

College Mathematics Journal, **27** (1996) no. 3, 186-192.

Copyright
The Mathematical Association of America

Capturing the Origin with Random Points: Generalizations of a Putnam Problem

Ralph Howard
Department of Mathematics
University of South Carolina
Columbia SC 29208

and

Paul Sisson
Department of Mathematics
LSU - Shreveport
Shreveport LA 71115

1 Introduction

Problem A-6 of the 53rd Putnam Competition read as follows:

Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

The problem has a geometric immediacy that makes it tantalizing: the tetrahedron so formed is readily visualized and no great mathematical background is necessary to understand the question being asked. Further, it is almost impossible to resist the urge to generalize the problem. Some of the variants that spring to mind quickly are:

- (1) Suppose $n+1$ points are chosen at random from the surface of a ball in \mathbf{R}^n . What is the probability that the center of the ball lies inside the simplex in \mathbf{R}^n whose vertices are the $n+1$ points (i.e. the convex hull of the $n+1$ points)?
- (2) Four points are chosen at random from within a ball in \mathbf{R}^3 (or $n+1$ points from an n -ball in \mathbf{R}^n). What is the probability that the center of the ball lies within the convex hull of the points?

(3)

Four points are chosen at random from the surface of some other object in \mathbf{R}^3 (or $n+1$ points from the surface of some object in \mathbf{R}^n). What is the probability that a fixed interior point of the object lies inside the convex hull of the four (respectively, $n+1$) points?

(4)

More vaguely, assume the action is centered about the origin in \mathbf{R}^n , and that $n+1$ points are chosen "at random" in \mathbf{R}^n . What is the probability that the convex hull of the $n+1$ points contains the origin?

The list can easily be extended, but as question **(4)** demonstrates we have already reached the point where the questions need to be more carefully posed.

Despite the fact that the original Putnam question is so easily understood, the solution is (not surprisingly) not arrived at with equal ease. This sentiment is supported by the fact that 123 of the top 203 scorers on the Putnam exam submitted no solution at all to problem A-6, and a relatively low number of 9 of the top scorers received a full 10 points for the problem. This difficulty in answering such an easily grasped problem just makes it more intriguing, of course, and suggests that the problem and its generalizations are worth investigating. In this paper we will develop a surprisingly simple answer to questions **(1)** and **(2)**. In addition, our result answers rather general forms of questions **(3)** and **(4)**.

In [3], Klosinski, Alexanderson and Larson offer the following solution to A-6. Assume the sphere is centered at the origin, and that the first point P_0 is located at the north pole of the sphere, with the three remaining points then located at random locations on the sphere. We can assume that these remaining points are chosen in a two-step process: first a diameter $P_{i1}P_{i2}$ ($i \in \{1,2,3\}$) is fixed and then one of the two end-points $\{P_{i1}, P_{i2}\}$ is selected as a vertex of the tetrahedron. Figure 1 below illustrates a typical orientation of the choices. The eight possible tetrahedra $P_0P_{1j_1}P_{2j_2}P_{3j_3}$ (with each j_i being 1 or 2) are equally likely. Further, we can assume that the result is an honest tetrahedron and that the origin does not lie on any face. (Recall that the plane through three noncollinear points P_1 , P_2 and P_3 consists of all affine combinations

$$\alpha_1 \vec{P_1} + \alpha_2 \vec{P_2} + \alpha_3 \vec{P_3},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. With probability one, neither the fourth vertex nor the origin lies in the plane through any three vertices.)

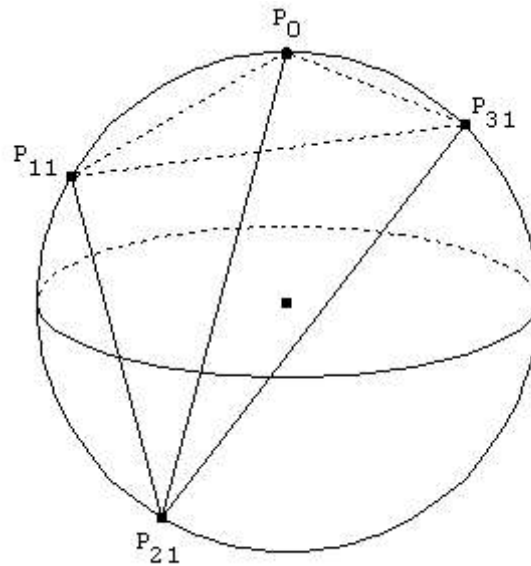


Figure 1: Typical choice of vertices.

In particular, the four vertex vectors

$$\vec{P}_0, \vec{P}_{11}, \vec{P}_{21} \text{ and } \vec{P}_{31}$$

must be linearly dependent, so there exists a 4-tuple (w,x,y,z) for which

$$\vec{0} = w \vec{P}_0 + x \vec{P}_{11} + y \vec{P}_{21} + z \vec{P}_{31}$$

and for which w,x,y and z are all non-zero. Then since

$$\vec{P}_{i1} = -\vec{P}_{i2},$$

the eight equations

$$\vec{0} = w \vec{P}_0 + x \vec{P}_{1j_1} + y \vec{P}_{2j_2} + z \vec{P}_{3j_3}$$

have the solutions

$$(w,x,y,z), (w,x,y,-z), (w,x,-y,z), (w,-x,y,z), \\ (w,x,-y,-z), (w,-x,-y,z), (w,-x,y,-z), (w,-x,-y,-z).$$

Each point in the tetrahedron with vertices $P_0, P_{1j_1}, P_{2j_2}, P_{3j_3}$ can be uniquely represented as a convex combination

$$\beta_0 \vec{P}_0 + \beta_1 \vec{P}_{1j_1} + \beta_2 \vec{P}_{2j_2} + \beta_3 \vec{P}_{3j_3}$$

(where each $\beta_i \geq 0$ and $\beta_0 + \beta_1 + \beta_2 + \beta_3 = 1$), so the origin is contained in the tetrahedron $P_0P_{1j_1}P_{2j_2}P_{3j_3}$ if and only if the 4-tuple solving the associated vector equation

$$\vec{0} = w \vec{P}_0 + x \vec{P}_{1j_1} + y \vec{P}_{2j_2} + z \vec{P}_{3j_3}$$

consists of four coordinates of the same sign. Since only one of the above eight solutions has this property, only one of the eight equally likely tetrahedra contains the origin, and hence the probability that the origin is contained in the randomly chosen tetrahedron is $1/8$.

2 First Generalization

So far, so good. This solution generalizes in the obvious way and gives us the answer of $1/2^n$ to question **(1)** in the above list. But what of question **(2)**? The above approach seems inadequate in this case, since points can now be chosen anywhere along the randomly chosen diameters.

Let us employ one of the standard procedures when faced with a difficult problem: that of changing the problem to something easier. We will attempt first to answer question **(2)** in \mathbf{R}^2 . Specifically, if three points are chosen at random from the unit disk B^2 , what is the probability that the triangle thus formed contains the origin? Let us further simplify the problem by assuming that we are choosing three points at random with respect to a probability measure P on B^2 which is rotationally invariant ; that is, measures of subsets of B^2 are unchanged under rotational translations. We will also continue to assume the appropriate degree of non-degeneracy of the measure (more on this in the next section).

Since we are assuming rotational invariance, we can assume that the first point P_1 is fixed between 0 and 1 on the positive x-axis. With probability one, the second point P_2 of the triangle is not located at the origin, and we can form the ray starting at the origin and passing through P_2 . Let θ be the angle between the positive x-axis and this ray. The question can now be posed as a conditional probability problem: given θ , what is the probability that the third point P_3 defines a triangle which contains the origin? Integrating this probability over all possible θ 's will then give us the answer we seek.

In order to simplify our work, let us agree upon some notation. Given a point P in $B^2 - \{(0,0)\}$, let $\Theta(P)$ denote the angle from the positive x-axis to the ray beginning at the origin and passing through P (see Figure 2). Thus, $\theta_1 \leq \Theta(P) \leq \theta_2$ will indicate that P lies in the sector of B^2 defined by the angles θ_1 and θ_2 . Let $P(\text{capture})$ denote the probability that the origin is captured within the triangle formed by the three points P_1 , P_2 and P_3 . Thus, the first task is to calculate $P(\text{capture} \mid \Theta(P_2) = \theta)$, for each $\theta \in [0, 2\pi]$.

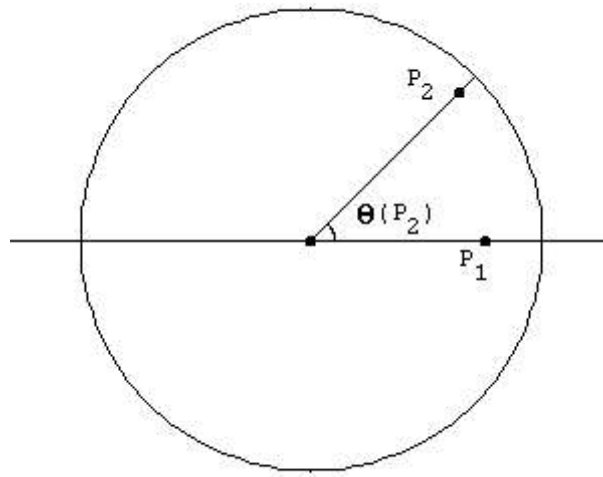


Figure 2: Illustration of $\Theta(P_2)$ for a typical P_2 .

Suppose first that $0 \leq \theta \leq \pi$. It is not difficult to see that a necessary and sufficient condition for capture is that $\pi \leq \Theta(P_3) \leq \pi + \theta$. That is, the ray from the origin to P_3 must pass through S^1 (the boundary of B^2) at a point between π units and $\pi + \theta$ units, as measured from the positive x-axis. Since the length of this arc is θ , this conditional probability is $\theta/2\pi$, i.e. $P(\text{capture} \mid \Theta(P_2) = \theta) = \theta/2\pi$. Similarly, if $\pi \leq \theta \leq 2\pi$, $P(\text{capture} \mid \Theta(P_2) = \theta) = 1 - \theta/2\pi$.

We can now approximate our solution with an appropriate Riemann sum. Let $\{\theta_0, \dots, \theta_n\}$ be a partition of $[0, 2\pi]$. Then

$$\begin{aligned} P(\text{capture}) &\approx \sum_{i=0}^{n-1} P(\text{capture} \mid \theta_i \leq \Theta(P_2) \leq \theta_{i+1}) P(\theta_i \leq \Theta(P_2) \leq \theta_{i+1}) \\ &= \sum_{i=0}^{n-1} P(\text{capture} \mid \theta_i \leq \Theta(P_2) \leq \theta_{i+1}) \frac{\Delta\theta_i}{2\pi}. \end{aligned}$$

In the limit of finer and finer partitions, we obtain

$$\begin{aligned} P(\text{capture}) &= \int_0^{2\pi} P(\text{capture} \mid \Theta(P_2) = \theta) \frac{d\theta}{2\pi} \\ &= \int_0^{\pi} \frac{\theta}{2\pi} \frac{d\theta}{2\pi} + \int_{\pi}^{2\pi} \left(1 - \frac{\theta}{2\pi}\right) \frac{d\theta}{2\pi} \\ &= 1/4. \end{aligned}$$

Examination of this argument shows that we have answered more than we set out to, since the fact that P is a probability measure on B^2 is really irrelevant. As long as P is a probability measure on \mathbf{R}^2 which is rotationally invariant and suitably non-degenerate, the result is the same. We are already aware of one consequence of this: if P is a uniformly distributed probability measure on S^1 , the method of Klosinski, Alexanderson and Larson tells us that with probability $1/4$ the origin will be contained in a randomly chosen triangle. We can also begin to make sense of question (4) by noting that if P is the usual Gaussian probability measure on all of \mathbf{R}^2 , the probability that three randomly chosen points captures the origin is again $1/4$.

A related problem in geometric probability, whose many variants are dealt with in [1], [2], [4], [5] and [6], is to find the probability that three points chosen at random from a region in the plane will form an acute triangle. One version can be easily answered here. Since the origin is captured by three points chosen at random from the unit circle if and only if the three points form an acute triangle, the probability that an acute triangle is formed by three points chosen at random from S^1 is also $1/4$.

The results above suggest that under rather general circumstances $n+1$ points chosen randomly from a region in \mathbf{R}^n which is symmetric with respect to the origin will capture the origin with probability $1/2^n$. Our main result gives conditions which guarantee the validity of this conclusion, and thus provides answers to questions (2), (3) and (4).

3 Second Generalization

We begin with a theorem, in which we finally describe the amount of non-degeneracy of the measures that we require. We also discard rotational-invariance for a weaker condition.

Theorem 1 Let \mathbf{R}^n be endowed with a probability measure μ which is symmetric with respect to the origin and such that when $n+1$ points are chosen independently with respect to μ , with probability one their convex hull is a simplex. Then the probability that the origin is contained in the simplex generated by $n+1$ such random points is $1/2^n$.

Recall that a measure μ is symmetric if $\mu(-S) = \mu(S)$ for any measurable set S . Also, $n+1$ points x_1, x_2, \dots, x_{n+1} from \mathbf{R}^n are vertices of a simplex if and only if none of these points lies on a hyperplane containing the other n points. Thus simplexes in \mathbf{R}^2 are triangles and simplexes in \mathbf{R}^3 are tetrahedra.

As examples of how the theorem can be applied, we could let μ be a uniformly distributed probability measure on a rectangle centered at the origin in \mathbf{R}^2 , or on the boundary of such a rectangle. In either case, if three points are chosen at random with respect to the measure, the probability that the origin is contained in the triangle thus formed is $1/4$. In \mathbf{R}^3 , the original Putnam answer of $1/8$ applies to four points chosen at random from a cube, or from the surface of a cube, as well as from the sphere. More generally, let D be any domain in \mathbf{R}^n with finite volume and which is symmetric with respect to the origin. Then the probability that the origin is in the convex hull of $n+1$ points chosen uniformly and independently from D is $1/2^n$. Note that, as in Figure 3, it is not necessary for D to contain the origin, or even for D to be connected.

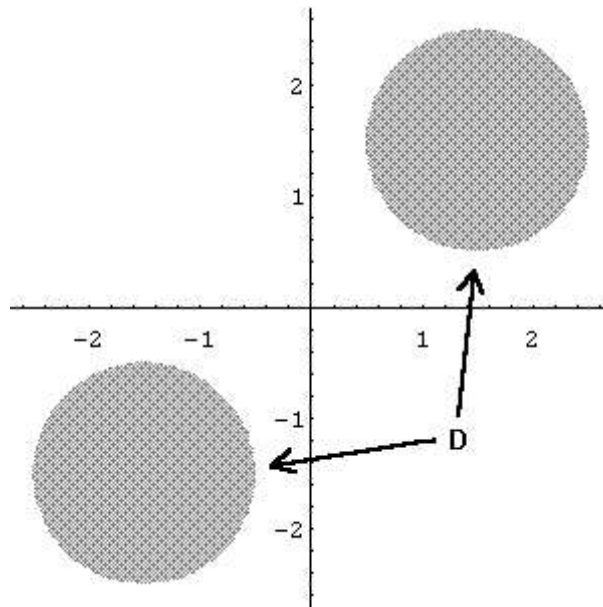


Figure 3: A disconnected domain D to which the theorem applies.

We will now proceed with the proof of the theorem.

Proof: Begin by defining a new probability space $\Omega = \mathbf{R}^n \times \dots \times \mathbf{R}^n$ ($n+1$ factors) with the product measure $\mu \times \dots \times \mu$. Let

$$A = \{(x_1, \dots, x_{n+1}) \in \Omega \mid \text{the origin of } \mathbf{R}^n \text{ is in the convex hull of } x_1, \dots, x_{n+1}\}$$

For each set of indices $I \subset \{1, \dots, n+1\}$ let

$$A_I = \{\omega \in \Omega \mid \omega_I \in A\},$$

where for $\omega = (x_1, x_2, \dots, x_{n+1})$ we define $\omega_I = (y_1, y_2, \dots, y_{n+1})$ with

$$y_i = \begin{cases} -x_i & \text{if } i \in I \\ x_i & \text{if } i \in I^c. \end{cases}$$

Thus $A = A_{\emptyset}$, where \emptyset is the empty set of indices.

Our proof now rests on four observations.

(i)

$A_I = A_{I^c}$ where I^c is the complement of I in $\{1, \dots, n+1\}$

(ii)

A_I and A_J are essentially disjoint if J is neither I nor I^c

(iii)

$$\Omega = \cup \{A_I \mid I \subset \{1, \dots, n+1\}\}$$

(iv)

The sets A_I all have equal measure.

To prove (i), suppose $\omega \in A_I$, with $\omega = (x_1, x_2, \dots, x_{n+1})$. Then $\omega_I \in A$, i.e.

$$\sum \beta_i y_i = \vec{0},$$

with $0 \leq \beta_i \leq 1$ for each i and $\sum \beta_i = 1$. But then

$$\sum \beta_i (-y_i) = \vec{0},$$

and since

$$-y_i = \begin{cases} -x_i & \text{if } i \in I^c \\ x_i & \text{if } i \in I \end{cases}$$

it follows that $\omega_{I^c} \in A$, so $\omega \in A_{I^c}$. Of course, the same argument shows that $A_{I^c} \subset A_I$ as well, so $A_I = A_{I^c}$.

Next, let x_1, \dots, x_{n+1} be chosen independently with respect to μ from \mathbf{R}^n , and let $\omega = (x_1, \dots, x_{n+1})$. Because of the hypothesis that with probability one none of the random points x_i is in the hyperplane spanned by the other n points, the origin has a representation as an affine combination of x_1, x_2, \dots, x_{n+1} that is almost surely unique, say

$$\vec{0} = \sum \alpha_i x_i,$$

where $\sum \alpha_i = 1$. Thus, except possibly for ω in a set of measure zero in Ω , the set $I = \{i \mid \alpha_i \leq 0\}$ for the above representation is uniquely determined by ω , and picks out the set $A_I = A_{I^c}$ of which ω is a member. Thus observation **(ii)** is proved.

Observation **(iii)** follows from noting that for any $n+1$ randomly chosen points x_1, x_2, \dots, x_{n+1} from \mathbf{R}^n , the zero vector can be expressed as an affine combination of these points, and the set $A_I = A_{I^c}$ which contains $\omega = (x_1, x_2, \dots, x_{n+1})$ is essentially uniquely determined by this expression. Finally, **(iv)** follows from the hypothesis that μ is invariant under reflection through the origin, so that for any index set I the mapping $\omega \rightarrow \omega_I$ on Ω that changes the signs of the coordinates in I preserves the product measure. This mapping sends A_I onto A , so $\mu(A_I) = \mu(A)$ for all I .

The index set $\mathfrak{I} = \{I \mid I \subset \{1, \dots, n+1\}\}$ has 2^{n+1} elements, but $A_I = A_{I^c}$ so our claims show that Ω can be written as an essentially disjoint union of 2^n subsets each with the same measure. Thus each of these subsets has measure $1/2^n$. In particular A has measure $1/2^n$, completing the proof of the theorem. ♦

After discovering the above proof, we learned that our result for points chosen from the surface of the unit sphere in \mathbf{R}^n could be deduced from a result due to J. G. Wendel and found in [8]. Wendel showed that if N points are chosen at random from the surface of the unit sphere in \mathbf{R}^n , the probability that all N points lie in the same hemisphere is given by

$$p_{n,N} = 2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k}.$$

$$\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

If we let $N = n+1$, then the probability that the origin is contained in the convex hull of $n+1$ points chosen at random from the unit sphere in \mathbf{R}^n is $1 - p_{n,n+1}$, which as the reader can verify is $1/2^n$. A referee pointed out another interestingly odd result related to our investigation that can be found on page 124 in [Z]: If five points are chosen at random from a ball in \mathbf{R}^3 , the probability that one of them is contained in the tetrahedron generated by the other four is $9/143$.

The authors are grateful to the editor and referees for many useful suggestions.

References

- [1] G. R. Hall, Acute Triangles in the n -Ball, *J. Appl. Prob.*, **19** (1982) pp 712-715.
- [2] D. G. Kendall and W. S. Kendall, Alignments in Two-Dimensional Random Sets of Points, *Adv. Appl. Prob.*, **12** (1980) pp 380-424.
- [3] L.F. Klosinski, G.L. Alexanderson and L.C. Larson, The Fifty-Third William Lowell Putnam Mathematical Competition, *Am. Math. Monthly*, **100** (1993) pp 755-767.
- [4] E. Langford, The Probability that a Random Triangle is Obtuse, *Biometrika*, **56** (1969) pp 689-690.
- [5] E. Langford, A problem in Geometrical Probability, *Math Mag.*, **43** (1970) pp 237-244.
- [6] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, Massachusetts (1976).
- [7] H. Solomon *Geometric Probability*, SIAM, Philadelphia, Pennsylvania (1978).
- [8] J. G. Wendel, A Problem in Geometric Probability, *Math. Scand.*, **11** (1962) pp 109-111.

File translated from T_EX by L^AT_EX version 2.32.
On 10 Oct 1999, 13:00.