From Online Optimization to PID Controllers: Mirror Descent with Momentum

Santiago R. Balseiro* Haihao Lu[†] Vahab Mirrokni[‡] Balasubramanian Sivan[§]

February 15, 2022

Abstract

We study a family of first-order methods with momentum based on mirror descent for online convex optimization, which we dub online mirror descent with momentum (OMDM). Our algorithms include as special cases gradient descent and exponential weights update with momentum. We provide a new and simple analysis of momentum-based methods in a stochastic setting that yields a regret bound that decreases as momentum increases. This immediately establishes that momentum can help in the convergence of stochastic subgradient descent in convex nonsmooth optimization. We showcase the robustness of our algorithm by also providing an analysis in an adversarial setting that gives the first non-trivial regret bounds for OMDM. Our work aims to provide a better understanding of the benefits of momentum-based methods, which despite their recent empirical success, is incomplete.

Finally, we discuss how OMDM can be applied to stochastic online allocation problems, which are central problems in computer science and operations research. In doing so, we establish an important connection between OMDM and popular approaches from optimal control such as PID controllers, thereby providing regret bounds on the performance of PID controllers. The improvements of momentum are most pronounced when the step-size is large, thereby indicating that momentum provides a robustness to misspecification of tuning parameters. We provide a numerical evaluation that verifies the robustness of our algorithms.

1 Introduction

First-order methods such as Stochastic Gradient Descent (SGD) and its variants are among the most popular methods for convex optimization. A common technique to accelerate first-order methods is to incorporate momentum or inertia to the iterates. These methods date to Polyak (1964) and are known to provide faster convergence rates in deterministic settings. In the last decade, there has been a renewed interest in first-order methods as these have been particularly successful in solving challenging optimization problems such as training machine learning prediction methods Nesterov (2003); Sra et al. (2012); Beck (2017) and online allocation problems such as budget pacing in ad markets (Devanur et al., 2019; Gupta and Molinaro, 2016; Balseiro and Gur, 2017; Conitzer et al., 2021). Momentum-based methods, in particular, have been used extensively in machine learning (Kingma and Ba, 2015; Sutskever et al., 2013). Despite their empirical success, the theoretical

^{*}Columbia Business School and Google Research (srb2155@columbia.edu).

[†]The University of Chicago, Booth School of Business (haihao.lu@chicagobooth.edu). Part of the work was done at Google Research.

[‡]Google Research (mirrokni@google.com)

[§]Google Research (balusivan@google.com)

understanding of the benefits of momentum is, to the best of our knowledge, incomplete. In particular, we lack results that provably establish better convergence rates for momentum methods for convex stochastic non-smooth optimization.

Existing results that analyze first-order methods with momentum fall under the following categories. First, the analysis assumes that the objective function is smooth. In many problems, such as online allocation problems, objectives are non-smooth so those results are not directly applicable. Second, the state-of-the-art analysis of momentum algorithms in stochastic settings with smooth objective functions mostly yield convergence rates that at most match those of vanilla SGD, rather than showing improved performance.

Our goal in this paper is to study a family of first-order methods with momentum based on mirror descent, which we dub online mirror descent with momentum (OMDM). We provide the first analysis that establishes a constant factor better asymptotic convergence guarantee compared to SGD, for convex non-smooth optimization in stochastic settings. Moreover, we showcase the robustness of OMDM by proving it attains vanishing regret even when inputs are adversarially chosen. Finally, we discuss how these methods can be applied to stochastic online allocation problems, which are central problems in computer science and operations research with wide applications in practice. In doing so, we establish an important connection between OMDM and popular approaches from optimal control such as PID controllers and exhibit the robustness of momentum to misspecifications of the tuning parameters.

1.1 Main Contributions

We consider the general setting of online convex optimization (OCO), which is a powerful framework for online decision making (Hazan, 2008). In each period, a decision maker chooses an action and then Nature selects a convex loss function. The payoff of the decision maker is given by the cumulative losses over T time periods. The goal is to design algorithms that attain low regret compared to the best static action in hindsight, i.e., to an oracle that knows all the loss functions in advance but who is restricted to choose the same action across time.

We present a general algorithm that endows mirror descent with momentum. In a nutshell, momentum incorporates memory to the mirror descent by taking an exponentially smooth averaged of the past gradients, where $\beta \in [0,1)$ is the momentum parameter / smoothing factor. In the case of $\beta=0$ only the last gradient is used and as β is increased, the past gradients are given larger weights. Mirror descent (without any momentum) is the workhorse algorithm of OCO as it allows to better capture the geometry of the feasible sets. A benefit of the more general OMDM algorithm is that various instantiations of the reference function can now recover several special cases. For example, using the squared-Euclidean reference function, we can recover online gradient descent with momentum and using the negative entropy reference function we obtain multiplicative weights update with momentum.

We provide three different interpretations of momentum. The first is a statistical interpretation that hinges on the popularity of exponential smoothing in time series analysis. The basic idea is that by smoothing gradients, their variance should reduce at the expense of introducing bias. The second interpretation, which provides a central role in our application to online allocation problems, is from the optimal control perspective: our momentum algorithm can be interpreted as a sequential controller with a proportional and integral term. This brings to the foreground a connection between online learning and optimal control, which has attracted a considerable amount of attention recently in the reinforcement learning literature (Recht, 2019). Our final interpretation is from physics, as

momentum can of course be thought of as adding inertia to the iterates.

We begin by studying an adversarial setting in which Nature chooses the losses adaptively. Previous results for similar algorithms require that momentum vanishes as the length of the horizon increases, which is unacceptable in practice as typical values of momentum are quite high, like $\beta = 0.9$ (Alacaoglu et al., 2020). We provide the first adversarial regret bounds for OMDM and show that our algorithm attains $T^{1/2}$ regret with constant step-sizes and for every level of momentum $\beta \in [0,1)$ (Theorem 1). While these results highlight the robustness of our algorithm to assumptions on the input, unfortunately, they suggest that momentum provides no improvement in performance when the inputs are adversarial. We show using a simple example that this phenomenon is not a limitation of our analysis but rather an information theoretical impossibility: the fundamental reason that momentum fails to improve convergence is that statistical averaging grants no benefits when the input is adversarial.

We then consider a stochastic setting in which Nature is restricted to drawing the loss functions independently from an identical distribution. Importantly, we assume that the distribution is unknown to the algorithm and can be chosen adversarially by nature at time zero. For the case of stochastic inputs we show better convergence guarantees by leveraging the variance-reduction effect of averaging past gradients. We show that small amounts of momentum actually lead to smaller regret bounds (Theorem 2), i.e., our results establish a factor $\sqrt{1+\beta}$ better convergence rate for stochastic gradient descent with momentum (SGDM) over SGD. Besides, our proof is distinctly simpler / shorter than earlier proofs, and can serve as the basis for further investigation into SGDM. We remark that a limitation of our analysis, however, is that it does not allow for constraints on the feasible region and can only handle certain reference functions, which notably include the squared-Euclidean norm, i.e., SGDM.

We then explore the application of our algorithms to online allocation problems, which have applications to revenue management, online matching, online advertising, online retailing, etc. In an online allocation problem, a decision maker needs to repeatedly make actions that consume resources and generate rewards. The objective is to maximize cumulative rewards subject to constraints on resource consumption. A popular paradigm to solve this problem is to consider dual-descent algorithms that operate in the Lagrangian dual: actions are the Lagrange multipliers of the resource constraints. Because the dual problems are always convex (even if the primal problem is not), our algorithm can naturally be used to tackle these problems. In particular, we provide a simple algorithm based on OMDM that attains vanishing regret relative to the offline problem when the inputs are stochastic.

In an online allocation problem, our algorithm yields a clear PID interpretation: it can be thought as a controller acting on the error as measured by the difference between the target spend (i.e., how much resources should we spend per time period to evenly spread them over time) and actual amount spent. This interpretation is appealing as PID controllers have been recently proposed in the online allocation literature as heuristic methods without any theoretical guarantees (Tashman et al., 2020; Zhang et al., 2016; Smirnov et al., 2016; Yang et al., 2019). Ours is the first paper that couches controllers in an online framework and provides a formal analysis of these methods.

We conclude by performing some numerical experiments on online linear programming with resource constraints, which confirm our theoretical predictions. An interesting takeaway from our experiments is that some degree of momentum increases the robustness of the algorithm. In practice, it is usually impossible to pick the optimal step-size since this depends on the parameters of the problem, which are usually hard to estimate. While momentum has little impact if step-sizes are underesti-

mated, it can lead to better performance when step-sizes are overestimated. Therefore, momentum can increase the operating range in which first-order algorithms attain good performance.

1.2 Related Literature

Our work contributes to various streams of literature, which we overview below.

Momentum and Accelerated Methods in Deterministic Convex Optimization. Momentum is a very useful technique in continuous optimization. It dates back to Polyak's heavy-ball method Polyak (1964), which was shown to enjoy fast linear convergence rate for convex quadratic minimization. In 1980s, Nesterov (1983) proposed a variant of momentum methods (a.k.a. Nesterov's accelerated methods), and showed its improved convergence guarantee for solving general convex smooth optimization. Nesterov's accelerated methods turn out to achieve the optimal convergence rate for smooth convex optimization as a first-order method Nemirovsky and Yudin (1983). While the convergence guarantee was proved in Nesterov (1983), there was no clear intuition behind the reason momentum is helpful to speed up the convergence of gradient descent. In the last ten years, there have been many attempts trying to explain the intuition behind Nesterov's accelerated methods, for example, a continuous time approach Su et al. (2016), a geometric perspective Bubeck et al. (2015), and a linear coupling viewpoint of gradient descent and mirror descent Allen-Zhu and Orecchia (2014).

Stochastic Gradient Descent with Momentum. While the role of momentum is better understood now in the deterministic world, the story for stochastic case is still incomplete. For smooth convex optimization, there have been a few variants of momentum methods that can achieve improved complexity compared to vanilla SGD. For example, Lin et al. (2015) proposed a catalyst framework that is based on approximated accelerated proximal point method and used stochastic algorithms to solve the subproblems. Allen-Zhu (2017) proposed a "negative momentum" on top of Nesterov's momentum. Both algorithms achieve improved complexity compared to SGD, however, they require evaluating the exact value of the gradient of the underlining objective, thus only work for the finite sum scheme. For general convex stochastic optimization beyond a finite sum structure, Lan (2012) presented the first accelerated method (AC-SA algorithm) that achieves the optimal convergence rate for stochastic algorithms. All the above are variants of SGDM, but the update rules are indeed a bit different from the vanilla version of SGDM.

More recently, there have been researches on analyzing the vanilla version of SGDM, due to its great success in training neural networks. In particular, Sebbouh et al. (2020) showed that SGDM converges to the minimum of the objective in expectation, and the convergence rate is similar to SGD for convex smooth objectives. Liu et al. (2020) showed that SGDM enjoys a similar convergence rate as SGD for strongly-convex smooth objectives and for non-convex smooth objectives.

All of the above works rely on the differentiability of the objective. In contrast, our results in Section 4 imply that the vanilla SGDM enjoys a constant better asymptotic convergence guarantee compared to the best known guarantee of SGD for non-smooth convex optimization.

Online Gradient Descent with Momentum. Online convex optimization studies online decision making with convex loss functions. Some classic online first-order methods include online gradient descent, online mirror descent, regularized-follow-the-leader, online adaptive methods, etc Hazan et al. (2016); Duchi et al. (2011). The major difference of online optimization compared to stochas-

tic optimization is that the loss functions can be chosen adversarially in online optimization, while the loss functions in stochastic optimization come from an unknown stochastic i.i.d. distribution.

Online algorithms have achieved great success in machine learning training. In particular, the adaptive gradient methods (ADAGRAD) Duchi et al. (2011) adaptively scale the step-size for each dimension and can speed up the training of many machine learning problems. Later on, Kingma and Ba (2015) proposes ADAM, and its major novelty is to apply the exponential weighted average to gradient estimates, which is equivalent to momentum method for unconstrained problems. While ADAM is one of the most successful algorithms for training neural networks nowadays, Reddi et al. (2019) shows that the theoretical analysis of ADAM contains major technical issues and indeed ADAM may not even converge. After that, many variants of ADAM have been proposed Reddi et al. (2019); Chen et al. (2018); Huang et al. (2018), but most of the analysis requires the momentum parameter $\beta \to 0$, and their analyses are more complicated than ours. More recently, Alacaoglu et al. (2020) proposed a simple analysis for variants of ADAM that allows arbitrary momentum parameter β , but the regret bound is $1/(1-\beta)$ times larger than vanilla OGD.

Our analysis is inspired by Alacaoglu et al. (2020), whose analysis also does not require that $\beta \to 0$. Our paper extends their work to online mirror descent with momentum (OMDM), where we obtain a similar regret bound with a term $1/(1-\beta)$ times larger than vanilla OMD. Furthermore, we show that if the loss functions stochastic and drawn i.i.d., OGDM can indeed enjoy a *smaller regret bound* than vanilla OGD.

Online Allocation. Online allocation is an important problem in computer science and operations research with wide applications in practice. Many previous works of online allocation focus on the stochastic i.i.d. input model. In particular, Devanur and Hayes (2009) present a two-phase dual training algorithm for the AdWords problem (a special case of online allocation problem): estimating the dual variables by solving a linear program in the first exploration phase; and taking actions using the estimated dual variables in the second exploitation phase. They show that the proposed algorithm obtains regret of order $O(T^{2/3})$. Soon after, Feldman et al. (2010) present similar two-phases algorithms for more general linear online allocation problems with similar regret guarantees. Later on, Agrawal et al. (2014), Devanur et al. (2019) and Kesselheim et al. (2014) propose primal- and/or dual-based algorithms that dynamically update decisions by periodically solving a linear program using all data collected so far, and show that the proposed algorithms have $O(T^{1/2})$ regret, which turns out to be the optimal regret rate in T. More recently, Balseiro et al. (2020b,a); Li et al. (2020) propose simple dual descent algorithms for online allocation problems with stochastic inputs, which attain $O(T^{1/2})$ regret. The algorithm updates dual variables in each period in linear time and avoids solving large auxiliary programs. Our proposed algorithms fall into this category: the update per iteration can be efficiently computed and there is no need to solve large convex optimization problems.

2 Online Mirror Descent with Momentum

We consider the online convex optimization problem with T time horizons over a feasible set $\mathcal{U} \subseteq \mathbb{R}^m$, which is assumed to be convex and closed. For each time step t, an algorithm selects a solution $\mu_t \in \mathcal{U}$; following this, a convex loss function $w_t : \mathcal{U} \to \mathbb{R}$ is revealed, and the algorithm incurs a cost $w_t(\mu_t)$. We assume that the algorithm has access to subgradients $g_t \in \partial w_t(\mu_t)$. We remark that the subgradients $g_t \in \mathbb{R}^m$ are observed after taking the action. Then, for any $\mu \in \mathcal{U}$, we denote

Algorithm 1: Online Mirror Descent with Momentum (OMDM)

```
Input: initial solution \mu_1, momentum parameter \beta, step-size \eta, z_0 = 0 for t = 1, ..., T do

Play action \mu_t and receive subgradient g_t.

Compute z_t = \beta z_{t-1} + (1 - \beta)g_t.

Compute \mu_{t+1} = \min_{\mu \in \mathcal{U}} \left\{ z_t^\top \mu + \frac{1}{\eta} V_h(\mu, \mu_t) \right\} end
```

the regret of the online algorithm against a fixed static action $\mu \in \mathcal{U}$ by

$$\operatorname{Regret}(\mu) := \sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu).$$

We shall consider two different models for how the convex functions and their subgradients are generated. In the *adversarial model*, the functions $w_t(\mu)$ and the g_t subgradients are generated adversarially. The adversary can choose the functions adaptively and after observing the action chosen by the algorithm. In the *stochastic model*, the functions are drawn independently from a distribution that is unknown to the algorithm.

Algorithm 1 presents our main algorithm, which we dub online mirror descent algorithm with momentum. As we discuss below, our algorithm can be interpreted as adding momentum or inertia to the solution and can be thought of combining Polyak's heavy-ball Polyak (1964) with mirror descent. The algorithm has three parameters: a step size $\eta \geq 0$, a momentum parameter $\beta \in [0, 1)$, and a reference function $h: \mathcal{U} \to \mathbb{R}$. We assume the following for the reference function:

Assumption 1 (Assumptions on reference function). We assume

- 1. $h(\mu)$ is either differentiable or essentially smooth (Bauschke et al., 2001) in \mathbb{U} .
- 2. $h(\mu)$ is σ -strongly convex in $\|\cdot\|_2$ -norm in \mathbb{U} , i.e., $h(\mu_1) \geq h(\mu_2) + \nabla h(\mu_2)^{\top} (\mu_1 \mu_2) + \frac{\sigma}{2} \|\mu_1 \mu_2\|_2^2$ for any $\mu_1, \mu_2 \in \mathbb{U}$.

The algorithm maintains at each point in time an incumbent solution μ_t and an exponential average of past gradients z_{t-1} . The algorithm plays the action μ_t and receives a subgradient g_t . The exponential average of the gradients is updated and then the iterate moves in the opposite direction of the averaged gradient. Movement from the incumbent solution is penalized using the Bregman divergence $V_h(x,y) = h(x) - h(y) - \nabla h(y)^{\top}(x-y)$ as a notion of distance. Moreover, solutions are projected to the feasible set \mathcal{U} to guarantee feasibility. The advantage of using the Bregman divergence in the projection step is that, in many cases, it allows to better capture the geometry of the feasible set and, in some settings, leads to closed-form formulas for the projection.

We conclude by discussing some instantiations of the algorithm for some common reference functions. First, consider the case when the reference function is the squared-Euclidean norm. That is, suppose $h(\mu) = \frac{1}{2} \|\mu\|_2^2$. In this case, the update rule can be written as

$$\mu_{t+1} = \operatorname{Proj}_{\mathcal{U}}(\mu_t - \eta z_t),$$

where $\operatorname{Proj}_{\mathcal{U}}(x) = \min_{\mu \in \mathcal{U}} \|x - \mu\|_2$ denotes the projection of a point $x \in \mathbb{R}^m$ to the feasible set \mathcal{U} . In the absence of momentum, i.e., $\beta = 0$, we have that $z_t = g_t$ and our algorithms reduce to online sub-

gradient descent. Instead of taking a direction in the opposite direction of the current subgradient, when $\beta > 0$ our algorithm takes an exponentially smoothed average of the past subgradients.

Second, suppose the feasible set is the unit simplex $\mathcal{U} = \left\{ \mu \in \mathbb{R}_+^m : \sum_{j=1}^m \mu_j = 1 \right\}$, where \mathbb{R}_+ denotes the set of non-negative real numbers. In this case, a well-known good choice for the reference function is the negative entropy $h(\mu) = \sum_{j=1}^m \mu_j \log(\mu_j)$. When the initial solution lies in the relative interior of the feasible set, the update rule is given by

$$(\mu_{t+1})_j = \frac{(\mu_t)_j \exp(-\eta(z_t)_j)}{\sum_{i=1}^m (\mu_t)_i \exp(-\eta(z_t)_i)},$$

which can be efficiently computed. In the case when $\beta = 0$, the algorithm reduces to online entropic descent (Arora et al., 2012).

Finally, consider the case when the feasible set is the non-negative orthant, i.e., $\mathcal{U} = \mathbb{R}_+^m$. This case would play a key role in the online allocation application discussed in Section 5. In this case, when the reference function is the squared-Euclidean norm, we obtain that

$$\mu_{t+1} = \max(\mu_t - \eta z_t, 0),$$

while, in the case of the negative entropy reference function, we obtain whenever the initial solution is strictly positive that

$$(\mu_{t+1})_j = (\mu_t)_j \exp(-\eta(z_t)_j)$$
,

which leads to a multiplicative weights update algorithm with momentum.

2.1 Interpretations

We provide three possible interpretations of our algorithm. The first is a statistical interpretation and hinges on the fact that, as we discussed, the algorithm averages past gradients. In particular, applying the update repeatedly together with the fact that $z_0 = 0$ we obtain that

$$z_t = (1 - \beta) \sum_{s=1}^t \beta^{t-s} g_s.$$
 (1)

Therefore, z_t is the exponential average of past gradients. In the absence of momentum, i.e., $\beta = 0$, we have that $z_t = g_t$ and our algorithm reduces to online mirror descent. Instead of taking a direction in the opposite direction of the current subgradient, when $\beta > 0$ our algorithm takes an exponentially smoothed average of the past subgradients. In the case of stochastic input, averaging introduces a variance versus bias trade-off. Intuitively, on the one hand, averaging past gradients can reduce variance as independent noises accumulated in past periods would tend to cancel each other. On the other hand, as the solution changes so does the gradients and averaging gradients from distant points introduce bias. In Section 4, we shall show that in some circumstances the variance-reduction effect can dominate and averaging can lead to better regret bounds.

The second interpretation is from the *optimal control* perspective. The dual mirror descent update in Algorithm 1 can be written as

$$\nabla h(\tilde{\mu}_t) = \nabla h(\mu_t) - \eta z_t , \quad \mu_{t+1} = \arg\min_{\mu \in \mathcal{U}} V_h(\mu, \tilde{\mu}_t) , \qquad (2)$$

which can be easily verified by noticing that the optimality condition of Algorithm 1 and (19) are both $0 \in \nabla h(\mu_{t+1}) - \nabla h(\mu_t) + \eta z_t + \mathcal{N}_{\mathcal{U}}(\mu_{t+1})$, where $\mathcal{N}_{\mathcal{U}}(\mu_{t+1}) = \{g \in \mathbb{R}^m | g^{\top}(\mu - \mu_{t+1}) \leq 1\}$

0 for all $\mu \in \mathcal{U}$ } is the normal cone of \mathcal{U} at μ_{t+1} . Therefore, the solution μ_t is first mapped to the "dual" space using the gradient of the reference function to a solution $y_t = \nabla h(\mu_t)$, then the dual solution is updated in the direction of the average gradient to obtain a new solution $\tilde{y}_t = \nabla h(\tilde{\mu}_t)$, and solutions are finally projected back to the feasible set \mathcal{U} using the Bregman divergence as a notion of distance. Expanding the dual update, we can write

$$\tilde{y}_t = y_t - \underbrace{\eta(1-\beta)g_t}_{\text{proportional term}} + \underbrace{\beta(1-\beta)\sum_{s=1}^{t-1}\beta^{t-1-s}g_s}_{\text{integral term}}.$$

At the time t, the gradient g_t can be interpreted as the "error" of the system: the algorithm seeks to reduce the error to zero to minimize the function by adjusting the dual control variable y_t . Therefore, we can interpret the algorithm as a sequential PI controller with a proportional and an integral term. The P term is proportional to the current value of error: if the error is positive, we need to reduce the control to decrease the error. The I term takes an exponential average of all past errors. Two observations are in error. First, we do not consider a derivative term since this is rarely used in practice because it can negatively impact the stability of the system. Second, our is a sequential controller as it adjusts the rate of change of the control variable as opposed to acting directly on the control variable. We remark that adjusting the rate of change is common in practice, so we adopt this interpretation (see, e.g., Zhang et al. 2016; Smirnov et al. 2016; Conitzer et al. 2021; Balseiro and Gur 2019). If we follow the textbook definition of a PI controller our P term is actually an I term and our I term can be thought of as a iterated integral term. We provide further explanations in the context of our online allocation application in Section 5.

The third interpretation is from *physics* and relies on the fact that when $\beta \in (0,1)$ we are adding momentum or inertia to the current solution. To see this, consider the case where the feasible set is $\mathcal{U} = \mathbb{R}^m$ so that there is no projection. In this case, we can write the update rule of the solution as

$$\nabla h(\mu_{t+1}) = \nabla h(\mu_t) - \eta z_t.$$

Therefore, because $-\eta z_{t-1} = \nabla h(\mu_t) - \nabla h(\mu_{t-1})$. We obtain that

$$\nabla h(\mu_{t+1}) = \nabla h(\mu_t) + \beta \left(\nabla h(\mu_t) - \nabla h(\mu_{t-1}) \right) - \eta (1 - \beta) g_t.$$

The term $\nabla h(\mu_t) - \nabla h(\mu_{t-1})$ can be interpreted as inertia or momentum: once the solution is moving it will tend to move in the same direction. To enhance this physical interpretation, let $\Delta_t = \nabla h(\mu_t) - \nabla h(\mu_{t-1})$ be the velocity of the solution. Some algebra yields that

$$\underline{\Delta_{t-1} - \Delta_t}_{\text{acceleration}} = -\underbrace{(1-\beta)\Delta_{t-1}}_{\text{friction}} - \underbrace{\eta(1-\beta)g_t}_{\text{conservative force}}.$$

Interpreting the left-hand side as the acceleration of the solution, i.e., the change of velocity, by Newtown's second law, we can interpret the right-hand size as the net force acting on the solution. The solution can thus be interpreted as a "heavy ball" subject to two forces. The first is friction, which increases proportionally to the speed and tends to bring the ball to rest. The second is a conservative force induced by the field w_t . The ball can be thought as moving through the graph of the function w_t with its acceleration determined by how steep the function is at the current point. In particular, if $w_t(\mu) = \mu^2$ the acting force behaves like a spring and the algorithm traces the dynamics of a damped oscillator. Therefore, while momentum can reduce the time taken to approach the orbit of an optimal solution, it can sometimes lead to undesirable oscillations.

3 Theoretical Results for Adversarial Inputs

The next theorem presents our main regret bound on Algorithm 1 for online convex optimization with adversarial inputs. We make no assumptions on the input other than the fact that the subgradients are bounded.

Theorem 1. Consider the sequence of convex functions $w_t(\mu)$. Let $g_t \in \partial_{\mu} w_t(\mu_t)$ be a sub-gradient and suppose that the dual variables are updated according to Algorithm 1. Suppose sub-gradients are bounded by $||g_t||_2 \leq G$ and the reference function is σ -strongly convex with respect to $||\cdot||_2$ -norm. Then, for every $\mu \in \mathcal{U}$ we have

Regret
$$(\mu) := \sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu) \le \sum_{t=1}^{T} g_t^{\top}(\mu_t - \mu)$$

$$\le \frac{1}{\eta} V_h(\mu, \mu_1) + \frac{G^2}{(1-\beta)\sigma} \eta(T-1) + \frac{G^2}{2(1-\beta)^2 \sigma} \eta. \tag{3}$$

The proof of Theorem 1 can be found in Section 3.1. We first remark that by choosing a step size of order $\eta \sim T^{-1/2}$ we obtain a regret bound of $O(T^{1/2})$, which matches the worst-case bounds for adversarial online convex optimization (see, e.g., Theorem 3.2 of Hazan et al. (2016)). To put our results in perspective, it is instructive to compare our bound to that of online mirror descent without momentum. When $\beta = 0$, it is known that (Balseiro et al., 2020a, Appendix G)

Regret
$$(\mu) \le \frac{1}{\eta} V_h(\mu, \mu_1) + \frac{G^2}{\sigma} \eta T$$
.

Therefore, compared to the case without momentum we obtain an extra factor $(1-\beta)$ in the second term's denominator together with an extra third term. When step-sizes are chosen to be of order $\eta \sim T^{-1/2}$ the third term is negligible for large enough T. The second term, however, meaningfully contributes to regret and is increasing in β . This suggests that, in the case of adversarial input, momentum deteriorates performance.¹ This follows because averaging past gradients does not necessarily reduce variance and, instead, averaging can increase the bias and force the algorithm to overshoot. Interestingly, in Section 4 we show that momentum can improve performance when the input is stochastic. In appendix B we present an example showing that the performance of momentum method can be worse than without it in the adversarial case.

As a byproduct of the analysis, we can provide the following data-dependent bound, which is useful for the analysis of stochastic inputs and online allocation algorithm in the next sections:

Regret
$$(\mu) \le \frac{1}{\eta} V_h(\mu, \mu_1) + \frac{\eta}{(1-\beta)\sigma} \sum_{t=1}^{T-1} ||z_t||_2^2 + \frac{G^2 \eta}{2(1-\beta)^2 \sigma}.$$
 (4)

Finally, when the feasible set is bounded, we can provide an alternative bound that dispenses with the $(1-\beta)^2$ term in the denominator of the third term. This alternative bound is

Regret
$$(\mu) \le \frac{1}{\eta} V_h(\mu, \mu_1) + \frac{\beta G^2}{(1-\beta)\sigma} \eta T + \frac{\beta G}{1-\beta} \|\mu - \mu_T\|_2$$
.

The factor $T/(1-\beta)$ in the second term suggests that the regret bound grows unbounded as $\beta \to 1$. This term can be improved to $\sum_{t=1}^{T} (1-\beta^t)^2/(1-\beta)$, which goes to zero as $\beta \to 1$ by strengthening (6) to $||z_t||_2^2 \le (1-\beta^t)^2 G^2$. With this modification, the second term in the regret bound is single-peaked in β .

3.1 Proof of Theorem 1

We first show two properties of mirror descent that are useful for the analysis. The first property is that the dual solutions produced by our algorithm are stable in the sense that the change in the multipliers from one period to the next is proportional to the step size and the norm of z_t . Proofs of missing results are located in the appendix.

Lemma 1. It holds for any $t \ge 1$ that $\|\mu_{t+1} - \mu_t\|_2 \le \frac{\sqrt{2}}{\sigma} \eta \|z_t\|_2$.

The next lemma is an extension of the three-point property in convex optimization.

Lemma 2. It holds for any $t \ge 1$ that

$$\langle z_t, \mu_t - \mu \rangle \le \frac{\eta}{2\sigma} \|z_t\|_2^2 + \frac{1}{\eta} V_h(\mu, \mu_t) - \frac{1}{\eta} V_h(\mu, \mu_{t+1}) .$$

We now are ready to prove Theorem 1.

Proof of Theorem 1. Because the functions $w_t(\mu)$ are convex we obtain that

$$\sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu) \le \sum_{t=1}^{T} g_t^{\top} (\mu_t - \mu).$$

We bound the latter expression in the sequel. By Lemma 1 of Alacaoglu et al. (2020) we have that

$$g_t^{\top}(\mu_t - \mu) = z_{t-1}^{\top}(\mu_{t-1} - \mu) - \frac{\beta}{1-\beta}z_{t-1}^{\top}(\mu_t - \mu_{t-1}) + \frac{1}{1-\beta}\left(z_t^{\top}(\mu_t - \mu) - z_{t-1}^{\top}(\mu_{t-1} - \mu)\right).$$

Summing from t = 1 to t = T and using that $z_0 = 0$ we obtain that

$$\sum_{t=1}^{T} g_{t}^{\top}(\mu_{t} - \mu) = \underbrace{\sum_{t=1}^{T-1} z_{t}^{\top}(\mu_{t} - \mu)}_{S_{1}} + \underbrace{\frac{\beta}{1-\beta}}_{1-\beta} \underbrace{\sum_{t=1}^{T} z_{t-1}^{\top}(\mu_{t-1} - \mu_{t})}_{S_{2}} + \underbrace{\frac{1}{1-\beta}}_{S_{3}} \underbrace{z_{T}^{\top}(\mu_{T} - \mu)}_{S_{3}}.$$

We bound each term at a time. For the second term, we have

$$z_t^{\top}(\mu_t - \mu_{t+1}) \leq ||z_t||_2 ||\mu_t - \mu_{t+1}||_2 \leq \frac{\sqrt{2}}{\sigma} \eta ||z_t||_2^2$$
.

by Cauchy-Schwartz and Lemma 1. Therefore, we obtain that

$$S_2 = \sum_{t=1}^{T} z_{t-1}^{\top} (\mu_{t-1} - \mu_t) = \sum_{t=1}^{T-1} z_{t}^{\top} (\mu_{t} - \mu_{t+1}) \le \frac{\sqrt{2}}{\sigma} \eta \sum_{t=1}^{T-1} \|z_{t}\|_{2}^{2}.$$

For the first term, we use Lemma 2 to obtain

$$S_1 = \sum_{t=1}^{T-1} z_t^{\top} (\mu_t - \mu) \le \frac{1}{\eta} V_h(\mu, \mu_1) - \frac{1}{\eta} V_h(\mu, \mu_T) + \frac{\eta}{2\sigma} \sum_{t=1}^{T-1} \|z_t\|_2^2.$$

For the last term, we have that

$$S_{3} = z_{T}^{\top}(\mu_{T} - \mu) \leq ||z_{T}||_{2} ||\mu_{T} - \mu||_{2}$$

$$\leq \frac{\eta}{2(1 - \beta)\sigma} ||z_{T}||_{2}^{2} + \frac{(1 - \beta)\sigma}{2\eta} ||\mu_{T} - \mu||_{2}^{2}$$

$$\leq \frac{\eta}{2(1 - \beta)\sigma} ||z_{T}||_{2}^{2} + \frac{(1 - \beta)}{\eta} V_{h}(\mu, \mu_{T}),$$

where the first inequality follows by Hölder, the second by the Peter-Paul inequality, and the third inequality is because h is strongly convex.

Putting everything together, we obtain that

$$S_1 + \frac{\beta}{1-\beta}S_2 + \frac{1}{1-\beta}S_3 \le \frac{1}{\eta}V_h(\mu,\mu_1) + \frac{\eta}{\sigma}\left(\frac{\beta}{1-\beta} + \frac{1}{2}\right)\sum_{t=1}^{T-1}\|z_t\|_2^2 + \frac{\eta}{2(1-\beta)^2\sigma}\|z_T\|_2^2.$$
 (5)

We conclude by bounding the term $||z_t||_2^2$. We have that

$$||z_t||_2^2 = \left\| (1-\beta) \sum_{s=1}^t \beta^{t-s} g_s \right\|_2^2 = (1-\beta)^2 \sum_{j=1}^m \left(\sum_{s=1}^t \beta^{t-s} (g_s)_j \right)^2$$

$$\leq (1-\beta)^2 \sum_{j=1}^m \left(\sum_{s=1}^t \beta^{t-s} \right) \left(\sum_{s=1}^t \beta^{t-s} (g_s)_j^2 \right) = (1-\beta)(1-\beta^t) \sum_{s=1}^t \beta^{t-s} ||g_s||_2^2$$

$$\leq (1-\beta^t)^2 G^2 \leq G^2, \tag{6}$$

where the first inequality follows by Cauchy-Schwartz, the third equation by exchanging the order of summation and using the formula for the geometric sum $\sum_{s=1}^{t} \beta^{t-s} = (1-\beta^t)/(1-\beta)$, the second inequality because $||g_s||_2 \leq G$, and the last because $\beta \in [0,1)$. The result follows by using the previous bound on (5) together with the fact that $\beta/(1-\beta) + 1/2 \leq 1/(1-\beta)$.

4 Theoretical Results for Stochastic Inputs

In the previous section, we obtained a regret bound that deteriorates as we increase the level of momentum. In this section, we provide better regret bounds for online gradient descent with stochastic inputs that can take advantage of the alluded averaging effects of exponential smoothing.

Here we consider unconstrained online gradient descent, where the iterations have an update rule

$$z_t = \beta z_{t-1} + (1 - \beta)g_t, \mu_{t+1} = \mu_t - \eta z_t.$$
 (7)

Furthermore, we consider the stochastic inputs model, where the convex function $w_t(\mu)$ is chosen randomly from a probability distribution that is unknown to the algorithm. We assume the expectation of the convex function exists for any μ , i.e., $\bar{w}(\mu) = \mathbb{E}[w_t(\mu)]$ exists and has a finite minimal value $\bar{w}^* = \inf_{\mu \in \mathbb{R}^m} \bar{w}(\mu)$.

Theorem 2. Consider online gradient descent (7) with stochastic input model, then we have

$$\mathbb{E}\left[\sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu)\right] \leq \frac{1}{2\eta} \|\mu - \mu_1\|_2^2 + \frac{\eta G^2 T}{1+\beta} + \frac{2\beta \left(\bar{w}(\mu_0) - \bar{w}^*\right)}{1-\beta^2} + \frac{G^2 \eta}{2(1-\beta)^2}.$$

In particular, when we choose $\eta \sim T^{-1/2}$, we have

$$\mathbb{E}\left[\sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu)\right] \le \frac{1}{2\eta} \|\mu - \mu_1\|_2^2 + \frac{\eta G^2 T}{1+\beta} + O(1) . \tag{8}$$

For the case of stochastic inputs we are able to show better convergence guarantees by leveraging the variance-reduction effect of averaging past gradients. In particular, our analysis revolves around obtaining tighter bounds for the cumulative errors of the squared averages $||z_t||_2^2$. In the adversarial case, the best bound we can show is $||z_t||_2^2 \leq G^2$ as given in equation (6). Below we give an informal derivation of our bound.

Denote by $\bar{g}_t = \partial w(\mu_t) = \mathbb{E}[g_t|\mu_t]$ the expected value of the gradient. Then, using (1) we can write

$$\mathbb{E} \|z_{t}\|^{2} = (1 - \beta)^{2} \mathbb{E} \left\| \sum_{s=1}^{t} \beta^{t-s} g_{s} \right\|^{2} = (1 - \beta)^{2} \mathbb{E} \left\| \sum_{s=1}^{t} \beta^{t-s} \bar{g}_{s} + \sum_{s=1}^{t} \beta^{t-s} (g_{s} - \bar{g}_{s}) \right\|^{2}$$

$$= (1 - \beta)^{2} \mathbb{E} \left\| \sum_{s=1}^{t} \beta^{t-s} \bar{g}_{s} \right\|^{2} + (1 - \beta)^{2} \mathbb{E} \left\| \sum_{s=1}^{t} \beta^{t-s} (g_{s} - \bar{g}_{s}) \right\|^{2}.$$
variance

By using that martingale differences are orthogonal, we can bound the variance term as follows

$$\mathbb{E}\left\|\sum_{s=1}^{t} \beta^{t-s} (g_s - \bar{g}_s)\right\|^2 = \sum_{s=1}^{t} \beta^{2(t-s)} \mathbb{E}\left\|g_s - \bar{g}_s\right\|^2 \le G^2 \sum_{s=0}^{t-1} \beta^{2s} \le G^2 \sum_{s=0}^{\infty} \beta^{2s} = \frac{G^2}{1 - \beta^2},$$

where the first inequality follows because $\mathbb{E} \|g_s - \bar{g}_s\|^2 \leq G^2$ together with performing a change of variables in the summation and the second inequality follows from extending the sum to infinity.

The bound on the bias term is a trickier, since the time series of expected gradients \bar{g}_t is serially correlated. Notice that eventually as t grows large, the iterates μ_t will get close enough to the optimal solution μ^* and remain in the orbit of the optimal solution performing a mean-reverting random walk of radius proportional $\eta \|z_t\|_2$. Assuming that gradients are bounded, it should take the algorithm $\|\mu^0 - \mu^*\|/\eta$ time periods to get to the orbit of the optimal solution. When $\eta \sim T^{-1/2}$ the algorithm should spend a large fraction of the time in the orbit of the optimal solution performing a random walk. As a result, for t large enough the expected gradients \bar{g}_t should be almost independent from each other. Similar to the variance term, we can bound the bias term as follows

$$\mathbb{E} \left\| \sum_{s=1}^{t} \beta^{t-s} \bar{g}_s \right\|^2 \approx \sum_{s=1}^{t} \beta^{2(t-s)} \mathbb{E} \left\| \bar{g}_s \right\|^2 \le \frac{G^2}{1 - \beta^2}.$$

Using the above bounds, we obtain that

$$\mathbb{E} \|z_t\|^2 \lesssim 2 \frac{G^2 (1-\beta)^2}{1-\beta^2} = 2 \frac{G^2 (1-\beta)}{1+\beta}.$$

As a result, we obtain that the squared averaged gradients should converge to zero at rate of $(1-\beta)$ for sufficiently large t. Theorem 2 formalizes this analysis and, in the process, introduces an extra term to account for the "transient" time it takes to reach the orbit.

It is instructive to compare our regret bound to that of standard stochastic gradient descent without momentum (see, e.g., Hazan 2008):

$$\sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu) \le \frac{1}{2\eta} \|\mu - \mu_1\|_2^2 + \eta G^2 T , \qquad (9)$$

Here we can see that momentum can provide a constant $1 + \beta$ better factor in the second term. When η is chosen optimally according to $\eta \sim (1+\beta)^{1/2}T^{-1/2}$, this leads to a $\sqrt{1+\beta}$ factor better regret bound than standard online gradient descent.

Remark 1. Theorem 2 automatically provides a convergence guarantee for offline stochastic gradient descent with momentum. In the offline setting, the goal is to solve the minimization problem:

$$\min_{\mu} \bar{w}(\mu)$$

Taking the expectation (by noticing w_t is chosen after μ_t) and choosing $\mu = \mu^*$ as the minimizer of $\bar{w}(\mu)$ in (8), we obtain by convexity that

$$\bar{w}\left(\frac{1}{T}\sum_{t=1}^{T}\mu_{t}\right) - \bar{w}(\mu^{*}) \leq \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}w_{t}(\mu_{t}) - w_{t}(\mu)\right] \leq \frac{1}{2T\eta}\|\mu - \mu_{1}\|_{2}^{2} + \frac{\eta G^{2}}{(1+\beta)} + O\left(\frac{1}{T}\right).$$

Recall that the stochastic gradient descent for convex nonsmooth optimization has convergence guarantees as in (9). This showcases that stochastic gradient descent has $\sqrt{1+\beta}$ better convergence rate with an optimal choice of step-size of order $\eta \sim (1+\beta)^{1/2}T^{-1/2}$. As far as we know, this is the first result to showcase that momentum can be helpful for the convergence of stochastic gradient descent for convex nonsmooth optimization.

4.1 Proof of Theorem 2

From the update rule (7) we have:

$$||z_t||_2^2 = ||\beta z_{t-1} + (1-\beta)g_t||_2^2 = \beta^2 ||z_{t-1}||^2 + (1-\beta)^2 ||g_t||_2^2 + 2\beta(1-\beta)z_{t-1}^\top g_t.$$

Subtracting $\beta^2 ||z_t||_2^2$ from both sides, using that $(1 - \beta^2) = (1 - \beta)(1 + \beta)$, and dividing by $1 - \beta$ we obtain that

$$(1+\beta)\|z_t\|_2^2 = \frac{\beta^2}{1-\beta} \left(\|z_{t-1}\|^2 - \|z_t\|^2 \right) + (1-\beta)\|g_t\|_2^2 + 2\beta z_{t-1}^\top g_t.$$

Summing over t = 1, ..., T and telescoping the first term, we obtain that

$$(1+\beta) \sum_{t=1}^{T} \|z_{t}\|_{2}^{2} = \frac{\beta^{2}}{1-\beta} (\|z_{0}\|^{2} - \|z_{T}\|^{2}) + (1-\beta) \sum_{t=1}^{T} \|g_{t}\|_{2}^{2} + 2\beta \sum_{t=1}^{T} z_{t-1}^{\top} g_{t}$$

$$\leq (1-\beta) \sum_{t=1}^{T} \|g_{t}\|_{2}^{2} + 2\beta \sum_{t=2}^{T} z_{t-1}^{\top} g_{t}, \qquad (10)$$

where we used that $z_0 = 0$ and norms are non-negative.

On the other hand, notice that w_t is convex, thus its expectation \bar{w} is also convex. Let $\bar{g}_t = \mathbb{E}[g_t] \in \partial \bar{w}(\mu_t)$, then it holds that

$$\bar{w}(\mu_{t-1}) \ge \bar{w}(\mu_t) + \bar{g}_t^{\top}(\mu_{t-1} - \mu_t) = \bar{w}(\mu_t) + \eta \bar{g}_t^{\top} z_{t-1},$$

which implies after reorganizing that

$$\bar{g}_t^{\top} z_{t-1} \le \frac{1}{\eta} \left(\bar{w}(\mu_{t-1}) - \bar{w}(\mu_t) \right) .$$
 (11)

Note that by the tower rule of expectation we have that

$$\mathbb{E}[z_{t-1}^{\top}g_t] = \mathbb{E}[\mathbb{E}[z_{t-1}^{\top}g_t|(z_{t-1},\mu_t)]] = \mathbb{E}[z_{t-1}^{\top}\mathbb{E}[g_t|(z_{t-1},\mu_t)]] = \mathbb{E}[z_{t-1}^{\top}\bar{g}_t],$$

because $g_t \in \partial w_t(\mu_t)$ and w_t is independent of (z_{t-1}, μ_t) . Therefore, taking expectations in (10), using (11), and rearranging we obtain that

$$(1+\beta)\sum_{t=1}^{T} \mathbb{E}\left[\|z_{t}\|_{2}^{2}\right] \leq (1-\beta)\sum_{t=1}^{T} \mathbb{E}[\|g_{t}\|_{2}^{2} + \frac{2\beta}{\eta}\left(\bar{w}(\mu_{1}) - \mathbb{E}[\bar{w}(\mu_{T})]\right),$$

Using that gradients are bounded by $||g_t||_2 \leq G$, we conclude that

$$\sum_{t=1}^{T} \mathbb{E}\left[\|z_{t}\|_{2}^{2}\right] \leq \frac{(1-\beta)G^{2}T}{1+\beta} + \frac{2\beta}{\eta(1+\beta)} \left(\bar{w}(\mu_{1}) - \mathbb{E}[\bar{w}(\mu_{T})]\right).$$

The result follows from substituting into (4) and noticing that $\sigma = 1$, $V_h(\mu, \mu_1) = \frac{1}{2} \|\mu - \mu_1\|^2$ and $\mathbb{E}[\bar{w}(\mu_T)] \geq \bar{w}^*$.

Remark 2. The analysis above can be extended to mirror descent, but require additional assumptions on $\bar{w}(\mu)$ and the reference function $h(\mu)$, that is, $\bar{w}(\mu)$ and $h(\mu)$ are second-order differentiable and the matrix $\nabla^2(\bar{w}(\nabla h^*(y))\nabla^2 h^*(y))$ is normal for any y in the domain, where h^* is the conjugate function of h. This condition holds, for example, when $h(\mu)$ is squared-Euclidean norm square, i.e., for gradient descent case, or when both \bar{w}_t and h are coordinate-wisely separable functions.

To see this, we know under the above condition that

$$\nabla(\nabla \bar{w}(\nabla h^*(y))) = \nabla^2(\bar{w}(\nabla h^*(y))\nabla^2 h^*(y))$$

is a positive semidefinite matrix, because the product of two positive semidefinite matrices is positive semidefinite iff the product is normal Meenakshi and Rajian (1999). Thus, there exists a function $\phi: \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \phi(y) = \nabla \bar{w}(\nabla h^*(y))$ by Theorem 10.9 from Apostol (1969). Moreover, the function $\phi(y)$ is convex since $\nabla(\nabla \bar{w}(\nabla h^*(y)))$ is positive semidefinite.

Then, we have

$$\mathbb{E}\left[\bar{g}_{t}^{\top} z_{t-1}\right] = \frac{1}{\eta} \nabla \bar{w}(\mu_{t})^{\top} \left(\nabla h(\mu_{t-1}) - \nabla h(\mu_{t})\right)$$
$$= \frac{1}{\eta} \nabla \bar{w}(\nabla h^{*}(y_{t}))^{\top} \left(y_{t-1} - y_{t}\right) \leq \frac{1}{\eta} (\phi(y_{t-1}) - \phi(y_{t})) , \qquad (12)$$

where $y_t = \nabla h(\mu_t)$, and the last inequality uses the convexity of ϕ and $\nabla \phi(y) = \nabla \bar{w}(\nabla h^*(y))$. We can replace (11) with (12), and everything else in the proof of Theorem 2 follows.

5 Application: Online Allocation Problem

As an application of our algorithm, we consider a generic online allocation problem with a finite horizon of T time periods and resource constraints. Our setup is similar to that of Balseiro et al. (2020a). At time t, the decision maker receives a request $\gamma_t = (f_t, b_t, \mathcal{X}_t) \in \mathbb{S}$ where $f_t : \mathcal{X}_t \to \mathbb{R}_+$ is a non-negative reward function, $b_t : \mathcal{X}_t \to \mathbb{R}_+^m$ is an non-negative resource consumption function, and $\mathcal{X}_t \subset \mathbb{R}_+^d$ is a compact set. We denote by \mathbb{S} the set of all possible requests that can be received. After observing the request, the decision maker takes an action $x_t \in \mathcal{X}_t \subseteq \mathbb{R}^d$ that leads to reward $f_t(x_t)$ and consumes $b_t(x_t)$ resources. The total amount of resources is $B \in \mathbb{R}_+^m$ with $B_j > 0$ for all j. Because $b_t(\cdot) \geq 0$ resource availability only reduces with time, we also assume that there exists a void action $0 \in \mathcal{X}_t$ that consumes no resources, i.e., $b_t(0) = 0$. This guarantees that the problem always admits a feasible solution. We denote by $\vec{\gamma} = (\gamma_1, \dots, \gamma_T)$ the vector of inputs over time $1, \dots, T$.

Online allocation problems have a plethora of applications in practice, including capacity allocation problems in airline and hotel revenue management (Bitran and Caldentey, 2003; Talluri and Van Ryzin, 2006; Gallego et al., 2019), allocation of advertising opportunities in internet advertising markets (Karp et al., 1990; Devanur and Hayes, 2009; Feldman et al., 2009, 2010), bidding in repeated auctions with budgets (Balseiro and Gur, 2019), personalized assortment optimization in online retailing marketplaces (Bernstein et al., 2015; Golrezaei et al., 2014), etc.

In contrast to the online convex optimization setting, we benchmark our algorithm against the reward of the optimal solution when the request sequence $\vec{\gamma}$ is known in advance, i.e., the offline optimum problem:

$$OPT(\vec{\gamma}) = \max_{x:x_t \in \mathcal{X}_t} \sum_{t=1}^{T} f_t(x_t)$$

$$s.t. \sum_{t=1}^{T} b_t(x_t) \le B.$$
(13)

This benchmark is more reasonable for online allocation problems because it captures the best possible performance under full information of all requests. We remark that, in contrast to the online convex optimization setting, the optimal actions of the offline problem are time dependent. Therefore, the benchmark for online allocation is different to that of OCO since a fixed static action can perform poorly in this setting.

We consider a data-driven regime in which requests are drawn independently from a probability distribution $\mathcal{P} \in \Delta(\mathcal{S})$ that is unknown to the decision maker, where $\Delta(\mathcal{S})$ is the space of all probability distributions over support set \mathcal{S} . An online algorithm A makes, at time t, a real-time decision x_t based on the current request $(f_t, b_t, \mathcal{X}_t)$ and the history. We define the reward of an algorithm for input $\vec{\gamma}$ as

$$R(A|\vec{\gamma}) = \sum_{t=1}^{T} f_t(x_t),$$

where x_t is the action taken by algorithm A at time t. Moreover, the algorithm A must satisfy constraints $\sum_{t=1}^{T} b_t(x_t) \leq B$ and $x_t \in \mathcal{X}$ for every $t \leq T$.

We measure the regret of an algorithm as the worst-case difference over distributions in $\Delta(S)$,

Algorithm 2: Dual-Based Online Algorithm for Online Allocation

Input: Total time periods T, initial resources $B_1 = B$, step-size η , momentum parameter β , $z_0 = 0$, initial dual solution μ_1 .

for $t = 1, \dots, T$ do

Receive request $(f_t, b_t, \mathfrak{X}_t)$.

Make the primal decision x_t and update the remaining resources B_t :

$$\tilde{x}_t = \arg\max_{x \in \mathcal{X}_t} \left\{ f_t(x) - \mu_t^\top b_t(x) \right\} ,$$

$$x_t = \begin{cases} \tilde{x}_t & \text{if } b_t(\tilde{x}_t) \le B_t \\ 0 & \text{otherwise} \end{cases} ,$$

$$B_{t+1} = B_t - b_t(x_t).$$
(14)

Obtain a sub-gradient of the dual function:

$$g_t = -b_t(x_t) + \frac{B}{T} .$$

Compute the weighted average sub-gradient: $z_t = \beta z_{t-1} + (1-\beta)g_t$. Update the dual variable by a mirror step: $\mu_{t+1} = \min_{\mu \geq 0} \left\{ z_t^\top \mu + \frac{1}{\eta} V_h(\mu, \mu_t) \right\}$.

end

between the expected performance of the benchmark and the algorithm:

$$\operatorname{Regret}\left(A\right) = \sup_{\mathcal{P} \in \Delta(\mathcal{S})} \left\{ \mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} \left[\operatorname{OPT}(\vec{\gamma}) - R(A|\vec{\gamma}) \right] \right\} \,.$$

5.1 Algorithm

Algorithm 2 presents an algorithm for the online allocation, which is inspired by other dual-descent algorithms considered in the literature (see, e.g., Gupta and Molinaro 2016; Agrawal and Devanur 2014; Devanur et al. 2019; Li and Ye 2019; Balseiro et al. 2020b; Li et al. 2020). The algorithm maintains a dual variable $\mu_t \in \mathbb{R}^m$ for each resource that is updated using online mirror descent with momentum.

To motivate our algorithm, we briefly describe the dual problem of OPT in which we move the constraints to the objective using a vector of Lagrange multipliers $\mu \geq 0$. For $\mu \in \mathbb{R}_+^m$ we define

$$f_t^*(\mu) := \sup_{x \in \mathcal{X}_t} \{ f_t(x) - \mu^\top b_t(x) \},$$
 (15)

as the optimal opportunity-cost-adjusted reward of request γ_t . For a fixed input $\vec{\gamma}$, define the Lagrangian dual function $D(\mu|\vec{\gamma}):\mathbb{R}^m_+\to\mathbb{R}$ as $D(\mu|\vec{\gamma}):=\sum_{t=1}^T f_t^*(\mu)+B^\top\mu$. Weak duality implies that $D(\mu|\vec{\gamma})$ provides an upper bound on $\mathrm{OPT}(\vec{\gamma})$, i.e., it holds for every $\mu\in\mathbb{R}^m_+$ that $\mathrm{OPT}(\vec{\gamma})\leq D(\mu|\vec{\gamma})$. Algorithm 2 thus runs online mirror descent with momentum on the sequence of functions

$$w_t(\mu) := f_t^*(\mu) + (B/T)^\top \mu.$$
 (16)

At time t, the algorithm receives a request $(f_t, b_t, \mathcal{X}_t)$, and computes the optimal response \tilde{x}_t that maximizes the reward adjusted by the opportunity cost of consuming resources. It then takes this

action (i.e., $x_t = \tilde{x}_t$) if the action does not exceed the resource constraint, otherwise it takes a void action (i.e., $x_t = 0$). By Danskin's theorem, it follows that $g_t := -b_t(\tilde{x}_t) + B/T$ is a sub-gradient of $w_t(\mu)$ at μ_t . The algorithm then utilizes the subgradients g_t to update the dual variable using online mirror descent with momentum. Because dual variables are non-negative, we take the the feasible set for OMDM to be $\mathcal{U} = \mathbb{R}^m_+$.

Before presenting the performance guarantees of our algorithm, we discuss the connection to PID controllers. For simplicity, consider the squared-Euclidean norm reference function—a similar discussion applies to other popular choices such as the negative entropy. For this case, the update rule can be written as

$$\mu_{t+1} = \max\left(\mu_t - \underbrace{\eta(1-\beta)g_t}_{\text{proportional term}} + \underbrace{\beta(1-\beta)\sum_{s=1}^{t-1}\beta^{t-1-s}g_s}_{\text{integral term}}, 0\right).$$

Here, the error terms have a clear physical interpretation:

$$g_t = \underbrace{B/T}_{\text{target spend}} - \underbrace{b_t(x_t)}_{\text{actual spend}}.$$

When resource constraints are binding, the algorithm should aim to deplete resources evenly over time, so B/T can be interpreted as the target spend, i.e., how much resources should be spend on average per time period. The second term is the actual spend, i.e., how much resources are actually spend per time period. The difference of these two terms gives the error in expenditures. As a result, our algorithm can be understood as running a PI controller on the difference between target and actual spend: it adjusts the control variables μ_t to make the error g_t as small as possible. We remark that PID controllers have been extensively used in online markets, albeit with no theoretical guarantees (Tashman et al., 2020; Zhang et al., 2016; Smirnov et al., 2016; Yang et al., 2019).

In many circumstances, it is more appropriate that the control variables non-linearly. Let ℓ_j : $\mathbb{R}_+ \to \mathbb{R}$ be continuous and strictly increasing functions. Then, one could consider running the PI controller on the transformed control variables $\ell_j((\mu_t)_j)$, which would yield the update rule

$$(\mu_{t+1})_j = \ell_j^{-1} \left(\max \left(\ell_j(\mu_t)_j \right) - \eta(z_t)_j, 0 \right) . \tag{17}$$

By comparing (17) to (19) it is easy to see that such non-linear controllers can be incorporated to our model by considering the reference function $h(\mu) = \sum_{j=1}^{j} \int_{0}^{\mu_{j}} \ell_{j}(s) ds$ in online mirror descent with momentum. (In particular, setting $\ell_{j}(\mu) = \log(\mu)$ recovers the multiplicative weights update algorithm with momentum.) The reference function is always convex because the functions ℓ are increasing and differentiable by the fundamental theorem of calculus. Note, however, that in many cases—such as the case of multiplicative weights—the resulting reference function might not be strongly convex over the non-negative orthant. In the next section, we provide a refined analysis of our algorithm that allows us to bypass this issue by restricting attention to a compact subset (see Lemma 3 and the ensuing discussion).

5.2 Performance Analysis

We present a worst-case regret bound Algorithm 2 by extending the analysis of Balseiro et al. (2020a), which introduces a meta-algorithm that can use any online linear optimization algorithm as a black box to solve online allocation problems. Their analysis, however, requires adversarial guarantees for the algorithm and works with the linear functions $g_t^{\mathsf{T}}\mu$. While this allows us to

readily invoke Theorem 1 to obtain a regret bound, it does not allow to exploit the improved regret bounds that are attainable when the inputs are stochastic. The following result provides a refinement when we used an online convex optimization with the functions $w_t(\mu)$ given in (16).

In the following, we denote by $E(G, T, \mu | \vec{\gamma})$ the worst-case regret of OMDM under sample path $\vec{\gamma}$ when the functions w_t have gradients bounded by G against a fixed static action $\mu \in \mathcal{U}$.

Theorem 3. Let $\bar{b} = \sup_{x \in \mathcal{X}} \|b(x)\|_2$, $\bar{b}_{\infty} = \sup_{x \in \mathcal{X}} \|b(x)\|_{\infty}$, and $OPT = \mathbb{E}_{\vec{\gamma} \sim \mathbb{P}^T}[OPT(\vec{\gamma})]$. Suppose requests come from an i.i.d. model with unknown distribution. Then, for any $T \geq 1$, Algorithm 2 that

Regret
$$(A) \le C + \max_{\mu \in \mathcal{D}} \mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} \left[E(G, T, \mu | \vec{\gamma}) \right],$$
 (18)

where

$$C = \frac{\bar{b}_{\infty} \text{ OPT}}{\min_{i} B_{i}}, \qquad G = \bar{b} + \frac{\max B_{j}}{T}, \quad and \quad \mathcal{D} = \left\{0, \frac{\text{OPT}}{B_{1}} e_{1}, \dots, \frac{\text{OPT}}{B_{m}} e_{m}\right\},$$

and where $e_j \in \mathbb{R}^m$ is the j-th unit vector.

The proof of the previous result follows that of Corollary 1 of Balseiro et al. (2020a) and is omitted. As we mentioned above, the key changes are to apply OMDM to the convex functions $w_t(\mu)$ and postponing taking expectations of the regret of OMDM until the very end (as opposed to using an adversarial regret bound).

If we employ the adversarial regret bound in Theorem 1 with a step-size of the order $\eta \sim T^{-1/2}$ we obtain a regret bound of $O(T^{1/2})$ when the expected offline performance satisfies OPT = O(T) and initial resources scale linearly with T, i.e., they satisfy $B_j = \rho_j T$ for some fixed $\rho_j > 0$. The expected offline performance satisfies grows linearly, for example, when the reward functions f_t are bounded from above.

One drawback of applying the adversarial regret bounds is that we cannot show any improvement with momentum. When the algorithm does not project the dual solutions, then Theorem 2 would yield a similar regret bound of $T^{1/2}$ but with a factor $(1+\beta)$ in the denominator of the second term in the regret bound, which suggests an improvement when introducing momentum. Unfortunately, we cannot apply this result directly because dual variables need to be restricted to the non-negative orthant. We remark, however, that online allocation problems are the most interesting when resources are scarce, in which case we would expect the optimal dual variables to be strictly positive and projection to play a lesser role. Nevertheless, we conjecture that our guarantees for the stochastic case continue to hold even with projections, which are supported by our numerical experiments in Section 5.4, where momentum is shown to provide some improvements on performance.

5.3 Reference Functions and Analysis of the Iterates

Finally, we conclude by discussing the implications of different choices for the reference functions. One natural choice is the squared-Euclidean norm, which would deal with online gradient descent with momentum. Because the reference function $h(\mu) = \|\mu\|_2^2/2$ is strongly convex over \mathbb{R}_+^m , then Assumption 1 is satisfied and our results apply.

Another common choice is using the negative entropy $h(\mu) = \sum_{j=1}^{m} \mu_j \log(\mu_j)$, which would lead to exponential weights with momentum. Unfortunately, this function is not strongly convex over \mathbb{R}^m_+ because the curvature of the negative entropy goes to zero for large values of μ , i.e., it is

asymptotically linear. To apply this algorithm we need a more sophisticated analysis of the evolution of the iterates.

Below we present an analysis of the evolution of the iterates with the goal of proving that these remain bounded through the run of the algorithm. We present our analysis when the reference function is separable.

Assumption 2 (Separability of the reference function h). The reference function $h(\mu)$ is coordinatewisely separable, i.e., $h(\mu) = \sum_{j=1}^{m} h_j(\mu_j)$ where $h_j : \mathbb{R}_+ \to \mathbb{R}$ is an univariate function. Moreover, for every resource j the function h_j is σ -strongly convex over $[0, \mu_j^{\max}]$ with $\mu_j^{\max} := \bar{f}/\rho_j + \eta(\bar{b}_\infty + \rho_j)/(\sigma(1-\beta))$ and $\rho_j = B_j/T$.

The following result shows that the dual variables remain bounded when the reference function is separable and strongly convex on a compact set.

Lemma 3. Suppose that Assumption 2 holds and for resource j the initial conditions satisfy $\mu_{1,j} \leq \bar{f}/\rho_j$ and $z_{1,j} = 0$. Then we have that $\mu_{t,j} \leq \mu_j^{\max}$ for all $t \geq 1$.

Using the previous result, we can now apply Algorithm 2 to exponential weights with momentum because the function $\mu_j \log(\mu_j)$ is (μ_j^{\max}) -strongly convex over the set $[0, \mu_j^{\max}]$. Therefore, we can restrict the algorithm to the box $\mathcal{U} := \prod_{j=1}^m [0, \mu_j^{\max}]$ without loss of optimality. A similar analysis can be extended to other non-linear updates such as the one described in (17), which do not necessarily induce strongly convex reference functions over the non-negative orthant.

5.4 Numerical Experiments

We evaluate the performance of our algorithm on a set of random online linear programming problems borrowed from Balseiro et al. (2020b), which are motivated by budget pacing of advertising campaigns. The action set is the unit simplex $\mathfrak{X} = \{x \in \mathbb{R}_+^d : \sum_{i=1}^d x_i \leq 1\}$. The reward and consumption functions are both linear and given by $f_t(x) = r_t^\top x$ and $b_t(x) = c_t x$, respectively, where $r_t \in \mathbb{R}_+^d$ is a reward vector and $c_t \in \mathbb{R}_+^{m \times d}$ is a consumption matrix. The reward vectors and consumption matrices are generated using truncated normal and Bernoulli distributions, respectively. The underlying parameters of these distributions together with the initial amount of resources $B \in \mathbb{R}_+^m$ are also generated randomly. That is, there are two layers of randomness in the experiments: we first draw the parameters of the underlying model and then sample reward vectors and resource consumption matrices from these distributions.

For our experiments we chose the number of resources to m=10 and the dimension of the action set to d=5. We generated 10 different set of parameters for the first layer and then simulated 10 trials for each set of parameters. We tried different lengths of horizon but report results for T=10,000—the results for other lengths of horizons were similar and thus omitted. In our algorithm, we chose the squared-Euclidean norm, yielding the online gradient descent algorithm with momentum. We tried different combinations for the step-size η and the momentum parameter β . To wit, we set the step-size to $\eta=sT^{1/2}$, where s is a multiplier of the step-size chosen to be $s\in\{0.01,0.1,1,10,100,1000\}$, and $\beta\in\{0,0.5,0.9,0.95,0.99,0.995,0.999\}$. The expected rewards, calculated by taking sample averages over all trials, are reported in Figure 1. The black error bars provide 95% confidence intervals.

Some observations are in order. First, the optimal multiplier of the step-size is in the range 1-10. Interestingly, around the optimal step-size momentum, yields small improvements on performance. Second, when the step-size is small, introducing momentum has little impact on performance. Large

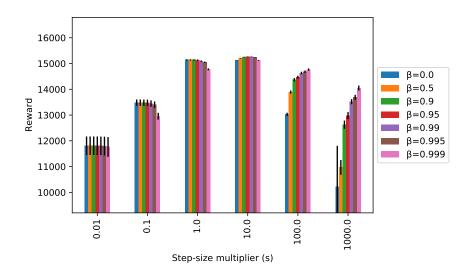


Figure 1: Expected performance of our algorithm for different values of β and step-sizes multipliers s.

values of β , however, can decrease performance. This is expected as momentum can add too much friction to the algorithm and the dual variables might take too long to converge to optimal values. Third, momentum has non-trivial influence on performance when the step-size is large. As we discussed in Section 4, after a transient period, the dual variables would perform a random walk around the orbit of the dual variables. When the step-size is large, the time to reach this orbit is small, but the steps of the random walk would be of order $\eta \|z_t\|_2$. Introducing momentum averages the values of the gradient, which reduces the size of the orbit to order $\eta(1-\beta)^{1/2}$. This leads to increased expected performance when the step-sizes are large and reduces the variance, which is also attractive in practice.

An interesting takeaway is that some degree of momentum increases the robustness of the algorithm. In practice, it is usually impossible to pick the optimal step-size since this depends on the parameters of the problem, which are usually unknown, and even if the estimates are known, these can change unexpectedly in the real world. Therefore, one could end up overestimating or underestimating the true optimal step-size. While momentum has little impact if step-sizes are underestimated, it can hedge against the risk of overestimating step-sizes, thus increasing the operating range in which the algorithm attains good performance.

6 Conclusion and Future Directions

In this paper, we present a new and simple analysis of OMDM for convex nonsmooth optimization. In the stochastic i.i.d. setting, we show that OGDM has improved regret compared with OGD. In the adversarial setting, we show that OMDG enjoys $O(T^{1/2})$ regret. We then build up the connections between OGDM and PID controllers, and apply these results to online allocation problems such as the budget pacing problem of advertisers' campaigns. This provides the first theoretical analysis for the performance of PID controllers, one of the most popular practical algorithms for budget pacing.

We conclude by listing a few open questions and future directions. As we can see in the numerical experiments, the revenue performance of momentum methods is robust to the step-size choice. An interesting future research direction is to formalize this observation and provide a theoretical

framework of robustness. Furthermore, in many online allocation problems (taking budget pacing as an example), the data is usually neither adversarial nor i.i.d. in practice, but shares certain non-stationary patterns. It is interesting to study the performance of online momentum methods in a non-stationary setting. Finally, it remains open to show whether momentum can be helpful for general online mirror descent with projections in the stochastic i.i.d. setting.

References

- Shipra Agrawal and Nikhil R Devanur. Fast algorithms for online stochastic convex programming. In *Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms*, pages 1405–1424, 2014.
- Shipra Agrawal, Zizhuo Wang, and Yinyu Ye. A dynamic near-optimal algorithm for online linear programming. *Operations Research*, 62(4):876–890, 2014.
- Ahmet Alacaoglu, Yura Malitsky, Panayotis Mertikopoulos, and Volkan Cevher. A new regret analysis for Adam-type algorithms. In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 202–210. PMLR, 13–18 Jul 2020.
- Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1200–1205. ACM, 2017.
- Zeyuan Allen-Zhu and Lorenzo Orecchia. Linear coupling: An ultimate unification of gradient and mirror descent. arXiv preprint arXiv:1407.1537, 2014.
- Tom M Apostol. Calculus, Volume 2. John Wiley & Sons, second edition, 1969.
- Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8(1):121–164, 2012.
- Santiago Balseiro, Haihao Lu, and Vahab Mirrokni. The best of many worlds: Dual mirror descent for online allocation problems. arXiv preprint arXiv:2011.10124, 2020a.
- Santiago Balseiro, Haihao Lu, and Vahab Mirrokni. Dual mirror descent for online allocation problems. In *International Conference on Machine Learning*, pages 613–628. PMLR, 2020b.
- Santiago R. Balseiro and Yonatan Gur. Learning in repeated auctions with budgets: Regret minimization and equilibrium. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, EC '17, page 609, New York, NY, USA, 2017. Association for Computing Machinery. ISBN 9781450345279.
- Santiago R. Balseiro and Yonatan Gur. Learning in repeated auctions with budgets: Regret minimization and equilibrium. *Management Science*, 65(9):3952–3968, 2019.
- Heinz H Bauschke, Jonathan M Borwein, and Patrick L Combettes. Essential smoothness, essential strict convexity, and legendre functions in banach spaces. *Communications in Contemporary Mathematics*, 3(04):615–647, 2001.
- Amir Beck. First-order methods in optimization. SIAM, 2017.

- Fernando Bernstein, A Gürhan Kök, and Lei Xie. Dynamic assortment customization with limited inventories. *Manufacturing & Service Operations Management*, 17(4):538–553, 2015.
- D. Bertsekas. Nonlinear Programming. Athena Scientific, Belmont, MA, 1999.
- Gabriel Bitran and René Caldentey. An overview of pricing models for revenue management. *Manufacturing & Service Operations Management*, 5(3):203–229, 2003.
- Sébastien Bubeck, Yin Tat Lee, and Mohit Singh. A geometric alternative to nesterov's accelerated gradient descent. arXiv preprint arXiv:1506.08187, 2015.
- Gong Chen and Marc Teboulle. Convergence analysis of a proximal-like minimization algorithm using bregman functions. SIAM Journal on Optimization, 3(3):538–543, 1993.
- Jinghui Chen, Dongruo Zhou, Yiqi Tang, Ziyan Yang, Yuan Cao, and Quanquan Gu. Closing the generalization gap of adaptive gradient methods in training deep neural networks. arXiv preprint arXiv:1806.06763, 2018.
- Vincent Conitzer, Christian Kroer, Eric Sodomka, and Nicolas E Stier-Moses. Multiplicative pacing equilibria in auction markets. *Operations Research*, 2021.
- Nikhil R Devanur and Thomas P Hayes. The adwords problem: online keyword matching with budgeted bidders under random permutations. In *Proceedings of the 10th ACM conference on Electronic commerce*, pages 71–78, 2009.
- Nikhil R Devanur, Kamal Jain, Balasubramanian Sivan, and Christopher A Wilkens. Near optimal online algorithms and fast approximation algorithms for resource allocation problems. *Journal of the ACM (JACM)*, 66(1):1–41, 2019.
- John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(7), 2011.
- Jon Feldman, Aranyak Mehta, Vahab Mirrokni, and Shan Muthukrishnan. Online stochastic matching: Beating 1-1/e. In 50th Annual IEEE Symposium on Foundations of Computer Science, pages 117–126, 2009.
- Jon Feldman, Monika Henzinger, Nitish Korula, Vahab S. Mirrokni, and Cliff Stein. Online stochastic packing applied to display ad allocation. In *Proceedings of the 18th annual European conference on Algorithms*, 2010.
- Guillermo Gallego, Huseyin Topaloglu, et al. Revenue management and pricing analytics, volume 209. Springer, 2019.
- Negin Golrezaei, Hamid Nazerzadeh, and Paat Rusmevichientong. Real-time optimization of personalized assortments. *Management Science*, 60(6):1532–1551, 2014.
- Anupam Gupta and Marco Molinaro. How the experts algorithm can help solve lps online. *Mathematics of Operations Research*, 41(4):1404–1431, 2016.
- E. Hazan. Sparse approximate solutions to semidefinite programs. *Proceedings of Theoretical Informatics*, 8th Latin American Symposium (LATIN), pages 306–316, 2008.
- Elad Hazan et al. Introduction to online convex optimization. Foundations and Trends® in Optimization, 2(3-4):157–325, 2016.

- Haiwen Huang, Chang Wang, and Bin Dong. Nostalgic adam: Weighting more of the past gradients when designing the adaptive learning rate. arXiv preprint arXiv:1805.07557, 2018.
- Sham Kakade, Shai Shalev-Shwartz, and Ambuj Tewari. On the duality of strong convexity and strong smoothness: Learning applications and matrix regularization. 2009.
- Richard M Karp, Umesh V Vazirani, and Vijay V Vazirani. An optimal algorithm for on-line bipartite matching. In *Proceedings of the twenty-second annual ACM symposium on Theory of computing*, pages 352–358, 1990.
- Thomas Kesselheim, Andreas Tönnis, Klaus Radke, and Berthold Vöcking. Primal beats dual on online packing lps in the random-order model. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pages 303–312, 2014.
- Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization. In 3rd International Conference on Learning Representations, 2015.
- Guanghui Lan. An optimal method for stochastic composite optimization. *Mathematical Program-ming*, 133(1):365–397, 2012.
- Xiaocheng Li and Yinyu Ye. Online linear programming: Dual convergence, new algorithms, and regret bounds. arXiv preprint arXiv:1909.05499, 2019.
- Xiaocheng Li, Chunlin Sun, and Yinyu Ye. Simple and fast algorithm for binary integer and online linear programming. arXiv preprint arXiv:2003.02513, 2020.
- Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. A universal catalyst for first-order optimization. In Advances in Neural Information Processing Systems, 2015.
- Yanli Liu, Yuan Gao, and Wotao Yin. An improved analysis of stochastic gradient descent with momentum. Advances in Neural Information Processing Systems, 33:18261–18271, 2020.
- AR Meenakshi and C Rajian. On a product of positive semidefinite matrices. *Linear algebra and its applications*, 295(1-3):3–6, 1999.
- A. S. Nemirovsky and D. B. Yudin. *Problem Complexity and Method Efficiency in Optimization*. Wiley, New York, 1983.
- Y. Nesterov. *Introductory lectures on convex optimization: a basic course*. Kluwer Academic Publishers, Boston, 2003.
- Yurii Nesterov. A method of solving a convex programming problem with convergence rate $O(1/k^2)$. In Soviet Mathematics Doklady, volume 27, pages 372–376, 1983.
- Frank Nielsen, Jean-Daniel Boissonnat, and Richard Nock. Bregman voronoi diagrams: Properties, algorithms and applications. arXiv preprint arXiv:0709.2196, 2007.
- Boris T Polyak. Some methods of speeding up the convergence of iteration methods. *Ussr computational mathematics and mathematical physics*, 4(5):1–17, 1964.
- Benjamin Recht. A tour of reinforcement learning: The view from continuous control. *Annual Review of Control, Robotics, and Autonomous Systems*, 2:253–279, 2019.

- Sashank J Reddi, Satyen Kale, and Sanjiv Kumar. On the convergence of adam and beyond. arXiv preprint arXiv:1904.09237, 2019.
- Othmane Sebbouh, Robert M Gower, and Aaron Defazio. On the convergence of the stochastic heavy ball method. arXiv preprint arXiv:2006.07867, page 4, 2020.
- Yury Smirnov, Quan Lu, and Kuang-chih Lee. Online ad campaign tuning with pid controllers, April 21 2016. US Patent App. 14/518,601.
- Suvrit Sra, Sebastian Nowozin, and Stephen J Wright. *Optimization for machine learning*. Mit Press, 2012.
- Weijie Su, Stephen Boyd, and Emmanuel J Candes. A differential equation for modeling Nesterov's accelerated gradient method: theory and insights. *Journal of Machine Learning Research*, 17 (153):1–43, 2016.
- Ilya Sutskever, James Martens, George Dahl, and Geoffrey Hinton. On the importance of initialization and momentum in deep learning. In *International conference on machine learning*, pages 1139–1147. PMLR, 2013.
- Kalyan T Talluri and Garrett J Van Ryzin. The theory and practice of revenue management, volume 68. Springer Science & Business Media, 2006.
- Michael Tashman, Jiayi Xie, John Hoffman, Lee Winikor, and Rouzbeh Gerami. Dynamic bidding strategies with multivariate feedback control for multiple goals in display advertising. arXiv preprint arXiv:2007.00426, 2020.
- Xun Yang, Yasong Li, Hao Wang, Di Wu, Qing Tan, Jian Xu, and Kun Gai. Bid optimization by multivariable control in display advertising. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 1966–1974, 2019.
- Weinan Zhang, Yifei Rong, Jun Wang, Tianchi Zhu, and Xiaofan Wang. Feedback control of real-time display advertising. In *Proceedings of the Ninth ACM International Conference on Web Search and Data Mining*, pages 407–416, 2016.

A Proofs of Missing Results

A.1 Proof of Lemma 1

The dual mirror descent update in Algorithm 1 can be written as

$$\nabla h(\tilde{\mu}_t) = \nabla h(\mu_t) - \eta z_t , \quad \mu_{t+1} = \arg\min_{\mu \in \mathcal{H}} V_h(\mu, \tilde{\mu}_t) , \qquad (19)$$

which can be easily verified by noticing that the optimality condition of Algorithm 1 and (19) are both $0 \in \nabla h(\mu_{t+1}) - \nabla h(\mu_t) + \eta z_t + \mathcal{N}_{\mathcal{U}}(\mu_{t+1})$, where $\mathcal{N}_{\mathcal{U}}(\mu_{t+1}) = \{g \in \mathbb{R}^m | g^{\top}(\mu - \mu_{t+1}) \leq 0 \text{ for all } \mu \in \mathcal{U}\}$ is the normal cone of \mathcal{U} at μ_{t+1} . Let h^* be the convex conjugate of h, then $\nabla h^* = (\nabla h)^{-1}$, and ∇h^* is $1/\sigma$ -Lipschitz continuous in the primal norm by recalling that h is σ -strongly-convex in the dual norm (see, e.g., Kakade et al. 2009). Thus,

$$\|\tilde{\mu}_{t} - \mu_{t}\|_{2} = \|\nabla h^{*}(\nabla h(\tilde{\mu}_{t})) - \mu_{t}\|_{2} = \|\nabla h^{*}(\nabla h(\mu_{t}) - \eta z_{t}) - \mu_{t}\|_{2}$$

$$= \|\nabla h^{*}(\nabla h(\mu_{t}) - \eta z_{t}) - \nabla h^{*}(\nabla h(\mu_{t}))\|_{2} \le \frac{\eta}{\sigma} \|z_{t}\|_{2}.$$
(20)

Meanwhile, it follows by the generalized Pythagorean Theorem of Bregman projection (Nielsen et al., 2007) that

$$V_h(\mu_t, \tilde{\mu}_t) \ge V_h(\mu_t, \mu_{t+1}) + V_h(\mu_{t+1}, \tilde{\mu}_t) \ge V_h(\mu_t, \mu_{t+1}) \ge \frac{\sigma}{2} \|\mu_{t+1} - \mu_t\|_2^2 , \qquad (21)$$

where the second inequality is from the non-negativity of Bregman divergence and the last inequality uses the strong-convexity of h w.r.t. dual norm. On the other hand, we have

$$V_{h}(\mu_{t}, \tilde{\mu}_{t}) \leq V_{h}(\mu_{t}, \tilde{\mu}_{t}) + V_{h}(\tilde{\mu}_{t}, \mu_{t}) = (\nabla h(\tilde{\mu}_{t}) - \nabla h(\mu_{t}))^{\top} (\tilde{\mu}_{t} - \mu_{t})$$

$$\leq \|\nabla h(\tilde{\mu}_{t}) - \nabla h(\mu_{t})\| \|\tilde{\mu}_{t} - \mu_{t}\|_{2} \leq \frac{\eta^{2}}{\sigma} \|z_{t}\|_{2}^{2},$$
(22)

where the second inequality follows from Cauchy-Schwartz, and the third inequality utilizes (19) and (20). We finish the proof by combining (21) and (22). \Box

A.2 Proof of Lemma 2

Because the Bregman divergence V_h is differentiable and convex in its first argument, and \mathcal{U} is convex, the first order conditions for the Bregman projection (see, e.g., Proposition 2.1.2 in Bertsekas 1999) are given by

$$\left(z_t + \frac{1}{\eta} \left(\nabla h(\mu_{t+1}) - \nabla h(\mu_t)\right)\right)^{\top} (\mu - \mu_{t+1}) \ge 0, \quad \forall \mu \in \mathcal{U}.$$
(23)

Therefore, it holds for any $\mu \in \mathcal{U}$ that

$$\langle z_{t}, \mu_{t} - \mu \rangle = \langle z_{t}, \mu_{t} - \mu_{t+1} \rangle + \langle z_{t}, \mu_{t+1} - \mu \rangle$$

$$\leq \langle z_{t}, \mu_{t} - \mu_{t+1} \rangle + \frac{1}{\eta} (\nabla h(\mu_{t+1}) - \nabla h(\mu_{t}))^{\top} (\mu - \mu_{t+1})$$

$$= \langle z_{t}, \mu_{t} - \mu_{t+1} \rangle + \frac{1}{\eta} V_{h}(\mu, \mu_{t}) - \frac{1}{\eta} V_{h}(\mu, \mu_{t+1}) - \frac{1}{\eta} V_{h}(\mu_{t+1}, \mu_{t})$$

$$\leq \langle z_{t}, \mu_{t} - \mu_{t+1} \rangle + \frac{1}{\eta} V_{h}(\mu, \mu_{t}) - \frac{1}{\eta} V_{h}(\mu, \mu_{t+1}) - \frac{\sigma}{2\eta} \|\mu_{t+1} - \mu_{t}\|_{2}^{2}$$

$$\leq \frac{\eta}{2\sigma} \|z_{t}\|_{2}^{2} + \frac{1}{\eta} V_{h}(\mu, \mu_{t}) - \frac{1}{\eta} V_{h}(\mu, \mu_{t+1}) ,$$
(24)

where the first inequality follows from (23); the second equality follows from Three-Point Property stated in Lemma 3.1 of Chen and Teboulle (1993); the second inequality is by strong convexity of h; and the third inequality uses that $a^2 + b^2 \ge 2ab$ for $a, b \in \mathbb{R}$ and Cauchy-Schwarz to obtain

$$\frac{\sigma}{2\eta} \|\mu_{t+1} - \mu_t\|_2^2 + \frac{\eta}{2\sigma} \|z_t\|_2^2 \ge \|\mu_{t+1} - \mu_t\|_2 \|z_t\|_2 \ge |\langle z_t, \mu_t - \mu_{t+1} \rangle|,$$

The proof follows.

A.3 Proof of Lemma 3

Fix a resource j. We assume that $B_j/T < \bar{b}_{\infty}$ as otherwise, the dual variables are monotonically decreasing and the bound is trivial. Let $\Delta_{t,j} = \dot{h}_j(\mu_t) - \dot{h}_j(\mu_{t-1})$ be the difference of the iterates evaluated at the derivative of the reference function.

We first argue that $|\Delta_{t,j}| \leq 2\eta \bar{b}_{\infty}$ for all $t \geq 1$. We can write the update as follows: $\dot{h}_j(\tilde{\mu}_{t,j}) = \dot{h}_j(\mu_{t,j}) - \eta z_{t,j}$ and $\mu_{t+1,j} = \max(\tilde{\mu}_{t,j}, 0)$. Therefore,

$$|\Delta_{t+1,j}| = \left| \dot{h}_j(\mu_t) - \dot{h}_j(\mu_{t-1}) \right| \le \left| \dot{h}_j(\tilde{\mu}_t) - \dot{h}_j(\mu_{t-1}) \right| = \eta |z_{t,j}| = \eta \left| (1 - \beta) \sum_{s=1}^t \beta^{t-s} g_{s,j} \right|$$

$$\le \eta(\bar{b}_{\infty} + \rho_j)(1 - \beta) \sum_{s=1}^t \beta^{t-s} = \eta(\bar{b}_{\infty} + \rho_j)(1 - \beta^t) \le \eta(\bar{b}_{\infty} + \rho_j), \qquad (25)$$

where the first inequality follows because the projection is contractive and \dot{h}_j is monotone, the second inequality because $|g_{t,j}| \leq |b_{t,j}(x_t)| + |B_j/T| \leq \bar{b}_{\infty} + \rho_j$ together with the triangle inequality, the fourth equality using the formula for the geometric sum $\sum_{s=1}^{t} \beta^{t-s} = (1-\beta^t)/(1-\beta)$, and the last inequality because $\beta \in (0,1)$.

Define $h_j^*(c) = \max_{\mu_j} \{c\mu_j - h_j(\mu_j)\}$ as the conjugate function of $h_j(\mu_j)$, then by Assumption 2 it holds that $h_j^*(\cdot)$ is a $\frac{1}{\sigma}$ -smooth univariate convex function (Kakade et al., 2009). Furthermore, $\dot{h}_j^*(\cdot)$ is increasing, and $\dot{h}_j^*(\dot{h}_j(\mu)) = \mu$.

Consider a time period $\underline{\tau}$ such that $\mu_{\underline{\tau}-1,j} < \bar{f}/\rho_j < \mu_{\underline{\tau},j}$. If such time does not exist, the bound trivially holds because $\mu_{1,j} \leq \bar{f}/\rho_j \leq \mu^{\max}$. Consider a sequence of consecutive time periods $t = \underline{\tau}, \dots, \bar{\tau}$ in which the dual variables are never projected. If the dual variables are projected at $\bar{\tau} + 1$ we have that $\mu_{\bar{\tau}+1,j} = 0$ and the result follows.

We have that for all $t = \underline{\tau}, \dots, \bar{\tau} - 1$

$$\Delta_{t+1,j} = \dot{h}_{j}(\mu_{t+1}) - \dot{h}_{j}(\mu_{t}) = -\eta z_{t,j} = \eta(\beta z_{t-1,j} + (1-\beta)g_{t,j})
= \beta(\dot{h}_{j}(\mu_{t}) - \dot{h}_{j}(\mu_{t-1})) - \eta(1-\beta)g_{t,j}
= \beta \Delta_{t,j} - \eta(1-\beta)g_{t,j},$$
(26)

where the second equation follows because the dual variables are not projected, the third from the definition of the momentum term, and the fourth because we are not projecting.

Because rewards are bounded by $f_t(x) \leq \bar{f}$ and using that $0 \in \mathcal{X}$ is feasible, it holds that $0 = f_t(0) \leq f_t(x_t) - \mu_t^{\top} b_t(x_t) \leq \bar{f} - \mu_t^{\top} b_t(x_t)$, whereby $\mu^{\top} b_t(x_t) \leq \bar{f}$. Since $\mu_t \geq 0, b_t(x) \geq 0, x_t \in \mathcal{X} \subseteq \mathbb{R}^d_+$, it holds for any $j \in J$ that $b_{t,j}(x_t) \leq \frac{\bar{f}}{\mu_{t,j}}$. Meanwhile, it follows by the definition of \bar{b}_{∞} that $b_{t,j}(x_t) \leq \bar{b}_{\infty}$. This implies that

$$b_{t,j}(x_t) \le \min\left(\frac{\bar{f}}{\mu_{t,j}}, \bar{b}_{\infty}\right) \le \frac{\bar{f}}{\mu_{t,j}} \le \rho_j$$

because $\mu_{t,j} \geq \bar{f}/\rho_j \geq \bar{f}/\bar{b}_{\infty}$. Therefore $-g_{t,j} = b_{t,j}(x_t) - \rho_j \leq 0$ and $\Delta_{t+1,j} \leq \beta \Delta_{t,j}$, which implies that $\Delta_{t,j} \leq \beta^{t-\underline{\tau}} \Delta_{\underline{\tau},j}$ for $\underline{\tau} \leq t \leq \bar{\tau}$. Therefore, for $\underline{\tau} \leq t \leq \bar{\tau}$ we have that

$$\dot{h}_{j}(\mu_{t,j}) = \dot{h}_{j}(\mu_{\underline{\tau}-1,j}) + \sum_{s=\tau}^{t} \Delta_{s,j} \leq \dot{h}_{j}(\mu_{\underline{\tau}-1,j}) + \Delta_{\underline{\tau},j} \sum_{s=\tau}^{t} \beta^{t-\underline{\tau}} \leq \dot{h}_{j}(\mu_{\underline{\tau}-1,j}) + \frac{\eta(\bar{b}_{\infty} + \rho_{j})}{1 - \beta}$$

where the least inequality follows from (25) together with $\sum_{s=\underline{\tau}}^{t} \beta^{t-\underline{\tau}} \leq 1/(1-\beta)$. This implies that

$$\mu_{t,j} = \dot{h}_j^*(\dot{h}_j(\mu_{t,j})) \le \dot{h}_j^* \left(\dot{h}_j(\mu_{\underline{\tau}-1,j}) + \frac{\eta(\bar{b}_{\infty} + \rho_j)}{1 - \beta} \right) \le \mu_{\underline{\tau}-1,j} + \frac{\eta(\bar{b}_{\infty} + \rho_j)}{(1 - \beta)\sigma} \le \mu_j^{\max}$$

where the first inequality is the monotonicity of $\dot{h}_{j}^{*}(\cdot)$, the second inequality is from $\dot{h}_{j}^{*}(\dot{h}_{j}(\mu_{j})) = \mu_{j}$ and the $\frac{1}{\sigma}$ -smoothness of $h_{j}^{*}(\cdot)$, and the last inequality utilizes $\mu_{\underline{\tau}-1,j} \leq \bar{f}/\rho_{j}$ and and the definition of μ^{\max} . The result follows.

B An Example Showing that OGDM Has Worse Regret than OGD

In this part, we discuss when OGDM has worse regret than OGD in the adversarial case. Here we consider the unconstrained case of OGD. Then we know that for OGD, the iterate is given by

$$\mu_t = \mu_1 - \eta \left(\sum_{s=1}^{t-1} g_s \right) ,$$

and the iterate of OGDM is given by

$$\mu_t = \mu_1 - \eta \left(\sum_{s=1}^{t-1} z_s \right) = \mu_1 - \eta \left(\sum_{s=1}^{t-1} (1 - \beta) \sum_{i=1}^{s} \beta^{s-i} g_i \right) = \mu_1 - \eta \left(\sum_{s=1}^{t-1} (1 - \beta^{t-s}) g_s \right) .$$

As we can see, the difference of OGDM iterate and OGD iterate is fairly minimal. In particular, if g_s in both algorithms are the same, then the only difference is the $\beta^{t-s}g_s$ term, which is exponentially in t-s. Effectively, one can view OGDM as OGD with smaller step-size. In the light of this observation, we next present an example where OGDM can have $(1+\beta)$ times worse performance than OGD in the adversarial case.

Let $\mu \in \mathbb{R}^d$. With some abuse of notation, we use μ_k^t to represent the k-th coordinate of the t-th iteration of μ . Consider the case when d is large enough, in particular $d \geq (T+1)/2$. For any given $T = 2T' \leq 2d$, $0 < \beta < 1$ and $\eta = 1/\sqrt{T}$, we set

$$w_{2t+b}(\mu) = \max\{0, \eta - \mu_t\} \text{ for } t = 0, ... T' - 1, b = 1, 2 .$$

Then we know that

$$\min_{\mu} \sum_{t=1}^{T} w_t(\mu) = 0 ,$$

where an optimal solution is $\mu = \eta e$ and e is the all-one vector.

Suppose the initial solution is $\mu^1=0$. For OGD, we know that for any t=0,...,T'-1 that $\mu_t^{2t+b}=\eta(b-1)=(b-1)/\sqrt{T}$, thus

$$w_{2t+b}(\mu^{2t+b}) = 1 - \mu_t^{2t+b} = \begin{cases} \eta & \text{if } b = 1\\ 0 & \text{if } b = 2 \end{cases}$$
.

As a result, $w_t(\mu^t) = 1/\sqrt{T}$ if t is odd and $w_t(\mu^t) = 0$ if t is even. Thus the regret of OGD is

$$\sum_{t=1}^{T} w_t(\mu^t) - \min_{\mu} \sum_{t=1}^{T} w_t(\mu) = \sqrt{T}/2.$$

On the other hand, for OGDM, we know that for any t=0,...,T'-1 that $\mu_t^{2t+b}=\eta(1-\beta)(b-1)=(b-1)(1-\beta)/\sqrt{T}$, thus

$$w_{2t+b}(\mu^{2t+b}) = \eta - \mu_t^{2t+b} = \begin{cases} \eta & \text{if } b = 1\\ \eta \beta & \text{if } b = 2 \end{cases}$$
.

As a result $w_t(\mu^t) = 1/\sqrt{T}$ if t is odd and $w_t(\mu^t) = \beta/\sqrt{T}$ if t is even. Thus the regret of OGDM is

$$\sum_{t=1}^{T} w_t(\mu^t) - \min_{\mu} \sum_{t=1}^{T} w_t(\mu) = \sqrt{T}(1+\beta)/2.$$

For this example, the regret of OGDM is $(1 + \beta)$ larger than the regret of OGD. Notice that $(1 + \beta) \sim \frac{1}{1 - \beta}$ when β is small. This shows that the regret of OGDM may be a constant time worse than the regret of OGD in the adversarial case.