Written Assignment 2 – Solutions

1. [10] Let $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

a) Find the unique vector $\vec{z}$ with length $\|\vec{x} + 3\vec{y}\|$ that is in the same direction as $-2\vec{x} + \vec{y}$.

b) Find, without using the cross product, a unit vector that is orthogonal to both $\vec{x}$ and $\vec{y}$. How many possible such vectors are there?

Solution:

(a) The easiest approach is to write down $-2\vec{x} + \vec{y}$ and then re-scale it so it has the appropriate length.

$$-2\vec{x} + \vec{y} = -2 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -9 \end{bmatrix}$$

If we divide this by its length (normalise it) and then multiply by $\|\vec{x} + 3\vec{y}\|$, we get a vector of length $\|\vec{x} + 3\vec{y}\|$. We have $\|-2\vec{x} + \vec{y}\| = \sqrt{4^2 + 4^2 + 9^2} = \sqrt{113}$ and

$$\vec{x} + 3\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

so $\|\vec{x} + 3\vec{y}\| = \sqrt{2^2 + 5^2 + 8^2} = \sqrt{93}$.

The answer is

$$\frac{\|\vec{x} + 3\vec{y}\|}{\|-2\vec{x} + \vec{y}\|}(-2\vec{x} + \vec{y}) = \sqrt{\frac{93}{113}} \begin{bmatrix} 4 \\ 0 \\ 11 \end{bmatrix}$$

(b) A vector $\vec{z}$ is orthogonal to $\vec{x}$ and $\vec{y}$ if and only if $\vec{x} \cdot \vec{z} = 0$ and $\vec{y} \cdot \vec{z} = 0$. So we require

$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 2z_1 - z_2 + 5z_3 = 0 \qquad \text{and} \qquad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 2z_2 + z_3 = 0$$

We must find a solution to

$$\begin{aligned} 2z_1 - z_2 + 5z_3 &= 0 \\ 2z_2 + z_3 &= 0 \end{aligned}$$

Setting $z_3 = t$, we have $z_2 = -\frac{1}{2}t$ and $z_1 = z_2 - 5t = -\frac{11}{2}t$.

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = t \begin{bmatrix} -11/2 \\ -1/2 \\ 1 \end{bmatrix}$$

We need $\vec{z}$ to be a unit vector, so we need

$$\begin{aligned}
& \left\| t \begin{bmatrix} -11/2 \\ -1/2 \\ 1 \end{bmatrix} \right\| = 1 \\
\Leftrightarrow \quad & \frac{|t|}{2} \left\| \begin{bmatrix} -11 \\ -1 \\ 2 \end{bmatrix} \right\| = 1 \\
\Leftrightarrow \quad & \frac{|t|}{2} \sqrt{11^2 + 1^2 + 2^2} = 1 \\
\Leftrightarrow \quad & |t| = \frac{2}{\sqrt{126}}
\end{aligned}$$

So there are two solutions:

$$\pm \frac{1}{\sqrt{126}} \begin{bmatrix} -11 \\ -1 \\ 2 \end{bmatrix}$$

Intuitively, we can see that there should be two solutions. Consider the line orthogonal to $\vec{x}$ and $\vec{y}$: there is one unit vector in one direction along this line and another in the opposite direction.

2. [6] Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be non-zero vectors in $\mathbb{R}^n$ where each pair of distinct vectors $\vec{v}_i, \vec{v}_j$ are orthogonal. Prove that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent. (*Hint:* if $\vec{v}$ is a linear combination of the vectors what is $\vec{v} \cdot \vec{v}_i$?)

Solution:

To show that the set of vectors are linearly independent we must show that the equation

$$c_1\vec{v_1} + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0} \tag{1}$$

has only the trivial solution $c_1 = c_2 = \ldots = c_k = 0$.

Let $1 \le i \le k$ and consider the dot product of both sides of the above equation with $\vec{v_i}$. We get

$$(c_1\vec{v_1} + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k) \cdot \vec{v_i} = \vec{0} \cdot \vec{v_i}.$$

Expanding out we see

$$c_1\vec{v_1} \cdot \vec{v_i} + c_2\vec{v}_2 \cdot \vec{v_i} + \ldots + c_k\vec{v}_k \cdot \vec{v_i} = 0.$$

Now, if $j \neq i$ then $\vec{v_j} \cdot \vec{v_i} = 0$, by orthogonality. Thus all but the ith term become zero and the equation is

$$c_i\vec{v_i} \cdot \vec{v_i} = 0.$$

Since all of the vectors are non-zero, we have $\vec{v_i} \cdot \vec{v_i} = \|\vec{v_i}\|^2 \neq 0$, and so we can divide both sides by $\|\vec{v_i}\|^2$ to get $c_i = 0$.

This holds for all i. Thus the only solution to equation (1) is the trivial one $c_1 = c_2 = \ldots = c_k = 0$ and the vectors are linearly independent.

3. [10] Let P be a plane in $\mathbb{R}^3$ that passes through the points $A = (1, 2, 1)$, $B = (2, 4, 3)$ and $C = (1, 2, -1)$.

a) Find a vector equation for P.

b) Prove that P is a subspace of $\mathbb{R}^3$ and find a basis for P.

c) Find a scalar equation for P.

Solution:

(a) First we find two vectors parallel to the plane:

$$\overrightarrow{AB} = \begin{bmatrix} 2-1 \\ 4-2 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \overrightarrow{AC} = \begin{bmatrix} 1-1 \\ 2-2 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Any point on the plane is given by adding scalar multiples of these two vectors to a point on the plane. So a vector equation for the plane is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}$$

Note that this is not unique.

(b) We first check that $\vec{0}$ is in the plane P. In particular, we must find a solution to

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

It is straightforward to see that this has a solution when $t_1 = -1$ and $t_2 = -1/2$, that is,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

Since $\vec{0}$ lies in P, we can rewrite the vector equation for P as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}.$$

We immediately see that any vector in P can be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$. In other words, P is the span of these two vectors. We know that the span of any set of vectors is a subspace, hence P is a subspace.

The vectors $\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\begin{bmatrix}0\\0\\-2\end{bmatrix}$ are a spanning set for P and it is clear that they are linearly independent as neither is a scalar multiple of the other. Thus they are a basis for P.

Note that this basis is not unique.

(c) We need to find a normal vector $\vec{n}$ to the plane P. We have

$$\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix} = n_1 + 2n_2 + 2n_3 = 0 \qquad \text{and} \qquad \begin{bmatrix}0\\0\\-2\end{bmatrix} \cdot \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix} = -2n_3 = 0$$

Solving this, we get $n_3 = 0$ and $n_1 = -2n_2$, so one choice of $\vec{n}$ is

$$\vec{n} = \begin{bmatrix}-2\\1\\0\end{bmatrix}$$

(any re-scaling of this is also a valid choice for $\vec{n}$).

The scalar equation of a plane is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$, where $\vec{p}$ is a vector that lies on the plane. Taking $\vec{p} = \vec{0}$ to be the vector lying on the plane, we get the scalar equation

$$-2x_1 + x_2 = 0.$$

Note that this is unique up to re-scaling.

4. [14] Let ℓ be a line in $\mathbb{R}^3$ with vector equation

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

a) Find the distance from the point $P = (-s, s+3, s+2)$ to ℓ, in terms of the parameter s.

b) Let ℓ' be the line with vector equation

$$\vec{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

Find the minimum distance between the lines ℓ' and ℓ.

c) Which point on ℓ' is closest to ℓ?

d) Which point on ℓ is closest to ℓ'?

Solution:

(a) Let $B = (-1, 2, 3)$ be a point on the line ℓ and let $\vec{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ be the direction vector for ℓ.

The vector from the point P to the line ℓ orthogonal to ℓ is given by

$$perp_{\vec{d}}(\overrightarrow{PB}) = \overrightarrow{PB} - proj_{\vec{d}}(\overrightarrow{PB})$$

We have

$$\overrightarrow{PB} = \begin{bmatrix} (-1) - (-s) \\ 2 - (s+3) \\ 3 - (s+2) \end{bmatrix} = \begin{bmatrix} s - 1 \\ -s - 1 \\ -s + 1 \end{bmatrix}$$

and

$$\begin{aligned} proj_{\vec{d}}(\overrightarrow{PB}) &= \frac{\overrightarrow{PB} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \\ &= \frac{0(s-1) + 1(-s-1) + 1(-s+1)}{0^2 + 1^2 + 1^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{-2s}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ -s \end{bmatrix} \end{aligned}$$

So

$$perp_{\vec{d}}(\overrightarrow{PB}) = \begin{bmatrix} s-1 \\ -s-1 \\ -s+1 \end{bmatrix} - \begin{bmatrix} 0 \\ -s \\ -s \end{bmatrix} = \begin{bmatrix} s-1 \\ -1 \\ 1 \end{bmatrix}$$

The distance from P to l is the length of this vector:

$$\sqrt{(s-1)^2 + (-1)^2 + 1^2} = \sqrt{s^2 - 2s + 3}.$$

(b) Observe that a point on the line ℓ' has the form $(-s, s+3, s+2)$ for some $s \in \mathbb{R}$. Thus we can use our answer to part (a). The minimum distance from ℓ' to ℓ is the minimum value of $\sqrt{s^2 - 2s + 3}$ over all $s \in \mathbb{R}$. You can calculate this minimum either by completing the square or by differentiation.

Completing the square: $\sqrt{s^2 - 2s + 3} = \sqrt{(s-1)^2 + 2}$ and since $(s-1)^2$ is a square, it is always ≥ 0. We see that $\sqrt{(s-1)^2 + 2} \geq \sqrt{2}$ with minimum attained when $s = 1$.

Differentiation: The quadratic function $f(s) = s^2 - 2s + 3$ attains its minimum when $\frac{df}{ds} = 2s - 2 = 0$. Thus the minimum is attained when $s = 1$ and we have $\sqrt{s^2 - 2s + 3} \geq \sqrt{2}$.

(c) We have that the minimum distance from a point of the form $(-s, s+3, s+2)$ on the line ℓ' to the line l was attained when $s = 1$. Thus the point on l closest to l' is $(-1, 1+3, 1+2) = (-1, 4, 3)$.

(d) Using the notation and calculations from part (a), if Q is the closest point on l to the point $P = (-s, s+3, s+2)$, we have

$$\overrightarrow{OQ} = \overrightarrow{OP} + perp_{\vec{d}}(\overrightarrow{PB}) = \begin{bmatrix} -s \\ s+3 \\ s+2 \end{bmatrix} + \begin{bmatrix} s-1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ s+2 \\ s+3 \end{bmatrix}.$$

We know that at the point P with minimum distance to ℓ the value of s is 1. In particular, the closest point on ℓ to the line ℓ' is $(-1, 3, 4)$.

5. [10] Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ x_1 & + & & & -x_3 & + & 2x_4 & = & 1 \\ -2x_1 & + & x_2 & + & 4x_3 & & & = & -2 \end{array}$$

a) Write down the augmented matrix of the system and find a row echelon form. Show your working, clearly indicate the row operations you are using at each step.

b) Find the solution set of the system of linear equations, expressing your answer in vector form.

c) What is the rank of the augmented matrix for this system of linear equations?

Solution:
(a) The augmented matrix is

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 & 1 \\ -2 & 1 & 4 & 0 & -2 \end{array}\right].$$

We use Gaussian elimination to put this into a row echelon form.

$$\left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 & 1 \\ -2 & 1 & 4 & 0 & -2 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-R_1} \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -2 & -4 \\ 0 & 5 & 10 & 8 & 8 \end{array}\right]$$

$$\xrightarrow{R_2\times\frac{-1}{2}} \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 5 & 10 & 8 & 8 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & -2 \end{array}\right].$$

This is a row echelon form.
We can go further and put it into reduced row echelon form if we like.

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & -2 \end{array}\right] \xrightarrow{R_3\times\frac{1}{3}} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2/3 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-2R_3} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 7/3 \\ 0 & 1 & 2 & 0 & 8/3 \\ 0 & 0 & 0 & 1 & -2/3 \end{array}\right]$$

(b) By part (a), the solution set to the system of the linear equations is the same as the solution set to

$$\begin{array}{rcrcrcr} x_1 & & & + & -x_3 & & = & 7/3 \\ & & x_2 & + & 2x_3 & & = & 8/3 \\ & & & & & x_4 & = & -2/3 \end{array}$$

The leading variables are x_1, x_2 and x_4 and the free variable is x_3. Set $x_3 = t$ and use back-substitution.

$$
\begin{aligned}
x_4 &= -2/3 \\
x_3 &= t \\
x_2 &= 8/3 - 2x_3 = 8/3 - 2t \\
x_1 &= 7/3 + x_3 = 7/3 + t
\end{aligned}
$$

The general solution is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7/3 + t \\ 8/3 - 2t \\ t \\ -2/3 \end{bmatrix}
$$

and you can check that this satisfies the original equations.

(c) The rank of the augmented matrix is the number of non-zero rows of a row echelon form of it. We can see that the rank is 3.