Exercises week 7

Overview

Week 7: Readout

Keywords: Dispersive shift, C-shunted flux qubit, Wigner functions.

[1, Chapter: VA-VB1] Philip Krantz, Morten Kjaergaard, Fei Yan, Terry P. Orlando, Simon Gustavsson, and William D. Oliver. A Quantum Engineer's Guide to Superconducting Qubits. *Applied Physics Reviews*, 6(2):021318, June 2019. arXiv: 1904.06560

[2, Chapters: III, App. D] Jens Koch, Terri M. Yu, Jay Gambetta, A. A. Houck, D. I. Schuster, J. Majer, Alexandre Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Charge-insensitive qubit design derived from the Cooper pair box. *Physical Review A*, 76(4):042319, October 2007

E1

Exercises concerning [2]. Here, we will generalize the expression for the dispersive shift and learn about the details of the Schrieffer-Wolff transformation.

(a) We begin by reviewing the Schrieffer-Wolff transformation. Assume that you have a diagonalized Hamiltonian H_0 and a small pertubation ηH_1 where $\eta \ll 1$ such that the full system is

$$H = H_0 + \eta H_1.$$

In order to diagonalize the full Hamiltonian up to second order in η, we need to find an antihermitian matrix ηS which can transform the Hamiltonian,

$$H' = e^{\eta S} H e^{-\eta S}.$$

By using a version of the Baker-Campbell-Hausdorff formula,

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots,$$

expand $H' = e^{\eta S} H e^{-\eta S}$ to second order in η.

(b) By demanding that the first order term in your expansion is zero, further reduce your expression until you get

$$H' = H_0 + \frac{\eta^2}{2}[S, H_1],$$

where $[S, H_0] = -H_1$. In general, the challenging part is to find S such that this last expression is satisfied, but if you know S, the Schrieffer-Wolff transformation gives a simple expression for the Hamiltonian to second order.

(c) We now consider the generalized Jaynes-Cummings model

$$H = H_0 + H_1, \quad H_0 = \sum_k \omega_k \left|k\right\rangle \left\langle k\right| + \omega_r a^\dagger a, \quad H_1 = \sum_{i,j} g_{ij} \left|i\right\rangle \left\langle j\right| (a + a^\dagger).$$

We make the educated guess that S is similar to H_1, (the commutator $[S, H_0]$ is a linear operator afterall),

$$S = \sum_{i,j} |i\rangle \langle j| \, (\nu_{ij} a + \mu_{ij} a^\dagger).$$

Determine ν_{ij} and μ_{ij} such that $[S, H_0] = -H_1$ is fulfilled. You should find that

$$\nu_{ij} = \frac{g_{ij}}{\omega_i - \omega_j - \omega_r}, \quad \mu_{ij} = \frac{g_{ij}}{\omega_i - \omega_j + \omega_r}.$$

(d) Show that the frequency shift due to $[S, H_1]/2$ is

$$H_{shift} = \sum_{ij} |i\rangle \langle i| \, |g_{ij}|^2 \left(\frac{1}{\omega_{ij} - \omega_r} + \left(\frac{1}{\omega_{ij} - \omega_r} + \frac{1}{\omega_{ij} + \omega_r} \right) a^\dagger a \right),$$

where $\omega_{ij} = \omega_i - \omega_j$. There will also be other terms, but we are only interested in the terms proportional to $\sum_i |i\rangle \langle i|$.

(e) Reduce the expressions from (c) and (d), to show that the effective model describing the qubit and resonator is

$$H = \left(\omega_{01} + \delta + \chi \, a^\dagger a\right) \frac{\sigma_z}{2} + \left(\omega_r + \frac{\rho}{2}\right) a^\dagger a,$$

$$\begin{aligned}
\delta &= \sum_j \delta_{0j} - \delta_{1j}, \\
\chi &= \sum_j \chi_{0j} - \chi_{1j}, \\
\rho &= \sum_j \chi_{0j} + \chi_{1j}, \\
\delta_{ij} &= \frac{|g_{ij}|^2}{\omega_{ij} - \omega_r}, \\
\chi_{ij} &= |g_{ij}|^2 \left(\frac{1}{\omega_{ij} - \omega_r} + \frac{1}{\omega_{ij} + \omega_r} \right),
\end{aligned}$$

where $\sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1|$. We thus see three effects: a shift of the qubit frequency (δ), a shift of the resonator frequency (ρ) and a qubit-resonator dependent shift (χ).

E2

In this exercise, we will try to plot the qubit dependent resonator shift χ from E1(e). We consider the generic transmon ($E_J/E_C = 50$, $E_J = 10\, h\,$GHz).

(a) Make sure that you have the transmon Hamiltonian at hand as a sparse matrix (remember: transmon, periodic potential, discrete charges, charge basis is preferable).

(b) Compute the eigenenergies and eigenstates of the transmon Hamiltonian and plot χ as found in E1(e) as a function of the detuning $\Delta = \omega_1 - \omega_0 - \omega_r$. Note that the coupling elements are given by $|g_{ij}|^2 = g^2 \, | \langle i| \, n \, |j\rangle \, |^2$ where g is the capacitive coupling strength between the transmon and the resonator and we may choose $g = 50\, h\,$MHz. Do you see the straddling regime as in Fig. 20 in Ref. [1] and Fig. 9 in Ref. [2]?

E3

In this exercise, we will try drive the readout resonator in order to see the qubit dependent shift.

(a) Consider the Hamiltonian in E1(e). We may add a drive term to the resonator

$$H_{drive} = \epsilon \left(a e^{i\omega_d t} + a^\dagger e^{-i\omega_d t} \right),$$

where ϵ is the drive amplitude which we set to $\epsilon = 10\,h\,\mathrm{MHz}$. We may choose a rotating frame for both the qubit and the resonator such that the combined Hamiltonian becomes

$$H_{rot} = \Omega\, a^\dagger a + \frac{\chi}{2}\, a^\dagger a\, \sigma_z + \epsilon \left(a + a^\dagger \right),$$

where $\Omega = \omega_r - \omega_d$ is the detuning between the drive and the resonator and such that H_{rot} is time independent.

We begin this exercise by defining a sparse matrix that represents a (and by conjugation $a^\dagger$). If we choose a basis for the resonator states as $|0\rangle = (1, 0, 0, \dots)$, $|1\rangle = (0, 1, 0, \dots)$ and so on, define a sparse matrix representing a that act as $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$. This matrix is formally infinite, but let us truncate it to the lowest 20 states.

(b) In order to proceed, define the full system H_{rot} (remember the implicit $\otimes$ and identities). The resulting matrix should then be 40x40. We may choose $\chi = 30\,h\,\mathrm{MHz}$ or you may choose a value in the straddling regime of E2(b).

(c) We may now time-evolve an initial state $|\psi(t=0)\rangle = |0\rangle_r \otimes |s\rangle_q$ for a time $t = 500\mathrm{ns}$ to get the final state $|\psi(t)\rangle = e^{-iH_{rot}t}\,|\psi(t=0)\rangle$. For initial qubit states $|s\rangle = |0\rangle, |1\rangle, (|0\rangle + |1\rangle)/\sqrt{2}$, plot the expectation value of the number of photons in the resonator $\left\langle a^\dagger a \right\rangle = \langle \psi(t)|\, a^\dagger a \otimes I_q\, |\psi(t)\rangle$ as a function of the detuning Ω.

(d) Compare the resulting figure from E3(c) to Fig. 19 from Ref. [1]. If you increase the number of photons in the resonator, then the reflected signal that you measure is decreasing.

References

[1] Philip Krantz, Morten Kjaergaard, Fei Yan, Terry P. Orlando, Simon Gustavsson, and William D. Oliver. A Quantum Engineer's Guide to Superconducting Qubits. *Applied Physics Reviews*, 6(2):021318, June 2019. arXiv: 1904.06560.

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