DMA Modulus Equations

Dual Cantilever Flexure Mode

The stiffness model equation for a rectangular cross section sample is:

$$K = \frac{24 \cdot E \cdot I}{L^3 \cdot \left[1 + \frac{12}{5} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]}$$

ν - Poisson's Ratio
L - Sample Length
t - Sample Thickness
I - Sample Moment of Inertia
E - Elastic Modulus

The sample moment of inertia is,

$$I = \frac{w \cdot t^3}{12}$$

w- Sample Width

The stiffness model equation assumes that the ends of the sample are fixed, or that there is no deformation of the sample beyond where the sample enters the clamp. This is never achieved in practice; to do so would require a discontinuity in the strains within the sample at the clamp face. Thus, the sample does deform within the clamped region and the sample stiffness equation is in error. A sample stiffness correction factor can be defined as,

$$F = \frac{Ks}{K}$$

Ks – Measured Stiffness

Substituting for K in the model stiffness equation and solving for the Modulus,

$$E = \frac{Ks}{F} \cdot \frac{L^3}{24 \cdot I} \cdot \left[1 + \frac{12}{5} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]$$

The clamping correction factor was determined by finding the sample stiffness using a Finite Element Analysis of the sample deformation and calculating F using the sample stiffness equation and the FEA stiffness. Curve fitting of the results gives the equation for F as a function of L/t,

$$F = 0.7616 - 0.0271 \cdot \sqrt{\frac{L}{t}} + 0.1083 \cdot \mathrm{In}\left(\frac{L}{t}\right)$$

The stiffness model equation for a cylindrical cross section sample is,

$$K = \frac{E}{\frac{L^3}{24 \cdot I} + S \cdot (1+\nu) \cdot \frac{L}{A}}$$

S – Shear Shape Factor for Cylindrical Beam

The sample moment of inertia is,

$$I = \frac{\pi}{64} \cdot \left(D^4 - d^4\right)$$

D – Outside Diameter
d – Inside Diameter, 0 for a solid cylinder

The shear shape factor for a cylinder beam is found by using Energy Methods and numerical integration; a curve fit of the shape factor is,

$$S = 1.112 - 0.04086 \cdot \frac{d}{D} + 1.033 \cdot \left(\frac{d}{D}\right)^2 + 0.5935 \cdot \left(\frac{d}{D}\right)^3 - 2.226 \cdot \left(\frac{d}{D}\right)^4 + 1.029 \cdot \left(\frac{d}{D}\right)^5$$

As for the rectangular cross section model above, a sample stiffness correction is defined as,

$$F = \frac{Ks}{K}$$

Substituting for K in the model stiffness equation and solving for the Modulus,

$$E = \frac{Ks}{F} \cdot \left[\frac{L^3}{24 \cdot I} + S \cdot (1+\nu) \cdot \frac{L}{A}\right]$$

The same clamping correction factor is used for both the cylindrical and the rectangular cross sections.

Single Cantilever Flexure Mode

The equations for modulus for the rectangular single cantilever flexure modes are the same as for the dual cantilever with the exception of the factor; 12 should replace the factor 24. The clamping correction factor is the same as for the dual cantilever.

Single Cantilever Cylindrical Sample

$$E = \frac{Ks}{F} \cdot \left[\frac{L^3}{12 \cdot I} + 2 \cdot S \cdot (1+\nu) \cdot \frac{L}{A}\right]$$

Parallel Plate Compression Mode

The model equation for sample stiffness is,

$$K = \frac{A \cdot E}{t}$$

The model equation assumes that transverse strains are negligible. In fact, because of dilatation effects, transverse strains in the sample are significant, so that the measured stiffness will be lower than the sample stiffness. A stiffness correction factor can be defined as,

$$F = \frac{Ks}{K}$$ Ks – Measured Stiffness

Substituting for K in the model equation and solving for the Modulus,

$$E = \frac{Ks}{F} \cdot \frac{t}{A}$$

Redefine the correction factor as Modulus correction factor, instead of stiffness correction factor

$$Fe = \frac{1}{F}$$

With the redefined correction factor, the Modulus is found from,

$$E = Ks \cdot Fe \cdot \frac{t}{A}$$

The clamping correction factor was determined by finding the sample stiffness using a Finite Element Analysis of the sample deformation and calculating F using the sample stiffness equation and the FEA stiffness. Correction factors were determined for solid circular, square and ring shaped samples. Curve fitting of the results gives the equation for F as a function of t/D and Poisson's ratio,

Solid Circular Sample

$$\frac{1}{Fe} = \frac{1.009 - 1.814 \cdot \nu + 0.8257 \cdot \nu^2 - 0.1303 \cdot \left(\frac{t}{D}\right) + 2.776 \cdot \left(\frac{t}{D}\right)^2 - 1.461 \cdot \left(\frac{t}{D}\right)^3}{1 - 1.999 \cdot \nu + 0.03745 \cdot \left(\frac{t}{D}\right) + 2.457 \cdot \left(\frac{t}{D}\right)^2 - 1.244 \cdot \left(\frac{t}{D}\right)^3}$$

Square Sample

$$\frac{1}{Fe} = \frac{1.004 - 1.793 \cdot \nu + 0.7736 \cdot \nu^2 - 0.05294 \cdot \left(\frac{t}{D}\right) + 2.006 \cdot \left(\frac{t}{D}\right)^2 - 1.027 \cdot \left(\frac{t}{D}\right)^3}{1 - 2.001 \cdot \nu + 0.06237 \cdot \left(\frac{t}{D}\right) + 1.838 \cdot \left(\frac{t}{D}\right)^2 - 0.9334 \cdot \left(\frac{t}{D}\right)^3}$$

Ring Sample

$$\frac{1}{Fe} = \frac{0.9405 - 0.8639 \cdot \nu - 0.3012 \cdot \ln\left(\frac{D-d}{t}\right) + 0.0944 \cdot \ln\left(\frac{D-d}{t}\right)^2 - 0.01019 \cdot \ln\left(\frac{D-d}{t}\right)^3}{1 - 1.296 \cdot \nu - 0.2895 \cdot \ln\left(\frac{D-d}{t}\right) + 0.07989 \cdot \ln\left(\frac{D-d}{t}\right)^2 - 0.007355 \cdot \ln\left(\frac{D-d}{t}\right)^3}$$

3-Point Flexure

The model equation for 3-point flexure of a rectangular cross section sample is,

$$K = \frac{6 \cdot E \cdot I}{L^3 \cdot \left[1 + \frac{6}{10} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]}$$

Assuming that local deformations of the sample in the region of the supports is negligibly small, no correction factor is needed so that the modulus is,

$$E = Ks \cdot \frac{L^3}{6 \cdot I} \cdot \left[1 + \frac{6}{10} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]$$

The model equation for 3-point flexure of a cylindrical cross section sample is,

$$K = \frac{E}{\frac{L^3}{6 \cdot I} + S \cdot (1+\nu) \cdot \frac{L}{A}}$$

Assuming that local deformations of the sample in the region of the supports is negligibly small, no correction factor is needed so that the modulus is,

$$E = Ks \cdot \left[\frac{L^3}{6 \cdot I} + S \cdot (1+\nu) \cdot \frac{L}{A}\right]$$

The shear shape factor is the same as that of the cantilever modes.

Shear Sandwich Mode

The model equation for sample stiffness of a rectangular cross section in pure shear is,

$$K = \frac{5 \cdot G \cdot w \cdot h}{3 \cdot t}$$

G- Shear Modulus of the sample
w- Width of Sample, i.e. horizontal dimension
h - Height of Sample, i.e. vertical dimension
t - Sample Thickness, i.e. between clamp faces

Dilatation of the sample due to compression when the screws are tightened causes the measured stiffness to differ from the model equation. The amount of dilatation is dependent upon the degree of compression of the sample, which we cannot quantify so a correction factor for this effect will be ignored. The equation for shear modulus is,

$$G = Ks \frac{3 \cdot t}{5 \cdot w \cdot h}$$

Film and Fiber Tension Mode

The model equation for sample stiffness is,

$$K = \frac{A \cdot E}{L}$$

A - Sample Cross-Section Area
L - Sample Length

Because the sample will have a very small area as compared to it's length, no end effects correction is needed, so the modulus equation is,

$$E = Ks \cdot \frac{L}{A}$$

Stress-Strain Equations

Film Fiber Mode

Nominal or Engineering Strain

$$\varepsilon_0 = \frac{\Delta L}{L_0}$$

ε_0 - Nominal Strain
ΔL - Cumulative Change in Sample Length
L_0 - Initial Sample Length

True Strain

$$\varepsilon = \mathrm{In}\left(\frac{L_0 + \Delta L}{L_0}\right)$$

ε - True Strain

Nominal or Engineering Stress

$$\sigma = \frac{P}{A_0}$$

σ_0 - Nominal Stress
P - Applied Load
A_0 - Initial Area

True Stress

$$\sigma_0 = \left(\frac{P}{A_0}\right) \cdot \left(\frac{L_0 + \Delta L}{L_0}\right)$$

σ - True Stress

The nominal or engineering stress and strain are based on the assumption that the cross sectional area of the sample maintains the initial value and that the change in sample length is small for the duration of the experiment (these two statements are equivalent). For experiments where the total displacement is a very small percentage of the initial length, this gives good results. True stress and strain account for the reduction of area and increase in the sample length that occurs as the sample is deformed. True stress and strain must be used for experiments where the sample stretches or shrinks considerably.

Parallel Plate Compression Mode

Nominal or Engineering Strain

$$\varepsilon_0 = \frac{\Delta L}{L_0}$$

Nominal or Engineering Stress

$$\sigma = Fe \cdot \frac{P}{A_0}$$

Fe - Correction factor that accounts for sample dilatation effects

Shear Sandwich

Nominal Shear Strain

$$\gamma_0 = \frac{\delta}{T}$$

δ - Displacement Amplitude
T - Separation of the Clamp Surfaces

Nominal Shear Stress

$$\tau_0 = \frac{P}{2 \cdot A}$$

A - Sample Cross-Sectional Area

Dual and Single Cantilever Flexure Mode

Maximum Nominal Normal Stress Magnitude (Rectangular Cross Section)

$$\sigma_x = \frac{3 \cdot P \cdot L}{w \cdot t^2}$$

L - Length Between Clamps, One Side
w- Sample Width
t - Sample Thickness
P - ½ Applied Force for Dual cantilever, Full Applied Force for Single Cantilever

Maximum Nominal Normal Strain Magnitude (Rectangular Cross Section)

$$\varepsilon_x = \frac{3 \cdot \delta \cdot t \cdot Fc}{L^2 \cdot \left[1 + \frac{12}{5} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]}$$

Fc - Dual Cantilever Clamping Correction Factor
ν - Poisson's Ratio

Maximum Nominal Normal Stress Magnitude (Cylindrical Cross Section)

$$\sigma_x = \frac{P \cdot L \cdot D}{4 \cdot I}$$

D – Sample Outside Diameter

Maximum Nominal Normal Strain Magnitude (Cylindrical Cross Section)

$$\varepsilon_x = \frac{\delta \cdot L \cdot D \cdot Fc}{8 \cdot \left[\frac{L^3}{24} + S \cdot I \cdot (1+\nu) \cdot \left(\frac{L}{A}\right)\right]}$$

S – Shear Correction Factor for Cylindrical Beam

Note that the maximum stress and strain occur at the clamp faces, so both stress and strain can have positive or negative sense depending on whether it is on the top or bottom surface of the sample and at the fixed or moving clamp. The stress and strain do not include any contribution from the clamp.

3-Point Flexure

Maximum Nominal Normal Stress Magnitude (Rectangular Cross Section)

$$\sigma_x = \frac{P \cdot L \cdot t}{4 \cdot I}$$

P – Full Load
L – ½ Length

Maximum Nominal Normal Strain Magnitude (Rectangular Cross Section)

$$\varepsilon_x = \frac{3 \cdot \delta \cdot t}{2 \cdot L^2 \cdot \left[1 + \frac{6}{10} \cdot (1+\nu) \cdot \left(\frac{t}{L}\right)^2\right]}$$

Maximum Nominal Normal Stress Magnitude (Cylindrical Cross Section)

$$\sigma_x = \frac{P \cdot L \cdot D}{4 \cdot I}$$

Maximum Nominal Normal Strain Magnitude (Cylindrical Cross Section)

$$\varepsilon_x = \frac{\delta \cdot D}{4 \cdot \left[\frac{L^2}{6} + \frac{S}{A} \cdot I \cdot (1+\nu)\right]}$$

Note that the maximum stress and strain occur at the mid-span of the sample where the drive applies the load, so both stress and strain can have positive or negative sense depending on whether it is on the top or bottom surface of the sample.

Caution must be exercised in using these equations. One may be tempted to calculate stress and strain and take the ratio to determine the Elastic Modulus, for instance to check the Storage Modulus. If this is done, the results will probably not agree with the Storage Modulus computed by the DMA program, this will be particularly true if the sample stretches or shrinks significantly during the experiment. The reason is that the calculated Storage Modulus assumes that the sample behaves in a linearly elastic manner, when samples, which behave in accordance with other constitutive laws are run, the program calculated Storage Modulus may be greatly in error and in fact is meaningless. The stress and strain equations are correct for any material but cannot be used to compute any type of modulus, unless the constitutive law for the sample is known.