MATH 211 Fall 2022

Written Assignment 3 – Solutions

1. [10] Find, with justification, a homogeneous system of linear equations in four variables whose solution set is equal to

$$\operatorname{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -3 \\ 5 \end{bmatrix}\right\}$$

(*Hint*: Take a generic vector $\vec{x} \in \mathbb{R}^4$. When is $\vec{x}$ a linear combination of the four given vectors?).

<u>Solution</u>: Let $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ and $\vec{v_4}$ be the four given vectors. Let

$$A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 5 & 5 \\ -1 & 0 & -3 & -2 \\ -1 & -1 & -2 & -3 \\ 1 & 3 & 0 & 5 \end{bmatrix},$$

and let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

be an arbitrary vector in $\mathbb{R}^4$. By a result from lectures, $\vec{x}$ lies in $\operatorname{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ if and only if the linear system with augmented matrix $[A|\vec{x}]$ is consistent (see pages 115–117 in the textbook). In other words, $\vec{x}$ lies in $\operatorname{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ if and only if A and $[A|\vec{x}]$ have the same rank (see Theorem 2.2.2 in the textbook). To determine when this holds, we use Gaussian elimination to find a row echelon form of $[A|\vec{x}]$:

$$[A|\vec{x}] = \left[\begin{array}{cccc|c} 2 & 1 & 5 & 5 & x_1 \\ -1 & 0 & -3 & -2 & x_2 \\ -1 & -1 & -2 & -3 & x_3 \\ 1 & 3 & 0 & 5 & x_4 \end{array}\right] \longrightarrow \begin{array}{c} R_1 \leftrightarrow R_4 \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & x_4 \\ -1 & 0 & -3 & -2 & x_2 \\ -1 & -1 & -2 & -3 & x_3 \\ 2 & 1 & 5 & 5 & x_1 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \\ R_2 + R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & x_4 \\ 0 & 3 & -3 & 3 & x_2 + x_4 \\ 0 & 2 & -2 & 2 & x_3 + x_4 \\ 0 & -5 & 5 & -5 & x_1 - 2x_4 \end{array}\right] \longrightarrow R_2 - R_3 \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & x_4 \\ 0 & 1 & -1 & 1 & x_2 - x_3 \\ 0 & 2 & -2 & 2 & x_3 + x_4 \\ 0 & -5 & 5 & -5 & x_1 - 2x_4 \end{array}\right]$$

$$\longrightarrow \begin{array}{r} \\ \\ R_3 - 2R_2 \\ R_4 + 5R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & x_4 \\ 0 & 1 & -1 & 1 & x_2 - x_3 \\ 0 & 0 & 0 & 0 & -2x_2 + 3x_3 + x_4 \\ 0 & 0 & 0 & 0 & x_1 + 5x_2 - 5x_3 - 2x_4 \end{array}\right]$$

From this computation, we see that the rank of A is 2, while the rank of $[A|\vec{x}]$ is 2 if and only if both $-2x_2 + 3x_3 + x_4$ and $x_1 + 5x_2 - 5x_3 - 2x_4$ are equal to 0. Thus,

$$\begin{array}{rcrcrcrcl} & - & 2x_2 & + & 3x_3 & + & x_4 & = & 0 \\ x_1 & + & 5x_2 & - & 5x_3 & - & 2x_4 & = & 0 \end{array}$$

is a homogeneous linear system in four variables whose solution set is precisely the subspace of $\mathbb{R}^4$ spanned by $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ and $\vec{v_4}$.

Remark: This isn't the only correct answer. The coefficient matrix of the system we found is

$$A = \begin{bmatrix} 0 & -2 & 3 & 1 \\ 1 & 5 & -5 & -2 \end{bmatrix}.$$

If B is any matrix row equivalent to A (or a matrix obtained by appending finitely many rows of zeros to A), then the homogeneous linear system with coefficient matrix B has the same solution set as A, and thus gives another correct answer.

2. [12] A square matrix A is said to be *idempotent* if $A^2 = A$.

(a) Find all idempotent 2×2 matrices whose second column is the zero vector. Show your work.

(b) By a result from lectures, the mapping

$$L\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \vec{x} \mapsto \operatorname{proj}_{\vec{e_1}} \vec{x}$$

is linear. Find the standard matrix of L, and verify that it is idempotent.

(c) Is it true that for any non-zero vector $\vec{a} \in \mathbb{R}^n$, the standard matrix of the linear mapping

$$L\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \vec{x} \mapsto \operatorname{proj}_{\vec{a}} \vec{x}$$

is idempotent? Give a brief explanation of your answer.

Solution: (a) If $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ for some $a, b \in \mathbb{R}$, then

$$A^2 = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ ab & 0 \end{bmatrix},$$

and so A is idempotent if and only if $a^2 = a$ and $ab = b$. Now $a^2 = a$ if and only if $a \in \{0, 1\}$. If $a = 1$, then it is certainly true that $ab = b$. If $a = 0$, on the other hand, then $ab = b$ if and only if $b = 0$. The complete set of idempotent 2×2 matrices whose second column is the zero vector is therefore

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}.$$

(b) Since $\vec{e_1}$ is parallel to itself, we have

$$L(\vec{e_1}) = \operatorname{proj}_{\vec{e_1}} \vec{e_1} = \vec{e_1}.$$

On the other hand, since $\vec{e_2}$ is orthogonal to $\vec{e_1}$, we have

$$L(\vec{e_2}) = \operatorname{proj}_{\vec{e_1}} \vec{e_2} = \vec{0},$$

and so

$$[L] = \begin{bmatrix} L(\vec{e_1}) & L(\vec{e_2}) \end{bmatrix} = \begin{bmatrix} \vec{e_1} & \vec{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that this matrix is an element of the set found in part (a), and is therefore idempotent.

(c) Yes, this *is* true: Let $\vec{x} \in \mathbb{R}^n$, and let $\vec{y} = L(\vec{x}) = \operatorname{proj}_{\vec{a}} \vec{x}$. By definition, $\vec{y}$ is then parallel to $\vec{a}$, and so $L(\vec{y}) = \operatorname{proj}_{\vec{a}} \vec{y} = \vec{y}$. Thus,

$$(L \circ L)(\vec{x}) = L\big(L(\vec{x})\big) = L(\vec{y}) = \vec{y} = L(\vec{x}),$$

and so $L \circ L = L$. Recalling that the standard matrix of a composition of linear mappings is the product of the respective standard matrices (Theorem 3.2.5 in the textbook), we then obtain that $[L]^2 = [L \circ L] = [L]$, i.e., $[L]$ is idempotent.

3. [10] In each of the following cases, determine (with justification) whether the given mapping L is linear. In the cases where it is, find the standard matrix of L.

(i) $L\colon \mathbb{R}^2 \to \mathbb{R}$, $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{cases} x_1 & \text{if } x_1x_2 \geq 0 \\ x_2 & \text{if } x_1x_2 < 0 \end{cases}.$

(ii) $L\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1\vec{u} + x_2\vec{v}$, where $\vec{u}$ and $\vec{v}$ are fixed (but unknown) vectors in $\mathbb{R}^2$.

Solution: (i) In this case, L is *not* linear. Indeed, although L preserves scalar multiplication, it doesn't preserve addition, i.e., it is not true that $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$. For instance, if

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

then

$$L(\vec{x}+\vec{y}) = L\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = 2$$

(because $2 \cdot 0 = 0$), whereas

$$L(\vec{x}) + L(\vec{y}) = L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 1 + (-1) = 0$$

(because $1 \cdot 1 = 1 \geq 0$ and $1 \cdot (-1) = -1 < 0$).

(b) In this case L *is* linear. Indeed, consider the 2×2 matrix $A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, then

$$L(\vec{x}) = x_1\vec{u} + x_2\vec{v} = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x},$$

and so L is the linear operator on $\mathbb{R}^2$ with standard matrix A.

<u>*Remark:*</u> Alternatively, we can show the linearity of L by direct verification of the defining conditions, namely:

(L1) $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$;
(L2) $L(t\vec{x}) = tL(\vec{x})$ for all $t \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$.

Let's treat these separately:

(L1) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Then

$$\begin{aligned} L(\vec{x}+\vec{y}) &= L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) \\ &= (x_1+y_1)\vec{u} + (x_2+y_2)\vec{v} \\ &= (x_1\vec{u} + x_2\vec{v}) + (y_1\vec{u} + y_2\vec{v}) \\ &= L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \\ &= L(\vec{x}) + L(\vec{y}), \end{aligned}$$

and so (L1) holds.

(L2) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. If $t \in \mathbb{R}$, then

$$\begin{aligned}
L(t\vec{x}) &= L\left(\begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}\right) \\
&= (tx_1)\vec{u} + (tx_2)\vec{v} \\
&= t(x_1\vec{u} + x_2\vec{v}) \\
&= tL\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\
&= tL(\vec{x}),
\end{aligned}$$

and so (L2) holds.

4. [10] There is a unique linear mapping $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad L\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find, with justification, the standard matrix of L.

<u>**Solution:**</u> Let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

To solve the problem, we first need to express each of the standard unit vectors $\vec{e_1}$, $\vec{e_2}$ and $\vec{e_3}$ as linear combinations of $\vec{u}$, $\vec{v}$ and $\vec{w}$. To do this, we find the reduced row echelon form of the matrix $\left[\begin{array}{ccc|c|c|c} \vec{u} & \vec{v} & \vec{w} & \vec{e_1} & \vec{e_2} & \vec{e_3} \end{array}\right]$ using Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c|c|c} \vec{u} & \vec{v} & \vec{w} & \vec{e_1} & \vec{e_2} & \vec{e_3} \end{array}\right] = \left[\begin{array}{rrr|r|r|r} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|r|r|r} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \times (-1)} \left[\begin{array}{rrr|r|r|r} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + R_2 \\ R_3 - R_2}} \left[\begin{array}{rrr|r|r|r} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \times (-1)} \left[\begin{array}{rrr|r|r|r} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

From the computation, we see that

$$\begin{array}{rcrcrcr} \vec{e_1} & = & \vec{u} & + & \vec{v} & + & \vec{w} \\ \vec{e_2} & = & & - & \vec{v} & - & \vec{w} \\ \vec{e_3} & = & \vec{u} & & & - & \vec{w} \end{array}$$

Since L is a linear mapping, it follows that

$$\begin{aligned} L(\vec{e_1}) = L(\vec{u}) + L(\vec{v}) + L(\vec{w}) &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} L(\vec{e_2}) = -L(\vec{v}) - L(\vec{w}) &= -\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} L(\vec{e_3}) =)L(\vec{u}) - L(\vec{w}) &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -3 \end{bmatrix}. \end{aligned}$$

By definition, we then have that

$$[L] = \begin{bmatrix} L(\vec{e_1}) & L(\vec{e_2}) & L(\vec{e_3}) \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -4 & -3 \end{bmatrix}.$$

Remark: We outline another approach based on the concept of invertibility for square matrices (discussed in this week's lectures): If $[L]$ is the standard matrix of L, then the information given the question amounts to the matrix equation

$$[L] \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \end{bmatrix}.$$

Now the 3×3 matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

has rank 3, and is therefore invertible. If we can determine its inverse, we can then solve for $[L]$ using the identity

$$[L] = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1}.$$

In fact, we found the needed inverse in the solution above: It is the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

(see the algorithm for matrix inversion discussed in class), and so

$$[L] = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -4 & -3 \end{bmatrix}.$$

5. [8] The matrix

$$\left(\frac{1}{3}\right) \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

is the standard matrix of the linear mapping $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by reflection about a plane $\mathcal{P}$ in 3-space that passes through the origin. Find a vector equation for $\mathcal{P}$ (*Hint*: What does L do to the vectors on $\mathcal{P}$?).

Solution: Let A be the 3×3 matrix given in the equation, and let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be an arbitrary vector in $\mathbb{R}^3$. Note that $\vec{x}$ lies on $\mathcal{P}$ if and only if $L(\vec{x}) = \vec{x}$ (the vectors fixed by L are precisely those in the plane of reflection).[1] Since A is the standard matrix of L, it follows that $\vec{x}$ lies on $\mathcal{P}$ if and only if $A\vec{x} = \vec{x}$. Now

[1]When we say that $\vec{x}$ lies on $\mathcal{P}$, we mean that it gives the displacement from the origin to a point on $\mathcal{P}$.

$$
\begin{aligned}
A\vec{x} = \vec{x} \quad &\Leftrightarrow \quad \left(\frac{1}{3}\right)\begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&\Leftrightarrow \quad \left(\begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

In other words, $A\vec{x} = \vec{x}$ if and only if $\vec{x}$ is a solution of the homogeneous linear system with augmented matrix

$$
\left[\begin{array}{ccc|c} -1 & -2 & 1 & 0 \\ -2 & -4 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right].
$$

Now the first and second rows of this matrix are non-zero multiples of the third row, and so $A\vec{x} = \vec{x}$ if and only if

$$
x_1 + 2x_2 - x_3 = 0
$$

(the equation corresponding to the third row of the matrix). By the preceding remarks, the latter is therefore a scalar equation for $\mathcal{P}$.

Remark: As usual, this is not the only correct answer here: Scalar equations of planes are only unique up to scaling (multiplying through by a non-zero constant).

Alternative Approaches: We outline two alternative approaches. Let

$$
\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

be an unspecified normal vector for $\mathcal{P}$. Then $L(\vec{n}) = -\vec{n}$, and so

$$\left(\frac{1}{3}\right)\begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Expanding, we obtain a system of three linear equations in a, b and c. Solving the system reveals that $b = 2a$ and $c = -a$, from which it follows that

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

is a normal vector for $\mathcal{P}$, and so $x_1 + 2x_2 - x_3 = 0$ is a scalar equation for $\mathcal{P}$.

Alternatively, we can work with an explicit formula for L: As discussed in class, we have

$$L(\vec{x}) = \vec{x} - \operatorname{proj}_{\vec{n}} \vec{x}$$

for all $\vec{x} \in \mathbb{R}^3$. Since $[L] = \begin{bmatrix} L(\vec{e_1}) & L(\vec{e_2}) & L(\vec{e_3}) \end{bmatrix}$, we therefore have that

$$\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = L(\vec{e_1}) = \vec{e_1} - \operatorname{proj}_{\vec{n}} \vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\left(\frac{a}{a^2+b^2+c^2}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

$$\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = L(\vec{e_2}) = \vec{e_2} - \operatorname{proj}_{\vec{n}} \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\left(\frac{b}{a^2+b^2+c^2}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = L(\vec{e_3}) = \vec{e_3} - \operatorname{proj}_{\vec{n}} \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\left(\frac{c}{a^2+b^2+c^2}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Solving these equations reveals that $b = 2a$ and $c = -a$, allowing us to solve the problem as before.