NOTE ON HYDRODYNAMICS

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ABSTRACT. A study is made of the actual trajectories of fluid particles in certain motions of classical hydrodynamics. When a solid body moves through an incompressible fluid, it induces a *drift* in the fluid, such that the final positions of the particles are further on than those from which they started. The *drift-volume* enclosed between the initial and final positions is equal to the volume corresponding to *hydrodynamic mass*, that is, the mass of fluid to be added to that of the solid in calculating its kinetic energy. This result is proved quite generally. The work involves integrals which are not absolutely convergent, and these are discussed in relation to the general mechanics of fluids. When the trajectories are considered of the fluid surrounding a rotating body, it is shown that the fluid particles slowly drift round the body, even though the motion is irrotational and without circulation. There seems to be in some respects a closer resemblance between the behaviour of the idealized hydrodynamic fluid and a real fluid than might be expected from the well-known discrepancies between them.

1. Much of elementary hydrodynamics is concerned with the flow of fluid past a solid body, or alternatively with the motion of a solid body through a fluid, but in the latter case little attention has been given to the actual trajectories of the particles of the fluid. The present work arose from a wish to study these trajectories and the results were found to be rather unexpected. Though they are concerned exclusively with the idealized incompressible fluid of classical hydrodynamics, and not with real fluids, they may be worth placing on record, since they seem to increase our physical insight into the behaviour of a fluid.

The typical problem to be considered is this. A solid body is moving uniformly through an infinite incompressible fluid. An infinite thin plane of the fluid, at right angles to the motion, is marked so as to be made recognizable, perhaps by means of some dye-stuff. After the passage of the body, what is the form assumed by this infinite plane? Elementary considerations suggested that in moving towards the right the body would displace an equal volume of fluid, so that there should be a *reflux* of fluid towards the left in compensation. It might therefore be expected that the wall of dye ought to end further to the left than its original position, and it was rather surprising to find by the solution of a particular problem that in fact it goes towards the right. The resolution of this paradox has been one of the interesting points in the work. It will appear that the motion of the fluid is hardly affected by the reflux, but is closely associated with the phenomenon of *hydrodynamic mass*, the mass that is derived from the kinetic energy of the fluid surrounding the body.

I have not succeeded in finding a discussion of these matters in the text-books. It is true that Lamb (1) formulates the *Lagrangian equations*, but he makes no use of them, and indeed the form of these equations is not at all convenient for most purposes. In two or three places he discussed the flow round moving bodies, and he gives drawings of their stream-lines, but the figures are little help, and indeed are almost misleading, since he does not draw attention to the difference between the instantaneous stream-lines and the actual lines of flow of the fluid, and these are entirely different. In Milne-

Thomson (2), p. 233, there is a brief account of one of the problems to be discussed here, but it does not go very deeply into the subject.

2. When the moving body is a circular cylinder travelling at right angles to its length the problem admits of complete solution. The body is supposed to be moving from left to right in the positive direction of x. Take axes fixed in it, so that with these axes the fluid is moving from right to left. The velocity may be taken as unity without loss of generality. The velocity potential then is

$$\phi = x + \frac{a^2 x}{r^2}, \tag{2·1}$$

and the flow is given by

$$\frac{dx}{dt} = -1 + a^2 \frac{x^2 - y^2}{r^4}, \quad \frac{dy}{dt} = a^2 \frac{2xy}{r^4}. \tag{2·2}$$

One integral of these equations is the ordinary stream function, which may be written as

$$y\left(1 - \frac{a^2}{r^2}\right) = Y, \tag{2·3}$$

so that the constant Y corresponds to the initial and final position of the stream-line with reference to the central line of motion.

The quantity required, which will be called the *drift*, is the total displacement of a particle in the x direction, referred to axes in which the infinite parts of the fluid are at rest. For this what is required is not x, but $X = x + t$. By transforming (2·2) so that the polar coordinate θ becomes the independent variable, it is easy to derive

$$X = \int_{-\infty}^{\infty} (\dot{x} + 1)\, dt = \int_0^{\pi} \frac{a^2 \cos 2\theta\, d\theta}{\sqrt{(Y^2 + 4a^2 \sin^2\theta)}}. \tag{2·4}$$

The complete solution calls for elliptic functions. Let

$$k = 2a/\sqrt{(Y^2 + 4a^2)}, \tag{2·5}$$

and set $\cos\theta = -\operatorname{sn} v$, so that v ranges from $-K$ to K. Then the whole course of the motion is given by

$$\left.\begin{aligned} y(v) &= \frac{a}{k}(\operatorname{dn} v + k'), \quad X(v) = \frac{a}{k}\left[\left(1 - \frac{k^2}{2}\right) v - E(v)\right], \\ t(v) &= \frac{a}{k}\frac{\operatorname{sn} v}{\operatorname{cn} v}(\operatorname{dn} v + k') + X(v). \end{aligned}\right\} \tag{2·6}$$

As pointed out by Milne-Thomson, this curve is in fact one of the 'elasticas'. The total drift is

$$X = \frac{2a}{k}\left[\left(1 - \frac{k^2}{2}\right) K - E\right]. \tag{2·7}$$

A few of the lines of flow are shown in Fig. 1. The origin of time has been taken at the moment of central passage of the cylinder. The numbers on the curves record the times at those points in the suitable unit; thus a point marked 2 gives the position of the water particle when the cylinder has moved forward by 2 of its radii from the central position.

For a large value of Y the total drift is easily found to be $\pi a^4/2Y^3$. In reaching its final position the particle, however, describes a much larger curve; it moves approximately in a circle of radius $a^2/2Y$. For small Y there is a logarithmic infinity, such that approximately the total drift is

$$a[\ln(8a/Y) - 2]. \tag{2·8}$$

Near the central line the dye is drawn out into a pair of narrow sheets making a wake leading up to the cylinder. It is this that makes it convenient to draw curves based on the central position of the cylinder, rather than on a plane far in front of it as was

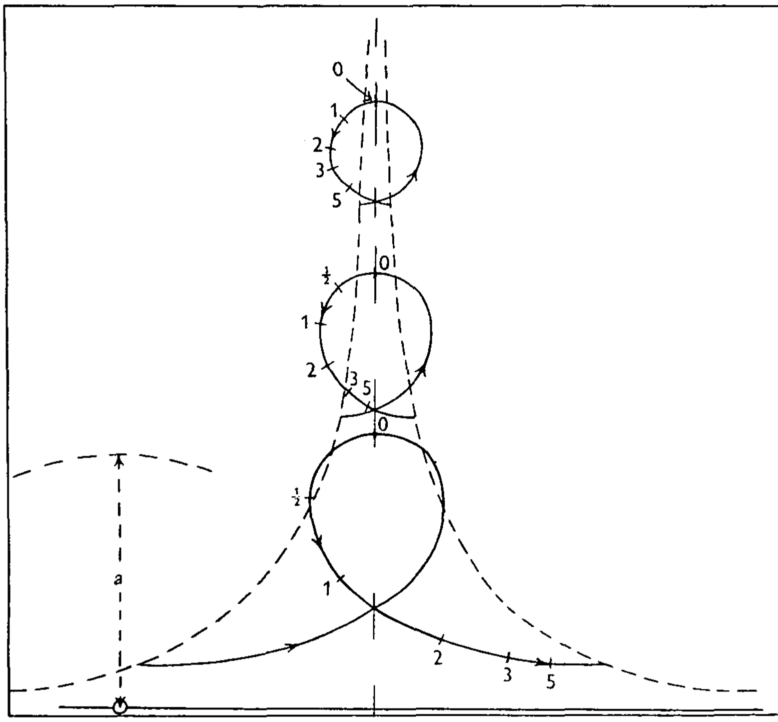

Fig. 1. Broken line on left is upper half of wall of dye before passage of cylinder, that on right after passage; a is cylinder's radius. Firm lines show three trajectories. Numbers mark times of passage to these points in units in which cylinder moves distance a.

proposed in § 1. The occurrence of these infinities is easily understood from the consideration that, in the system of coordinates in which the body is at rest, there are two points in contact with the body at which the fluid is also at rest; therefore the moving body must push fluid in front of it and drag fluid behind it. Though it need hardly be said that the actual motion of fluids is very different, it may be worth giving the magnitudes that this motion would imply. Take a cylinder of 1 cm. radius, then when it has gone 10 cm. the wake of dye would be about 10^{-4} cm. across, and at 20 cm. it would be about 10^{-8} cm.

The most interesting quantity to consider is D, the *drift-volume*—in the two-dimensional case it is an area. This is the volume enclosed between the initial and final position of the wall of dye. Then

$$D = \int_{-\infty}^{\infty} X\,dY. \tag{2·9}$$

It is more easily derived from (2·4) than (2·7) and is found to be

$$D = \pi a^2.$$

It must be noted that this is positive. It is in no way connected with the reflux of fluid displaced by the cylinder, which as it happens would be exactly the negative of this quantity. It is the hydrodynamic mass associated with a circular cylinder, a result which will later be established generally for any form of body.

3. A very similar calculation can be done for the motion of a sphere. It is more troublesome, because X can only be expressed as an integral, and not in terms of any tabulated functions. Thus the actual value of the drift X, as depending on Y, the distance from the central line, could only be given by elaborate quadratures. Nevertheless, it is possible to do the double integration so as to obtain the *drift-volume*, and this is found to be

$$D = \tfrac{2}{3}\pi a^3, \tag{3·1}$$

and so once again equal to the hydrodynamic mass.

Since this equality will be proved quite generally I shall not give any of the work for the particular case of the sphere, but I may mention two details. For small values of Y there is again a logarithmic infinity. The point of mentioning this is that in some matters there is a difference between two and three dimensions, in that logarithmic infinities that occur in two dimensions disappear in three; this is not true for the drift, which always has a logarithmic infinity. For large values of Y the trajectory is a curve on the scale a^3/Y^2, ending in a drift proportional to a^6/Y^5.

4. In deriving general formulae there are certain differences between the cases of two and three dimensions, and in view of the simpler relations between potential and stream function for two dimensions it will be best to work out this case first.

In coordinates fixed with reference to a cylindrical body of any form the flow will be represented by

$$\phi = x + f. \tag{4·1}$$

If the body is in free space f will be expressible as a series of circular harmonics in negative powers of r, but the case of flow in a channel will also have to be examined, where that would not be true. Associated with ϕ is the stream-function ψ, which at the surface of the body is constant. Alternatively the boundary condition may be expressed as

$$n_x f_x + n_y f_y = -n_x, \tag{4·2}$$

where n_x, n_y is the normal drawn towards the fluid.

The flow is given by

$$\dot{x} = -1 - f_x, \quad \dot{y} = -f_y. \tag{4·3}$$

The stream function gives one integral of these equations. It may be written as

$$\psi = y + g = Y, \tag{4·4}$$

so that Y defines the initial and final asymptotic line of flow.

To get the second integration the most natural variable to use is ϕ rather than t. Then

$$\frac{d\phi}{dt} = -(\phi_x^2 + \phi_y^2) = -\frac{\partial(\phi, \psi)}{\partial(x, y)}. \tag{4·5}$$

The drift is given by

$$X(Y) = \int_{-\infty}^{\infty} (\dot{x} + 1)\, dt = \int_{-\infty}^{\infty} (-f_x)\, d\phi \Big/ \frac{\partial(\phi, \psi)}{\partial(x, y)}. \tag{4·6}$$

Here the integrand is to be transformed from its expression in x and y, into terms of ϕ and ψ, and it is then to be integrated for ϕ with ψ held constant.

Problems of two-dimensional flow are frequently expressed by giving the relation of the complex $z = x+iy$, to the complex $w = \phi+i\psi$. In this form (4·6) becomes

$$X(\psi) = \left[\int\left|\frac{dz}{dw}\right|^2 d\phi - \Re z\right]_{-\infty}^{\infty}, \tag{4·7}$$

which is often the best formula for use in specific problems of flow.

The *drift-volume* is given by

$$D = \int_{-\infty}^{\infty} X\,dY = \int d\psi \int d\phi(-f_x)\Big/\frac{\partial(\phi,\psi)}{\partial(x,y)} = \iint(-f_x)\,dx\,dy. \tag{4·8}$$

Here the field of integration of course excludes the body, and is to be taken over the whole of the fluid. But it is important to notice that the integral is not absolutely convergent and can attain quite different values according to which integration is done first. In the present case there is no doubt of the order, for the ϕ integration was to be performed before the ψ integration. Since ϕ and ψ tend respectively to the same values as x and y in the distant parts of the fluid, this signifies that in (4·8) the x integration is to be done first. An alternative way of stating this is that if the integrations are taken respectively between the infinite values $x = \pm l$ and $y = \pm m$, then l is to be much greater than m. However, this matter of the convergence of the integral calls for deeper consideration, since the expression (4·8) comes up in a variety of ways in studying the mechanics of the fluid.

5. Before considering how the expression

$$J = \iint(1-\phi_x)\,dx\,dy \tag{5·1}$$

arises in the mechanics of the fluid, it will be best to know the various values it may assume.

Associated with ϕ there is the stream function ψ, and applying Green's theorem

$$J = \iint(1-\psi_y)\,dx\,dy = -\int^{(0)}(y-\psi)\,n_y\,ds - \int^{(\infty)}(y-\psi)\,n_y\,ds. \tag{5·2}$$

Here the symbol (0) signifies integration over the body, and (∞) over the surface at infinity. ψ is constant on the body, so that the associated integral vanishes and $\int^{(0)} yn_y\,ds = V$, the volume of the body. Thus the first term always gives $-V$.

As regards the second term in (5·2) consider first the case of free space. To define the field of integration, suppose the boundary is a box $x = \pm l$, $y = \pm m$, where both l and m are to be infinite, but in an unspecified ratio. In the more distant parts of the field ϕ and ψ can be expanded in circular harmonics in the form

$$\phi = x + A\frac{x}{r^2} + B\frac{y}{r^2} + C\frac{x^2-y^2}{r^4} + \ldots, \tag{5·3}$$

$$\psi = y - A\frac{y}{r^2} + B\frac{x}{r^2} - C\frac{2xy}{r^4} + \ldots. \tag{5·4}$$

Substituting (5·4) in (5·2) it is easy to see that only one term contributes and the result is

$$2Am\int_{-l}^{l}\frac{dx}{x^2+m^2} = 4A\tan^{-1}(l/m). \tag{5·5}$$

Thus the extreme values that J can assume are given by

$$J(m \gg l) = -V, \tag{5·6}$$

$$J(l \gg m) = 2\pi A - V. \tag{5·7}$$

As will appear, (5·7) is in fact the hydrodynamic mass.

A different case is where the flow is through a finite channel of breadth $y = \pm m$. On the two faces of this channel $\psi = y$, so that the second term in (5·2) vanishes and $J = -V$ always.

There is thus superficially a paradox. For open space $J = -V$ only if m is much greater than l, whereas it is not true if m is much smaller than l, the value being changed even as to sign. But if m is so much smaller than l as to be actually finite, then once again $J = -V$. This matter will be discussed later.

6. The integral (5·1) has at least three different significances in the mechanics of the fluid. One, the drift-volume, has already been developed, but the others must be examined also.

In the system in which the body is moving, the fluid's velocity in the x-direction at any point is $u = 1-\phi_x$, and the total flux of fluid is at rate $\int u\,dy$ across any transverse plane. The total transfer of fluid is the time integral of this; this is the same as the x-integral, and so is J. Here the y integration was done first, so that $m \gg l$, and the answer is of course $-V$, representing the reflux of fluid displaced by the body.

On the other hand, if ρ is the density of the fluid,

$$\iint \rho u\,dx\,dy = \rho J \tag{6·1}$$

represents its total momentum, but here there is no very evident specification whether l or m should be the greater.

The kinetic energy of the fluid is

$$\iint \tfrac{1}{2}\rho[(1-\phi_x)^2+\phi_y^2]\,dx\,dy = \tfrac{1}{2}\rho H, \tag{6·2}$$

which is an absolutely convergent integral, defining H the *hydrodynamic mass*. But if H can yield energy, it should also be able to yield momentum, so that it must be possible to relate (6·1) and (6·2).

Now
$$\begin{aligned} H-J &= \iint[\phi_x(\phi_x-1)+\phi_y^2]\,dx\,dy \\ &= \iint\left[\frac{\partial}{\partial x}(\phi-x)\,\phi_x+\frac{\partial}{\partial y}(\phi-x)\,\phi_y\right]dx\,dy \\ &= -\int^{(0,\infty)}(\phi-x)\,[n_x\phi_x+n_y\phi_y]\,ds. \end{aligned}$$

The second factor vanishes on the body. At infinity $\phi_x = 1$, $\phi_y = 0$; only the leading term in (5·3) contributes and it yields $4A\tan^{-1}(m/l)$. Thus

$$J(l \gg m) = H. \tag{6·3}$$

This establishes that for free space the *drift-volume* is equal to the *hydrodynamic mass*, which was the main proposition to be proved, but it leaves open the question about why it is necessary to take $l \gg m$ in order to get the momentum, and though it falls a little outside the main purpose of this paper it may be well to clear the matter up.

The usual derivation of hydrodynamic mass is from kinetic energy, but this suffers from the fundamental weakness that it assumes that the body's velocity is constant, whereas the very essence of mass is that it should be associated with acceleration. To study it therefore the velocity must be allowed to vary. Let a central point of the body have coordinate x_0, a function of the time. Then the flow is given by

$$\phi = \dot{x}_0 f(x - x_0, y). \tag{6·4}$$

Here $\nabla^2 f = 0$ throughout the fluid; on the body f satisfies (4·2), and near infinity it is of order r^{-1}. It must be noted that in an incompressible fluid effects are propagated to infinity instantaneously, so that it will be necessary to specify the conditions at infinity as well as the forces on the body, in order that this system of flow may be maintained.

Bernoulli's equation gives

$$p/\rho = \ddot{x}_0 f - \dot{x}_0^2 f_x - \tfrac{1}{2}\dot{x}_0^2(f_x^2 + f_y^2) + \text{constant}. \tag{6·5}$$

The total force on the fluid must then be

$$\iint \rho \frac{Du}{Dt}\,dx\,dy = -\iint \frac{\partial p}{\partial x}\,dx\,dy = \int^{(0)} p n_x\,ds + \int^{(\infty)} p n_x\,ds. \tag{6·6}$$

The first integral represents the force exerted by the body on the fluid. By the use of (4·2) it may be rewritten

$$\rho \int^{(0)} [(\ddot{x}_0 f - \dot{x}_0^2 f_x)(-n_x f_x - n_y f_y) - \tfrac{1}{2}\dot{x}_0^2(f_x^2 + f_y^2)\,n_x]\,ds, \tag{6·7}$$

and in this form it can be retransformed by Green's theorem, since the integrand in (6·7) now converges to zero on the surface at infinity. It is easily seen to yield $\rho H \ddot{x}_0$.

In the second integral only the first term of (6·5) survives, so that the fluid must experience a force $\rho \ddot{x}_0 \int^{(\infty)} f n_x\,ds$ in order to maintain the acceleration. In the case of free space f will have form given by (5·3) and this yields $-\rho\ddot{x}_0 4A\tan^{-1}(m/l)$. Thus if $l \gg m$ the motion will be accelerated without need for the introduction of a force at infinity. This explains why J only yields the momentum in the form $J(l \gg m)$. If it were assumed that $m \gg l$, the two walls $x = \pm l$ would have to exert forces in order that the motion should be correctly maintained.

When the flow is in a finite channel $y = \pm m$ the matter is different. Here it is not hard to see that, for positive x, f must assume the asymptotic form

$$f = A_0 + A_1 e^{-\pi(x - x_0)/2m} \sin \pi y/2m + \ldots, \tag{6·8}$$

and for negative x

$$f = -B_0 + B_1 e^{\pi(x - x_0)/2m} \sin \pi y/2m + \ldots. \tag{6·9}$$

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Then there must be a force from infinity of amount $-\rho 2m(A_0+B_0)\ddot{x}_0$. In other words, the motion demands a pressure difference $\rho(A_0+B_0)\ddot{x}_0$ between the ends of the channel, in addition to the force exerted by the body.

7. In § 5 it was shown that the drift-volume had different forms for free space and for a channel, and these results must be reconciled. This can be done by the consideration of a small body in a wide channel in the following manner.

Suppose the complex potential $w = \phi + i\psi$ is known as a function of $z = x+iy$. Let a be the radius of the smallest circle that completely encloses the body, and let m be the radius of a concentric circle just not large enough to cut a wall of the channel. In the ring-shaped region between, w can be expanded in a Laurent series in powers of z, and the orders of magnitude of the coefficients of this series can be assigned. The series will be

$$w = z + \sum_1^\infty c_n a^{n+1} z^{-n} + \sum_2^\infty d_n a^2 m^{-n-1} z^n. \tag{7·1}$$

As to the negative powers, the series is to diverge for $|z| < a$. The coefficient c_1 is of order unity, as exemplified in § 2, and in order that the series should begin to diverge there, some at least of the c_n's must also be of order unity and none of higher order. For the positive series, the type of expression required may be seen by taking an infinite series of images in the planes $y = \pm m$, since such a series of images will give a result satisfying the boundary conditions at the two faces, even though it will somewhat disturb the form at the inner circle. Thus the formula will be of type

$$\begin{aligned} w &= z + a^2[z^{-1} + (z-2mi)^{-1} + (z+2mi)^{-1} + (z-4mi)^{-1} + \ldots] \\ &= z + \frac{a^2\pi}{2m}\coth\frac{\pi z}{2m} = z + a^2[z^{-1} + \Sigma d_n m^{-n-1} z^n], \end{aligned}$$

and the d_n's are of order unity, since the cotangent series diverges for $|z| > m$. This form has only been derived for a special case, but it is easy to see that any later terms in the first series would give less important contributions to the images from which the second is derived. It must therefore be generally true that in (7·1) some of the d_n's will be of order unity.

Now take a length l, still inside the outer circle and consider how nearly complete the drift is at that point. The degree of completeness is given by (4·6) if the upper limit is taken as l instead of ∞. At this upper limit $\phi_x^2 + \phi_y^2$ is nearly unity, and $d\phi$ is nearly the same as dx, so that the drift there will depart from its final value by an amount of order $f(l, Y)$. Thus the incompleteness of the drift at l will be of order

$$\Re[c_1 a^2 (l+iY)^{-1} + a^2 d_2 m^{-3} (l+iY)^2].$$

Since c_1 and d_2 are of order unity these terms are respectively of orders $a^2/\sqrt{(l^2+Y^2)}$ and $a^2(l^2+Y^2)/m^3$.

The first of these corresponds to the drift in free space, which is practically complete if l is any considerable multiple of a. The second is of no comparable importance as long as $m \gg \sqrt{(l^2+Y^2)}$. Thus in a wide channel the central part of the drift surface is exactly the same as for free space. On the other hand, at the sides, where Y becomes comparable with m, the inequality fails, and the second term becomes important.

The general result for a body in a wide channel is that the central part of the drift surface is unaffected, so that the drift-volume derived from it will be H. Since the total drift-volume has to be $-V$, this central drift-volume is compensated at the edges by a negative drift, small in length but extending over a very wide area, giving exactly a volume $-H-V$. In this manner the contrast between free space and a channel is reconciled.

8. The drift in three dimensions calls for slightly different formulation, because there is no longer the same simple relation between potential and stream function. It will suffice to consider the case of free space; considerations like those of §§ 6, 7 are obviously applicable to the case of a channel.

The velocity potential in the system with the body at rest is

$$\phi = x + f, \tag{8·1}$$

where f is now of order r^{-2}. It is expansible in spherical harmonics of which the leading terms are

$$f = \frac{Ax}{r^3} + \frac{By}{r^3} + \frac{Cz}{r^3} + \frac{D(3x^2-r^2)}{r^5} + \dots. \tag{8·2}$$

The equations of motion are

$$\dot{x} = -\phi_x = -1 - f_x, \quad \dot{y} = -\phi_y, \quad \dot{z} = -\phi_z. \tag{8·3}$$

There will be two integrals independent of the time, say ψ and χ, representing two families of surfaces in which the stream-lines lie. Any functional combinations of ψ and χ will do equally well, and in particular a pair can be chosen so that near positive infinity

$$\psi = y + g = Y, \quad \chi = z + h = Z, \tag{8·4}$$

where g and h are of order r^{-2}. Note that though the ψ's and χ's are orthogonal to one another near infinity, they will not in general continue to be so nearer the origin. Nor will they in general return to the same lines as x approaches negative infinity. For example a screw-shaped body moving in pure translation may give rise to a rotational drift.

The conditions for ψ and χ to be integrals of (8·3) are

$$\psi_x\phi_x + \psi_y\phi_y + \psi_z\phi_z = 0, \quad \chi_x\phi_x + \chi_y\phi_y + \chi_z\phi_z = 0.$$

Hence

$$\phi_x = \kappa\frac{\partial(\psi,\chi)}{\partial(y,z)}, \quad \phi_y = \kappa\frac{\partial(\psi,\chi)}{\partial(z,x)}, \quad \phi_z = \kappa\frac{\partial(\psi,\chi)}{\partial(x,y)}. \tag{8·5}$$

Then in view of the relation $\nabla^2\phi = 0$

$$\frac{\partial(\kappa,\psi,\chi)}{\partial(x,y,z)} = 0.$$

It follows from this that κ is independent of ϕ, and constant along any stream-line. With the choice of (8·4) for the stream lines, $\kappa = 1$ at infinity, and therefore it is so everywhere.

Since

$$\frac{d\phi}{dt} = -(\phi_x^2 + \phi_y^2 + \phi_z^2) = -\frac{\partial(\phi,\psi,\chi)}{\partial(x,y,z)}, \tag{8·6}$$

the drift along the line Y, Z is

$$X(Y,Z) = \int_{-\infty}^{\infty} (\dot{x}+1)\,dt = \int_{-\infty}^{\infty} (-f_x)\,d\phi \bigg/ \frac{\partial(\phi,\psi,\chi)}{\partial(x,y,z)}, \tag{8·7}$$

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where the integrand is to be expressed in terms of ϕ, ψ, χ before the integration. The drift-volume is

$$D = \iint X\,dY\,dZ = \iiint d\psi\,d\chi(-f_x)\,d\phi \Big/ \frac{\partial(\phi,\psi,\chi)}{\partial(x,y,z)} = \iiint (-f_x)\,dx\,dy\,dz. \tag{8·8}$$

This integral is not absolutely convergent, and in evaluating it the x-integration is to be taken to infinity, before the y- and z-integrations are done. The bounds must thus be taken in the infinite box as $\pm l$, $\pm m$, $\pm n$ with $l \gg m, n$. From the analogy with (6·1) this is evidently the hydrodynamic mass as measured by the linear momentum of the fluid.

The mass as derived from the kinetic energy is of course the same. For

$$\begin{aligned} H - D &= \iiint [(\phi_x - 1)^2 + \phi_y^2 + \phi_z^2 - (1-\phi_x)]\,dx\,dy\,dz \\ &= \iiint \left[\frac{\partial}{\partial x}(f\phi_x) + \frac{\partial}{\partial y}(f\phi_y) + \frac{\partial}{\partial z}(f\phi_z)\right] dx\,dy\,dz \\ &= -\iint^{(0,\,\infty)} f[n_x\phi_x + n_y\phi_y + n_z\phi_z]\,dS. \end{aligned}$$

On the body the second factor vanishes. Everywhere at infinity $\phi_x = 1$ and $\phi_y = \phi_z = 0$, so that

$$H - D = \int_{-m}^{m}\int_{-n}^{n} [f(l,y,z) - f(-l,y,z)]\,dy\,dz. \tag{8·9}$$

The only term that can contribute is the first in (8·2). Then

$$H - D = 8A \tan^{-1}(mn/l\sqrt{\{l^2+m^2+n^2\}})$$

so that since $l \gg m, n$ once again $H = D$.

There is no need to pursue the various questions about reflux, motion in a channel and so on, since though a little more troublesome to prove they will evidently be just the same as in the case of two dimensions.

9. An entirely different type of motion will now be examined, the motion of the fluid associated with a rotating elliptic cylinder. The motion is irrotational and without circulation. For the motion inside an ellipse Lamb gives stream function

$$\psi = \tfrac{1}{2}\omega \frac{a^2-b^2}{a^2+b^2}(x^2-y^2), \tag{9·1}$$

where x, y are moving axes pointing along the two axes of the rotating ellipse.

Referred to these moving axes the fluid particles obey the equations

$$\dot{x} - \omega y = -\psi_y, \quad \dot{y} + \omega x = \psi_x, \tag{9·2}$$

from which

$$\dot{x} = \omega\frac{2a^2}{a^2+b^2}y, \quad \dot{y} = -\omega\frac{2b^2}{a^2+b^2}x. \tag{9·3}$$

Take a particle at zero time at distance κa along the major axis. Then later

$$x = \kappa a \cos\frac{2ab}{a^2+b^2}\omega t, \quad y = -\kappa b \sin\frac{2ab}{a^2+b^2}\omega t.$$

The particle thus describes an ellipse, being at any moment at the place of eccentric angle $-\dfrac{2ab}{a^2+b^2}\omega t$. Such a motion is oscillatory, but superposed on the oscillation there is angular velocity $-\dfrac{2ab}{a^2+b^2}\omega$. This refers to the moving axes of x and y. Referred to fixed axes the particle will have an angular velocity of drift $\dfrac{(a-b)^2}{a^2+b^2}\omega$ superposed on its oscillatory motion.

Suppose, for example, that the positive half of the major axis is initially marked by dye. This line will remain a radius of the ellipse, and periodically it will coincide with the major axis again. Each time it does this, it will have advanced its absolute direction in space. In the course of $(a^2+b^2)/(a-b)^2$ complete turns of the ellipse it will have rotated completely once right round. It may well be to repeat that this drift occurs in spite of the fact that the motion is irrotational.

10. In the case of motion outside an elliptic cylinder, the fluid also goes round even though the motion is irrotational and without circulation. The problem was fully solved by Lamb, but its expression in elliptic coordinates has concealed some of the qualities of the motion. Lamb gives the stream function

$$\psi = \tfrac{1}{4}\omega c^2 e^{2\beta-2\xi}\cos 2\eta \tag{10·1}$$

for flow round an ellipse of axes $c\cosh\beta$, $c\sinh\beta$ turning with angular velocity ω. The elliptic coordinates ξ, η are referred to axes rotating with the ellipse. Then the particle motion is given by

$$\dot{x}-y\omega = -\psi_y, \quad \dot{y}+x\omega = \psi_x, \tag{10·2}$$

and these can be transformed into

$$\dot{\xi} = -\Psi_\eta/\tfrac{1}{2}c^2(\cosh 2\xi-\cos 2\eta), \quad \dot{\eta} = \Psi_\xi/\tfrac{1}{2}c^2(\cosh 2\xi-\cos 2\eta), \tag{10·3}$$

where
$$\Psi = \tfrac{1}{4}\omega c^2[e^{2\beta-2\xi}\cos 2\eta-\cosh 2\xi-\cos 2\eta]. \tag{10·4}$$

One integral is therefore $\Psi =$ constant, and for the rest, since it is the rotation that is of interest, the best equation to use is

$$\dot{\eta} = -\omega\frac{\sinh 2\xi+e^{2\beta-2\xi}\cos 2\eta}{\cosh 2\xi-\cos 2\eta}. \tag{10·5}$$

For a particle at a great distance ξ is large and nearly constant. Then $\dot{\eta}+\omega$ is nearly $2\omega e^{-4\xi} = \omega\frac{1}{8}c^4/r^4$, which shows the average rate at which a distant particle drifts round in a circle.

Next consider the special stream-line

$$(e^{2\beta-2\xi}-1)\cos 2\eta-\cosh 2\xi = -\cosh 2\beta, \tag{10·6}$$

which is satisfied by $\xi = \beta$ and so represents the flow at the surface of the ellipse. On this line

$$\dot{\eta} = -\omega\frac{\sinh 2\beta+\cos 2\eta}{\cosh 2\beta-\cos 2\eta}. \tag{10·7}$$

Then if $\sinh 2\beta > 1$, $\dot{\eta}$ is negative everywhere on the surface. The fluid travels relatively in the negative direction, but with an angular velocity that is less than ω, so that once again the drift is forwards. The detailed result has no great interest and need not be given.

If $\sinh 2\beta < 1$, there will be points on the surface where $\dot{\eta} = 0$, so that at these points the fluid is at rest relative to the solid. This holds if the axes of the ellipse are in a ratio less than $(\sqrt{2}-1)$. The general results are quite complicated, and to illustrate them it will suffice to take the special case of a rotating flat plate, for which $\beta = 0$. There will then of course be infinite velocities of flow round the sharp edges, but the consequent cavitation will be disregarded by imagining the pressure to be infinite.

The surface stream-line now factorizes into two parts,

$$\sinh \xi = 0 \quad \text{and} \quad \cos 2\eta = -e^{\xi} \sinh \xi.$$

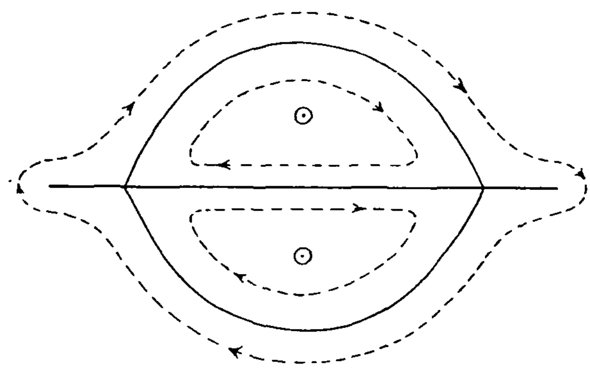

Fig. 2. Plate rotating counter clockwise entrains water inside firm line. Arrows show directions of water's motion relative to plate.

Expressed in polar coordinates the second part is the curve

$$r^2 = c^2 \frac{1+\sin\theta}{2+4\sin\theta}. \tag{10·8}$$

Its form is shown in Fig. 2. It cuts the plate at $r = c/\sqrt{2}$ and the axis $\theta = \frac{1}{2}\pi$ at $r = c/\sqrt{3}$. The fluid inside this curve is trapped and cannot escape. It goes round in an 'eddy'—speaking relatively, for, of course, the actual motion is irrotational. At the point where $\eta = \frac{1}{2}\pi$ and $\sinh 2\xi = e^{-2\xi}$, it may be seen from (10·5) that $\dot{\eta}$ vanishes. This is the point $r = \frac{1}{2}c[3^{\frac{1}{4}} - 3^{-\frac{1}{4}}] = 0{\cdot}278c$. If the particle at this point is marked by dye, it will be seen to stay there permanently as though rigidly attached to the plate. Round it the 'eddy' circulates relatively to the plate in the manner shown by the arrows.

11. There is a special and a general conclusion that may be drawn from this work.

The special conclusion relates to hydrodynamic mass. This conception is by no means a mathematical fiction, but is a genuine physical phenomenon. An amount of fluid corresponding to the hydrodynamic mass is being really carried along by the solid body. A crude analogy will make the matter clearer. A cart is going along a muddy road, and its wheels pick up mud. This mud is carried round for one turn of the wheel, and is then deposited on the ground again. Neither when picked up nor when set down does the mud receive any energy or momentum from the cart, nor does it give any to it. The only effect is that the mud is displaced to a new position farther forward along the road and set down there at rest. Yet evidently the mass of the mud on the wheel must be counted in estimating the forces needed to move the cart.

Much the same is true in the case of the hydrodynamic moment of inertia of the rotating ellipse. This moment of inertia can be equally well derived from the energy or from the angular momentum, for here there is no difficulty about the convergence of integrals, as there was in the case of the momentum of a body moving in translation. The present work has shown that the fluid does actually drift round with the rotating body. To relate the moment of inertia to this drift directly would hardly give any new information, since in fact it would merely be to recalculate the moment of inertia from the angular momentum. So in this case it will suffice to notice that there is this actual circular drift, and that it does occur even though the motion of the fluid is irrotational and without circulation.

As to the general conclusion, every student of hydrodynamics is familiar with the striking differences between the behaviour of the ideal hydrodynamic fluid and real fluids. In particular there is the completely unresisted motion of a solid body through the ideal fluid. This result is so contrary to reality that it makes the student sceptical of everything connected with problems of flow. Though of course the present work in no way alters this question of unresisted motion, yet it does show that the ideal fluid nevertheless has features very like those of a real fluid. Thus the passage of a solid body is accompanied by a 'wake', and in the case of a rotating body, a certain amount of fluid may be entrained permanently in an 'eddy' in much the way that would occur in a real fluid. There must be many similar questions to which we have at present no answer, of which it would be interesting to have the solutions.

In view of the great progress made in recent years in the study of the motion of real fluids, these matters may seem of secondary importance, but a knowledge of them does help in understanding some of the important features of fluid motion.

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