Superconducting Qubits

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Introduction

The code for all the plots and simulations can be found in the following github repository https://github.com/Shellf1sh/Superconducting-Qubits-Course. There are some of the exercises that I've not done, either because i wouldn't figure them out or I simply didn't have time. For them I've still added the "title" (i.e. E2) c) or E1 e)), but just left the text underneath empty.

Contents

Week 1: Circuit quantization and the transmon (Done)

E1

a)

Equation (5) can be found by simply substituting $I = CdV/dt$ into equation (3) and using $V = \dot{\Phi}$

$$\mathcal{T}_C = \int V \cdot I dt = \int VC\frac{dV}{dt}dt = C\int VdV = \frac{1}{2}CV^2 = \frac{1}{2}C\dot{\Phi}^2 \tag{1.1}$$

For equation (6) the potential energy can use partial integration

$$\mathcal{U}_L = \int V \cdot I dt = \int \frac{d\Phi}{dt} I dt = \Phi I - \int \Phi \frac{dI}{dt}dt = \Phi I - \int \Phi dI \tag{1.2}$$

We can then use the relation from the text $V = LdI/dt \Rightarrow dI = Vdt/L$.

$$\mathcal{U}_L = \Phi \cdot I - \int \Phi dI = \Phi \cdot I - \frac{1}{L}\int \Phi V dt \tag{1.3}$$

Now we can use $\Phi = \int V dt = L\int dI = LI \Rightarrow I = \Phi/L$ and $d\Phi/dt = V \Rightarrow d\Phi = Vdt$

$$\mathcal{U}_L = \frac{\Phi^2}{L} - \frac{1}{L}\int \Phi d\Phi = \frac{\Phi^2}{L} - \frac{\Phi^2}{2L} = \frac{\Phi^2}{2L} \tag{1.4}$$

With the kinetic and the potential energy we can now write the Langrangian in accordance with equation (7)

$$\mathcal{L} = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2L}\Phi^2 \tag{1.5}$$

Now we can find the conjugate variable to Phi which will be our "momentum" by doing a Legendre transformation as in equation (8)

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} \tag{1.6}$$

and we can then insert the conjugate variable and the the following expression for the Langrangian,

$$\mathcal{L} = \frac{Q^2}{2C} - \frac{\Phi^2}{2L}, \tag{1.7}$$

and the Hamiltonian becomes,

$$H = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{C} - \frac{Q^2}{2C} + \frac{\Phi^2}{2L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}. \tag{1.8}$$

The final step is to use the relations $Q = C\dot{\Phi} = CV$ and $\Phi = LI$ to get equation (9)

$$H = \frac{1}{2}CV^2 + \frac{1}{2}LI^2 \tag{1.9}$$

To find equation (13) we use the fact that Q is the charge and can therefore be written as $Q = 2e\hat{n}$. We also define $\phi = 2\pi\Phi/\Phi_0$. In that case the Hamiltonian becomes

$$H = \frac{(2e)^2\hat{n}^2}{2C} + \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\right)^2\phi^2 = 4E_C\hat{n}^2 + \frac{1}{2}E_L \tag{1.10}$$

Where $E_C = e^2/2C$ and $E_L = (\Phi_0/2\pi)^2/L$

b)

To compare the Hamiltonian for the LC-circuit to the harmonic oscillator we first let n correspond to p and ϕ correspond to x. Then we can compare the factors in front. In the first term $4E_c = 1/2m \Rightarrow m = 1/8E_c$. In the second term we get $E_L = m\omega_r^2 \Rightarrow \omega_r = \sqrt{E_L/m}$ inserting our previous expression for m we get $\omega_r = \sqrt{8E_cE_L}$.

E2

a)

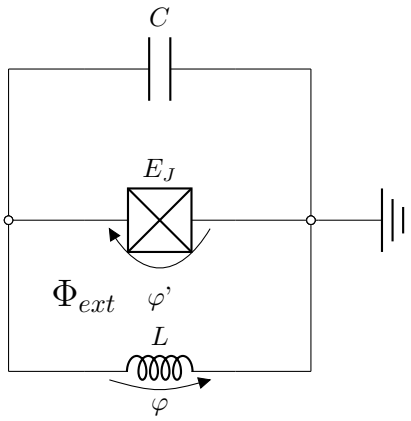


Figure 1: Transmon circuit with the branch flux's and the external flux labelled.

To find the Hamiltonian of the transmon circuit we use the same procedure as in the harmonic oscillator above, but the potential energy is different this time.

$$E_{kin} = \frac{C}{2}\dot{\varphi}^2 \quad , \quad E_{pot} = -E_J\cos(\varphi') + \frac{1}{2L}\varphi^2, \tag{1.11}$$

where the minus in front of the Josephson energy is a convention. We can then again find the Lagrangian,

$$\mathcal{L} = E_{kin} - E_{pot} = \frac{C}{2}\dot{\varphi}^2 + E_J\cos(\varphi') - \frac{1}{2L}\varphi^2. \tag{1.12}$$

We then find the conjugate variable which once again is

$$q = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = C\dot{\varphi}. \tag{1.13}$$

With the conjugate the variable the Hamiltonian then becomes

$$H = q\dot{\varphi} - \mathcal{L} = \frac{q^2}{C} - \frac{q^2}{2C} - E_J\cos(\varphi') + \frac{1}{2L}\varphi^2 \tag{1.14}$$

$$= 4E_Cn^2 - E_J\cos(\varphi') + \frac{1}{2}E_L\varphi^2 \tag{1.15}$$

$$= 4E_Cn^2 - E_J\cos(\varphi - \varphi_{ext}) + \frac{1}{2}E_L\varphi^2. \tag{1.16}$$

In the last line we have used $\varphi + \varphi' = \varphi_e$, where $\varphi_e = 2\pi\Phi_{ext}/\phi_0$, and the fact the cosine is an even function so that we can switch the sign inside.

b)

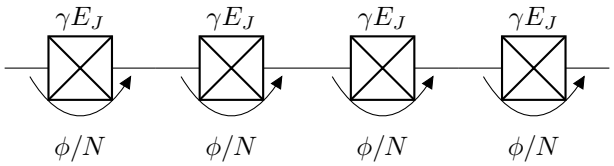


Figure 2: A row of N Josephson junctions with the distributed phase approximation so the phase drop over each Josephson junction is ϕ/N.

With many Josephson junctions in series like this then the term in the Hamiltonian that describes them will be given by $H_{row} = -N\gamma E_J \cos(\phi/N)$. For large N the ϕ/N will be small and we can Taylor expand the cosine $\cos(\phi/N) \approx 1 - \phi^2/N^2 + \mathcal{O}(\phi^3/N^3)$. Inserting that in the previous expression we find that

$$H_{row} = -N\gamma E_J \left(\frac{\phi^2}{N^2}\right) = \frac{1}{2}\frac{\gamma E_J}{N}\phi^2, \tag{1.17}$$

where we have ignored the constant term from the 1. We can then see that this is the same form as the inductive term with $\gamma E_J/N = E_L$.

c)

The first thing to do is to Taylor expand the cosine to fourth order, but this time around $\varphi = \pi$.

$$\cos(\varphi + \pi) \approx -1 + 0 + \frac{\varphi^2}{2!} + 0 - \frac{\varphi^4}{4!} + \mathcal{O}(\varphi^5). \tag{1.18}$$

For the quadratic terms to cancel our we get the following relation

$$-\frac{E_J}{2} + \frac{E_L}{2} = 0 \Rightarrow E_J = E_L \xrightarrow[E_L = \frac{\gamma}{N} E_J]{} E_J = \frac{\gamma}{N} E_J. \tag{1.19}$$

So the quadratic terms exactly cancel each other when $\gamma/N = 1$

d)

For the $\gamma/N > 1$ case then the potential is almost quadratic (at least for $\gamma/N \gg 1$) and then we have low anharmonicity since the pure quadratic potential has $\alpha = 0$. If $\gamma/N = 1$ then quadratic parts cancel out and ϕ^4 potential (with small higher order terms), which has some anharmonicity. In the case of $\gamma/N < 1$ then we have 2 minima and there will be a large anharmonicity, since there will be 2 states with almost equal energy and then some higher states.

e)

Quartic Potential: To calculate the anharmonicity we will use the WKB formula given in the problem set to first calculate the different frequencies. For the quadratic case (ϕ^2) then the exponent is $2 \cdot 4/(4+2) = 4/3$ so

$$\omega_{0,q} \propto (1/2)^{4/3} = 0.4 \quad , \quad \omega_{1,q} \propto (3/2)^{4/3} = 1.7 \quad , \quad \omega_{2,q} \propto (5/2)^{4/3} = 3.4. \tag{1.20}$$

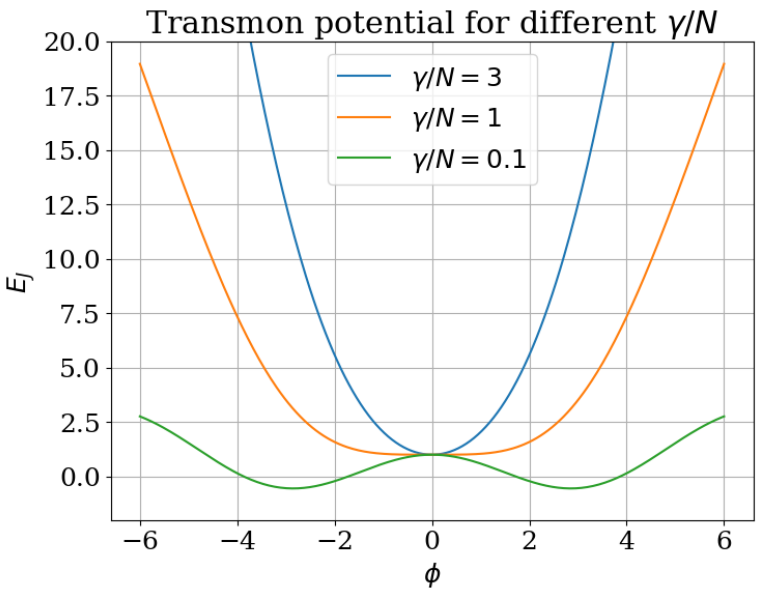


Figure 3: The transmon potential plottet fo 3 different regimes of γ/N.

The energy differences are then

$$\omega_{01,q} = 1.3 \quad , \quad \omega_{12,q} = 1.7, \tag{1.21}$$

and then we can find the relative anharmonicity to be

$$\alpha_{r,q} = \frac{\omega_{12,q} - \omega_{01,q}}{\omega_{01,q}} = 0.3 \tag{1.22}$$

Infinite Square Well: In the infinite square well then the frequencies go as $\omega_{n,\square} = (n+1)^2$. So in this case the frequencies are

$$\omega_{0,\square} = (0+1)^2 = 1 \quad , \quad \omega_{1,\square} = (1+1)^2 = 4 \quad , \quad \omega_{2,\square} = (2+1)^2 = 9 \tag{1.23}$$

$$\alpha_{r,\square} = \frac{\omega_{12,\square} - \omega_{01,\square}}{\omega_{01,\square}} = \frac{2}{3} \tag{1.24}$$

E3

a)

Since the current can only run through Josephson junctions and inductors then there is one complete loop that the current can run through. In this circuit there is three degrees of freedom. This is because there is four elements in the one loop and the flux's of the elements have to sum to the external flux which is fixed and that leaves three degrees of freedom.

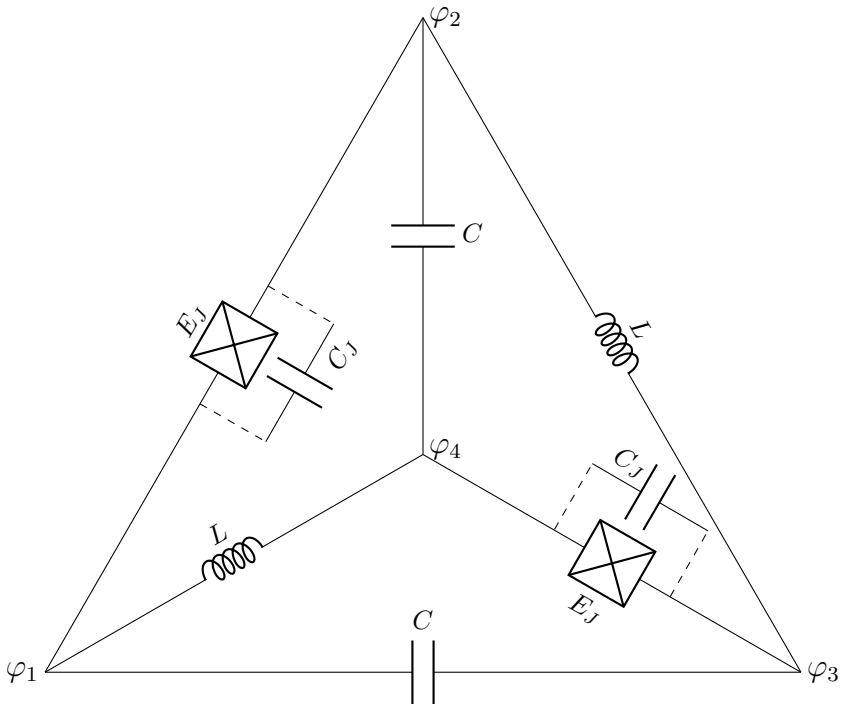


Figure 4: The circuit of a $\pi - 0$ qubit with the Josephson junctions capacitance shown with capacitors connected by dotted lines. The φ's are the node flux's.

b)

For this we define 3 new variables (we only need 3 since that is how many degrees of freedom we have) given by,

$$\theta = \varphi_2 + \varphi_3 - (\varphi_1 + \varphi_4) = \varphi_2 + \varphi_3 - \varphi_1 - \varphi_4 \tag{1.25}$$
$$\phi = \varphi_1 + \varphi_3 - (\varphi_2 + \varphi_4) = \varphi_1 + \varphi_3 - \varphi_2 - \varphi_4 \tag{1.26}$$
$$\xi = \varphi_1 + \varphi_2 - (\varphi_3 + \varphi_4) = \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 \tag{1.27}$$

We can then use these equations to write the branch flux through each component as a function of θ, ϕ and ξ. Doing that we get the following six equations

$$\varphi_1 - \varphi_2 = \frac{\phi - \theta}{2} \quad , \quad \varphi_1 - \varphi_4 = \frac{\phi + \xi}{2} \quad , \quad \varphi_2 - \varphi_4 = \frac{\xi + \theta}{2} \tag{1.28}$$
$$\varphi_1 - \varphi_3 = \frac{\xi - \theta}{2} \quad , \quad \varphi_2 - \varphi_3 = \frac{\xi - \phi}{2} \quad , \quad \varphi_3 - \varphi_4 = \frac{\phi + \theta}{2}. \tag{1.29}$$

We can now proceed as with the transmon circuit and write the kinetic and potential energy.

$$E_{kin} = \frac{C}{2}\left(\frac{\dot{\xi} - \dot{\theta}}{2}\right)^2 + \frac{C}{2}\left(\frac{\dot{\xi} + \dot{\theta}}{2}\right)^2 + \frac{C_J}{2}\left(\frac{\dot{\phi} - \dot{\theta}}{2}\right)^2 + \frac{C_J}{2}\left(\frac{\dot{\phi} + \dot{\theta}}{2}\right)^2$$
$$= \frac{C}{4}\dot{\xi}^2 + \frac{C}{4}\dot{\theta}^2 + \frac{C_J}{4}\dot{\phi}^2 + \frac{C_J}{4}\dot{\phi}^2 = \frac{C_\Sigma}{4}\dot{\theta}^2 + \frac{C}{4}\dot{\xi}^2 + \frac{C_J}{4}\dot{\phi}^2, \tag{1.30}$$

where we have define $C_\Sigma = C + C_J$. The potential energy is given by,

$$E_{pot} = \frac{1}{2L}\left(\frac{\xi+\phi}{2}\right)^2 + \frac{1}{2L}\left(\frac{\xi-\phi}{2}\right)^2 - E_J\cos\left(\frac{\phi-\theta}{2}\right) - E_J\cos\left(\frac{\phi+\theta}{2}\right)$$
$$= \frac{1}{2}\frac{1}{2L}(\xi^2+\phi^2) - E_J\cos(\phi)\cos(\theta+\varphi_e), \quad (1.31)$$

where in the last line we have added the external flux through the loop by making the substitution $\theta \to \theta + \varphi_e$. The full Lagrangian then becomes,

$$\mathcal{L} = \frac{C_\Sigma}{4}\dot{\theta}^2 + \frac{C}{4}\dot{\xi}^2 + \frac{C_J}{4}\dot{\phi}^2 - \frac{1}{2}\frac{1}{2L}\xi^2 - \frac{1}{2}\frac{1}{2L}\phi^2 + E_J\cos(\phi)\cos(\theta+\varphi_e). \quad (1.32)$$

The conjugate variables are then

$$q_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{C_\Sigma}{2}\dot{\theta} \quad , \quad q_\xi = \frac{\partial\mathcal{L}}{\partial\dot{\xi}} = \frac{C}{2}\dot{\xi} \quad , \quad q_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{C_J}{2}\dot{\phi} \quad (1.33)$$

We can then use the conjugate variables to write the Lagrangian

$$\mathcal{L} = \frac{1}{C_\Sigma}q_\theta^2 + \frac{1}{C}q_\xi^2 + \frac{1}{C_J}q_\phi^2 - \frac{1}{2}\frac{1}{2L}\phi^2 - \frac{1}{2}\frac{1}{2L}\xi^2 + E_J\cos(\phi)\cos(\theta+\varphi_e) \quad (1.34)$$

Then the Hamiltonian becomes

$$H = \frac{1}{C_\Sigma}q_\theta^2 + \frac{1}{C}q_\xi^2 + \frac{1}{C_J}q_\phi^2 + \frac{1}{2}\frac{1}{2L}\phi^2 + \frac{1}{2}\frac{1}{2L}\xi^2 - E_J\cos(\phi)\cos(\theta+\varphi_e) \quad (1.35)$$

To get the usual E_C and E_L constants we can make the following rewrite

$$H = 4\frac{e^2}{4C_\Sigma}n_\theta^2 + 4\frac{e^2}{C}n_\xi^2 + 4\frac{e^2}{C_J}q_\phi^2 + \frac{1}{2}\frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\right)^2\phi^2 + \frac{1}{2}\frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\right)^2\xi^2 - E_J\cos(\phi)\cos(\theta+\varphi_e) \quad (1.36)$$

With $E_C^\theta = e^2/4C_\Sigma$, $E_C^\xi = e^2/4C$, $E_C^\phi = e^2/4C_J$ and $E_L = (\Phi_0/2\pi)/2L$.

$$H = 4E_C^\theta(n_\theta - n_g^\theta)^2 + 4E_C^\phi(n_\phi - n^\phi)^2 + \frac{1}{2}E_L\phi^2 - E_J\cos(\phi)\cos(\theta+\varphi_e) + H_\xi \quad (1.37)$$

Where $H_\xi = 4E_C^\xi(n_\xi - n_g^\xi)^2 + E_L\xi^2/2$

c)

C and C_Σ are the heavy since the capacitance is the effective "mass" of the system. The main difference is that the heavy modes are highly localized compared to the more spread out light modes.

d)

The main difference between the two cases is the number states with almost the same energy. With zero flux bias there's one clear ground state in the lowest "valley". In the π flux bias case there will be 4 states with similar energies, but different parities. In reality two of them will be raised energy wise because the light modes are located lower in the potential. So we'll have 2 states very close in energy with different parity and then a bit above we'll have two other states, very close and with opposite parity.

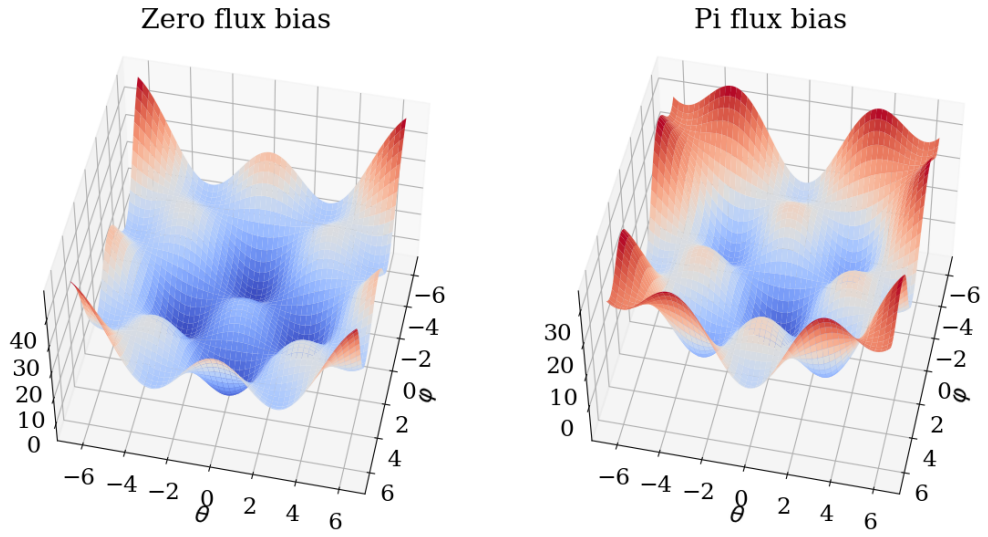


Figure 5: The potential "landscape" for two different external flux bias' for the $\pi - 0$ qubit.

Week 2: Numerical experiments and decoherence

E1

a)

Defining the sparse matrix for the cosine potential can be done in the following way

```
    phi = np.linspace(-np.pi, np.pi, N)
    cosMatrix = np.diag(np.cos(phi))
```

b)

The q^2 matrix is also simply to define. The one thing to note is the corner elements that ensures the periodicity.

```
    q2Matrix = 1/delta**2 * (2*np.diag(diagArr) - np.diag(offDiagArr, k = 1) -
    np.diag(offDiagArr, k = -1))
    q2Matrix[1][N-1], q2Matrix[N-1][1] = -1/delta**2, -1/delta**2  #Periodic
    boundary conditions
```

c)

The Hamiltonian we are trying to solve is

$$H = 4E_C n^2 - E_J \cos(\varphi_e)\cos(\varphi). \tag{2.1}$$

Which can be done in the following way,

```
    def hamiltonian(E_C, E_J, phi_e):
        E_kin = 4*E_C*q2Matrix
        E_pot = -E_J*np.abs(np.cos(phi_e))*cosMatrix
        return E_kin + E_pot, E_J*np.cos(phi_e)
```

The second element that the function returns is the potential and that is so that it can be plotted.

d)

Before being able to compute the wavefunctions and recreate Fig. 1a then we need to find the right parameters. On page 9 in the circuitQ paper we see that the Josephson energy is $E_J = 9.69GHz \cdot h$ and the capacitance is $C = 100fF$, which using $E_C = e^2/2C$ we find that $E_C = 0.194GHz \cdot h$. We can then plug these values into our Hamiltonian and solve for the eigenvalues and eigenvectors with the lowest energies and we find

E2

a)

The Γ_1 relaxation rate is given by

$$\Gamma_1 = \frac{1}{\hbar^2}\left|\langle 0|\frac{\partial H}{\partial \lambda}|1\rangle\right|^2 S(\omega_{01}). \tag{2.2}$$

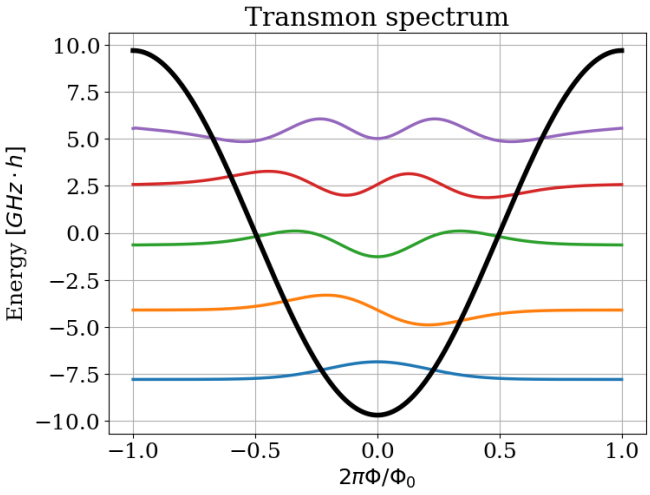


Figure 6: The first 5 wavefunction of the transmon plotted in the cosine potential.

Using equation (2.1) then we can take the derivative we respect to the external flux Psi_e and get,

$$\frac{\partial H}{\partial \Phi_e} = E_J \frac{2\pi}{\Phi_0} \sin\left(\frac{2\pi}{\Phi_0}\Phi_e\right) \cos(\phi) \tag{2.3}$$

Inserting that and the expression for the spectral function we get

$$\Gamma_1 = (2\pi)^2 \frac{E_J^2}{h^2} \frac{(2\pi)^2}{\Phi_0^2} \frac{2\pi 10^{-12} \frac{\Phi_0^2}{Hz} Hz}{|\omega_{01}|} |\langle 0|\cos(\phi)|1\rangle|^2 = (2\pi)^5 \tilde{E_J}^2 \frac{10^{-12}}{|\omega_{01}|} |\langle 0|\cos(\phi)|1\rangle|^2 \tag{2.4}$$

Here the $\tilde{E_J}^2$ has a tilde because it is now in units of $GHZ \cdot h$ as opposed to before.

b)

We can then sweep the external flux and plot the result to get figure 7. There are sweet spots at 0 and π as expected from fig. 2(b) in quantum engineers guide where we can see that the freqency should be invariant to first order noise.Unfortunately there seems to be a problem with the scaling. For a $T_1 \approx 100\mu s$ then $\Gamma_1 \approx 10^4 Hz$. This means that I'm roughly $10^9 - 10^{10}$ from a sensible result. That's about a factor GHz which could be the problem, but i couldn't find it.

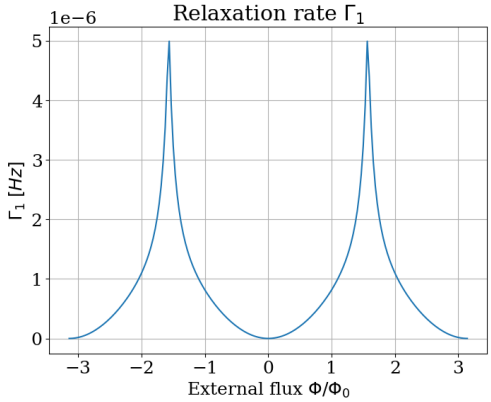


Figure 7: The relaxation rate as a function of the external flux, though probably with an incorrect scaling factor.

c)

For ohmic relaxation we take the derivative of the Hamiltonian with respect to q

$$\frac{\partial H}{\partial q} = 8E_C n \frac{1}{e} \tag{2.5}$$

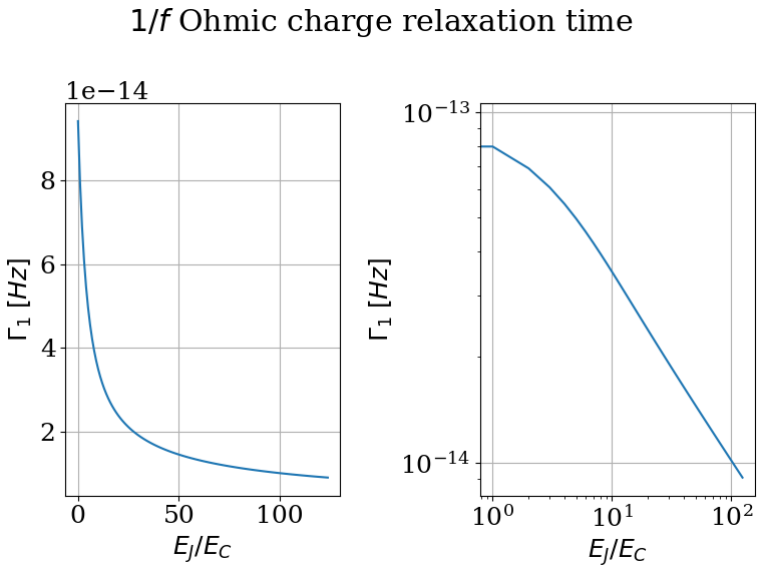


Figure 8: The Γ_1 relaxation rate as a function of E_J/E_C plotted on both a regular plot and a log-log plot. The linearity of the log-log plot suggests that it a power law.

We can then use equation (2.2) again to find
the relaxation rate Γ_1. Then we can now look
at how operating the transmon in different regimes affect the relaxation time. Fixing $E_C = 1GHz \cdot h$ and varying E_J we get figure 8. We do see a heavy suppression as expected, but the charge noise should be exponentially suppressed not power law suppressed as we see here. It's the anharmonicity that should be power law suppressed.

E3

a)

b)

c)

E4

Week 3: Single qubit gates

E1

a)

Firstly when going to the rotation frame then the H_0 term vanishes as mentioned in Quantum Engineer's Guide. A thing to note is how the unitary transformation matrix can be written

$$\mathcal{U}_{rf} = e^{iH_0 t} = \begin{pmatrix} e^{\frac{-i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{pmatrix} \tag{3.1}$$

So transforming the drive Hamiltonian we get

$$\tilde{H}_d = \mathcal{U}_{rf}\Omega V_d(t)\sigma_y \mathcal{U}_{rf}^\dagger = \Omega V_d(t) \begin{pmatrix} e^{\frac{-i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{-i\omega t}{2}} \end{pmatrix} \tag{3.2}$$

$$= \Omega V_d(t) \begin{pmatrix} 0 & ie^{i\omega t} \\ -ie^{-i\omega t} & 0 \end{pmatrix} \tag{3.3}$$

From here it's not obvious what to do so we'll work backwards from the result we want to show, which is equation (84) in Quantum Engineer's Guide.

$$\begin{aligned} \Omega V_d(t)(\cos(\omega t)\sigma_y - \sin(\omega t)\sigma_x) &= \Omega V_d(t) \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\sigma_y - \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\sigma_x \right] \\ &= \Omega V_d(t) \frac{i}{2} \left[\begin{pmatrix} 0 & -e^{i\omega t} - e^{-i\omega t} \\ e^{i\omega t} + e^{-i\omega t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{i\omega t} - e^{-i\omega t} \\ e^{i\omega t} - e^{-i\omega t} & 0 \end{pmatrix} \right] \end{aligned} \tag{3.4}$$

Which adds up to be

$$\Omega V_d(t)(\cos(\omega t)\sigma_y - \sin(\omega t)\sigma_x) = \Omega V_d(t) \begin{pmatrix} 0 & ie^{i\omega t} \\ -ie^{-i\omega t} & 0 \end{pmatrix} = \mathcal{U}_{rf}\Omega V_d(t)\sigma_y = \tilde{H}_d \tag{3.5}$$

With our new expression for $\tilde{H}_d = \Omega V_d(t)(\cos(\omega t)\sigma_y - \sin(\omega t)\sigma_x)$ we can insert a generic drive function $V_d(t) = V_0 v(t)$ which can be written as,

$$v_(t) = s(t)\left(\cos(\phi)\sin(\omega_d t) + \sin(\phi)\cos(\omega_d t)\right) = s(t)\left(I\sin(\omega_d t) - Q\cos(\omega_d t)\right), \tag{3.6}$$

where $I = \cos(\phi)$ and $Q = -\sin(\phi)$. We can now insert this expression and we'll drop the constants for now and insert them again later

$$\begin{aligned} &\left(I\sin(\omega_d t) - Q\cos(\omega_d t)\right)\left(\cos(\omega t)\sigma_y - \sin(\omega t)\sigma_x\right) \\ &= \frac{1}{2}\left[I\sigma_y \sin(\omega_d t)\cos(\omega t) - I\sigma_x \sin(\omega_d t)\sin(\omega t) - Q\sigma_y \cos(\omega_d t)\cos(\omega t) + Q\sigma_x \cos(\omega_d t)\sin(\omega t)\right] \end{aligned} \tag{3.7}$$

This can be simplified using trigonometric identities which for example can be found here. When using these equations we immediately drop the fast rotating term with $\omega + \omega_d$. (We'll insert the factor 1/2 again later.)

$$= I\sigma_y \sin(\delta\omega t) - I\sigma_x \cos(\delta\omega t) - Q\sigma_y \cos(\delta\omega t) - Q\sigma_x \sin(\delta\omega t) \tag{3.8}$$

$$= I(\cos(\delta\omega t) - \sigma_y \sin(\delta\omega t)) + Q(\sigma_y \cos(\delta\omega t) + \sin(\delta\omega t)) \tag{3.9}$$

In the upper right corner of the matrix $\sigma_x = 1$ and $\sigma_y = -i$ and so we get

$$\Rightarrow -I(\cos(\delta\omega t) + i\sin(\delta\omega t)) - Q(i\cos(\delta\omega t) + \sin(\delta\omega t)) \tag{3.10}$$

$$= -(Ie^{i\delta\omega t} - iQe^{i\delta\omega t}) = -e^{i(\delta\omega t+\phi)}. \tag{3.11}$$

In the last step we've used $I - iQ = e^{i\phi}$. Likewise for the lower left corner we can insert $\sigma_x = 1$ and $\sigma_y = i$ and get that there the element is $-e^{-i(\delta\omega t+\phi)}$. Writing the matrix explicitly and inserting the neglected factors back in we find that the driving Hamiltonian can written on the following form

$$\tilde{H}_d = -\frac{\Omega V_0 s(t)}{2}\begin{pmatrix} 0 & e^{i(\delta\omega t+\phi)} \\ e^{-i(\delta\omega t+\phi)} & 0 \end{pmatrix} \tag{3.12}$$

If there's no detuning then $\delta\omega = 0$ and we can simplify the Hamiltonian further

$$\tilde{H}_d = -\frac{\Omega V_0 s(t)}{2}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} = -\frac{\Omega V_0 s(t)}{2}\begin{pmatrix} 0 & \cos(\phi) + i\sin(\phi) \\ \cos(\phi) - i\sin(\phi) & 0 \end{pmatrix} \tag{3.13}$$

$$= -\frac{\Omega V_0 s(t)}{2}\begin{pmatrix} 0 & I - iQ \\ I + iQ & 0 \end{pmatrix} = -\frac{\Omega V_0 s(t)}{2}(I\sigma_x + Q\sigma_y) \tag{3.14}$$

We can know look at a concrete example of applying this Hamiltonian. We'll look at the case for $\phi = 0$ so $I = 1$ and $Q = 0$, so the Hamiltonian becomes $\tilde{H}_d = -\Omega V_0 s(t)\sigma_x/2$. Writing the time evolution operator for this Hamiltonian it is (see equation 2.31 in Sakurai).

$$\mathcal{U}_{\nabla\{,\tilde{\mathcal{H}}_\lceil} = \exp\left(\frac{-i}{2}(-\Omega V_0)\int_0^t s(t')dt'\sigma_x\right). \tag{3.15}$$

But this also looks like a rotation operator (see equation 3.60 in Sakurai), where we can define the rotation angle as

$$\Theta(t) = -\Omega V_0 \int_0^t s(t')dt' \tag{3.16}$$

b)

E2

a)

The symmetric tunable transmon is given by

$$H = 4E_C n^2 - E_J \cos(\phi_e)\cos(\phi) \tag{3.17}$$

Writing this in the charge basis the n^2 matrix becomes diagonal with n^2 on the diagonals and $\cos(\phi_e)$ has off-diagonal elements.

$$n = \begin{pmatrix} -n^2_{coutoff} & & \\ & \ddots & \\ & & n^2_{coutoff} \end{pmatrix} \quad , \quad \cos(\phi) = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & \\ 1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & 1 & 0 \end{pmatrix} \tag{3.18}$$

In python these can be constructed as following

```
nList = np.array((range(-n_cut, n_cut+1)))

nMatrix = np.diag(nList)
n2Matrix = np.matmul(nMatrix, nMatrix)

offDiag = np.ones(2*n_cut)
cosMatrix = 0.5 *(np.diag(offDiag, k = 1) + np.diag(offDiag, k = -1))
```

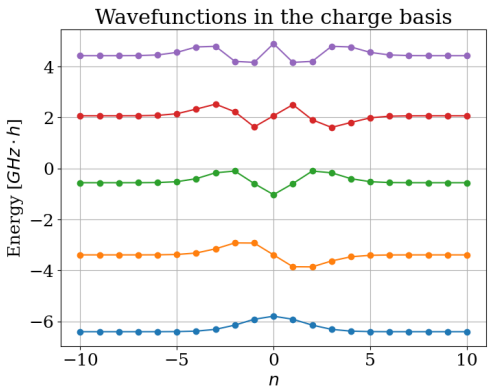
b)

We can then use these matrices to construct the Hamiltonian

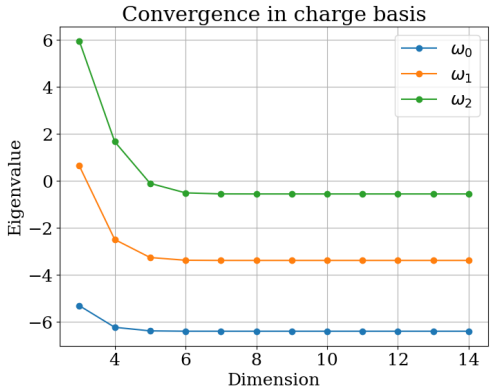
```
E_kin = 4*E_C*n2Matrix
E_pot = -E_J*np.sqrt(np.cos(phi_e)**2+d**2*np.sin(phi_e)**2)*cosMatrix
return E_kin + E_pot
```

c)

With the Hamiltonian constructed then we can compute the eigenvalue and eigenvectors. To



(a) The lowest 5 states in the transmon plotted at their respective energies.



(b) The 3 lowest energy states in the transmon plotted as a function of the cutoff value.

Figure 9: As we can see in the above plots our choice of cutoff is more than adequate.

check whether 10 is an appropriate cutoff we can plot the energies as a function of the cutoff to see when they converge. As we can see in figure 9b 10 is more than good enough for the simulations that we are doing here.

d)

E3

a)

Since we've already calculated the energies then it's straight forward to construct the Hamiltonian as written in the exercise

```
H_Q = np.zeros((3,3))
H_Q[1,1] = eigenvalues[1]-eigenvalues[0]
H_Q[2,2] = eigenvalues[2]-eigenvalues[0]
```

We can know move on to the coupling Hamiltonian H_C. With a capacitive coupling the elements in the coupling Hamiltonian are given by the number/charge operator $H_C^{ij} = \langle \Psi_i | \hat{n} | \Psi_j \rangle$. Implementing this it looks like

```
H_C = np.zeros_like(H_Q)

nMatrix = np.diag(nList)

for i in range(3):
    for j in range(3):
        H_C[i,j] = (eigenvectors[:,j].T)@nMatrix@eigenvectors[:,i]

g1, g2 = H_C[0,1], H_C[1,2]
```

Where *nlist* is a list over the possible n's (i.e. from $-n_{cutoff}$ to n_{cutoff}). Doing this the coupling Hamiltonian in the 3x3 subspace is

$$H_C = \begin{pmatrix} 0 & -1.09 & 0 \\ -1.09 & 0 & 1.49 \\ 0 & 1.49 & 0 \end{pmatrix} \tag{3.19}$$

So $g_1 = -1.09$ and $g_2 = 1.49$ for our chosen parameters.

b)

For our envelope function I've chosen a normalized Gaussian so the integral in the expression for A disappears.

c)

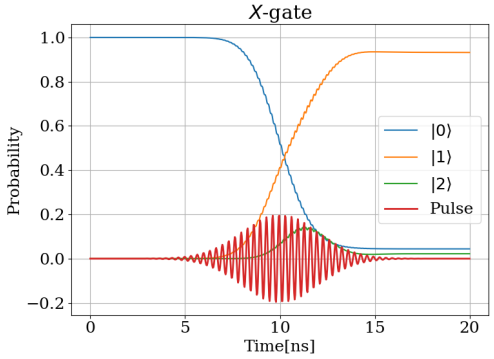
A is then simply $A = \pi/g_1$ and the function to construct the full Hamiltonian is

```
T = 20
pulse_mean, pulse_sigma = T/2, 2

A=np.pi/(g1)

def H_t(t):
    return H_Q + A * envolope(t, pulse_mean, pulse_sigma) * np.cos(omega*t) *
    H_C
```

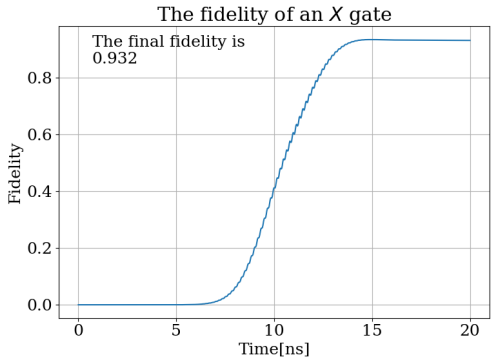
d)

Now that we have the Hamiltonian we can through it into `ivp_solve` and solve it for an initial state. In our case, since we are making an X-gate we start with $|\Psi_{init}\rangle = (1, 0, 0)^T$. Doing that we can see what the fidelity of our gate would be in figure 10b.

e)



(a) The probability of being in either $|0\rangle$, $|1\rangle$ or $|2\rangle$ as the X-gate is being applied to our qubit.



(b) The fidelity of the X-gate (i.e. probability of being the the target state of $(0, 1, 0)^T$)

Figure 10

Week 4: The fluxonium qubit

E1

a)

The fluxuniom qubit's Hamiltonian is,

$$H = 4E_C n^2 - E_J \cos(\phi) + \frac{1}{2} E_L (\phi + \phi_e)^2. \tag{4.1}$$

The two last terms is what plays the role of the potential. For this I've decided to plot the potential for a range of external flux's instead of a slider so it can be but in this PDF, but one could also easily use the slider widget in matplotlib to get a feel for how the potential reacts to different external potentials.

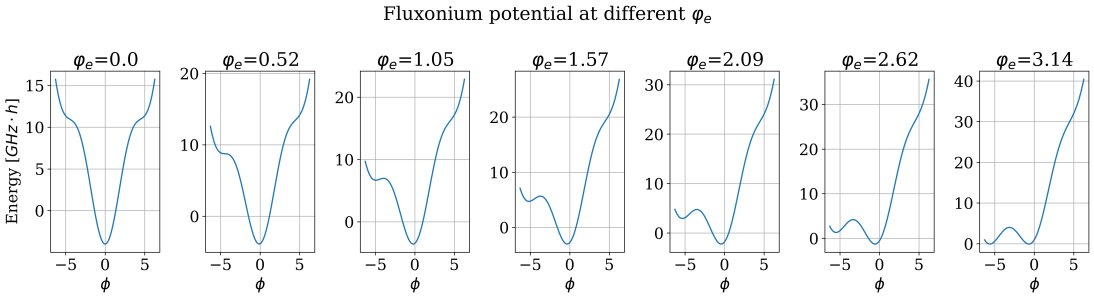


Figure 11: The "potential" part of the fluxonium qubit at 7 different external flux's. The parameters for the plot are $E_J = E_L = 1$.

b)

We can then do the coordinate transformation $\phi \to \phi - \phi_e$. Now the potential is $-E_J \cos(\phi - \phi_e) + E_L \phi^2/2$. We can then plot this along with the previous plot to see how they differ. Unsurprisingly such a coordinate transformation simply shifts the potential to the side. In one case the quadratic part gets shifted to one side, and in the other case the cosine part gets shifted to the other side, but they are physically the same potential.

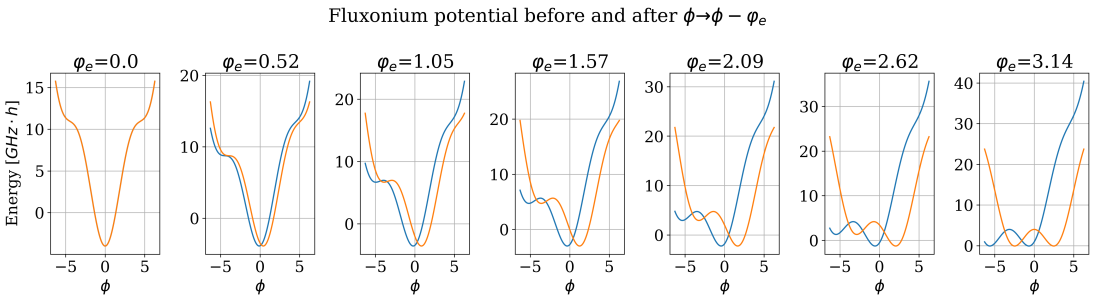


Figure 12: The two different versions of the potential plotted on top of each other. Here it's clear to see that they are the same simply shifted.

E2

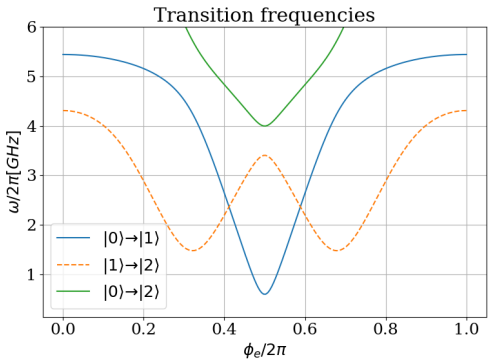
a)

Since we are working in the flux basis here then the potential is a diagonal matrix and then the Hamiltonian can be constructed as following

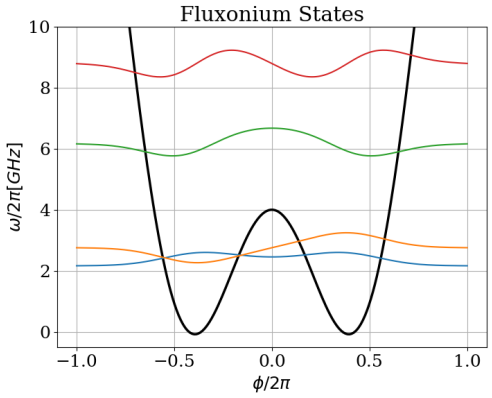
```
def Hamiltonian(E_Ch, E_Lh, E_Jh):
    diagArr = np.ones(N)
    offDiagArr = np.ones(N-1)
    q2Matrix = 1/delta**2 * (2*np.diag(diagArr) - np.diag(offDiagArr, k = 1) -
    np.diag(offDiagArr, k = -1))

    #Potential matrix
    pot = fluxonium_potential(phi, np.pi, E_J=E_Jh, E_L=E_Lh)
    potMatrix = np.diag(pot)

    return 4*E_Ch* q2Matrix + potMatrix, pot, E_Jh
```

b)

Solving this Hamiltonian we can recreate figure 2(a) and 2(b) from "Blueprint for High-Performance Fluxonium Quantum Processor"



(a) The three lowest transition frequencies dependency on external flux.



(b) The four lowest eigenstates of the fluxonium qubit.

Figure 13

c)

d)

e)

Week 5: Two-qubit gates

E1

a)

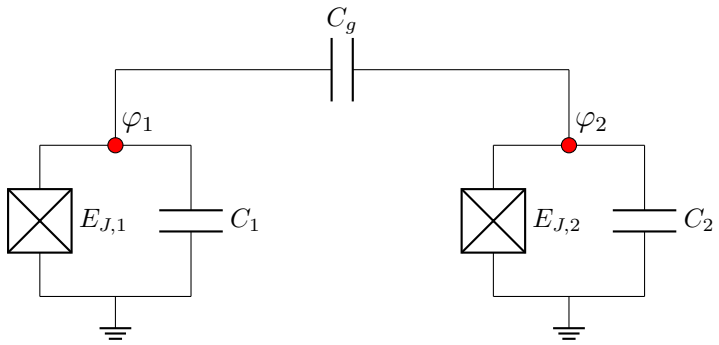


Figure 14: Capacitive coupling between two transmon qubits where the red dots are the non-zero node fluxs.

b)

Using the node flux's we can write the kinetic and potential energy

$$E_{kin} = \frac{C_1}{2}\dot{\varphi}_1^2 + \frac{C_2}{2}\dot{\varphi}_2^2 + \frac{C_g}{2}(\dot{\varphi}_1 - \dot{\varphi}_2)^2 \tag{5.1}$$

$$E_{pot} = -E_{J,1}\cos(\phi_1) - E_{J,2}\cos(\phi_2) \tag{5.2}$$

We can then write the Langrangian

$$\mathcal{L} = \frac{C_1}{2}\dot{\varphi}_1^2 + \frac{C_2}{2}\dot{\varphi}_2^2 + \frac{C_g}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 - 2\dot{\varphi}_1\dot{\varphi}_2) - E_{pot} \tag{5.3}$$

$$= \left(\frac{C_1}{2} + \frac{C_g}{2}\right)\dot{\varphi}_1^2 + \left(\frac{C_2}{2} + \frac{C_g}{2}\right)\dot{\varphi}_2^2 - C_g\dot{\varphi}_1\dot{\varphi}_2 - E_{pot} \tag{5.4}$$

$$\approx \frac{C_1}{2}\dot{\varphi}_1^2 + \frac{C_2}{2}\dot{\varphi}_2^2 - C_g\dot{\varphi}_1\dot{\varphi}_2 - E_{pot} \tag{5.5}$$

Where we've used the fact that $C_g \ll C_{1/2}$ so $C_{1/2} + C_g \approx C_{1/2}$. We then find the conjugate variables q_1 and q_2

$$q_1 = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = C_1\dot{\varphi}_1 + C_g\dot{\varphi}_2 \quad \text{and} \quad q_2 = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = C_2\dot{\varphi}_2 + C_g\dot{\varphi}_1 \tag{5.6}$$

We can now find that capacitance matrix and invert it to get and expression for $\dot{\varphi}_1$ and $\dot{\varphi}_2$ expressed by q_1 and q_2.

$$\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix} = \begin{pmatrix}C_1 & -C_g\\ -C_g & C_2\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} \Rightarrow \mathbf{C}^{-1} = \frac{1}{C_1C_2 - C_g^2}\begin{pmatrix}C_2 & C_g\\ C_g & C_1\end{pmatrix} \approx \frac{1}{C_1C_2}\begin{pmatrix}C_2 & C_g\\ C_g & C_1\end{pmatrix} \tag{5.7}$$

Now we have two equations for $\dot{\varphi}_1$ and $\dot{\varphi}_2$

$$\dot{\varphi}_1 = \frac{1}{C_1C_2}(C_2q_1 + C_gq_2) \quad \text{and} \quad \dot{\varphi}_2 = \frac{1}{C_1C_2}(C_1q_2 + C_gq_1) \tag{5.8}$$

The Hamiltonian is given by $H = \sum_i \dot{\varphi}_i q_i - \mathcal{L}$, so we'll first calculate the sum

$$\sum_i \dot{\varphi}_i q_i = \frac{1}{C_1C_2}\left(C_2q_1^2 + C_gq_1q_2 + C_1q_2^2 + C_gq_1q_2\right) = \frac{1}{C_1}q_1^2 + \frac{1}{C_2}q_2^2 + \frac{2C_g}{C_1C_2}q_1q_2 \tag{5.9}$$

Next we have to rewrite the kinetic term so it depends on q, we'll start with the first term

$$\frac{C_1}{2}\dot{\varphi}_1^2 = \frac{C_1}{2}\left(\frac{1}{C_1C_2}(C_2q_1 + C_gq_2)\right)^2 = \frac{1}{2C_1}q_1^2 + \frac{C_g}{C_1C_2}q_1q_2, \tag{5.10}$$

where we've dropped a term that is proportional to C_g^2/C_1C_2. The second term is almost the same,

$$\frac{C_2}{2}\dot{\varphi}_2^2 = \frac{C_2}{2}\left(\frac{1}{C_1C_2}(C_1q_2 + C_gq_1)\right)^2 = \frac{1}{2C_2}q_2^2 + \frac{C_g}{C_1C_2}q_1q_2. \tag{5.11}$$

The third term in the kinetic energy can be written as

$$C_g\dot{\varphi}_1\dot{\varphi}_2 = \frac{C_g}{(C_1C_2)^2}(C_2q_1 + C_gq_2)(C_1q_2 + C_gq_1) \approx \frac{C_g}{C_1C_2}q_1q_2 \tag{5.12}$$

In the $\dot{\varphi}_1\dot{\varphi}_2$ we've dropped any term of order C_g^2 or higher since they would be extremely small. Combining those three equations the kinetic energy becomes

$$E_{kin} = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2 + \frac{C_g}{C_1C_2}q_1q_2 \tag{5.13}$$

The Hamiltonian then becomes

$$H = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2 + \frac{C_g}{C_1C_2}q_1q_2 + E_{pot} \tag{5.14}$$

Using $q_i = 2en_i$, $E_{C,1} = e^2/2C_i$ and inserting the potential energy we recover the correct Hamiltoninan

$$H = 4E_{C,1}n_1^2 + 4E_{C,2}n_2^2 - E_{J,1}\cos(\phi_1) - E_{J,2}\cos(\phi_2) + \frac{4e^2C_g}{C_1C_2}n_1n_2 \tag{5.15}$$

c)

d)

E2

a)

We can rewrite the condition given in the problem as $c_1c_4 = c_2c_3 \Rightarrow c_2 = c_1c_4/c_3$, with this we can substitute in to the general state $|c\rangle$

$$|c\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1\frac{c_4}{c_3} \\ c_3 \\ c_3\frac{c_4}{c_3} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{c_4}{c_3} \end{pmatrix} \tag{5.16}$$

So $|c\rangle$ is a product state.

b)

Using the iSWAP gate on our trial function we get

$$\begin{pmatrix}1&0&0&0\\0&0&-i&0\\0&-i&0&0\\0&0&0&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\-i\\-i\\1\end{pmatrix} \tag{5.17}$$

We can then see that $c_1c_4 = 1/4$ and $c_2c_3 = -1/4$ so the resulting state is not a product state the the two qubits are know entangled. Likewise for the $\sqrt{\text{iSWAP}}$ we get

$$\begin{pmatrix}1&0&0&0\\0&1/\sqrt{2}&-i/\sqrt{2}&0\\0&-i/\sqrt{2}&1/\sqrt{2}&0\\0&0&0&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\\frac{1-i}{\sqrt{2}}\\\frac{1-i}{\sqrt{2}}\\1\end{pmatrix} \tag{5.18}$$

Again using the condition from a) we see that $c_1c_4 = 1/4$ and $c_2c_3 = -i/4$ and again we have an entangled state.

c)

With the general iSWAP gate from (109) we end up with the state

$$\begin{pmatrix}1&0&0&0\\0&\cos(gt)&-i\sin(gt)&0\\0&-i\sin(gt)&\cos(gt)&0\\0&0&0&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\\cos(gt)-i\sin(gt)\\\cos(gt)-i\sin(gt)\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\e^{-igt}\\e^{-igt}\\1\end{pmatrix} \tag{5.19}$$

Since the iSWAP-gate is maximally entangled and the $\sqrt{\text{iSWAP}}$ is "half-entangled" then one way to measure entanglement could the angle between c_1c_4 and c_2c_3. In the iSWAP case the angle is π whereas in the $\sqrt{\text{iSWAP}}$ it's only $\pi/2$.

d)

We can now try different gates and check whether this measure of entanglement also holds for other gates and states. First let's try the CPHASE gate,

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}. \tag{5.20}$$

Here $c_1c_4 = -1/4$ and $c_2c_3 = 1/4$, so the measure still holds up so far. We can also try the CNOT gate

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}. \tag{5.21}$$

Now $c_1c_4 = 1/4$ and $c_2c_3 = 1/4$ and they aren't entangled, but so far we've used the same trial function, let's use another trial function with the CNOT gate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.22}$$

$c_1c_4 = 1/4$ and $c_2c_3 = 0$ so they are entangled, but the angle measure doesn't work here so the can't be used as a general measure of entanglement.

Week 6: Two-qubit gates (fluxonium)

E1

a)

The relevant constants for the coupled qubits are

$$E_{J,A} = 4\text{GHz} \cdot h \qquad E_{C,A} = 1\text{GHz} \cdot h \qquad E_{L,A} = 0.9\text{GHz} \cdot h \tag{6.1}$$

$$E_{J,B} = 4\text{GHz} \cdot h \qquad E_{C,B} = 1\text{GHz} \cdot h \qquad E_{L,B} = 0.9\text{GHz} \cdot h \quad , \quad J_C = 0.002\text{GHz} \cdot h \tag{6.2}$$

b)

The Hamiltonian for the single fluxonium qubit is,

$$H_f = 4E_C\hat{n}^2 - E_J\cos(\phi) + \frac{E_L}{2}(\phi + \phi_e)^2. \tag{6.3}$$

Which as python code can be written as

```
dim = 30
cutoff = 2*np.pi

def qubit_H(E_C, E_J, E_L, dimension = dim, cut = cutoff):
    E_kin = 4*E_C*n2Matrix(cut, dimension)
    E_pot, potential, phi = potMatrix(cut, dimension, E_J, E_L)
    return E_kin + E_pot, potential, phi
```

The second part is the coupling between the two qubits. That Hamiltonian is given by

$$H_{coup} = J_C\hat{n}_A\hat{n}_B - J_L\hat{\phi}_A\hat{\phi}_B. \tag{6.4}$$

This part of the Hamiltonian can be codes as following

```
def coupling_H(J_c, dim = minimal_dim, cutoff = minimal_cut, J_Lc = J_L):
    n = nMatrix(cutoff, dim)
    phiMatrix = np.diag(np.linspace(-cutoff, cutoff, dim))
    coupled_matrix = J_c * np.kron(n, n) - J_Lc * np.kron(phiMatrix, phiMatrix)
    return coupled_matrix
```

These two (really three because there's two of the first) Hamiltonians to make the full Hamiltonian following

$$H = H_{f,A} \otimes I_B + I_A \otimes H_{f,B} + H_{coup} \tag{6.5}$$

c)

With the full Hamiltonian we can simply calculate the eigenvalues and eigenvectors to find ξ_{ZZ} and μ_φ. They don't look exactly like in the paper, but they have the right shape. The longitudinal coupling has two minima, they are just not quite in the right spot, and the hybridization is a straight line with a negative slop, it's just to negative.

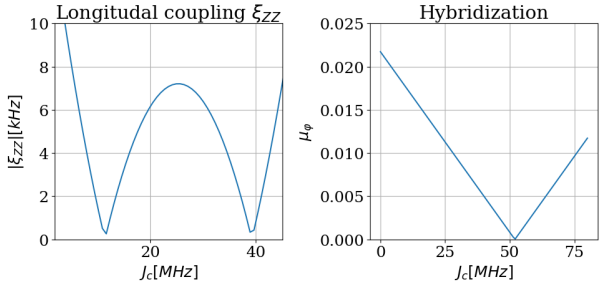


Figure 15: A recreation of figure 5(c) froom Quantum Engineer's Guide.

So i suspect that the method is correct, but there's a conversion factor or a constant somewhere that's wrong.

E2

a)

The first thing to do is to define the pulse shape which is given by

$$\mathcal{E}(t) = \begin{cases} \frac{\epsilon}{2}\left[1 - \cos\left(\frac{\pi t}{\tau_{ramp}}\right)\right] & 0 \leq t \leq \tau_{ramp} \\ \epsilon & \tau_{ramp} \leq t \leq \tau_g - \tau_{ramp} \\ \frac{\epsilon}{2}\left[1 - \cos\left(\frac{\pi(\tau_g - t)}{\tau_{ramp}}\right)\right] & \tau_g - \tau_{ramp} \leq t \leq \tau_g \end{cases} \tag{6.6}$$

It's then quite straight forward to construct the drive Hamiltonian as it is given in the exercise sheet. We can then add the two Hamiltonians as we did when we constructed the single qubit gates in week 3. Having done that we will restrict our selves to a subspace of the original Hamiltonian for computational speed. To do that we remember that we can write an operator in a specific representation with $\langle\Psi_i|M|\Psi_j\rangle = M_{ij}$. We then let $\Psi_{i/j}$ run over the first 20 or however many we choose eigenstates. In python that can be done in the following way,

```
def convert_to_subspace(operator, dimension):
    eigenval, eigenvect = sc.sparse.linalg.eigsh(operator, k = dimension, which
    = "SA")

    new_operator = np.ones((dimension, dimension), dtype=complex)

    for i in range(dimension):
        for j in range(dimension):
            new_operator[i][j] = eigenvect[:,i]@operator@eigenvect[:,j]

    return new_operator

```

and with that we can express the Hamiltonian as a 20x20 matrix.

b)

Week 7: Readout and the C-shunted flux qubit

E1

a)

First we expand the H_0 term to second order

$$e^{\eta S} H_0 e^{-\eta S} \approx= H_0 + \eta[S, H_0] + \frac{\eta^2}{2}[S, [S, H_0]] \tag{7.1}$$

Expanding the ηH_1 term to second order in η we get

$$\eta e^{\eta S} H_1 e^{-\eta S} \approx \eta H_1 + \eta^2 [S, H_1] \tag{7.2}$$

So the full transformed Hamiltonian is,

$$H' = H_0 + \eta\left([S, H_0] + H_1\right) + \eta^2 \left(\frac{1}{2}[S, [S, H_0]] + [S, H_0]\right) \tag{7.3}$$

b)

The first order term is zero if $[S, H_1] = -H_1$. Inserting that we get,

$$H' = H_0 + \eta^2 \left(\frac{1}{2}[S, -H_1] + [S, H_1]\right) = H_0 + \frac{\eta^2}{2}[S, H_1] \tag{7.4}$$

c)

We start by calculating the commutator $[S, H_0]$. To do that we calculate ech part seperately

$$SH_0 = \sum_{ij} |i\rangle\langle j| \left(\nu_{ij} a + \mu_{ij} a^\dagger\right) \sum_k \omega_k |k\rangle\langle k| + \omega_r a^\dagger a \tag{7.5}$$

$$= \sum_{ij} |i\rangle\langle j| \, \omega_j (\nu_{ij} a + \mu_{ij} a^\dagger) + \omega_r (\nu_{ij} a a^\dagger a + \mu_{ij} a^\dagger a^\dagger a) \tag{7.6}$$

Where we've used that $\langle j|k\rangle = \delta_{jk}$. Similarly the other part of the commutator becomes

$$H_0 S = \sum_{ij} |i\rangle\langle j| \, \omega_i (\nu_{ij} a + \mu_{ij} a^\dagger) + \omega_r (\nu_{ij} a^\dagger a a + \mu_{ij} a^\dagger a a^\dagger) \tag{7.7}$$

Combining these to equations we get

$$[S, H_0] = \sum_{ij} |i\rangle\langle j| \left(\omega_j - \omega_i\right)(\nu_{ij} a + \mu_{ij} a^\dagger) + \omega_r \left[\nu_{ij}(a a^\dagger a - a^\dagger a a) + \mu_{ij}(a^\dagger a^\dagger a - a^\dagger a a^\dagger)\right] \tag{7.8}$$

$$= \sum_{ij} |i\rangle\langle j| \left(\omega_j - \omega_i\right)(\nu_{ij} a + \mu_{ij} a^\dagger) + \omega_r (\nu_{ij} a - \mu_{ij} a^\dagger) \tag{7.9}$$

$$= \sum_{ij} |i\rangle\langle j| \left[(\omega_j - \omega_i + \omega_r)\nu_{ij} a + (\omega_j - \omega_i - \omega_r)\mu_{ij} a^\dagger\right] \tag{7.10}$$

Comparing this to H_1 and using $[S, H_0] = -H_1$ then it's clear that,

$$\nu_{ij} = \frac{g_{ij}}{\omega_i - \omega_j - \omega_r} \quad \text{and} \quad \nu_{ij} = \frac{g_{ij}}{\omega_i - \omega_j + \omega_r}. \tag{7.11}$$

d)

We can then calculate the shift to the energies caused by the second order term $[S, H_1]/2$. Once again we split the calculation up into more manageable pieces, but first will introduce some new notation for brevity,

$$\frac{1}{\omega_{ij}^{\pm}} = \frac{1}{\omega_i - \omega_j \pm \omega_r} \quad \text{and} \quad \frac{1}{\omega_{ij}^{-}} = -\frac{1}{\omega_{ij}^{+}} \tag{7.12}$$

$$SH_1 = \sum_{ij} |i\rangle\langle j| \left(\frac{1}{\omega_{ij}^{-}} a + \frac{1}{\omega_{ij}^{+}} a^\dagger\right) \sum_{lm} |l\rangle\langle m| \, g_{lm}(a + a^\dagger) \tag{7.13}$$

$$= \sum_{ijm} |i\rangle\langle m| \, g_{ij} g_{jm} \left(\frac{1}{\omega_{ij}^{-}} a + \frac{1}{\omega_{ij}^{+}} a^\dagger\right)(a + a^\dagger) \tag{7.14}$$

$$\Rightarrow \sum_{ij} |i\rangle\langle i| \, |g_{ij}|^2 \left(\frac{1}{\omega_{ij}^{-}} aa + \frac{1}{\omega_{ij}^{-}} aa^\dagger + \frac{1}{\omega_{ij}^{+}} a^\dagger a + \frac{1}{\omega_{ij}^{+}} a^\dagger a^\dagger\right) \tag{7.15}$$

Likewise the second part becomes

$$H_1 S = \sum_{ij} |i\rangle\langle i| \, |g_{ij}|^2 \left(\frac{1}{\omega_{ji}^{-}} aa + \frac{1}{\omega_{ji}^{-}} a^\dagger a + \frac{1}{\omega_{ji}^{+}} aa^\dagger + \frac{1}{\omega_{ji}^{+}} a^\dagger a^\dagger\right) \tag{7.16}$$

Dropping the two photon processes the commutator becomes

$$\frac{[S, H_1]}{2} = \frac{1}{2} \sum_{ij} |i\rangle\langle i| \, |g_{ij}|^2 \left[aa^\dagger \left(\frac{1}{\omega_{ij}^{-}} - \frac{1}{\omega_{ji}^{+}}\right) + a^\dagger a \left(\frac{1}{\omega_{ij}^{+}} - \frac{1}{\omega_{ji}^{-}}\right)\right] \tag{7.17}$$

$$= \frac{1}{2} \sum_{ij} |i\rangle\langle i| \, |g_{ij}|^2 2 \left[\frac{1}{\omega_{ij}^{-}} + \left(\frac{1}{\omega_{ij}^{-}} + \frac{1}{\omega_{ij}^{+}}\right) a^\dagger a\right] \tag{7.18}$$

$$= \sum_{ij} |i\rangle\langle i| \, |g_{ij}|^2 \left[\frac{1}{\omega_{ij} - \omega_r} + \left(\frac{1}{\omega_{ij} - \omega_r} + \frac{1}{\omega_{ij} + \omega_r}\right) a^\dagger a\right] \tag{7.19}$$

e)

E2

a)

The transmon Hamiltonian is that same as has we've done previously

```
def hamiltonian(E_C = 0.2, E_J = 10, phi_e = np.pi):
    E_kin = 4*E_C*n2Matrix
    E_pot = -E_J*np.abs(np.cos(phi_e))*cosMatrix
    return E_kin + E_pot, E_J*np.cos(phi_e)
```

b)

Coupling a transmon to a resonator will shift it's energy the shift is denoted χ and can be found with the following with following equation,

$$\chi = \sum_j \chi_{0j} - \chi_{1j} = \sum_j \left[|g_{0j}|^2 \left(\frac{1}{\omega_{0j} - \omega_r} + \frac{1}{\omega_{0j} + \omega_r} \right) - |g_{1j}|^2 \left(\frac{1}{\omega_{1j} - \omega_r} + \frac{1}{\omega_{1j} + \omega_r} \right) \right] \tag{7.20}$$

Where $|g_{ij}|^2 = g^2 |\langle i|n|j\rangle|^2$. To calculate χ we first need all the energies and states so we can calculate ω_{oj}, ω_{1j}, $|g_{0j}|^2$ and $|g_{1j}|^2$.

```
omega01 = transmon_energy[2]-transmon_energy[1]

for i in range(dim-2):
    g0[i] = g**2 * np.abs(transmon_state[:,1]@nMatrix@transmon_state[:,i+1])**2
    g1[i] = g**2 * np.abs(transmon_state[:,2]@nMatrix@transmon_state[:,i+1])**2
    omega0[i] = transmon_energy[i+1]-transmon_energy[1]
    omega1[i] = transmon_energy[i+1]-transmon_energy[2]

```

For this calculation we've truncated the sum at $j = 10$ since for hugh j the transition rates $|g_{0j}|^2$ and $|g_{1j}|^2$ will becomes essentially zeros. With all these values we can know calculate χ.

```
def Xi(delta_arr):
    def xi_val(delta):
        Xi_arr = g0* (1/(omega0-omega01+delta) + 1/(omega0-omega01+delta)) - g1
    * (1/(omega1-omega01+delta) + 1/(omega1-omega01+delta))
        return np.sum(Xi_arr)

    Xi_arr = [xi_val(x) for x in delta_arr]

    return Xi_arr

```

Which we can then plot as a function of δ in figure 16 ($\delta = \omega_1 - \omega_0 - \omega_r$).

E3

a)

b)

c)

d)

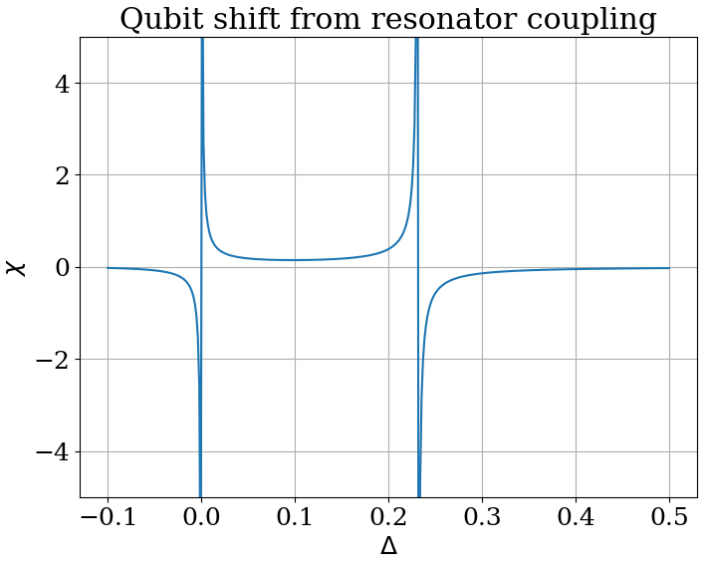


Figure 16: The resonator-qubit coupling as a function of the detuning. We can clearly see the straddling regime in the middle.

Week 8: The generalized flux qubit framework and beyond transmon qubit

There where no exercises for week 8.

Week 9: Surface code

E1

a)

The X_{ab} and X_{cd} clearly commute, $[X_{ab}, X_{cd}] = 0$, since they operate on different qubits. When it comes to Z_{abcd} we'll use the fact that X and Z anticommute so $ZX = -XZ$.

$$\begin{aligned}[Z_{abcd}, X_{ab}] =& [Z_a Z_b, X_a X_b] Z_c Z_d && (9.1)\\ =& (Z_a Z_b X_a X_b - X_a X_b Z_a Z_b) Z_c Z_d && (9.2)\\ =& (Z_a Z_b X_a X_b - Z_a Z_b X_a X_b) Z_c Z_d = 0. && (9.3)\end{aligned}$$

The exact same calculation can be preformed with X_{cd} and so $[Z_{abcd}, X_{cd}] = 0$.

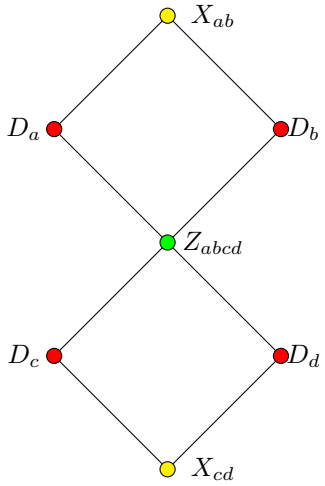


Figure 17: Schematic of distance 2 surface code with 4 data qubits and 3 measurement qubits. X measurements qubits are marked yellow, Z measurements are green and data qubits are marked red.

b)

First it's convenient to convert all the states to the same basis, so using $|\pm\rangle = |0\rangle \pm |1\rangle$. (We'll suppress normalization constant since they are not important for the following discussion because in all states the probabilities are equally distributed.) With this result we can rewrite $|{+}{+}\rangle$ and $|{-}{-}\rangle$.

$$|\pm\pm\rangle = |\pm\rangle \otimes |\pm\rangle = |00\rangle + |11\rangle \pm |01\rangle \pm |10\rangle \qquad (9.4)$$

From this we can see that the state $|{+}{+}\rangle + |{-}{-}\rangle = |00\rangle + |11\rangle$ which is a +1 eigenstate of $Z_{ab(cd)}$. Similarly we can construct $|{+}{+}\rangle - |{-}{-}\rangle = |01\rangle + |10\rangle$ which is a -1 eigenstate of $Z_{ab(cd)}$, and both states are +1 eigenstates of $X_{ab(cd)}$.

c)

We can now use the results from the previous section to construct +1 eigenstates of $Z_{abcd} = Z_{ab} Z_{cd}$. The two starts are

$$|0_L\rangle = (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) = |0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle \qquad (9.5)$$

and

$$|1_L\rangle = (|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle) = |0110\rangle + |1010\rangle + |0101\rangle + |1001\rangle \qquad (9.6)$$

d)

We can then operator on our states with $X_L = X_{ac}$ and $Z_L = Z_{ab}$ on our states to check that they actually are our logical state.

$$X_L |0_L\rangle = X_{ac} |0_L\rangle = |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle = |1_L\rangle \qquad (9.7)$$

and

$$Z_L |1_L\rangle = Z_{ab} |1_L\rangle = (-1) |0110\rangle + (-1) |1010\rangle + (-1) |0101\rangle + (-1) |1001\rangle = - |1_L\rangle \qquad (9.8)$$

e)

Starting with X errors, any X error can be detected by the Z_{abcd} stabilizer. For example if there's an X error on D_a then the result of the Z_{abcd} stabilizer will be

$$\begin{aligned} Z_{abcd}\left|\psi\right\rangle &= Z_{abcd}(\left|1000\right\rangle + \left|1011\right\rangle + \left|0100\right\rangle + \left|0111\right\rangle) \\ &= (-1)^1\left|1000\right\rangle + (-1)^3\left|1011\right\rangle + (-1)^1\left|0100\right\rangle + (-1)^3\left|0111\right\rangle = (-1)\left|\psi\right\rangle. \end{aligned} \tag{9.9}$$

Here we get an eigenvalue of (-1) which is not an eigenvalue of either $\left|0_L\right\rangle$ or $\left|1_L\right\rangle$ and so we know that we have an X error. Likewise a Z error on for example D_c will be detected by X_{cd}. To work with in the $\left|\pm\right\rangle$ basis, so we'll convert our logical states to that basis. Since $\left|{+}{+}\right\rangle + \left|{-}{-}\right\rangle = \left|00\right\rangle + \left|11\right\rangle$ so the zero state becomes

$$\begin{aligned} \left|0_L\right\rangle &= (\left|{+}{+}\right\rangle + \left|{-}{-}\right\rangle) \otimes (\left|{+}{+}\right\rangle + \left|{-}{-}\right\rangle) \\ &= \left|{+}{+}{+}{+}\right\rangle + \left|{+}{+}{-}{-}\right\rangle + \left|{-}{-}{+}{+}\right\rangle + \left|{-}{-}{-}{-}\right\rangle \end{aligned} \tag{9.10}$$

A Z error will be realized as a change from + to - or vice versa

$$\left|+\right\rangle = \left|0\right\rangle + \left|1\right\rangle \xrightarrow{Z \text{ error}} \left|0\right\rangle - \left|1\right\rangle = \left|-\right\rangle \tag{9.11}$$

So now a Z error on D_c from $\left|0_L\right\rangle$ will be

$$\begin{aligned} X_{cd}\left|\psi\right\rangle &= X_{cd}(\left|{+}{+}{-}{+}\right\rangle + \left|{+}{+}{+}{-}\right\rangle + \left|{-}{-}{-}{+}\right\rangle + \left|{-}{-}{+}{-}\right\rangle) \\ &= (-1)\left|{+}{+}{-}{+}\right\rangle + (-1)\left|{+}{+}{+}{-}\right\rangle + (-1)\left|{-}{-}{-}{+}\right\rangle + (-1)\left|{-}{-}{+}{-}\right\rangle = (-1)\left|\psi\right\rangle, \end{aligned} \tag{9.12}$$

and once again we see that there has been an error because we measure a -1 on our stabilizer.

f)

If we know that the there are never errors on D_a and D_c then we still can't say which data qubit a X error occured since an error on either D_b or D_d will give measurement of -1 and therefore we can't correct them. Z error on the other hand can be corrected since if there's an error on D_b then it will give a -1 measurement on X_{ab}, but not on X_{cd} and so we can correct it, and vice versa.