Written Assignment 5 – Solutions

1. [12] Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 3 & -5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) Determine the eigenvalues of A, showing all your work.
(b) Find a basis for each eigenspace of A.
(c) Is A diagonalizable? Explain your answer.
(d) Compute (with justification) the vector

$$A^k \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where k is a fixed (but unknown) positive integer.

Solution: (a) The characteristic polynomial of A is

$$\begin{aligned} C_A(\lambda) = |A - \lambda I_4| &= \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 3 & -5 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \qquad \text{(expand along column 2)} \\ &= (1-\lambda)\big(2-\lambda\big)^3 \end{aligned}$$

(to compute the 3×3 determinant, we have used that the determinant of a triangular matrix is the product of the entries on its main diagonal – see Theorem 5.1.3 in the textbook). The eigenvalues of A are therefore 1 and 2 (the solutions of the characteristic equation $C_A(\lambda) = 0$).

(b) As in lectures, we write $E_\lambda(A)$ for the eigenspace of A corresponding to the eigenvalue λ. By definition, $E_\lambda(A)$ is the null space of the matrix $A - \lambda I_4$. In particular, $E_1(A)$ is the null space of $A - I_4$, i.e., the solution space of the linear system with augmented matrix

$$[A - I_4|\vec{0}] = \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Note that we can solve this system without doing any elimination: Rows 1 and 4 of the matrix tell us that we must have $x_1 = x_4 = 0$, and row 3 then tells us that $x_3 = -x_4 = 0$. Thus, any solution of the system must have the form

$$t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

for some $t \in \mathbb{R}$. On the other hand, since $E_1(A)$ is not the zero space, we know that the system has non-zero solutions, and so every vector of the above form is in fact a solution of the system. In other words,

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of the eigenspace $E_1(A)$. The other eigenspace, $E_2(A)$, is the null space of the matrix $A - 2I_4$. To find a basis for it, we use elimination to solve the homogeneous linear system $(A - 2I_4)\vec{x} = \vec{0}$:

$$[A - 2I_4|\vec{0}] = \left[\begin{array}{rrrr|r} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{row exchanges}} \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + 5R_2} \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Noting that the general solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (t_1, t_2 \in \mathbb{R}),$$

we obtain that

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of the eigenspace $E_2(A)$.

<u>*Remark*</u>: As usual, these are not the only correct answers, since a non-zero subspace of $\mathbb{R}^4$ has infinitely many bases.

(c) Recall that for any $n \times n$ matrix B, the following are equivalent (see Theorems 6.2.2 and 6.2.3 in the textbook):

- B is diagonalizable;
- The sum of the geometric multiplicities of the eigenvalues of B is n;
- The geometric multiplicity of each eigenvalue of B is equal to its algebraic multiplicity.

In light of these equivalences, the solutions to (a) and (b) show that A is *n*ot diagonalizable. We can argue here in two ways:

- By the solution to (b), the sum of the geometric multiplicities of the eigenvalues of A is 3 ($\mathcal{B}_1 \cup \mathcal{B}_2$ consists of exactly 3 vectors). Since A is a 4×4 matrix, the equivalence of the first two statements above then tells us that A is not diagonalizable.
- By the solution to (b), the geometric multiplicity of the eigenvalue 2 is 2 ($\mathcal{B}_2$ consists of 2 vectors). By the solution to (a), however, the algebraic multiplicity of this eigenvalue is 3 ($C_A(\lambda)$ is divisible by $(\lambda-2)^3$). By the equivalence of the first and third statements above, this again shows that A is not diagonalizable.

(d) Observe first that

$$\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

By the solution to (b), the three vectors on the right-hand side of the equation are eigenvectors of A with eigenvalues 1, 2 and 2, respectively. We therefore have that

$$A^k \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = A^k \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + A^k \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + A^k \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1^k \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2^k \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2^k \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2^{k+1} \\ 1+2^k \\ 2^k \\ 0 \end{bmatrix}.$$

2. [18] Let $A = \begin{bmatrix} 0 & -3/4 & -1/4 \\ -1/2 & 1/4 & 1/4 \\ 1/2 & 3/4 & 3/4 \end{bmatrix}$.

(a) Determine the eigenvalues of A, showing all your work.
(b) Find a basis for each eigenspace of A.
(c) Explain why A is diagonalizable, and then find an invertible 3×3 matrix P and a diagonal 3×3 matrix D such that $P^{-1}AP = D$.
(d) Show that the matrix A^{2022} is approximately equal to $(1/2) \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

<u>Solution</u>: (a) The characteristic polynomial of A is

$$\begin{aligned}
C_A(\lambda) = |A - \lambda I_3| &= \begin{vmatrix} -\lambda & -3/4 & -1/4 \\ -1/2 & 1/4-\lambda & 1/4 \\ 1/2 & 3/4 & 3/4-\lambda \end{vmatrix} \\
&= \begin{vmatrix} -\lambda & -3/4 & -1/4 \\ 0 & 1-\lambda & 1-\lambda \\ 1/2 & 3/4 & 3/4-\lambda \end{vmatrix} \qquad (R_2 + R_3) \\
&= \begin{vmatrix} -\lambda & -3/4 & 1/2 \\ 0 & 1-\lambda & 0 \\ 1/2 & 3/4 & -\lambda \end{vmatrix} \qquad (C_3 - C_2) \\
&= (1-\lambda) \begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} \qquad \text{(expand along row 2)} \\
&= (1-\lambda)(\lambda^2 - 1/4) \\
&= (1-\lambda)(\lambda + 1/2)(\lambda - 1/2).
\end{aligned}$$

Thus, the eigenvalues of A are $-1/2$, $1/2$ and 1 (the solutions of the characteristic equation $C_A(\lambda) = 0$).

(b) Again, we write $E_\lambda(A)$ for the eigenspace of A corresponding to the eigenvalue λ, i.e., the null space of the matrix $A - \lambda I_3$. To find a basis of $E_{-1/2}(A)$, we solve the system of linear equations $(A + (1/2)I_3)\vec{x} = \vec{0}$:

$$[A+(1/2)I_3|\vec{0}] = \left[\begin{array}{rrr|r} 1/2 & -3/4 & -1/4 & 0 \\ -1/2 & 3/4 & 1/4 & 0 \\ 1/2 & 3/4 & 5/4 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \\ R_2 + R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{rrr|r} 1/2 & -3/4 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3/2 & 3/2 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} R_1 \times 2 \\ R_2 \leftrightarrow R_3 \\ \end{array} \left[\begin{array}{rrr|r} 1 & -3/2 & -1/2 & 0 \\ 0 & 3/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 3/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow R_2 \times (2/3) \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Noting that the general solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}),$$

we obtain that

$$\mathcal{B}_{-1/2} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis of $E_{-1/2}(A)$. Next, we find a basis of $E_{1/2}(A)$ by solving the linear system $(A - (1/2)I_3)\vec{x} = \vec{0}$:

$$[A-(1/2)I_3|\vec{0}] = \left[\begin{array}{rrr|r} -1/2 & -3/4 & -1/4 & 0 \\ -1/2 & -1/4 & 1/4 & 0 \\ 1/2 & 3/4 & 1/4 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \left[\begin{array}{rrr|r} -1/2 & -3/4 & -1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} R_1 \times (-2) \\ R_2 \times 2 \\ \end{array} \left[\begin{array}{rrr|r} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} R_1 - (3/2)R_2 \\ \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Noting that the general solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}),$$

we obtain that

$$\mathcal{B}_{1/2} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis of $E_{1/2}(A)$. Finally, we find a basis of $E_1(A)$ by solving the linear system $(A - I_3)\vec{x} = \vec{0}$:

$$[A - (1/2)I_3|\vec{0}] = \left[\begin{array}{rrr|r} -1 & -3/4 & -1/4 & 0 \\ -1/2 & -3/4 & 1/4 & 0 \\ 1/2 & 3/4 & -1/4 & 0 \end{array}\right] \longrightarrow \begin{array}{c} R_1 \times (-1) \\ R_2 \times 2 \\ R_3 \times 2 \end{array} \left[\begin{array}{rrr|r} 1 & 3/4 & 1/4 & 0 \\ -1 & -3/2 & 1/2 & 0 \\ 1 & 3/2 & -1/2 & \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \\ R_2 + R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{rrr|r} 1 & 3/4 & 1/4 & 0 \\ 0 & -3/4 & 3/4 & 0 \\ 0 & 3/4 & -3/4 & 0 \end{array}\right] \longrightarrow \begin{array}{c} R_1 + R_2 \\ \\ R_3 + R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & -3/4 & 3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \quad R_2 \times (-4/3) \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Noting that the general solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}),$$

we obtain that

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis of $E_1(A)$.

(c) The solutions to (a) and (b) above show that A is diagonalizable. We can argue here in any of the following ways (see Theorem 6.2.4 in the textbook for the first one, and the solution to Q1 for the others):

- A is diagonalizable since it's a 3×3 matrix with 3 distinct eigenvalues;
- A is diagonalizable since it's a 3×3 matrix and the sum of the geometric multiplicities of it's eigenvalues is 3 ($\mathcal{B}_{-1/2} \cup \mathcal{B}_{1/2} \cup \mathcal{B}_1$ consists of exactly 3 vectors).
- A is diagonalizable, since the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity (all multiplicities are 1 here).

Moreover, by the solution to (b), we have $P^{-1}AP = D$, where

$$P = \begin{bmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark: These are not the only valid answers. Each eigenspace of A has infinitely many bases, so there are infinitely many (ordered) bases of $\mathbb{R}^3$ consisting of eigenvectors of A, and hence infinitely many valid choices of P. The matrix D is also not unique, since two different choices of P may result in two different arrangements of the eigenvalues along the main diagonal of $P^{-1}AP$.

(d) Let

$$B = (1/2)\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We want to show that $A^{2022} \approx B$. Let P and D be the matrices from the solution to (c) satisfying the equation $P^{-1}AP = D$. Then $P^{-1}A^{2022}P = D^{2022}$, and so $A^{2022}P = PD^{2022}$. Now

$$BP = (1/2)\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and so

$$\begin{aligned}
A^{2022}P = PD^{2022} &= P\begin{bmatrix} (-1/2)^{2022} & 0 & 0 \\ 0 & (1/2)^{2022} & 0 \\ 0 & 0 & 1^{2022} \end{bmatrix} \\
&= P\begin{bmatrix} (1/2)^{2022} & 0 & 0 \\ 0 & (1/2)^{2022} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= P\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1/2)^{2022}P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (1/2)^{2022}\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\
&= BP + (1/2)^{2022}\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Multiplying through on the right by P^{-1}, we obtain that

$$A^{2022} = B + (1/2)^{2022} \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} P^{-1}.$$

Thus, to show that $A^{2022} \approx B$, we have to show that the matrix

$$C = (1/2)^{2022} \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} P^{-1}$$

is approximately zero. This may be done by explicit computation of P^{-1}. Alternatively, we can recall that the entries of P^{-1} are of the $\frac{C}{\det A}$, where C is a cofactor of P. Now the determinant of P is 4, and each of its cofactors are integers in the interval $[-2, 2]$. The entries of P^{-1} therefore lie in the interval $[-1/2, 1/2]$, from which it follows that the entries of

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} P^{-1}$$

lie in the interval $[-1, 1]$. But the entries of C then lie in the interval $[-(1/2)^{2022}, (1/2)^{2022}]$, so C is approximately zero.

Alternative Solutions: More directly, we can proceed by showing that

$$P^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix},$$

and then explicitly calculating A^{2022} as the product $PD^{2022}P^{-1}$. Another approach is as follows: Let

$$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

By the solution to (b), $\vec{u}$, $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues $-1/2$, $1/2$ and 1, respectively. Since $(1/2)^{2022}$ is approximately 0, the vectors $A^{2022}\vec{u} = (-1/2)^{2022}\vec{u}$ and $A^{2022}\vec{v} = (1/2)^{2022}\vec{v}$ are approximately equal to the zero vector. At the same time, $A^{2022}\vec{w} = 1^{2022}\vec{w} = \vec{w}$. Observe now that

$$\vec{e_1} = (1/2)(\vec{v} - \vec{u}), \quad \vec{e_2} = (1/2)(\vec{w} - \vec{u}) \quad \text{and} \quad \vec{e_3} = (1/2)(\vec{v} + \vec{w}).$$

Thus

$$A^{2022}\vec{e_1} = (1/2)(A^{2022}\vec{v} - A^{2022}\vec{u}) \approx \vec{0},$$

$$A^{2022}\vec{e_2} = (1/2)(A^{2022}\vec{w} - A^{2022}\vec{u}) \approx (1/2)\vec{w}$$

and

$$A^{2022}\vec{e_3} = (1/2)(A^{2022}\vec{v} + A^{2022}\vec{w}) \approx (1/2)\vec{w},$$

and so

$$A^{2022} = \begin{bmatrix} A^{2002}\vec{e_1} & A^{2022}\vec{e_2} & A^{2022}\vec{e_3} \end{bmatrix} \approx \begin{bmatrix} \vec{0} & (1/2)\vec{w} & (1/2)\vec{w} \end{bmatrix} = (1/2)\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

3. [10] Let $\vec{u}$ be a unit vector in $\mathbb{R}^3$, and let A be the 3×3 matrix $\vec{u}\vec{u}^T$.

(a) Show that $\vec{u}$ is an eigenvector of A with eigenvalue 1.
(b) Show that 0 is an eigenvalue of A, and that its associated eigenspace is a plane through the origin.
(c) Is A diagonalizable? Explain your answer.

Solution: We first recall that if $\vec{v} \in \mathbb{R}^3$, then the dot product $\vec{u} \cdot \vec{v}$ is equal to the matrix product $\vec{u}^T\vec{v}$ (viewed as a scalar). In particular, we have

$$A\vec{v} = (\vec{u}\vec{u}^T)\vec{v} = \vec{u}(\vec{u}^T\vec{v}) = (\vec{u} \cdot \vec{v})\vec{u}.$$

With this noted, we can now answer the questions at hand:

(a) Since $\vec{u}$ is a unit vector (i.e., has length 1), we have

$$A\vec{u} = (\vec{u} \cdot \vec{u})\vec{u} = \|\vec{u}\|^2\vec{u} = \vec{u},$$

and so $\vec{u}$ is an eigenvector of A with eigenvalue 1.

(b) The eigenvectors of A are with eigenvalue 0 are the non-zero vectors $\vec{v} \in \mathbb{R}^3$ for which $A\vec{v} = \vec{0}$. In other words, they are the non-zero elements of $\mathrm{Null}(A)$. But since $\vec{u} \neq \vec{0}$, we have

$$A\vec{v} = \vec{0} \quad \Leftrightarrow \quad (\vec{u} \cdot \vec{v})\vec{u} = \vec{0} \quad \Leftrightarrow \quad \vec{u} \cdot \vec{v} = 0,$$

and so $\mathrm{Null}(A)$ is the plane $\mathcal{P}$ through the origin in $\mathbb{R}^3$ with normal vector $\vec{u}$ (i.e., with scalar equation $\vec{u} \cdot \vec{x} = 0$). Thus, 0 is an eigenvalue of A, and its associated eigenspace is the plane $\mathcal{P}$.

(c) Yes, A is diagonalizable. To prove this, we need to show that the sum of the geometric multiplicities of the eigenvalues of A is 3 (see the solution to Q1). But part (b) shows that 0 is an eigenvalue of A with geometric multiplicity 2, and since 1 is also an eigenvalue of A (part (a)), we see that the sum of the geometric multiplicities of the eigenvalues of A is at least 3, and hence exactly 3.

Remark: The solution of (c) reveals that 0 and 1 are the only eigenvalues of A, with geometric (and hence algebraic) multiplicities 1 and 2, respectively. In fact, A is nothing else but the standard matrix of the linear operator given by orthogonal projection onto the vector $\vec{u}$.

4. [10] Let A be an orthogonal $n \times n$ matrix.

(a) Show that $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.

(b) Using (a), show that the following statemnts hold:

  (i) The only possible eigenvalues of A are 1 and -1;

  (ii) If $\vec{v}$ is an eigenvector of A with eigenvalue 1, and $\vec{w}$ is an eigenvector of A with eigenvalue -1, then $\vec{v}$ and $\vec{w}$ are orthogonal.

(c) Is it true that 1 and -1 are always eigenvalues of A? Justify your answer with a proof or counterexample.

Solution: (a) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. By the solution to Q3, we have

$$(A\vec{v}) \cdot (A\vec{w}) = (A\vec{v})^T(A\vec{w}) = (\vec{v}^T A^T)(A\vec{w}) = \vec{v}^T(A^T A)\vec{w}$$

and

$$\vec{v} \cdot \vec{w} = \vec{v}^T\vec{w}.$$

But since A is orthogonal, $A^T A = I_n$, and so $(A\vec{v}) \cdot (A\vec{w}) = \vec{v}^T\vec{w} = \vec{v} \cdot \vec{w}$.

(b) Suppose that $\vec{v}$ is an eigenvector of A with eigenvalue λ, and $\vec{w}$ is an eigenvector of A with eigenvalue μ. By part (a), we then have that

$$\vec{v} \cdot \vec{w} = (A\vec{v}) \cdot (A\vec{w}) = (\lambda\vec{v}) \cdot (\mu\vec{w}) = \lambda\mu(\vec{v} \cdot \vec{w}) \tag{$\star$}$$

With this noted, we can now address the questions at hand:

(i) Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. By ($\star$) (with $\vec{w} = \vec{v}$), we then have that

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = \lambda^2(\vec{v} \cdot \vec{v}) = \lambda^2\|\vec{v}\|^2.$$

Since $\vec{v} \neq \vec{0}$, $\|\vec{v}\| \neq 0$, and so $\lambda^2 = 1$, i.e., $\lambda \in \{\pm 1\}$.

(ii) Let $\vec{v}$ be an eigenvector of A with eigenvalue 1 and $\vec{w}$ an eigenvector of A with eigenvector -1. By ($\star$), we then have that

$$\vec{v} \cdot \vec{w} = 1 \cdot (-1)(\vec{v} \cdot \vec{w}) = -\vec{v} \cdot \vec{w},$$

and so $\vec{v} \cdot \vec{w} = 0$, i.e., $\vec{v}$ and $\vec{w}$ are orthogonal.

(c) No, it is *not* true that 1 and -1 must be eigenvalues of A. For instance, if $0 < \theta < \pi$, then the standard matrix of the linear operator on $\mathbb{R}^2$ that rotates vectors counterclockwise about the origin is an orthogonal matrix with no (real) eigenvalues. For example, taking $\theta = \pi/2$ gives the orthogonal matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which has characteristic polynomial

$$C_A(\lambda) = |A - \lambda I_2| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

But the characteristic equation $\lambda^2 + 1 = 0$ has no solutions in the real numbers, and so neither 1 nor -1 are eigenvalues of A. Of course, there are also orthogonal matrices having one of ± 1 as an eigenvalue but not the other. For instance, 1 is an eigenvalue of the orthogonal matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

but -1 is not. Similarly, -1 is an eigenvalue of $-I_2$, but 1 is not.