PART TWO
THE THEORETICAL TOOLS

6
Establishing a pricing framework

6.1 INTRODUCTION AND MOTIVATION

This and the following chapter will outline the general pricing strategy that will be used in the context of the valuation of interest-rate products, and introduce the no-arbitrage conditions and the statistical and financial concepts needed for the task. Many of the theorems that will be used in the practical implementations in later chapters are often more simply (and elegantly) obtained in continuous time. The degree of mathematical sophistication needed to treat 'properly' the continuous-time case is considerably higher than the level needed for the discrete case. The analysis will therefore be carried out in detail for the latter both in the text (where financial intuition will be stressed) and in Appendix A (where a mathematically more satisfactory treatment is presented). The transfer of the results to the continuous case will be accomplished in a somewhat heuristic manner. The distinction between a heuristic argument and sloppy reasoning is a rather subtle one, especially when the issues, as is the case for the convergence of the discrete results to the continuous case, are actually rather delicate (see, e.g., Willinger and Taqqu (1991)). It was, however, felt that by explaining in good detail the discrete-time set-up, and in a more cursory way the continuous limit, more could be gained in intuition than would be lost in rigour. As usual, the link with computational methodologies, such as lattices, has been emphasised throughout. If, ultimately, the reader wants to be able to produce, for his original research, fully rigorous and mathematically 'well-justifiable' results, there is probably no alternative but to undertake the study of stochastic calculus applied to finance and option pricing in particular, as treated, to name a few, in Harrison and Pliska (1981), Gardiner (1990), Lamberton and Lapeyre (1991), Oksendal (1995), Duffie (1992) or Dothan (1990). Baxter and Rennie (1996) provide an excellent compromise between mathematical rigour and financial intuition. Their book, unfortunately not focused on interest rate derivatives, is warmly recommended. For those readers interested in developing a 'working relationship' with the concepts involved, so as to enable them to understand the gist of proofs and arguments found in the literature based on the martingale formalism and to carry out simple calculations when needed, this chapter and the next are meant to provide the insight in the underlying financial motivation and further the intuition behind the mathematical theorems, whilst the above-mentioned Appendix A is designed to supply the actual mathematical tools. A later section of the next chapter (Section 7.6), in particular, will attempt to make clear the link between the more traditional way of looking at no-arbitrage (as presented in Section 7.1), and the more modern approach. A few worked-out examples will be provided, in order to show how the formalism introduced can easily be brought to yield useful results of direct financial relevance.

In order to add a further motivation towards the undertaking of the study of the material presented in Appendix A, one can fairly say that a large proportion of the modern literature on option pricing, even published in journals, such as *Risk* magazine, aimed mainly at a practitioners' audience, is so deeply couched in the martingale formalism, that many readers tend to perceive the terminology used as an insurmountable barrier to entry. Therefore the attempt has been made in this book to introduce and define as simply as possible those terms that tend to create the greatest problems. The goal of the writer will have been satisfactorily fulfilled if at least a few readers, scared away in the past by finding in an article that a trading strategy had to be predictable, will continue to plough through their reading, reassured that the expression can be interpreted as asking of the strategy the rather reasonable requirement that it should be put together without knowledge of future, and therefore unknown, events. The wisdom of couching even relatively simple and non-technical results in the formalism borrowed from continuous-time stochastic processes could be debated at length, especially since the need for introducing rather complex concepts often stems from rather 'technical' reasons (Kolmogorov's axiomatic approach to probability is as powerful as it is elegant, but does not recommend itself for great intuitional appeal). However one might feel about this state of affairs, this tendency seems to be here to stay, and, therefore, at least a 'working acquaintance' with the terminology and the underlying ideas is probably indispensable in order to be able to keep abreast with what is published in virtually all the interesting journals.

Let us begin by considering the problem of the evaluation of the simplest possible contingent claim. We shall start from the knowledge of the state of the world prevailing today, ω_0, and we postulate that this state of the world will evolve, over one time interval, to either of two possible states: ω_1 or ω_2. More precisely, we know that a security S, of price today S_0, will have values S_1 or S_2 according to whether states of the world ω_1 or ω_2 will prevail after the time interval. The possible values S_1 and S_2 are known before the move, but we do not know the probability of reaching ω_1 or ω_2 from ω_0. In addition, a default-free discount bond, P, is also available, which pays £1 with certainty in both states of the world, and which trades in the market today at £P_0. The price of this bond

implicitly defines the riskless rate of return over the time period: if continuous compounding is chosen, this riskless rate can be expressed as

$$P_0 = \exp[-r\Delta t] \Rightarrow r = -\ln[P_0]/\Delta t \tag{6.1}$$

Finally, a contingent claim C trades in the economy, and we know with certainty that the contingent claim will be worth £C_1 or £C_2 if state ω_1 or ω_2 prevails, respectively. We want to determine the 'fair' price today, C_0, of the contingent claim. By 'fair' we mean that, were the claim to trade at any other price, an unlimited profit could be made by entering suitable strategies in S, P and C. Notice that, by the way the problem has been set up, the value of C after one time interval is deemed to depend directly on the states of the world, rather than on the security S. Since ω_1 and ω_2, however, are uniquely associated with the values S_1 and S_2, one could have equivalently required C to be a function of S. If this had been the case, the common description of S and C as the 'underlying' and 'derivative' security would have been obviously justified.

We shall approach the evaluation of the fair price of this contingent claim in four distinct (albeit related) ways: the first, third and fourth approaches will provide the same (correct) answer, whilst the second, perhaps the most intuitively 'plausible' one, will be shown to give a wrong value.

6.2 FIRST APPROACH — 'REPLICATION STRATEGY'

Let us form a portfolio Π made up of α units of the stock S and β units of the bond. Therefore, at time 0,

$$\Pi_0 = \alpha S_0 + \beta P_0. \tag{6.2}$$

If state ω_1 prevails, then the portfolio will be worth

$$\Pi_1 = \alpha S_1 + \beta P_1, \tag{6.3}$$

and similarly, if state ω_2 occurs,

$$\Pi_2 = \alpha S_2 + \beta P_2. \tag{6.3'}$$

Let us now impose the condition that this portfolio should have the same value as the contingent claim C after one time interval, irrespective of which state of the world might prevail, i.e.:

$$C_1 = \Pi_1 = \alpha S_1 + \beta P_1 \tag{6.4}$$

$$C_2 = \Pi_2 = \alpha S_2 + \beta P_2. \tag{6.4'}$$

For the special case considered here, i.e. for the case where the riskless security is a bond maturing after the time step, $P_1 = P_2 = 1$ with certainty, and therefore

Equations (6.4) and (6.4′) simplify to

$$C_1 = \Pi_1 = \alpha S_1 + \beta \qquad (6.4'')$$

$$C_2 = \Pi_2 = \alpha S_2 + \beta. \qquad (6.4''')$$

For this particularly simple case the solution can be trivially found to be given by

$$\alpha = (C_1 - C_2)/(S_1 - S_2) \qquad (6.5)$$

$$\beta = (C_2 S_1 - C_1 S_2)/(S_1 - S_2) \qquad (6.5')$$

In general, however (i.e. irrespective of the values of P in the states ω_1 and ω_2), since C_1, C_2, S_1, S_2, P_1 and P_2 are all known quantities, the system of two linear equations in two unknowns (6.4) and (6.4′) will always admit a unique solution as long as the determinant of the associated matrix is not equal to 0, i.e. as long as

$$\det \begin{vmatrix} S_1 & P_1 \\ S_2 & P_2 \end{vmatrix} \neq 0 \qquad (6.6)$$

Provided that $P_1 = P_2 \neq 0$, the determinant requirement simply translates into the condition that

$$S_1 P_2 - S_2 P_1 \neq 0 \Rightarrow S_1 - S_2 \neq 0, \qquad (6.6')$$

i.e., that S_1 should be different from S_2:

$$\text{if } P_1 = P_2 \neq 0 \Rightarrow \det \begin{vmatrix} S_1 & P_1 \\ S_2 & P_2 \end{vmatrix} \neq 0 \Leftrightarrow S_1 \neq S_2$$

In matrix form the problem can be written as

$$\underset{(2,2)}{\mathbf{A}} \; \underset{(2,1)}{\mathbf{b}} = \underset{(2,1)}{\mathbf{C}} \qquad (6.7)$$

with

$$\mathbf{A} = \begin{vmatrix} S_1 & P_1 \\ S_2 & P_2 \end{vmatrix}$$

$$\mathbf{b} = \begin{vmatrix} \alpha \\ \beta \end{vmatrix}$$

and

$$\mathbf{C} = \begin{vmatrix} C_1 \\ C_2 \end{vmatrix}.$$

The solution vector **b**, obtainable as

$$\mathbf{b} = \mathbf{A}^{-1}\mathbf{C}, \qquad (6.8)$$

(where $\mathbf{A}^{-1}$ denotes the inverse of matrix **A**) contains the holdings α and β of the security S and of the bond P, respectively, needed to ensure that Π will assume exactly the same values as the contingent claim in each of the two possible states of the world. But, if this is the case, the portfolio thus constructed must be worth today no more and no less than the contingent claim itself, i.e.

$$C_0 = \Pi_0 = \alpha S_0 + \beta P_0. \qquad (6.9)$$

Substituting the values for α and β, one obtains

$$C_0 = [(C_1 - C_2)/(S_1 - S_2)]S_0 + [(C_2 S_1 - S_2 C_1)/(S_1 - S_2)]P_0. \qquad (6.9')$$

This value for the claim is 'fair' in the sense that unlimited profits could be made if C traded at any other price by buying (selling) the claim and entering at the same time a short (long) position in the portfolio (which, by construction, replicates all the possible C-payoffs at time 1). The solution that we have found therefore gives a unique and certainly correct value for C_0, against which all other pricing methodologies can be benchmarked.

Before leaving this evaluation procedure, it is just as important to highlight on which quantities the solution does depend ($S_0, S_1, S_2, P_0, P_1, P_2, C_1$ and C_2), as to stress on what quantities it does *not* depend: one should notice carefully, in fact, that no mention was made about the likelihood of occurrence of ω_1 or ω_2, apart from implicitly requiring that both probabilities of occurrence, $p(\omega_1)$ and $p(\omega_2)$, were non-zero. Therefore, perhaps surprisingly, knowledge of these probabilities turns out not to be necessary for the evaluation of the fair price of the contingent claim. This crucially important observation will be revisited at the end of the next section.

6.3 SECOND APPROACH — 'NAIVE EXPECTATION'

Let us now make the important additional assumption that we can avail ourselves of the extra piece of information that the probability of occurrence of state ω_1 is, say, $\frac{1}{2}$ (and, therefore, also $p(\omega_2) = \frac{1}{2}$). Notice that this extra information drastically changes our knowledge with respect to the 'replicating-portfolio' scenario, since we are now in a position to evaluate the expectation of the return from the security S. Plausibly, knowledge of this additional important piece of information should allow one to obtain the value of the contingent claim today more directly and easily than we managed to do with the replicating strategy. Notice, however, that, even with this added piece of information, before solving the problem, i.e. before finding C_0, we *do* know the expectation of C at time 1, but we do not know (yet) the expected *return* from the contingent claim.

One could be tempted to speculate, on the basis of the extra piece of information we now have, that the 'fair' price of the contingent claim today should be equal to the weighted average of the two possible outcomes C_1 and C_2, appropriately discounted by the rate implied by the riskless bond. After all, one might argue, this 'naive expectation' would indeed be the 'average' discounted payoff from the contingent claim one would obtain if one were to trade in the same contingent claim (i.e. in a contingent claim with the same possible outcomes and the same known probabilities of occurrence) over and over again. However plausible this reasoning might be, a moment's reflection shows that the expectation calculated on the basis of the probabilities we know to apply in the real world to events ω_1 and ω_2 must in general produce a value different from the fair price obtained using the replication strategy. Since, in fact, the exogenous probabilities p_1 and p_2 do not depend in any way on the replication strategy set up in Section 6.2, it would be an incredible coincidence if the 'real-world' expectation and the value today of the replicating portfolio turned out to have the same value. To look at the matter from a different angle, if, due perhaps to the arrival of new information, the probabilities p_1 and p_2 were to change over time, the expectation of the two possible payoffs would also change; the replication-strategy result, however, would remain exactly the same. On the other hand, we also know from the previous discussion that, if the replication price were not enforced, unlimited profits could easily be made by simultaneous trading in the replicating portfolio and the contingent claim. Therefore, unless we want to grant our counterparty the possibility of a (potentially infinitely large) free lunch, the 'naive expectation' price must be not simply different, but unquestionably *wrong*.

If this is the case, what went wrong in the reasoning that led to the free lunch? The pitfalls and the fallacies implicit in this naive line of argument are excellently exposed in Baxter and Rennie (1996). Without repeating their arguments, one can approach the problem from a different angle, and begin to notice that, even if *on average* the payoff from the contingent claim might well be given by a precise and certain number (that can be evaluated on the basis of the real-world probabilities), the holder of the claim (unlike the holder of the bond P) will experience for any given outcome a return which is *not* certain, but displays a certain variance. It is plausible to assume that, in the real world, the return that an investor will, in general, demand from a security should be a function not only of its expected return, but also of the uncertainty connected with it. The precise return will depend on the investor's appetite for risk: if he is risk-avoiding, then he will demand an extra 'compensation' from a risky security, on top of the return that he would earn holding the riskless bond P; if he is risk-seeking, he will be happy with a lower return than the riskless rate; if, finally, he is risk-neutral then he will accept a return exactly identical to the one obtainable from the bond. If, in addition, one is prepared to assume[1] that the uncertainty in returns can be satisfactorily described by the percentage standard deviations of the returns themselves, then the behavioural attitude towards risk of the 'average investor' could be expressed by a simple linear relationship of the type

$$\mu_x = r + \lambda\sigma_x \tag{6.10}$$

where μ_x is the required expected return from x, r is the riskless rate, σ_x is the percentage standard deviation per unit time (volatility) of security x, and λ is the (security-independent!) compensation per unit risk above the riskless rate. Notice that λ describes a behavioural feature of the investor, and has therefore nothing to do with any specific security *per se*. Notice also that, much as the various assumptions made (about, for instance, the linearity of the dependence of the extra return on the unit risk) might seem reasonable, they are no more and no less than hypotheses, and that any result derived from them will enjoy no greater validity than the assumptions themselves. This was not the case for the replicating strategy, that required us to make no assumptions at all on top of our information set. With this caveat in mind, we shall attempt a third line of attack towards the evaluation of the contingent claim along the lines of thought just outlined.

6.4 THIRD APPROACH — 'MARKET PRICE OF RISK'

Let us recall, first, that we now know *both* the values attained by S in the two states of the world *and* the probabilities of reaching these states. If we assume the security S to be log-normally distributed, from the values S_1 and S_2 we can therefore compute both the percentage standard deviation of S (normally referred to as the volatility of S) and its expectation. Armed with the knowledge of these quantities, of the riskless rate r and of the return expected from security S (obtainable from its expectation), we are in a position to estimate the market price of risk, λ:

$$E_{t_0}[S] = (p_1S_1 + p_2S_2) = S_0 + \mu_S\Delta t \tag{6.11}$$

$$\sigma_S^2\Delta t = p_1(S_1 - E[S])^2 + p_2(S_2 - E[S])^2 \tag{6.11$'$}$$

$$\lambda = (\mu_S - r)/\sigma_S. \tag{6.11$''$}$$

But this market price of risk, estimated from security S, is, by the way it is defined, security-independent. As such it must apply to the contingent claim as well, which, in turn, must also be priced in a 'fair' market so as to yield a return

$$\mu_C = r + \lambda\sigma_C. \tag{6.12}$$

The only quantity that remains to be evaluated is therefore σ_C. *If* the further assumption is now made that the returns from the claim are also log-normally distributed, we can estimate a *percentage* volatility using expressions (6.11) and (6.11$'$) above, with C now replacing S.

Armed with the knowledge of σ_C, we can now attempt to value the contingent claim today by averaging the two possible claim values after the time step and discounting them by the factor $\exp(-\mu_C\Delta t)$ (rather than $\exp(-r\Delta t)$!). By so doing we do obtain a value which is, in general, different from the naive-expectation result, but *very* similar (indeed, almost identical) to the value obtained with the replication approach, which we know must give the right answer. (A numerical example that clarifies the procedure is presented at the end of this section.)

Therefore, we can indeed recover the fair price of the contingent claim also by making use of the real-world probabilities, but only if these are used in conjunction with the market price of risk. Arriving at the answer following this route will, in general, be much more complex: there will be, in fact, as many market prices of risk as there are independent sources of shocks for the economy, and all these risk prices will be 'hidden' in the prices of traded securities. If we manage, however, to give a convincing, albeit perhaps simplified, description of the behavioural attitudes towards different types of risk of the 'average investor', then we have indeed achieved something richer and more informative than what we managed to accomplish using the replication argument. In a nutshell, this is the difference between equilibrium models, such as CIR or LS (Chapters 11 and 14, respectively), and no-arbitrage models, such as BDT, HJM, HW or GBS (Chapters 12, 17, 13, 16, respectively).

Why is the market-price-of-risk result *almost* right? As mentioned before, several assumptions had to be introduced in the course of the derivation. Recall, first of all, that we estimated the market price of risk using one security (S), invoked its security independence and then applied it to another security (C). Therefore, any inaccuracy in our estimation of the market price of risk stemming from the imperfection of our assumptions (for instance, that 'riskiness' for which investors seek compensation should be exactly described by the *percentage* standard deviation of the product) or of our estimations would transfer directly into a wrong estimate for μ_C. More importantly, we estimated the percentage standard deviations of S and C under the assumption that both should be log-normally distributed. Not only may this not be truly the case, but the joint assumptions might even be incompatible. Therefore, even if the behaviour of investors were perfectly described by Equation (6.12), the estimate of the standard deviation of C could be sufficiently imprecise to give rise to slightly inaccurate values for C_0.

Notice carefully that we would have obtained an (almost) identical result even if our exogenous probabilities p_1 and p_2 had been different. These different probabilities would, in fact, have given rise to a different market price of risk, but the latter would then have been consistently used for both S and C. We seem, therefore, to have reached a somewhat paradoxical conclusion: on the one hand, in fact, the market price of risk has been shown to be necessary in order to estimate the fair value C_0, if use is to be made of the real-world probabilities; on the other hand, it turns out that, had these probabilities been different, a new market price of risk would have resulted, but (almost) the same price for C_0 would have been obtained. Within the estimation caveats outlined above, the price for the contingent claim is therefore truly independent of the market price of

risk! In other words, estimating the 'true' rate of return from asset S is not really necessary if we 'just' want to establish a fair value for C. (If anything, it seems to make things more complicated.) This conclusion, after all, should by now be only mildly surprising, since in the exact-replication result no mention was made of the probabilities of occurrence of ω_1 and ω_2, and therefore no expectations could be computed. This important observation can lead to the fourth, and (as the reader might be relieved to know) last, line of attack towards the evaluation of the fair value of C_0.

A worked-out example

Let the stock, the bond and the contingent claim have the values shown in Figure 6.1. The time step Δt is 0.1 (in years). The price of the bond implies a continuously compounded riskless rate of return of 10.00%. We assume that the probabilities of transition from ω_0 to ω_1 and ω_2 are both $\frac{1}{2}$. From the values of the contingent claim in the two states we can estimate a percentage volatility for the contingent claim σ_C of 97.281%, and for the security σ_S of 20.00%, as long as we are prepared to make the assumption of log-normal distribution for both S and C. The relative expression, valid for $p_1 = p_2 = \frac{1}{2}$, is

$$\sigma_S\sqrt{\Delta t} = [\ln(S_2) - \ln(S_1)]/2,$$

where Δt is the length in years of the time step. A similar expression holds for σ_C. From the knowledge of the values S_0, S_1 and S_2 we can obtain the expected return from S. Since the total expected return should be equal to the sum of the riskless rate (10.00%) and a compensation per unit risk times the percentage volatility of the stock (see Equation (6.10)), we can estimate the market price of risk to be $\lambda = 24.00\%$. This value can now be used in conjunction with the estimated percentage volatility of the claim, σ_C, in order to determine (Equation (6.12)) the expected return from the claim. This turns out to be given by $\mu_C = 33.348\%$.

		Stock	*Bond*	*Claim*
		107.9012	100	27.90117
Stock	*Bond*			
100	99.00498			
		95.08057	100	15.08057

Figure 6.1 Values of stock, bond and contingent claim in worked-out example

This gives for C_0 the value of 20.78602, which compares favourably with the replicating-portfolio value of 20.79601. Were the contingent claim to be given by the function $f(S) = \mathrm{Max}(S - K, 0)$, then Table 6.1 would show the 'true' (Replication) and 'market-price-of-risk' (MPR) result for several values of the strike K.

Notice that the more the option is in-the-money, the more the claim resembles a stock, and the better the joint log-normal assumption for stock and claim holds, producing a better estimate of σ_C, and, consequently, a closer agreement for the market-price-of-risk result with the correct replicating-portfolio result. Conversely, the higher the strike the greater the discrepancy between the two results. In particular, if the strike were higher than S_1, giving a value of $C_1 = 0$, the estimate of the percentage volatility of the claim would quickly become progressively unsatisfactory, producing significantly different values for the estimated fair value of the contingent claim.

Table 6.1 Values of the Contingent Claim C_0 Obtained as Indicated in the Worked-out Example

Strike	MPR	Replication
94	6.7312	6.9352
92	8.8291	8.9153
90	10.8456	10.8954
88	12.8430	12.8755
86	14.8328	14.8556
84	16.8190	16.8357
82	18.8031	18.8158
80	20.7860	20.7959
78	22.7682	22.7760
76	24.7498	24.7561
74	26.7310	26.7362
72	28.7120	28.7163
70	30.6928	30.6964
68	32.6735	32.6765
66	34.6541	34.6566
64	36.6346	36.6367
62	38.6150	38.6168
60	40.5953	40.5969
58	42.5757	42.5770
56	44.5559	44.5571
54	46.5362	46.5372
52	48.5164	48.5173
50	50.4966	50.4974
48	52.4768	52.4775
46	54.4570	54.4576

6.5 FOURTH APPROACH — RISK-NEUTRAL VALUATION

We saw before that the sources of 'imperfection' in the market-price-of-risk evaluation were to be found (i) in the possible mis-specification of the behavioural relationship between risk and expected reward, and (ii) in the imprecision in assigning a percentage volatility to C from the two known values C_1 and C_2. But, in the view of these difficulties, why not make use of the independence from the market-price-of-risk of the fair value of C_0, and choose a market price of risk of zero, which corresponds to indifference on the part of the investor towards risk? This would obviously be inadequate if we wanted to provide a 'true' description of the economy (as an equilibrium approach attempts to achieve), but would serve our purposes perfectly well if we were 'just' interested in pricing a contingent claim (as no-arbitrage approaches are designed to do). If indeed we were to choose a market price of risk equal to zero, neither would we encounter the problems linked with 'distilling' the 'true' market price of risk from the known values of S_1, S_2, and the probabilities p_1 and p_2, nor would we meet the difficulties connected with the estimation of σ_C. In other terms, *for the purpose of the evaluation of the fair value of* C_0, one can assume that the returns from S, C, P, or, for that matter, *any* security are simply equal to the riskless rate and one can obtain the probabilities p_1 and p_2 from this choice. Notice, however, that, since we have chosen risk neutrality purely for reasons of computational expedience, the implied probabilities p_1 and p_2 are also a pure computational device, and bear no relationship to the true ('real-world') probabilities. If one performs the calculation with this value of λ, the risk-neutral price of C_0 is, within numerical noise, indeed identical to the replication-portfolio value (that we know must be exact). That the risk-neutral valuation should bring about exactly the same result as the replicating-portfolio approach should, upon reflection, come as little surprise: the replication strategies taught us how to build a portfolio giving exactly the same payoffs as the contingent claim we wanted to value; therefore, by combining the portfolio and the contingent claim itself (with long and short positions, respectively) one is creating a new portfolio with identical and known payoffs in all states of the world; but this, apart from a scaling factor, is simply a pure discount bond, which by definition must earn the riskless return.

We have, in a way, come full circle: we started from an exact valuation procedure that made no use of probabilities; we then introduced these probabilities and the accompanying expectations, and showed that a 'naive', albeit plausible, use of these quantities actually yielded a wrong result; we identified the missing ingredient in the expectation approach, i.e. we recognised the fact that, since the returns from the non-bond traded securities are uncertain, risk-averse, risk-neutral or risk-seeking investors will demand different expected returns; finally, we showed how this difficult-to-estimate quantity, the market price of risk, could actually be dispensed with by using a risk-neutral valuation. The purpose of the exercise was not to repeat a procedure that is after all very similar to the well-known Black-and-Scholes original reasoning, but to show both the strengths and the shortcomings of the successful approaches (the replicating strategy and the risk-neutral valuation, on the one hand, and the market-price-of-risk line of approach on the other) so as to motivate the more formal and precise treatment of probability, stochastic calculus and financial theory needed to treat quantitatively realistically complex cases. The three successful approaches dealt with above contain *in nuce* a surprising wealth of mathematical and financial concepts, that will be more fully explored at an intuitive, semi-quantitative level in the following sections of this chapter, and in a more precise and formal manner in Appendix B. Before dealing explicitly with the set-up necessary to describe in a more precise way the financial markets, however, a few more comments on the simple example treated above can supply very useful intuitive guidance for future developments.

6.6 PSEUDO-PROBABILITIES

From Section 6.1 we know that, in the case when P_1 and P_2 are both equal to 1, the holdings of stock (α) and riskless bond (β) needed to replicate the payoffs of the contingent claim are

$$\alpha = (C_1 - C_2)/(S_1 - S_2) \tag{6.13}$$

$$\beta = (C_2 S_1 - C_1 S_2)/(S_1 - S_2) \tag{6.13'}$$

Therefore, as we showed before,

$$C_0 = [(C_1 - C_2)/(S_1 - S_2)]S_0 + [(C_2 S_1 - S_2 C_1)/(S_1 - S_2)]P_0. \tag{6.14}$$

Solving for C_1 and C_2 the expression above can be rearranged to give

$$C_0 = C_1[S_0/(S_1 - S_2) - S_2 P_0/(S_1 - S_2)] + C_2[S_1 P_0/(S_1 - S_2) - S_0/(S_1 - S_2)] \tag{6.15}$$

If one now defines

$$\pi_1 = [S_0/P_0 - S_2]/(S_1 - S_2) \tag{6.16}$$

$$\pi_2 = [S_1 - S_0/P_0]/(S_1 - S_2) \tag{6.16'}$$

one can write

$$\boldsymbol{C_0 = [C_1\pi_1 + C_2\pi_2]P_0}. \tag{6.17}$$

Formally, looking at Equation (6.17), one might be tempted to 'interpret' the value today of the contingent claim as if it were given by a discounted expectation of the two possible terminal values, taken with weights or 'probabilities' π_1 and π_2. But, in order to see whether this probabilistic interpretation can be warranted, the two quantities π_1 and π_2 must be examined more closely.

The first encouraging observation is that both these weights do not depend on the initial or terminal values of the contingent claim itself, and that they are

therefore truly state-dependent, rather than security-dependent, as state probabilities should be. Furthermore, as one can directly check from Equations (6.16) and (6.16′), their sum always adds up to 1:

$$\pi_1 + \pi_2 = 1 \tag{6.18}$$

As for their sign and magnitude (required to be positive and smaller than 1, respectively, for the probabilistic interpretation to hold), let us begin by considering the case where $S_2 > S_1$. (The opposite case can be dealt with by following exactly the same reasoning.) As long as $S_2 > S_0/P_0$, or, conversely, $S_1 < S_0/P_0$, then it is clear from Equations (6.16) and (6.16′) that both π_1 and π_2 are guaranteed to be positive. But these requirements are not as arbitrary as they might at first appear: let us assume, for instance, that $S_2 < S_0/P_0 \equiv S_0 \exp[r\Delta t]$. If that were the case, then the risky security S would earn, even in the most favourable case, a return $(S_2 - S_0)/S_0$ below the return r obtainable by holding the riskless bond P (remember that $S_2 > S_1$). One could therefore enter a strategy consisting of selling one unit of S at time 0, receiving £S_0, and investing the proceeds from the sale in S_0/P_0 units of the bond P. At time 0 the strategy, Σ, would be worth

$$\Sigma = (-1)S_0 + (S_0/P_0)P_0 = 0. \tag{6.19}$$

After the price move the strategy will be worth

$$\Sigma_1 = (-1)S_1 + (S_0/P_0) \tag{6.20}$$

$$\Sigma_2 = (-1)S_2 + (S_0/P_0) \tag{6.20'}$$

(since $P_2 = P_1 = 1$) according to whether state 1 or 2 prevails, respectively. But, given our assumptions about the relative magnitudes of S_1, S_2 and S_0/P_0, if S_2 is smaller than S_0/P_0 (as we assumed), so *a fortiori* is S_1 (which is smaller than S_2); in both states of the world we would therefore have obtained a strictly positive payoff: we would, in other terms, have devised a strategy which has cost us nothing to put together (Equation (6.19)) and that certainly pays a positive amount: we would therefore be the beneficiaries of a free lunch, that we could make as large as desired by going short not just one, but an arbitrarily large number of units of S. A similar reasoning obviously applies to the case when $S_1 > S_0/P_0$, when the strategy, this time, would be to go long the stock and short the bond. Putting the two constraints together, one can see that, to prevent free lunches, the return from the riskless bond must be 'in between' the two possible uncertain returns from the stock. But if this is the case, looking back at Equations (6.16) and (6.16′), one can establish that the following relationships must hold true:

$$\pi_1 > 0 \tag{6.21}$$

$$\pi_2 > 0 \tag{6.21'}$$

$$\pi_1 < 1 \qquad (6.21'')$$

$$\pi_2 < 1 \qquad (6.21''')$$

$$\pi_1 + \pi_2 = 1 \qquad (6.21'''')$$

If we couple Equations (6.21) with the contingent-claim-independence of π_1 and π_2, no further conditions are needed to warrant these two quantities the interpretation of probabilities, and, given the generality of the treatment (recall that we have said nothing specific about the payoffs C_1, C_2, S_1 or S_2, and that we found the two quantities π_1 and π_2 to be state-dependent but claim-independent), this must be true for *any* contingent claim. This finding can therefore be formalised as follows: given the one-step, two-state universe described above, **the fair value at time 0 of any contingent claim can always be evaluated as a discounted expectation of the possible values of the contingent claim at time 1, with probabilities defined by Equations (6.16) and (6.16′)**:

$$\mathbf{C_0 = [C_1\pi_1 + C_2\pi_2]P_0.} \qquad (6.17)$$

Alternatively, and just as importantly, the reasoning outlined above shows that (for the simple universe here examined) **one can always find a couple of numbers that have all the properties of probabilities, and such that, if used as prescribed by Equation (6.17) to calculate the value of a contingent claim, no free lunches can take place.** These probabilities, needless to say, have nothing to do with the 'real-world' probabilities that we assumed in the 'naive-expectation' section (and that proved useless or, at best, cumbersome and imprecise in determining the fair value of the contingent claim). The couple of numbers π_1 and π_2 can therefore be aptly described as 'pseudo-probabilities'. These pseudo-probabilities are simply a computational construct underpinned by the requirement that no free-lunch strategies should be allowed. Therefore, even if the naive expectation has been shown to be of no use in computing a fair value for C_0, an 'expectation' of sorts has reappeared from the back door: the securities market (S and P) described above completely defines a contingent-claim-independent pseudo-probability distribution (as described by π_1 and π_2) by means of which the fair value of any claim can be obtained as a simple discounted expectation.

Still more insight can be reaped by extending the analysis a bit further. Let us consider a quantity, Z, defined, at all times and in each state of the world, as the ratio of the price of the contingent claim to the price of the riskless bond:

$$Z \equiv C/P. \qquad (6.22)$$

Formally, since the pseudo-probabilities are security-independent, after dividing through by P_0 one can rewrite Equation (6.17) as

$$C_0/P_0 = C_1\pi_1 + C_2\pi_2, \qquad (6.23)$$

or, remembering that $P_1 = P_2 = 1$,

$$C_0/P_0 = (C_1/P_1)\pi_1 + (C_2/P_2)\pi_2 \equiv Z_0 = Z_1\pi_1 + Z_2\pi_2. \qquad (6.23')$$

Therefore the value today of the ratio Z is given by the simple (i.e. undiscounted) expectation of the values of Z at time 1 obtainable with the same probabilities obtained in the previous discussion. The same result would have been obtained if we had defined Z to be the ratio of the security S (which is, after all, a special case of contingent claim) to the riskless bond. But, since the contingent claim payoffs C_1 and C_2 are almost completely general, the same result applies to the stock, to the bond (trivially), and to any contingent claim. Therefore, for $Z \equiv X/P$ (with $X = S, C$ or P) it is always true that

$$Z_0 = Z_1\pi_1 + Z_2\pi_2 = E[X(1)/P(1)] = E[Z(1)] = X_0/P_0. \qquad (6.24)$$

(the notation $X(1)$ indicates the possible values of X at time 1, and $E[\cdot]$ is the expectation operator for the pseudo-probabilities π_1 and π_2). This result is particularly interesting when one recognises that the dimensionless ratio Z expresses the price of a given security X as the price of X in terms of security P, or, more precisely, as the number of units of P that correspond to a given price of X. Therefore, if we are prepared to work with this 'normalised' price Z, we can dispense with discounting altogether. More importantly, by simply invoking the absence of free lunches, we can say that **there must always exist a set of probabilities π_1 and π_2, determined as above, such that the π-expectation of future realisations of the normalised price of any security is simply equal to the value of the normalised security today**. Therefore, in the absence of free lunches, **no return should be expected from this normalised security**.

One might fear that the result that we obtained hinged on the security P having a certain value of 1 in all states of the world at time 1, i.e. in its being a riskless asset. As we noticed before, however, the choice of $P_1 = P_2 = 1$ was expedient, but not necessary: the same reasoning would have followed for any security P (with some restrictions about the sign of its possible payoffs that will be explored later on). The 2×2 system of Equations (6.4) would have been replaced by

$$C_1 = \Pi_1 = \alpha S_1 + \beta P_1 \qquad (6.25)$$

$$C_2 = \Pi_2 = \alpha S_2 + \beta P_2, \qquad (6.25')$$

giving as a solution for α and β

$$\alpha = (P_2C_1 - P_1C_2)/(P_2S_1 - P_1S_2) \qquad (6.26)$$

$$\beta = (C_1/P_1) - (S_1/P_1)(C_2P_1 - C_1P_2)/(S_2P_1 - S_1P_2)$$
$$= (C_1/P_1) - (S_1/P_1)\alpha = (C_1 - S_1\alpha)/P_1. \qquad (6.26')$$

It can at this point be left as the proverbial exercise for the reader the task of showing that, following exactly the same reasoning as before, for any security P with strictly positive payoffs P_1 and P_2 at time 1,

(i) new but similar contingent-claim-independent pseudo-probabilities would have been derived;
(ii) the prices today of contingent claims can be obtained as a discounted expectation taken over these pseudo-probabilities;
(iii) the expectation of relative prices Z (now normalised by the new asset P) are again simply equal to the value of the ratio today, Z_0;
(iv) exactly zero return should be expected (in the π-probability world) from the normalised assets Z.

In other terms, the choice of a different instrument (not necessarily a riskless bond) not only still allows a replication strategy, but also gives rise to pseudo-probabilities $\{\pi\}$ (different for each choice of P) in terms of which contingent claims can be valued as discounted expectations.

It is important to notice that the result just obtained gives us an alternative and equivalent route to find the pseudo-probabilities. Rather than solving the system of equations presented above, one can simply choose as unit of account an arbitrary security P, construct the relative price $Z = S/P$, and work out the probability π_1 that ensures that Z should display no expected growth over the time step. Since Z_0 is simply given by

$$Z_0 = Z_1\pi_1 + Z_2\pi_2 = Z_1\pi_1 + Z_2(1 - \pi_1) \tag{6.27}$$

and $\pi_2 = 1 - \pi_1$, the probability π_1 can be found as

$$\pi_1 = (Z_0 - Z_2)/(Z_1 - Z_2). \tag{6.28}$$

This useful result will be used in Section 6.8, and actually constitutes the standard procedure in order to ensure that the prices produced by a given model are consistent with the absence of free lunches. It will be shown in the following (see, e.g., Chapter 12) that, in practice, pseudo-probabilities are only rarely, if ever, explicitly obtained using expressions of the type shown in Equations (6.16) and (6.16′), and the route followed is actually more similar to the reasoning that leads to Equation (6.27). In the meantime, Equation (6.28), together with the definition of normalised price Z, clearly shows that changing the security P changes the pseudo-probabilities that ensure the absence of free lunches. This result will be revisited over and over again.

6.7 A PRICING FRAMEWORK

The example treated in considerable detail in the previous section might seem to have been unnecessarily laboured. It has been analysed in some depth, however, because the reasoning and the concepts can be used as a blueprint for the more precise treatment to be found in the following sections, and in Appendix A. The concepts and the definitions encountered in these later parts can indeed allow very general and elegant axiomatic treatments, but do not recommend themselves for great intuitional transparency. It is therefore useful at this point (a) to illustrate how the methodology employed so far can be extended into a usable pricing tool; and (b) to give a preview of the correspondence between concepts introduced in the previous section and the financial and mathematical terminology that one encounters in the literature. Needless to say, the attentive reader would recognise by himself these correspondences after the material has been understood and properly digested; but a few preliminary comments at this stage could furnish helpful pointers and assist the financial intuition.

Interesting and illuminating as the previous example might have been, the reader might feel that too much hinged on the rather artificial features of the financial universe: why, for example, were only two states allowed? Had there been three states, then another independent security would have been needed to replicate the payoffs of the contingent claim in states ω_1, ω_2 and ω_3. The keen reader is most welcome to repeat the previous treatment for the case of three states and three securities, but surprisingly little new conceptual insight is to be gained: only the algebra becomes somewhat more involved. Nonetheless the fact that a rather artificial requirement (i.e. the fact that we should allow for three rather than two states) should have a 'financial' implication (i.e. the fact that we seem to have to employ an additional security) is clearly unsatisfactory. Despite these understandable perplexities, the road mapped out in the previous sections is indeed the correct one, and only needs some 'widening' and some 'reinforcement of the banks'. As an example of the type of 'fixing' that will be needed, it will be shown later on that, under reasonably wide assumptions about the nature of the arrival of new information in the economy, and *moving towards the continuous limit*, the two-state case can indeed constitute a general, rather than an *ad hoc*, description of the evolution of the financial markets. More generally, the tasks ahead of us can therefore be outlined as follows:

(1) We shall first of all describe the securities market, and define a contingent claim and the concept of arbitrage.
(2) We shall then show that it is always possible to construct a special trading strategy that *instantaneously* mimics the behaviour of a given contingent claim; by re-adjusting this strategy in an 'admissible' way we shall therefore replicate the claim all the way to its expiry.
(3) If this procedure is successful, we shall then argue that the set-up cost of the replicating strategy must be identical, under penalty of arbitrage, to the value today of the contingent claim.
(4) Looking more carefully at these strategies, we shall find that they are unique, given a set of 'hedging' instruments, but that they can be set up using different building blocks (discount bonds, or money-market accounts, for instance).
(5) We shall show that some of these replicating instruments can be more 'natural' than others in taking out the risk exposure from the contingent claim; and that each choice of hedging instruments will give rise to the

same price for the contingent claim, but to different pseudo-probability distributions (of the type encountered before); since different probability distributions will have to give the same (unique) price for C_0, some other features of the process describing the underlying securities will have to change when moving from one replicating strategy to another.

As promised above, before embarking on this project a preview of the 'correspondences' between the intuitively plausible concepts introduced in Sections 6.1 to 6.5 and the more precise terms that will be introduced later on is provided below. For the moment most of the terms in the right-hand column might be just names, but the list could be referred to at a later stage to confirm or refresh the intuition.

free lunch	$\Rightarrow$	arbitrage
real-world probabilities	$\Rightarrow$	real-world measure
pseudo-probabilities	$\Rightarrow$	risk-neutral measure
replicating strategy	$\Rightarrow$	self-financing trading strategy
'normalising' asset (e.g. P in $Z = X/P$)	$\Rightarrow$	numeraire
no expected return from relative prices	$\Rightarrow$	relative prices are martingales

Apart from these term-by-term correspondences, deeper parallels can be drawn: we argued in Section 6.6, for instance, that the choice of a riskless bond for security P was not unique, and that any security (with suitable restrictions on the sign of its payoffs) could have been used instead; this remark is just a case of the switching of numeraires that will be dealt with at length in the next chapter (see Section 7.8) and provides a powerful tool for tackling practical valuation problems in the most efficient manner. The accompanying change in probabilities, alluded to after Equations (6.26), will be shown to stem in general from Girsanov's theorem. The dimensionless prices of 'normalised securities', Z, will be more precisely defined as the relative prices, which will be shown to be martingales under the risk-neutral measure induced by a particular choice of numeraire. The 'financially sound' procedure whereby the replicating strategy was constructed *before* the actual prices prevailing at the end of the time step are revealed will appear to be an instance of a predictable process, and the importance of these processes in stochastic integration and in creating self-financing trading strategies will become apparent.

With the path so signposted, the next section will begin to analyse in some detail the case of the valuation of a contingent claim in a multi-period setting.

6.8 EVALUATION OF A CONTINGENT CLAIM IN A MULTI-PERIOD SETTING

The case we would like to be able to tackle in this section is considerably more complex than the one dealt with (in four different ways!) in Sections 6.1 to 6.5.

Namely, we would like to be able to assign a fair price in the sense discussed above to a contingent claim that depends on the states of the world prevailing after many (rather than one) steps. We want to allow for as many finite steps as we may wish between today and the terminal date in the trading horizon, i.e., in this context, the expiry of the contingent claim. Notice that we have not moved to the continuous limit, but we are simply allowing for an arbitrarily large, but still finite, number of trading opportunities and of times when prices of securities will be revealed. As it turns out, the 'technical' complications which arise in moving from the multi-period discrete case to the continuous case are not at all trivial, but the incremental financial intuition that can be reaped by making such a step is not as significant as in moving from the one-period to the multi-period discrete case.

As for the geometry of the multi-branch tree, we choose, for clarity of exposition, a non-recombining binomial tree, of the type depicted in Figure 6.2; whilst, in practice, recombining trees are far preferable for computational purposes (see Chapter 8), 'bushy' trees are much handier for expositional purposes, since to each node there corresponds one and only one path. As in the one-period case, we do not assume to know *a priori* the probabilities of moving from one parent node to the two connected arrival nodes.

Formally more precise formulations and definitions of such concepts as 'arbitrage', 'contingent claim', 'trading strategy', etc. are presented in Appendix B. In order to avoid missing the proverbial wood for the trees, however, the choice has been made to appeal in this chapter to the reader's intuition as to the meaning of many of these terms, and to define the remaining in as non-technical a fashion

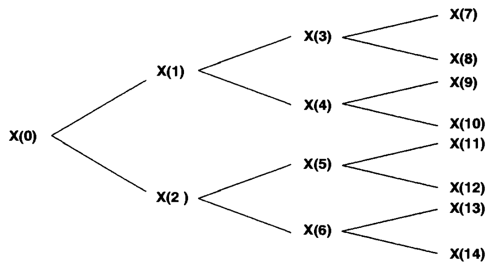

Figure 6.2 The geometry of a non-recombining ('bushy') tree over three time steps, showing the evolution of a generic quantity X (which could be the 'asset' S, the 'bond' P, the contingent claim, C, or any relative price, Z). Notice that a single path can be associated with any node: for instance, the path $X(0)-X(1)-X(4)-X(10)$ is *the* only path associated with node $X(10)$

as possible. Frequent pointers are dispersed in the qualitative discussion to allow for a more formal and precise discussion.

The winning strategy in the case of the one-period two-state contingent claim consisted of:

(i) replicating exactly over one time step the payoffs of the claim with a suitably built portfolio; and
(ii) arguing that the set-up cost of this trading strategy had to be equal, under penalty of arbitrage, to the fair value of the contingent claim.

We want to attempt to employ a similar procedure in the multi-period case, but two main problems stand in our way: whilst it is easy enough to build a non-recombining multi-step tree if one assumes that the initial and each intermediate state can only evolve to exactly two possible states, this 'two-branch' assumption might appear both restrictive and artificial. If we could prove that this procedure enjoys an acceptable degree of generality, one could then be tempted to argue that, since each intermediate node 'looks' exactly like the original (time-0) node in the one-period case, one should be able to repeat the old reasoning on a node-by-node basis. The crux of the argument in the one-period case, however, was that, after *the* time step, all the uncertainty in the economy was resolved, and the final value of the contingent claim known with certainty. Therefore one was able to rest assured that, after paying the set-up cost for the replicating portfolio, no more *net* cash would have to be received or paid. In the multi-period case, however, even if it could in general be set up by means of a lot of one-period-like problems strung together, for all we know at this stage, we might have to make new, and possibly state-dependent, injections of cash to replicate locally the claim over the two possible next states. If this were the case, speaking of a unique set-up cost for *a* replicating portfolio would be meaningless, and we would seem to be led down the unappealing market-price-of-risk route in order to assign a fair value to the contingent claim. Therefore, the second question we have to answer is: 'how do we know that the cost of setting a trading strategy such that it will replicate the contingent claim over the *first* time step will allow us to replicate the claim all the way to expiry?'.

These two questions have brought in a third one, important although not quite as fundamental. In the one-period case, speaking about the claim after one step or at expiry was one and the same thing. In the multi-period instance, however, we know exactly what we mean by the contingent claim at its expiry, since this is exactly how a contingent claim is defined (i.e. by specifying the value it assumes in the various possible states of the world, or, more technically, its random profile). In the multi-period setting, however, what do we mean when we speak of the contingent claim after two, or three, or 47 steps? In technical terms, a contingent claim is a random variable (a random payoff) and not a stochastic process. Despite the fact that this latter question is somewhat less 'fundamental' than the previous two, we shall begin by addressing it first.

6.9 SELF-FINANCING TRADING STRATEGIES

Let us consider the last two steps of a multi-period problem. At the last time step, that we shall label by the index n, we know without doubt what the contingent claim is worth in any of the 2^n states: this is, after all, the very definition of a contingent claim. Let us move backwards in time to time step $(n-1)$, and consider any of the $2^{(n-1)}$ states, say, the jth. This state, by our construction, is 'connected' with two and only two states at time step n, that we can label as $(n, j+1)$ and $(n, j-1)$. We do not know, for the moment, what transactions and what cash flows might have occurred to get to state j after $(n-1)$ steps. We do know, however, that, *given the fact that we are at node* $(n-1, j)$, and that at time step n we reach expiry time, the problem now looks *exactly* like the simple one-period problem of the previous sections. Neglecting, once again, earlier transactions, all the results we have obtained in Sections 6.1 to 6.4 still apply. In particular, we are in the position to set up a local replicating portfolio that will certainly pay *at expiry* either $C(n, j+1)$ or $C(n, j-1)$, i.e. either of the values of the claim in the two states of the world attainable from node $(n-1, j)$. Therefore one can meaningfully speak of the fair value of the claim contingent upon one's being at node $(n-1, j)$. In more concrete terms, we can unambiguously speak of how much one should be prepared to pay in this state of the world in order to obtain an entitlement to $C(n, j-1)$ or $C(n, j+1)$. And, since there was nothing special about state j, one can repeat exactly the same reasoning for all the states at time step $n-1$. Therefore we have succeeded in bringing back the whole contingent claim by one time slice, with the simple proviso of adding to the expression 'fair value of the contingent claim' the clause 'contingent upon state $(n-1, j)$ having been reached'. If one looks at the problem in this light, the terminal step looks less 'special': also the terminal (time n) values of the contingent claim are, in fact, trivially 'fair' values, and also, in a sense, conditional, since they apply only if a particular state obtains. Therefore, even if it would have appeared somewhat pedantic, the terminal values of the contingent claim could have been described as 'fair values of the claim contingent upon a particular state at time n having been reached'. Seen from this perspective, the situation at time slice $n-1$ is *exactly* identical to the situation at time n. We have therefore succeeded in 'bringing back' the conditional contingent claim by one time step. By bootstrapping, i.e. by repeating the same procedure backwards $n-1$ times, we can hope to be able to work our way back to the root of the tree. Let us examine carefully what this 'brought-back' conditional contingent claim 'means' at state (0,0) for the simple case of a two-state tree (Figure 6.3).

By paying $C(0)$ today one can set up a portfolio made up of $\alpha(0)$ and $\beta(0)$ units of S and P that will certainly produce $C(1)$ or $C(2)$ according to which state of the world will prevail. But $C(1)$ can be chosen to be the amount of money needed to construct (with obvious notation) the portfolio of $\alpha(1)$ and $\beta(1)$ units of S and P that certainly replicates the payoffs $C(3)$ and $C(4)$. Similarly $C(2)$ can be chosen to be the amount of money required to purchase holdings

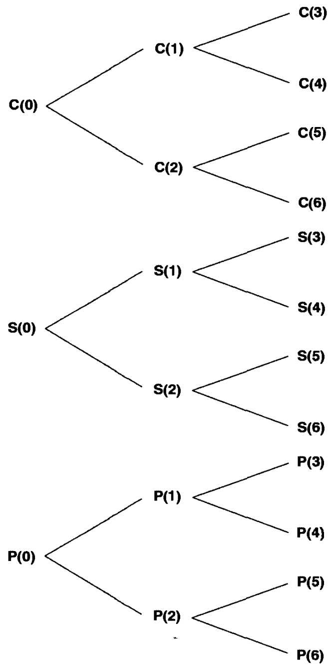

Figure 6.3 The values of the 'asset', S, the 'bond', P, and the contingent claim, C, after two steps; geometrically equivalent nodes on the three non-recombining trees, i.e. nodes labelled by the same number, correspond to the same state of the world

$\alpha(2)$ and $\beta(2)$ needed to produce $C(5)$ and $C(6)$ with certainty. **Therefore, as long as one is prepared to alter, *at no extra cost*, the composition of the portfolio at each time step, one can certainly replicate the payoffs of the contingent claim at time step 2 by paying $C(0)$ today!** Notice the importance of the 'at no extra cost' qualifier: $C(1)$ and $C(2)$ are needed to replicate $[C(3)\ C(4)]$ and $[C(5)\ C(6)]$, respectively, but $C(0)$ is all that is required to provide $C(1)$ or $C(2)$, as needed. Notice also that it is not necessary to 'peek ahead': one starts with an allocation $\alpha(0)$, $\beta(0)$ which will not, in general, be the one that will be needed at time step $n-1$, i.e. at the time step before the expiry of the contingent claim, and therefore the hedging strategy is dynamic and not at all static. (Notice that, in the one-period case, there is no distinction between a static and a dynamic hedging strategy.) We can rest assured, however, that, by re-balancing the portfolio as and when needed, the contingent claim's payoff at expiry will be replicated. As a consequence, also in the multi-period case, we do not need any foreknowledge of prices not yet realised in the market, and the procedure, that we could aptly described as 'wait-and-see', can indeed be put in place in practice. (A more technical term for the holdings α and β adjusted over time in the manner just described is a 'previsible process'. See Appendix A, Section A.2.)

Therefore, *en route* to answering the third question, regarding the meaningfulness of speaking of the contingent claim at times earlier than its expiry, we have also answered the second important question mentioned above: we have in fact established that the price we have to pay in order to set up an appropriate replicating portfolio *over the first time step* is all we need to replicate the claim at expiry. Therefore this amount of money must equal the fair (no-arbitrage) price of the contingent claim. By this reasoning we have shown that, also in the multi-period case, the fair price of a contingent claim paying a random payoff after n time steps is equal to the set-up cost of the replicating portfolio over the *first* time step.

The issue that still remains to be addressed is whether the two-step branching really constitutes a reasonably general set-up rather than a convenient but financially artificial construct. Before tackling this question, however, far deeper insight can be obtained into the valuation of contingent claims in a multi-period setting by dwelling a little longer on the implications of what we have just discovered.

6.10 FAIR PRICES AS EXPECTATIONS

Before proceeding, it is first of all worth stressing once again the importance of the fact that the suggested procedure can be carried out all the way up to time step $n-1$ without any further injections or withdrawals of funds: if this were not the case, and future net cash movements were of a non-deterministic nature (i.e., dependent on states not yet realised), then the whole argument would collapse, at least in the form here presented. The fact that, in net cash terms, the 'wait-and-see' procedure suitably takes care of itself is important enough to warrant it the special name of 'self-financing trading strategy'. A more precise description can be found in Appendix B, Section B.1.

Second, we know that, in the one-period case, the fair value of a contingent claim can be expressed as an expectation over the possible terminal payoffs calculated using pseudo-probabilities. These probabilities, it will be recalled, had nothing to do with real-world probabilities of events, but depended on the values of S and on the type of instrument, P, chosen as 'bond'. The law associating a pseudo-probability to each possible state of the world is normally referred to as a measure (see Section A.2). Loosely speaking, one can think of a measure as a probability density. Therefore, whenever one speaks of expectations, it is essential to specify under what probability distribution (measure) these expectations are taken. This is far from being a pedantic proviso, since, as we have seen, each different type of instrument P chosen as numeraire will give rise to a different set of pseudo-probabilities; hence to different measures; hence to different expectations. More precisely, if, for instance, the instrument P is chosen to be the discount bond maturing at the payoff time of the contingent claim, then the corresponding measure is normally called the forward-neutral measure; or, to make another common choice, if the instrument P is taken to be a money-market account, rolled over at each time step at the then prevailing short rate, then the associated measure is often called the 'risk-neutral measure'. Notice that, in the one-period setting, there is no distinction between a discount bond maturing at claim expiry and a one-period money-market account.

In the multi-period setting, as discussed above, we can repeat the one-step reasoning for each node taken with its own two branching states. We can therefore also say that, for the example above, the value of the contingent claim is equal to the expectation in the appropriate measure of the two values $C(1)$ and $C(2)$. Also $C(1)$ and $C(2)$, however, can be seen as expectations, over [$C(3)$ and $C(4)$] and [$C(5)$ and $C(6)$], respectively. Therefore $C(0)$ is an expectation of an expectation. Needless to say, in the case of n steps, the fair value $C(0)$ will turn out to be given, by extending the same reasoning, by n nested conditional expectations. Since, in general, the evaluation of an expectation can be reduced to performing (analytically or numerically) an integral, it would seem that evaluating a contingent claim should always be tantamount to performing an n-dimensional integral, with n potentially very large. This is indeed the route implicitly taken when using computational lattices or trees (see Chapters 8 and 9). If, however, the contingent claim is truly European, i.e. if it only depends on the states of the world at one point in time, then a powerful result, known as the *tower law*, can come to the rescue, by drastically reducing the dimensionality of the underlying integral from n to one.[2] More precisely, the tower law states that the expectation at time i of conditional expectations taken at a *later* time j is simply equal to the expectation taken at the earlier time i: see Section A.3. But, if this is the case, this result can give us a powerful computational aid, by freeing us from the need to think always in terms of locally replicating portfolios. The reasoning goes as follows: the fair price $C(0)$ can be equivalently regarded as either the set-up cost of the first locally replicating portfolio (that will furnish us with sufficient holdings of instruments and cash to build on the way $n-1$ further portfolios) or as the (discounted) expectation taken under the appropriate measure of a discounted expectation of a discounted expectation ...

of a discounted expectation. But, by the tower law, **this is simply the discounted expectation at time 0 over the terminal distribution.**

Furthermore, as we saw in Section 6.4, if we work in terms of the normalised (relative) prices Z we can dispense with discounting altogether. Therefore, under the appropriate measure, the fair value $Z(0) = C(0)/P(0)$ can be evaluated as the *simple expectation* over the terminal distribution of the relative prices $Z(T)$. Needless to say, since $P(0)$ is a known market price today, from the knowledge of $Z(0)$ we can immediately extract the fair value of the unnormalised (i.e. cash) contingent claim.

The only piece of information we still seem to be lacking is how to find this 'appropriate measure'. In order to tackle this task, let us start again from what we know how to do, i.e. from the strung-together replicating portfolios, and let us look again at the results previously obtained immediately after Equation (6.24). We there established that

(i) **there must always exist a set of probabilities π_1 and π_2, determined as shown in Section 6.5, such that the π-expectation of future realisations of the normalised price of any security is simply equal to the value of the normalised security today;**
(ii) **no return should be expected from this normalised security.**

The second result implies that, over a single time-step, the relative price Z should display no expected growth. (By getting slightly ahead of ourselves we can already say that Z should display no drift.) A process possessing this property is called a martingale: see Section A.5. But the tower law comes to the rescue again: the property of the process Z of being a martingale, i.e. of displaying no drift whatsoever, embodies an expectation condition and, by the tower law, must apply not only locally but also globally: in other terms, the expectation of a given relative price at time step n must be equal to the relative price today. This requirement gives us a powerful tool to determine the 'appropriate' measure: that set of pseudo-probabilities that make relative prices martingales (i.e. driftless) will certainly produce the 'appropriate' measure under which the expectations have to be taken. This is a generalisation of the result obtained in the one-period case at the end of Section 6.5, which could be profitably revisited at this point. This might all seem rather abstract at the moment, but the treatment will be repeated over and over again in the course of the analysis of specific models (for instance, Section 12.2 will show in practice how to extract the measure from the requirement that relative prices should be martingales in the case of the BDT model). The important thing to remember is that the martingale condition should apply locally and not just globally: the tower law does provide us with a powerful tool, but does not dispense us from the need to ensure that, under the evaluation measure, the relative price should be a martingale at each individual node.

6.11 SWITCHING NUMERAIRES AND RELATING EXPECTATIONS UNDER DIFFERENT MEASURES[3]

Despite the fact that still quite a lot of work lies ahead of us, we have already achieved some very important results towards the evaluation of the no-arbitrage price of a contingent claim in a multi-period setting. Namely we have established:

(i) that the replicating-strategy argument still holds on a node-by-node basis;
(ii) that the cost of setting up the node-(0,0) replicating portfolio is all we have to pay in order to construct, by means of a self-financing trading strategy, the terminal payoff of the claim;
(iii) that this initial set-up cost can be expressed in terms of n nested conditional expectations;
(iv) that, via the tower law, these nested conditional expectations can in turn all be evaluated as a *simple* expectation taken at the origin over the terminal values of the relative prices Z, as long as the expectation is taken under the appropriate measure;
(v) that this appropriate measure must always exist if we want to prevent arbitrage;
(vi) that this measure can be implicitly determined by imposing that, globally and locally, relative prices should be martingales.

We still have to justify the reasonableness of the two-state branching. If the reader can bear a further delay, one more important aspect can be profitably explored at this point. As has been repeatedly emphasised, changing the instrument P changes, for a given instrument S, the no-arbitrage pseudo-probabilities on a node-by-node basis: see, e.g., Equations (6.27) and (6.28). If one makes the identification of P as the numeraire, i.e. as the normalising security that turns cash prices into dimensionless relative prices, it is clear why changing instrument P is referred to as 'switching of numeraires'. Therefore switching numeraire induces the change in the measure (see Equation (6.28)) required to turn the new relative prices into martingales. Let us examine more closely the nature of this change.

Let us assume that we have already found a measure Q, associated with numeraire P, under which relative prices are martingales. In concrete terms, the measure Q is 'represented' by the collection of the pseudo-probabilities $\pi(i, j)$ connecting the generic node j at time step i with its two neighbours at time step $i+1$. Let us now choose a different numeraire, P', which will induce a new no-arbitrage measure Q', with pseudo-probabilities $\pi'(i, j)$. At each node we can create the ratio of $\pi(i, j)$ to $\pi'(i, j)$, and call this ratio $\psi(i, j)$. A pseudo-probability connects a parent (starting) and an arrival point. By convention, we can assign this pseudo-probability to the 'arrival' node. So, if $\pi(1, 1)$ and $\pi(1, -1)$ are the pseudo-probabilities connected with measure Q to go from the root of the tree to nodes (1,1) and $(1, -1)$, respectively, and similarly if $\pi'(1, 1)$ and $\pi'(1, -1)$ are the pseudo-probabilities connected with measure Q' to go from the root of the tree to nodes (1,1) and $(1, -1)$, then we shall associate the value $\psi(1, 1) = \pi(1, 1)/\pi'(1, 1)$ to node (1,1), and $\psi(1, -1) = \pi(1, -1)/\pi'(1, -1)$ to node $(1, -1)$. This leaves undetermined what value of ψ to assign to the node (0,0). Let us choose, in what might seem at the moment to be a purely arbitrary way, the value $\psi(0, 0) = 1$. Given a node (r, s) we can easily evaluate the total probability of reaching it from the root after r steps under measure Q or Q': it will simply be given by the product of the probabilities $\pi(i, j)$ or $\pi'(i, j)$ encountered along *the* path connecting the root with node (r, s).

Given the definitions above we can introduce a new useful quantity, namely the product, for a given path, of the ratios $\psi(i, j)$ out to node (r, s), and denote it by the symbol $\Psi(r, s)$. (Notice again that, since our tree does not recombine, speaking of a terminal node such as (r, s) is tantamount to speaking of a specific path.) Conversely, one could have similarly defined $\phi(i, j)$ as the ratio $\pi'(i, j)/\pi(i, j)$, and $\Phi(r, s)$ as the product of the terms $\phi(i, j)$ out to node (r, s). For the construction to be feasible, however, all the $\pi(i, j)$ and all the $\pi'(i, j)$ must be different from zero, or either ratio will not be defined. But we saw before, when we discussed the determinant condition (see Equation (6.6) in Section 6.2), that indeed both pseudo-probabilities propagating from a given node had to be different from zero for the replicating strategy to be feasible. Therefore, despite the fact that switching numeraires can affect measures in a way that we have not discovered yet, we can already say that this transformation must be such that events impossible under one measure will also be impossible under the new measure. (In technical terms this property of two measures is referred to as the equivalence of measures Q and Q', or as the fact that Q and Q' share the same null set. See Section A.2.) With this proviso in mind, we can rest assured that the quantities ψ, ϕ, Ψ and Φ will all be well defined.

What use can be made of the products of ratios Ψ and Φ, now that we know that we can meaningfully speak about them? To begin with, the new quantities $\psi(i, j)$ can immediately tell us how to convert an expectation taken under one measure into an expectation taken under a different measure (and, after all, we are always interested in expectations taken under given measures, not in the measures themselves). To see how this 'switch of expectations' can be accomplished, let us begin by considering the simplest expectation, i.e. the one taken over the nodes (1,1), $(1, -1)$ of the first time step. In keeping with the notation above, $\pi(1, 1)$, $\pi'(1, 1)$, $\pi(1, -1)$ and $\pi'(1, -1)$ are the pseudo-probabilities, and

$$\Psi(1, 1) = \psi(1, 1) = \pi(1, 1)/\pi'(1, 1) \tag{6.29}$$

$$\Psi(1, -1) = \psi(1, -1) = \pi(1, -1)/\pi'(1, -1) \tag{6.29'}$$

The expectation under Q of any quantity X over the two values occurring at time 1 is simply given by

$$E_0^Q[X] = \pi(1, 1)X(1, 1) + \pi(1, -1)X(1, -1) = \sum_i \pi(1, i)X(1, i); \tag{6.30}$$

with $i = -1, 1$; but, for this particularly simple case, $\pi(1, i)$ can be rewritten as $\pi'(1, i)\Psi(1, i)$ and therefore

$$E_0^Q[X] = \sum_i \pi(1, i)X(1, i) = \sum_i \pi'(1, i)[X(1, i)\Psi(1, i)]. \qquad (6.30')$$

Therefore, for the one-step case, **the expectation under Q** (i.e. with pseudo-probabilities $\pi(i, j)$) **can be replaced by an expectation under Q'** (i.e. using pseudo-probabilities $\pi'(i, j)$) **simply by multiplying each quantity $X(i,j)$ by the ratio $\Psi(i,j)$, i.e. by taking the Q' expectation not of X, but of the product $X\Psi$:**

$$E_0^Q[X] = E_0^{Q'}[X\Psi] \qquad (6.31)$$

A moment's thought will show that this is true not only for the first time step, but after an arbitrary number of moves. In fact, the expectation under Q of a quantity X at time step s is given by

$$E_0^Q[X_s] = \sum_{i=1,\ldots,2^s} [\Pi_{k=1,s}\pi_{k,i}]X(s, i), \qquad (6.32)$$

where the i sum runs over all the possible paths from the tree root (0,0) to each of the 2^s nodes at time s, and the new notation $\pi_{k,i}$ has been introduced to denote the probability of transition from time $k-1$ to time k for the path originating at the tree root (0,0) and ending at node i at time s: see the figures below. (Remember that there is one and only one path to any such node in a non-recombining tree.) But, given the definitions above, each $\pi_{k,i}$ can be rewritten as $\pi'_{k,i}\psi_{k,i}$ and therefore

$$E_0^Q[X_s] = \sum_i [\Pi_{k=1,s}\pi'_{k,i}\psi_{k,i}]X(s, i) = \sum_i [\Pi_{k=1,s}\pi'_{k,i}\,\Pi_{k=1,s}\psi_{k,i}]X(s, i)$$
$$= \sum_i [\Pi_{k=1,s}\pi'_{k,i}\,\Psi_{s,i}X(s, i)] = E_0^{Q'}[\Psi_s X_s], \qquad (6.33)$$

where the subscript notation used for the quantities $\psi_{k,i}$ has the same meaning as the subscripts for $\pi_{k,i}$.

Notice that the product of the pseudo-probabilities encountered along a given path simply gives the pseudo-probability of occurrence of a particular path, as long as the various moves are independent. It is easy to check that

$$\sum_i [\Pi_{k=1,s}\pi'_{k,i}] = 1 \qquad (6.34)$$

Therefore, also in the multi-step case, it is indeed true that the expectation taken at time 0 under measure Q of a process X after s time steps is equal to the expectation taken at time 0 under measure Q' of the new process $X\Psi$. This result is important enough to warrant giving the quantity Ψ the special name

of the Radon–Nikodym derivative, often symbolically denoted by dQ/dQ'. The following worked-out example should make these concepts clearer.

A worked-out example

Let us start from the case of a security S displaying the values shown in Figure 6.4 over the first two time steps. The first numeraire, P, can assume the possible values shown in Figure 6.5 over the same time steps.

With this numeraire, Figure 6.6 shows the relative prices Z (in bold) and pseudo-probabilities π that are obtained. As mentioned before, the pseudo-probabilities are determined by enforcing the requirement that the relative prices should be martingales. See Equation (6.28); for the bottom corner of the bushy tree after the first step, for instance, this translates to the condition that $1.021103 = (0.973913 \times 0.812058) + (1.225000 \times 0.187942)$.

A second numeraire, P', and the accompanying relative prices, Z', and pseudo-probabilities, π', are then shown in Figures 6.7 and 6.8. Once again, the pseudo-probabilities π' are obtained by imposing that the Z' should be martingales.

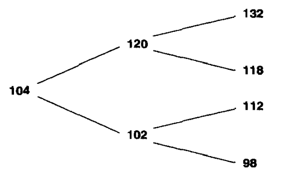

Figure 6.4 Values of S over the first two time steps

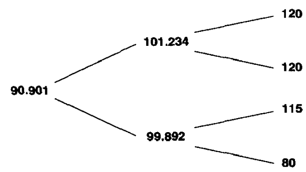

Figure 6.5 Values of P over the first two time steps

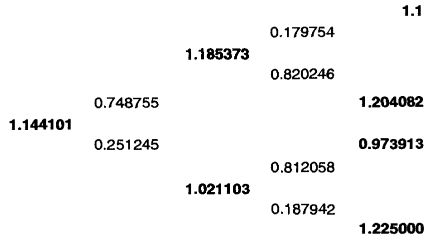

Figure 6.6 Relative prices Z (in bold) and pseudo-probabilities π

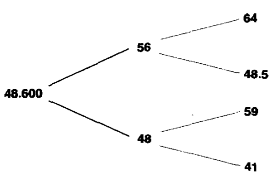

Figure 6.7 Values of P' over the first two time steps

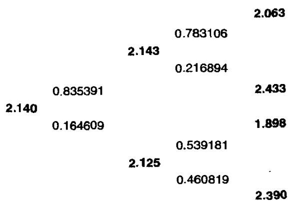

Figure 6.8 Relative prices Z' (in bold) and pseudo-probabilities π'

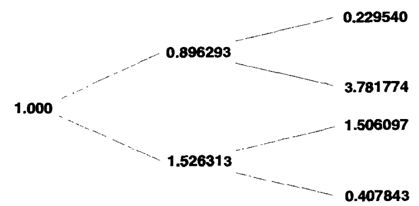

Figure 6.9 Values of $\psi = \pi/\pi'$ over the first two time steps

With this information we are in a position to obtain the ratios $\psi = \pi/\pi'$ (see Figure 6.9).

Figure 6.10(a) and (b) then show the products of the pseudo-probabilities π and π', respectively, along the different possible paths (the integers 0 to 6 label the paths, or, equivalently, the states of the world after two time steps). For both trees, one can check that the sum over all the paths of the products of the pseudo-probabilities does add up to 1 (bottom cell). We are now in a position to build the tree for the Radon–Nikodym derivative Ψ (Figure 6.11).

As shown above, we can at this point equivalently calculate the expectation of, say, S after two steps using either the pseudo-probabilities π, or the pseudo-probabilities π' and the new quantity $S\Psi$. The resulting arrays of quantities S, $S\Psi$, $\Pi_i\pi_i$ and $\Pi_i\pi'_i$ corresponding to the various paths (states of the world) after two steps are shown in Table 6.2. The two entries on the bottom line show that indeed the same expectation is obtained for S, *in the measure Q associated with numeraire P*, either by using the probabilities π, or by using the probabilities π' associated with the measure Q' induced by numeraire P' and by taking the expectation of the product of the variable S and the Radon–Nikodym derivative.

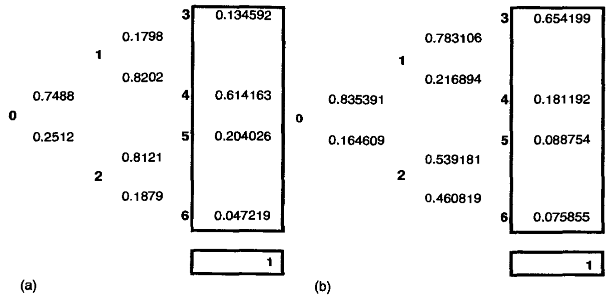

Figure 6.10 Products of the pseudo-probabilities (a) π, and (b) π', along paths 1 to 6

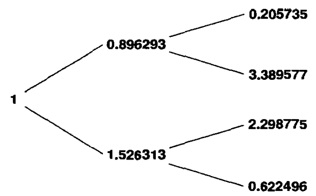

Figure 6.11 Values of the Radon–Nikodym derivative Ψ

Table 6.2 Values Corresponding to Paths 1–6 after Two Time Steps

State	$\Pi_i\pi_i$	S	$S\Psi$	$\Pi_i\pi'_i$
3	0.134592	132	27.15708	0.654199
4	0.614163	118	399.9701	0.181192
5	0.204026	112	257.4628	0.0887S4
6	0.047219	98	61.00459	0.075855
Expectation		117.7157		117.7157

Apart from establishing that events possible in the measure implied by a numeraire must be possible also in the measure induced by a different numeraire, i.e. apart from the equivalence of the two measures, we have not been able to establish anything more precise about the nature of the transformation of measure. In order to accomplish this task, the time has finally come to address the issue of the justifiability of the two-state branching procedure. The following section will deal with this topic.

6.12 JUSTIFYING THE TWO-STATE BRANCHING PROCEDURE

The two 'up' and 'down' values reachable from any given node have so far been allowed to be totally arbitrary (apart from the constraint that the return from no security should in both states be above or below the riskless return). Let us now decompose the two moves into two components, the first common to both states, and the second state specific; more precisely, let us choose the common move to be such that, were it subtracted from the arrival values, these would be symmetrically positioned above and below the parent node,[4] as in Figure 6.12. We can therefore write

$$S(\mathrm{u}) = S(0) + m + s$$
$$S(\mathrm{d}) = S(0) + m - s$$

with $m = 5$ and $s = 7$. Similar constructions (with obviously different numerical values) can be carried out for the 'bond' and the contingent claim. The replicating-portfolio strategy discussed at length in Section 6.2 can then be used to determine the pseudo-probabilities of reaching the upper and lower states. (In the following the qualifier 'pseudo' will often be omitted to lighten the description of the

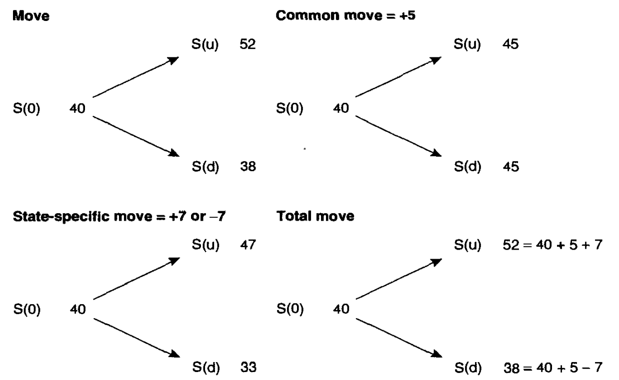

Figure 6.12 Decomposition into common and state-specific moves

procedure.) With these probabilities the expected value and the variance of the change in S, ΔS, over the time step can be easily evaluated:

$$E[\Delta S] = E[S(1) - S(0)] = (m + s)\pi + (m - s)(1 - \pi) \quad (6.35)$$
$$\mathrm{Var}[\Delta S] = ((m + s) - E[\Delta S])^2\pi + ((m - s) - E[\Delta S])^2(1 - \pi)$$
$$= 4s^2\pi(1 - \pi) \quad (6.36)$$

where, to lighten notation, $\pi = \pi^{\mathrm{up}}$ and $(1 - \pi) = \pi^{\mathrm{down}}$. Nothing has been said so far about the length of the time step, but, as mentioned before, we would like to be able to make it as small as desired, and to add as many further steps as necessary in order to reach the same terminal date. So, if the time step in Figure 6.12 was of length one (in some units), we might want, to begin with, to reduce it to half this length. Before doing so, however, notice that, if we wanted to be able to assign exogenously the variance of the increment of S, perhaps by making it time- and/or state-dependent, this would uniquely determine the quantity s, since π is uniquely determined by the no-arbitrage replicating-portfolio strategy. Therefore, if we want to assign 'from the outside' the variance of the increment, we are not at liberty to choose the values $S(\mathrm{u})$ and $S(\mathrm{d})$ in a completely arbitrary way (as we had so far assumed), but we must do so in such a way that $|S(\mathrm{u}) - S(\mathrm{d})| = 2s$.

Let us now make the extremely powerful assumptions that the magnitude of the common move, m, should depend linearly on the length of the time step, but that the state-specific part of the move, s, should be linear *in the square root* of the time step (see also Section A.4 about this assumption):

$$m \equiv \mu\Delta t \quad (6.37)$$
$$s \equiv \sigma\sqrt{\Delta t} \quad (6.37')$$

Notice that the exogenous standard deviation per unit time, or volatility, σ, can be made state- and time-dependent, as can the expectation per unit time, or drift, μ, but once these external choices have been made, assumptions (6.37) and (6.37′) prescribe the possible values of S from a given parent node.

What would have been obtained for the expectation (6.35) and the variance (6.36) if assumptions (6.37) and (6.37′) had been enforced? To begin with, it is straightforward to rewrite Equations (6.35) and (6.36) as

$$E[\Delta S] = \mu\Delta t + (\sigma\sqrt{\Delta t})(2\pi - 1) \quad (6.35')$$
$$\mathrm{Var}[\Delta S] = 4\sigma^2\Delta t[\pi(1 - \pi)] \quad (6.36')$$

But, from Section 6.6, Equations (6.16) and (6.16′), we know the values for $\pi = \pi_1$ and $(1 - \pi) = \pi_2$ necessary to prevent arbitrage:

$$\pi_1 = [S_0/P_0 - S(\mathrm{d})]/(S(\mathrm{u}) - S(\mathrm{d})) \quad (6.38)$$
$$\pi_2 = [S(\mathrm{u}) - S_0/P_0]/(S(\mathrm{u}) - S(\mathrm{d})) \quad (6.38')$$

Therefore

$$S(\mathrm{u}) - S(\mathrm{d}) = 2s = 2\sigma\sqrt{\Delta t} \tag{6.39}$$

$$S_0/P_0 = S_0 \exp[r\Delta t] \cong S_0(1 + r\Delta t) \tag{6.40}$$

and, after substituting,

$$\pi_1 = \tfrac{1}{2} + (S_0 r - \mu)(\sqrt{\Delta t})/(2\sigma) \tag{6.41}$$

$$\pi_2 = \tfrac{1}{2} - (S_0 r - \mu)(\sqrt{\Delta t})/(2\sigma) \tag{6.41$'$}$$

In the limit as Δt goes to zero, $\pi_1 = \pi_2 = \frac{1}{2}$, and therefore

$$E[\Delta S] = \mu \Delta t \tag{6.35$''$}$$

$$\mathrm{Var}[\Delta S] = \sigma^2 \Delta t \tag{6.36$''$}$$

But Equations (6.35″) and (6.36″) describe a process ΔS that grows over time with a drift and a variance per unit time μ and σ^2, respectively; we have therefore reached the conclusion that, **if indeed the increments of S are such that Equations (6.37) and (6.37′) are satisfied, then, in the limit as Δt approaches zero, the sample variance and expectation obtainable from the values S(u) and S(d) are the same as the variance and expectation of a random walk (Brownian motion in continuous time) with drift μ and variance per unit time σ^2.** (See Section A.4 for a mathematically more satisfactory treatment.) Conversely, **if the shocks are generated by the increments of a non-symmetric random walk, then a two-state branching process is adequate to recover its first two moments, at least in the limit as Δt approaches zero.**

It is now clear that the conditions introduced above are not as arbitrary as they might have appeared when first introduced, but they are both necessary and sufficient for the shocks to the economy to be generated by a non-symmetric random walk. That the latter should be the case is, of course, an assumption rather than a 'provable truth', and its virtual ubiquity in option pricing should not make one think otherwise. A true jump process — a process, that is, such that, for any 'well-behaved' volatility, the scaling property (6.37′) does not apply — cannot be analysed along these lines, and the whole no-arbitrage argument, that hinges on the two-state-branching procedure, does not carry through.[5] If we are happy, however, with the Brownian assumption, i.e. if we believe that an equation of the type

$$\Delta S = \mu \Delta t + \sigma\varepsilon\sqrt{\Delta t} \tag{6.42}$$

(with ε a draw from a standardised normal distribution) can adequately describe the evolution of the price of securities, then we can rest assured that the arguments presented so far all apply — in particular, we know, for instance, that a three-state evolution is unnecessary and that, therefore, the financially unappealing introduction of a third security (needed to solve the 3 × 3 system) is not required.

6.13 THE NATURE OF THE TRANSFORMATION BETWEEN MEASURES — GIRSANOV'S THEOREM

There is still an important conceptual step to make: we know that, in the absence of arbitrage, the choice of a numeraire induces a set of probabilities (a measure Q') such that normalised prices are martingales. We can think of this new measure as the result of a change from the (unknown) real-world probabilities to a new set of pseudo-probabilities. In the previous section we discovered that, if the shocks to the economy are produced by a Brownian motion of known variance, then, in the measure Q', the two-state branching procedure is all that is needed to give an adequate description of the process. But, since we can at best have access to the variance of this Brownian motion in the *real* world, how do we know that, when we work with the 'scrambled' probabilities of the measure Q', the resulting process variance will turn out to be the same as the variance that we would like to be able to assign exogenously? Indeed, how do we know that, in Q', a Brownian motion is still a Brownian motion at all? In other words, what is the nature of the transformation of a given Brownian motion when we move from the real-world measure Q to the new measure Q'?

We shall not attempt to answer this question by providing a proof, at least in this section (though see Section A.7). We shall, instead, work through a specific example that should clarify the intuition behind the relevant mathematical theorem (Girsanov's theorem).

Let us consider again the now-familiar case of a stock, S, a claim, C, and a bond, P, which move from an initial parent state to two possible states after a time step Δt. Using P as numeraire, let us determine the pseudo-probabilities π_1 and π_2 that make the relative price $Z' = S/P$ a martingale. From Equations (6.16) and (6.16′) of Section 6.6 we know that these pseudo-probabilities are given by

$$\pi_1 = [S_0/P_0 - S_2]/(S_1 - S_2) \tag{6.43}$$

$$\pi_2 = [S_1 - S_0/P_0]/(S_1 - S_2) \tag{6.43$'$}$$

Let us now consider a different asset, T, that we can choose as a second numeraire; this new unit of account will give rise to new relative prices Z'' $= S/T$ and new probabilities π'_1 and π'_2 under which the prices Z'' are driftless. We want to ask the following question: what happens to the variance and drift of S as the interval Δt approaches zero when we move from the probabilities $\{\pi\}$ to the probabilities $\{\pi'\}$, i.e. when we switch numeraire?

Answering this question in complete generality would require rather complex mathematical tools (see Section A.7); as mentioned above, however, in this section we can look at a specific example, and examine the nature of the transformation for this case. To this effect, let us begin with a Δt of 0.25 years, and consider the prices in the two possible states for the stock and the bond obtainable using real-world drifts of 10% S and 12% P, and volatilities of 20% S and 16% P, respectively. The resulting values are shown in Figure 6.13.

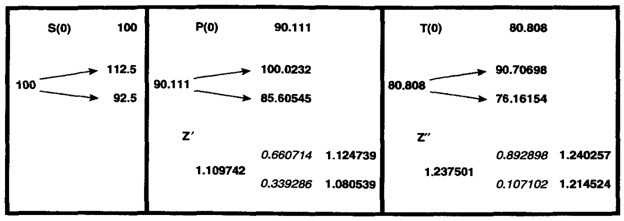

Figure 6.13 Prices after $\Delta t = 0.25$

With these values one can then obtain the relative price of S using numeraire P or T, denoted by Z' and Z'', respectively. Imposing the requirement that these relative prices should be martingales then determines the two sets of equivalent pseudo-probabilities, by means of which the expectation and variance of S in the two measures can be calculated. The values are shown in italics in Figure 6.13. For the chosen time step both the variance and the expectation differ in moving from one measure to the other: the expectation, for instance, is 105.71 and 110.358 in the measure Q and Q', respectively.

If one wanted to reduce the time step, more significant than the expectations and variances would be the drift (i.e. the expectation divided by the time step) and the volatility (i.e. the standard deviation per unit time). These quantities are reported for the two numeraires in Table 6.3.

When this scaling is carried out, a very interesting feature becomes apparent: whilst the drifts converge in the two measures to different values, **the variance per unit time is actually the same irrespective of whether one is working in the measure Q or in the measure Q', at least as long as the time step is allowed to become sufficiently small.** Therefore the transformation of the

Table 6.3 The Standard Deviations per Unit Time in Measures Q' and Q'', Respectively (STD'[S] and STD''[S]) and the Expectations per Unit Time ($E'[S]$ and $E''[S]$), also in Measures Q' and Q'', with the Percentage Difference (Diff) between the Two.

Dt	STD'[S]	STD''[S]	Diff	$E'[S]$	$E''[S]$	Diff
0.250000	18.9387	12.3697	34.69%	22.8571	41.4318	44.83%
0.125000	19.4627	16.3786	15.85%	23.0244	42.4643	45.78%
0.062500	19.7296	18.2171	7.67%	23.1111	43.0183	46.28%
0.031250	19.8643	19.1138	3.78%	23.1553	43.3056	46.53%
0.015625	19.9321	19.5580	1.88%	23.1776	43.4519	46.66%
0.007813	19.9660	19.7793	0.94%	23.1888	43.5257	46.72%
0.003906	19.9830	19.8897	0.47%	23.1944	43.5628	46.76%

probabilities induced by a change of numeraire is such that the 'new' drifts are measure-dependent, but the variances are not, and can therefore be estimated (and will apply with equal validity) in *any* measure (in particular, the real-world one).

This observation is of fundamental importance, since otherwise any estimation or prediction about the volatility that will be experienced by an underlying asset during the course of the option (estimation or prediction necessarily made in the real world) would have no bearing on the option price.

With this last example we have virtually exhausted the possible insights that can be reaped from a discrete multi-period setting without entering into virtually any of the mathematical details. Hopefully, the treatment in Appendix A will now be clearer, if the reader wants immediately to proceed with a more quantitative approach at this stage. For those who would prefer to move on to modelling and pricing issues, sufficient instruments should have been furnished to undertake the task.

ENDNOTES

1. A more general treatment than the one presented here can actually show that a functional relationship between the drift μ_x of an asset x and the riskless rate r of the type of Equation (6.10) must always hold true for an arbitrage-free economy shocked by s Brownian motions, and where $s+1$ independent assets are tradable. See, e.g., Heath, Jarrow and Morton (1989) or Anderson *et al.* (1996). Nonetheless, other, albeit more general, assumptions still have to be made to obtain this result, about, for instance, the distributional properties of the underlying assets. For the limited purposes of the present discussion, Equation (6.10) is therefore taken as an adequate starting assumption.
2. Notice carefully that the claim must be *truly* European for this reduction to apply: it is not sufficient that the payoff should be at one and only one point in time, but it must also depend on the realisation of a state of the world at the same point in time. Therefore, for the second example treated in Chapter 5, the reduction in dimensionality of the integral could only be accomplished to dimension two.
3. Section 6.11 is somewhat more complex than the other parts of the chapter, and could be omitted on first reading.
4. The reasoning could be equally well carried out for prices, as is done here, or for their logarithms. The price formulation is used here for simplicity.
5. In special cases, no-arbitrage arguments can still be produced for some jump processes. See Merton (1990).