MATH 211 Fall 2022

Sample Final Exam – Answers

<u>Note</u>: The following is just an answer key for the computational problems, along with a few additional comments. For the actual exam, your answers will need to be clearly justified in order to earn full credit.

1. [12] Let $A = \begin{bmatrix} 2 & 5 & 6 \\ -2 & -7 & -9 \\ 1 & 3 & 4 \end{bmatrix}$.

(a) Show that A is invertible, and find its inverse A^{-1}.

(b) Find the unique matrix X satisfying the equation $AX = B$, where $B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 3 \end{bmatrix}$.

<u>Answer</u>: (a) We bring the partitioned matrix $[A|I_3]$ to RREF, obtaining a partitioned matrix $[B|C]$. If $B = I_3$, then A is invertible with inverse C. Doing the computation in this case, we obtain that A is invertible with inverse

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -6 \\ -1 & 1 & 4 \end{bmatrix}.$$

(b) $X = \begin{bmatrix} 3 & 10 \\ -2 & -17 \\ 1 & 11 \end{bmatrix}$.

2. [10] Let ℓ be the line in 3-space that passes through the points $A = (1, -1, 1)$ and $B = (2, 7, 6)$, and let $\mathcal{P}$ be the plane in 3-space given by the equation $x_1 + x_2 + x_3 = 8$.

(a) Find a set of parametric equations for ℓ.

(b) Find the unique point C at which ℓ intersects $\mathcal{P}$.

(c) Express $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$ as the sum of a vector parallel to $\mathcal{P}$ and a vector perpendicular to $\mathcal{P}$.

<u>Answer</u>: (a) The equations

$$\begin{array}{rcrcrl} x_1 & = & 1 & + & t & \\ x_2 & = & -1 & + & 8t & \quad (t \in \mathbb{R}) \\ x_3 & = & 1 & + & 5t & \end{array}$$

are a set of parametric equations for ℓ (*Note*: There are infinitely many ways to parametrize the points on ℓ, so this isn't the only correct answer).

(b) $C = (3/2, 3, 7/2)$.

(c) The desired expression is

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

(the first (resp. second) vector on the right is parallel (resp. perpendicular) to $\mathcal{P}$).

3. [10] Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & -1 & k^2-5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 12 \\ k \end{bmatrix}$, where k is a real number.

(a) Find a row echelon form of the matrix $[A|\vec{b}]$.
(b) Determine all values of k for which the vector $\vec{b}$ can be expressed as a linear combination of the columns of A.
(c) If $\vec{b}$ can be expressed as a linear combination of the columns of A in more than one way, what is the value of k? Explain your answer.

<u>Answer</u>: (a) One of the row echelon forms of $[A|\vec{b}]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & k^2-9 & k+3 \end{array}\right]$$

(*Note*: A non-zero matrix has infinitely many row echelon forms, so this isn't the only correct answer).

(b) All real numbers other than 3.

(c) $k = -3$.

4. [13] Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$.

(a) Evaluate the determinant of A (you may use whatever method(s) you wish to do this).
(b) Show that A^T is invertible.
(c) Compute the determinant of $B = -A^3(A^T)^{-1}$.

Answer: (a) $\det(A) = -2$.

(b) A^T is invertible if and only if $\det(A^T) \neq 0$. But $\det(A^T) = \det(A) = -2$.

(c) $\det(B) = 4$.

5. [12] Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{rcl} x_1 & = & x_3 + 3x_4 - x_2 \\ x_3 & = & 2x_1 + x_2 - 2x_4 \end{array} \right\}$.

(a) Show that S is a subspace of $\mathbb{R}^4$.
(b) What is meant by the **dimension** of a subspace of $\mathbb{R}^n$?
(c) Determine, with justification, the dimension of S.

Answer: (a) Show directly that S contains the zero vector and is closed under addition and scalar multiplication. Alternatively, explain that S is the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & -3 \\ 2 & 1 & -1 & -2 \end{bmatrix}$$

(as discussed in class, the null space of any matrix is a subspace of $\mathbb{R}^n$, where n is the number of columns in the matrix).

(b) The number of vectors in any basis of that subspace.

(c) $\dim(S) = 2$.

6. [13] Consider the matrices

$$A = \begin{bmatrix} -2 & -1 & -3 & -8 & -1 & -7 \\ 3 & 3 & 3 & 15 & 3 & 9 \\ -3 & -1 & -5 & k & -1 & k \\ -1 & 1 & -3 & -1 & 3 & -9 \\ 3 & 2 & 4 & 13 & 3 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where k is a real number. To answer the following questions, you may use that ***B* is the reduced row echelon form of *A***.

(a) Find a basis of the null space of A.
(b) Find a basis of the subspace of $\mathbb{R}^6$ spanned by the rows of A.

(c) Find a basis of the subspace of $\mathbb{R}^5$ spanned by the columns of A.
(d) Determine the nullity of A^T.
(e) Determine, with justification, the value of k.

Solution: (a) One basis of $\mathrm{Null}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(*Note*: $\mathrm{Null}(A)$ has infinitely many bases, so this isn't the only correct answer).

(b) We are being asked for a basis of $\mathrm{Row}(A)$. One basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(*Note*: $\mathrm{Row}(A)$ has infinitely many bases, so this isn't the only correct answer).

(c) We are being asked for a basis of $\mathrm{Col}(A)$. One basis is

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ -3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \\ 3 \end{bmatrix} \right\}$$

(*Note*: $\mathrm{Col}(A)$ has infinitely many bases, so this isn't the only correct answer).

(d) $\mathrm{nullity}(A^T) = 2$.

(e) $k = -11$.

7. [18] Let $A = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 2 & 3 \\ 1 & -4 & -5 \end{bmatrix}$.

(a) Determine the characteristic polynomial of A.

(b) The eigenvalues of A are -1 and -2. What are their algebraic multiplicities? Explain your answer.
(c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.
(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
(e) Without computing A^5, determine the vector $A^5\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right)$.

Answer: (a) $C_A(\lambda) = -(\lambda+1)(\lambda+2)^2$.

(b) -1 has algebraic multiplicity 1, and -2 has algebraic multiplicity 2.

(c) One basis of $\mathbb{R}^3$ consisting of eigenvectors of A is

$$\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

(*Note*: $\mathbb{R}^3$ has infinitely many such bases, so this isn't the only correct answer).

(d) One solution is

$$P = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(*Note*: There are infinitely many pairs (P, D) satisfying the desired conditions, so this isn't the only correct answer).

(e) $A^5\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -96 \\ 1 \\ -33 \end{bmatrix}$.

8. [12] Let

$$\vec{u} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix},$$

and let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping given by the formula $L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{v}$.

(a) Show that L is a linear mapping.
(b) Find $[L]$, the standard matrix of L.

(c) Find the vector $\vec{w} \in \mathbb{R}^3$ obtained by rotating $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ counter-clockwise about the positive x_3-axis through an angle of $\pi/2$ radians and then applying the mapping L.

Answer: (a) Show directly that L preserves addition and scalar multiplication. Alternatively, show that L is the matrix mapping given by the matrix $[L]$ below.

(b) $[L] = \begin{bmatrix} 2 & -3 & -1 \\ 4 & -6 & -2 \\ -10 & 15 & 5 \end{bmatrix}$.

(c) $\vec{w} = \begin{bmatrix} -9 \\ -18 \\ 45 \end{bmatrix}$.

9. [10] Let

$$\mathcal{C} = \left\{ \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(a) Show that $\mathcal{C}$ is an orthogonal set.
(b) Show that $\mathcal{C}$ is a basis of $\mathbb{R}^3$.
(c) Let $\vec{x} \in \mathbb{R}^3$. Given that

$$\vec{x} \cdot \vec{u} = -6, \quad \vec{x} \cdot \vec{v} = 4 \quad \text{and} \quad \vec{x} \cdot \vec{w} = 12,$$

determine $[\vec{x}]_{\mathcal{C}}$, the coordinate vector of $\vec{x}$ with respect to $\mathcal{C}$.
(d) Suppose now that $\mathcal{B}$ is another basis of $\mathbb{R}^3$, and that

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

If $[2\vec{v} - \vec{y}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, what is $[\vec{y}]_{\mathcal{B}}$? Justify your answer.

Answer: (a) Verify that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$.

(b) $\mathcal{C}$ is an orthogonal set of non-zero vectors, and is hence linearly independent. But any set of 3 linearly independent vectors in $\mathbb{R}^3$ is a basis of $\mathbb{R}^3$. (

(c) $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$.

(d) $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$.

10. [10] Let A be a **diagonalizable** 5×5 matrix with characteristic polynomial

$$C_A(\lambda) = -\lambda^2(\lambda - 1)(\lambda^2 + 2\lambda + 1).$$

(a) What do we mean by the **geometric multiplicity** of an eigenvalue of A?
(b) Show that -1 is an eigenvalue of A, and determine its geometric multiplicity.
(c) Determine, with justification, the rank of the matrix $B = A - I_5$.
(d) Show that $A^5 = A$.

<u>Answer</u>: (a) If λ is an eigenvalue of A, then its geometric multiplicity is the dimension of the eigenspace $E_\lambda(A) = \text{Null}(A - \lambda I_5)$.

(b) $C_A(-1) = 0$, and so -1 is an eigenvalue of A. Its geometric multiplicity is 2.

(c) $\text{rank}(B) = 4$.

(d) Use diagonalizability: There exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. The diagonal entries of D are the eigenvalues of A, which are 0, -1 and 1. Then $D^5 = D$ ($x^5 = x$ for all $x \in \{0, -1, 1\}$), and so

$$A^5 = (PDP^{-1})^5 = PD^5P^{-1} = PDP^{-1} = A.$$