The impact of covariance misspecification in risk-based portfolios☆

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Abstract

The equal-risk-contribution, inverse-volatility weighted, maximum-diversification and minimum-variance portfolio weights are all direct functions of the estimated covariance matrix. We perform a Monte Carlo study to assess the impact of covariance matrix misspecification to these risk-based portfolios. Our results show that the equal-risk-contribution and inverse-volatility weighted portfolio weights are relatively robust to covariance misspecification, but that the minimum-variance and maximum-diversification portfolios are highly sensitive to errors in the estimated variance and correlation, respectively.

Keywords: Covariance misspecification, Monte Carlo study, Risk-based portfolios

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1. Introduction

We study the sensitivity of risk-based portfolio optimization methods to covariance matrix misspecification. We conduct this analysis for four distinct well-known risk-based portfolios, namely the minimum-variance portfolio, the inverse-volatility weighted portfolio (Leote De Carvalho et al., 2012), the equal-risk-contribution portfolio (Maillard et al., 2010), and the maximum-diversification portfolio (Choueifaty and Coignard, 2008). We consider risk-based portfolios that are constructed with six distinct investment universes, which include factor-based equity portfolios, industry-based equity portfolios, multi-asset portfolios and portfolios made of single equities.

Using Engle (2002)'s dynamic conditional correlation (DCC) model as the true data generating process, our Monte Carlo study reveals substantial differences in the sensitivity to covariance misspecifications between the various allocation methodologies. First, the impact of covariance matrix misspecification is substantial for the minimum-variance and the maximum-diversification portfolios. The least sensitive portfolios are the equal-risk-contribution and the inverse-volatility weighted portfolios. Second, by decomposing the impact of covariance misspecifications into variances and correlations components, we show that, with the exception of maximum diversification, risk-based strategies are especially sensitive to misspecifications in the variance component. Third, we show that EWMA covariance matrix estimators are both statistically and economically closer to the true optimal allocations than both the (Ledoit and Wolf, 2003) and the sample-based estimators. Practically, our results imply that investors that are willing to invest in risk-based strategies should devote a particular attention to the methodology that is used by the investment manager to estimate the covariance matrix of the assets and, more particularly, its diagonal elements.

Risk-based allocation strategies have become extremely popular among investors during the last decade. For instance, a recent research published by JP Morgan (Kolanovic et al., 2015) indicates that the total amount managed with a risk parity approach is close to $500 BN as of August 2015. This number does not include the assets managed by investment vehicles that are pursuing minimum-variance and maximum-diversification strategies. Furthermore, most Commodity Trading Advisors (CTA funds) are also using a risk-based weighting scheme to perform their asset allocation. Despite the increasing popularity of risk-based investment strategies, there has been a shortage of scientific evidence evaluating the impact of second moment forecasting errors on the outcome of risk-based portfolio optimizations.

Our work aims at filling this gap by extending the work of Chan et al. (1999) and Ledoit and Wolf (2003) to risk-based asset allocation methodologies. Indeed, the aforementioned studies focus exclusively on the impact of the accuracy of the covariance matrix forecasts on the performance of mean-variance or benchmark-tracking portfolios. More recently, Zakamulin (2015) investigates the impact of various covariance matrix forecasting methodologies on the performance of both mean-variance and target volatility strategies but he does not pay attention to the other three very popular risk-based strategies that we investigate in this paper. Note that the objective of this study is to assess the impact of covariance matrix misspecification on the optimal weights that result from the different risk-based optimization methods. Therefore, we leave the question of the impact of covariance matrix misspecification on both portfolio performance and turnover open for further research.

The rest of this document is structured as follows. Section 2 provides a description of the various risk-based allocation methodologies. Section 3 proposes some numerical illustrations aimed at assessing the impact of second moment misspecification on the optimal allocations. The Monte Carlo study is presented in Section 4. Section 5 concludes.

2. Risk-based portfolios

We consider a market with N risky securities and denote a generic portfolio in this market by the $(N \times 1)$ vector $\mathbf{w} \equiv (w_1, \dots, w_N)'$. The $(N \times N)$ covariance matrix of the $(N \times 1)$ arithmetic returns $\mathbf{r} \equiv (r_1, \dots, r_N)'$ at the desired holding horizon is denoted by $\mathbf{\Sigma}$. We consider long-only portfolios in our analysis. Moreover, we define $\boldsymbol{\iota}_N$ as a $(N \times 1)$ vector of ones and $\mathbf{0}_N$ as a $(N \times 1)$ vector of zeros.

We consider four risk-based portfolios in our study.

Minimum-variance portfolio. The minimum-variance portfolio is obtained as:

$$\mathbf{w}_{\text{min}} \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \right\} \,, \tag{1}$$

where $\mathcal{C} \equiv \left\{ \mathbf{w} \in \mathbb{R}_+^N | \mathbf{w}'\boldsymbol{\iota}_N = 1 \right\}$ is the long-only full investment constraint.

Inverse-volatility weighted portfolio. The inverse-volatility weighted portfolio, called equal-risk-budget portfolio in Leote De Carvalho et al. (2012), allocates to the N stocks with volatility $\sigma_1, \dots, \sigma_N$ the weight:

$$\mathbf{w}_{\text{iv}} \equiv \left(\frac{1/\sigma_1}{\sum_{j=1}^N 1/\sigma_j}, \dots, \frac{1/\sigma_N}{\sum_{j=1}^N 1/\sigma_j} \right)' \,. \tag{2}$$

Equal-risk-contribution portfolio. The equal-risk-contribution portfolio is the portfolio for which all assets contribute equally to the overall portfolio volatility. Or equivalently, it is the portfolio for which the percentage volatility risk contribution of all N assets equals $1/N$, where percentage volatility risk contribution of the ith asset is given by $\%RC_i \equiv \frac{w_i [\mathbf{\Sigma}\mathbf{w}]_i}{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}$. It is computed by solving the following optimization problem:[1]

$$\mathbf{w}_{\text{erc}} \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \sum_{i=1}^{N} (\%RC_i - \tfrac{1}{N})^2 \right\} \,. \tag{3}$$

Maximum-diversification portfolio. Let $\boldsymbol{\sigma} \equiv \sqrt{\text{diag}(\mathbf{\Sigma})}$ be the $(N \times 1)$ vector of standard deviations of arithmetic returns of the N assets in the universe. An important property of the portfolio

[1] If a numerical solution is not found, we follow the recommendation in Maillard et al. (2010) and slightly modify the problem and optimize over the N-dimensional vector $\mathbf{u}$ such that $\mathbf{w} \equiv \mathbf{u}/(\mathbf{u}'\boldsymbol{\iota}_N)$, under the constraint that $\mathbf{u} \geq \mathbf{0}_N$ and $\mathbf{u}'\boldsymbol{\iota}_N > 0$. This new optimization problem is easier to solve numerically as an inequality constraint is less restrictive than the full investment equality constraint.

standard deviation is its sub-additivity, $\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}} \leq \mathbf{w}'\boldsymbol{\sigma}$, or equivalently that the ratio between the weighted average volatility and the portfolio volatility exceeds one:

$$DR(\mathbf{w}) \equiv \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \geq 1\,. \tag{4}$$

Choueifaty and Coignard (2008) call (4) the *portfolio's diversification ratio* and define the maximum-diversification portfolio as the portfolio that has the highest diversification ratio:

$$\mathbf{w}_{\text{md}} \equiv \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}}\{DR(\mathbf{w})\}\,. \tag{5}$$

3. Numerical illustrations

Consider a universe of $N = 3$ risky assets with the following covariance structure:

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ & 1 & \rho_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \tag{6}$$

with $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$. Assets #1 and #2 have a volatility of 10% and asset #3 has volatility of 20%. We purposely set a higher volatility for asset #3 to illustrate the case of a balanced portfolio that can contain bonds and equities. The correlations are chosen to illustrate the case of a balanced portfolio that may contain sovereign bonds, corporate bonds and equities. Sovereign bonds have been negatively correlated with both corporate bonds and equities in the past. On the other hand, the correlation between corporate bonds and equities has been positive. Table 1 reports the resulting allocations and risk contributions of each risk-based portfolio.

[Table 1 here]

We can see that the minimum-variance allocation is highly concentrated both in terms of weights and risk. The weights of the maximum-diversification allocation are concentrated as well but the risk of the portfolio is better diversified than the one of the minimum-variance allocation. In order to assess the impact of misspecified volatility estimates on the optimal allocations, we let the volatility of asset #3 vary between 10% and 30% and compute the L^1 distance between the weights of optimal portfolios computed with the true parameters, $\mathbf{w}$, and the ones computed with the misspecified ones, $\hat{\mathbf{w}}$. The L^1 distance writes:

$$\|\mathbf{w} - \hat{\mathbf{w}}\|_1 \equiv \sum_{i=1}^{N} |w_i - \hat{w}_i|\,. \tag{7}$$

If we consider Figure 1, which reports the distance of each portfolio as a function of the volatility estimates of asset #3, we can see significant differences between the distances of the mean-variance portfolio and those of the other 3 risk-based portfolios. The minimum-variance portfolio appears to be very sensitive to the misspecification of asset #3 volatility. Its distance from the optimal weights increases almost exponentially with the level of volatility under-estimation. The minimum-variance portfolio is less sensitive to over-estimations of asset #3 volatility as its weight stays at zero as soon as its estimated volatility is above 18%. The remaining three risk-based portfolios appear to be less sensitive to volatility misspecifications than the minimum variance portfolio. Finally, note that even if the optimization methodology differs across the three remaining risk-based portfolios, we cannot denote significant differences between the distances of the three allocations.

[Figure 1 here]

Our next numerical illustration investigates the sensitivity of the risk-based portfolios to correlation misspecification. In order to do so, we shift the value of the correlation between asset #2 and asset #3, $\rho_{2,3}$, and leave all other parameters constant. As before, we use the distance measure (7) to quantify the impact of the correlation misspecification on the optimal weights. Given the fact that the inverse-volatility weighted portfolio does not use correlations, this latter is excluded from our analysis.

If we consider Figure 2, we can see that both the minimum-variance and the maximum-diversification portfolios are highly sensitive to misspecification of the correlation coefficients. This is not surprising as those strategies are the ones with the most concentrated allocations. As before, the fact that asset #3 is twice as volatile as the other two assets and therefore very often excluded from the minimum-variance optimal portfolio reduces the sensitivity of this allocation to over-estimation of $\rho_{2,3}$. On the other hand, the maximum-diversification portfolio is sensitive to both under- and over-estimations of the parameter $\rho_{2,3}$.

[Figure 2 here]

Overall, our numerical analysis reveals that the minimum-variance portfolio is sensitive to both volatility and correlation misspecifications. The maximum diversification portfolio appears to be highly sensitive to correlation misspecification but does not show higher sensitivity to volatility misspecification than the inverse-weighted and equal-risk-contribution portfolios, which are usually less concentrated. However, in order to enlarge the scope of our numerical illustration, we now turn to a larger scale investigation with a Monte Carlo study.

4. Monte Carlo study

This section performs a Monte Carlo study to assess the impact of covariance misspecification to risk-based portfolios. We first present the various covariance estimators used in our study, then describe the data and the Monte Carlo setup, and finally discuss the results.

4.1. Covariance matrix estimators

Suppose we have a time series of T past returns, $\mathbf{r}_1, \ldots, \mathbf{r}_T$, to estimate the covariance matrix $\mathbf{\Sigma}$ of $\mathbf{r}_{T+1}$. We consider the following estimators:

Sample-based (SMPL). The most well-known and simple estimator of covariance, which sets Σ to the sample covariance of the T historical returns.

Ledoit-Wolf (LW). Weighted average of the sample covariance matrix and a *prior*. The prior is given by a one-factor model and the factor is equal to the cross-sectional average of all returns. The shrinkage intensity is the plug-in estimate of the mean square optimal one. See Ledoit and Wolf (2003). Like the sample-based covariance estimator, the LW estimate of covariance ignores the potential time-variation in the conditional covariance matrix.

Exponentially weighed moving average (EWMA). Simple dynamic model in which recent returns have more weights than past returns in the estimation. The decay parameter is set to 0.94 as advocated by RiskMetrics Group (1996) for daily returns.

Dynamic conditional correlation (DCC). Sophisticated model by Engle (2002) in which the variances and correlations are of GARCH(1,1) type, allowing persistence in the variance and the correlation dynamics. The model is estimated via composite likelihood technique as suggested by Engle et al. (2008).[2]

4.2. Setup

Previous research shows that the impact of comovements on optimal weights depends on the general level of correlation between the assets of the investment universe (e.g., see Schumann, 2013). Therefore, we conduct our study on six investment universes (datasets) of various variance/correlation structure and dimensions; see Table 2. For five out of six datasets, we use value-weighted portfolios obtained from the data library of Kenneth French.[3] This includes the well-known size, book-to-market and past performance (momentum) sorted portfolios, as well as industry portfolios based on the firms' four-digit SIC code. The Fama-French research portfolios contain US equities that are listed on NYSE, AMEX or NASDAQ. Our fifth universe covers seven major asset classes whose prices are obtained from the Thomson Reuters Datastream database. The sixth dataset is formed by the 30 constituents of the Dow Jones Industrial index as of June 2014. For each dataset, we retrieve the daily adjusted closing prices from December 2008 to November 2014.

[Table 2 here]

For each universe, we set up our Monte Carlo experiment by first fitting a DCC model by maximum likelihood to $T = 1500$ daily arithmetic (about six years of data) returns. For each dataset, our estimated parameters indicate persistence in the conditional variances and persistence in correlations; as expected for daily data.

For each Monte Carlo replication, we generate DCC multivariate normal scenarios of length $T + h$, where h is the forecasting (rebalancing) horizon. For our experiment, we consider a forecasting horizon of $h = 1$ day. We estimate the various covariance matrix estimators from the first

[2]Estimations are performed with an adapted version of the `R` package **rmgarch** (Ghalanos, 2014).

[3]Data are available at `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

T simulated returns and project matrices at horizon h. In order to decompose the sensitivity of the allocation methodologies into their variance and correlation components, we proceed as follows. For the variance component, we fit each variance estimator from the T simulated log-returns, which are then projected at horizon h together with the supposedly true DCC correlations in order to get the covariance matrix. For the correlation component, we fit correlations for each estimator from the T simulated log-returns and we project them together with the true DCC volatilities (i.e., GARCH(1,1)) at horizon h to get the covariance matrix.

For each risk-based portfolio, we assess the performance of the various covariance methods by using two distinct metrics. First, for each Monte Carlo replication, the L^1 distance in (7) is computed. Second, we compute objective-based distances to assess the economic/financial impact of misspecification. For the minimum-variance portfolio, we compute the ratio $\sigma(\hat{\mathbf{w}})/\sigma(\mathbf{w})$ which is larger than one when misspecification is observed. For the inverse-volatility portfolio, we compute $H^*(\%RC(\hat{\mathbf{w}})) - H^*(\%RC(\mathbf{w}))$, the difference between the normalized Herfindahl index on risk contributions for the misspecified and true portfolio. This difference will be larger than zero if the estimated portfolio weights are misspecified. For the equal-risk-contribution portfolio we simply compute $H^*(\%RC(\hat{\mathbf{w}}))$, as the normalized Herfindahl index is zero for the true portfolio. Finally, for the maximum-diversification portfolio, we compute the ratio $DR(\hat{\mathbf{w}})/DR(\mathbf{w})$ which is lower than one when misspecification is observed.

4.3. Results

The results of the Monte Carlo study are presented in Table 3. For each universe and covariance estimator, the average (over one hundred Monte Carlo replications) for the L^1 and objective-based distances are reported. Overall, the results are in line with the conclusions of the numerical illustrations performed in Section 3. Indeed, we clearly notice that the minimum-variance portfolio is the most sensitive one to covariance misspecification. It displays a substantially higher average distance. On top, the objective-based distance is substantially higher than one for all universes and covariance matrix estimators. The maximum diversification allocation methodology is the second most sensitive one to covariance misspecifications. This confirms the fact that higher portfolio concentration increases the sensitivity to covariance misspecifications. The inverse-volatility weighted and equal-risk-contribution allocations under misspecified covariance are closer to the true DCC allocations. Interestingly, in five cases out of six, the inverse-volatility weighted method does not display a much lower sensitivity to covariance misspecification than the equal-risk-contribution method even if this latter requires $N(N-1)$ additional parameter estimates. The maximum-diversification portfolio lies between the second best performer and the minimum-variance portfolio.

Three very interesting facts emerge from the examination of the results for each different asset universe. First, the core portfolio (universe #5) appears to be the most challenging one for the equal-risk-contribution allocation. Indeed, as shown in Table 3, such a balanced universe is characterized by both a low average level of correlations between the assets and a high level of dispersion between the correlation coefficients. In that context, the equal-risk-contribution allocation underperforms all allocations but the minimum-variance one. The objective-based distance of the equal-risk-contribution allocation is substantially higher than zero for the core portfolio universe. Note also that the objective-based metric of the maximum diversification allocation deteriorates when the investment universe is characterized by lowly-correlated asset classes. The least sensitive allocation is the inverse-volatility weighted allocation, which does not make use of correlation coefficients. This result has practical implications as most equal-risk-contribution investment funds (risk-parity funds) invest in a mix of asset classes made of sovereign bonds, corporate bonds, real estate, equities and commodities.

Second, there does not seem to be any relationship between the number of assets in the universe and the portfolios sensitivity to covariance misspecification. Indeed, the distance of the portfolios in the DJIA universe (universe #6), which is made of 30 assets is not higher than the ones of the other universes that is made of fewer assets.

Third, the statistical and economic distances of the portfolios that are constructed with the EWMA covariance matrix are much smaller than those that are constructed with the Ledoit and Wolf (2003) and the sample-based estimates.

[Table 3 here]

The Monte-Carlo results for the decomposition of sensitivities into the variance and correlation components are reported in Table 4 and Table 5, respectively. By comparing the results of Table 4 with those of Table 5, we see that both the minimum-variance and equal-risk-contribution methodologies display a much higher sensitivity (both statistically and economically) to variance misspecifications than to correlation misspecifications. However, this finding is not verified for all estimation methodology/universe pairs. Indeed, we note that both the balanced (portfolio #5) and DJIA stocks (portfolio #6) portfolios constructed with the EWMA estimators display more sensitivity to correlation than to variance misspecifications. In fact, for the two aforementioned portfolios, the statistical and economic distances of the EWMA allocations are higher than those of both the Ledoit and Wolf (2003) and sample-based estimators. Finally, the maximum-diversification methodology, seems to be much more sensitive to correlation misspecifications than to variance ones. This is true for all six portfolio universes and all estimators.

[Tables 4 and 5 here]

5. Conclusions

Risk-based allocation methodologies have become increasingly popular among investors. These strategies have two main advantages compared to other widespread quantitative asset allocation strategies such as mean-variance. First, they do not require the estimation of expected returns. Second, they place risk management at the center of the allocation process. As a consequence, the superiority of risk-based investment methods is highly dependent on the quality of the risk parameter estimates, which are used as input to construct the portfolios.

In this paper, we first document differences between the sensitivities of the various methodologies to estimation risk. Minimum-variance and the maximum diversification portfolios are much more sensitive to misspecifications of the covariance matrix than the other risk-based allocation methodologies. Their vulnerability to a misspecified covariance matrix is due to the higher concentration of their portfolio weights. Second, by decomposing the sensitivity of the allocation methodologies into their variance and correlation components, we show that, for both the minimum-variance and the equal-risk-contribution methodologies, sampling errors in variances have a higher impact on the portfolio allocation than correlation misspecifications. The reverse is true for maximum diversification strategies. Third, we show that the allocations obtained with the EWMA covariance matrix estimators are closer to the true allocations than those obtained with both the Ledoit and Wolf (2003) and sample estimates. Finally, we show that the composition of the opportunity set impacts the degree of sensitivity of some allocation strategies to covariance misspecifications. In particular, equal-risk-contribution strategies are very sensitive to covariance misspecifications when the opportunity set is characterized by low and dispersed correlations between its comprising assets.

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Table 1: Optimal weights and risk contribution of risk-based portfolios. We consider a universe of $N = 3$ assets with covariance matrix $\mathbf{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$. For the four risk-based portfolios, we report the optimal allocation $\mathbf{w}$, the percentage risk contribution $\%RC(\mathbf{w})$, the normalized Herfindahl index $H^*(\mathbf{w}) \equiv \frac{H(\mathbf{w})-1/N}{1-1/N}$ where $H(\mathbf{w}) \equiv \sum_{i=1}^{N} w_i^2$ as well as the portfolio's volatility $\sigma(\mathbf{w}) \equiv \sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}$. The normalized Herfindahl has value in $[0, 1]$ with a value of zero indicating perfect diversification and a value of one indicating perfect concentration. All quantities are reported in percentages.

	Min-var		Inv-vol		ERC		Max-div	
	$\mathbf{w}_{\text{min}}$	$\%RC$	$\mathbf{w}_{\text{iv}}$	$\%RC$	$\mathbf{w}_{\text{erc}}$	$\%RC$	$\mathbf{w}_{\text{md}}$	$\%RC$
Asset #1	50.0	50.0	40.0	18.4	48.3	33.3	56.6	46.9
Asset #2	49.9	49.9	40.0	42.1	33.6	33.3	22.6	18.8
Asset #3	0.0	0.1	20.0	39.5	18.1	33.3	20.8	34.4
H^*	25.0	25.0	4.0	5.1	6.8	0.0	12.2	6.0
σ	6.7		7.8		7.4		7.5	

Table 2: List of datasets / universes considered. N denotes the number of "asset" in each universe. Min, med, max report the minimum, median and maximum values of the unconditional volatilities (annualized, in percent) and the unconditional pairwise correlations (in percent) in the universe. For the core portfolio, we consider the following assets from the Thomson Reuters Datastream database: US Benchmark 30 Year Govt. Bond Index (BMUS30Y), MSCI EUROPE Stock Index (MSEROP), MSCI Emerging Markets Stock Index (MSEMKF), MSCI PACIFIC Stock Index (MSPACF), MSCI US Stock Index (MSUSAML), S&P US REIT Index (SBBRUSL), S&P Goldman Sachs Commodity Index (GSCITOT). For the DJIA portfolio, we consider the 30 constituents belonging to the Dow Jones universe as of July 2014.

			Volatility			Correlation		
#	Dataset / universe	N	min	med	max	min	med	max
1	Portfolios formed on size	10	17.2	21.6	27.3	87.9	96.2	98.8
2	Portfolios formed on book-to-market	10	17.2	20.4	28.2	83.0	92.6	95.8
3	Portfolios formed on momentum	10	18.5	21.7	37.2	58.0	87.9	96.0
4	Industry portfolios	10	13.7	18.9	28.5	68.2	79.0	92.1
5	Core portfolio	7	13.0	16.3	27.5	-50.2	38.7	78.2
6	DJIA portfolio	30	14.6	25.0	44.1	27.1	48.3	84.5

Table 3: Monte Carlo results for misspecification on the covariance matrix $\mathbf{\Sigma}$. Left part: Average L^1 distances (multiplied by 100 for convenience) of weights (7) obtained with the various universes and covariance matrix estimators, for the four risk-based portfolios (minimum-variance, inverse-volatility weighted, equal-risk contribution and maximum-diversification). Right: Average objective-based distances. For the Min-vol portfolio: $\sigma(\hat{\mathbf{w}})/\sigma(\mathbf{w})$, for the Inv-vol portfolio: $H^*(\%RC(\hat{\mathbf{w}})) - H^*(\%RC(\mathbf{w}))$, for the ERC portfolio: $H^*(\%RC(\hat{\mathbf{w}}))$, and for the Max-div portfolio: $DR(\hat{\mathbf{w}})/DR(\mathbf{w})$. Averages are computed over one hundred Monte Carlo replications. See Table 1 for details.

		L^1 distance				Objective-based distance			
#	Cov type	Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
1	DCC	14.235	1.145	1.156	8.488	1.001	0.003	0.003	1.000
1	EWMA	42.837	2.700	2.757	21.030	1.008	0.015	0.015	0.999
1	LW	96.069	8.885	9.026	59.799	1.067	0.150	0.160	0.995
1	SMPL	96.074	8.885	9.026	59.795	1.067	0.150	0.160	0.995
2	DCC	25.148	1.422	1.464	10.246	1.003	0.004	0.004	1.000
2	EWMA	49.856	3.590	3.702	25.131	1.013	0.026	0.029	0.999
2	LW	137.930	11.931	12.205	80.188	1.118	0.307	0.331	0.988
2	SMPL	137.942	11.931	12.205	80.178	1.118	0.307	0.331	0.988
3	DCC	21.663	1.621	1.685	9.884	1.003	0.005	0.007	1.000
3	EWMA	46.434	4.270	4.382	25.256	1.013	0.042	0.044	0.998
3	LW	158.960	17.500	17.923	88.793	1.240	0.669	0.712	0.978
3	SMPL	158.961	17.500	17.922	88.807	1.240	0.669	0.712	0.978
4	DCC	17.845	1.864	2.020	9.538	1.003	0.006	0.008	1.000
4	EWMA	37.477	5.057	5.895	24.655	1.013	0.037	0.077	0.997
4	LW	95.422	15.226	16.275	78.715	1.117	0.530	0.620	0.970
4	SMPL	95.448	15.226	16.275	78.759	1.117	0.530	0.620	0.970
5	DCC	10.175	2.090	2.498	9.237	1.004	0.016	0.057	0.997
5	EWMA	28.430	5.322	7.964	29.015	1.028	0.097	0.506	0.975
5	LW	51.367	18.259	19.552	31.050	1.124	2.065	3.351	0.958
5	SMPL	51.383	18.259	19.556	31.078	1.124	2.065	3.361	0.958
6	DCC	21.005	2.346	3.128	21.153	1.006	0.005	0.006	0.997
6	EWMA	60.260	6.197	12.475	69.264	1.048	0.018	0.122	0.966
6	LW	86.844	17.192	18.331	58.348	1.139	0.245	0.271	0.971
6	SMPL	87.178	17.192	18.326	58.640	1.140	0.245	0.271	0.971

Table 4: Monte Carlo results for misspecification on the correlation matrix. See Table 3 for details.

		L^1 distance				Objective-based distance			
#	Cov type	Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
1	DCC	2.057	0.000	0.153	8.066	1.000	0.000	0.000	1.000
1	EWMA	5.350	0.000	0.449	20.153	1.000	0.000	0.000	1.000
1	LW	11.484	0.000	0.759	58.264	1.001	0.000	0.001	0.996
1	SMPL	11.481	0.000	0.759	58.260	1.001	0.000	0.001	0.996
2	DCC	2.917	0.000	0.222	9.935	1.000	0.000	0.000	1.000
2	EWMA	7.770	0.000	0.693	24.758	1.001	0.000	0.001	0.999
2	LW	18.030	0.000	1.564	78.275	1.002	0.000	0.006	0.990
2	SMPL	18.029	0.000	1.564	78.261	1.002	0.000	0.006	0.990
3	DCC	3.005	0.000	0.312	9.375	1.000	0.000	0.000	1.000
3	EWMA	10.347	0.000	0.979	24.111	1.001	0.000	0.003	0.999
3	LW	18.968	0.000	2.319	84.508	1.003	0.000	0.017	0.983
3	SMPL	18.974	0.000	2.319	84.515	1.003	0.000	0.017	0.983
4	DCC	4.806	0.000	0.577	9.043	1.000	0.000	0.001	1.000
4	EWMA	13.847	0.000	1.999	23.510	1.003	0.000	0.011	0.998
4	LW	24.454	0.000	3.985	77.310	1.008	0.000	0.035	0.973
4	SMPL	24.487	0.000	3.986	77.352	1.008	0.000	0.035	0.973
5	DCC	6.868	0.000	1.406	8.739	1.002	0.000	0.024	0.998
5	EWMA	27.138	0.000	5.782	28.645	1.025	0.000	0.318	0.977
5	LW	22.879	0.000	4.840	26.176	1.019	0.000	0.257	0.979
5	SMPL	22.895	0.000	4.838	26.209	1.019	0.000	0.257	0.979
6	DCC	10.038	0.000	1.605	20.670	1.001	0.000	0.001	0.997
6	EWMA	49.359	0.000	9.931	68.494	1.035	0.000	0.084	0.967
6	LW	24.283	0.000	4.926	56.050	1.009	0.000	0.015	0.976
6	SMPL	24.681	0.000	4.941	56.514	1.009	0.000	0.015	0.975

Table 5: Monte Carlo results for misspecification on the variances. See Table 3 for details.

		L^1 distance				Objective-based distance			
#	Cov type	Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
1	DCC	13.731	1.145	1.151	1.285	1.001	0.003	0.003	1.000
1	EWMA	42.611	2.700	2.719	3.350	1.007	0.015	0.015	1.000
1	LW	97.321	8.885	8.932	10.616	1.068	0.150	0.155	0.999
1	SMPL	97.321	8.885	8.932	10.616	1.068	0.150	0.155	0.999
2	DCC	24.283	1.422	1.429	1.424	1.003	0.004	0.004	1.000
2	EWMA	49.635	3.590	3.610	3.822	1.012	0.026	0.027	1.000
2	LW	139.127	11.931	11.971	12.908	1.116	0.307	0.317	0.998
2	SMPL	139.127	11.931	11.971	12.908	1.116	0.307	0.317	0.998
3	DCC	21.362	1.621	1.644	1.806	1.002	0.005	0.006	1.000
3	EWMA	45.139	4.270	4.317	4.654	1.012	0.042	0.041	1.000
3	LW	160.810	17.500	17.727	20.976	1.231	0.669	0.682	0.994
3	SMPL	160.810	17.500	17.727	20.976	1.231	0.669	0.682	0.994
4	DCC	16.911	1.864	1.876	1.835	1.003	0.006	0.007	1.000
4	EWMA	35.974	5.057	5.142	5.008	1.012	0.037	0.053	1.000
4	LW	96.367	15.226	15.288	15.347	1.107	0.530	0.560	0.996
4	SMPL	96.367	15.226	15.288	15.347	1.107	0.530	0.560	0.996
5	DCC	6.536	2.090	1.926	1.995	1.002	0.016	0.032	0.999
5	EWMA	15.911	5.322	5.172	4.741	1.010	0.097	0.178	0.998
5	LW	48.402	18.259	18.460	15.081	1.109	2.065	3.332	0.975
5	SMPL	48.402	18.259	18.460	15.081	1.109	2.065	3.332	0.975
6	DCC	16.702	2.346	2.358	2.393	1.004	0.005	0.004	1.000
6	EWMA	42.635	6.197	6.250	6.265	1.023	0.018	0.024	0.999
6	LW	84.598	17.192	16.889	14.391	1.122	0.245	0.234	0.996
6	SMPL	84.598	17.192	16.889	14.391	1.122	0.245	0.234	0.996

Figure 1: Illustration of the impact of volatility misspecification. The graph reports the L^1 distance of weights (7) for the four risk-based portfolios as a function of σ_3, the volatility estimates of asset #3. The universe consists of $N = 3$ assets with covariance matrix $\mathbf{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$.

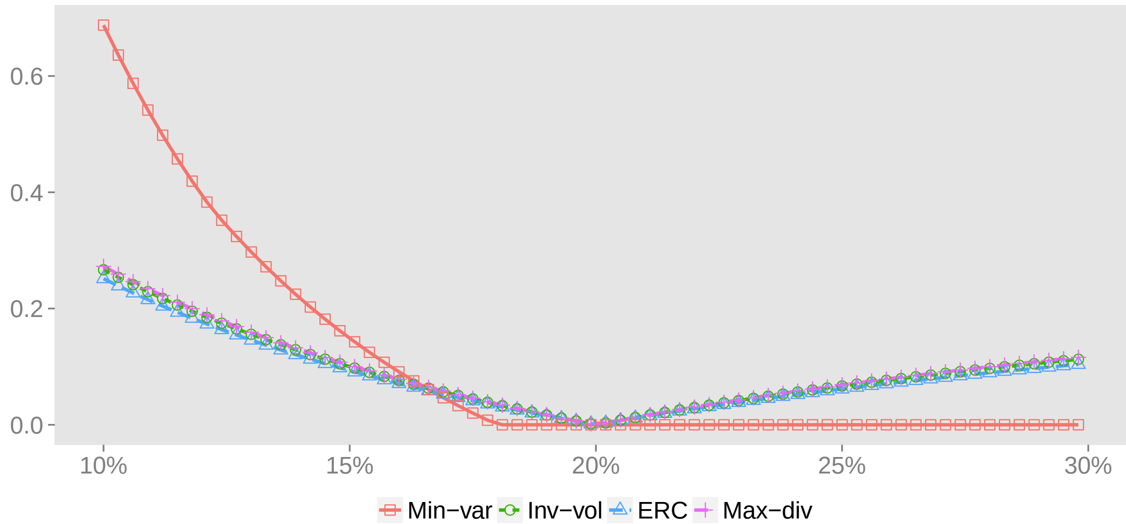

Figure 2: Illustration of the impact of correlation misspecification. The graph reports the L^1 distance of weights (7) for the three risk-based portfolios (as the inverse-volatility weighted portfolio does not depend on the correlations) as a function of $\rho_{2,3}$, the correlation between asset #2 and asset #3. The universe consists of $N = 3$ assets with covariance matrix $\mathbf{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$.