_{xy}) \\ &= c_1({}_tp_x + {}_tp_y) + (c_2 - 2c_1)\,{}_tp_{xy}. \end{aligned}$$

Example 18.2.2

Express the actuarial present value of the annuity described in Example 18.2.1 in terms of actuarial present values for annuities on single- and joint-life statuses.

Solution:

The actuarial present value is

$$\int_0^\infty v^t \left[\sum_{j=1}^{4} \left(\frac{1}{2}\right)^{4-j} {}_tp_{\overline{wxyz}}^{[j]} \right] dt.$$

The coefficients and their differences are given below.

j	c_j	Δc_j	$\Delta^2 c_j$	$\Delta^3 c_j$	$\Delta^3 c_j$
0	0	1/8	0	1/8	0
1	1/8	1/8	1/8	1/8	—
2	1/4	1/4	1/4	—	—
3	1/2	1/2	—	—	—
4	1	—	—	—	—

Thus, the integral is equal to

$$\int_0^\infty v^t \left(\frac{1}{8}\, {}_tD_1 + \frac{1}{8}\, {}_tD_3 \right) dt = \frac{1}{8}\,(\bar{a}_w + \bar{a}_x + \bar{a}_y + \bar{a}_z) + \frac{1}{8}\,(\bar{a}_{wxy} + \bar{a}_{wxz} + \bar{a}_{wyz} + \bar{a}_{xyz}).$$

Such expressions can be examined for reasonableness by interpreting the final form in terms of a collection of annuities for which, at any outcome, the total of their rates of payment equals the rate of payment of the original annuity. For example, in this particular case the original annuity commences payment at rate 1, while the final form relates to a collection of four single-life and four joint-life annuities all paying at the rate 1/8. At a time between the first and second deaths, the original annuity's rate would be 1/2, while three of the single-life annuities and one of the joint-life annuities would still be in payment status, each with rate 1/8. The rates at other times can be compared in a similar way. ▼

We have an expression for ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[k]}$ as a special case of Theorem 18.2.1.

Corollary 18.2.1

$$ {}_tp_{\overline{x_1x_2\cdots x_m}}^{[k]} = \sum_{j=k}^{m} (-1)^{j-k} \binom{j}{k} {}_tD_j. \tag{18.2.4}$$

Proof:

In Theorem 18.2.1, set $c_k = 1$ and $c_j = 0$ for $j \neq k$. For these c_j's, $\Delta^j c_0 = (\mathrm{E} - 1)^j c_0 = (-1)^{j-k} \binom{j}{k}$, $j = k, k+1, \ldots, m$. ■

Example 18.2.3

Express the actuarial present value of a continuous annuity of 1 per annum while exactly three of five lives survive, in terms of actuarial present values of joint-life annuities.

Solution:

Using (5.2.4) and then (18.2.4), we have

$$\begin{aligned}
\bar{a}_{\overline{x_1x_2x_3x_4x_5}}^{[3]} &= \int_0^\infty v^t \, {}_tp_{\overline{x_1x_2x_3x_4x_5}}^{[3]} \, dt \\
&= \int_0^\infty v^t \, ({}_tD_3 - 4\, {}_tD_4 + 10\, {}_tD_5) \, dt \\
&= \bar{a}_{x_1x_2x_3} + \bar{a}_{x_1x_2x_4} + 8 \text{ more joint three-life annuity values} \\
&\quad - 4(a_{x_1x_2x_3x_4} + \bar{a}_{x_1x_2x_3x_5} + 3 \text{ more joint four-life annuity values}) \\
&\quad + 10\bar{a}_{x_1x_2x_3x_4x_5}.
\end{aligned}$$

▼

From the relationship

$$ {}_tp_{\overline{x_1x_2\cdots x_m}}^{\,h} = \sum_{j=h}^{m} {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}, \tag{18.2.5}$$

we have the following corollary to Theorem 18.2.1.

Corollary 18.2.2

For arbitrary numbers $d_0, d_1, d_m, \ldots, d_m$,

$$\sum_{j=0}^{m} d_j \, {}_tp_{\overline{x_1x_2\cdots x_m}}^{\,j} = d_0 + \sum_{j=1}^{m} \Delta^{j-1} d_1 \, {}_tD_j. \tag{18.2.6}$$

Proof:

Using (18.2.5), we start with

$$\sum_{h=0}^{m} d_h \, {}_tp_{\overline{x_1x_2\cdots x_m}}^{\,h} = \sum_{h=0}^{m} \sum_{j=h}^{m} d_h \, {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}.$$

Interchanging the summations, we can write

$$\sum_{j=0}^{m} d_j \, {}_tp_{\overline{x_1x_2\cdots x_m}}^{\,j} = \sum_{j=0}^{m} \left(\sum_{h=0}^{j} d_h \right) {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]},$$

which, by defining $c_j = \Sigma_{h=0}^{j} d_h$ for $j = 0, 1, \ldots, m$, is in the form of (18.2.3). For these c's, $c_0 = d_0$ and $\Delta c_j = d_{j+1}$ for $j = 0, 1, \ldots, m - 1$, thus $\Delta^j c_0 = \Delta^{j-1}(\Delta c_0) = \Delta^{j-1} d_1$ for $j = 1, 2, \ldots, m$. Then we have, from the right-hand side of (18.2.3),

$$\sum_{j=0}^{m} d_j \; {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;j} = d_0 + \sum_{j=1}^{m} \Delta^{j-1} d_1 \; {}_tD_j. \qquad \blacksquare$$

Corollary 18.2.1 can be used to express the survival function of the k-survivor status in terms of joint- and single-life survival functions.

Corollary 18.2.3

$$ {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;k} = \sum_{j=k}^{m} \left[(-1)^{j-k} \binom{j-1}{k-1}\right] {}_tD_j. \tag{18.2.7}$$

Proof:

In Corollary 18.2.1, set $d_k = 1$ and $d_j = 0$, for $j \neq k$. For these d's, $\Delta^{j-1} d_1 = (\mathrm{E} - 1)^{j-1} d_1 = (-1)^{j-k} \binom{j-1}{k-1}$, $j = k, k + 1, \ldots, m$. $\blacksquare$

From the expression for its survival function in (18.2.7) we can obtain, by differentiation, a parallel expression for the p.d.f. of the future lifetime of the k-survivor status, T, as

$$f_T(t) = \frac{d}{dt}\left(1 - {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;k}\right) = \sum_{j=k}^{m} (-1)^{j-k} \binom{j-1}{k-1} (-{}_tD_j'). \tag{18.2.8}$$

The actuarial present value and other characteristics of the probability distribution of the present value of a set of payments that depend on T can be determined using (18.2.7) or (18.2.8). In such determinations we use the fact that $-{}_tD_j'$ is the sum of the p.d.f.'s of the future lifetimes of the $\binom{m}{j}$ joint j-life statuses of the m lives.

Example 18.2.4

Let T denote the future lifetime of the last-survivor status of three lives. Exhibit in terms of joint- and single-life functions

a. The survival function

b. $\mathrm{E}[v^T]$

c. $\mathrm{E}[\bar{a}_{\overline{T|}}]$.

Solution:

a. By (18.2.7),

$$\begin{aligned} {}_tp_{\overline{x_1 x_2 x_3}} &= \sum_{j=1}^{3} (-1)^{j-1} \binom{j-1}{0} \, {}_tD_j \\ &= {}_tD_1 + (-1) \, {}_tD_2 + {}_tD_3 \end{aligned}$$

where

$$ {}_tD_1 = {}_tp_{x_1} + {}_tp_{x_2} + {}_tp_{x_3}, $$

$$ {}_tD_2 = {}_tp_{x_1x_2} + {}_tp_{x_1x_3} + {}_tp_{x_2x_3}, $$

$$ {}_tD_3 = {}_tp_{x_1x_2x_3}. $$

b. We denote $E[v^T]$ by $\bar{A}_{\overline{x_1x_2x_3}}$ and use (18.2.8) to obtain

$$ \bar{A}_{\overline{x_1x_2x_3}} = \int_0^\infty v^t(-1)({}_tD_1' - {}_tD_2' + {}_tD_3')\,dt $$

$$ = \bar{A}_{x_1} + \bar{A}_{x_2} + \bar{A}_{x_3} - (\bar{A}_{x_1x_2} + \bar{A}_{x_1x_3} + \bar{A}_{x_2x_3}) + \bar{A}_{x_1x_2x_3}. $$

c. Replacing v^T by $\bar{a}_{\overline{T}|}$ in part (b) and denoting $E[\bar{a}_{\overline{T}|}]$ by $\bar{a}_{\overline{x_1x_2x_3}}$, we have

$$ \bar{a}_{\overline{x_1x_2x_3}} = \bar{a}_{x_1} + \bar{a}_{x_2} + \bar{a}_{x_3} - (\bar{a}_{x_1x_2} + \bar{a}_{x_1x_3} + \bar{a}_{x_2x_3}) + \bar{a}_{x_1x_2x_3}. $$

For any survival status, $v^T + \delta\bar{a}_{\overline{T}|} = 1$, so we can calculate either of the expected values from the other using $\bar{A}_{\overline{x_1x_2x_3}} + \delta\bar{a}_{\overline{x_1x_2x_3}} = 1$. ▼

By differentiating both sides of (18.2.6), we can extend the relationship to the corresponding p.d.f.'s. This can be used for insurances paying an amount upon each death among the m lives.

Example 18.2.5

Consider an insurance on (x), (y), and (z) paying 1 on the first death, 2 on the second death, and 3 on the third death. Express the actuarial present value for the insurance in terms of actuarial present values for unit amount insurances on single- and joint-life statuses.

Solution:

Let $f_j(t)$ be the p.d.f. for the future lifetime of the j-survivor status. The actuarial present value is

$$ \int_0^\infty v^t[1f_3(t) + 2f_2(t) + 3f_1(t)]\,dt. $$

In the notation of (18.2.6) we have the following:

j	d_j	Δd_j	$\Delta^2 d_j$	$\Delta^3 d_j$
0	0	3	−4	4
1	3	−1	0	—
2	2	−1	—	—
3	1	—	—	—

Hence the actuarial present value is

$$ \int_0^\infty v^t(-1)(3\,{}_tD_1' - {}_tD_2')\,dt = 3\,(\bar{A}_x + \bar{A}_y + \bar{A}_z) - (\bar{A}_{xy} + \bar{A}_{xz} + \bar{A}_{yz}). $$ ▼

18.3 Compound Statuses

In the previous section we defined statuses for several lives by means of the k-survivor status. Others statuses can be defined by compounding. A ***compound status*** is said to exist if the status is based on a combination of statuses, at least one of which is itself a status involving more than one life. We examine some possibilities in Example 18.3.1.

Example 18.3.1

Describe the conditions of payment for the annuities and insurances corresponding to the following actuarial present-value symbols:

a. $\bar{a}_{\overline{wx}:\overline{yz}}$ b. $\bar{a}_{\overline{\overline{wx}:(yz)}}$ c. $\bar{a}_{\overline{(x:\overline{n|}):(yz:\overline{m|})}}$

d. $\bar{A}_{\overline{wx}:yz}$ e. $\bar{A}_{\overline{(wx):(yz)}}^{\,2}$ f. $\bar{A}_{\overline{(wx):y:z}}$.

Solution:

a. The annuity is payable continuously at the rate of 1 per year while at least one of (w) and (x) and at least one of (y) and (z) survive. Thus the annuity is payable while three or four of the lives survive and while two survive if one is from the pair (w), (x) and the other from the pair (y), (z).
b. The annuity is payable continuously at the rate of 1 per year while at least two of the four lives survive and also while only one survives if that survivor is either (w) or (x).
c. The annuity is payable continuously at the rate of 1 per year while either (x) is alive and an n-year period has not elapsed, or while both (y) and (z) are alive and an m-year period has not elapsed.
d. A unit amount is payable at the moment of the first death if (y) or (z) dies first, and otherwise on the second death.
e. A unit amount is payable at the moment of the second death if two deaths consist of one from the (w), (x) group and the other from the (y), (z) group. If not, the payment is made at the time of the third death.
f. A unit amount is payable only after (y), (z), and one of (w) and (x) have died. In other words, it is payable at the moment of the third death if (w) or (x) remains alive, but otherwise at the moment of the fourth death. ▼

In applications, a numerical value for any one of these actuarial present values would most likely be obtained by first expressing it in terms of those for single- and joint-life statuses. The relationships in Section 9.4 among $T(xy)$, $T(\overline{xy})$, $T(x)$, and $T(y)$, and among $K(xy)$, $K(\overline{xy})$, $K(x)$, and $K(y)$, hold for survival statuses (u), (v). For example,

$$v^{T(uv)} + v^{T(\overline{uv})} = v^{T(u)} + v^{T(v)}. \tag{18.3.1}$$

Employing parts of Example 18.3.1, we will illustrate the process of using (18.3.1) and similar identities. First, we consider part (e):

$$\bar{A}_{\overline{(wx):(yz)}} = \bar{A}_{wx} + \bar{A}_{yz} - \bar{A}_{wxyz}.$$

Here, $(u) = (wx)$ and $(v) = (yz)$. To write $\bar{A}_{(wx):(yz)}$ as $\bar{A}_{wxyz}$ we have used

$$\min\{\min[T(w), T(x)], \min[T(y), T(z)]\} = \min[T(w), T(x), T(y), T(z)]. \quad (18.3.2)$$

For part (c) of Example 18.3.1, we have

$$\bar{a}_{\overline{(x:\overline{n|})(yz:\overline{m|})}} = \bar{a}_{x:\overline{n|}} + \bar{a}_{yz:\overline{m|}} - \bar{a}_{xyz:\overline{n|}}$$

where the last term is obtained from

$$\min[T(x), T(y), T(z), T(\overline{n|}), T(\overline{m|})] = \min[T(x), T(y), T(z), T(\overline{n|})]$$

for the case $n \le m$.

Other arrangements, as in parts (a), (b), (d), and (f) of Example 18.3.1, require the use of other relationships. For part (a), we want

$$\bar{a}_{\overline{wx}:\overline{yz}} = \mathrm{E}[\bar{a}_{\overline{T|}}] \quad (18.3.3)$$

where

$$\begin{aligned} T &= \min[T(\overline{wx}), T(\overline{yz})] \\ &= \min\{\max[T(w), T(x)], \max[T(y), T(z)]\}. \end{aligned}$$

A simple answer, like (18.3.2), is not available for this random variable. To proceed, let us first assume that $T(\overline{wx})$ and $T(\overline{yz})$ are independent and look at $s(t)$, the survival function of T. Thus

$$\begin{aligned} s(t) &= \Pr(T > t) = \Pr(\min[T(\overline{wx}), T(\overline{yz})] > t) \\ &= \Pr[T(\overline{wx}) > t, T(\overline{yz}) > t] \\ &= \Pr[T(\overline{wx}) > t] \Pr[T(\overline{yz}) > t] \\ &= {}_tp_{\overline{wx}}\ {}_tp_{\overline{yz}} \\ &= ({}_tp_w + {}_tp_x - {}_tp_{wx})({}_tp_y + {}_tp_z - {}_tp_{yz}) \\ &= {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} - {}_tp_{wyz} - {}_tp_{xyx} \\ &\quad - {}_tp_{wxy} - {}_tp_{wxz} + {}_tp_{wxyz} \end{aligned} \quad (18.3.4)$$

for the independent case. Now, using (18.3.4), we obtain

$$\begin{aligned} \bar{a}_{\overline{wx}:\overline{yz}} &= \int_0^\infty v^t\ s(t)\ dt \\ &= \bar{a}_{wy} + \bar{a}_{wz} + \bar{a}_{xy} + \bar{a}_{xz} - \bar{a}_{wyz} - \bar{a}_{xyz} - \bar{a}_{wxy} - \bar{a}_{wxz} + \bar{a}_{wxyz}. \end{aligned} \quad (18.3.5)$$

We return to (18.3.4) and show that a parallel relationship for the random variables holds, and then that (18.3.4) is true without the independence assumption. We start with the assertion that for all possible outcomes,

$$\begin{aligned} T(\overline{wx}:\overline{yz}) &= T(wy) + T(wz) + T(xy) + T(xz) - T(wyz) \\ &\quad - T(xyz) - T(wxy) - T(wxz) + T(wxyz). \end{aligned} \quad (18.3.6)$$

The outcomes can be collected into 24 mutually exclusive events according to the order of $T(w)$, $T(x)$, $T(y)$, and $T(z)$. Since the given assertion is symmetric in w and

x and symmetric in y and z, only six different outcomes require verification. As an example, consider $T(w) < T(x) < T(y) < T(z)$ for which the left-hand side of (18.3.6) is $T(\overline{wx}{:}\overline{yz}) = T(x)$ and the right-hand side is, on a term-by-term basis,

$$T(w) + T(w) + T(x) + T(x) - T(w) - T(x) - T(w) - T(w) + T(w) = T(x)$$

as required. The other cases can be verified in the same way.

An expression in annuities that is parallel to (18.3.6) can be established by similar reasoning. Thus,

$$\begin{aligned} \bar{a}_{\overline{T(\overline{wx}:\overline{yz})}|} &= \bar{a}_{\overline{T(wy)}|} + \bar{a}_{\overline{T(wz)}|} + \bar{a}_{\overline{T(xy)}|} + \bar{a}_{\overline{T(xz)}|} \\ &\quad - \bar{a}_{\overline{T(wyz)}|} - \bar{a}_{\overline{T(xyz)}|} - \bar{a}_{\overline{T(wxy)}|} - \bar{a}_{\overline{T(wxz)}|} + \bar{a}_{\overline{T(wxyz)}|} . \end{aligned} \tag{18.3.7}$$

Taking expectations of both sides of this expression we have (18.3.5).

We emphasize two aspects of the independence assumption for this case. It would not be used to establish (18.3.7), nor is it required in the expectation calculation used to obtain (18.3.5) from (18.3.7). Again, however, to obtain joint-life status functions from single-life life tables, for convenience, we do assume that individual future lifetimes are independent.

18.4 Contingent Probabilities and Insurances

In this section we extend the notion of contingent functions (Section 9.9) to more than two lives. We start with an integral expression for the required probability, or actuarial present value, which can then be rewritten in terms of probabilities or actuarial present value defined on the first death. It is then possible to use some of the techniques of Section 9.10 to complete the evaluation. In any case, numerical integration methods can be used.

To obtain an integral expression for a probability, we use

$$\Pr(A) = \int_{-\infty}^{\infty} \Pr(A|T = t)\, f_T(t)\, dt \tag{18.4.1}$$

where T will usually mean the time of death of an individual life.

Example 18.4.1

Express ${}_nq_{wx\overset{2}{y}z}$ in terms of functions contingent on the first death.

Solution:

Here A is the event that (y) is the second life among (w), (x), (y), and (z) to die and does so within n years. Since A is defined by $T(y)$, we use $T(y)$ as T in (18.4.1) to obtain

$${}_nq_{wx\overset{2}{y}z} = \int_0^n \Pr(A|T(y) = t)\, {}_tp_y\, \mu_y(t)\, dt.$$

The integral's limits follow from

$$f_{T(y)}(t) = 0 \qquad t < 0$$

and

$$\Pr[A \mid T(y) = t] = 0 \qquad t > n.$$

Now, (y) will be the second to die if and only if there are exactly two of (w), (x), and (z) surviving at that time. If we assume that $T(y)$ is independent of $T(w)$, $T(x)$, and $T(z)$, then

$$\Pr[A \mid T(y) = t] = {}_tp_{\overline{wxz}}^{[2]} \qquad t < n$$

and

$$\begin{aligned} {}_nq_{wx\overset{2}{y}z} &= \int_0^n {}_tp_{\overline{wxz}}^{[2]}\, {}_tp_y\, \mu_y(t)\, dt \\ &= \int_0^n ({}_tD_2 - 3\, {}_tD_3)\, {}_tp_y\, \mu_y(t)\, dt \\ &= {}_nq_{wx\overset{1}{y}} + {}_nq_{w\overset{1}{y}z} + {}_nq_{x\overset{1}{y}z} - 3\, {}_nq_{wx\overset{1}{y}z}. \end{aligned}$$

(The second integral comes from applying Theorem 18.2.1.) ▼

The similarity of the final expression of Example 18.4.1 to previous results that did not require independence suggests the assumed independence was not necessary. Alternative derivations in Exercises 18.18 and 18.38 will verify that this is the case.

A contingent insurance can be analyzed by a similar procedure based on

$$\mathrm{E}[Z] = \int_{-\infty}^{\infty} \mathrm{E}[Z \mid T = t]\, f_T(t)\, dt. \tag{18.4.2}$$

Example 18.4.2

Express $\bar{A}_{wx\overset{2}{y}}$ in terms of actuarial present values for insurances contingent on the first death only.

Solution:

Let Z be the random variable representing the present value at issue of the insurance benefit. Since the insurance is payable on the death of (y), we choose $T(y)$ to play the role of T in the conditional expectation of (18.4.2):

$$\bar{A}_{wx\overset{2}{y}} = \mathrm{E}[Z] = \int_0^{\infty} \mathrm{E}[Z \mid T(y) = t]\, {}_tp_y\, \mu_y(t)\, dt.$$

If, at the death of (y) at duration t, there is exactly one of (w) and (x) surviving, the unit benefit will be paid; otherwise no benefit will be paid. Thus, we have

$$\mathrm{E}[Z \mid T(y) = t] = v^t\, {}_tp_{\overline{wx}}^{[1]}$$

and

$$
\begin{aligned}
\bar{A}_{wx\overset{2}{y}} &= \int_0^\infty v^t\, {}_tp_{\overline{wx}}^{[1]}\, {}_tp_y\, \mu_y(t)\, dt \\
&= \int_0^\infty v^t({}_tD_1 - 2\, {}_tD_2)\, {}_tp_y\, \mu_y(t)\, dt \\
&= \bar{A}_{x\overset{1}{y}} + \bar{A}_{w\overset{1}{y}} - 2\, \bar{A}_{wx\overset{1}{y}}\,.
\end{aligned}
$$

Because the benefit is 1, Var(Z) can be obtained by the rule of moments. ▼

18.5 Compound Contingent Functions

The functions in this section are distinguished from those in the previous section by specifications on the order of deaths prior to the death when the benefits are paid, or the event is defined. These specifications on the prior deaths are indicated by numbers placed below the symbols for the lives involved. We examine two such symbols and note the distinctions possible in the notation.

The symbols ${}_nq_{\underset{1}{x}\overset{2}{y}z}$ and ${}_nq_{\underset{1}{x}y\overset{3}{z}}$ both refer to events in which $T(x) < T(y) < T(z)$. They differ, though, in that the second death must occur before time n in the first event, while the third death must precede time n in the second event.

It is not always possible to express compound contingent functions completely in terms of functions depending only on the first death. On the other hand, the function can always be expressed as one or more multiple integrals of the joint p.d.f. of the future-lifetime random variables of the lives involved. The example of this general procedure (Example 18.5.1) is more complex than others in this section.

Example 18.5.1

Derive an expression for the probability that (w), (x), (y), and (z) die in that order with less than 10 years between the deaths of (w) and (z) and less than 5 years between the deaths of (x) and (y).

Solution:

We first define the event A for use in a multivariate version of (18.4.1):

$$
A = \left\{ \begin{matrix} T(w) < T(x) < T(y) < T(z) \\ T(z) - T(w) < 10 \\ T(y) - T(x) < 5 \end{matrix} \right\}. \tag{18.5.1}
$$

We choose to condition on $T(w)$ and $T(x)$ because these are involved in both the upper and lower bounds for $T(y)$ and $T(z)$. That is,

$$
\Pr(A) = \int_0^\infty \int_0^\infty \Pr(A \mid [T(w) = r] \cap [T(x) = s])\, g_{T(w),T(x)}(r, s)\, ds\, dr \tag{18.5.2}
$$

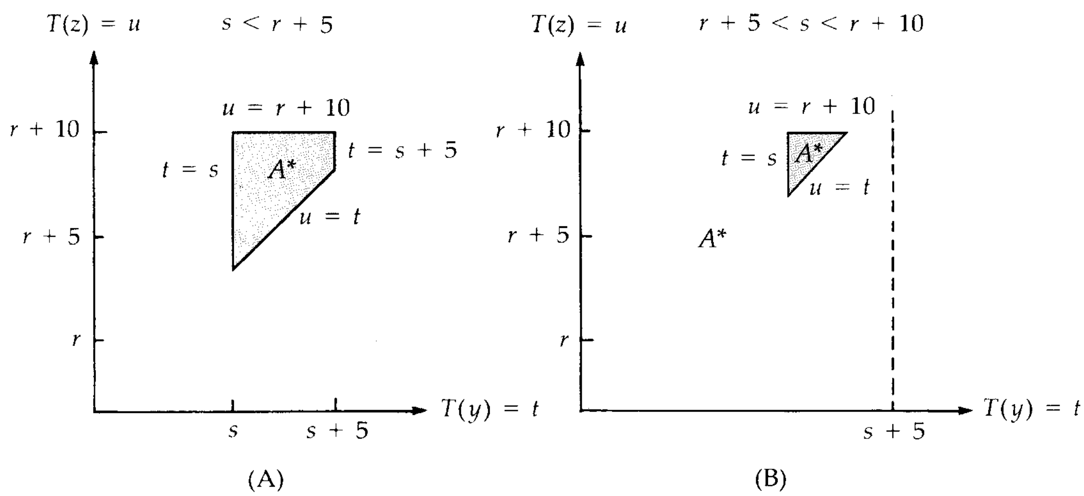
where $g_{T(w),T(x)}(r, s)$ is the joint p.d.f. of $T(w)$ and $T(x)$. Now, $\Pr[A|(T(w) = r) \cap (T(x) = s)]$ is equal to $\Pr(A^*)$ where

$$A^* = \begin{Bmatrix} r < s < T(y) < T(z) < r + 10 \\ T(y) < s + 5 \end{Bmatrix},$$

and the probability is calculated by the conditional distribution of $T(y)$ and $T(z)$, given $T(x) = s$ and $T(w) = r$. Thus, $\Pr(A^*)$ can be set up in the sample space of the random variables $T(y)$ and $T(z)$. Two cases are displayed in Figure 18.5.1.

FIGURE 18.5.1

Cases (A), $s < r + 5$, and (B), $r + 5 < s < r + 10$



Using the abbreviated notation $h(t, u)$ for the conditional p.d.f. of $T(y)$ and $T(z)$ given $T(w) = r$ and $T(x) = s$, we have

$$\Pr(A^*) = \begin{cases} \displaystyle\int_s^{s+5} \int_t^{r+10} h(t, u)\, du\, dt & r < s < r + 5 \\ \displaystyle\int_s^{r+10} \int_t^{r+10} h(t, u)\, du\, dt & r + 5 < s < r + 10 \\ 0 & s > r + 10 \text{ or } s < r. \end{cases}$$

Substituting into (18.5.2), we have

$$\Pr(A) = \int_0^{\infty} \int_r^{r+5} \left[\int_s^{s+5} \int_t^{r+10} h(t, u) g_{T(w),T(x)}(r, s)\, du\, dt \right] ds\, dr$$
$$+ \int_0^{\infty} \int_{r+5}^{r+10} \left[\int_s^{r+10} \int_t^{r+10} h(t, u) g_{T(w),T(x)}(r, s)\, du\, dt \right] ds\, dr.$$

Under the assumption of mutually independent future lifetimes, the integrand can be replaced by

$${}_rp_w\ \mu_w(r)\ {}_sp_x\ \mu_x(s)\ {}_tp_y\ \mu_y(t)\ {}_up_z\ \mu_z(u).$$ ▼

We now examine some compound contingent probabilities that can be written in terms of single integrals. We will first obtain equivalent forms for a probability by applying (18.4.1).

Example 18.5.2

Write three different integrals for ${}_nq_{\underset{1}{x}y\overset{3}{z}}$ and reduce one of them to probability functions dependent on only the first death. Assume mutually independent future lifetimes.

Solution:

Here $A = \{T(x) < T(y) < T(z) < n\}$. We set three integrals by conditioning on each of the future lifetimes:

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^\infty \Pr[A|T(x) = t]\ {}_tp_x\ \mu_x(t)\ dt $$

and

$$ \Pr[A|T(x) = t] = \begin{cases} 0 & t > n \\ {}_tp_{yz}\ {}_{n-t}q_{y+t:\overset{2}{z+t}} & t \le n; \end{cases} $$

thus

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^n {}_tp_{yz}\ {}_{n-t}q_{y+t:\overset{2}{z+t}}\ {}_tp_x\ \mu_x(t)\ dt. $$

Similarly,

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^\infty \Pr[A|T(y) = t]\ {}_tp_y\ \mu_y(t)\ dt $$

$$ = \int_0^n {}_tq_x\ {}_tp_z\ {}_{n-t}q_{z+t}\ {}_tp_y\ \mu_y(t)\ dt $$

and

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^\infty \Pr[A|T(z) = t]\ {}_tp_z\ \mu_z(t)\ dt $$

$$ = \int_0^n {}_tq_{x\overset{2}{y}}\ {}_tp_z\ \mu_z(t)\ dt. $$

The second of these integrals can be expressed in terms of first-death probabilities as follows:

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^n (1 - {}_tp_x)({}_tp_z - {}_np_z)\ {}_tp_y\ \mu_y(t)\ dt $$

$$ = {}_nq_{\overset{1}{y}z} - {}_nq_{x\overset{1}{y}z} - {}_np_z\ ({}_nq_y - {}_nq_{x\overset{1}{y}}). $$

▼

Example 18.5.3

Use (18.4.1) to write four different integral expressions for ${}_nq_{\underset{1}{w}\underset{2}{x}\overset{3}{y}z}$. Assume mutually independent future lifetimes.

Solution:

Here $A = \{T(w) < T(x) < T(y) < T(z) \text{ and } T(y) < n\}$. Then

$$
\begin{aligned}
{}_nq_{\underset{1}{w}\underset{2}{x}\overset{3}{y}z} &= \int_0^n {}_tp_{xyz}\ {}_{n-t}q_{\underset{1}{x+t}:\overset{2}{y+t}:z+t}\ {}_tp_w\ \mu_w(t)\ dt \\
&= \int_0^n {}_tq_w\ {}_tp_{yz}\ {}_{n-t}q_{\overset{1}{y+t}:z+t}\ {}_tp_x\ \mu_x(t)\ dt \\
&= \int_0^n {}_tq_{w\overset{2}{x}}\ {}_tp_z\ {}_tp_y\ \mu_y(t)\ dt \\
&= \int_0^n {}_tq_{w\underset{1}{x}\overset{3}{y}}\ {}_tp_z\ \mu_z(t)\ dt + {}_nq_{w\underset{1}{x}\overset{3}{y}}\ {}_np_z. \qquad (18.5.3)
\end{aligned}
$$

The last line, obtained by conditioning on $T(z)$, requires one expression for $T(z) < n$ and another for $T(z) > n$. ▼

In the application of (18.4.1) to the examples of this section we have used the assumption of independent future lifetimes in writing the $\Pr[A|T = t]$ factors of the integrands. We now consider the numerical evaluation of these probabilities when a single Gompertz mortality law is used for each life involved.

Example 18.5.4

Under a Gompertz law, show that

$$
{}_\infty q_{\underset{1}{w}\underset{2}{x}\overset{3}{y}z} = {}_\infty q_{\overset{1}{w}xyz}\ {}_\infty q_{\overset{1}{x}yz}\ {}_\infty q_{\overset{1}{y}z}.
$$

Solution:

Letting $n \to \infty$ in (18.5.3), we have

$$
{}_\infty q_{\underset{1}{w}\underset{2}{x}\overset{3}{y}z} = \int_0^\infty {}_tq_w\ {}_tp_{yz}\ {}_\infty q_{\overset{1}{y+t}:z+t}\ {}_tp_x\ \mu_x(t)\ dt. \qquad (18.5.4)
$$

In Example 9.10.1(b) it was shown that, under the Gompertz mortality law,

$$
{}_nq_{\overset{1}{x}y} = \frac{c^x}{c^w}\ {}_nq_w \qquad (18.5.5)
$$

where $c^w = c^x + c^y$. Adapting this and substituting it for ${}_\infty q_{\overset{1}{\overline{y+t}}:z+t}$ in the integrand of (18.5.4), we obtain

$$
\begin{aligned}
{}_{\infty}q_{\substack{wxyz\\ 12}}^{\,3} &= \int_0^{\infty} \frac{c^{y+t}}{c^{y+t} + c^{z+t}}\, {}_tq_w\, {}_tp_{yz}\, {}_tp_x\, \mu_x(t)\, dt \\
&= \frac{c^y}{c^y + c^z} \left({}_{\infty}q^{1}_{xyz} - {}_{\infty}q_{w\overset{1}{x}yz}\right).
\end{aligned}
$$

Formula (18.5.5) can be extended to more than two lives and then used in this expression; therefore,

$$
\begin{aligned}
{}_{\infty}q_{\substack{wxyz\\ 12}}^{\,3} &= \frac{c^y}{c^y + c^z} \left(\frac{c^x}{c^x + c^y + c^z} - \frac{c^x}{c^w + c^x + c^y + c^z}\right) \\
&= \left(\frac{c^w}{c^w + c^x + c^y + c^z}\right)\left(\frac{c^x}{c^x + c^y + c^z}\right)\left(\frac{c^y}{c^y + c^z}\right) \\
&= {}_{\infty}q^{1}_{wxyz}\; {}_{\infty}q^{1}_{xyz}\; {}_{\infty}q^{1}_{yz}.
\end{aligned}
$$

▼

18.6 More Reversionary Annuities

In Section 9.7 we examined a number of insurance and annuity contracts involving more than one life. Included in that discussion were the more common types of reversionary annuities, those involving only two lives and some examples with terms certain. We consider examples with terms certain measured from a date of death and examples with contingent events defining the start of the annuity payments. We also restrict our discussion to continuous annuities.

Let us examine two reversionary annuities with a term certain measured from the date of death. For a reversionary annuity paying an n-year temporary annuity to (y) after the death of (x), the term certain is a deferred status, so we go back to first principles. The present value at policy issue, Z, is

$$
Z = \begin{cases} 0 & T(y) \le T(x) \\ v^{T(x)}\, \bar{a}_{\overline{T(y)-T(x)}|} & T(x) < T(y) \le T(x) + n \\ v^{T(x)}\, \bar{a}_{\overline{n}|} & T(x) + n \le T(y). \end{cases}
$$

Using (18.4.2) with conditioning on $T(x) = t$, we can write the actuarial present value as

$$
\begin{aligned}
\mathrm{E}[Z] &= \int_0^{\infty} \mathrm{E}[Z \mid T(x) = t]\; {}_tp_x\, \mu_x(t)\, dt \\
&= \int_0^{\infty} {}_tp_y\, v^t\, \bar{a}_{y+t:\overline{n}|}\; {}_tp_x\, \mu_x(t)\, dt. \qquad (18.6.1)
\end{aligned}
$$

By substituting

$$
\bar{a}_{y+t:\overline{n}|} = \int_t^{t+n} v^{s-t}\; {}_{s-t}p_{y+t}\, ds
$$

into (18.6.1) we obtain

$$E[Z] = \int_0^\infty \int_t^{t+n} v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, ds \, dt.$$

Next we interchange the order of integration so that

$$\begin{aligned}
E[Z] &= \int_0^n \int_0^s v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds + \int_n^\infty \int_{s-n}^s v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds \\
&= \int_0^n v^s \, {}_sp_y \, (1 - {}_sp_x) \, ds + \int_n^\infty v^s \, {}_sp_y \, ({}_{s-n}p_x - {}_sp_x) \, ds \\
&= \bar{a}_{y:\overline{n}|} - \bar{a}_{xy} + v^n \, {}_np_y \, \bar{a}_{x:y+n}. \qquad (18.6.2)
\end{aligned}$$

The second display in (18.6.2) is the current payment form for this actuarial present value.

Another reversionary annuity of this type would be one where the annuity starts n years after the death of (x) and pays only as long as (y) remains alive. The present value at policy issue, Z, is

$$Z = \begin{cases} 0 & T(y) \le T(x) + n \\ (v^{T(x)+n})\bar{a}_{\overline{T(y)-T(x)-n}|} & T(x) + n < T(y). \end{cases}$$

Using (18.4.2) with conditioning on $T(x) = t$, we can write the actuarial present value as

$$\begin{aligned}
E[Z] &= \int_0^\infty E[Z \mid T(x) = t] \, {}_tp_x \, \mu_x(t) \, dt \\
&= \int_0^\infty {}_{t+n}p_y \, v^{t+n} \, \bar{a}_{y+n+t} \, {}_tp_x \, \mu_x(t) \, dt. \qquad (18.6.3)
\end{aligned}$$

By substituting

$${}_{t+n}p_y \, v^{t+n} \, \bar{a}_{y+n+t} = \int_{t+n}^\infty v^s \, {}_sp_y \, ds$$

into (18.6.3), we obtain

$$E[Z] = \int_0^\infty \int_{t+n}^\infty v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, ds \, dt.$$

Next we interchange the order of integration, so that

$$\begin{aligned}
E[Z] &= \int_n^\infty \int_0^{s-n} v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds \\
&= \int_n^\infty v^s \, {}_sp_y \, (1 - {}_{s-n}p_x) \, ds \\
&= v^n \, {}_np_y \, (\bar{a}_{y+n} - \bar{a}_{x:y+n}) = v^n \, {}_np_y \, \bar{a}_{x|y+n}. \qquad (18.6.4)
\end{aligned}$$

Another class of reversionary annuities that we consider is of, perhaps, limited commercial interest: those where some contingent event must occur before payments start. We consider two such examples and proceed from first principles.

Example 18.6.1

Express the reversionary annuity's actuarial present value, which has symbol $\bar{a}^{1}_{xy|z}$, (a) by definition and (b) in the current payment form by interchanging the order of integration in your answer to (a).

Solution:

Using (18.4.2) and conditioning on $T(x) = t$,

$$\begin{aligned}
\bar{a}^{1}_{xy|z} &= \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, {}_tp_y \, {}_tp_z \, \bar{a}_{z+t} \, dt \\
&= \int_0^\infty {}_tp_x \, \mu_x(t) \, {}_tp_y \left(\int_t^\infty v^s \, {}_sp_z \, ds \right) dt \\
&= \int_0^\infty v^s \, {}_sp_z \left[\int_0^s {}_tp_x \, \mu_x(t) \, {}_tp_y \, dt \right] ds \\
&= \int_0^\infty v^s \, {}_sp_z \, {}_sq^{1}_{xy} \, ds.
\end{aligned}$$

▼

This result can be considered as the current payment form of the actuarial present value. It shows that the general form for reversionary annuities can be interpreted quite broadly with the possibility that the failure of status (u) can involve a contingent probability. In general, we have

$$\bar{a}_{u|v} = \int_0^\infty v^t \, {}_tp_v \, {}_tq_u \, dt. \tag{18.6.5}$$

Our next example shows a particularly simple case involving two lives where the actuarial present value can be reduced to a form not involving integrals.

Example 18.6.2

Express the actuarial present-value symbol $\bar{a}^{1}_{x:\overline{n}|\,|y}$ in a form free of integrals.

Solution:

By (18.6.5),

$$\begin{aligned}
\bar{a}^{1}_{x:\overline{n}|\,|y} &= \int_0^\infty v^t \, {}_tp_y \, {}_tq^{1}_{x:\overline{n}|} \, dt \\
&= \int_0^n v^t \, {}_tp_y \left[\int_0^t {}_sp_x \, \mu_x(s) \, ds \right] dt + \int_n^\infty v^t \, {}_tp_y \left[\int_0^n {}_sp_x \, \mu_x(s) \, ds \right] dt \\
&= \int_0^n v^t \, {}_tp_y \, (1 - {}_tp_x) \, dt + (1 - {}_np_x) \int_n^\infty v^t \, {}_tp_y \, dt \\
&= \bar{a}_y - \bar{a}_{xy:\overline{n}|} - v^n \, {}_np_{xy} \, \bar{a}_{y+n}.
\end{aligned}$$

▼

18.7 Benefit Premiums and Reserves

Here we examine benefit premiums and benefit reserves for the insurances of this chapter. As in Chapter 6, the benefit premium is defined by the equivalence principle. Following the development in Chapter 7, benefit reserves are defined prospectively as the conditional expectation of the future loss, given survival to the duration of the reserve.

The premium payment period must end no later than the time of claim payment and, in the case of contingent insurances, when it is clear that no claim payment can be made. The period may be shorter.

In the case of insurances payable on the first death, the premiums are payable only while all lives survive. Using the equivalence principle we have, for example, the following:

$$P_{xy}\,\ddot{a}_{xy} = A_{xy},$$

$$_{10}P^{\{4\}}(\bar{A}_{\overset{1}{xy}:\overline{20|}})\,\ddot{a}^{\{4\}}_{xy:\overline{10|}} = \bar{A}_{\overset{1}{xy}:\overline{20|}},$$

and

$$P(\bar{A}_{\overset{1}{x}yz})\ddot{a}_{xyz} = \bar{A}_{\overset{1}{x}yz}.$$

Insurances payable on the second or a later death give rise to more than one possible premium payment period. To minimize the benefit premium that can be charged for a particular insurance benefit, we use the longest period. The following example illustrates the process for a number of cases.

Example 18.7.1

Using the equivalence principle, write the equation for the following benefit premiums:

a. $P_{\overline{xy}}$ b. $P(\bar{A}_{\overset{2}{\overline{xyz}}})$ c. $P(\bar{A}_{\overline{wx}:yz})$

d. $P(\bar{A}_{x\overset{2}{y}z})$ e. $P(\bar{A}_{x\overset{2}{\underset{1}{y}}z})$.

Solution:

a. $P_{\overline{xy}}\,\ddot{a}_{\overline{xy}} = A_{\overline{xy}}$
b. $P(\bar{A}_{\overset{2}{\overline{xyz}}})\ddot{a}_{\overset{2}{\overline{xyz}}} = \bar{A}_{\overset{2}{\overline{xyz}}}$
c. $P(\bar{A}_{\overline{wx}:yz})\ddot{a}_{\overline{wx}:yz} = \bar{A}_{\overline{wx}:yz}$
d. As long as (y) and at least one of (x) and (z) are alive, payment of the benefit is still possible. Therefore,

$$P(\bar{A}_{x\overset{2}{y}z})\,\ddot{a}_{y:\overline{xz}} = \bar{A}_{x\overset{2}{y}z}.$$

e. In this case payment of the benefit is still possible if all are alive or if only (y) and (z) are alive. Thus the appropriate premium payment period is the lifetime of (yz), and

$$P(\bar{A}_{\substack{xyz\\1}}^{\;2})\,\ddot{a}_{yz} = \bar{A}_{\substack{xyz\\1}}^{\;2}. \qquad \blacktriangledown$$

As the conditional expectation of the future loss, the benefit reserve will depend on the condition of the status used in the calculation. The reserve is unique for an insurance payable on the first death because all lives must survive until termination of the insurance. We illustrate reserve formulas for two of these insurances:

$${}_5V^{\;1}_{\overline{xy}:\overline{10|}} = A^{\;1}_{\overline{x+5:y+5}:\overline{5|}} - P^{\;1}_{\overline{xy}:\overline{10|}}\,\ddot{a}_{x+5:y+5:\overline{5|}}$$

where

$$P^{\;1}_{\overline{xy}:\overline{10|}}\,\ddot{a}_{xy:\overline{10|}} = A^{\;1}_{\overline{xy}:\overline{10|}}$$

and

$${}_5V^1_{xyz} = A^{\;1}_{x+5:y+5:z+5} - P^1_{xyz}\,\ddot{a}_{x+5:y+5:z+5}\,.$$

For an insurance payable on the second or later death, the benefit reserve can be calculated with the given condition of the expectation being either (a) which lives are surviving or (b) only that the insurance has not terminated through the last death.

Consider the simple case of a fully continuous unit insurance payable upon the failure of $(\overline{xy})$ with premiums payable until the second death. Let ${}_tL$ be the future loss at t. Given the information about which of x and y (or both) are surviving at t, we would have

$$\mathrm{E}[{}_tL\,|\,T(x) > t \cap T(y) > t] = \bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\,\bar{a}_{\overline{x+t:y+t}}, \tag{18.7.1}$$

$$\mathrm{E}[{}_tL\,|\,T(x) > t \cap T(y) \le t] = \bar{A}_{x+t} - \bar{P}(\bar{A}_{\overline{xy}})\,\bar{a}_{x+t}, \tag{18.7.2}$$

or

$$\mathrm{E}[{}_tL\,|\,T(x) \le t \cap T(y) > t] = \bar{A}_{y+t} - \bar{P}(\bar{A}_{\overline{xy}})\,\bar{a}_{y+t}. \tag{18.7.3}$$

On the other hand if the given information is only that the survival status $(\overline{xy})$ has not failed, the benefit reserve is

$${}_t\bar{V}(\bar{A}_{\overline{xy}}) = \mathrm{E}[{}_tL\,|\,T(\overline{xy}) > t]$$

which we can calculate by the law of total probability as the sum

$$\begin{aligned}&\mathrm{E}[{}_tL\,|\,T(x) > t \cap T(y) \le t]\,\Pr[T(x) > t \cap T(y) \le t]\\&+ \mathrm{E}[{}_tL\,|\,T(x) \le t \cap T(y) > t]\,\Pr[T(x) \le t \cap T(y) > t]\\&+ \mathrm{E}[{}_tL\,|\,T(x) > t \cap T(y) > t]\,\Pr[T(x) > t \cap T(y) > t].\end{aligned} \tag{18.7.4}$$

In this expression the conditional expectations are given by (18.7.1)–(18.7.3). On the assumption of independent $T(x)$ and $T(y)$, the probabilities are of the form

$$\Pr[T(x) > t \cap T(y) \le t\,|\,T(\overline{xy}) > t] = \frac{{}_tp_x\,(1 - {}_tp_y)}{{}_tp_x\,(1 - {}_tp_y) + {}_tp_y\,(1 - {}_tp_x) + {}_tp_x\,{}_tp_y}.$$

Combining these we have

$$
{}_t\bar{V}(\bar{A}_{\overline{xy}}) = \left[\frac{1}{{}_tp_x\,(1 - {}_tp_y) + {}_tp_y\,(1 - {}_tp_x) + {}_tp_x\,{}_tp_y}\right] \{{}_tp_x(1 - {}_tp_y)[\bar{A}_{x+t} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{x+t}]
$$

$$
+ {}_tp_y(1 - {}_tp_x)[\bar{A}_{y+t} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{y+t}] + {}_tp_x\,{}_tp_y\,[\bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{\overline{x+t:y+t}}]\}. \tag{18.7.5}
$$

We now use the results from Section 9.7,

$$
\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}
$$

and

$$
\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}\,,
$$

in the final bracketed term of (18.7.5) to establish the equality

$$
{}_tp_x\,{}_tp_y[\bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{\overline{x+t:y+t}}]
$$

$$
= {}_tp_x\,{}_tp_y[\bar{A}_{x+t} + \bar{A}_{y+t} - \bar{A}_{x+t:y+t} - \bar{P}(\bar{A}_{\overline{xy}})(\bar{a}_{x+t} + \bar{a}_{y+t} - \bar{a}_{x+t:y+t})].
$$

Substituting this into (18.7.5) we have

$$
{}_t\bar{V}(\bar{A}_{\overline{xy}}) = [({}_tp_x\,\bar{A}_{x+t} + {}_tp_y\,\bar{A}_{y+t} - {}_tp_x\,{}_tp_y\,\bar{A}_{x+t:y+t})
$$

$$
- \bar{P}(\bar{A}_{\overline{xy}})({}_tp_x\,\bar{a}_{x+t} + {}_tp_y\,\bar{a}_{y+t} - {}_tp_x\,{}_tp_y\,\bar{a}_{x+t:y+t})] \,/\, ({}_tp_x + {}_tp_y - {}_tp_{xy}) \tag{18.7.6}
$$

Because $(\overline{xy})$ is a survival status it has a proper conditional survival function, given that it has survived to t, which we will denote by ${}_up_{\overline{xy}+t}$. The benefit reserve for the insurance that was just discussed above can be calculated directly from the conditional survival function if it is first calculated. More precisely,

$$
{}_up_{\overline{xy}+t} = \Pr[T(xy) > u + t \,|\, T(xy) > t]
$$

$$
= \frac{{}_{t+u}p_x + {}_{t+u}p_y - {}_{t+u}p_{xy}}{{}_tp_x + {}_tp_y - {}_tp_{xy}}. \tag{18.7.7}
$$

We emphasize that only when $t = 0$ is it known that both (x) and (y) are alive. If we assume independence between the future lifetimes of (x) and (y), we have as the corresponding conditional p.d.f. for the last survivor status, given that it has survived to t,

$$
\frac{{}_{u+t}p_x\,\mu_x(t + u) + {}_{u+t}p_y\,\mu_y(t + u) - {}_{u+t}p_{xy}[\mu_x(t + u) + \mu_y(t + u)]}{{}_tp_x + {}_tp_y - {}_tp_{xy}}. \tag{18.7.8}
$$

If each of the ${}_{u+t}p$ factors in the numerators of (18.7.7) and (18.7.8) is factored as ${}_{u+t}p_x = {}_up_{x+t}\,{}_tp_x$, for example, then expressions in those equations will appear as weighted averages with the weights being the probabilities of survival to t.

When (18.7.8) is used to calculate $\mathrm{E}[v^{T(xy)-t} - \bar{P}(\bar{A}_{\overline{xy}})\,\bar{a}_{\overline{T(xy)-t}|} \,|\, T(xy) > t]$, then (18.7.6) is obtained again.

18.8 Notes and References

The practical applications of the ideas of this chapter have not been as numerous as those in some of the others. Nevertheless, extensive actuarial literature exists on various topics in multiple-life theory. Parts of Chapters 10, 11, 12, and 13 in Jordan (1967), and parts of Chapters 7 and 8 in Neill (1977) contain material on these topics.

Theorem 18.2.2 is a basic theorem of probability. It combines many of the ideas in Chapter 4 of Feller (1968). The technique used in proving results of this type is often called the ***method of inclusion and exclusion.*** The main results in the field are summarized and an extensive reference list provided by Takács (1967). Credit for the application of these algebraic methods in calculating life annuity values has been given to Waring. Earlier actuarial textbooks gave the results of Corollaries 18.2.1 and 18.2.3 by the so-called ***Z method.*** This was an algebraic mnemonic based on the observation that the coefficients of ${}_tD_j$ in ${}_tp_{\overline{x_1 \cdots x_m}}^{[k]}$ and in ${}_tp_{\overline{x_1 \cdots x_m}}^{k}$ are those in the expansions of $Z^k/(1 + Z)^{k+1}$ and $Z^k/(1 + Z)^k$, respectively.

An earlier version of Theorem 18.2.2 is contained in a discussion by Schuette and Nesbitt of a paper by White and Greville (1959). The use of these methods to determine the actuarial present value of a share in a share-and-share-alike last-survivor annuity is the subject of Exercise 18.36 and a paper by Rasor and Myers (1952). Another proof of Theorem 18.2.2 that avoids the use of ideas from probability is given by Buchta (1994).

Some of the issues regarding premiums and reserves on last-survivor insurances were discussed by Frasier in *The Actuary* (1978).

Life insurance policies with nonforfeiture values contain an embedded option. At each policy anniversary the insured has the option to take the nonforfeiture value and negotiate a new insurance contract, using health and market information available at that time, in an attempt to increase the actuarial present value of life insurance wealth. Reynolds (1994) discusses the cost implications of this option with respect to last survivor policies for which premiums, reserves, and nonforfeiture values are determined, using a conditional survival function as in (18.7.7). Reynolds develops the proposition that mortality antiselection, in the sense that those statuses exercising the withdrawal option will be in "better health," that is, the statuses will have a higher probability of long survival than those continuing, will be significant. Provision for the expected cost of this option should be built into the design of the policy. The argument depends on the observation that nonforfeiture values, like reserves, that are derived from conditional survival functions that assume only the survival of the status will tend to be larger than those that incorporate additional information about the survival of (x) and (y) as in (18.7.1), (18.7.2), and (18.7.3).

Appendix

Theorem 18.2.2

Let $A_1, A_2, \ldots, A_n$ represent the events of interest, and let $P_{[j]}$ denote the probability that exactly j of the n events take place. Further, let D_j be the sum, for all combinations of j events out of the n, of the probabilities that j specified events will occur, irrespective of the occurrence of the other $n - j$ events. Then, for any choice of numbers $c_0, c_1, \ldots, c_n$,

$$c_0 P_{[0]} + c_1 P_{[1]} + c_2 P_{[2]} + \cdots + c_n P_{[n]} = c_0 + D_1 \Delta c_0 + D_2 \Delta^2 c_0 + \cdots + D_n \Delta^n c_0.$$

Proof:

Let X_i denote the indicator for the event A_i, that is, $X_i = 1$ for sample points in A_i and $X_i = 0$ for sample points not in A_i. Let Y_j be the indicator such that $Y_j = 1$ for sample points in exactly j of the n events $A_1, A_2, \ldots, A_n$ and $Y_j = 0$ for the other sample points. We note that the expectation of Y_j is $P_{[j]}$. Finally, we define an operator, $\phi(\mathrm{E})$, a function of the shift operator, $\mathrm{E} = 1 + \Delta$, by

$$\phi(\mathrm{E}) = (X_1\mathrm{E} + 1 - X_1)(X_2\mathrm{E} + 1 - X_2) \cdots (X_n\mathrm{E} + 1 - X_n).$$

We note that any factor equals E if the corresponding $X_t = 1$ and equals 1 if $X_i = 0$. After multiplying, we have for any single point

$$\phi(\mathrm{E}) = Y_0 + Y_1\mathrm{E} + Y_2\mathrm{E}^2 + \cdots + Y_n\mathrm{E}^n$$

since, in the expansion of the product, the power of E is equal to the number of the X_i equaling 1. Thus the exponent of E is equal to the number of the events $A_1, A_2, A_3, \ldots, A_n$ containing the sample point.

Since a power of the shift operator, E^j, applied to c_0 yields c_j, we obtain

$$\phi(\mathrm{E})c_0 = c_0Y_0 + c_1Y_1 + \cdots + c_nY_n,$$

and then the expectation of $\phi(\mathrm{E})c_0$ is

$$c_0P_{[0]} + c_1P_{[1]} + \cdots + c_nP_{[n]}.$$

Since $\mathrm{E} = 1 + \Delta$, we can also write $\phi(\mathrm{E})$ as

$$\begin{aligned} \phi(\mathrm{E}) &= (1 + X_1\Delta)(1 + X_2\Delta) \cdots (1 + X_n\Delta) \\ &= 1 + \sum_{j=1}^{n} \left(\sum_{i_1, i_2, \ldots, i_j} X_{i_1} X_{i_2} \cdots X_{i_j} \right) \Delta^j, \end{aligned}$$

which displays the coefficient of Δ^j as the sum of all possible products, $\binom{n}{j}$ in number, of the X_i taken j at a time. Since $X_{i_1} X_{i_2} \cdots X_{i_j} = 1$ only if the sample point is in $A_{i_1} A_{i_2} \cdots A_{i_j}$, the expectation of $X_{i_1} X_{i_2} \cdots X_{i_j}$ is $\Pr(A_{i_1} A_{i_2} \cdots A_{i_j})$ and the expectation of

$$\sum_{i_1, i_2, \ldots, i_j} X_{i_1} X_{i_2} \cdots X_{i_j}$$

is D_j. Hence the expectation of $\phi(\mathrm{E})c_0$ can also be written as

$$c_0 + D_1\Delta c_0 + D_2\Delta^2 c_0 + \cdots + D_n\Delta^n c_0.$$

Equating the two forms for the expectation of $\phi(E)c_0$ completes the proof of the theorem. ■

The familiar inclusion-exclusion theorem of probability provides an example of applying Theorem 18.2.2. For $n = 4$,

$$\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) = P_{[1]} + P_{[2]} + P_{[3]} + P_{[4]}.$$

Here $c_0 = 0$, and $c_1 = c_2 = c_3 = c_4 = 1$ in the first form of the expectation of $\phi(E)c_0$. From the table

i	c_i	Δc_i	$\Delta^2 c_i$	$\Delta^3 c_i$	$\Delta^4 c_i$
0	0	1	−1	1	−1
1	1	0	0	0	—
2	1	0	0	—	—
3	1	0	—	—	—
4	1	—	—	—	—

we see that the second form of the expectation is

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) &= D_1 - D_2 + D_3 - D_4 \\ &= \sum_{i=1}^{4} \Pr(A_i) - \sum_{\substack{\text{all combinations}\\ \text{of two of}\\ 1,2,3,4}} \Pr(A_iA_j) \\ &\quad + \sum_{\substack{\text{all combinations}\\ \text{of three of}\\ 1,2,3,4}} \Pr(A_iA_jA_k) - \Pr(A_1A_2A_3A_4).\end{aligned}$$

Exercises

Unless otherwise indicated, all lives are subject to the same table of mortality rates, and their time-until-death random variables are independent.

Section 18.2

18.1. Describe the events having probabilities given by the following expressions:

a. ${}_tp_{wx} + {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} + {}_tp_{yz} - 3({}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}) + 7\,{}_tp_{wxyz}$

b. ${}_tp_w + {}_tp_x + {}_tp_y + {}_tp_z - 2({}_tp_{wx} + {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} + {}_tp_{yz}) + 4({}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}) - 8\,{}_tp_{wxyz}$.

18.2. Use the corollaries of Section 18.2 to verify that ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[0]} = 1 - {}_tp_{\overline{x_1x_2\cdots x_m}}^{1}$.

18.3. An extract from a table of joint-life annuities valued at 3-1/2% interest reads as follows.

Joint-Life Status	Actuarial Present Value of Joint-Life Annuity-Immediate
20:26:28	14.4
20:26:29	14.3
20:28:29	14.0
26:28:29	13.8
20:26:28:29	12.5

a. Calculate the actuarial present value of an annuity payable at the end of each year while exactly three of (20), (26), (28), and (29) are alive.
b. Calculate the actuarial present value for an insurance of 10,000 payable at the end of the year of death of the second life to fail out of (20), (26), (28), and (29).

18.4. Express ${}_tp_{\overline{wxyz}}^{\,2} - {}_tp_{\overline{wxyz}}^{\,[2]}$ in terms of ${}_tD_j$, $j = 1, 2, 3, 4$.

18.5. Express, in terms of annuity symbols, the actuarial present value of an annuity of 1 per year payable at the end of each year while (w) and at most one of (x), (y), and (z) are alive.

18.6. If $\mu_{40}(t) = 0.002$, $0 \le t \le 10$, and $\delta = 0.05$, calculate the value of $\bar{A}_{40:40:40:40:40:\overline{10|}}$.

18.7. A trust is set up to provide income to (x), (y), and (z). The fund is to provide a continuous income at the rate of 8 per year to each while all three are alive, at a rate of 10 per year to each while two are alive, and at a rate of 15 per year to a sole survivor. Calculate the actuarial present values of
a. All the payments to be made
b. All the payments to be made to (x).

18.8. An insurance provides a death benefit of 4 payable immediately upon the first death among four lives age x, a benefit of 3 payable upon the second death, a benefit of 2 payable upon the third death, and a benefit of 1 payable upon the last death. If $\bar{A}_x = 0.4$ and $\bar{A}_{xx} = 0.5$, evaluate the actuarial present value of this insurance.

Section 18.3

18.9. Develop an expression in terms of single- and joint-life annuity symbols for the actuarial present value of an annuity-immediate of 1,000 per month payable
a. While exactly one of (40) and (35) is surviving during the next 25 years
b. While at least one of (40) and (35) survives at an age less than age 65.

18.10. Express the following in terms of symbols of annuities certain and single- and joint-life annuities:

a. $\bar{a}_{\overline{x:y:\overline{n}|}}$

b. $\bar{a}_{\overline{(25:\overline{40}|):\overline{30}|}}$.

Section 18.4

18.11. If at each duration the force of mortality for (x) is $1/2$ that for (y) while the force of mortality for (z) is twice that for (y), what is the probability that of the three lives (x) will die

a. First

b. Second

c. Third.

18.12. Which of the following statements are true? Correct the others as necessary.

a. $\bar{A}_{\overline{wxyz}}^{\,1} = \bar{A}_{\overset{1}{w}xyz} + \bar{A}_{w\overset{1}{x}yz} + \bar{A}_{wx\overset{1}{y}z} + \bar{A}_{wxy\overset{1}{z}}$

b. $\bar{A}_{\overline{wxyz}}^{\,3} = \bar{A}_{\overset{2}{w}xyz} + \bar{A}_{w\overset{2}{x}yz} + \bar{A}_{wx\overset{2}{y}z} + \bar{A}_{wxy\overset{2}{z}}$

c. $\bar{A}_{wxy\overset{3}{z}} = \bar{A}_{w\overset{1}{z}} + \bar{A}_{x\overset{1}{z}} + \bar{A}_{y\overset{1}{z}} - (\bar{A}_{wx\overset{1}{z}} + \bar{A}_{wy\overset{1}{z}} + \bar{A}_{xy\overset{1}{z}}) + \bar{A}_{wxy\overset{1}{z}}$.

18.13. Write, as a definite integral, the actuarial present value for an insurance to be paid at the moment of death of (x) if (x) survives (y). The benefit amount is equal to the time elapsed between the issue of the policy and the date of death of (y).

18.14. If Gompertz's law applies with $\mu(40) = 0.003$ and $\mu(56) = 0.012$, calculate

a. ${}_{\infty}q_{40:\overset{2:3}{48}:56}$ b. ${}_{\infty}q_{40:\overset{2}{48}:56}$.

[Note: In part (a) the notation 2:3 indicates the event that (48) dies second or third among the lives involved.]

18.15. An insurance of 1 issued on the lives (x), (y), and (z) is payable at the moment of death of (z) only if (x) has been dead at least 10 years and (y) has been dead less than 10 years. Express the actuarial present value of this insurance in terms of actuarial present values for insurances and pure endowments.

18.16. Develop an expression that does not involve integrals for the actuarial present value of an insurance of 1 payable 10 years after the death of (x), provided that either or both of (y) and (z) survive (x) and both are dead before the end of the 10-year period.

18.17. Obtain a formula for the single contract premium for a special contingent, unit insurance payable if (30) dies before (60), or within 5 years after the death of the latter, with return of the contract premium, without interest, 5 years after the death of (60) if no claim under the insurance arises by the death of (30). Assume the loading is 7-1/2% of the benefit premium.

18.18. Without using the independence assumption, establish relations such as

$$ {}_nq_{wxy}^{\;\;1} = {}_nq_{wxyz}^{\;\;1} + {}_nq_{xwx\underset{1}{y}z}^{\;\;\;\;2}, $$

and use them to obtain the result of Example 18.4.1.

18.19. Without using the independence assumption, establish the relations

$$ \bar{A}_{xy}^{\;1} = \bar{A}_{xyz}^{\;1} + \bar{A}_{x\underset{1}{y}z}^{\;2} $$

$$ \bar{A}_{yz}^{1} = \bar{A}_{xyz}^{\;1} + \bar{A}_{x\underset{1}{y}z}^{\;2}, $$

and use them to obtain the result of Example 18.4.2.

18.20. Express ${}_\infty q_{w\underset{1}{x}yz}^{\;2}$

a. As a definite integral
b. In terms of simple contingent probabilities.

18.21. Assuming that Gompertz's law applies, show that

a. ${}_tq_{xy}^{\;2} = {}_tq_y - \dfrac{c^y}{c^x + c^y}\, {}_tq_{xy}$

b. $\bar{A}_{x\underset{1}{y}\overset{3}{z}} = \dfrac{c^x}{c^x + c^y}\,\bar{A}_z - \dfrac{c^z}{c^y + c^z}\,\bar{A}_{yz} + \dfrac{c^y}{c^x + c^y}\,\dfrac{c^z}{c^x + c^y + c^z}\,\bar{A}_{xyz}.$

18.22. If $\mu(x) = 1/(100 - x)$ for $0 < x < 100$ applies for (20), (40), and (60), evaluate

a. ${}_\infty q_{20:\underset{1}{40}:\overset{2}{60}}$ b. ${}_\infty q_{20:40:\overset{1}{60}}$ c. ${}_\infty q_{20:\overset{1}{40}}$.

This illustrates that ${}_\infty q_{x\underset{1}{y}z}^{\;2} = {}_\infty q_{xyz}^{\;1}\, {}_\infty q_{yz}^{\;1}$, which holds on the basis of Gompertz's law, does not hold in general.

18.23. On the basis of a mortality table following Gompertz's law (with $c^8 = 2$), $\bar{A}_{54} = 0.3$, $\bar{A}_{62} = 0.4$, and $\bar{A}_{70} = 0.52$. Determine $\bar{A}_{54:54:\underset{1}{62}}^{\;\;2}$.

18.24. Given $\bar{A}_w = 0.6$, $\bar{A}_{wx}^{1} = 0.3$, $\bar{A}_{wxx}^{1} = 0.2$, and $\bar{A}_{wxxx}^{1} = 0.1$, evaluate

a. $\bar{A}_{wxxx}^{2}$ b. $\bar{A}_{wxxx}^{4}$ c. $\bar{A}_{\underset{122}{wxxx}}^{4}$.

18.25. Express in integral form the probability that (x), (y), and (z) will die in that order within the next 25 years with at least 10 years separating the times of any pair of deaths.

18.26. Express in integral form the probability that (10), (20), and (30) will all die before attaining age 60 with (20) being the second to die.