Risk-Parity Portfolio

Inrtoduction

Practical flaws of Markowitz Model:

1.variance is not a good risk measure in practice. The solution is use alternative measure for risk, such as downside risk (VaR, CVaR, Sortino ratio). The theoreticians described a number of properties a good risk measure should satify: **a coherent risk measure**. Such properties include monotonicity, sub-additivity, homogeneity, and translational invariance. *Ref* coherent risk measure (https://en.wikipedia.org/wiki/Coherent_risk_measure)

2.Sensitivity w.r.t estimators (parameter estimation of covariance matrix and mean vector by using historic data). If we use historic data, we introduce another assumption: no time-varying of estimator(and in practice Smaple size cannot be large enough due to lack of stationarity of data). Some improvements includes modeling r_t conditional on $\mathcal{F}_{t-1}$ and give different weights to observation data like **Garch** model by not assuming that conditional covariance and conditional mean are constant. Solutions include **shrinkage estimators**, **Black-Litterman estimators** (similar to a reverse engineering: do things backwards).

3.Lack of diversification (this seems quite contradictive; but in most case we allocate a large weight to only few assets): concentrates risk too much in few assets (2008 financial crisis): solution is the **risk-parity portfolio**.

Conclusion

Simply speaking, Markowitz framework is two-objectives optimiation problem: the mean return $\mathbf{w^T}\boldsymbol{\mu}$ and the variance $\mathbf{w^T}\boldsymbol{\Sigma}\mathbf{w^T}$. The idea is to find a trade-off between these two regarding how risk-averse a investor is.

Risk parity portfolio

- Less senstive to parameter estimation errors
- Diversifies the risk

Review: Emprical demonstration of flaws of Markowiz framwork

$$
\begin{array}{ll}
\underset{\mathbf{w}}{maximize} & \boldsymbol{\mu}^T - \lambda \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \\
subject\ to & \mathbf{1}^T \mathbf{w} = 1 \\
 & \mathbf{w} \geqslant 0
\end{array}
$$

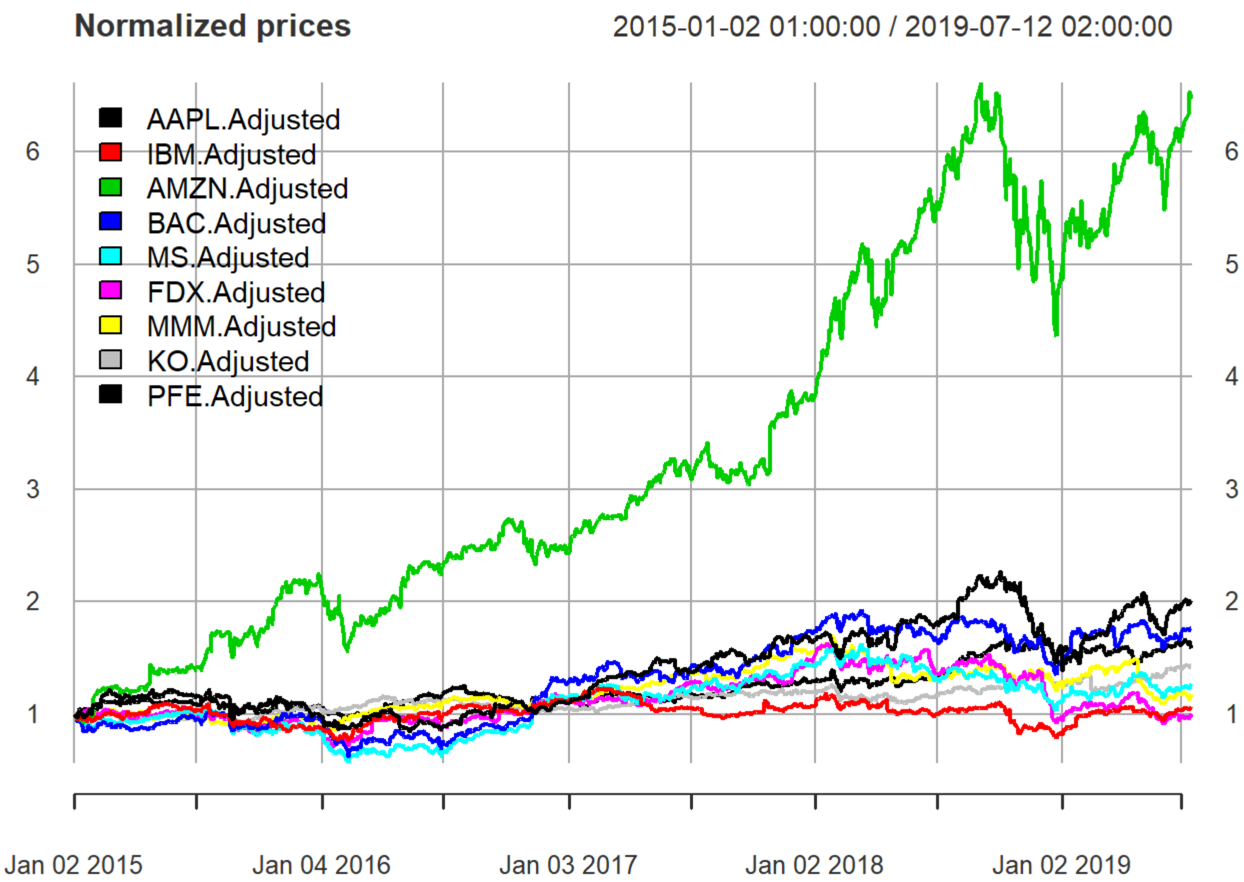
For convience, we define a function to find the solution using *CVXR* package for convex quadratic optimization problem.

```
library("CVXR")
weights_Markowitz <- function(lambda, Sigma, mu){

  w <- Variable(length(mu))
  res <- solve(Problem(Maximize(t(mu) %*% w - lambda*quad_form(w, Sigma)),
                       constraints = list(w >= 0, sum(w) == 1)))
  w_Markowitz <- as.matrix(res$getValue(w))
  rownames(w_Markowitz) <- stock_namels
  return(w_Markowitz)
}
```

```
library(xts)
library(quantmod)
begin_date <- "2015-01-01"
end_date <- "2019-07-13"
stock_namels <- c("AAPL", "IBM", "AMZN", "BAC", "MS", "FDX", "MMM", "KO", "PFE")
prices <- xts()
stock = "IBM"
for (stock in stock_namels){
  prices <- cbind(prices, Ad(getSymbols(stock, from = begin_date, to = end_date, auto.assign = FALSE
)))
}

plot(prices/rep(prices[1, ], each = nrow(prices)), legend.loc = "topleft",
     main = "Normalized prices")
```



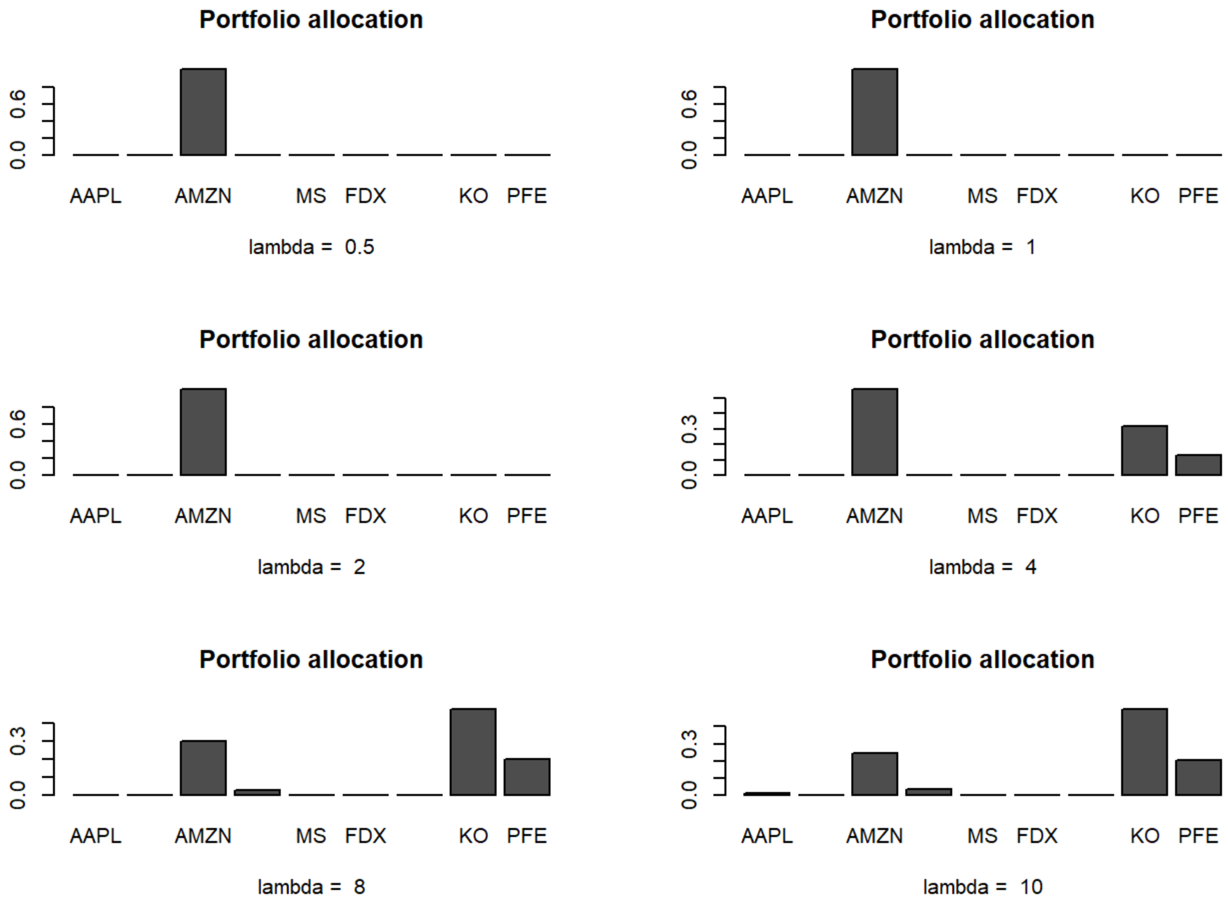
```
logret <- diff(log(prices))
colnames(logret) <- stock_namels
mu <- colMeans(logret, na.rm = TRUE)
Sigma <- cov(na.omit(logret))
```

Lack of diversification

Generally, λ is between 0 and 4. The larger λ, the more risk-averse.

```
par(mfrow = c(3,2))

for (lmd in c(0.5,1,2,4,8,10)){
  barplot(t(weights_Markowitz(lmd, Sigma, mu)), main = "Portfolio allocation", xlab = paste("lambda =
 ", lmd))
}
```




```
par(mfrow = c(1,1))
```

Remark: if λ goes to positive infinity, it will converge to GMVP(Global Minimum Variance Portfolio) situation

#Risk-Parity Portfolio – most simple case

More complicated case might be added if I study further

$$\sigma(\mathbf{w}) = \sqrt{\mathbf{w}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{w}}$$

$$= \sum_{i=1}^{N} w_i \frac{\partial \sigma}{\partial w_i}$$

$$= \sum_{i=1}^{N} \frac{w_i (\mathbf{\Sigma w})_i}{\sqrt{\mathbf{w^T \Sigma w}}}$$

Normalized risk contribution: $\frac{w_i(\mathbf{\Sigma w})_i}{\sqrt{\mathbf{w^T\Sigma w}}}$

risk budgeting equations

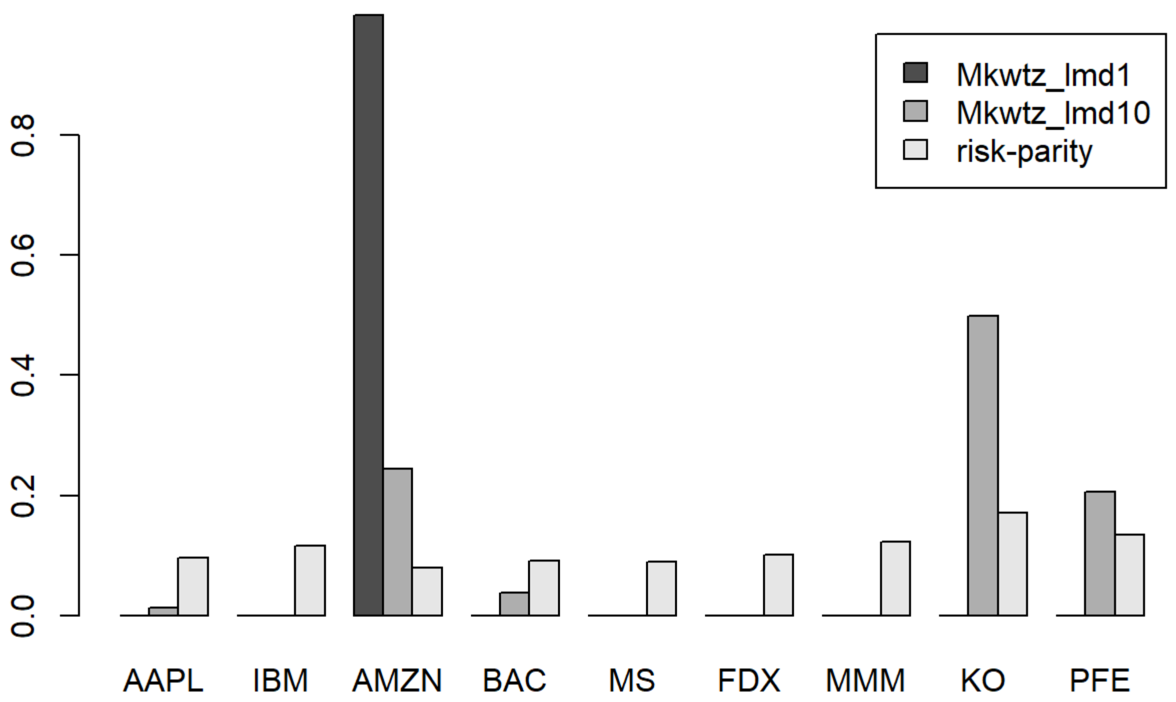
$$w_i (\mathbf{\Sigma w})_{\mathbf{i}} = b_i \mathbf{w^T \Sigma w}, \qquad i = 1, \ldots, N$$

Risk parity portfolio is a special case given the budgeting portfolio with $b = 1/N$, more specificly, the risk budgeting portfolio (no short position)is:

$$w_i = \frac{\sqrt{b_i}/\sqrt{\Sigma_{ii}}}{\sum_{k=1}^{N} \sqrt{b_k}/\sqrt{\Sigma_{kk}}}, \qquad i = 1, \ldots, N.$$

Closed-form solution for most simple case

```
w_naive <- 1/sqrt(diag(Sigma))
w_naive <- w_naive/sum(w_naive)
w_Markowitz_lmd1 <- weights_Markowitz(1 , Sigma, mu)
w_Markowitz_lmd10 <- weights_Markowitz(10 , Sigma, mu)
w_all <-  cbind(w_Markowitz_lmd1,w_Markowitz_lmd10, w_naive)
colnames(w_all) <- c("Mkwtz_lmd1", "Mkwtz_lmd10","risk-parity" )
barplot(t(w_all), beside = TRUE, legend = colnames(w_all))
```



Test for outsample performance (waited to be added)

Sensitivity to parameters (particularly to μ) (waited to be added)