Written Assignment 4 – Solutions

1. [10] Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 4 & 1 & 2 & -1 \\ -1 & 3 & 2 & 0 \\ 3 & 4 & 2 & 5 \end{bmatrix}$$

a) Find, with justification, a basis for the null space of A.

b) Find, with justification, a basis for the column space of A.

c) Is A an invertible matrix? Justify your answer.

d) Let

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \qquad \vec{v}_4 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 5 \end{bmatrix}.$$

Find, with justification, a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that forms a basis of the subspace S of $\mathbb{R}^4$ spanned by these four vectors. Write each of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ as a linear combination of these basis vectors.

Solution:

(a) The null space of A contains all solutions to $A\vec{x} = \vec{0}$. Solving this using row reduction:

$$\left[\begin{array}{cccc|c} 0 & 0 & -1 & 3 & 0 \\ 4 & 1 & 2 & -1 & 0 \\ -1 & 3 & 2 & 0 & 0 \\ 3 & 4 & 2 & 5 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} -1 & 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 4 & 1 & 2 & -1 & 0 \\ 3 & 4 & 2 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \times (-1)} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 4 & 1 & 2 & -1 & 0 \\ 3 & 4 & 2 & 5 & 0 \end{array}\right] \xrightarrow[R_4 - 3R_1]{R_3 - 4R_1} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 13 & 10 & -1 & 0 \\ 0 & 13 & 8 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & 0 \\ 0 & 13 & 10 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 13 & 8 & 5 & 0 \end{array}\right] \xrightarrow{R_3 \times (1/13)} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & 0 \\ 0 & 1 & 10/13 & -1/13 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 13 & 8 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[R_4-13R_2]{R_1+3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 4/13 & -3/13 & 0 \\ 0 & 1 & 10/13 & -1/13 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -2 & 6 & 0 \end{array}\right] \xrightarrow{R_3\times(-1)} \left[\begin{array}{cccc|c} 1 & 0 & 4/13 & -3/13 & 0 \\ 0 & 1 & 10/13 & -1/13 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & -2 & 6 & 0 \end{array}\right]$$

$$\xrightarrow[R_4+2R_3]{\substack{R_1 - 4/13R_3 \\ R_2 - 10/13R_3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9/13 & 0 \\ 0 & 1 & 0 & 29/13 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The only free variable is $x_4 = t$ and we have $x_1 = \frac{-9}{13}t$, $x_2 = \frac{-29}{13}t$ and $x_3 = 3t$.

The general solution is $t \begin{bmatrix} -9/13 \\ -29/13 \\ 3 \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$ and thus a basis of $\mathrm{Null}(A)$ is

$$\left\{ \begin{bmatrix} -9/13 \\ -29/13 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(b) We can see from part (a) that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 9/13 \\ 0 & 1 & 0 & 29/13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second and third columns of $\mathrm{RREF}(A)$ form a basis of the column space of $\mathrm{RREF}(A)$, and so the first, second and third columns of A form a basis of the column space of A. In particular, a basis of $\mathrm{Col}(A)$ is

$$\left\{ \begin{bmatrix} 0 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

(c) From part (a), we know $\mathrm{Null}(A) \neq \{\vec{0}\}$. This implies that A is not invertible.

Alternatively, one can observe from $\mathrm{RREF}(A)$ that $\mathrm{rank}(A) = 3$. In particular, this is not equal to the number of rows of A, and so A is not invertible.

Alternatively to this, one could observe that since the basis for $\mathrm{Col}(A)$ contains only three elements, $\mathrm{Col}(A) \neq \mathbb{R}^4$ and so A is not invertible.

(d) The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ are the columns of A. Thus by definition of column space, $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \mathrm{Col}(A)$. By part (b), we deduce that a basis for $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

$$\left\{ \begin{bmatrix} 0 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in the basis, these are immediately written as linear combinations of the basis vectors.

We know that linear dependencies between columns of $\mathrm{RREF}(A)$ correspond to linear dependencies between columns of A. If we label the columns of $\mathrm{RREF}(A)$ by $\vec{u}_1, \vec{u}_2, \vec{u}_3$ and $\vec{u}_4$ we immediately see that $\vec{u}_4 = \frac{9}{13}\vec{u}_1 + \frac{29}{13}\vec{u}_2 - 3\vec{u}_3$. Thus the same relationship holds between $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$:

$$\vec{v}_4 = \frac{9}{13}\vec{v}_1 + \frac{29}{13}\vec{v}_2 - 3\vec{v}_3.$$

2. [10] Let A be an $n \times n$ matrix.

a) Suppose that $B = A^3$ is invertible. Prove that A must also be invertible. *(Hint: Find a matrix X such that $AX = I_n$.)*

b) Suppose that A is invertible and C is a non-zero $n \times n$ matrix that satisfies $AC = CA$. Must C be invertible? Justify your answer.

Solution:

(a) Since B is invertible, B^{-1} exists. We have that

$$\begin{aligned} BB^{-1} &= I_3 \\ A^3B^{-1} &= I_3 \\ A\left(A^2B^{-1}\right) &= I_3 \end{aligned}$$

Thus the matrix A^2B^{-1} is the inverse of A, and A is invertible.

Alternatively, one can note that since A^3 is invertible, $\det(A^3) \neq 0$. However we know that $\det(A^3) = (\det(A))^3$, and so $\det(A) \neq 0$. This implies that A is invertible.

(b) It is not the case that C must be invertible. For example, take $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is invertible, and take $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Since $A = I_2$, we have $AC = C = CA$, but $\operatorname{rank}(C) = 1$ and so C is not invertible.

3. [10] Consider the matrix

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

a) Prove that B is invertible and find its inverse.

b) Express B as a product of elementary matrices.

Solution:

(a) We must put the following matrix into reduced row echelon form.

$$[B \mid I_3] = \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -3 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \times 1/3} \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1/3 & 1/3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 + R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 2/3 & 1/3 & 0 \\ 0 & 1 & -1 & -1/3 & 1/3 & 0 \\ 0 & 0 & 3 & 1/3 & -1/3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \times 1/3} \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 2/3 & 1/3 & 0 \\ 0 & 1 & -1 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 1/9 & -1/9 & 1/3 \end{array}\right] \xrightarrow[R_2 + R_3]{R_1 - 2R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 4/9 & 5/9 & -2/3 \\ 0 & 1 & 0 & -2/9 & 2/9 & 1/3 \\ 0 & 0 & 1 & 1/9 & -1/9 & 1/3 \end{array}\right]$$

As $\mathrm{RREF}(B) = I_3$, we know that B is invertible. Moreover,

$$B^{-1} = \begin{bmatrix} 4/9 & 5/9 & -2/3 \\ -2/9 & 2/9 & 1/3 \\ 1/9 & -1/9 & 1/3 \end{bmatrix}$$

You can check that $BB^{-1} = B^{-1}B = I_3$.

(b) We must write down the elementary matrices corresponding to the elementary row operations that we did to put B into its reduced row echelon form.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 - R_1) \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 + R_1)$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 \times 1/3) \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 \times 3)$$

$$E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_1 + R_2) \qquad E_3^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_1 - R_2)$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (R_3 - R_2) \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (R_3 + R_2)$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad (R_3 \times 1/3) \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (R_3 \times 3)$$

$$E_6 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_1 - 2R_3) \qquad E_6^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_1 + 2R_3)$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 + R_3) \qquad E_7^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 - R_3)$$

Then $E_7E_6E_5E_4E_3E_2E_1B = I_3$ and so $B = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$. (Note that the order matters!)

This is not the unique way to write B as a product of elementary matrices.

4. [10] Consider the following two bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}.$$

a) Find the change of coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ that transforms $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

b) Let

$$\vec{x} = 3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}.$$

Find $[\vec{x}]_{\mathcal{C}}$ and $[\vec{x}]_{\mathcal{B}}$, the expressions of $\vec{x}$ in terms of $\mathcal{C}$-coordinates and $\mathcal{B}$-coordinates respectively.

Solution:

(a) We must put the following matrix into reduced row echelon form.

$$[B \mid C] = \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & -3 & 5 \\ 0 & 1 & 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & -3 & 5 \\ 0 & 1 & 2 & 4 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2\times 1/3} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 5/3 \\ 0 & 1 & 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 5/3 \\ 0 & 1 & -1 & 0 & -1 & 5/3 \\ 0 & 0 & 3 & 4 & 2 & -5/3 \end{array}\right]$$

$$\xrightarrow{R_3\times 1/3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 5/3 \\ 0 & 1 & -1 & 0 & -1 & 5/3 \\ 0 & 0 & 3 & 4/3 & 2/3 & -5/9 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/3 & -7/3 & 25/9 \\ 0 & 1 & 0 & 4/3 & -1/3 & 10/9 \\ 0 & 0 & 3 & 4/3 & 2/3 & -5/9 \end{array}\right]$$

Alternatively, you may spot that B is the matrix from question 3 and we already calculated B^{-1} in that question. We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = B^{-1}C = \begin{bmatrix} 4/9 & 5/9 & -2/3 \\ -2/9 & 2/9 & 1/3 \\ 1/9 & -1/9 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 5 \\ 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -5/3 & -7/3 & 25/9 \\ 4/3 & -1/3 & 10/9 \\ 4/3 & 2/3 & -5/9 \end{bmatrix}$$

(b) We can see from the definition of $\mathcal{C}$-coordinates that the expression of $\vec{x}$ in terms of $\mathcal{C}$-coordinates is

$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

To calculate $\vec{x}$ in terms of $\mathcal{B}$-coordinates, we use our change of coordinates matrix.

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} -{}^{5}\!/_{3} & -{}^{7}\!/_{3} & {}^{25}\!/_{9} \\ {}^{4}\!/_{3} & -{}^{1}\!/_{3} & {}^{10}\!/_{9} \\ {}^{4}\!/_{3} & {}^{2}\!/_{3} & -{}^{5}\!/_{9} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -{}^{70}\!/_{9} \\ {}^{26}\!/_{9} \\ {}^{41}\!/_{9} \end{bmatrix}$$

5. [10] Consider the matrix

$$M = \begin{bmatrix} 7 & -3 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -2 & 0 & 2 & 0 & 3 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 2 \end{bmatrix}$$

Use iterated cofactor expansion to evaluate the determinant of M. Make it clear which row or column you are expanding along at each step.

Solution:

Pick a row or column with at most one non-zero entry. We will first expand along the second column, not forgetting the sign.

$$\det(M) = \begin{vmatrix} 7 & -3 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -2 & 0 & 2 & 0 & 3 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 2 \end{vmatrix} = -(-3) \begin{vmatrix} 0 & 1 & 0 & -1 \\ -2 & 2 & 0 & 3 \\ 5 & 0 & 0 & 0 \\ 0 & -1 & -3 & 2 \end{vmatrix}$$

Then we will pick the third row of the above 4×4 matrix, followed by the second column of the resulting 3×3 matrix.

$$= 3 \times 5 \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 3 \times 5 \times \big(-(-3)\big) \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 3 \times 5 \times 3\big((1)(3) - (-1)(2)\big) = 3 \times 5 \times 3 \times 5 = 225$$