Written Assignment 1 – Solutions

1. [10] Let ℓ be the line in 3-space that passes through the points $A = (3, -1, 2)$ and $B = (5, 4, -1)$.

(a) Find a set of parametric equations for ℓ.
(b) Determine the set of all points on ℓ whose distance from A is at most half the distance from A to B.
(c) Determine the value of k for which the point $C = (k, k-1, k-6)$ lies on ℓ.
(d) Determine the unique point D at which ℓ intersects the plane with scalar equation $-2x_1 + x_2 - x_3 = 3$.

Solution: (a) The vector

$$\overrightarrow{AB} = \begin{bmatrix} 5-3 \\ 4-(-1) \\ -1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

is a direction vector for ℓ, and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \quad (t \in \mathbb{R})$$

is a vector equation for ℓ. In particular,

$$\begin{aligned} x_1 &= 3 + 2t \\ x_2 &= -1 + 5t \qquad (t \in \mathbb{R}) \\ x_3 &= 2 - 3t \end{aligned}$$

is a set of parametric equations for ℓ.

Remark: There are infinitely many ways to parametrize the points on a line, so this is far from the only correct answer: To the find the given set of equations, we needed to choose a point on ℓ and a direction vector for ℓ. The choice of another point or direction vector would have led to a different set of equations.

(b) If Q is a point on ℓ, then its distance from A is at most half the distance from A to B if and only if $\overrightarrow{OQ} = \overrightarrow{OA} + t\overrightarrow{AB}$ for some $t \in [-\frac{1}{2}, \frac{1}{2}]$ (the reader should convince themselves of this by drawing a picture). By the solution to (a), this holds if and only if $Q = (3 + 2t, -1 + 5t, 2 - 3t)$ for some $t \in [-\frac{1}{2}, \frac{1}{2}]$.

(c) By the solution to (a), every point on ℓ is equal to $(3 + 2t, -1 + 5t, 2 - 3t)$ for some $t \in \mathbb{R}$. In particular, if $C = (k, k-1, k-6)$ lies on ℓ, then

$$k = 3t + 2, \quad k - 1 = -1 + 5t \quad \text{and} \quad k - 6 = 2 - 3t$$

for some $t \in \mathbb{R}$. Combining the first two equations, we get that

$$3t + 2 = k = 1 + (k - 1) = 1 + (-1 + 5t) = 5t,$$

so $t = 1$ and $k = 3 \cdot 1 + 2 = 5$ (and $C = B$!).

(d) Again, the solution to (a) tells us that every point on ℓ is equal to $(3+2t, -1+5t, 2-3t)$ for some $t \in \mathbb{R}$. Now if $t \in \mathbb{R}$, then $(3+2t, -1+5t, 2-3t)$ lies on the plane with equation $-2x_1 + x_2 - x_3 = 3$ if and only if

$$-2(3 + 2t) + (-1 + 5t) - (2 - 3t) = 3.$$

Solving, we find that $t = 3$, and so $D = (3 + 2 \cdot 3, -1 + 5 \cdot 3, 2 - 3 \cdot 3) = (9, 14, -7)$ is the unique point of intersection of ℓ and the given plane.

2. [10] Let ℓ' be the line in 3-space that passes through the point $(1, -1, 1)$ and is parallel to the line ℓ from Problem 1 above.

(a) Find a set of parametric equations for the plane $\mathcal{P}$ in 3-space that passes through the origin and contains the line ℓ'.
(b) Find a vector equation for the line of intersection of the plane $\mathcal{P}$ from part (a) and the plane $\mathcal{P}'$ with scalar equation $x_1 + x_2 + x_3 = 0$.

Solution: (a) Since ℓ' is parallel to ℓ, it has the same direction vectors as ℓ. By the solution to Problem 1(a), it follows that

$$\vec{d_1} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

is a direction vector for ℓ', and hence $\mathcal{P}$ (because $\mathcal{P}$ contains ℓ). Since $\mathcal{P}$ contains the origin and the point $P = (1, -1, 1)$, a second direction vector for $\mathcal{P}$ is given by

$$\vec{d_2} = \overrightarrow{OP} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Since there is no scalar $t \in \mathbb{R}$ for which $\vec{d_1} = t\vec{d_2}$, $\vec{d_1}$ and $\vec{d_2}$ are not parallel (i.e., linearly independent), and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (t_1, t_2 \in \mathbb{R})$$

is a vector equation for $\mathcal{P}$ (again, since $\mathcal{P}$ passes through the origin, we can choose the zero vector as the position vector in our equation). In particular,

$$\begin{array}{rcl} x_1 & = & 2t_1 + t_2 \\ x_2 & = & 5t_1 - t_2 \\ x_3 & = & -3t_1 + t_2 \end{array} \quad (t_1, t_2 \in \mathbb{R})$$

is a set of parametric equations for $\mathcal{P}$.

Remark: As in Problem 1(a), this is not the only correct answer. Finding a set of parametric equations for $\mathcal{P}$ requires choosing a point on $\mathcal{P}$ and a pair of non-parallel direction vectors, and different choices will result in different sets of equations.

(b) By the solution to (a), every point on $\mathcal{P}$ is equal to $(2t_1+t_2, 5t_1-t_2, -3t_1+t_2)$ for some $t_1, t_2 \in \mathbb{R}$. Now, if $t_1, t_2 \in \mathbb{R}$, then $(2t_1+t_2, 5t_1-t_2, -3t_1+t_2)$ lies on the plane $\mathcal{P}'$ (with equation $x_1+x_2+x_3=0$) if and only if

$$(2t_1+t_2)+(5t_1-t_2)+(-3t_1+t_2)=0.$$

Simplifying, we see that this holds if and only if $t_2=-4t_1$. The points lying the line of intersection of the two planes are therefore those of the form

$$\big(2t_1+(-4t_1), 5t_1-(-4t_1), -3t_1+(-4t_1)\big)=(-2t_1, 9t_1, -7t_1)$$

for some $t_1 \in \mathbb{R}$. In other words, the equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 9 \\ -7 \end{bmatrix} \quad (t \in \mathbb{R})$$

is a vector equation for the line of intersection of $\mathcal{P}$ and $\mathcal{P}'$.

Remark: Again, this isn't the only correct answer for the reasons given in the remark following the solution to Problem 1(a).

3. [10] In each of the following cases, determine whether the given set S is a subspace of $\mathbb{R}^3$ (if it is a subspace, give a proof; if not, show by example that one of the defining conditions fails to hold in general):

(a) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1x_2 = x_1x_3 \right\}$;

(b) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 - 2x_2 + x_3 = 0 \text{ and } x_2 - 3x_3 = 0 \right\}$.

Solution: (a) In this case, S is *not* a subspace of $\mathbb{R}^3$. Indeed, although S is non-empty and closed under scalar multiplication, it is not closed under addition. For example, the vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

both lie in S ($1 \cdot 0 = 0 = 0 \cdot 0$ and $0 \cdot 1 = 0 = 0 \cdot 0$), but

$$\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

does not ($1 \cdot 1 = 1 \neq 0 = 1 \cdot 0$).

(b) In this case, S *is* a subspace of $\mathbb{R}^3$. We give two different proofs (the reader is encouraged to examine both approaches):

First Proof: In this first proof, we will check directly that the defining conditions are satisfied. First, since $0 - 2 \cdot 0 + 0 = 0 = 0 - 3 \cdot 0$, S contains the zero vector (and is hence non-empty). To prove that S is a subspace of $\mathbb{R}^3$, we now have to check that S is closed under addition and scalar multiplication.

Closure under addition: Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

be elements of S. By definition,

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Now, since $\vec{x}$ and $\vec{y}$ are elements of S, we have that

$$x_1 - 2x_2 + x_3 = 0, \quad x_2 - 3x_3 = 0, \quad y_1 - 2y_2 + y_3 = 0 \quad \text{and} \quad y_2 - 3y_3 = 0.$$

In particular,

$$(x_1 + y_1) - 2(x_2 + y_2) + (x_3 + y_3) = (x_1 - 2x_2 + x_3) + (y_1 - 2y_2 + y_3) = 0 + 0 = 0$$

and

$$(x_2 + y_2) - 3(x_3 + y_3) = (x_2 - 3x_3) + (y_2 - 3y_3) = 0 + 0 = 0,$$

and so $\vec{x} + \vec{y}$ is also an element of S.

Closure under scalar multiplication: Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be an element of S, and let $t \in \mathbb{R}$ By definition,

$$t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix}.$$

Now, since $\vec{x}$ is an element of S, we have that

$$x_1 - 2x_2 + x_3 = 0 \quad \text{and} \quad x_2 - 3x_3 = 0.$$

But then

$$tx_1 - 2(tx_3) + tx_3 = t(x_1 - 2x_3 + x_3) = t \cdot 0 = 0$$

and

$$tx_2 - 3(tx_3) = t(x_2 - 3x_3) = t \cdot 0 = 0,$$

and so $t\vec{x}$ is also an element of S.

Second Proof: In this second proof, we will show that S is equal to the span of some finite subset of $\mathbb{R}^3$, and is hence a subspace of $\mathbb{R}^3$ by a key result discussed in lectures (Theorem 1.4.2 in the textbook). Explicitly, substituting $3x_3$ for x_2 in the equation $x_1 - 2x_2 + x_3 = 0$ gives the equation $x_1 = 5x_3$. The system of equations

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 3x_3 & = & 0 \end{array}$$

is therefore equivalent to the system

$$\begin{array}{rcl} x_1 & = & 5x_3 \\ x_2 & = & 3x_3 \end{array},$$

and so S is the set of all vectors of the form

$$\begin{bmatrix} 5x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

for some $x_3 \in \mathbb{R}$. By definition, this means that

$$S = \text{Span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \right\},$$

and so S is a subspace of $\mathbb{R}^3$ (geometrically, S is identified with a line passing through the origin in 3-space).

4. [12]

(a) Let S be a subspace of $\mathbb{R}^n$ for some positive integer n. Show that if S contains the vectors $\vec{e_1}, \ldots, \vec{e_n}$, then it equals $\mathbb{R}^n$ (recall that $\vec{e_i}$ denotes the vector in $\mathbb{R}^n$ whose ith entry is 1 and whose remaining entries are all 0).

(b) Show that $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3$.

(c) Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$? Justify your answer.

Solution: (a) Since S is a subspace of $\mathbb{R}^n$, it is closed under taking combinations of any its elements. In particular, since $\vec{e_1}, \ldots, \vec{e_n} \in S$, every linear combination of $\vec{e_1}, \ldots, \vec{e_n}$ is an element of S (i.e., S contains $\text{Span}\{\vec{e_1}, \ldots, \vec{e_n}\}$). But if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is an element of $\mathbb{R}^n$, then

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\vec{e_1} + x_2\vec{e_2} + \cdots + x_n\vec{e_n},$$

and so $\vec{x}$ lies in S. This shows that $S = \mathbb{R}^n$.

(b) Let

$$S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

By the key result cited in the solution to Problem 3(b) (Theorem 1.4.2 in the textbook), S is a subspace of $\mathbb{R}^3$. In view of part (a), proving that $S = \mathbb{R}^3$ therefore amounts to showing that the vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

lie in S. In other words, we have to show that each of these vectors is a linear combination of

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

But

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}\vec{v_1} + \frac{1}{2}\vec{v_2},$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}\vec{v_1} - \frac{1}{2}\vec{v_2}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{v_3} - \vec{v_1},$$

so the claim holds.

Remark: A more direct approach would be to simply observe that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left(\frac{x_1+x_2}{2} - x_3\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{x_1-x_2}{2}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for any $x_1, x_2, x_3 \in \mathbb{R}$.

(c) The answer is *yes*: By part (b), the given set of vectors is a spanning set for $\mathbb{R}^3$. To show that it is a *basis* of $\mathbb{R}^3$, it therefore only remains to show that it is linearly independent. As discussed in lectures, this amounts to showing that the vector equation

$$t_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution $t_1 = t_2 = t_3 = 0$. To see that this is the case, note that the given vector equation is equivalent to the following system of scalar equations:

$$\begin{array}{ccccccc} t_1 & + & t_2 & + & t_3 & = & 0 \\ t_1 & - & t_2 & + & t_3 & = & 0 \\ & & & & t_3 & = & 0 \end{array}$$

But if $t_3 = 0$, then the first two equations say that $t_1 = -t_2 = t_2$, which is possible only if $t_1 = t_2 = 0$. Thus, the system has only the trivial solution, and so the given set of vectors is indeed a basis of $\mathbb{R}^3$.

5. [8] Let S be a 2-dimensional subspace of $\mathbb{R}^n$ for some positive integer n. Show that if $\{\vec{u}, \vec{v}\}$ is a basis of S, then $\{\vec{u}, \vec{u}+\vec{v}\}$ is also a basis of S.

Solution: To prove that $\{\vec{u}, \vec{u}+\vec{v}\}$ is a basis of S, we have to show that:

(i) $\text{Span}\{\vec{u}, \vec{u}+\vec{v}\} = S$, i.e., every vector in S is a linear combination of $\vec{u}$ and $\vec{u}+\vec{v}$;

(ii) $\{\vec{u}, \vec{u}+\vec{v}\}$ is linearly independent.

Let's check these conditions separately:

(i) Let $\vec{x}$ be an arbitrary element of S. Since $\{\vec{u}, \vec{v}\}$ is a basis of S (and hence a spanning set for S), we can write

$$\vec{x} = a\vec{u} + b\vec{v}$$

for some $a, b \in \mathbb{R}$. Then

$$\vec{x} = (a-b)\vec{u} + b(\vec{u} + \vec{v}),$$

and so $\vec{x}$ is a linear combination of $\vec{u}$ and $\vec{u} + \vec{v}$.

(ii) Let $t_1, t_2, \in \mathbb{R}$ be such that

$$t_1\vec{u} + t_2(\vec{u} + \vec{v}) = \vec{0}.$$

As discussed in the solution to Problem 4(c), proving that $\{\vec{u}, \vec{u} + \vec{v}\}$ is linearly independent amounts to showing that $t_1 = t_2 = 0$. Note however, that the given equation can be re-written as

$$(t_1 + t_2)\vec{u} + t_2\vec{v} = \vec{0}.$$

Since $\{\vec{u}, \vec{v}\}$ is a basis of S (and hence linearly independent), the only way to write $\vec{0}$ as a linear combination of $\vec{u}$ and $\vec{v}$ is the trivial way, i.e., $0\vec{u} + 0\vec{v} = \vec{0}$. We must therefore have that $t_1 + t_2 = t_2 = 0$, and so $t_1 = t_2 = 0$, as desired.