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Alberto Isidori

Lectures in Feedback Design for Multivariable Systems

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Lectures in Feedback Design for Multivariable Systems

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ISSN 1439-2232
Advanced Textbooks in Control and Signal Processing
ISBN 978-3-319-42030-1 ISBN 978-3-319-42031-8 (eBook)
DOI 10.1007/978-3-319-42031-8

Library of Congress Control Number: 2016945769

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Series Editors' Foreword

The *Advanced Textbooks in Control and Signal Processing* series is designed as a vehicle for the systematic textbook presentation of both fundamental and innovative topics in the control and signal processing disciplines. It is hoped that prospective authors will welcome the opportunity to publish a more rounded and structured presentation of some of the newer emerging control and signal processing technologies in this textbook series. However, there will always be a place in the series for contemporary presentations of foundational material in these important engineering areas.

The science of control comprises two interdependent activities: control theory and control engineering. Control theory provides an understanding of how and why controllers work, while control engineering is concerned with transforming theoretical solutions into controller designs that can be implemented in real applications. Sometimes it is control theory that leads the way, suggesting new methods or providing a deeper understanding of how to solve control problems. At other times it is an engineering solution or a technical innovation that creates the means of doing control differently, or opens up a new applications field, and then control theorist has to catch up. Either way control has been a very vibrant and expanding field for engineers and researchers since the middle of the last century. It has been a fascinating journey that continues today, exploiting links to areas in computer science and networks.

Alberto Isidori has traveled this path not only as an eminent control theorist and engineer, but also as a professional, contributing to the development of the important international institutions of the control community. In many ways this set of lectures is his personal reflection on the methods of feedback design for multivariable systems. He seeks to present an overarching view of design for both linear and nonlinear multivariable systems, identifying structural similarities within the two forms. A linear form can of course be viewed as a special case of a nonlinear system and it should not be surprising that strong theoretical links exist between the two.

Also found in this textbook is some work on coordinated control in what are often termed "leader–follower" systems. It is the introduction of restricted information transfer between the participating systems that makes the control design challenging. Leader–follower systems and the coordinated control design problem are one continuing interest at today's international control conferences.

The insightful, reflective nature of the textbook assumes that the reader has already studied the basics of control theory. A guide to the knowledge that is assumed of the reader can be found in the two useful appendices.

The text is based on Prof. Isidori's considerable teaching experience, including many years of lecturing at the Sapienza University in Rome, and at various international guest lecture courses. It provides a thoughtful and insightful exposition of where we stand today in control theory and is a very welcome and valuable addition to the *Advanced Textbooks in Control and Signal Processing* series.

About the Author

Alberto Isidori has been a Professor of Automatic Control at the University Sapienza, Rome since 1975. He has been a visiting academic at most of the foremost international universities and institutions where control theory and control engineering is preeminent as a subject of study.

He was President of the European Union Control Association (EUCA) between 1995 and 1997. And for the period 2008–2011, he was President of the International Federation of Automatic Control (IFAC).

Professor Isidori is noted for his books on nonlinear systems some of which were published by Springer, notably:

- *Nonlinear Control Systems* by Alberto Isidori (ISBN 978-3-540-19916-8, 1995)
- *Nonlinear Control Systems II* by Alberto Isidori (ISBN 978-1-85233-188-7, 1999); and
- *Robust Autonomous Guidance* by Alberto Isidori, Lorenzo Marconi and Andrea Serrani (ISBN 978-1-85233-695-0, 2003); a volume in the *Advances in Industrial Control* monograph series.

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Preface

The purpose of this book is to present, in organized form, a number of "selected topics" in feedback design, for linear and nonlinear systems. The choice of topics reflects a teaching activity held in the past 15 years at the University Sapienza in Rome as well as at various other academic institutions. The focus of the book is on methods for achieving asymptotic stability, and disturbance rejection, in the presence of model uncertainties. Among various possible options to deal with such design problem, the emphasis is on methods that, in one form or another, appeal to the classical "small-gain" principle for robust stability. In this setting, the aim is to offer a presentation as much as possible similar for linear and nonlinear systems. Of course, for pedagogical reason, linear systems are handled first. Even though the focus of the book is multi-input multi-output systems, for pedagogical reasons, single-input single-output nonlinear systems are handled first in some detail. As a result, the book begins with a rather tutorial flavor (Chaps. 2 to 8) and ends with a more monographic nature (Chaps. 9 to 12). A more detailed description of the topics covered in the book can be found in the introductory Chap. 1. The book presumes some familiarity with the theory of linear systems and with a few basic concepts associated with the analysis of the stability of equilibrium in a nonlinear system. For the reader's convenience, a sketchy summary of the relevant background concepts is offered in the two appendices.

The monographic portion of the book reflects in part my own research activity, conducted in collaboration with Lorenzo Marconi and Laurenty Praly, whose unselfish friendship, competence, and patience is gratefully acknowledged. The topics covered in this book have been taught during the past few years at the University Sapienza in Rome as well as at the EECI Graduate School on Control, in Paris, and at the Institute of Cybersystems and Control of Zhejiang University, in

Hangzhou. Encouragement and support from Francoise Lamnabhi and Hongye Su, respectively, are gratefully acknowledged. In particular, while at Zhejiang University, I have been able to establish a fruitful collaboration with Lei Wang, the outcome of which is reflected in the material presented in Chaps. 10 and 11.

Rome, Italy
May 2016

Alberto Isidori

Contents

Chapter 1
An Overview

1.1 Introduction

This is a book intended for readers, familiar with the fundamentals of linear system and control theory, who are interested in learning methods for the design of feedback laws for (linear and nonlinear) multivariable systems, in the presence of model uncertainties. One of the main purposes of the book is to offer a parallel presentation for linear and nonlinear systems. Linear systems are dealt with in Chaps. 2–5, while nonlinear systems are dealt with in Chaps. 6–12. Among the various design options in the problem of handling model uncertainties, the focus of the book is on methods that appeal—in one form or in another—to the so-called "Small-Gain Theorem." In this respect, it should be stressed that, while some of such methods may require, for their practical implementation, a "high-gain feedback" on selected measured variables, their effectiveness is proven anyway with the aid of the Small-Gain Theorem. Methods of this kind lend themselves to a presentation that is pretty similar for linear and nonlinear systems and this is the viewpoint adopted in the book. While the target of the book are multi-input multi-output (MIMO) systems, for pedagogical reasons in some cases (notably for nonlinear systems) the case of single-input single-output (SISO) systems is handled first in detail. Two major design problems are addressed (both in the presence of model uncertainties): asymptotic stabilization "in the large" (that is, with a "guaranteed region of attraction") of a given equilibrium point and asymptotic rejection of the effect of exogenous (disturbance and/or commands) inputs on selected regulated outputs. This second problem, via the design of an "internal model" of the exogenous inputs, is reduced to the problem of asymptotic stabilization of a closed invariant set.

The book presupposes, from the reader, some familiarity with basic concepts in linear system theory (such as reachability, observability, minimality), elementary control theory (transfer functions, feedback loop analysis), and with the basic concepts associated with the notion of stability of an equilibrium in a nonlinear system. Two Appendices collect some relevant background material in this respect, to the

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_1

purpose of fixing the appropriate notations and making the presentation—to some extent—self-contained.

1.2 Robust Stabilization of Linear Systems

Chapters 2 and 3 of this book deal with the design of control laws by means of which a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.1}$$

can be stabilized in spite of model uncertainties. In this respect, it should be stressed first of all that, while a simple method for stabilizing a system of this form consists in the use of the "observer-based" controller[1]

$$\begin{aligned} \dot{\hat{x}} &= (A + BF - GC)\hat{x} + Gy \\ u &= F\hat{x} \end{aligned}$$

with F and G such that the matrices $A + BF$ and, respectively, $A - GC$ have all eigenvalues with negative real part, this method is not suitable in the presence model uncertainties, because the design of the controller in question requires an accurate knowledge of the matrices A, B, C that characterize the model (1.1).

There are various design methods by means of which stabilization in spite of model uncertainties can be achieved, that depend—among other things—on the form in which the model uncertainties are characterized. The methods described in these two chapters are based on a classical result known as *Small-Gain Theorem* according to which, if two stable systems are coupled to each other and the coupling is "weak" (in a precise sense that will be defined) the resulting system is still stable. The reason why these methods are preferred is that they find a natural counterpart in the same design problem for nonlinear systems, as shown from Chap. 6 on.

The simpler example of a linear system subject to model uncertainties consists in a system modeled by equations of the form (1.1), in which the state vector x, the input vector u, and the output vector y have fixed dimensions but (some of) the entries of the matrices A, B, C are known only within given (and possibly not negligible) tolerance bounds. This case is usually referred to as the case of *structured uncertainties*. A simple method by means of which—if the system has just one input and just one output—the effect of such uncertainties can be overcome is based on the following elementary and well-known property. Suppose the transfer function

$$T(s) = \frac{N(s)}{D(s)}$$

[1] See Sect. A.4 in Appendix A.

of the system has n poles and $n-1$ zeros and let

$$b = \lim_{s\to\infty} \frac{sN(s)}{D(s)}.$$

Then, it is well known from the theory of the *root locus* that if k is a real number with the same sign as that of b, the n roots of polynomial

$$P(s) = D(s) + kN(s),$$

have the following property: as $|k| \to \infty$, one (real) root diverges to $-\infty$ while the remaining $n-1$ roots converge to the roots of $N(s)$ (that is to the zeros of $T(s)$). Thus, if all zeros of $T(s)$ have negative real part, the system can be made stable by means of the simple control law

$$u = -ky,$$

provided that $|k|$ is sufficiently large. It is also well known that, if $T(s)$ has $n-r$ zeros (now with $r > 1$) all of which have negative real part, the same stability result can be achieved by means of an output feedback controller characterized by the transfer function

$$T_{\mathrm{c}}(s) = -k\frac{N'(s)}{(1+\varepsilon s)^{r-1}}$$

in which $N'(s)$ is a monic polynomial of degree $r-1$ having all roots with negative real part and $\varepsilon > 0$ is a small number. In fact, the law in question is classically interpreted as the "addition of $r-1$ new zeros" having negative real part, which results in a system with transfer function

$$T(s) = \frac{N(s)N'(s)}{D(s)}$$

whose $n-1$ zeros have negative real part—to which the previous stability result can be applied—and in the "addition of $r-1$ poles" which are real, negative, and "far away" from the origin in the complex plane (so as to preserve the stability of the closed-loop system), obtaining in this way a controller characterized by a *proper* transfer function.

It is natural to guess that, if the coefficients of $T(s)$ are subject to uncertainties but these uncertainties range over *closed and bounded sets*, one ought to be able to find one single (large) value of k and one single (small) value of ε, both *not depending on the uncertainties as such but only on the sets in which such uncertainties* are confined, for which the resulting closed-loop system is stable. The purpose of Chap. 2 is essentially to provide a formal proof of this result. This will be done considering, for the controlled system, a state-space model as in (1.1). The state-space model framework is particularly convenient because in this way a specific formalism and appropriate concepts can be developed that prove very useful in the more general

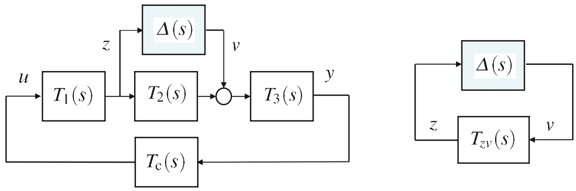

Fig. 1.1 The control of a system seen as interconnection of an accurate model and of a poor model and its interpretation as a pure feedback interconnection

context of solving problems of asymptotic disturbance rejection (in Chap. 4), of consensus in a decentralized control of a network of systems (in Chap. 5), as well as in the extension of such design methods to the case of nonlinear systems, addressed later in Chap. 6.

A more general setting in which uncertainties can be handled is when the system to be controlled can be seen as interconnection of a *nominal system*, whose structure and parameters are accurately known and of a *poorly modeled system*, whose structure and parameters may not be known except for some "loosely defined" properties. This setup covers indeed the case of parameter uncertainties[2] but also extends the range of model uncertainties to the case in which only the "dominant" dynamical effects have been explicitly (and possibly accurately) modeled, while other dynamical effects of lesser relevance have been deliberately neglected. The setup in question is usually referred to as the case of *unstructured uncertainties*. As an example, this is the case when the transfer function $T(s)$ of a single-input single-output system can be expressed in the form

$$T(s) = T_3(s)[T_2(s) + \Delta(s)]T_1(s) \tag{1.2}$$

in which $T_1(s)$, $T_2(s)$, $T_3(s)$ are accurately known while only some "loose" properties of $\Delta(s)$ are known. Typically, such properties are expressed as follows: all poles of $\Delta(s)$ have negative real part and a number δ is known such that

$$|\Delta(j\omega)| \le \delta \qquad \text{for all } \omega \in \mathbb{R}.$$

Suppose now that a system characterized by a transfer function of the form (1.2) is controlled by a system having transfer function $T_c(s)$. This yields the closed-loop system depicted in Fig. 1.1 on the left. An elementary manipulation shows that such closed-loop system coincides with the pure feedback interconnection (see Fig. 1.1 on the right) of a subsystem with input v and output z characterized by the transfer function

$$T_{zv}(s) = \frac{T_3(s)T_1(s)T_c(s)}{1 - T_3(s)T_2(s)T_1(s)T_c(s)}$$

and a subsystem with input z and output v, characterized by the transfer function $\Delta(s)$. Observe that the various $T_i(s)$'s appearing in $T_{zv}(s)$ are by assumption accurately

[2] As shown later in Chap. 3, see (3.42) and subsequent discussion.

known. Thus, it makes sense to consider the problem of designing $T_c(s)$ in such a way that the subsystem having transfer function $T_{zv}(s)$ is stable. If this is the case, the system of Fig. 1.1 is seen as the pure feedback interconnection of two stable subsystems. If, *in addition*, for some number $\gamma < (1/\delta)$ the function $T_{zv}(s)$ satisfies the inequality

$$|T_{zv}(j\omega)| \leq \gamma \qquad \text{for all } \omega \in \mathbb{R} \tag{1.3}$$

then the inequality

$$|T_{zv}(j\omega)\Delta(j\omega)| \leq c_0 \qquad \text{for all } \omega \in \mathbb{R}$$

holds for some $c_0 < 1$ and, by a well-known stability criterion, it can be concluded that the closed-loop system of Fig. 1.1 is stable, actually regardless of what $\Delta(s)$ is, so long as its indicated "loose" properties hold.

It is seen in this way that a systematic, although conservative, method of achieving stability in spite of unmodeled dynamics consists in looking at a system modeled by equations of the form

$$\begin{aligned}\dot{x} &= Ax + B_1 v + B_2 u\\ z &= C_1 x + D_{11} v + D_{12} u\\ y &= C_2 x + D_{21} v\end{aligned}$$

(see Fig. 1.2 and compare with Fig. 1.1), and—regarding y as an output available for measurement and u as an input available for control—seeking a controller yielding a *stable* closed-loop system, having a transfer function—between input v and output z—such that a *constraint* of the form (1.3) holds. A design problem of this type, which is known as the *γ-suboptimal H_∞ control problem*, is addressed in Chap. 3 for a general multi-input multi-output linear system. The solution of such problem is based on a thorough analysis of the relations existing between the parameters that characterize the state-space model of the system and the least upper bound of the norm of its transfer function over the imaginary axis in the complex plane, which is presented in the first section of the chapter. The design of a controller is presented in the last section of the chapter, and is based on the feasibility analysis of appropriate *linear matrix inequalities*.

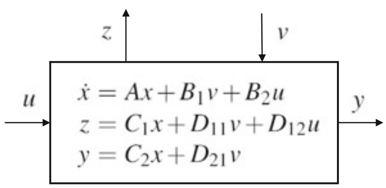

Fig. 1.2 The system to be controlled

1.3 Regulation and Tracking in Linear Systems

In the previous section, we have seen that a method for dealing with unstructured uncertainties consists in looking at a system such as the one described in Fig. 1.2 and seeking a controller yielding a stable closed-loop system having a transfer function—between input v and output z—that satisfies a constraint of the form (1.3). The problem in question, motivated by a problem of robust stability, has also an independent interest per se. In fact, regarding v as a vector of external disturbances, affecting the behavior of the controlled plant, and z as a set of selected variables of special interest, the design problem outlined above can be seen as the problem of finding a controller that—while keeping the closed-loop stable—enforces a prescribed *attenuation* of the effect of the disturbance v on the variable of interest z.

In Chap. 4, the problem of designing a control so as to influence the effect of certain external signals on some selected variables is studied in more detail. Specifically, the problem is addressed of finding a control law that yields a closed-loop system in which the influence of such external signals on the variables of interest ultimately *decays to zero*. This requirement is substantially stronger than the requirement of imposing a prescribed attenuation and its is not likely to be feasible in general. It can be enforced, though, if *a model of the external signals* is available. This gives rise to what is known as the *problem of asymptotic tracking and/or disturbance rejection*, or more commonly the *problem of output regulation*. The problem in question can be viewed as a generalized version of a classical problem in the elementary theory of servomechanisms: the so-called "set point control" problem. In fact, this problem consists in the design of a control ensuring that the output $y(t)$ of the controlled plant asymptotically tracks *any* prescribed *constant* reference $y_{\text{ref}}(t)$, and does this in spite of *any* possible *constant* disturbance $d(t)$. In this case, the external signals are the prescribed reference $y_{\text{ref}}(t)$ and the disturbance $d(t)$. Even if their (constant) value is not available to the controller, a precise model for such signals is known. In fact, they are solution of the (trivial) differential equations $\dot{y}_{\text{ref}}(t) = 0$ and $\dot{d}(t) = 0$. It is well known that such problem is solved if the control guarantees that the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ is subject to an *integral* action. It is also known that a similar design goal can be achieved—by means of similar design techniques—if the external signals are *oscillations* of a fixed frequency but unknown amplitude.

The setup considered in Chap. 4 is that of a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= C_{\text{e}}x + Q_{\text{e}}w \\ y &= Cx + Qw, \end{aligned} \tag{1.4}$$

the first of which describes a plant with state x and *control input* u, subject to a set of *exogenous input* variables w which includes *disturbances* (to be rejected) and/or *references* (to be tracked). The second equation defines a set of *regulated* (or *error*) variables e, the variables of interest that need to be "protected" from the effect of the exogenous variables w, while the third equation defines a set of *measured* variables

y, which are assumed to be available for feedback. As mentioned before, a model is assumed to exist for the exogenous variables that takes the form of a homogeneous linear differential equation

$$\dot{w} = Sw. \tag{1.5}$$

The problem is to design a controller, with input y and output u, yielding a *stable* closed-loop system in which $e(t)$ *asymptotically decays to zero* as t tends to ∞.

Consistently with the approach pursued in the previous two chapters, it will be required that the design goal in question be achieved in spite of model uncertainties, be they structured or unstructured. In this setup, the results established in Chaps. 2 and 3 prove of fundamental importance. A particularly delicate robustness problem arises when the model of the exogenous signal is uncertain, which is the case when the parameters that characterize the matrix S in (1.5) are not known. This problem can be handled by means of techniques borrowed from the theory of adaptive control, as shown in Sect. 4.8.

1.4 From Regulation to Consensus

Having learned, in Chap. 4, how to design a controller in such a way that the output $y(t)$ of a plant tracks a reference output $y_{\text{ref}}(t)$ generated by an autonomous system of the form

$$\begin{aligned} \dot{w} &= Sw \\ y_{\text{ref}} &= -Qw, \end{aligned} \tag{1.6}$$

consider now the problem in which a *large* number of systems

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \tag{1.7}$$

is given ($k = 1, \ldots, N$ and N large) and all such systems have to be controlled in such a way that each output $y_k(t)$ asymptotically tracks the output $y_{\text{ref}}(t)$ of (1.6), i.e., in such a way that all tracking errors

$$e_k(t) = y_k(t) - y_{\text{ref}}(t) = C_k x_k + Qw$$

decay to zero as $t \to \infty$. A problem of this kind is usually referred to as a problem of *leader–followers coordination*. Trivially, the problem could be solved by means of the design procedure described in Chap. 4. The problem with such approach, though, is that—in this framework—the controller that generates u_k is supposed to be driven by the corresponding tracking error e_k, i.e.,—for each k—the controller that generates u_k is supposed to *have access to* the reference *output* $y_{\text{ref}}(t)$. If the number N is very large, this entails an excessively large exchange of information, which could be prohibitive if the followers are spatially distributed over a very large (geographical) region.

The challenge is to achieve the desired tracking goal with a limited exchange of information, consisting only in the differences between the output of each individual system and those of a limited number of other systems (that, with the idea of a spatial distribution in mind, are usually called the *neighbors*), one of which could be, possibly but not necessarily, the reference generator (1.6). Such restricted information pattern renders the problem a little bit more challenging. In Chap. 5, the problem is addressed under the assumption that the information available for the (decentralized) controller of the kth system consists of a *fixed* linear combination the differences $(y_j - y_k)$, of the form

$$v_k = \sum_{j=0}^{N} a_{kj}(y_j - y_k) \qquad k = 1, \ldots, N \tag{1.8}$$

in which the a_{kj}'s are elements of a $N \times (N+1)$ *sparse* matrix and $y_0 = y_{\mathrm{ref}}$. Typically, only few of the coefficients a_{kj} are nonzero and only those systems for which $a_{k0} \neq 0$ have access to $y_{\mathrm{ref}}(t)$.

If a problem of this kind is solved, the outputs $y_k(t)$ of all systems (1.7)—the followers—asymptotically converge to a single function of time, the output $y_{\mathrm{ref}}(t)$ of system (1.6)—the leader. In this case, it is said that the outputs of all systems, leader and followers altogether, *achieve consensus*. A more general version of such design problem arises when a set of N system modeled by equations of the form (1.7) is given, *no leader is specified*, and (decentralized) controllers are to be designed, which receive information modeled as in (1.8),[3] such that all outputs $y_k(t)$ achieve consensus, i.e., asymptotically converge to a single function of time. A problem of this kind is called a problem of *leaderless coordination* or, more commonly, a *consensus problem*. This problem also will be addressed in Chap. 5. As expected, instrumental in the solution of such problems is the theory developed in Chap. 4.

1.5 Feedback Stabilization and State Observers for Nonlinear Systems

Chapter 6 deals with the problem of designing state feedback laws for single-input single-output nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{1.9}$$

Such systems are usually referred to as *input-affine systems*. The first part of the chapter follows very closely the viewpoint presented in Sect. 2.1 of Chap. 2. An integer r is defined, the *relative degree of the system*, that plays a role similar to

[3] With j now ranging from 1 to N.

that of the difference between the number of poles and the number of zeros in the transfer function of a linear system. On the basis of this concept and of some related properties, a change of coordinates can be defined, by means of which the system can be expressed in the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z, \xi) + b(z, \xi)u \\ y &= \xi_1, \end{aligned} \tag{1.10}$$

in which $z \in \mathbb{R}^{n-r}$ and $\xi = \text{col}(\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$. This form is known as *normal form* of system's equations. The form is guaranteed to exist locally in some open subsets in the state space, but under appropriate conditions the existence of a globally defined normal form can be established.

The special interest in this form resides in the role played by the upper equation

$$\dot{z} = f_0(z, \xi) \tag{1.11}$$

in problems of asymptotic stabilization. In fact, if the system characterized by such equation, viewed as a system with state z and input ξ is *input-to-state stable*,[4] then the full system (1.10) can be *globally stabilized* by means of a *state feedback* control law of the form

$$u(z, \xi) = \frac{1}{b(z, \xi)}(-q(z, \xi) + \hat{K}\xi), \tag{1.12}$$

in which

$$\hat{K} = \begin{pmatrix} k_0 & k_1 & \ldots & k_{r-1} \end{pmatrix}$$

is a vector of design parameters chosen in such a way that the roots of the polynomial

$$p(\lambda) = \lambda^r - k_{r-1}\lambda^{r-1} - \cdots - k_1\lambda - k_0$$

have negative real part. In fact, system (1.10) subject to the feedback law (1.12) can be seen as a stable linear system cascaded with an input-to-state stable system and this, by known properties,[5] is an asymptotically stable system (see Fig. 1.3). This result, discussed in Sect. 6.4, is perhaps one of simplest results available that deal with global stabilization via state feedback. It is observed, in this respect, that if the system in question were a linear system, the dynamics of (1.11) would reduce to that

[4]See Sect. B.2 of Appendix B for the definition of the concept of input-to-state stability and its properties.

[5]See Sect. B.3.

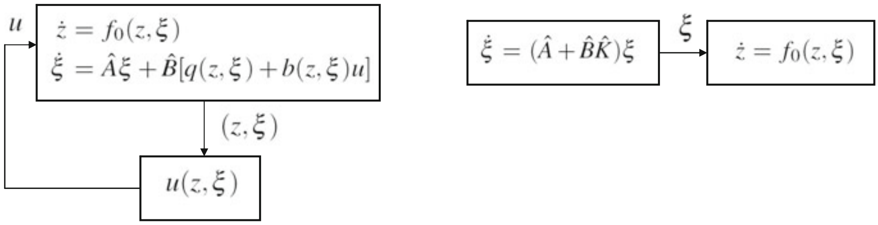

Fig. 1.3 System (1.10) controlled by (1.12) and its equivalent form

of a linear system of the form

$$\dot{z} = F_0 z + G_0 \xi$$

with F_0 a matrix whose eigenvalue coincide[6] with the roots of the numerator of the transfer function, i.e., with the zeros of the system. Thus, the assumption that the dynamics of (1.11) are input-to-state stable can be seen as a nonlinear equivalent of the assumption that the zeros of the system have negative real part, assumption that is at the basis of the methods for robust stability presented in Chap. 2. In analogy, the dynamics of (1.11) are called the *zero dynamics* of the nonlinear system (1.9).

If system (1.11) does not have the indicated input-to-state stability properties, global stabilization of system (1.10) is still possible, under the assumption that: (i) the map $f_0(z, \xi)$ only depends on z and on the first component ξ_1 of ξ, i.e.,

$$\dot{z} = f_0(z, \xi_1)$$

and, (ii) a function $\xi_1^*(z)$ exists such that the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, \xi_1(z))$$

is globally asymptotically stable. The underlying design method, known as *back-stepping*, is described in Sect. 6.4 as well.

The two stabilization methods outlined above require, as the detailed analysis in Sect. 6.4 highlights, accurate knowledge of the nonlinear functions $q(z, \xi)$, $b(z, \xi)$ (and even of $f_0(z, \xi_1)$ in the second method) that need to be exactly "compensated" or "cancelled" for the design of the appropriate stabilizing feedback law. Thus, as such, they are not suitable for solving global stabilization problems in the presence of model uncertainties. However, if a less ambitious design goal is pursued, a more robust feedback law can be easily designed. The less ambitious design goal consists in seeking, instead of global asymptotic stability, just (local) asymptotic stability but with "guarantee" on the region of attraction. More precisely the stability result that can be obtained is as follows: a compact set $\mathscr{C}$ in the state space is given, and then a prescribed equilibrium is made asymptotically stable, with a domain of

[6]Under the assumption that the system is reachable and observable, see Sect. 2.1 for more details.

attraction $\mathscr{A}$ that is guaranteed to contain the given set $\mathscr{C}$. Clearly, if this is feasible, a feedback law will result that depends on the choice of $\mathscr{C}$. Such design goal can be achieved, still under the assumption that the dynamics of (1.11) are input-to-state stable, by means of a simple law of the form $u = K\xi$ in which the entries of the matrix K are appropriate design parameters. A feedback law of this type is a *partial-state* feedback law, identical to the one introduced in Sect. 2.4, and enjoys the same robustness properties. All relevant results are presented in Sect. 6.5.

Having shown how, under suitable assumptions, a nonlinear system can be globally asymptotically (or only locally asymptotically, but with a guaranteed region of attraction) stabilized by means of full *state* or *partial-state* feedback, in Chap. 7 the design of an asymptotic *observer* is considered. The existence of such observer reposes on conditions that guarantee the existence of a change of variables leading to a special *observability* normal *form*. Thus, also in the problem of asymptotic state estimation, the passage to special forms plays an important role in the design. As in the case of linear systems, the observer of a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{1.13}$$

consists in a *copy* of the dynamics of the observed system, *corrected* by a term proportional to the mismatch between the actual output and its estimated value, i.e., is a system of the form

$$\dot{\hat{x}} = f(\hat{x}, u) + G[y - h(\hat{x}, u)].$$

The difference with the case of linear systems is that the "output injection" vector G needs to have a special form. In fact, G is chosen as

$$G = \mathrm{col}(\kappa c_{n-1}, \kappa^2 c_{n-2}, \ldots, \kappa^n c_0).$$

In Sects. 7.3 and 7.4, it is shown that, under reasonable hypotheses, there is a choice of the design parameters $c_0, c_1, \ldots, c_{n-1}$ and a number κ^* such that, if $\kappa > \kappa^*$, the estimation error $x(t) - \hat{x}(t)$ asymptotically decays to zero, for any value of $x(0) - \hat{x}(0)$. Since the parameter κ needs to be large enough, the observer in question is usually referred to as a *high-gain observer*.

Finally, in the last section of the chapter, a nonlinear version of the *separation principle* of linear system theory[7] is discussed. Namely, assuming that the dynamics of (1.13) can be stabilized by means of a state feedback of the form $u = u^*(x)$, the effect of a (dynamical) output feedback of the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u^*(\hat{x})) + G[y - h(\hat{x}, u^*(\hat{x}))] \\ u &= u^*(\hat{x}) \end{aligned}$$

[7] See Theorem A.4 and in particular (A.18) in Appendix A.

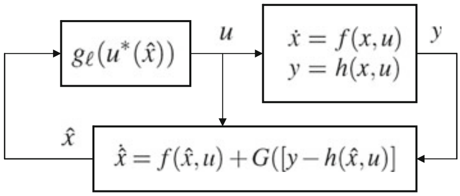

Fig. 1.4 Control via separation principle

is analyzed. In this context, it is stressed that—since the controlled system is nonlinear—a feedback law of this form might induce finite escape times. A precautionary measure to avoid this drawback is to "clip" control function $u^*(\hat{x})$ when it takes large values. As a result of this measure, in the previous formula the control $u^*(\hat{x})$ is replaced by a control of the form

$$u = g_\ell(u^*(\hat{x}))$$

in which $g_\ell(s)$ is a *saturation* function, that is a continuously differentiable function that coincides with s when $|s| \le \ell$, is odd and monotonically increasing, and satisfies $\lim_{s\to\infty} g_\ell(s) = \ell(1+c)$ with $c \ll 1$. This leads to a control scheme described as in Fig. 1.4. In this way though, global asymptotic stability no longer can be assured and only asymptotic stability with a guaranteed region of attraction can be obtained. All details are presented in Sect. 7.5.

1.6 Robust Stabilization of Nonlinear Systems

The output feedback, observer-based, stabilization result provided at the end of Chap. 7 is indeed relevant but cannot be seen as robust, because—as it is the case for linear systems—the construction of the observer requires the knowledge of an accurate model of the plant. Even if the state feedback law is robust (as it is the case when the feedback law can be expressed as a linear function of the partial state ξ), the estimation of ξ by means of an observer such as the one discussed in Chap. 7 may fail to be robust. The problem therefore arises of finding, if possible, a robust estimate of the state of the system and, specifically, of the partial state ξ that would be used for robust stabilization (in case system (1.11) is input-to-state stable). Since, for each $i = 1,\dots,r$, the ith component ξ_i of the state $\xi(t)$ coincides with the derivative of order $i-1$, with respect to t, of the output $y(t)$ of the system, it is reasonable to guess that at least a rough estimate of $\xi(t)$ could be obtained from a dynamical system of order r, driven by the output $y(t)$. This, in fact, is what happens in the case of a linear system, as shown in Sect. 2.5 (see in particular (2.26)). This problem has been addressed in various forms in the literature, yielding to methods for robust stabilization that can be seen as counterparts, for nonlinear systems, of the method discussed in Chap. 2.[8] In a relatively recent version of such methods, it has been shown that *all* ingredients needed to implement the feedback law (1.12), that is all components of ξ as well the functions $b(z, \xi)$ and $q(z, \xi)$, can be robustly estimated. The design of a robust observer that accomplishes this goal, known as *extended observer*, is described—in the more general context of a multi-input multi-output system—in Sect. 10.3, while its asymptotic properties as well as the resulting separation principle are discussed in Sect. 10.4.

Chapter 8 deals with a nonlinear version of the *Small-Gain Theorem*, used in Chap. 3 to address robust stabilization problems in the presence of unmodeled dynamics. In a nutshell, the theorem in question says that the pure feedback interconnection of two input-to-state stable systems is globally asymptotically stable if the composition of the respective "gain functions" is a *contraction*.[9] This theorem can be used as a tool to analyze the stability of interconnected nonlinear systems and, like in the case of linear systems, to address problems of robust stability.

1.7 Multi-input Multi-output Nonlinear Systems

The methods for feedback stabilization described in Chap. 2, for linear systems, and in Chap. 6, for nonlinear systems, are extended in Chaps. 9–11 to the case of multi-input multi-output systems. To avoid duplications, only nonlinear systems are considered, and—for convenience—only the case of systems having the same number of inputs and output channels is addressed. Extension of the design methods described in Chap. 2 to the case of a (linear) system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.14}$$

having the same number m of inputs and output components is relatively straightforward in the special case in which there exists a set of integers $r_1, r_2, \ldots, r_m$ (one per each component of the output) such that, for each $i = 1, \ldots, m$, the ith row c_i of C satisfies

$$c_i B = c_i A B = \cdots = c_i A^{r_i - 2} B = 0, \qquad c_i A^{r_i - 1} B \neq 0 \tag{1.15}$$

and, *in addition*, the $m \times m$ matrix

[8] See references in Chaps. 7 and 10.

[9] See Sect. B.2 of Appendix B for a precise definition of the concept of gain function and Sect. 8.1 of Chap. 8 for the definition of the concept of contraction.

$$J = \begin{pmatrix} c_1 A^{r_1 - 1} B \\ c_2 A^{r_2 - 1} B \\ \cdots \\ c_m A^{r_m - 1} B \end{pmatrix}$$

is nonsingular, in which case

$$\begin{pmatrix} y_1^{(r_1)} \\ \cdots \\ y_m^{(r_m)} \end{pmatrix} = \begin{pmatrix} c_1 A^{r_1} \\ \cdots \\ c_m A^{r_m} \end{pmatrix} x + Ju, \tag{1.16}$$

with

$$\det[J] \neq 0. \tag{1.17}$$

Systems for which property (1.15)–(1.17) holds are said to have a *vector relative degree* $\{r_1, \ldots, r_m\}$, and it is known that the subclass of multivariable systems that have such property coincides with the class of systems for which there exist a matrix F and a nonsingular matrix G that render the transfer function matrix

$$T_{\text{cl}}(s) = C(sI - A - BF)^{-1} BG$$

purely *diagonal*.

A substantially more general (and hence more interesting) class of systems to deal with is the class of systems whose transfer function matrix $T(s) = C(sI - A)^{-1}B$ is *invertible*, which in what follows are referred to as *invertible systems*.[10] This class of systems is more general because, as it is easy to check, if the properties (1.15)–(1.17) that hold the transfer function of the system is necessarily invertible, but the converse is not true. This is shown in the following elementary example.

Example 1.1 Consider a linear system with two inputs and two outputs modeled by the Eq. (1.14) in which

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $c_1 B \neq 0$ and $c_2 B \neq 0$, it is seen that the two integers defined in (1.15) are $r_1 = r_2 = 1$. However, the matrix

$$J = \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

[10] Note that there is no ambiguity between *left* and *right* invertibility, because the matrix $T(s)$ is square.

is singular and hence condition (1.17) does not hold. This system, though, is invertible, as it can be checked by looking at its transfer function matrix

$$T(s) = \frac{1}{s^3 - s^2 - s} \begin{pmatrix} s^2 - s & s \\ s^2 - s + 1 & 2s \end{pmatrix}.$$

$\triangleleft$

The extension of the design methods presented in Chaps. 2 and 6 to the more general class of multivariable systems that are invertible (but possibly do not possess a vector relative degree) is not as straightforward as one might imagine. In fact, such extension requires (notably in the case of nonlinear systems) a preliminary detailed study—that will be conducted in Chap. 9—of the consequences of the property of invertibility. In Sects. 9.2 and 9.3, the concept of *invertibility* for a multivariable nonlinear system, affine in the inputs, is discussed. The property of being invertible is indirectly defined by means of a classical algorithm, known as *structure algorithm*, that was originally conceived as a tool to characterize—in a multivariable linear system—certain integers[11] that can be associated with the limit of $T(s)$ as $s \to \infty$ and was then extended by various authors to nonlinear systems,[12] as a tool to characterize the property of when the input of a system is uniquely determined by its output (and by its state). A by-product of paramount importance of the algorithm in question is the possibility of defining new coordinates functions by means of which the system is described in a special normal form (see Sect. 9.4). This normal form highlights—as the (simpler) normal form (1.10) of a single-input single-output system does—special relations between individual components of the state vector, and provides the background for the development of feedback stabilizing laws. The algorithm in question is also useful to obtain a *classification* of multivariable nonlinear systems, on the basis of which—in order of increasing complexity—the stabilization problem will be handled in Chaps. 10 and 11.

Example 1.2 The following elementary example shows how multivariable nonlinear systems can be classified. Consider a two-input two-output system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, x) \\ \dot{x}_1 &= a_1(z, x) + b_{11}(z, x)u_1 + b_{12}(z, x)u_2 \\ \dot{x}_2 &= x_3 + \delta(z, x)[a_1(z, x) + b_{11}(z, x)u_1 + b_{12}(z, x)u_2] \\ \dot{x}_3 &= a_2(z, x) + b_{21}(z, x)u_1 + b_{22}(z, x)u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned} \tag{1.18}$$

in which $x = \mathrm{col}(x_1, x_2, x_3)$, where it is assumed that the matrix

[11] Such integers characterize what is known as the *zero structure* at $s = \infty$ of $T(s)$.

[12] See references of Chap. 9.

$$B(z,x) = \begin{pmatrix} b_{11}(z,x) & b_{12}(z,x) \\ b_{21}(z,x) & b_{22}(z,x) \end{pmatrix}$$

is nonsingular for all (z, x).

If $\delta(z, x)$ is identically zero, it is seen that $x_3 = y_2^{(1)}$ and, as a consequence,

$$\begin{pmatrix} y_1^{(1)} \\ y_2^{(2)} \end{pmatrix} = \begin{pmatrix} a_1(z,x) \\ a_2(z,x) \end{pmatrix} + B(z,x)u.$$

This equation can be considered as a nonlinear equivalent of Eq. (1.16), written for $m = 2$ and $\{r_1, r_2\} = \{1, 2\}$, and the system is said to have vector relative degree $\{1, 2\}$.

If $\delta(z, x)$ is nonzero, while the first two components of x coincide with y_1 and y_2, the third component x_3 is a combination (with a state-dependent coefficients) of the first derivatives of y_1 and y_2, namely

$$x_3 = y_2^{(1)} - \delta(z,x)y_1^{(1)}.$$

Something special, though, occurs if $\delta(z, x)$ is a constant. If this is the case, in fact, the system can be changed, via feedback, into a system whose input–output behavior is the same as that of a linear system. This is achieved using the control

$$u = [B(z,x)]^{-1} \begin{pmatrix} -a_1(z,x) + v_1 \\ -a_2(z,x) + v_2 \end{pmatrix}$$

that modifies system (1.18) into

$$\begin{aligned} \dot{z} &= f_0(z,x) \\ \dot{x}_1 &= v_1 \\ \dot{x}_2 &= x_3 + \delta(z,x)v_1 \\ \dot{x}_3 &= v_2 \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned}$$

If the "multiplier" $\delta(z, x)$ is constant, the behavior of the latter—seen as a system with input v and output y—is that of a linear system having transfer function

$$T(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ \frac{\delta}{s} & \frac{1}{s^2} \end{pmatrix}.$$

The system is said to be *input–output* linearizable. Note also that, if $\delta \neq 0$, the matrix $T(s)$ is invertible, but property (1.15)–(1.17) fails to hold.

Finally, consider the (more general) case in which the multiplier $\delta(z, x)$ only depends on z and x_1. If this is the case, the system is invertible, i.e., the input $u(t)$ can be uniquely recovered from $y(t)$ and $x(t)$. To see why this is the case, observe that, while both $y_1^{(1)}$ and $y_2^{(1)}$ explicitly depend on u, the relations

$$\begin{aligned} y_1^{(1)} &= a_1(z,x) + b_{11}(z,x)u_1 + b_{12}(z,x)u_2 \\ y_2^{(1)} &= x_3 + \delta(z,x_1)[a_1(z,x) + b_{11}(z,x)u_1 + b_{12}(z,\xi)u_2] \end{aligned}$$

do not uniquely determine u because—if $\delta(z,x_1) \neq 0$—the matrix that multiplies u in such system of equations is singular. Rather, u can be uniquely determined by a system consisting of first of such identities and of the identity obtained by taking the derivative of the second one with respect to time. In fact, in the system[13]

$$\begin{aligned} y_1^{(1)} &= a_1(z,x) + b_{11}(z,x)u_1 + b_{12}(z,x)u_2 \\ y_2^{(2)} &= a_2(z,x) + b_{21}(z,x)u_1 \\ &\qquad + b_{22}(z,x)u_2 + \left[\frac{\partial\delta}{\partial z} f_0(z,x) + \frac{\partial\delta}{\partial x_1} y_1^{(1)}\right] y_1^{(1)} + \delta(z,x_1) y_1^{(2)} \end{aligned}$$

the matrix that multiplies u is nonsingular, as a consequence of the nonsingularity of the matrix $B(z,x)$. This system, like the linear system in the previous example, is a "prototype" of an *invertible* system which fails to satisfy the (nonlinear version of) condition (1.17). $\triangleleft$

In Chap. 10 we will address the problem of (robust) asymptotic stabilization of a multivariable nonlinear system having a vector relative degree, while in Chap. 11 we address the same problem for the more general class of system that are input–output linearizable[14] and, under appropriate assumptions, also a more general class of systems that are invertible but not input–output linearizable.

1.8 Regulation and Tracking in Nonlinear Systems

Finally, in Chap. 12 the nonlinear analogues of the design methods described in Chap. 4 are discussed. The problem is to control a system that can be seen as a nonlinear analogue of (1.5)–(1.4), namely a system modeled by equations of the form

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[13] In the equation that follows and in the consequences drawn immediately after, we take explicit advantage of the fact that the multiplier $\delta(z,x_1)$ *does not* depend on x_2.

[14] The relevance of this special class of systems resides not so much in the fact that there is a feedback law that can force a linear input–output behavior (law that is not likely to be used in practice since a law that cancels the nonlinearities necessarily requires the exact knowledge of the system model and hence is not suitable in the presence of model uncertainties) but, rather, in the special properties of the normal forms of a system for which such law exists. In fact, the properties of such special normal form can be fruitfully exploited in the design of robust output feedback stabilizing controls for invertible multivariable systems.

$$\begin{aligned}\dot{w} &= s(w)\\ \dot{x} &= f(w, x, u)\\ e &= h_{\mathrm{e}}(w, x)\\ y &= h(w, x),\end{aligned}$$

in such a way that all trajectories are bounded and $e(t)$ asymptotically decays to zero as time tends to ∞. The analysis is cast in a form that follows, as much as possible, the analysis carried out in Chap. 4. Prerequisite to this is an extension to nonlinear systems of the notion of *steady-state* response, extension that is discussed in detail in Sect. B.6 of Appendix B.

The design methods presented in the chapter only deal with the special case in which $y = e$, i.e., the case in which measured output and regulated output coincides. The more general case of systems in which y includes, in addition to e, an extra set of measured variables y_{r}, covered for linear systems in Sect. 4.4, is not covered here for the simple reason that a full and satisfactory solution to such problem is not known yet for nonlinear systems. In the case of systems having $y = e$, however, the design methods covered in Sects. 4.6 and 4.7 can be extended to nonlinear systems, and the resulting theory is presented in Chap. 12.

Part I
Linear Systems

Chapter 2
Stabilization of Minimum-Phase Linear Systems

2.1 Normal Form and System Zeroes

The purpose of this section is to show how single-input single-output linear systems can be given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which certain fundamental properties of the system are highlighted, that plays a relevant role in the method for robust stabilization discussed in the following sections.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally defined as follows. Given a single-input single-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{2.1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, consider the sequence of real numbers CB, $CAB,\ CA^2B, \ldots,\ CA^kB, \ldots$ Let r denote the least integer for which $CA^{r-1}B \neq 0$. This integer is called the *relative degree* of system (2.1). In other words, r is the integer uniquely characterized by the conditions

$$\begin{aligned} &CB = CAB = \ldots = CA^{r-2}B = 0 \\ &CA^{r-1}B \neq 0. \end{aligned} \tag{2.2}$$

The relative degree of a system can be easily identified with an integer associated with the transfer function of the system. In fact, consider the transfer function of (2.1)

$$T(s) = C(sI - A)^{-1}B$$

and recall that $(sI - A)^{-1}$ can be expanded, in negative powers of s, as

$$(sI - A)^{-1} = \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots$$

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_2

This yields, for $T(s)$, the expansion

$$T(s) = \frac{1}{s}CB + \frac{1}{s^2}CAB + \frac{1}{s^3}CA^2B + \cdots$$

Using the definition of r, it is seen that the expansion in question actually reduces to

$$\begin{aligned} T(s) &= \frac{1}{s^r}CA^{r-1}B + \frac{1}{s^{r+1}}CA^rB + \frac{1}{s^{r+2}}CA^{r+1}B + \cdots \\ &= \frac{1}{s^r}[CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots] \end{aligned}$$

Thus

$$s^rT(s) = CA^{r-1}B + \frac{1}{s}CA^rB + \frac{1}{s^2}CA^{r+1}B + \cdots$$

from which it is deduced that

$$\lim_{s\to\infty} s^rT(s) = CA^{r-1}B. \tag{2.3}$$

Recall now that $T(s)$ is a rational function of s, the ratio between a numerator polynomial $N(s)$ and a denominator polynomial $D(s)$

$$T(s) = \frac{N(s)}{D(s)}.$$

Since the limit (2.3) is finite and (by definition of r) *nonzero*, it is concluded that r is necessarily the difference between the degree of $D(s)$ and the degree of $N(s)$. This motivates the terminology "relative degree." Finally, note that, if $T(s)$ is expressed (as it is always possible) in the form

$$T(s) = K' \frac{\Pi_{i=1}^{n-r}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)},$$

it necessarily follows that[1]

$$K' = CA^{r-1}B.$$

We use now the concept of relative degree to derive a change of variables in the state space, yielding a form of special interest. The following facts are easy consequences of the definition of relative degree.[2]

Proposition 2.1 *The r rows of the $r \times n$ matrix*

[1]The parameter K' is sometimes referred to as the *high-frequency gain* of the system.

[2]For a proof see, e.g., [1, pp. 142–144].

$$T_1 = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.4}$$

are linearly independent. As a consequence, $r \le n$.

We see from this that, if r is strictly less than n, it is possible to find—in many ways—a matrix $T_0 \in \mathbb{R}^{(n-r)\times n}$ such that the resulting $n \times n$ matrix

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} T_0 \\ C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} \tag{2.5}$$

is *nonsingular*. Moreover, the following result also holds.

Proposition 2.2 *It is always possible to pick T_0 in such a way that the matrix (2.5) is nonsingular and $T_0 B = 0$.*

We use now the matrix T introduced above to define a change of variables. To this end, in view of the natural partition of the rows of T in two blocks (the upper block consisting of the $n-r$ rows of T_0 and the lower block consisting of the r rows of the matrix (2.4)) it is natural to choose different notations for the first $n-r$ new state variables and for the last r new state variables, setting

$$z = T_0 x, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} x.$$

To determine the equations describing the system in the new coordinates, we take the derivatives of z and ξ with respect to time. For the former, no special structure is found and we simply obtain

$$\dot{z} = T_0 \dot{x} = T_0(Ax + Bu) = T_0 Ax + T_0 Bu. \tag{2.6}$$

On the contrary, for the latter, a special structure can be displayed. In fact, observing that $\xi_i = CA^{i-1}x$, for $i = 1, \ldots, r$, and using the defining properties (2.2), we obtain

$$\begin{aligned} \dot{\xi}_1 &= C\dot{x} = C(Ax + Bu) = CAx = \xi_2 \\ \dot{\xi}_2 &= CA\dot{x} = CA(Ax + Bu) = CA^2 x = \xi_3 \\ &\cdots \\ \dot{\xi}_{r-1} &= CA^{r-2}\dot{x} = CA^{r-2}(Ax + Bu) = CA^{r-1}x = \xi_r \end{aligned}$$

and

$$\dot{\xi}_r = CA^{r-1}\dot{x} = CA^{r-1}(Ax + Bu) = CA^r x + CA^{r-1}Bu.$$

The equations thus found can be cast in a compact form. To this end, it is convenient to introduce a special triplet of matrices, $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times 1}$, $\hat{C} \in \mathbb{R}^{1\times r}$, which are defined as[3]

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.7}$$

With the help of such matrices, it is easy to obtain

$$\begin{aligned} \dot{\xi} &= \hat{A}\xi + \hat{B}(CA^r x + CA^{r-1}Bu) \\ y &= Cx = \xi_1 = \hat{C}\xi. \end{aligned} \tag{2.8}$$

Note that the right-hand sides of (2.6) and (2.8) are still expressed in terms of the "original" set of state variables x. To complete the change of coordinates, x should be expressed as a (linear) function of the new state variables z and ξ, that is as a function of the form

$$x = M_0 z + M_1 \xi$$

in which M_0 and M_1 are partitions of the inverse of T, implicitly defined by

$$\begin{pmatrix} M_0 & M_1 \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = I.$$

Setting

$$A_{00} = T_0 A M_0, \qquad A_{01} = T_0 A M_1, \qquad B_0 = T_0 B,$$

$$A_{10} = CA^r M_0, \qquad A_{11} = CA^r M_1, \qquad b = CA^{r-1}B,$$

the equations in question can be cast in the form

$$\begin{aligned} \dot{z} &= A_{00}z + A_{01}\xi + B_0 u \\ \dot{\xi} &= \hat{A}\xi + \hat{B}(A_{10}z + A_{11}\xi + bu) \\ y &= \hat{C}\xi. \end{aligned} \tag{2.9}$$

These equations characterize the so-called *normal form* of the Eq. (2.1) describing the system. Note that the matrix B_0 can be made equal to 0 if the option described in Proposition 2.2 is used. If this is the case, the corresponding normal form is said

[3] A triplet of matrices of this kind is often referred to as a triplet in *prime form*.

to be *strict*. On the contrary, the coefficient b, which is equal to the so-called high-frequency gain of the system, is always nonzero.

In summary, by means of the change of variables indicated above, the original system is transformed into a system described by matrices having the following structure

$$TAT^{-1} = \begin{pmatrix} A_{00} & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & \hat{C} \end{pmatrix}. \tag{2.10}$$

One of the most relevant features of the normal form of the equations describing the system is the possibility of establishing a relation between the zeros of the transfer function of the system and certain submatrices appearing in (2.9).

We begin by observing that

$$T(s) = \frac{\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix}}{\det(A - sI)}. \tag{2.11}$$

This is a simple consequence of a well-known formula for the determinant of a partitioned matrix.[4] Using the latter, we obtain

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det(A - sI)\det(-C(A - sI)^{-1}B)$$

from which, bearing in mind the fact that $C(A - sI)^{-1}B$ is a scalar quantity, the identity (2.11) immediately follows. Note that in this way we have identified simple and appealing expressions for the numerator and denominator polynomial of $T(s)$.

Next, we determine an expansion of the numerator polynomial.

Proposition 2.3 *The following expansion holds*

$$\det \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = (-1)^r CA^{r-1}B \det([A_{00} - \frac{1}{b}B_0A_{10}] - sI). \tag{2.12}$$

Proof Observe that the left-hand side of (2.12) remains unchanged if A, B, C is replaced by TAT^{-1}, TB, CT^{-1}. In fact

$$\begin{pmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

[4] The formula in question is

$$\det \begin{pmatrix} S & P \\ Q & R \end{pmatrix} = \det(S)\det(R - QS^{-1}P).$$

.

from which, using the fact that the determinant of a product is the product of the determinants and that

$$\det\begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} T^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} TT^{-1} & 0 \\ 0 & 1 \end{pmatrix} = 1$$

we obtain

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det\begin{pmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & 0 \end{pmatrix}.$$

Furthermore, observe also that the left-hand side of (2.12) remains unchanged if A is replaced by $A + BF$, regardless of how the matrix $F \in \mathbb{R}^{1\times n}$ is chosen. This derives from the expansion

$$\begin{pmatrix} A + BF - sI & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ F & 1 \end{pmatrix}$$

and from the fact that the determinant of the right-hand factor in the product above is simply equal to 1.

With these observations in mind, use for T the transformation that generates the normal form (2.10), to arrive at

$$\begin{aligned}
\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} &= \det\begin{pmatrix} \begin{pmatrix} A_{00} - sI & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} - sI \end{pmatrix} & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix} \\
&= \det\begin{pmatrix} \begin{pmatrix} A_{00} - sI & A_{01} \\ \hat{B}A_{10} & \hat{A} + \hat{B}A_{11} - sI \end{pmatrix} + \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F & \begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{C} \end{pmatrix} & 0 \end{pmatrix}.
\end{aligned}$$

The choice

$$F = \frac{1}{b}\begin{pmatrix} -A_{10} & -A_{11} \end{pmatrix}$$

yields

$$\begin{pmatrix} B_0 \\ \hat{B}b \end{pmatrix} F = \begin{pmatrix} -\frac{1}{b}B_0A_{10} & -\frac{1}{b}B_0A_{11} \\ -\hat{B}A_{10} & -\hat{B}A_{11} \end{pmatrix}.$$

and hence it follows that

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det\begin{pmatrix} [A_{00} - \frac{1}{b}B_0A_{10}] - sI & A_{01} - \frac{1}{b}B_0A_{11} & 0 \\ 0 & \hat{A} - sI & \hat{B}b \\ 0 & \hat{C} & 0 \end{pmatrix}.$$

The matrix on the right-hand side is block-triangular, hence

$$\det\begin{pmatrix} A - sI & B \\ C & 0 \end{pmatrix} = \det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)\det\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix}.$$

Finally, observe that

$$\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = \begin{pmatrix} -s & 1 & 0 \cdots & 0 & 0 \\ 0 & -s & 1 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 & 0 \\ 0 & 0 & 0 \cdots & -s & b \\ 1 & 0 & 0 \cdots & 0 & 0 \end{pmatrix}$$

Expanding the determinant according to the entries of last row we obtain

$$\det\begin{pmatrix} \hat{A} - sI & \hat{B}b \\ \hat{C} & 0 \end{pmatrix} = (-1)^{r+2}\det\begin{pmatrix} 1 & 0 \cdots & 0 & 0 \\ -s & 1 \cdots & 0 & 0 \\ \cdot & \cdot \cdots & \cdot & \cdot \\ 0 & 0 \cdots & 1 & 0 \\ 0 & 0 \cdots & -s & b \end{pmatrix} = (-1)^r b$$

which, bearing in mind the definition of b, yields (2.12). ◁

With this expansion in mind, we return to the formula (2.11), from which we deduce that

$$T(s) = \frac{(-1)^r CA^{r-1}B\ \det([A_{00} - \frac{1}{b}B_0A_{10}] - sI)}{\det(A - sI)}.$$

Changing the signs of both matrices in the numerator and denominator yields the final expression

$$T(s) = CA^{r-1}B\,\frac{\det(sI - [A_{00} - \frac{1}{b}B_0A_{10}])}{\det(sI - A)}. \tag{2.13}$$

If the triplet A, B, C is a minimal realization of its transfer function, i.e., if the pair (A, B) is reachable and the pair (A, C) is observable, the numerator and denominator polynomial of this fraction cannot have common factors.[5] Thus, we can conclude that if the pair (A, B) is reachable and the pair (A, C) is observable, the $n - r$ eigenvalues of the matrix

$$[A_{00} - \frac{1}{b}B_0A_{10}].$$

can be identified with *zeros of the transfer function* $T(s)$.

[5]Otherwise, $T(s)$ could be written as strictly proper rational function in which the denominator is a polynomial of degree *strictly less* than n. This would imply the existence of a realization of dimension strictly less than n, contradicting the minimality of A, B, C.

Remark 2.1 Note that if, in the transformation T used to obtain the normal form, the matrix T_0 has been chosen so as to satisfy $TB = 0$, the structure of the normal form (2.10) is simplified, and $B_0 = 0$. In this case, the zeros of the transfer function coincide with the eigenvalues of the matrix A_{00}. ◁

2.2 The Hypothesis of Minimum-Phase

We consider in this chapter the case of a linear single-input single-output system of fixed dimension n, whose coefficient matrices may depend on a *vector* μ of (possibly) *uncertain parameters*. The value of μ is not known, nor is available for measurement (neither directly nor indirectly through an estimation filter) but it is assumed to be *constant* and to range over a fixed, and *known*, *compact set* $\mathbb{M}$.

Accordingly, the Eq. (2.1) will be written in the form

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x. \end{aligned} \tag{2.14}$$

The theory described in what follows is based on the following basic hypothesis.

Assumption 2.1 $A(\mu), B(\mu), C(\mu)$ are matrices of continuous functions of μ. For every $\mu \in \mathbb{M}$, the pair $(A(\mu), B(\mu))$ is reachable and the pair $(A(\mu), C(\mu))$ is observable. Moreover:

(i) The relative degree of the system is the same for all $\mu \in \mathbb{M}$.
(ii) The zeros of the transfer function $C(\mu)(sI - A(\mu))^{-1}B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$.

In the classical theory of servomechanisms, systems whose transfer function has zeros only in the (closed) left-half complex plane have been often referred to as *minimum-phase systems*. This a terminology that dates back more or less to the works of H.W. Bode.[6] For convenience, we keep this terminology to express the property that a system satisfies condition (ii) of Assumption 2.1, even though the latter (which, as it will be seen, is instrumental in the proposed robust stabilization strategy) excludes the occurrence of zeros on the imaginary axis. Thus, in what follows, a (linear) system satisfying condition (ii) of Assumption 2.1 will be referred to as a minimum-phase system.

Letting r denote the relative degree, system (2.14) can be put in *strict* normal form, by means of a change of variables

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0(\mu) \\ T_1(\mu) \end{pmatrix} x$$

in which

[6] See [2] and also [3, p. 283].

$$T_1(\mu) = \begin{pmatrix} C(\mu) \\ C(\mu)A(\mu) \\ \cdots \\ C(\mu)A^{r-1}(\mu) \end{pmatrix}$$

and $T_0(\mu)$ satisfies $T_0(\mu)B(\mu) = 0$. Note that $T_1(\mu)$ is by construction a continuous function of μ and $T_0(\mu)$ can always be chosen as a continuous function of μ.

The normal form in question is written as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u \\ y &= \xi_1, \end{aligned}$$

with $z \in \mathbb{R}^{n-r}$ or, alternatively, as

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u] \\ y &= \hat{C}\xi \end{aligned}$$

in which $\hat{A}$, $\hat{B}$, $\hat{C}$ are the matrices introduced in (2.7). Moreover,

$$b(\mu) = C(\mu)A^{r-1}(\mu)B(\mu).$$

As a consequence of Assumption 2.1:

(i) $b(\mu) \neq 0$ for all $\mu \in \mathbb{M}$. By continuity, $b(\mu)$ it is either positive or negative for all $\mu \in \mathbb{M}$. In what follows, without loss of generality, it will be assumed that

$$b(\mu) > 0 \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.15}$$

(ii) The eigenvalues of $A_{00}(\mu)$ have negative real part for all $\mu \in \mathbb{M}$. This being the case, it is known from the converse Lyapunov Theorem[7] that there exists a unique, symmetric, and *positive definite*, matrix $P(\mu)$, of dimension $(n-r) \times (n-r)$, of *continuous* functions of μ such that

$$P(\mu)A_{00}(\mu) + A_{00}^{\mathrm{T}}(\mu)P(\mu) = -I \qquad \text{for all } \mu \in \mathbb{M}. \tag{2.16}$$

[7] See Theorem A.3 in Appendix A.

2.3 The Case of Relative Degree 1

Perturbed systems that belong to the class of systems characterized by Assumption 2.1 can be *robustly stabilized* by means of a very simple-minded feedback strategy, as it will be described in what follows. The Eq. (2.14) define a *set* of systems, one for each value of μ. A *robust stabilizer* is a feedback law that stabilizes each member of this set. The feedback law in question is not allowed to know which one is the individual member of the set that is being controlled, i.e., it is has to be the same for each member of the set. In the present context of systems modeled as in (2.14), the robustly stabilizing feedback must be a fixed dynamical system not depending on the value of μ.

For simplicity, we address first in this section the case of a system having relative degree is 1. In this case ξ is a vector of dimension 1 (i.e. a scalar quantity) and the normal form reduces to

$$\begin{aligned}\dot{z} &= A_{00}(\mu)z + A_{10}(\mu)\xi\\ \dot{\xi} &= A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)u\\ y &= \xi.\end{aligned} \tag{2.17}$$

Consider the control law[8]

$$u = -ky. \tag{2.18}$$

This yields a closed-loop system of the form

$$\begin{aligned}\dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi\\ \dot{\xi} &= A_{10}(\mu)z + [A_{11}(\mu) - b(\mu)k]\xi,\end{aligned}$$

or, what is the same

$$\begin{pmatrix}\dot{z}\\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}A_{00}(\mu) & A_{01}(\mu)\\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k]\end{pmatrix}\begin{pmatrix}z\\ \xi\end{pmatrix}. \tag{2.19}$$

We want to prove that, under the standing assumptions, if k is large enough this system is stable for all $\mu \in \mathbb{M}$. To this end, consider the positive definite $n \times n$ matrix

$$\begin{pmatrix}P(\mu) & 0\\ 0 & 1\end{pmatrix}$$

in which $P(\mu)$ is the matrix defined in (2.16). If we are able to show that the matrix

[8] The negative sign is a consequence of the standing hypothesis (2.15). If $b(\mu) < 0$, the sign must be reversed.

$$Q(\mu) = \begin{pmatrix} P(\mu) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{00}(\mu) & A_{01}(\mu) \\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k] \end{pmatrix} + \begin{pmatrix} A_{00}(\mu) & A_{01}(\mu) \\ A_{10}(\mu) & [A_{11}(\mu) - b(\mu)k] \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

is *negative definite*, then by the direct Lyapunov Theorem,[9] we can assert that system (2.19) has all eigenvalues with negative real part. A simple calculation shows that, because of (2.16),

$$Q(\mu) = \begin{pmatrix} -I & [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)] \\ [P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]^{\mathrm{T}} & 2[A_{11}(\mu) - b(\mu)k] \end{pmatrix} \tag{2.20}$$

To check positive definiteness, we change sign to $Q(\mu)$ and appeal to Sylvester's criterion for positive definiteness, i.e., we check the sign of all leading principal minors. Because of the special form of $-Q(\mu)$, all its leading principal minors of order $1, 2, \ldots, n-1$ are equal to 1 (and hence positive). Thus the matrix in question is positive definite if (and only if) its determinant is positive. To compute the determinant, we observe that the matrix in question has the form

$$-Q(\mu) = \begin{pmatrix} I & d(\mu) \\ d^{\mathrm{T}}(\mu) & q(\mu) \end{pmatrix}.$$

Hence, thanks to the formula for the determinant of a partitioned matrix

$$\det[-Q(\mu)] = \det[I]\det[q(\mu) - d^{\mathrm{T}}(\mu)I^{-1}d(\mu)] = q(\mu) - d^{\mathrm{T}}(\mu)d(\mu) = q(\mu) - \|d(\mu)\|^2.$$

Thus, the conclusion is that $Q(\mu)$ is negative definite for all $\mu \in \mathbb{M}$ if (and only if)

$$q(\mu) - \|d(\mu)\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Reverting to the notations of (2.20), we can say that $Q(\mu)$ is negative definite, or—what is the same—system (2.19) has all eigenvalues with negative real part, for all $\mu \in \mathbb{M}$ if

$$-2[A_{11}(\mu) - b(\mu)k] - \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Since $b(\mu) > 0$, the inequality is equivalent to

$$k > \frac{1}{2b(\mu)}\Big(2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2\Big) \qquad \text{for all } \mu \in \mathbb{M}.$$

Now, set

[9] See Theorem A.2 in Appendix A.

$$k^* := \max_{\mu\in\mathbb{M}}\Big(\frac{2A_{11}(\mu) + \|[P(\mu)A_{01}(\mu) + A_{10}^{\mathrm{T}}(\mu)]\|^2}{2b(\mu)}\Big).$$

This maximum exists because the functions are continuous functions of μ and $\mathbb{M}$ is a compact set. Then we are able to conclude that if

$$k > k^*$$

the control law

$$u = -ky$$

stabilizes the closed-loop system, regardless of what the particular value of $\mu \in \mathbb{M}$ is. In other words, this control *robustly* stabilizes the given set of systems.

Remark 2.2 The control law $u = -ky$ is usually referred to as a *high-gain* output feedback. As it is seen from the previous analysis, a large value of k makes the matrix $Q(\mu)$ negative definite. The matrix in question has the form

$$Q(\mu) = \begin{pmatrix} -I & -d(\mu) \\ -d^{\mathrm{T}}(\mu) & -2b(\mu)k + q_0(\mu) \end{pmatrix}.$$

The role of a large k is to render the term $-2b(\mu)k$ *sufficiently negative*, so as: (i) to overcome the uncertain term $q_0(\mu)$, and (ii) to overcome the effect of the uncertain off-diagonal terms. Reverting to the Eq. (2.19) that describe the closed-loop system, one may observe that the upper equation can be seen as a stable subsystem with state z and input ξ, while the lower equation can be seen as a subsystem with state ξ and input z. The role of a *large* k is: (i) to render the lower subsystem stable, and (ii) to *lower* the effect of the *coupling* between the two subsystems. This second role, which is usually referred to as a *small-gain property*, will be described and interpreted in full generality in the next chapter. ◁

2.4 The Case of Higher Relative Degree: Partial State Feedback

Consider now the case of a system having higher relative degree $r > 1$. This system can be "artificially" reduced to a system to which the stabilization procedure described in the previous section is applicable, by means of a simple strategy. Let the variable ξ_r of the normal form be replaced by a new state variable defined as

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} \tag{2.21}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters. With this change of variable (it is only a change of variables, no control has been chosen yet !), a system is obtained which has the form

$$
\begin{aligned}
\dot{z} &= A_{00}(\mu)z + \tilde{a}_{01}(\mu)\xi_1 + \cdots + \tilde{a}_{0,r-1}(\mu)\xi_{r-1} + \tilde{a}_{0r}(\mu)\theta \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= -(a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1}) + \theta \\
\dot{\theta} &= A_{10}(\mu)z + \tilde{a}_{11}(\mu)\xi_1 + \cdots + \tilde{a}_{1,r-1}(\mu)\xi_{r-1} + \tilde{a}_{1r}(\mu)\theta + b(\mu)u \\
y &= \xi_1,
\end{aligned}
$$

in which the $\tilde{a}_{0i}(\mu)$'s and $\tilde{a}_{1i}(\mu)$'s are appropriate coefficients.[10]

This system can be formally viewed as a system having relative degree 1, with input u and output θ. To this end, in fact, it suffices to set

$$
\zeta = \begin{pmatrix} z \\ \xi_1 \\ \cdots \\ \xi_{r-1} \end{pmatrix}
$$

and rewrite the system as

$$
\dot{\zeta} = \begin{pmatrix} A_{00}(\mu) & \tilde{a}_{01}(\mu) & \tilde{a}_{02}(\mu) & \cdots & \tilde{a}_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} \zeta + \begin{pmatrix} \tilde{a}_{0r}(\mu) \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \theta
$$

$$
\dot{\theta} = \begin{pmatrix} A_{10}(\mu) & \tilde{a}_{11}(\mu) & \tilde{a}_{12}(\mu) & \cdots & \tilde{a}_{1,r-1}(\mu) \end{pmatrix} \zeta + \tilde{a}_{1r}(\mu)\theta + b(\mu)u.
$$

The latter has the structure of a system in normal form

[10] These coefficients can be easily derived as follows. Let

$$
A_{01}(\mu) = \begin{pmatrix} a_{01,1} & a_{01,2} & \cdots & a_{01,r} \end{pmatrix}.
$$

Hence

$$
A_{01}(\mu)\xi = a_{01,1}\xi_1 + a_{01,2}\xi_2 + \cdots + a_{01,r}\xi_r.
$$

Since

$$
\xi_r = \theta - (a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1})
$$

we see that

$$
A_{01}(\mu)\xi = [a_{01,1} - a_{01,r}a_0]\xi_1 + [a_{01,2} - a_{01,r}a_1]\xi_2 + \cdots + [a_{01,r-1} - a_{01,r}a_{r-2}]\xi_{r-1} + a_{01,r}\theta
$$

The latter can be rewritten as

$$
\tilde{a}_{01}(\mu)\xi_1 + \cdots + \tilde{a}_{0,r-1}(\mu)\xi_{r-1} + \tilde{a}_{0r}(\mu)\theta.
$$

A similar procedure is followed to transform $A_{11}(\mu)\xi + a_0\dot{\xi}_1 + \cdots + a_{r-1}\dot{\xi}_{r-1}$.

$$
\begin{aligned}
\dot{\zeta} &= F_{00}(\mu)\zeta + F_{01}(\mu)\theta \\
\dot{\theta} &= F_{10}(\mu)\zeta + F_{11}(\mu)\theta + b(\mu)u
\end{aligned}
\tag{2.22}
$$

in which $F_{00}(\mu)$ is a $(n-1)\times(n-1)$ block-triangular matrix

$$
F_{00}(\mu) = \begin{pmatrix} A_{00}(\mu) & \tilde{a}_{01}(\mu) & \tilde{a}_{02}(\mu) & \cdots & \tilde{a}_{0,r-1}(\mu) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix} := \begin{pmatrix} A_{00}(\mu) & * \\ 0 & A_0 \end{pmatrix}
$$

with

$$
A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{r-2} \end{pmatrix}.
$$

By Assumption 2.1, all eigenvalues of the submatrix $A_{00}(\mu)$ have negative real part for all μ. On the other hand, the characteristic polynomial of the submatrix A_0, which is a matrix in companion form, coincides with the polynomial

$$
p_a(\lambda) = a_0 + a_1\lambda + \cdots + a_{r-2}\lambda^{r-2} + \lambda^{r-1}. \tag{2.23}
$$

The design parameters $a_0, a_1, \ldots, a_{r-2}$ can be chosen in such a way that all eigenvalues of A_0 have negative real part. If this is the case, we can conclude that *all* the $n-1$ eigenvalues of $F_{00}(\mu)$ have negative real part, for all μ.

Thus, system (2.22) can be seen as a system having relative degree 1 which satisfies all the assumptions used in the previous section to obtain robust stability. In view of this, it immediately follows that there exists a number k^* such that, if $k > k^*$, the control law

$$
u = -k\theta
$$

robustly stabilizes such system.

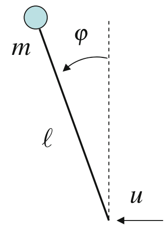
Note that the control thus found, expressed in the original coordinates, reads as

$$
u = -k[a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r]
$$

that is as a linear combination of the components of the vector ξ. This is a *partial state* feedback, which can be written, in compact form, as

$$
u = H\xi. \tag{2.24}
$$

Fig. 2.1 An inverted pendulum



Remark 2.3 It it worth to observe that, by definition,

$$\xi_1(t) = y(t), \quad \xi_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \quad \dots \quad , \quad \xi_r(t) = \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}.$$

Thus, the variable θ is seen to satisfy

$$\theta(t) = a_0 y(t) + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cdots + a_{r-2}\frac{\mathrm{d}^{r-2}y(t)}{\mathrm{d}t^{r-2}} + \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}.$$

In other words, θ can be seen as output of a system with input y and transfer function

$$D(s) = a_0 + a_1 s + \cdots + a_{r-2}s^{r-2} + s^{r-1}.$$

It readily follows that, if $T(s, \mu) = C(\mu)(sI - A(\mu))^{-1}B(\mu)$ is the transfer function of (2.14), the transfer function of system (2.22), seen as a system with input u and output θ, is equal to $D(s)T(s, \mu)$. As confirmed by the state-space analysis and in particular from the structure of $F_{00}(\mu)$, this new system has $n - 1$ zeros, $n - r$ of which coincide with the original zeros of (2.14), while the additional $r - 1$ zeros coincide with the roots of the polynomial (2.23). In other words, the indicated design method can be interpreted as an addition of $r - 1$ zeros having negative real part (so as to lower the relative degree to the value 1 while keeping the property that all zeros have negative real part) followed by high-gain output feedback on the resulting output.

Example 2.1 Consider the problem of robustly stabilizing a rocket's upright orientation in the initial phase of the launch. The equation describing the motion on a vertical plane is similar to those that describes the motion of an inverted pendulum (see Fig. 2.1) and has the form[11]

$$J_t \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = mg\ell\sin(\varphi) - \gamma\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \ell\cos(\varphi)u.$$

[11] See [3, pp. 36–37].

in which ℓ is the length of the pendulum, m is the mass concentrated at the tip of the pendulum, $J_t = J + m\ell^2$ is the total moment of inertia, γ is a coefficient of rotational viscous friction and u is a force applied at the base.

If the angle φ is sufficiently small, one can use the approximations $\sin(\varphi) \approx \varphi$ and $\cos(\varphi) \approx 1$ and obtain a linear model. Setting $\xi_1 = \varphi$ and $\xi_2 = \dot{\varphi}$, the equation can be put in state space form as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= q_1\xi_1 + q_2\xi_2 + bu \end{aligned}$$

in which

$$q_1 = \frac{mg\ell}{J_t}, \qquad q_2 = -\frac{\gamma}{J_t}, \qquad b = \frac{\ell}{J_t}.$$

Note that the system is unstable (because q_1 is positive) and that, if φ is considered as output, the system has relative degree 2, and hence is trivially minimum-phase.

According to the procedure described above, we pick (compare with (2.21))

$$\theta = \xi_2 + a_0\xi_1$$

with $a_0 > 0$, and obtain (compare with (2.22))

$$\begin{aligned} \dot{\xi}_1 &= -a_0\xi_1 + \theta \\ \dot{\theta} &= (q_1 - a_0q_2 - a_0^2)\xi_1 + (q_2 + a_0)\theta + bu. \end{aligned}$$

This system is going to be controlled by

$$u = -k\theta,$$

which results in (compare with (2.19))

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk \end{pmatrix} \begin{pmatrix} \xi_1 \\ \theta \end{pmatrix}.$$

It is known from the theory described above that, if k is sufficiently large, the system is stable. To determine the minimal value of k as a function of the parameters, we impose that the matrix

$$\begin{pmatrix} -a_0 & 1 \\ (q_1 - a_0q_2 - a_0^2) & (q_2 + a_0) - bk \end{pmatrix}$$

has eigenvalues with negative real part. This is the case if

$$\begin{aligned} &(q_2 + a_0) - bk < 0 \\ &(-a_0)(q_2 + a_0 - bk) - (q_1 - a_0q_2 - a_0^2) > 0, \end{aligned}$$

that is

$$k > \max\left\{\frac{q_2 + a_0}{b}, \frac{q_1}{a_0 b}\right\}.$$

Reverting to the original parameters, this yields

$$k > \max\left\{\frac{a_0 J_t - \gamma}{\ell}, \frac{mg}{a_0}\right\}.$$

A conservative estimate is obtained if the term—γ is neglected (which is reasonable, since the value of γ, the coefficient of viscous friction, may be subject to large variations), obtaining

$$k > \max\left\{\frac{a_0(J + m\ell^2)}{\ell}, \frac{mg}{a_0}\right\}.$$

Once the ranges of the parameters J, m, ℓ are specified, this expression can be used to determine the design parameters a_0 and k. Expressed in the original variable the stabilizing control is

$$u = -k(\xi_2 + a_0\xi_1) = -k\dot{\varphi} - ka_0\varphi, \tag{2.25}$$

that is, the classical "proportional-derivative" feedback. ◁

2.5 The Case of Higher Relative Degree: Output Feedback

We have seen in the previous section that a system satisfying Assumption 2.1 can be robustly stabilized by means of a feedback law which is a linear form in the states $\xi_1, \ldots, \xi_r$ that characterize its normal form. In general, the components of the state ξ are not directly available for feedback, nor they can be retrieved from the original state x, since the transformation that defines ξ in terms of x depends on the uncertain parameter μ. We see now how this problem can be overcome, by designing a dynamic controller that provides appropriate "replacements" for the components of ξ in the control law (2.24). Observing that these variables coincide, by definition, with the measured output y and with its first $r-1$ derivatives with respect to time, it seems reasonable to try to generate the latter by means of a dynamical system of the form

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \kappa c_{r-1}(y - \hat{\xi}_1)\\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \kappa^2 c_{r-2}(y - \hat{\xi}_1)\\
&\cdots\\
\dot{\hat{\xi}}_{r-1} &= \hat{\xi}_r + \kappa^{r-1} c_1(y - \hat{\xi}_1)\\
\dot{\hat{\xi}}_r &= \kappa^r c_0(y - \hat{\xi}_1).
\end{aligned} \tag{2.26}$$

In fact, if $\hat{\xi}_1(t)$ where identical to $y(t)$, it would follow that $\hat{\xi}_i(t)$ coincides with $y^{(i-1)}(t)$, that is with $\xi_i(t)$, for all $i = 1, 2, \ldots, r$. In compact form, the system thus defined can be rewritten as

$$\dot{\hat{\xi}} = \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi}),$$

in which

$$G_0 = \begin{pmatrix} c_{r-1} \\ c_{r-2} \\ \cdots \\ c_1 \\ c_0 \end{pmatrix}, \qquad D_\kappa = \begin{pmatrix} \kappa & 0 & \cdots & 0 \\ 0 & \kappa^2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \kappa^r \end{pmatrix},$$

and $\hat{A}$, $\hat{C}$ are the matrices defined in (2.7).

Let now ξ be replaced by $\hat{\xi}$ in the expression of the control law (2.24). In this way, we obtain a dynamic controller, described by equations of the form

$$\begin{aligned} \dot{\hat{\xi}} &= \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi}) \\ u &= H\hat{\xi}. \end{aligned} \tag{2.27}$$

It will be shown in what follows that, if the parameters κ and $c_0, \ldots, c_{r-1}$ which characterize (2.27) are chosen appropriately, this *dynamic—output feedback—control* law does actually robustly stabilize the system.

Controlling the system (assumed to be expressed in strict normal form) by means of the control (2.27) yields a closed-loop system

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat{\xi}] \\ \dot{\hat{\xi}} &= \hat{A}\hat{\xi} + D_\kappa G_0(y - \hat{C}\hat{\xi}). \end{aligned}$$

To analyze this closed-loop system, we first perform a change of coordinates, letting the $\hat{\xi}_i$'s be replaced by variables e_i's defined as

$$e_i = \kappa^{r-i}(\xi_i - \hat{\xi}_i), \qquad i = 1, \ldots, r.$$

According to the definition of the matrix D_κ, it is observed that

$$e = \kappa^r D_\kappa^{-1}(\xi - \hat{\xi}),$$

that is

$$\hat{\xi} = \xi - \kappa^{-r} D_\kappa e.$$

The next step in the analysis is to determine the differential equations for the new variables e_i, for $i = 1, \ldots, r$. Setting

$$e = \mathrm{col}(e_1, \dots, e_r).$$

a simple calculation yields

$$\dot{e} = \kappa(\hat{A} - G_0\hat{C})e + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H\hat{\xi}].$$

Replacing also $\hat{\xi}$ with its expression in terms of ξ and e we obtain, at the end, a description of the closed-loop system in the form

$$\begin{aligned}
\dot{z} &= A_{00}(\mu)z + A_{01}(\mu)\xi \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)] \\
\dot{e} &= \kappa(\hat{A} - G_0\hat{C})e + \hat{B}[A_{10}(\mu)z + A_{11}(\mu)\xi + b(\mu)H(\xi - \kappa^{-r}D_\kappa e)].
\end{aligned}$$

To simplify this system further, we set

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and define the matrices

$$\begin{aligned}
F_{00}(\mu) &= \begin{pmatrix} A_{00}(\mu) & A_{01}(\mu) \\ \hat{B}A_{10}(\mu) & \hat{A} + \hat{B}[A_{11}(\mu) + b(\mu)H] \end{pmatrix} \\
F_{01}(\mu) &= -\begin{pmatrix} 0 \\ \hat{B}b(\mu)H \end{pmatrix} \\
F_{10}(\mu) &= \begin{pmatrix} \hat{B}A_{10}(\mu) & \hat{B}[A_{11}(\mu) + b(\mu)H] \end{pmatrix} \\
F_{11}(\mu) &= -\hat{B}b(\mu)H
\end{aligned}$$

in which case the equations of the closed-loop system will be rewritten as

$$\begin{aligned}
\dot{\tilde{x}} &= F_{00}(\mu)\tilde{x} + F_{01}(\mu)\kappa^{-r}D_\kappa e \\
\dot{e} &= F_{10}(\mu)\tilde{x} + [\kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa]e.
\end{aligned}$$

The advantage of having the system written in this form is that we know that the matrix $F_{00}(\mu)$, if H has been chosen as described in the earlier section, has eigenvalues with negative real part for all μ. Hence, there is a positive definite symmetric matrix $P(\mu)$ such that

$$P(\mu)F_{00}(\mu) + F_{00}(\mu)^{\mathrm{T}}P(\mu) = -I.$$

Moreover, it is readily seen that the characteristic polynomial of the matrix $(\hat{A} - G_0\hat{C})$ coincides with the polynomial

$$p_c(\lambda) = c_0 + c_1\lambda + \cdots + c_{r-1}\lambda^{r-1} + \lambda^r. \tag{2.28}$$

Thus, the coefficients $c_0, c_1, \ldots, c_{r-1}$ can be chosen in such a way that all eigenvalues of $(\hat{A} - G_0\hat{C})$ have negative real part. If this is done, there exists a positive definite symmetric matrix $\hat{P}$ such that

$$\hat{P}(\hat{A} - G_0\hat{C}) + (\hat{A} - G_0\hat{C})^{\mathrm{T}}\hat{P} = -I.$$

This being the case, we proceed now to show that the direct criterion of Lyapunov is fulfilled, for the positive definite matrix

$$\begin{pmatrix} P(\mu) & 0 \\ 0 & \hat{P} \end{pmatrix}$$

if the number κ is large enough. To this end, we need to check that the matrix

$$Q = \begin{pmatrix} P(\mu) & 0 \\ 0 & \hat{P} \end{pmatrix} \begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa \\ F_{10}(\mu) & \kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa \end{pmatrix}$$
$$+ \begin{pmatrix} F_{00}(\mu) & F_{01}(\mu)\kappa^{-r}D_\kappa \\ F_{10}(\mu) & \kappa(\hat{A} - G_0\hat{C}) + F_{11}(\mu)\kappa^{-r}D_\kappa \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P(\mu) & 0 \\ 0 & \hat{P} \end{pmatrix}$$

is negative definite. In view of the definitions of $P(\mu)$ and $\hat{P}$, we see that $-Q$ has the form

$$-Q = \begin{pmatrix} I & -d(\mu,\kappa) \\ -d^{\mathrm{T}}(\mu,\kappa) & \kappa I - q(\mu,\kappa) \end{pmatrix}.$$

in which

$$\begin{aligned} d(\mu,\kappa) &= [P(\mu)F_{01}(\mu)\kappa^{-r}D_\kappa + F_{10}^{\mathrm{T}}(\mu)\hat{P}] \\ q(\mu,\kappa) &= [\hat{P}F_{11}(\mu)\kappa^{-r}D_\kappa + \kappa^{-r}D_\kappa F_{11}^{\mathrm{T}}(\mu)\hat{P}]. \end{aligned}$$

We want the matrix $-Q$ to be positive definite. According to Schur's Lemma,[12] this is the case if and only if the Schur's complement

$$\kappa I - q(\mu,\kappa) - d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa) \tag{2.29}$$

is positive definite. This is actually the case if κ is large enough. To check this claim, assume, without loss of generality, that $\kappa \geq 1$ and observe that in this case the diagonal matrix

$$\kappa^{-r}D_\kappa = \mathrm{diag}(\kappa^{-r+1}, \ldots, \kappa^{-1}, 1),$$

has norm 1. If this is the case, the positive number

$$\kappa^* = \sup_{\substack{\mu\in\mathbb{M} \\ \kappa\geq 1}} \|q(\mu,\kappa) + d^{\mathrm{T}}(\mu,\kappa)d(\mu,\kappa)\|$$

[12] See (A.1) in Appendix A.

is well-defined. To say that the quadratic form (2.29) is positive definite is to say that, for any nonzero $z \in \mathbb{R}^r$,

$$\kappa z^{\mathrm{T}} z > z^{\mathrm{T}}[q(\mu, \kappa) + d^{\mathrm{T}}(\mu, \kappa) d(\mu, \kappa)] z.$$

Clearly, if $\kappa > \max\{1, \kappa^*\}$, this inequality holds and the matrix (2.29) is positive definite.

It is therefore concluded that if system (2.14) is controlled by (2.27), with H chosen as indicated in the previous section, and $\kappa > \max\{1, \kappa^*\}$, the resulting closed-loop system has all eigenvalues with negative real part, for any $\mu \in \mathbb{M}$.

In summary, we have shown that the uncertain system (2.1), under Assumption 2.1, can be *robustly stabilized* by means of a dynamic output-feedback control law of the form

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_{r-1} & 1 & 0 & \cdots & 0 \\ -\kappa^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\kappa^{r-1} c_1 & 0 & 0 & \cdots & 1 \\ -\kappa^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} \kappa c_{r-1} \\ \kappa^2 c_{r-2} \\ \cdots \\ \kappa^{r-1} c_1 \\ \kappa^r c_0 \end{pmatrix} y$$

$$u = -k \left(a_0 \; a_1 \; a_2 \; \cdots \; a_{r-2} \; 1 \right) \hat{\xi},$$

in which $c_0, c_1, \ldots, c_{r-1}$ and, respectively, $a_0, a_1, \ldots, a_{r-2}$ are such that the polynomials (2.28) and (2.23) have negative real part, and κ and k are large (positive) parameters.

Remark 2.4 Note the striking similarity of the arguments used to show the negative definiteness of Q with those used in Sect. 2.3. The large value of κ is instrumental in overcoming the effects of an additive term in the bottom-right block and of the off-diagonal terms. Both these terms depend now on κ (this was not the case in Sect. 2.3) but fortunately, if $\kappa \geq 1$, such terms have bounds that are independent of κ. Also in this case, we can interpret the resulting system as interconnection of a stable subsystem with state x and input e, connected to a subsystem with state e and input z. The role of a large κ is: (i) to render the lower subsystem stable, and (ii) to lower the effect of the coupling between the two subsystems. ◁

Example 2.2 Consider again the system of Example 2.1 and suppose that only the variable φ is available for feedback. In this case, we use a control

$$u = -k(a_0 \hat{\xi}_1 + \hat{\xi}_2),$$

in which $\hat{\xi}_1, \hat{\xi}_2$ are provided by the dynamical system

$$\dot{\hat{\xi}} = \begin{pmatrix} -\kappa c_1 & 1 \\ -\kappa^2 c_0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} \kappa c_1 \\ \kappa^2 c_0 \end{pmatrix} \varphi.$$

For convenience, we can take $c_0 = 1$ and $c_1 = 2$, in which case the characteristic polynomial of this system becomes $(s + \kappa)^2$. Setting $\epsilon = 1/\kappa$ and computing the transfer function $T_c(s)$ of the controller, between φ and u, it is seen that

$$T_c(s) = -k \, \frac{(1 + 2a_0\varepsilon)s + a_0}{(\varepsilon s + 1)^2}.$$

Or course, as $\varepsilon \to 0$, the control approaches the proportional-derivative control (2.25). $\triangleleft$

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Chapter 3
The Small-Gain Theorem for Linear Systems and Its Applications to Robust Stability

3.1 The $\mathscr{L}_2$ Gain of a Stable Linear System

In this section,[1] we analyze some properties of the forced response of a linear system to piecewise continuous input functions—defined on the time interval $[0, \infty)$—which have the following property

$$\lim_{T\to\infty} \int_0^T \|u(t)\|^2 dt < \infty.$$

The space of all such functions, endowed with the so-called $\mathscr{L}_2$ norm, which is defined as

$$\|u(\cdot)\|_{\mathscr{L}_2} := \Big(\int_0^\infty \|u(t)\|^2 dt\Big)^{\frac{1}{2}},$$

is denoted by $\mathscr{U}_{\mathscr{L}_2}$. The main purpose of the analysis is to show that, if the system is *stable*, the forced output response from the initial state $x(0) = 0$ has a similar property, i.e.,

$$\lim_{T\to\infty} \int_0^T \|y(t)\|^2 dt < \infty. \tag{3.1}$$

This makes it possible to compare the $\mathscr{L}_2$ norms of input and output functions and define a concept of "gain" accordingly.

Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.2}$$

[1]For an extended and more detailed coverage of the topics handled here in sections 3.1 through 3.5, see the monographs [1–6], [11] and the survey [12].

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_3

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$. Suppose the matrix A has all eigenvalues with negative real part. Let α be a positive number. According to the direct criterion of Lyapunov, the equation

$$PA + A^{\mathrm{T}}P = -\alpha I \tag{3.3}$$

has a unique solution P, which is a symmetric and positive definite matrix.

Let $V(x) = x^{\mathrm{T}}Px$ be the associated positive definite quadratic form. If $x(t)$ is any trajectory of (2.2),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)Px(t)] &= \frac{\partial V}{\partial x}\bigg|_{x=x(t)} \dot{x}(t) = 2x^{\mathrm{T}}(t)P(Ax(t)+Bu(t))\\ &= x^{\mathrm{T}}(t)(PA + A^{\mathrm{T}}P)x(t) + 2x^{\mathrm{T}}(t)PBu(t).\end{aligned}$$

Adding and subtracting $d^2u(t)^{\mathrm{T}}u(t)$ to the right-hand side, and dropping—for convenience—the dependence on t, we obtain successively

$$\begin{aligned}\frac{\partial V}{\partial x}(Ax+Bu) &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + d^2u^{\mathrm{T}}u - d^2u^{\mathrm{T}}u + 2x^{\mathrm{T}}PBu\\ &= -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u - d^2\left(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px\right)^{\mathrm{T}}\left(u - \tfrac{1}{d^2}B^{\mathrm{T}}Px\right) + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px\\ &\le -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px.\end{aligned}$$

Subtracting and adding to the right-hand side the quantity $y^{\mathrm{T}}y$ and using the inequality

$$y^{\mathrm{T}}y = (Cx+Du)^{\mathrm{T}}(Cx+Du) \le 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du$$

it is seen that

$$\begin{aligned}\frac{\partial V}{\partial x}(Ax+Bu) &\le -\alpha x^{\mathrm{T}}x + d^2u^{\mathrm{T}}u + \tfrac{1}{d^2}x^{\mathrm{T}}PBB^{\mathrm{T}}Px - y^{\mathrm{T}}y + 2x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2u^{\mathrm{T}}D^{\mathrm{T}}Du\\ &= x^{\mathrm{T}}\left(-\alpha I + \tfrac{1}{d^2}PBB^{\mathrm{T}}P + 2C^{\mathrm{T}}C\right)x + u^{\mathrm{T}}(d^2I + 2D^{\mathrm{T}}D)u - y^{\mathrm{T}}y.\end{aligned}$$

Clearly, for any choice of $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$-\alpha I + 2C^{\mathrm{T}}C \le -2\varepsilon I.$$

Pick one such α and let P be determined accordingly as a solution of (3.3). With P fixed in this way, it is seen that, if the number d is sufficiently large, the inequality

$$\frac{1}{d^2}PBB^{\mathrm{T}}P \le \varepsilon I$$

holds, and hence

$$x^{\mathrm{T}}\left(-\alpha I + \frac{1}{d^2} PBB^{\mathrm{T}}P + 2C^{\mathrm{T}}C\right)x \le -\varepsilon x^{\mathrm{T}}x.$$

Finally, let $\bar{\gamma} > 0$ be any number satisfying

$$d^2 I + 2D^{\mathrm{T}}D \le \bar{\gamma}^2 I.$$

Using the last two inequalities, it is concluded that the derivative of $V(x(t))$ *along the trajectories of* (3.2) satisfies an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \left.\frac{\partial V}{\partial x}\right|_{x=x(t)}(Ax(t) + Bu(t)) \le -\varepsilon\|x(t)\|^2 + \bar{\gamma}^2\|u(t)\|^2 - \|y(t)\|^2. \tag{3.4}$$

This inequality is called a *dissipation inequality*.[2] Summarizing the discussion up to this point, one can claim that for *any stable linear system*, given any positive real number ε, it is always possible to find a positive definite symmetric matrix P and a coefficient $\bar{\gamma}$ such that a dissipation inequality of the form (3.4), which—dropping for convenience the dependence on t—can be simply written as

$$\frac{\partial V(x)}{\partial x}(Ax + Bu) \le -\varepsilon\|x\|^2 + \bar{\gamma}^2\|u\|^2 - \|y\|^2, \tag{3.5}$$

is satisfied.

The inequality thus established plays a fundamental role in characterizing a parameter, associated with a stable linear system, which is called the $\mathscr{L}_2$ *gain*. As a matter of fact, suppose the input $u(\cdot)$ of (3.2) is a function in $\mathscr{U}_{\mathscr{L}_2}$. Integration of the inequality (3.4) on the interval $[0, t]$ yields, for any initial state $x(0)$,

$$\begin{aligned} V(x(t)) &\le V(x(0)) + \bar{\gamma}^2 \int_0^t \|u(\tau)\|^2 d\tau - \int_0^t \|y(\tau)\|^2 d\tau \\ &\le V(x(0)) + \bar{\gamma}^2 \int_0^\infty \|u(\tau)\|^2 d\tau = V(x(0)) + \bar{\gamma}^2\Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2, \end{aligned}$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t \in [0, \infty)$ and bounded. Now, suppose $x(0) = 0$ and observe that the previous inequality yields

$$V(x(t)) \le \bar{\gamma}^2 \int_0^t \|u(\tau)\|^2 d\tau - \int_0^t \|y(\tau)\|^2 d\tau$$

for any $t > 0$. Since $V(x(t)) \ge 0$, it is seen that

[2]The concept of dissipation inequality was introduced by J.C. Willems in [13], to which the reader is referred for more details.

$$\int_0^t \|y(\tau)\|^2 d\tau \le \bar{\gamma}^2 \int_0^t \|u(\tau)\|^2 dt \le \bar{\gamma}^2 \Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2$$

for any $t > 0$ and therefore the property (3.1) holds. In particular,

$$\Big[\|y(\cdot)\|_{\mathscr{L}_2}\Big]^2 \le \bar{\gamma}^2 \Big[\|u(\cdot)\|_{\mathscr{L}_2}\Big]^2,$$

i.e.,

$$\|y(\cdot)\|_{\mathscr{L}_2} \le \bar{\gamma}\, \|u(\cdot)\|_{\mathscr{L}_2}.$$

In summary, for any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, the response of a stable linear system from the initial state $x(0) = 0$ is defined for all $t \ge 0$ and produces an output $y(\cdot)$ which has the property (3.1). Moreover the *ratio* between the $\mathscr{L}_2$ norm of the output and the $\mathscr{L}_2$ norm of the input is bounded by the number $\bar{\gamma}$ which appears in the dissipation inequality (3.5).

Having seen that, in stable linear system, an input having finite $\mathscr{L}_2$ norm produces, from the initial state $x(0) = 0$, an output response which also has a finite $\mathscr{L}_2$ norm, suggests to look at the ratios between such norms, for all possible $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$, and to seek the least upper bound of such ratios. The quantity thus defined is called the $\mathscr{L}_2$ *gain* of the (stable) linear system. Formally, the gain in question is defined as follows: pick any $u(\cdot) \in \mathscr{U}_{\mathscr{L}_2}$ and let $y_{0,u}(\cdot)$ be the resulting response from the initial state $x(0) = 0$; the $\mathscr{L}_2$ gain of the system is the quantity

$$\mathscr{L}_2 \; gain = \sup_{\|u(\cdot)\|_{\mathscr{L}_2} \ne 0} \frac{\|y_{0,u}(\cdot)\|_{\mathscr{L}_2}}{\|u(\cdot)\|_{\mathscr{L}_2}}.$$

With this definition in mind, return to the dissipation inequality (3.5). Suppose that a system of the form (3.2) is given and that an inequality of the form (3.5) holds for a positive definite matrix P. We have in particular (set $u = 0$)

$$\frac{\partial V(x)}{\partial x} A x \le -\varepsilon \|x\|^2 ,$$

from which it is seen that the system is stable. Since the system is stable, the $\mathscr{L}_2$ gain can be defined and, as a consequence of the previous discussion, it is seen that

$$\mathscr{L}_2 \; gain \le \bar{\gamma}. \tag{3.6}$$

Thus, in summary, if an inequality of the form (3.5) holds for a positive definite matrix P, the system is stable and *its* $\mathscr{L}_2$ *gain is bounded from above by the number* $\bar{\gamma}$.

We will see in the next section that the fulfillment of an inequality of the form (3.5) is equivalent to the fulfillment of a *linear matrix inequality* involving the system data A, B, C, D.

3.2 An LMI Characterization of the $\mathscr{L}_2$ Gain

In this section, we derive alternative characterizations of the inequality (3.5).

Lemma 3.1 *Let $V(x) = x^{\mathrm{T}}Px$, with P a positive definite symmetric matrix. Suppose that, for some $\varepsilon > 0$ and some $\bar{\gamma} > 0$, the inequality*

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2 + \bar{\gamma}^2\|u\|^2 - \|Cx + Du\|^2 \tag{3.7}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Then, for any $\gamma > \bar{\gamma}$,

$$D^{\mathrm{T}}D - \gamma^2 I < 0 \tag{3.8}$$

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0. \tag{3.9}$$

Conversely, suppose (3.8) and (3.9) hold for some γ. Then there exists $\varepsilon > 0$ such that, for all $\bar{\gamma}$ satisfying $0 < \gamma - \varepsilon < \bar{\gamma} < \gamma$, the inequality (3.7) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Proof Suppose that the inequality (3.7), that can be written as

$$2x^{\mathrm{T}}P[Ax + Bu] + \varepsilon x^{\mathrm{T}}x - \bar{\gamma}^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \leq 0, \tag{3.10}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. For $x = 0$ this implies, in particular,

$$-\bar{\gamma}^2 I + D^{\mathrm{T}}D \leq 0.$$

Since $\gamma > \bar{\gamma}$, it follows that $D^{\mathrm{T}}D < \gamma^2 I$, which is precisely condition (3.8). Moreover, since $\gamma > \bar{\gamma}$, (3.7) implies

$$2x^{\mathrm{T}}P[Ax + Bu] + \varepsilon x^{\mathrm{T}}x - \gamma^2 u^{\mathrm{T}}u + x^{\mathrm{T}}C^{\mathrm{T}}Cx + 2x^{\mathrm{T}}C^{\mathrm{T}}Du + u^{\mathrm{T}}D^{\mathrm{T}}Du \leq 0 \tag{3.11}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Now, observe that, for each fixed x, the left-hand side of (3.11) is a quadratic form in u, expressible as

$$M(x) + N(x)u - u^{\mathrm{T}}Wu. \tag{3.12}$$

in which

$$W = \gamma^2 I - D^{\mathrm{T}}D$$

and

$$M(x) = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I)x\,, \qquad N(x) = 2x^{\mathrm{T}}(PB + C^{\mathrm{T}}D).$$

Since W is positive definite, this form has a unique maximum point, at

$$u_{\max}(x) = \frac{1}{2} W^{-1} N^{\mathrm{T}}(x).$$

Hence, (3.11) holds if and only if the value of the form (3.12) at $u = u_{\max}(x)$ is nonpositive, that is if and only if

$$M(x) + \frac{1}{4} N(x) W^{-1} N(x)^{\mathrm{T}} \leq 0.$$

Using the expressions $M(x)$, $N(x)$, W, the latter reads as

$$x^{\mathrm{T}}(PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}})x \leq 0$$

and this, since $\varepsilon > 0$, implies condition (3.9).

To prove the converse claim, observe that (3.9) implies

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon_1 I + [PB + C^{\mathrm{T}}D][\gamma^2 I - D^{\mathrm{T}}D]^{-1}[PB + C^{\mathrm{T}}D]^{\mathrm{T}} < 0 \tag{3.13}$$

provided that $\varepsilon_1 > 0$ is small enough. The left-hand sides of (3.8) and (3.13), which are negative definite, by continuity remain negative definite if γ is replaced by any $\bar{\gamma}$ satisfying $\gamma - \varepsilon_2 < \bar{\gamma} < \gamma$, provided that $\varepsilon_2 > 0$ is small enough. Take now $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. The resulting form $M(x) + N(x)u - u^{\mathrm{T}}\bar{W}u$, in which $\bar{W} = \bar{\gamma}^2 I - D^{\mathrm{T}}D$, is non positive and (3.7) holds.

Remark 3.1 Note that the inequalities (3.8) and (3.9) take a substantially simpler form in the case of a system in which $D = 0$ (i.e., systems with no direct feed-through between input and output). In this case, in fact, (3.8) becomes irrelevant and (3.9) reduces to

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \frac{1}{\gamma^2} PBB^{\mathrm{T}}P < 0.$$

Lemma 3.2 *Let γ be a fixed positive number. The inequality (3.8) holds* and *there exists a positive definite symmetric matrix P satisfying (3.9) if and only if there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{3.14}$$

Proof Consider the matrix inequality

$$\begin{pmatrix} A^{\mathrm{T}}X + XA + \frac{1}{\gamma}C^{\mathrm{T}}C & XB + \frac{1}{\gamma}C^{\mathrm{T}}D \\ B^{\mathrm{T}}X + \frac{1}{\gamma}D^{\mathrm{T}}C & -\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D \end{pmatrix} < 0. \tag{3.15}$$

This inequality holds if and only if the lower right block

$$-\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D$$

is negative definite, which is equivalent to condition (3.8), and so is its the Schur's complement

$$A^{\mathrm{T}}X + XA + \frac{1}{\gamma}C^{\mathrm{T}}C - \left[XB + \frac{1}{\gamma}C^{\mathrm{T}}D\right]\left[-\gamma I + \frac{1}{\gamma}D^{\mathrm{T}}D\right]^{-1}\left[XB + \frac{1}{\gamma}C^{\mathrm{T}}D\right]^{\mathrm{T}}.$$

This, having replaced X by $\frac{1}{\gamma}P$, is identical to condition (3.9).

Rewrite now (3.15) as

$$\begin{pmatrix} A^{\mathrm{T}}X + XA & XB \\ B^{\mathrm{T}}X & -\gamma I \end{pmatrix} + \begin{pmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{pmatrix} \frac{1}{\gamma} \begin{pmatrix} C & D \end{pmatrix} < 0$$

and use again (backward) Schur's complement to arrive at (3.14). ◁

3.3 The H_∞ Norm of a Transfer Function

Functions having finite $\mathscr{L}_2$ norm may be seen as signals having *finite energy*[3] over the infinite time interval $[0, \infty)$, and therefore the $\mathscr{L}_2$ gain can be given the interpretation of (an upper bound of the) ratio between energies of output and input. Another similar interpretation, in terms of energies associated with input and output, is possible, which does not necessarily require the consideration of the case of finite energy over the infinite time interval $[0, \infty)$. Suppose the input is a *periodic* function of time, with period T, i.e., that

$$u(t + kT) = \bar{u}(t), \quad \text{for all } t \in [0, T), \text{ all integer } k$$

for some piecewise continuous function $\bar{u}(t)$, defined on $[0, T)$. Also, suppose that, for some suitable initial state $x(0) = \bar{x}$, the state response $x(t)$ of the system is defined for all $t \in [0, T]$ and satisfies

$$x(T) = \bar{x}.$$

[3] If, in the actual physical system, the components of the input $u(t)$ are *voltages* (or *currents*), the quantity $\|u(t)\|^2$ can be seen as instantaneous *power*, at time t, associated with such input and its integral over a time interval $[t_0, t_1]$ as *energy* associated with such input over this time interval.

Then, it is obvious that $x(t)$ exists for all $t \geq 0$, and is a *periodic* function, having the same period T of the input, namely

$$x(t + kT) = x(t), \quad \text{for all } t \in [0, T), k \geq 0$$

and so is the corresponding output response $y(t)$.

For the triplet $\{u(t), x(t), y(t)\}$ thus defined, integration of the inequality (3.4) over an interval $[t_0, t_0 + T]$, with arbitrary $t_0 \geq 0$, yields

$$V(x(t_0 + T)) - V(x(t_0)) \leq \bar{\gamma}^2 \int_{t_0}^{t_0+T} \|u(\tau)\|^2 d\tau - \int_{t_0}^{t_0+T} \|y(\tau)\|^2 d\tau,$$

i.e., since $V(x(t_0 + T)) = V(x(t_0))$,

$$\int_{t_0}^{t_0+T} \|y(\tau)\|^2 d\tau \leq \bar{\gamma}^2 \int_{t_0}^{t_0+T} \|u(\tau)\|^2 d\tau. \tag{3.16}$$

Observe that the integrals on both sides of this inequality are independent of t_0, because the integrands are periodic functions having period T, and recall that the *root mean square* value of any (possibly vector-valued) periodic function $f(t)$ (which is usually abbreviated as “r.m.s.” and characterizes the *average power* of the signal represented by $f(t)$) is defined as

$$\|f(\cdot)\|_{\text{r.m.s.}} = \left(\frac{1}{T} \int_{t_0}^{t_0+T} \|f(\tau)\|^2 d\tau\right)^{\frac{1}{2}}.$$

With this in mind, (3.16) yields

$$\|y(\cdot)\|_{\text{r.m.s.}} \leq \bar{\gamma} \, \|u(\cdot)\|_{\text{r.m.s.}}. \tag{3.17}$$

In other words, the number $\bar{\gamma}$ that appears in the inequality (3.5) can be seen also as an upper bound for the ratio between the r.m.s. value of the output and the r.m.s value of the input, whenever a periodic input is producing (from an appropriate initial state) a periodic (state and output) response.

Consider now the special case in which the input signal is a *harmonic function* of time, i.e.,

$$u(t) = u_0 \cos(\omega_0 t)$$

It is known that, if the state $x(0)$ is appropriately chosen,[4] the output response of the system coincides with the so-called *steady-state response*, which is the harmonic function

$$y_{\text{ss}}(t) = \text{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \text{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t),$$

[4]See Sect. A.5 in Appendix A. This is the case if $x(0) = \Pi_1$, in which Π_1 is the first column of the solution Π of the Sylvester equation (A.29).

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D.$$

Recall that

$$\int_0^{\frac{2\pi}{\omega_0}} \|u(t)\|^2 dt = \frac{\pi}{\omega_0}\|u_0\|^2$$

and therefore

$$\int_0^{\frac{2\pi}{\omega_0}} \|y_{\rm ss}(t)\|^2 dt = \frac{\pi}{\omega_0}\|T(j\omega_0)u_0\|^2.$$

In other words,

$$\begin{array}{rcl} \|u(\cdot)\|^2_{\rm r.m.s.} & = & \frac{1}{2}\|u_0\|^2 \\ \|y_{\rm ss}(\cdot)\|^2_{\rm r.m.s.} & = & \frac{1}{2}\|T(j\omega_0)u_0\|^2. \end{array}$$

Thus, from the interpretation illustrated above one can conclude that, if the system satisfies (3.5), then

$$\|T(j\omega_0)u_0\|^2 = 2\|y_{\rm ss}(\cdot)\|^2_{\rm r.m.s.} \le \bar{\gamma}^2 2\|u(\cdot)\|^2_{\rm r.m.s.} = \bar{\gamma}^2\|u_0\|^2$$

i.e.,

$$\|T(j\omega_0)u_0\| \le \bar{\gamma}\ \|u_0\|,$$

or, bearing in mind the definition of norm of a matrix,[5]

$$\|T(j\omega_0)\| \le \bar{\gamma}.$$

Define now the quantity

$$\|T\|_{H_\infty} := \sup_{\omega\in\mathbb{R}} \|T(j\omega)\|,$$

which is called the H_∞ *norm* of the matrix $T(j\omega)$. Observing that ω_0 in the above inequality is arbitrary, it is concluded

$$\|T\|_{H_\infty} \le \bar{\gamma}\ . \tag{3.18}$$

Therefore, a linear system that satisfies (3.5) is stable and *the H_∞ norm of its frequency response matrix is bounded from above by the number $\bar{\gamma}$.*

---

[5]Recall that the norm of a matrix $T \in \mathbb{R}^{p\times m}$ is defined as

$$\|T\| = \sup_{\|u\|\neq 0} \frac{\|Tu\|}{\|u\|} = \max_{\|u\|=1} \|Tu\|.$$

3.4 The Bounded Real Lemma

We have seen in the previous sections that, for a system of the form (3.2), if there exists a number $\gamma > 0$ and a symmetric positive definite matrix X satisfying (3.14), then there exists a number $\bar{\gamma} < \gamma$ and a positive definite quadratic form $V(x)$ satisfying (3.5) for some $\varepsilon > 0$. This, in view of the interpretation provided above, proves that the fulfillment of (3.14) for some γ (with X positive definite) *implies* that

(i) the system is asymptotically stable,

(ii) its $\mathscr{L}_2$ gain is strictly less than γ,

(iii) the H_∞ norm of its transfer function is strictly less than γ.

However, put in these terms, we have only learned that (3.5) implies both (ii) and (iii) and we have not investigated yet whether converse implications might hold. In this section, we complete the analysis, by showing that the two properties (ii) and (iii) are, in fact, two different manifestations of the same property and both imply (3.14).

This will be done by means of a circular proof involving another equivalent version of the property that the number γ is an upper bound for the H_∞ norm of the transfer function matrix of the system, which is very useful for practical purposes, since it can be easily checked. More precisely, the fact that γ is an upper bound for the H_∞ norm of the transfer function matrix of the system can be checked by looking at the spectrum of a matrix of the form

$$H = \begin{pmatrix} A_0 & R_0 \\ -Q_0 & -A_0^{\mathrm{T}} \end{pmatrix} , \tag{3.19}$$

in which R_0 and Q_0 are *symmetric* matrices which, together with A_0, depend on the matrices A, B, C, D which characterize the system and on the number γ.[6]

As a matter of fact the following result, known in the literature as *Bounded Real Lemma*, holds.

Theorem 3.1 *Consider the linear system (3.2) and let $\gamma > 0$ be a fixed number. The following are equivalent:*

(i) *there exists $\bar{\gamma} < \gamma$, $\varepsilon > 0$ and a symmetric positive definite matrix P such that (3.7) holds for $V(x) = x^{\mathrm{T}}Px$,*

(ii) *all the eigenvalues of A have negative real part and the frequency response matrix of the system $T(j\omega) = C(j\omega I - A)^{-1}B + D$ satisfies*

$$\|T\|_{H_\infty} := \sup_{\omega\in\mathbb{R}} \|T(j\omega)\| < \gamma, \tag{3.20}$$

[6] A matrix (of real numbers) with this structure is called an *Hamiltonian matrix* and has the property that its spectrum is symmetric with respect to the imaginary axis (see Lemma A.5 in Appendix A).

(iii) *all the eigenvalues of A have negative real part, the matrix* $W = \gamma^2 I - D^T D$ *is positive definite, and the Hamiltonian matrix*

$$H = \begin{pmatrix} A + BW^{-1}D^T C & BW^{-1}B^T \\ -C^T C - C^T D W^{-1} D^T C & -A^T - C^T D W^{-1} B^T \end{pmatrix} \tag{3.21}$$

has no eigenvalues on the imaginary axis,
(iv) *there exists a positive definite symmetric matrix X satisfying*

$$\begin{pmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{3.22}$$

Proof We have already shown, in the previous sections, that if (i) holds, then (3.2) is an asymptotically stable system, with a frequency response matrix satisfying

$$\|T\|_{H_\infty} \le \bar{\gamma}.$$

Thus, (i) $\Rightarrow$ (ii).

To show that (ii) $\Rightarrow$ (iii), first of all that note, since

$$\lim_{\omega \to \infty} T(j\omega) = D$$

it necessarily follows that $\|Du\| < \gamma$ for all u with $\|u\| = 1$ and this implies $\gamma^2 I > D^T D$, i.e., the matrix W is positive definite.

Now observe that the Hamiltonian matrix (3.21) can be expressed in the form

$$H = L + MN$$

for

$$L = \begin{pmatrix} A & 0 \\ -C^T C & -A^T \end{pmatrix}, \quad M = \begin{pmatrix} B \\ -C^T D \end{pmatrix},$$

$$N = \begin{pmatrix} W^{-1} D^T C & W^{-1} B^T \end{pmatrix}.$$

Suppose, by contradiction, that the matrix H has eigenvalues on the imaginary axis. By definition, there exist a $2n$-dimensional vector x_0 and a number $\omega_0 \in \mathbb{R}$ such that

$$(j\omega_0 I - L)x_0 = MNx_0.$$

Observe now that the matrix L has no eigenvalues on the imaginary axis, because its eigenvalues coincide with those of A and $-A^T$, and A is by hypothesis stable. Thus $(j\omega_0 I - L)$ is nonsingular. Observe also that the vector $u_0 = Nx_0$ is nonzero because otherwise x_0 would be an eigenvector of L associated with an eigenvalue at $j\omega_0$, which is a contradiction. A simple manipulation yields

$$u_0 = N(j\omega_0 I - L)^{-1} M u_0. \tag{3.23}$$

It is easy to check that

$$N(j\omega_0 I - L)^{-1} M = W^{-1}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D] \tag{3.24}$$

where $T(s) = C(sI - A)^{-1}B + D$. In fact, it suffices to compute the transfer function of

$$\begin{aligned} \dot{x} &= Lx + Mu \\ y &= Nx \end{aligned}$$

and observe that $N(sI - L)^{-1}M = W^{-1}[T^{\mathrm{T}}(-s)T(s) - D^{\mathrm{T}}D]$.

Multiply (3.24) on the left by $u_0^{\mathrm{T}} W$ and on the right by u_0, and use (3.23), to obtain

$$u_0^{\mathrm{T}} W u_0 = u_0^{\mathrm{T}}[T^{\mathrm{T}}(-j\omega_0)T(j\omega_0) - D^{\mathrm{T}}D]u_0,$$

which in turn, in view of the definition of W, yields

$$\gamma^2 \|u_0\|^2 = \|T(j\omega_0)u_0\|^2.$$

This implies

$$\|T(j\omega_0)\| = \sup_{\|u_0\| \neq 0} \frac{\|T(j\omega_0)u_0\|}{\|u_0\|} \geq \gamma$$

and this contradicts (ii), thus completing the proof.

To show that (iii) $\Rightarrow$ (iv), set

$$\begin{aligned} F &= (A + BW^{-1}D^{\mathrm{T}}C)^{\mathrm{T}} \\ Q &= -BW^{-1}B^{\mathrm{T}} \\ GG^{\mathrm{T}} &= C^{\mathrm{T}}(I + DW^{-1}D^{\mathrm{T}})C \end{aligned}$$

(the latter is indeed possible because $I + DW^{-1}D^{\mathrm{T}}$ is a positive definite matrix: in fact, it is a sum of the positive definite matrix I and of the positive semidefinite matrix $DW^{-1}D^{\mathrm{T}}$; hence there exists a nonsingular matrix M such that $I + DW^{-1}D^{\mathrm{T}} = M^{\mathrm{T}}M$; in view of this, the previous expression holds with $G^{\mathrm{T}} = MC$).

It is easy to check that

$$H^{\mathrm{T}} = \begin{pmatrix} F & -GG^{\mathrm{T}} \\ -Q & -F^{\mathrm{T}} \end{pmatrix}$$

and this matrix by hypothesis has no eigenvalues on the imaginary axis. Moreover, it is also possible to show that the pair $(F, -GG^{\mathrm{T}})$ thus defined is stabilizable. In fact, suppose that this is not the case. Then, there is a vector $x \neq 0$ such that

$$x^{\mathrm{T}}\left(F - \lambda I \;\; -GG^{\mathrm{T}}\right) = 0$$

for some λ with non-negative real part. Then,

$$0 = \begin{pmatrix} A + BW^{-1}D^{\mathrm{T}}C - \lambda I \\ -C^{\mathrm{T}}M^{\mathrm{T}}MC \end{pmatrix} x.$$

This implies in particular $0 = x^{\mathrm{T}}C^{\mathrm{T}}M^{\mathrm{T}}MCx = \|MCx\|^2$ and hence $Cx = 0$, because M is nonsingular. This in turn implies $Ax = \lambda x$, and this is a contradiction because all the eigenvalues of A have negative real part.

Thus[7] there is a unique solution Y^- of the Riccati equation

$$Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q = 0\ , \tag{3.25}$$

satisfying $\sigma(F - GG^{\mathrm{T}}Y^-) \subset \mathbb{C}^-$. Moreover,[8] the set of solutions Y of the inequality

$$YF + F^{\mathrm{T}}Y - YGG^{\mathrm{T}}Y + Q > 0 \tag{3.26}$$

is nonempty and any Y in this set is such that $Y < Y^-$.

Observe now that

$$\begin{aligned}
&Y^-F + F^{\mathrm{T}}Y^- - Y^-GG^{\mathrm{T}}Y^- + Q \\
&= Y^-(A^{\mathrm{T}} + C^{\mathrm{T}}DW^{-1}B^{\mathrm{T}}) + (A + BW^{-1}D^{\mathrm{T}}C)Y^- - Y^-C^{\mathrm{T}}(I + DW^{-1}D^{\mathrm{T}})CY^- - BW^{-1}B^{\mathrm{T}} \\
&= Y^-A^{\mathrm{T}} + AY^- - [Y^-C^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY^- - B^{\mathrm{T}}] - Y^-C^{\mathrm{T}}CY^-,
\end{aligned}$$

and therefore (3.25) yields

$$Y^-A^{\mathrm{T}} + AY^- \geq 0.$$

Set now $U(z) = z^{\mathrm{T}}Y^-z$, let $z(t)$ denote (any) integral curve of

$$\dot{z} = A^{\mathrm{T}}z\ , \tag{3.27}$$

and observe that the function $U(z(t))$ satisfies

$$\frac{\partial U(z(t))}{\partial t} = 2z^{\mathrm{T}}(t)Y^-A^{\mathrm{T}}z(t) = z^{\mathrm{T}}(t)[Y^-A^{\mathrm{T}} + AY^-]z(t) \geq 0.$$

This inequality shows that the function $V(z(t))$ is non-decreasing, i.e., $V(z(t)) \geq V(z(0))$ for any $z(0)$ and any $t \geq 0$. On the other hand, system (3.27) is by hypothesis asymptotically stable, i.e., $\lim_{t\to\infty} z(t) = 0$. Therefore, necessarily, $V(z(0)) \leq 0$, i.e., the matrix Y^- is negative semi-definite. From this, it is concluded that any solution Y of (3.26), that is of the inequality

[7] See Proposition A.1 in Appendix A.

[8] See Proposition A.2 in Appendix A.

$$YA^{\mathrm{T}} + AY - [YC^{\mathrm{T}}D - B]W^{-1}[D^{\mathrm{T}}CY - B^{\mathrm{T}}] - YC^{\mathrm{T}}CY > 0 \,, \tag{3.28}$$

which necessarily satisfies $Y < Y^{-} \leq 0$, is a negative definite matrix.

Take any of the solutions Y of (3.28) and consider $P = -Y^{-1}$. By construction, this matrix is a positive definite solution of the inequality in (3.9). Thus, by Lemma 3.2, (iv) holds.

The proof that (iv) $\Rightarrow$ (i) is provided by Lemmas 3.2 and 3.1. ◁

Example 3.1 As an elementary example of what the criterion described in (iii) means, consider the case of the single-input single-output linear system

$$\begin{aligned} \dot{x} &= -ax + bu \\ y &= x + du \end{aligned} \tag{3.29}$$

in which it is assumed that $a > 0$, so that the system is stable. The transfer function of this system is

$$T(s) = \frac{ds + (ad + b)}{s + a}.$$

This function has a pole at $p = -a$ and a zero at $z = -(ad + b)/d$. Thus, bearing in mind the possible Bode plots of a function having one pole and one zero (see Fig. 3.1), it is seen that

$$\begin{aligned} |d| < |\frac{ad+b}{a}| \qquad &\Rightarrow \quad \|T\|_{H_\infty} = |T(0)| = |\frac{ad+b}{a}| \\ |d| > |\frac{ad+b}{a}| \qquad &\Rightarrow \quad \|T\|_{H_\infty} = \lim_{\omega\to\infty} |T(j\omega)| = |d|. \end{aligned}$$

Thus,

$$\|T\|_{H_\infty} = \max\left\{|d|, |\frac{ad+b}{a}|\right\}. \tag{3.30}$$

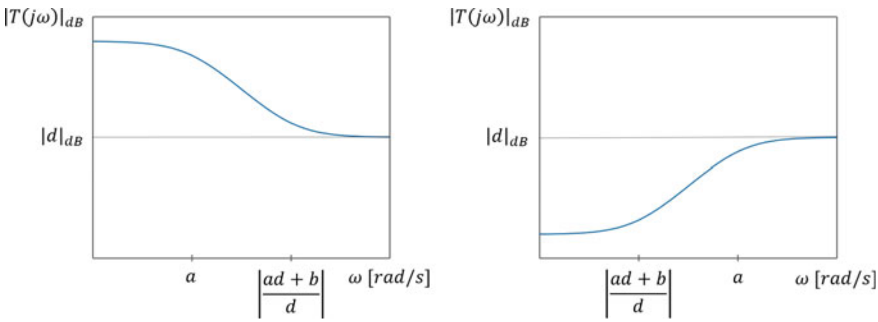


Fig. 3.1 The Bode plots of the transfer function of (3.29). *Left* the case $|T(0)| > |d|$. *Right* the case $|T(0)| < |d|$

The criterion in the Bounded Real Lemma says that the H_∞ norm of $T(s)$ is strictly less than γ if and only if $\gamma^2 I - D^{\mathrm{T}} D > 0$, which in this case becomes

$$\gamma^2 > d^2 \tag{3.31}$$

and the Hamiltonian matrix (3.21), which in this case becomes

$$H = \begin{pmatrix} -a + \dfrac{bd}{\gamma^2 - d^2} & \dfrac{b^2}{\gamma^2 - d^2} \\ -1 - \dfrac{d^2}{\gamma^2 - d^2} & a - \dfrac{bd}{\gamma^2 - d^2} \end{pmatrix},$$

has no eigenvalues with zero real part. A simple calculation shows that the characteristic polynomial of H is

$$p(\lambda) = \lambda^2 - a^2 + \frac{2abd}{\gamma^2 - d^2} + \frac{b^2}{\gamma^2 - d^2}.$$

This polynomial has no roots with zero real part if and only if

$$a^2 - \frac{2abd}{\gamma^2 - d^2} - \frac{b^2}{\gamma^2 - d^2} > 0$$

which, since $\gamma^2 - d^2 > 0$, is the case if and only if

$$\gamma^2 a^2 > (ad + b)^2. \tag{3.32}$$

Using both (3.31) and (3.32), it is concluded that, according to the Bounded Real Lemma, $\|T\|_{H_\infty} < \gamma$ if and only if

$$\gamma > \max\left\{ |d|, \left|\frac{ad + b}{a}\right| \right\}$$

which is exactly what (3.30) shows. $\lhd$

3.5 Small-Gain Theorem and Robust Stability

The various characterizations of the $\mathscr{L}_2$ gain of a system given in the previous sections provide a powerful tool for the study of the stability properties of feedback interconnected systems. To see why this is the case, consider two systems Σ_1 and Σ_2, described by equations of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + D_i u_i \end{aligned} \tag{3.33}$$

with $i = 1, 2$, in which we assume that

$$\begin{aligned} \dim(u_2) &= \dim(y_1) \\ \dim(u_1) &= \dim(y_2). \end{aligned}$$

Suppose that the matrices D_1 and D_2 are such that the *interconnections*

$$\begin{aligned} u_2 &= y_1 \\ u_1 &= y_2 \end{aligned} \tag{3.34}$$

makes sense (see Fig. 3.2).

This will be the case if, for each x_1, x_2, there is a unique pair u_1, u_2 satisfying

$$\begin{array}{rcl} u_1 &=& C_2 x_2 + D_2 u_2 \\ u_2 &=& C_1 x_1 + D_1 u_1 \ , \end{array}$$

i.e., if the system of equations

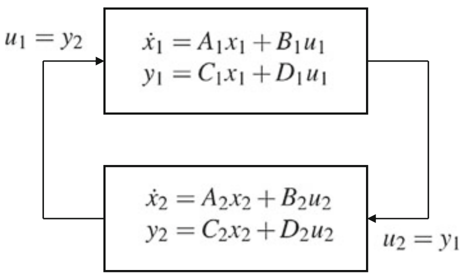
$$\begin{array}{rcl} u_1 - D_2 u_2 &=& C_2 x_2 \\ -D_1 u_1 + u_2 &=& C_1 x_1 \end{array}$$

has a *unique* solution u_1, u_2. This occurs if and only if the matrix

$$\begin{pmatrix} I & -D_2 \\ -D_1 & I \end{pmatrix}$$

is invertible, i.e., the matrix $I - D_2 D_1$ is nonsingular.[9] The (autonomous) system defined by (3.33) together with (3.34) is the *pure feedback interconnection* of Σ_1 and Σ_2.

Fig. 3.2 A pure feedback interconnection of two systems Σ_1 and Σ_2



[9]Note that an equivalent condition is that the matrix $I - D_1 D_2$ is nonsingular.

Suppose now that both systems Σ_1 and Σ_2 are stable. As shown in Sect. 3.1, there exist two positive definite matrices P_1, P_2, two positive numbers $\varepsilon_1, \varepsilon_2$ and two real numbers $\bar{\gamma}_1, \bar{\gamma}_2$ such that Σ_1 and Σ_2 satisfy inequalities of the form (3.5), namely

$$\frac{\partial V_i}{\partial x_i}(A_i x_i + B_i u_i) \le -\varepsilon_i \|x_i\|^2 + \bar{\gamma}_i^2 \|u_i\|^2 - \|y_i\|^2 \tag{3.35}$$

in which $V_i(x_i) = x_i^{\mathrm{T}} P_i x_i$.

Consider now the quadratic form

$$W(x_1, x_2) = V(x_1) + aV(x_2) = \begin{pmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & aP_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

which, if $a > 0$, is positive definite. A simple calculation shows that, for any trajectory $(x_1(t), x_2(t))$ of the pure feedback interconnection of Σ_1 and Σ_2, the function $W(x_1(t), x_2(t))$ satisfies

$$\begin{aligned}
\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 &\le -\varepsilon_1\|x_1\|^2 - a\varepsilon_2\|x_1\|^2 + \bar{\gamma}_1^2\|u_1\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2\|u_2\|^2 - a\|y_2\|^2 \\
&\le -\varepsilon_1\|x_1\|^2 - a\varepsilon_2\|x_1\|^2 + \bar{\gamma}_1^2\|y_2\|^2 - \|y_1\|^2 + a\bar{\gamma}_2^2\|y_1\|^2 - a\|y_2\|^2 \\
&= -\varepsilon_1\|x_1\|^2 - a\varepsilon_2\|x_1\|^2 + \begin{pmatrix} y_1^{\mathrm{T}} & y_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\end{aligned}$$

If

$$\begin{pmatrix} (-1 + a\bar{\gamma}_2^2)I & 0 \\ 0 & (\bar{\gamma}_1^2 - a)I \end{pmatrix} \le 0, \tag{3.36}$$

we have

$$\frac{\partial W}{\partial x_1}\dot{x}_1 + \frac{\partial W}{\partial x_2}\dot{x}_2 \le -\varepsilon_1\|x_1\| - a\varepsilon_2\|x_2\| = \begin{pmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} -\varepsilon_1 I & 0 \\ 0 & -a\varepsilon_2 I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The quadratic form on the right-hand side is negative definite and hence, according to the criterion of Lyapunov, the interconnected system is asymptotically stable. Condition (3.36), on the other hand, can be fulfilled for some $a > 0$ if (and only if)

$$-1 + a\bar{\gamma}_2^2 \le 0, \qquad \bar{\gamma}_1^2 - a \le 0$$

i.e., if

$$\bar{\gamma}_1^2 \le a \le \frac{1}{\bar{\gamma}_2^2}.$$

A number $a > 0$ satisfying these inequalities exists if and only if $\bar{\gamma}_2^2 \bar{\gamma}_1^2 \le 1$.

All the above yields the following important result.

Theorem 3.2 *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose (3.33) satisfy inequalities of the form (3.35), with P_1, P_2 positive definite and*

$$\bar{\gamma}_1\bar{\gamma}_2 \leq 1.$$

Then, the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Remark 3.2 In the proof of the above statement, we have considered the general case in which both component subsystems have an internal dynamics. To cover the special case in which one of the two subsystem is a memoryless system, a slightly different (actually simpler) argument is needed. Suppose the second component subsystem is modeled as

$$y_2 = D_2u_2$$

and set $\bar{\gamma}_2 = \|D_2\|$. Then,

$$\|u_1\|^2 = \|y_2\|^2 \leq \bar{\gamma}_2^2\|u_2\|^2 = \bar{\gamma}_2^2\|y_1\|^2.$$

As a consequence, for the interconnection, we obtain

$$\begin{aligned}\frac{\partial V_1}{\partial x_1}(A_1x_1 + B_1u_1) &\leq -\varepsilon_1\|x_1\|^2 + \bar{\gamma}_1^2\|u_1\|^2 - \|y_1\|^2 \\ &\leq -\varepsilon_1\|x_1\|^2 + (\bar{\gamma}_1^2\bar{\gamma}_2^2 - 1)\|y_1\|^2.\end{aligned}$$

If $\bar{\gamma}_1\bar{\gamma}_2 \leq 1$, the quantity above is negative definite and the interconnected system is stable. ◁

Note also that, in view of the Bounded Real Lemma, the theorem above can be rephrased in terms of H_∞ norms of the transfer functions of the two component subsystems, as follows:

Corollary 3.1 *Consider a pair of systems (3.33) and suppose that the matrix $I - D_2D_1$ is nonsingular. Suppose both systems are asymptotically stable. Let*

$$T_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$$

denote the respective transfer functions. If

$$\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} < 1 \tag{3.37}$$

the pure feedback interconnection of Σ_1 and Σ_2 is asymptotically stable.

Proof Suppose that condition (3.37) holds, i.e., that

$$\|T_1\|_{H_\infty} < \frac{1}{\|T_2\|_{H_\infty}}.$$

Then, it is possible to find a positive number γ_1 satisfying

$$\|T_1\|_{H_\infty} < \gamma_1 < \frac{1}{\|T_2\|_{H_\infty}}.$$

Set $\gamma_2 = 1/\gamma_1$. Then

$$\|T_i\|_{H_\infty} < \gamma_i\,, \quad i = 1, 2.$$

From the Bounded Real Lemma, it is seen that—for both $i = 1, 2$—there exists $\varepsilon_i > 0$, $\bar{\gamma}_i < \gamma_i$ and positive definite P_i such that (3.33) satisfy inequalities of the form (3.35), with $V_i(x) = x_i^{\mathrm{T}} P_i x_i$. Moreover, $\bar{\gamma}_1\bar{\gamma}_2 < \gamma_1\gamma_2 = 1$. Hence, from Theorem 3.2 and Remark 3.2, the result follows. $\triangleleft$

The result just proven is known as the *Small-Gain Theorem* (of linear systems). In a nutshell, it says that if both component systems are stable, a *sufficient* condition for the stability of their (pure feedback) interconnection is that the product of the H_∞ norms of the transfer functions of the two component subsystems is *strictly less than 1*. This theorem is the point of departure for the study of robust stability via H_∞ methods.

It should be stressed that the "small gain condition" (3.37) provided by this corollary is only sufficient for stability of the interconnection and that the strict inequality in (3.37) cannot be replaced, in general, by a loose inequality. Both these facts are explained in the example which follows.

Example 3.2 Let the two component subsystems be modeled by

$$\begin{aligned} \dot{x}_1 &= -ax_1 + u_1 \\ y_1 &= x_1 \end{aligned}$$

and

$$y_2 = du_2.$$

Suppose $a > 0$ so that the first subsystem is stable. The stability of the interconnection can be trivially analyzed by direct computation. In fact, the interconnection is modeled as

$$\dot{x}_1 = -ax_1 + dx_1$$

from which is it is seen that a necessary and sufficient condition for the interconnection to be stable is that $d < a$.

On the other hand, the Small-Gain Theorem yields a more conservative estimate. In fact

$$T_1(s) = \frac{1}{s+a} \qquad \text{and} \qquad T_2(s) = d$$

and hence

$$\|T_1\|_{H_\infty} = T_1(0) = \frac{1}{a} \qquad \text{and} \qquad \|T_2\|_{H_\infty} = |d|$$

The (sufficient) condition (3.37) becomes $|d| < a$, i.e., $-a < d < a$.

This example also shows that the inequality in (3.37) has to be strict. In fact, the condition $\|T_1\|_{H_\infty} \cdot \|T_2\|_{H_\infty} = 1$ would yield $|d| = a$, which is not admissible because, if $d = a$ the interconnection is not (asymptotically) stable (while it would be asymptotically stable if $d = -a$).

Example 3.3 It is worth observing that a special version of such sufficient condition was implicit in the arguments used in Chap. 2 to prove asymptotic stability. Take, for instance, the case studied in Sect. 2.3. System (2.19) can be seen as pure feedback interconnection of a subsystem modeled by

$$\begin{aligned} \dot{z} &= A_{00}(\mu)z + A_{01}(\mu)u_1 \\ y_1 &= A_{10}(\mu)z \end{aligned} \tag{3.38}$$

and of a subsystem modeled by

$$\begin{aligned} \dot{\xi} &= [A_{11}(\mu) - b(\mu)k]\xi + u_2 \\ y_2 &= \xi. \end{aligned} \tag{3.39}$$

The first of two such subsystems is a stable system, because $A_{00}(\mu)$ has all eigenvalues in $\mathbb{C}^-$. Let

$$T_1(s) = A_{10}(\mu)[sI - A_{00}(\mu)]^{-1}A_{01}(\mu).$$

denote its transfer function. Its H_∞ norm depends on μ but, since μ ranges over a compact set $\mathbb{M}$, it is possible to find a number $\gamma_1 > 0$ such that

$$\max_{\mu \in \mathbb{M}} \|T_1\|_{H_\infty} < \gamma_1.$$

Note that this number γ_1 depends only on the data that characterize the controlled system (2.17) and not on the coefficient k that characterizes the feedback law (2.18).

Consider now the second subsystem. If $b(\mu)k - A_{11}(\mu) > 0$, this is a stable system with transfer function

$$T_2(s) = \frac{1}{s + (b(\mu)k - A_{11}(\mu))}$$

Clearly,

$$\|T_2\|_{H_\infty} = T_2(0) = \frac{1}{b(\mu)k - A_{11}(\mu)}.$$

It is seen from this that $\|T_2\|_{H_\infty}$ can be arbitrarily *decreased* by *increasing* the coefficient k. In other words: a *large* value of the output feedback gain coefficient k in (2.18) forces a *small* value of $\|T_2\|_{H_\infty}$.

As a consequence of the Small-Gain Theorem, the interconnected system (namely, system (2.19)) is stable if

$$\frac{\gamma_1}{b(\mu)k - A_{11}(\mu)} < 1$$

which is indeed the case if

$$k > \max_{\mu\in\mathbb{M}} \frac{\gamma_1 + A_{11}(\mu)}{b(\mu)}.$$

In summary, the result established in Sect. 2.3 can be re-interpreted in the following terms. Subsystem (3.38) is a stable system, with a transfer function whose H_∞ norm has some fixed bound γ_1 (on which the control has no influence, though). By increasing the value of the gain coefficient k in (2.18), the subsystem (3.39) can be rendered stable, with a H_∞ norm that can be made arbitrarily small, in particular smaller than the (fixed) number $1/\gamma_1$. This makes the small-gain condition (3.37) fulfilled and guarantees the (robust) stability of system (2.19). ◁

We turn now to the discussion of the problem of robust stabilization. The problem in question can be cast, in rather general terms, as follows. A plant with *control* input u and *measurement* output y whose model is uncertain can be, from a rather general viewpoint, thought of as the interconnection of a *nominal system* modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v \end{aligned} \tag{3.40}$$

in which the "additional" input v and the "additional" output z are seen as output and, respectively, input of a system

$$\begin{aligned} \dot{x}_\mathrm{p} &= A_\mathrm{p} x_\mathrm{p} + B_\mathrm{p} z \\ v &= C_\mathrm{p} x_\mathrm{p} + D_\mathrm{p} z \end{aligned} \tag{3.41}$$

whose parameters are *uncertain* (Fig. 3.3).[10]

This setup includes the special case of a plant of fixed dimension whose parameters are uncertain, that is a plant modeled as

$$\begin{aligned} \dot{x} &= (A_0 + \delta A)x + (B_0 + \delta B)u \\ y &= (C_0 + \delta C)x \end{aligned} \tag{3.42}$$

[10]Note that, for the interconnection (3.40)–(3.41) to be well-defined, the matrix $I - D_\mathrm{p} D_{11}$ is required to be invertible.

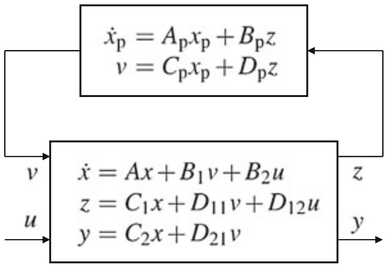


Fig. 3.3 A controlled system seen as interconnection of an accurate model and of a lousy model

in which A_0, B_0, C_0 represent nominal values and δA, δB, δC uncertain perturbations. In fact, the latter can be seen as interconnection of a system of the form (3.40) in which

$$A = A_0 \qquad B_1 = \begin{pmatrix} I & 0 \end{pmatrix} \quad B_2 = B_0$$
$$C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix} \; D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$
$$C_2 = C_0 \quad D_{21} = \begin{pmatrix} 0 & I \end{pmatrix}$$

with a system of the form

$$v = \begin{pmatrix} \delta A & \delta B \\ \delta C & 0 \end{pmatrix} z$$

which is indeed a special case of a system of the form (3.41). The interconnection (3.40)–(3.41) of a *nominal model* and of a *dynamic perturbation* is more general, though, because it accommodates for perturbations which possibly include *unmodeled dynamics* (see example at the end of the section). In this setup, all modeling uncertainties are confined in the model (3.41), including the dimension itself of x_p.

Suppose now that, in (3.41), A_p is a Hurwitz matrix and that the transfer function

$$T_p(s) = C_p(sI - A_p)^{-1}B_p + D_p$$

has an H_∞ norm which is bounded by a known number γ_p. That is, *assume* that, no matter what the perturbations are, the perturbing system (3.41) is a stable system satisfying

$$\|T_p\|_{H_\infty} < \gamma_p. \tag{3.43}$$

for some γ_p.

Let

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \tag{3.44}$$

be a controller for the nominal plant (3.40) yielding a closed-loop system

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} &= \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix} v \\
z &= \begin{pmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11} + D_{12} D_c D_{21})\, v
\end{aligned}$$

which is asymptotically stable and whose transfer function, between the input v and output z has an H_∞ norm bounded by a number γ satisfying

$$\gamma\, \gamma_p < 1. \tag{3.45}$$

If this is the case, thanks to the Small-Gain Theorem, it can be concluded that the controller (3.44) stabilizes any of the perturbed plants (3.40) and (3.41), so long as the perturbation is such that (3.41) is asymptotically stable and the bound (3.43) holds (Fig. 3.4).

In this way, the problem of robust stabilization is reduced to a problem of stabilizing a nominal plant and to simultaneously enforce a bound on the H_∞ norm of its transfer function.

Example 3.4 A simplified model describing the motion of a vertical takeoff and landing (VTOL) aircraft in the vertical-lateral plane can be obtained in the following way. Let y, h and θ denote, respectively, the horizontal and vertical position of the center of mass C and the roll angle of the aircraft with respect to the horizon, as in Fig. 3.5. The control inputs are the thrust directed out the bottom of the aircraft, denoted by T, and the rolling moment produced by a couple of equal forces, denoted by F, acting at the wingtips. Their direction is not perpendicular to the horizontal body axis, but tilted by some fixed angle α. Letting M denote the mass of the aircraft, J the moment of inertia about the center of mass, ℓ the distance between the wingtips

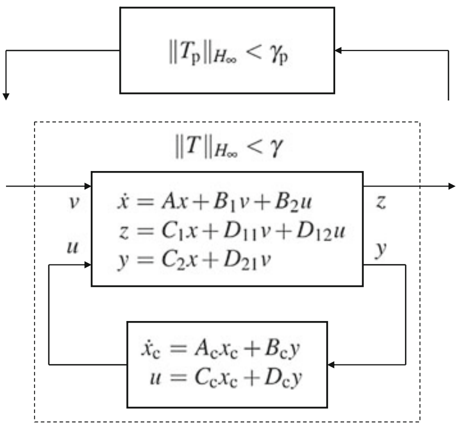


Fig. 3.4 The H_∞ approach to robust stabilization

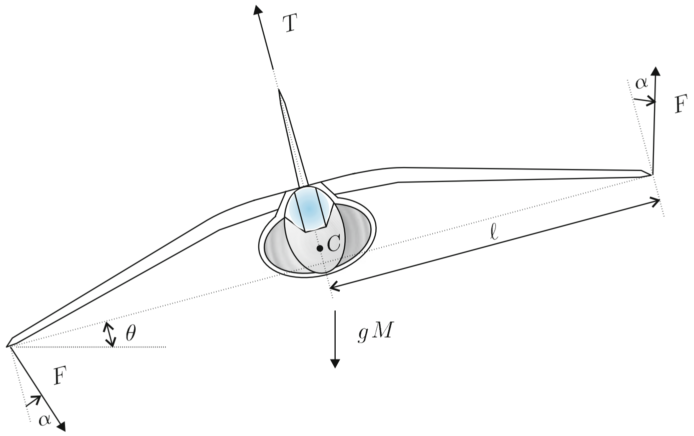


Fig. 3.5 A VTOL moving in the lateral-vertical plane

and g the gravitational acceleration, it is seen that the motion of the aircraft is modeled by the equations

$$\begin{aligned} M\ddot{y} &= -\sin(\theta)T + 2\cos(\theta)F\sin\alpha \\ M\ddot{h} &= \cos(\theta)T + 2\sin(\theta)F\sin\alpha - gM \\ J\ddot{\theta} &= 2\ell F\cos\alpha. \end{aligned}$$

The purpose of the control input T is that of moving the center of mass of the airplane in the vertical-lateral plane, while that of the input F is to control the airplane's attitude, i.e., the roll angle θ.

In the hovering maneuver, the thrust T is expected to compensate exactly the force of gravity. Thus, T can be expressed as $T = gM + \delta T$ in which δT is a residual input use to control the attitude. If the angle θ is sufficiently small one can use the approximations $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 1$ and neglect the nonlinear terms, so as to obtain a linear simplified model

$$\begin{aligned} M\ddot{y} &= -gM\theta + 2F\sin\alpha \\ M\ddot{h} &= \delta T \\ J\ddot{\theta} &= 2\ell F\cos\alpha. \end{aligned}$$

In this simplified model, the motion in the vertical direction is totally decoupled, and controlled by δT. On the other hand, the motion in the lateral direction and the roll angle are coupled, and controlled by F. We concentrate on the analysis of the latter.

The system with input F and output y is a four-dimensional system, having relative degree 2. Setting $\zeta_1 = \theta$, $\zeta_2 = \dot{\theta}$, $\xi_1 = y$, $\xi_2 = \dot{y}$ and $u = F$, it can be expressed, in state-space form, as

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \tfrac{2\ell}{J}(\cos\alpha)u \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -g\zeta_1 + \tfrac{2}{M}(\sin\alpha)u.
\end{aligned}
$$

Note that this is not a strict normal form. It can be put in strict normal form, though, changing ζ_2 into

$$\tilde{\zeta}_2 = \zeta_2 - \frac{\ell M \cos\alpha}{J \sin\alpha}\xi_2,$$

as the reader can easily verify. The strict normal form in question is given by

$$
\begin{aligned}
\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\tilde{\zeta}}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{\ell Mg\cos\alpha}{J\sin\alpha} & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \tilde{\zeta}_2 \end{pmatrix} + \begin{pmatrix} \frac{\ell M\cos\alpha}{J\sin\alpha} \\ 0 \end{pmatrix} \xi_2 \\
\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} \zeta_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tfrac{2}{M}(\sin\alpha)u.
\end{aligned}
\tag{3.46}
$$

It is seen from this that the zeros of the system are the roots of

$$N(\lambda) = \lambda^2 - \frac{\ell Mg\cos\alpha}{J\sin\alpha},$$

and hence the system *is not* minimum phase, because one zero has positive real part. Therefore the (elementary) methods for robust stabilization described in the previous section cannot be applied.

However, the approach presented in this section is applicable. To this end, observe that the characteristic polynomial of the system is the polynomial $D(\lambda) = \lambda^4$ and consequently its transfer function has the form

$$T(s) = \frac{\frac{2}{M}(\sin\alpha)s^2 - \frac{2\ell g}{J}(\cos\alpha)}{s^4}.$$

From this—by known facts—it is seen that an equivalent realization of (3.46) is given by

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u := A_0 + B_0 u \\
y &= \begin{pmatrix} -\frac{2\ell g}{J}(\cos\alpha) & 0 & \frac{2}{M}(\sin\alpha) & 0 \end{pmatrix} x := Cx.
\end{aligned}
\tag{3.47}
$$

The matrices A_0 and B_0 are not subject to perturbations. Thus, the only uncertain parameters are the entries of the matrix C. We may write Cx in the form

$$Cx = C_0 x + D_{\mathrm{p}} z$$

in which $C_0 \in \mathbb{R}^4$ is a vector of nominal values, $z = \mathrm{col}(x_1, x_3)$ and $D_{\mathrm{p}} \in \mathbb{R}^2$ is an (uncertain) term whose elements represent the deviations, from the assumed nominal values, of the first and third entry of C.[11]

This being the case, it is concluded that the (perturbed) model can be expressed as interconnection of an accurately known system, and of a (memoryless) uncertain system, with the former modeled as

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u \\ z &= \mathrm{col}(x_1, x_3) \\ y &= C_0 x + v, \end{aligned}$$

and the latter modeled as

$$v = D_{\mathrm{p}} z \,. \qquad \triangleleft$$

Example 3.5 Consider a d.c. motor, in which the stator field is kept constant, and the control is provided by the rotor voltage. The system in question can be characterized by equations expressing the mechanical balance and the electrical balance. The mechanical balance, in the hypothesis of viscous friction only (namely friction torque purely proportional to the angular velocity) has the form

$$J\dot{\Omega} + F\Omega = T$$

in which Ω denotes the angular velocity of the motor shaft, J the inertia of the rotor, F the viscous friction coefficient, and T the torque developed at the shaft. The torque T is proportional to the rotor current I, namely,

$$T = k_m I.$$

The rotor current, in turn, is determined by the electrical balance of the rotor circuit. This circuit, with reasonable approximation, can be modeled as in Fig. 3.6, in which R is the resistance of the rotor winding, L is the inductance of the rotor winding, R_b is the contact resistance of the brushes, C_s is a stray capacitance. The voltage V is the control input and the voltage E is the so-called "back electromotive force (e.m.f.)" which is proportional to the angular velocity of the motor shaft, namely

[11] Since α is a small angle, it makes sense to take

$$C_0 = \begin{pmatrix} -\frac{2\ell_0 g}{J_0} & 0 & 0 & 0 \end{pmatrix},$$

where ℓ_0 and J_0 are the nominal values of ℓ and J, in which case

$$D_{\mathrm{p}} = \mathrm{row}\left(2g\left(\frac{\ell_0}{J_0} - \frac{\ell}{J}(\cos\alpha)\right), \frac{2}{M}(\sin\alpha)\right).$$

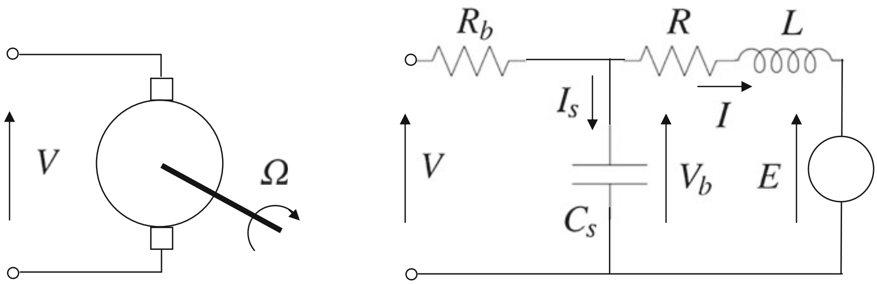


Fig. 3.6 A d.c. motor and its equivalent circuit

$$E = k_e \Omega .$$

The equations expressing the electrical balance are

$$\begin{aligned} R_b(I + I_s) &= V - V_b \\ C_s \frac{\mathrm{d}V_b}{\mathrm{d}t} &= I_s \\ L\frac{\mathrm{d}I}{\mathrm{d}t} + RI &= V_b - E . \end{aligned}$$

These can be put in state-space form by setting

$$\xi = \mathrm{col}(V_b,\ I)\,, \qquad z = \mathrm{col}(V,\ \Omega)$$

to obtain (recall that $E = k_e \Omega$)

$$\begin{aligned} \dot{\xi} &= F\xi + Gz \\ I &= H\xi \end{aligned}$$

in which

$$F = \begin{pmatrix} \dfrac{-1}{C_s R_b} & \dfrac{-1}{C_s} \\ \\ \dfrac{1}{L} & \dfrac{-R}{L} \end{pmatrix}, \quad G = \begin{pmatrix} \dfrac{1}{C_s R_b} & 0 \\ 0 & \dfrac{-k_e}{L} \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

In this way, the rotor current I is seen as output of a system with input z. Note that this system is stable, because characteristic polynomial of the matrix F

$$d(\lambda) = \lambda^2 + \left(\frac{1}{C_s R_b} + \frac{R}{L} \right) \lambda + \frac{1}{LC_s} + \frac{R}{LC_s R_b}$$

has roots with negative real part. A simple calculation shows that the two entries of the transfer function of this system

$$T(s) = \big(T_1(s)\ T_2(s)\big)$$

have the expressions

$$T_1(s) = \frac{1}{(C_s R_b\, s + 1)(L\, s + R) + R_b}, \qquad T_2(s) = \frac{-(C_s R_b\, s + 1)k_e}{(C_s R_b\, s + 1)(L\, s + R) + R_b}.$$

Note that, if R_b is negligible, the two functions can be approximated as

$$T_1(s) \cong \frac{1}{(L\, s + R)}, \qquad T_2(s) \cong \frac{-k_e}{(L\, s + R)}.$$

and also that, if $(L/R) \ll 1$, the functions can be further approximated, over a reasonable range of frequencies, as

$$T_1(s) \cong \frac{1}{R}, \qquad T_2(s) = \frac{-k_e}{R}.$$

This shows that, neglecting the dynamics of the rotor circuit, the rotor current can be approximately expressed as

$$I_0 \cong \frac{1}{R}(V - k_e \Omega) = Kz,$$

in which K is the row vector

$$K = \begin{pmatrix} \frac{1}{R} & \frac{-k_e}{R} \end{pmatrix}.$$

With this in mind, the (full) expression of the rotor current can be written as

$$I = I_0 + v$$

where v is the output of a system

$$\begin{aligned} \dot{\xi} &= F\xi + Gz \\ v &= H\xi - Kz. \end{aligned}$$

Replacing this expression into the equation of the mechanical balance, letting x_1 denote the angular position of the rotor shaft (in which case it is seen that $\dot{x}_1 = \Omega$), setting

$$x_2 = \Omega\,, \qquad u = V$$

one obtains a model of the form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\Big(\frac{F}{J}+\frac{k_m k_e}{JR}\Big)x_2+\frac{k_m}{J}v+\frac{k_m}{JR}u.\end{aligned}$$

In summary, letting $y = x_1$ denote the measured output of the system, the (full) model of the system in question can be seen as a system of the form (3.40), with

$$A=\begin{pmatrix}0 & 1\\ 0 & -\Big(\dfrac{F}{J}+\dfrac{k_m k_e}{JR}\Big)\end{pmatrix},\quad B_1=\begin{pmatrix}0\\ \dfrac{k_m}{J}\end{pmatrix},\quad B_2=\begin{pmatrix}0\\ \dfrac{k_m}{JR}\end{pmatrix}$$

$$C_1=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\qquad D_{11}=\begin{pmatrix}0\\ 0\end{pmatrix},\qquad D_{12}=\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$C_2=\begin{pmatrix}1 & 0\end{pmatrix},\qquad D_{21}=0\,,$$

interconnected with a system of the form (3.41) in which $A_\mathrm{p}, B_\mathrm{p}, C_\mathrm{p}, D_\mathrm{p}$ coincide, respectively, with the matrices $F, G, H, -K$ defined above. The system is modeled as the interconnection of a two subsystems: a low-dimensional subsystem, that models only the dominant dynamical phenomena (the dynamics of the motion of the rotor shaft) and whose parameters are assumed to be known with sufficient accuracy, and a subsystem which may include highly uncertain parameters (the parameters R_b and C_s) and whose dynamics are neglected when a model valid only on a low range of frequencies is sought. The design philosophy described above is that of seeking a feedback law, acting on the system that models the dominant dynamical phenomena, so as to obtain robust stability in spite of parameter uncertainties and un-modeled dynamics.

3.6 The Coupled LMIs Approach to the Problem of γ-Suboptimal H_∞ Feedback Design

Motivated by the results discussed at the end of the previous section, we consider now a design problem which goes under the name of *problem of γ-suboptimal H_∞ feedback design*.[12] Consider a linear system described by equations of the form

[12] In this section, the exposition closely follows the approach of [7, 9, 10] to the problem of γ-suboptimal H_∞ feedback design. For the numerical implementation of the design methods, see also [8].

$$\begin{aligned} \dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v. \end{aligned} \tag{3.48}$$

Let $\gamma > 0$ be a fixed number. The problem of γ-suboptimal H_∞ feedback design consists in finding a controller

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \tag{3.49}$$

yielding a closed-loop system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} &= \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix} v \\ z &= \begin{pmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11} + D_{12} D_c D_{21})\, v \end{aligned} \tag{3.50}$$

which is *asymptotically stable* and whose transfer function, between the input v and output z, has an H_∞ norm *strictly less* than γ.

The interest in such problem, in view of the results discussed earlier, is obvious. In fact, if this problem is solved, the controller (3.49) robustly stabilizes any perturbed system that can be seen as pure feedback interconnection of (3.48) and of an uncertain system of the form (3.41), so long as the latter is a stable system having a transfer function whose H_∞ norm is less than or equal to the inverse of γ. Of course, the smaller is the value of γ for which the problem is solvable, the "larger" is the set of perturbations against which robust stability can be achieved.

Rewrite system (3.50) as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathscr{A}\mathbf{x} + \mathscr{B}v \\ z &= \mathscr{C}\mathbf{x} + \mathscr{D}v \end{aligned}$$

where

$$\mathscr{A} = \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix}$$

$$\mathscr{C} = \begin{pmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{pmatrix}, \qquad \mathscr{D} = D_{11} + D_{12} D_c D_{21}.$$

In view of the Bounded Real Lemma, the closed-loop system has the desired properties if and only if there exists a symmetric matrix $\mathscr{X} > 0$ satisfying

$$\mathscr{X} > 0 \tag{3.51}$$

$$\begin{pmatrix} \mathscr{A}^T \mathscr{X} + \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} & \mathscr{C}^T \\ \mathscr{B}^T \mathscr{X} & -\gamma I & \mathscr{D}^T \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{pmatrix} < 0. \tag{3.52}$$

Thus, the problem is to try to find a quadruplet $\{A_c, B_c, C_c, D_c\}$ such that (3.51) and (3.52) hold for some symmetric $\mathscr{X}$.

The basic inequality (3.52) will be now transformed as follows. Let

$$x \in \mathbb{R}^n, \quad x_c \in \mathbb{R}^k, \quad v \in \mathbb{R}^{m_1}, \quad u \in \mathbb{R}^{m_2}, \quad z \in \mathbb{R}^{p_1}, \quad y \in \mathbb{R}^{p_2}.$$

Set

$$\mathbf{A}_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad \mathbf{C}_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix}, \tag{3.53}$$

$$\Psi(\mathscr{X}) = \begin{pmatrix} \mathbf{A}_0^{\mathrm{T}}\mathscr{X} + \mathscr{X}\mathbf{A}_0 & \mathscr{X}\mathbf{B}_0 & \mathbf{C}_0^{\mathrm{T}} \\ \mathbf{B}_0^{\mathrm{T}}\mathscr{X} & -\gamma I & D_{11}^{\mathrm{T}} \\ \mathbf{C_0} & D_{11} & -\gamma I \end{pmatrix}, \tag{3.54}$$

$$\mathscr{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^{\mathrm{T}} & 0 & 0_{m_2\times m_1} & D_{12}^{\mathrm{T}} \end{pmatrix},$$

$$\mathscr{Q} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ C_2 & 0 & D_{21} & 0_{p_2\times p_1} \end{pmatrix},$$

and

$$\Xi(\mathscr{X}) = \begin{pmatrix} \mathscr{X} & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{p_1} \end{pmatrix}.$$

Collecting the parameters of the controller (3.49) in the $(n + m_2) \times (n + p_2)$ matrix

$$\mathbf{K} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \tag{3.55}$$

the inequality (3.52) can be rewritten as

$$\Psi(\mathscr{X}) + \mathscr{Q}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}[\mathscr{P}\Xi(\mathscr{X})] + [\mathscr{P}\Xi(\mathscr{X})]^{\mathrm{T}}\mathbf{K}\mathscr{Q} < 0. \tag{3.56}$$

Thus, the problem of γ-suboptimal H_∞ feedback design can be cast as the problem of finding a symmetric matrix $\mathscr{X} > 0$ and a matrix $\mathbf{K}$ such that (3.56) holds. Note that this inequality, is not a linear matrix inequality in the unknowns $\mathscr{X}$ and $\mathbf{K}$. Rather, it is a *bilinear* matrix inequality.[13] However, the problem of finding a matrix $\mathscr{X} > 0$ for which (3.56) is solved by some $\mathbf{K}$ can be cast in terms of linear matrix inequalities. In this context, the following result is very useful.[14]

[13]The inequality (3.56), for each fixed $\mathscr{X}$ is a linear matrix inequality in $\mathbf{K}$ and, for each fixed $\mathbf{K}$ is a linear matrix inequality in $\mathscr{X}$.

[14]The proof of Lemma 3.3 can be found in [7].

Lemma 3.3 *Given a symmetric $m \times m$ matrix Ψ and two matrices P, Q having m columns, consider the problem of finding some matrix K of compatible dimensions such that*

$$\Psi + Q^{\mathrm{T}} K P + P^{\mathrm{T}} K^{\mathrm{T}} Q < 0. \tag{3.57}$$

Let W_P and W_Q be two matrices such that[15]

$$\begin{aligned}\mathrm{Im}(W_P) &= \mathrm{Ker}(P)\\ \mathrm{Im}(W_Q) &= \mathrm{Ker}(Q).\end{aligned}$$

Then (3.57) is solvable in K if and only if

$$\begin{aligned}W_Q^{\mathrm{T}} \Psi W_Q &< 0\\ W_P^{\mathrm{T}} \Psi W_P &< 0.\end{aligned} \tag{3.58}$$

This lemma can be used to eliminate $\mathbf{K}$ from (3.56) and obtain existence conditions depending only on $\mathscr{X}$ and on the parameters of the plant (3.48). Let $W_{\mathscr{P}\Xi(\mathscr{X})}$ be a matrix whose columns span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$ and let $W_{\mathscr{Q}}$ be a matrix whose columns span $\mathrm{Ker}(\mathscr{Q})$. According to Lemma 3.3, there exists $\mathbf{K}$ for which (3.56) holds if and only if

$$\begin{aligned}W_{\mathscr{Q}}^{\mathrm{T}} \Psi(\mathscr{X}) W_{\mathscr{Q}} &< 0\\ W_{\mathscr{P}\Xi(\mathscr{X})}^{\mathrm{T}} \Psi(\mathscr{X}) W_{\mathscr{P}\Xi(\mathscr{X})} &< 0.\end{aligned} \tag{3.59}$$

The second of these two inequalities can be further manipulated observing that if $W_{\mathscr{P}}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$, the columns of the matrix $[\Xi(\mathscr{X})]^{-1} W_{\mathscr{P}}$ span $\mathrm{Ker}(\mathscr{P}\Xi(\mathscr{X}))$. Thus, having set

$$\Phi(\mathscr{X}) = \begin{pmatrix} \mathbf{A}_0 \mathscr{X}^{-1} + \mathscr{X}^{-1} \mathbf{A}_0^{\mathrm{T}} & \mathbf{B}_0 & \mathscr{X}^{-1} \mathbf{C}_0^{\mathrm{T}} \\ \mathbf{B}_0^{\mathrm{T}} & -\gamma I & D_{11}^{\mathrm{T}} \\ \mathbf{C}_0 \mathscr{X}^{-1} & D_{11} & -\gamma I \end{pmatrix}, \tag{3.60}$$

the second of (3.59) can be rewritten as

$$W_{\mathscr{P}}^{\mathrm{T}} [\Xi(\mathscr{X})]^{-1} \Psi(\mathscr{X}) [\Xi(\mathscr{X})]^{-1} W_{\mathscr{P}} = W_{\mathscr{P}}^{\mathrm{T}} \Phi(\mathscr{X}) W_{\mathscr{P}} < 0. \tag{3.61}$$

It could therefore be concluded that (3.56) is solvable for some $\mathbf{K}$ if and only if the matrix $\mathscr{X}$ satisfies the first of (3.59) and (3.61).

In view of this, the following (intermediate) conclusion can be drawn.

Proposition 3.1 *There exists a k-dimensional controller that solves the problem of γ-suboptimal H_∞ feedback design if and only if there exists a $(n+k) \times (n+k)$ symmetric matrix $\mathscr{X} > 0$ satisfying the following set of linear matrix inequalities*

[15] Observe that, since $\mathrm{Ker}(P)$ and $\mathrm{Ker}(Q)$ are subspaces of $\mathbb{R}^m$, then W_P and W_Q are matrices having m rows.

$$\begin{aligned} W_{\mathscr{Q}}^{\mathrm{T}} \Psi(\mathscr{X}) W_{\mathscr{Q}} &< 0 \\ W_{\mathscr{P}}^{\mathrm{T}} \Phi(\mathscr{X}) W_{\mathscr{P}} &< 0, \end{aligned} \tag{3.62}$$

in which $W_{\mathscr{P}}$ is a matrix whose columns span $\mathrm{Ker}(\mathscr{P})$*, $W_{\mathscr{Q}}$ is a matrix whose columns span* $\mathrm{Ker}(\mathscr{Q})$*, and $\Psi(\mathscr{X})$ and $\Phi(\mathscr{X})$ are the matrices defined in (3.54) and (3.60). For any of such $\mathscr{X}$'s, a solution* $\mathbf{K}$ *of the resulting linear matrix inequality (3.56) exists and provides a solution of the problem of γ-suboptimal H_∞ feedback design.*

The two inequalities (3.62) thus found can be further simplified. To this end, it is convenient to partition $\mathscr{X}$ and $\mathscr{X}^{-1}$ as

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & * \end{pmatrix}, \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix} \tag{3.63}$$

in which R and S are $n \times n$ and N and M are $n \times k$. With the partition (3.63), the matrix (3.54) becomes

$$\Psi(\mathscr{X}) = \begin{pmatrix} A^{\mathrm{T}} S + SA & A^{\mathrm{T}} N & SB_1 & C_1^{\mathrm{T}} \\ N^{\mathrm{T}} A & 0 & N^{\mathrm{T}} B_1 & 0 \\ B_1^{\mathrm{T}} S & B_1^{\mathrm{T}} N & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & 0 & D_{11} & -\gamma I \end{pmatrix} \tag{3.64}$$

and the matrix (3.60) becomes

$$\Phi(\mathscr{X}) = \begin{pmatrix} AR + RA^{\mathrm{T}} & AM & B_1 & RC_1^{\mathrm{T}} \\ M^{\mathrm{T}} A^{\mathrm{T}} & 0 & 0 & M^{\mathrm{T}} C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 R & C_1 M & D_{11} & -\gamma I \end{pmatrix}. \tag{3.65}$$

From the definition of $\mathscr{Q}$, it is readily seen that a matrix $W_{\mathscr{Q}}$ whose columns span $\mathrm{Ker}(\mathscr{Q})$ can be expressed as

$$W_{\mathscr{Q}} = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \\ Z_2 & 0 \\ 0 & I_{p_1} \end{pmatrix}$$

in which

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\begin{pmatrix} C_2 & D_{21} \end{pmatrix}.$$

Then, an easy calculation shows that the first inequality in (3.62) can be rewritten as

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}S + SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

Likewise, from the definition of $\mathscr{P}$, it is readily seen that a matrix $W_{\mathscr{P}}$ whose columns span $\mathrm{Ker}(\mathscr{P})$ can be expressed as

$$W_{\mathscr{P}} = \begin{pmatrix} V_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ V_2 & 0 \end{pmatrix}$$

in which

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

is a matrix whose columns span

$$\mathrm{Ker}\left(B_2^{\mathrm{T}} \ D_{12}^{\mathrm{T}}\right).$$

Then, an easy calculation shows that the second inequality in (3.62) can be rewritten as

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} AR + RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0.$$

In this way, the two inequalities (3.62) have been transformed into two inequalities involving the submatrix S of $\mathscr{X}$ and the submatrix R of $\mathscr{X}^{-1}$. To complete the analysis, it remains to connect these matrices to each other. This is achieved via the following lemma.

Lemma 3.4 *Let S and R be symmetric $n \times n$ matrices. There exist a $n \times k$ matrix N and $k \times k$ symmetric matrix Z a such that*

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} > 0 \tag{3.66}$$

and

$$\begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix}^{-1} = \begin{pmatrix} R & * \\ * & * \end{pmatrix} \tag{3.67}$$

if and only if

$$\mathrm{rank}(I - SR) \le k \tag{3.68}$$

and

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \geq 0. \tag{3.69}$$

Proof To prove necessity, write

$$\mathscr{X} = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & Z \end{pmatrix} \qquad \text{and} \qquad \mathscr{X}^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix}.$$

Then, by definition we have

$$\begin{aligned} SR + NM^{\mathrm{T}} &= I \\ N^{\mathrm{T}}R + ZM^{\mathrm{T}} &= 0\,. \end{aligned} \tag{3.70}$$

Thus $I - SR = NM^{\mathrm{T}}$ and this implies (3.68), because N has k columns. Now, set

$$\mathscr{T} = \begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

Using (3.70), the first of which implies $MN^{\mathrm{T}} = I - RS$ because S and R are symmetric, observe that

$$\mathscr{T}^{\mathrm{T}} \mathscr{X} \mathscr{T} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Pick $y \in \mathbb{R}^{2n}$ and define $x \in \mathbb{R}^{n+k}$ as $x = \mathscr{T} y$. Then, it is seen that

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y = x^{\mathrm{T}} \mathscr{X} x \geq 0$$

because the matrix $\mathscr{X}$ is positive definite by assumption. This concludes the proof of the necessity.

To prove sufficiency, let $\hat{k} \leq k$ be the rank of $I - SR$ and let $\hat{N}, \hat{M}$ two $n \times \hat{k}$ matrices of rank $\hat{k}$ such that

$$I - SR = \hat{N}\hat{M}^{\mathrm{T}}. \tag{3.71}$$

Using the property (3.71) it is possible to show that the equation

$$\hat{N}^{\mathrm{T}}R + \hat{Z}\hat{M}^{\mathrm{T}} = 0 \tag{3.72}$$

has a solution $\hat{Z}$. In fact, observe that

$$\hat{M}\hat{N}^{\mathrm{T}}R - R\hat{N}\hat{M}^{\mathrm{T}} = (I - RS)R - R(I - SR) = 0.$$

Let L be any matrix such that $\hat{M}^{\mathrm{T}}L = I$ (which exists because the $\hat{k}$ rows of $\hat{M}^{\mathrm{T}}$ are independent) and, from the identity above, obtain

$$\hat{M}\hat{N}^{\mathrm{T}}RL + R\hat{N} = 0.$$

This shows that the matrix $\hat{Z} = (\hat{N}^{\mathrm{T}}RL)^{\mathrm{T}}$ satisfies (3.72). The matrix $\hat{Z}$ is symmetric. In fact, note that

$$0 = \hat{M}(\hat{N}^{\mathrm{T}}R + \hat{Z}\hat{M}^{\mathrm{T}}) = (I - RS)R + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} = R - RSR + \hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$$

and hence $\hat{M}\hat{Z}\hat{M}^{\mathrm{T}}$ is symmetric. This yields

$$0 = \hat{M}\hat{Z}\hat{M}^{\mathrm{T}} - \hat{M}\hat{Z}^{\mathrm{T}}\hat{M}^{\mathrm{T}} = \hat{M}(\hat{Z} - \hat{Z}^{\mathrm{T}})\hat{M}^{\mathrm{T}}$$

from which it is seen that $\hat{Z} = \hat{Z}^{\mathrm{T}}$ because $\hat{M}^{\mathrm{T}}$ has independent rows.

As a consequence of (3.71) and(3.72), the symmetric matrix

$$\hat{\mathscr{X}} = \begin{pmatrix} S & \hat{N} \\ \hat{N}^{\mathrm{T}} & \hat{Z} \end{pmatrix} \tag{3.73}$$

is a solution of

$$\begin{pmatrix} S & I \\ \hat{N}^{\mathrm{T}} & 0 \end{pmatrix} = \hat{\mathscr{X}} \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}. \tag{3.74}$$

The symmetric matrix $\hat{\mathscr{X}}$ thus found is invertible because, otherwise, the independence of the rows of the matrix on the left-hand side of (3.74) would be contradicted. It is easily seen that

$$\hat{\mathscr{X}}^{-1} = \begin{pmatrix} R & * \\ \hat{M}^{\mathrm{T}} & * \end{pmatrix}. \tag{3.75}$$

Moreover, it is also possible to prove that $\hat{\mathscr{X}} > 0$. In fact, letting

$$\hat{\mathscr{T}} = \begin{pmatrix} I & R \\ 0 & \hat{M}^{\mathrm{T}} \end{pmatrix}$$

observe that

$$\hat{\mathscr{T}}^{\mathrm{T}}\hat{\mathscr{X}}\hat{\mathscr{T}} = \begin{pmatrix} S & I \\ I & R \end{pmatrix}.$$

Suppose $x^{\mathrm{T}}\hat{\mathscr{X}}x < 0$ for some $x \neq 0$. Using the fact that the rows of $\hat{\mathscr{T}}$ are independent, find y such that $x = \hat{\mathscr{T}}y$. This would make

$$y^{\mathrm{T}} \begin{pmatrix} S & I \\ I & R \end{pmatrix} y < 0\,,$$

which contradicts (3.69). Thus, $x^{\mathrm{T}}\hat{\mathscr{X}}x \geq 0$ for all x, i.e., $\hat{\mathscr{X}} \geq 0$. But since $\hat{\mathscr{X}}$ is nonsingular, we conclude that $\hat{\mathscr{X}}$ is positive definite.

If $\hat{k} = k$ the sufficiency is proven. Otherwise, set $\ell = k - \hat{k}$ and

$$\mathscr{X} = \begin{pmatrix} S & \hat{N} & 0 \\ \hat{N}^{\mathrm{T}} & \hat{Z} & 0 \\ 0 & 0 & I_{\ell\times\ell} \end{pmatrix},$$

observe that $\mathscr{X}$ is positive definite and that

$$\mathscr{X}^{-1} = \begin{pmatrix} R & * & 0 \\ \hat{M}^{\mathrm{T}} & * & 0 \\ 0 & 0 & I_{\ell\times\ell} \end{pmatrix}$$

has the required structure. $\triangleleft$

Remark 3.3 Note that condition (3.69) implies $S > 0$ and $R > 0$. This is the consequence of the arguments used in the proof of the previous lemma, but can also be proven directly as follows. Condition (3.69) implies that the two diagonal blocks S and R are positive semidefinite. Consider the quadratic form associated with the matrix on left-hand side of (3.69),

$$V(x, z) = \begin{pmatrix} x^{\mathrm{T}} & z^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} S & I \\ I & R \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = x^{\mathrm{T}} S x + z^{\mathrm{T}} R z + 2 z^{\mathrm{T}} x.$$

We prove that R is nonsingular (and thus positive definite). By contradiction, suppose this is not the case. Then, there is a vector $\bar{z} \neq 0$ such that $\bar{z}^{\mathrm{T}} R \bar{z} = 0$. Pick any vector $\bar{x}$ such that $\bar{z}^{\mathrm{T}} \bar{x} \neq 0$. Then, for any $c \in \mathbb{R}$,

$$V(\bar{x}, c\bar{z}) \leq \lambda_{\max}(S) \|\bar{x}\|^2 + 2c\bar{z}^{\mathrm{T}} \bar{x}.$$

Clearly, by choosing an appropriate c, the right hand side can be made strictly negative. Thus, $V(x, z)$ is not positive semidefinite and this contradicts (3.69). The same arguments are used to show that also S is nonsingular. $\triangleleft$

This lemma establishes a *coupling condition* between the two submatrices S and R identified in the previous analysis that determines the positivity of the matrix $\mathscr{X}$. Using this lemma it is therefore possible to arrive at the following conclusion.

Theorem 3.3 *Consider a plant of modeled by equations of the form (3.48). Let V_1, V_2, Z_1, Z_2 be matrices such that*

$$\operatorname{Im} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} C_2 & D_{21} \end{pmatrix}, \qquad \operatorname{Im} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} \end{pmatrix}.$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S and R satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_1^{\mathrm{T}} & Z_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}S + SA & SB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.76}$$

$$\begin{pmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} AR + RA^{\mathrm{T}} & RC_1^{\mathrm{T}} & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{3.77}$$

$$\begin{pmatrix} S & I \\ I & R \end{pmatrix} \geq 0. \tag{3.78}$$

In particular, there exists a solution of dimension k if and only if there exist R and S satisfying (3.76)–(3.78) and, in addition,

$$\operatorname{rank}(I - RS) \leq k. \tag{3.79}$$

The result above describes necessary and sufficient conditions for the existence of a controller that solves the problem of γ-suboptimal H_∞ feedback design. For the actual *construction* of a controller, one may proceed as follows. Assuming that S and R are positive definite symmetric matrices satisfying the system of linear matrix inequalities (3.76)–(3.78), construct a matrix $\mathscr{X}$ as indicated in the proof of Lemma 3.4, that is, find two $n \times k$ matrices N and M such that

$$I - SR = NM^{\mathrm{T}}$$

with $k = \operatorname{rank}(I - SR)$, and solve for $\mathscr{X}$ the linear equation

$$\begin{pmatrix} S & I \\ N^{\mathrm{T}} & 0 \end{pmatrix} = \mathscr{X} \begin{pmatrix} I & R \\ 0 & M^{\mathrm{T}} \end{pmatrix}.$$

By construction, the matrix in question is positive definite which satisfies (3.62) and their equivalent versions (3.59). Thus, according to Lemma 3.3, there exists a matrix $\mathbf{K}$ satisfying (3.56). This is a linear matrix inequality in the only unknown $\mathbf{K}$. The solution of such inequality provides the required controller as

$$\mathbf{K} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}.$$

Remark 3.4 It is worth observing that if the problem of γ-suboptimal H_∞ feedback design has *any* solution at all, it does have a solution in which the dimension of the controller, i.e., the dimension of the vector x_c in (3.49), does not exceed n, i.e., the dimension of the vector x in (3.48). This is an immediate consequence of the construction shown above, in view of the fact that the rank of $(I - SR)$ cannot exceed n.

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Chapter 4
Regulation and Tracking in Linear Systems

4.1 The Problem of Asymptotic Tracking and Disturbance Rejection

In the last part of the previous chapter, we have considered a plant modeled as in (3.48), and—regarding y as an output available for *measurement* and u as an input available for *control*—we have studied the problem of finding a controller of the form (3.49) yielding a *stable* closed-loop system having a transfer function—between input v and output z—whose H_∞ norm does not exceed a given number γ. The problem in question was motivated by the interest in solving a problem of robust stability, but it has an independent interest *per se*. In fact, regarding v as a vector of external *disturbances*, affecting the behavior of the controlled plant, and z as set of variables of special interest, the problem considered in Sect. 3.6 can be regarded as the problem of finding a controller that—while keeping the closed-loop stable—enforces a prescribed *attenuation* of the effect of the disturbance v on the variable of interest z, the attenuation being expressed in terms of the H_∞ norm (or of the $\mathscr{L}_2$ gain, if desired) of the resulting system.

In the present chapter, we continue to study a problem of this kind, i.e., the control of a plant affected by external disturbances, from which certain variables of interest have to be "protected," but with some differences. Specifically, while in Sect. 3.6 we have considered the case in which the influence of the disturbances on the variables of interest had to be attenuated by a given factor, we consider in this section the case in which the influence of the disturbances on the variables of interest should ultimately vanish. This is indeed a much stronger requirement and it is unlikely that it might be enforced in general. It can be enforced, though, if the disturbances happen to belong to a specific (well-defined) family of signals. This gives rise to a specific setup, known as *problem of asymptotic disturbance rejection and/or tracking*, or more commonly *problem of output regulation*, that will be explained in more detail hereafter.

For consistency with the notations currently used in the literature dealing with the specific problem addressed in this chapter, the controlled plant (compare with

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_4

(3.48)) is assumed to be modeled by equations written in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w \\ y &= Cx + Qw. \end{aligned} \tag{4.1}$$

The first equation of this system describes a *plant* with *state* $x \in \mathbb{R}^n$ and *control input* $u \in \mathbb{R}^m$, subject to a set of *exogenous input* variables $w \in \mathbb{R}^{n_w}$ which includes *disturbances* (to be rejected) and/or *references* (to be tracked). The second equation defines a set of *regulated* (or *error*) variables $e \in \mathbb{R}^p$, which are expressed as a linear combination of the plant state x and of the exogenous input w.[1] The third equation defines a set of *measured* variables $y \in \mathbb{R}^q$, which are assumed to be available for feedback, and are also expressed as a linear combination of the plant state x and of the exogenous input w.

The control action to (4.1) is to be provided by a feedback *controller* which processes the measured information y and generates the appropriate control input. In general, the controller in question is a system modeled by equations of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}y \\ u &= C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y, \end{aligned} \tag{4.2}$$

with state $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, which yields a closed-loop system modeled by equations of the form

$$\begin{aligned} \dot{x} &= (A + BD_{\mathrm{c}}C)x + BC_{\mathrm{c}}x_{\mathrm{c}} + (P + BD_{\mathrm{c}}Q)w \\ \dot{x}_{\mathrm{c}} &= B_{\mathrm{c}}Cx + A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}Qw \\ e &= C_{\mathrm{e}}x + Q_{\mathrm{e}}w. \end{aligned} \tag{4.3}$$

This system is seen as a system with *input* w and *output* e. The purpose of the control is to make sure that the closed-loop system be *asymptotically stable* and that the regulated variable e, viewed as a function of time, *asymptotically decays to zero* as time tends to ∞, for every possible initial state and for every possible *exogenous input* in a prescribed family of functions of time. This requirement is also known as property of *output regulation*. For the sake of mathematical simplicity, and also because in this way a large number of relevant practical situations can be covered, it is assumed that the family of the exogenous inputs $w(\cdot)$ which affect the plant, and for which asymptotic decay of the regulated variable is to be achieved, is the family of all functions of time which is the solution of a homogeneous linear differential equation

$$\dot{w} = Sw \tag{4.4}$$

[1] If some components of w are (external) commands that certain variables of interest are required to track, then some of the components of e can be seen as *tracking errors*, that is differences between the actual values of those variables of interest and their expected reference values. Overall, the components of e can simply be seen as variables on which the effect of w is expected to vanish asymptotically.

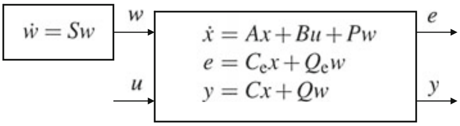

Fig. 4.1 The controlled plant and the exosystem

with state $w \in \mathbb{R}^{n_w}$, for all possible initial conditions $w(0)$. This system, which is viewed as a mathematical model of a "generator" for all possible exogenous input functions, is called the *exosystem* (Fig. 4.1). In this chapter, we will discuss this design problem at various levels of generality.

Note that, without loss of generality, in the analysis of this problem it can be assumed that all the eigenvalues of S have nonnegative real part. In fact, if there were eigenvalues having negative real part, system (4.4) could be split (after a similarity transformation) as

$$\begin{aligned} \dot{w}_s &= S_s w_s \\ \dot{w}_u &= S_u w_u \end{aligned}$$

with S_s having *all* eigenvalues in $\mathbb{C}^-$ and S_u having *no* eigenvalue in $\mathbb{C}^-$. The input $w_s(t)$ generated by the upper subsystem is asymptotically vanishing. Thus, once system (4.3) has been rendered stable, the influence of $w_s(t)$ on the regulated variable $e(t)$ asymptotically vanishes as well. In other words, the only exogenous inputs that matter in the problem under consideration are those generated by the lower subsystem.

This being said, it must be observed—though—that if some of the eigenvalues of S have *positive* real part, there will be initial conditions from which the exosystem (4.4) generates signals that grow unbounded as time increases. Thus, also the associated response of the state of (4.3) can grow unbounded as time increases and this is not a reasonable setting. In view of all of this, in the analysis which follows we will proceed under the assumption that all eigenvalues of S have zero real part (and are simple roots of its minimal polynomial), even though all results that will be presented are also valid in the more general setting in which S has eigenvalues with positive real part (and/or has eigenvalues with zero real part which are multiple roots of its minimal polynomial).

4.2 The Case of Full Information and Francis' Equations

For expository reasons we consider first the (nonrealistic) case in which the full state x of the plant and the full state w of the exosystem are available for measurement, i.e., the case in which the measured variable y in (4.1) is $y = \mathrm{col}(x, w)$. This is called the case of "full information". We also assume that all system parameters are known exactly.

In this setup, we consider the special version of the controller (4.2) in which $u = D_c y$, that we rewrite for convenience as

$$u = Fx + Lw. \tag{4.5}$$

Proposition 4.1 *The problem of output regulation in the case of full information has a solution if and only if*

(i) the matrix pair (A, B) *is stabilizable*
(ii) there exists a solution pair (Π, Ψ) *of the linear matrix equations*

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C_e\Pi + Q_e. \end{aligned} \tag{4.6}$$

Proof (Necessity) System (4.1) controlled by (4.5) can be regarded as an autonomous linear system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + BL & A + BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}. \tag{4.7}$$

Suppose the pair of matrices F, L is such the controller (4.5) solves the problem of output regulation. Then, the matrix $A + BF$ has all eigenvalues with negative real part and, hence, the matrix pair (A, B) *must be stabilizable*.

The eigenvalues of system (4.7) are those of $(A + BF)$ and those of S. The former are all in $\mathbb{C}^-$ while none of the latter, in view of the standing assumption described at the end of the previous section, is in $\mathbb{C}^-$. Hence this system possesses a *stable* eigenspace and a *center* eigenspace. The former can be described as

$$\mathscr{V}^s = \mathrm{Im} \begin{pmatrix} 0 \\ I \end{pmatrix},$$

while the latter, which is complementary to the stable eigenspace, can be described as

$$\mathscr{V}^c = \mathrm{Im} \begin{pmatrix} I \\ \Pi \end{pmatrix},$$

in which Π is a solution of the Sylvester equation

$$\Pi S = (A + BF)\Pi + (P + BL). \tag{4.8}$$

Setting

$$\Psi = L + F\Pi$$

we deduce that the pair Π, Ψ satisfies the first equation of (4.6).

Any trajectory of (4.7) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, on the contrary, is persistent: it is the *steady-state component* of the trajectory. We denote it as

$$x_{ss}(t) = \Pi w(t).$$

As a consequence, the steady-state component of the regulated output $e(t)$ is

$$e_{ss}(t) = (C_e \Pi + Q_e) w(t).$$

If the controller (4.5) solves the problem of output regulation, we must have $e_{ss}(t) = 0$ and this can only occur if Π satisfies $C_e \Pi + Q_e = 0$. In terms of state trajectories, this is equivalent to say that the steady-state component of any state trajectory *must be contained in the kernel of the map* $e = Q_e w + C_e x$. We have shown in this way that the solution Π of the Sylvester equation (4.8) necessarily satisfies the second equation of (4.6). In summary, if the pair of matrices F, L is such that the controller (4.5) solves the problem of output regulation, Eq. (4.6) *must have a solution* pair (Π, Ψ).

[Sufficiency] Suppose that (A, B) is stabilizable and pick F such that the eigenvalues of $A + BF$ have negative real part. Suppose Eq. (4.6) have a solution (Π, Ψ) and pick L as

$$L = \Psi - F\Pi,$$

that is, consider a control law of the form

$$u = \Psi w + F(x - \Pi w).$$

The corresponding closed-loop system is

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + B(\Psi - F\Pi) & A + BF \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}.$$

Consider the change of variables

$$\tilde{x} = x - \Pi w$$

and, bearing in mind the first equation of (4.6) and the choice of L, observe that, in the new coordinates,

$$\dot{\tilde{x}} = (A + BF)\tilde{x}.$$

Hence $\lim_{t\to\infty} \tilde{x}(t) = 0$, because the eigenvalues of $A + BF$ have negative real part.

In the new coordinates,

$$e(t) = C_e \tilde{x}(t) + (C_e \Pi + Q_e) w(t) = C_e \tilde{x}(t).$$

Therefore, using the second equation of (4.6), we conclude that

$$\lim_{t\to\infty} e(t) = 0$$

and we see that the proposed law solves the problem. ◃

Equation (4.6) are called the *linear regulator equations* or, also, the *Francis's equations*.[2] The following result is useful to determine the existence of solutions.[3]

Lemma 4.1 *The Francis' equation (4.6) have a solution for any* (P, Q_e) *if and only if*

$$\text{rank}\begin{pmatrix} A-\lambda I & B \\ C_e & 0 \end{pmatrix} = \#\ \text{rows} \qquad \forall \lambda \in \sigma(S). \tag{4.9}$$

If this is the case and the matrix on the left-hand side of (4.9) is square (i.e., the control input u and the regulated output e have the same number of components), the solution (Π, Ψ) *is unique.*

The condition (4.9) is usually referred to as the *nonresonance condition*. Note that the condition is necessary and sufficient if the existence of a solution *for any* (P, Q_e) is sought. Otherwise, it is simply a sufficient condition. Note also that such condition requires $m \geq p$, i.e., that the number of components of the control input u be larger than or equal to the number of components of the regulated output.

Example 4.1 As an example of what the nonresonance condition (4.9) means and why it plays a crucial role in the solution of (4.6), consider the case in which $\dim(e) = \dim(u) = 1$. Observe that the first two equations of (4.1), together with (4.4), can be written in the form of a composite system with input u and output e as

$$\begin{aligned} \begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} &= \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}\begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \\ e &= \begin{pmatrix} Q_e & C_e \end{pmatrix}\begin{pmatrix} w \\ x \end{pmatrix}. \end{aligned} \tag{4.10}$$

Bearing in mind the conditions (2.2) used to identify the value of the relative degree of a system, let r be such that $C_e A^k B = 0$ for all $k < r-1$ and $C_e A^{r-1} B \neq 0$. A simple calculation shows that, for any $k \geq 0$,

$$\begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^k \begin{pmatrix} 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ A^k B \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} Q_e & C_e \end{pmatrix}\begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^k \begin{pmatrix} 0 \\ B \end{pmatrix} = 0$$

for $k = 1, \ldots, r-2$ and

[2] See [1].

[3] See Appendix A.2.

$$\begin{pmatrix} Q_e & C_e \end{pmatrix} \begin{pmatrix} S & 0 \\ P & A \end{pmatrix}^{r-1} \begin{pmatrix} 0 \\ B \end{pmatrix} = C_e A^{r-1} B.$$

Thus, system (4.10), viewed as system with input u and output e has relative degree r and can be brought to (strict) normal form, by means of a suitable change of variables. In order to identify such change of variables, let

$$\xi_1 = Q_e w + C_e x$$

and define recursively $\xi_2, \ldots, \xi_r$ so that

$$\xi_{i+1} = \dot{\xi}_i$$

for $i = 1, \ldots, r-1$, as expected. It is easy to see that ξ_i can be given an expression of the form

$$\xi_i = R_i w + C_e A^{i-1} x.$$

in which R_i is a suitable matrix. This is indeed the case for $i = 1$ if we set $R_1 = Q_e$. Assuming that it is the case for a generic i, it is immediate to check that

$$\begin{aligned} \xi_{i+1} &= \dot{\xi}_i = R_i \dot{w} + C_e A^{i-1} \dot{x} \\ &= R_i S w + C_e A^{i-1}(Ax + Bu + Pw) = (R_i S + C_e A^{i-1} P) w + C_e A^{i-1} x, \end{aligned}$$

which has the required form with $R_{i+1} = (R_i S + C_e A^{i-1} P)$.

Then, set (compare with (2.4))

$$T_1 = \begin{pmatrix} C_e \\ C_e A \\ \cdots \\ C_e A^{r-1} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 \\ R_2 \\ \cdots \\ R_r \end{pmatrix},$$

let T_0 be a matrix, satisfying $T_0 B = 0$, such that (compare with (2.5))

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}$$

is nonsingular and define the new variables as

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0 x \\ Rw + T_1 x \end{pmatrix}$$

(the variable w is left unchanged).

This being the case, it is readily seen that the system, in the new coordinates, reads as

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{z} &= T_0Ax + T_0Pw \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= C_eA^rx + bu + (R_rS + C_eA^{r-1}P)w \\
e &= \xi_1,
\end{aligned}$$

in which

$$b = C_eA^{r-1}B \neq 0.$$

To complete the transformation, it remains to replace x by its expression in terms of z, ξ, w. Proceeding as in Sect. 3.1, let M_0 and M_1 be partitions of the inverse of T and observe that

$$x = M_0z + M_1\xi - M_1Rw.$$

Then, it can be concluded that the (strict) normal form of such system has the following expression

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{z} &= A_{00}z + A_{01}\xi + P_0w \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= A_{10}z + A_{11}\xi + bu + P_1w \\
e &= \xi_1,
\end{aligned} \tag{4.11}$$

in which the four matrices $A_{00}, A_{01}, A_{10}, A_{11}$ are *precisely* the matrices that characterize the (strict) normal form of a system defined as

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
e &= C_ex,
\end{aligned}$$

and P_0, P_1 are suitable matrices. Hence, as shown in Sect. 2.1, the matrix A_{00} is a matrix whose eigenvalues coincide with the zeros of the transfer function

$$T_e(s) = C_e(sI - A)^{-1}B. \tag{4.12}$$

The normal form (4.11) can be used to determine a solution of Francis' equations. To this end, let Π be partitioned as

$$\Pi = \begin{pmatrix} \Pi_0 \\ \Pi_1 \end{pmatrix}$$

consistently with the partition of $\tilde{x}$, and let $\pi_{1,i}$ denote the ith row of Π_1.

Then, it is immediate to check that Francis's equations are rewritten as

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + A_{01}\Pi_1 + P_0 \\
\pi_{1,1} S &= \pi_{1,2} \\
&\cdots \\
\pi_{1,r-1} S &= \pi_{1,r} \\
\pi_{1,r} S &= A_{10}\Pi_0 + A_{11}\Pi_1 + b\Psi + P_1 \\
0 &= \pi_{1,1}.
\end{aligned}
\tag{4.13}
$$

The last equation, along with the second, third, and rth, yield $\pi_{1,i} = 0$ for all i, i.e.,

$$\Pi_1 = 0.$$

As a consequence, the remaining equations reduce to

$$
\begin{aligned}
\Pi_0 S &= A_{00}\Pi_0 + P_0 \\
0 &= A_{10}\Pi_0 + b\Psi + P_1.
\end{aligned}
\tag{4.14}
$$

The first of these equations is a Sylvester equation in Π_0, that has a unique solution if and only if the none of the eigenvalues of S is an eigenvalue of the matrix A_{00}. Since the eigenvalues of A_{00} are the zeros of (4.12), this equation has a unique solution if and only if none of the eigenvalues of S is a zero of $T_e(s)$. Entering the solution Π_0 of this equation into the second one yields an equation that, since $b \neq 0$, can be solved for Ψ, yielding

$$\Psi = \frac{-1}{b}[A_{10}\Pi_0 + P_1].$$

In summary, it can be concluded that Francis' equations have a unique solution if and only if *none of the eigenvalues of S coincides with a zero of the transfer function* $T_e(s)$.

Of course the condition thus found must be consistent with the condition resulting from Lemma 4.1, specialized to the present context in which $m = p = 1$. In this case, the condition (4.9) becomes

$$\det\begin{pmatrix} A - \lambda I & B \\ C_e & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S).$$

Now, it is known (see Sect. 2.1) that the roots of the polynomial

$$\det\begin{pmatrix} A - \lambda I & B \\ C_e & 0 \end{pmatrix}$$

coincide with the zeros of the transfer function $T_e(s)$. Thus, the condition indicated in Lemma 4.1 is the condition that none of the eigenvalues of S is a zero of $T_e(s)$, which is precisely the condition obtained from the construction above. ◁

4.3 The Case of Measurement Feedback: Steady-State Analysis

Consider now the case in which the feedback law is provided by a controller that does not have access to the full state x of the plant and the full state w of the exosystem. We assume that the controller has access to the regulated output e and, possibly, to an additional *supplementary* set of independent measured variables. In other words, we assume that the vector

$$y = Cx + Qw$$

of *measured outputs* of the plant can be split into two parts as in

$$y = \begin{pmatrix} e \\ y_r \end{pmatrix} = \begin{pmatrix} C_e \\ C_r \end{pmatrix} x + \begin{pmatrix} Q_e \\ Q_r \end{pmatrix} w$$

in which e is the *regulated* output and $y_r \in \mathbb{R}^{p_r}$. As opposite to the case considered in the previous section, this is usually referred to as the case of "measurement feedback." In this setup the controlled plant, together with the exosystem, is modeled by equations of the form

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= Ax + Bu + Pw \\ e &= C_e x + Q_e w \\ y_r &= C_r x + Q_r w. \end{aligned} \tag{4.15}$$

The control is provided by a generic dynamical system with input y and output u, modeled as in (4.2). Proceeding as in the first part of the proof of Proposition 4.1, it is easy to deduce the following *necessary* conditions for the solution of the problem of output regulation.

Proposition 4.2 *The problem of output regulation in the case of measurement feedback has a solution* only if

(i) the triplet $\{A, B, C\}$ *is stabilizable and detectable*
(ii) there exists a solution pair (Π, Ψ) *of the Francis' equation (4.6).*

Proof System (4.1) controlled by (4.2) can be regarded as an autonomous linear system (see Fig. 4.2)

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} S & 0 & 0 \\ P + BD_cQ & A + BD_cC & BC_c \\ B_cQ & B_cC & A_c \end{pmatrix} \begin{pmatrix} w \\ x \\ x_c \end{pmatrix}. \tag{4.16}$$

If a controller of the form (4.2) solves the problem of output regulation, all the eigenvalues of the matrix

$$\begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}$$

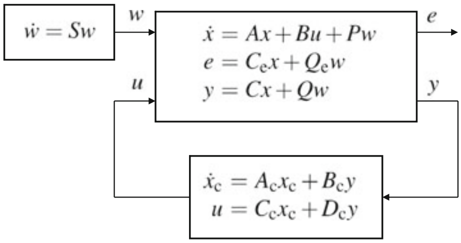

Fig. 4.2 The closed-loop system, augmented with the exosystem

have negative real part. Hence, the triplet $\{A, B, C\}$ *must be stabilizable and detectable*.

Since by assumption S has all eigenvalues on the imaginary axis, system (4.16) possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$\mathscr{V}^c = \mathrm{I} \begin{pmatrix} I \\ \Pi \\ \Pi_c \end{pmatrix}$$

in which the pair (Π, Π_c) is a solution of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_cQ \\ B_cQ \end{pmatrix}. \tag{4.17}$$

Setting

$$\Psi = D_cC\Pi + C_c\Pi_c + D_cQ$$

it is observed that the pair (Π, Ψ) satisfies the first equation of (4.6).

Any trajectory of (4.16) has a unique decomposition into a component entirely contained in the stable eigenspace and a component entirely contained in the center eigenspace. The former, which asymptotically decays to 0 as $t \to \infty$, is the *transient component* of the trajectory. The latter, in which $x(t)$ and $x_c(t)$ have, respectively, the form

$$x_{ss}(t) = \Pi w(t) \qquad x_{c,ss}(t) = \Pi_c w(t), \tag{4.18}$$

is the *steady-state component* of the trajectory.

If the controller (4.2) solves the problem of output regulation, the steady-state component of any trajectory must be contained in the kernel of the map $e = Q_e w + C_e x$ and hence the solution (Π, Π_c) of the Sylvester equation (4.17) necessarily satisfies the second equation of (4.6). This shows that if the controller (4.2) solves the problem of output regulation, Eq. (4.6) necessarily have a solution pair (Π, Ψ). $\triangleleft$

In order to better understand the steady-state behavior of the closed-loop system (4.16), it is convenient to split B_c and D_c consistently with the partition adopted for y, as

$$B_c = \begin{pmatrix} B_{ce} & B_{cr} \end{pmatrix} \qquad D_c = \begin{pmatrix} D_{ce} & D_{cr} \end{pmatrix},$$

and rewrite the controller (4.2) as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_{ce} e + B_{cr} y_r \\ u &= C_c x_c + D_{ce} e + D_{cr} y_r . \end{aligned}$$

In these notations, the closed-loop system (4.16) becomes

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} S & 0 & 0 \\ P + BD_{ce}Q_e + BD_{cr}Q_r & A + BD_{ce}C_e + BD_{cr}C_r & BC_c \\ B_{ce}Q_e + B_{cr}Q_r & B_{ce}C_e + B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} w \\ x \\ x_c \end{pmatrix}, \tag{4.19}$$

and the Sylvester equation (4.17) becomes

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_{ce}C_e + BD_{cr}C_r & BC_c \\ B_{ce}C_e + B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_{ce}Q_e + BD_{cr}Q_r \\ B_{ce}Q_e + B_{cr}Q_r \end{pmatrix}.$$

Bearing in mind the fact that, if the controller solves the problem of output regulation, the matrix Π must satisfy the second of (4.6), the equation above reduces to

$$\begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} S = \begin{pmatrix} A + BD_{cr}C_r & BC_c \\ B_{cr}C_r & A_c \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_c \end{pmatrix} + \begin{pmatrix} P + BD_{cr}Q_r \\ B_{cr}Q_r \end{pmatrix},$$

and we observe that, in particular, the matrix Π_c satisfies

$$\begin{aligned} \Pi_c S &= A_c \Pi_c + B_{cr}(C_r \Pi + Q_r) \\ \Psi &= C_c \Pi_c + D_{cr}(C_r \Pi + Q_r), \end{aligned} \tag{4.20}$$

in which (Π, Ψ) is the solution pair of (4.6).

Equation (4.20), from a general viewpoint, could be regarded as a constraint on the solution (Π, Ψ) of (4.6). These equations interpret the ability, of the controller, to generate the *feedforward input* necessary to keep the regulated variable identically zero in steady state. In steady state, the state $x(t)$ of the plant evolves (see (4.18)) as

$$x_{ss}(t) = \Pi w(t),$$

the regulated output $e(t)$ is identically zero, because

$$e_{ss}(t) = (C_e \Pi + Q_e) w(t) = 0, \tag{4.21}$$

the additional measured output $y_r(t)$ evolves as

$$y_{r,ss}(t) = (C_r \Pi + Q_r) w(t) \tag{4.22}$$

and the state $x_c(t)$ of the controller evolves (see again (4.18)) as

$$x_{c,ss}(t) = \Pi_c w(t).$$

The first equation of (4.20) expresses precisely the property that $\Pi_c w(t)$ is the steady-state response of the controller, when the latter is forced by a steady-state input of the form (4.21)–(4.22). The second equation of (4.20), in turn, shows that the output of the controller, in steady state, is a function of the form

$$\begin{aligned} u_{ss}(t) &= C_c x_{c,ss}(t) + D_{ce} e_{ss}(t) + D_{cr} y_{r,ss}(t) \\ &= C_c \Pi_c w(t) + D_{cr}(C_r \Pi + Q_r) w(t) = \Psi w(t). \end{aligned}$$

The latter, as predicted by Francis' equations, is a control able to force in the controlled plant a steady-state trajectory of the form $x_{ss}(t) = \Pi w(t)$ and consequently to keep $e_{ss}(t)$ identically zero. The property thus described is usually referred to as the *internal model property*: any controller that solves the problem of output regulation necessarily embeds a model of the feedforward inputs needed to keep $e(t)$ identically zero.

4.4 The Case of Measurement Feedback: Construction of a Controller

The possibility of constructing a controller that solves the problem in the case of measurement feedback reposes on the following preliminary result. Let

$$\psi(\lambda) = s_0 + s_1\lambda + \cdots + s_{d-1}\lambda^{d-1} + \lambda^d$$

denote the *minimal* polynomial of S and set

$$\Phi = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -s_0 I & -s_1 I & -s_2 I & \cdots & -s_{d-1} I \end{pmatrix}, \tag{4.23}$$

in which all blocks are $p \times p$. Set also

$$G = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \end{pmatrix}^{\mathrm{T}} \tag{4.24}$$

in which all blocks are $p \times p$.

Note that there exists a nonsingular matrix T such that

$$T\Phi T^{-1} = \begin{pmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & S_0 \end{pmatrix}, \qquad TG = \begin{pmatrix} G_0 & 0 & \cdots & 0 \\ 0 & G_0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & G_0 \end{pmatrix} \tag{4.25}$$

in which S_0 is the $d \times d$ matrix

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0 & -s_1 & -s_2 & \cdots & -s_{d-1} \end{pmatrix}. \tag{4.26}$$

and

$$G_0 = \begin{pmatrix} 0 \; 0 \cdots 0 \; 1 \end{pmatrix}^{\mathrm{T}}.$$

From this, we see in particular that $\psi(\lambda)$ is also the minimal polynomial of Φ. As a consequence

$$\begin{aligned} s_0 I + s_1 S + \cdots + s_{d-1} S^{d-1} + S^d &= 0 \\ s_0 I + s_1 \Phi + \cdots + s_{d-1} \Phi^{d-1} + \Phi^d &= 0. \end{aligned} \tag{4.27}$$

With Φ and G constructed in this way, consider now the system obtained by letting the regulated output of (4.1) drive a postprocessor characterized by the equations

$$\dot{\eta} = \Phi\eta + Ge. \tag{4.28}$$

Since the minimal polynomial of the matrix Φ in (4.28) coincides with the minimal polynomial of the matrix S that characterizes the exosystem, system (4.28) is usually called an *internal model* (of the exosystem). Note that, in the coordinates $\tilde{\eta} = T\eta$, with T such that (4.25) hold, system (4.28) can be seen as a bench of p identical subsystems of the form

$$\dot{\tilde{\eta}}_i = S_0 \tilde{\eta}_i + G_0 e_i$$

each one driven by the ith component e_i of e.

In what follows, we are going to show that the problem of output regulation can be solved by means of a controller having the following structure[4]

$$\begin{aligned} \dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_{\mathrm{s}}\xi + B_{\mathrm{s}}y + J_{\mathrm{s}}\eta \\ u &= C_{\mathrm{s}}\xi + D_{\mathrm{s}}y + H_{\mathrm{s}}\eta \end{aligned} \tag{4.29}$$

in which $A_{\mathrm{s}}, B_{\mathrm{s}}, C_{\mathrm{s}}, D_{\mathrm{s}}, J_{\mathrm{s}}, H_{\mathrm{s}}$, are suitable matrices (see Fig. 4.3). This controller consists of the postprocessor (4.28) whose state η drives, along with the full measured

[4]The arguments uses hereafter are essentially the same as those used in [2–4].

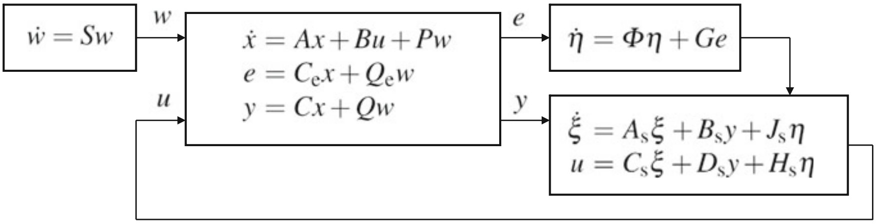

Fig. 4.3 The plant, augmented with the exosystem, controlled by (4.29)

output y, the system

$$\begin{aligned}\dot{\xi} &= A_s\xi + B_s y + J_s\eta\\ u &= C_s\xi + D_s y + H_s\eta,\end{aligned}$$

and can be seen as a system of the general form (4.2) if we set

$$x_c = \begin{pmatrix}\eta\\ \xi\end{pmatrix} \qquad A_c = \begin{pmatrix}\Phi & 0\\ J_s & A_s\end{pmatrix} \qquad B_c = \begin{pmatrix}(G\ 0)\\ B_s\end{pmatrix}$$

$$C_c = \begin{pmatrix}H_s & C_s\end{pmatrix} \qquad D_c = D_s.$$

The first step in proving that a controller of the form (4.29) can solve the problem of output regulation consists in the analysis of the properties of stabilizability and detectability of a system—with state (x, η), input u and output y_a—defined as follows

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{\eta}\end{pmatrix} &= \begin{pmatrix}A & 0\\ GC_e & \Phi\end{pmatrix}\begin{pmatrix}x\\ \eta\end{pmatrix} + \begin{pmatrix}B\\ 0\end{pmatrix}u\\ y_a &= \begin{pmatrix}C & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}x\\ \eta\end{pmatrix},\end{aligned} \tag{4.30}$$

that will be referred to as the *augmented* system. As a matter of fact, it turns out that the system in question has the following important property.

Lemma 4.2 *The augmented system (4.30) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C\}$ is stabilizable and detectable
(ii) the nonresonance condition

$$\text{rank}\begin{pmatrix}A - \lambda I & B\\ C_e & 0\end{pmatrix} = n + p \qquad \forall\lambda \in \sigma(S). \tag{4.31}$$

holds.

Proof To check detectability we look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A-\lambda I & 0 \\ GC_{\mathrm{e}} & \Phi-\lambda I \\ C & 0 \\ 0 & I \end{pmatrix}$$

for any λ having nonnegative real part. Taking linear combinations of rows (and using the fact that the rows of C_{e} are part of the rows of C), this matrix can be easily reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 \\ 0 & 0 \\ C & 0 \\ 0 & I \end{pmatrix}$$

from which it is seen that the columns are linearly independent if and only if so are those of the submatrix

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}.$$

Hence, we conclude that system (4.30) is detectable if and only if so is pair A, C.

To check stabilizability, let Φ and G be defined as above, and look at the linear independence of the rows of the matrix

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & -\lambda I & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1}+\lambda)I & 0 \end{pmatrix}$$

for any λ having nonnegative real part. Taking linear combinations of columns, this matrix is initially reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & * & * & \cdots & * & 0 \end{pmatrix}$$

in which $\psi(\lambda)$ is the minimal polynomial of S. Then, taking linear combinations of rows, this matrix is reduced to a matrix of the form

$$\begin{pmatrix} A-\lambda I & 0 & 0\ 0 \cdots 0\ B \\ 0 & 0 & I\ 0 \cdots 0\ 0 \\ 0 & 0 & 0\ I \cdots 0\ 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0\ 0 \cdots I\ 0 \\ C_{\mathrm{e}} & -\psi(\lambda)I & 0\ 0 \cdots 0\ 0 \end{pmatrix}.$$

After permutation of rows and columns, one finally obtains a matrix of the form

$$\begin{pmatrix} A-\lambda I & B & 0 & 0 \\ C_{\mathrm{e}} & 0 & -\psi(\lambda)I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

(where the lower right block is an identity matrix of dimension $(d-1)p$). If λ is not an eigenvalue of S, $\psi(\lambda) \neq 0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix}.$$

On the contrary, if λ is an eigenvalue of S, $\psi(\lambda) = 0$ and the rows are independent if and only if so are those of

$$\begin{pmatrix} A-\lambda I & B \\ C_{\mathrm{e}} & 0 \end{pmatrix}.$$

Thus, it is concluded that system (4.30) is stabilizable if and only if so is the pair (A, B) and the *nonresonance* condition (4.31) holds. ◁

If the assumptions of this lemma hold, the system (4.30) is stabilizable by means of a (dynamic) feedback. In other words, there exists matrices $A_{\mathrm{s}}, B_{\mathrm{s}}, C_{\mathrm{s}}, D_{\mathrm{s}}, J_{\mathrm{s}}, H_{\mathrm{s}}$, such that the closed-loop system obtained controlling (4.30) by means of a system of the form

$$\begin{aligned} \dot{\xi} &= A_{\mathrm{s}}\xi + B_{\mathrm{s}}y'_{\mathrm{a}} + J_{\mathrm{s}}y''_{\mathrm{a}} \\ u &= C_{\mathrm{s}}\xi + D_{\mathrm{s}}y'_{\mathrm{a}} + H_{\mathrm{s}}y''_{\mathrm{a}}, \end{aligned} \tag{4.32}$$

in which y'_{a} and y''_{a} are the upper and—respectively—lower blocks of the output y_{a} of (4.30), is stable. In what follows system (4.32) will be referred to as a *stabilizer*.

Remark 4.1 A simple expression of such stabilizer can be found in this way. By Lemma 4.2 the augmented system (4.30) is stabilizable. Hence, there exist matrices $L \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times pd}$ such that the system

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \left(\begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} L & M \end{pmatrix} \right) \begin{pmatrix} x \\ \eta \end{pmatrix}$$

is stable. Moreover, since the pair (A, C) is detectable, there is a matrix N such that $(A - NC)$ has all eigenvalues in $\mathbb{C}^-$. Let the augmented system (4.30) be controlled by

$$
\begin{aligned}
\dot{\xi} &= A\xi + N(y'_{\mathrm{a}} - C\xi) + B(L\xi + My''_{\mathrm{a}}) \\
u &= L\xi + My''_{\mathrm{a}}.
\end{aligned}
\tag{4.33}
$$

Bearing in mind that $y'_{\mathrm{a}} = Cx$, that $y''_{\mathrm{a}} = \eta$, and using arguments identical to those used in the proof of the sufficiency in Theorem A.4, it is seen that the resulting closed-loop system is stable. ◁

With this result in mind, consider, for the solution of the problem of output regulation, a candidate controller of the form (4.29), in which the matrices $A_{\mathrm{s}}, B_{\mathrm{s}}, C_{\mathrm{s}}, D_{\mathrm{s}}, J_{\mathrm{s}}, H_{\mathrm{s}}$, are chosen in such a way that (4.32) stabilizes system (4.30). This yields a closed-loop system of the form

$$
\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P + BD_{\mathrm{s}}Q & A + BD_{\mathrm{s}}C & BH_{\mathrm{s}} & BC_{\mathrm{s}} \\ GQ_{\mathrm{e}} & GC_{\mathrm{e}} & \Phi & 0 \\ B_{\mathrm{s}}Q & B_{\mathrm{s}}C & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.34}
$$

If the stabilizer (4.32) stabilizes system (4.30), the matrix

$$
\begin{pmatrix} A + BD_{\mathrm{s}}C & BH_{\mathrm{s}} & BC_{\mathrm{s}} \\ GC_{\mathrm{e}} & \Phi & 0 \\ B_{\mathrm{s}}C & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \tag{4.35}
$$

has all eigenvalues with negative real part. Hence, the closed-loop system possesses a *stable eigenspace* and a *center eigenspace*. The latter, in particular, can be expressed as

$$
\mathscr{V}^{\mathrm{c}} = \mathrm{Im} \begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} \tag{4.36}
$$

in which $(\Pi_x, \Pi_\eta, \Pi_\xi)$ is a solution of the Sylvester equation

$$
\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A + BD_{\mathrm{s}}C & BH_{\mathrm{s}} & BC_{\mathrm{s}} \\ GC_{\mathrm{e}} & \Phi & 0 \\ B_{\mathrm{s}}C & J_{\mathrm{s}} & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P + BD_{\mathrm{s}}Q \\ GQ_{\mathrm{e}} \\ B_{\mathrm{s}}Q \end{pmatrix}. \tag{4.37}
$$

In particular, it is seen that Π_η and Π_x satisfy

$$
\Pi_\eta S = \Phi \Pi_\eta + G(C_{\mathrm{e}}\Pi_x + Q_{\mathrm{e}}). \tag{4.38}
$$

This is a *key* property, from which it will be deduced that the proposed controller solves the problem of output regulation. In fact, the following result holds.

Lemma 4.3 *If Φ is the matrix defined in (4.23) and G is the matrix defined in (4.24), Eq. (4.38)* implies

$$C_e \Pi_x + Q_e = 0. \tag{4.39}$$

Proof Let Π_η be partitioned consistently with the partition of Φ, as

$$\Pi_\eta = \begin{pmatrix} \Pi_{\eta,1} \\ \Pi_{\eta,2} \\ \cdots \\ \Pi_{\eta,d} \end{pmatrix}.$$

Bearing in mind the special structure of Φ and G, Eq. (4.38) becomes

$$\begin{aligned}
\Pi_{\eta,1} S &= \Pi_{\eta,2} \\
\Pi_{\eta,2} S &= \Pi_{\eta,3} \\
&\cdots \\
\Pi_{\eta,d-1} S &= \Pi_{\eta,d} \\
\Pi_{\eta,d} S &= -s_0 \Pi_{\eta,1} - s_1 \Pi_{\eta,2} - \cdots - s_{d-1} \Pi_{\eta,d} + C_e \Pi_x + Q_e.
\end{aligned}$$

The first $d-1$ of these yield, for $i = 1, 2, \ldots d$,

$$\Pi_{\eta,i} = \Pi_{\eta,1} S^{i-1},$$

which, replaced in the last one, yield in turn

$$\Pi_{\eta,1} S^d = \Pi_{\eta,1}(-s_0 I - s_1 S - \cdots - s_{d-1} S^{d-1}) + C_e \Pi_x + Q_e. \tag{4.40}$$

By definition, $\psi(\lambda)$ satisfies the first of (4.27). Hence, (4.40) implies (4.39). ◁

We have proven in this way that if the matrix (4.35) has all eigenvalues with negative real part, and the matrices Φ and G have the form (4.23) and (4.24), the center eigenspace of system (4.1) controlled by (4.29), whose expression is given in (4.36), is such that Π_x satisfies (4.39). Since in steady state $x_{ss}(t) = \Pi_x w(t)$, we see that $e_{ss}(t) = C_e \Pi_x w(t) + Q_e w(t) = 0$ and conclude that the proposed controller solves the problem of output regulation. We summarize this result as follows.

Proposition 4.3 *Suppose that*

(i) the triplet $\{A, B, C\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.31) holds.

Then, the problem of output regulation is solvable, in particular by means of a controller of the form (4.29), in which Φ, G have the form (4.23), (4.24) and $A_s, B_s, C_s, D_s, J_s, H_s$ are such that (4.32) stabilizes the augmented plant (4.30).

Remark 4.2 Note, that, setting

$$\Psi = C_s \Pi_\xi + D_s(C \Pi_x + Q) + H_s \Pi_\eta,$$

it is seen that the pair (Π_x, Ψ) is a solution of Francis's equation (4.6). ◁

4.5 Robust Output Regulation

We consider in this section the case in which the plant is affected by *structured uncertainties*, that is the case in which the coefficient matrices that characterize the model of the plant depend on a vector μ of uncertain parameters, as in

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C_{\mathrm{e}}(\mu)x + Q_{\mathrm{e}}(\mu)w \\ y_{\mathrm{r}} &= C_{\mathrm{r}}(\mu)x + Q_{\mathrm{r}}(\mu)w \end{aligned} \tag{4.41}$$

(note that S is assumed to be *independent* of μ). Coherently with the notations adopted before, we set

$$C(\mu) = \begin{pmatrix} C_{\mathrm{e}}(\mu) \\ C_{\mathrm{r}}(\mu) \end{pmatrix}, \qquad Q(\mu) = \begin{pmatrix} Q_{\mathrm{e}}(\mu) \\ Q_{\mathrm{r}}(\mu) \end{pmatrix}.$$

We show in what follows how the results discussed in the previous section can be enhanced to obtain a robust controller. First of all, observe that if a robust controller exists, this controller must solve the problem of output regulation for each of μ. Hence, for each of such values, the necessary conditions for existence of a controller determined in the earlier sections must hold: the triplet $\{A(\mu), B(\mu), C(\mu)\}$ must be stabilizable and detectable and, above all, the μ-dependent Francis equations

$$\begin{aligned} \Pi(\mu)S &= A(\mu)\Pi(\mu) + B(\mu)\Psi(\mu) + P(\mu) \\ 0 \quad &= C_{\mathrm{e}}(\mu)\Pi(\mu) + Q_{\mathrm{e}}(\mu). \end{aligned} \tag{4.42}$$

must have a solution pair $\Pi(\mu)$, $\Psi(\mu)$ (that, in general, we expect to be μ-dependent).

The design method discussed in the previous section was based on the possibility of stabilizing system (4.30) by means of a (dynamic) feedback driven by y_{a}. The existence of such stabilizer was guaranteed by the fulfillment of the assumptions of Lemma 4.2. In the present setting, in which the parameters of the plant depend on a vector μ of uncertain parameters, one might pursue a similar approach, seeking a controller (which should be μ-independent) that stabilizes the augmented system[5]

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A(\mu) & 0 \\ GC_{\mathrm{e}}(\mu) & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix} u \\ y_{\mathrm{a}} &= \begin{pmatrix} C(\mu) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.43}$$

It should be stressed, though, that a result such as that of the Lemma 4.2 cannot be invoked anymore. In fact, the (necessary) assumption that the triplet

[5]It is worth observing that, since by assumption the matrix S is not affected by parameter uncertainties, so is its minimal polynomial $\psi(\lambda)$ and consequently so is the matrix Φ defined in (4.23).

$\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ no longer guarantees the existence of a *robust* stabilizer for (4.43). For example, a stabilizer having the structure (4.33) cannot be used, because the latter presumes a precise knowledge of the matrices A, B, C that characterize the model of the plant, and this is no longer the case when such matrices depend on a vector μ of uncertain parameters.

We will show later in the chapter how (and under what assumptions) such a robust stabilizer may be found. In the preset section, we take this as an hypothesis, i.e., we *suppose* that there exists a dynamical system, modeled as in (4.32), that robustly stabilizes (4.43). To say that the plant (4.43), controlled by (4.32), is robustly stable is the same thing as to say that the matrix

$$\begin{pmatrix} A(\mu) + B(\mu)D_sC(\mu) & B(\mu)H_s & B(\mu)C_s \\ GC_e(\mu) & \Phi & 0 \\ B_sC(\mu) & J_s & A_s \end{pmatrix} \tag{4.44}$$

is Hurwitz for every value of μ.

Let now the uncertain plant (4.41) be controlled by a system of the form

$$\begin{aligned} \dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_s\xi + B_sy + J_s\eta \\ u &= C_s\xi + D_sy + H_s\eta. \end{aligned} \tag{4.45}$$

The resulting closed-loop system is an autonomous linear system having the form

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P(\mu) + B(\mu)D_sQ(\mu) & A + B(\mu)D_sC(\mu) & B(\mu)H_s & B(\mu)C_s \\ GQ_e(\mu) & GC_e(\mu) & \Phi & 0 \\ B_sQ(\mu) & B_sC(\mu) & J_s & A_s \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.46}$$

Since the matrix (4.44) is Hurwitz for every value of μ and S has all eigenvalues on the imaginary axis, the closed-loop system possesses two complementary invariant subspaces: a *stable eigenspace* and a *center eigenspace*. The latter, in particular, has the expression

$$\mathscr{V}^c = \mathrm{Im} \begin{pmatrix} I \\ \Pi_x(\mu) \\ \Pi_\eta(\mu) \\ \Pi_\xi(\mu) \end{pmatrix}$$

in which $(\Pi_x(\mu), \Pi_\eta(\mu), \Pi_\xi(\mu))$ is the (unique) solution of the Sylvester equation (compare with (4.37))

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A(\mu) + B(\mu)D_\mathrm{s}C(\mu) & B(\mu)H_\mathrm{s} & B(\mu)C_\mathrm{s} \\ GC_\mathrm{e}(\mu) & \Phi & 0 \\ B_\mathrm{s}C(\mu) & J_\mathrm{s} & A_\mathrm{s} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P(\mu) + B(\mu)D_\mathrm{s}Q(\mu) \\ GQ_\mathrm{e}(\mu) \\ B_\mathrm{s}Q(\mu) \end{pmatrix}.$$

From this Sylvester equation, one deduces (compare with (4.38)) that

$$\Pi_\eta(\mu)S = \Phi\Pi_\eta(\mu) + G(C_\mathrm{e}(\mu)\Pi_x(\mu) + Q_\mathrm{e}(\mu)).$$

From this, using Lemma 4.3, it is concluded that

$$C_\mathrm{e}(\mu)\Pi_x(\mu) + Q_\mathrm{e}(\mu) = 0.$$

In steady state $x_\mathrm{ss}(t) = \Pi_x(\mu)w(t)$ and, in view of equation above, we conclude that $e_\mathrm{ss}(t) = 0$. We summarize the discussion as follows.

Proposition 4.4 *Let Φ be a matrix of the form (4.23) and G a matrix of the form (4.24). Suppose the system (4.43) is robustly stabilized by a stabilizer of the form (4.32). Then the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.45).*

Remark 4.3 One might be puzzled by the absence of the nonresonance condition in the statement of the proposition above. It turns out, though, that this condition is implied by the assumption of robust stabilizability of the augmented system. As a matter of fact, in this statement it is assumed that there exists a stabilizer that stabilizes the augmented plant (4.43) for every μ. As a trivial consequence, the latter is stabilizable and detectable for every μ. This being the case, it is seen from Lemma 4.2 that, if the system (4.43) is robustly stabilized by a stabilizer of the form (4.32), necessarily the triplet $\{A(\mu), B(\mu), C(\mu)\}$ is stabilizable and detectable for every μ and the nonresonance condition must hold for every μ. We stress also that the nonresonance condition, which we have seen is necessary, also implies the existence of a solution of Francis's equations (4.42). This is an immediate consequence of Lemma 4.1.

4.6 The Special Case in Which $m = p$ and $p_\mathrm{r} = 0$

We discuss in this section the design of regulators in the special case of a plant in which the number of regulated outputs is equal to the number of control inputs, and no additional measurements outputs are available. Of course, the design of a regulator could be achieved by following the general construction described in the previous section, but in this special case alternative (and somewhat simpler) design procedures are available, which will be described in what follows.

Immediate consequences of the assumption $m = p$ are the fact that the nonresonance condition can be rewritten as

$$\det \begin{pmatrix} A - \lambda I & B \\ C_e & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S), \tag{4.47}$$

and the fact that, if this is the case, the solution Π, Ψ of Francis's equations (4.6) is *unique*. If, in addition, $p_r = 0$, the construction described above can be simplified and an alternative structure of the controller is possible.

If $p_r = 0$, in fact, the structure of the controller (4.29) becomes

$$\begin{aligned} \dot\eta &= \Phi\eta + Ge \\ \dot\xi &= A_s\xi + B_s e + J_s\eta \\ u &= C_s\xi + D_s e + H_s\eta. \end{aligned} \tag{4.48}$$

Suppose that J_s and H_s are chosen as

$$\begin{aligned} J_s &= B_s\Gamma \\ H_s &= D_s\Gamma \end{aligned}$$

in which Γ is a matrix to be determined. In this case, the proposed controller can be seen as a pure *cascade connection* of a postprocessing *filter* modeled by

$$\begin{aligned} \dot\eta &= \Phi\eta + Ge \\ \tilde e &= \Gamma\eta + e, \end{aligned} \tag{4.49}$$

whose output $\tilde e$ drives a stabilizer modeled by

$$\begin{aligned} \dot\xi &= A_s\xi + B_s\tilde e \\ u &= C_s\xi + D_s\tilde e \end{aligned} \tag{4.50}$$

as shown in Fig. 4.4.

Such special form of J_s and H_s is admissible if a system of the form (4.50) exists that stabilizes augmented plant

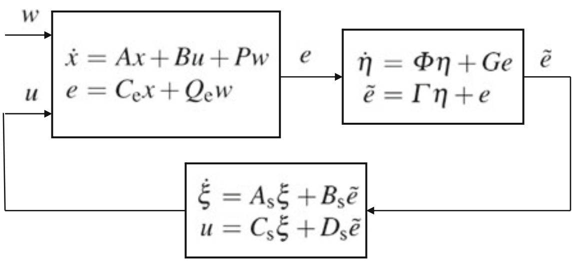

Fig. 4.4 The control consists in a *postprocessing* internal model cascaded with a stabilizer

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ \tilde{e} &= \begin{pmatrix} C_{\mathrm{e}} & \Gamma \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \end{aligned} \tag{4.51}$$

i.e., if the latter is stabilizable and detectable. In this respect, the following result is useful.

Lemma 4.4 *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.51) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_{\mathrm{e}}\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Proof Stabilizability is a straightforward consequence of Lemma 4.2. Detectability holds if the columns of the matrix

$$\begin{pmatrix} A - \lambda I & 0 \\ GC_{\mathrm{e}} & \Phi - \lambda I \\ C_{\mathrm{e}} & \Gamma \end{pmatrix}$$

are independent for all λ having nonnegative real part. Taking linear combinations of rows, we transform the latter into

$$\begin{pmatrix} A - \lambda I & 0 \\ 0 & \Phi - G\Gamma - \lambda I \\ C_{\mathrm{e}} & \Gamma \end{pmatrix},$$

and we observe that, if Γ is such that the matrix $\Phi - G\Gamma$ is Hurwitz, the augmented system (4.51) is detectable if and only if so is the pair (A, C_{e}). ◁

Note that a matrix Γ that makes $\Phi - G\Gamma$ Hurwitz certainly exists because by construction the pair Φ, G is reachable. We can therefore conclude that, if the triplet $\{A, B, C_{\mathrm{e}}\}$ is stabilizable and detectable, if the nonresonance condition holds and if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz (as it is always possibile), the problem of output regulation can be solved by means of a controller consisting of the cascade of (4.49) and (4.50).[6]

We summarize this in the following statement.

[6]Note that the filter (4.49) is an invertible system, the inverse being given by

$$\begin{aligned} \dot{\eta} &= (\Phi - G\Gamma)\eta + G\tilde{e} \\ e &= -\Gamma\eta + \tilde{e}. \end{aligned}$$

Hence, if Γ is chosen in such that a way $\Phi - G\Gamma$ is Hurwitz, the inverse of (4.49) is a stable system.

Proposition 4.5 *Suppose that*

(i) the triplet $\{A, B, C_e\}$ *is stabilizable and detectable*
(ii) the nonresonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned}\dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_s\xi + B_s(\Gamma\eta + e) \\ u &= C_s\xi + D_s(\Gamma\eta + e),\end{aligned} \tag{4.52}$$

in which Φ, G *have the form (4.23), (4.24),* Γ *is such that a* $\Phi - G\Gamma$ *is Hurwitz and* A_s, B_s, C_s, D_s *are such that (4.50) stabilizes the augmented plant (4.51).*

The controller (4.52) is, as observed, the cascade of two subsystems. It would be nice to check whether these subsystems could be "swapped," i.e., whether the same result could be obtained by means of a controller consisting of a preprocessing filter of the form

$$\begin{aligned}\dot{\eta} &= \Phi\eta + G\tilde{u} \\ u &= \Gamma\eta + \tilde{u}\end{aligned} \tag{4.53}$$

whose input $\tilde{u}$ is provided by a stabilizer of the form

$$\begin{aligned}\dot{\xi} &= A_s\xi + B_s e \\ \tilde{u} &= C_s\xi + D_s e\end{aligned} \tag{4.54}$$

as shown in Fig. 4.5.

This is trivially possible if $m = 1$, because the two subsystems in question are single-input single-output. However, as it is shown now, this is possible also if $m > 1$. To this end, observe that controlling the plant (4.1) by means of (4.53) and (4.54) yields an overall system modeled by

$$\begin{pmatrix}\dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi}\end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P + BD_sQ_e & A + BD_sC_e & B\Gamma & BC_s \\ GD_sQ_e & GD_sC_e & \Phi & GC_s \\ B_sQ_e & B_sC_e & 0 & A_s \end{pmatrix}\begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}. \tag{4.55}$$

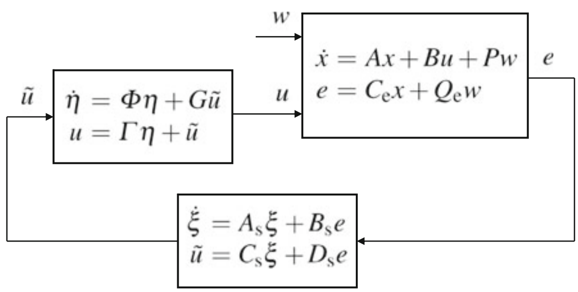

Fig. 4.5 The control consists in a stabilizer cascaded with a *preprocessing* internal model

On the basis of our earlier discussions, it can be claimed that the problem of output regulation is solved if the matrix

$$\begin{pmatrix} A + BD_\mathrm{s}C_\mathrm{e} & B\Gamma & BC_\mathrm{s} \\ GD_\mathrm{s}C_\mathrm{e} & \Phi & GC_\mathrm{s} \\ B_\mathrm{s}C_\mathrm{e} & 0 & A_\mathrm{s} \end{pmatrix} \tag{4.56}$$

is Hurwitz and the associated center eigenspace of (4.55) is contained in the kernel of the error map $e = C_\mathrm{e}x + Q_\mathrm{e}w$.

The matrix (4.56) is Hurwitz if (and only if) the stabilizer (4.54) stabilizes the augmented plant

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B \\ G \end{pmatrix} \tilde{u} \\ e &= \begin{pmatrix} C_\mathrm{e} & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}. \end{aligned} \tag{4.57}$$

In what follows the conditions under which this is possible are discussed.

Lemma 4.5 *Let Φ and G be defined as in (4.23) and (4.24). Let Γ be such that the matrix $\Phi - G\Gamma$ is Hurwitz. Then, the pair (Φ, Γ) is observable.*

Proof Since $\Phi - G\Gamma$ is Hurwitz, the pair (Φ, Γ) is by definition detectable. But Φ has by assumption all eigenvalues with nonnegative real part. In this case, detectability (of the pair (Φ, Γ)) is equivalent to observability. ◃

Lemma 4.6 *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, there exists a nonsingular matrix T such that*

$$\Gamma = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} T$$

and

$$T\Phi = \Phi T.$$

Proof If the pair (Φ, Γ) is observable, the square $pd \times pd$ matrix (recall that, by construction, the minimal polynomial of Φ has degree d)

$$T = \begin{pmatrix} \Gamma \\ \Gamma\Phi \\ \cdots \\ \Gamma\Phi^{d-1} \end{pmatrix}$$

is invertible. In view of the second of (4.27), a simple calculation shows that this matrix renders the two identities in the lemma fulfilled. ◃

Lemma 4.7 *Let Γ be such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, the augmented system (4.57) is stabilizable and detectable if and only if*

(i) the triplet $\{A, B, C_e\}$ *is stabilizable and detectable*
(ii) the nonresonance condition (4.47) holds.

Proof To check stabilizability, observe that the rows of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma & B \\ 0 & \Phi - \lambda I & G \end{pmatrix}$$

which, via combination of columns, can be reduced to the matrix

$$\begin{pmatrix} A - \lambda I & 0 & B \\ 0 & \Phi - G\Gamma - \lambda I & G \end{pmatrix},$$

are linearly independent for each λ having nonnegative real part if and only if the pair (A, B) is stabilizable. To check detectabililty, we need to look at the linear independence of the columns of the matrix

$$\begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_e & 0 \end{pmatrix}. \tag{4.58}$$

By Lemma 4.5, the pair (Φ, Γ) is observable. Let T be a matrix that renders the two identities in Lemma 4.6 fulfilled, and have the matrix (4.58) replaced by the matrix

$$\begin{pmatrix} I & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & B\Gamma \\ 0 & \Phi - \lambda I \\ C_e & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T^{-1} \end{pmatrix} = \begin{pmatrix} A - \lambda I & B\Gamma T^{-1} \\ 0 & \Phi - \lambda I \\ C_e & 0 \end{pmatrix},$$

which, in view of the forms of ΓT^{-1} and Φ can be expressed in more detail as

$$\begin{pmatrix} A - \lambda I & B & 0 & 0 & \cdots & 0 \\ 0 & -\lambda I & I & 0 & \cdots & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & -s_0 I & -s_1 I & -s_2 I & \cdots & -(s_{d-1} + \lambda)I \\ C_e & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From this, the check of detectability condition proceeds essentially as in the check of stabilizability condition in Lemma 4.2. Taking linear combinations and permutation of columns and rows, one ends up with a matrix of the form

$$\begin{pmatrix} A - \lambda I & B & 0 \\ C_e & 0 & 0 \\ 0 & -\psi(\lambda) I & 0 \\ 0 & 0 & I \end{pmatrix}$$

from which the claim follows. ◁

If the matrix (4.56) is Hurwitz, the center eigenspace of (4.55) can be expressed as

$$\mathscr{V}^{\mathrm{c}} = \mathrm{Im}\begin{pmatrix} I \\ \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix}$$

in which Π_x, Π_η, Π_ξ are solutions of the Sylvester equation

$$\begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} S = \begin{pmatrix} A + BD_{\mathrm{s}}C_{\mathrm{e}} & B\Gamma & BC_{\mathrm{s}} \\ GD_{\mathrm{s}}C_{\mathrm{e}} & \Phi & GC_{\mathrm{s}} \\ B_{\mathrm{s}}C_{\mathrm{e}} & 0 & A_{\mathrm{s}} \end{pmatrix} \begin{pmatrix} \Pi_x \\ \Pi_\eta \\ \Pi_\xi \end{pmatrix} + \begin{pmatrix} P + BD_{\mathrm{s}}Q_{\mathrm{e}} \\ GD_{\mathrm{s}}Q_{\mathrm{e}} \\ B_{\mathrm{s}}Q_{\mathrm{e}} \end{pmatrix}.$$

The matrices Π_x, Π_η, Π_ξ that characterize $\mathscr{V}^{\mathrm{c}}$ can, in this particular setting, be easily determined. To this end, recall that, if $m = p$ and the nonresonance condition holds, the solution Π, Ψ of Francis' equations (4.6) is *unique*. As shown in the next lemma, a special relation between the matrix Ψ and the matrices Φ and Γ that characterize the filter (4.53) can be established.

Lemma 4.8 *Let Φ be defined as in (4.23). Let Γ be a $(p \times dp)$ matrix such that the pair (Φ, Γ) is observable. Then, given any matrix Ψ, there exists a matrix Σ such that*

$$\begin{aligned} \Sigma S &= \Phi \Sigma \\ \Psi &= \Gamma \Sigma. \end{aligned} \tag{4.59}$$

Proof Let T be a matrix such that the two identities in Lemma 4.6 hold. Change (4.59) in

$$\begin{aligned} (T\Sigma)S &= T\Phi T^{-1}(T\Sigma) = \Phi(T\Sigma) \\ \Psi &= \Gamma T^{-1}(T\Sigma) = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} (T\Sigma), \end{aligned}$$

and then check that

$$T\Sigma = \begin{pmatrix} \Psi \\ \Psi S \\ \cdot \\ \Psi S^{d-1} \end{pmatrix}$$

is a solution, thanks to the first one of (4.27). ◁

Using (4.59) and (4.6) it is easily seen that the triplet

$$\Pi_x = \Pi, \qquad \Pi_\eta = \Sigma, \qquad \Pi_\xi = 0$$

is a solution of the Sylvester equation above, as a matter of fact, the *unique* solution of that equation. Thus, in particular, the steady state of the closed-loop system (4.55) is such that $x_{ss}(t) = \Pi w(t)$ with Π obeying the second equation of (4.6). As a consequence, $e_{ss}(t) = 0$ and the problem of output regulation is solved. We summarize the discussion as follows.

Proposition 4.6 *Suppose that*

(i) the triplet $\{A, B, C_e\}$ is stabilizable and detectable
(ii) the nonresonance condition (4.47) holds.

Then, the problem of output regulation is solvable by means of a controller of the form

$$\begin{aligned} \dot{\xi} &= A_s\xi + B_s e \\ \dot{\eta} &= \Phi\eta + G(C_s\xi + D_s e) \\ u &= \Gamma\eta + (C_s\xi + D_s e), \end{aligned} \tag{4.60}$$

in which Φ, G have the form (4.23), (4.24), Γ is such that a $\Phi - G\Gamma$ is Hurwitz and A_s, B_s, C_s, D_s are such that

$$\begin{aligned} \dot{\xi} &= A_s\xi + B_s e \\ \tilde{u} &= C_s\xi + D_s e. \end{aligned} \tag{4.61}$$

stabilizes the augmented plant (4.57).

We have seen in this section that, if $m = p$ and $p_r = 0$, the problem of output regulation can be approached, under identical assumptions, in two equivalent ways. In the first one of these, the regulated output e is *postprocessed* by an *internal model* of the form (4.49) and the resulting augmented system is subsequently stabilized. In the second one of these, the control input u is *preprocessed* by an *internal model* of the form (4.53) and the resulting augmented system is stabilized (see Figs. 4.4 and 4.5).

While both modes of control yield the same result, it should be stressed that the steady-state behaviors of the state variables are different. In the first mode of control, it is seen from the analysis above that in steady state

$$x_{ss}(t) = \Pi_x w(t), \qquad \eta_{ss}(t) = \Pi_\eta w(t), \qquad \xi_{ss}(t) = \Pi_\xi w(t).$$

in which, using in particular Lemma 4.3, it is observed that

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi\Pi_\eta \\ \Pi_\xi S &= A_s\Pi_\xi + B_s\Gamma\Pi_\eta \\ \Psi &= C_s\Pi_\xi + D_s\Gamma\Pi_\eta \end{aligned}$$

where Π, Ψ is the unique solution of Francis's equations (4.6). In the second mode of control, on the other hand, in steady state we have

$$x_{ss}(t) = \Pi_x w(t), \qquad \eta_{ss}(t) = \Pi_\eta w(t), \qquad \xi_{ss}(t) = 0,$$

in which

$$\begin{aligned} \Pi_x &= \Pi \\ \Pi_\eta S &= \Phi \Pi_\eta \\ \Psi &= \Gamma \Pi_\eta \end{aligned}$$

with Π, Ψ the unique solution of Francis's equations (4.6). In particular, in the second mode of control, in steady state the stabilizer (4.61) is *at rest*.

In both modes of control the *internal model* has an identical structure, that of the system

$$\begin{aligned} \dot{\eta} &= \Phi \eta + G\hat{u} \\ \hat{y} &= \Gamma \eta + \hat{u}, \end{aligned} \tag{4.62}$$

in which Φ is the matrix defined in (4.23). In the discussion above, the matrix G has been taken as in (4.24), and the matrix Γ was any matrix rendering $\Phi - G\Gamma$ a Hurwitz matrix. However, it is easy to check that one can reverse the roles of G and Γ. In fact, using the state transformation

$$\tilde{\eta} = T\eta$$

with T defined as in the proof of Lemma 4.6 it is seen that an equivalent realization of (4.62) is

$$\begin{aligned} \dot{\tilde{\eta}} &= \Phi \tilde{\eta} + \tilde{G}\hat{u} \\ \hat{y} &= \tilde{\Gamma} \eta + \hat{u}, \end{aligned} \tag{4.63}$$

in which $\tilde{G} = TG$ and

$$\tilde{\Gamma} = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} \tag{4.64}$$

(here all blocks are $p \times p$, with $p = m$ by hypothesis). Thus, one can design the internal model picking Φ as in (4.23), Γ as in (4.64) and then choosing a G that makes $\Phi - G\Gamma$ a Hurwitz matrix.

Finally, note that if we let F denote the Hurwitz matrix $F = \Phi - G\Gamma$, the internal model can be put in the form

$$\begin{aligned} \dot{\eta} &= F \eta + G\hat{y} \\ \hat{y} &= \Gamma \eta + \hat{u}. \end{aligned} \tag{4.65}$$

The two modes of control lend themselves to solve also a problem of *robust* regulation. From the entire discussion above, in fact, one can arrive at the following conclusion.

Proposition 4.7 *Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24), and Γ a matrix such that $\Phi - G\Gamma$ is Hurwitz (alternatively: Γ a matrix of the form (4.64) and G a matrix such that $\Phi - G\Gamma$ is Hurwitz). If system (4.51) is robustly stabilized by a stabilizer of the form (4.50), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.52). If system (4.57)* *is robustly stabilized by a stabilizer of the form (4.61), the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60).*

4.7 The Case of SISO Systems

We consider now the case in which $m = p = 1$ and $p_r = 0$, and the coefficient matrices that characterize the controlled plant depend on a vector μ of uncertain parameters, as in (4.41), which we will rewrite for convenience as[7]

$$\begin{aligned}\dot{w} &= Sw\\ \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w\\ e &= C(\mu)x + Q(\mu)w.\end{aligned} \tag{4.66}$$

Let the system be controlled by a controller of the form (4.60). We know from the previous analysis that the controller in question solves the problem of output regulation if there exists a stabilizer of the form (4.61) that stabilizes the augmented system

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{\eta}\end{pmatrix} &= \begin{pmatrix}A(\mu) & B(\mu)\Gamma\\ 0 & \Phi\end{pmatrix}\begin{pmatrix}x\\ \eta\end{pmatrix} + \begin{pmatrix}B(\mu)\\ G\end{pmatrix}\tilde{u}\\ e &= \begin{pmatrix}C(\mu) & 0\end{pmatrix}\begin{pmatrix}x\\ \eta\end{pmatrix}.\end{aligned} \tag{4.67}$$

Suppose the triplet in question has a well-defined relative degree r between control input $\tilde{u}$ and regulated output e, independent of μ. It is known from Chap. 2 that a single-input single-output system having well-defined relative degree and *all zeros with negative real part* can be robustly stabilized by (dynamic) output feedback. Thus, we seek assumptions ensuring that the augmented system so defined has a well-defined relative degree and all zeros with negative real part. An easy calculation shows that, if the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree r,[8] then the augmented system (4.67) still has relative degree r, between the input $\tilde{u}$ and the output e. In fact, for all $k \le r-1$

$$\begin{pmatrix}C(\mu) & 0\end{pmatrix}\begin{pmatrix}A(\mu) & B(\mu)\Gamma\\ 0 & \Phi\end{pmatrix}^k = \begin{pmatrix}C(\mu)A^k(\mu) & 0\end{pmatrix}$$

from which it is seen that the relative degree is r, with high-frequency gain

$$\begin{pmatrix}C(\mu) & 0\end{pmatrix}\begin{pmatrix}A(\mu) & B(\mu)\Gamma\\ 0 & \Phi\end{pmatrix}^{r-1}\begin{pmatrix}B(\mu)\\ G\end{pmatrix} = C(\mu)A^{r-1}(\mu)B(\mu).$$

[7] Note that the subscript "e" has been dropped.

[8] Here and in the following we use the abbreviation "the triplet $\{A, B, C\}$" to mean the associated system (2.1).

To evaluate the zeros, we look at the roots of the polynomial

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu)\Gamma & B(\mu) \\ 0 & \Phi - \lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0.$$

The determinant is unchanged if we multiply the matrix, on the right, by a matrix of the form

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Gamma & I \end{pmatrix}$$

no matter what Γ is. Thus, the zeros of the system are the roots of the polynomial

$$\det \begin{pmatrix} A(\mu) - \lambda I & 0 & B(\mu) \\ 0 & \Phi - G\Gamma - \lambda I & G \\ C(\mu) & 0 & 0 \end{pmatrix} = 0.$$

The latter clearly coincide with the roots of

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \cdot \det(\Phi - G\Gamma - \lambda I) = 0.$$

Thus, the $n + d - r$ zeros of the augmented plant are given by the $n - r$ zeros of the triplet $\{A(\mu), B(\mu), C(\mu)\}$ and by the d eigenvalues of the matrix $\Phi - G\Gamma$. If, as indicated above, the matrix Γ is chosen in such a way that the matrix $\Phi - G\Gamma$ is Hurwitz (as it is always possible), we see that the zeros of the augmented system have negative real part if so are the zeros of the triplet $\{A(\mu), B(\mu), C(\mu)\}$.

We summarize the conclusion as follows.

Proposition 4.8 *Consider an uncertain system of the form (4.66). Suppose that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has a well-defined relative degree and all its $n - r$ zeros have negative real part, for every value of μ. Let Φ be a matrix of the form (4.23), G a matrix of the form (4.24), and Γ a matrix such that $\Phi - G\Gamma$ is Hurwitz. Then, the problem of robust output regulation is solvable, in particular by means of a controller of the form (4.60), in which (4.61) is a robust stabilizer of the augmented system (4.67).*

Example 4.2 To make this result more explicit, it is useful to examine in more detail the case of a system having relative degree 1. As a by-product, additional insight in the design procedure is gained, that proves to be useful in similar contexts in the next sections. Consider the case in which the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has relative degree 1, i.e., is such that $C(\mu)B(\mu) \neq 0$. Then, as shown in Example 4.1, system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P(\mu) & A(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B(\mu) \end{pmatrix} u$$

$$e = \begin{pmatrix} Q(\mu) & C(\mu) \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}$$

has relative degree 1. It can be put in normal form, obtaining a system (see again Example 4.1)

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu)w \\ \dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu)w \\ e &= \xi, \end{aligned} \tag{4.68}$$

in which it is assumed that $b(\mu) > 0$. As shown in Example 4.1, since by hypothesis the eigenvalues of $A_{00}(\mu)$ have negative real part, the Sylvester equation[9]

$$\Pi_0(\mu)S = A_{00}(\mu)\Pi_0(\mu) + P_0(\mu) \tag{4.69}$$

has a solution $\Pi_0(\mu)$ and therefore the regulator equations (4.6) have (unique) solution

$$\Pi(\mu) = \begin{pmatrix} \Pi_0(\mu) \\ 0 \end{pmatrix}, \qquad \Psi(\mu) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu)].$$

Changing z into $\tilde{z} = z - \Pi_0(\mu)w$ yields the simplified system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\ \dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[u - \Psi(\mu)w]. \end{aligned}$$

Let now this system be controlled by a preprocessor of the form (4.53), with Γ such that the matrix $F = \Phi - G\Gamma$ is Hurwitz and $\tilde{u}$ provided by a stabilizer (4.54). As shown in Lemma 4.8, there always exists a matrix $\Sigma(\mu)$ such that (see (4.59) in this respect)

$$\begin{aligned} \Sigma(\mu)S &= \Phi\Sigma(\mu) \\ \Psi(\mu) &= \Gamma\Sigma(\mu). \end{aligned} \tag{4.70}$$

Using these identities and changing η into $\tilde{\eta} = \eta - \Sigma(\mu)w$ yields the following (augmented) system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi \\ \dot{\tilde{\eta}} &= \Phi\tilde{\eta} + G\tilde{u} \\ \dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma\tilde{\eta} + \tilde{u}], \end{aligned} \tag{4.71}$$

[9]Since the parameters of the equation are μ-dependent so is expected to be its solution.

in which, as expected, the subsystem characterized by the last three equations is *independent* of w. If such subsystem is (robustly) stabilized, then in particular $\xi \to 0$ as $t \to \infty$. Since $e = \xi$, the problem of output regulation is (robustly) solved. Now, the system in question has relative degree 1 (between input $\tilde{u}$ and output e) and its $n-1+d$ zeros all have negative real part. Hence the stabilizer (4.61) can be a pure output feedback, i.e., $\tilde{u} = -ke$, with large k. To double-check that this is the case, let $\tilde{u}$ be fixed in this way and observe that the resulting closed-loop system is a system of the form

$$\dot{x} = A(\mu)x$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \qquad A(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma & a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.72}$$

A similarity transformation $\bar{x} = T(\mu)x$, with

$$T(\mu) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\frac{1}{b(\mu)}G \\ 0 & 0 & 1 \end{pmatrix} \tag{4.73}$$

changes $A(\mu)$ into the matrix (set here $F = \Phi - G\Gamma$)

$$\bar{A}(\mu) = T(\mu)A(\mu)T^{-1}(\mu) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F & \frac{1}{b(\mu)}(FG - Ga_{11}(\mu)) \\ A_{10}(\mu) & b(\mu)\Gamma & \Gamma G + a_{11}(\mu) - b(\mu)k \end{pmatrix}. \tag{4.74}$$

Since the eigenvalues of $A_{00}(\mu)$ and those of F have negative real part, the converse Lyapunov Theorem says that there exists a $(n-1+d) \times (n-1+d)$ positive-definite symmetric matrix $Z(\mu)$ such that

$$Z(\mu)\begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix} + \begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{10}(\mu) & F \end{pmatrix}^{\mathrm{T}} Z(\mu) < 0.$$

Then, it is an easy matter to show that the $(n+d) \times (n+d)$ positive-definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

is such that

$$P(\mu)\bar{A}(\mu) + \bar{A}^{\mathrm{T}}(\mu)P(\mu) < 0 \tag{4.75}$$

if k is large enough.[10] From this, using the direct Theorem of Lyapunov, we conclude that, if k is large enough, the matrix (4.74) has all eigenvalues with negative real part. If this is the case, the lower subsystem of (4.71) is robustly stabilized and the problem of output regulation is robustly solved. ◁

4.8 Internal Model Adaptation

The remarkable feature of the controller discussed in the previous section is the ability of securing asymptotic decay of the regulated output $e(t)$ in spite of parameter uncertainties.

Thus, control schemes consisting of an internal model and of a robust stabilizer efficiently address the problem of rejecting all exogenous inputs generated by the exosystem. In this sense, they generalize the classical way in which integral-control-based schemes cope with constant but unknown disturbances, even in the presence of parameter uncertainties. There still is a limitation, though, in these schemes: the necessity of a precise model of the exosystem. As a matter of fact, the controller considered above contains a pair of matrices Φ, Γ whose construction (see above) requires the knowledge of the precise values of the coefficients of the minimal polynomial of S. This limitation is not sensed in a problem of set point control, where the uncertain exogenous input is constant and thus obeys a trivial, parameter independent, differential equation, but becomes immediately evident in the problem of rejecting, e.g., a *sinusoidal* disturbance of unknown amplitude and phase. A robust controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoidal signal, but the *frequency* at which the internal model oscillates must exactly match the frequency of the exogenous signal: any mismatch in such frequencies results in a nonzero steady-state error.

In what follows we show how this limitation can be removed, by automatically tuning the "natural frequencies" of the robust controller. For the sake of simplicity, we limit ourselves to sketch here the main philosophy of the design method.[11]

Consider again the single-input single-output system (4.66) for which we have learned how to design a robust regulator but suppose, now, that the model of the exosystem that generates the disturbance w depends on a vector ρ of uncertain parameters, ranging on a prescribed compact set $\mathcal{Q}$, as in

$$\dot{w} = S(\rho)w. \tag{4.76}$$

We retain the assumption that the exosystem is neutrally stable, in which case $S(\rho)$ can only have eigenvalues on the imaginary axis (with simple multiplicity in the

[10] See Sect. 2.3.

[11] The approach in this section closely follows the approach described, in a more general context, in [5].

minimal polynomial). Therefore, uncertainty in the value of ρ is reflected in the uncertainty in the value of the imaginary part of these eigenvalues.

Let

$$\psi_\rho(\lambda) = s_0(\rho) + s_1(\rho)\lambda + \cdots + s_{d-1}(\rho)\lambda^{d-1} + \lambda^d$$

denote the minimal polynomial of $S(\rho)$ and assume that the coefficients $s_0(\rho), s_1(\rho), \ldots, s_{d-1}(\rho)$ are continuous functions of ρ. Following the design procedure illustrated in the previous sections, consider a pair of matrices Φ_ρ, G defined as

$$\Phi_\rho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -s_0(\rho) & -s_1(\rho) & -s_2(\rho) & \cdots & -s_{d-1}(\rho) \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

the former of which is a continuous function of ρ.

We know from the discussion above that, if the parameter ρ were known, a controller consisting of a preprocessing internal model of the form

$$\begin{aligned} \dot\eta &= \Phi_\rho \eta + G\tilde u \\ u &= \Gamma\eta + \tilde u, \end{aligned} \tag{4.77}$$

with $\tilde u$ provided by a robust stabilizer of the form (4.61), would solve the problem of output regulation. We have also seen that such a stabilizer exists if Γ is *any* matrix that renders $\Phi_\rho - G\Gamma$ a Hurwitz matrix and if, in addition, all $n - r$ zeros of the triplet $\{A(\mu), B(\mu), C(\mu)\}$ have negative real part for all μ.

Note that in this context the choice of Γ is arbitrary, so long as the matrix $\Phi_\rho - G\Gamma$ is Hurwitz. In what follows, we choose this matrix as follows. Let F be a *fixed* $d \times d$ matrix

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix} \tag{4.78}$$

with a characteristic polynomial

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_{d-1}\lambda^{d-1} + \lambda^d,$$

having all roots with negative real part, let G be defined as above, i.e.,

$$G = \begin{pmatrix} 0 \; 0 \cdots 0 \; 1 \end{pmatrix}^{\mathrm T}, \tag{4.79}$$

and choose, for Γ, a matrix of the form

$$\Gamma_\rho = \big((a_0 - s_0(\rho)) \ (a_1 - s_1(\rho)) \ \cdots \ (a_{d-1} - s_{d-1}(\rho))\big), \tag{4.80}$$

(note that we have added the subscript "ρ" to stress the dependence of such Γ on the vector ρ of possibly uncertain parameters). This choice is clearly such that

$$\Phi_\rho - G\Gamma_\rho = F$$

is a Hurwitz matrix. Hence, if ρ were known, this choice would be admissible. With this choice, the preprocessing internal model (4.77) can be rewritten as

$$\begin{aligned} \dot\eta &= F\eta + G(\Gamma_\rho\eta + \tilde u) \\ u &= \Gamma_\rho\eta + \tilde u. \end{aligned} \tag{4.81}$$

Essentially, what we have done is to "shift" the uncertain data from the matrix Φ_ρ to the vector Γ_ρ. The realization (4.81) of the internal model, though, lends itself to the implementation of some easy (and standard) adaptive control techniques.

If ρ were known, the controller (4.81), with $\tilde u$ provided by a robust stabilizer of the form (4.61) would be a robust controller. In case ρ is not known, one may wish to replace the vector Γ_ρ with an *estimate* $\hat\Gamma$, *to be tuned* by means of an appropriate adaptation law.

We illustrate how this can be achieved in the simple situation in which the system has relative degree 1. To facilitate the analysis, we assume that the controlled plant has been initially put in normal form (see Example 4.1), which in the present case will be[12]

$$\begin{aligned} \dot w &= S(\rho)w \\ \dot z &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w \\ \dot\xi &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)u + P_1(\mu,\rho)w \\ e &= \xi. \end{aligned} \tag{4.82}$$

By assumption, the $n-1$ eigenvalues of the matrix $A_{00}(\mu)$ have negative real part for all μ.

Consider a *tunable* preprocessing internal model of the form

$$\begin{aligned} \dot\eta &= F\eta + G(\hat\Gamma\eta + \tilde u) \\ u &= \hat\Gamma\eta + \tilde u \end{aligned} \tag{4.83}$$

in which $\hat\Gamma$ is a $1 \times d$ vector to be tuned. The associated augmented system becomes

[12] It is seen from the construction in Example 4.1 that, in the normal form (4.11), the matrices P_0 and P_1 are found by means of transformations involving A, B, C, P and also S. Thus, if the former are functions of μ and the latter is a function of ρ, so are expected to be P_0 and P_1.

$$\begin{aligned}
\dot{w} &= S(\rho)w\\
\dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w\\
\dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\hat{\Gamma}\eta + \tilde{u}] + P_1(\mu,\rho)w\\
\dot{\eta} &= F\eta + G[\hat{\Gamma}\eta + \tilde{u}]\\
e &= \xi.
\end{aligned}$$

Define an *estimation error*

$$\tilde{\Gamma} = \hat{\Gamma} - \Gamma_\rho,$$

and rewrite the system in question as (recall that $F + G\Gamma_\rho = \Phi_\rho$)

$$\begin{aligned}
\dot{w} &= S(\rho)w\\
\dot{z} &= A_{00}(\mu)z + a_{01}(\mu)\xi + P_0(\mu,\rho)w\\
\dot{\eta} &= \Phi_\rho\eta + G\tilde{u} + G\tilde{\Gamma}\eta\\
\dot{\xi} &= A_{10}(\mu)z + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho\eta + \tilde{u}] + b(\mu)\tilde{\Gamma}\eta + P_1(\mu,\rho)w\\
e &= \xi.
\end{aligned}$$

We know from the analysis in Example 4.2 that, if $\tilde{\Gamma}$ were zero, the choice of a stabilizing control

$$\tilde{u} = -k\xi \tag{4.84}$$

(with $k > 0$ and large) would solve the problem of robust output regulation. Let $\tilde{u}$ be chosen in this way and consider, for the resulting closed-loop system, a change of coordinates (see again Example 4.2)

$$\tilde{z} = z - \Pi_0(\mu,\rho)w, \qquad \tilde{\eta} = \eta - \Sigma(\mu,\rho)w$$

in which $\Pi_0(\mu,\rho)$ is a solution of

$$\Pi_0(\mu,\rho)S(\rho) = A_{00}(\mu)\Pi_0(\mu,\rho) + P_0(\mu,\rho)$$

and $\Sigma(\mu,\rho)$ satisfies

$$\begin{aligned}
\Sigma(\mu,\rho)S(\rho) &= \Phi_\rho\Sigma(\mu,\rho)\\
\Psi(\mu,\rho) &= \Gamma_\rho\Sigma(\mu,\rho),
\end{aligned}$$

in which

$$\Psi(\mu,\rho) = \frac{-1}{b(\mu)}[A_{10}(\mu)\Pi_0(\mu) + P_1(\mu,\rho)].$$

This yields a system of the form

$$\begin{aligned}
\dot{w} &= S(\rho)w\\
\dot{\tilde{z}} &= A_{00}(\mu)\tilde{z} + a_{01}(\mu)\xi\\
\dot{\tilde{\eta}} &= \Phi_\rho\tilde{\eta} - Gk\xi + G\tilde{\Gamma}\eta\\
\dot{\xi} &= A_{10}(\mu)\tilde{z} + a_{11}(\mu)\xi + b(\mu)[\Gamma_\rho\tilde{\eta} - k\xi] + b(\mu)\,\tilde{\Gamma}\eta
\end{aligned}$$

(note that we *have not* modified the terms $\tilde{\Gamma}\eta$ for reasons that will become clear in a moment).

The dynamics of w is now completely decoupled, so that we can concentrate on the lower subsystem, that can be put in the form (compare with (4.72))

$$\dot{x} = A(\mu, \rho)x + B(\mu)\,\tilde{\Gamma}\eta \tag{4.85}$$

in which

$$x = \begin{pmatrix} \tilde{z} \\ \tilde{\eta} \\ \xi \end{pmatrix}, \quad A(\mu, \rho) = \begin{pmatrix} A_{00}(\mu) & 0 & a_{01}(\mu) \\ 0 & \Phi_\rho & -Gk \\ A_{10}(\mu) & b(\mu)\Gamma_\rho & a_{11}(\mu) - b(\mu)k \end{pmatrix}, \quad B(\mu) = \begin{pmatrix} 0 \\ G \\ b(\mu) \end{pmatrix}.$$

It is known from Example 4.2 that—since the eigenvalues of $A_{00}(\mu)$ and those of $F = \Phi_\rho - G\Gamma_\rho$ have negative real part—a matrix such as $A(\mu, \rho)$ is Hurwitz, provided that k is large enough. Specifically, let x be changed in

$$\bar{x} = T(\mu)x$$

with $T(\mu)$ defined as in (4.73), which changes (4.85) in a system of the form

$$\dot{\bar{x}} = \bar{A}(\mu, \rho)\bar{x} + \bar{B}(\mu)\,\tilde{\Gamma}\eta, \tag{4.86}$$

in which $\bar{A}(\mu, \rho) = T(\mu)A(\mu, \rho)T^{-1}(\mu)$, and

$$\bar{B}(\mu) = T(\mu)B(\mu) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} b(\mu).$$

It has been shown in Example 4.2 that there exists a positive-definite matrix

$$P(\mu) = \begin{pmatrix} Z(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

and a number k^* such that

$$P(\mu)\bar{A}(\mu, \rho) + \bar{A}^{\mathrm{T}}(\mu, \rho)P(\mu) < 0 \tag{4.87}$$

if $k > k^*$.[13]

Consider now the positive-definite quadratic form

$$U(\bar{x}, \tilde{\Gamma}) = \bar{x}^{\mathrm{T}}P(\mu)\bar{x} + b(\mu)\tilde{\Gamma}\tilde{\Gamma}^{\mathrm{T}},$$

[13] Recall, in this respect, that both the uncertain vectors μ and ρ range on compact sets.

and compute the derivative of this function along the trajectories of system (4.86). Letting $Q(\mu, \rho)$ denote the negative definite matrix on the left-hand side of (4.87) and observing that

$$\bar{x}^{\mathrm{T}} P(\mu)\bar{B}(\mu) = \xi b(\mu),$$

this yields

$$\begin{aligned}
\dot{U} &= \bar{x}^{\mathrm{T}}[P(\mu)\bar{A}(\mu,\rho) + \bar{A}^{\mathrm{T}}(\mu,\rho)P(\mu)]\bar{x} + 2\bar{x}^{\mathrm{T}}P(\mu)\bar{B}(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2\xi b(\mu)\tilde{\Gamma}\eta + 2b(\mu)\tilde{\Gamma}\dot{\tilde{\Gamma}}^{\mathrm{T}} \\
&= \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} + 2b(\mu)\tilde{\Gamma}[\xi\eta + \dot{\tilde{\Gamma}}^{\mathrm{T}}].
\end{aligned}$$

Observing that Γ_ρ is constant, we see that

$$\dot{\tilde{\Gamma}}^{\mathrm{T}} = \dot{\hat{\Gamma}}^{\mathrm{T}}.$$

The function $\dot{U}$ cannot be made negative definite, but it can be made *negative semidefinite*, by simply taking

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -\xi\eta$$

so as to obtain

$$\dot{U}(\bar{x}, \tilde{\Gamma}) = \bar{x}^{\mathrm{T}}Q(\mu,\rho)\bar{x} \leq 0.$$

Thus, since $U(\bar{x}, \tilde{\Gamma})$ is positive definite and $\dot{U}(\bar{x}, \tilde{\Gamma}) \leq 0$, the trajectories of the closed-loop system are *bounded*. In addition, appealing to La Salle's invariance principle,[14] we can claim that the trajectories asymptotically converge to an invariant set contained in the locus where $\dot{U}(\bar{x}, \tilde{\Gamma}) = 0$.

Clearly, from the expression above, since $Q(\mu, \rho)$ is a definite matrix, we see that

$$\dot{U}(\bar{x}, \tilde{\Gamma}) = 0 \qquad \Rightarrow \qquad \bar{x} = 0.$$

Hence, $\lim_{t\to\infty} \bar{x}(t) = 0$, which in particular implies $\lim_{t\to\infty} \xi(t) = 0$. Therefore, the problem of robust output regulation (in the presence of exosystem uncertainties) is solved.

We summarize the result as follows.

Proposition 4.9 *Consider an uncertain single-input single-output system*

$$\begin{aligned}
\dot{w} &= S(\rho)w \\
\dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\
e &= C(\mu)x + Q(\mu)w.
\end{aligned}$$

[14] See Theorem B.6 in Appendix B.

Suppose the system has relative degree 1 and, without loss of generality, $C(\mu)B(\mu) > 0$. *Suppose the* $n-1$ *zeros of the triplet* $\{A(\mu), B(\mu), C(\mu)\}$ *have negative real part, for every value of* μ. *Then, the problem of robust output regulation is solved by a controller of the form*

$$\begin{aligned}\dot{\eta} &= F\eta + G(\hat{\Gamma}\eta - ke)\\ u &= \hat{\Gamma}\eta - ke\end{aligned}$$

in which F, G *are matrices of the form (4.78) and (4.79),* $k > 0$ *is a large number and* $\hat{\Gamma}$ *is provided by the* adaptation law

$$\dot{\hat{\Gamma}}^{\mathrm{T}} = -e\eta.$$

The extension to systems having higher relative degree, which relies upon arguments similar to those presented in Sect. 2.5, is relatively straightforward, but it will not be covered here.

Example 4.3 A classical control problem arising in the steel industry is the control of the steel thickness in a rolling mill. As shown schematically in Fig. 4.6, a strip of steel of thickness H goes in on one side, and a thinner strip of steel of thickness h comes out on the other side. The exit thickness h is determined from the balance of two forces: a force proportional to the difference between the incoming and outgoing thicknesses

$$F_H = W(H - h)$$

and a force proportional to the gap between the rolls, that can be expressed as

$$F_s = M(h - s)$$

in which s, known as "unloaded screw position," is seen as a control input. The expression of F_s presumes that the rolls are perfectly round. However, this is seldom the case. The effect of rolls that are not perfectly round (problem known as

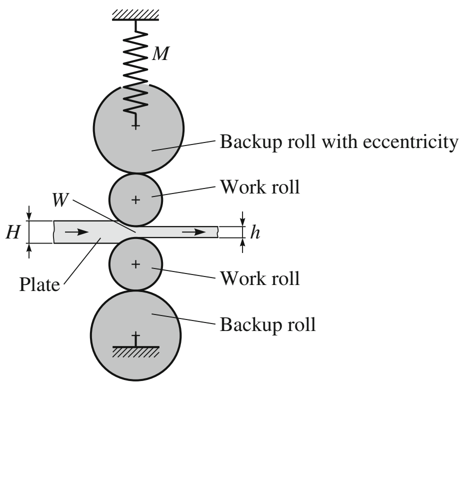

Fig. 4.6 A schematic representation of the process of thickness reduction

"eccentricity") can be modeled by adding a perturbation d to the gap $h - s$ in the expression of F_s, which yields

$$F_s^d = M(h - s - d).$$

Rolls that are not perfectly round can be thought of as rolls of variable radius and this radius—if the rolls rotate at constant speed—is a periodically varying function of time, the period being equal to the time needed for the roll to perform a complete revolution. Thus, the term d that models the perturbation of the nominal gap $h - s$ is a periodic function of time. The period of such function is not fixed though, because it depends on the rotation speed of the rolls.

Balancing the two forces F_H and F_s^d yields

$$h = \frac{1}{M + W}(Ms + WH + Md).$$

The purpose of the design is to control the thickness h. This, if h_{ref} is the prescribed reference value for h, yields a tracking *error* defined as

$$e = \frac{1}{M + W}(Ms + WH + Md) - h_{\text{ref}}.$$

The unloaded screw position s is, in turn, proportional to the angular position of the shaft of a servomotor which, neglecting friction and mechanical losses, can be modeled as

$$\ddot{s} = bu$$

in which u is seen as a control. Setting $x_1 = s$ and $x_2 = \dot{s}$, we obtain a model of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= bu \\ e &= c_1 x_1 + q_1 h_{\text{ref}} + q_2 H + q_3 d \end{aligned}$$

in which c_1, q_1, q_2, q_3 are fixed coefficients.

In this expression, h_{ref} and H are *constant* exogenous inputs, while d is a periodic function which, for simplicity, we assume to be a *sinusoidal* function. Thus, setting $w_1 = h_{\text{ref}}$, $w_2 = H$, $w_3 = d$, the tracking error can be rewritten as

$$e = Cx + Qw$$

in which

$$C = \begin{pmatrix} c_1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1 & q_2 & q_3 & 0 \end{pmatrix}$$

and $w \in \mathbb{R}^4$ satisfies

$$\dot{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & -\rho & 0 \end{pmatrix} := S_\rho w.$$

In normal form, the model thus found is given by

$$\begin{aligned} \dot{w} &= S_\rho w \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= c_1 bu + QS_\rho^2 w \\ e &= \xi_1 . \end{aligned}$$

This system has relative degree 2. Thus, according to the method described in Sect. 2.4, we define a new variable θ as

$$\theta = \xi_2 + a_0\xi_1 = \dot{e} + a_0 e \tag{4.88}$$

in which $a_0 > 0$, and set $z = \xi_1$ to obtain a system which, viewed as a system with input u and output θ, has relative degree 1 and one zero with negative real part

$$\begin{aligned} \dot{w} &= S_\rho w \\ \dot{z} &= -a_0 z + \theta \\ \dot{\theta} &= -a_0^2 z + a_0\theta + c_1 bu + QS_\rho^2 w. \end{aligned}$$

The theory developed above can be used to design a control law, driven by the "regulated variable" θ, that will steer $\theta(t)$ to 0 as $t \to \infty$. This suffices to steer also the actual tracking error $e(t)$ to zero. In fact, in view of (4.88), it is seen that $e(t)$ is the output of a stable one-dimensional linear system

$$\dot{e} = -a_0 e + \theta$$

driven by the input θ. If $\theta(t)$ asymptotically vanishes, so does $e(t)$.

For the design of the internal model, it is observed that the minimal polynomial of S_ρ is the polynomial of degree 3

$$\psi_\rho(\lambda) = \lambda^3 + \rho^2\lambda,$$

which has a fixed root at $\lambda = 0$. Thus, it is natural to seek a setting in which only two parameters are adapted (those that correspond to the uncertain roots in $\pm j\rho$). This can be achieved in this way. Pick

$$F = \begin{pmatrix} 0 & H_2 \\ -G_2 & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}, \qquad \Gamma_\rho = \begin{pmatrix} 1 & \Gamma_{2,\rho} \end{pmatrix}$$

in which F_2, G_2 is the pair of matrices

$$F_2 = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with a_0 and a_1 both positive (so that F_2 is Hurwitz) and $\Gamma_{2,\rho}$ such that

$$F_2 + G_2 \Gamma_{2,\rho} = \begin{pmatrix} 0 & 1 \\ -\rho^2 & 0 \end{pmatrix}.$$

Finally, let P_2 be a positive-definite solution of the Lyapunov equation

$$P_2 F_2 + F_2^{\mathrm{T}} P_2 = -I$$

and pick $H_2 = G_2^{\mathrm{T}} P_2$. Then, the following properties hold[15]:

(i) the matrix F is Hurwitz
(ii) the minimal polynomial of $F + G\Gamma_\rho$ is $\psi_\rho(\lambda)$
(iii) the pair $(F + G\Gamma_\rho, \Gamma_\rho)$ is observable
(iv) for any $\Psi \in \mathbb{R}^{1\times 3}$ there exists a matrix $\Sigma(\rho)$ such that

$$\Sigma(\rho) S_\rho = (F + G\Gamma_\rho)\Sigma(\rho), \qquad \Psi = \Gamma_\rho \Sigma(\rho).$$

As shown above, if ρ were known, the controller (4.81) with $\tilde{u} = -k\theta$, namely the controller

$$\begin{aligned} \dot{\eta} &= F\eta + G[\Gamma_\rho \eta - k\theta] \\ u &= \Gamma_\rho \eta - k\theta \end{aligned}$$

would solve the problem of output regulation. Since ρ is not know, Γ_ρ has to be replaced by a vector of tunable parameters. Such vector, though, needs not to be a (1×3) vector because the first component of Γ_ρ, being equal to 1, is not uncertain. Accordingly, in the above controller, this vector is replaced by a vector of the form

$$\hat{\Gamma} = \begin{pmatrix} 1 & \hat{\Gamma}_2 \end{pmatrix}$$

in which only $\hat{\Gamma}_2$ is a vector of tunable parameters.

The analysis that the suggested controller is able to solve the problem of output regulation in spite of the uncertainty about the value of ρ, if an appropriate adaptation

[15]To prove (i), it suffices to observe that the positive-definite matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix}$$

satisfies

$$QF + F^{\mathrm{T}} Q = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} \leq 0,$$

and use LaSalle's invariance principle. The proof of (ii) is achieved by direct substitution. Property (iii) is a consequence of (i) and of (ii), which says that all eigenvalues of $F + G\Gamma_\rho$ have zero real part. Property (iv) follows from Lemma 4.8.

law is chosen for $\hat{\Gamma}_2$, is identical to the one presented above, and will not repeated here. We limit ourself to conclude with the complete model of the controller which, in view of all of the above, reads as

$$\begin{aligned} \dot{\eta}_1 &= H_2\eta_2 \\ \dot{\eta}_2 &= F_2\eta_2 + G_2(\hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0 e)) \\ \dot{\hat{\Gamma}}_2^{\mathrm{T}} &= -\eta_2(\dot{e} + a_0 e) \\ u &= \eta_1 + \hat{\Gamma}_2\eta_2 - k(\dot{e} + a_0 e). \end{aligned} \qquad \triangleleft$$

4.9 Robust Regulation via H_∞ Methods

In Sect. 4.7, appealing to the results presented in Chap. 2, we have shown how robust regulation can be achieved in the special case $m = p = 1$, under the assumption that the triplet $\{A(\mu), B(\mu), C(\mu)\}$ has a well-defined relative degree and all its $n - r$ zeros have negative real part, for every value of μ. In this section, we discuss how robust regulation can be achieved in a more general setting, appealing to the method for robust stabilization presented in Sects. 3.5 and 3.6.[16]

For consistency with the notation used in the context of robust stabilization via H_∞ methods, we denote the controlled plant as

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= Ax + B_1 v + B_2 u + Pw \\ z &= C_1 x + D_{11} v + D_{12} u + Q_1 w \\ y &= C_2 x + D_{21} v + Q_2 w \\ e &= C_{\mathrm{e}} x + D_{\mathrm{e}1} v + Q_{\mathrm{e}} w, \end{aligned} \tag{4.89}$$

in which

$$C_{\mathrm{e}} = EC_2, \quad D_{\mathrm{e}1} = ED_{21}, \quad Q_{\mathrm{e}} = EQ_2$$

with

$$E = \begin{pmatrix} I_p & 0 \end{pmatrix}.$$

According to the theory presented in Sects. 4.4 and 4.5, we consider a controller that has the standard structure of a *postprocessing internal model*

$$\dot{\eta} = \Phi\eta + Ge, \tag{4.90}$$

in which Φ, G have the form (4.23) and (4.24), cascaded with a *robust stabilizer*.

The purpose of such stabilizer is to solve the problem of γ-suboptimal H_∞ feedback design for an *augmented* plant defined as

[16]The approach in this section essentially follows the approach of [6]. See also [7, 8] for further reading.

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{\mathrm{e1}} \end{pmatrix} v + \begin{pmatrix} B_2 \\ 0 \end{pmatrix} u \\
z &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + D_{11} v + D_{12} u \\
y_{\mathrm{a}} &= \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} D_{21} \\ 0 \end{pmatrix} v.
\end{aligned}
\tag{4.91}
$$

If this is the case, in fact, on the basis of the theory of robust stabilization via H_∞ methods, one can claim that the problem of output regulation is solved, *robustly* with respect to dynamic perturbations that can be expressed as

$$
v = P(s)z,
$$

in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

For convenience, let system (4.91) be rewritten as

$$
\begin{aligned}
\dot{x}_{\mathrm{a}} &= A_{\mathrm{a}} x_{\mathrm{a}} + B_{\mathrm{a1}} v + B_{\mathrm{a2}} u \\
z &= C_{\mathrm{a1}} x + D_{\mathrm{a11}} v + D_{\mathrm{a12}} u \\
y_{\mathrm{a}} &= C_{\mathrm{a2}} x + D_{\mathrm{a21}} v,
\end{aligned}
\tag{4.92}
$$

in which

$$
A_{\mathrm{a}} = \begin{pmatrix} A & 0 \\ GC_{\mathrm{e}} & \Phi \end{pmatrix}, \quad B_{\mathrm{a1}} = \begin{pmatrix} B_1 \\ GD_{\mathrm{e1}} \end{pmatrix}, \quad B_{\mathrm{a2}} = \begin{pmatrix} B_2 \\ 0 \end{pmatrix},
$$

$$
C_{\mathrm{a1}} = \begin{pmatrix} C_1 & 0 \end{pmatrix}, \quad D_{\mathrm{a11}} = D_{11}, \quad D_{\mathrm{a12}} = D_{12},
$$

$$
C_{\mathrm{a2}} = \begin{pmatrix} C_2 & 0 \\ 0 & I \end{pmatrix}, \quad D_{\mathrm{a21}} = \begin{pmatrix} D_{21} \\ 0 \end{pmatrix}.
$$

Observe, in particular, that the state x_{a} has dimension $n + dp$.

The necessary and sufficient conditions for the solution of a γ-suboptimal feedback design problem are those determined in Theorem 3.3. With reference to system (4.92), the conditions in question are rewritten as follows.

Theorem 4.1 *Consider a plant of modeled by equations of the form (4.92). Let* $V_{\mathrm{a1}}, V_{\mathrm{a2}}, Z_{\mathrm{a1}}, Z_{\mathrm{a2}}$ *be matrices such that*

$$
\mathrm{Im} \begin{pmatrix} Z_{\mathrm{a1}} \\ Z_{\mathrm{a2}} \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} C_{\mathrm{a2}} & D_{\mathrm{a21}} \end{pmatrix}, \qquad \mathrm{Im} \begin{pmatrix} V_{\mathrm{a1}} \\ V_{\mathrm{a2}} \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} B_{\mathrm{a2}}^{\mathrm{T}} & D_{\mathrm{a12}}^{\mathrm{T}} \end{pmatrix}.
$$

The problem of γ-suboptimal H_∞ feedback design has a solution if and only if there exist symmetric matrices S_{a} and R_{a} satisfying the following system of linear matrix inequalities

$$\begin{pmatrix} Z_{a1}^T & Z_{a2}^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_a^T S_a + S_a A_a & S_a B_{a1} & C_{a1}^T \\ B_{a1}^T S_a & -\gamma I & D_{a11}^T \\ C_{a1} & D_{a11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_{a1} & 0 \\ Z_{a2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.93}$$

$$\begin{pmatrix} V_{a1}^T & V_{a2}^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_a R_a + R_a A_a^T & R_a C_{a1}^T & B_{a1} \\ C_{a1} R_a & -\gamma I & D_{a11} \\ B_{a1}^T & D_{a11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} V_{a1} & 0 \\ V_{a2} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.94}$$

$$\begin{pmatrix} S_a & I \\ I & R_a \end{pmatrix} \geq 0. \tag{4.95}$$

In particular, there exists a solution of dimension k if and only if there exist R_a and S_a satisfying (4.93)–(4.95) and, in addition,

$$\operatorname{rank}(I - R_a S_a) \leq k. \tag{4.96}$$

In view of the special structure of the matrices that characterize (4.92), the conditions above can be somewhat simplified. Observe that the inequality (4.93) can be rewritten as

$$\begin{pmatrix} Z_{a1}^T S_a (A_a Z_{a1} + B_{a1} Z_{a2}) + (A_a Z_{a1} + B_{a1} Z_{a2})^T S_a Z_{a1} - \gamma Z_{a2}^T Z_{a2} & Z_{a1}^T C_{a1}^T + Z_{a2}^T D_{a11}^T \\ C_{a1} Z_{a1} + D_{a11} Z_{a2} & -\gamma I \end{pmatrix} < 0. \tag{4.97}$$

The kernel of the matrix

$$\begin{pmatrix} C_{a2} & D_{a21} \end{pmatrix} = \begin{pmatrix} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} Z_{a1} \\ Z_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} \\ Z_2 \end{pmatrix}$$

in which Z_1 and Z_2 are such that

$$\operatorname{Im} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} C_2 & D_{21} \end{pmatrix}.$$

Therefore, since $GC_e Z_1 + GD_{e1} Z_2 = GE(C_2 Z_1 + D_{21} Z_2) = 0$,

$$(A_a Z_{a1} + B_{a1} Z_{a2}) = \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{e1} \end{pmatrix} Z_2 = \begin{pmatrix} AZ_1 + B_1 Z_2 \\ 0 \end{pmatrix}.$$

Thus, if S_a is partitioned as

$$S_a = \begin{pmatrix} S & * \\ * & * \end{pmatrix},$$

in which S is $n \times n$, we see that

$$Z_{a1}^{T} S_a (A_a Z_{a1} + B_{a1} Z_{a2}) = Z_1^{T} S (A Z_1 + B_1 Z_2).$$

Moreover, $Z_{a2}^{T} Z_{a2} = Z_2^{T} Z_2$ and

$$C_{a1} Z_{a1} + D_{a11} Z_{a2} = C_1 Z_1 + D_{11} Z_2.$$

It is therefore concluded that the inequality (4.97) reduces to the inequality

$$\begin{pmatrix} Z_1^{T} & Z_2^{T} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{T} S + SA & SB_1 & C_1^{T} \\ B_1^{T} S & -\gamma I & D_{11}^{T} \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \\ 0 & I \end{pmatrix} < 0 \tag{4.98}$$

which is *identical* to the inequality (3.76) determined in Theorem 3.3.

The inequality (4.94), in general, does not lend itself to any special simplification. We simply observe that the kernel of the matrix

$$\begin{pmatrix} B_{a2}^{T} & D_{a12}^{T} \end{pmatrix} = \begin{pmatrix} B_2^{T} & 0 & D_{12}^{T} \end{pmatrix}$$

is spanned by the columns of a matrix

$$\begin{pmatrix} V_{a1} \\ V_{a2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} V_2 & 0 \end{pmatrix} \end{pmatrix}$$

in which V_1 and V_2 are such that

$$\mathrm{Im} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} B_2^{T} & D_{12}^{T} \end{pmatrix},$$

and hence (4.94), in the actual plant parameters, can be rewritten as

$$\begin{pmatrix} V_1^{T} & 0 & V_2^{T} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A & 0 \\ GC_e & \Phi \end{pmatrix} R_a + R_a \begin{pmatrix} A^{T} & C_e^{T} G^{T} \\ 0 & \Phi^{T} \end{pmatrix} & R_a \begin{pmatrix} C_1^{T} \\ 0 \end{pmatrix} & \begin{pmatrix} B_1 \\ GD_{e1} \end{pmatrix} \\ \begin{pmatrix} C_1 & 0 \end{pmatrix} R_a & -\gamma I & D_{11} \\ \begin{pmatrix} B_1^{T} & D_{e1}^{T} G^{T} \end{pmatrix} & D_{11}^{T} & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ V_2 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} < 0. \tag{4.99}$$

Thanks to the simplified version of (4.93), it is possible to show—as a corollary of Theorem 4.1—that the problem in question can be solved by a controller of dimension not exceeding n.

Corollary 4.1 *Consider the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). Suppose there exists positive-definite symmetric matrices S and R_a satisfying the system of linear matrix equations (4.98), (4.99) and*

$$\begin{pmatrix} S & \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} & R_{\mathrm{a}} \end{pmatrix} > 0. \tag{4.100}$$

Then the problem can be solved, by a controller of dimension not exceeding n.

Proof As a consequence of (4.100), the matrix R_{a} is positive definite and hence nonsingular. Let R_{a} be partitioned as

$$R_{\mathrm{a}} = \begin{pmatrix} R_{\mathrm{a}11} & R_{\mathrm{a}12} \\ R_{\mathrm{a}12}^{\mathrm{T}} & R_{\mathrm{a}22} \end{pmatrix}$$

in which $R_{\mathrm{a}11}$ is $n \times n$, and let R_{a}^{-1} be partitioned as

$$R_{\mathrm{a}}^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix}$$

in which Y_{11} is $n \times n$. As shown above, a $(n+dp) \times (n+dp)$ matrix S_{a} of the form

$$S_{\mathrm{a}} = \begin{pmatrix} S & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix} \tag{4.101}$$

is a solution of (4.93). It is possible to show that condition (4.100) implies (4.95). In fact, with

$$T = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ -Y_{12} & 0 & I & 0 \\ -Y_{22} & 0 & 0 & I \end{pmatrix}$$

one obtains

$$T^{\mathrm{T}} \begin{pmatrix} S_{\mathrm{a}} & I \\ I & R_{\mathrm{a}} \end{pmatrix} T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S & I & 0 \\ 0 & I & R_{\mathrm{a}11} & R_{\mathrm{a}12} \\ 0 & 0 & R_{\mathrm{a}12}^{\mathrm{T}} & R_{\mathrm{a}22} \end{pmatrix}.$$

By (4.100), the matrix on the right is positive semidefinite and this implies (4.95). Thus, (4.98)–(4.100) altogether imply (4.93)–(4.95).

Finally, observe that

$$R_{\mathrm{a}} S_{\mathrm{a}} = R_{\mathrm{a}} \begin{pmatrix} S & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & I \end{pmatrix}$$

and hence the last dp columns of $(I - R_{\mathrm{a}} S_{\mathrm{a}})$ are zero. As a consequence

$$\operatorname{rank}(I - S_{\mathrm{a}} R_{\mathrm{a}}) \le n,$$

from which it is concluded that the problem can be solved by a controller dimension not exceeding n. ◁

Remark 4.4 It is worth stressing the fact that the dimension of the robust stabilizer does not exceed the dimension n of the controlled plant, and this despite of the fact that the robust stabilizer is designed for an augmented plant of dimension $n + dp$. This is essentially due to the structure of the augmented plant and in particular to the fact that the component η of the state of such augmented plant is not affected by the exogenous input v and is directly available for feedback. ◁

If the hypotheses of this corollary are fulfilled, there exists a controller

$$\begin{aligned} \dot{\xi} &= A_c\xi + B_c y'_a + J_s y''_a \\ u &= C_c\xi + D_c y'_a + H_s y''_a, \end{aligned}$$

in which y'_a and y''_a are the upper and—respectively—lower blocks of the output y_a of (4.91), that solves the problem of γ-suboptimal H_∞ feedback design for the augmented plant (4.91). The matrices $A_c, B_c, C_c, D_c, J_s, H_s$ can be found solving an appropriate linear matrix inequality.[17] With these matrices, one can build a controller of the form

$$\begin{aligned} \dot{\eta} &= \Phi\eta + Ge \\ \dot{\xi} &= A_c\xi + B_c y + J_s\eta \\ u &= C_c\xi + D_c y + H_s\eta \end{aligned}$$

that solves the problem of output regulation for the perturbed plant (4.89), robustly with respect to dynamic perturbations that can be expressed as $v = P(s)z$, in which $P(s)$ is the transfer function of a stable uncertain system, satisfying $\|P\|_{H_\infty} < 1/\gamma$.

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[17] See Sect. 3.6.

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Chapter 5
Coordination and Consensus of Linear Systems

5.1 Control of a Network of Systems

In this chapter, we study control problems that can be seen as extensions of the problem of asymptotic tracking discussed in the previous chapter and, in fact, can be addressed by means of appropriate enhancements of the design techniques presented therein. One of such problems is the so-called *leader–follower coordination problem*, which can be cast in the following terms.

Suppose a *set* of linear systems is given, modeled by equations of the form

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \tag{5.1}$$

with state $x_k \in \mathbb{R}^{n_k}$, control input $u_k \in \mathbb{R}^m$ and controlled (and also measured) output $y_k \in \mathbb{R}^p$, for $k = 1, \ldots, N$.[1] The number N of such systems is *large*. The problem is to design a control law under which the output y_k of each system asymptotically tracks the output y_0 of a *single* autonomous system

$$\begin{aligned} \dot{x}_0 &= A_0 x_0 \\ y_0 &= C_0 x_0 . \end{aligned} \tag{5.2}$$

In this context, system (5.2) is called the *leader* and each of the N systems (5.1) is called a *follower*. Altogether, the N systems (5.1) and (5.2) are often called the *agents*.

The theory presented in the previous section could provide an elementary approach to the solution of such problem. If fact, it would suffice to define a *tracking error*

$$e_k = y_k - y_0 \tag{5.3}$$

[1]Note that all such systems have the same number of input and output components.

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_5

and to solve a set of N problems of output regulation for N systems, each one of which is modeled by equations of the form

$$\begin{aligned}\dot{x}_0 &= A_0 x_0\\ \dot{x}_k &= A_k x_k + B_k u_k\\ e_k &= C_k x_k - C_0 x_0,\end{aligned}$$

where the role of exogenous input is now taken by the state of x_0 of the leader. This approach will end up with the design of a set of N controllers, each one of which is modeled by equations of the form

$$\begin{aligned}\dot{x}_{c,k} &= A_{c,k} x_{c,k} + B_{c,k} v_k\\ u_k &= C_{c,k} x_{c,k} + D_{c,k} v_k,\end{aligned} \tag{5.4}$$

whose input v_k is provided by the kth tracking error e_k, i.e., in which

$$v_k = e_k.$$

The problem with this approach, though, is that *each controller* is supposed to be driven by the corresponding tracking error e_k, i.e., each controller needs to *have access to* the (reference) *output of the leader*. If the number N of agents is very large, this entails an excessively large exchange of information, which could be prohibitive if the followers are spatially distributed over a very large (geographical) region. The challenge is to achieve the desired *coordination* (between the output of each follower and the output of the leader) with a limited exchange of information, consisting only in the differences between the output of each agent and those of a limited number of other agents (one of which could be, possibly but not necessarily, the leader).

Such (more restricted) information pattern indeed makes the problem more challenging. In what follows we will consider the case in which the input v_k to each *local* controller is a *fixed* linear combination of the differences between the outputs of the agents, namely a function of the form

$$v_k = \sum_{j=0}^{N} a_{kj}(y_j - y_k) \qquad k = 1, \ldots, N \tag{5.5}$$

in which the a_{kj}'s are elements of a $N \times (N+1)$ sparse matrix. Note that the coefficient $-a_{k0}$ is the *weight* by means of which the difference $(y_k - y_0)$, that is the k-th tracking error e_k, is weighted in the linear combination v_k, which characterizes the information provided to the control of the kth follower. In this chapter, we address the case in which only a few of such coefficients are nonzero, which means that only a few of the local controllers have access to the output of the leader. A problem of this kind is called a problem of *leader–followers coordination*.

If the problem in question is solved (and in the next sections we will discuss how such problem can be solved), eventually all outputs of the followers asymptotically converge to a single function of time, the output of the leader. In this case, it is said that the outputs of all agents, leader and followers altogether, *achieve consensus*. A more general version of the problem is when a set of N agents modeled by equations of the form (5.1) is given, *no leader is specified*, and a control law is sought, similar to the one described above (that is consisting of a set of N local controllers of the form (5.4) with v_k a *fixed* linear combination of the differences between the outputs of the agents) under which the outputs of all the agents achieve consensus, i.e., asymptotically converge to a single function of time. A problem of this kind is called a problem of *leaderless coordination* or, more commonly, a *consensus problem*.[2]

5.2 Communication Graphs

As specified in the previous section, the local controllers (5.4) are driven by a *fixed* linear combination of the differences between the outputs of the agents. This pattern of communication between individual agents is conveniently described by means of a few concepts borrowed from the theory of graphs. A *communication graph*[3] is a triplet $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, A\}$ in which:

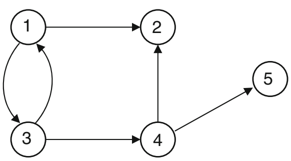
- $\mathscr{V}$ is a set of N *nodes* (or *vertices*), $\mathscr{V} = \{\upsilon_1, \upsilon_2, \dots, \upsilon_N\}$, one for each of the N agents in the set.
- $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is a set of *edges* that models the interconnection between nodes, according to the following convention: (υ_k, υ_j) belongs to $\mathscr{E}$ if there is a flow of information *from node υ_j to node υ_k*.
- A is a $N \times N$ matrix whose entry a_{kj} represents a *weight* in the flow of information from node υ_j to node υ_k. The matrix A is called the *adjacency matrix* of the graph and its elements are *nonnegative*.

It is assumed that there are no self-loops, i.e., that $(\upsilon_k, \upsilon_k) \notin \mathscr{E}$ or, what is the same, $a_{kk} = 0$ for all $k = 1, \dots, N$. The graph is *undirected* if $a_{kj} = a_{jk}$, for all (k, j) (with $k \neq j$). Otherwise, the graph is said to be *directed*. In what follows, we will consider the (more general) case of directed graphs. The set of *neighbors* of node υ_k is the set $\mathscr{N}_k = \{\upsilon_j \in \mathscr{V} : a_{kj} \neq 0\}$, i.e., the set of nodes from which there is a direct flow of information to node υ_k. A *path* from node υ_j to node υ_k is a sequence of r distinct nodes $\{\upsilon_{\ell_1}, \dots, \upsilon_{\ell_r}\}$ with $\upsilon_{\ell_1} = \upsilon_j$ and $\upsilon_{\ell_r} = \upsilon_k$ such that $(\upsilon_{i+1}, \upsilon_i) \in \mathscr{E}$ (or, what is the same, a sequence $\{\upsilon_{\ell_1}, \dots, \upsilon_{\ell_r}\}$ such that, for each i, υ_{ℓ_i} is a neighbor of $\upsilon_{\ell_{i+1}}$).

---

[2]Further motivations for the interest of an information pattern such as the one described by (5.5) can be found in [1–6] and [20].

[3]Background material on graphs can be found, e.g., in [7, 8]. All objects defined below are assumed to be independent of time. In this case the communication graph is said to be *time-invariant*. Extensions of definitions, properties and results to the case of time-varying graphs are not covered in this book and the reader is referred, e.g. to [9, 10].



$$A = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} & 0 \\ a_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{pmatrix}$$

Fig. 5.1 A connected communication graph and its adjacency matrix

Definition 5.1 A graph $\mathcal{G}$ is *connected* if there is a node υ such that, for any other node $\upsilon_k \in \mathcal{V} \setminus \{\upsilon\}$, there is a path from υ to υ_k.

In other words, a graph is connected if there is a node from which information can propagate to all other nodes along paths. This node is sometimes called a *root*.[4]

In the *leaderless coordination* problem, we will consider the case in which the information available for control purpose at the kth agent has the form

$$v_k = \sum_{j=1}^{N} a_{kj}(y_j - y_k) \qquad k = 1, \ldots, N \tag{5.6}$$

in which y_i, for $i = 1, \ldots, N$, is a measurement taken from agent i. This can be equivalently expressed as

$$v_k = \sum_{j \in \mathcal{N}_k} a_{kj}(y_j - y_k) \qquad k = 1, \ldots, N$$

viewing v_k as a weighted sum of the relative values of the measurements taken from agent k and its neighbors.

Letting L denote the matrix defined by

$$\begin{aligned} \ell_{kj} &= -a_{kj} & \qquad \text{for } k \neq j \\ \ell_{kk} &= \textstyle\sum_{i=1}^{N} a_{ki} & \end{aligned}$$

the expression (5.6) can be rewritten as

$$v_k = -\sum_{j=1}^{N} \ell_{kj} y_j \qquad k = 1, \ldots, N. \tag{5.7}$$

The matrix L is called the *Laplacian matrix* of the graph. Observe that, by definition, the diagonal entries of L are nonnegative, the off-diagonal elements are nonpositive and that, for each row, the sum of all elements on this row is zero. Letting $\mathbf{1}_N$ denote the "all-ones" N-vector

[4] Note that in a connected graph there may be more than one node with such property. See the example in Fig. 5.1.

$$\mathbf{1}_N = \text{col}(1, 1, \ldots, 1),$$

this latter property can be written as

$$L\,\mathbf{1}_N = 0.$$

It is seen from the above that the matrix L is singular or, what is the same, that $\lambda = 0$ is always an eigenvalue of L. Such eigenvalue is referred to as the *trivial* eigenvalue of L. Let the other (possibly nonzero) $N-1$ eigenvalues of L be denoted as $\lambda_2(L), \ldots, \lambda_N(L)$. The real parts of these eigenvalues play an important role with respect to the property of connectivity.

Theorem 5.1 *A graph is connected if and only if its Laplacian matrix L has only one trivial eigenvalue $\lambda_1 = 0$ and all other eigenvalues $\lambda_2(L), \ldots, \lambda_N(L)$ have positive real parts.*

In the *leader–followers coordination* problem, we will consider the case in which the information available for control purpose at the kth agent has the form (5.5). The only (marginal) difference between this and (5.6) is that in (5.5) the sum extends from 0 to N (the leader being the agent associated with the index "0") while k ranges from 1 to N. The adjacency matrix is now a $(N+1) \times (N+1)$ matrix, the first row and column being indexed by "0", in which all elements on the first row are zero, because the leader receives no information from the followers. The nonzero entries on the first column correspond to followers which do receive information *directly* from the leader. The expression (5.5) can be rewritten in terms of the tracking errors (5.3) as

$$v_k = \sum_{j=0}^{N} a_{kj}[y_j - y_0 - (y_k - y_0)] = \sum_{j=0}^{N} a_{kj}(e_j - e_k) \qquad k = 1, \ldots, N,$$

and this, in view of the definition of the Laplacian matrix L (associated with the $(N+1) \times (N+1)$ adjacency matrix that characterizes the expression (5.5)) and of the fact that $e_0 = 0$, can be rewritten as (compare with (5.7))

$$v_k = -\sum_{j=1}^{N} \ell_{kj} e_j \qquad k = 1, \ldots, N. \tag{5.8}$$

5.3 Leader–Follower Coordination

In what follows, consistently with a basic assumption in the theory of output regulation, it is assumed that the matrix A_0 which characterizes the leader (5.2) has a purely diagonal Jordan form and all eigenvalues with zero real part. As far as the followers (5.1) are concerned, to simplify matters we assume $m = p = 1$ and, for the time being, that each of them has relative degree 1. We also assume that each follower has *all zeros with negative real part*.

For convenience, the model of each follower is expressed directly in normal form, i.e., as

$$\begin{aligned} \dot{z}_k &= A_{k,00} z_k + A_{k,01} y_k \\ \dot{y}_k &= A_{k,10} z_k + A_{k,11} y_k + b_k u_k \end{aligned} \tag{5.9}$$

in which, as a consequence of the assumption on the zeros, $A_{k,00}$ is a matrix having all eigenvalues in $\mathbb{C}^-$. We also assume, without loss of generality, that $b_k > 0$.

Having defined a tracking error as in (5.3), the equations that characterize the follower can be rewritten in the form (change y_k into $e_k = y_k - C_0 x_0$)

$$\begin{aligned} \dot{z}_k &= A_{k,00} z_k + A_{k,01} e_k + A_{k,01} C_0 x_0 \\ \dot{e}_k &= A_{k,10} z_k + A_{k,11} e_k + [A_{k,11} C_0 - C_0 A_0] x_0 + b_k u_k. \end{aligned} \tag{5.10}$$

For the purpose of using the theory of output regulation, we check first the existence of a solution of the Francis' equations. A standard calculation shows that if $\Pi_{k,0}$ is a solution of the equation

$$\Pi_{k,0} A_0 = A_{k,00} \Pi_{k,0} + A_{k,01} C_0$$

(solution which exists because the spectra of A_0 and $A_{k,00}$ are disjoint), the solution of Francis' equation is given by

$$\Pi_k = \begin{pmatrix} \Pi_{k,0} \\ 0 \end{pmatrix}, \qquad \Psi_k = -\frac{1}{b_k} [A_{k,10} \Pi_{k,0} + A_{k,11} C_0 - C_0 A_0].$$

Changing z_k into $\tilde{z}_k = z_k - \Pi_{k,0} x_0$ yields the simplified system

$$\begin{aligned} \dot{\tilde{z}}_k &= A_{k,00} \tilde{z}_k + A_{k,01} e_k \\ \dot{e}_k &= A_{k,10} \tilde{z}_k + A_{k,11} e_k + b_k [u_k - \Psi_k x_0]. \end{aligned} \tag{5.11}$$

With the results discussed in Sect. 4.7 in mind, we consider for each agent a preprocessing internal model of the form

$$\begin{aligned} \dot{\eta}_k &= \Phi \eta_k + G \bar{u}_k \\ u_k &= \Gamma \eta_k + \bar{u}_k \end{aligned} \tag{5.12}$$

in which Φ and G are as in (4.23) and (4.24), where we take $p = 1$ and pick $s_0, s_1, \ldots, s_{d-1}$ as the coefficients of the minimal polynomial of A_0, and where Γ is such that $\Phi - G\Gamma$ is a Hurwitz matrix. As shown in Sect. 4.6 there always exists a matrix Σ_k such that (see (4.59) in this respect)

$$\Sigma_k A_0 = \Phi \Sigma_k$$
$$\Psi_k = \Gamma \Sigma_k .$$

Using these identities and changing η_k into $\tilde{\eta}_k = \eta_k - \Sigma_k x_0$ yields the following (augmented) system

$$\begin{aligned} \dot{\tilde{z}}_k &= A_{k,00}\tilde{z}_k + A_{k,01}e_k \\ \dot{\tilde{\eta}}_k &= \Phi\tilde{\eta}_k + G\bar{u}_k \\ \dot{e}_k &= A_{k,10}\tilde{z}_k + A_{k,11}e_k + b_k[\Gamma\tilde{\eta}_k + \bar{u}_k], \end{aligned} \tag{5.13}$$

which, as expected, is independent of x_0. This is a system having relative degree 1 between input $\bar{u}_k$ and output e_k. To put it in normal form, define

$$\zeta_k = \begin{pmatrix} \tilde{z}_k \\ \tilde{\eta}_k - \frac{1}{b_k}Ge_k \end{pmatrix}$$

and obtain, after simple manipulations, a system having the following structure

$$\begin{aligned} \dot{\zeta}_k &= F_{k,00}\zeta_k + F_{k,01}e_k \\ \dot{e}_k &= F_{k,10}\zeta_k + F_{k,11}e_k + b_k\bar{u}_k, \end{aligned} \tag{5.14}$$

in which

$$F_{k,00} = \begin{pmatrix} A_{k,00} & 0 \\ -\frac{1}{b_k}GA_{k,10} & \Phi - G\Gamma \end{pmatrix}.$$

By assumption the eigenvalues of $A_{k,00}$ are in $\mathbb{C}^-$, and by construction the eigenvalues of $\Phi - G\Gamma$ are in $\mathbb{C}^-$. Thus, all eigenvalues of $F_{k,00}$ are in $\mathbb{C}^-$. It follows from the theory developed in Sect. 2.1 that, if it were possible to pick

$$\bar{u}_k = -ge_k$$

with a sufficiently large value of g, the state of (5.14) would asymptotically converge to 0. In particular e_k would converge to 0 and this would imply that the desired tracking goal is achieved. This control mode, though, in general is not feasible, as anticipated in Sect. 5.1, because this control mode requires that, for each $k = 1, \ldots, N$, the controller of agent k has access to the tracking error e_k. This would entail an excessively large exchange of information between the leader and the followers.

Fortunately, the desired control goal can still be achieved if $\bar{u}_k$ is taken as (compare with (5.5))

$$\bar{u}_k = g\sum_{j=0}^{N} a_{kj}(y_j - y_k) \qquad k = 1, \ldots, N \tag{5.15}$$

in which g is a suitable gain parameter and the a_{kj} are the entries of the adjacency matrix A of a *connected* graph modeling the pattern of communication between leader and followers. To see why this is the case, let us *stack* all systems (5.14) together, to obtain a system modeled as

$$\begin{aligned} \dot{\zeta} &= F_{00}\zeta + F_{01}e \\ \dot{e} &= F_{10}\zeta + F_{11}e + B\bar{u} \end{aligned} \tag{5.16}$$

in which

$$\begin{aligned} \zeta &= \mathrm{col}(\zeta_1, \dots, \zeta_N) \\ e &= \mathrm{col}(e_1, \dots, e_N) \\ \bar{u} &= \mathrm{col}(\bar{u}_1, \dots, \bar{u}_N) \end{aligned}$$

and

$$F_{00} = \begin{pmatrix} F_{1,00} & 0 & \cdots & 0 \\ 0 & F_{2,00} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & F_{N,00} \end{pmatrix} \qquad F_{01} = \begin{pmatrix} F_{1,01} & 0 & \cdots & 0 \\ 0 & F_{2,01} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & F_{N,01} \end{pmatrix}$$

$$F_{10} = \begin{pmatrix} F_{1,10} & 0 & \cdots & 0 \\ 0 & F_{2,10} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & F_{N,10} \end{pmatrix} \qquad F_{11} = \begin{pmatrix} F_{1,11} & 0 & \cdots & 0 \\ 0 & F_{2,11} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & F_{N,11} \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & b_N \end{pmatrix}.$$

Note that the vector $\bar{u}$, whose entries are defined in (5.15), can be expressed as (see (5.8) in this respect)

$$\bar{u} = -gL_{22}e$$

in which g is a gain parameter and L_{22} is the lower-right $(N \times N)$ submatrix of the Laplacian matrix L associated with the adjacency matrix A that characterizes the (fixed) communication protocol. Thus, system (5.16) controlled by (5.15) is modeled by

$$\begin{aligned} \dot{\zeta} &= F_{00}\zeta + F_{01}e \\ \dot{e} &= F_{10}\zeta + (F_{11} - gBL_{22})e \end{aligned} \tag{5.17}$$

in which the matrix BL_{22} is fixed and g is still a free design parameter.

We discuss now the properties of the matrix BL_{22}. Bearing in mind the fact that B is a purely diagonal matrix with positive entries, consider the $(N+1) \times (N+1)$ matrix $\hat{A}$ in which the element $\hat{a}_{kj}$ is defined as

$$\begin{aligned} \hat{a}_{0j} &= 0 & & j = 0, 1, \dots, N \\ \hat{a}_{kj} &= b_k a_{kj}, & & j = 0, 1, \dots, N \quad k = 1, \dots, N. \end{aligned}$$

Clearly, the matrix $\hat{A}$ has been obtained from the adjacency matrix A, associated with communication graph $\mathcal{G}$ that characterizes the communication protocol between leader and followers, by multiplying all elements on the kth row (for $k = 1, \ldots, N$) of A by the *positive* coefficient b_k. The matrix $\hat{A}$ can be regarded as an adjacency matrix of a communication graph, that will be denoted as $\hat{\mathcal{G}}$. Since all coefficients b_k are nonzero, it is readily seen that if the graph $\mathcal{G}$ is connected, so is the graph $\hat{\mathcal{G}}$.[5] Bearing in mind the definition of the Laplacian matrix of a graph, observe that the matrix BL_{22} coincides with the lower-right $(N \times N)$ submatrix of the Laplacian matrix $\hat{L}$ of the graph $\hat{\mathcal{G}}$. By virtue of Theorem 5.1, if the graph $\hat{\mathcal{G}}$ is connected, the matrix $\hat{L}$ has one trivial eigenvalue at 0 while all other eigenvalues have positive real part. But, in the present case, the first row of $\hat{L}$ has all zero entries and hence the eigenvalues of BL_{22} coincide with the N nontrivial eigenvalues of $\hat{L}$. In this way, it is concluded that, *if the graph $\mathcal{G}$ is connected, the matrix BL_{22} has all eigenvalues with positive real part.*

Return now to the system (5.17) in which we know that, by assumption (and also by construction), all eigenvalues of F_{00} have negative real part and that, if the graph $\mathcal{G}$ is connected, the *negative* of the matrix BL_{22} has all eigenvalues with negative real part as well. Let P_0 be a positive definite symmetric solution of the Lyapunov equation

$$P_0 F_{00} + F_{00}^{\mathrm{T}} P_0 = -I$$

and let P_1 be a positive definite symmetric solution of the Lyapunov equation

$$P_1(-BL_{22}) + (-BL_{22})^{\mathrm{T}} P_1 = -I.$$

Set

$$P = \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix}$$

and observe that

$$P \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} - gBL_{22} \end{pmatrix} + \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} - gBL_{22} \end{pmatrix}^{\mathrm{T}} P$$

$$= \begin{pmatrix} -I & P_0 F_{01} + F_{10}^{\mathrm{T}} P_1 \\ P_1 F_{10} + F_{01}^{\mathrm{T}} P_0 & -gI + P_1 F_{11} + F_{11}^{\mathrm{T}} P_1 \end{pmatrix}.$$

It is readily seen that, if g is large enough, the matrix on the right-hand-side is negative definite. To this end, in fact, it suffices to look at the Schur's complement of the upper-left block, namely

[5] In fact, if the graph $\mathcal{G}$ possesses a root from which information can propagate to all other nodes along paths, the same is true for the graph $\hat{\mathcal{G}}$, because the nonzero entries of $\hat{A}$ coincide with the nonzero entries of A.

$$-gI + P_1 F_{11} + F_{11}^{\mathrm{T}} P_1 + [P_1 F_{10} + F_{01}^{\mathrm{T}} P_0][P_0 F_{01} + F_{10}^{\mathrm{T}} P_1]\,,$$

which is negative definite if g is large enough.

If this is the case, system (5.17) is stable and in particular e asymptotically decays to 0, which means that the desired coordination goal is achieved. We summarize the result as follows.

Proposition 5.1 *Consider a leader–followers coordination problem. Suppose that each of the followers has relative degree 1 and all zeros in $\mathbb{C}^-$. Let each follower be controlled by a local controller of the form (5.12), in which Φ and G are as in (4.23) and (4.24), where we take $p = 1$ and pick $s_0, s_1, \dots, s_{d-1}$ as the coefficients of the minimal polynomial of A_0, and where Γ is such that $\Phi - G\Gamma$ is a Hurwitz matrix. Let the control $\bar{u}_k$ provided to such local controller be expressed as in (5.15), in which g is a gain parameter and the a_{kj} are the entries of the adjacency matrix A, associated with the communication graph $\mathcal{G}$ that characterizes the communication protocol between leader and followers. Suppose the graph $\mathcal{G}$ is connected. Then, there is a number g^* such that, if $g > g^*$, the output of the followers achieve consensus with the output of the leader, i.e.,*

$$\lim_{t\to\infty} (y_k(t) - y_0(t)) = 0$$

for all $k = 1, \dots, N$.

The construction described above can easily be extended to the case in which the agents have a relative degree higher than 1. Without loss of generality, we can assume that all agents in (5.1) have *the same* relative degree r. In fact, it is always possible to achieve such property by adding a suitable number of integrators on the input channel of each agent. This being the case, the model of each agent can be expressed in (strict) normal form as (compare with (2.9))

$$\begin{aligned}
\dot{z}_k &= A_{k,00} z_k + A_{k,01} \xi_k \\
\dot{\xi}_k &= \hat{A} \xi_k + \hat{B}[A_{k,10} z_k + A_{k,11} \xi_k + b_k u_k] \\
y_k &= \hat{C} \xi_k
\end{aligned} \tag{5.18}$$

in which

$$\xi_k = \mathrm{col}(\xi_{k1}, \xi_{k2}, \dots, \xi_{kr}).$$

As before, it is assumed that $A_{k,00}$ is a matrix having all eigenvalues in $\mathbb{C}^-$ and $b_k > 0$.

It is known from Sect. 2.4 that the dynamics of (5.18), if ξ_{kr} is replaced by

$$\theta_k = \xi_{kr} + a_0 \xi_{k1} + a_1 \xi_{k2} + \cdots + a_{r-2} \xi_{k,r-1} \tag{5.19}$$

can be seen as a system having relative degree 1 between input u_k and output θ_k, modeled in normal form as (compare with (2.22))

$$\begin{aligned}\dot{\zeta}_k &= F_{k,00}\zeta_k + F_{k,10}\theta_k \\ \dot{\theta}_k &= F_{k,10}\zeta_k + F_{k,11}\theta_k + b_k u_k.\end{aligned} \tag{5.20}$$

Moreover, if the coefficients $a_0, a_1, \ldots, a_{r-2}$ are such that the polynomial (2.23) has all roots in $\mathbb{C}^-$, all eigenvalues of $F_{k,00}$ are in $\mathbb{C}^-$.

Recall that, by definition, $\xi_{ki}(t)$ coincides with the derivative of order $(i-1)$ of $y_k(t)$ with respect to time (for $i = 1, \ldots, r$). Thus, the function $\theta_k(t)$, in terms of the original output $y_k(t)$, can be expressed as

$$\theta_k(t) = y_k^{(r-1)}(t) + a_0 y_k(t) + a_1 y_k^{(1)}(t) + \cdots + a_{r-2} y_k^{(r-2)}(t).$$

For consistency, also the output y_0 of the leader (5.2) is replaced by variable having a similar structure, namely

$$\theta_0 = C_0 A_0^{r-1} x_0 + a_0 C_0 x_0 + a_1 C_0 A_0 x_0 + \cdots a_{r-2} C_0 A_0^{r-2} x_0$$

which yields

$$\theta_0(t) = y_0^{(r-1)}(t) + a_0 y_0(t) + a_1 y_0^{(1)}(t) + \cdots a_{r-2} y_0^{(r-2)}(t).$$

The systems defined in this way satisfy the assumptions of Proposition 5.1. Thus, it can be claimed that if each agent is controlled by a controller of the form (5.12), with parameters chosen as specified in the proposition, with $\bar{u}_k$ given by

$$\bar{u}_k = g \sum_{j=0}^{N} a_{kj}(\theta_j - \theta_k) \qquad k = 1, \ldots, N, \tag{5.21}$$

and the communication graph is connected, there is g^* such that, if $g > g^*$, the (modified) outputs of the followers achieve consensus with the (modified) output of the leader, namely

$$\lim_{t\to\infty} (\theta_k(t) - \theta_0(t)) = 0$$

for all $k = 1, \ldots, N$. In other words, for each $k = 1, \ldots, N$, the difference

$$\chi_k(t) = \theta_k(t) - \theta_0(t)$$

satisfies $\lim_{t\to\infty} \chi_k(t) = 0$.

Recalling the definition of tracking error e_k given in (5.3), and the definitions of all θ_k's (including that of θ_0), it is seen that $\chi_k(t)$ can be expressed as

$$\chi_k(t) = e_k^{(r-1)}(t) + a_0 e_k(t) + a_1 e_k^{(1)}(t) + \cdots a_{r-2} e_k^{(r-2)}(t), \tag{5.22}$$

which shows that the vector

$$\bar{e}_k = \mathrm{col}(e_k, e_k^{(1)}, \dots, e_k^{(r-2)})$$

satisfies a linear equation of the form

$$\dot{\bar{e}}_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-2} \end{pmatrix} \bar{e}_k + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \chi_k.$$

Since the $a_0, a_1, \dots, a_{r-2}$ are such that the polynomial (2.23) has all roots in $\mathbb{C}^-$, this is a stable system, driven by an input $\chi_k(t)$ that asymptotically decays to 0. Thus $\bar{e}_k(t)$ asymptotically decays to zero and so does the tracking error $e_k(t)$.

In conclusion, if each agent is controlled by a controller of the form (5.12), with parameters chosen as specified in the Proposition 5.1, with $\bar{u}_k$ given by (5.21), and the communication graph is connected, for large enough values of g the tracking errors (5.3) asymptotically decay to zero, i.e., outputs of the followers (5.1) achieve consensus with the output of the leader (5.2).

The communication protocol (5.21) presumes the availability of the variables θ_k's. If this is not the case, one could replace them by quantities of the form

$$\hat{\theta}_k = \hat{\xi}_{kr} + a_0\hat{\xi}_{k1} + a_1\hat{\xi}_{k2} + \cdots a_{r-2}\hat{\xi}_{k,r-1},$$

in which the $\hat{\xi}_{ki}$'s, for $i = 1, \dots, r$ and $k = 0, 1, \dots, N$, are estimates provided by systems of the form (2.26), the kth one of which ($k = 0, 1, \dots, N$) is modeled by equations of the form

$$\begin{aligned} \dot{\hat{\xi}}_{k1} &= \hat{\xi}_{k2} + \kappa c_{r-1}(y_k - \hat{\xi}_{k1}) \\ \dot{\hat{\xi}}_{k2} &= \hat{\xi}_{k3} + \kappa^2 c_{r-2}(y_k - \hat{\xi}_{k1}) \\ &\cdots \\ \dot{\hat{\xi}}_{k,r-1} &= \hat{\xi}_{kr} + \kappa^{r-1} c_1(y_k - \hat{\xi}_{k1}) \\ \dot{\hat{\xi}}_{k,r} &= \kappa^r c_0(y_k - \hat{\xi}_{k1}), \end{aligned} \tag{5.23}$$

in which the c_i's are such that the polynomial $p(\lambda) = \lambda^{r-1} + c_{r-1}\lambda^{r-2} + \cdots + c_2\lambda + c_0$ has all eigenvalues in $\mathbb{C}^-$ and κ is a design parameter. Arguments similar to those used in Sect. 2.5, which are not repeated here, can be used to show that if κ is large enough the desired consensus goal is achieved.

5.4 Consensus in a Homogeneous Network: Preliminaries

In this and in the following sections, we will consider the problem of achieving consensus among the outputs of a set of N agents, modeled as in (5.1).[6] The input u_k to each of such systems is provided by a local controller modeled as in (5.4), in which v_k represents (relative) information received from other agents, and has the form (5.6), consisting in a weighted linear combination of the relative values of the outputs of the *neighboring* agents. The weights a_{kj} are the entries of the adjacency matrix A that characterizes the communication pattern among agents. It is said that the set of (controlled) agents *achieve consensus*, if for some function of time, that will be denoted by $y_{\text{cons}}(t)$, it occurs that

$$\lim_{t\to\infty} (y_k(t) - y_{\text{cons}}(t)) = 0,$$

for all $k = 1, \dots, N$.

There is no leader in this setting, thus the consensus output $y_{\text{cons}}(t)$ is not the output of a separate autonomous system. Note, in this respect, that if all outputs of the (controlled) agents coincide, $v_k = 0$, and hence the control u_k generated by the controller (5.4) is the output of the autonomous system

$$\begin{aligned} \dot{x}_{c,k} &= A_{c,k} x_{c,k} \\ u_k &= C_{c,k} x_{c,k}. \end{aligned}$$

As a consequence, the output $y_k(t)$ of the kth agent is necessarily the output of the autonomous system

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k C_{c,k} x_{c,k} \\ \dot{x}_{c,k} &= A_{c,k} x_{c,k} \\ y_k &= C_k x_k. \end{aligned}$$

In this and in the following section we will study the simpler case in which all agents are identical, i.e.,

$$\begin{aligned} \dot{x}_k &= A x_k + B u_k \\ y_k &= C x_k, \end{aligned} \qquad \text{for all} \quad k = 1, \dots, N \tag{5.24}$$

in which $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are fixed matrices. In this case the network of controlled agents is said to be *homogeneous*. In the last section of the chapter we will discuss the general case in which the agents may be different, which is called the case of a *heterogeneous* network.

We will consider two modes of control. A *static* control mode, in which the control u_k is given by

$$u_k = K v_k \tag{5.25}$$

[6] From now on, we drop the assumption—considered in the previous section—that $p = m = 1$.

with $K \in \mathbb{R}^{m\times p}$ a matrix of feedback gains to be found, and a *dynamic* control mode, in which the control u_k is the output of a dynamical system

$$\begin{aligned}\dot{\xi}_k &= A_c\xi_k + B_c v_k \\ u_k &= C_c\xi_k\end{aligned} \tag{5.26}$$

in which $\xi_k \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c are parameters characterizing the dynamic control law to be found.

Bearing in mind the expression (5.7) of v_k, it is readily seen that the first mode of control results in a overall closed-loop system modeled by a set of equations of the form

$$\dot{x}_k = Ax_k - \sum_{j=1}^{N} \ell_{kj} BKCx_j \qquad i = 1, \dots, N \tag{5.27}$$

while the second mode of control results in a closed-loop system modeled by a set of equations of the form

$$\begin{pmatrix}\dot{x}_k \\ \dot{\xi}_k\end{pmatrix} = \begin{pmatrix}A & BC_c \\ 0 & A_c\end{pmatrix}\begin{pmatrix}x_k \\ \xi_k\end{pmatrix} - \sum_{j=1}^{N} \ell_{kj}\begin{pmatrix}0 \\ B_c\end{pmatrix}\begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x_j \\ \xi_j\end{pmatrix} \qquad i = 1, \dots, N. \tag{5.28}$$

Both these sets have the same structure, that of a set of systems of the form

$$\dot{\bar{x}}_k = F\bar{x}_k - \sum_{j=1}^{N} \ell_{kj} GH\bar{x}_j \qquad i = 1, \dots, N. \tag{5.29}$$

In fact, (5.27) derives from (5.29) by picking

$$\bar{x}_k = x_k, \qquad F = A, \qquad G = BK, \qquad H = C,$$

while (5.28) derives from (5.29) by picking

$$\bar{x}_k = \begin{pmatrix}x_k \\ \xi_k\end{pmatrix}, \qquad F = \begin{pmatrix}A & BC_c \\ 0 & A_c\end{pmatrix}, \qquad G = \begin{pmatrix}0 \\ B_c\end{pmatrix}, \qquad H = \begin{pmatrix}C & 0\end{pmatrix}.$$

Letting $\bar{n}$ denote the dimension of $\bar{x}_k$, letting x denote the $N\bar{n}$-vector

$$x = \mathrm{col}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N)$$

and using the Kroeneker product of matrices,[7] the set (5.29) of systems can be expressed as

$$\dot{x} = (I_N \otimes F)x - (L \otimes GH)x. \tag{5.30}$$

[7]For a couple of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, their Kroeneker product—denoted by $A \otimes B$—is the $mp \times nq$ matrix defined as

In what follows, it is assumed that the communication graph is *connected*. If this is the case, the Laplacian matrix L of the graph has one trivial eigenvalue $\lambda_1 = 0$, for which $\mathbf{1}_N$ is an eigenvector, while all other eigenvalues have positive real part, i.e., there exists a number $\mu > 0$ such that

$$\mathrm{Re}[\lambda_i(L)] \geq \mu \qquad \text{for all } i = 2, \ldots, N. \tag{5.31}$$

Consider the nonsingular matrix

$$T = \begin{pmatrix} 1 & 0_{1\times(N-1)} \\ \mathbf{1}_{(N-1)} & I_{(N-1)} \end{pmatrix}. \tag{5.32}$$

It is readily seen that the matrix $\tilde{L} = T^{-1}LT$ has the following structure

$$\tilde{L} = \begin{pmatrix} 0 & \tilde{L}_{12} \\ 0 & \tilde{L}_{22} \end{pmatrix}$$

in which $\tilde{L}_{12}$ is a $1 \times (N-1)$ row vector and $\tilde{L}_{22}$ is a $(N-1) \times (N-1)$ matrix, whose eigenvalues evidently coincide with the nontrivial eigenvalues $\lambda_2(L), \ldots, \lambda_N(L)$ of L.

Consider now the controlled system (5.30) and change of coordinates as $\tilde{x} = (T^{-1} \otimes I_n)x$. Then,[8]

$$\begin{aligned}\dot{\tilde{x}} &= (T^{-1} \otimes I_n)[(I_N \otimes F) - (L \otimes GH)](T \otimes I_n)\tilde{x} \\ &= (T^{-1} \otimes I_n)[(T \otimes F) - (LT \otimes GH)]\tilde{x} \\ &= [(I_N \otimes F) - (\tilde{L} \otimes GH)]\tilde{x}.\end{aligned}$$

Observe that, by definition of T, the vector $\tilde{x}$ has the form

$$\tilde{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 - \bar{x}_1 \\ \cdots \\ \bar{x}_N - \bar{x}_1 \end{pmatrix}$$

and hence can be split be split as

(Footnote 7 continued)

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

[8] We use in what follows the property $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, which—in particular—implies $(T^{-1} \otimes I_n)^{-1} = (T \otimes I_n)$.

$$\tilde{x} = \begin{pmatrix} \bar{x}_1 \\ x_\delta \end{pmatrix}$$

in which

$$x_\delta = \begin{pmatrix} \bar{x}_2 - \bar{x}_1 \\ \cdots \\ \bar{x}_N - \bar{x}_1 \end{pmatrix}.$$

Then, it is easily checked that the system thus obtained has a *block-triangular* structure, of the following form

$$\begin{aligned} \dot{\bar{x}}_1 &= F\bar{x}_1 - (\tilde{L}_{12} \otimes GH)x_\delta \\ \dot{x}_\delta &= [(I_{(N-1)} \otimes F) - (\tilde{L}_{22} \otimes GH)]x_\delta. \end{aligned} \tag{5.33}$$

From this, the following conclusion is immediately obtained.[9]

Lemma 5.1 *Consider the controlled system (5.30) and suppose that all the eigenvalues of the matrix*

$$(I_{(N-1)} \otimes F) - (\tilde{L}_{22} \otimes GH) \tag{5.34}$$

have negative real part. Then, for all $k = 2, \ldots, N$

$$\lim_{t\to\infty} [\bar{x}_k(t) - \bar{x}_1(t)] = 0$$

Thus, the states $\bar{x}_k(t)$ of (5.29) achieve consensus.

This result stresses the importance of placing the eigenvalues of the matrix (5.34) in the open left-half plane, so as to reach consensus among the states $\bar{x}_k(t)$ of (5.29). In order to show how this goal can be pursued, let $R \in \mathbb{C}^{(N-1)\times(N-1)}$ be a nonsingular matrix, define

$$\Lambda = R\tilde{L}_{22}R^{-1}$$

and choose R in such a way that Λ is a triangular matrix.[10] If this is the case, the entries on the diagonal of Λ coincide with the nontrivial eigenvalues $\lambda_2(L), \ldots, \lambda_N(L)$ of L. The similarity transformation $R \otimes I_n$ changes the matrix (5.34) in

$$(I_{(N-1)} \otimes F) - (\Lambda \otimes GH).$$

Since the matrix Λ is diagonal, the matrix in question is *block-diagonal*, the diagonal blocks being $\bar{n} \times \bar{n}$ matrices of the form

$$F - \lambda_i(L)GH, \qquad i = 2, \ldots, N. \tag{5.35}$$

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[9]See, e.g., [11].

[10]Note that in this case the elements of R may be complex numbers.

Thus, it can be claimed that the eigenvalues of the matrix (5.34) have negative real part if and only if, for each $i = 2, \ldots, N$, the eigenvalues of the matrix (5.35) have negative real part. This yields the conclusion summarized as follows.

Proposition 5.2 *Consider the controlled system (5.30). Suppose that for each $i = 2, \ldots, N$, the eigenvalues of the matrix (5.35) have negative real part. Then, the states $\bar{x}_k(t)$ of (5.29) achieve consensus.*

In the next section we will discuss how the property indicated in this proposition can be fulfilled, separately for the static and for the dynamic control mode. Note also that, in both cases, the output y_k of the kth agent (5.24) can be expressed as a linear function of the state $\bar{x}_k$ of (5.29). In fact, in both cases

$$y_k = H\bar{x}_k.$$

Thus, if the conditions of Proposition 5.2 hold, the outputs y_k of (5.24) achieve consensus, as desired.

One may wonder how the consensus trajectory looks like. To this end, it is convenient to recall the following basic result.

Lemma 5.2 *Consider the block-triangular system*

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 \\ \dot{z}_2 &= A_{22}z_2. \end{aligned} \tag{5.36}$$

with $z_1 \in \mathbb{R}^{n_1}$ and $z_2 \in \mathbb{R}^{n_2}$. Suppose that all the eigenvalues of A_{22} have negative real part. Then, there exists numbers $M > 0$, $\alpha > 0$ and a matrix $R \in \mathbb{R}^{n_1 \times (n_1+n_2)}$ such that, for all $(z_1(0), z_2(0)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

$$\|z_1(t) - e^{A_{11}t}z^\circ\| \leq Me^{-\alpha t}\|z_2(0)\|,$$

in which

$$z^\circ = R\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}.$$

This lemma can be used to estimate the asymptotic behavior of the response $\bar{x}_1(t)$ in system (5.33). In fact, because of this lemma, it is easily seen that, if all the eigenvalues of the matrix (5.34) have negative real part, there exist numbers $M > 0$, $\alpha > 0$ and a matrix $R \in \mathbb{R}^{\bar{n} \times N\bar{n}}$ such that, for all $x(0)$, the state $\bar{x}_1(t)$ satisfies

$$\|\bar{x}_1(t) - e^{Ft}\bar{x}^\circ\| \leq Me^{-\alpha t}\|x_\delta(0)\|.$$

in which $\bar{x}^\circ = Rx(0)$. Since all $\bar{x}_k(t)$'s asymptotically converge to $\bar{x}_1(t)$, it follows that

$$\lim_{t\to\infty}[\bar{x}_k(t) - e^{Ft}\bar{x}^\circ] = 0 \qquad \text{for all } k = 1, 2, \ldots, N.$$

In other words, if all the eigenvalues of the matrix (5.34) have negative real part, the states of the individual subsystems of (5.30) reach consensus along a particular solution of

$$\dot{\bar{x}} = F\bar{x},$$

the solution resulting from an initial state $\bar{x}^\circ$ which is a (fixed) linear function of the initial states $\bar{x}_1(0), \dots, \bar{x}_N(0)$ of the individual subsystems.

In the original consensus problem in static control mode, $\bar{x}_k = x_k$ and the matrix F coincides with the matrix A that describes the dynamics of the agents. Thus, if the hypotheses of the previous proposition are fulfilled, we conclude that

$$\lim_{t\to\infty} [x_k(t) - e^{At}x^\circ] = 0 \qquad \text{for all } k = 1, 2, \dots, N,$$

for some x° of the form $x^\circ = Rx(0)$, with $x(0) = \mathrm{col}(x_1(0), \dots, x_N(0))$. In particular, the outputs of the individual systems reach consensus along the function

$$y_{\mathrm{cons}}(t) = Ce^{At}x^\circ.$$

In the consensus problem in dynamic control mode,

$$\bar{x}_k = \begin{pmatrix} x_k \\ \xi_k \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} A & BC_{\mathrm{c}} \\ 0 & A_{\mathrm{c}} \end{pmatrix},$$

If the hypotheses of the previous proposition are fulfilled, the states $x_k(t)$ of the individual subsystems reach consensus along a particular solution of

$$\begin{pmatrix} \dot{x}_k \\ \dot{\xi}_k \end{pmatrix} = \begin{pmatrix} A & BC_{\mathrm{c}} \\ 0 & A_{\mathrm{c}} \end{pmatrix} \begin{pmatrix} x_k \\ \xi_k \end{pmatrix}.$$

If, *in addition*, the matrix A_{c} is a Hurwitz matrix, the upper component $x_k(t)$ of this solution satisfies (again, use Lemma 5.2)

$$\|x_k(t) - e^{At}x_k^\circ\| \le Ke^{-\lambda t}\|\xi_k(0)\|,$$

for some x°. Hence, it can be concluded that the states x_k of the individual agents still satisfy a inequality of the form

$$\lim_{t\to\infty} [x_k(t) - e^{At}x^\circ] = 0 \qquad \text{for all } k = 1, 2, \dots, N,$$

for some x° of the form $x^\circ = Rx(0)$, with $x(0) = \mathrm{col}(\bar{x}_1(0), \dots, \bar{x}_N(0))$ and $\bar{x}_k(0) = \mathrm{col}(x_k(0),\ \xi_k(0))$. Once again, the outputs of the individual agents reach consensus along a function

$$y_{\mathrm{cons}}(t) = Ce^{At}x^\circ.$$

Remark 5.1 Note that the consensus output is determined by the set of *initial states* of *all* the individual agents (and controllers). In fact, the state x° is determined by all such initial states. In other words, there is not a unique consensus trajectory, but rather one consensus trajectory for each choice of initial states. ◁

5.5 Consensus in a Homogeneous Network: Design

In the static control mode, we consider the special case of agents (5.24) in which $n = m$ and having $B = I$ (or, equivalently, having a nonsingular B). In this case the matrix (5.35) reduces to a matrix of the form

$$A - \lambda_i(L)KC, \qquad i = 2, \dots, N.$$

We also assume that the pair (A, C) is *observable*.

This being the case, let $\mu > 0$ be a number satisfying (5.31), pick $a > 0$ and let P be the unique positive definite symmetric solution of the algebraic Riccati equation[11]

$$AP + PA^{\mathrm{T}} - 2\mu PC^{\mathrm{T}}CP + aI = 0. \tag{5.37}$$

Set

$$K = PC^{\mathrm{T}}.$$

To determine whether the resulting matrix

$$A_i = A - \lambda_i(L)PC^{\mathrm{T}}C$$

has all the eigenvalues with negative real part, it suffices to check that[12]

$$x^*(PA_i^* + A_iP)x < 0, \qquad \text{for all } x \neq 0$$

in which the superscript " $*$ " denotes conjugate transpose. By construction

$$\begin{aligned} x^*(PA_i^* + A_iP)x &= x^*(PA^{\mathrm{T}} - \lambda_i^*(L)PC^{\mathrm{T}}CP + AP - \lambda_i(L)PC^{\mathrm{T}}CP)x \\ &= x^*(PA^{\mathrm{T}} + AP - 2\mathrm{Re}[\lambda_i(L)]PC^{\mathrm{T}}CP)x \\ &\leq x^*(PA^{\mathrm{T}} + AP - 2\mu PC^{\mathrm{T}}CP)x = -a\|x\|^2 \end{aligned}$$

and hence it is concluded that the choice $K = PC^{\mathrm{T}}$ solves the problem. For convenience, we summarize this result a follows.[13]

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[11] The existence of such $P > 0$ is guaranteed by the assumption that the pair (A, C) is observable. See, e.g., [12].

[12] See proof of Theorem A.2 in Appendix A.

[13] See, e.g., [13]. See also [14, 15] and [21–23] for further reading.

Proposition 5.3 *Consider a set of agents of the form (5.24), with $B = I$. Suppose the pair (A, C) is observable. Suppose the communication graph is connected. Let the control be given by*

$$u_k = PC^{\mathrm{T}}\Big(\sum_{j=1}^{N} a_{kj}(y_i - y_k)\Big) \qquad k = 1, \dots, N,$$

with P the unique positive definite symmetric solution of the algebraic Riccati equation (5.37). Then, output consensus is reached.

In the dynamic control mode, the matrix (5.35) reduces to a matrix of the form

$$\begin{pmatrix} A & BC_{\mathrm{c}} \\ -\lambda_i(L)B_{\mathrm{c}}C & A_{\mathrm{c}} \end{pmatrix}$$

which is similar to a matrix of the form

$$\begin{pmatrix} A & -\lambda_i(L)BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}} \end{pmatrix}. \tag{5.38}$$

To place the eigenvalues of this matrix in $\mathbb{C}^-$, the idea is to choose for $A_{\mathrm{c}}, B_{\mathrm{c}}$ the structure of an observer-based stabilizer, with

$$A_{\mathrm{c}} = A + GC - BC_{\mathrm{c}}, \qquad B_{\mathrm{c}} = -G,$$

and G such that $A + GC$ has all eigenvalues in $\mathbb{C}^-$ (which is possible under the assumption that (A, C) is observable).

For the design of the matrix C_{c}, consider the the family if algebraic Riccati equations

$$P_\varepsilon A + A^{\mathrm{T}}P_\varepsilon - \mu P_\varepsilon BB^{\mathrm{T}}P_\varepsilon + \varepsilon I = 0 \tag{5.39}$$

in which $\varepsilon > 0$.[14] It is known[15] that if the matrix A has eigenvalues with non-positive real parts and the pair (A, B) is reachable, this equation has a unique positive definite symmetric solution P_ε. The matrix P_ε depends continuously on ε and $\lim_{\varepsilon\downarrow 0} P_\varepsilon = 0$. This being the case, set $C_{\mathrm{c}} = B^{\mathrm{T}}P_\varepsilon$.

In summary, the triplet $\{A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}\}$ is chosen as

$$\begin{aligned} A_{\mathrm{c}} &= A + GC - BB^{\mathrm{T}}P_\varepsilon \\ B_{\mathrm{c}} &= -G \\ C_{\mathrm{c}} &= B^{\mathrm{T}}P_\varepsilon \end{aligned}$$

[14] The approach described hereafter is based on the work [16].

[15] See, e.g., [17, p. 22].

with sufficiently small ε. Note that, since $\lim_{\varepsilon\downarrow 0} P_\varepsilon = 0$, for small ε the eigenvalues of A_c have negative real part, as desired. In what follows, it is shown that a controller defined in this way is capable of solving the problem of output consensus for the homogeneous network of systems (5.24)

Proposition 5.4 *Consider a set of agents of the form (5.24). Suppose that A has eigenvalues with non-positive real parts, that the pair (A, B) is reachable and the pair (A, C) is observable. Suppose the communication graph is connected. Let the control be given by*

$$\begin{aligned} \dot{\xi}_k &= (A + GC - BB^\mathrm{T} P_\varepsilon)\xi_k - G\Big(\textstyle\sum_{j=1}^N a_{kj}(y_j - y_k)\Big) \\ u_k &= (B^\mathrm{T} P_\varepsilon)\xi_k \end{aligned} \qquad k = 1, \ldots, N,$$

with G such that $\sigma(A + GC) \in \mathbb{C}^-$ and P_ε the unique positive definite symmetric solution of the algebraic Riccati equation (5.39). Then, there is a number ε^* such that, if $0 < \varepsilon < \varepsilon^*$, output consensus is reached.

Proof Having chosen A_c, B_c, C_c as indicated above, taking a similarity transformation with

$$T = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$$

changes matrix (5.38) in

$$A_i = \begin{pmatrix} A - \lambda_i BB^\mathrm{T} P_\varepsilon & -\lambda_i BB^\mathrm{T} P_\varepsilon \\ (\lambda_i - 1)BB^\mathrm{T} P_\varepsilon & A + GC + (\lambda_i - 1)BB^\mathrm{T} P_\varepsilon \end{pmatrix},$$

in which, for convenience we have omitted the argument "L" of $\lambda_i(L)$.

Let Q be the unique positive definite symmetric solution of

$$Q(A + GC) + (A + GC)^\mathrm{T} Q = -I$$

and pick

$$\mathscr{P} = \begin{pmatrix} P_\varepsilon & 0 \\ 0 & cQ \end{pmatrix}$$

with $c > 0$. In what follows, we check that

$$\bar{x}^*(\mathscr{P} A_i + A_i^* \mathscr{P})\bar{x} < 0, \qquad \text{for all } \bar{x} \neq 0.$$

To this end, split $\bar{x}$ as $\bar{x} = \mathrm{col}(x, \xi)$ and, using (5.39), note that

$$\begin{aligned}\bar{x}^*(\mathscr{P}A_i + A_i^*\mathscr{P})\bar{x} = &-\varepsilon x^* x + \mu x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x - 2\mathrm{Re}[\lambda_i] x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x \\ &- \lambda_i x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon \xi - \lambda_i^* \xi^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x \\ &+ c[(\lambda_i - 1)\xi^* QBB^{\mathrm{T}} P_\varepsilon x + (\lambda_i^* - 1) x^* P_\varepsilon BB^{\mathrm{T}} Q\xi] \\ &- c\xi^*\xi + c[(\lambda_i - 1)\xi^* QBB^{\mathrm{T}} P_\varepsilon \xi + (\lambda_i^* - 1)\xi^* P_\varepsilon BB^{\mathrm{T}} Q\xi].\end{aligned}$$

The cross-terms can be estimated via Young's inequality,[16] yielding

$$-\lambda_i x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon \xi - \lambda_i^* \xi^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x \leq \frac{\mu}{2} x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x + \frac{2}{\mu}|\lambda_i|^2 \xi^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon \xi$$

and

$$\begin{aligned}&c[(\lambda_i - 1)\xi^* QBB^{\mathrm{T}} P_\varepsilon x + (\lambda_i^* - 1) x^* P_\varepsilon BB^{\mathrm{T}} Q\xi] \\ &\qquad \leq \tfrac{\mu}{2} x^* P_\varepsilon BB^{\mathrm{T}} P_\varepsilon x + \tfrac{2c^2}{\mu}|\lambda_i - 1|^2 \xi^* QBB^{\mathrm{T}} Q\xi.\end{aligned}$$

Since $\mu \leq \mathrm{Re}[\lambda_i]$, it is seen that

$$\begin{aligned}\bar{x}^*(\mathscr{P}A_i + A_i^*\mathscr{P})\bar{x} \leq &-\varepsilon\|x\|^2 - c\|\xi\|^2 + \tfrac{2}{\mu}|\lambda_i|^2\|BB^{\mathrm{T}}\|\|P_\varepsilon\|^2\|\xi\|^2 \\ &+ \tfrac{2c^2}{\mu}|\lambda_i - 1|^2\|QB\|^2\|\xi\|^2 + 2c|\lambda_i - 1|^2\|QBB^{\mathrm{T}}\|\|P_\varepsilon\|\|\xi\|^2.\end{aligned}$$

Set now

$$\alpha = \frac{2}{\mu}(\max_i |\lambda_i|^2)\|BB^{\mathrm{T}}\|, \qquad \beta = \frac{2}{\mu}(\max_i |\lambda_i - 1|^2)\|QB\|^2,$$

$$\gamma = 2(\max_i |\lambda_i - 1|^2)\|QBB^{\mathrm{T}}\|.$$

Then,

$$\bar{x}^*(\mathscr{P}A_i + A_i^*\mathscr{P})\bar{x} \leq -\varepsilon\|x\|^2 - (c - \alpha\|P_\varepsilon\|^2 - \beta c^2 - \gamma c\|P_\varepsilon\|)\|\xi\|^2.$$

Picking $c = \|P_\varepsilon\| := c(\varepsilon)$ and setting

$$\delta(\varepsilon) = c(\varepsilon) - (\alpha + \beta + \gamma)c^2(\varepsilon)$$

yields

$$\bar{x}^*(\mathscr{P}A_i + A_i^*\mathscr{P})\bar{x} \leq -\varepsilon\|x\|^2 - \delta(\varepsilon)\|\xi\|^2.$$

[16] That is, using $x^* X^* Y y + y^* Y^* X x \leq d x^* X^* X x + \frac{1}{d} y^* Y^* Y y$, which holds for any $d > 0$.

Knowing that P_ε depends continuously on ε and that $\lim_{\varepsilon\downarrow 0} P_\varepsilon = 0$, it is deduced that there exists a number ε^* such that

$$0 < \varepsilon < \varepsilon^* \qquad \Rightarrow \qquad \delta(\varepsilon) > 0.$$

For all such ε,

$$\bar{x}^*(\mathscr{P}A_i + A_i^*\mathscr{P})\bar{x} < 0 \qquad \text{for all } x \neq 0,$$

and, as a consequence, all eigenvalues of (5.38) have negative real part. This, in view of the results discussed in the earlier section, shows that the proposed control structure solves the consensus problem. ◁

5.6 Consensus in a Heterogeneous Network

The analysis carried out in the previous sections can be easily extended, so as to handle also the problem of consensus in a heterogeneous network. The (simple) idea is to use the result of Proposition 5.3 to induce consensus among the outputs in a set of N *identical reference generators*, modeled—for $k = 1, \dots, N$—as

$$\begin{aligned} \dot{w}_k &= Sw_k + Kv_k \\ \theta_k &= Qw_k, \end{aligned} \tag{5.40}$$

in which $w_k \in \mathbb{R}^{n_0}$ and $\theta_k \in \mathbb{R}^p$, controlled by

$$v_k = \sum_{j=1}^{N} a_{kj}(\theta_j - \theta_k), \tag{5.41}$$

and then to design a (local) regulator, so as to force the output $y_k(t)$ of the kth agent in the set (5.1) to asymptotically track the output $\theta_k(t)$ of the corresponding local reference generator (5.40).[17]

This yields a two-step design procedure. The first step can be simply completed by exploiting the result of Proposition 5.3, as shown below in more detail. The second step consists in considering, for each of the (nonidentical) agents (5.1), a *tracking error* e_k defined as

$$e_k = y_k - \theta_k$$

[17]The approach described in this section is motivated by the works of [18] and [19]. In particular, the work [18] shows that the approach outline above is in some sense necessary for the solution of the problem in question.

and seeking a *local controller*, driven by the tracking error e_k,

$$\begin{aligned} \dot{\xi}_k &= F_k \xi_k + G_k e_k \\ u_k &= H_k \xi_k \end{aligned}$$

that solves a problem of output regulation for a system modeled as

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ e_k &= C_k x_k - Q w_k. \end{aligned} \tag{5.42}$$

All of this is depicted in Fig. 5.2.

To the purposes of completing the first step of the design, assume that the pair (S, Q) is observable and that the coefficients a_{kj} in (5.41) are the entries of an adjacency matrix A associated with a *connected* communication graph. Let $\mu > 0$ be a number satisfying (5.31), pick $a > 0$ and let P be the unique positive definite symmetric solution of the algebraic Riccati equation

$$SP + PS^{\mathrm{T}} - 2\mu PQ^{\mathrm{T}}QP + aI = 0. \tag{5.43}$$

Then, it follows from Proposition 5.3 that, if the matrix K in (5.40) is chosen as $K = PQ^{\mathrm{T}}$, the outputs of the set (5.40) of local reference generators controlled by (5.41) reach consensus. In particular, as shown at the end of Sect. 5.4, the consensus is reached along a function of the form

$$\theta_{\mathrm{cons}}(t) = Qe^{St}w^{\circ} \tag{5.44}$$

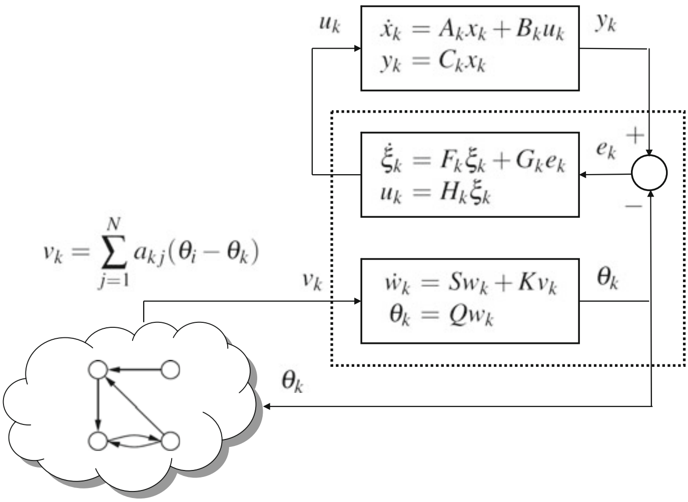

Fig. 5.2 Control of a heterogeneous network

in which w° is a vector depending on the initial conditions $w_k(0)$ of the various systems in the set (5.40). In other words, there exists w° such that

$$\lim_{t\to\infty}(\theta_k(t) - Qe^{St}w^\circ) = 0$$

for all $k = 1, \dots, N$.

Then, we proceed with the second step of the design. In the analysis of the (local) regulation problem thus defined, a little extra care is need, to correctly characterize the exosystem (which, as in any regulation problem, is supposed to generate the exogenous input: in this case, the input w_k in (5.42)). In fact, the various w_k's are not independent of each other. Thus, the correct approach to the problem is to consider—as exosystem—the entire set local reference generators (5.40) controlled by (5.41). Setting

$$w = \mathrm{col}(w_1, w_2, \dots, w_N),$$

such interconnection can be modeled as

$$\dot{w} = [(I_N \otimes S) - (L \otimes KQ)]w.$$

It is known from the analysis of Sect. 5.4 that the system thus defined can be transformed, by similarity, into a system in block-triangular form. More precisely, changing w into $(T^{-1} \otimes I_{n_0})w$, with T defined as in (5.32), the system can be changed into a system of the form

$$\begin{aligned} \dot{w}_1 &= Sw_1 - (\tilde{L}_{12} \otimes KQ)w_\delta \\ \dot{w}_\delta &= [(I_{(N-1)} \otimes S) - (\tilde{L}_{22} \otimes KQ)]w_\delta, \end{aligned} \tag{5.45}$$

where

$$w_\delta = \begin{pmatrix} w_2 - w_1 \\ \cdots \\ w_N - w_1 \end{pmatrix}.$$

Moreover, thanks to the results of the analysis in Sect. 5.5, it is known that if K is taken as indicated above, i.e., $K = PQ^{\mathrm{T}}$ with P solution of (5.43), the matrix

$$[(I_{(N-1)} \otimes S) - (\tilde{L}_{22} \otimes KQ)]$$

has all eigenvalues in $\mathbb{C}^-$.

For convenience, system (5.45) is rewritten in the form

$$\begin{aligned} \dot{w}_1 &= Sw_1 + Rw_\delta \\ \dot{w}_\delta &= S_\delta w_\delta. \end{aligned}$$

in which S_δ is a matrix having all eigenvalues in $\mathbb{C}^-$. Moreover, it is observed that the exogenous input w_k in (5.42) can be expressed as

$$w_k = w_1 + H_k w_\delta$$

in which $H_k \in \mathbb{R}^{n_0 \times (N-1)n_0}$. In fact, $H_1 = 0$ and, for $k = 2, \dots, N$,

$$H_k = \begin{pmatrix} 0 \cdots 0 \; I_{n_0} \; 0 \cdots 0 \end{pmatrix}$$

with the identity block in the $(k-1)$th place.

In view of all of this, it follows that the second step of the design consists in solving a problem of output regulation for a plant that can be seen as modeled by equations of the form

$$\begin{aligned} \dot{w}_1 &= S w_1 + R w_\delta \\ \dot{w}_\delta &= S_\delta w_\delta \\ \dot{x}_k &= A_k x_k + B_k u_k \\ e_k &= C_k x_k - Q w_1 - Q H_k w_\delta . \end{aligned} \tag{5.46}$$

The design of the regulator can be accomplished along lines similar (but not quite identical) to those followed in the construction described in Sect. 4.6. Specifically, consider the case in which $m = p$ and assume that—for each k—there is a solution (Π_k, Ψ_k) of the Francis' equations

$$\begin{aligned} \Pi_k S &= A_k \Pi_k + B_k \Psi_k \\ 0 &= C_k \Pi_k - Q . \end{aligned} \tag{5.47}$$

Let Φ and G be as in (4.23) and (4.24), where we pick $s_0, s_1, \dots, s_{d-1}$ as the coefficients of the minimal polynomial of S, and where Γ is such that $\Phi - G\Gamma$ is a Hurwitz matrix. Then, it is known from Sect. 4.6 that there exists a matrix Σ_k satisfying

$$\begin{aligned} \Sigma_k S &= \Phi \Sigma_k \\ \Psi_k &= \Gamma \Sigma_k . \end{aligned} \tag{5.48}$$

The regulator for (5.46) is a system modeled as (compare with (4.60))

$$\begin{aligned} \dot{\xi}_k &= A_{\mathrm{s},k} \xi_k + B_{\mathrm{s},k} e_k \\ \dot{\eta}_k &= \Phi \eta_k + G(C_{\mathrm{s},k} \xi_k + D_{\mathrm{s},k} e_k) \\ u_k &= \Gamma \eta_k + (C_{\mathrm{s},k} \xi_k + D_{\mathrm{s},k} e_k) . \end{aligned} \tag{5.49}$$

To check that such controller is able to achieve the desired goal, consider the interconnection of (5.46) and (5.49), and change the variables as

$$\begin{aligned} \tilde{x}_k &= x_k - \Pi_k w_1 \\ \tilde{\eta}_k &= \eta_k - \Sigma_k w_1 . \end{aligned}$$

This, using (5.47) and (5.48), after a few standard manipulation yields

$$\begin{aligned}
\dot{w}_1 &= Sw_1 + Rw_\delta \\
\dot{w}_\delta &= S_\delta w_\delta \\
\dot{\tilde{x}}_k &= (A_k + B_k D_{\mathrm{s},k} C_k)\tilde{x}_k + B_k \Gamma \tilde{\eta}_k + B_k C_{\mathrm{s},k}\xi_k - (B_k D_{\mathrm{s},k} Q H_k + \Pi_k R) w_\delta \\
\dot{\tilde{\eta}}_k &= \Phi \tilde{\eta}_k + G C_{\mathrm{s},k}\xi_k + G D_{\mathrm{s},k} C_k \tilde{x}_k - (G D_{\mathrm{s},k} Q H_k + \Sigma_k R) w_\delta \\
\dot{\xi}_k &= A_{\mathrm{s},k}\xi_k + B_{\mathrm{s},k} C_k \tilde{x}_k - B_{\mathrm{s},k} Q H_k w_\delta
\end{aligned} \tag{5.50}$$

and

$$e_k = C_k \tilde{x}_k - Q H_k w_\delta.$$

Note that the four bottom equations of (5.50) are not affected by w_1. They can be organized as a block-triangular system of the form

$$\begin{pmatrix} c\dot{w}_\delta \\ \dot{\tilde{x}}_k \\ \dot{\tilde{\eta}}_k \\ \dot{\xi}_k \end{pmatrix} = \begin{pmatrix} ccccS_\delta & 0 & 0 & 0 \\ * & A_k + B_k D_{\mathrm{s},k} C_k & B_k \Gamma & B_k C_{\mathrm{s},k} \\ * & G D_{\mathrm{s},k} C_k & \Phi & G C_{\mathrm{s},k} \\ * & B_{\mathrm{s},k} C_k & 0 & A_{\mathrm{s},k} \end{pmatrix} \begin{pmatrix} cw_\delta \\ \tilde{x}_k \\ \tilde{\eta}_k \\ \xi_k \end{pmatrix}.$$

It is known from the analysis in Sect. 4.6 that, if (A_k, B_k) is stabilizable, if (A_k, C_k) is detectable, if the nonresonance condition

$$\det \begin{pmatrix} A_k - \lambda I & B_k \\ C_k & 0 \end{pmatrix} \neq 0 \qquad \forall \lambda \in \sigma(S) \tag{5.51}$$

holds,[18] and if Γ is such that $\Phi - G\Gamma$ is a Hurwitz matrix, it is possible to find a quadruplet $\{A_{\mathrm{s},k}, B_{\mathrm{s},k}, C_{\mathrm{s},k}, D_{\mathrm{s},k}\}$ such that the matrix (compare with (4.56))

$$\begin{pmatrix} A_k + B_k D_{\mathrm{s},k} C_k & B_k \Gamma & B_k C_{\mathrm{s},k} \\ G D_{\mathrm{s},k} C_k & \Phi & G C_{\mathrm{s},k} \\ B_{\mathrm{s},k} C_k & 0 & A_{\mathrm{s},k} \end{pmatrix}$$

has all eigenvalues in $\mathbb{C}^-$. Let this be the case. The matrix S_δ, on the other hand, has all eigenvalues in $\mathbb{C}^-$ because of the choice $K = PQ^{\mathrm{T}}$ in (5.40). Hence, it is concluded that the system above—which is a block-triangular system—is stable. In particular, both $\tilde{x}_k(t)$ and $w_\delta(t)$ tend to 0 as $t \to \infty$, and this yields

$$\lim_{t\to\infty} e_k(t) = \lim_{t\to\infty} [C_k \tilde{x}_k(t) - Q H_k w_\delta(t)] = 0.$$

As a consequence

$$\lim_{t\to\infty} [y_k(t) - \theta_k(t)] = 0.$$

[18] Note that the latter guarantees the existence a (unique) the solution pair of (5.47).

This, since all $\theta_k(t)$s achieve consensus, shows that under the proposed control scheme also all $y_k(t)$s achieve consensus, in particular along a function of the form (5.44).

To summarize, we conclude that the consensus problem for a heterogeneous network can be solved by means of a controller having the following structure

$$\begin{aligned}
\dot{w}_k &= Sw_k + PQ^{\mathrm{T}}\Big(\textstyle\sum_{j=1}^{N} a_{kj}(\theta_j - \theta_k)\Big) \\
\theta_k &= Qw_k \\
\dot{\eta}_k &= \Phi\eta_k + G(C_{\mathrm{s},k}\xi_k + D_{\mathrm{s},k}[y_k - \theta_k]) \\
\dot{\xi}_k &= A_{\mathrm{s},k}\xi_k + B_{\mathrm{s},k}[y_k - \theta_k] \\
u_k &= \Gamma\eta_k + (C_{\mathrm{s},k}\xi_k + D_{\mathrm{s},k}[y_k - \theta_k]).
\end{aligned}$$

The assumptions under which the parameters of this controller can be chosen so as to induce output consensus are that the communication graph is connected, the pair (S, Q) is observable, the pairs (A_k, B_k) are stabilizable, the pairs (A_k, C_k) are detectable and the nonresonance conditions (5.51) hold. Note also that an extension that covers the cases in which the models of the agents (5.1) are affected by uncertainties is possible, appealing to the methods for the design of robust regulators discussed in Sects. 4.7 and 4.9. Details are left to the reader.

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Part II
Nonlinear Systems

Chapter 6
Stabilization of Nonlinear Systems via State Feedback

6.1 Relative Degree and Local Normal Forms

In this section, we consider the class of single-input single-output nonlinear systems that can be modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{6.1}$$

in which $x \in \mathbb{R}^n$ and in which $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth maps and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.[1] Such systems are usually referred to as *input-affine* systems.[2]

As shown in Sect. 2.1 for a linear system, if the coordinates in which the state space description is provided are appropriately transformed, the equations describing the system can be brought to a form in which the design of feedback laws is facilitated. In this section, we develop a similar approach for the analysis of systems of the form (6.1) and we derive the nonlinear counterpart of the normal form (2.9).[3]

In the case of a linear system, a change of coordinates consists in replacing the original state vector x with a new vector $\tilde{x}$ related to x by means of linear transformation $\tilde{x} = Tx$ in which T is a nonsingular matrix. If the system is nonlinear, it is more appropriate to allow also for *nonlinear* changes of coordinates. A nonlinear change of coordinates is a transformation $\tilde{x} = \Phi(x)$ in which $\Phi(\cdot)$ is a map $\mathbb{R}^n \to \mathbb{R}^n$. Such a map qualifies for a change of coordinates if:

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[1]By "smooth map" we mean a C^∞ map, i.e., a map for which partial derivatives of any order are defined and continuous. In what follows, $f(x)$ and $g(x)$ will be sometimes regarded as smooth *vector fields* of $\mathbb{R}^n$.

[2]In fact, the right-hand side of the upper equation for each x is an affine function of u.

[3]Some of the topics presented in this chapter are covered in various textbooks on nonlinear systems, such as [1–6]. The approach here follows that of [2]. For further reading, see also [14–16].

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_6

(i) $\Phi(\cdot)$ is invertible, i.e. there exists a map $\Phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi^{-1}(\Phi(x)) = x$ for all $x \in \mathbb{R}^n$ and $\Phi(\Phi^{-1}(\tilde{x})) = \tilde{x}$ for all $\tilde{x} \in \mathbb{R}^n$.
(ii) $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ are both smooth mappings, i.e., have continuous partial derivatives of any order.

A transformation of this type is called a *global diffeomorphism* on $\mathbb{R}^n$.[4] To deduce the model of the system in the $\tilde{x}$ coordinates, set $\tilde{x}(t) = \Phi(x(t))$ and differentiate both sides with respect to time. This yields

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \frac{\partial \Phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial \Phi}{\partial x}\big[f(x(t)) + g(x(t))u(t)\big].$$

Then, expressing $x(t)$ as $x(t) = \Phi^{-1}(\tilde{x}(t))$, one obtains

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{f}(\tilde{x}(t)) + \tilde{g}(\tilde{x}(t))u(t)\\ y(t) &= \tilde{h}(\tilde{x}(t))\end{aligned}$$

where

$$\tilde{f}(\tilde{x}) = \left[\frac{\partial \Phi}{\partial x} f(x)\right]_{x=\Phi^{-1}(\tilde{x})} \qquad \tilde{g}(\tilde{x}) = \left[\frac{\partial \Phi}{\partial x} g(x)\right]_{x=\Phi^{-1}(\tilde{x})} \qquad \tilde{h}(\tilde{x}) = [h(x)]_{x=\Phi^{-1}(\tilde{x})}.$$

Sometimes, a transformation possessing properties (i) and (ii) and defined for all x is difficult to find. Thus, in some cases one rather looks at transformations defined only in a *neighborhood* of a given point. A transformation of this type is called a *local diffeomorphism*. A sufficient condition for a map $\Phi(\cdot)$ to be a local diffeomorphism is described below.

Lemma 6.1 *Suppose $\Phi(\cdot)$ is a smooth map defined on some subset U of $\mathbb{R}^n$. Suppose the jacobian matrix of Φ is nonsingular at a point $x = x^\circ \in U$. Then, on a suitable neighborhood $U^\circ \subset U$ of x°, $\Phi(\cdot)$ defines a local diffeomorphism.*

Having understood how coordinates are transformed, we proceed now with the definition of an integer that plays a role identical to that of the relative degree defined in Sect. 2.1 for a linear system. The nonlinear system (6.1) is said to have relative degree r at a point x° if[5]:
(i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r - 1$
(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

---

[4]The first of the two properties is clearly needed in order to have the possibility of reversing the transformation and recovering the original state vector as $x = \Phi^{-1}(\tilde{x})$, while the second one guarantees that the description of the system in the new coordinates is still a smooth one.

[5]Let λ be real-valued function and f a vector field, both defined on a subset U of $\mathbb{R}^n$. The function $L_f\lambda$ is the real-valued function defined as

$$L_f\lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x) := \frac{\partial \lambda}{\partial x} f(x).$$

Note that the concept thus introduced is a *local concept*, namely r may depend on the specific point x° where the functions $L_g L_f^k h(x)$ are evaluated. The value of r may be different at different points of $\mathbb{R}^n$ and there may be points where a relative degree cannot be defined. This occurs when the first function of the sequence

$$L_g h(x),\ L_g L_f h(x), \ldots,\ L_g L_f^k h(x), \ldots$$

which is not identically zero (in a neighborhood of x°) is zero exactly at the point $x = x^\circ$. However, since $f(x)$, $g(x)$, $h(x)$ are smooth, the set of points where a relative degree can be defined is an open and dense subset of $\mathbb{R}^n$.

Remark 6.1 A single-input single-output linear system is a special case of a system of the form (6.1), obtained by setting $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$. In this case $L_f^k h(x) = CA^k x$ and therefore $L_g L_f^k h(x) = CA^k B$. Thus, the integer r defined above coincides with the integer r defined in Sect. 2.1 by the conditions (2.2). ◁

We have seen in Sect. 2.1 that the relative degree of a linear system can be interpreted as a simple property of the transfer function, namely the difference between the degrees of the denominator and the numerator polynomials of such function. However, an alternative interpretation is also possible, that does not appeal to the notion of transfer function and also holds for a nonlinear system of the form (6.1). Assume the system at time $t = 0$ is in the state $x(0) = x^\circ$ and let us calculate the value of the output $y(t)$ and of its derivatives with respect to time $y^{(k)}(t)$, for $k = 1, 2, \ldots$, at $t = 0$. We obtain

$$y(0) = h(x(0)) = h(x^\circ)$$

and

$$y^{(1)}(t) = \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial h}{\partial x}[f(x(t)) + g(x(t))u(t)] = L_f h(x(t)) + L_g h(x(t))u(t).$$

At time $t = 0$,

$$y^{(1)}(0) = L_f h(x^\circ) + L_g h(x^\circ)u(0),$$

from which it is seen that, if $r = 1$, the value $y^{(1)}(0)$ is an affine function of $u(0)$. Otherwise, suppose r is larger than 1. If $|t|$ is small, $x(t)$ remains in a neighborhood of x° and hence $L_g h(x(t)) = 0$ for all such t. As a consequence

$$y^{(1)}(t) = L_f h(x(t)).$$

(Footnote 5 continued)
This function is sometimes called the *derivative of* λ *along* f. If g is another vector field, the notation $L_g L_f \lambda(x)$ stands for the derivative of the real-valued function $L_f\lambda$ along g and the notation $L_f^k\lambda(x)$ stands for the derivative of the real-valued function $L_f^{k-1}\lambda$ along f.

This yields

$$y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial L_f h}{\partial x}[f(x(t)) + g(x(t))u(t)] = L_f^2 h(x(t)) + L_g L_f h(x(t))u(t).$$

At time $t = 0$,

$$y^{(2)}(0) = L_f^2 h(x^\circ) + L_g L_f h(x^\circ)u(0),$$

from which it is seen that, if $r = 2$, the value $y^{(2)}(0)$ is an affine function of $u(0)$. Otherwise, if r larger than 2, for all t near $t = 0$ we have $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t)).$$

Continuing in this way, we get

$$\begin{aligned} y^{(k)}(t) &= L_f^k h(x(t)) \qquad \text{for all } k < r \text{ and all } t \text{ near } t = 0 \\ y^{(r)}(0) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(0). \end{aligned}$$

Thus, the integer r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = 0$ in order to have the value $u(0)$ of the input explicitly appearing.

The calculations above suggest that the functions $h(x), L_f h(x), \ldots, L_f^{r-1}h(x)$ must have a special importance. As a matter of fact, such functions can be used in order to define, at least partially, a local coordinates transformation around x°. This fact is based on the following property, which extends Proposition 2.1 to the case of systems of the form (6.1).[6]

Proposition 6.1 *The differentials*[7] *of the r functions* $h(x)$, $L_f h(x), \ldots, L_f^{r-1}h(x)$ *are linearly independent at* $x = x^\circ$.

Proposition 6.1 shows that necessarily $r \leq n$ and that the r functions $h(x)$, $L_f h(x), \ldots, L_f^{r-1}h(x)$ qualify as a partial set of new coordinate functions around the point x°. If $r = n$ these functions define a full change of coordinates. Otherwise, if $r < n$ the change of coordinates can be completed by picking $n - r$ additional functions, yielding a full change of coordinates that can be seen as equivalent of the change of coordinates introduced in Proposition 2.2.

Proposition 6.2 *Suppose the system has relative degree* r *at* x°. *Then* $r \leq n$. *If* r *is strictly less than* n, *it is always possible to find* $n - r$ *more functions* $\psi_1(x), \ldots, \psi_{n-r}(x)$ *such that the mapping*

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[6] See [2, p. 140–142] for a proof of Propositions 6.1 and 6.2.

[7] Let λ be a real-valued function defined on a subset U of $\mathbb{R}^n$. Its *differential*, denoted $d\lambda(x)$, is the row vector

$$d\lambda(x) = \left(\frac{\partial\lambda}{\partial x_1}\,\frac{\partial\lambda}{\partial x_2}\cdots\frac{\partial\lambda}{\partial x_n}\right) := \frac{\partial\lambda}{\partial x}.$$

$$\Phi(x) = \begin{pmatrix} \psi_1(x) \\ \cdots \\ \psi_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

has a jacobian matrix which is nonsingular at x° and therefore qualifies as a local coordinates transformation in a neighborhood of x°. The value at x° of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\psi_1(x), \ldots, \psi_{n-r}(x)$ in such a way that

$$L_g \psi_i(x) = 0 \qquad \textit{for all } 1 \le i \le n-r \textit{ and all } x \textit{ around } x^\circ.$$

The description of the system in the new coordinates is found very easily. Set

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_{n-r} \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdots \\ \psi_{n-r}(x) \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{r-1} h(x) \end{pmatrix}$$

and

$$\tilde{x} = \mathrm{col}(z_1, \ldots, z_{n-r}, \xi_1, \ldots, \xi_r) := \Phi(x).$$

Bearing in mind the previous calculations, it is seen that

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = L_f h(x(t)) = \xi_2(t) \\ &\cdots \\ \frac{\mathrm{d}\xi_{r-1}}{\mathrm{d}t} &= \frac{\partial (L_f^{r-2} h)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = L_f^{r-1} h(x(t)) = \xi_r(t). \end{aligned}$$

while for ξ_r we obtain

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}t} = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t).$$

On the right-hand side of this equation, x must be replaced by its expression as a function of $\tilde{x}$, which will be written as $x = \Phi^{-1}(z, \xi)$. Thus, setting

$$\begin{aligned} q(z, \xi) &= L_f^r h(\Phi^{-1}(z, \xi)) \\ b(z, \xi) &= L_g L_f^{r-1} h(\Phi^{-1}(z, \xi)) \end{aligned}$$

the equation in question can be rewritten as

$$\frac{d\xi_r}{dt} = q(z(t), \xi(t)) + b(z(t), \xi(t))u(t).$$

Note that, by definition of relative degree, at the point $\tilde{x}^\circ = \mathrm{col}(z^\circ, \xi^\circ) = \Phi(x^\circ)$, we have $b(z^\circ, \xi^\circ) = L_g L_f^{r-1} h(x^\circ) \neq 0$. Thus, the coefficient $b(z, \xi)$ is nonzero for all (z, ξ) in a neighborhood of (z°, ξ°).

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However, if $\psi_1(x), \ldots, \psi_{n-r}(x)$ have been chosen in such a way that $L_g\psi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \psi_i}{\partial x}[f(x(t)) + g(x(t))u(t)] = L_f\psi_i(x(t)) + L_g\psi_i(x(t))u(t) = L_f\psi_i(x(t)).$$

Setting

$$f_0(z, \xi) = \begin{pmatrix} L_f\psi_1(\Phi^{-1}(z, \xi)) \\ \cdots \\ L_f\psi_{n-r}(\Phi^{-1}(z, \xi)) \end{pmatrix}$$

the latter can be rewritten as

$$\frac{dz}{dt} = f_0(z(t), \xi(t)).$$

Thus, in summary, in the new (local) coordinates the system is described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z, \xi) + b(z, \xi)u. \end{aligned} \tag{6.2}$$

In addition to these equations, one has to specify how the output of the system is related to the new state variables. Being $y = h(x)$, it is immediately seen that

$$y = \xi_1. \tag{6.3}$$

The equations thus introduced are said to be in *strict* normal form. They are useful in understanding how certain control problems can be solved. The equations in question can be given a compact expression if we use the three matrices $\hat{A} \in \mathbb{R}^r \times \mathbb{R}^r$, $\hat{B} \in \mathbb{R}^r \times \mathbb{R}$ and $\hat{C} \in \mathbb{R} \times \mathbb{R}^r$ defined in (2.7). With the aid of such matrices, the Eqs. (6.2) and (6.3) can be rewritten as

$$
\begin{aligned}
\dot{z} &= f_0(z, \xi) \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)u] \\
y &= \hat{C}\xi,
\end{aligned}
\tag{6.4}
$$

which can be seen as the nonlinear counterpart of (2.9).

Remark 6.2 Note that, if $f(0) = 0$, that is if $x = 0$ is an equilibrium of the autonomous system $\dot{x} = f(x)$, and if $h(0) = 0$, the functions $L_f^k h(x)$ are zero at $x = 0$. Since the values of the complementary functions $\psi_j(x)$ at $x = 0$ are arbitrary, one can pick them in such a way that $\psi_j(0) = 0$. As a result, $\Phi(0) = 0$. Accordingly, in system (6.4) we have

$$f_0(0, 0) = 0 \qquad q(0, 0) = 0. \qquad \triangleleft$$

6.2 Global Normal Forms

We address, in this section, the problem of deriving the global version of the coordinates transformation and normal form introduced in Sect. 6.1. The exposition is limited to the introduction of a few basic concepts involved and to the main results. A detailed analysis and the appropriate differential-geometric background can be found in the literature.[8]

Consider again a single-input single-output nonlinear system described by equations of the form (6.1), in which $f(x)$ and $g(x)$ are smooth vector fields, and $h(x)$ is a smooth function, defined on $\mathbb{R}^n$. Assume also that $f(0) = 0$ and $h(0) = 0$. This system is said to have *uniform* relative degree r if it has relative degree r at each $x^\circ \in \mathbb{R}^n$.

If system (6.1) has uniform relative degree r, the r differentials

$$dh(x),\ dL_f h(x),\ \ldots,\ dL_f^{r-1} h(x)$$

are linearly independent at each $x \in \mathbb{R}^n$ and therefore the set

$$Z^* = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0\}$$

(which is nonempty in view of the hypothesis that $f(0) = 0$ and $h(0) = 0$) is a smooth embedded submanifold of $\mathbb{R}^n$, of codimension r. In particular, each connected component of Z^* is a maximal integral manifold of the (nonsingular and involutive) distribution

$$\Delta^* = (\mathrm{span}\{dh, dL_f h, \ldots, dL_f^{r-1} h\})^\perp.$$

[8] See, e.g., [2].

The submanifold Z^* plays an important role in the construction a globally defined version of the coordinates transformation considered in Sect. 6.1. The transformation in question is defined in the proposition which follows.[9]

Proposition 6.3 *Suppose (6.1) has uniform relative degree r. Set*

$$\alpha(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)} \qquad \beta(x) = \frac{1}{L_g L_f^{r-1} h(x)}$$

and consider the (globally defined) vector fields

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \qquad \tilde{g}(x) = g(x)\beta(x).$$

Suppose the vector fields[10]

$$\tau_i(x) = (-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), \qquad 1 \le i \le r \tag{6.5}$$

are complete.[11]

Then Z^* is connected. Suppose Z^* is diffeormorphic to $\mathbb{R}^{n-r}$. Then, the smooth mapping

$$\begin{aligned} \Phi^{-1}: \quad & Z^* \times \mathbb{R}^r \quad \to \mathbb{R}^n \\ & (z, (\xi_1, \ldots, \xi_r)) \mapsto \Phi_{\xi_r}^{\tau_1} \circ \Phi_{\xi_{r-1}}^{\tau_2} \circ \cdots \circ \Phi_{\xi_1}^{\tau_r}(z), \end{aligned} \tag{6.6}$$

in which $\Phi_t^\tau(x)$ denotes the flow of the vector field τ, has a globally defined smooth inverse

$$(z, (\xi_1, \ldots, \xi_r)) = \Phi(x) \tag{6.7}$$

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[9] See [7] and also [2] for the proof.

[10] Let f and g be vector fields of $\mathbb{R}^n$. Their *Lie bracket*, denoted $[f, g]$, is the vector field of $\mathbb{R}^n$ defined as

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

The vector field $ad_f^k g$ is recursively defined as follows

$$ad_f^0 g = g \qquad ad_f^k g = [f, ad_f^{k-1} g].$$

[11] If τ is a vector field of $\mathbb{R}^n$, the *flow* of τ is the map $\Phi_t^\tau(x)$, in which $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, defined by the following properties: $\Phi_0^\tau(x) = x$ and

$$\frac{\mathrm{d}\Phi_t^\tau(x)}{\mathrm{d}t} = \tau(\Phi_t^\tau(x)).$$

In other words, $\Phi_t^\tau(x)$ is the value at time $t \in \mathbb{R}$ of the integral curve of the o.d.e. $\dot{x} = \tau(x)$ passing through $x = 0$ at time $t = 0$. The vector field τ is said to be *complete* if $\Phi_t^\tau(x)$ is defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

$$\begin{aligned} z &= \Phi^{\tau_r}_{-h(x)} \circ \cdots \circ \Phi^{\tau_2}_{-L^{r-2}_{\tilde{f}} h(x)} \circ \Phi^{\tau_1}_{-L^{r-1}_{\tilde{f}} h(x)}(x) \\ \xi_i &= L^{i-1}_{\tilde{f}} h(x) \qquad 1 \le i \le r. \end{aligned}$$

The globally defined diffeomorphism (6.7) changes system (6.1) into a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)u] \\ y &= \hat{C}\xi \end{aligned} \tag{6.8}$$

in which $\hat{A}, \hat{B}, \hat{C}$ *are matrices of the form (2.7) and*

$$\begin{aligned} \xi &= \mathrm{col}(\xi_1, \dots, \xi_r) \\ q(z, \xi) &= L^r_f h(\Phi^{-1}(z, (\xi_1, \dots, \xi_r))) \\ b(z, \xi) &= L_g L^{r-1}_f h(\Phi^{-1}(z, (\xi_1, \dots, \xi_r))). \end{aligned}$$

If, and only if, the vector fields (6.5) are such that

$$[\tau_i, \tau_j] = 0 \qquad \text{for all } 1 \le i, j \le r,$$

then the globally defined diffeomorphism (6.7) changes system (6.1) into a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)u] \\ y &= \hat{C}\xi. \end{aligned} \tag{6.9}$$

Remark 6.3 Note that, if $r < n$, the submanifold Z^* is the largest (with respect to inclusion) smooth submanifold of $h^{-1}(0)$ with the property that, at each $x \in Z^*$, there is $u^*(x)$ such that the vector field

$$f^*(x) = f(x) + g(x)u^*(x)$$

is tangent to Z^*. Actually, for each $x \in Z^*$ there is only one value of $u^*(x)$ that renders this condition satisfied, which is

$$u^*(x) = \frac{-L^r_f h(x)}{L_g L^{r-1}_f h(x)}. \tag{6.10}$$

The submanifold Z^* is an *invariant* manifold of the (autonomous) system

$$\dot{x} = f^*(x) \tag{6.11}$$

and the restriction of this system to Z^* can be identified with $(n-r)$-dimensional autonomous system

$$\dot{z} = f_0(z, 0). \qquad \triangleleft$$

Remark 6.4 Note that (6.9) differs from (6.8) in the fact that while the map $f_0(\cdot, \cdot)$ in (6.8) possibly depends on all components $\xi_1, \ldots, \xi_r$ of ξ, in (6.8) the map in question only depends on its first component ξ_1. $\triangleleft$

6.3 The Zero Dynamics

In this section, we introduce and discuss an important concept, that in many instances plays a role exactly similar to that of the "zeros" of the transfer function in a linear system.[12] In order to motivate this concept, consider again for a moment a linear system in (strict) normal form (2.9). Pick any $z^\circ \in \mathbb{R}^{n-r}$, set $\xi^\circ = 0$, and note that there is a (unique) input function $u^\circ(\cdot)$ such that the output response $y(t)$ of (2.9) from the initial state $(z(0), \xi(0)) = (z^\circ, \xi^\circ) = (z^\circ, 0)$ is *identically zero* for all $t \in \mathbb{R}$. This is the function

$$u^\circ(t) = \frac{-1}{b} A_{10} e^{A_{00}t} z^\circ.$$

A simple calculation, in fact, shows that the pair

$$z(t) = e^{A_{00}t} z^\circ, \qquad \xi(t) = 0$$

is a solution pair of (2.9) when $u(t) = u^\circ(t)$, and this solution passes through $(z^\circ, 0)$ at time $t = 0$. The output $y(t)$ associated with such pair is clearly *identically zero*. It is seen in this way that the dynamics of the autonomous system

$$\dot{z} = A_{00} z$$

characterize the *forced* state behavior of (2.9), whenever input and initial conditions are chosen in such a way that the output is identically zero. Bearing in mind the fact that the eigenvalues of A_{00} are the zeros of the transfer function,[13] this characterization could be pushed a little bit further. In fact, if $\bar{z}$ is an eigenvector of A_{00}, associated with an eigenvalue $\bar{\lambda}$, it is seen that the forced output response of the system to the input

$$\bar{u}(t) = \frac{-1}{b} A_{10} e^{\bar{\lambda}t} \bar{z},$$

[12] For further reading, see [8] and the references cited thererein.

[13] We have tacitly assumed, as in Sect. 2.1, that the triplet A, B, C is a minimal realization of its transfer function.

from the initial condition $(z(0), \xi(0)) = (\bar{z}, 0)$, is identically zero. In other words, as it is well known, for each zero of the transfer function one can find an input and an initial condition yielding an output which is identically zero. In this section, we extend such interpretation to the case of a nonlinear system of the form (6.1).

Consider again system (6.1), assume that $f(0) = 0$ and $h(0) = 0$, suppose that the system has relative degree r at $x^\circ = 0$ and consider its normal form (6.4) in which, in view of Remark 6.2, $f_0(0, 0) = 0$ and $q(0, 0) = 0$. For such system we study the problem of finding all pairs consisting of an initial state x° and of an input function $u(\cdot)$, for which the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t = 0$.[14]

Recalling that in the normal form $y(t) = \xi_1(t)$, it is seen that if $y(t) = 0$ for all t, then

$$\xi_1(t) = \xi_2(t) = \cdots = \xi_r(t) = 0,$$

that is $\xi(t) = 0$ for all t. Thus, when the output of the system is identically zero its state is constrained to evolve in such a way that also $\xi(t)$ is identically zero. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = q(z(t), 0) + b(z(t), 0)u(t)$$

(recall that $b(z(t), 0) \neq 0$ if $z(t)$ is close to 0). As far as the variable $z(t)$ is concerned, it is seen that

$$\dot{z}(t) = f_0(z(t), 0).$$

From this analysis we deduce the following facts. If the output $y(t)$ is identically zero, then necessarily the initial state of the system must be such that $\xi(0) = 0$, whereas $z(0) = z^\circ$ can be arbitrary. According to the value of z°, the input must be

$$u(t) = -\frac{q(z(t), 0)}{b(z(t), 0)}$$

where $z(t)$ denotes the solution of the differential equation

$$\dot{z} = f_0(z, 0) \qquad z(0) = z^\circ. \tag{6.12}$$

The dynamics of (6.12) characterize the forced state behavior of the system when input and initial conditions are chosen in such a way as to constrain the output to remain identically zero. These dynamics, which are rather important in many of the subsequent developments, are called the *zero dynamics* of the system.

Remark 6.5 Note the striking analogy with the corresponding properties of a linear system. Note also that the same analysis could be performed in the original coor-

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[14] If we deal with a locally defined normal form, all functions of time are to be seen as functions defined in a neighborhood of $t = 0$. Otherwise, if the normal form is globally defined, such functions are defined for all t for which the solution of (6.12) is defined.

dinates. In fact, bearing in mind the results presented in Sect. 6.2, it is seen that if $y(t) = 0$ for all $t \in \mathbb{R}$, then $x(t) \in Z^*$. Moreover,

$$u(t) = u^*(x(t))$$

where $u^*(x)$ is the function (6.10), and $x(t)$ is a trajectory of the autonomous system (6.11). $\triangleleft$

As we will see in the sequel of this Chapter, the asymptotic properties of the system identified by the upper equation in (6.8) play a role of paramount importance in the design of stabilizing feedback laws. In this context, the properties expressed by the following definition are relevant.[15]

Definition 6.1 Consider a system of the form (6.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has uniform relative degree r and possesses a globally defined normal. The system is *globally minimum-phase* if the equilibrium $z = 0$ of the zero dynamics

$$\dot{z} = f_0(z, 0) \tag{6.13}$$

is globally asymptotically stable. The system is *strongly minimum-phase* if system

$$\dot{z} = f_0(z, \xi), \tag{6.14}$$

viewed as a system with input ξ and state z, is input-to-state stable.

By definition—then—a system is strongly minimum-phase if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any $z(0) \in \mathbb{R}^{n-r}$ and any piecewise-continuous bounded function $\xi(\cdot) : [0, \infty) \to \mathbb{R}^r$, the response $z(t)$ of (6.14) from the initial state $z(0)$ at time $t = 0$ satisfies,

$$\|z(t)\| \le \beta(\|z(0)\|, t) + \gamma(\|\xi(\cdot)\|_{[0,t]}) \qquad \text{for all } t \ge 0. \tag{6.15}$$

According to the criterion of Sontag,[16] a system is strongly minimum-phase if there exist a continuously differentiable function $V_0 : \mathbb{R}^{n-r} \to \mathbb{R}$, three class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ such that[17]

$$\underline{\alpha}(\|z\|) \le V_0(z) \le \overline{\alpha}(\|z\|) \qquad \text{for all } z \in \mathbb{R}^{n-r} \tag{6.16}$$

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[15] In the second part of the definition, of the property of *input-of-state stability* is invoked. For a summary of the main characterizations of such property and a number of related results, see Sect. B.2 of Appendix B. For further reading about the property of input-to-state stability, see references [10, 11, 14] of Appendix B. For further reading about the notion of a minimum-phase system, see also [8, 9].

[16] See Sect. B.2 in Appendix B.

[17] In what follows, property (6.16) will be sometime expressed in the equivalent form: $V(x)$ is *positive definite and proper*.

and

$$\frac{\partial V_0}{\partial z} f(z, \xi) \leq -\alpha(\|z\|) \qquad \text{for all } (z, \xi) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \text{ such that } \|z\| \geq \chi(\|\xi\|). \tag{6.17}$$

Of course, this criterion, written for $\xi = 0$, includes as a special case the criterion of Lyapunov to determine when a system is globally minimum-phase.

In what follows, it will be useful to consider the special case of systems that are globally minimum-phase and the convergence to the equilibrium of the trajectories of (6.13) is locally exponential and—also—the special case of systems that are strongly minimum-phase and the functions $\gamma(\cdot)$ and $\beta(\cdot, \cdot)$ of the estimate (6.15), for some $d > 0$ and some $\ell > 0$, $M > 0$, $\alpha > 0$, are bounded as

$$\begin{aligned} \gamma(r) &\leq \ell\, r \\ \beta(r, t) &\leq M e^{-\alpha t} r \end{aligned} \qquad \text{for } |r| \leq d. \tag{6.18}$$

Such stronger properties are summarized in the following Definition.

Definition 6.2 A system is *globally—and also locally exponentially—minimum-phase* if the equilibrium $z = 0$ of (6.13) is globally asymptotically and locally exponentially stable. A system is *strongly—and also locally exponentially—minimum-phase* if for any $z_0 \in \mathbb{R}^{n-r}$ and any piecewise-continuous bounded function $\xi_0(\cdot) : [0, \infty) \to \mathbb{R}^r$, an estimate of the form (6.15) holds, where $\beta(\cdot, \cdot)$ and $\gamma(\cdot)$ are a class $\mathscr{K}\mathscr{L}$ function and, respectively, a class $\mathscr{K}$ function bounded as in (6.18).

Remark 6.6 It is seen from the analysis above that a linear system is globally minimum-phase if the system $\dot{z} = A_{00} z$ is asymptotically stable, i.e., if it is minimum-phase in the sense defined in Sect. 2.2.[18] Of course, this property automatically implies that the linear system

$$\dot{z} = A_{00} z + A_{01} \xi$$

is input-to-state stable and hence the system is strongly and also locally exponentially minimum-phase. ◁

6.4 Stabilization via Full State Feedback

A typical setting in which normal forms are useful is the derivation of systematic methods for stabilization *in the large* of certain classes of nonlinear system, even in the presence of parameter uncertainties. We begin this analysis with the observation that, if the system is *strongly minimum-phase*, it is quite easy to design a globally stabilizing *state feedback* law. To be precise, consider again a system in normal form

[18] This analogy is the motivation for the introduction of the term "minimum-phase" to indicate the asymptotic properties considered in Definition 6.1.

(6.8), which we assume to be globally defined, and assume that the system is strongly minimum-phase, i.e., assume that $f_0(0,0) = 0$ and that

$$\dot{z} = f_0(z, \xi),$$

viewed as a system with input ξ and state z, is input-to-state stable. Bearing in mind the fact that the coefficient $b(z, \xi)$ is nowhere zero, consider the feedback law

$$u = \frac{1}{b(z, \xi)} (-q(z, \xi) + \hat{K}\xi), \tag{6.19}$$

in which $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is a vector of design parameters. Under this feedback law, the system becomes

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= (\hat{A} + \hat{B}\hat{K})\xi. \end{aligned} \tag{6.20}$$

Since the pair $(\hat{A}, \hat{B})$ is reachable, it is possible to pick $\hat{K}$ so that the matrix $(\hat{A} + \hat{B}\hat{K})$ is a Hurwitz matrix. If this is the case, system (6.20) appears as a cascade-connection in which a globally asymptotically stable system (the lower subsystem) drives an input-to-state stable system (the upper subsystem). According to Theorem B.5 of Appendix B, such cascade-connection is globally asymptotically stable. In other words, if $\hat{K}$ is chosen in this way, the feedback law (6.19) *globally asymptotically* stabilizes the equilibrium $(z, \xi) = (0, 0)$ of the closed-loop system.

The feedback law (6.19) is expressed in the (z, ξ) coordinates that characterize the normal form (6.8). To express it in the original coordinates that characterize the model (6.1), it suffices to bear in mind that

$$b(z, \xi) = L_g L_f^{r-1} h(x), \qquad q(z, \xi) = L_f^r h(x)$$

and to observe that, since $\xi_i = L_f^{i-1} h(x)$ for $i = 1, \ldots, r$, then

$$\hat{K}\xi = \sum_{i=1}^{r} \hat{k}_i L_f^{i-1} h(x)$$

in which $\hat{k}_1, \ldots, \hat{k}_r$ are the entries of the row vector $\hat{K}$. Thus, we can conclude what follows.

Proposition 6.4 *Consider a system of the form (6.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has uniform relative degree r and possesses a globally defined normal form. Suppose the system is strongly minimum-phase. If $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is any vector such that $\sigma(\hat{A} + \hat{B}\hat{K}) \in \mathbb{C}^-$, the* state feedback *law*

$$u(x) = \frac{1}{L_g L_f^{r-1} h(x)} \Big(-L_f^r h(x) + \sum_{i=1}^{r} \hat{k}_i L_f^{i-1} h(x) \Big), \tag{6.21}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

This feedback strategy, although very intuitive and elementary, is not useful in a practical context because it relies upon exact cancelation of certain nonlinear function and, as such, possibly *non-robust*. Uncertainties in $q(z, \xi)$ and $b(z, \xi)$ would make this strategy unapplicable. Moreover, the implementation of such control law requires the availability, for feedback purposes, of the *full* state (z, ξ) of the system, a condition that might be hard to ensure. Thus, motivated by these considerations, we readdress the problem in what follows, by seeking feedback laws depending on fewer measurements (hopefully only on the measured output y) and possibly robust with respect to model uncertainties. Of course, in return, some price has to be paid.

We conduct this analysis in the next section and in the following Chapters. For the time being we conclude by showing how, in the context of full state feedback, the assumption that the system is *strongly* minimum-phase can be weakened. This is possible, to some extent, if the normal form of the system has the special structure (6.9). However, the expression of the control law becomes more involved.

The design of such feedback law is based on a recursive procedure, known as *backstepping*, by means of which it is possible to construct, for a system having the special structure (6.9), a state feedback stabilizing law as well as a Lyapunov function.[19] The procedure in question reposes on the following results.

Lemma 6.2 *Consider a system described by equations of the form*

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= u \end{aligned} \tag{6.22}$$

in which $(z, \xi) \in \mathbb{R}^{n-r} \times \mathbb{R}$, *and* $f(0, 0) = 0$. *Suppose the equilibrium* $z = 0$ *of* $\dot{z} = f(z, 0)$ *is globally asymptotically stable, with Lyapunov function* $V(z)$. *Express* $f(z, \xi)$ *in the form*[20]

$$f(z, \xi) = f(z, 0) + p(z, \xi)\xi \tag{6.23}$$

[19] For additional reading on such design procedure, as well as on its use in problems of adaptive control, see [3, 10].

[20] To check that this is always possible, observe that the difference

$$\bar{f}(z, \xi) = f(z, \xi) - f(z, 0)$$

is a smooth function vanishing at $\xi = 0$, and express $\bar{f}(z, \xi)$ as

$$\bar{f}(z, \xi) = \int_0^1 \frac{\partial \bar{f}(z, s\xi)}{\partial s} ds = \int_0^1 \left[\frac{\partial \bar{f}(z, \zeta)}{\partial \zeta}\right]_{\zeta = s\xi} \xi ds.$$

where $p(z,\xi)$ is a smooth function. Set

$$u(z,\xi) = -\xi - \frac{\partial V}{\partial z} p(z,\xi). \tag{6.24}$$

Then, the equilibrium $(z,\xi)=(0,0)$ of (6.22) controlled by (6.24) is globally asymptotically stable, with Lyapunov function

$$W(z,\xi) = V(z) + \frac{1}{2}\xi^2.$$

Proof By assumption, $V(z)$ is positive definite and proper, which implies that the function $W(z,\xi)$ in the lemma is positive definite and proper as well. Moreover,

$$\frac{\partial V}{\partial z} f(z,0) \le -\alpha(\|z\|) \qquad \forall z \in \mathbb{R}^{n-r}$$

for some class $\mathscr{K}$ function $\alpha(\cdot)$. Observe that

$$\dot W = \frac{\partial W}{\partial z} f(z,\xi) + \frac{\partial W}{\partial \xi} u = \frac{\partial V}{\partial z} f(z,0) + \frac{\partial V}{\partial z} p(z,\xi)\xi + \xi u.$$

Choosing u as in (6.24) yields

$$\dot W \le -\alpha(\|z\|) - \xi^2 \qquad \forall (z,\xi) \in \mathbb{R}^{n-r} \times \mathbb{R}.$$

The quantity on the right-hand side is negative for all nonzero (z,ξ) and this proves the lemma. ◁

In the next lemma (which contains Lemma 6.2 as a particular case), this result is extended by showing that to the purpose of stabilizing the equilibrium $(z,\xi)=(0,0)$ of system (6.22), it suffices to assume that the equilibrium $z=0$ of

$$\dot z = f(z,\xi),$$

viewed as a system with state z and input ξ, is *stabilizable* by means of a smooth control law $\xi = \xi^*(z)$. To see that this is the case, change the variable ξ of (6.22) into

$$\zeta = \xi - \xi^*(z),$$

which transforms (6.22) into a system

$$\begin{aligned}\dot z &= f(z,\xi^*(z)+\zeta)\\ \dot\zeta &= -\frac{\partial \xi^*}{\partial z} f(z,\xi^*(z)+\zeta) + u,\end{aligned} \tag{6.25}$$

Pick now

$$u = -\frac{\partial \xi^*}{\partial z} f(z, \xi^*(z) + \zeta) + \bar{u}$$

so as to obtain a system of the form

$$\begin{aligned} \dot{z} &= f(z, \xi^*(z) + \zeta) \\ \dot{\zeta} &= \bar{u}. \end{aligned}$$

This system has the same structure as that of system (6.22), and—by construction—satisfies the assumptions of Lemma 6.2. Thus, this system can be globally stabilized by means of a control $\bar{u}$ having the structure of the control indicated in this lemma. This yields the following consequence.

Lemma 6.3 *Consider a system described by equations of the form (6.22), in which* $(z, \xi) \in \mathbb{R}^{n-r} \times \mathbb{R}$, *and* $f(0, 0) = 0$. *Suppose there exists a smooth real-valued function* $\xi^* : \mathbb{R}^{n-r} \to \mathbb{R}$, *with* $\xi^*(0) = 0$, *such that the equilibrium* $z = 0$ *of*

$$\dot{z} = f(z, \xi^*(z))$$

is globally asymptotically stable, with Lyapunov function $V(z)$. *Set* $\zeta = \xi - \xi^*(z)$ *and express* $f(z, \xi^*(z) + \zeta)$ *in the form*

$$f(z, \xi^*(z) + \zeta) = f(z, \xi^*(z)) + p^*(z, \zeta)\zeta$$

Set

$$u^*(z, \zeta) = \frac{\partial \xi^*}{\partial z} f(z, \xi^*(z) + \zeta) - \zeta - \frac{\partial V}{\partial z} p^*(z, \zeta) \tag{6.26}$$

Then, the equilibrium $(z, \zeta) = (0, 0)$ *of (6.25) controlled by (6.26) is globally asymptotically stable, with Lyapunov function*

$$W^*(z, \zeta) = V(z) + \frac{1}{2}\zeta^2.$$

As a consequence, the equilibrium $(z, \xi) = (0, 0)$ *of (6.22) controlled by*

$$u(z, \xi) = \frac{\partial \xi^*}{\partial z} f(z, \xi) - \xi + \xi^*(z) - \frac{\partial V}{\partial z} p^*(z, \xi - \xi^*(z))$$

is globally asymptotically stable, with Lyapunov function

$$W(z, \xi) = V(z) + \frac{1}{2}(\xi - \xi^*(z))^2.$$

The function $\xi^*(z)$, which is seen as a "control" imposed on the upper subsystem of (6.22), is usually called a *virtual control*. The property indicated in Lemma 6.3

can be used repeatedly, to address the problem of stabilizing a system of the form (6.9). In the first iteration, beginning from a virtual control $\xi_1^*(z)$ that stabilizes the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, \xi_1^*(z)),$$

using this lemma one finds a virtual control $\xi_2^*(z, \xi_1)$ that stabilizes the equilibrium $(z, \xi_1) = (0, 0)$ of

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2^*(z, \xi_1). \end{aligned}$$

Then, using the lemma again, one finds a virtual control $\xi_3^*(z, \xi_1, \xi_2)$ that stabilizes the equilibrium $(z, \xi_1, \xi_2) = (0, 0, 0)$ of

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3^*(z, \xi_1). \end{aligned}$$

and so on. In summary, it is straightforward to derive the following stabilization result about a system in the form (6.9).

Proposition 6.5 *Consider a system of the form (6.9), in which* $f_0(0, 0) = 0$. *Suppose there exists a smooth real-valued function* $\xi_1^* : \mathbb{R}^{n-r} \to \mathbb{R}$, *with* $\xi_1^*(0) = 0$, *such that the equilibrium* $z = 0$ *of*

$$\dot{z} = f_0(z, \xi_1^*(z))$$

is globally asymptotically stable. Then, there exists a smooth state feedback law

$$u = u(z, \xi_1, \ldots, \xi_r)$$

with $u(0, 0, \ldots, 0) = 0$, *which globally asymptotically stabilizes the equilibrium* $(z, \xi_1, \ldots, \xi_r) = (0, 0, \ldots, 0)$ *of the corresponding closed-loop system.*

Of course, a special case in which the result of this proposition holds is when $\dot{z} = f_0(z, 0)$ has a globally asymptotically stable equilibrium at $z = 0$, i.e., when the system is globally minimum-phase. In this case, in fact, the assumption of the proposition holds with $\xi_1^*(z) = 0$.

Remark 6.7 Note that, while the feedback law (6.19) has a very simple expression, the feedback law derived above cannot be easily expressed in closed form. Rather, it can only be derived by means of a recursive procedure. The actual expression of the law in question also requires the explicit knowledge of the function $V(z)$. In return, the stabilization method just described does not require the upper subsystem of the form (6.9) to be input-to-state stable, as assumed in the case of the law (6.19), but only relies upon the assumption that the subsystem in question is stabilizable, by means of an appropriate virtual control $\xi_1^*(z)$. $\triangleleft$

Remark 6.8 It is immediate to extend the stabilization result of Proposition 6.5 to a system of the form

$$
\begin{aligned}
\dot{z} &= f_0(z, \xi_1) \\
\dot{\xi}_1 &= q_1(z, \xi_1) + b_1(z, \xi_1)\xi_2 \\
\dot{\xi}_2 &= q_2(z, \xi_1, \xi_2) + b_2(z, \xi_1, \xi_2)\xi_3 \\
& \cdots \\
\dot{\xi}_r &= q_r(z, \xi_1, \ldots, \xi_r) + b_r(z, \xi_1, \ldots, \xi_r)u.
\end{aligned}
$$

so long as the coefficients $b_i(z, \xi_1, \ldots, \xi_i)$ for all $i = 1, \ldots, r$ are nonzero. In this case, in fact, this system is globally diffeomorphic to a system of the form (6.9), as the reader can easily verify. ◁

6.5 Stabilization via Partial State Feedback

In the previous section, we have described how certain classes of nonlinear systems (namely systems that, after a globally defined change of coordinates can be expressed in the form (6.8) or (6.9)), under suitable hypotheses on the subsystem $\dot{z} = f_0(z, \xi)$ (input-to-state stability, or global stabilizability in the special case of a system of the form (6.9)) can be globally stabilized via *full state feedback*. In this section, we show how similar results can be obtained using a *partial state* feedback. Of course, if limited state information are available, we expect that weaker results will be obtained.

We consider first the case of a system having relative degree 1, which in normal form is written as

$$
\begin{aligned}
\dot{z} &= f_0(z, \xi) \\
\dot{\xi} &= q(z, \xi) + b(z, \xi)u \\
y &= \xi
\end{aligned}
\tag{6.27}
$$

in which $z \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}$. As before, we assume that

$$
\begin{aligned}
f_0(0, 0) &= 0 \\
q(0, 0) &= 0.
\end{aligned}
$$

The coefficient $b(z, \xi)$ is by definition nowhere zero. Being a continuous function of (z, ξ), it is either always positive or always negative. Without loss of generality, we assume

$$
b(z, \xi) > 0 \qquad \text{for all } (z, \xi).
$$

We retain the assumption that the system is *globally minimum-phase*.

The system is controlled by the very simple feedback law

$$
u = -ky \tag{6.28}
$$

with $k > 0$, which yields a closed-loop system

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= q(z, \xi) - b(z, \xi)k\xi. \end{aligned} \tag{6.29}$$

Set[21]

$$x = \mathrm{col}(z, \xi)$$

and rewrite (6.29) as

$$\dot{x} = F_k(x) \tag{6.30}$$

in which

$$F_k(x) = \begin{pmatrix} f_0(z, \xi) \\ q(z, \xi) - b(z, \xi)k\xi \end{pmatrix}.$$

Proposition 6.6 *Consider system (6.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has uniform relative degree 1 and possesses a globally defined normal form. Suppose the system is globally minimum-phase. Let the control be provided by the output feedback $u = -ky$ so that a closed-loop system modeled as in (6.30) is obtained. Then, for every choice of a compact set $\mathscr{C}$ and of a number $\varepsilon > 0$, there is a number k^* and a finite time T such that, if $k \geq k^*$, all trajectories of the closed-loop system with initial condition $x(0) \in \mathscr{C}$ remain bounded and satisfy $\|x(t)\| < \varepsilon$ for all $t \geq T$.*

Proof Consider, for system (6.30), the candidate Lyapunov function[22]

$$W(x) = V(z) + \frac{1}{2}\xi^2$$

which is positive definite and proper. For any real number $a > 0$, let Ω_a denote the sublevel set of $W(x)$

$$\Omega_a = \{x \in \mathbb{R}^n : W(x) \leq a\}$$

and let

$$B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$$

denote the (open) ball radius ε. Assume, without loss of generality that $\mathscr{C}$ is such that $B_\varepsilon \subset \mathscr{C}$. Since $W(x)$ is positive definite and proper, there exist numbers $0 < d < c$ such that

$$\Omega_d \subset B_\varepsilon \subset \mathscr{C} \subset \Omega_c.$$

---

[21] With a mild abuse of notation, we use here x instead of $\tilde{x}$, to denote the vector of coordinates that characterize the normal form of the system.

[22] The arguments used in this proof are essentially those originally proposed in [11] and frequently reused in the literature.

Consider also the compact "annular" region

$$S_d^c = \{x \in \mathbb{R}^n : d \le W(x) \le c\}.$$

It will be shown that—if the gain coefficient k is large enough—the function

$$\dot{W}(x) := \frac{\partial W}{\partial x} F_k(x) = \frac{\partial V}{\partial z} f_0(z, \xi) + \xi q(z, \xi) - b(z, \xi) k \xi^2$$

is *negative* at each point of S_d^c. To this end, proceed as follows. Consider the compact set

$$S_0 = \{x \in S_d^c : \xi = 0\}.$$

At each point of S_0

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z, 0) \le -\alpha(\|z\|)$$

Since $\min_{x \in S_0} \|z\| > 0$, there is a number $a > 0$ such that

$$\dot{W}(x) \le -a \qquad \forall x \in S_0.$$

Hence, by continuity, there is an open set $S' \supset S_0$ such

$$\dot{W}(x) \le -a/2 \qquad \forall x \in S'. \tag{6.31}$$

Consider now the set

$$S'' = \{x \in S_d^c : x \notin S'\}.$$

which is a compact set (and note that $S'' \cup S' = S_d^c$), let

$$M = \max_{x \in S''} \{\frac{\partial V}{\partial z} f_0(z, \xi) + \xi q(z, \xi)\} \qquad m = \min_{x \in S''} \{b(z, \xi) \xi^2\}$$

and observe that $m > 0$ because $b(z, \xi) > 0$ and ξ cannot vanish at any point of S''.[23] Thus, since $k > 0$, we obtain

$$\dot{W}(x) \le M - km \qquad \forall x \in S''.$$

Let k_1 be such that $M - k_1 m = -a/2$. Then, if $k \ge k_1$,

$$\dot{W}(x) \le -a/2 \qquad \forall x \in S''. \tag{6.32}$$

[23] In fact, let x be a point of S_d^c for which $\xi = 0$. Then $x \in S_0$, which implies $x \in S'$. Any of such x cannot be in S''.

This, together with (6.31) shows that

$$k \geq k_1 \qquad \Rightarrow \qquad \dot{W}(x) \leq -a/2 \qquad \forall x \in S_d^c,$$

as anticipated.

This being the case, suppose the initial condition $x(0)$ of (6.30) is in S_d^c. It follows from known arguments[24] that $x(t) \in \Omega_c$ for all $t \geq 0$ and, at some time

$$T \leq 2(c-d)/a,$$

$x(T)$ is on the boundary of the set Ω_d.[25] On the boundary of Ω_d the derivative of $W(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_d and remains there for all $t \geq T$. Since all $x \in \Omega_d$ are such that $\|x\| < \varepsilon$, this completes the proof. $\triangleleft$

This proposition shows that, no matter how large the set $\mathscr{C}$ of initial conditions is chosen and no matter how small a "target set" B_ε is chosen, there is a value k^* of the gain in (6.28) and a finite time T such that, if the actual gain parameter k is larger than or equal to k^*, all trajectories of the closed-loop system with origin in $\mathscr{C}$ are bounded and for all $t \geq T$ remain in the set B_ε. This property is commonly referred to by saying that the control law (6.28) is *able* to *semiglobally and practically* stabilize the point $(z, \xi) = (0, 0)$. The term "practical" (as opposite to *asymptotic*) is meant to stress the fact that the convergence is not to a point, but rather to a neighborhood of that point, that can be chosen *arbitrarily small*, while the term "semiglobal" (as opposite to *global*) is meant to stress the fact that the convergence to the target set is not for all initial conditions, but rather for a compact set of initial conditions, that can be chosen *arbitrarily large*.[26]

The result presented in Proposition 6.6 can be seen as nonlinear analogue of the stabilization result presented in Sect. 2.3 of Chap. 2. The standing assumption in both cases is that the system is minimum-phase and the stability result is obtained via *high-gain* output feedback. In the case of the nonlinear system, the minimal value k^* of the feedback gain k is determined by the choice of the set $\mathscr{C}$ and by the value of ε. In particular, as it is clear from the proof of Proposition 6.6, k^* increases as $\mathscr{C}$ "increases" (in the sense of set inclusion) and also increases as ε decreases.

To obtain *asymptotic* stability, either a nonlinear control law $u = -\kappa(y)$ is needed or, if one insists in using a linear law $u = -ky$, extra assumptions are necessary. The first option shall be briefly covered in Sect. 8.2 of Chap. 8.[27] For the time being, we address here the second option.

[24]See Sect. B.1 of Appendix B, in this respect.

[25]Note that such T only depends on the choice of $\mathscr{C}$ and ε and not on the value of k.

[26]For further reading, see also [12].

[27]See also [13], in this respect.

Proposition 6.7 *Consider system (6.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the system is globally—and also locally exponentially—minimum-phase. Let the control be provided by the output feedback $u = -ky$. Then, for every choice of a compact set $\mathscr{C}$ there is a number k^* such that, if $k \geq k^*$, the equilibrium $x = 0$ of the resulting closed-loop system is asymptotically (and locally exponentially) stable, with a domain of attraction $\mathscr{A}$ that contains the set $\mathscr{C}$.*

Proof Expand $f_0(z, 0)$ as

$$f_0(z, 0) = F_0 z + g(z)$$

in which

$$F_0 = \frac{\partial f_0}{\partial z}(0, 0) \qquad \text{and} \qquad \lim_{z \to 0} \frac{\|g(z)\|}{\|z\|} = 0. \tag{6.33}$$

If the zero dynamics are locally exponentially stable, all eigenvalues of F_0 have negative real part and there exists a positive-definite solution P of

$$PF_0 + F_0^{\mathrm{T}} P = -2I.$$

Consider, for the system, the candidate quadratic Lyapunov function

$$U(x) = z^{\mathrm{T}} P z + \frac{1}{2} \xi^2$$

yielding

$$\dot{U}(x) = 2z^{\mathrm{T}} P f_0(z, \xi) + \xi q(z, \xi) - b(z, \xi) k \xi^2.$$

Write $f_0(z, \xi)$ as

$$f_0(z, \xi) = F_0 z + g(z) + [f_0(z, \xi) - f_0(z, 0)].$$

Because of (6.33), there is a number δ such that

$$z \in B_\delta \qquad \Rightarrow \qquad \|g(z)\| \leq \frac{1}{2\|P\|} \|z\|$$

and this yields

$$2z^{\mathrm{T}} P [F_0 z + g(z)] = -2\|z\|^2 + 2z^{\mathrm{T}} P g(z) \leq -\|z\|^2 \qquad \forall z \in B_\delta.$$

Set

$$S_\delta = \{(z, \xi) : \|z\| < \delta, |\xi| < \delta\}.$$

Since the function $[f_0(z, \xi) - f_0(z, 0)]$ is a continuously differentiable function that vanish at $\xi = 0$, there is a number M_1 such that

$$\|2z^{\mathrm{T}}P[f_0(z,\xi)-f_0(z,0)]\| \leq M_1\|z\||\xi| \qquad \text{for all } (z,\xi)\in S_\delta.$$

Likewise, since $q(z,\xi)$ is a continuously differentiable function that vanish at $(z,\xi) = (0,0)$, there are numbers N_1 and N_2 such that

$$|\xi q(z,\xi)| \leq N_1\|z\||\xi| + N_2|\xi|^2 \qquad \text{for all } (z,\xi)\in S_\delta.$$

Finally, since $b(z,\xi)$ is positive and nowhere zero, there is a number b_0 such that

$$-kb(z,\xi) \leq -kb_0 \qquad \text{for all } (z,\xi)\in S_\delta.$$

Putting all these inequalities together, one finds that, for all for all $(z,\xi)\in S_\delta$

$$\dot{U}(x) \leq -\|z\|^2 + (M_1+N_1)\|z\||\xi| - (kb_0 - N_2)|\xi|^2.$$

It is easy to check that if k is such that

$$(2kb_0 - 2N_2 - 1) > (M_1+N_1)^2,$$

then

$$\dot{U}(x) \leq -\frac{1}{2}\|x\|^2.$$

This shows that there is a number k_2 such that, if $k \geq k_2$, the function $\dot{U}(x)$ is negative definite for all $(z,\xi)\in S_\delta$. Pick now any (nontrivial) sublevel set Ω_d of $U(x)$ entirely contained in the set S_δ and let r be such that $B_r \subset \Omega_d$. Then, the argument above shows that, for all $k \geq k_2$, the equilibrium $x = 0$ is asymptotically stable with a domain of attraction that contains B_r. Note that the set B_r depends only by the value of δ and on the matrix P, with both δ and P determined only by the function $f_0(z,0)$. In particular, the set B_r does *not* depend on the value chosen for k.

Pick $\varepsilon < r$ and any $\mathscr{C}$. By Proposition 6.6, we know that there is a number k_1 such that, if $k \geq k_1$, all trajectories with initial condition in $\mathscr{C}$ in finite time enter the (closure of) the set B_ε, and hence enter the region of attraction of $x = 0$. Setting $k^* = \max\{k_1, k_2\}$, it can be concluded that for all $k \geq k^*$ the equilibrium $x = 0$ of (6.30) is asymptotically (in fact, locally exponentially) stable, with a domain of attraction that contains $\mathscr{C}$. ◁

The case of a system having relative degree $r > 1$ can be reduced to the case discussed above by means of a technique which is reminiscent of the technique introduced in Sect. 2.4. Suppose the normal form

$$\begin{aligned}
\dot{z} &= f_0(z,\xi_1,\ldots,\xi_{r-1},\xi_r)\\
\dot{\xi}_1 &= \xi_2\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r\\
\dot{\xi}_r &= q(z,\xi_1,\ldots,\xi_{r-1},\xi_r) + b(z,\xi_1,\ldots,\xi_{r-1},\xi_r)u.
\end{aligned} \tag{6.34}$$

is globally defined and let the variable ξ_r be replaced by a new state variable defined as (compare with (2.21))

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters.

After this change of coordinates, the system becomes

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\
\dot{\xi}_1 &= \xi_2\\
&\cdots\\
\dot{\xi}_{r-2} &= \xi_{r-1}\\
\dot{\xi}_{r-1} &= -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta\\
\dot{\theta} &= a_0\xi_2 + a_1\xi_3 + \cdots + a_{r-2}(-\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\
&\qquad + q(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\
&\qquad + b(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)u.
\end{aligned}$$

This system, with θ regarded as output, has a structure which is identical to that of system (6.27). In fact, if we set

$$\begin{aligned}
\zeta &= \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}) \in \mathbb{R}^{n-1}\\
\bar{y} &= \theta
\end{aligned}$$

and define $\bar{f}_0(\zeta, \theta)$, $\bar{q}(\zeta, \theta)$, $\bar{b}(\zeta, \theta)$, $\bar{y}$ as

$$\bar{f}_0(\zeta, \theta) = \begin{pmatrix} f_0(z, \xi_1, \ldots, \xi_{r-1}, -\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\ \xi_2\\ \cdots\\ \xi_{r-1}\\ -\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta \end{pmatrix}$$

$$\begin{aligned}
\bar{q}(\zeta, \theta) &= a_0\xi_2 + a_1\xi_3 + \cdots + a_{r-2}(-\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\
&\qquad + q(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)\\
\bar{b}(\zeta, \theta) &= b(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta),
\end{aligned}$$

the system can be rewritten in the form

$$\begin{aligned}
\dot{\zeta} &= \bar{f}_0(\zeta, \theta)\\
\dot{\theta} &= \bar{q}(\zeta, \theta) + \bar{b}(\zeta, \theta)u\\
\bar{y} &= \theta,
\end{aligned} \tag{6.35}$$

identical to that of (6.27). Thus, identical stability results can be obtained, if $\bar{f}_0(\zeta, \theta)$, $\bar{q}(\zeta, \theta)$, $\bar{b}(\zeta, \theta)$ satisfy conditions corresponding to those assumed on $f_0(z, \xi)$, $q(z, \xi)$, $b(z, \xi)$.

From this viewpoint, it is trivial to check that if the functions $f_0(z, \xi)$, $q(z, \xi)$, and $b(z, \xi)$ satisfy

$$\begin{aligned} f_0(0,0) &= 0 \\ q(0,0) &= 0 \\ b(z,\xi) &> 0 \qquad \text{for all } (z,\xi) \end{aligned}$$

then also

$$\begin{aligned} \bar{f}_0(0,0) &= 0 \\ \bar{q}(0,0) &= 0 \\ \bar{b}(\zeta,\theta) &> 0 \qquad \text{for all } (\zeta,\theta). \end{aligned}$$

In order to be able to use the stabilization results indicated in Proposition 6.6 (respectively, in Proposition 6.7), it remains to check whether system (6.35) is globally minimum-phase (respectively, globally—and also locally exponentially—minimum-phase), i.e., whether the equilibrium $\zeta = 0$ of

$$\dot{\zeta} = \bar{f}_0(\zeta, 0)$$

is globally asymptotically stable (respectively, globally asymptotically and locally exponentially stable). This system has the structure of a cascade interconnection

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, -\sum_{i=1}^{r-1} a_{i-1}\xi_i) \\ \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \cdots \\ \dot{\xi}_{r-2} \\ \dot{\xi}_{r-1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_{r-2} \\ \xi_{r-1} \end{pmatrix}. \end{aligned} \tag{6.36}$$

If the a_i's are such that the polynomial

$$d(\lambda) = \lambda^{r-1} + a_{r-2}\lambda^{r-2} + \cdots + a_1\lambda + a_0 \tag{6.37}$$

is Hurwitz, the lower subsystem of the cascade is (globally) asymptotically stable. If system (6.34) is strongly minimum-phase, the upper subsystem of the cascade, viewed as a system with input $(\xi_1, \ldots, \xi_{r-1})$ and state z is input-to-state stable. Thus, system (6.36) is globally asymptotically stable. If, in addition, the zero dynamics of (6.34) are also locally exponentially stable, then system (6.36) is also locally exponentially stable.

Under such hypotheses, observing that—in the present context—the stabilizing feedback (6.28) becomes

$$u = -k\theta = -k(a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r)$$

and bearing in mind the fact that $\xi_i = L_f^{i-1}h(x)$ for $i = 1, \ldots, r$, one can conclude the following stabilization result.

Proposition 6.8 *Consider system (6.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has uniform relative degree r and possesses a globally defined normal form. Suppose the system is strongly minimum-phase. Let the control be provided by a feedback of the form*

$$u = -k\Big(\sum_{i=1}^{r} a_i L_f^{i-1}h(x)\Big). \tag{6.38}$$

with the a_i's such that the polynomial (6.37) is Hurwitz and $a_{r-1} = 1$. Then, for every choice of a compact set $\mathscr{C}$ and of a number $\varepsilon > 0$, there is a number k^* and a finite time T such that, if $k \geq k^*$, all trajectories of the closed-loop system with initial condition $x(0) \in \mathscr{C}$ remain bounded and satisfy $\|x(t)\| < \varepsilon$ for all $t \geq T$. If the system is strongly—and also locally exponentially—minimum-phase, for every choice of a compact set $\mathscr{C}$ there is a number k^* such that, if $k \geq k^*$, the equilibrium $x = 0$ of the resulting closed-loop system is asymptotically (and locally exponentially) stable, with a domain of attraction $\mathscr{A}$ that contains the set $\mathscr{C}$.

Remark 6.9 Note that in Proposition 6.8 we have used the assumption that the system is *strongly* minimum-phase while in Propositions 6.6 and 6.7 we had used the weaker assumption that the system is globally minimum-phase. This is due to the fact that we are considering here the general case in which the normal form of the system has the structure (6.8), where the dynamics of the upper "subsystem"

$$\dot{z} = f_0(z, \xi)$$

are (possibly) affected by *all* the components of ξ. If the normal form of the system had the special structure (6.9), in which the dynamics of z are affected *only* by ξ_1, one might have weakened the assumption, requiring only the system to be globally minimum-phase. Details are not covered here and can be found in the literature.[28] If $r = 1$ the difference between the structure (6.8) and (6.9) ceases to exists and this explains the weaker assumption used in Propositions 6.6 and 6.7. $\triangleleft$

Remark 6.10 It is worth comparing the feedback laws (6.21) and (6.38) which, in the coordinates of the normal form, read as

$$u = \frac{1}{b(z, \xi)}\Big(-q(z, \xi) + \sum_{i=1}^{r} k_i \xi_i\Big)$$

and, respectively,

$$u = -k\Big(\sum_{i=1}^{r} a_i \xi_i\Big).$$

[28] See, e.g., [7, 12], [2, pp. 439–448].

The effect of the former is to cancel certain nonlinear terms, yielding

$$\dot{\xi}_r = \sum_{i=1}^{r} k_i \xi_i,$$

while if the latter is used such cancelation does not takes place and

$$\dot{\xi}_r = q(z, \xi) - b(z, \xi) k \sum_{i=1}^{r} a_i \xi_i. \tag{6.39}$$

The first law is indeed sensitive to uncertainties in $b(z, \xi)$ and $q(z, \xi)$ while the second is not. If the second law is used, the presence of the nonlinear terms in (6.39) is made negligible by taking a large value of the "gain parameter" k. Such "simpler" and "more robust" feedback law, though, is not expected to yield global stability, but only practical stability with a guaranteed region of attraction. If asymptotic stability is sought, the additional assumption that the equilibrium $z = 0$ of the zero dynamics is locally exponentially stable is needed. ◁

The stabilizing feedback laws discussed in this section are, to some extent, nonlinear counterparts of the feedback law considered in Sect. 2.4. Such laws presume the availability of all the components $\xi_1, \ldots, \xi_r$ of the *partial state* ξ that characterizes the normal form (6.8). One might wonder, at this point, whether procedures similar to those described in Sect. 2.5 could be set up, and whether the goal of achieving asymptotic stability could be obtained using, instead of ξ, an appropriate "replacement" generated by a dynamical systems driven only by the measured output y. As a matter of fact, this is actually feasible. The extension of the method presented in Sect. 2.5 and the associated convergence proof involve some subtleties and require appropriate care. Since one of the purposes of the book is to discuss problems of stabilization also for multi-input multi-output systems, in order to reduce duplications we defer the discussion of this topic to Chap. 10, where the case of multi-input multi-output systems is addressed. In that Chapter, it will be shown not only how a suitable replacement of the partial state ξ can be generated, but also how a "robust" version of the feedback law (6.21) can be obtained, if semiglobal stabilization is sought.

6.6 Examples and Counterexamples

Example 6.1 In this example it is shown that, if the relative degree of the system is higher than 1 and the normal form *does not* have the special structure (6.9), the assumption that system is globally minimum phase may not suffice to guarantee the existence of a feedback law that stabilizes the equilibrium $x = 0$ with an arbitrarily large region of attraction. Consider the system having relative degree 2

$$\begin{aligned}\dot{z} &= -z + z^2\xi_1 + \varphi(z, \xi_2)\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= u\end{aligned}$$

and suppose

$$\varphi(z, \xi_2) = \frac{1}{4}(z - \xi_2)^2 z.$$

This system is globally minimum phase. However, it is not strongly minimum phase.

Set $\eta = z\xi_1$ and observe that

$$\dot{\eta} = -\eta + \eta^2 + \varphi(z, \xi_2)\xi_1 + z\xi_2.$$

Consider the set

$$S = \{(z, \xi_1, \xi_2) \in \mathbb{R}^3 : \eta \geq 2.\}$$

At each point of S

$$\varphi(z, \xi_2)\xi_1 + z\xi_2 = \frac{1}{4}(z - \xi_2)^2\eta + z\xi_2 \geq \frac{1}{2}(z - \xi_2)^2 + z\xi_2 = \frac{1}{2}(z^2 + \xi_2^2) \geq 0.$$

Therefore, at each point of the set S,

$$\dot{\eta} \geq -\eta + \eta^2 \geq 2.$$

This shows that if $\eta(0) = z(0)\xi_1(0) \geq 2$, then $z(t)\xi_1(t) \geq 2$ for all times for which the trajectory is defined. As a consequence, it is concluded that—no matter how the control u is chosen—trajectories with initial conditions satisfying $z(0)\xi_1(0) \geq 2$ can never converge to the equilibrium point $(z, \xi_1, \xi_2) = (0, 0, 0)$. ◁

Example 6.2 Consider the system

$$\begin{aligned}\dot{z} &= -z + z^2\xi\\ \dot{\xi} &= u.\end{aligned}$$

If $u = -\xi$, the system reduces exactly to the system considered in Example B.3, which is known to have finite escape times. However, this system fulfills the assumptions of Proposition 6.7. In fact, it is globally and also locally exponentially minimum phase. Thus, for any arbitrary compact set $\mathscr{C}$ of initial conditions, there exists a number k^* such that, if $k \geq k^*$, the equilibrium $(z, \xi) = (0, 0)$ of

$$\begin{aligned}\dot{z} &= -z + z^2\xi\\ \dot{\xi} &= -k\xi.\end{aligned}$$

is locally exponentially stable, with a domain of attraction that contains the set $\mathscr{C}$. This fact is not in contradiction with the conclusions of the Example B.3. In fact, as shown in that example, for each $z_0 > 1$, the solution would escape to infinity in finite time if

$$\xi(t) = \exp(-kt)\xi_0 \geq 1 \qquad \text{for all } t \in [0, t_{\max}(z_0)) \tag{6.40}$$

where $t_{\max}(z_0) = \ln(z_0) - \ln(z_0 - 1)$. Now, for any choice of (z_0, ξ_0) it is always possible to find a value of k such that $\xi(t)$ decreases fast enough so that the estimate (6.40) is violated. Thus, the arguments used in the example to conclude that the system has finite escape time are no longer applicable. From the example we learn, though, that even if by choosing a suitable k the system can be made asymptotically stable, with a domain of attraction that contains the set $\mathscr{C}$, there are initial conditions outside $\mathscr{C}$ from which the trajectory escapes to infinity in finite time. ◁

Example 6.3 This example is meant to show that, in a system like (6.29), asymptotic stability of the equilibrium $z = 0$ of $\dot{z} = f_0(z, 0)$ is not sufficient to guarantee, even for large $k > 0$, local asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$. Consider the system

$$\begin{aligned}\dot{z} &= -z^3 + \xi \\ \dot{\xi} &= z + u.\end{aligned}$$

This system is globally minimum phase but not globally *and also locally exponentially* minimum phase. Set $u = -k\xi$ to obtain the system (compare with (6.29))

$$\begin{aligned}\dot{z} &= -z^3 + \xi \\ \dot{\xi} &= z - k\xi.\end{aligned} \tag{6.41}$$

No matter how k is chosen, the equilibrium $(z, \xi) = (0, 0)$ cannot be locally asymptotically stable. In fact, the linear approximation at this equilibrium

$$\begin{pmatrix}\dot{z} \\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & -k\end{pmatrix}\begin{pmatrix}z \\ \xi\end{pmatrix}$$

has a characteristic polynomial $\lambda^2 + k\lambda - 1$ with one root having positive real part for any k.

This is not in contradiction, though, with Proposition 6.6. In fact, system (6.41), for $k > 0$, has three equilibria

$$\begin{aligned}(z, \xi) &= (0, 0) \\ (z, \xi) &= (1/\sqrt{k}, 1/\sqrt{k^3}) \\ (z, \xi) &= (-1/\sqrt{k}, -1/\sqrt{k^3}).\end{aligned}$$

The former of these is unstable, but the other two are locally asymptotically stable. The unstable equilibrium, which is a saddle point, has a stable manifold $\mathscr{M}^{\mathrm{s}}$ (the set of all points x with the property that the integral curve satisfying $x(0) = x$ is defined for all $t > 0$ and $\lim_{t\to\infty} x(t) = 0$) and an unstable manifold $\mathscr{M}^{\mathrm{u}}$ (the set of all points x with the property that the integral curve satisfying $x(0) = x$ is defined for all $t < 0$ and $\lim_{t\to-\infty} x(t) = 0$). Trajectories with initial conditions that are not in $\mathscr{M}^s$ asymptotically converge, as $t \to \infty$, to either one of the two locally stable equilibria. From the characterization above, it is seen that, for any ε, there is a number k^* such that, if $k > k^*$, both the stable equilibria are in B_ε. Thus, all trajectories enter in finite time the target set B_ε (but, with the exception of those with origin in $\mathscr{M}^s$, do not converge to $x = 0$). ◁

Example 6.4 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1^3 + x_2\\ \dot{x}_2 &= x_2 + x_3\\ \dot{x}_3 &= x_1 + x_2^2 + u\\ y &= x_2.\end{aligned} \tag{6.42}$$

This system has relative degree 2. Its normal form is obtained setting

$$\xi_1 = x_2, \quad \xi_2 = x_2 + x_3, \quad z = x_1$$

and reads as

$$\begin{aligned}\dot{z} &= z - z^3 + \xi_1\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= z + \xi_1^2 + u\\ y &= \xi_1.\end{aligned}$$

Its zero dynamics

$$\dot{z} = z - z^3$$

are unstable. Hence, stabilization methods based on high-gain feedback from ξ_1, ξ_2 cannot be used. However, the system can be stabilized by full state feedback, using the procedure described in the second part of Sect. 6.4.

The dynamics of z can be stabilized by means of the virtual control $\xi_1^*(z) = -2z$. This yields a system

$$\dot{z} = -z - z^3$$

with Lyapunov function $W_1(z) = \frac{1}{2}z^2$.

Change ξ_1 into $\zeta_1 = \xi_1 - \xi_1^*(z) = \xi_1 + 2z$, to obtain

$$\begin{aligned}\dot{z} &= -z - z^3 + \zeta_1\\ \dot{\zeta}_1 &= \xi_2 - 2z - 2z^3 + 2\zeta_1\\ \dot{\xi}_2 &= z + (\zeta_1 - 2z)^2 + u.\end{aligned}$$

The subsystem consisting of the two upper equations, viewed as a system with state (z, ζ_1) and control ξ_2, can be stabilized by means of a virtual control

$$\xi_2^*(z, \zeta_1) = -(-2z - 2z^3 + 2\zeta_1) - \zeta_1 - \frac{\partial W_1}{\partial z} = -3\zeta_1 + z + 2z^3.$$

This yields a system

$$\begin{aligned}\dot{z} &= -z - z^3 + \zeta_1\\ \dot{\zeta}_1 &= -z - \zeta_1\end{aligned}$$

with Lyapunov function

$$W_2(z, \zeta_1) = \frac{1}{2}(z^2 + \zeta_1^2).$$

Change now ξ_2 into

$$\zeta_2 = \xi_2 - \xi_2^*(z, \zeta_1) = \xi_2 + 3\zeta_1 - z - 2z^3$$

to obtain a system of the form

$$\begin{aligned}\dot{z} &= -z - z^3 + \zeta_1\\ \dot{\zeta}_1 &= -z - \zeta_1 + \zeta_2\\ \dot{\zeta}_2 &= a(z, \zeta_1, \zeta_2) + u\end{aligned}$$

in which

$$a(z, \zeta_1, \zeta_2) = (\zeta_1 - 2z)^2 - z - 4\zeta_1 - 3\zeta_2 + 3z^3 + 2z^5 - 2z^2\zeta_1.$$

This system is stabilized by means of the control

$$u(z, \zeta_1, \zeta_2) = -a(z, \zeta_1, \zeta_2) - \zeta_2 - \frac{\partial W_2}{\partial \zeta_1} = -a(z, \zeta_1, \zeta_2) - \zeta_2 - \zeta_1,$$

and the corresponding closed-loop system has Lyapunov function

$$W_3(z, \zeta_1, \zeta_2) = \frac{1}{2}(z^2 + \zeta_1^2 + \zeta_2^2).$$

Reversing all changes of coordinates used in the above construction, one may find a stabilizing law $u = \psi(x)$ expressed in the original coordinates (x_1, x_2, x_2). ◁

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Chapter 7
Nonlinear Observers and Separation Principle

7.1 The Observability Canonical Form

In this chapter we discuss the design of observers for nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{7.1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$. In particular, we focus our discussion on the design of the so-called *high-gain observers*.[1]

As it happens for the design of state feedback laws, where—as shown in Chap. 6—if the equations that describe the system are put in special forms the design of such feedback is greatly facilitated, also the design of observers is simplified if the equations describing the system are changed into equations having a special structure. In this spirit, it will be assumed that there exists a *globally* defined diffeomorphism

$$\begin{aligned} \Phi : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto z \end{aligned}$$

that carries system (7.1) into a system described by equations of the form

$$\begin{aligned} \dot{z}_1 &= \tilde{f}_1(z_1, z_2, u) \\ \dot{z}_2 &= \tilde{f}_2(z_1, z_2, z_3, u) \\ &\cdots \\ \dot{z}_{n-1} &= \tilde{f}_{n-1}(z_1, z_2, \ldots, z_n, u) \\ \dot{z}_n &= \tilde{f}_n(z_1, z_2, \ldots, z_n, u) \\ y &= \tilde{h}(z_1, u) \end{aligned} \tag{7.2}$$

[1] High-gain observers have been considered by various authors in the literature. Here we closely follow the approach of Gauthier and Kupca, who have thoroughly investigated the design of high-gain observers in [1].

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_7

in which $\tilde{h}(z_1, u)$ and the $\tilde{f}_i(z_1, z_2, \ldots, z_{i+1}, u)$'s *have the following properties*

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0, \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0, \quad \text{for all} \quad i = 1, \ldots, n-1 \tag{7.3}$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$. Equations having this structure and such properties are said to be in *Gauthier–Kupca's uniform observability canonical form*.

Remark 7.1 The term "uniform" here is meant to stress the fact that the possibility of "observing" the state is not influenced by the actual input $u(\cdot)$ affecting the system. In fact, in contrast with the case of a linear system, in which the property of observability is independent of the input, this might not be the case for a nonlinear system, as it will also appear from the analysis below. The use of the term "observability" canonical form can be motivated by various arguments. To begin with, consider—as a special case—a single-output linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}$. Suppose that (A, C) is an *observable pair*, and consider a vector $z \in \mathbb{R}^n$ of new state variables defined by

$$z = Tx = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} x.$$

This is an admissible change of variables because, as a consequence of the observability assumption, the matrix T is nonsingular. A trivial calculation shows that, in the new coordinates, the system is expressed by equations of the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \cdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-2} \\ CA^{n-1} \end{pmatrix} \dot{x} = \begin{pmatrix} CAx + CBu \\ CA^2x + CABu \\ \cdots \\ CA^{n-1}x + CA^{n-2}Bu \\ CA^n x + CA^{n-1}Bu \end{pmatrix} = \begin{pmatrix} z_2 + CBu \\ z_3 + CABu \\ \cdots \\ z_n + CA^{n-2}Bu \\ CA^n T^{-1} z + CA^{n-1}Bu \end{pmatrix}$$

$$y = z_1 + Du$$

which clearly have the form (7.2) and in which the properties (7.3) hold.

For a general nonlinear system, suppose equations of the form (7.2), with properties (7.3), hold. Pick an input $u(\cdot)$, fix an initial condition $z(0)$ and let $z(t)$ and $y(t)$ denote the corresponding state and output trajectories. Define a function $F_1(z_1, u, y)$ as

$$F_1(z_1, u, y) = y - \tilde{h}(z_1, u).$$

By construction, the triplet $\{z_1(t), u(t), y(t)\}$ satisfies

$$F_1(z_1(t), u(t), y(t)) = 0. \tag{7.4}$$

The first condition in (7.3) implies

$$\frac{\partial F_1}{\partial z_1} \neq 0 \qquad \forall (z_1, u, y).$$

Therefore, by the implicit function theorem, Eq. (7.4)—that can be seen as defining $z_1(t)$ as *implicit function* of $(u(t), y(t))$—can be solved for $z_1(t)$, at least in a neighborhood of a point $\{z_1(0), u(0), y(0)\}$ satisfying

$$y(0) = \tilde{h}(z_1(0), u(0)).$$

In other words, there is a map $z_1 = H_1(u, y)$ such that

$$F_1(H_1(u, y), u, y) = 0$$

for all (u, y) in a neighborhood of $(u(0), y(0))$ and $z_1(t)$ can be expressed as

$$z_1(t) = H_1(u(t), y(t)). \tag{7.5}$$

In summary, the value of z_1 at time t can be expressed as an *explicit* function of the values of u and y at the same time t.

Next, consider a function $F_2(z_2, z_1, \dot{z}_1, u)$ defined as

$$F_2(z_2, z_1, \dot{z}_1, u) = \dot{z}_1 - \tilde{f}_1(z_1, z_2, u).$$

By construction, the triplet $\{z_1(t), z_2(t), u(t)\}$ satisfies

$$F_2(z_2(t), z_1(t), \dot{z}_1(t), u(t)) = 0. \tag{7.6}$$

The second condition in (7.3) implies

$$\frac{\partial F_2}{\partial z_2} \neq 0 \qquad \forall (z_2, z_1, \dot{z}_1, u).$$

Therefore Eq. (7.6)—that can be seen as defining $z_2(t)$ as *implicit function* of $(z_1(t), \dot{z}_1(t), u(t))$—can be solved for $z_2(t)$, at least in a neighborhood of a point $(z_2(0), z_1(0), \dot{z}_1(0), u(0))$ satisfying

$$\dot{z}_1(0) = \tilde{f}_1(z_1(0), z_2(0), u(0)).$$

In other words, there is a map $z_2 = H_2(z_1, \dot{z}_1, u)$ such that

$$F_2(H_2(z_1, \dot{z}_1, u), z_1, \dot{z}_1, u) = 0$$

for all $(z_1, \dot{z}_1, u)$ in a neighborhood of $(z_1(0), \dot{z}_1(0), u(0))$ and $z_2(t)$ can be expressed as

$$z_2(t) = H_2(z_1(t), \dot{z}_1(t), u(t)). \tag{7.7}$$

In view of the previous calculation, which has shown that $z_1(t)$ can be expressed as in (7.5), define a function $\dot{H}_1(u, u^{(1)}, y, y^{(1)})$ as

$$\dot{H}_1(u, u^{(1)}, y, y^{(1)}) = \frac{\partial H_1}{\partial u} u^{(1)} + \frac{\partial H_1}{\partial y} y^{(1)}$$

and observe that

$$\dot{z}_1(t) = \dot{H}_1(u(t), u^{(1)}(t), y(t), y^{(1)}(t)).$$

Entering this in (7.7), it is seen that $z_2(t)$ can be expressed as

$$z_2(t) = H_2(H_1(u(t), y(t)), \dot{H}_1(u(t), u^{(1)}(t), y(t), y^{(1)}(t)), u(t)).$$

In summary, the value of z_2 at time t can be expressed as an *explicit* function of the values of $u, u^{(1)}$ and $y, y^{(1)}$ at the same time t.

Proceeding in this way and using all of the (7.3), it can be concluded that, at least locally around a point $z(0), u(0), y(0)$, *all* the components $z_i(t)$ of $z(t)$ can be explicitly expressed as functions of the values $u(t), u^{(1)}(t), \ldots, u^{(i-1)}(t)$ and $y(t), y^{(1)}(t), \ldots, y^{(i-1)}(t)$ of input, output and their higher-order derivatives with respect to time. In other words, *the values of input, output and their derivatives* with respect to time up to order $n-1$, at any given time t, *uniquely determine the value of the state* at this time t. This being the case, it can be legitimately claimed that, if the system possesses a form such as (7.2), with properties (7.3), this system is *observable*. $\triangleleft$

We review in what follows how the existence of canonical forms of this kind can be checked and the canonical form itself can be constructed. We begin with the description of a set of *necessary* conditions for the existence of such canonical form.

Consider again system (7.1), suppose that $f(0,0) = 0$, $h(0,0) = 0$, and define—recursively—a sequence of real-valued functions $\varphi_i(x, u)$ as follows

$$\varphi_1(x, u) := h(x, u), \qquad \varphi_i(x, u) := \frac{\partial \varphi_{i-1}}{\partial x} f(x, u), \tag{7.8}$$

for $i = 2, \ldots, n$. Using such functions, define a sequence of $\mathbb{R}^i$-valued functions $\Phi_i(x, u)$ as follows

$$\Phi_i(x,u) = \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}$$

for $i = 1, \ldots, n$. Finally, with each of such $\Phi_i(x,u)$'s, associate the subspace

$$\mathscr{K}_i(x,u) = \mathrm{Ker}\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}.$$

Note that the map

$$D_i(u) \quad : \quad x \quad \mapsto \quad \mathscr{K}_i(x,u)$$

identifies a *distribution* on $\mathbb{R}^n$. The collection of all these distributions has been called[2] the *canonical flag* of (7.1). The notation chosen stresses the fact that the map in question, in general, depends on the parameter $u \in \mathbb{R}^m$. The canonical flag is said to be *uniform* if:

(i) for all $i = 1, \ldots, n$, for all $u \in \mathbb{R}^m$ and for all $x \in \mathbb{R}^n$

$$\dim \mathscr{K}_i(x,u) = n - i.$$

(ii) for all $i = 1, \ldots, n$ and for all $x \in \mathbb{R}^n$

$$\mathscr{K}_i(x,u) = \text{independent of } u.$$

In other words, condition (i) says that the distribution $D_i(u)$ has constant dimension $n - i$. Condition (ii) says that, for each x, the subspace $\mathscr{K}_i(x,u)$ is always the same, regardless of what $u \in \mathbb{R}^m$ is.[3]

Proposition 7.1 *System (7.1) is globally diffeomorphic to a system in Gauthier–Kupca's observability canonical form only if its canonical flag is uniform.*

Proof (Sketch of) Suppose a system is already in observability canonical form and compute its canonical flag. A simple calculation shows that the each of the functions $\varphi_i(x,u)$ is a function of the form

$$\tilde{\varphi}_i(z_1, \ldots, z_i, u),$$

and

$$\frac{\partial \tilde{\varphi}_i}{\partial z_i} \neq 0, \qquad \text{for all } z_1, \ldots, z_i, u.$$

---

[2]See [1].

[3] Condition (i) is a "regularity" condition. Condition expresses the independence of $\mathscr{K}_i(x,u)$ on the parameter u.

Thus, for each $i = 1, \dots, n$

$$\mathscr{K}_i(z, u) = \mathrm{Im}\begin{pmatrix} 0 \\ I_{n-i} \end{pmatrix}.$$

This shows that the canonical flag of a system in observability canonical form is uniform. This property is not altered by a diffeomorphism and hence the condition in question is a necessary condition for an observability canonical form to exist. ◁

Remark 7.2 The necessary condition thus identified is also sufficient for the existence of a *local* diffeomorphism carrying system (7.1) into a system in observability canonical form.[4] ◁

We describe now a set of *sufficient* conditions for a system to be globally diffeomorphic to a system in Gauthier–Kupca's uniform observability canonical form.

Proposition 7.2 *Consider the nonlinear system (7.1) and define a map*

$$\begin{aligned} \Phi : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto z = \Phi(x) \end{aligned}$$

as

$$\Phi(x) = \begin{pmatrix} \varphi_1(x, 0) \\ \varphi_2(x, 0) \\ \cdots \\ \varphi_n(x, 0) \end{pmatrix}.$$

Suppose that:

(i) the canonical flag of (7.1) is uniform,
(ii) $\Phi(x)$ is a global diffeomorphism.

Then, system (7.1) is globally diffeomorphic, via $z = \Phi(x)$, to a system in Gauthier–Kupca's uniform observability canonical form.

Proof By assumption,

$$\mathrm{Ker}\Big[\frac{\partial \Phi_i}{\partial x}\Big]_{(x,u)}$$

has constant dimension $n - i$ and it is independent of u. Now denote by $T(z)$ the inverse of the diffeomorphism $\Phi(x)$. Since $\Phi(x) = \Phi_n(x, 0)$, $T(z)$ is a globally defined mapping which satisfies

$$\Phi_n(T(z), 0) = z.$$

[4] See [1], Chap. 3, Theorem 2.1.

Then

$$\left[\frac{\partial \Phi_n}{\partial x}\right]_{\substack{x=T(z)\\u=0}} \frac{\partial T}{\partial z} = I$$

for all $z \in \mathbb{R}^n$. This implies, for all $j > i$,

$$\left[\frac{\partial \Phi_i}{\partial x}\right]_{\substack{x=T(z)\\u=0}} \frac{\partial T}{\partial z_j} = 0,$$

or, what is the same,

$$\frac{\partial T}{\partial z_j} \in \mathrm{Ker}\left[\frac{\partial \Phi_i}{\partial x}\right]_{\substack{x=T(z)\\u=0}}, \qquad \forall z \in \mathbb{R}^n.$$

But this, because the $\mathscr{K}_i(x, u)$'s are independent of u, implies

$$\frac{\partial T}{\partial z_j} \in \mathrm{Ker}\left[\frac{\partial \Phi_i}{\partial x}\right]_{\substack{x=T(z)\\u=u}}, \qquad \forall j > i, \forall z \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{7.9}$$

Suppose the map $z = \Phi(x)$ is used to change coordinates in (7.1) and consider the system in the new coordinates

$$\begin{aligned} \dot{z} &= \tilde{f}(z, u) \\ y &= \tilde{h}(z, u), \end{aligned}$$

where

$$\tilde{h}(z, u) = h(T(z), u), \qquad \tilde{f}(z, u) = \left[\frac{\partial \Phi_n}{\partial x}\right]_{\substack{x=T(z)\\u=0}} f(T(z), u).$$

Define

$$\tilde{\varphi}_1(z, u) := \tilde{h}(z, u), \qquad \tilde{\varphi}_i(z, u) := \frac{\partial \tilde{\varphi}_{i-1}}{\partial z} \tilde{f}(z, u) \qquad \text{for } 2 = 1, \ldots, n.$$

and note that

$$\tilde{\varphi}_1(z, u) = \varphi_1(T(z), u), \qquad \tilde{\varphi}_i(z, u) = \varphi_i(T(z), u),$$

which implies

$$\tilde{\Phi}_i(z, u) := \begin{pmatrix} \tilde{\varphi}_1(z, u) \\ \vdots \\ \tilde{\varphi}_i(z, u) \end{pmatrix} = \Phi_i(T(z), u).$$

Using (7.9) for $i = 1$, we obtain

$$\frac{\partial \tilde{\varphi}_1(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_1}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 1$ which means that $\tilde{h}(z,u) = \tilde{\varphi}_1(z,u)$ only depends on z_1. Note also that

$$\frac{\partial \tilde{\varphi}_1}{\partial z} = \begin{pmatrix} \frac{\partial \tilde{\varphi}_1}{\partial z_1} & 0 \cdots 0 \end{pmatrix}$$

and hence, by the uniformity assumption,

$$\frac{\partial \tilde{h}}{\partial z_1} = \frac{\partial \tilde{\varphi}_1}{\partial z} \neq 0.$$

This being the case,

$$\tilde{\varphi}_2(z,u) = \frac{\partial \tilde{h}}{\partial z_1} \tilde{f}_1(z,u).$$

Use now (7.9) for $i = 2$, to obtain (in particular)

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z_j} = \left[\frac{\partial \varphi_2}{\partial x}\right]_{\substack{x=T(z)\\u=u}} \frac{\partial T}{\partial z_j} = 0$$

for all $j > 2$ which means that $\tilde{\varphi}_2(z,u)$ only depends on z_1, z_2. Looking at the form of $\tilde{\varphi}_2(z,u)$, we deduce that $\tilde{f}_1(z,u)$ only depends on z_1, z_2. Moreover, an easy check shows that

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z} = \begin{pmatrix} * & \frac{\partial \tilde{h}}{\partial z_1} \frac{\partial \tilde{f}_1}{\partial z_2} & 0 \cdots 0 \end{pmatrix}$$

and hence

$$\frac{\partial \tilde{f}_1}{\partial z_2} \neq 0$$

because otherwise the uniformity assumption would be contradicted. Continuing in the same way, the result follows. ◁

7.2 The Case of Input-Affine Systems

In this section we specialize the results discussed in the previous section to the case in which the model of the system is *input-affine*, i.e., the system is modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{7.10}$$

It is easy to check that, in a system of the form (7.10), the functions $\varphi_i(x, u)$ defined by (7.8) have the following expressions:

$$\begin{aligned} \varphi_1(x, u) &= h(x) \\ \varphi_2(x, u) &= L_f h(x) + L_g h(x) u \\ \varphi_3(x, u) &= L_f^2 h(x) + [L_g L_f h(x) + L_f L_g h(x)]u + L_g^2 h(x) u^2 \\ \varphi_4(x, u) &= L_f^3 h(x) + [L_g L_f^2 h(x) + L_f L_g L_f h(x) + L_f^2 L_g h(x)]u + \\ &\quad + [L_g^2 L_f h(x) + L_g L_f L_g h(x) + L_f L_g^2 h(x)]u^2 + L_g^3 h(x) u^3 \\ \varphi_5(x, u) &= \cdots \end{aligned}$$

Hence

$$\Phi_n(x, 0) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{pmatrix} := \Phi(x).$$

It is known from Proposition 7.2 that if the canonical flag of (7.10) is uniform and if $\Phi_n(x, 0)$ is a global diffeomorphism, the system is transformable into uniform observability canonical form. The form in question is

$$\begin{aligned} \dot{z} &= \tilde{f}(z) + \tilde{g}(z)u \\ y &= \tilde{h}(z) \end{aligned}$$

in which

$$\tilde{f}(z) = \left[\frac{\partial \Phi(x)}{\partial x} f(x)\right]_{x=\Phi^{-1}(z)}, \quad \tilde{g}(z) = \left[\frac{\partial \Phi(x)}{\partial x} g(x)\right]_{x=\Phi^{-1}(z)},$$
$$\tilde{h}(z) = h(\Phi^{-1}(z)).$$

By construction,

$$\tilde{h}(z) = z_1.$$

Moreover, bearing in mind the special structure of $\Phi(x)$, it is easy to check that $\tilde{f}(z)$ has the following form:

$$\tilde{f}(z) = \begin{pmatrix} z_2 \\ z_3 \\ \cdots \\ z_n \\ \tilde{f}_n(z_1, \ldots, z_n) \end{pmatrix}.$$

Finally, since we know that the ith entry of $\tilde{f}(z) + \tilde{g}(z)u$ can only depend on $z_1, z_2, \dots, z_{i+1}$, we deduce that $\tilde{g}(z)$ must necessarily be of the form

$$\tilde{g}(z) = \begin{pmatrix} \tilde{g}_1(z_1, z_2) \\ \tilde{g}_2(z_1, z_2, z_3) \\ \cdots \\ \tilde{g}_{n-1}(z_1, z_2, \dots, z_n) \\ \tilde{g}_n(z_1, z_2, \dots, z_n) \end{pmatrix}.$$

However, it is also possible to show that g_i actually is independent of z_{i+1}. We check this for $i = 1$ (the other checks are similar). Observe that

$$\begin{aligned} \tilde{\varphi}_1(z, u) &= z_1 \\ \tilde{\varphi}_2(z, u) &= z_2 + \tilde{g}_1(z_1, z_2)u \end{aligned}$$

Computing Jacobians we obtain

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z, u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z, u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ \dfrac{\partial \tilde{g}_1}{\partial z_1} u & (1 + \dfrac{\partial \tilde{g}_1}{\partial z_2} u) & 0 \cdots 0 \end{pmatrix}.$$

If, at some point (z_1, z_2),

$$\frac{\partial \tilde{g}_1}{\partial z_2} \neq 0$$

it is possible to find u such that

$$1 + \frac{\partial \tilde{g}_1}{\partial z_2} u = 0$$

in which case

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}_1(z, u)}{\partial z} \\ \dfrac{\partial \tilde{\varphi}_2(z, u)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 \; 0 \; 0 \cdots 0 \\ * \; 0 \; 0 \cdots 0 \end{pmatrix}.$$

At this value of (z, u) the subspace $\mathscr{K}_2(z, u)$ has dimension $n - 1$ and not $n - 2$ as prescribed. Hence the uniformity conditions are violated. We conclude that

$$\frac{\partial \tilde{g}_1}{\partial z_2} = 0$$

for all (z_1, z_2), i.e., that $\tilde{g}_1$ is independent of z_2.

Proceeding in this way, it can be concluded that the uniform observability canonical form of an input-affine system has the following structure:

$$\begin{aligned}
\dot{z}_1 &= z_2 + \tilde{g}_1(z_1)u \\
\dot{z}_2 &= z_3 + \tilde{g}_2(z_1, z_2)u \\
&\cdots \\
\dot{z}_{n-1} &= z_n + \tilde{g}_{n-1}(z_1, z_2, \ldots, z_{n-1})u \\
\dot{z}_n &= \tilde{f}_n(z_1, z_2, \ldots, z_n) + \tilde{g}_n(z_1, z_2, \ldots, z_n)u \\
y &= z_1.
\end{aligned}$$

7.3 High-Gain Nonlinear Observers

In this section, we describe how to design a *global* asymptotic state observer for a system in Gauthier–Kupca's observability canonical form. Letting $\underline{z}_i$ denote the vector

$$\underline{z}_i = \text{col}(z_1, \ldots, z_i)$$

the canonical form in question can be rewritten in more concise form as[5]

$$\begin{aligned}
\dot{z}_1 &= f_1(\underline{z}_1, z_2, u) \\
\dot{z}_2 &= f_2(\underline{z}_2, z_3, u) \\
&\cdots \\
\dot{z}_{n-1} &= f_{n-1}(\underline{z}_{n-1}, z_n, u) \\
\dot{z}_n &= f_n(\underline{z}_n, u) \\
y &= h(z_1, u).
\end{aligned} \tag{7.11}$$

The construction described below reposes on the following two additional technical assumptions:

(i) each of the maps $f_i(\underline{z}_i, z_{i+1}, u)$, for $i = 1, \ldots, n-1$, is globally Lipschitz with respect to $\underline{z}_i$, uniformly in z_{i+1} and u, and the map $f_n(\underline{z}_n, u)$ is globally Lipschitz with respect to $\underline{z}_n$, uniformly in u.

(ii) there exist two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \leq \left|\frac{\partial h}{\partial z_1}\right| \leq \beta, \quad \text{and} \quad \alpha \leq \left|\frac{\partial f_i}{\partial z_{i+1}}\right| \leq \beta, \quad \text{for all} \quad i = 1, \ldots, n-1$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$.

Remark 7.3 Note that the properties (i) and (ii) can be assumed without loss of generality if it is known—a priori—that $z(t)$ remains in a compact set $\mathscr{C}$. ◁

[5] For convenience, we drop the "tilde" above $h(\cdot)$ and the $f_i(\cdot)$'s.

The observer for (7.11) consists of a *copy* of the dynamics of (7.11) corrected by an *innovation* term proportional to the difference between the output of (7.11) and the output of the copy. More precisely, the observer in question is a system of the form

$$\begin{aligned}
\dot{\hat{z}}_1 &= f_1(\underline{\hat{z}}_1, \hat{z}_2, u) + \kappa c_{n-1}(y - h(\hat{z}_1, u)) \\
\dot{\hat{z}}_2 &= f_2(\underline{\hat{z}}_2, \hat{z}_3, u) + \kappa^2 c_{n-2}(y - h(\hat{z}_1, u)) \\
&\cdots \\
\dot{\hat{z}}_{n-1} &= f_{n-1}(\underline{\hat{z}}_{n-1}, \hat{z}_n, u) + \kappa^{n-1} c_1(y - h(\hat{z}_1, u)) \\
\dot{\hat{z}}_n &= f_n(\underline{\hat{z}}_n, u) + \kappa^n c_0(y - h(\hat{z}_1, u)),
\end{aligned} \tag{7.12}$$

in which κ and $c_{n-1}, c_{n-2}, \dots, c_0$ are design parameters.

Letting the "observation error" be defined as

$$\eta_i = z_i - \hat{z}_i, \qquad i = 1, 2, \dots, n,$$

an estimate of this error can be obtained as follows. Observe that, using the mean value theorem, one can write

$$\begin{aligned}
f_i(\underline{z}_i, z_{i+1}, u) - f_i(\underline{\hat{z}}_i, \hat{z}_{i+1}, u) &= f_i(\underline{z}_i(t), z_{i+1}(t), u(t)) - f_i(\underline{z}_i(t), \hat{z}_{i+1}(t), u(t)) \\
&\quad + f_i(\underline{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\underline{\hat{z}}_i(t), \hat{z}_{i+1}(t), u(t)) \\
&= \frac{\partial f_i}{\partial z_{i+1}}(\underline{z}_i(t), \delta_i(t), u(t))\, \eta_{i+1}(t) \\
&\quad + f_i(\underline{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\underline{\hat{z}}_i(t), \hat{z}_{i+1}(t), u(t))
\end{aligned}$$

in which $\delta_i(t)$ is a number in the interval $[\hat{z}_{i+1}(t), z_{i+1}(t)]$. Note also that

$$y - h(\hat{z}_1, u) = h(z_1(t), u(t)) - h(\hat{z}_1(t), u(t)) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))\, \eta_1$$

in which $\delta_0(t)$ is a number in the interval $[\hat{z}_1(t), z_1(t)]$. Setting

$$\begin{aligned}
g_1(t) &= \frac{\partial h}{\partial z_1}(\delta_0(t), u(t)) \\
g_{i+1}(t) &= \frac{\partial f_i}{\partial z_{i+1}}(\underline{z}_i(t), \delta_i(t), u(t)) \qquad \text{for } i = 1, \dots, n-1,
\end{aligned} \tag{7.13}$$

the relation above yields

$$\begin{aligned}
\dot{\eta}_i &= f_i(\underline{z}_i(t), z_{i+1}(t), u(t)) - f_i(\underline{\hat{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - \kappa^i c_{n-i}(y - h(\hat{z}_1(t), u(t))) \\
&= g_{i+1}(t)\eta_{i+1} + f_i(\underline{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\underline{\hat{z}}_i(t), \hat{z}_{i+1}(t), u(t)) - \kappa^i c_{n-i} g_1(t)\eta_1.
\end{aligned}$$

The equations thus found can be organized in matrix form as

$$\begin{pmatrix} \dot\eta_1 \\ \dot\eta_2 \\ \cdots \\ \dot\eta_{n-1} \\ \dot\eta_n \end{pmatrix} = \begin{pmatrix} -\kappa c_{n-1} g_1(t) & g_2(t) & 0 & \cdots 0 & 0 \\ -\kappa^2 c_{n-2} g_1(t) & 0 & g_3(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ -\kappa^{n-1} c_1 g_1(t) & 0 & 0 & \cdots 0 & g_n(t) \\ -\kappa^n c_0 g_1(t) & 0 & 0 & \cdots 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_{n-1} \\ \eta_n \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ \cdots \\ F_{n-1} \\ F_n \end{pmatrix}$$

in which

$$F_i = f_i(\underline{z}_i(t), \hat z_{i+1}(t), u(t)) - f_i(\underline{\hat z}_i(t), \hat z_{i+1}(t), u(t))$$

for $i = 1, 2, \dots, n-1$ and

$$F_n = f_n(\underline{z}_n(t), u(t)) - f_n(\underline{\hat z}_n(t), u(t)).$$

Consider now a "rescaled" observation error defined as

$$e_i = \kappa^{n-i}\,\eta_i = \kappa^{n-i}(z_i - \hat z_i), \qquad i = 1, 2, \dots, n. \tag{7.14}$$

Simple calculations show that

$$\begin{pmatrix} \dot e_1 \\ \dot e_2 \\ \cdots \\ \dot e_{n-1} \\ \dot e_n \end{pmatrix} = \kappa \begin{pmatrix} -c_{n-1} g_1(t) & g_2(t) & 0 & \cdots 0 & 0 \\ -c_{n-2} g_1(t) & 0 & g_3(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{n-1} \\ e_n \end{pmatrix} + \begin{pmatrix} \kappa^{n-1} F_1 \\ \kappa^{n-2} F_2 \\ \cdots \\ \kappa F_{n-1} \\ F_n \end{pmatrix}. \tag{7.15}$$

The right-hand side of this equation consists of a term which is linear in the vector

$$e = \mathrm{col}(e_1, e_2, \dots, e_n)$$

and of a nonlinear term. The nonlinear term, though, can be bounded by a quantity which is linear in $\|e\|$. In fact, observe that, because of Assumption (i), there is a number L such that

$$|F_i| \le L\|\underline{\hat z}_i - \underline{z}_i\| \qquad \text{for } i = 1, 2, \dots, n.$$

Clearly,

$$\|\underline{\hat z}_i - \underline{z}_i\| = \sqrt{\eta_1^2 + \eta_2^2 + \cdots + \eta_i^2} = \sqrt{\frac{e_1^2}{(\kappa^{n-1})^2} + \frac{e_2^2}{(\kappa^{n-2})^2} + \cdots + \frac{e_i^2}{(\kappa^{n-i})^2}}$$

from which we see that, if $\kappa > 1$,

$$|\kappa^{n-i} F_i| \le L\sqrt{\frac{e_1^2}{(\kappa^{i-1})^2} + \frac{e_2^2}{(\kappa^{i-2})^2} + \cdots + e_i^2} \le L\,\|e\|. \tag{7.16}$$

As far as the properties of the linear part are concerned, the following useful lemma can be invoked.

Lemma 7.1 *Consider a matrix of the form*

$$A(t) = \begin{pmatrix} -c_{n-1}g_1(t) & g_2(t) & 0 & \cdots 0 & 0 \\ -c_{n-2}g_1(t) & 0 & g_3(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots 0 & 0 \end{pmatrix}$$

and suppose there exists two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \le g_i(t) \le \beta \qquad \textit{for all } t \ge 0 \textit{ and } i = 1, 2, \ldots, n. \tag{7.17}$$

Then, there is a set of real numbers $c_0, c_1, \ldots, c_{n-1}$, a real number $\lambda > 0$ and a symmetric positive-definite $n \times n$ matrix S, all depending only on α and β, such that

$$SA(t) + A^{\mathrm{T}}(t)S \le -\lambda I. \tag{7.18}$$

Remark 7.4 Note that, if the $g_i(t)$'s were constant, the matrix in question could be seen as a matrix of the form

$$A - GC = \begin{pmatrix} 0 & g_2 & 0 & \cdots 0 & 0 \\ 0 & 0 & g_3 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ 0 & 0 & 0 & \cdots 0 & g_n \\ 0 & 0 & 0 & \cdots 0 & 0 \end{pmatrix} - \begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \cdots \\ c_1 \\ c_0 \end{pmatrix} \begin{pmatrix} g_1 & 0 \cdots 0 & 0 \end{pmatrix}.$$

If all g_i's are nonzero, the pair A, C is observable and hence there exists a vector G, namely a set of coefficients $c_0, c_1, \ldots, c_{n-1}$, such that the matrix $A - GC$ is Hurwitz and an inequality of the form (7.18) holds. Lemma 7.1 considers the more general case in which the $g_i(t)$'s are functions of time, but bounded as in (7.17). It is stressed that, even if in this case the matrix $A(t)$ is time dependent, the matrix S is not, and only depends on the numbers α and β. The proof of the lemma, which also describes how the coefficients c_i's are to be chosen, is given in Sect. 7.4. ◁

Note that, in view of (7.13), the hypothesis that the $g_i(t)$'s satisfy (7.17) is in the present case fulfilled as a straightforward consequence of Assumption (ii). With this result in mind consider, for system (7.15), the candidate Lyapunov function

$$V(e) = e^{\mathrm{T}} S e.$$

Then, we have

$$\dot{V}(e(t)) = \kappa e^{\mathrm{T}}(t)[A(t)S + SA^{\mathrm{T}}(t)]e(t) + 2e^{\mathrm{T}}(t)S\tilde{F}(t)$$

having denoted by $\tilde{F}(t)$ the vector

$$\tilde{F}(t) = \mathrm{col}(\kappa^{n-1}F_1, \kappa^{n-2}F_2, \dots, F_n).$$

Using the estimates (7.16) and (7.18), and setting

$$a_0 = 2\|S\|\, L\, \sqrt{n}$$

it is seen that

$$\dot{V}(e(t)) \le -(\kappa\lambda - a_0)\|e\|^2.$$

Set $\kappa^* = (a_0/\lambda)$. If $\kappa > \kappa^*$, the estimate

$$\dot{V}(e(t)) \le -aV(e(t))$$

holds, for some $a > 0$. As a consequence, by standard results, it follows that

$$\lim_{t\to\infty} e(t) = 0.$$

We summarize the entire discussion in the following statement.

Theorem 7.1 *Consider a nonlinear in uniform observability canonical form (7.11) and an observer defined as in (7.12). Suppose assumptions (i) and (ii) hold. Then, there is a choice of the coefficients $c_0, c_1, \dots, c_{n-1}$ and a value κ^* such that, if $\kappa > \kappa^*$,*

$$\lim_{t\to\infty} (z_i(t) - \hat{z}_i(t)) = 0 \qquad \text{for all } i = 1, 2, \dots, n$$

for any pair of initial conditions $(z(0), \hat{z}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$.

In summary, the observer (7.12) asymptotically tracks the state of system (7.11) if the coefficients $c_0, c_1, \dots, c_{n-1}$ are such that the property indicated in Lemma 7.1 holds (which is always possible by virtue of Assumption (ii)) and if the number κ is sufficiently large. This is why the observer in question is called a "high-gain" observer.

7.4 The Gains of the Nonlinear Observer

In this section, we give a proof Lemma 7.1.[6] The result is obviously true in case $n = 1$. In this case, in fact, the matrix $A(t)$ reduces to the scalar quantity

$$A(t) = -c_0 g_1(t).$$

[6] See also [1] and reference to Dayawansa therein.

Taking $S = 1$ we need to prove that

$$-2c_0 g_1(t) \leq -\lambda$$

for some $\lambda > 0$, which is indeed possible if the sign[7] of c_0 is the same as that of $g_1(t)$.

The proof for $n \geq 2$ is by induction on n, based on the following result.

Lemma 7.2 *Let α and β be fixed real numbers satisfying $0 < \alpha < \beta$. Suppose there exists a choice of numbers $a_0, a_1, \ldots, a_{i-1}$, a positive-definite matrix S_i and a number $\lambda_i > 0$, all dependent only on α and β, such that, for any set of continuous functions $\{g_1(t), g_2(t), \ldots, g_i(t)\}$ bounded as*

$$\alpha \leq |g_j(t)| \leq \beta \quad \text{for all } t \geq 0 \quad \text{and all } 1 \leq j \leq i,$$

the matrix

$$A_i(t) = \begin{pmatrix} -a_{i-1}g_1(t) & g_2(t) & 0 & \cdots 0 & 0 \\ -a_{i-2}g_1(t) & 0 & g_3(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ -a_1 g_1(t) & 0 & 0 & \cdots 0 & g_i(t) \\ -a_0 g_1(t) & 0 & 0 & \cdots 0 & 0 \end{pmatrix}$$

satisfies

$$S_i A_i(t) + A_i^{\mathrm{T}}(t) S_i \leq -\lambda_i I.$$

Then, there exists numbers $b_0, b_1, \ldots, b_{i-1}, b_i$, a positive-definite matrix S_{i+1} and a number $\lambda_{i+1} > 0$, all dependent only on α and β, such that, for any set of continuous functions $\{g_1(t), g_2(t), \ldots, g_i(t), g_{i+1}(t)\}$ bounded as

$$\alpha \leq |g_j(t)| \leq \beta \quad \text{for all } t \geq 0 \quad \text{and all } 1 \leq j \leq i+1, \tag{7.19}$$

the matrix

$$A_{i+1}(t) = \begin{pmatrix} -b_i g_1(t) & g_2(t) & 0 & \cdots 0 & 0 \\ -b_{i-1}g_1(t) & 0 & g_3(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ -b_1 g_1(t) & 0 & 0 & \cdots 0 & g_{i+1}(t) \\ -b_0 g_1(t) & 0 & 0 & \cdots 0 & 0 \end{pmatrix}$$

satisfies

$$S_{i+1} A_{i+1}(t) + A_{i+1}^{\mathrm{T}}(t) S_{i+1} \leq -\lambda_{i+1} I. \tag{7.20}$$

Proof Let $\{g_1(t), g_2(t), \ldots, g_i(t), g_{i+1}(t)\}$ be any set of continuous functions bounded as in (7.19). Set

[7] The $g_i(t)$'s are continuous functions that never vanish. Thus, each of them has a well-defined sign.

$$A(t) = \begin{pmatrix} 0 & g_3(t) & 0 & \cdots 0 & 0 \\ 0 & 0 & g_4(t) & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ 0 & 0 & 0 & \cdots 0 & g_{i+1}(t) \\ 0 & 0 & 0 & \cdots 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b_{i-1} \\ b_{i-2} \\ \cdots \\ b_1 \\ b_0 \end{pmatrix}$$

$$C(t) = \begin{pmatrix} g_2(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad K = -\begin{pmatrix} a_{i-1} \\ a_{i-2} \\ \cdots \\ a_1 \\ a_0 \end{pmatrix}$$

and note that

$$A_{i+1}(t) = \begin{pmatrix} -b_i g_1(t) & C(t) \\ -B g_1(t) & A(t) \end{pmatrix}.$$

Using the nonsingular matrix

$$T = \begin{pmatrix} 1 & 0 \\ K & I_i \end{pmatrix},$$

obtain

$$\tilde{A}_{i+1}(t) = T A_{i+1}(t) T^{-1} = \begin{pmatrix} -b_i g_1(t) - C(t)K & C(t) \\ -(B + K b_i) g_1(t) - L(t)K & L(t) \end{pmatrix}$$

in which

$$L(t) = A(t) + KC(t).$$

The matrix $L(t)$ has a structure identical to that of $A_i(t)$, with $g_j(t)$ replaced by the $g_{j+1}(t)$, for $j = 1, \ldots, i$. Thus, by the hypothesis of the lemma, the matrix $L(t)$ satisfies

$$S_i L(t) + L^{\mathrm{T}}(t) S_i \le -\lambda_i I.$$

Take now the positive-definite quadratic form

$$z^{\mathrm{T}} \tilde{S}_{i+1} z = \begin{pmatrix} z_1^{\mathrm{T}} & z_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S_i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and set

$$Q(z) = z^{\mathrm{T}} (\tilde{S}_{i+1} \tilde{A}_{i+1}(t) + \tilde{A}_{i+1}^{\mathrm{T}}(t) \tilde{S}_{i+1}) z.$$

We prove that there exist a choice of B and a number b_i such that

$$Q(z) \le -\frac{\lambda_i}{2} \|z\|^2. \tag{7.21}$$

Simple calculations show that

$$\begin{aligned} Q(z) = {} & 2(-b_i g_1(t) - C(t)K)z_1^2 + 2C(t)z_1 z_2 \\ & + 2z_2^{\mathrm{T}} S_i[-(B + Kb_i)g_1(t) - L(t)K]z_1 + 2z_2^{\mathrm{T}} S_i L(t)z_2. \end{aligned}$$

Choose

$$B = -Kb_i$$

and observe that

$$Q(z) \le 2(-b_i g_1(t) - C(t)K)|z_1|^2 + 2(\|C(t)\| + \|S_i L(t)K\|)|z_1|\,\|z_2\| - \lambda_i \|z_2\|^2.$$

Clearly, (7.21) holds if

$$\left[-2(b_i g_1(t) + C(t)K) + \frac{1}{2}\right]|z_1|^2 + 2(\|C(t)\| + \|S_i L(t)K\|)|z_1|\,\|z_2\| - \frac{\lambda_i}{2}\|z_2\|^2 < 0$$

which is the case if the matrix

$$M = \begin{pmatrix} -2(b_i g_1(t) + C(t)K) + \frac{1}{2} & (\|C(t)\| + \|S_i L(t)K\|) \\ (\|C(t)\| + \|S_i L(t)K\|) & -\frac{\lambda_i}{2} \end{pmatrix}$$

is negative definite. The matrix M is negative definite if and only if

$$\begin{aligned} & (b_i g_1(t) + C(t)K) - \tfrac{1}{4} > 0 \\ & \left[(b_i g_1(t) + C(t)K) - \tfrac{1}{4}\right]\lambda_i > (\|C(t)\| + \|S_i L(t)K\|)^2. \end{aligned}$$

Since $g_1(t)$ by assumption has a well-defined sign, let b_i have the same sign as that of $g_1(t)$. Then, the previous inequalities hold if

$$|b_i|\,|g_1(t)| > \frac{1}{4} + |C(t)K| + \frac{1}{\lambda_i}(\|C(t)\| + \|S_i\|\,\|L(t)\|\,\|K\|)^2. \tag{7.22}$$

In this expression, $|g_1(t)|$ is bounded from below by α, $\|C(t)\| = |g_2(t)|$ is bounded from above by β, the quantities K, λ_i, and S_i are fixed and depend only on α and β, $L(t)$ is a matrix bounded by $(1 + \|K\|)\beta$. Thus, if $|b_i|$ is sufficiently large, the inequality in question holds, that is (7.21) is true.

This being the case, it is immediate to check that the matrix $S_{i+1} = T^{\mathrm{T}}\tilde{S}_{i+1}T$ satisfies (7.20) with $\lambda_{i+1} = (\lambda_i/2)\lambda_{\min}(T^{\mathrm{T}}T)$. Since λ_i, T and S_{i+1} only depend on α and β, the lemma is proven. ◁

The proof above provides a recursive procedure for the calculations of the coefficients $c_0, c_1, \ldots, c_{n-1}$ for which the result of Lemma 7.1 holds, as illustrated in the following simple example.

Example 7.1 Consider a system of the form (7.2) with $n = 3$ and $m = 1$

$$\begin{aligned}\dot{z}_1 &= f_1(z_1, z_2, u)\\ \dot{z}_2 &= f_2(z_1, z_2, z_3, u)\\ \dot{z}_2 &= f_3(z_1, z_2, z_3, u)\\ y &= h(z_1, u)\end{aligned}$$

and suppose there exist numbers $0 < \alpha < \beta$ such that

$$\alpha < |\frac{\partial h}{\partial z_1}| < \beta, \quad \alpha < |\frac{\partial f_1}{\partial z_2}| < \beta, \quad \alpha < |\frac{\partial f_2}{\partial z_3}| < \beta,$$

for all $z \in \mathbb{R}^3$ and $u \in \mathbb{R}$. Thus, the (unavailable) functions $g_1(t)$, $g_2(t)$, $g_3(t)$ satisfy

$$\alpha < |g_i(t)| < \beta, \qquad i = 1, 2, 3.$$

At the first iteration, consider the matrix

$$A_1(t) = -a_0 g_1(t).$$

Suppose, without loss of generality, that $g_1(t) > 0$. Take $S_1 = 1$, that yields

$$S_1 A_1(t) + A_1^{\mathrm{T}}(t) S_1 \leq -2a_0\alpha.$$

Thus, the assumption of the lemma is fulfilled with $a_0 = 1$, $\lambda_1 = 2\alpha$ and $S_1 = 1$.

The lemma is now used to deduce the desired property for the matrix

$$A_2(t) = \begin{pmatrix} -b_1 g_1(t) & g_2(t) \\ -b_0 g_1(t) & 0 \end{pmatrix}.$$

Following the procedure indicated in the proof, observe that (recall that $a_0 = 1$)

$$B = b_0, \quad K = -1, \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad L(t) = -g_2(t), \quad \tilde{S}_2 = I_2.$$

As indicated, take $B = -K b_1$, which means

$$b_0 = b_1.$$

The magnitude of b_1 should satisfy (7.22). Since $K = -1$, $\lambda_1 = 2\alpha$, $S_1 = 1$, and $L(t) = -g_2(t)$, the inequality in question becomes

$$|b_1 g_1(t)| > \frac{1}{4} + |g_2(t)| + \frac{1}{2\alpha}(2|g_2(t)|)^2,$$

which is fulfilled if

$$b_1 > b_1^* := \frac{1}{4} + \frac{\beta}{\alpha} + 2\,\frac{\beta^2}{\alpha^2}.$$

As shown in the lemma, an inequality of the form (7.20) holds for $i = 2$, with

$$S_2 = T^{\mathrm{T}} \tilde{S}_2 T = T^{\mathrm{T}} T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

and $\lambda_2 = \alpha \lambda_{\min}(T^{\mathrm{T}} T)$. Letting the pair (b_0, b_1) thus found by denoted by (a_0, a_1), we see that (a_0, a_1), together with λ_2 and S_2 are such that the assumption of the lemma holds for $i = 2$. Observe that $a_0 = a_1 = a$, with $a > b_1^*$.

The next (and final) step is to find numbers b_0, b_1, b_2 such that the 3×3 matrix

$$A_3(t) = \begin{pmatrix} -b_2 g_1(t) & g_2(t) & 0 \\ -b_1 g_1(t) & 0 & g_3(t) \\ -b_0 g_1(t) & 0 & 0 \end{pmatrix}$$

has the desired properties. Following the procedure indicated in the proof, observe that

$$B = \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}, \quad K = \begin{pmatrix} -a \\ -a \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -a & 0 & 1 \end{pmatrix},$$

$$L(t) = \begin{pmatrix} 0 & g_3(t) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -a \\ -a \end{pmatrix} \begin{pmatrix} g_2(t) & 0 \end{pmatrix}, \quad \tilde{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

As indicated, take $B = -K b_2$, which means

$$\begin{pmatrix} b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} b_2.$$

The magnitude of b_2 should satisfy the inequality (7.22), which in the present case is fulfilled if (observe that $\|K\| = 2a$, $\|S_2\| < 3$ and $\lambda_2 > 0.25\alpha$)

$$b_2 g_1(t) > \frac{1}{4} + |g_2(t)|\, 2a + \frac{1}{0.25\alpha} \left(|g_2(t)| + 3(|g_3(t)| + 2a\, |g_2(t)|)\right)^2.$$

Therefore, it suffices that

$$b_2 > b_2^* = \frac{1}{4} + \frac{2b_1^* \beta}{\alpha} + \frac{4(4\beta + 6b_1^* \beta)^2}{\alpha^2}.$$

In this way, we have found a triplet of numbers b_0, b_1, b_2, a number λ_3 and a positive-definite matrix S_3 such that (7.20) holds for $i = 3$. Observe that

$$b_0 = ab_2 \qquad b_1 = ab_2 \qquad b_2 > b_2^*$$

while $S_3 = T^{\mathrm{T}}\tilde{S}_3 T$ and $\lambda_3 = (\lambda_2/2)\lambda_{\min}(T^{\mathrm{T}}T)$. Note that all such quantities only depend on the bounds α and β and not on the specific functions $g_1(t)$, $g_2(t)$, $g_3(t)$. $\triangleleft$

7.5 A Nonlinear Separation Principle

In this section, we show how the theory of nonlinear observers presented earlier can be used to the purpose of achieving asymptotic stability via dynamic output feedback.[8] Consider a single-input single-output nonlinear system in observability canonical form (7.2), which we rewrite in compact form as

$$\begin{aligned} \dot{z} &= f(z, u) \\ y &= h(z, u), \end{aligned} \tag{7.23}$$

with $f(0, 0) = 0$ and $h(0, 0) = 0$ and suppose there exists a feedback law $u = u^*(z)$, with $u^*(0) = 0$, such that the equilibrium $z = 0$ of

$$\dot{z} = f(z, u^*(z)) \tag{7.24}$$

is globally asymptotically stable.

Assume, for the time being, that the Assumptions (i) and (ii) of Sect. 7.3 hold (we shall see later how these can be removed) and consider an observer of the form (7.12), which we rewrite in compact form as

$$\dot{\hat{z}} = f(\hat{z}, u) + D_\kappa G_0(y - h(\hat{z}, u)), \tag{7.25}$$

in which D_κ is the matrix (compare with Sect. 2.5)

$$D_\kappa = \operatorname{diag}(\kappa, \kappa^2, \ldots, \kappa^n) \tag{7.26}$$

and G_0 is the vector

$$G_0 = \operatorname{col}(c_{n-1}, \ldots, c_1, c_0).$$

An obvious choice to achieve asymptotic stability, suggested by the analogy with linear systems, would be to replace z by its estimate $\hat{z}$ in the map $u^*(z)$. However, this simple choice may prove to be dangerous, for the following reason. Bearing in mind the analysis carried out in Sect. 7.3, recall that

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[8] In this section, we continue to essentially follow the approach of [1]. A slightly alternative approach and additional relevant results can be found in [2, 3].

$$e = \begin{pmatrix} \kappa^{n-1} & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & \kappa & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} (z - \hat{z}) = \kappa^n D_\kappa^{-1}(z - \hat{z}).$$

We have seen that, to secure asymptotic convergence of the observation error $e(t)$ to zero is necessary to increase κ. This, even if the initial conditions $z(0)$ and $\hat{z}(0)$ of the plant and of the observer are taken in a compact set, may entail large values of $e(t)$ for (positive) times close to $t = 0$. In fact,

$$e(0) = \kappa^n D_\kappa^{-1}(z(0) - \hat{z}(0))$$

and $\|e(0)\|$ grows unbounded with increasing κ. Since $e(t)$ is a continuous function of t, we should expect that, if κ is large, there is an initial interval of time on which $\|e(t)\|$ is large.[9]

Now, note that feeding the system (7.23) with a control $u = u^*(\hat{z})$ would result in a system

$$\dot{z} = f(z, u^*(z - \kappa^{-n} D_\kappa e)).$$

This is viewed as a system with state z subject to an input $\kappa^{-n} D_\kappa e(t)$. Now, if κ is large, the matrix $\kappa^{-n} D_\kappa$ remains bounded, because all elements of this (diagonal) matrix are non positive powers of κ. In fact $\|\kappa^{-n} D_\kappa\| = 1$ if $\kappa \geq 1$, as an easy calculation shows. However, has remarked above, $\|e(t)\|$ may become large for small values of t, if κ is large. Since the system is nonlinear, this may result in a finite escape time.[10] To avoid such inconvenience, as a precautionary measure, it is appropriate to "saturate" the control, by choosing instead a law of the form

$$u = g_\ell(u^*(\hat{z}))) \tag{7.27}$$

in which $g_\ell : \mathbb{R} \to \mathbb{R}$ is a smooth *saturation* function, that is a function characterized by the following properties:

(i) $g_\ell(s) = s$ if $|s| \leq \ell$,
(ii) $g_\ell(s)$ is odd and monotonically increasing, with $0 < g'_\ell(s) \leq 1$,
(iii) $\lim_{s\to\infty} g_\ell(s) = \ell(1 + c)$ with $0 < c \ll 1$.

The real number $\ell > 0$ is usually referred to as the "saturation level."

The consequence of choosing the control u as in (7.27) is that *global* asymptotic stability is no longer assured. However, as it will be shown, *semiglobal stabilizability* is still possible.[11] In what follows, it will be shown that, for every compact set $\mathscr{C}$ of initial conditions in the state space, there is a choice of design parameters such that the

[9] This phenomenon is sometimes referred to as "peaking".

[10] See Example B.3 in Appendix B in this respect.

[11] The idea of saturating the control, outside a fixed region on which the trajectories are expected to range, so as to prevent finite escape times, has been originally suggested by Khalil in [4, 5], see also [6].

equilibrium $(z, \hat{z}) = (0, 0)$ of the closed-loop system is asymptotically stable, with a domain of attraction that contains $\mathscr{C}$.

Consider now the aggregate of (7.23), (7.25), and (7.27), namely

$$\begin{aligned}\dot{z} &= f(z, g_\ell(u^*(\hat{z})))\\ \dot{\hat{z}} &= f(\hat{z}, g_\ell(u^*(\hat{z}))) + D_\kappa G_0(h(z, g_\ell(u^*(\hat{z}))) - h(\hat{z}, g_\ell(u^*(\hat{z})))).\end{aligned} \tag{7.28}$$

Replacing $\hat{z}$ by its expression in terms of z and e, we obtain for the first equation a system that can be written as

$$\begin{aligned}\dot{z} &= f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e)))\\ &= f(z, u^*(z)) - f(z, u^*(z)) + f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e)))\\ &= F(z) + \Delta(z, e)\end{aligned}$$

in which

$$\begin{aligned}F(z) &= f(z, u^*(z))\\ \Delta(z, e) &= f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e))) - f(z, u^*(z)).\end{aligned}$$

Note that, by the inverse Lyapunov Theorem, since the equilibrium $z = 0$ of (7.24) is globally asymptotically stable, there exists a smooth function $V(z)$, satisfying

$$\underline{\alpha}(\|z\|) \le V(z) \le \overline{\alpha}(\|z\|) \qquad \text{for all } z,$$

and

$$\frac{\partial V}{\partial z} F(z) \le -\alpha(\|z\|) \qquad \text{for all } z,$$

for some class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$.

The analysis proceeds as follows. For simplicity let the compact set $\mathscr{C}$ in which initial conditions are taken be a set of the form $\mathscr{C} = \overline{B}_R \times \overline{B}_R$ in which $\overline{B}_R$ denotes the closure of B_R, the ball of radius R in $\mathbb{R}^n$. Choose a number c such that

$$\Omega_c = \{z \in \mathbb{R}^n : V(z) \le c\} \supset \overline{B}_R,$$

and then choose the parameter ℓ in the definition of $g_\ell(\cdot)$ as

$$\ell = \max_{z \in \Omega_{c+1}} u^*(z) + 1.$$

Since e enters in $\Delta(z, e)$ through the bounded function $g_\ell(\cdot)$, it is easy to realize that there is a number δ_1 such that

$$\|\Delta(z, e)\| \le \delta_1, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^n$$

Note also that, if $z \in \Omega_{c+1}$ and $\|\kappa^{-n} D_\kappa e\|$ are small,

$$g_\ell(u^*(z - \kappa^{-n} D_\kappa e)) = u^*(z - \kappa^{-n} D_\kappa e) \tag{7.29}$$

and hence $\Delta(z, 0) = 0$. Assuming, without loss of generality, $\kappa \geq 1$, it is seen that $\|\kappa^{-n} D_\kappa\| = 1$ and hence $\|e\|$ small implies $\|\kappa^{-n} D_\kappa e\|$ small. Therefore, there are numbers δ_2, ε such that

$$\|\Delta(z, e)\| \leq \delta_2 \|e\|, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } \|e\| \leq \varepsilon.$$

These numbers $\delta_1, \delta_2, \varepsilon$ are independent of κ (so long as $\kappa \geq 1$) and only depend on the number R that characterizes the radius of the ball $\overline{B}_R$ in which $z(0)$ is taken.

Let $z(0) \in \overline{B}_R \subset \Omega_c$. Regardless of what $e(t)$ is, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{V}(z(t)) = \frac{\partial V}{\partial z} [F(z) + \Delta(z, e)] \leq -\alpha(\|z\|) + \|\frac{\partial V}{\partial z}\| \delta_1.$$

Setting

$$M = \max_{z \in \Omega_{c+1}} \|\frac{\partial V}{\partial z}\|$$

we observe that the previous estimate yields, in particular

$$\dot{V}(z(t)) \leq M\delta_1$$

which in turn yields

$$V(z(t)) \leq V(z(0)) + M\delta_1 t \leq c + M\delta_1 t.$$

From this it is deduced that $z(t)$ remains in Ω_{c+1} at least until time $T_0 = 1/M\delta_1$. This time may be very small but, because of the presence of the saturation function $g_\ell(\cdot)$, it is independent of κ. It rather only depends on the number R that characterizes the radius of the ball $\overline{B}_R$ in which $z(0)$ is taken.

Recall now that the variable e satisfies the estimate established in Sect. 7.3. Letting $V(e)$ denote the quadratic form $V(e) = e^{\mathrm{T}} S e$, we know that

$$\dot{V}(e(t)) \leq -2\alpha_\kappa V(e(t))$$

in which

$$\alpha_\kappa = \frac{1}{2}\left(\kappa\lambda - 2\|S\| L\sqrt{n}\right) := \frac{1}{2}(\kappa\lambda - a_0)$$

is a number that can be made arbitrarily large by increasing κ (recall that λ and $\|S\|$ only depend of the bounds α and β in Assumption (ii) and L on the Lipschitz constants in Assumption (i)). From this inequality, bearing in mind the fact that

$$a_1 \|e\|^2 \leq V(e) \leq a_2 \|e\|^2$$

in which a_1, a_2 are numbers depending on S and hence only on α, β, we obtain

$$\|e(t)\| \leq A\, e^{-\alpha_\kappa t}\, \|e(0)\|, \qquad \text{with } A = \sqrt{\frac{a_2}{a_1}},$$

which is valid for all t, so long as $z(t)$ exists.

Recall now that $e(0) = \kappa^n D_\kappa^{-1}(z(0) - \hat{z}(0))$. Since $\kappa \geq 1$, it is seen that $\|\kappa^n D_\kappa^{-1}\| = \kappa^{n-1}$. Thus, if initial conditions are such that $(z(0), \hat{z}(0)) \in \overline{B}_R \times \overline{B}_R$, we have

$$\|e(0)\| \leq 2\kappa^{n-1} R.$$

Consequently

$$\|e(t)\| \leq 2AR\, e^{-\alpha_\kappa t} \kappa^{n-1},$$

which is valid so long as $z(t)$ is defined. Note that

$$2AR\, e^{-\alpha_\kappa T_0} \kappa^{n-1} = 2AR\, e^{\frac{a_0 T_0}{2}} e^{-\frac{\lambda T_0}{2}\kappa} \kappa^{n-1}.$$

The function $e^{-\frac{\lambda T_0}{2}\kappa} \kappa^{n-1}$ is a polynomial function of κ multiplied by an exponentially decaying function of κ (recall that $T_0 > 0$). Thus, it tends to 0 as $\kappa \to \infty$. As a consequence, for any ε there is a number κ^* such that, if $\kappa > \kappa^*$,

$$\|e(T_0)\| \leq \varepsilon.$$

This also implies (since $\alpha_\kappa > 0$)

$$\|e(t)\| \leq 2AR\, e^{-\alpha_\kappa (t - T_0)} e^{-\alpha_\kappa T_0} \kappa^{n-1} \leq \varepsilon,$$

for all $t > T_0$, so long as $z(t)$ is defined.

Return now to the inequality

$$\dot{V}(z(t)) = \frac{\partial V}{\partial z}[F(z) + \Delta(z, e)] \leq -\alpha(\|z\|) + \|\frac{\partial V}{\partial z}\| \|\Delta(z, e)\|.$$

Pick $\kappa \geq \kappa^*$, so that $\| e(t)\| \leq \varepsilon$ for all $t \geq T_0$ and hence, so long as $z(t) \in \Omega_{c+1}$,

$$\dot{V}(z(t)) \leq -\alpha(\|z(t)\|) + M\delta_2\varepsilon.$$

Pick any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \leq V(z) \leq c + 1\}.$$

Let r be

$$r = \min_{z \in S_d^{c+1}} \|z\|.$$

By construction

$$\alpha(\|z\|) \geq \alpha(r) \qquad \text{for all } z \in S_d^{c+1}.$$

If ε is small enough

$$M\delta_2\varepsilon \leq \frac{1}{2}\alpha(r),$$

and hence

$$\dot{V}(z(t)) \leq -\frac{1}{2}\alpha(r),$$

so long as $z(t) \in S_d^{c+1}$. By standard arguments,[12] this proves that any trajectory $z(t)$ which starts in $\overline{B}_R$, in a finite time (which only depends on the choice of R and d), enters the set Ω_d and remains in this set thereafter. Observing that for any (small) ε' there is a number d such that $\Omega_d \subset B_{\varepsilon'}$, it can be concluded that, *for any choice of* $\varepsilon' \ll R$ there exist a number κ^* and a time T^* such that, if $\kappa > \kappa^*$, all trajectories with initial conditions $(z(0), \hat{z}(0)) \in \overline{B}_R \times \overline{B}_R$ are bounded and satisfy

$$(z(t), \hat{z}(t)) \in B_{\varepsilon'} \times B_{\varepsilon'} \qquad \text{for all } t \geq T^*.$$

Moreover, $\lim_{t\to\infty} e(t) = 0$. If ε' is small enough, on the set $B_{\varepsilon'} \times B_{\varepsilon'}$, the first equation of (7.28) becomes (see (7.29))

$$\dot{z} = F(z) + f(z, u^*(z - \kappa^{-n} D_\kappa e)) - f(z, u^*(z).$$

Thus, using Lemma B.4 of Appendix B, it is concluded that also $\lim_{t\to\infty} z(t) = 0$. This proves the following result.

Proposition 7.3 *Consider system (7.23), assumed to be expressed in uniform observability canonical form, and suppose Assumptions (i) and (ii) hold. Suppose that a state feedback law $u = u^*(z)$ globally asymptotically stabilizes the equilibrium $z = 0$ of (7.24). Let the system be controlled by (7.27), in which $\hat{z}$ is provided by the observer (7.25). Then, for every choice of R, there exist a number ℓ and a number κ^* such that, if $\kappa > \kappa^*$, all trajectories of the closed-loop system with initial conditions in $\overline{B}_R \times \overline{B}_R$ are bounded and* $\lim_{t\to\infty}(z(t), \hat{z}(t)) = (0, 0)$.

It remains to discuss the role of the Assumptions (i) and (ii). Having proven that the trajectories of the system starting in $B_R \times B_R$ remain in a bounded region, it suffices to look for numbers α and β and a Lipschitz constant L making Assumptions (i) and (ii) valid *only on this bounded region*. To this end, recall that the parameters c and ℓ considered in the proof of the above proposition only depend on the number R and on the Lyapunov function $V(z)$ of (7.24), and not on the observer. Let these parameters be fixed as in the previous proof, define

$$\mathscr{S}_{c,\ell} = \{(z, u) : z \in \Omega_{c+1}, \|u\| \leq 2\ell\},$$

[12] See, in this respect, Sect. B.1 of Appendix B.

and let $f_c(z,u)$ and $h_c(z,u)$ be smooth functions, expressed in observability canonical form, satisfying

$$\begin{aligned} f_c(z,u) &= f(z,u), \qquad \forall (z,u) \in \mathscr{S}_{c,\ell} \\ h_c(z,u) &= h(z,u), \qquad \forall (z,u) \in \mathscr{S}_{c,\ell}. \end{aligned}$$

In other words, system

$$\begin{aligned} \dot{z} &= f_c(z,u) \\ y &= h_c(z,u) \end{aligned} \tag{7.30}$$

agrees with system (7.23) on the set $\mathscr{S}_{c,\ell}$. The assumption that system (7.23) is expressed in uniform observability canonical form implies that properties (7.3) hold for such system. Thus, since $\mathscr{S}_{c,\ell}$ is a compact set, it is always possible to define $f_c(z,u)$ and $h_c(z,u)$, at points $(z,u) \notin \mathscr{S}_{c,\ell}$, in such a way that Assumptions (i) and (ii) are fulfilled.[13] Note also that the system

$$\dot{z} = f_c(z, u^*(z))$$

coincides with system (7.24) for all $z \in \Omega_{c+1}$. Thus, in particular, the Lyapunov function $V(z)$ introduced in the previous discussion satisfies

$$\frac{\partial V}{\partial z} f_c(z, u^*(z)) \le -\alpha(\|z\|) \qquad \forall z \in \Omega_{c+1}.$$

From this, repeating the proof of Proposition 7.3, one reaches the conclusion that if system (7.30) is controlled by

$$\begin{aligned} \dot{\hat{z}} &= f_c(\hat{z}, g_\ell(u^*(\hat{z}))) + D_\kappa G_0(h_c(z, g_\ell(u^*(\hat{z}))) - h_c(\hat{z}, g_\ell(u^*(\hat{z}))) \\ u &= g_\ell(u^*(\hat{z})) \end{aligned} \tag{7.31}$$

and initial conditions are in $\overline{B}_R \times \overline{B}_R$, there is a number κ^* such that if $\kappa > \kappa^*$ all trajectories are bounded and converge to the equilibrium $(z, \hat{z}) = (0, 0)$. This controller generates an input always satisfying $\|u(t)\| \le 2\ell$ and induces in (7.30) a trajectory which always satisfies $z(t) \in \Omega_{c+1}$. Since (7.30) and the original plant agree on the set $\mathscr{S}_{c,\ell}$, the controller constructed in this way achieves the same result if used for the original plant.

7.6 Examples

Example 7.2 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_1 + x_1^3 + x_2)^3 u \\ \dot{x}_2 &= x_1 - 3x_1^2 x_3 + (x_2 + x_1^3)u - 3x_1^2(x_1 + x_1^3 + x_2)^3 u \\ \dot{x}_3 &= (x_2 + x_1^3)^2 \\ y &= x_2 + x_1^3. \end{aligned}$$

[13] Form more details, see [1].

We want to see whether or not it is transformable into a system in uniform observability canonical form. To this end, we compute the functions $\varphi_i(x, u)$ and obtain

$$\begin{aligned}
\varphi_1(x, u) &= x_2 + x_1^3 \\
\varphi_2(x, u) &= x_1 + (x_2 + x_1^3)u \\
\varphi_3(x, u) &= x_3 + (x_1 + x_1^3 + x_2)^3 u + (x_1 + (x_2 + x_1^3)u)u.
\end{aligned}$$

Passing to the Jacobians, we have

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x, u)}{\partial x} \\ \dfrac{\partial \varphi_2(x, u)}{\partial x} \\ \dfrac{\partial \varphi_3(x, u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 3x_1^2 & 1 & 0 \\ 1 + 3x_1^2 u & u & 0 \\ * & * & 1 \end{pmatrix}$$

from which we see that

$$\mathscr{K}_1(x, u) = \operatorname{Im} \begin{pmatrix} 1 & 0 \\ -3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathscr{K}_2(x, u) = \operatorname{Im} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathscr{K}_3(x, u) = \{0\}.$$

Hence, the "uniformity conditions" hold:

- $\mathscr{K}_1(x, u)$ has constant dimension 2 and is independent of u.
- $\mathscr{K}_2(x, u)$ has constant dimension 1 and is independent of u.
- $\mathscr{K}_3(x, u)$ has constant dimension 0 and (trivially) is independent of u.

Moreover

$$z = \Phi_n(x, 0) = \begin{pmatrix} x_2 + x_1^3 \\ x_1 \\ x_3 \end{pmatrix}$$

is a global diffeomorphism. Its inverse is

$$x = \begin{pmatrix} z_2 \\ z_1 - z_2^3 \\ z_3 \end{pmatrix}.$$

In the new coordinates z, the system reads as

$$\begin{aligned}
\dot{z}_1 &= z_2 + z_1 u \\
\dot{z}_2 &= z_3 + (z_2 + z_1)^3 u \\
\dot{z}_3 &= z_1^2 \\
y &= z_1,
\end{aligned}$$

which is a uniform observability canonical form.

Example 7.3 Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1 x_2 u \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= (x_2 + x_1^3)^2 + u \\
y &= x_1
\end{aligned}$$

This system is not transformable into a system in uniform observability canonical form. In fact, we have

$$\begin{aligned}
\varphi_1(x,u) &= x_1 \\
\varphi_2(x,u) &= x_2 + x_1 x_2 u
\end{aligned}$$

and

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2 u & 1 + x_1 u & 0 \end{pmatrix}.$$

Hence

$$\mathscr{K}_1(x,u) = \mathrm{Im} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$\mathscr{K}_2(x,u) = \mathrm{Im} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } 1 + x_1 u \neq 0$$

$$\mathscr{K}_2(x,u) = \mathrm{Im} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } 1 + x_1 u = 0$$

and the uniformity conditions are not fulfilled. The transformation is not possible.

Example 7.4 Consider again the system (6.42) of Example 6.4. This system can be transformed into uniform observability canonical form. Note that

$$\begin{aligned}
\varphi_1(x,u) &= x_2 \\
\varphi_2(x,u) &= x_2 + x_3 \\
\varphi_3(x,u) &= x_1 + x_2 + x_2^2 + x_3 + u
\end{aligned}$$

and therefore

$$\begin{pmatrix} \dfrac{\partial \varphi_1(x,u)}{\partial x} \\ \dfrac{\partial \varphi_2(x,u)}{\partial x} \\ \dfrac{\partial \varphi_3(x,u)}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 + 2x_2 & 1 \end{pmatrix}.$$

The uniformity conditions are fulfilled. Moreover

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Phi_n(x,0) = \begin{pmatrix} x_2 \\ x_2 + x_3 \\ x_1 + x_2 + x_2^2 + x_3 \end{pmatrix}$$

is a global diffeomorphism, with inverse given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_3 - z_1^2 - z_2 \\ z_1 \\ z_2 - z_1 \end{pmatrix} := \Phi^{-1}(z).$$

In the new coordinates, the system is described by the equations

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + u \\ \dot{z}_3 &= f_3(z_1, z_2, z_3) + u \\ y &= z_1 \end{aligned} \tag{7.32}$$

in which

$$f_3(z_1, z_2, z_3) = z_1 - z_2 + 2z_3 - z_1^2 + 2z_1z_2 - (z_3 - z_2 - z_1^2)^3.$$

For such system, one can build the observer

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + \kappa c_2(y - \hat{z}_1) \\ \dot{\hat{z}}_2 &= \hat{z}_3 + \kappa^2 c_1(y - \hat{z}_1) + u \\ \dot{\hat{z}}_3 &= f_3(\hat{z}_1, \hat{z}_2, \hat{z}_3) + \kappa^3 c_0(y - \hat{z}_1) + u. \end{aligned} \tag{7.33}$$

Now, recall that the equilibrium $x = 0$ of system (6.42) can be globally asymptotically stabilized by means of a state feedback law $u = \psi(x)$, whose expression was determined in Example 6.4. Passing to the z coordinates, the equilibrium $z = 0$ of (7.32) is globally asymptotically stabilized by the feedback law $u^*(z) = \psi(\Phi^{-1}(z))$. In view of the results presented in Sect. 7.5, it is concluded that a dynamic output feedback consisting of (7.33) and of

$$u = g_\ell(\psi(\Phi^{-1}(z)))$$

is able to asymptotically stabilize the equilibrium $(z, \hat{z}) = (0, 0)$ of the associated closed-loop system, with a region of attraction that contains an arbitrarily fixed compact set. ◁

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Chapter 8
The Small-Gain Theorem for Nonlinear Systems and Its Applications to Robust Stability

8.1 The Small-Gain Theorem for Input-to-State Stable Systems

In Sect. 3.5, we have determined conditions under which the pure feedback interconnection of two stable linear systems is stable. As a matter of fact, we have shown (see Theorem 3.2 and also Corollary 3.1) that the interconnection of two stable linear systems is stable if the product of the respective $\mathscr{L}_2$ gains is small.[1] In this section, we discuss a similar property for the pure feedback interconnection of two nonlinear systems. Instrumental, in this analysis, is the notion of *input-to-state stability* and the associated notion of *gain function*.[2]

Consider a nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned} \tag{8.1}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and $f_1(0,0) = 0$, $f_2(0,0) = 0$ (see Fig. 8.1). This is seen as interconnection of a system Σ_1 with internal state x_1 and input x_2 and of a system Σ_2 with internal state x_2 and input x_1.

We assume that both Σ_1 and Σ_2 are input-to-state stable. According to Definition B.1, this means that there exists two class $\mathscr{KL}$ functions $\beta_1(\cdot,\cdot)$, $\beta_2(\cdot,\cdot)$ and two class $\mathscr{K}$ functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ such that, for any bounded input $x_2(\cdot)$ and any $x_1(0) \in \mathbb{R}^{n_1}$, the response $x_1(t)$ of

[1] More precisely, if the $\mathscr{L}_2$ gains of the two component systems are upper-bounded by two numbers $\bar{\gamma}_1$ and $\bar{\gamma}_2$ satisfying $\bar{\gamma}_1\bar{\gamma}_2 \le 1$.

[2] See [9] for an introduction to the property of input-to-state stability.

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_8

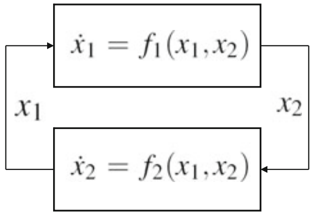

Fig. 8.1 A pure feedback interconnection of two nonlinear systems

$$\dot{x}_1 = f_1(x_1, x_2)$$

in the initial state $x_1(0)$ satisfies

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(0)\|, t), \gamma_1(\|x_2(\cdot)\|_{[0,t]})\} \tag{8.2}$$

for all $t \geq 0$, and for any bounded input $x_1(\cdot)$ and any $x_2(0) \in \mathbb{R}^{n_2}$, the response $x_2(t)$ of

$$\dot{x}_2 = f_2(x_1, x_2)$$

in the initial state $x_2(0)$ satisfies

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(0)\|, t), \gamma_2(\|x_1(\cdot)\|_{[0,t]})\} \tag{8.3}$$

for all $t \geq 0$.

In what follows we shall prove that if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ satisfies[3]

$$\gamma_1 \circ \gamma_2(r) < r \quad \text{for all } r > 0, \tag{8.4}$$

[3] A function $\gamma : [0, \infty) \to [0, \infty)$ satisfying $\gamma(0) = 0$ and $\gamma(r) < r$ for all $r > 0$ is called a *simple contraction*. Observe that if $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then also $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction. In fact, let $\gamma_1^{-1}(\cdot)$ denote the inverse of the function $\gamma_1(\cdot)$, which is defined on an interval of the form $[0, r_1^*)$ where

$$r_1^* = \lim_{r \to \infty} \gamma_1(r).$$

If $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then

$$\gamma_2(r) < \gamma_1^{-1}(r) \quad \text{for all} \quad 0 < r < r_1^*,$$

and this shows that

$$\gamma_2(\gamma_1(r)) < r \quad \text{for all} \quad r > 0,$$

i.e., $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction.

the pure feedback interconnection of Σ_1 and Σ_2 is globally asymptotically stable. This result is usually referred to as the *Small-Gain* Theorem for input-to-state stable systems.[4]

Theorem 8.1 *Suppose Σ_1 and Σ_2 are two input-to-state stable systems. If the condition (8.4) holds, system (8.1) is globally asymptotically stable.*

Proof The first step in the proof consists in showing that $x_2(\cdot)$ and $x_1(\cdot)$ are defined on the entire time interval $[0, \infty)$, and bounded. Pick any pair of arbitrary initial conditions $x_1(0) = x_1^\circ \in \mathbb{R}^{n_1}$, $x_2(0) = x_2^\circ \in \mathbb{R}^{n_2}$. To prove that trajectories are bounded, we proceed by contradiction. Suppose that this is not the case. Then, for every number $R > 0$, there exists a time $T > 0$, such that both trajectories are defined on $[0, T]$, and

$$\text{either} \quad \|x_1(T)\| > R \quad \text{or} \quad \|x_2(T)\| > R. \tag{8.5}$$

Choose R such that

$$\begin{aligned} R &> \max\{\beta_1(\|x_1^\circ\|, 0), \gamma_1 \circ \beta_2(\|x_2^\circ\|, 0)\} \\ R &> \max\{\beta_2(\|x_2^\circ\|, 0), \gamma_2 \circ \beta_1(\|x_1^\circ\|, 0)\}, \end{aligned} \tag{8.6}$$

and let T be such that (8.5) holds. Consider, for system Σ_1, an input $\bar{x}_2(\cdot)$ defined as $\bar{x}_2(t) = x_2(t)$ for $t \in [0, T]$ and $\bar{x}_2(t) = 0$ for $t > T$. This input is bounded. Using (8.2), it is seen that

$$\|x_1(\cdot)\|_{[0,T]} \le \max\{\beta_1(\|x_1^\circ\|, 0), \gamma_1(\|x_2(\cdot)\|_{[0,T]})\}. \tag{8.7}$$

Likewise, consider for system Σ_2 an input $\bar{x}_1(\cdot)$ defined as $\bar{x}_1(t) = x_1(t)$ for $t \in [0, T]$ and $\bar{x}_1(t) = 0$ for $t > T$. This input is bounded. Using (8.3) it is seen that

$$\|x_2(\cdot)\|_{[0,T]} \le \max\{\beta_2(\|x_2^\circ\|, 0), \gamma_2(\|x_1(\cdot)\|_{[0,T]})\}. \tag{8.8}$$

Let us now replace the estimate (8.8) into (8.7) and observe that, if $a \le \max\{b, c, \theta(a)\}$ and $\theta(a) < a$, then necessarily $a \le \max\{b, c\}$. This, using the hypothesis that $\gamma_1 \circ \gamma_2(r) < r$, yields

$$\|x_1(\cdot)\|_{[0,T]} \le \max\{\beta_1(\|x_1^\circ\|, 0), \gamma_1 \circ \beta_2(\|x_2^\circ\|, 0)\}. \tag{8.9}$$

The hypothesis (8.4) also implies that $\gamma_2 \circ \gamma_1(r) < r$ for $r > 0$, and therefore an argument identical to the one used before shows that

$$\|x_2(\cdot)\|_{[0,T]} \le \max\{\beta_2(\|x_2^\circ\|, 0), \gamma_2 \circ \beta_1(\|x_1^\circ\|, 0)\}. \tag{8.10}$$

From (8.9) and (8.10), using the definition (8.6), we obtain

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[4] Related results can be found in [6, 7, 10]. See also [2–4] and [11] for earlier versions of a Small-Gain Theorem for nonlinear systems.

$$\|x_1(\cdot)\|_{[0,T]} \le R \quad \text{and} \quad \|x_2(\cdot)\|_{[0,T]} \le R$$

which contradicts (8.5).

Having shown that the trajectories are defined for all $t \ge 0$ and bounded, (8.2) and (8.3) yield

$$\begin{aligned} \|x_1(\cdot)\|_\infty &\le \max\{\beta_1(\|x_1^\circ\|, 0), \gamma_1(\|x_2(\cdot)\|_\infty)\} \\ \|x_2(\cdot)\|_\infty &\le \max\{\beta_2(\|x_2^\circ\|, 0), \gamma_2(\|x_1(\cdot)\|_\infty)\}, \end{aligned}$$

combining which, and using the property that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, one obtains

$$\begin{aligned} \|x_1(t)\| &\le \max\{\beta_1(\|x_1^\circ\|, 0), \gamma_1 \circ \beta_2(\|x_2^\circ\|, 0)\} \\ \|x_2(t)\| &\le \max\{\beta_2(\|x_2^\circ\|, 0), \gamma_2 \circ \beta_1(\|x_1^\circ\|, 0)\}, \end{aligned}$$

for all $t \ge 0$. This, since $\beta_i(r, 0)$ and $\gamma_i(r)$ are continuous functions vanishing at $r = 0$, shows that the equilibrium $(x_1, x_2) = (0, 0)$ of (8.1) is stable in the sense of Lyapunov.

To prove asymptotic stability, knowing that $\|x_2(t)\|$ is bounded on $[0, \infty)$, let

$$r_2 = \limsup_{t\to\infty} \|x_2(t)\|.$$

Pick $\varepsilon > 0$ and let $h > 0$ be such that

$$\gamma_1(r_2 + h) = \gamma_1(r_2) + \varepsilon.$$

By definition of r_2, there is a time $T > 0$ such that $\|x_2(t)\| \le r_2 + h$ for all $t \ge T$. Using (8.2), observe that

$$\|x_1(t)\| \le \max\{\beta_1(\|x_2(T)\|, t - T), \gamma_1(\|x_2(\cdot)\|_{[T,T+t]})\}$$

for all $t \ge T$. Since $\beta_1(\|x_2(T)\|, t - T)$ decays to 0 as $t \to \infty$, there exists a time $T' \ge T$ such that $\beta_1(\|x_2(T)\|, t - T) \le \varepsilon$ for all $t \ge T'$. Thus, for all $t \ge T'$ we have

$$\begin{aligned} \|x_1(t)\| &\le \max\{\varepsilon, \gamma_1(\|x_2(\cdot)\|_{[T,T+t]})\} \le \max\{\varepsilon, \gamma_1(r_2 + h)\} \\ &= \max\{\varepsilon, \gamma_1(r_2) + \varepsilon\} = \gamma_1(r_2) + \varepsilon = \gamma_1(\limsup_{t\to\infty} \|x_2(t)\|) + \varepsilon. \end{aligned} \tag{8.11}$$

In this way, we have shown that for any ε there is a time T' such that, for all $t \ge T'$, the estimate (8.11) holds. Thus,

$$\limsup_{t\to\infty} \|x_1(t)\| \le \gamma_1(\limsup_{t\to\infty} \|x_2(t)\|). \tag{8.12}$$

In a similar way one finds that

$$\limsup_{t\to\infty} \|x_2(t)\| \leq \gamma_2(\limsup_{t\to\infty} \|x_1(t)\|). \tag{8.13}$$

Combining (8.12) with (8.13) and using the fact that $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, we obtain

$$\limsup_{t\to\infty} \|x_1(t)\| \leq \gamma_1 \circ \gamma_2(\limsup_{t\to\infty} \|x_1(t)\|) < \limsup_{t\to\infty} \|x_1(t)\|$$

which shows that necessarily

$$\limsup_{t\to\infty} \|x_1(t)\| = 0 = \limsup_{t\to\infty} \|x_2(t)\|.$$

This proves that the equilibrium is asymptotically stable. $\triangleleft$

The condition (8.4), i.e., the condition that the composed function $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, is the nonlinear analogue of the condition determined in Theorem 3.2 and Corollary 3.1 for the analysis of the stability of the pure feedback interconnection of two stable linear systems. As it was the case in the context of linear systems, the condition in question is only a *sufficient* condition for global asymptotic stability. From a practical viewpoint, the result of Theorem 8.1 can be implemented as follows. First of all, one has to check that both Σ_1 and Σ_2 are input-to-state stable and find estimates of the two gain functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$. This can be obtained by seeking the existence (see Sect. B.2 in Appendix B) of an ISS-Lyapunov function for each system, i.e., a pair of positive definite and proper functions $V_1(x_1)$, $V_2(x_2)$ such that, for some $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\chi_1(\cdot)$, $\chi_2(\cdot)$

$$\|x_1\| \geq \chi_1(\|x_2\|) \quad \Rightarrow \quad \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \leq -\alpha_1(\|x_1\|)$$

and

$$\|x_2\| \geq \chi_2(\|x_1\|) \quad \Rightarrow \quad \frac{\partial V_2}{\partial x_2} f_2(x_1, x_2) \leq -\alpha_2(\|x_2\|).$$

If such functions can be found, in fact, Σ_1 and Σ_2 are input-to-state stable and, as shown in Theorem B.3, estimates of the gain functions can be obtained.

Example 8.1 Consider the pure feedback interconnection of two systems Σ_1 and Σ_2, modeled by equations of the form

$$\Sigma_1: \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= x_1 x_2 - x_2^3 + x_2 \varphi(u_1) \\ y_1 &= x_2, \end{aligned} \tag{8.14}$$

in which $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\varphi(0) = 0$, and

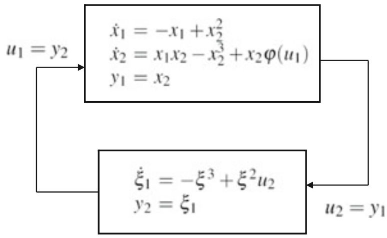

Fig. 8.2 The system of Example 8.1

$$\Sigma_2 : \quad \begin{aligned} \dot{\xi}_1 &= -\xi^3 + \xi^2 u_2 \\ y_2 &= \xi_1 . \end{aligned} \tag{8.15}$$

(see Fig. 8.2).

It follows from Example B.2 in Appendix B that system Σ_2 is input-to-state stable, with a gain function than can be estimated as

$$\gamma_2(r) = \ell_2 r,$$

in which ℓ_2 is a number larger than 1 (but otherwise arbitrary). To determine the input-to-state stability properties of Σ_1, we consider the candidate ISS-Lyapunov function

$$V(x_1, x_1) = \frac{1}{2}(x_1^2 + x_2^2),$$

for which we obtain

$$\begin{aligned} \dot{V} &= -x_1^2 - x_2^4 + x_2^2\varphi(u) \leq -x_1^2 - x_2^4 + \tfrac{1}{2}x_2^4 + \tfrac{1}{2}\varphi^2(u_1) \\ &\leq -\tfrac{1}{2}(x_1^2 + x_2^4) + \tfrac{1}{2}\varphi^2(u_1). \end{aligned}$$

The function $(x_1^2 + x_2^4)$ is positive definite and proper. Hence, there exists a class $\mathscr{K}_\infty$ function $\alpha(r)$ such that

$$x_1^2 + x_2^4 \geq \alpha(\|x\|)$$

for all $x \in \mathbb{R}^2$.[5] Let $\sigma(r)$ be a class $\mathscr{K}$ function satisfying

$$\sigma(r) \geq \max\{\varphi^2(r), \varphi^2(-r)\}$$

for all $r \geq 0$. Then, we obtain

[5]The reader should have no difficulty in checking that a function defined as

$$\begin{aligned} \alpha(r) &= r^4 && \text{for } 0 \leq r \leq \tfrac{1}{\sqrt{2}} \\ \alpha(r) &= r^2 - \tfrac{1}{4} && \text{for } r \geq \tfrac{1}{\sqrt{2}} \end{aligned} \tag{8.16}$$

has the indicated property.

$$\dot{V} \le -\frac{1}{2}\alpha(\|x\|) + \frac{1}{2}\sigma(|u_1|),$$

from which it can be deduced that also system Σ_1 is input-to-state stable. In fact, with $\chi(r)$ defined as

$$\chi(r) = \alpha^{-1}(\ell_1\sigma(r))$$

and $\ell_1 > 1$, we obtain

$$\|x\| \ge \chi(|u_1|) \quad \Rightarrow \quad \dot{V} \le -\frac{\ell_1 - 1}{2\ell_1}\alpha(\|x\|).$$

So long as an estimate of the gain is concerned, bearing in mind the fact that $V(x) = \frac{1}{2}\|x\|^2$ and using Theorem B.3, we obtain

$$\gamma_1(r) = \chi(r).$$

The interconnected system is globally asymptotically stable if the small-gain condition, which in this case reads as

$$\ell_2\alpha^{-1}(\ell_1\sigma(r)) < r \quad \text{for all } r > 0,$$

holds. This condition can be written in equivalent form as

$$\sigma(r) < \frac{1}{\ell_1}\alpha\left(\frac{1}{\ell_2}r\right) \quad \text{for all } r > 0,$$

in which, we recall, ℓ_1 and ℓ_2 are larger than 1 (but otherwise arbitrary). Thus, it is concluded that a sufficient condition for the interconnected system to be asymptotically stable is the existence of a class $\mathscr{K}$ function $\sigma(\cdot)$ and a pair of numbers $\ell_1 > 1$, $\ell_2 > 1$ such that the function $\varphi(r)$ in (8.14) satisfies

$$\max\{\varphi^2(r), \varphi^2(-r)\} < \frac{1}{\ell_1}\alpha\left(\frac{1}{\ell_2}r\right) \quad \text{for all } r > 0,$$

where $\alpha(r)$ is the function (8.16). ◁

8.2 Gain Assignment

The Small-Gain Theorem for input-to-state stable systems can be used in various ways when dealing with the problem of globally asymptotically stabilizing a nonlinear system. The simplest application is found in the context of the stabilization problem discussed in Sect. 6.5, namely the problem of globally asymptotically stabilizing, using output feedback, a system having relative degree 1. Consider again system (6.27), rewritten here for convenience

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= q(z, \xi) + b(z, \xi)u \\ y &= \xi, \end{aligned} \tag{8.17}$$

in which $z \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}$ and, as usual, assume that

$$\begin{aligned} f_0(0, 0) &= 0 \\ q(0, 0) &= 0. \end{aligned}$$

Assume also that the system is *strongly minimum phase*, which means that system

$$\dot{z} = f_0(z, \xi) \tag{8.18}$$

viewed as a system with state z and input ξ is input-to-state stable. Suppose a control $u = k(\xi)$ can be found such that

$$\dot{\xi} = q(z, \xi) + b(z, \xi)k(\xi) \tag{8.19}$$

viewed as a system with state ξ and input z is input-to-state stable. If this were the case and if the two gain functions of (8.18) and (8.19), which we denote as $\gamma_z(\cdot)$ and $\gamma_\xi(\cdot)$, respectively, were such that

$$\gamma_z \circ \gamma_\xi(r) < r \quad \text{for all } r > 0, \tag{8.20}$$

it would follow from Theorem 8.1 that the resulting closed-loop system, namely system (8.17) controlled by $u = k(\xi)$, is globally asymptotically stable.

In this context, it is observed that the gain function $\gamma_z(\cdot)$ associated with (8.18) is fixed, i.e., there is not control on it. Thus, the proposed design strategy is applicable if it is possible to find a control $u = k(\xi)$ such that system (8.19) is rendered input-to-state stable, *with a gain function* $\gamma_\xi(\cdot)$ *that respects the constraint* (8.20). As a matter of fact, this is actually possible, as shown below.

For convenience, let the lower subsystem of (8.17) be rewritten in more general terms as

$$\dot{x} = q(z, x) + b(z, x)u, \tag{8.21}$$

in which $x \in \mathbb{R}$ and $z \in \mathbb{R}^k$. Then, the following result holds.[6]

Lemma 8.1 *Consider system (8.21). Suppose there exists a number* $b_0 > 0$ *such that*

$$b(z, x) \geq b_0 \quad \textit{for all } (z, x) \in \mathbb{R}^k \times \mathbb{R}$$

[6]See [1] for further results of this kind.

and two class $\mathscr{K}$ functions $\rho_0(\cdot)$ and $\rho_1(\cdot)$ such that

$$|q(z,x)| \le \max\{\rho_0(|x|), \rho_1(\|z\|)\} \quad \textit{for all } (z,x) \in \mathbb{R}^k \times \mathbb{R}.$$

Let $\gamma(\cdot)$ be a class $\mathscr{K}_\infty$ function. Then there exists a strictly increasing function $k(x)$, with $k(0) = 0$ and $k(-x) = -k(x)$ which is continuously differentiable everywhere except at $x = 0$, where it is only continuous, such that the system

$$\dot{x} = q(z,x) + b(z,x)k(x) \tag{8.22}$$

viewed as a system with state x and input z is input-to-state stable, with gain function $\gamma(\cdot)$. If, in addition, $\rho_0(\cdot)$ and $\rho_1 \circ \gamma^{-1}(\cdot)$ are locally Lipschitz at the origin, the result holds with a function $k(x)$ which is continuously differentiable everywhere.

Proof Choose

$$k(x) = -\frac{1}{b_0}(x + \alpha(x))$$

in which $\alpha(\cdot)$ is a continuous strictly increasing function, with $\alpha(0) = 0$ and $\alpha(-x) = -\alpha(x)$, satisfying

$$\alpha(r) \ge \max\{\rho_0(r), \rho_1 \circ \gamma^{-1}(r)\} \quad \text{for all } r \ge 0. \tag{8.23}$$

If the two functions $\rho_0(\cdot)$ and $\rho_1 \circ \gamma^{-1}(\cdot)$ are locally Lipschitz at the origin, a continuously differentiable function $\alpha(\cdot)$ exists that satisfies (8.23).

Now, consider, for the controlled system (8.22), the candidate ISS-Lyapunov function $V(x) = x^2$ and observe that[7]

$$\begin{aligned}\frac{\partial V}{\partial x}[q(z,x) + b(z,x)k(x)] &= 2x\left[q(z,x) - \frac{b(z,x)}{b_0}(x + \alpha(x))\right] \\ &\le -2|x|^2 + 2|x|[\max\{\rho_0(|x|), \rho_1(\|z\|)\} - \alpha(|x|)].\end{aligned}$$

We want to prove that if

$$|x| \ge \gamma(\|z\|) \tag{8.24}$$

the right-hand side of the previous inequality is bounded by $-2|x|^2$. To this end, observe that (8.23) implies

$$\alpha \circ \gamma(r) \ge \rho_1(r),$$

and therefore (8.24) implies

$$\alpha(|x|) \ge \rho_1(\|z\|).$$

Since $\alpha(|x|) \ge \rho_0(|x|)$, we conclude that

[7] Use the fact that $x\alpha(x) = |x|\alpha(|x|)$.

$$|x| \geq \gamma(\|z\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}[q(z,x) + b(z,x)k(x)] \leq -2|x|^2. \tag{8.25}$$

Bearing in mind Theorem B.3 and the fact that, in this case, $\underline{\alpha}(r) = \overline{\alpha}(r) = r^2$, it is concluded that system (8.22) is input-to-state stable, with gain function $\gamma(\cdot)$. ◁

This result provides the desired answer to the question posed at the beginning, i.e., to what extent it is possible to find a control $u = k(\xi)$ that renders system (8.19) input-to-state stable, with a gain function $\gamma_\xi(\cdot)$ that respects the constraint (8.20). To this end, in fact, it suffices to pick any function $\gamma(\cdot)$ satisfying

$$\gamma^{-1}(r) > \gamma_z(r) \quad \text{for all } r > 0,$$

and find $k(\xi)$ accordingly. This leads to the following conclusion.

Proposition 8.1 *Consider a system of the form (8.17). Suppose the system is strongly minimum phase, with gain function $\gamma_z(\cdot)$. Suppose there exists a number $b_0 > 0$ and two class $\mathscr{K}$ functions $\rho_0(\cdot)$ and $\rho_1(\cdot)$ such that*

$$\begin{array}{l} b(z,\xi) \leq b_0 \\ |q(z,\xi)| \leq \max\{\rho_0(|\xi|), \rho_1(\|z\|)\} \end{array} \quad \textit{for all } (z,\xi) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Then there exists a strictly increasing function $k(\xi)$, with $k(0) = 0$ and $k(-\xi) = -k(\xi)$ which is continuously differentiable everywhere except at $\xi = 0$, where it is only continuous, such that the control law $u = k(\xi)$ globally asymptotically stabilizes the equilibrium $(z,\xi) = (0,0)$ of the resulting closed-loop system. If, in addition, $\rho_0(\cdot)$ and $\rho_1 \circ \gamma_z(\cdot)$ are locally Lipschitz at the origin, the result holds with a function $k(\xi)$ which is continuously differentiable everywhere.

The design method outlined in the proof of Lemma 8.1 is applied in the following simple example.

Example 8.2 In the light of the result obtained above, we revisit Example 6.3, namely the system

$$\begin{aligned} \dot{z} &= -z^3 + \xi \\ \dot{\xi} &= z + u, \end{aligned} \tag{8.26}$$

which we have shown to be semiglobally and practically stabilizable using a linear feedback law $u = -k\xi$.

The system is strongly minimum phase. To check that this is the case, consider the candidate ISS-Lyapunov function $V(z) = \frac{1}{2}z^2$ for which we obtain

$$\frac{\partial V}{\partial z}[-z^3 + \xi] = -z^4 + z\xi.$$

To determine an estimate of the gain function, we try to enforce the constraint

$$-z^4 + |z|\,|\xi| < -\frac{1}{2}z^4.$$

Such constraint holds if

$$\frac{1}{2}z^4 > |z|\,|\xi|,$$

which is the case if

$$|z| \geq (2|\xi|)^{\frac{1}{3}}.$$

Thus, the gain function of the upper subsystem of (8.26) can be estimated by

$$\gamma_z(r) = (2r)^{\frac{1}{3}}.$$

From this it is seen that the small-gain condition (8.20) is fulfilled if

$$\gamma_\xi(r) = \frac{1}{4}r^3.$$

We proceed now with the design of the function $k(\xi)$ according to the construction shown in the proof of Lemma 8.1. In the present case, $\rho_0(r) = 0$ and $\rho_1(r) = r$. Thus, we can pick the function $\alpha(\cdot)$ as

$$\alpha(r) = \rho_1 \circ \gamma^{-1}(r) = \gamma^{-1}(r) = (4r)^{\frac{1}{3}}, \quad r > 0.$$

The resulting feedback law is a function $k(\xi)$ which satisfies $k(0) = 0$, $k(-\xi) = -k(\xi)$ and

$$k(\xi) = -\xi - (4\xi)^{\frac{1}{3}}, \quad \text{for } \xi > 0.$$

Note that the function $k(\xi)$ is only continuous, and not even locally Lipschitz, at $\xi = 0$.

This result should be compared with Example 6.3, in which it was shown that, using a *linear* control law $u = k\xi$, system (8.26) could only be semiglobally and practically stabilized. The current example shows that the system in question can actually be *globally asymptotically* stabilized, but at the price of using a *nonlinear* control law $u = k(\xi)$, which moreover happens to be only continuous at $\xi = 0$. ◁

8.3 An Application to Robust Stability

The Small-Gain Theorem for input-to-state stable systems can be used to the purpose of robustly stabilizing a system in the presence of model uncertainties in the same way as the Small-Gain Theorem for linear systems was used to this purpose in Sect. 3.5. This is the case when the system to be controlled can be seen as interconnection of an accurately modeled system

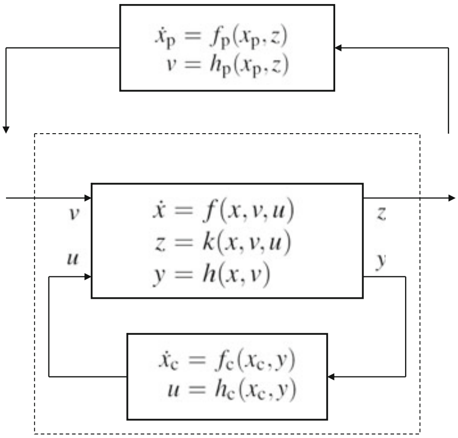

Fig. 8.3 The small-gain approach to robust stabilization

$$\begin{aligned}\dot{x} &= f(x, v, u)\\ z &= k(x, v, u)\\ y &= h(x, v)\end{aligned}$$

and of a poorly modeled system

$$\begin{aligned}\dot{x}_p &= f_p(x_p, z)\\ v &= h_p(x_p, z).\end{aligned}$$

In this case, the problem is addressed by seeking a control

$$\begin{aligned}\dot{x}_c &= f_c(x_c, y)\\ u &= h_c(x_c, y),\end{aligned}$$

that forces the fulfillment of the small-gain condition (see Fig. 8.3).

The difference with the case studied in Chap. 4, though, is that general methods such as the method described in Sect. 3.6 are not available, and one must proceed on a case-by-case basis. In what follows, we describe a simple application, which essentially extends to the case of a multi-input system, the method for gain assignment described in the previous section. Consider a system modeled as in

$$\begin{aligned}\dot{x}_p &= f_p(x_p, x)\\ \dot{x} &= q(x_p, x) + b(\mu)u\end{aligned} \tag{8.27}$$

in which $x_p \in \mathbb{R}^{n_p}$, $x \in \mathbb{R}^m$, $u \in \mathbb{R}^m$ and μ is a vector of uncertain parameters, ranging over a compact set $\mathbb{M}$. It is assumed that the upper subsystem, seen as a system with state x_p and input x, is input-to-state stable, with a gain function $\gamma_p(\cdot)$ which is known. The matter is to find a control u that renders the lower subsystem, seen as a system with state x and input x_{p}, is input-to-state stable with a gain function $\gamma(\cdot)$ such that the composition $\gamma \circ \gamma_{\mathrm{p}}(\cdot)$ is a simple contraction.

Note that, in this example of application, in the controlled part of the system it is assumed that state and input have the same number m of components, which indeed renders the design problem substantially simpler. It is also assumed that the full state x is available for feedback, in which case a memoryless control $u = k_{\mathrm{c}}(x)$ is sought. In this setting, the matter then is to find a function $k_{\mathrm{c}}(x)$ such that the system

$$\dot{x} = q(x_{\mathrm{p}}, x) + b(\mu)k_{\mathrm{c}}(x) \tag{8.28}$$

is input-to-state stable, with a specified gain function.[8]

To address this problem, a couple of preliminary results is needed.

Lemma 8.2 [9] *Let $b(\mu)$ be a $m \times m$ matrix of continuous functions of μ, in which μ is a vector of uncertain parameters ranging over a compact set $\mathbb{M}$. Suppose all leading principal minors $\Delta_i(\mu)$ of $b(\mu)$ satisfy*

$$0 < b_0 \le |\Delta_i(\mu)| \quad \textit{for all } \mu \in \mathbb{M} \tag{8.29}$$

for some b_0. Then there exist a nonsingular diagonal $m \times m$ matrix E, whose entries are equal either to 1 or to -1, a symmetric positive definite $m \times m$ matrix $S(\mu)$ of continuous functions of μ and positive numbers a_1, a_2 satisfying

$$a_1\|x\|^2 \le x^{\mathrm{T}}S(\mu)x \le a_2\|x\|^2 \quad \textit{for all } \mu \in \mathbb{M} \tag{8.30}$$

and a strictly upper triangular $m \times m$ matrix $U(\mu)$ of continuous functions of μ such that

$$b(\mu) = ES(\mu)(I + U(\mu)) \quad \textit{for all } \mu \in \mathbb{M}. \tag{8.31}$$

Proof It is well known that a nonsingular $m \times m$ matrix $b(\mu)$ admits the decomposition

$$b(\mu) = (I + L_1(\mu))D(\mu)(I + L_2^{\mathrm{T}}(\mu)) \tag{8.32}$$

in which

$$D = \mathrm{diag}\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_m}{\Delta_{m-1}}\right\},$$

[8] Note that the resulting system is a special case of the system in Fig. 8.2, namely the interconnection of

$$\dot{x} = q(v_1, x) + b(v_2)u \quad z = x \quad y = x$$

and

$$\begin{aligned} \dot{x}_{\mathrm{p}} &= f_{\mathrm{p}}(x_{\mathrm{p}}, z) \qquad & v_1 &= x_{\mathrm{p}} \\ \dot{\mu} &= 0 & v_2 &= \mu \end{aligned}$$

with control $u = h_{\mathrm{c}}(y)$.

[9] For similar results, see also [5, 8].

and both $L_1(\mu)$ and $L_2(\mu)$ are strictly lower triangular matrices, whose entries are rational functions of the entries of $b(\mu)$. Since μ ranges over a compact set and all $|\Delta_i(\mu)|$'s are bounded from below, the diagonal elements $d_{ii}(\mu)$ of $D(\mu)$ satisfy

$$d_0 \leq |d_{ii}(\mu)| \leq d_1,$$

for some pair $0 < d_0 < d_1$. Having defined $D_+(\mu) = \text{diag}(|d_{11}(\mu)|, \dots, |d_{mm}(\mu)|)$, the matrix $D(\mu)$ can be written as $D(\mu) = D_+(\mu)E$ in which E is a diagonal matrix whose elements can only be 1 or -1. Then, consider the matrix

$$S(\mu) = E^{\mathrm{T}}(I + L_1(\mu))D_+(\mu)(I + L_1(\mu))^{\mathrm{T}} E,$$

which by construction is positive definite. Since μ ranges over a compact set, an estimate of the form (8.30) holds, with $0 < a_1 < a_2$ independent of μ.

Bearing in mind that $EE^{\mathrm{T}} = I$, rewrite (8.32) as

$$\begin{aligned}
b(\mu) &= (I + L_1(\mu))D_+(\mu)E(I + L_2^{\mathrm{T}}(\mu)) \\
&= (I + L_1(\mu))D_+(\mu)(I + L_1^{\mathrm{T}}(\mu))(I + L_1^{\mathrm{T}}(\mu))^{-1}E(I + L_2^{\mathrm{T}}(\mu)) \\
&= EE^{\mathrm{T}}(I + L_1(\mu))D_+(\mu)(I + L_1^{\mathrm{T}}(\mu))EE^{\mathrm{T}}(I + L_1^{\mathrm{T}}(\mu))^{-1}E(I + L_2^{\mathrm{T}}(\mu)) \\
&= ES(\mu)E^{\mathrm{T}}(I + L_1^{\mathrm{T}}(\mu))^{-1}E(I + L_2^{\mathrm{T}}(\mu)).
\end{aligned}$$

Clearly, $E^{\mathrm{T}}(I + L_1^{\mathrm{T}}(\mu))^{-1}E$ is an upper triangular matrix whose diagonal elements are all 1 and so is $E^{\mathrm{T}}(I + L_1^{\mathrm{T}}(\mu))^{-1}E(I + L_2^{\mathrm{T}}(\mu))$. Therefore, there is a strictly upper triangular matrix $U(\mu)$ such that

$$E^{\mathrm{T}}(I + L_1^{\mathrm{T}}(\mu))^{-1}E(I + L_2^{\mathrm{T}}(\mu)) = I + U(\mu)$$

and this completes the proof. ◁

Lemma 8.3 *Let $U(\mu)$ be a strictly upper triangular $m \times m$ matrix of continuous functions of μ, with μ ranging over a compact set $\mathbb{M}$. Let*

$$G_k = \text{diag}(k^m, k^{m-1}, \dots, k). \tag{8.33}$$

Then, there is a number k_0 such that, if $k > k_0$,

$$[I + U(\mu)]G_k + G_k^{\mathrm{T}}[I + U(\mu)]^{\mathrm{T}} \geq kI \quad \textit{for all } \mu.$$

Proof Since

$$[I + U(\mu)]G_k = \begin{pmatrix} k^m & u_{12}(\mu)k^{m-1} & \dots & u_{1m}(\mu)k \\ 0 & k^{m-1} & \dots & u_{2m}(\mu)k \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & k \end{pmatrix}$$

it is seen that

$$[I+U(\mu)]G_k + G_k^{\mathrm{T}}[I+U(\mu)]^{\mathrm{T}} - kI = \begin{pmatrix} 2k^m - k & u_{12}(\mu)k^{m-1} & \dots & u_{1m}(\mu)k \\ u_{12}(\mu)k^{m-1} & 2k^{m-1}-k & \dots & u_{2m}(\mu)k \\ \cdot & \cdot & \dots & \cdot \\ u_{1m}(\mu)k & u_{2m}(\mu)k & \dots & k \end{pmatrix}.$$

If this matrix is positive definite, the lemma is proven. To see that this is the case, we compute the leading principal minors. The first leading principal minor is

$$M_1 = 2k^m - k$$

Clearly, there exist $k_1 > 0$ such that, for any $k > k_1$, this number is positive. The second leading principal minor is

$$\begin{aligned} M_2 &= (2k^m - k)(2k^{m-1} - k) - u_{12}^2(\mu)k^{2(m-1)} \\ &= 4k^{2m-1} - 2k^{m+1} - 2k^m + k^2 - u_{12}^2(\mu)k^{2(m-1)}. \end{aligned}$$

The coefficient of the highest power of k is positive and μ ranges over a compact set. Hence, there exists $k_2 > 0$ such that if $k > k_2$, then $M_2 > 0$ for all μ. Proceeding in this way, we can find a set of positive numbers $k_1, k_2, \dots, k_m$, and a number $k_0 = \max\{k_1, k_2, \dots, k_m\}$ such that, if $k > k_0$,

$$[I+U(\mu)]G_k + G_k^{\mathrm{T}}[I+U^{\mathrm{T}}(\mu)] - kI > 0 \quad \text{for all } \mu$$

and this completes the proof. ◁

We return now to the lower subsystem of (8.27), which—if the assumptions of Lemma 8.2 hold—can be rewritten as

$$\dot{x} = q(x_{\mathrm{p}}, x) + ES(\mu)(I + U(\mu))u. \tag{8.34}$$

For this system we choose a control of the form

$$u = k_{\mathrm{p}}(x) = -K(\|x\|)Ex, \tag{8.35}$$

where $K(r)$ is the $m \times m$ diagonal matrix

$$K(r) = G_{\kappa(r)} = \mathrm{diag}([\kappa(r)]^m, [\kappa(r)]^{m-1}, \dots, \kappa(r)) \tag{8.36}$$

in which $\kappa : [0, \infty) \to [0, \infty)$ is a continuous function. This yields

$$\dot{x} = q(x_{\mathrm{p}}, x) - ES(\mu)(I + U(\mu))K(\|x\|)Ex. \tag{8.37}$$

Then, a result essentially identical to that of Lemma 8.1 holds.[10]

[10] See also [12].

Lemma 8.4 *Consider system (8.28). Suppose $b(\mu)$ is such that the assumptions of Lemma 8.2 hold. Suppose also that there exists two class $\mathscr{K}$ functions $\rho_0(\cdot)$ and $\rho_1(\cdot)$ such that*

$$\|q(x_\mathrm{p}, x)\| \leq \max\{\rho_0(\|x\|), \rho_1(\|x_\mathrm{p}\|)\} \quad \textit{for all } (x_\mathrm{p}, x).$$

Let $\chi(\cdot)$ be a class $\mathscr{K}_\infty$ function and suppose that $\rho_0(\cdot)$ and $\rho_1 \circ \chi^{-1}(\cdot)$ are locally Lipschitz at the origin. Then there exists a continuous function $\kappa(s)$ such that system (8.37), viewed as a system with state x and input x_p is input-to-state stable, with gain function

$$\gamma(s) = \sqrt{\frac{a_2}{a_1}}\chi(s), \tag{8.38}$$

in which a_1, a_2 are numbers for which (8.30) holds.

Proof Consider the candidate ISS-Lyapunov function $V(x) = x^\mathrm{T} E^\mathrm{T} S^{-1}(\mu) E x$. Let $\kappa(r) > k_0$ for all $r \geq 0$, with k_0 defined as in Lemma 8.3. Then, using the result of this lemma, observe that

$$\begin{aligned}\frac{\partial V}{\partial x}\dot{x} &= \frac{\partial V}{\partial x}[q(x_\mathrm{p}, x) - ES(\mu)(I + U(\mu))K(\|x\|)Ex] \\ &= 2x^\mathrm{T} E^\mathrm{T} S^{-1}(\mu) E q(x_\mathrm{p}, x) - 2x^\mathrm{T} E^\mathrm{T}(I + U(\mu)) G_{\kappa(\|x\|)} E x \\ &\leq 2x^\mathrm{T} E^\mathrm{T} S^{-1}(\mu) E q(x_\mathrm{p}, x) - \kappa(\|x\|)\|Ex\|^2 \\ &\leq 2\|x\|\,\|S^{-1}(\mu)\|\,\|q(x_\mathrm{p}, x)\| - \kappa(\|x\|)\|x\|^2.\end{aligned}$$

As in the proof of Lemma 8.1, let $\alpha(\cdot)$ be a class $\mathscr{K}$ function satisfying

$$\alpha(r) \geq 2\|S^{-1}(\mu)\| \max\{\rho_0(r), \rho_1 \circ \chi^{-1}(r)\} \quad \text{for all } r \geq 0 \text{ and all } \mu \in \mathbb{M}.$$

Since $\rho_0(\cdot)$ and $\rho_1 \circ \chi^{-1}(\cdot)$ are locally Lipschitz at the origin, one can assume that such $\alpha(\cdot)$ is continuously differentiable at the origin and therefore the limit of $\alpha(r)/r$ as $r \to 0^+$ exists, is finite and positive. Define $\kappa(r)$ as

$$\kappa(r) = \bar{k} + \frac{\alpha(r)}{r},$$

in which $\bar{k} > k_0$. Then, the previous estimate becomes

$$\frac{\partial V}{\partial x}\dot{x} \leq -k_0\|x\|^2 + \|x\|\Big[2\|S^{-1}(\mu)\| \; \max\{\rho_0(\|x\|), \rho_1(\|x_\mathrm{p}\|)\} - \alpha(\|x\|)\Big]$$

from which arguments identical to those used in the proof of Lemma 8.1 prove that

$$\|x\| \geq \chi(\|x_\mathrm{p}\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}\dot{x} \leq -k_0\|x\|^2.$$

Now, recall that

$$\frac{1}{a_2}\|x\|^2 \le x^{\mathrm{T}} S^{-1}(\mu) x \le \frac{1}{a_1}\|x\|^2.$$

Therefore, using Theorem B.3, it is concluded that system (8.37) is input-to-state stable, with gain function (8.38). ◁

From this, a result essentially identical to that of Proposition 8.1 follows.

Proposition 8.2 *Consider system (8.27). Suppose the upper subsystem, viewed as a system with state x_{p} and input x, is input-to-state stable, with a gain function $\gamma_{\mathrm{p}}(\cdot)$. Suppose the assumptions of Lemma 8.2 hold. Suppose also that there exists two class $\mathscr{K}$ functions $\rho_0(\cdot)$ and $\rho_1(\cdot)$ such that*

$$\|q(x_{\mathrm{p}}, x)\| \le \max\{\rho_0(\|x\|), \rho_1(\|x_{\mathrm{p}}\|)\} \quad \textit{for all } (x_{\mathrm{p}}, x),$$

and $\rho_0(\cdot)$ and $\rho_1 \circ \gamma_{\mathrm{p}}(\cdot)$ are locally Lipschitz at the origin. Then there exists a continuous function $\kappa(r)$ such that system (8.27) with control $u = -G_{\kappa(\|x\|)}Ex$ is globally asymptotically stable.

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Chapter 9
The Structure of Multivariable Nonlinear Systems

9.1 Preliminaries

In this chapter, we shall see how the theory developed for single-input single-output systems can be extended to nonlinear systems having many inputs and many outputs, usually referred to as *multivariable* systems. To simplify matters, we shall restrict our analysis to the consideration of systems having the same number m of input and output channels. The multivariable nonlinear systems we consider are described in state-space form by equations of the following kind

$$\begin{aligned}\dot{x} = &\ f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y_1 = &\ h_1(x) \\ &\ \cdots \\ y_m = &\ h_m(x)\end{aligned}$$

in which $f(x), g_1(x), \ldots, g_m(x)$ are smooth vector fields, and $h_1(x), \ldots, h_m(x)$ smooth functions, defined on $\mathbb{R}^n$. Most of the times, these equations will be rewritten in the more compact form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{9.1}$$

having set

$$\begin{aligned}u &= \text{col}(u_1, \ldots, u_m) \\ y &= \text{col}(y_1, \ldots, y_m)\end{aligned}$$

and where

$$\begin{aligned}g(x) &= (g_1(x) \ldots g_m(x)) \\ h(x) &= \text{col}(h_1(x), \ldots, h_m(x))\end{aligned}$$

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_9

are respectively an $n \times m$ matrix and an m-dimensional vector.

In the previous chapters, we have seen that the design of feedback laws for single-input single-output nonlinear systems is facilitated by a preliminary analysis aimed at understanding the structure of the relation between input, state, and higher order derivatives of the output. This analysis has been, in particular, taken as a point of departure for the introduction of changes of variables that played a considerable role in understanding the design of feedback laws. A similar, yet substantially more elaborate, analysis will be conducted in this chapter for multivariable nonlinear systems. In the course of this analysis it turns out to be useful, occasionally, to consider the system subject to a control input u that is a function of the state x and of a vector v of auxiliary variables (to be seen as "new inputs" for the resulting controlled system). In order to keep the structure of (9.1), affine in the control, we consider in what follows the case in which such control is expressed in the form

$$u = \alpha(x) + \beta(x)v, \tag{9.2}$$

where $\alpha(x)$ is an m-dimensional vector of smooth functions, $\beta(x)$ is a $m \times m$ matrix of smooth functions and

$$v = \mathrm{col}(v_1, \ldots, v_m)$$

is the m-dimensional vector of new inputs. Setting

$$\begin{aligned} \tilde{f}(x) &= f(x) + g(x)\alpha(x) \\ \tilde{\beta}(x) &= g(x)\beta(x) \end{aligned} \tag{9.3}$$

the composition of (9.1) and (9.2) can be written in the form

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + \tilde{g}(x)v \\ y &= h(x), \end{aligned} \tag{9.4}$$

which has the same structure as (9.1). As a nontriviality condition, it will be always assumed that $\beta(x)$ is *nonsingular* for all x.

9.2 The Basic Inversion Algorithm

In this section, we describe a recursive algorithm, known as the *Structure Algorithm*, which plays a crucial role in the analysis of multivariable nonlinear systems.[1] This algorithm is a powerful tool in the derivation of the multivariable version of the

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[1]The algorithm in question, originally introduced by Silverman in [4] to the purpose of analyzing the structure of the "zeros at infinity" of the transfer function matrix of a multivariable linear system, was extended in [5, 6] to the purpose of analyzing the property of invertibility of a nonlinear system. Additional features of such algorithm are discussed in [7] and [10]. Related results on nonlinear invertibiliy can be found in [8, 9].

normal forms determined earlier in Chap. 6 and in the study of the problem of *left invertibility*. The term "left invertibility," generally speaking, denotes the possibility of uniquely recovering the input of a system from the knowledge of its output and of its internal state. For a single-input single-output system, a sufficient condition for left invertibility is that the system has uniform relative degree. In this case in fact, it is possible to write

$$y^{(r)}(t) = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t)$$

and the coefficient $L_g L_f^{r-1} h(x(t))$ is nowhere zero, so that the input $u(t)$ can be uniquely recovered as

$$u(t) = \frac{1}{L_g L_f^{r-1} h(x(t))} [y^{(r)}(t) - L_f^r h(x(t))].$$

In the case of a multivariable system, though, the issue is much more involved and a property such as the one described above can only be decided at the end of a sequence of recursive calculations, described in what follows, known as nonlinear *Structure Algorithm* or also as nonlinear *Inversion Algorithm* A byproduct of such calculations is also the possibility of determining functions of x that can be used for a change of coordinates in the state space, by means of which the system is brought to a form that can be considered as the multivariable version of the normal form of a single-input single-output system.

Step 1. Set $\rho_0 = 0$, consider the function

$$S_0(y, x) := -y + h(x),$$

and define

$$\dot{S}_0(y^{(1)}, x, u) := \frac{\partial S_0}{\partial y} y^{(1)} + \frac{\partial S_0}{\partial x} [f(x) + g(x)u].$$

The function $\dot{S}_0(y^{(1)}, x, u)$ thus defined is an affine function of u and can be written as

$$\dot{S}_0(y^{(1)}, x, u) = L_1(y^{(1)}, x) + M_1(x)u$$

in which

$$L_1(y^{(1)}, x) = -y^{(1)} + \frac{\partial S_0}{\partial x} f(x) \qquad M_1(x) = \frac{\partial S_0}{\partial x} g(x).$$

Assume that $M_1(x)$ has constant rank ρ_1 and that there exists a fixed set of ρ_1 rows (empty if $\rho_1 = 0$) that are linearly independent for all x. After a permutation of rows in $S_0(y, x)$ if necessary, let such rows be the first ρ_1 rows of $M_1(x)$. Partition $L_1(y^{(1)}, x)$ and $M_1(x)$ as

$$L_1(y^{(1)}, x) = \begin{pmatrix} L_1'(y^{(1)}, x) \\ L_1''(y^{(1)}, x) \end{pmatrix} \qquad M_1(x) = \begin{pmatrix} M_1'(x) \\ M_1''(x) \end{pmatrix} \tag{9.5}$$

where in both cases the upper blocks have ρ_1 rows (and hence the lower blocks have $m - \rho_1$ rows). Consistently, let $\dot{S}_0(y^{(1)}, x, u)$ be partitioned as

$$\dot{S}_0(y^{(1)}, x, u) = \begin{pmatrix} \dot{S}_0'(y^{(1)}, x, u) \\ \dot{S}_0''(y^{(1)}, x, u) \end{pmatrix}.$$

As a consequence of the assumption on $M_1(x)$, if $\rho_1 > 0$ there exists a $(m - \rho_1) \times \rho_1$ matrix $F_1(x)$ such that

$$M_1''(x) = -F_1(x)M_1'(x). \tag{9.6}$$

Define

$$S_1(y^{(1)}, x) := L_1''(y^{(1)}, x) + F_1(x)L_1'(y^{(1)}, x),$$

and observe that

$$\begin{pmatrix} I_{\rho_1} & 0 \\ F_1(x) & I_{m-\rho_1} \end{pmatrix} \begin{pmatrix} \dot{S}_0'(y^{(1)}, x, u) \\ \dot{S}_0''(y^{(1)}, x, u) \end{pmatrix} = \begin{pmatrix} L_1'(y^{(1)}, x) \\ S_1(y^{(1)}, x) \end{pmatrix} + \begin{pmatrix} M_1'(x) \\ 0 \end{pmatrix} u. \tag{9.7}$$

If $\rho_1 = 0$, the upper blocks in (9.5) do not exist and consequently

$$S_1(y^{(1)}, x) = L_1(y^{(1)}, x).$$

Step 2. Consider the function $S_1(y^{(1)}, x)$ defined at step 1 and define

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) := \frac{\partial S_1}{\partial y^{(1)}} y^{(2)} + \frac{\partial S_1}{\partial x}[f(x) + g(x)u].$$

The function $\dot{S}_1(y^{(1)}, y^{(2)}, x, u)$ thus defined is an affine function of u and can be written as

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) = L_2(y^{(1)}, y^{(2)}, x) + M_2(y^{(1)}, x)u$$

in which

$$L_2(y^{(1)}, y^{(2)}, x) = \frac{\partial S_1}{\partial y^{(1)}} y^{(2)} + \frac{\partial S_1}{\partial x} f(x) \qquad M_2(y^{(1)}, x) = \frac{\partial S_1}{\partial x} g(x).$$

Define[2]

$$J_2(y^{(1)}, x) := \begin{pmatrix} M_1'(x) \\ M_2(y^{(1)}, x) \end{pmatrix} \tag{9.8}$$

[2]For consistency, set also $J_1(x) = M_1(x)$.

and assume that this (square) matrix has constant rank ρ_2 and that there exists a fixed set of $\rho_2 - \rho_1$ rows (empty if $\rho_2 = \rho_1$) of $M_2(y^{(1)}, x)$ that, together with the first ρ_1 rows (that are linearly independent by assumption) form a linearly independent set for all $(y^{(1)}, x)$. After a permutation of rows in $S_1(y^{(1)}, x)$ if necessary, let such rows be the first $\rho_2 - \rho_1$ rows of $M_2(y^{(1)}, x)$. Partition $L_2(y^{(1)}, y^{(2)}, x)$ and $M_2(y^{(1)}, x)$ as

$$L_2(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} L_2'(y^{(1)}, y^{(2)}, x) \\ L_2''(y^{(1)}, y^{(2)}, x) \end{pmatrix} \qquad M_2(y^{(1)}, x) = \begin{pmatrix} M_2'(y^{(1)}, x) \\ M_2''(y^{(1)}, x) \end{pmatrix} \tag{9.9}$$

where in both cases the upper blocks have $\rho_2 - \rho_1$ rows (and hence the lower blocks have $m - \rho_2$ rows). Consistently, let $\dot{S}_1(y^{(1)}, y^{(2)}, x, u)$ be partitioned as

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) = \begin{pmatrix} \dot{S}_1'(y^{(1)}, y^{(2)}, x, u) \\ \dot{S}_1''(y^{(1)}, y^{(2)}, x, u) \end{pmatrix}.$$

As a consequence of the assumption on the matrix (9.8), if $\rho_2 > \rho_1 > 0$ there exist a $(m - \rho_2) \times \rho_1$ matrix $F_{21}(y^{(1)}, x)$ and a $(m - \rho_2) \times (\rho_2 - \rho_1)$ matrix $F_{22}(y^{(1)}, x)$ such that

$$M_2''(y^{(1)}, x) = -F_{21}(y^{(1)}, x)M_1'(x) - F_{22}(y^{(1)}, x)M_2'(y^{(1)}, x). \tag{9.10}$$

Define

$$\begin{aligned} S_2(y^{(1)}, y^{(2)}, x) := {} & L_2''(y^{(1)}, y^{(2)}, x) + F_{21}(y^{(1)}, x)L_1'(y^{(1)}, x) \\ & + F_{22}(y^{(1)}, x)L_2'(y^{(1)}, y^{(2)}, x), \end{aligned}$$

and observe that

$$\begin{aligned} & \begin{pmatrix} I_{\rho_1} & 0 & 0 \\ 0 & I_{\rho_2-\rho_1} & 0 \\ F_{21}(y^{(1)}, x) & F_{22}(y^{(1)}, x) & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} \dot{S}_0'(y^{(1)}, x, u) \\ \dot{S}_1'(y^{(1)}, y^{(2)}, x, u) \\ \dot{S}_1''(y^{(1)}, y^{(2)}, x, u) \end{pmatrix} = \\ & = \begin{pmatrix} L_1'(y^{(1)}, x) \\ L_2'(y^{(1)}, y^{(2)}, x) \\ S_2(y^{(1)}, y^{(2)}, x) \end{pmatrix} + \begin{pmatrix} M_1'(x) \\ M_2'(y^{(1)}, x) \\ 0 \end{pmatrix} u. \end{aligned} \tag{9.11}$$

If $\rho_2 > \rho_1$ but $\rho_1 = 0$, the term $F_{21}(y^{(1)}, x)M_1'(x)$ is missing in (9.10) and so is the term $F_{21}(y^{(1)}, x)L_1'(y^{(1)}, x)$ in the definition of $S_2(y^{(1)}, y^{(2)}, x)$. Moreover, formula (9.11) must be adapted in a obvious manner, removing the terms $\dot{S}_0'(y^{(1)}, x, u)$, $L_1'(y^{(1)}, x)$ and $M_1'(x)$ that do not exist. If $\rho_2 - \rho_1 = 0$ and $\rho_1 > 0$, the upper blocks in the partition of $L_2(y^{(1)}, y^{(2)}, x)$ and $M_2(y^{(1)}, x)$ do not exist. Consequently, (9.10) is rewritten as

$$M_2(y^{(1)}, x) = -F_{21}(y^{(1)}, x)M_1'(x)$$

and

$$S_2(y^{(1)}, y^{(2)}, x) = L_2(y^{(1)}, y^{(2)}, x) + F_{21}(y^{(1)}, x)L_1'(y^{(1)}, x).$$

If $\rho_2 = \rho_1 = 0$, also the upper blocks in (9.5) do not exist and consequently

$$S_2(y^{(1)}, y^{(2)}, x) = L_2(y^{(1)}, y^{(2)}, x).$$

Step k+1. Consider the function $S_k(y^{(1)}, \ldots, y^{(k)}, x)$ defined at step k and define

$$\dot{S}_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) := \frac{\partial S_k}{\partial y^{(1)}} y^{(2)} + \cdots + \frac{\partial S_k}{\partial y^{(k)}} y^{(k+1)} + \frac{\partial S_k}{\partial x}[f(x) + g(x)u].$$

The function $\dot{S}_k(y^{(1)}, \ldots, y^{(k+1)}, x, u)$ thus defined is an affine function of u and can be written as

$$\dot{S}_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) = L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) + M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)u$$

in which

$$L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) = \frac{\partial S_k}{\partial y^{(1)}} y^{(2)} + \cdots + \frac{\partial S_k}{\partial y^{(k)}} y^{(k+1)} + \frac{\partial S_k}{\partial x} f(x)$$

$$M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x) = \frac{\partial S_k}{\partial x} g(x).$$

Define

$$J_{k+1}(y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}, x) := \begin{pmatrix} M_1'(x) \\ \cdots \\ M_k'(y^{(1)}, \ldots, y^{(k-1)}, x) \\ M_{k+1}(y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}, x) \end{pmatrix} \tag{9.12}$$

and assume that this (square) matrix has constant rank ρ_{k+1} and that there exists a fixed set of $\rho_{k+1} - \rho_k$ rows (empty if $\rho_{k+1} = \rho_k$) of $M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)$ that, together with the first ρ_k rows (that are linearly independent by assumption) form a linearly independent set for all $(y^{(1)}, \ldots, y^{(k)}, x)$. After a permutation of rows in $S_k(y^{(1)}, \ldots, y^{(k)}, x)$ if necessary, let such rows be the first $\rho_{k+1} - \rho_k$ rows of $M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)$. Partition $L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x)$ and $M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)$ as

$$L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) = \begin{pmatrix} L_{k+1}'(y^{(1)}, \ldots, y^{(k+1)}, x) \\ L_{k+1}''(y^{(1)}, \ldots, y^{(k+1)}, x) \end{pmatrix}$$

$$M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x) = \begin{pmatrix} M_{k+1}'(y^{(1)}, \ldots, y^{(k)}, x) \\ M_{k+1}''(y^{(1)}, \ldots, y^{(k)}, x) \end{pmatrix}$$

where in both cases the upper blocks have $\rho_{k+1} - \rho_k$ rows (and hence the lower blocks have $m - \rho_{k+1}$ rows). Consistently, let $\dot{S}_k(y^{(1)}, \ldots, y^{(k+1)}, x, u)$ be partitioned as

$$\dot{S}_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) = \begin{pmatrix} \dot{S}'_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) \\ \dot{S}''_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) \end{pmatrix}.$$

As a consequence of the assumption on the matrix (9.12), if $\rho_{k+1} - \rho_k > 0$ there exist $(m - \rho_{k+1}) \times (\rho_i - \rho_{i-1})$ matrices $F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x)$, for $i = 1, \ldots, k+1$ (recall that $\rho_0 = 0$), such that

$$M''_{k+1}(y^{(1)}, \ldots, y^{(k)}, x) = -\sum_{i=1}^{k+1} F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) M'_i(y^{(1)}, \ldots, y^{(i-1)}, x). \tag{9.13}$$

Define

$$\begin{aligned} &S_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) \\ &\quad := L''_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) + \sum_{i=1}^{k+1} F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) L'_i(y^{(1)}, \ldots, y^{(i)}, x), \end{aligned}$$

and observe that

$$\begin{pmatrix} I_{\rho_1} & 0 & \cdots & 0 & 0 \\ 0 & I_{\rho_2-\rho_1} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & I_{\rho_{k+1}-\rho_k} & 0 \\ F_{k+1,1}(\cdot) & F_{k+1,2}(\cdot) & \cdots & F_{k+1,k+1}(\cdot) & I_{m-\rho_{k+1}} \end{pmatrix} \begin{pmatrix} \dot{S}'_0(y^{(1)}, x, u) \\ \dot{S}'_1(y^{(1)}, y^{(2)}, x, u) \\ \cdots \\ \dot{S}'_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) \\ \dot{S}''_k(y^{(1)}, \ldots, y^{(k+1)}, x, u) \end{pmatrix}$$
$$= \begin{pmatrix} L'_1(y^{(1)}, x) \\ L'_2(y^{(1)}, y^{(2)}, x) \\ \cdots \\ L'_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) \\ S_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) \end{pmatrix} + \begin{pmatrix} M'_1(x) \\ M'_2(y^{(1)}, x) \\ \cdots \\ M'_{k+1}(y^{(1)}, \ldots, y^{(k)}, x) \\ 0 \end{pmatrix} u. \tag{9.14}$$

If $\rho_{k+1} > \rho_k$ but $\rho_i - \rho_{i-1} = 0$ for some $i \leq k$, the term

$$F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) M'_i(y^{(1)}, \ldots, y^{(i-1)}, x)$$

is missing in (9.13) and so is the corresponding term in the definition of

$$S_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x).$$

Moreover, formula (9.14) must be adapted in a obvious manner, removing the terms $\dot{S}'_{i-1}(y^{(1)}, \ldots, y^{(i)}, x, u)$, $L'_i(y^{(1)}, \ldots, y^{(i)}, x)$ and $M'_i(y^{(1)}, \ldots, y^{(i-1)}, x)$ that do not exist. If $\rho_{k+1} - \rho_k = 0$, then the upper blocks in the partition of $L_{k+1}(y^{(1)}, \ldots,$

$y^{(k+1)}, x)$ and $M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)$ do not exist,

$$M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x) = -\sum_{i=1}^{k} F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) M_i'(y^{(1)}, \ldots, y^{(i-1)}, x),$$

and

$$\begin{aligned} & S_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) \\ & \quad := L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) + \sum_{i=1}^{k} F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) L_i'(y^{(1)}, \ldots, y^{(i)}, x), \end{aligned}$$

where the term $F_{k+1,i}(y^{(1)}, \ldots, y^{(k)}, x) L_i'(y^{(1)}, \ldots, y^{(i)}, x)$ may be missing if $\rho_i - \rho_{i-1} = 0$. In particular, if $\rho_1 = \cdots = \rho_{k+1} = 0$, then

$$S_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x) := L_{k+1}(y^{(1)}, \ldots, y^{(k+1)}, x).$$

Definition 9.1 The structure algorithm *has no singularities* if the matrix $J_1(x) := M_1(x)$ has constant rank ρ_1 and there exists a fixed set of ρ_1 rows (empty if $\rho_1 = 0$) that are linearly independent for all x and, for all $k = 1, 2, \ldots$, the matrix (9.12) has constant rank ρ_{k+1} and there exists a fixed set of $\rho_{k+1} - \rho_k$ rows of $M_{k+1}(y^{(1)}, \ldots, y^{(k)}, x)$ (empty if $\rho_{k+1} = \rho_k$) that, together with the first ρ_k rows of (9.12), form a linearly independent set for all $(y^{(1)}, \ldots, y^{(k)}, x)$.

Definition 9.2 The system (9.1) is *uniformly invertible* (in the sense of Singh) if the structure algorithm has no singularities and there exists a integer $k^* \leq n$ such that $\rho_{k^*} = m$.

Remark 9.1 Note that, in the above definition, the integer k^* is taken to be less than or equal to n, the dimension of the state space of (9.1). This is not a restriction, though. In fact, it will be shown later (see Proposition 9.2) that, if the structure algorithm has no singularities and $\rho_{k^*} = m$ for some k^*, then this integer k^* is necessarily less than or equal to n. Knowing such bound on k^* it is useful because, if the algorithm is run up to step n and it is found that $\rho_n < m$, then it is concluded that the system *is not* uniformly invertible. ◃

Note that, if the system is invertible in the sense of this definition, it is possible to write[3]

[3]Clearly, if at step k^* one finds that $\rho_{k^*} = m$, then the partitions of $L_{k^*}(\cdot)$ and $M_{k^*}(\cdot)$ are trivial partitions, in which the upper blocks coincide with the matrices themselves, and this explains the notation used in the following formula.

$$\begin{pmatrix} \dot{S}'_0(y^{(1)}, x, u) \\ \dot{S}'_1(y^{(1)}, y^{(2)}, x, u) \\ \cdots \\ \dot{S}'_{k^*-1}(y^{(1)}, \ldots, y^{(k^*)}, x, u) \end{pmatrix} = \begin{pmatrix} L'_1(y^{(1)}, x) \\ L'_2(y^{(1)}, y^{(2)}, x) \\ \cdots \\ L'_{k^*}(y^{(1)}, \ldots, y^{(k^*)}, x) \end{pmatrix} + \begin{pmatrix} M'_1(x) \\ M'_2(y^{(1)}, x) \\ \cdots \\ M'_{k^*}(y^{(1)}, \ldots, y^{(k^*-1)}, x) \end{pmatrix} u, \tag{9.15}$$

and the matrix that multiplies u on the right-hand side is a $m \times m$ matrix that is nonsingular for all $(y^{(1)}, \ldots, y^{(k^*-1)}, x)$.

With this in mind, consider again system (9.1), in which we assume that $f(x), g(x), h(x)$ are smooth, let $u : \mathbb{R} \to \mathbb{R}^m$ be a smooth[4] input function, let $x(0)$ be the value of the state at time $t = 0$, let $x(t)$ denote the value at time t of the resulting state trajectory and let $y(t) = h(x(t))$ denote the value at time t of resulting output. Clearly, for all t for which $x(t)$ is defined,

$$S_0(y(t), x(t)) = 0.$$

Since the derivative of $S_0(y(t), x(t))$ with respect to time coincides, by construction, with the function $\dot{S}_0(y^{(1)}(t), x(t), u(t))$, it is seen that also

$$\dot{S}_0(y^{(1)}(t), x(t), u(t)) = 0$$

for all t for which $x(t)$ is defined. This, in turn, in view of (9.7) implies that

$$S_1(y^{(1)}(t), x(t)) = 0$$

for all t for which $x(t)$ is defined. Continuing in this way, it is seen that, for any $k \geq 2$, the function $S_k(y^{(1)}(t), \ldots, y^{(k)}(t), x(t))$ vanishes for all t for which $x(t)$ is defined and so does its derivative with respect to time.

Thus, it is concluded from (9.15) that, for all t for which $x(t)$ is defined,

$$0 = \begin{pmatrix} L'_1(y^{(1)}(t), x(t)) \\ L'_2(y^{(1)}(t), y^{(2)}(t), x(t)) \\ \cdots \\ L'_{k^*}(y^{(1)}(t), \ldots, y^{(k^*)}(t), x(t)) \end{pmatrix} + \begin{pmatrix} M'_1(x(t)) \\ M'_2(y^{(1)}(t), x(t)) \\ \cdots \\ M'_{k^*}(y^{(1)}(t), \ldots, y^{(k^*-1)}(t), x(t)) \end{pmatrix} u(t).$$

Since the matrix that multiplies $u(t)$ is nonsingular, this relation can be used to recover explicitly the value of $u(t)$ as a function of the value of $x(t)$ and of the values of $y(t)$ and its derivatives with respect to time, up to order k^*. It is for this reason

[4] It suffices to choose a C^k function with k large enough so that all derivatives of $y(t)$ appearing in the developments which follow are defined and continuous.

that a system having the property indicated in Definition 9.2 is considered to be *left invertible*.

Remark 9.2 Observe that, in the case of a single-input single output-system, the structure algorithm has no singularities if and only if the system has uniform relative degree. In this case,

$$S_k(y^{(1)}, \dots, y^{(k)}, x) = -y^{(k)} + L_f^k h(x)$$

for all $k \le r - 1$,

$$\dot{S}_{r-1}(y^{(1)}, \dots, y^{(r)}, x) = -y^{(r)} + L_f^r h(x) + L_g L_f^{r-1} h(x) u,$$

and $J_r(y^{(1)}, \dots, y^{(r-1)}, x) = M_r(y^{(1)}, \dots, y^{(r-1)}, x) = L_g L_f^{r-1} h(x)$ has rank 1 for all x. ◁

Example 9.1 Consider the system with three inputs and three outputs

$$\begin{aligned}
\dot{x}_1 &= x_6 + u_1 \\
\dot{x}_2 &= x_5 + x_2 x_6 + x_2 u_1 \\
\dot{x}_3 &= x_6 + x_1 x_6 + x_1 u_1 \\
\dot{x}_4 &= u_2 \\
\dot{x}_5 &= x_4 \\
\dot{x}_6 &= x_8 \\
\dot{x}_7 &= u_3 \\
\dot{x}_8 &= x_7 + x_5 u_2,
\end{aligned}$$

$$\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_3.
\end{aligned}$$

The inversion algorithm proceeds as follows. In step 1, define

$$S_0(x, y) = \begin{pmatrix} -y_1 + x_1 \\ -y_2 + x_2 \\ -y_3 + x_3 \end{pmatrix}$$

and obtain

$$\dot{S}_0(y^{(1)}, x, u) = L_1(y^{(1)}, x) + M_1(x) u = \begin{pmatrix} -y_1^{(1)} + x_6 \\ -y_2^{(1)} + x_5 + x_2 x_6 \\ -y_3^{(1)} + x_6 + x_1 x_6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_1 & 0 & 0 \end{pmatrix} u.$$

The matrix

$$J_1(x) = M_1(x) = \begin{pmatrix} 1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_1 & 0 & 0 \end{pmatrix}$$

has rank $\rho_1 = 1$. Then

$$M_1'(x) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad M_1''(x) = \begin{pmatrix} x_2 & 0 & 0 \\ x_1 & 0 & 0 \end{pmatrix}$$

$$L_1'(y^{(1)}, x) = \begin{pmatrix} -y_1^{(1)} + x_6 \end{pmatrix}, \qquad L_1''(y^{(1)}) = \begin{pmatrix} -y_2^{(1)} + x_5 + x_2 x_6 \\ -y_3^{(1)} + x_6 + x_1 x_6 \end{pmatrix}$$

and

$$F_1(x) = \begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix},$$

so that

$$S_1(y^{(1)}, x) = \begin{pmatrix} -y_2^{(1)} + x_5 + x_2 x_6 \\ -y_3^{(1)} + x_6 + x_1 x_6 \end{pmatrix} + \begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix} (-y_1^{(1)} + x_6) = \begin{pmatrix} -y_2^{(1)} + x_5 + x_2 y_1^{(1)} \\ -y_3^{(1)} + x_6 + x_1 y_1^{(1)} \end{pmatrix}.$$

In step 2, we see that

$$\begin{aligned} \dot{S}_1(y^{(1)}, y^{(2)}, x, u) &= L_2(y^{(1)}, y^{(2)}, x) + M_2(y^{(1)}, x)u \\ &= \begin{pmatrix} -y_2^{(2)} + x_4 + (x_5 + x_2 x_6) y_1^{(1)} + x_2 y_1^{(2)} \\ -y_3^{(2)} + x_8 + x_6 y_1^{(1)} + x_1 y_1^{(2)} \end{pmatrix} + \begin{pmatrix} x_2 y_1^{(1)} & 0 & 0 \\ y_1^{(1)} & 0 & 0 \end{pmatrix} u. \end{aligned}$$

The matrix

$$J_2(y^{(1)}, x) = \begin{pmatrix} M_1'(x) \\ M_2(y^{(1)}, x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2 y_1^{(1)} & 0 & 0 \\ y_1^{(1)} & 0 & 0 \end{pmatrix}$$

has still rank $\rho_2 = \rho_1 = 1$. Hence, $M_2'(y^{(1)}, x)$ and $L_2'(y^{(1)}, y^{(2)}, x)$ do not exist,

$$M_2''(y^{(1)}, x) = \begin{pmatrix} x_2 y_1^{(1)} & 0 & 0 \\ y_1^{(1)} & 0 & 0 \end{pmatrix}$$

$$L_2''(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} -y_2^{(2)} + x_4 + (x_5 + x_2 x_6) y_1^{(1)} + x_2 y_1^{(2)} \\ -y_3^{(2)} + x_8 + x_6 y_1^{(1)} + x_1 y_1^{(2)} \end{pmatrix}$$

and

$$F_{21}(y^{(1)}, x) = \begin{pmatrix} -x_2 y_1^{(1)} \\ -y_1^{(1)} \end{pmatrix}.$$

This yields

$$S_2(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} -y_2^{(2)} + x_4 + (x_5 + x_2x_6)y_1^{(1)} + x_2y_1^{(2)} \\ -y_3^{(2)} + x_8 + x_6y_1^{(1)} + x_1y_1^{(2)} \end{pmatrix} + \begin{pmatrix} -x_2y_1^{(1)} \\ -y_1^{(1)} \end{pmatrix} (-y_1^{(1)} + x_6),$$

i.e.

$$S_2(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} -y_2^{(2)} + x_4 + x_5y_1^{(1)} + x_2y_1^{(2)} + x_2[y_1^{(1)}]^2 \\ -y_3^{(2)} + x_8 + x_1y_1^{(2)} + [y_1^{(1)}]^2 \end{pmatrix}.$$

In step 3, we observe that

$$\begin{aligned} \dot{S}_2(y^{(1)}, y^{(2)}, y^{(3)}, x, u) &= L_3(y^{(1)}, y^{(2)}, y^{(3)}, x) + M_3(y^{(1)}, y^{(2)}, x)u \\ &= L_3(y^{(1)}, y^{(2)}, y^{(3)}, x) + \begin{pmatrix} x_2y_1^{(2)} + x_2[y_1^{(1)}]^2 & 1 & 0 \\ y_1^{(2)} & x_5 & 0 \end{pmatrix} u. \end{aligned}$$

The matrix

$$J_3(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} M_1'(x) \\ M_3(y^{(1)}, y^{(2)}, x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2y_1^{(2)} + x_2[y_1^{(1)}]^2 & 1 & 0 \\ y_1^{(2)} & x_5 & 0 \end{pmatrix}$$

has now rank $\rho_3 = 2$. Hence

$$\begin{aligned} M_3'(y^{(1)}, y^{(2)}, x) &= \left(x_2y_1^{(2)} + x_2[y_1^{(1)}]^2 \ 1 \ 0\right) \\ M_3''(y^{(1)}, y^{(2)}, x) &= \left(y_1^{(2)} \ x_5 \ 0\right) \end{aligned}$$

and

$$\begin{aligned} F_{31}(y^{(1)}, y^{(2)}, x) &= x_5(x_2y_1^{(2)} + x_2[y_1^{(1)}]^2) - y_1^{(2)} \\ F_{33}(y^{(1)}, y^{(2)}, x) &= -x_5. \end{aligned}$$

This yields for $S_3(y^{(1)}, y^{(2)}, y^{(2)}, x)$ an expression of the form

$$\begin{aligned} S_3(y^{(1)}, y^{(2)}, y^{(2)}, x) = L_3''(y^{(1)}, y^{(2)}, y^{(3)}, x) &+ F_{31}(y^{(1)}, y^{(2)}, x)L_1'(y^{(1)}, x) \\ &+ F_{33}(y^{(1)}, y^{(2)}, x)L_3'(y^{(1)}, y^{(2)}, y^{(3)}, x). \end{aligned}$$

In step 4, one obtains

$$\dot{S}_3(y^{(1)}, \ldots, y^{(4)}, x, u) = L_4(y^{(1)}, \ldots, y^{(4)}, x) + M_4(y^{(1)}, y^{(2)}, y^{(3)}, x)u,$$

in which the (1×3) row vector $M_4(y^{(1)}, y^{(2)}, y^{(3)}, x)$ has the form

$$M_4(y^{(1)}, y^{(2)}, y^{(3)}, x) = \left(* \ * \ 1\right).$$

Hence, the matrix

$$J_4(y^{(1)}, y^{(2)}, y^{(3)}, x) = \begin{pmatrix} M_1'(x) \\ M_3'(y^{(1)}, y^{(2)}, x) \\ M_4(y^{(1)}, y^{(2)}, y^{(3)}, x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_2 y_1^{(2)} + x_2 [y_1^{(1)}]^2 & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

has rank $\rho_4 = 3$ and the algorithm terminates. The system is uniformly invertible. ◁

9.3 An Underlying Recursive Structure of the Derivatives of the Output

In the previous section, we have described the structure algorithm in rather general terms, to the purpose of identifying the required regularity assumptions and to introduce and motivate the property of invertibility. In this section, we revisit the various steps of the algorithm in more detail, showing that the various functions generated at each step can be given more explicit expressions, highlighting—in particular—the role of the individual components of the output and of their higher order derivatives with respect to time.

In what follows, given a vector $y \in \mathbb{R}^m$, for any pair of indices $1 \le i \le j \le m$, we use $y_{[i,j]}$ to denote the vector

$$y_{[i,j]} = \mathrm{col}(y_i, y_{i+1}, \ldots, y_j).$$

Set $H_1(x) = \dfrac{\partial S_0}{\partial x} f(x)$ and split it as

$$H_1(x) = \begin{pmatrix} H_1'(x) \\ H_1''(x) \end{pmatrix}$$

consistently with the partition used in (9.5). It follows that

$$\begin{aligned} L_1'(y^{(1)}, x) &= -y^{(1)}_{[1,\rho_1]} + H_1'(x) \\ L_1''(y^{(1)}, x) &= -y^{(1)}_{[\rho_1+1,m]} + H_1''(x). \end{aligned}$$

As a consequence, it is seen that $S_1(y^{(1)}, x)$ can be expressed in the form

$$S_1(y^{(1)}, x) = -y^{(1)}_{[\rho_1+1,m]} + Q_1(y^{(1)}_{[1,\rho_1]}, x)$$

in which[5]

$$Q_1(y^{(1)}_{[1,\rho_1]}, x) = H_1''(x) + F_1(x)[y^{(1)}_{[1,\rho_1]} + H_1'(x)].$$

Accordingly, $L_2(y^{(1)}, y^{(2)}, x)$ can be expressed in the form

$$L_2(y^{(1)}, y^{(2)}, x) = -y^{(2)}_{[\rho_1+1,m]} + H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x)$$

in which

$$H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) = \frac{\partial Q_1}{\partial y^{(1)}_{[1,\rho_1]}} y^{(2)}_{[1,\rho_1]} + \frac{\partial Q_1}{\partial x} f(x)$$

while $M_2(y^{(1)}, x)$ is seen to be independent of $y^{(1)}_{[\rho_1+1,m]}$, and accordingly rewritten as $\tilde{M}_2(y^{(1)}_{[1,\rho_1]}, x)$. Note that

$$\tilde{M}_2(y^{(1)}_{[1,\rho_1]}, x) = \frac{\partial Q_1}{\partial x} g(x).$$

All of the above yields

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) = -y^{(2)}_{[\rho_1+1,m]} + H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) + \tilde{M}_2(y^{(1)}_{[1,\rho_1]}, x)u.$$

Let now $H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x)$ and $\tilde{M}_2(y^{(1)}_{[1,\rho_1]}, x)$ be partitioned as in (9.9), which yields

$$L_2(y^{(1)}, y^{(2)}, x) = \begin{pmatrix} L_2'(y^{(1)}, y^{(2)}, x) \\ L_2''(y^{(1)}, y^{(2)}, x) \end{pmatrix} = \begin{pmatrix} -y^{(2)}_{[\rho_1+1,\rho_2]} + H_2'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) \\ -y^{(2)}_{[\rho_2+1,m]} + H_2''(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) \end{pmatrix}$$

and

$$M_2(y^{(1)}, x) = \begin{pmatrix} \tilde{M}_2'(y^{(1)}_{[1,\rho_1]}, x) \\ \tilde{M}_2''(y^{(1)}_{[1,\rho_1]}, x) \end{pmatrix}.$$

Since $M_2(y^{(1)}, x)$ is independent of $y^{(1)}_{[\rho_1+1,m]}$, so are the matrices $F_{21}(y^{(1)}, x)$ and $F_{22}(y^{(1)}, x)$. Letting the latter be denoted, respectively, as $\tilde{F}_{21}(y^{(1)}_{[1,\rho_1]}, x)$ and $\tilde{F}_{22}(y^{(1)}_{[1,\rho_1]}, x)$, it is concluded that

$$\tilde{M}_2''(y^{(1)}_{[1,\rho_1]}, x) = -\tilde{F}_{21}(y^{(1)}_{[1,\rho_1]}, x)M_1'(x) - \tilde{F}_{22}(y^{(1)}_{[1,\rho_1]}, x)\tilde{M}_2'(y^{(1)}_{[1,\rho_1]}, x)$$

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[5]If $\rho_1 = 0$, the formula that follows must be replaced by

$$Q_1(y^{(1)}_{[1,\rho_1]}, x) = H_1(x).$$

while $S_2(y^{(1)}, y^{(2)}, x)$ can be written as

$$S_2(y^{(1)}, y^{(2)}, x) = -y^{(2)}_{[\rho_2+1,m]} + Q_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x)$$

in which[6]

$$\begin{aligned} Q_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x) = H_2''(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) + \tilde{F}_{21}(y^{(1)}_{[1,\rho_1]}, x)[-y^{(1)}_{[1,\rho_1]} + H_1'(x)] \\ + \tilde{F}_{22}(y^{(1)}_{[1,\rho_1]}, x)[-y^{(2)}_{[\rho_1+1,\rho_2]} + H_2'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x)]. \end{aligned}$$

Continuing in this way, it is seen that $L_3(y^{(1)}, y^{(2)}, y^{(3)}, x)$ can be expressed in the form

$$L_3(y^{(1)}, y^{(2)}, y^{(3)}, x) = -y^{(3)}_{[\rho_2+1,m]} + H_3(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, y^{(3)}_{[1,\rho_2]}, x),$$

while $M_3(y^{(1)}, y^{(2)}, x)$ is seen to be independent of $y^{(1)}_{[\rho_1+1,m]}$ and of $y^{(2)}_{[\rho_2+1,m]}$, and accordingly rewritten as $\tilde{M}_3(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x)$. As a consequence $S_3(y^{(1)}, y^{(2)}, y^{(3)}, x)$ can be expressed in the form

$$S_3(y^{(1)}, y^{(2)}, y^{(3)}, x) = -y^{(3)}_{[\rho_3+1,m]} + Q_3(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, y^{(3)}_{[1,\rho_3]}, x).$$

Similar expressions can be derived for higher values of k, yielding

$$\begin{aligned} \dot{S}'_{k-1}(y^{(1)}, \dots, y^{(k)}, x, u) = -y^{(k)}_{[\rho_{k-1}+1,\rho_k]} + H_k'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \dots, y^{(k-1)}_{[1,\rho_{k-1}]}, y^{(k)}_{[1,\rho_{k-1}]}, x) \\ + \tilde{M}_k'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \dots, y^{(k-1)}_{[1,\rho_{k-1}]}, x)u \end{aligned}$$

and

$$S_k(y^{(1)}, \dots, y^{(k)}, x) = -y^{(k)}_{[\rho_k+1,m]} + Q_k(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \dots, y^{(k)}_{[1,\rho_k]}, x).$$

Suppose now the system is invertible in the sense of the Definition 9.2. As explained in the conclusions of the last section, for each k the functions $\dot{S}'_{k-1}(y^{(1)}, \dots, y^{(k)}, x, u)$ and $S_k(y^{(1)}, \dots, y^{(k)}, x)$ vanish along the trajectories of system (9.1). Therefore, for each k

$$\begin{aligned} y^{(k)}_{[\rho_{k-1}+1,\rho_k]} = H_k'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \dots, y^{(k-1)}_{[1,\rho_{k-1}]}, y^{(k)}_{[1,\rho_{k-1}]}, x) \\ + \tilde{M}_k'(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \dots, y^{(k-1)}_{[1,\rho_{k-1}]}, x)u \end{aligned} \tag{9.16}$$

[6] If $\rho_2 - \rho_1 = 0$, the formula that follows must be replaced by

$$Q_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x) = H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) + \tilde{F}_{21}(y^{(1)}_{[1,\rho_1]}, x)[-y^{(1)}_{[1,\rho_1]} + H_1'(x)],$$

while if $\rho_2 = \rho_1 = 0$ we have

$$Q_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x) = H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x).$$

and

$$y^{(k)}_{[\rho_k+1,m]} = Q_k(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, \ldots, y^{(k)}_{[1,\rho_k]}, x). \tag{9.17}$$

In particular, writing all (9.16) together, it can be concluded that the identity

$$\begin{pmatrix} y^{(1)}_{[1,\rho_1]} \\ y^{(2)}_{[\rho_1+1,\rho_2]} \\ y^{(3)}_{[\rho_2+1,\rho_3]} \\ \vdots \\ y^{(k^*)}_{[\rho_{k*-1}+1,\rho_k*]} \end{pmatrix} = \begin{pmatrix} H'_1(x) \\ H'_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) \\ H'_3(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, y^{(3)}_{[1,\rho_2]}, x) \\ \vdots \\ H'_{k^*}(y^{(1)}_{[1,\rho_1]}, \ldots, y^{(k^*-1)}_{[1,\rho_{k^*-1}]}, y^{(k^*)}_{[1,\rho_{k^*-1}]}, x) \end{pmatrix}$$
$$+ \begin{pmatrix} M'_1(x) \\ \tilde{M}'_2(y^{(1)}_{[1,\rho_1]}, x) \\ \tilde{M}'_3(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x) \\ \vdots \\ \tilde{M}'_{k^*}(y^{(1)}_{[1,\rho_1]}, \ldots, y^{(k^*-1)}_{[1,\rho_{k^*-1}]}, x) \end{pmatrix} u$$

holds, in which the $m \times m$ matrix that multiplies u on the right-hand side is invertible for all values of the arguments.

Remark 9.3 It is worth stressing the underlying structure involved in the expressions of the higher order derivatives of y. In fact, the first derivative is expressed as

$$y^{(1)} = \begin{pmatrix} y^{(1)}_{[1,\rho_1]} \\ y^{(1)}_{[\rho_1+1,m]} \end{pmatrix} = \begin{pmatrix} H'_1(x) + M'_1(x)u \\ Q_1(y^{(1)}_{[1,\rho_1]}, x) \end{pmatrix}.$$

The lower $m - \rho_1$ components of the second derivative are expressed as

$$y^{(2)}_{[\rho_1+1,m]} = \begin{pmatrix} y^{(2)}_{[\rho_1+1,\rho_2]} \\ y^{(2)}_{[\rho_2+1,m]} \end{pmatrix} = \begin{pmatrix} H_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_1]}, x) + \tilde{M}'_2(y^{(1)}_{[1,\rho_1]}, x)u \\ Q_2(y^{(1)}_{[1,\rho_1]}, y^{(2)}_{[1,\rho_2]}, x) \end{pmatrix}.$$

and so on. ◁

9.4 Partial and Full Normal Forms

In this section, it is shown that if a system of the form (9.1) is invertible, in the sense of Definition 9.2, it is possible to define a special change of coordinates, by means of which the equations that describe the system can be transformed in equations having a special structure, that can be regarded as a multivariable version of the normal forms introduced in Chap. 6. The changes of coordinates in question are based on an important property of the functions $S_k(y^{(1)}, \ldots, y^{(k)}, x)$ constructed at the various steps of the structure algorithm.

More precisely, suppose a system is invertible in the sense of Definition 9.2, set

$$T_k(x) = S_{k-1}(0, \ldots, 0, x) \qquad \text{for } k = 1, 2, \ldots, k^*$$

and define

$$\Xi(x) = \begin{pmatrix} T_1(x) \\ T_2(x) \\ \cdots \\ T_{k^*}(x) \end{pmatrix}. \tag{9.18}$$

Note that the vector $\Xi(x)$ has

$$d = m + (m - \rho_1) + (m - \rho_2) + \cdots + (m - \rho_{k^*-1})$$

components. It can be proven that the smooth map

$$\begin{aligned} \Xi : \mathbb{R}^n &\to \mathbb{R}^d \\ x &\mapsto \xi = \Xi(x) \end{aligned}$$

has the following property.[7]

Proposition 9.1 *If system (9.1) is uniformly invertible, then the d components of $\Xi(x)$ have linearly independent differentials at each $x \in \mathbb{R}^n$.*

An immediate consequence of this is the following property.

Proposition 9.2 *If system (9.1) is uniformly invertible, then $d \leq n$ and $k^* \leq n$.*

Proof The first part of the proposition is a trivial consequence of Proposition 9.1. To prove the second part, observe that $m - \rho_i \geq 1$ for all $i = 0, \ldots, k^* - 1$. Thus,

$$k^* \leq \sum_{i=0}^{k^*-1} (m - \rho_i) = d \leq n.$$

$\triangleleft$

As a consequence of this property the map $\Xi(\cdot)$ can be used to define, at least locally, a partial set of new coordinates in the state space. We strengthen this property by requiring that the set in question can be completed in such a way that a globally defined change of coordinates is obtained.

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[7] See [1, pp. 109–112].

Assumption 9.1 System (9.1) is uniformly invertible, in the sense of Definition 9.2, and there exists a map

$$\begin{aligned} Z : \mathbb{R}^n &\to \mathbb{R}^{n-d} \\ x &\mapsto z = Z(x) \end{aligned}$$

such that the resulting map

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} Z(x) \\ \Xi(x) \end{pmatrix} := \Phi(x)$$

is a globally defined diffeomorphism.

We will determine in the sequel sufficient conditions under which the map $Z(\cdot)$ exists. For the time being, we observe that this Assumption implies that the map $\Xi(\cdot)$ is onto $\mathbb{R}^d$ and we proceed by showing how the d components of $\Xi(\cdot)$ can be used as a partial set of new coordinates.

To begin with, consider the function $T_1(x)$ and define

$$\dot{T}_1(x, u) := \frac{\partial T_1}{\partial x}[f(x) + g(x)u]$$

and observe that, by definition of $\dot{S}_0(y^{(1)}, x, u)$, we have

$$\dot{T}_1(x, u) = \dot{S}_0(0, x, u) = L_1(0, x) + M_1(x)u.$$

Assuming $\rho_1 > 0$, split $T_1(x)$ as

$$T_1(x) = \begin{pmatrix} T_1'(x) \\ T_1''(x) \end{pmatrix}$$

consistently with the partition used in (9.5), recall that the upper block has ρ_1 rows while the lower block has $m - \rho_1$ rows, and observe that

$$\dot{T}_1'(x, u) = L_1'(0, x) + M_1'(x)u, \tag{9.19}$$

in which the $\rho_1 \times m$ matrix $M_1'(x)$, by hypothesis, has rank ρ_1 for all x. Observe also that, by definition,

$$T_2(x) = S_1(0, x) = L_1''(0, x) + F_1(x)L_1'(0, x).$$

This formula can be fruitfully used to determine the expression of $\dot{T}_1''(x, u)$. In fact, using this formula and property (9.6), it is seen that

$$\begin{aligned} \dot{T}_1''(x, u) = \dot{S}_0''(0, x, u) &= L_1''(0, x) + M_1''(x)u \\ &= T_2(x) - F_1(x)L_1'(0, x) + M_1''(x)u \\ &= T_2(x) - F_1(x)[L_1'(0, x) + M_1'(x)u]. \end{aligned}$$

This expression relates the derivative with respect to time of the *lower* block of the set of new coordinates $T_1(x)$ to the set of new coordinates $T_2(x)$ and to the term $[L_1'(0,x) + M_1'(x)u]$ that, as shown above, expresses the derivative with respect to time of the *upper* block of the set of new coordinates $T_1(x)$. In other words

$$\dot{T}_1''(x,u) = T_2(x) - F_1(x)\dot{T}_1'(x,u). \tag{9.20}$$

The pair of expressions (9.19) and (9.20) hold if $\rho_1 > 0$. If $\rho_1 = 0$, we have instead

$$\dot{T}_1(x,u) = L_1(0,x)$$

and

$$S_1(0,x) = L_1(0,x)$$

from which it is seen that

$$\dot{T}_1(x,u) = T_2(x). \tag{9.21}$$

Consider now the function $T_2(x)$ and define

$$\dot{T}_2(x,u) := \frac{\partial T_2}{\partial x}[f(x) + g(x)u]$$

and observe that, by definition of $\dot{S}_1(y^{(1)}, y^{(2)}, x, u)$, we have

$$\dot{T}_2(x,u) = \dot{S}_1(0,0,x,u) = L_2(0,0,x) + M_2(0,x)u.$$

Assuming $\rho_2 > \rho_1 > 0$, split $T_2(x)$ as

$$T_2(x) = \begin{pmatrix} T_2'(x) \\ T_2''(x) \end{pmatrix}$$

consistently with the partition used in (9.9), recall that the upper block has $\rho_2 - \rho_1$ rows while the lower block has $m - \rho_2$ rows, and observe that

$$\dot{T}_2'(x,u) = L_2'(0,0,x) + M_2'(0,x)u, \tag{9.22}$$

and, by hypothesis, the $\rho_2 \times m$ matrix

$$\begin{pmatrix} M_1(x) \\ M_2'(0,x) \end{pmatrix}$$

has rank ρ_2 for all x. Observe also that, by definition,

$$T_3(x) = S_2(0,0,x) = L_2''(0,0,x) + F_{21}(0,x)L_1'(0,x) + F_{22}(0,x)L_2'(0,0,x).$$

This formula can be fruitfully used to determine the expression of $\dot{T}_2''(x,u)$. In fact, using this formula and property (9.10), it is seen that

$$\begin{aligned}\dot{T}_2''(x,u) &= L_2''(0,0,x) + M_2''(0,x)u\\ &= T_3(x) - F_{21}(0,x)L_1'(0,x) - F_{22}(0,x)L_2'(0,0,x) + M_2''(0,x)u\\ &= T_3(x) - F_{21}(0,x)[L_1'(0,x) + M_1'(x)u]\\ &\quad - F_{22}(0,x)[L_2'(0,0,x) + M_2'(0,x)u].\end{aligned}$$

Again, this establishes a relation between the derivative with respect to time of the *lower* block of the set of new coordinates $T_2(x)$, the set of new coordinates $T_3(x)$ and the terms $[L_1'(0,x) + M_1'(x)u]$ and $[L_2'(0,0,x) + M_2'(0,x)u]$ that express the derivatives with respect to time of the *upper* block of the set $T_1(x)$ and the *upper* block of the set $T_2(x)$. In other words

$$\dot{T}_2''(x,u) = T_3(x) - F_{21}(0,x)\dot{T}_1'(x,u) - F_{22}(0,x)\dot{T}_2'(x,u). \tag{9.23}$$

The pair of expressions (9.19) and (9.20) hold if $\rho_2 > \rho_1 > 0$. If $\rho_2 - \rho_1 > 0$ but $\rho_1 = 0$, the term $F_{21}(0,x)\dot{T}_1'(x,u)$ in the formula above is missing. If $\rho_2 - \rho_1 = 0$ and $\rho_1 > 0$, we have

$$S_2(0,0,x) = L_2(0,0,x) + F_{21}(0,x)L_1'(0,x)$$

and hence

$$\begin{aligned}\dot{T}_2(x,u) &= L_2(0,0,x) + M_2(0,x)u\\ &= T_3(x) - F_{21}(0,x)L_1'(0,x) + M_2(0,x)u\\ &= T_3(x) - F_{21}(0,x)[L_1'(0,x) + M_1'(x)u],\end{aligned}$$

that is

$$\dot{T}_2(x,u) = T_3(x) - F_{21}(0,x)\dot{T}_1'(x,u). \tag{9.24}$$

If $\rho_2 = \rho_1 = 0$, we have

$$\dot{T}_2(x,u) = T_3(x). \tag{9.25}$$

Clearly, similar results can be derived for all k's, up to $k = k^*$. In particular, so long as the function $T_{k^*}(x)$ is concerned, after having set

$$\dot{T}_{k^*}(x,u) = \frac{\partial T_{k^*}(x)}{\partial x}[f(x) + g(x)u]$$

it is found that (see (9.15))

$$\dot{T}_{k^*}(x,u) = L_{k^*}'(0,\ldots,0,x) + M_{k^*}'(0,\ldots,0,x)u. \tag{9.26}$$

The equations found in this way can be organized in various formats. Let

$$X_k = T_k(x), \qquad \text{for } k = 1, \dots, k^*,$$

define the new (sets of) coordinates. The vector X_k has dimension $m - \rho_{k-1}$, in which ρ_{k-1}—the rank of the matrix $J_{k-1}(\cdot)$—is an integer increasing (but not necessarily *strictly* increasing) with k (with ρ_1 not necessarily positive). With the sequence of integers $\rho_1, \rho_2, \dots$ we associate a set of integers $r_1, r_2, \dots, r_\ell$, with $r_\ell = k^*$, defined as follows. Let r_1 be the smallest integer such that $\rho_{r_1} > 0$. Let $r_2 > r_1$ be the smallest integer such that

$$\begin{aligned}
&\rho_{r_1} = \rho_{r_1+1} = \cdots = \rho_{r_2-1} \\
&\rho_{r_2} > \rho_{r_2-1},
\end{aligned}$$

and so on, until an integer $r_{\ell-1}$ is determined such that

$$\begin{aligned}
&\rho_{r_{\ell-1}} > \rho_{r_{\ell-1}-1} \\
&\rho_{r_{\ell-1}} = \rho_{r_{\ell-1}+1} = \cdots = \rho_{r_\ell-1} \\
&m = \rho_{r_\ell} > \rho_{r_\ell-1}.
\end{aligned}$$

Counting the number of components of the vectors X_k, it is seen that all r_1 vectors

$$X_1(x), X_2(x), \dots, X_{r_1}(x) \tag{9.27}$$

have m components, all $r_2 - r_1$ vectors

$$X_{r_1+1}(x), X_{r_1+2}(x), \dots, X_{r_2}(x) \tag{9.28}$$

have $m - \rho_1$ components, and so on, ending with the conclusion that the $r_\ell - r_{\ell-1}$ vectors

$$X_{r_{\ell-1}+1}(x), X_{r_{\ell-1}+2}(x), \dots, X_{r_\ell}(x) \tag{9.29}$$

have $m - \rho_{r_\ell-1}$ components.

Then, using arguments identical to those used above to deduce the expressions of $\dot{T}_1(x, u)$ and $\dot{T}_2(x, u)$, it is found that the sets of new coordinates $X_1, \dots, X_{r_1}$ satisfy equations of the form

$$\begin{aligned}
\dot{X}_1 &= X_2 \\
\dot{X}_2 &= X_3 \\
&\cdots \\
\dot{X}_{r_1-1} &= X_{r_1} \\
\dot{X}'_{r_1} &= a_1(x) + b_1(x)u \\
\dot{X}''_{r_1} &= X_{r_1+1} + R_{r_1+1,1}(x)[a_1(x) + b_1(x)u],
\end{aligned} \tag{9.30}$$

the sets of new coordinates $X_{r_1+1}, \ldots, X_{r_2}$ satisfy equations of the form

$$
\begin{aligned}
\dot{X}_{r_1+1} &= X_{r_1+2} + R_{r_1+2,1}(x)[a_1(x) + b_1(x)u] \\
\dot{X}_{r_1+2} &= X_{r_1+3} + R_{r_1+3,1}(x)[a_1(x) + b_1(x)u] \\
&\cdots \\
\dot{X}_{r_2-1} &= X_{r_2} + R_{r_2,1}(x)[a_1(x) + b_1(x)u] \\
\dot{X}'_{r_2} &= a_2(x) + b_2(x)u \\
\dot{X}''_{r_2} &= X_{r_2+1} + R_{r_2+1,1}(x)[a_1(x) + b_1(x)u] + R_{r_2+1,2}(x)[a_2(x) + b_2(x)u],
\end{aligned}
\tag{9.31}
$$

and so on, until it is found that the sets of new coordinates $X_{r_{\ell-1}+1}, \ldots, X_{r_\ell}$ satisfy equations of the form

$$
\begin{aligned}
\dot{X}_{r_{\ell-1}+1} &= X_{r_{\ell-1}+2} + \sum_{j=1}^{\ell-1} R_{r_{\ell-1}+2,j}(x)[a_j(x) + b_j(x)u] \\
\dot{X}_{r_{\ell-1}+2} &= X_{r_{\ell-1}+3} + \sum_{j=1}^{\ell-1} R_{r_{\ell-1}+3,j}(x)[a_j(x) + b_j(x)u] \\
&\cdots \\
\dot{X}_{r_\ell-1} &= X_{r_\ell} + \sum_{j=1}^{\ell-1} R_{r_\ell,j}(x)[a_j(x) + b_j(x)u] \\
\dot{X}_{r_\ell} &= a_\ell(x) + b_\ell(x)u.
\end{aligned}
\tag{9.32}
$$

Moreover, the output of the system is by definition expressed as

$$y = X_1.$$

In these equations, the coefficient matrices $R_{i,j}(x)$ coincide with the values at $(y^{(1)}, \ldots, y^{(i-2)}) = (0, \ldots, 0)$ of the matrices $-F_{i-1,r_j}(y^{(1)}, \ldots, y^{(i-2)}, x)$ determined at the various stages of the structure algorithm. Moreover, the matrices $b_i(x)$ are partitions of $J_{k^*}(0, \ldots, 0, x)$, that is

$$
\begin{pmatrix} b_1(x) \\ \cdots \\ b_\ell(x) \end{pmatrix} = J_{k^*}(0, \ldots, 0, x).
\tag{9.33}
$$

An alternative, a little more detailed, form can be obtained as follows. Define a set of integers $m_1, m_2, \ldots, m_\ell$ as

$$m_i := \rho_{r_i} - \rho_{r_{i-1}}$$

and note that $m_1 + m_2 + \cdots + m_\ell = m$. Then, for every $r_{i-1} + 1 \le j \le r_i$, split the vector X_j into $\ell - i + 1$ blocks as

$$X_j = \mathrm{col}(\xi_{i,j}, \dots, \xi_{\ell,j}).$$

in which $\xi_{k,j}$, for $i \le k \le \ell$, has m_k components. The splitting thus defined can be organized as shown in the following table, where all vectors on ith row have the same number m_i of components,

$$\begin{array}{llllllllllll}
\xi_{1,1} & \cdots & \xi_{1,r_1} & & & & & & & & & \\
\xi_{2,1} & \cdots & \xi_{2,r_1} & \xi_{2,r_1+1} & \cdots & \xi_{2,r_2} & & & & & & \\
\xi_{3,1} & \cdots & \xi_{3,r_1} & \xi_{3,r_1+1} & \cdots & \xi_{3,r_2} & \xi_{3,r_2+1} & \cdots & \xi_{3,r_3} & & & \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdots & & \\
\xi_{\ell,1} & \cdots & \xi_{\ell,r_1} & \xi_{\ell,r_1+1} & \cdots & \xi_{\ell,r_2} & \xi_{\ell,r_2+1} & \cdots & \xi_{\ell,r_3} & \cdots & \xi_{\ell,r_{\ell-1}+1} \cdots & \xi_{\ell,r_\ell}
\end{array}$$

Moreover, consistently with the partition of $X_1 = y$, split y into ℓ blocks as

$$y = \mathrm{col}(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_\ell),$$

in which $\bar{y}_i$ has m_i components.

It is easily seen that the ith "string" of (sets of) coordinates $\xi_{i,1} \dots, \xi_{i,r_i}$, all of which have m_i components, obeys equations of the form

$$\begin{aligned}
\dot{\xi}_{i,1} &= \xi_{i,2} \\
&\cdots \\
\dot{\xi}_{i,r_1-1} &= \xi_{i,r_1} \\
\dot{\xi}_{i,r_1} &= \xi_{i,r_1+1} + \delta^1_{i,r_1+1}(x)[a_1(x) + b_1(x)u] \\
&\cdots \\
\dot{\xi}_{i,r_2-1} &= \xi_{i,r_2} + \delta^1_{i,r_2}(x)[a_1(x) + b_1(x)u] \\
\dot{\xi}_{i,r_2} &= \xi_{i,r_2+1} + \delta^1_{i,r_2+1}(x)[a_1(x) + b_1(x)u] + \delta^2_{i,r_2+1}(x)[a_2(x) + b_2(x)u] \\
&\cdots \\
\dot{\xi}_{i,r_{i-1}} &= \xi_{i,r_{i-1}+1} + \sum_{j=1}^{i-1} \delta^j_{i,r_{i-1}+1}(x)[a_j(x) + b_j(x)u] \\
&\cdots \\
\dot{\xi}_{i,r_i-1} &= \xi_{i,r_i} + \sum_{j=1}^{i-1} \delta^j_{i,r_i}(x)[a_j(x) + b_j(x)u] \\
\dot{\xi}_{i,r_i} &= a_i(x) + b_i(x)u,
\end{aligned} \tag{9.34}$$

while

$$\bar{y}_i = \xi_{i,1}.$$

It is worth observing that, for any choice of vectors $v_i \in \mathbb{R}^{m_i}$, the set of equations

$$v_i = a_i(x) + b_i(x)u \qquad i = 1, \dots, \ell \tag{9.35}$$

has a unique solution u. In fact, setting

$$A(x) = \begin{pmatrix} a_1(x) \\ \cdots \\ a_\ell(x) \end{pmatrix}, \qquad B(x) = \begin{pmatrix} b_1(x) \\ \cdots \\ b_\ell(x) \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ \cdots \\ v_\ell \end{pmatrix} \tag{9.36}$$

the set in question is written as

$$v = A(x) + B(x)u,$$

where the matrix $B(x)$ is invertible. In fact (see (9.33)) the matrix $B(x)$ coincides with $J_{k^*}(0, \ldots, 0, x)$ and the latter is nonsingular a consequence of the assumption that the system is invertible. Clearly, the solution u of (9.35) is given by a control of the form (9.2), in which

$$\alpha(x) = -B^{-1}(x)A(x) \qquad \beta(x) = B^{-1}(x). \tag{9.37}$$

To conclude the analysis, it remains to discuss conditions under which the map $Z(\cdot)$ considered in Assumption 9.1 exists. We do this in the special case in which all m_i's in the partitions above are equal to 1, i.e., in the special case in which all $\xi_{j,k}$'s are one-dimensional, and hence $\ell = m$.[8] Using the vector $\alpha(x)$ and the matrix $\beta(x)$ defined in (9.37), define $\tilde{f}(x)$ and $\tilde{g}(x)$ as in (9.3). Then, define recursively a set of d vector fields as follows

$$\begin{aligned} Y_m^1(x) &= \tilde{g}_m(x) \\ Y_m^i(x) &= (-1)^{i-1}\mathrm{ad}_{\tilde{f}}^{i-1}\tilde{g}_m(x) \qquad \text{for } i = 2, \ldots, r_m, \end{aligned}$$

$$Y_{m-1}^1(x) = \tilde{g}_{m-1}(x) - \sum_{i=1}^{r_m - r_{m-1}} \delta_{m, r_m - i+1}^{m-1}(x) Y_m^{i+1}$$

$$Y_{m-1}^i = (-1)^{i-1}\mathrm{ad}_{\tilde{f}}^{i-1} Y_{m-1}^1(x) \qquad \text{for } i = 2, \ldots, r_{m-1},$$

$$Y_{m-2}^1(x) = \tilde{g}_{m-2}(x) - \sum_{i=1}^{r_m - r_{m-2}} \delta_{m, r_m - i+1}^{m-2}(x) Y_m^{i+1} - \sum_{i=1}^{r_{m-1} - r_{m-2}} \delta_{m-1, r_{m-1} - i+1}^{m-2}(x) Y_{m-1}^{i+1}$$

$$Y_{m-2}^i = (-1)^{i-1}\mathrm{ad}_{\tilde{f}}^{i-1} Y_{m-2}^1(x) \qquad \text{for } i = 2, \ldots, r_{m-2},$$

and so on.

[8] The general case only requires appropriate notational adaptations.

Then, the following result holds, which is the multivariable version of Proposition 6.3.[9]

Proposition 9.3 *Suppose that the vector fields*

$$Y_j^i(x), \qquad 1 \le j \le m, \qquad 1 \le i \le r_j$$

are complete. Then, the set

$$Z^* = \{x \in \mathbb{R}^n : \Xi(x) = 0\} \tag{9.38}$$

is a connected smooth embedded submanifold of $\mathbb{R}^n$, of codimension d. Suppose Z^* is diffeomorphic to $\mathbb{R}^{n-d}$. Then, there exists a map

$$\begin{aligned} Z : \mathbb{R}^n &\to \mathbb{R}^{n-d} \\ x &\mapsto z = Z(x) \end{aligned}$$

such that the resulting map

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} Z(x) \\ \Xi(x) \end{pmatrix} := \Phi(x)$$

is a globally defined diffeomorphism.

In what follows, the sets of equations (9.30), (9.31), ..., (9.32), or—alternatively—the sets of equations (9.34) for $i = 1, \ldots, \ell$ are referred to as a *partial normal form*, because—if $d < n$—the components of $\Xi(x)$ are simply a partial set of new coordinates.

Remark 9.4 The partial normal form, among other things, plays an important role in the problem of finding the input that forces the output to remain identically zero. In fact, from the set (9.30), in which $X_1(t) = y(t)$, it is seen that if $y(t)$ is identically zero, so are $X_1(t)$ and $X_2(t), \ldots, X_{r_1}(t)$. Then, from the last two equations of this set, in which $X'_{r_1}(t)$ and $X''_{r_1}(t)$ are partitions of $X_{r_1}(t)$, it is seen that

$$a_1(x) + b_1(x)u = 0,$$

and also that $X_{r_1+1}(t)$ is identically zero. Using these two facts in the set (9.31), it is seen that $X_{r_1+2}(t), \ldots, X_{r_2}(t)$ are all identically zero. Then, from the last two equations of this set it is seen that

$$a_2(x) + b_2(x)u = 0,$$

[9] For a proof, see [2] and also [1, pp. 114–115]. Related results can also be found in [11].

and that $X_{r_2+1}(t)$ is also identically zero. Continuing in this way, it is concluded that, if $y(t)$ is identically zero, then $\Xi(x(t)) = 0$, i.e., $x(t)$ evolves on the set Z^* defined by (9.38) and the input u necessarily satisfies

$$0 = A(x) + B(x)u.$$

Thus, $u(t) = u^*(x(t))$ in which $u^*(x)$ is the function (compare with (9.37))

$$u^*(x) = -B^{-1}(x)A(x).$$

The submanifold Z^* is an invariant manifold of the autonomous system

$$\dot{x} = f(x) + g(x)u^*(x)$$

and the restriction of this system to Z^* is a $n - d$ dimensional system whose dynamics characterize the internal dynamics forced in (9.1) when input (and initial conditions) are chosen in such a way as to force the output to remain identically zero. These dynamics are the *zero dynamics* of the multivariable system (9.1). ◁

If a *full* normal form is sought, it is necessary to consider also the complementary coordinates $z = Z(x)$, which clearly obey equations of the form[10]

$$\dot{z} = L_f Z(x) + L_g Z(x)u. \tag{9.39}$$

The latter, for consistence, can be conveniently rewritten as

$$\dot{z} = f_0(z, \xi) + g_0(z, \xi)u, \tag{9.40}$$

in which $f_0(z, \xi)$ and $g_0(z, \xi)$ are defined by

$$f_0(z, \xi) = L_f Z(x)\Big|_{x=\Phi^{-1}(z,\xi)}, \qquad g_0(z, \xi) = L_g Z(x)\Big|_{x=\Phi^{-1}(z,\xi)}.$$

The union of (9.40) and all (9.34) characterizes the normal form of the system. Note that if a complementary set $z = Z(x)$ of coordinates can be found such that $L_g Z(x) = 0$, then Eq. (9.40) reduces to

$$\dot{z} = f_0(z, \xi), \tag{9.41}$$

and the associated normal form will be referred to as a *strict* normal form.

[10]The notation used in this formula is defined as follows. Recall that $Z(x)$ is a $(n - d) \times 1$ vector, whose ith component is the function $z_i(x)$, and that $g(x)$ is a $n \times m$ matrix, whose jth column is the vector field $g_j(x)$. The vector $L_f Z(x)$ is the $(n - d) \times 1$ vector whose ith entry is $L_f z_i(x)$ and the matrix $L_g Z(x)$ is the $(n - d) \times m$ matrix whose entry on the ith row and jth column is $L_{g_j} z_i(x)$.

Remark 9.5 Note that, while Assumption 9.1, which includes the assumption that the system is uniformly invertible, implies the existence of a globally defined normal form (9.40)–(9.34), the converse is not true in general. In fact, a system possessing a normal form having the structure indicated above may fail to be uniformly invertible. This is shown, for instance, in the following simple example. ◁

Example 9.2 Consider the system, with $x \in \mathbb{R}^3$, $u \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, described by

$$
\begin{aligned}
\dot{x}_1 &= a_1(x) + b_1(x)u\\
\dot{x}_2 &= x_3 + \delta(x)[a_1(x) + b_1(x)u]\\
\dot{x}_3 &= a_2(x) + b_2(x)u\\
y_1 &= x_1\\
y_2 &= x_2
\end{aligned}
$$

in which

$$B(x) = \begin{pmatrix} b_1(x)\\ b_2(x) \end{pmatrix}$$

is a 2×2 matrix, nonsingular for all x. This system is already expressed in normal form, which can be simply seen by setting $\xi_{11} = x_1, \xi_{21} = x_2, \xi_{22} = x_3$. To check whether the system is uniformly invertible, we run the inversion algorithm. Beginning with

$$S_0 = \begin{pmatrix} -y_1 + x_1\\ -y_2 + x_2 \end{pmatrix}$$

we see that

$$L_1 = \frac{\partial S_0}{\partial y} y^{(1)} + \frac{\partial S_0}{\partial x} f(x) = \begin{pmatrix} -y_1^{(1)} + a_1(x)\\ -y_2^{(1)} + x_3 + \delta(x)a_1(x) \end{pmatrix}$$

$$M_1 = \frac{\partial S_0}{\partial x} g(x) = \begin{pmatrix} b_1(x)\\ \delta(x)b_1(x) \end{pmatrix}.$$

The matrix M_1 has rank $\rho_1 = 1$, and its the first row is nowhere zero. Thus,

$$
\begin{aligned}
M_1' &= b_1(x)\\
M_2'' &= \delta(x)b_1(x)\\
L_1' &= -y_1^{(1)} + a_1(x)\\
L_2'' &= -y_2^{(1)} + x_3 + \delta(x)a_1(x)
\end{aligned}
$$

and we can take

$$F_1(x) = -\delta(x).$$

Accordingly

$$\begin{aligned} S_1 &= L_2'' + F_1 L_1' = -y_2^{(1)} + x_3 + \delta(x)a_1(x) - \delta(x)[-y_1^{(1)} + a_1(x)] \\ &= -y_2^{(1)} + x_3 + \delta(x)y_1^{(1)}. \end{aligned}$$

At step 2, we have

$$L_2 = \frac{\partial S_1}{\partial y^{(1)}} y^{(2)} + \frac{\partial S_1}{\partial x} f(x)$$

and

$$\begin{aligned} M_2 = \frac{\partial S_1}{\partial x} g(x) &= b_2(x) + \left[\frac{\partial \delta}{\partial x_1} b_1(x) + \frac{\partial \delta}{\partial x_2}\delta(x) b_1(x) + \frac{\partial \delta}{\partial x_3} b_2(x)\right] y_1^{(1)} \\ &= \left[\frac{\partial \delta}{\partial x_1} + \frac{\partial \delta}{\partial x_2}\delta(x)\right] y_1^{(1)} b_1(x) + \left[1 + \frac{\partial \delta}{\partial x_3} y_1^{(1)}\right] b_2(x). \end{aligned}$$

As a consequence, the matrix (9.8) can be expressed as

$$J_2 = \begin{pmatrix} 1 & 0 \\ \left[\frac{\partial \delta}{\partial x_1} + \frac{\partial \delta}{\partial x_2}\delta(x)\right] y_1^{(1)} & \left[1 + \frac{\partial \delta}{\partial x_3} y_1^{(1)}\right] \end{pmatrix} B(x).$$

The rank of this matrix *is not* constant for all $x, y_1^{(1)}$. In fact, for $y_1^{(1)} = 0$ the rank is 2 (recall that $B(x)$ has rank 2), while if

$$\frac{\partial \delta}{\partial x_3} y_1^{(1)} = -1$$

the rank is 1. Thus, in general, the structure algorithm may have *singularities* and it cannot be claimed that the system is uniformly invertible in the sense of definition (9.36).

From this analysis it can also be concluded that the system is uniformly invertible *if and only if*

$$\frac{\partial \delta}{\partial x_3} = 0$$

i.e., if the coefficient $\delta(x)$ is *independent* of the component x_3 of the state vector x. We will return in more detail on this property in Sect. 11.4. ◁

Remark 9.6 Note that, while in a single-input single-output system the hypotheses of Proposition 6.3 guarantee the existence of a *strict* normal form, in a multi-input multi-output system the hypotheses of Proposition 9.3 do not guarantee the existence of a strict normal form. As a matter of fact, much stronger assumptions are needed to this end.[11] ◁

[11] See, again, [2] and also [1, pp. 115–118].

Example 9.3 Consider the case of a two-input two-output system, with $n = 4$. Suppose $L_g h(x)$ is nonsingular, so that the inversion algorithm terminates at step 1, with $\rho_1 = 2$. The partial normal form is simply a set of two equations

$$\begin{aligned}\dot{\xi}_{11} &= a_1(x) + b_1(x)u\\ \dot{\xi}_{21} &= a_2(x) + b_2(x)u\end{aligned}$$

in which

$$\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix} = y, \qquad \begin{pmatrix}a_1(x)\\ a_2(x)\end{pmatrix} = L_f h(x) \qquad \begin{pmatrix}b_1(x)\\ b_2(x)\end{pmatrix} = L_g h(x).$$

The existence of a strict normal form implies the existence of a map $Z : \mathbb{R}^4 \to \mathbb{R}^2$, with the properties indicated in Assumption 9.1, such that

$$L_g Z(x) = 0.$$

Letting $z_1(x), z_2(x)$ denote the two components of $Z(x)$, the condition in question reads

$$L_{g_j} z_i(x) = \frac{\partial z_i(x)}{\partial x} g_j(x) = 0 \qquad \text{for } i = 1, 2 \text{ and } j = 1, 2. \tag{9.42}$$

By virtue of a known property of the Lie bracket of vector fields, these conditions imply[12]

$$\frac{\partial z_i(x)}{\partial x}[g_1, g_2](x) = 0 \qquad \text{for } i = 1, 2.$$

which, together with both (9.42), yields

$$\frac{\partial Z(x)}{\partial x}\big(g_1(x)\ g_2(x)\ [g_1, g_2](x)\big) = 0. \tag{9.43}$$

This expression shows that the three columns of

$$G(x) = \big(g_1(x)\ g_2(x)\ [g_1, g_2](x)\big)$$

are vectors in the kernel of the 2×4 matrix

$$K(x) = \frac{\partial Z(x)}{\partial x}.$$

---

[12] If $\tau_1(x)$ and $\tau_2(x)$ are vector fields defined on $\mathbb{R}^n$ and $\lambda(x)$ is a real-valued function defined on $\mathbb{R}^n$, the property in question consists in the identity

$$L_{[\tau_1,\tau_2]}\lambda(x) = L_{\tau_1}L_{\tau_2}\lambda(x) - L_{\tau_2}L_{\tau_1}\lambda(x).$$

Therefore, if $L_{\tau_i}\lambda(x) = 0$ for $i = 1, 2$, then $L_{[\tau_1,\tau_2]}\lambda(x) = 0$ also.

If $\Phi(x)$ is a globally defined diffeomorphism, the two rows of the matrix $K(x)$ are linearly independent and hence its kernel has dimension 2. Thus, (9.43) can be fulfilled only if the three columns of the matrix $G(x)$ are linearly *dependent*, for each $x \in \mathbb{R}^4$. If these columns are independent, a map $Z(x)$ with the properties indicated above cannot exist. Now, it is easy to find examples of systems for which this is the case. Consider for instance the case of a system in which

$$g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} x_3 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad [g_1, g_2](x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad h(x) = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

and $f(x)$ is unspecified. In this system $L_g h(x)$ is the identity matrix, but the system does not admit globally defined *strict* normal forms. ◁

9.5 The Case of Input–Output Linearizable Systems

As observed at the end of Sect. 9.2, a single-input single-output system is invertible, in the sense of Definition 9.2, if and only if the system has uniform relative degree, in which case the derivative of order r of the output $y(t)$ with respect to time can be expressed in the form

$$y^{(r)}(t) = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t),$$

where the coefficient $L_g L_f^{r-1} h(x)$ in nonzero for all x. It is seen from this that if the system is controlled by a law of the form (9.2) in which

$$\alpha(x) = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)}, \qquad \beta(x) = \frac{1}{L_g L_f^{r-1} h(x)},$$

one obtains

$$y^{(r)}(t) = v(t).$$

In other words, the output $y(t)$ of the controlled system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ y &= h(x) \end{aligned}$$

is related to its input $v(t)$ by a *linear* differential equation. The effect of the feedback law is to force a *linear input–output* behavior on the resulting closed-loop system. For this reason, the law in question is usually referred to as an *input–output linearizing* feedback law.

The possibility of forcing, by means of an appropriate control, a linear input–output behavior on a nonlinear system is not limited to single-input single-output systems but holds in a more general context. In this section, we show that such possibility holds for any multivariable uniformly invertible system, provided that the functions determined at each step of the structure algorithm have suitable properties. To be able to address the problem in general terms, we begin by recalling a condition that identifies those nonlinear systems whose input–output behavior can be regarded as input–output behavior of a linear system. Such condition, that has been derived in the context of the realization theory for nonlinear systems,[13] can be expressed as follows. With system (9.1), which we recall is a system having m inputs and m outputs, let us associate a series of $m \times m$ matrices $W_k(x)$ defined as follows[14]

$$W_k(x) = L_g L_f^{k-1} h(x).$$

Proposition 9.4 *The input–output behavior of system (9.1) is that of a linear system if and only if, for $k = 1, \ldots, 2n$, the matrices $W_k(x)$ are independent of x.*

In view of this result, the problem of forcing—by means of a control—a linear input–output behavior on a nonlinear system can be cast as follows. Consider a control of the form (9.2), with nonsingular $\beta(x)$, and the associated closed-loop system (9.4), with $\tilde{f}(x)$ and $\tilde{g}(x)$ given by (9.3). The problem is to find a pair $(\alpha(x), \beta(x))$ such that, for $k = 1, \ldots, 2n$, the $m \times m$ matrix

$$\tilde{W}_k(x) = L_{\tilde{g}} L_{\tilde{f}}^{k-1} h(x)$$

is independent of x. Conditions for the existence of a solution to this problem are given in the following result.

Proposition 9.5 *Suppose system (9.1) satisfies Assumption 9.1. There exists a feedback law of the form (9.2), in which $\beta(x)$ is a $m \times m$ matrix invertible for all x, such that the resulting system (9.4) has a linear input–output behavior if and only if all matrices $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, determined by the structure algorithm are constant matrices.*[15]

Proof (*Sufficiency*) Suppose all $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, are constant matrices. In this case, for $i = 2, \ldots, \ell$, all $\delta^j_{i,k}(x)$ multipliers in (9.34), for $k = r_1 + 1, \ldots, r_i$ and $j = 1, \ldots, i - 1$, are simply real numbers, independent of x. Pick a control (9.2) with $\alpha(x)$ and $\beta(x)$ defined as in (9.37). Under such control law, system (9.34) becomes

[13] See [3].

[14] The matrix $W_k(x)$ is the $m \times m$ matrix whose entry on the ith row and jth column is $L_{g_j} L_f^{k-1} h_i(x)$. With reference to the functions $\tilde{f}(x)$ and $\tilde{\beta}(x)$ defined in (9.3), note that, for any $m \times 1$ vector $\lambda(x)$, we have $L_{\tilde{f}}\lambda(x) = L_f\lambda(x) + [L_g\lambda(x)]\alpha(x)$ and $L_{\tilde{g}}\lambda(x) = [L_g\lambda(x)]\beta(x)$.

[15] For consistency, we rewrite the matrix $F_1(x)$ determined at the first step of the algorithm as $F_{11}(x)$.

$$\begin{aligned}
\dot{\xi}_{i,1} &= \xi_{i,1} \\
&\cdots \\
\dot{\xi}_{i,r_1-1} &= \xi_{i,r_1} \\
\dot{\xi}_{i,r_1} &= \xi_{i,r_1+1} + \delta^1_{i,r_1+1} v_1 \\
&\cdots \\
\dot{\xi}_{i,r_2-1} &= \xi_{i,r_2} + \delta^1_{i,r_2} v_1 \\
\dot{\xi}_{i,r_2} &= \xi_{i,r_2+1} + \delta^1_{i,r_2+1} v_1 + \delta^2_{i,r_2+1} v_2 \\
&\cdots \\
\dot{\xi}_{i,r_{i-1}} &= \xi_{i,r_{i-1}+1} + \sum_{j=1}^{i-1} \delta^j_{i,r_{i-1}+1} v_j \\
&\cdots \\
\dot{\xi}_{i,r_i-1} &= \xi_{i,r_i} + \sum_{j=1}^{i-1} \delta^j_{i,r_i} v_j + \delta^2_{i,r_2+1} v_j \\
\dot{\xi}_{i,r_i} &= v_i,
\end{aligned} \tag{9.44}$$

with

$$\bar{y} = \xi_{i,1}.$$

The aggregate of all such systems, for $i = 1, \ldots, \ell$, is a linear d-dimensional time-invariant system with input v and output y. Thus, the input–output behavior of the controlled system is that of a linear system.

(*Necessity*) Suppose the feedback pair $(\alpha(x), \beta(x))$ solves the problem, set

$$\begin{aligned}
\hat{\beta}(x) &= \beta^{-1}(x) \\
\hat{\alpha}(x) &= -\beta^{-1}(x)\alpha(x)
\end{aligned}$$

and note that

$$\begin{aligned}
f(x) &= \tilde{f}(x) + \tilde{g}(x)\hat{\alpha}(x) \\
g(x) &= \tilde{g}(x)\hat{\beta}(x).
\end{aligned}$$

By assumption

$$K_k := L_{\tilde{g}} L_{\tilde{f}}^{k-1} h(x)$$

is a constant matrix for all $k = 1, \ldots 2n$. A simple induction argument[16] shows that

$$L_f^k h(x) = L_{\tilde{f}}^k h(x) + K_k \hat{\alpha}(x) + K_{k-1} L_f \hat{\alpha}(x) + \cdots + K_1 L_f^{k-1} \hat{\alpha}(x).$$

From this, taking the derivatives of both sides along $g(x)$ it follows that, for any $k > 1$,

[16]Simply take the derivative of both sides along $f(x)$, use the fact that $L_f L_{\tilde{f}}^k h(x) = L_{\tilde{f}}^{k+1} h(x) + L_{\tilde{g}} L_{\tilde{f}}^k h(x) \hat{\alpha}(x)$ and the fact that the K_i's are constant.

$$L_gL_f^{k-1}h(x) = L_{\tilde{g}}L_{\tilde{f}}^{k-1}h(x)\hat{\beta}(x) + K_{k-1}L_g\hat{\alpha}(x)$$
$$+ K_{k-2}L_gL_f\hat{\alpha}(x) + \cdots + K_1L_gL_f^{k-2}\hat{\alpha}(x)$$

i.e.,

$$L_gL_f^{k-1}h(x) = K_k\hat{\beta}(x) + K_{k-1}L_g\hat{\alpha}(x) + K_{k-2}L_g\hat{L}_f\alpha(x) + \cdots + K_1L_gL_f^{k-2}\hat{\alpha}(x). \tag{9.45}$$

The consequences of this expression on the structure algorithm are now examined.[17] Observe that, by definition

$$\dot{S}_0(y^{(1)}, x, u) = -y^{(1)} + L_fh(x) + L_gh(x)u$$

and that

$$M_1(x) = L_gh(x) = L_{\tilde{g}}h(x)\hat{\beta}(x) = K_1\hat{\beta}(x).$$

The matrix K_1 is constant, while the matrix $\hat{\beta}(x)$ is nonsingular for all x. Therefore, the matrix $M_1(x)$ has constant rank $\rho_1 = \text{rank}(K_1)$. After a permutation of rows if necessary, let the first ρ_1 rows of K_1 be linearly independent, partition K_1 as

$$K_1 = \begin{pmatrix} K_1' \\ K_1'' \end{pmatrix}$$

and let F_1 be a $(m - \rho_1) \times \rho_1$ *constant* matrix such that

$$K_1'' = -F_1K_1'.$$

Then, clearly,

$$\begin{pmatrix} M_1'(x) \\ M_1''(x) \end{pmatrix} = \begin{pmatrix} K_1'\hat{\beta}(x) \\ K_1''\hat{\beta}(x) \end{pmatrix}$$

and

$$S_1(y^{(1)}, x) = -\left(F_1 \ I_{m-\rho_1}\right) y^{(1)} + \left(F_1 \ I_{m-\rho_1}\right) L_fh(x).$$

As a consequence of this, since F_1 is constant, we have

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) = -\left(F_1 \ I_{m-\rho_1}\right) y^{(2)} + \left(F_1 \ I_{m-\rho_1}\right) L_f^2h(x) + \left(F_1 \ I_{m-\rho_1}\right) L_gL_fh(x)u$$

i.e.,

$$\begin{aligned} L_2(y^{(1)}, y^{(2)}, x) &= -\left(F_1 \ I_{m-\rho_1}\right) y^{(2)} + \left(F_1 \ I_{m-\rho_1}\right) L_f^2h(x) \\ M_2(y^{(1)}, x) &= \left(F_1 \ I_{m-\rho_1}\right) L_gL_fh(x)\,. \end{aligned}$$

---

[17] We consider in what follows the case in which $\rho_1 > 0$ and $\rho_2 > 0$. The other cases require simple adaptations of the arguments.

In the second of these identities, use (9.45) written for $k = 2$, to obtain

$$M_2(y^{(1)}, x) = \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} [K_2\hat{\beta}(x) + K_1 L_g \hat{\alpha}(x)] = \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_2 \hat{\beta}(x)$$

where the last passage is a consequence of the fact that, by definition

$$\begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_1 = 0.$$

Setting, to simplify notations,

$$\bar{M}_2 = \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_2.$$

it follows that the matrix (9.8) has the following expression

$$\begin{pmatrix} M_1'(x) \\ M_2(y^{(1)}, x) \end{pmatrix} = \begin{pmatrix} K_1' \\ \bar{M}_2 \end{pmatrix} \hat{\beta}(x),$$

from which it is seen that, since the matrix $\hat{\beta}(x)$ is nonsingular for all x, the matrix (9.8) has constant rank

$$\rho_2 = \text{rank} \begin{pmatrix} K_1' \\ \bar{M}_2 \end{pmatrix}. \tag{9.46}$$

Take a permutation the rows of $\bar{M}_2$, if necessary, so that the upper ρ_2 rows of the matrix in (9.46) are independent and split $\bar{M}_2$ as in (9.9). Then, there exist a $(m - \rho_2) \times \rho_1$ matrix F_{21} and a $(m - \rho_2) \times (\rho_2 - \rho_1)$ matrix F_{22}, both clearly independent of x, such that

$$\bar{M}_2'' = -F_{21}K_1' - F_{22}\bar{M}_2'.$$

Observe now that

$$S_2(\cdot) = \begin{pmatrix} F_{21} & 0_{(m-\rho_2)\times(m-\rho_1)} \end{pmatrix} [-y^{(1)} + L_f h(x)] + \begin{pmatrix} F_{22} & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} [-y^{(2)} + L_f^2 h(x)]$$

from which it follows that

$$M_3(\cdot) = \begin{pmatrix} F_{21} & 0_{(m-\rho_2)\times(m-\rho_1)} \end{pmatrix} L_g L_f h(x) + \begin{pmatrix} F_{22} & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} L_g L_f^2 h(x).$$

Using (9.45) written for $k = 2$ and $k = 3$ it is seen that

$$\begin{aligned} M_3(\cdot) = {} & [\begin{pmatrix} F_{21} & 0_{(m-\rho_2)\times(m-\rho_1)} \end{pmatrix} K_2 + \begin{pmatrix} F_{22} & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_3]\hat{\beta}(x) \\ & + [\begin{pmatrix} F_{21} & 0_{(m-\rho_2)\times(m-\rho_1)} \end{pmatrix} K_1 + \begin{pmatrix} F_{22} & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_2] L_g \hat{\alpha}(x) \\ & + \begin{pmatrix} F_{22} & I_{m-\rho_2} \end{pmatrix} \begin{pmatrix} F_1 & I_{m-\rho_1} \end{pmatrix} K_1 L_g L_f \hat{\alpha}(x) \end{aligned}$$

By construction

$$\begin{aligned}
&\left(F_{21}\ 0_{(m-\rho_2)\times(m-\rho_1)}\right) K_1 + \left(F_{22}\ I_{m-\rho_2}\right)\left(F_1\ I_{m-\rho_1}\right) K_2 \\
&\quad = F_{21}K_1' + \left(F_{22}\ I_{m-\rho_2}\right)\bar{M}_2 = 0 \\
&\left(F_1\ I_{m-\rho_1}\right) K_1 = 0
\end{aligned}$$

and therefore

$$M_3(\cdot) = \bar{M}_3\hat{\beta}(x)$$

for some constant matrix $\bar{M}_3$. As a consequence, the matrix (9.12), written for $k = 2$, is a matrix of the form

$$\begin{pmatrix} M_1'(x) \\ M_2'(y^{(1)}, x) \\ M_3(y^{(1)}, y^{(2)}, x) \end{pmatrix} = \begin{pmatrix} K_1' \\ \bar{M}_2' \\ \bar{M}_3 \end{pmatrix} \hat{\beta}(x),$$

from which it is deduced that $F_{31}(\cdot)$, $F_{32}(\cdot)$, $F_{33}(\cdot)$ are constant matrices. Continuing in the same way, it is not difficult to arrive at the conclusion that all matrices $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, determined by the structure algorithm are constant matrices. $\triangleleft$

Example 9.4 Consider the system with three inputs and three outputs

$$\begin{aligned}
\dot{x}_1 &= x_5^2 + u_1 \\
\dot{x}_2 &= x_4 + x_5^2 + u_1 \\
\dot{x}_3 &= x_6 + 3x_5^2 + 3u_1 \\
\dot{x}_4 &= u_3 \\
\dot{x}_5 &= x_1^3 + u_1 + (1 + x_2^2)u_2 \\
\dot{x}_6 &= x_5 + 2u_3,
\end{aligned}$$

$$\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_3.
\end{aligned}$$

The inversion algorithm proceeds as follows. In step 1, define

$$S_0(x, y) = \begin{pmatrix} -y_1 + x_1 \\ -y_2 + x_2 \\ -y_3 + x_3 \end{pmatrix}$$

and obtain

$$\dot{S}_0(y^{(1)}, x, u) = L_1(y^{(1)}, x) + M_1(x)u = \begin{pmatrix} -y_1^{(1)} + x_5^2 \\ -y_2^{(1)} + x_4 + x_5^2 \\ -y_3^{(1)} + x_6 + 3x_5^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} u.$$

The matrix

$$J_1(x) = M_1(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

has rank $\rho_1 = 1$. Then

$$M_1' = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad M_1'' = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$L_1'(y^{(1)}, x) = \begin{pmatrix} -y_1^{(1)} + x_5^2 \end{pmatrix}, \qquad L_1''(y^{(1)}) = \begin{pmatrix} -y_2^{(1)} + x_4 + x_5^2 \\ -y_3^{(1)} + x_6 + 3x_5^2 \end{pmatrix}$$

and

$$F_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

so that

$$S_1(y^{(1)}, x) = \begin{pmatrix} -y_2^{(1)} + x_4 + x_5^2 \\ -y_3^{(1)} + x_6 + 3x_5^2 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} (-y_1^{(1)} + x_5^2) = \begin{pmatrix} -y_2^{(1)} + y_1^{(1)} + x_4 \\ -y_3^{(1)} + 3y_1^{(1)} + x_6 \end{pmatrix}.$$

In step 2, we see that

$$\dot{S}_1(y^{(1)}, y^{(2)}, x, u) = L_2(y^{(1)}, y^{(2)}, x) + M_2 u = \begin{pmatrix} -y_2^{(2)} + y_1^{(2)} \\ -y_3^{(2)} + 3y_1^{(2)} + x_5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} u.$$

The matrix

$$J_2 = \begin{pmatrix} M_1' \\ M_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

has rank $\rho_2 = 2$. Hence,

$$M_2' = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \qquad M_2'' = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}$$

$$L_2'(y^{(1)}, y^{(2)} x) = \begin{pmatrix} -y_2^{(2)} + y_1^{(2)} \end{pmatrix}, \qquad L_2''(y^{(1)}, y^{(2)} x) = \begin{pmatrix} -y_3^{(2)} + 3y_1^{(2)} + x_5 \end{pmatrix}$$

and

$$F_{21} = 0, \qquad F_{22} = -2.$$

This yields

$$
\begin{aligned}
S_2(y^{(1)}, y^{(2)}, x) &= \left(-y_3^{(2)} + 3y_1^{(2)} + x_5\right) - 2\left(-y_2^{(2)} + y_1^{(2)}\right)\\
&= -y_3^{(2)} + y_1^{(2)} + 2y_2^{(2)} + x_5.
\end{aligned}
$$

In step 3, we obtain

$$
\begin{aligned}
\dot{S}_2(y^{(1)}, y^{(2)}, y^{(3)}, x, u) &= -y_3^{(3)} + y_1^{(3)} + 2y_2^{(3)} + x_1^3 + M_3(x)u\\
&= -y_3^{(3)} + y_1^{(3)} + 2y_2^{(3)} + x_1^3 + \begin{pmatrix}1 & (1+x_2)^2 & 0\end{pmatrix} u
\end{aligned}
$$

The matrix

$$
J_3(x) = \begin{pmatrix} M_1' \\ M_2' \\ M_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & (1+x_2)^2 & 0 \end{pmatrix}
$$

has rank $\rho_3 = 3$ for all $x \in \mathbb{R}^6$ and the algorithm terminates. The system is uniformly invertible. Moreover, since F_1 and F_{22} are constant matrices, the system can be transformed, by means of a feedback of the form (9.2), into a system whose input–output behavior is that of a linear system.

To see how this can be achieved, we determine the normal form of the system. According to the procedure described in Sect. 9.4, we define

$$
\begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{31} \end{pmatrix} = X_1 = S_0(0, x), \qquad \begin{pmatrix} \xi_{22} \\ \xi_{32} \end{pmatrix} = X_2 = S_1(0, x), \qquad \xi_{33} = X_3 = S_2(0, 0, x),
$$

In these coordinates, the system is described by the equations

$$
\begin{aligned}
\dot{\xi}_{11} &= x_5^2 + u_1\\
\dot{\xi}_{21} &= \xi_{22} + [x_5^2 + u_1]\\
\dot{\xi}_{22} &= u_3\\
\dot{\xi}_{31} &= \xi_{32} + 3[x_5^2 + u_1]\\
\dot{\xi}_{32} &= \xi_{33} + 2u_3\\
\dot{\xi}_{33} &= x_1^3 + u_1 + (1 + x_2^2)u_2,
\end{aligned}
$$

The feedback law obtained solving for u the equation

$$
\begin{pmatrix} x_5^2 + u_1 \\ u_3 \\ x_1^3 + u_1 + (1 + x_2^2)u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}
$$

forces a linear input–output behavior. ◁

9.6 The Special Case of Systems Having Vector Relative Degree

A multivariable nonlinear system of the form (9.1) is said to have *vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ at a point x° if:

(i) $L_{g_j} L_f^k h_i(x) = 0$ for all $1 \le i \le m$, for all $k < r_i - 1$, for all $1 \le j \le m$, and for all x in a neighborhood of x°,

(ii) the $m \times m$ matrix

$$B(x) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{pmatrix} \tag{9.47}$$

is nonsingular at $x = x^\circ$.[18]

Remark 9.7 It is clearly seen that this definition includes the one given in Chap. 6 for a single-input single-output nonlinear system. As far as the sequence of integers $\{r_1, r_2, \ldots, r_m\}$ is concerned, observe that each r_i is associated with the ith output channel of the system. By definition, for all $k < r_i - 1$, all m entries of the row vector

$$\left(L_{g_1} L_f^k h_i(x) \; L_{g_2} L_f^k h_i(x) \; \ldots \; L_{g_m} L_f^k h_i(x)\right)$$

are zero for all x in a neighborhood of x° and, for $k = r_i - 1$, in this row vector at least one entry is nonzero at x°. Thus, in view of condition (i), it can be said that, for each i, there is at least one choice of j such that the (single-input single-output) system having output y_i and input u_j has exactly relative degree r_i at x°. However, such property is only implied by (i) and (ii). As a matter of fact, property (ii) is much stronger, as it includes the assumption that the matrix $B(x^\circ)$ is *nonsingular*. $\triangleleft$

Remark 9.8 The nonsingularity of $B(x^\circ)$ may be interpreted as the appropriate multivariable version of the assumption that the coefficient $L_g L_f^{r-1} h(x^\circ)$ is nonzero in a single-input single-output system having vector relative degree at $x = x^\circ$. $\triangleleft$

System (9.1) is said to have *uniform vector relative degree* if there exists a sequence $r_1, r_2, \ldots, r_m$ such that system (9.1) has vector relative degree $\{r_1, r_2, \ldots, r_m\}$ at each $x^\circ \in \mathbb{R}^n$.

Proposition 9.6 *System (9.1) has uniform vector relative degree if and only if it is uniformly invertible, in the sense of Definition 9.2, and all matrices $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, determined by the structure algorithm are zero.*

[18] There is no abuse of notation in using here symbols (specifically $r_1, \ldots, r_m$ in (i) and (ii), and $B(x)$ in (9.47)) identical to symbols used earlier in Sect. 9.4. As a matter of fact, we will show in a moment that the notations are consistent.

Proof Suppose the system has uniform vector relative degree and let the outputs be reordered in such a way that $r_1 \le r_2 \le \cdots \le r_m$. Then, it is straightforward to see that, if $r_1 = 1$, then $J_1(x) = L_g h(x)$ while, if $r_1 > 1$,

$$J_1(\cdot) = \cdots = J_{r_1-1}(\cdot) = 0$$

and

$$J_{r_1}(\cdot) = L_g L_f^{r_1-1} h(x).$$

Let m_1 be the number of nonzero rows in $L_g L_f^{r_1-1} h(x)$. Then $r_1 = r_2 = \cdots = r_{m_1}$ and $r_{m_1+1} > r_{m_1}$. By assumption (ii), the upper m_1 rows of $J_{r_1}(\cdot)$ are linearly independent at each x, while the lower $m - m_1$ rows are identically zero. Therefore, all matrices $F_{i,j}(\cdot)$ with $i = 1, \ldots, r_1$ and $j = 1, \ldots, i$ are missing, with the exception of $F_{r_1,r_1}(\cdot)$ which is zero. As a consequence

$$S_{r_1}(\cdot) = L''_{r_1}(\cdot) = -[y^{(r_1)}]'' + [L_f^{r_1} h(x)]''.$$

Continuing in a similar way, it is easily concluded that the structure algorithm has no singularities, that all matrices $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, determined by the structure algorithm are either missing or zero, and that $J_{r_m}(\cdot)$ coincides with the matrix $B(x)$ defined in (9.47), which is nonsingular for all x. Hence the system is uniformly invertible.

Conversely, suppose all matrices $F_{ij}(\cdot)$, with $i = 1, \ldots, k^* - 1$ and $j = 1, \ldots i$, determined by the structure algorithm are zero. Then, in the set of equations (9.34), all multipliers $\delta^j_{i,k}(x)$, for $k = r_1 + 1, \ldots, r_i$ and $j = 1, \ldots, i - 1$, are zero. As a consequence,

$$\bar{y}_i^{(r_i)} = \xi_{i,1}^{(r_i)} = a_i(x) + b_i(x)u.$$

From this, using the fact that the matrix $B(x)$ in (9.36) is nonsingular for all x, it is immediate to conclude that the system has uniform vector relative degree, with

$$r_{m_1+\cdots+m_i+1} = \cdots = r_{m_1+\cdots+m_i+m_{i+1}}. \qquad \triangleleft$$

It is seen from this proposition that the class of those multivariable systems that have uniform vector relative degree is a subclass of those multivariable systems that can be input–output linearized, which in turn is a subclass of those multivariable systems that are invertible.

If Assumption 9.1 holds, the equations describing a system having uniform vector relative degree $\{r_1, r_2, \ldots, r_m\}$ can be transformed, by means of a globally defined diffeomorphism, into equations of the form

$$
\begin{aligned}
\dot{z} &= f_0(z,\xi) + g_0(z,\xi)u \\
\dot{\xi}_{1,1} &= \xi_{1,2} \\
&\cdots \\
\dot{\xi}_{1,r_1-1} &= \xi_{1,r_1} \\
\dot{\xi}_{1,r_1} &= a_1(x) + b_1(x)u \\
&\cdots \\
\dot{\xi}_{m,1} &= \xi_{m,2} \\
&\cdots \\
\dot{\xi}_{m,r_m-1} &= \xi_{m,r_m} \\
\dot{\xi}_{m,r_m} &= a_m(x) + b_m(x)u \\
y_1 &= \xi_{1,1} \\
&\cdots \\
y_m &= \xi_{m,1},
\end{aligned}
\tag{9.48}
$$

where the vector ξ in the upper equation is defined as

$$
\begin{aligned}
\xi &= \mathrm{col}(\xi_1, \xi_2, \ldots, \xi_m) \\
\xi_i &= \mathrm{col}(\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,r_i}) \qquad i = 1, 2, \ldots, m.
\end{aligned}
$$

Such equations can be put in compact form, by letting $\hat{A}$, $\hat{B}$, $\hat{C}$ denote the matrices

$$
\begin{aligned}
\hat{A} &= \mathrm{diag}(\hat{A}_1, \ldots, \hat{A}_m) \\
\hat{B} &= \mathrm{diag}(\hat{B}_1, \ldots, \hat{B}_m) \\
\hat{C} &= \mathrm{diag}(\hat{C}_1, \ldots, \hat{C}_m)
\end{aligned}
\tag{9.49}
$$

in which $\hat{A}_i \in \mathbb{R}^{r_i \times r_i}$, $\hat{B}_i \in \mathbb{R}^{r_i \times 1}$, $\hat{C}_i \in \mathbb{R}^{1 \times r_i}$ are defined as

$$
\hat{A}_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B}_i = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C}_i \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{9.50}
$$

With such notations, in fact, the equations in question can be rewritten in the form

$$
\begin{aligned}
\dot{z} &= f_0(z,\xi) + g_0(z,\xi)u \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[A(x) + B(x)u] \\
y &= \hat{C}\xi,
\end{aligned}
\tag{9.51}
$$

where $A(x)$ and $B(x)$ are the $m \times 1$ vector and, respectively, the $m \times m$ matrix defined in (9.36). Finally, setting

$$
\begin{aligned}
q(z,\xi) &= A(\Phi^{-1}(z,\xi)) \\
b(z,\xi) &= B(\Phi^{-1}(z,\xi))
\end{aligned}
$$

with $x = \Phi^{-1}(z, \xi)$ the inverse of the map $\Phi(x)$ considered in Assumption 9.1, the equations in question can be further rewritten as

$$\begin{aligned} \dot{z} &= f_0(z, \xi) + g_0(z, \xi)u \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)u] \\ y &= \hat{C}\xi. \end{aligned} \tag{9.52}$$

Note that the $m \times m$ matrix $b(z, \xi)$ is *nonsingular* for all (z, ξ).

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Chapter 10
Stabilization of Multivariable Nonlinear Systems: Part I

10.1 The Hypothesis of Strong Minimum-Phase

In this and in the following chapter we consider nonlinear systems having m inputs and m outputs, modeled by equations of the form (9.1) and we suppose that Assumption 9.1 holds. This assumption guarantees the existence of a globally defined change of coordinates $\tilde{x} = \Phi(x)$ in which $\tilde{x}$ can be split in two subsets $z \in \mathbb{R}^{n-d}$ and $\xi \in \mathbb{R}^d$. The set ξ—in turn—is split in ℓ subsets, each one of which consists of a string of r_i of components

$$\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,r_i}$$

which are seen to satisfy equations of the form (9.34).[1] The collection of all (9.34), for $i = 1, \ldots, \ell$, is a set of $m_1r_1 + m_2r_2 + \cdots + m_\ell r_\ell = d$ equations that characterizes what has been called a *partial* normal form. A *full* normal form is obtained by adding the Eq. (9.40) that models the flow of the complementary set z of new coordinates. It should be observed, in this respect, that a strict normal form, namely a normal form in which (9.40) takes the simpler structure (9.41), exists only under additional (and strong) assumptions (see Remark 9.6).

The problem addressed in the present and in the following chapter is the design of feedback laws yielding global stability or stability with guaranteed domain of attraction. To this end it is appropriate to consider a property that can be viewed as an extension, to multivariable systems, of the property of *strong minimum phase* considered in Definition 6.1 and that in fact reduces to such property in the case of a single-input single-output system.

Definition 10.1 Consider a system of the form (9.1), with $f(0) = 0$ and $h(0) = 0$. Suppose the system satisfies Assumption 9.1 so that a normal form can be globally

[1] Recall that $\xi_{ij} \in \mathbb{R}^{m_i}$ and that $m_1 + m_2 + \cdots + m_\ell = m$.

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_10

defined. The system is *strongly minimum-phase* if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any $x(0) \in \mathbb{R}^n$ and any admissible input function $u(\cdot) : [0,\infty) \to \mathbb{R}^m$, so long as $x(t)$ is defined the estimate

$$\|z(t)\| \le \beta(\|z(0)\|, t) + \gamma(\|\xi(\cdot)\|_{[0,t]}) \tag{10.1}$$

holds.

It is seen from the previous definition that, if a strict normal form exists, the role of the input $u(\cdot)$ in the previous definition is irrelevant and the indicated property reduces to the property that system (9.41), viewed as a system with state z and input ξ, is input-to-state stable. As in Chap. 6, it will be useful to consider also the stronger version of the property in question in which $\gamma(\cdot)$ and $\beta(\cdot)$ are bounded as in (6.18).

Definition 10.2 A system is *strongly—and also locally exponentially—minimum-phase* if, for any $x(0) \in \mathbb{R}^n$ and any admissible input function $u(\cdot) : [0,\infty) \to \mathbb{R}^m$, so long as $x(t)$ is defined an estimate of the form (10.1) holds, where $\beta(\cdot,\cdot)$ and $\gamma(\cdot)$ are a class $\mathcal{KL}$ function and, respectively, a class $\mathcal{K}$ function bounded as in (6.18).

Remark 10.1 If a strict normal form exists, in which (see (9.41))

$$\dot z = f_0(z, \xi),$$

the property of strong minimum phase can be—in principle—checked by seeking the existence of a continuously differentiable function $V_0 : \mathbb{R}^{n-d} \to \mathbb{R}$, three class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|z\|) \le V_0(z) \le \overline{\alpha}(\|z\|) \qquad \text{for all } z \in \mathbb{R}^{n-d}$$

and

$$\frac{\partial V_0}{\partial z} f_0(z,\xi) \le -\alpha(\|z\|) \qquad \text{for all } (z,\xi) \in \mathbb{R}^{n-d} \times \mathbb{R}^d \text{ such that } \|z\| \ge \chi(\|\xi\|).$$

However, if the system does not possess a strict normal form (for instance because conditions analogous to those outlined in Example 9.3 do not hold), a similar criterion for determining the property of strong minimum phase is not available. ◁

Example 10.1 Consider the system

$$\begin{aligned}
\dot z_1 &= -z_1 + z_2\xi_1 u_2\\
\dot z_2 &= -z_2 + \xi_2 - z_1\xi_1 u_2\\
\dot\xi_1 &= u_1\\
\dot\xi_2 &= u_2.
\end{aligned}$$

This is a system having vector relative degree $\{1, 1\}$ expressed in normal form. However, this system cannot be expressed in *strict* normal form. In fact, since

$$g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} z_2\xi_1 \\ -z_1\xi_1 \\ 0 \\ 1 \end{pmatrix}, \quad [g_1, g_2](x) = \begin{pmatrix} z_2 \\ -z_1 \\ 0 \\ 0 \end{pmatrix}$$

it is seen that the matrix

$$G(x) = \big(g_1(x)\ g_2(x)\ [g_1, g_2](x)\big)$$

has rank 3 for all nonzero z. Hence, as shown in Example 9.3, a strict normal form cannot exist.

This system is strongly minimum phase. In fact, taking the positive definite function $V(z) = z_1^2 + z_2^2$, it is seen that along the trajectories of (9.40)

$$\dot{V} = -2(z_1^2 + z_2^2) + 2z_2\xi_2.$$

Form this, it is readily seen that a bound of the form (10.1) holds, regardless of what the input $u(\cdot)$ is.[2] ◁

10.2 Systems Having Vector Relative Degree: Stabilization via Full State Feedback

In this section we consider a straightforward extension, to multivariable systems having vector relative degree, of the stabilization method presented in Sect. 6.4.

It follows from the analysis presented in Chap. 9 that, if a system has *uniform vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ and Assumption 9.1 holds, the equations describing such system can be transformed, by means of a globally defined diffeomorphism, into equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) + g_0(z, \xi)u \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)u] \\ y &= \hat{C}\xi, \end{aligned} \tag{10.2}$$

where the matrices $\hat{A}, \hat{B}, \hat{C}$ are defined as in (9.49)–(9.50) and $b(z, \xi)$ is a $m \times m$ matrix that is *nonsingular* for all (z, ξ). It is also known that, if $f(0) = 0$ and $h(0) = 0$, then

$$f_0(0, 0) = 0, \qquad q(0, 0) = 0,$$

so that the point $(z, \xi) = (0, 0)$ is an equilibrium.

Consider now the feedback law (compare with (6.19))

[2] Compare with Example B.4 in Appendix B.

$$u = [b(z, \xi)]^{-1}(-q(z, \xi) + \hat{K}\xi), \tag{10.3}$$

in which

$$\hat{K} = \mathrm{diag}(\hat{K}_1, \ldots, \hat{K}_m),$$

yielding the closed-loop system (compare with (6.20))

$$\begin{aligned} \dot{z} &= f_0(z, \xi) + g_0(z, \xi)[b(z, \xi)]^{-1}(-q(z, \xi) + \hat{K}\xi) \\ \dot{\xi} &= (\hat{A} + \hat{B}\hat{K})\xi, \end{aligned} \tag{10.4}$$

in which $(\hat{A} + \hat{B}\hat{K})$ is the $d \times d$ matrix

$$\hat{A} + \hat{B}\hat{K} = \mathrm{diag}(\hat{A}_1 + \hat{B}_1\hat{K}_1, \ldots, \hat{A}_m + \hat{B}_m\hat{K}_m).$$

Note also that, since each pair $(\hat{A}_i, \hat{B}_i)$ is a reachable pair, the eigenvalues of $\hat{A} + \hat{B}\hat{K}$ can be freely assigned by properly choosing the matrix $\hat{K}$.

Then, the following result—a multivariable version of Proposition 6.4—holds.

Proposition 10.1 *Suppose system (10.2) is strongly minimum phase. If the matrix $\hat{K}$ is such that $\sigma(\hat{A} + \hat{B}\hat{K}) \in \mathbb{C}^-$, the* state feedback *law (10.3) globally asymptotically stabilizes the equilibrium* $(z, \xi) = (0, 0)$.

Proof We begin by showing that, for any initial condition $(z(0), \xi(0))$ the trajectory of the closed loop systems is defined (and, actually, bounded) for all $t \geq 0$. As a matter of fact, suppose that this is not the case. Then, for any R, there exists a time T_R such that the trajectory is defined on the interval $[0, T_R]$ and

$$\text{either} \quad \|\xi(T_R)\| > R \qquad \text{or} \quad \|z(T_R)\| > R.$$

Since, on the time interval $[0, T_R]$, $\xi(t)$ is bounded as

$$\|\xi(t)\| \leq Me^{-\alpha t}\|\xi(0)\|$$

for some suitable $M > 0$ and $\alpha > 0$, if we pick $R > M\|\xi(0)\|$ the first inequality is contradicted. As far as the second one is concerned, using the property that the system is strongly minimum phase, it is seen that—since the trajectory is defined on the interval $[0, T_R]$—the estimate

$$\|z(T_R)\| \leq \beta(\|z(0)\|, T_R) + \gamma(\|\xi(\cdot)\|_{[0,T_R]}) \leq \beta(\|z(0)\|, 0) + \gamma(M\|\xi(0)\|)$$

holds. Thus, if

$$R > \beta(\|z(0)\|, 0) + \gamma(M\|\xi(0)\|),$$

also the second inequality is contradicted.

Having proven that the trajectory is defined for all $t \geq 0$, we use the estimate (10.1) to claim that the upper subsystem of (10.4), viewed as a system with state z and input ξ, is input-to-state stable. Since the lower subsystem is globally asymptotically stable, the conclusion follows from standard results.[3] ◁

Remark 10.2 From the arguments used in the previous proof, it is also readily seen that, if system (10.2) is strongly—and also locally exponentially—minimum phase, the equilibrium $(z, \xi) = (0, 0)$ of the closed loop system (10.4) is globally asymptotically and also *locally exponentially* stable. ◁

As observed right after Proposition 6.4, the feedback law (10.3), although intuitive and simple, is not useful in a practical context because it relies upon exact cancellation of nonlinear functions and requires the availability of the *full* state (z, ξ) of the system. We will see in the subsequent sections of this chapter how such limitations can—to some extent—be overcome.

10.3 A Robust "Observer"

We describe now a design method, based a work of Khalil and Freidovich,[4] according to which the feedback law (10.3) can be effectively replaced—to the purpose of achieving stability with a guaranteed domain of attraction—by a (dynamic) feedback law in which the availability of (z, ξ) and the knowledge of the functions $b(z, \xi)$ and $q(z, \xi)$ are not required.

To this end, we need to put a restriction on the matrix $b(z, \xi)$ appearing in (10.2). Specifically, we consider the case in which the matrix in question has the property indicated in the following assumption.

Assumption 10.1 There exist a constant nonsingular matrix $\mathbf{b} \in \mathbb{R}^{m \times m}$ and a number $0 < \delta_0 < 1$ such that

$$\max_{\Lambda \text{ diagonal}, \ \|\Lambda\| \leq 1} \|[b(z, \xi) - \mathbf{b}]\Lambda \mathbf{b}^{-1}\| \leq \delta_0 \qquad \text{for all } (z, \xi). \tag{10.5}$$

Remark 10.3 It is easy to see that Assumption 10.1 implies

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[3] See Theorem B.5 of Appendix B, and also [1].

[4] See [2] and also [3, 4]. The work [2] extends and improves the pioneering work of [5, 14], in which the idea of a "rough" estimation of higher derivatives of the output was first introduced to the purpose of *robustly* stabilizing via *dynamic output feedback*, with a guaranteed region of attraction, a system that is globally stabilizable via *partial state feedback*. Substantial extensions, in various directions, of this earlier work can be found in [6]. One of the improvements shown in [2] is that, with the aid of what is called "an extended observer", the design is rendered robust with respect to uncertainties in the terms $q(z, \xi)$ and $b(z, \xi)$. The exposition given here of the method of [2] follows the pattern of proof proposed in [7], for single-input single-output systems, and extended in [8] to the case of multi-input multi-output systems. For an extension to the case in which the hypothesis of minimum-phase does not hold, see [9].

$$b_{\min} \le \|b(z,\xi)\| \le b_{\max} \qquad \text{for all } (z,\xi) \tag{10.6}$$

for some pair $b_{\min} \le b_{\max}$. Actually, in case $m = 1$, Assumption 10.1 is *equivalent* to (10.6). In fact, if $m = 1$, Assumption 10.1 reduces to the assumption of existence of a number $\mathbf{b}$ such that

$$\left|\frac{b(z,\xi) - \mathbf{b}}{\mathbf{b}}\right| \le \delta_0 < 1$$

which is always possible if $b(z,\xi)$ is bounded as in (10.6). Thus, Assumption 10.1 can be regarded as a multivariable version of the (very reasonable) assumption that the so-called high-frequency gain coefficient $b(z,\xi)$, of a single-input single-output nonlinear system, is bounded from below and from above.[5] $\triangleleft$

With $\hat{K}$ defined as in Proposition 10.1, let $\psi(\xi,\sigma)$ be the function defined as

$$\psi(\xi,\sigma) = \mathbf{b}^{-1}[\hat{K}\xi - \sigma],$$

in which $\xi \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^m$, and $\mathbf{b}$ is a nonsingular matrix such that Assumption 10.1 holds.

Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function, that is a function characterized by the following properties (see also Sect. 7.5 in this respect):

(i) $g(s) = s$ if $|s| \le L$,
(ii) $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \le 1$,
(iii) $\lim_{s\to\infty} g(s) = L(1 + c)$ with $0 < c \ll 1$.

Note that, in order to simplify matters, the parameter L that characterizes the "saturation level" is not explicitly indicated in the proposed notation. To handle the case in which the argument is a vector $s \in \mathbb{R}^m$, we consider a function $G : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$G(s) = \mathrm{col}(g(s_1), g(s_2), \ldots, g(s_m))$$

in which $g(\cdot)$ is a fixed saturation function.

System (10.2) will be controlled by a control law of the form

$$u = G(\psi(\hat{\xi},\sigma)) = \begin{pmatrix} g(\psi_1(\hat{\xi},\sigma)) \\ g(\psi_2(\hat{\xi},\sigma)) \\ \cdots \\ g(\psi_m(\hat{\xi},\sigma)) \end{pmatrix} \tag{10.7}$$

in which

[5] In the control of systems having a vector relative degree, it is inevitable to put assumptions on the matrix $b(z,\xi)$, the so-called high-frequency gain matrix. Such assumptions are essentially meant to make sure that the various components of $b(x,\xi)u$ have the right sign. Different hypotheses have been proposed in the literature to this purpose, see e.g., [10–13]. We consider Assumption 10.1 a reasonable alternative.

$$\begin{aligned}\hat{\xi} &= \mathrm{col}(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_m) \\ \sigma &= \mathrm{col}(\sigma_1, \sigma_1, \ldots, \sigma_m)\end{aligned}$$

where, for $i = 1, 2, \ldots, m$,

$$\hat{\xi}_i = \mathrm{col}(\hat{\xi}_{i,1}, \hat{\xi}_{i,2}, \ldots, \hat{\xi}_{i,r_i}) \qquad \text{and} \qquad \sigma_i$$

are states of a dynamical system described by equations of the form

$$\begin{aligned}\dot{\hat{\xi}}_{i,1} &= \hat{\xi}_{i,2} + \kappa c_{i,r_i}(y_i - \hat{\xi}_{i,1}) \\ \dot{\hat{\xi}}_{i,2} &= \hat{\xi}_{i,3} + \kappa^2 c_{i,r_i-1}(y_i - \hat{\xi}_{i,1}) \\ &\cdots \\ \dot{\hat{\xi}}_{i,r_i-1} &= \hat{\xi}_{i,r_i} + \kappa^{r_i-1} c_{i,2}(y_i - \hat{\xi}_{i,1}) \\ \dot{\hat{\xi}}_{i,r_i} &= \sigma_i + \mathbf{b}_i G(\psi(\hat{\xi}, \sigma)) + \kappa^{r_i} c_{i,1}(y_i - \hat{\xi}_{i,1}) \\ \dot{\sigma}_i &= \kappa^{r_i+1} c_{i,0}(y_i - \hat{\xi}_{i,1}).\end{aligned} \tag{10.8}$$

In these equations, the coefficient κ and $c_{i,0}, c_{i,1}, \ldots, c_{i,r_i}$ are design parameters, while $\mathbf{b}_i$ is the ith row of the matrix $\mathbf{b}$.

The dynamical system thus defined has a structure similar—to some extent—to that of the observer considered in Chap. 7. In view of performing a similar convergence analysis, the states $\hat{\xi}_{i,1}, \hat{\xi}_{i,2}, \ldots, \hat{\xi}_{i,r_i}, \sigma_i$ will be replaced by variables defined as (compare with (7.14))

$$\begin{aligned}e_{i,1} &= \kappa^{r_i}(\xi_{i,1} - \hat{\xi}_{i,1}) \\ e_{i,2} &= \kappa^{r_i-1}(\xi_{i,2} - \hat{\xi}_{i,2}) \\ &\cdots \\ e_{i,r_i} &= \kappa(\xi_{i,r_i} - \hat{\xi}_{i,r_i}) \\ e_{i,r_i+1} &= q_i(z, \xi) + [b_i(z, \xi) - \mathbf{b}_i]G(\psi(\xi, \sigma)) - \sigma_i.\end{aligned} \tag{10.9}$$

Setting

$$\begin{aligned}e &= \mathrm{col}(e_1, \ldots, e_m) \\ e_i &= \mathrm{col}(e_{i,1}, e_{i,2}, \ldots, e_{i,r_i+1}), \qquad i = 1, \ldots, m\end{aligned}$$

and

$$\hat{\xi}^{\mathrm{ext}} = \mathrm{col}(\hat{\xi}, \sigma),$$

Equation (10.9) define a map

$$\begin{aligned}T : \mathbb{R}^{d+m} &\to \mathbb{R}^{d+m} \\ \hat{\xi}^{\mathrm{ext}} &\mapsto e = T(z, \xi, \hat{\xi}^{\mathrm{ext}})\end{aligned} \tag{10.10}$$

which we are going to show is *globally invertible*.

Lemma 10.1 *Suppose Assumption 10.1 holds. Then the map (10.10) is globally invertible.*

Proof Observe that the first r_i of the identities (10.9) can be trivially solved for $\hat{\xi}_{i,j}$, expressing it as a function of $e_{i,j}$ and $\xi_{i,j}$. To complete the inversion, it remains to find σ from the last identities. This can be achieved if $\mathbf{b}$ has suitable properties. Setting

$$\varsigma = \text{col}(e_{1,r_1+1}, e_{2,r_2+1}, \dots, e_{m,r_m+1}), \tag{10.11}$$

the last equations of (10.9), written for $i = 1, \dots, m$, can be organized in the single equation

$$\varsigma = q(z, \xi) + [b(z, \xi) - \mathbf{b}]G(\psi(\xi, \sigma)) - \sigma.$$

Adding and subtracting $\hat{K}\xi$ on the right-hand side and bearing in mind the definition of $\psi(\xi, \sigma)$, this becomes

$$\varsigma = q(z, \xi) - \hat{K}\xi + [b(z, \xi) - \mathbf{b}]G(\psi(\xi, \sigma)) + \mathbf{b}\psi(\xi, \sigma),$$

which in turn can be rewritten as

$$\mathbf{b}^{-1}[\hat{K}\xi - q(z, \xi) + \varsigma] = \mathbf{b}^{-1}[b(z, \xi) - \mathbf{b}]G(\psi(\xi, \sigma)) + \psi(\xi, \sigma). \tag{10.12}$$

Set now

$$\psi^*(z, \xi, \varsigma) = \mathbf{b}^{-1}[\hat{K}\xi - q(z, \xi) + \varsigma]$$

and consider a map $F : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$F(s) = \mathbf{b}^{-1}[b(z, \xi) - \mathbf{b}]G(s) + s. \tag{10.13}$$

With such notations, the relation (10.12) can be simply rewritten as

$$\psi^* = F(\psi).$$

Since $G(s)$ is bounded and so is $\|b(z, \xi)\|$, the map $F(s)$ is proper (that is, $\|F(s)\| \to \infty$ as $\|s\| \to \infty$). Thus, according to Hadamard's Theorem, the map $F(s)$ has a globally defined inverse if the Jacobian of $F(s)$ is everywhere nonsingular. The Jacobian of $F(s)$, in turn, has the following expression

$$\frac{\partial F}{\partial s} = \mathbf{b}^{-1}[b(z, \xi) - \mathbf{b}]G'(s) + I \tag{10.14}$$

in which $G'(s)$ denotes the matrix

$$G'(s) = \text{diag}(g'(s_1), g'(s_2), \dots, g'(s_m)).$$

The expression (10.14) can be further rewritten as

$$\frac{\partial F}{\partial s} = \mathbf{b}^{-1}\Big([b(z,\xi) - \mathbf{b}]G'(s) + \mathbf{b}\Big) = \mathbf{b}^{-1}(M(z,\xi,s) + I)\mathbf{b}$$

in which

$$M(z,\xi,s) = [b(z,\xi) - \mathbf{b}]G'(s)\mathbf{b}^{-1}. \tag{10.15}$$

Thus, the jacobian of $F(s)$ is nonsingular if and only if so is the matrix

$$M(z,\xi,s) + I. \tag{10.16}$$

Clearly, a sufficient condition for (10.16) to be invertible is that $\|M(z,\xi,s)\| < 1$.[6]

The property that $\|M(z,\xi,s)\| < 1$ is *implied* by Assumption 10.1. In fact, since $G'(s)$ is diagonal and satisfies $|G'(s)| \le 1$, we have (under such Assumption)

$$\|M(z,\xi,s)\| \le \max_{\Lambda \text{ diagonal, } |\Lambda| \le 1} \|[b(z,\xi) - \mathbf{b}]\Lambda\mathbf{b}^{-1}\| \le \delta_0 < 1 \qquad \text{for all } (z,\xi,s).$$

In other words, Assumption 10.1 guarantees that the Jacobian matrix (10.14) of the map $F(s)$ is nowhere singular and hence the map in question is globally invertible.

Having shown that $F(s)$ is globally invertible, we can express $\psi(\xi,\sigma)$ as

$$\psi(\xi,\sigma) = F^{-1}(\psi^*(z,\xi,\varsigma)),$$

which—bearing in mind the definition of $\psi(\xi,\sigma)$—yields

$$\sigma = \hat{K}\xi - \mathbf{b}F^{-1}(\psi^*(z,\xi,\varsigma))$$

showing how σ can be expressed as a function of e (and, of course, of (z,ξ)). This completes the proof that the map $e = T(z,\xi,\hat{\xi}^{\text{ext}})$ is globally invertible in $\hat{\xi}^{\text{ext}}$. $\triangleleft$

Having shown that (10.9) define an admissible change of variables, we proceed now to transform the Eq. (10.8) in the new coordinates. To simplify the notations, in what follows we find it useful—occasionally—to denote the pair of variables (z,ξ), that characterize the controlled plant, by

$$x = \mathrm{col}(z,\xi)$$

[6] By contradiction, suppose $M(z,\xi,s) + I$ were singular. Then, there would exist a vector x satisfying $M(z,\xi,s)x = -x$ which would imply $\|M(z,\xi,s)\| \ge 1$ which contradicts the condition $\|M(z,\xi,s)\| < 1$.

in which case the vector e is seen as a function of $(x, \hat{\xi}^{\text{ext}})$ and, conversely, the vector $\hat{\xi}^{\text{ext}}$ is seen as a function of (x, e).

Remark 10.4 Using the various expressions introduced above, the fact that $q(0,0) = 0$ and the fact that $F(0) = 0$, it is easy to check that, if $x = 0$, then $e = 0$ if and only if $\hat{\xi}^{\text{ext}} = 0$. ◁

So long as $e_{i,1}, \dots, e_{i,r_i-1}$ are concerned, it is immediate to see that

$$\begin{aligned} \dot{e}_{i,1} &= \kappa(e_{i,2} - c_{i,r_i} e_{i,1}) \\ \dot{e}_{i,2} &= \kappa(e_{i,3} - c_{i,r_i-1} e_{i,1}) \\ &\cdots \\ \dot{e}_{i,r_i-1} &= \kappa(e_{i,r_i} - c_{i,2} e_{i,1}). \end{aligned} \tag{10.17}$$

For the variable e_{i,r_i}, we get

$$\begin{aligned} \dot{e}_{i,r_i} &= \kappa[q_i(z,\xi) + (b_i(z,\xi) - \mathbf{b}_i)G(\psi(\hat{\xi},\sigma)) - \sigma_i - c_{i,r_i} e_{i,1}] \\ &= \kappa[e_{i,r_i+1} - c_{i,1} e_{i,1}] + \kappa[b_i(z,\xi) - \mathbf{b}_i][G(\psi(\hat{\xi},\sigma)) - G(\psi(\xi,\sigma))] \\ &:= \kappa[e_{i,r_i+1} - c_{i,1} e_{i,1}] + \Delta_{1,i}(x,e), \end{aligned} \tag{10.18}$$

in which we have denoted by $\Delta_{1,i}(x, e)$ the ith row of the vector

$$\Delta_1(x,e) = \kappa[b(z,\xi) - \mathbf{b}][G(\psi(\hat{\xi},\sigma)) - G(\psi(\xi,\sigma))].$$

Finally, for e_{i,r_i+1} we obtain

$$\begin{aligned} \dot{e}_{i,r_i+1} = \dot{q}_i(z,\xi) + \dot{b}_i(z,\xi)G(\psi(\xi,\sigma)) + [b_i(z,\xi) - \mathbf{b}_i]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}[\hat{K}\dot{\xi} - \dot{\sigma}] \\ - \kappa c_{i,0} e_{i,1}, \end{aligned}$$

in which $\dot{q}_i(z, \xi)$ and $\dot{b}_i(z, \xi)$ denote the derivatives with respect to time of $q_i(z, \xi)$ and $b_i(z, \xi)$. In view of the developments which follow, it is useful to separate, in this expression, the terms that depend on the design parameters. This is achieved by setting

$$\Delta_2(x,e) = \dot{q}(z,\xi) + \dot{b}(z,\xi)G(\psi(\xi,\sigma)) + [b(z,\xi) - \mathbf{b}]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}\hat{K}\dot{\xi}.$$

which yields

$$\dot{e}_{i,r_i+1} = \Delta_{2,i}(x,e) + [b_i(z,\xi) - \mathbf{b}_i]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}[-\dot{\sigma}] - \kappa c_{i,0} e_{i,1},$$

in which $\Delta_{2,i}(x, e)$ denotes the ith row of the vector $\Delta_2(x, e)$. Looking at the expression of $\dot{\sigma}$, we can write

$$
\begin{aligned}
[b_i(z,\xi) - \mathbf{b}_i]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}\dot{\sigma} &= [b_i(z,\xi) - \mathbf{b}_i]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}\begin{pmatrix} \kappa c_{1,0}e_{1,1} \\ \kappa c_{2,0}e_{2,1} \\ \vdots \\ \kappa c_{m,0}e_{m,1} \end{pmatrix} \\
&:= \kappa \Delta_{0,i}(x,e)\begin{pmatrix} c_{1,0}e_{1,1} \\ c_{2,0}e_{2,1} \\ \vdots \\ c_{m,0}e_{m,1} \end{pmatrix}
\end{aligned}
$$

in which $\Delta_{0,i}(x,e)$ denotes the ith row of the matrix

$$\Delta_0(x,e) = M(z,\xi,\psi(\xi,\sigma)),$$

with $M(z,\xi,s)$ the matrix defined in (10.15). Thus, in summary

$$\dot{e}_{i,r_i+1} = -\kappa c_{i,0}e_{i,1} - \kappa \Delta_{0,i}(x,e)\begin{pmatrix} c_{1,0}e_{1,1} \\ c_{2,0}e_{2,1} \\ \vdots \\ c_{m,0}e_{m,1} \end{pmatrix} + \Delta_{2,i}(x,e). \tag{10.19}$$

Putting together (10.17), (10.18) and (10.19) we obtain, for $i = 1,\dots,m$, a system of the form

$$\dot{e}_i = \kappa A_i e_i - \kappa B_{2,i}\Delta_{0,i}(x,e)\begin{pmatrix} C_1e_1 \\ C_2e_2 \\ \vdots \\ C_me_m \end{pmatrix} + B_{1,i}\Delta_{1,i}(x,e) + B_{2,i}\Delta_{2,i}(x,e) \tag{10.20}$$

in which $A_i \in \mathbb{R}^{(r_i+1)\times(r_i+1)}$, $B_{1,i} \in \mathbb{R}^{(r_i+1)}$, $B_{2,i} \in \mathbb{R}^{(r_i+1)}$, $C_i^{\mathrm{T}} \in \mathbb{R}^{(r_i+1)}$ are matrices defined as

$$A_i = \begin{pmatrix} -c_{i,r_i} & 1\;0 \cdots 0\;0 \\ -c_{i,r_i-1} & 0\;1 \cdots 0\;0 \\ \cdot & \cdot\;\cdot \cdots \cdot\;\cdot \\ -c_{i,1} & 0\;0 \cdots 0\;1 \\ -c_{i,0} & 0\;0 \cdots 0\;0 \end{pmatrix}, \quad B_{1,i} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 0 \end{pmatrix}, \quad B_{2,i} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix},$$

$$C_i = \begin{pmatrix} c_{i,0} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Putting the dynamics of all e_i's together yields a final equation

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(x,e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(x,e) + \mathbf{B}_2\Delta_2(x,e) \tag{10.21}$$

in which

$$\begin{aligned}
\mathbf{A} &= \mathrm{diag}(A_1, A_2, \ldots, A_m) \\
\mathbf{B}_1 &= \mathrm{diag}(B_{1,1}, B_{1,2}, \ldots, B_{1,m}) \\
\mathbf{B}_2 &= \mathrm{diag}(B_{2,1}, B_{2,2}, \ldots, B_{2,m}) \\
\mathbf{C} &= \mathrm{diag}(C_1, C_2, \ldots, C_m)
\end{aligned}$$

and $\Delta_0(x, e)$, $\Delta_1(x, e)$, $\Delta_2(x, e)$ are the functions indicated above.

The special structure of this equation will be exploited in the next section, where a complete analysis of the asymptotic properties of the resulting closed-loop system is conducted. In this analysis, it is of paramount importance to take advantage of appropriate bounds on the functions $\Delta_0(x, e)$, $\Delta_1(x, e)$, $\Delta_2(x, e)$, described in what follows.

Lemma 10.2 *Suppose Assumption 10.1 holds and $\kappa \geq 1$. There exist numbers $\delta_0 < 1$ and δ_1 such that*

$$\begin{aligned}
\|\Delta_0(x, e)\| &\leq \delta_0 < 1 \qquad \text{for all } (x, e) \text{ and all } \kappa \\
\|\Delta_1(x, e)\| &\leq \delta_1 \|e\| \qquad \text{for all } (x, e) \text{ and all } \kappa.
\end{aligned} \tag{10.22}$$

Moreover, for each $R > 0$ there is a number M_R such that

$$\|x\| \leq R \quad \Rightarrow \quad \|\Delta_2(x, e)\| \leq M_R \qquad \text{for all } e \in \mathbb{R}^{d+m} \text{ and all } \kappa. \tag{10.23}$$

Proof The first bound in (10.22) is a straightforward consequence of the definition of $\Delta_0(x, e)$, in view of the fact that the matrix $M(z, \xi, s)$, as already shown in the proof of Lemma 10.1, is bounded by some $\delta_0 < 1$. So long as the second bound in (10.22) is concerned, observe first of all that $\Delta_1(x, e)$ is a smooth function that vanishes at $e = 0$, because in that case $\hat{\xi} = \xi$. Despite of the presence of the factor κ, it can be shown that this is function is also globally bounded as in (10.22), provided that $\kappa > 1$. To check that this is the case, recall that $[b(z, \xi) - \mathbf{b}]$ is globally bounded by assumption. Moreover $G(\cdot)$ is globally Lipschitz by construction. Therefore, for some L, we have (assume here that $\kappa \geq 1$)

$$\begin{aligned}
\|G(\psi(\hat{\xi}, \sigma)) - G(\psi(\xi, \sigma))\| &\leq L\|\psi(\hat{\xi}, \sigma) - \psi(\xi, \sigma)\| \leq L\|\mathbf{b}^{-1}\|\|\hat{K}\|\|\hat{\xi} - \xi\| \\
&\leq L\|\mathbf{b}^{-1}\|\|\hat{K}\|(\textstyle\sum_{k=1}^{m} \sum_{j=1}^{r_k} (\kappa^{-r_k - 1 + j} e_{k,j})^2)^{1/2} \\
&\leq L\|\mathbf{b}^{-1}\|\|\hat{K}\|\kappa^{-1}(\textstyle\sum_{k=1}^{m} \sum_{j=1}^{r_k} (e_{k,j})^2)^{1/2} \\
&\leq L\|\mathbf{b}^{-1}\|\|\hat{K}\|\kappa^{-1}\|e\|.
\end{aligned}$$

From this, it is seen that the second bound in (10.22) holds for some $\delta_1 > 0$.

So long as the bound in (10.23) is concerned, observe that, if system (10.2) is controlled by (10.7), we have

$$\begin{aligned}
\dot{z} &= f_0(z, \xi) + g_0(z, \xi)G(\psi(\hat{\xi}, \sigma)) \\
\dot{\xi} &= \hat{A}\xi + \hat{B}[q(z, \xi) + b(z, \xi)G(\psi(\hat{\xi}, \sigma))].
\end{aligned}$$

With this in mind, inspection of $\varDelta_{2,i}(x,e)$ shows that[7]

$$\begin{aligned}\varDelta_{2,i}(x,e) = &\tfrac{\partial q_i}{\partial z}[f_0(z,\xi)+g_0(z,\xi)G(\psi(\hat{\xi},\sigma)]\\ &+\tfrac{\partial q_i}{\partial \xi}[\hat{A}\xi+\hat{B}[q(z,\xi)+b(z,\xi)G(\psi(\hat{\xi},\sigma))]]\\ &+\tfrac{\partial b_i}{\partial z}[f_0(z,\xi)+g_0(z,\xi)G(\psi(\hat{\xi},\sigma)]G(\psi(\xi,\sigma))\\ &+\tfrac{\partial b_i}{\partial \xi}[\hat{A}\xi+\hat{B}[q(z,\xi)+b(z,\xi)G(\psi(\hat{\xi},\sigma))]]G(\psi(\xi,\sigma))\\ &+[b_i(z,\xi)-\mathbf{b}_i]G'(\psi(\xi,\sigma))\mathbf{b}^{-1}\hat{K}[\hat{A}\xi+\hat{B}[q(z,\xi)+b(z,\xi)G(\psi(\hat{\xi},\sigma))]].\end{aligned}$$

Bearing in mind the expressions of $G(\cdot)$ and $G'(\cdot)$ and the fact that $g(\cdot)$ and $g'(\cdot)$ are bounded functions, it is seen that, so long as $\|x\| \leq R$, the function $\varDelta_{2,i}(x,e)$ remains bounded—regardless of what e actually is[8]—by a number M_R that only depends on R and is independent of κ. This completes the proof of the lemma. $\lhd$

10.4 Convergence Analysis

We proceed now with the proof that, if system (10.2) is controlled by (10.7), in which $\hat{\xi}$ and σ are states of (10.8), the saturation level L in $g(\cdot)$ and the design parameters $c_{i,0}, c_{i,1}, \ldots, c_{i,r_i}$ and κ in (10.8) can be chosen in such a way that the resulting system has appropriate asymptotic properties. Specifically, it will be shown that the equilibrium point $(x, \xi^{\text{ext}}) = (0, 0)$ of the resulting closed-loop system can be *semiglobally and practically* stabilized and, under appropriate additional assumptions, also *semiglobally asymptotically* stabilized. The discussion below essentially follows the paradigm adopted in the proof of Proposition 7.3. However, adaptation to the present context requires number of non-negligible modifications.

We begin by rearranging the equations that describe the closed-loop system. First of all, as done in the previous section, we use e to replace the states $\hat{\xi}$ and σ of (10.8), in which case the Eq. (10.21) holds. Moreover, we add and subtract the function (10.3) to the control u defined in (10.7), which is therefore written as

$$u = [b(z,\xi)]^{-1}(-q(z,\xi)+\hat{K}\xi)+\varDelta_3(x,e)$$

in which

$$\varDelta_3(x,e) = G(\psi(\hat{\xi},\sigma)) - [b(z,\xi)]^{-1}(-q(z,\xi)+\hat{K}\xi),$$

[7] If $b(x)$ is a m-dimensional row vector, the notation $\frac{\partial b}{\partial x}\tau(x)$ used in the next formula stands for

$$\frac{\partial b}{\partial x}\tau(x) := \left(\frac{\partial b_1}{\partial x}\tau(x)\ \ \frac{\partial b_2}{\partial x}\tau(x)\ \cdots\ \frac{\partial b_m}{\partial x}\tau(x)\right).$$

[8] Recall that $\hat{\xi}$ and σ are functions of (x, e) and κ.

where, consistently with the notation used in the previous section, as arguments of $\Delta_3(\cdot)$ we have used x, e instead of $z, \xi, \hat{\xi}, \sigma$.

As a consequence, the equations of system (10.2) controlled by (10.7) can be regarded as equations of the form

$$\dot{x} = \mathbf{F}(x) + \mathbf{G}(x)\Delta_3(x, e) \tag{10.24}$$

in which

$$\mathbf{F}(x) = \begin{pmatrix} f_0(z,\xi) + g_0(z,\xi)[b(z,\xi)]^{-1}(-q(z,\xi) + \hat{K}\xi) \\ (\hat{A} + \hat{B}\hat{K})\xi \end{pmatrix} \qquad \mathbf{G}(x) = \begin{pmatrix} g_0(z,\xi) \\ \hat{B}b(z,\xi) \end{pmatrix}.$$

It is known from Sect. 10.2 that, if system (10.2) is strongly minimum phase and $\hat{K}$ is chosen as indicated in Proposition 10.1, the equilibrium $x = 0$ of

$$\dot{x} = \mathbf{F}(x) \tag{10.25}$$

is globally asymptotically stable. Hence, (10.24) can be regarded as a globally asymptotically stable system affected by a perturbation.

In this way, the entire closed-loop system is described by equations of the form

$$\begin{aligned} \dot{x} &= \mathbf{F}(x) + \mathbf{G}(x)\Delta_3(x, e) \\ \dot{e} &= \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(x, e) + \mathbf{B}_2\Delta_2(x, e). \end{aligned} \tag{10.26}$$

Note that $(x, e) = (0, 0)$ is an equilibrium point.[9] As anticipated, we are going to show how, given a fixed (but otherwise arbitrary large) compact set $\mathscr{C}$ of initial conditions, the free design parameters can be chosen in such a way that trajectories in finite time enter a fixed (but otherwise arbitrary) small target set having the equilibrium point $(x, e) = (0, 0)$ in its interior or even, under appropriate assumptions, converge to such equilibrium.

Remark 10.5 In this respect, it is important to stress that the set $\mathscr{C}$ is assigned in terms of the *original* state variables, namely the state x of the controlled plant (10.2) and the state $\hat{\xi}^{\text{ext}}$ of (10.8). In the Eq. (10.26), instead, the vector e replaces $\hat{\xi}^{\text{ext}}$. Thus, bearing in mind the change of variables defined in (10.9), it should be taken into account how the (compact) set to which the initial values of x and e belong is influenced by the choice of κ. We will return on this issue later on. ◁

With $\mathscr{C}$ being the fixed compact set of initial conditions of the system, pick a number $R > 0$ such that

$$(x, \hat{\xi}^{\text{ext}}) \in \mathscr{C} \qquad \Rightarrow \qquad (x, \hat{\xi}^{\text{ext}}) \in B_R \times B_R$$

---

[9]To this end, use the properties that $f_0(0,0) = 0$ and $q(0,0) = 0$ and the observation in Remark 10.4 to show that $\Delta_3(0,0) = 0$ and $\Delta_2(0,0) = 0$. From Lemma 10.2, observe that $\Delta_1(x, 0) = 0$.

in which B_R denotes an (open) ball of radius R.

The first design parameter which is being fixed is the "saturation" level L that characterizes the function $g(\cdot)$. The choice of L is based on a property of (10.25), which—as observed before—has a globally asymptotically stable equilibrium at $x = 0$. As a consequence of such property, it is known that there exists a positive definite and proper smooth $V(x)$ function satisfying

$$\frac{\partial V}{\partial x}\mathbf{F}(x) \leq -\alpha(\|x\|)$$

for some class $\mathscr{K}_\infty$ function $\alpha(\cdot)$. Pick a number $c > 0$ such that

$$\Omega_c = \{x : V(x) \leq c\} \supset B_R,$$

and choose for L the value

$$L = \max_{x \in \Omega_{c+1}} \Big[[b(z, \xi)]^{-1}(-q(z, \xi) + \hat{K}\xi) \Big] + 1.$$

We will show in what follows that, if the remaining design parameters are appropriately chosen, for all times $t \geq 0$ the vector $x(t)$ remains in Ω_{c+1}.

We begin with the analysis of the evolution of $x(t)$ for small values of $t \geq 0$. To this end observe that, since $G(\cdot)$ is a bounded function, so long as $x \in \Omega_{c+1}$ the perturbation term $\Delta_3(x, e)$ in the upper system of (10.26) remains bounded. In other words, there exists a number δ_3, such that

$$\|\Delta_3(x, e)\| \leq \delta_3 \qquad \text{for all } x \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{d+m}.$$

Note that the number δ_3 is independent of the choice of κ. Actually, it depends only on Ω_{c+1} and hence, indirectly (via the choice of R), on the choice of the set $\mathscr{C}$ in which the initial conditions are taken.

Consider now a trajectory with initial condition $x(0) \in B_R \subset \Omega_c$. In view of the above bound found for $\Delta_3(x, e)$, it can be claimed that, so long as $x(t) \in \Omega_{c+1}$,

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}(x)\Delta_3(x, e)] \leq -\alpha(\|x\|) + \|\frac{\partial V}{\partial x}\mathbf{G}(x)\|\delta_3.$$

Let

$$M = \max_{x \in \Omega_{c+1}} \|\frac{\partial V}{\partial x}\mathbf{G}(x)\|$$

and observe that the previous estimate yields, in particular,

$$\dot{V}(x(t)) \leq M\delta_3$$

from which it is seen that (recall that $x(0) \in \Omega_c$)

$$V(x(t)) \le V(x(0)) + M\delta_3 t \le c + M\delta_3 t.$$

This inequality shows that $x(t)$ remains in Ω_{c+1} at least until time $T_0 = 1/M\delta_3$. This time might be very small but it is independent of κ, because so are M and δ_3. It follows from the previous remarks that such T_0 only depends on Ω_{c+1} and hence only on the choice of the set $\mathscr{C}$.

During the time interval $[0, T_0]$ the state $e(t)$ remains bounded. This is seen from the bottom equation of (10.26), using the bounds determined for $\Delta_0(x, e)$, $\Delta_1(x, e)$, $\Delta_2(x, e)$ in Lemma 10.2 and the fact that $x(t) \in \Omega_{c+1}$ for all $t \in [0, T_0]$. The bound on $e(t)$, though, is affected by the value of κ. In fact, looking at the definitions of the various components of e, it is seen that $\|e(0)\|$ grows with κ (despite of the fact that, by assumption, $\|x(0)\| \le R$ and $\|\hat{\xi}^{\text{ext}}(0)\| \le R$).

Having established that trajectories of the system exists on the time interval $[0, T_0]$, we analyze next the behavior of $e(t)$ for times larger than T_0. To this end, the two results that follow are instrumental.

Lemma 10.3 *There exist a choice of the coefficients $c_{i,0}, \ldots, c_{i,r_i}$ for $1 \le i \le m$, a positive definite and symmetric $(d+m) \times (d+m)$ matrix P and a number $\lambda > 0$ such that*

$$P[\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}] + [\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}]^\top P \le -\lambda I. \tag{10.27}$$

Proof Consider the system

$$\dot{e} = [\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}]e, \tag{10.28}$$

which can be interpreted as a system, with m inputs and m outputs, modeled by

$$\begin{aligned} \dot{e} &= \mathbf{A}e + \mathbf{B}_2 u \\ y &= \mathbf{C}e \end{aligned} \tag{10.29}$$

controlled by $u = -\Delta_0(x, e)y$. Bearing in mind the (diagonal) structure of $\mathbf{A}$, $\mathbf{B}_2$, $\mathbf{C}$, and the expressions of A_i, $B_{2,i}$, C_i, it is seen that the transfer function matrix $T(s)$ of (10.29) is a diagonal matrix

$$T(s) = \text{diag}(T_1(s), T_2(s), \ldots, T_m(s))$$

whose diagonal elements are the following functions

$$T_i(s) = C_i(sI - A_i)^{-1}B_{i,2} = \frac{c_{i,0}}{s^{r_i+1} + c_{i,r_i}s^{r_i} + \cdots c_{i,1}s + c_{i,0}}.$$

Note that $T_i(0) = 1$. Therefore, if the $c_{i,j}$'s are chosen in such a way that the poles of $T_i(s)$ are all real and have negative real part, $|T_i(j\omega)| \le 1$ for all $\omega \in \mathbb{R}$. As a consequence

$$\|T(\cdot)\|_{H_\infty} = 1.$$

Consider now the number δ_0 defined in Assumption 10.1. Since this number is strictly less than 1, we can claim that

$$\|T(\cdot)\|_{H_\infty} < \delta_0^{-1}.$$

This being the case, it follows from the Bounded Real Lemma (see Theorem 3.1) that there exist a positive definite and symmetric matrix P and a number $\lambda > 0$ such that, along the trajectories of (10.29), the estimate

$$2e^{\mathrm{T}} P[\mathbf{A}e + \mathbf{B}_2 u] \le -\lambda\|e\|^2 + (\delta_0^{-1})^2\|u\|^2 - \|y\|^2$$

holds.

System (10.28) is obtained from (10.29) by picking $u = -\Delta_0(x, e)y$ for which, using the bound (10.22), we have

$$\|u\| \le \|\Delta_0(x, e)\|\|y\| \le \delta_0\|y\|.$$

Thus, along the trajectories of (10.28), the estimate

$$2e^{\mathrm{T}} P[\mathbf{A}e - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}e] \le -\lambda\|e\|^2 + (\delta_0^{-1})^2\delta_0^2\|y\|^2 - \|y\|^2 \le -\lambda\|e\|^2$$

holds, which proves the lemma. ◁

Lemma 10.4 *Let the $c_{i,j}$'s be chosen so as to make (10.27) satisfied. Suppose $x(t) \in \Omega_{c+1}$ for all $t \in [0, T_{\max})$ and suppose that $\|\hat{\xi}^{\mathrm{ext}}(0)\| \le R$. Then, for every $0 < T \le T_{\max}$ and every $\varepsilon > 0$, there is a κ^* such that, for all $\kappa \ge \kappa^*$,*

$$\|e(t)\| \le 2\varepsilon \quad for\ all\ t \in [T, T_{\max}).$$

Proof Set $U(e) = e^\top Pe$ and recall that

$$a_1\|e\|^2 \le U(e) \le a_2\|e\|^2, \tag{10.30}$$

for some pair a_1, a_2. Using the bound (10.22), we have

$$\begin{aligned}\dot{U}(e(t)) &= 2e^\top P\Big[\kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(x, e) + \mathbf{B}_2\Delta_2(x, e)\Big] \\ &\le -\kappa\lambda\|e\|^2 + 2e^\top P\mathbf{B}_1\Delta_1(x, e) + 2e^\top P\mathbf{B}_2\Delta_2(x, e) \\ &\le -(\kappa\lambda - 2\delta_1\|P\|)\|e\|^2 + 2\|e\|\|P\|\|\Delta_2(x, e)\| \\ &\le -(\kappa\lambda - 2\delta_1\|P\| - \mu\|P\|)\|e\|^2 + \tfrac{\|P\|}{\mu}\|\Delta_2(x, e)\|^2,\end{aligned}$$

in which μ is any arbitrary (positive) number. Let now R' be a number such that

$$\Omega_{c+1} \subset B_{R'}.$$

Using the bound (10.23) we see that, so long as $x(t) \in \Omega_{c+1}$,

$$\|\Delta_2(x, e)\| \le M_{R'}.$$

Choose μ as

$$\mu = \frac{M_{R'}^2 \|P\|}{\varepsilon^2}.$$

Bearing in mind the estimates in (10.30), set

$$\alpha_\kappa = \frac{\kappa\lambda - (2\delta_1 + \mu)\|P\|}{2a_2} \tag{10.31}$$

and suppose that κ is large enough so as to make

$$2\alpha_\kappa a_1 > 1,$$

which implies $\alpha_\kappa > 0$. Then, the inequality

$$\dot{U}(e(t)) \le -2\alpha_\kappa U(e(t)) + \varepsilon^2,$$

holds, for any $t \in [0, T_{\max})$. From this, by means of standard arguments, it can be concluded that[10]

$$\|e(t)\| \le Ae^{-\alpha_\kappa t}\|e(0)\| + \varepsilon, \qquad \text{in which} \quad A = \sqrt{\frac{a_2}{a_1}}.$$

At this point, it is necessary to obtain a bound for $\|e(0)\|$. Bearing in mind the Definitions (10.9), the fact that $\|x(0)\| \le R$ and $\|\hat{\xi}^{\text{ext}}(0)\| \le R$, and assuming $\kappa \ge 1$, it is seen that

$$\|e(0)\| \le \hat{R}\kappa^{r_{\max}}$$

in which $r_{\max} = \max\{r_1, r_2, \ldots, r_m\}$ and $\hat{R}$ is a number only depending on R. Therefore, $\|e(t)\|$ is bounded as

$$\|e(t)\| \le A\hat{R}e^{-\alpha_\kappa t}\kappa^{r_{\max}} + \varepsilon$$

for all $t \in [0, T_{\max})$. Let now T be any time satisfying $0 < T < T_{\max}$, and—using (10.31)—observe that

[10]Use the comparison lemma to get

$$U(t) \le e^{-2\alpha_\kappa t}U(0) + \frac{\varepsilon^2}{2\alpha_\kappa}$$

from which, using the estimates (10.30) and the fact that $2\alpha_\kappa a_1 > 1$, the claimed inequality follows.

$$A\hat{R}e^{-\alpha_\kappa T}\kappa^{r_{\max}} = A\hat{R}e^{\frac{(2\delta_1+\mu)\|P\|T}{2a_2}} e^{-\frac{\lambda T}{2a_2}\kappa}\kappa^{r_{\max}}.$$

Clearly, the function $f : \kappa \mapsto f(\kappa) = e^{-\frac{\lambda T}{2a_2}\kappa}\kappa^{r_{\max}}$ decays to 0 as $\kappa \to \infty$. Thus, there is a number κ^* such that

$$A\hat{R}e^{-\alpha_\kappa T}\kappa^{r_{\max}} = A\hat{R}e^{\frac{(2\delta_1+\mu)\|P\|T}{2a_2}} f(\kappa) \le \varepsilon.$$

for all $\kappa \ge \kappa^*$.[11] From this, it is concluded that, for all $\kappa \ge \kappa^*$,

$$\|e(T)\| \le 2\varepsilon.$$

Finally, bearing in mind the fact that $\alpha_\kappa > 0$, we see that for $t \in [T, T_{\max})$

$$\|e(t)\| \le A\hat{R}e^{-\alpha_\kappa(t-T)}e^{-\alpha_\kappa T}\kappa^{r_{\max}} + \varepsilon \le e^{-\alpha_\kappa(t-T)}\varepsilon + \varepsilon \le 2\varepsilon,$$

and this concludes the proof. $\triangleleft$

This lemma will now be used to show that the parameter κ can be chosen in such a way that $x(t) \in \Omega_{c+1}$ for all $t \in [0, \infty)$. We know from the previous analysis that $x(t) \in \Omega_{c+1}$ for $t \le T_0$. Suppose that, for some $T_{\max} > T_0$, $x(t) \in \Omega_{c+1}$ for all $t \in [T_0, T_{\max})$. From the lemma we know that for any choice of ε, there is a value of κ^* such that, if $\kappa \ge \kappa^*$, $\|e(t)\| \le 2\varepsilon$ for all $t \in [T_0, T_{\max})$.

Since $b(z, \xi)$ is nonsingular for all (z, ξ), there exists a number b_0 such that $\|[b(z, \xi)]^{-1}\| \le b_0$ for all $x \in \Omega_{c+1}$. Suppose ε satisfies

$$4\varepsilon b_0 < 1. \tag{10.32}$$

Then, bearing in mind the Definition (10.11) of ς, we obtain

$$\|[b(z, \xi)]^{-1}\varsigma\| \le \|[b(z, \xi)]^{-1}\|\,\|e\| \le b_0 2\varepsilon < \frac{1}{2},$$

for all $t \in [T_0, T_{\max})$, which in turn implies, because of the choice of L,

$$\|[b(z, \xi)]^{-1}[\hat{K}\xi - q(z, \xi) + \varsigma]\| \le L - \frac{1}{2}. \tag{10.33}$$

Using (10.12) and the definition of the function $F(s)$ in (10.13), observe that

$$[b(z, \xi)]^{-1}[\hat{K}\xi - q(z, \xi) + \varsigma] = [b(z, \xi)]^{-1}\mathbf{b}F(\psi(\xi, \sigma)).$$

Hence, if we consider the function

[11] Note that μ depends on ε and, actually, increases as ε decreases. This fact, however, does not affect the prior conclusion.

$$\tilde{F}(s) = [b(z,\xi)]^{-1}\mathbf{b}F(s) = [I - [b(z,\xi)]^{-1}\mathbf{b}]G(s) + [b(z,\xi)]^{-1}\mathbf{b}s,$$

the inequality (10.33) can be rewritten as

$$\|\tilde{F}(\psi(\xi,\sigma))\| \le L - \frac{1}{2}. \tag{10.34}$$

The function $\tilde{F}(s)$ is globally invertible (because so is $F(s)$, as shown earlier in the proof of Lemma 10.1). Moreover, it can be easily seen that the function $\tilde{F}(s)$ is an identity on the set

$$C_L := \{s \in \mathbb{R}^m : |s_i| \le L, \text{ for all } i = 1,\dots,m\}.$$

In fact,

$$s \in C_L \quad \Rightarrow \quad G(s) = s,$$

and this, using the expression shown above for $\tilde{F}(s)$, proves that $\tilde{F}(s) = s$ for all $s \in C_L$. As a consequence, the pre-image of any point p in the set C_L is the point p itself. Since the inequality (10.34) implies $\tilde{F}(\psi(\xi,\sigma)) \in C_L$, it is concluded that $\tilde{F}(\psi(\xi,\sigma)) = \psi(\xi,\sigma)$ and hence

$$\|\psi(\xi,\sigma)\| \le L - \frac{1}{2}, \tag{10.35}$$

for all $t \in [T_0, T_{\max})$.

We use this property to show that, for all $t \in [T_0, T_{\max})$,

$$G(\psi(\hat{\xi},\sigma)) = \psi(\hat{\xi},\sigma), \tag{10.36}$$

i.e. that on the time interval $t \in [T_0, T_{\max})$, none of the components of the control (10.7) is "saturated". To this end, observe that

$$\hat{\xi} = \xi - D(\kappa)e \tag{10.37}$$

in which $D(\kappa) = \text{diag}(D_1(\kappa),\dots,D_m(\kappa))$ where $D_i(\kappa)$ is the $r_i \times (r_i+1)$ matrix

$$D_i(\kappa) = \begin{pmatrix} \kappa^{-r_i} & 0 & \dots & 0 & 0 \\ 0 & \kappa^{-r_i-1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \dots & \kappa^{-1} & 0 \end{pmatrix}$$

and note that $\|D(\kappa)\| \le 1$ if, without loss of generality, it is assumed $\kappa \ge 1$. Since, by definition,

$$\psi(\hat{\xi},\sigma) = \psi(\xi,\sigma) - \mathbf{b}^{-1}\hat{K}D(\kappa)e,$$

if $\kappa > 1$ we have

$$\|\psi(\hat{\xi}, \sigma)\| \leq \|\psi(\xi, \sigma)\| + \|\mathbf{b}^{-1}\| \|\hat{K}\| \|e\|.$$

Suppose ε satisfies

$$4\|\mathbf{b}^{-1}\| \|\hat{K}\| \varepsilon < 1. \tag{10.38}$$

Then, using (10.35), we see that $\|\psi(\hat{\xi}, \sigma)\| < L$, and this proves that (10.36) holds on the time interval $[T_0, T_{\max})$.

We return now to Eq. (10.24) and observe that, for all $t \in [T_0, T_{\max})$,

$$\begin{aligned} \Delta_3(x, e) &= G(\psi(\hat{\xi}, \sigma)) - [b(z, \xi)]^{-1}(\hat{K}\xi - q(z, \xi)) \\ &= \psi(\hat{\xi}, \sigma) - [b(z, \xi)]^{-1}(\hat{K}\xi - q(z, \xi)) \\ &= \psi(\xi, \sigma) - \mathbf{b}^{-1}\hat{K} D(\kappa)e - [b(z, \xi)]^{-1}(\hat{K}\xi - q(z, \xi)). \end{aligned}$$

In this expression, $\psi(\xi, \sigma)$ can be replaced by

$$\psi(\xi, \sigma) = [b(z, \xi)]^{-1}(\hat{K}\xi - q(z, \xi) + \varsigma)$$

which is a consequence of (10.12), where we use the fact that $G(\psi(\xi, \sigma)) = \psi(\xi, \sigma)$. Hence

$$\Delta_3(x, e) = -\mathbf{b}^{-1}\hat{K} D(\kappa)e + [b(z, \xi)]^{-1}\varsigma$$

and, recalling that ς is part of e,

$$\|\Delta_3(x, e)\| \leq (\|\mathbf{b}^{-1}\| \, \|\hat{K}\| + b_0)\|e\|.$$

This estimate is now used in the evaluation of the derivative of $V(x)$ along the trajectories of (10.24). Setting

$$\delta_4 = M(\|\mathbf{b}^{-1}\| \, \|\hat{K}\| + b_0),$$

it is seen that, for all $t \in [T_0, T_{\max})$,

$$\frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}(x)\Delta_3(x, e)] \leq -\alpha(\|x\|) + \delta_4\|e\| \leq -\alpha(\|x\|) + 2\delta_4\varepsilon.$$

Suppose ε satisfies

$$2\delta_4\varepsilon < \min_{x \in \partial\Omega_{c+1}} \alpha(\|x\|). \tag{10.39}$$

If this is the case, then $\dot{V}(x(t))$ is negative on the boundary of Ω_{c+1}. As a consequence, $x(t) \in \Omega_{c+1}$ for all $t \geq 0$ and $T_{\max} = \infty$. In summary, from Lemma 10.4 we deduce that, given a number ε that satisfies (10.32), (10.38), (10.39), there exists a number

κ^* such that, if $\kappa \geq \kappa^*$, then for any initial condition in $\mathscr{C}$ the trajectories of the system are such that $x(t) \in \Omega_{c+1}$ for all $t \geq 0$ and $\|e(t)\| \leq 2\varepsilon$ for all $t \geq T_0$.

This being the case, arguments similar to those used in the proof of Proposition 6.6 can be invoked to show that also $x(t)$ enters, in finite time, an arbitrarily small target set. Let ε' be a (small) given number, pick d such that

$$\Omega_d \subset B_{\varepsilon'} \subset \Omega_{c+1},$$

consider the "annular" compact set

$$S_d^{c+1} = \{x : d \leq V(x) \leq c+1\},$$

and let

$$a = \min_{x \in S_d^{c+1}} \alpha(\|x\|).$$

If ε satisfies[12]

$$4\delta_4\varepsilon < a, \tag{10.40}$$

we see that

$$\dot{V}(x(t)) \leq -\frac{1}{2}a, \qquad \text{for all } x \in S_d^{c+1}.$$

This, as shown in a similar context in the proof of Proposition 6.6, implies the existence of a time $T_1 > T_0$ such that

$$x(t) \in B_{\varepsilon'} \qquad \text{for all } t \geq T_1.$$

In summary, we have proven the following result.

Proposition 10.2 *Consider system (10.2). Suppose the system is strongly minimum phase and Assumption 10.1 holds. Let the system be controlled by (10.7)–(10.8), with* $\mathbf{b}$ *chosen so as to satisfy the condition in Assumption 10.1. Let $\hat{K}$ such that $\hat{A} + \hat{B}\hat{K}$ is Hurwitz. For every choice of a compact set $\mathscr{C}$ and of a number $\bar{\varepsilon} > 0$, there is a choice of the design parameters L and $c_{i,0}, c_{i,1}, \ldots, c_{i,r_i}$, with $1 \leq i \leq m$, and a number κ^* such that, if $\kappa \geq \kappa^*$, there is finite time T_1 such that all trajectories of the closed-loop system with initial conditions $(x(0), \hat{\xi}^{\text{ext}}(0)) \in \mathscr{C}$ remain bounded and satisfy $\|x(t)\| \leq \bar{\varepsilon}$ and $\|\hat{\xi}^{\text{ext}}(t)\| \leq \bar{\varepsilon}$ for all $t \geq T_1$.*

Remark 10.6 The previous arguments show that, if the design parameter κ is large enough, $\|e(t)\| \leq 2\varepsilon$ for $t \geq T_0$, $\|x(t)\| \leq \varepsilon'$ for $t \geq T_1$ and, consequently (see 10.37), $\|\hat{\xi}(t)\| \leq \varepsilon' + 2\varepsilon$ for $t \geq T_1$. The value of ε is expected to satisfy the inequalities (10.32), (10.38), (10.40), in which–assuming that $\mathbf{b}$ and $\hat{K}$ are fixed—b_0 is determined by the choice of $\mathscr{C}$, M is determined by the choice of $\mathscr{C}$, and a is determined by the choice of ε'. With this in mind, it is easy to see how, once a single

[12]Note that the following inequality implies (10.39).

ultimate bound $\bar{\varepsilon}$ for $\|x(t)\|$ and $\|\xi^{\text{ext}}(t)\|$ has been chosen, ε can be fixed, which in turn determines the minimal value of κ. $\triangleleft$.

If asymptotic stability is sought, one can proceed as in the case considered in Sect. 6.5, by strengthening the assumption of strong minimum phase on the controlled system (10.2).

Proposition 10.3 *Consider system (10.2). Suppose the system is strongly—and also locally exponentially—minimum phase and Assumption 10.1 holds. Let the system be controlled by (10.7)–(10.8), with* $\mathbf{b}$ *chosen so as to satisfy the condition in Assumption 10.1. Let $\hat{K}$ such that $\hat{A} + \hat{B}\hat{K}$ is Hurwitz. For every choice of a compact set $\mathscr{C}$ there is a choice of the design parameters L and $c_{i,0}, c_{i,1}, \ldots, c_{i,r_i}$, with $1 \leq i \leq m$, and a number κ^* such that, if $\kappa \geq \kappa^*$, the equilibrium $(x, \hat{\xi}^{\text{ext}}) = (0, 0)$ is asymptotically stable, with a domain of attraction $\mathscr{A}$ that contains the set $\mathscr{C}$.*

Proof The proof uses arguments very similar to those used in the proof of Proposition 6.7. We limit ourselves to give only a sketch of such arguments. It is known from Remark 10.2 that the equilibrium $x = 0$ of (10.25) is locally exponentially stable. Hence, it can be assumed that the function $V(x)$, for some positive numbers $b, k_0, d > 0$ satisfies[13]

$$\frac{\partial V}{\partial x}\mathbf{F}(x) \leq -b\|x\|^2 \quad \text{and} \quad \|\frac{\partial V}{\partial x}\| \leq k_0\|x\| \qquad \text{for all } \|x\| \leq d.$$

Consider the set

$$S_d = \{(x, e) : \|x\| \leq d, \|e\| \leq d\}.$$

Looking at the definitions of $\Delta_3(x, e)$ and $\Delta_2(x, e)$ it is seen that, if d is small enough, there is a positive number k_1 (independent of κ if $\kappa \geq 1$) such that

$$\|\mathbf{G}(x)\Delta_3(x, e)\| \leq k_1\|e\| \qquad \text{for all } (x, e) \in S_d,$$

and two positive numbers k_2, k_3 (independent of κ if $\kappa \geq 1$) such that

$$\|\Delta_2(x, e)\| \leq k_2\|e\| + k_3\|x\| \qquad \text{for all } (x, e) \in S_d.$$

Hence, for all $(x, e) \in S_d$,

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}[\mathbf{F}(x) + \mathbf{G}(x)\Delta_3(x, e)] \leq -b\|x\|^2 + k_0k_1\|x\|\,\|e\|$$

and

$$\begin{aligned}\dot{U}(e(t)) &= 2e^\top P[\kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(x, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(x, e) + \mathbf{B}_2\Delta_2(x, e)] \\ &\leq -\kappa\lambda\|e\|^2 + 2e^\top P\mathbf{B}_1\Delta_1(x, e) + 2e^\top P\mathbf{B}_2\Delta_2(x, e) \\ &\leq -(\kappa\lambda - 2\delta_1\|P\| - 2k_2\|P\|)\|e\|^2 + 2k_3\|P\|\|e\|\|x\|.\end{aligned}$$

[13] See Lemma B.1 in Appendix B.

Thus, the function $W(x, e) = V(x) + U(e)$ satisfies

$$\dot{W}(x, e) \leq -b\|x\|^2 - (\kappa\lambda - 2\delta_1\|P\| - 2k_2\|P\|)\|e\|^2 + (k_0 k_1 + 2k_3\|P\|)\|x\|\,\|e\|,$$

so long as $(x, e) \in S_d$. Using standard estimates, it is seen from this that there is a number $\tilde{\kappa}$ such that, if $\kappa \geq \tilde{\kappa}$, the function $\dot{W}(x, e)$ is negative definite for all $(x, e) \in S_d$. From this, the conclusion follows as in the proof of Proposition 6.7. $\lhd$

As anticipated, the feedback law used to obtain the stabilization results described in the propositions above uses only the output y of the controlled system as measured variable, and does not rely upon exact knowledge of the functions of (z, ξ) that characterize the model (10.2). Rather, it only relies upon the knowledge of the values of the integers $r_1, \ldots, r_m$, on the assumption that the system is strongly minimum phase and on the assumption that the high-frequency gain matrix $b(z, \xi)$ satisfies Assumption 10.1. Note, in particular, that the construction of the "observer" (10.8) only requires knowing the values of $r_1, \ldots, r_m$: everything else only consists of design parameters. In this respect, it can be claimed that the feedback law in question is *robust*, as opposite to the feedback law discussed in Sect. 7.5.

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Chapter 11
Stabilization of Multivariable Nonlinear Systems: Part II

11.1 Handling Invertible Systems that Are Input–Output Linearizable

In this chapter, we consider the problem of achieving global stability, or stability with a guaranteed region of attraction, for systems that do not possess a vector relive degree but are *uniformly invertible*, in the sense of Definition 9.2 and satisfy Assumption 9.1. It has been shown in Chap. 9 that, under such assumptions, the equations describing the system can be transformed—by means of a globally defined diffeormorphism—into equations consisting of a set of the form (9.40) and of ℓ sets of the form (9.34).

In the first part of the chapter we address the case of a system that belongs to the special class considered in Sect. 9.5. The peculiar feature of such class of systems is that, in the Eq. (9.34), the coefficients $\delta^j_{i,k}(x)$ are *independent of* x. This property is of a great help in the design of stabilizing laws.

We begin by observing that, for this class of systems, the design of *full state* globally stabilizing feedback laws is relatively straightforward, if the system is assumed to be *strongly minimum phase*, in the sense of Definition 10.1. To this end, in fact, one could proceed as follows.[1] Suppose, to simplify matters, that $\ell = m$, in which case all m_i's are equal to 1 (in fact, the general case in which one or more of the m_i's is larger than 1 can be handled exactly in the same way). Consider the set (9.34) corresponding to $i = 1$, which is a system of the form

$$\begin{aligned}
\dot{\xi}_{1,1} &= \xi_{1,2} \\
&\cdots \\
\dot{\xi}_{1,r_1-1} &= \xi_{1,r_1} \\
\dot{\xi}_{1,r_1} &= a_1(x) + b_1(x)u.
\end{aligned}$$

[1] The design method described hereafter, that leads to the state feedback control (11.4), essentially follows the procedure suggested—in a more general context—in [1], where the property of minimum phase introduced in [5] is exploited.

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_11

Let u be such that

$$a_1(x) + b_1(x)u = \hat{K}_1\xi_1 \tag{11.1}$$

in which case the set in question reduces to the linear system

$$\dot{\xi}_1 = (\hat{A}_1 + \hat{B}_1\hat{K}_1)\xi_1$$

in which $\hat{A}_1$ and $\hat{B}_1$ are matrices whose structure is the same as that of the matrices (9.50). Clearly, it is possible to pick $\hat{K}_1$ so as to make $(\hat{A}_1 + \hat{B}_1\hat{K}_1)$ a Hurwitz matrix.

Next, consider the set (9.34) corresponding to $i = 2$, which—in view of (11.1)—is a system of the form

$$\begin{aligned}
\dot{\xi}_{2,1} &= \xi_{2,2} \\
&\cdots \\
\dot{\xi}_{2,r_1-1} &= \xi_{2,r_1} \\
\dot{\xi}_{2,r_1} &= \xi_{2,r_1+1} + \delta^1_{2,r_1+1}\hat{K}_1\xi_1 \\
&\cdots \\
\dot{\xi}_{i,r_2-1} &= \xi_{i,r_2} + \delta^1_{i,r_2}\hat{K}_1\xi_1 \\
\dot{\xi}_{i,r_2} &= a_2(x) + b_2(x)u.
\end{aligned}$$

Let u be such that

$$a_2(x) + b_2(x)u = \hat{K}_2\xi_2 \tag{11.2}$$

in which case the set in question reduces to a linear system

$$\dot{\xi}_2 = (\hat{A}_2 + \hat{B}_2\hat{K}_2)\xi_2 + A_{21}\xi_1$$

in which $\hat{A}_2$ and $\hat{B}_2$ are matrices whose structure is the same as that of the matrices (9.50) and A_{21} is a fixed matrix. Clearly, it is possible to pick $\hat{K}_2$ so as to make $(\hat{A}_2 + \hat{B}_2\hat{K}_2)$ a Hurwitz matrix.

This procedure can be iterated up to a last stage, corresponding to $i = \ell$, in which it is assumed that u is such that

$$a_\ell(x) + b_\ell(x)u = \hat{K}_\ell\xi_\ell. \tag{11.3}$$

The Eqs. (11.1), (11.2), ..., (11.3) altogether can be written as

$$A(x) + B(x)u = K\xi\ ,$$

in which $K = \mathrm{diag}(\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_\ell)$, and can be solved for u yielding

$$u = [B(x)]^{-1}(-A(x) + K\xi). \tag{11.4}$$

Under this control, the system reduces to a system of the form

$$\begin{aligned}\dot{z} &= f_0(z,\xi) + g_0(z,\xi)[B(x)]^{-1}(-A(x)+K\xi)\\ \dot{\xi} &= A_c\xi\end{aligned} \tag{11.5}$$

in which A_c has the following structure:

$$A_c = \begin{pmatrix} (\hat{A}_1+\hat{B}_1\hat{K}_1) & 0 & 0 & \cdots & 0\\ A_{21} & (\hat{A}_2+\hat{B}_2\hat{K}_2) & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & 0\\ A_{\ell 1} & A_{\ell 2} & A_{\ell 3} & \cdots & (\hat{A}_\ell+\hat{B}_\ell\hat{K}_\ell)\end{pmatrix}$$

and can be made a Hurwitz matrix by proper choice of $\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_\ell$. If this is the case, then

$$\|\xi(t)\| \le Me^{-\alpha t}\|\xi(0)\|$$

for some $M > 0$ and $\alpha > 0$. If the system is strongly minimum phase, this implies—as shown in the proof of Proposition 10.1 in a similar context—that the equilibrium $(z,\xi) = (0,0)$ is globally asymptotically stabilized.

The feedback law thus found, though, suffers of limitations that have already been outlined in similar contexts before: it relies upon exact cancelation of nonlinear terms and presupposes the availability of the full state (z,ξ). To overcome this limitation we proceed differently, by means of a design method consisting of a blend of the procedure used in Sect. 6.5 to derive an equation of the form (6.35), and of the robust stabilization method described in Chap. 10.

To this end, we show first how the equations that describe the system can be brought to a form that can be seen as a multivariable version of the form (6.35).[2] To this end, we begin by rewriting the equations (9.34) in more compact form.[3] Setting

$$\underline{\xi}_i = \begin{pmatrix}\xi_{i,1}\\ \xi_{i,2}\\ \cdots\\ \xi_{i,r_i-2}\\ \xi_{i,r_i-1}\end{pmatrix}, \quad \underline{A}_i = \begin{pmatrix}0 & I_{m_i} & 0 & \cdots & 0\\ 0 & 0 & I_{m_i} & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & I_{m_i}\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix} \quad \underline{B}_i = \begin{pmatrix}0\\ 0\\ \cdots\\ 0\\ I_{m_i}\end{pmatrix}.$$

we obtain[4]

$$\begin{aligned}\dot{\underline{\xi}}_1 &= \underline{A}_1\underline{\xi}_1 + \underline{B}_1\xi_{1,r_1}\\ \dot{\xi}_{1,r_1} &= a_1(x) + b_1(x)u\end{aligned}$$

and, for $i = 2, \ldots, \ell$,

---

[2]The transformations described hereafter, that lead to equations having the form (11.19), have been proposed in [2].

[3]In what follows we return, for completeness, to the general case in which one or more of the m_i's can be larger than 1.

[4]We tacitly assume that $r_1 > 1$. If $r_1 = 1$ the vector $\underline{\xi}_1$ is missing and so is the associated dynamics.

$$\begin{aligned} \dot{\underline{\xi}}_i &= \underline{A}_i \underline{\xi}_i + \underline{B}_i \xi_{i,r_i} + \textstyle\sum_{j=1}^{i-1} \Delta_{ij}[a_j(x) + b_j(x)u] \\ \dot{\xi}_{i,r_i} &= a_i(x) + b_i(x)u \end{aligned}$$

where the Δ_{ij}'s are $(r_i - 1)m_i \times m_j$ matrices, partitioned into $(m_i \times m_j)$-dimensional blocks, defined as follows

$$\begin{aligned} \Delta_{i1} &= \mathrm{col}(0, \dots, 0, \delta^1_{i,r_1+1}, \dots, \delta^1_{i,r_i}) \\ \Delta_{i2} &= \mathrm{col}(0, \dots, 0, \delta^2_{i,r_2+1}, \dots, \delta^2_{i,r_i}) \\ &\cdots \\ \Delta_{i,i-1} &= \mathrm{col}(0, \dots, 0, \delta^{i-1}_{i,r_{i-1}+1}, \dots, \delta^{i-1}_{i,r_i}). \end{aligned}$$

Next, we change variables as indicated in the following lemma.

Lemma 11.1 *Pick a set of ℓ Hurwitz polynomials*

$$d_i(\lambda) = \lambda^{r_i-1} + \hat{a}_{i,r_i-2}\lambda^{r_i-2} + \cdots + \hat{a}_{i,1}\lambda + \hat{a}_{i,0} \qquad i = 1, \dots, \ell.$$

Define, for $i = 1, \dots, \ell$, the $m_i \times (r_i - 1)m_i$ matrix

$$K_i = \left(\hat{a}_{i,0} I_{m_i} \; \hat{a}_{i,1} I_{m_i} \; \cdots \; \hat{a}_{i,r_i-2} I_{m_i} \right).$$

Define a $(d - m) \times d$ matrix T_0 as

$$T_0 = \begin{pmatrix} (I_{(r_1-1)m_1} \; 0) & 0 & 0 & \cdots & 0 \\ (0 \; -\Delta_{21}) & (I_{(r_2-1)m_2} \; 0) & 0 & \cdots & 0 \\ (0 \; -\Delta_{31}) & (0 \; -\Delta_{32}) & (I_{(r_3-1)m_3} \; 0) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (0 \; -\Delta_{\ell 1}) & (0 \; -\Delta_{\ell 2}) & (0 \; -\Delta_{\ell 3}) & \cdots & (I_{(r_\ell-1)m_\ell} \; 0) \end{pmatrix} \tag{11.6}$$

and a $m \times d$ matrix T_1 as

$$T_1 = \begin{pmatrix} (K_1 \; I_{m_1}) & 0 & 0 & \cdots & 0 \\ (0 \; -K_2\Delta_{21}) & (K_2 \; I_{m_2}) & 0 & \cdots & 0 \\ (0 \; -K_3\Delta_{31}) & (0 \; -K_3\Delta_{32}) & (K_3 \; I_{m_3}) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (0 \; -K_\ell\Delta_{\ell 1}) & (0 \; -K_\ell\Delta_{\ell 2}) & (0 \; -K_\ell\Delta_{\ell 3}) & \cdots & (K_\ell \; I_{m_\ell}) \end{pmatrix}. \tag{11.7}$$

Then, the $d \times d$ matrix

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} \tag{11.8}$$

is nonsingular. Set

$$\begin{aligned} \zeta &= T_0 \xi \\ \theta &= T_1 \xi. \end{aligned} \tag{11.9}$$

Then,

$$\begin{aligned}\dot{\zeta} &= F\zeta + G\theta \\ \dot{\theta} &= H\zeta + K\theta + [A(x) + B(x)u],\end{aligned} \tag{11.10}$$

in which

(i) the $(d-m) \times (d-m)$ *matrix* F *has a lower triangular structure*

$$F = \begin{pmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ F_{\ell 1} & F_{\ell 2} & \cdots & F_{\ell\ell} \end{pmatrix}$$

where, for $i = 1, \ldots, \ell$, *the matrix* F_{ii} *is a* $(r_i - 1)m_i \times (r_i - 1)m_i$ *matrix whose characteristic polynomial coincides with* $[d_i(\lambda)]^{m_i}$, *and hence is a Hurwitz matrix,*

(ii) the $m \times 1$ *vector* $A(x)$ *and, respectively, the* $m \times m$ *matrix* $B(x)$ *are those defined in (9.36).*

Proof Define

$$\begin{aligned}\zeta_1 &= \underline{\xi}_1 \\ \zeta_i &= \underline{\xi}_i - \textstyle\sum_{j=1}^{i-1} \Delta_{i,j}\xi_{j,r_j}, \qquad \text{for } i = 2, \ldots, \ell,\end{aligned} \tag{11.11}$$

This yields

$$\begin{aligned}\dot{\zeta}_1 &= \underline{A}_1\zeta_1 + \underline{B}_1\xi_{1,r_1} \\ \dot{\zeta}_i &= \underline{A}_i\zeta_i + \textstyle\sum_{j=1}^{i-1}\underline{A}_i\Delta_{i,j}\xi_{j,r_j} + \underline{B}_i\xi_{i,r_i}, \qquad \text{for } i = 2, \ldots, \ell.\end{aligned} \tag{11.12}$$

Define now, for $i = 1, \ldots, \ell$,

$$\theta_i = K_i\zeta_i + \xi_{i,r_i}, \tag{11.13}$$

where K_i is the matrix defined in the lemma. Then,

$$\begin{aligned}\dot{\zeta}_1 &= (\underline{A}_1 - \underline{B}_1K_1)\zeta_1 + \underline{B}_1\theta_1 \\ \dot{\zeta}_i &= (\underline{A}_i - \underline{B}_iK_i)\zeta_i - \textstyle\sum_{j=1}^{i-1}\underline{A}_i\Delta_{i,j}K_j\zeta_j + \sum_{j=1}^{i-1}\underline{A}_i\Delta_{i,j}\theta_j + \underline{B}_i\theta_i, \qquad \text{for } i = 2, \ldots, \ell.\end{aligned} \tag{11.14}$$

Setting, for $i = 1, \ldots \ell$,

$$F_{ii} = \underline{A}_i - \underline{B}_iK_i \tag{11.15}$$

(from which it is easily seen that the characteristic polynomial of the matrix F_{ii} coincides with $[d_i(\lambda)]^{m_i}$),

$$G_{ii} = \underline{B}_i$$

and, for $j = 1, \ldots, i-1$,

$$\begin{aligned} F_{ij} &= -\underline{A}_i \Delta_{i,j} K_j, \\ G_{ij} &= \underline{A}_i \Delta_{i,j}, \end{aligned}$$

Eq. (11.14) can be rewritten as

$$\begin{aligned} \dot{\zeta}_1 &= F_{11}\zeta_1 + G_{11}\theta_1 \\ \dot{\zeta}_i &= F_{ii}\zeta_i + \textstyle\sum_{j=1}^{i-1} F_{ij}\zeta_j + \sum_{j=1}^{i} G_{ij}\theta_j, \qquad \text{for } i = 2, \ldots, \ell. \end{aligned} \tag{11.16}$$

Moreover

$$\begin{aligned} \dot{\theta}_1 &= K_1[F_{11}\zeta_1 + G_{11}\theta_1] + a_1(x) + b_1(x)u \\ \dot{\theta}_i &= K_i[F_{ii}\zeta_i + \textstyle\sum_{j=1}^{i-1} F_{ij}\zeta_j + \sum_{j=1}^{i} G_{ij}\theta_j] + a_i(x) + b_i(x)u, \qquad \text{for } i = 2, \ldots, \ell. \end{aligned} \tag{11.17}$$

The equations thus found can be easily organized in compact form. Setting

$$\zeta = \mathrm{col}(\zeta_1, \zeta_2, \ldots, \zeta_\ell), \qquad \theta = \mathrm{col}(\theta_1, \theta_2, \ldots, \theta_\ell),$$

$$F = \begin{pmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ F_{\ell 1} & F_{\ell 2} & \cdots & F_{\ell\ell} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & 0 & \cdots & 0 \\ G_{21} & G_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ G_{\ell 1} & G_{\ell 2} & \cdots & G_{\ell\ell} \end{pmatrix},$$

we have in fact

$$\dot{\zeta} = F\zeta + G\theta, \tag{11.18}$$

and

$$\dot{\theta} = H\zeta + K\theta + [A(x) + B(x)u],$$

in which H and K are suitable matrices and $A(x)$ and $B(x)$ are those defined in (9.36). By construction $\zeta = T_0\xi$, in which T_0 is the matrix (11.6) and $\theta = T_1\xi$ in which T_1 is the matrix (11.7). Inspection of T_0 and T_1 shows that the matrix (11.8) is invertible and this concludes the proof of the lemma. ◁

It is seen from this lemma that the system under consideration, if all $\delta_{ik}^j(\cdot)$'s are independent of x, can be modeled by equations that have the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) + g_0(z, \xi)u \\ \dot{\zeta} &= F\zeta + G\theta \\ \dot{\theta} &= H\zeta + K\theta + [A(x) + B(x)u]. \end{aligned} \tag{11.19}$$

in which $x = (z, \xi)$ and ξ, in turn, is a linear function of (ζ, θ).

The output of the system is a linear function of ξ and hence, in the coordinates of (11.19), a linear function of (ζ, η). However, in the equations above we have deliberately omitted the indication of how the output of the system is expressed as a function of (ζ, θ) because, in the spirit of the construction used in Sect. 6.5 to derive an equation of the form (6.35), we proceed now by looking at the m-dimensional vector θ as if it were the "output" of (11.19). The first and more straightforward consequence of this viewpoint is that, since the matrix $B(x)$ is *nonsingular* for all x, if θ is regarded as "output" of a system whose dynamics have the structure (11.19), a system having *vector relative degree* $\{1, 1, \ldots, 1\}$ is obtained.

As a matter of fact, the equations thus found can be seen as characterizing a normal form for such system. In fact, if we set[5]

$$\bar{z} = \begin{pmatrix} z \\ \zeta \end{pmatrix}, \quad \bar{f}_0(\bar{z}, \theta) = \begin{pmatrix} f_0(z, \xi) \\ F\zeta + G\theta \end{pmatrix}, \quad \bar{g}_0(\bar{z}, \theta) = \begin{pmatrix} g_0(z, \xi) \\ 0 \end{pmatrix}$$

and

$$\bar{q}(\bar{z}, \theta) = H\zeta + K\theta + A(x), \qquad \bar{b}(\bar{z}, \theta) = B(x)$$

the Eq. (11.19) can be put in the form

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}_0(\bar{z}, \theta) + \bar{g}_0(\bar{z}, \theta)u \\ \dot{\theta} &= \bar{q}(\bar{z}, \theta) + \bar{b}(\bar{z}, \theta)u, \end{aligned} \tag{11.20}$$

which, since $\bar{b}(\bar{z}, \theta)$ is nonsingular for all $(\bar{z}, \theta)$, is precisely the standard structure of the normal form of a system—with input u and output θ—having vector relative degree $\{1, 1, \ldots, 1\}$. Note also that if, as always assumed, in the original system $h(0) = 0$ and $f(0) = 0$, then in the model (11.20) we have

$$\bar{f}_0(0, 0) \qquad \text{and} \qquad \bar{q}(0, 0) = 0.$$

The second relevant consequence is that, if system (11.19) is strongly minimum phase, so is system (11.20), as shown in the lemma below.

Lemma 11.2 *Let T_1 be defined as in Lemma 11.1. Consider system (11.19) and suppose there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any $x(0) \in \mathbb{R}^n$ and any admissible input function $u(\cdot) : [0, \infty) \to \mathbb{R}^m$, so long as $x(t)$ is defined the estimate*

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\|\xi(\cdot)\|_{[0,t]}) \tag{11.21}$$

holds. Then, there exist a class $\mathcal{KL}$ function $\bar{\beta}(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\bar{\gamma}(\cdot)$ such that, for any $x(0) \in \mathbb{R}^n$ and any admissible input function $u(\cdot) : [0, \infty) \to \mathbb{R}^m$, so long as $x(t)$ is defined the estimate

$$\|\bar{z}(t)\| \leq \beta(\|\bar{z}(0)\|, t) + \gamma(\|\theta(\cdot)\|_{[0,t]}) \tag{11.22}$$

holds.

[5]Recall that ξ is a linear function of (ζ, θ).

Proof As shown in Lemma 11.1, the matrix F in (11.18) is Hurwitz. If this is the case, the *linear* system (11.18), viewed as a system with input θ and state ζ, is input-to-state stable. In particular, there exist positive numbers M, α, L such that, for any piecewise-continuous bounded function $\theta(\cdot) : [0, \infty) \to \mathbb{R}^m$, the response $\zeta(t)$ of (11.18) from the initial state $\zeta(0)$ satisfies

$$\|\zeta(t)\| \leq Me^{-\alpha t}\|\zeta(0)\| + L\|\theta(\cdot)\|_{[0,t]}.$$

Bearing in mind the fact that ξ is a linear function of (ζ, θ), it follows that $\xi(t)$ satisfies an estimate of the form

$$\|\xi(t)\| \leq M'e^{-\alpha t}\|\zeta(0)\| + L'\|\theta(\cdot)\|_{[0,t]}, \tag{11.23}$$

so long as it is defined.

Combining the estimate (11.21) with the estimate (11.23), it is concluded that there exist a class $\mathscr{KL}$ function $\bar{\beta}(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\bar{\gamma}(\cdot)$ such that, for any $x(0) \in \mathbb{R}^n$ and any admissible input function $u(\cdot) : [0, \infty) \to \mathbb{R}^m$, so long as $x(t)$ is defined the an estimate of the form (11.22) holds. ◁

Remark 11.1 Using identical arguments it is concluded that if system (11.19) is strongly—and also locally exponentially—minimum phase, so is system (11.20). ◁

11.2 Stabilization by Partial-State Feedback

System (11.20), which is nothing else than the original system (11.19) with the sets of coordinates z and ζ grouped together, regarded as a system with output θ has vector relative degree $\{1, 1, \ldots, 1\}$. If (11.19) is strongly minimum phase and the matrix T (that defines the variable $\theta = T_1\xi$) is chosen as indicated in Lemma 11.1, then also system (11.20) is strongly minimum phase. Thus, the stabilization problem can be simply handled by means of the procedures described in Chap. 10.

Specifically, the system can be globally stabilized by means of the *full state* feedback law

$$u = [\bar{b}(\bar{z}, \theta)]^{-1}(-\bar{q}(\bar{z}, \theta) - \theta)$$

which, in the coordinates of (11.19), can be written as

$$u = [B(x)]^{-1}(-H\zeta - K\theta - A(x) - \theta).$$

Such feedback law, in fact, yields

$$\dot{\theta} = -\theta$$

and this, in view of the fact that system (11.20) is strongly minimum phase, suffices to obtain global asymptotic stability of the equilibrium $(\bar{z}, \theta) = (0, 0)$ (see Proposition 10.1 in this respect). The control law in question, though, is not useful as it relies upon exact cancelations and availability of the full state of the system. The results of Chap. 10 have shown, however, that a more "robust" version of such control law can be constructed, that yields asymptotic stability with a guaranteed region of attraction.

As a matter of fact, by appealing to the construction presented in Chap. 10, it possible to claim that the desired goal can be achieved by means of a dynamic control law of the form (10.7) and (10.8). In view of the fact that the vector relative degree is now $\{1, 1, \ldots, 1\}$, the argument $\psi(\xi, \sigma)$ of (10.7), in which ξ is replaced by θ, becomes $\mathbf{b}^{-1}[-\theta - \sigma]$ and the dynamic control law becomes[6]

$$\begin{aligned} u &= G(\mathbf{b}^{-1}[-\chi - \sigma]) \\ \dot{\chi} &= \sigma + \mathbf{b}G(\mathbf{b}^{-1}[-\chi - \sigma]) + \kappa c_1(\theta - \chi) \\ \dot{\sigma} &= \kappa^2 c_0(\theta - \chi). \end{aligned} \tag{11.24}$$

Then, results identical to those indicated in Propositions 10.2 and 10.3 hold. In particular, so long as the issue of asymptotic stability is concerned, the following result holds.

Assumption 11.1 There exist a constant nonsingular matrix $\mathbf{b} \in \mathbb{R}^{m\times m}$ and a number $0 < \delta_0 < 1$ such that the matrix $\bar{b}(\bar{z}, \theta)$ in (11.20) satisfies

$$\max_{\Lambda \text{ diagonal}, \ \|\|\leq 1} \|[\bar{b}(\bar{z}, \theta) - \mathbf{b}]\Lambda\mathbf{b}^{-1}\| \leq \delta_0 \qquad \text{for all } (\bar{z}, \theta). \tag{11.25}$$

Remark 11.2 Recall that the matrix $\bar{b}(\bar{z}, \theta)$ in (11.20) coincides with $B(x)$. Thus, the condition of Assumption 11.1 can also be written as

$$\max_{\Lambda \text{ diagonal}, \ \|\|\leq 1} \|[B(x) - \mathbf{b}]\Lambda\mathbf{b}^{-1}\| \leq \delta_0 \qquad \text{for all } x. \qquad \triangleleft$$

Proposition 11.1 *Consider system (11.19). Suppose the system is strongly—and also locally exponentially—minimum phase and Assumption 11.1 holds. Let the system be controlled by (11.24), with* $\mathbf{b}$ *chosen so as to satisfy the condition in Assumption 11.1. For every choice of a compact set* $\mathscr{C}$ *there is a choice of the design parameters,* L *and* $c_{i,1}, c_{i,2}$*, and a number* κ^* *such that, if* $\kappa \geq \kappa^*$*, the equilibrium* $(\bar{z}, \theta, \hat{\theta}, \sigma) = (0, 0, 0, 0)$ *is asymptotically stable, with a domain of attraction* $\mathscr{A}$ *that contains the set* $\mathscr{C}$.

The controller (11.24) is driven by the variable θ which, as shown before, has the form $\theta = T_1\xi$. Thus, the control in question is driven by a *partial state*. This control does not rely upon exact knowledge of the functions of (z, ξ) that characterize the model (11.19). It does require, though, availability of the entire partial state ξ that is necessary to build the function θ. The matrix T_1, as indicated in Lemma 11.1, is a matrix of design parameters and, as such, can be regarded as accurately known.

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[6]For convenience, we write here χ instead of $\hat{\theta}$.

Therefore, so long as all components of ξ are available, the control law in question can be regarded as *robust*. We will discuss in the next section how such partial state can be estimated.

11.3 Stabilization via Dynamic Output Feedback

The stabilization method described in the previous section reposes on the availability of the redesigned output θ. This, in turn, is a linear function of all d components of the vector $\xi = \Xi(x)$ defined in (9.18). In this section, we show how this vector can be approximately estimated, by means of a high-gain observer driven only by the actual measured output y. We retain the assumption that in the Eq. (9.34) the $\delta^j_{ik}(x)$'s are *constant* coefficients.

The matter is to estimate the vectors $\xi_1, \xi_2, \ldots, \xi_\ell$. As far as the vector ξ_1 is concerned, observe that its components $\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,r_1}$ satisfy

$$\begin{aligned}
\dot{\xi}_{1,1} &= \xi_{1,2}\\
\dot{\xi}_{1,2} &= \xi_{1,3}\\
&\cdots\\
\dot{\xi}_{1,r_1-1} &= \xi_{1,r_1}\\
\dot{\xi}_{1,r_1} &= a_1(x) + b_1(x)u.
\end{aligned}$$

Their estimate can be provided by a system of the form

$$\begin{aligned}
\dot{\hat{\xi}}_{1,1} &= \hat{\xi}_{1,2} + \kappa_1 c_{1,r_1-1}(\bar{y}_1 - \hat{\xi}_{1,1})\\
\dot{\hat{\xi}}_{1,2} &= \hat{\xi}_{1,3} + \kappa_1^2 c_{1,r_1-2}(\bar{y}_1 - \hat{\xi}_{1,1})\\
&\cdots\\
\dot{\hat{\xi}}_{1,r_1-1} &= \hat{\xi}_{1,r_1} + \kappa_1^{r_1-1} c_{1,1}(\bar{y}_1 - \hat{\xi}_{1,1})\\
\dot{\hat{\xi}}_{1,r_1} &= \kappa_1^{r_1} c_{1,0}(\bar{y}_1 - \hat{\xi}_{1,1})
\end{aligned} \tag{11.26}$$

in which the coefficients κ_1 and $c_{1,0}, c_{1,2}, \ldots, c_{1,r_1-1}$ are design parameters. Defining a vector e_1 of estimation errors as (recall that $\bar{y}_1 = \xi_{1,1}$)

$$e_1 = \begin{pmatrix} \kappa_1^{r_1-1} I_{m_1} & 0 & \cdots & 0\\ 0 & \kappa_1^{r_1-2} I_{m_1} & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdots & I_{m_1} \end{pmatrix} \begin{pmatrix} \xi_{1,1} - \hat{\xi}_{1,1}\\ \xi_{1,2} - \hat{\xi}_{1,2}\\ \cdots\\ \xi_{1,r_1} - \hat{\xi}_{1,r_1} \end{pmatrix} \tag{11.27}$$

it is readily seen that

$$\dot{e}_1 = \kappa_1 M_{11} e_1 + N_1[a_1(x) + b_1(x)u] \tag{11.28}$$

in which

$$M_{11} = \begin{pmatrix} -c_{1,r_1-1}I_{m_1} & I_{m_1} & 0 & \cdots & 0 \\ -c_{1,r_1-2}I_{m_1} & 0 & I_{m_1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -c_{1,1}I_{m_1} & 0 & 0 & \cdots & I_{m_1} \\ -c_{1,0}I_{m_1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ I_{m_1} \end{pmatrix}.$$

Let the parameters $c_{1,0}, c_{1,2}, \ldots, c_{1,r_1-1}$ be chosen in such a way that M_{11} is a Hurwitz matrix.

As far as the set ξ_2 is concerned, let this vector be split as $\mathrm{col}(\underline{\xi}_2, \xi_{2,r_2})$, as already done before Lemma 11.1, and recall that

$$\begin{aligned} \dot{\underline{\xi}}_2 &= \underline{A}_2\underline{\xi}_2 + \underline{B}_2\xi_{2,r_2} + \Delta_{21}[a_1(x) + b_1(x)u] \\ \dot{\xi}_{2,r_2} &= a_2(x) + b_2(x)u. \end{aligned}$$

As done in the proof of Lemma 11.1, in order to simplify these equations, it is convenient to replace the vector $\underline{\xi}_2$ by a vector ζ_2 defined as in (11.11), that is by

$$\zeta_2 = \underline{\xi}_2 - \Delta_{2,1}\xi_{1,r_1}.$$

This yields

$$\begin{aligned} \dot{\zeta}_2 &= \underline{A}_2\zeta_2 + \underline{A}_2\Delta_{2,1}\xi_{1,r_1} + \underline{B}_2\xi_{2,r_2} \\ \dot{\xi}_{2,r_2} &= a_2(x) + b_2(x)u. \end{aligned}$$

To the purpose of expressing these equations in detail, observe that Δ_{21} and $\underline{A}_2\Delta_{2,1}$ can be split, consistently with the partition of $\underline{\xi}_2$ in $r_2 - 1$ blocks, each one of dimension m_2, as

$$\Delta_{21} = \begin{pmatrix} \Delta_1 \\ \cdots \\ \Delta_{r_2-2} \\ \Delta_{r_2-1} \end{pmatrix}, \qquad \underline{A}_2\Delta_{21} = \begin{pmatrix} \Delta_2 \\ \cdots \\ \Delta_{r_2-1} \\ 0 \end{pmatrix}.$$

In this way, it is seen that the previous set of equations has the following structure:

$$\begin{pmatrix} \dot{\zeta}_{2,1} \\ \dot{\zeta}_{2,2} \\ \cdots \\ \dot{\zeta}_{2,r_2-1} \\ \dot{\xi}_{2,r_2} \end{pmatrix} = \begin{pmatrix} 0 & I_{m_2} & 0 & \cdots & 0 \\ 0 & 0 & I_{m_2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I_{m_2} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \zeta_{2,1} \\ \zeta_{2,2} \\ \cdots \\ \zeta_{2,r_2-1} \\ \xi_{2,r_2} \end{pmatrix} + \begin{pmatrix} \Delta_2\xi_{1,r_1} \\ \Delta_3\xi_{1,r_1} \\ \cdots \\ 0 \\ a_2(x) + b_2(x)u \end{pmatrix}.$$

This structure would suggests the use, for the estimation of $\zeta_{2,1}, , \ldots, \zeta_{2,r_2-1}, \xi_{2,r_2}$, of a system of the form

$$\begin{aligned}
\dot{\hat{\zeta}}_{2,1} &= \hat{\zeta}_{2,2} + \kappa_2 c_{2,r_2-1}(\zeta_{2,1} - \hat{\zeta}_{2,1}) + \Delta_2 \xi_{1,r_1} \\
\dot{\hat{\zeta}}_{2,2} &= \hat{\zeta}_{2,3} + \kappa_2^2 c_{2,r_2-2}(\zeta_{2,1} - \hat{\zeta}_{2,1}) + \Delta_3 \xi_{1,r_1} \\
&\cdots \\
\dot{\hat{\zeta}}_{2,r_2-1} &= \hat{\xi}_{2,r_2} + \kappa_2^{r_2-1} c_{2,1}(\zeta_{2,1} - \hat{\zeta}_{2,1}) \\
\dot{\hat{\xi}}_{2,r_2} &= \kappa_2^{r_2} c_{2,0}(\zeta_{2,1} - \hat{\zeta}_{2,1})
\end{aligned}$$

in which the coefficients κ_2 and $c_{2,0}, c_{2,1}, \dots, c_{2,r_2-1}$ are design parameters. In this expression, however, ξ_{1,r_1} and $\zeta_{2,1}$ are not directly available. Hence, we replace them by the estimates provided by (11.26), i.e., we replace ξ_{1,r_1} by $\hat{\xi}_{1,r_1}$ and $\zeta_{2,1}$, which by definition is equal to $\bar{y}_2 - \Delta_1 \xi_{1,r_1}$, by $\bar{y}_2 - \Delta_1 \hat{\xi}_{1,r_1}$. This yields

$$\begin{aligned}
\dot{\hat{\zeta}}_{2,1} &= \hat{\zeta}_{2,2} + \kappa_2 c_{2,r_2-1}(\bar{y}_2 - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1}) + \Delta_2 \hat{\xi}_{1,r_1} \\
\dot{\hat{\zeta}}_{2,2} &= \hat{\zeta}_{2,3} + \kappa_2^2 c_{2,r_2-2}(\bar{y}_2 - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1}) + \Delta_3 \hat{\xi}_{1,r_1} \\
&\cdots \\
\dot{\hat{\zeta}}_{2,r_2-1} &= \hat{\xi}_{2,r_2} + \kappa_2^{r_2-1} c_{2,1}(\bar{y}_2 - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1}) \\
\dot{\hat{\xi}}_{2,r_2} &= \kappa_2^{r_2} c_{2,0}(\bar{y}_2 - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1}).
\end{aligned} \tag{11.29}$$

Define a vector e_2 of estimation errors as

$$e_2 = \begin{pmatrix} \kappa_2^{r_2-1} I_{m_2} & 0 & \cdots & 0 & 0 \\ 0 & \kappa_2^{r_2-2} I_{m_2} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \kappa_2 I_{m_2} & 0 \\ 0 & 0 & \cdots & 0 & I_{m_2} \end{pmatrix} \begin{pmatrix} \zeta_{2,1} - \hat{\zeta}_{2,1} \\ \zeta_{2,2} - \hat{\zeta}_{2,2} \\ \cdots \\ \zeta_{2,r_2-1} - \hat{\zeta}_{2,r_2-1} \\ \xi_{2,r_2} - \hat{\xi}_{2,r_2} \end{pmatrix}. \tag{11.30}$$

Bearing in mind the fact that

$$\begin{aligned}
\bar{y}_2 - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1} &= \zeta_{2,1} + \Delta_1 \xi_{1,r_1} - \hat{\zeta}_{2,1} - \Delta_1 \hat{\xi}_{1,r_1} \\
&= \kappa_2^{-(r_2-1)} e_{2,1} + \Delta_1 e_{1,r_1}
\end{aligned}$$

this yields

$$\dot{e}_2 = \kappa_2 M_{22} e_2 + M_{21}(\kappa_2) e_1 + N_2[a_2(x) + b_2(x)u] \tag{11.31}$$

in which

$$M_{22} = \begin{pmatrix} -c_{2,r_2-1} I_{m_2} & I_{m_2} & 0 & \cdots & 0 \\ -c_{2,r_2-2} I_{m_2} & 0 & I_{m_2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -c_{2,1} I_{m_2} & 0 & 0 & \cdots & I_{m_2} \\ -c_{2,0} I_{m_2} & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ I_{m_2} \end{pmatrix},$$

and

$$M_{21}(\kappa_2) = \begin{pmatrix} 0 \cdots 0 & (\kappa_2^{r_2-1}\Delta_2 - \kappa_2^{r_2} c_{2,r_2-1}\Delta_1) \\ 0 \cdots 0 & (\kappa_2^{r_2-2}\Delta_3 - \kappa_2^{r_2} c_{2,r_2-2}\Delta_1) \\ \cdot \;\cdot \;\cdot \;\cdot \;\cdot & \cdot \\ 0 \cdots 0 & (-\kappa_2^{r_2} c_{2,1}\Delta_1) \\ 0 \cdots 0 & (-\kappa_2^{r_2} c_{2,0}\Delta_1) \end{pmatrix}.$$

Note that $M_{21}(\kappa_2)$ is a polynomial of degree r_2 in κ_2, i.e., can be expressed as

$$M_{21}(\kappa_2) = \overline{M}_{21,2}\kappa_2^2 + \cdots + \overline{M}_{21,r_2}\kappa^{r_2}.$$

Suppose, for the time being, that $\ell = 2$. In this case, the design of the estimator is concluded and we see that the estimation errors e_1, e_2 satisfy

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} \kappa_1 M_{11} & 0 \\ M_{21}(\kappa_2) & \kappa_2 M_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} [A(x) + B(x)u], \qquad (11.32)$$

in which M_{11} and M_{22} are Hurwitz matrices. System (11.32) can be seen as a linear system driven by the input $[A(x) + B(x)u]$. With the results of Sect. 10.4 in mind, it is important to prove a property similar to the property proven in Lemma 10.4, i.e., that, for any choice of a time $T > 0$ and a number ε, there is a choice of the "gain parameters" κ_1, κ_2 such that $\|e(t)\| \leq 2\varepsilon$ for all $t \geq T$. To this end, the following result is useful.

Lemma 11.3 *Consider a matrix of the form*

$$A(\rho_1, \rho_2) = \begin{pmatrix} A_1(\rho_1) & 0 \\ B(\rho_2) & \rho_2 A_2 \end{pmatrix}$$

in which the elements of $A_1(\rho_1)$ are polynomial functions of ρ_1, and the elements of $B(\rho_2)$ are polynomial functions of ρ_2. Assume that

(i) there exist a positive-definite matrix P_1, independent of ρ_1, and a number ρ_1^* such that, if $\rho_1 > \rho_1^*$,

$$P_1 A_1(\rho_1) + A_1^{\mathrm{T}}(\rho_1) P_1 \leq -\rho_1 I$$

(ii) the matrix A_2 is Hurwitz.

Then, there is a positive-definite matrix P, independent of ρ_2, an integer p and a number ρ_2^* such that, if $\rho_2 > \rho_2^*$,

$$PA(\rho_2^p, \rho_2) + A^{\mathrm{T}}(\rho_2^p, \rho_2)P \leq -\rho_2 I.$$

Proof Pick a positive-definite matrix P_2 satisfying $P_2 A_2 + A_2^{\mathrm{T}} P_2 = -2I$, and consider the positive-definite matrix

$$P = \begin{pmatrix} aP_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Suppose $\rho_1 > \rho_1^*$ and compute the derivative of $V(x) = x^{\mathrm{T}}Px$ along the solutions of $\dot{x} = A(\rho_1, \rho_2)x$, to obtain

$$\dot{V} \le -a\rho_1\|x_1\|^2 - 2\rho_2\|x_2\|^2 + 2\|P_2\|\,\|B(\rho_2)\|\,\|x_1\|\,\|x_2\|.$$

We seek a choice of ρ_1 that makes

$$\dot{V} \le -\rho_2(\|x_1\|^2 + \|x_2\|^2),$$

which is the case if the matrix

$$\begin{pmatrix} -a\rho_1 + \rho_2 & \|P_2\|\,\|B(\rho_2)\| \\ \|P_2\|\,\|B(\rho_2)\| & -\rho_2 \end{pmatrix}$$

is negative definite. To this end, we need

$$\begin{aligned} a\rho_1 &> \rho_2 \\ (a\rho_1 - \rho_2)\rho_2 &> [\|P_2\|\,\|B(\rho_2)\|]^2. \end{aligned} \tag{11.33}$$

By assumption, $B(\rho_2)$ is a polynomial in ρ_2, and therefore there exists an integer r such that

$$\|B(\rho_2)\|^2 = b_{2r}\rho_2^{2r} + b_{2r-1}\rho_2^{2r-1} + \cdots + b_1\rho_2 + b_0.$$

From this, it is seen that if $\rho_1 = \rho_2^{2r-1}$ and a is large enough, there exist a number ρ_2^* such that (11.33) hold for all $\rho_2 > \rho_2^*$, which proves the lemma. $\triangleleft$

Using this result, it is straightforward to conclude that, having chosen κ_1 as $\kappa_1 = \kappa_2^p$ (with $p = 2r_2 - 1$, as shown in the proof of the lemma), there is a positive-definite matrix P and a number κ_2^* such that, if $\kappa_2 > \kappa_2^*$

$$P\begin{pmatrix} \kappa_1 M_{11} & 0 \\ M_{21}(\kappa_2) & \kappa_2 M_{22} \end{pmatrix} + \begin{pmatrix} \kappa_1 M_{11} & 0 \\ M_{21}(\kappa_2) & \kappa_2 M_{22} \end{pmatrix}^{\mathrm{T}} P \le -\kappa_2 I.$$

From this, the reader should have no difficulties to prove a result similar to that of Lemma 10.4.

With this in mind, we return to the control law (11.24) which, as we have observed, is driven by a quantity

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} K_1\zeta_1 + \xi_{1,r_1} \\ K_2\zeta_2 + \xi_{2,r_2} \end{pmatrix}$$

that is not directly available for measurement. Following the design paradigm illustrated in Sect. 7.5, in the expression of θ_i the (unavailable) states ζ_i and ξ_{i,r_i} are to be replaced by the estimates $\hat{\zeta}_i$ and $\hat{\xi}_{i,r_i}$ provided by the estimators (11.26) and (11.29).

In the arguments used in Sect. 7.5, the actual control insisting on the system was obtained through appropriate saturation functions, whose role was essentially that of guaranteeing boundedness of the trajectories over the initial time interval $[0, T_0]$, an interval of time during which, as a consequence of the choice of large value of κ, the estimation error $e(t)$ could become excessively large (and hence, possibly, be the cause of finite escape times). The same strategy should be followed in the present context, and this suggests to replace the control (11.24) by a control of the form

$$\begin{aligned} u &= G(\mathbf{b}^{-1}[-\chi - \sigma]) \\ \dot{\chi} &= \sigma + \mathbf{b}G(\mathbf{b}^{-1}[-\chi - \sigma]) + \kappa c_1(S(\hat{\theta}) - \chi) \\ \dot{\sigma} &= \kappa^2 c_0(S(\hat{\theta}) - \chi), \end{aligned} \tag{11.34}$$

in which $S(\cdot)$ a saturation function and

$$\hat{\theta} = \begin{pmatrix} K_1\hat{\zeta}_1 + \hat{\xi}_{1,r_1} \\ K_2\hat{\zeta}_2 + \hat{\xi}_{2,r_2} \end{pmatrix},$$

with $\hat{\zeta}_1$, $\hat{\xi}_{1,r_1}$, $\hat{\zeta}_2$, $\hat{\xi}_{2,r_2}$ generated by (11.26) and (11.29). This results in a dynamical system driven only by the measured output y.

From this, the reader should have no difficulties in arriving at conclusions essentially identical to those of Propositions 10.2 and 10.3, i.e., to prove that the free design parameters can be chosen in such a was as to secure semiglobal practical stability, or even semiglobal asymptotic stability. Details are omitted and left to the reader.

In the discussion above we have assumed, for convenience, $\ell = 2$. The procedure can be recursively extended to cases in which $\ell > 2$. If $\ell = 3$, one considers the change of variables defined in (11.11) for $i = 3$, namely

$$\zeta_3 = \underline{\xi}_3 - \Delta_{31}\xi_{1,r_1} - \Delta_{32}\xi_{2,r_2}.$$

Then, an estimator having a structure similar to that of (11.29) can be designed. Finally, to show that the dynamics of the estimation error has the desired asymptotic properties, Lemma 11.3 can be used.[7] In the same manner one can treat the cases of higher values of ℓ.

11.4 Handling More General Classes of Invertible Systems

In this section we describe how, under suitable hypotheses, it is possible to handle cases in which the multipliers $\delta^j_{i,k}(x)$ in (9.34) are *not* constant. The first of these hypotheses is that the zero dynamics of the system are *trivial*, i.e., that $d = n$, in

[7] This lemma has been, in fact, stated in such a way that a recursive usage is possible.

which case the collection of all (9.34) characterizes a *full normal form*. The second of these hypotheses is that the multipliers $\delta^j_{i,k}(x)$ in (9.34) depend on the various components of x in a special "triangular" fashion. To simplify matters, we consider the case of a system having only two inputs and outputs[8] and we use $x_{i,k}$—instead of $\xi_{i,k}$—to denote the various components of the state vector x. For convenience, we let $[x_i]_k$ denote the vector

$$[x_i]_k = \mathrm{col}(x_{i,1}, \ldots, x_{i,k}),$$

and we let x_i denote the vector

$$x_i = \mathrm{col}(x_{i,1}, \ldots, x_{i,r_i}) = [x_i]_{r_i}.$$

The model of the system consists in the aggregate of a set of the form (9.34) written for $i = 1$, namely

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} \\
\dot{x}_{1,2} &= x_{1,3} \\
&\cdots \\
\dot{x}_{1,r_1-1} &= x_{1,r_1} \\
\dot{x}_{1,r_1} &= a_1(x) + b_1(x)u \\
y_1 &= x_{1,1}
\end{aligned} \tag{11.35}$$

and of a set of the form (9.34) written for $i = 2$ (note that $r_2 > r_1$). For the latter set, it is *assumed* that the multiplier $\delta^1_{2,k}(x)$ only depends on x_1 and on $[x_2]_{k-1}$. More explicitly, it is assumed that the set in question has the following structure:

$$\begin{aligned}
\dot{x}_{2,1} &= x_{2,2} \\
\dot{x}_{2,2} &= x_{2,3} \\
&\cdots \\
\dot{x}_{2,r_1-1} &= x_{2,r_1} \\
\dot{x}_{2,r_1} &= x_{2,r_1+1} + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})(a_1(x) + b_1(x)u) \\
&\cdots \\
\dot{x}_{2,r_2-1} &= x_{2,r_2} + \delta_{2,r_2}(x_1, [x_2]_{r_2-1})(a_1(x) + b_1(x)u) \\
\dot{x}_{2,r_2} &= a_2(x) + b_2(x)u \\
y_2 &= x_{2,1}.
\end{aligned} \tag{11.36}$$

The relevance of this particular structure, in the context of the problem of designing feedback stabilizing laws, resides in the facts that, for such special class of systems: (i) there exists a state feedback law that globally stabilizes the equilibrium $x = 0$, (ii) the state x can be expressed as a function of the derivatives of the output y with respect to time, up to order $r_2 - 1$. These two facts, together, make it possible to develop an output feedback stabilization scheme, as shown in the second part of this

[8]For the analysis of the general case, see [3].

section. For the time being, we proceed with the proof of the two properties outlined above.[9]

Lemma 11.4 *There exists a feedback law $u = \alpha(x)$ that globally asymptotically stabilizes the equilibrium $x = 0$ of (11.35) and (11.36).*

Proof Consider the set (11.35). Pick a function $v(x_1)$ so that the equilibrium $x_1 = 0$ of

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} \\
&\cdots \\
\dot{x}_{1,r_1-1} &= x_{1,r_1} \\
\dot{x}_{1,r_1} &= v_1(x_1)
\end{aligned}$$

is globally asymptotically stable and let u be such that

$$a_1(x) + b_1(x)u = v_1(x_1). \tag{11.37}$$

If this is the case, the set (11.36) becomes

$$\begin{aligned}
\dot{x}_{2,1} &= x_{2,2} \\
\dot{x}_{2,2} &= x_{2,3} \\
&\cdots \\
\dot{x}_{2,r_1-1} &= x_{2,r_1} \\
\dot{x}_{2,r_1} &= x_{2,r_1+1} + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})v_1(x_1) \\
&\cdots \\
\dot{x}_{2,r_2-1} &= x_{2,r_2} + \delta_{2,r_2}(x_1, [x_2]_{r_2-1})v_1(x_1) \\
\dot{x}_{2,r_2} &= a_2(x) + b_2(x)u.
\end{aligned}$$

This system has a structure similar to that of the system considered in Remark 6.8. In particular, in the jth equation, the variable $x_{2,j+1}$ can be regarded as a *virtual* control. Thus, the reader should have no difficulties in proving the existence of a function $v_2(x_1, x_2)$ such that, if

$$a_2(x) + b_2(x)u = v_2(x_1, x_2), \tag{11.38}$$

the equilibrium $(x_1, x_2) = (0, 0)$ of the entire system is globally asymptotically stable. Choosing u in such a way that both (11.37) and (11.38) hold, which is indeed possible because the matrix $B(x)$ is nonsingular, proves the lemma. ◁

Lemma 11.5 *Set, for $i = 1, 2$,*

$$\mathbf{y}_i^j = \mathrm{col}(y_i, y_i^{(1)}, \ldots, y_i^{(j-1)}).$$

---

[9] It is also easy to check that, if the $\delta_{2,k}(x)$'s have the indicated structure, the system is uniformly invertible (see also Remark 9.5).

There exists a map $\Psi : \mathbb{R}^{2r_2} \to \mathbb{R}^{r_1+r_2}$ *such that*

$$x = \Psi(\mathbf{y}_1^{r_2}, \mathbf{y}_2^{r_2}). \tag{11.39}$$

Proof Observe that, by definition

$$\begin{aligned} x_{1i} &= y_1^{(i-1)} \\ x_{2i} &= y_2^{(i-1)} \end{aligned} \qquad \text{for } i = 1, \dots, r_1. \tag{11.40}$$

So long as x_{2,r_1+1} is concerned, observe that

$$\begin{aligned} x_{2,r_1+1} &= \dot{x}_{2,r_1} - \delta_{2,r_1+1}(x_1, [x_2]_{r_1})[a_1(x) + b_1(x)u] \\ &= y_2^{(r_1)} - \delta_{2,r_1+1}(x_1, [x_2]_{r_1})y_1^{(r_1)}, \end{aligned}$$

in which the various components of the arguments x_1 and $[x_2]_{r_1}$ of $\delta_{2,r_1+1}(\cdot)$ coincide with $y_1, \dots, y_1^{(r_1-1)}$ and, respectively, with $y_2, \dots, y_2^{(r_1-1)}$, as shown in (11.40). Thus, it is concluded that there exists a function $\psi_{2,r_1+1}(\cdot)$ such that

$$x_{2,r_1+1} = \psi_{2,r_1+1}\left(y_1, \dots, y_1^{(r_1)}, y_2, \dots, y_2^{(r_1)}\right). \tag{11.41}$$

So long as x_{2,r_1+2} is concerned, observe that

$$\begin{aligned} x_{2,r_1+2} &= \dot{x}_{2,r_1+1} - \delta_{2,r_1+2}(x_1, [x_2]_{r_1+1})[a_1(x) + b_1(x)u] \\ &= \dot{\psi}_{2,r_1+1}\left(y_1, \dots, y_1^{(r_1)}, y_2, \dots, y_2^{(r_1)}\right) - \delta_{2,r_1+2}(x_1, [x_2]_{r_1+1})y_1^{(r_1)}. \end{aligned}$$

The first term $\dot{\psi}_{2,r_1+1}(\cdot)$ is a function $y_1, \dots, y_1^{(r_1+1)}, y_2, \dots, y_2^{(r_1+1)}$, while the arguments x_1 and $[x_2]_{r_1+1}$ of $\delta_{2,r_1+2}(\cdot)$ are functions of $y_1, \dots, y_1^{(r_1)}, y_2, \dots, y_2^{(r_1)}$, as shown in (11.40) and (11.41). Thus it is concluded that there exists a function $\psi_{2,r_1+2}(\cdot)$ such that

$$x_{2,r_1+2} = \psi_{2,r_1+2}\left(y_1, \dots, y_1^{(r_1+1)}, y_2, \dots, y_2^{(r_1+1)}\right). \tag{11.42}$$

The procedure can be iterated, until the existence of a map $\psi_{2,r_2}(\cdot)$ is shown, such that

$$x_{2,r_2} = \psi_{2,r_2}\left(y_1, \dots, y_1^{(r_2-1)}, y_2, \dots, y_2^{(r_2-1)}\right). \tag{11.43}$$

Putting all expressions (11.40), (11.41), (11.42), ... , (11.43) together proves the lemma. ◁

The two properties just proven show that the system can be stabilized by means of a feedback law of the form $u = \alpha(\Psi(\mathbf{y}_1^{r_2}, \mathbf{y}_2^{r_2}))$, in which $\mathbf{y}_1^{r_2}$ and $\mathbf{y}_2^{r_2}$ could be estimated by means of a high-gain observer. In this respect, though, it must be stressed that the arguments of $\Psi(\cdot)$ consist of y_1, y_2 and *all* their higher order derivatives up order $r_2 - 1$. In particular, this requires the estimation of the derivatives of y_1 from order

r_1 to order $r_2 - 1$ and such derivatives, in turn, depend on the input u and a few of its higher order derivatives, up to order $r_2 - r_1 - 1$. To circumvent this problem, it is convenient to dynamically extend the system, by adding a chain of $r_2 - r_1$ integrators on both input channels.[10] In other words, the system is extended by setting

$$\begin{aligned} u &= \zeta_1 \\ \dot{\zeta}_1 &= \zeta_2 \\ &\cdots \\ \dot{\zeta}_{r_2-r_1-1} &= \zeta_{r_2-r_1} \\ \dot{\zeta}_{r_2-r_1} &= v, \end{aligned} \tag{11.44}$$

in which $\zeta_i \in \mathbb{R}^2$ and where $v \in \mathbb{R}^2$ plays a role of new input.

In order to better understand the effect of such extension on the equations describing the system, it is convenient to replace the new (added) state variables $\zeta_1, \ldots, \zeta_{r_2-r_1}$ by a set of variables $\xi_1, \ldots, \xi_{r_2-r_1}$ recursively defined with the aid of the following construction. Set, for $i = 1, 2$,

$$\sigma_{i1}(x) = a_i(x)$$

and define recursively, for $j = 2, \ldots, r_2 - r_1$,

$$\begin{aligned} \sigma_{i2}(x, \zeta_1) &= \frac{\partial \sigma_{i1}}{\partial x}[f(x) + g(x)\zeta_1] + \frac{\partial [b_i(x)\zeta_1]}{\partial x}[f(x) + g(x)\zeta_1] \\ &\cdots \\ \sigma_{i,j+1}(x, \zeta_1, \ldots, \zeta_j) &= \frac{\partial \sigma_{i,j}}{\partial x}[f(x) + g(x)\zeta_1] + \sum_{k=1}^{j-1} \frac{\partial \sigma_{i,j}}{\partial \zeta_k}\zeta_{k+1} + \frac{\partial [b_i(x)\zeta_j]}{\partial x}[f(x) + g(x)\zeta_1]. \end{aligned}$$

Then, consider the set of variables $\xi_{i,j}$, with $i = 1, 2$ and $j = 1, \ldots, r_2 - r_1$, defined as

$$\xi_{i,j} = \sigma_{i,j}(x, \zeta_1, \ldots, \zeta_{j-1}) + b_i(x)\zeta_j.$$

It is easy to check that the change of variables thus defined is invertible. In fact, using the property that the matrix $B(x)$ is invertible, it is seen that[11]

$$\begin{aligned} \zeta_1 &= B^{-1}(x)[\xi_1 - \sigma_1(x)] \\ \zeta_2 &= B^{-1}(x)[\xi_2 - \sigma_2(x, \zeta_1)] \\ &\cdots \\ \zeta_{r_2-r_1} &= B^{-1}(x)[\xi_{r_2-r_1} - \sigma_{r_2-r_1}(x, \zeta_1, \ldots, \zeta_{r_2-r_1-1})]. \end{aligned}$$

The advantage in using such new variables is that, by construction

$$\xi_1 = A(x) + B(x)\zeta_1$$

[10] This extends, to the case of multivariable systems, a procedure suggested in [4].

[11] Here $\sigma_j(\cdot) = \mathrm{col}(\sigma_{1,j}(\cdot), \sigma_{2,j}(\cdot))$.

and, as one can easily check,

$$\dot{\xi}_j = \xi_{j+1}.$$

for $j = 1, \ldots, r_2 - r_1 - 1$. Moreover

$$\dot{\xi}_{r_2-r_1} = \sigma_{r_2-r_1+1}(x, \zeta_1, \ldots, \zeta_{r_2-r_1}) + B(x)v,$$

which we rewrite as

$$\dot{\xi}_{r_2-r_1} = \bar{\sigma}_{r_2-r_1+1}(x, \xi) + B(x)v$$

having replaced $\zeta_1, \ldots, \zeta_{r_2-r_1}$ by its expression as a function of (x, ξ).[12]

Thus, with these new variables, the dynamically extended system becomes the aggregate of

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} \\
&\cdots \\
\dot{x}_{1,r_1-1} &= x_{1,r_1} \\
\dot{x}_{1,r_1} &= \xi_{1,1} \\
\dot{\xi}_{1,1} &= \xi_{1,2} \\
&\cdots \\
\dot{\xi}_{1,r_2-r_1} &= \bar{\sigma}_{1,r_2-r_1+1}(x, \xi) + b_1(x)v \\
y_1 &= x_{1,1}
\end{aligned} \tag{11.45}$$

and

$$\begin{aligned}
\dot{x}_{2,1} &= x_{2,2} \\
&\cdots \\
\dot{x}_{2,r_1-1} &= x_{2,r_1} \\
\dot{x}_{2,r_1} &= x_{2,r_1+1} + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,1} \\
&\cdots \\
\dot{x}_{2,r_2-1} &= x_{2,r_2} + \delta_{2,r_2}(x_1, [x_2]_{r_2-1})\xi_{1,1} \\
\dot{x}_{2,r_2} &= \xi_{2,1} \\
\dot{\xi}_{2,1} &= \xi_{2,2} \\
&\cdots \\
\dot{\xi}_{2,r_2-r_1} &= \bar{\sigma}_{2,r_2-r_1+1}(x, \xi) + b_2(x)v \\
y_2 &= x_{2,1}
\end{aligned} \tag{11.46}$$

This system has a structure similar to that of (11.35) and (11.36). Therefore, a result similar to that of Lemma 11.4 holds.

Lemma 11.6 *There exists a feedback law $v = \alpha(x, \xi)$ that globally asymptotically stabilizes the equilibrium $(x, \xi) = 0$ of (11.45) and (11.46).*

---

[12] As usual, let

$$\begin{aligned}
\zeta &= \text{col}(\zeta_1, \ldots, \zeta_{r_2-r_1}) \\
\xi &= \text{col}(\xi_1, \ldots, \xi_{r_2-r_1}).
\end{aligned}$$

.

Note that since ξ can be expressed as a function of ζ and x, as shown above, the feedback law determined in this lemma can also be expressed as a function of x and ζ. In other words, the lemma says that there exists a function $\bar{v}(x, \zeta)$ such that, if system (11.35) and (11.36), extended by (11.44), is controlled by

$$v = \bar{v}(x, \zeta).$$

the equilibrium point $(x, \zeta) = (0, 0)$ is globally asymptotically stable.

The components of the vector ζ are states of the dynamic extension, hence available for feedback. Thus, to implement this feedback law, only the vector x has to be estimated. But we know, from the previous analysis, that $x = \Psi(\mathbf{y}_1^{r_2}, \mathbf{y}_2^{r_2})$. Hence, to implement this feedback law, estimates of $\mathbf{y}_1^{r_2}, \mathbf{y}_2^{r_2}$ suffice. Such estimates can be generated by means of a standard high-gain observer.

In this respect, by looking at (11.45), it is immediately seen that

$$y_1^{(r_2)} = \bar{\sigma}_{1,r_2-r_1+1}(x, \xi) + b_1(x)v.$$

A little computational effort is needed to find an equivalent relation for $y_2^{(r_2)}$. To this end, observe first of all that

$$y_2^{(r_1)} = \dot{x}_{2,r_1} = x_{2,r_1+1} + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,1}.$$

Thus

$$\begin{aligned} y_2^{(r_1+1)} &= x_{2,r_1+2} + \delta_{2,r_1+2}(x_1, [x_2]_{r_1+1})\xi_{1,1} + \\ &\quad + \dot{\delta}_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,1} + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,2}. \end{aligned}$$

The quantity $\dot{\delta}_{2,r_1+1}(x_1, [x_2]_{r_1})$ is a function of $x_1, [x_2]_{r_1+1}, \xi_{1,1}$. Thus

$$y_2^{(r_1+1)} = x_{2,r_1+2} + \varphi_{r_1+2}(x_1, [x_2]_{r_1+1}, \xi_{1,1}) + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,2}$$

for some function $\varphi_{r_1+2}(x_1, [x_2]_{r_1+1}, \xi_{1,1})$. Next, we see that

$$y_2^{(r_1+2)} = x_{2,r_1+3} + \varphi_{r_1+3}(x_1, [x_2]_{r_1+2}, \xi_{1,1}, \xi_{1,2}) + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})\xi_{1,3}$$

for some function $\varphi_{r_1+3}(x_1, [x_2]_{r_1+2}, \xi_{1,1}, \xi_{1,2})$. Continuing in the same way, one obtains for $y_2^{(r_2)}$ an expression of the form

$$\begin{aligned} y_2^{(r_2)} &= \xi_{2,1} + \varphi_{r_2+1}(x_1, x_2, \xi_{1,1}, \ldots, \xi_{1,r_2-r_1}) \\ &\quad + \delta_{2,r_1+1}(x_1, [x_2]_{r_1})[\bar{\sigma}_{1,r_2-r_1+1}(x, \xi) + b_1(x)v]. \end{aligned}$$

In summary, we find that $y_1^{(r_2)}$ and $y_2^{(r_2)}$ can be given expressions of the form (we have replaced ξ by its expression as function of (x, ζ))

$$\begin{aligned} y_1^{(r_2)} &= q_1(x, \zeta) + p_1(x)v \\ y_2^{(r_2)} &= q_2(x, \zeta) + p_2(x)v. \end{aligned}$$

Note that such expressions involve the input v but *not* its derivatives.

The quantities $y_i, y_i^{(1)}, \ldots, y_i^{(r_2-1)}$, needed for the implementation of the feedback law, are estimated by the states $\eta_{i,1}, \eta_{i,2}, \ldots, \eta_{i,r_2}$ of two "identical" high-gain observers modeled by equations of the form

$$\begin{aligned} \dot{\eta}_{i,1} &= \eta_{i,2} + \kappa c_{r_2-1}(y_i - \eta_{i,1}) \\ \dot{\eta}_{i,2} &= \eta_{i,3} + \kappa^2 c_{r_2-2}(y_i - \eta_{i,1}) \\ &\cdots \\ \dot{\eta}_{i,r_2-1} &= \eta_{i,r_2} + \kappa^{r_2-1} c_1(y_i - \eta_{i,1}) \\ \dot{\eta}_{i,r_2} &= q_i(\Psi(\eta_1, \eta_2), \zeta) + p_i(\Psi(\eta_1, \eta_2))v + \kappa^{r_2} c_0(y_i - \eta_{i,1}). \end{aligned}$$

The reader should have no difficulties in working out a semiglobal stabilization result and a separation principle similar to those obtained before in Chap. 7.

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Chapter 12
Regulation and Tracking in Nonlinear Systems

12.1 Preliminaries

The nonlinear equivalent of the design problem addressed in Chap. 4 can be cast in the following terms. The controlled plant is a finite-dimensional, time-invariant, nonlinear system modeled by equations of the form

$$\begin{aligned}\dot{x} &= f(w, x, u)\\ e &= h_{\mathrm{e}}(w, x)\\ y &= h(w, x),\end{aligned} \tag{12.1}$$

in which $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}^m$ is a vector of inputs used for *control* purposes, $w \in \mathbb{R}^{n_{\mathrm{w}}}$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances and uncertain model parameters, $e \in \mathbb{R}^p$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, $y \in \mathbb{R}^q$ is a vector of outputs that are available for *measurement*. The problem is to design a controller, which receives $y(t)$ as input and produces $u(t)$ as output in such a way that, in the resulting closed-loop system, $x(t)$ remains bounded and

$$\lim_{t\to\infty} e(t) = 0,$$

regardless of what the exogenous input $w(t)$ actually is.

As in the case of linear systems, the exogenous input $w(t)$ is assumed to be a (undefined) member of a fixed family of functions of time, the family of all solutions of a fixed ordinary differential equation of the form (the *exosystem*)

$$\dot{w} = s(w) \tag{12.2}$$

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8_12

obtained when its initial condition $w(0)$ is allowed to vary on a prescribed set W. For convenience, it is assumed that the set W on which the state of (12.2) is allowed to range is a *compact* set, *invariant* for the dynamics of (12.2). If this is the case, then the ω-limit set of W under the flow of (12.2) is the set W itself.[1] If this is the case, it can be said that the exosystem "is in steady-state" and this can be regarded as a nonlinear counterpart of the assumption that all the eigenvalues of a linear exosystem have zero real part.

The control law for (12.1) is to be provided by a system modeled by equations of the form

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y) \\ u &= h_c(x_c, y) \end{aligned} \tag{12.3}$$

with state $x_c \in \mathbb{R}^{n_c}$. The initial conditions $x(0)$ of the *plant* (12.1), and $x_c(0)$ of the *controller* (12.3) are allowed to range over a fixed *compact* sets $X \subset \mathbb{R}^n$ and, respectively $X_c \subset \mathbb{R}^{n_c}$.

In this setting, the problem of output regulation can be cast as follows. Consider the closed-loop system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, h_c(x_c, h(w, x))) \\ \dot{x}_c &= f_c(x_c, h(w, x)), \end{aligned} \tag{12.4}$$

regarded as an autonomous system with output

$$e = h_e(w, x).$$

The problem is to find $f_c(x_c, y)$ and $h_c(x_c, y)$ so that:

(i) the positive orbit of $W \times X \times X_c$ is bounded, i.e., the exists a bounded subset S of $W \times \mathbb{R}^n \times \mathbb{R}^{n_c}$ such that, for any $(w(0), x(0), x_c(0)) \in W \times X \times X_c$, the integral curve $(w(t), x(t), x_c(t))$ of (12.4) passing through $(w(0), x(0), x_c(0))$ at time $t = 0$ remains in S for all $t \geq 0$.

(ii) $\lim_{t\to\infty} e(t) = 0$, *uniformly* in the initial condition, i.e., for every $\varepsilon > 0$ there exists a time $\bar{t}$, depending only on ε and *not on* $(w(0), x(0), x_c(0))$, such that the integral curve $(w(t), x(t), x_c(t))$ of (12.4) yields $\|e(t)\| \leq \varepsilon$ for all $t \geq \bar{t}$.

Condition (i) replaces, and actually extends to a nonlinear setting, the requirement—considered in the case of linear systems—that the closed-loop system with $w = 0$ be asymptotically stable. Condition (ii) expresses the property of asymptotic regulation (or tracking). The property that convergence of the regulated variable $e(t)$ to zero be uniform in the initial conditions, which is granted in the case of a linear system, needs now to be explicitly requested, since in the case of nonlinear systems this may not be the case.[2] Form a practical viewpoint, in fact, uniformity in

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[1] See Lemma B.7 in Appendix B.

[2] Even if the initial conditions are taken in a compact set, see in this respect Example B.7 in Appendix B.

the initial conditions is essentially equivalent to the guarantee of a uniform "rate of decay" of the regulated variable to zero.

12.2 Steady-State Analysis

We begin by discussing—in rather general terms—the necessary conditions for the existence of a controller that solves the problem of output regulation. Suppose the nonlinear plant (12.1) is controlled by means of the nonlinear controller (12.3), which yields a closed-loop system modeled by the equations (12.4), and suppose the problem of output regulation is solved. Then, from item (i) in the characterization of the problem in question, it is seen that the positive orbit of the set $W \times X \times X_c$ of initial conditions is bounded and hence all trajectories asymptotically approach a steady-state locus $\omega(W \times X \times X_c)$. This set is the graph of a (possibly set-valued) map defined on W.[3] To streamline the analysis, we assume that this map is *single-valued*, i.e., that there exists a pair of maps $x = \pi(w)$ and $x_c = \pi_c(w)$, defined on W, such that

$$\omega(W \times X \times X_c) = \{(w, x, x_c) : w \in W, x = \pi(w), x_c = \pi_c(w)\}. \tag{12.5}$$

This is equivalent to assume that, in the closed-loop system, for each given exogenous input function $w(t)$, there exists a *unique* steady-state response, which therefore can be expressed as

$$\begin{aligned} x_{ss}(t) &= \pi(w(t)) \\ x_{c,ss}(t) &= \pi_c(w(t)). \end{aligned}$$

Moreover, for convenience, we also assume that the maps $\pi(w)$ and $\pi_c(w)$ are continuously differentiable. This enables us to characterize in simple terms the property that the steady-state locus is invariant under the flow of the closed-loop system (12.4). If this is the case, in fact, to say that the locus (12.5) is invariant under the flow of (12.4) is the same as to say that $\pi(w)$ and $\pi_c(w)$ satisfy[4]

$$\left.\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(w, \pi(w), h_c(\pi_c(w), h(w, \pi(w)))) \\ \frac{\partial \pi_c}{\partial w} s(w) &= f_c(\pi_c(w), h(w, \pi(w))) \end{aligned}\right. \qquad \forall w \in W. \tag{12.6}$$

These equations are the non-linear counterpart of the Sylvester equations (4.17).

From item (ii) in the characterization of the problem of output regulation, it is also seen that $\lim_{t\to\infty} e(t) = 0$, uniformly in the initial condition. This entails an important consequence on the set $\omega(W \times X \times X_c)$, expressed as follows.[5]

[3] See section B.6 in Appendix B.

[4] See again section B.6 in Appendix B.

[5] See [6] for a proof.

Lemma 12.1 *Suppose the positive orbit of* $W \times X \times X_c$ *is bounded. Then*

$$\lim_{t\to\infty} e(t) = 0,$$

uniformly in the initial conditions, if and only if

$$\omega(W \times X \times X_c) \subset \{(w, x, x_c) : h_e(w, x) = 0\}. \tag{12.7}$$

In other words, if the controller solves the problem of output regulation, the steady-state locus, which is asymptotically approached by the trajectories of the closed-loop system, must be a subset of the set of all pairs (w, x) for which $h_e(w, x) = 0$ and hence the map $\pi(w)$ necessarily satisfies

$$h_e(w, \pi(w)) = 0. \tag{12.8}$$

Proceeding as in the case of linear systems and setting

$$\psi(w) = h_c(\pi_c(w), h(w, \pi(w))), \tag{12.9}$$

the first equation of (12.6) and (12.8) can be rewritten in controller-independent form as

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(w, \pi(w), \psi(w)) \\ 0 &= h_e(w, \pi(w)) \end{aligned} \qquad \forall w \in W. \tag{12.10}$$

These equations, introduced in [2] and known as the *nonlinear regulator equations*, are the nonlinear counterpart of the Francis' equations (4.6).

In order to better understand the meaning of the second equation in (12.6), it is convenient—as done in a similar context for linear systems—to split the vector y of measured outputs in two parts as

$$y = \begin{pmatrix} e \\ y_r \end{pmatrix} = \begin{pmatrix} h_e(w, x) \\ h_r(w, x) \end{pmatrix}$$

and, accordingly, rewrite the controller (12.3) as

$$\begin{aligned} \dot{x}_c &= f_c(x_c, e, y_r) \\ u &= h_c(x_c, e, y_r). \end{aligned}$$

Using these notations, the second equation in (12.6) and (12.9), in the light of (12.8), become

$$\begin{aligned} \frac{\partial \pi_c}{\partial w} s(w) &= f_c(\pi_c(w), 0, h_r(w, \pi(w))) \\ \psi(w) &= h_c(\pi_c(w), 0, h_r(w, \pi(w))) \end{aligned} \qquad \forall w \in W. \tag{12.11}$$

These equations are the nonlinear counterpart of the equations (4.20) and have a similar interpretation. Observe also that, in the special case in which $y = e$, the equations (12.11) reduce to equations of the form

$$\begin{aligned}\frac{\partial \pi_\mathrm{c}}{\partial w} s(w) &= f_\mathrm{c}(\pi_\mathrm{c}(w), 0)\\ \psi(w) &= h_\mathrm{c}(\pi_\mathrm{c}(w), 0).\end{aligned} \tag{12.12}$$

At is was the case for linear systems, the conditions thus found can be exploited in the design of a controller that solves the problem of output regulation. The substantial difference with the case of linear systems, though, is that one no longer can appeal to results such as those used in Sect. 4.4, which were dependent on some fundamental properties of linear systems. In fact, the arguments used in the proof of Proposition 4.3 consisted of Lemma 4.2, expressing the properties of stabilizability and observability of the (linear) augmented system (4.30), and of Lemma 4.3, which was proved using (4.27). Both such arguments that have no general counterpart for nonlinear systems. In addition, it is stressed that, since the controlled system is affected by the exogenous input w and the latter is not available for feedback, stabilization methods that do not rely upon measurement of w, i.e., robust stabilization methods, are mandatory.

This being the case, we consider in what follows only the case in which $y = e$ (and, consistently, we identify $h_\mathrm{e}(w, x)$ with $h(w, x)$) and we pursue the method indicated in the second part of Sect. 4.6. In fact, the proof that the method in question is successful does not appeal to the result of Lemma 4.3. Mimicking the design philosophy used in that context, where the controller consisted of a preprocessing internal model of the form (4.53) with input $\bar{u}$ provided by a stabilizer of the form (4.54), we consider now a controller consisting of a preprocessing internal model described by equations of the form

$$\begin{aligned}\dot{\eta} &= \varphi(\eta) + G\bar{u}\\ u &= \gamma(\eta) + \bar{u}\end{aligned} \tag{12.13}$$

with state $\eta \in \mathbb{R}^d$, with input $\bar{u}$ provided by a suitable stabilizer. These equations, in fact, reduce to (4.53) if $\varphi(\eta)$ and $\gamma(\eta)$ are linear maps, i.e., if $\varphi(\eta) = \Phi\eta$ and $\gamma(\eta) = \Gamma\eta$.

In the case of linear systems, it was shown (see Lemma 4.8) that—so long as the triplet (Φ, G, Γ) is reachable and observable—given any matrix Ψ there always exist a matrix Σ satisfying

$$\begin{aligned}\Sigma S &= \Phi\Sigma\\ \Psi &= \Gamma\Sigma\end{aligned} \tag{12.14}$$

and this property was instrumental in determining—in the proof of Proposition 4.6—the expression of the center eigenspace in the associated closed-loop system (4.55).

In order to obtain a similar result in the present context, we *postulate* (for the time being) the fulfillment of equations that can be seen as a nonlinear counterpart of the two linear equations above, i.e., we assume the existence of a map $\sigma : W \to \mathbb{R}^d$

satisfying

$$\begin{aligned} \frac{\partial \sigma}{\partial w} s(w) &= \varphi(\sigma(w)) \\ \psi(w) &= \gamma(\sigma(w)) \end{aligned} \qquad \forall w \in W. \tag{12.15}$$

Consider now the composition of the plant (12.1), in which $e = y$ and w is modeled by the exosystem (12.2), with control u provided by a preprocessor described by equations of the form (12.13), in which $\bar{u}$ is—in turn—provided by a (linear)[6] stabilizer of the form (4.54), namely of the form

$$\begin{aligned} \dot{\xi} &= A_s \xi + B_s e \\ \bar{u} &= C_s \xi + D_s e. \end{aligned} \tag{12.16}$$

This yields an overall system described by the following equations

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, \gamma(\eta) + C_s \xi + D_s e) \\ \dot{\eta} &= \varphi(\eta) + G(C_s \xi + D_s e) \\ \dot{\xi} &= A_s \xi + B_s e \\ e &= h(w, x). \end{aligned} \tag{12.17}$$

Using (12.10) and (12.15), it is an easy matter to check that the graph of the (nonlinear) map

$$w \in W \quad \mapsto \begin{pmatrix} x \\ \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \pi(w) \\ \sigma(w) \\ 0 \end{pmatrix},$$

i.e., the set

$$\mathscr{A} = \{(w, x, \eta, \xi) : x = \pi(w), \eta = \sigma(w), \xi = 0\}, \tag{12.18}$$

is by construction an invariant manifold of (12.17).[7] Because of (12.10), the regulated variable e vanishes on the graph of the map (12.18). Thus, if the parameters of the stabilizer can be chosen in such a way that the (compact) manifold in question is asymptotically stable, all trajectories asymptotically converge to such manifold and the problem of output regulation is solved.

We can therefore formally conclude this section, dedicated to the steady-state analysis, in the case in which measured output and regulated output coincide, as follows. Consider the assumptions:

Assumption 12.1 The regulator equations (12.10) have a solution $(\pi(w), \psi(w))$.

[6]Strictly speaking, it is not necessary to consider here a linear stabilizer. In fact, as it will be seen in the sequel, it suffices that the stabilizer has an equilibrium state yielding $\bar{u} = 0$. However, since essentially all methods illustrated earlier in the book for nonlinear (robust) stabilization use *linear* stabilizers, in what follow we will consider a stabilizer of this form.

[7]Compare with a similar conclusion obtained in the proof of Proposition 4.6.

Assumption 12.2 There exist a pair $(\varphi(\eta), \gamma(\eta))$ such that (12.15) hold for some $\sigma(w)$.

Then, the problem of output regulation can be solved—by means of a controller consisting of (12.13) and (12.16)—if it is possible to choose the parameters of the stabilizer (12.16) in such a way that the graph of the map (12.18), a compact invariant manifold, is asymptotically stabilized. We will see in the next sections how this can be achieved.

12.3 The Case of SISO Systems

Essentially all methods for "robust" stabilization of nonlinear systems, presented in the previous chapters of the book, appeal to the property that the system is minimum phase. Thus, it is reasonable to stick with a similar assumption also in the present context, where the problem is to robustly stabilize a compact invariant manifold. For simplicity, we analyze in detail the case in which $\dim(e) = \dim(u) = 1$. The case in which $\dim(e) = \dim(u) = m > 1$ can be dealt with by means of similar techniques and will not be covered here.

The nonlinear regulator equations, introduced in the previous section, can be given a more tangible form if the model (12.1) of the plant is affine in the input u and, viewed as a system with input u and output e, has a globally defined *normal form*. This means that, in suitable (globally defined) coordinates, the composition of plant (12.1) and exosystem (12.2) can be modeled by equations of the form

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f_0(w, z, \xi_1, \dots, \xi_r)\\
\dot{\xi}_1 &= \xi_2\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r\\
\dot{\xi}_r &= q(w, z, \xi_1, \dots, \xi_r) + b(w, z, \xi_1, \dots, \xi_r)u\\
e &= \xi_1
\end{aligned} \tag{12.19}$$

in which r is the relative degree of the system, $z \in \mathbb{R}^{n-r}$ and $b(w, z, \xi_1, \dots, \xi_r)$, the so-called *high-frequency gain*, is nowhere zero.

If the model of the plant is available in normal form, the nonlinear regulator equations (12.10) can be analyzed by means a technique that extends the technique used in Example 4.1. Let $\pi(w)$ be partitioned, consistently with the partition of the state $(z, \xi_1, \dots, \xi_r)$ of (12.19), into

$$\pi(w) = \mathrm{col}(\pi_0(w), \pi_1(w), \dots, \pi_r(w)).$$

in which case the equations in question become

$$
\begin{aligned}
\frac{\partial \pi_0}{\partial w} s(w) &= f_0(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w)) \\
\frac{\partial \pi_i}{\partial w} s(w) &= \pi_{i+1}(w) \qquad i = 1, \ldots, r-1 \\
\frac{\partial \pi_r}{\partial w} s(w) &= q(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w)) + b(w, \pi_0(w), \pi_1(w), \ldots, \pi_r(w))\psi(w) \\
0 &= \pi_1(w).
\end{aligned}
$$

From these, we deduce that

$$\pi_1(w) = \cdots = \pi_r(w) = 0$$

while $\pi_0(w)$ satisfies

$$\frac{\partial \pi_0}{\partial w} s(w) = f_0(w, \pi_0(w), 0, \ldots, 0) \tag{12.20}$$

Moreover

$$\psi(w) = -\frac{q(w, \pi_0(w), 0, \ldots, 0)}{b(w, \pi_0(w), 0, \ldots, 0)}. \tag{12.21}$$

In summary, the nonlinear regulator equations have a solution if and only if there exists a map $\pi_0(w)$ that satisfies (12.20). If this is the case, the map $\psi(w)$ is given by (12.21).[8]

We are now in a position to appropriately formulate the assumption of minimum phase. The difference with the case dealt with in Sect. 6.3 is that now we should no longer consider a property of input-to-state stability to an *equilibrium*, but rather to a *compact invariant set*. In fact, as shown by (12.20) the set

$$\mathscr{A}_0 = \{(w, z) : w \in W, z = \pi_0(w)\}$$

is a compact invariant set of

$$
\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f_0(w, z, \xi_1, \ldots, \xi_r)
\end{aligned} \tag{12.22}
$$

if $\xi_1 = \cdots = \xi_r = 0$. With this in mind, we characterize the property of (strong) minimum phase as specified in the following definition.

Definition 12.1 Consider system (12.19) and suppose the nonlinear regulator equations have a solution, in which case there exists a map $\pi_0(w)$ that satisfies (12.20). Set $\bar{z}(t) = z(t) - \pi_0(w(t))$. The system is strongly minimum phase (with respect to the compact set $\mathscr{A}_0$) if there exist a class $\mathscr{KL}$ function $\beta_0(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\vartheta_0(\cdot)$ such that, for any $(w_0, z_0) \in W \times \mathbb{R}^{n-r}$ and any piecewise-continuous bounded function $\xi_0(\cdot) : [0, \infty) \to \mathbb{R}^r$, the response $(w(t), z(t))$ of (12.22) from the initial state $(w(0), z(0))$ satisfies

[8] It is instructive to compare these equations with (4.14).

$$\|\bar{z}(t)\| \leq \beta_0(\|\bar{z}(0)\|, t) + \vartheta_0(\|\xi_0(\cdot)\|_{[0,t]}) \qquad \text{for all } t \geq 0. \tag{12.23}$$

As described in the previous section, the control u of system (12.19) is provided by an internal model of the form (12.13), whose input $\bar{u}$ is in turn provided by a stabilizer. In order to be able to use the stabilization methods described earlier in the book, we should first should seek conditions under which the *augmented* system obtained from the composition of (12.19) and (12.13) is minimum phase (in the sense of Definition 12.1). Now, as it is easily seen, the augmented system in question, that can be written as (in what follows, wherever possible, we replace the string $(\xi_1, \ldots, \xi_r)$ by the vector ξ)

$$\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f_0(w, z, \xi) \\
\dot{\eta} &= \varphi(\eta) + G\bar{u} \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= q(w, z, \xi) + b(w, z, \xi)[\gamma(\eta) + \bar{u}] \\
e &= \xi_1,
\end{aligned} \tag{12.24}$$

still has relative degree r, but is not in strict normal form. In order to determine whether or not the property of being minimum phase holds it is appropriate to put first the system in strict normal form, which is not difficult.

To this end, set

$$F(\eta) = \varphi(\eta) - G\gamma(\eta) \tag{12.25}$$

in which case it is seen that (compare with (4.65))

$$\dot{\eta} = F(\eta) + G[\gamma(\eta) + \bar{u}].$$

Then, consider the change of variables

$$\zeta = \eta - GM(w, z, \xi) \tag{12.26}$$

in which

$$M(w, z, \xi) = \int_0^{\xi_r} \frac{1}{b(w, z, \xi_1, \ldots, \xi_{r-1}, s)} ds. \tag{12.27}$$

Note that $M(w, z, \xi)$ vanishes at $\xi_r = 0$ and that its derivative with respect to time has an expression of the form

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M(w, z, \xi) &= \frac{1}{b(w, z, \xi_1, \ldots, \xi_{r-1}, \xi_r)} \dot{\xi}_r + \int_0^{\xi_r} \frac{\mathrm{d}}{\mathrm{d}t} \Big[\frac{1}{b(w, z, \xi_1, \ldots, \xi_{r-1}, s)} \Big] ds \\
&= \frac{q(w, z, \xi)}{b(w, z, \xi)} + [\gamma(\eta) + \bar{u}] + N(w, x, \xi)
\end{aligned}$$

in which

$$N(w,x,\xi) = -\int_0^{\xi_r} \frac{1}{[b(w,z,\xi_1,\ldots,\xi_{r-1},s)]^2}\left[\frac{\partial b}{\partial w}s(w) + \frac{\partial b}{\partial z}f_0(w,z,\xi) + \sum_{i=1}^{r-1}\frac{\partial b}{\partial \xi_i}\xi_{i+1}\right]ds,$$

is a function that vanishes at $\xi_r = 0$.

In the new variables, we have

$$\dot{\zeta} = F(\zeta + GM(w,x,\xi)) - G\frac{q(w,z,\xi)}{b(w,z,\xi)} - GN(w,x,\xi),$$

and hence the composition of (12.19) and (12.13), written as

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f_0(w,z,\xi)\\
\dot{\zeta} &= F(\zeta + GM(w,x,\xi)) - G\frac{q(w,z,\xi)}{b(w,z,\xi)} - GN(w,x,\xi)\\
\dot{\xi}_1 &= \xi_2\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r\\
\dot{\xi}_r &= q(w,z,\xi) + b(w,z,\xi)[\gamma(\zeta + GM(w,x,\xi)) + \bar{u}]\\
e &= \xi_1,
\end{aligned} \tag{12.28}$$

is now in *strict* normal form.

To determine whether or not the property of being minimum phase holds, we concentrate on the subsystem

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f_0(w,z,\xi)\\
\dot{\zeta} &= F(\zeta + GM(w,x,\xi)) - G\frac{q(w,z,\xi)}{b(w,z,\xi)} - GN(w,x,\xi)
\end{aligned} \tag{12.29}$$

viewed as a system with state (w,z,ζ) and input ξ. We observe that, if Assumption 12.2 holds, the set

$$\mathscr{A}_{0\mathrm{a}} = \{(w,z,\zeta) : w \in W, z = \pi_0(w), \zeta = \sigma(w)\}$$

is a compact invariant manifold if $\xi = 0$. In fact, on such set, if $\xi = 0$ the right-hand side of the second equation becomes

$$f_0(w,\pi_0(w),0) = \frac{\partial \pi_0}{\partial w}s(w)$$

and, respectively, the right-hand side of the third equation becomes (use here the fact that $M(w,z,0) = 0$ and $N(w,z,0) = 0$ and (12.21) and (12.15))

$$F(\sigma(w)) - G\frac{q(w, \pi_0(w), 0)}{b(w, \pi_0(w), 0)} = \varphi(\sigma(w)) - G\gamma(\sigma(w)) + G\psi(w) = \frac{\partial \sigma}{\partial w}s(w).$$

Thus, it makes sense to consider the case in which the subsystem in question is input-to-state stable to the set $\mathscr{A}_{0a}$. As far as the z component is concerned, the required property is precisely the property considered in Definition 12.1. Thus, it remains to postulate a similar property for the ζ component, which we do in the form of an assumption.

Assumption 12.3 Set $\bar{\zeta}(t) = \zeta(t) - \sigma(w(t))$. There exist a class $\mathscr{KL}$ function $\beta_1(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\vartheta_1(\cdot)$ such that, for any $(w_0, \zeta_0) \in W \times \mathbb{R}^d$, any piecewise-continuous bounded function $(w_0(\cdot), z_0(\cdot)) : [0, \infty) \to W \times \mathbb{R}^{n-r}$ and any continuous bounded function $\xi_0(\cdot) : [0, \infty) \to \mathbb{R}^r$, the response $(w(t), \zeta(t))$ of (12.29) from the initial state $(w(0), \zeta(0))$ satisfies

$$\|\bar{\zeta}(t)\| \le \beta_1(\|\bar{\zeta}(0)\|, t) + \max\{\vartheta_1(\|\xi_0(\cdot)\|_{[0,t]}), \vartheta_1(\|\bar{z}(\cdot)\|_{[0,t]}\} \qquad \text{for all } t \ge 0. \tag{12.30}$$

Clearly, if the system (12.19) is strongly minimum phase—in the sense of Definition 12.1—and Assumption 12.3 holds, then also the augmented system (12.28) is minimum phase as well. As a matter of fact, composing the estimates in (12.23) and (12.30), it is easily seen that the quantity

$$\|w, z, \zeta\|_{\mathscr{A}_{0a}} := \| \begin{pmatrix} \bar{z} \\ \bar{\zeta} \end{pmatrix} \| = \| \begin{pmatrix} z - \pi_0(w) \\ \zeta - \sigma(w) \end{pmatrix} \|$$

satisfies an estimate of the form

$$\|(w, z, \zeta)\|_{\mathscr{A}_{0a}}(t) \le \beta(\|(w, z, \zeta)\|_{\mathscr{A}_{0a}}(0), t) + \vartheta(\|\xi_0(\cdot)\|_{[0,t]})\} \qquad \text{for all } t \ge 0$$

for some suitable class $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ and class $\mathscr{K}$ function $\vartheta(\cdot)$.

At this point, we can claim that, if Assumptions 12.1–12.3 hold and the system is strongly minimum phase, in the sense of Definition 12.1, it is possible—using the methods described earlier in the book—to design a (linear) stabilizer that makes the graph of the map (12.18), a compact invariant manifold of the closed-loop system (12.17), asymptotically attractive, completing in this way the design of a controller that solves the problem of output regulation. Clearly, the fulfillment of Assumption 12.1 and the hypothesis of strongly minimum phase, in the sense of Definition 12.1, are properties of the controlled plant. On the contrary, Assumptions 12.2 and 12.3 are properties of the "preprocessing" internal model (12.13), which is to be designed. Thus, to complete the analysis, it remains to examine to what extent the two assumptions in question can be fulfilled. This is discussed in the next section.

12.4 The Design of an Internal Model

As shown in Sect. 5.6, in the case of a linear system Assumptions 12.2 and 12.3 can always be fulfilled. In fact, choosing Φ and G as in (4.23) and (4.24) and picking Γ in such a way that the matrix F is Hurwitz, the (linear version of) Assumption 12.2 is fulfilled and the preprocessing internal model can be expressed as

$$\begin{aligned} \dot{\eta} &= F\eta + [\Gamma\eta + \bar{u}] \\ u &= \Gamma\eta + \bar{u}. \end{aligned}$$

The functions $M(w, z, \xi)$ and $N(w, z, \xi)$ introduced in the change of variables (12.26) become

$$M(w, z, \xi) = \frac{1}{b}\xi_r \qquad N = 0$$

while (see Example 4.1)

$$\begin{aligned} \frac{q(w, z, \xi)}{b(w, z, \xi)} &= \frac{1}{b}[P_1 w + A_{10} z + A_{11}\xi] = -\Psi w + \frac{1}{b}[A_{10}(z - \Pi_0 w) + A_{11}\xi] \\ &= -\Psi w + \frac{1}{b}[A_{10}\bar{z} + A_{11}\xi]. \end{aligned}$$

As a consequence, the dynamics of ζ become

$$\dot{\zeta} = F(\zeta + \frac{1}{b}\xi_r) + G\Psi w - G\frac{1}{b}[A_{10}\bar{z} + A_{11}\xi],$$

and, since (use here (12.14))

$$F\Sigma + G\Psi = (F + G\Gamma)\Sigma = \Phi\Sigma = \Sigma S,$$

the difference $\bar{\zeta} = \zeta - \Sigma w$ satisfies

$$\dot{\bar{\zeta}} = F\bar{\zeta} + \frac{1}{b}[F\xi_r - GA_{10}\bar{z} - GA_{11}\xi].$$

The matrix F is Hurwitz by construction and hence Assumption 12.3 trivially holds.

This construction has a nice nonlinear counterpart, which is based on the following important result, proven in [12].

Lemma 12.2 *Let $d \geq 2n_w + 2$. There exist an $\ell > 0$ and a subset $S \subset \mathbb{C}$ of zero Lebesgue measure such that if the eigenvalues of F are in $\{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] \leq -\ell\} \setminus S$, then there exist a differentiable function $\sigma : W \to \mathbb{R}^d$ and a continuous bounded function $\gamma : \mathbb{R}^d \to \mathbb{R}$ such that*[9]

[9]The function $\gamma(\cdot)$ is only guaranteed to be continuous and may fail to be continuously differentiable. Closed-forms expressions for $\gamma(\cdot)$ and other relevant constructive aspects are discussed in [13].

$$\begin{aligned} \frac{\partial \sigma}{\partial w} s(w) &= F\sigma(w) + G\psi(w) \\ \psi(w) &= \gamma(\sigma_(w)) \end{aligned} \qquad \textit{for all } w \in W. \tag{12.31}$$

With this result in mind it is easy to see how the properties indicated in Assumptions 12.2 and 12.3 can be fulfilled. In fact, choosing

$$\varphi(\eta) = F\eta + G\gamma(\eta)$$

the property in Assumption 12.2 trivially holds. With such choice, the function $F(\eta)$ in (12.25) becomes a linear function, i.e., $F(\eta) = F\eta$. Then, the dynamics of $\bar{\zeta} = \zeta - \sigma(w)$ become

$$\begin{aligned} \dot{\bar{\zeta}} &= F\bar{\zeta} + F\sigma(w) + FM(w,z,\xi) - G\frac{q(w,z,\xi)}{b(w,z,\xi)} - GN(w,z,\xi) - \frac{\partial \sigma}{\partial w} s(w) \\ &= F\bar{\zeta} + F\sigma(w) + FM(w,z,\xi) - G\frac{q(w,z,\xi)}{b(w,z,\xi)} - GN(w,z,\xi) - F\sigma(w) - G\psi(w) \\ &= F\bar{\zeta} + FM(w,z,\xi) - GN(w,z,\xi) - G[\frac{q(w,z,\xi)}{b(w,z,\xi)} + \psi(w)]. \end{aligned}$$

These are the dynamics of a linear stable system driven by an input

$$v(w,z,\xi) = FM(w,z,\xi) - GN(w,z,\xi) - G[\frac{q(w,z,\xi)}{b(w,z,\xi)} + \psi(w)] \tag{12.32}$$

that by construction vanishes when $\xi = 0$ and $\bar{z} = z - \pi_0(w) = 0$ (see (12.21)). Thus, under the mild technical assumption that

$$\|v(w,z,\xi)\| \le \max\{\hat{\vartheta}(\|\xi\|), \hat{\vartheta}(\|\bar{z}\|)\}, \tag{12.33}$$

for some class $\mathscr{K}$ function $\hat{\vartheta}(\cdot)$, also the property indicated in Assumption 12.2 holds.

The above setup is pretty general and, essentially, does not require specific assumptions (other than (12.33)). However, the construction of the function $\gamma(\cdot)$ is not immediate. To overcome this difficulty, another approach is available, which however requires a specific assumption, that essentially postulates the existence of a "regression-like" relation between the higher derivatives (with respect to time) of the function $\psi(w)$.

Assumption 12.4 There exists an integer d and a globally Lipschitz smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that

$$L_s^d \psi(w) = \phi(\psi(w), L_s\psi(w), \ldots, L_s^{d-1}\psi(w)) \qquad \text{for all } w \in W. \tag{12.34}$$

Under this assumption,[10] it is very easy to construct a pair of functions $(\varphi(\eta), \gamma(\eta))$ such that the properties indicated in Assumption 12.2 hold for some $\sigma(w)$. In fact, the reader should have no difficulties in checking that this is the case if

$$\varphi(\eta) = \begin{pmatrix} \eta_2 \\ \eta_3 \\ \cdots \\ \eta_d \\ \phi(\eta_1, \eta_2, \ldots, \eta_d) \end{pmatrix} = \hat{A}\eta + \hat{B}\phi(\eta)$$

$$\gamma(\eta) = \eta_1 = \hat{C}\eta$$

with $\hat{A}$, $\hat{B}$, $\hat{C}$ matrices of the form (3.8) and

$$\sigma(w) = \begin{pmatrix} \psi(w) \\ L_s\psi(w) \\ \cdots \\ L_s^{d-1}\psi(w) \end{pmatrix}.$$

This being the case, it remains to see whether the vector G can be chosen in such a way that also Assumption 12.3 holds. To this end observe that, if $\varphi(\eta)$ and $\gamma(\eta)$ are chosen in this way, then

$$F(\eta) = \varphi(\eta) - G\gamma(\eta) = (\hat{A} - G\hat{C})\eta + \hat{B}\phi(\eta).$$

As a consequence, the dynamics of $\bar{\zeta} = \zeta - \sigma(w)$ become

$$\begin{aligned} \dot{\bar{\zeta}} &= (\hat{A} - G\hat{C})(\bar{\zeta} + \sigma(w) + M(w, z, \xi)) + \hat{B}\phi(\bar{\zeta} + \sigma(w) + M(w, z, \xi)) \\ &\qquad - G\frac{q(w, z, \xi)}{b(w, z, \xi)} - GN(w, z, \xi) - \frac{\partial\sigma}{\partial w}s(w) \\ &= (\hat{A} - G\hat{C})(\bar{\zeta} + M(w, z, \xi)) + (\hat{A} - G\hat{C})\sigma(w) + \hat{B}\phi(\bar{\zeta} + \sigma(w) + M(w, z, \xi)) \\ &\qquad - G\frac{q(w, z, \xi)}{b(w, z, \xi)} - GN(w, z, \xi) - \hat{A}\sigma(w) - \hat{B}\phi(\sigma(w)) \\ &= (\hat{A} - G\hat{C})\bar{\zeta} + \hat{B}[\phi(\bar{\zeta} + \sigma(w) + M(w, z, \xi)) - \phi(\sigma(w)] \\ &\qquad + (\hat{A} - G\hat{C})M(w, z, \xi) - G[\frac{q(w, z, \xi)}{b(w, z, \xi)} + \psi(w)] - GN(w, z, \xi). \end{aligned}$$

For convenience, we rewrite the latter in the form

$$\dot{\bar{\zeta}} = (\hat{A} - G\hat{C})\bar{\zeta} + \hat{B}[\phi(\bar{\zeta} + \sigma(w) + M(w, z, \xi)) - \phi(\sigma(w)] + v(w, z, \xi),$$

in which $v(w, z, \xi)$ is a function identical to (12.32), if F is replaced by $(\hat{A} - G\hat{C})$.

Choose now

[10] Since W is a compact set, in the condition above only the values of $\phi(\cdot)$ on a compact set matter. Thus, the assumption that the function is globally Lipschitz can be taken without loss of generality.

$$G = D_\kappa G_0$$

in which D_κ is the matrix (compare with Sect. 2.5 and with Sect. 7.5)

$$D_\kappa = \mathrm{diag}(\kappa, \kappa^2, \dots, \kappa^d) \tag{12.35}$$

and $G_0 \in \mathbb{R}^d$ is such that the matrix $\hat{A} - G_0\hat{C}$ is Hurwitz, which is always possible since the pair $\hat{A}$, $\hat{C}$ is observable. Change variables as

$$\tilde{\zeta} = D_\kappa^{-1}\bar{\zeta}.$$

Bearing in mind the definition of $\hat{A}$, $\hat{B}$, $\hat{C}$ and D_κ, observe that

$$D_\kappa^{-1}\hat{A}D_\kappa = \kappa\hat{A}, \quad D_\kappa^{-1}\hat{B} = \frac{1}{\kappa^d}\hat{B}, \quad \hat{C}D_\kappa = \kappa\hat{C}, \tag{12.36}$$

from which we obtain

$$\dot{\tilde{\zeta}} = \kappa(\hat{A} - G_0\hat{C})\tilde{\zeta} + \frac{1}{\kappa^d}\hat{B}[\phi(D_\kappa\tilde{\zeta} + \sigma(w) + M(w,z,\xi)) - \phi(\sigma(w)] + D_\kappa^{-1}v(w,z,\xi).$$

Since $\hat{A} - G_0\hat{C}$ is Hurwitz, there exists a positive definite matrix P such that

$$P(\hat{A} - G_0\hat{C}) + (\hat{A} - G_0\hat{C})^{\mathrm{T}}P = -I.$$

Moreover, bearing in mind the fact that the function $\phi(\cdot)$ is globally Lypschitz and assuming, without loss of generality, that $\kappa \geq 1$, it is seen that

$$\begin{aligned} &\frac{1}{\kappa^d}\|\hat{B}[\phi(D_\kappa\tilde{\zeta} + \sigma(w) + M(w,z,\xi)) - \phi(\sigma(w)]\| \\ &\qquad \leq \frac{1}{\kappa^d}\bar{L}\|D_\kappa\tilde{\zeta} + M(w,z,\xi)\| \leq \bar{L}\|\tilde{\zeta}\| + \frac{1}{\kappa^d}\bar{L}\|M(w,z,\xi)\|, \end{aligned}$$

for some $\bar{L} > 0$. Thus, along trajectories of the system, the function $V(\tilde{\zeta}) = \tilde{\zeta}^{\mathrm{T}}P\tilde{\zeta}$ satisfies

$$\dot{V}(\tilde{\zeta}) \leq -\kappa\|\tilde{\zeta}\|^2 + 2\|\tilde{\zeta}\|\,\|P\|\Big[\bar{L}\|\tilde{\zeta}\| + \bar{L}\|M(w,z,\xi)\| + \|v(w,z,\xi)\|\Big].$$

This being the case, it is clear that, if κ is large enough, conclusions similar to those obtained in the previous case hold. More specifically, define

$$\bar{v}(w,z,\xi) = \bar{L}\|M(w,z,\xi)\| + \|v(w,z,\xi)\|,$$

observe that

$$2\|\tilde{\zeta}\|\,\|P\|\Big[\bar{L}\|M(w,z,\xi)\| + \|v(w,z,\xi)\|\Big] \le \|\tilde{\zeta}\|^2\|P\|^2 + \|\bar{v}(w,z,\xi)\|^2$$

choose κ so that

$$a = \kappa - 2\bar{L}\|P\| - \|P\|^2$$

is positive. Then,

$$\dot{V}(\tilde{\zeta}) \le -a\|\tilde{\zeta}\|^2 + \|\bar{v}(w,z,\xi)\|^2.$$

From this, it is concluded that if $\bar{v}(w,z,\xi)$ can be estimated as in (12.33), also the property indicated in Assumption 12.3 holds.[11]

Example 12.1 Consider a controlled Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 + u \end{aligned} \tag{12.37}$$

in which $\mu > 0$ is an uncertain fixed parameter, ranging on a compact set. It is required that its state x_1 asymptotically tracks the state w_1 of a fixed linear harmonic oscillator

$$\dot{w} = \begin{pmatrix} aw_2 \\ -aw_1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} w = Sw.$$

To this end, define

$$e = x_1 - w_1.$$

and consider the associated problem of output regulation.

To begin with the analysis, we put the system in normal form, obtaining

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= q(w,\xi) + u \\ e &= \xi_1. \end{aligned} \tag{12.38}$$

in which

$$q(w,\xi) = -(\xi_1 + w_1) + \mu(1 - (\xi_1 + w_1)^2)(\xi_2 + aw_2) + a^2 w_1.$$

Since in this case $\dim(z) = 0$, the nonlinear regulator equations do have a trivial solution, in which $\pi(w) = 0$ and

$$\psi(w) = -q(w,0) = (1 - a^2)w_1 - \mu(1 - w_1^2)aw_2.$$

[11] Note that G is present in the function $v(w,z,\xi)$ and hence in $\bar{v}(w,z,\xi)$. Thus, this function depends on κ and, actually, its magnitude grows with κ. However, this does not affect the conclusion. Once κ is fixed, the bound on $\bar{v}(w,z,\xi)$ is fixed as well. The "gain function" $\vartheta_1(\cdot)$ in (12.30) is influenced by the value of κ, but an estimate of this form holds anyway.

In this example, the condition requested in Assumption 12.4 happens to be true. To this end, observe that $\psi(w)$ can be expressed in the form

$$\psi(w) = \Psi \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix}$$

in which

$$\Psi = \big((1-a^2)\ -\mu a\ \mu a\ 0\ 0\ 0\big)\,.$$

Observe also that

$$L_s \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix} = \begin{pmatrix} a w_2 \\ -a w_1 \\ 2a w_1 w_2^2 - a w_1^3 \\ a w_2^3 - 2a w_2 w_1^2 \\ 3a w_1^2 w_2 \\ -3a w_2^2 w_1 \end{pmatrix} = S \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix},$$

in which

$$S = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & -a & 0 \\ 0 & 0 & -2a & 0 & 0 & a \\ 0 & 0 & 3a & 0 & 0 & 0 \\ 0 & 0 & 0 & -3a & 0 & 0 \end{pmatrix}.$$

From this, it is seen that, for any $k \geq 0$,

$$L_s^k \psi(w) = \Psi S^k \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix}.$$

Hence, by Cayley–Hamilton's Theorem, since S^6 is a linear combination of $I, S, \ldots, S^5$, it is clear that a regression-like relation such as (12.34) holds. Actually, it is not necessary to take $d = 6$ because the minimal polynomial of S is

$$m(\lambda) = (\lambda^2 + a^2)(\lambda^2 + 9a^2) = \lambda^4 + 10a^2\lambda^2 + 9a^4.$$

Thus, it is deduced that (12.34) holds with

$$\phi(\eta_1, \eta_2, \eta_3, \eta_4) = -9a^4\eta_1 - 10a^2\eta_3.$$

The composition of plant (12.38) and internal model is a system modeled by

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{\eta} &= \hat{A}\eta + \hat{B}\phi(\eta) + G\bar{u} \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= q(w, \xi) + [\hat{C}\eta + \bar{u}] \\
e &= \xi_1.
\end{aligned} \tag{12.39}$$

This is a system having relative degree 2, but not in strict normal form. To get a strict normal form, we change η into

$$\zeta = \eta - G\xi_2$$

and obtain

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{\zeta} &= (\hat{A} - G\hat{C})(\zeta + G\xi_2) + \hat{B}\phi(\zeta + G\xi_2) - Gq(w, \xi) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= q(w, \xi) + [\hat{C}(\zeta + G\xi_2) + \bar{u}] \\
e &= \xi_1.
\end{aligned} \tag{12.40}$$

To determine whether such system is minimum phase, in the sense of Definition 12.1, we look at the first two equations. It follows from the theory that $\sigma(w)$, which in this case is expressed as

$$\sigma(w) = \begin{pmatrix} \psi(w) \\ L_s\psi(w) \\ L_s^2\psi(w) \\ L_s^3\psi(w) \end{pmatrix} = \begin{pmatrix} \Psi \\ \Psi S \\ \Psi S^2 \\ \Psi S^3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix} := \Sigma \begin{pmatrix} w_1 \\ w_2 \\ w_1^2 w_2 \\ w_1 w_2^2 \\ w_1^3 \\ w_2^3 \end{pmatrix},$$

satisfies

$$\begin{aligned}
\frac{\partial \sigma}{\partial w} s(w) &= \hat{A}\sigma(w) + \hat{B}\phi(\sigma(w)) \\
\psi(w) &= \hat{C}\sigma(w).
\end{aligned}$$

Since in this case $\phi(\eta)$ is a linear function, for the dynamics of $\bar{\zeta} = \zeta - \sigma(w)$ we obtain

$$
\begin{aligned}
\dot{\bar{\zeta}} &= (\hat{A} - G\hat{C})(\bar{\zeta} + \sigma(w) + G\xi_2) + \hat{B}\phi(\bar{\zeta} + \sigma(w) + G\xi_2) - Gq(w,\xi) - \frac{\partial \sigma}{\partial w}s(w) \\
&= (\hat{A} - G\hat{C})(\bar{\zeta} + G\xi_2) + (\hat{A} - G\hat{C})\sigma(w) + \hat{B}\phi(\bar{\zeta} + \sigma(w) + G\xi_2) \\
&\quad - Gq(w,\xi) - \hat{A}\sigma(w) - \hat{B}\phi(\sigma(w)) \\
&= (\hat{A} - G\hat{C})\bar{\zeta} + \hat{B}\phi(\bar{\zeta}) + \hat{B}\phi(G\xi_2) + (\hat{A} - G\hat{C})G\xi_2 - G[q(w,\xi) + \psi(w)],
\end{aligned}
$$

which we rewrite as

$$
\dot{\bar{\zeta}} = (\hat{A} - G\hat{C})\bar{\zeta} + \hat{B}\phi(\bar{\zeta}) + v(w,\xi)
$$

in which $v(w,\xi)$ is a function that—by construction—vanishes at $\xi = 0$.

Now, it is clear that the (augmented) system is strongly minimum phase if one can pick G in such a way that the linear system

$$
\dot{\bar{\zeta}} = (\hat{A} - G\hat{C})\bar{\zeta} + \hat{B}\phi(\bar{\zeta})
$$

is asymptotically stable, and this is indeed possible, as shown before, if $G = D_\kappa G_0$, with G_0 such that $(\hat{A} - G_0\hat{C})$ is Hurwitz and κ is sufficiently large.

Having established that the augmented system is strongly minimum phase, it is possible to find a linear stabilizer of dimension 2 (namely equal to the relative degree of (12.39))

$$
\begin{aligned}
\dot{\xi} &= A_s\xi + B_s e \\
\bar{u} &= C_s\xi + D_s e,
\end{aligned}
$$

that stabilizes the invariant manifold

$$
\mathscr{A} = \{(w,\eta,\xi) : w \in W, \eta = \sigma(w), \xi = 0\}.
$$

On this manifold the regulation error e is zero and hence the problem of output regulation is solved, by the controller

$$
\begin{aligned}
\dot{\eta} &= \hat{A}\eta + \hat{B}\phi(\eta) + D_\kappa G_0[C_s\xi + D_s e] \\
\dot{\xi} &= A_s\xi + B_s e \\
u &= \hat{C}\eta + C_s\xi + D_s e.
\end{aligned}
$$

◁

12.5 Consensus in a Network of Nonlinear Systems

We conclude the chapter with a concise discussion of how the results presented in Chap. 5 can be extended to the case in which consensus is sought in a network of nonlinear agents. If the network is heterogeneous, mimicking the approach described in Sect. 5.6, the problem can be addressed in two steps: inducing consensus among a set of identical local reference generators and then using (decentralized) output regulation theory to synchronize the output of each agent with that of the local reference generator. The second step of this procedure only entails simple adaptations of the design method described earlier in this chapter. Therefore, in this section, we limit ourselves to show how the first step of the procedure can be completed. We refer the reader to Chap. 5 for all details and notations about networks of agents and interconnections.

Consider the problem of achieving consensus in a network of N identical nonlinear agents modeled by equations of the form[12]

$$\begin{aligned} \dot{x}_k &= f(x_k) + u_k \\ y_k &= h(x_k), \end{aligned} \tag{12.41}$$

in which $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$, $y_k \in \mathbb{R}$ and

$$f(x) = \hat{A}x + \hat{B}\phi(x) \qquad h(x) = \hat{C}x,$$

with $\hat{A}$, $\hat{B}$, $\hat{C}$ matrices of the form (2.7), and $\phi(x)$ a nonlinear function.

The control of such systems is chosen as

$$u_k = Kv_k$$

in which K is a vector of design parameters and (see (5.6))

$$v_k = \sum_{j=1}^{N} a_{kj}(y_j - y_k). \tag{12.42}$$

About the systems (12.41) we assume the following.

Assumption 12.5 The function $\phi(x)$ is globally Lipschitz and there exists a compact set $X \subset \mathbb{R}^n$, invariant for $\dot{x} = f(x)$, such that the system

$$\dot{x} = f(x) + v$$

is input-to-state stable with respect to v relative to X, namely there exist a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that[13]

$$\text{dist}(x(t), X) \le \beta(\text{dist}(x(0), X), t) + \gamma(\|v(\cdot)\|_{[0,t]}) \qquad \text{for all } t \ge 0.$$

To the purpose of solving the consensus problem, we choose the vector K as

---

[12] The model (12.41) should be compared with the case of a model (5.24) with B = I, that is precisely the case studied in the first part of Sect. 5.5.

[13] See section B.4 in Appendix B for the definition of distance of a point from a set.

$$K = D_\kappa K_0,$$

in which D_κ is the matrix (12.35), with d replaced by n, and K_0 a vector to be designed. By the definition of Laplacian matrix of a graph, the kth controlled agent of the network can be written as

$$\dot{x}_k = \hat{A}x_k + \hat{B}\phi(x_k) - D_\kappa K_0 \sum_{j=1}^{N} \ell_{kj}\hat{C}x_j. \tag{12.43}$$

Thus, setting $x = \mathrm{col}(x_1, \ldots, x_N)$ entire set of N controlled agents can be rewritten as

$$\dot{x} = \Big[(I_N \otimes \hat{A}) - (L \otimes D_\kappa K_0 \hat{C})\Big]x + (I_N \otimes \hat{B})\Phi(x)$$

where

$$\Phi(x) = \mathrm{col}(\phi(x_1), \ldots, \phi(x_N)).$$

Consider the change of variables $\tilde{x} = (T^{-1} \otimes I_n)x$, in which T is the matrix defined in (5.32). The system in the new coordinates becomes

$$\begin{aligned}\dot{\tilde{x}} &= (T^{-1} \otimes I_n)\Big[(I_N \otimes \hat{A}) - (L \otimes D_\kappa K_0 \hat{C})\Big](T \otimes I_n)\tilde{x} \;+\; (T^{-1} \otimes I_n)(I_N \otimes \hat{B})\Phi((T \otimes I_n)\tilde{x}) \\ &= \Big[(I_N \otimes \hat{A}) - (\tilde{L} \otimes D_\kappa K_0 \hat{C})\Big]\tilde{x} \;+\; (T^{-1} \otimes \hat{B})\Phi((T \otimes I_n)\tilde{x}).\end{aligned}$$

As in Sect. 5.4 observe that

$$\tilde{x} = \mathrm{col}(x_1, x_2 - x_1, \ldots, x_N - x_1)$$

and define

$$x_\delta = \mathrm{col}(x_2 - x_1, x_3 - x_1, \ldots, x_N - x_1).$$

Then, the system above exhibits a block-triangular structure, of the form

$$\begin{aligned}\dot{x}_1 &= \hat{A}x_1 + \hat{B}\phi(x_1) - (L_{12} \otimes D_\kappa K_0 \hat{C})x_\delta \\ \dot{x}_\delta &= \Big[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes D_\kappa K_0 \hat{C})\Big]x_\delta + \Delta\Phi(x_1, x_\delta)\end{aligned}$$

in which

$$\Delta\Phi(x_1, x_\delta) = (I_{N-1} \otimes \hat{B}) \begin{pmatrix} \phi(x_2) - \phi(x_1) \\ \phi(x_3) - \phi(x_1) \\ \cdots \\ \phi(x_N) - \phi(x_1) \end{pmatrix}.$$

Clearly, $\Delta\Phi(x_1, 0) \equiv 0$ for all $x_1 \in \mathbb{R}^n$. Moreover, since $\phi(x)$ is by assumption globally Lypschitz, it is easy to check that $\Delta\Phi(x_1, x_\delta)$ is globally Lipschitz in x_δ, uniformly in x_1.

It is clear that, if the equilibrium $x_\delta = 0$ of the lower subsystem is globally asymptotically stable, consensus is achieved. This is the case if κ is large enough. To prove this claim, assume that the interconnection graph is connected, in which case the Laplacian matrix L has only one trivial eigenvalue and all remaining eigenvalues have positive real part. As a consequence, there is a number $\mu > 0$ satisfying (5.31).

Lemma 12.3 *Suppose Assumption 12.5 hold. Suppose the interconnection graph is connected. Let P be the unique positive definite symmetric solution of the algebraic Riccati equation*

$$\hat{A}P + P\hat{A}^T - 2\mu P\hat{C}^T\hat{C}P + aI = 0$$

with $a > 0$, and μ satisfying (5.31). Take K_0 as

$$K_0 = PC^T. \tag{12.44}$$

Then, there is a number κ^* such that, if $\kappa \geq \kappa^*$, for any initial condition $x_1(t)$ is a bounded function and the equilibrium $x_\delta = 0$ of

$$\dot{x}_\delta = \left[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes D_\kappa K_0 \hat{C})\right] x_\delta + \Delta\Phi(x_1, x_\delta)$$

is globally exponentially stable. ◁

Proof Consider the rescaled state variable

$$\zeta = (I_{N-1} \otimes D_\kappa^{-1}) x_\delta,$$

from which—bearing in mind the properties (12.36)—it follows that

$$\dot{\zeta} = \kappa \left[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes K_0 \hat{C})\right] \zeta \ + \ \frac{1}{\kappa^n} \Delta\Phi(x_1, (I_{N-1} \otimes D_\kappa)\zeta). \tag{12.45}$$

It is known from Sect. 5.5 that the given choice of K_0 guarantees that the matrix $[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes K_0 \hat{C})]$ is Hurwitz. Therefore, there exists a positive definite matrix Q such that

$$Q\left[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes K_0 \hat{C})\right] + \left[(I_{N-1} \otimes \hat{A}) - (L_{22} \otimes K_0 \hat{C})\right]^{\mathrm{T}} Q = -I.$$

Moreover, since $\Delta\Phi(x_1, x_\delta)$ is globally Lipschitz in x_δ uniformly in x_1, we know that, for some $\bar{L} > 0$,

$$\|\Delta\Phi(x_1, (I_{N-1} \otimes D_\kappa)\zeta)\| \leq \bar{L}\|D_\kappa\| \, \|\zeta\| = \bar{L}\kappa^n \|\zeta\|.$$

Thus, along trajectories of the system, the function $V(\zeta) = \zeta^{\mathrm{T}} Q \zeta$ satisfies

$$\dot{V}(\zeta) \leq -\kappa\|\zeta\|^2 + 2\bar{L}\|Q\| \, \|\zeta\|^2.$$

Form this it is deduced that, if $\kappa > 2\bar{L}\|Q\|$, as long as trajectories are defined, $V(\zeta(t))$ is steadily decreasing. Let κ be fixed in this way. Since x_1 obeys

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}\phi(x_1) - (L_{12} \otimes D_\kappa K_0 \hat{C} D_\kappa)\zeta,$$

using Assumption 12.5, it is seen that $x_1(t)$ is bounded for all $t \geq 0$ (by a quantity that only depends on initial conditions and κ). Thus. trajectories of the entire system are defined for all $t \geq 0$ and the equilibrium $\zeta = 0$ is globally exponentially stable, with a (fixed) quadratic Lyapunov function. $\triangleleft$

This yields the following conclusion.

Proposition 12.1 *Let the hypotheses of the previous lemma hold and let K_0 be chosen as in (12.44). There is a number $\kappa^* > 0$ such that, if $\kappa \geq \kappa^*$, the states of the N systems (12.43) reach consensus, i.e., for every $x_k(0) \in \mathbb{R}^n$, $k = 1, \ldots, N$, there is a function $x^* : \mathbb{R} \to \mathbb{R}^n$ such that*

$$\lim_{t\to\infty} \|x_k(t) - x^*(t)\| = 0 \qquad \textit{for all } k = 1, \ldots, N.$$

$\triangleleft$

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Appendix A
Background Material in Linear Systems Theory

A.1 Quadratic Forms

In this section, a few fundamental facts about symmetric matrices and quadratic forms are reviewed.

Symmetric matrices. Let P be a $n \times n$ *symmetric* matrix of real numbers (that is, a matrix of real numbers satisfying $P = P^{\mathrm{T}}$). Then there exist an *orthogonal* matrix Q of real numbers[1] and a diagonal matrix Λ of real numbers

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

such that

$$Q^{-1}PQ = Q^{\mathrm{T}}PQ = \Lambda .$$

Indeed, the numbers $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of P. Thus, a symmetric matrix P of real numbers has real eigenvalues and a purely diagonal Jordan form.

Note that the previous identity can be rewritten as

$$PQ = Q\Lambda$$

from which it is seen that the ith column q_i of Q is an eigenvector of P, associated with the ith eigenvalue λ_i. If P is invertible, so is the matrix Λ, and

$$P^{-1}Q = Q\Lambda^{-1} ,$$

[1] That is, matrix Q of real numbers satisfying $QQ^{\mathrm{T}} = I$, or—what is the same—satisfying $Q^{-1} = Q^{\mathrm{T}}$.

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8

from which it is seen that Λ^{-1} is a Jordan form of P^{-1} and the ith column q_i of Q is also an eigenvector of P^{-1}, associated with the ith eigenvalue λ_i^{-1}.

Quadratic forms. Let P be a $n \times n$ matrix of real numbers and $x \in \mathbb{R}^n$. The expression

$$V(x) = x^{\mathrm{T}}Px = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_i x_j$$

is called a *quadratic form* in x. Without loss of generality, in the expression above we may assume that P is *symmetric*. In fact,

$$V(x) = x^{\mathrm{T}}Px = \frac{1}{2}[x^{\mathrm{T}}Px + x^{\mathrm{T}}Px] = x^{\mathrm{T}}\left[\frac{1}{2}(P + P^{\mathrm{T}})\right]x$$

and $\frac{1}{2}(P + P^{\mathrm{T}})$ is by construction symmetric.

Let P be symmetric, express it as $P = Q\Lambda Q^{\mathrm{T}}$ (see above) with eigenvalues sorted so that $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\begin{aligned} x^{\mathrm{T}}Px &= x^{\mathrm{T}}(Q\Lambda Q^{\mathrm{T}})x = (Q^{\mathrm{T}}x)^{\mathrm{T}}\Lambda(Q^{\mathrm{T}}x) = \textstyle\sum_{i=1}^{n}\lambda_i(Q^{\mathrm{T}}x)_i^2 \\ &\leq \lambda_1 \textstyle\sum_{i=1}^{n}(Q^{\mathrm{T}}x)_i^2 = \lambda_1(Q^{\mathrm{T}}x)^{\mathrm{T}}Q^{\mathrm{T}}x = \lambda_1 x^{\mathrm{T}}QQ^{\mathrm{T}}x = \lambda_1 x^{\mathrm{T}}x \\ &= \lambda_1\|x\|^2. \end{aligned}$$

With a similar argument we can show that $x^{\mathrm{T}}Px \geq \lambda_n\|x\|^2$. Usually, λ_n is denoted as $\lambda_{\min}(P)$ and λ_1 is denoted as $\lambda_{\max}(P)$. In summary, we conclude that

$$\lambda_{\min}(P)\|x\|^2 \leq x^{\mathrm{T}}Px \leq \lambda_{\max}(P)\|x\|^2.$$

Note that the inequalities are tight (*hint*: pick, as x, the last and, respectively, the first column of Q).

Sign-definite symmetric matrices. Let P be symmetric. The matrix P is said to be *positive semidefinite* if

$$x^{\mathrm{T}}Px \geq 0 \qquad \text{for all } x.$$

The matrix P is said to be *positive definite* if

$$x^{\mathrm{T}}Px > 0 \qquad \text{for all } x \neq 0.$$

We see from the above that P is positive semidefinite if and only if $\lambda_{\min} \geq 0$ and is positive definite if and only if $\lambda_{\min} > 0$ (which in turn implies the nonsingularity of P).

There is another criterion for a matrix to be positive definite, that does not require the computation of the eigenvalues of P, known as Sylvester's criterion. For a *symmetric* matrix P, the n minors

$$D_1 = \det(p_{11}), \quad D_2 = \det\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad D_3 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \ldots$$

are called the *leading principal minors*. Note that $D_n = \det(P)$.

Lemma A.1 *A symmetric matrix is positive definite if and only if all leading principal minors are positive, i.e.,* $D_1 > 0,\ D_2 > 0, \ldots, D_n > 0$.

Another alternative criterion, suited a for block-partitioned matrix, is the criterion due to Schur.

Lemma A.2 *The symmetric matrix*

$$\begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \tag{A.1}$$

is positive definite if and only if[2]

$$R > 0 \quad \text{and} \quad Q - SR^{-1}S^{\mathrm{T}} > 0. \tag{A.2}$$

Proof Observe that a necessary condition for (A.1) to be positive definite is $R > 0$. Hence R is nonsingular and (A.1) can be transformed, by congruence, as

$$\begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix} = \begin{pmatrix} Q - SR^{-1}S^{\mathrm{T}} & 0 \\ 0 & R \end{pmatrix}.$$

from which the condition (A.2) follows. ◁

A symmetric matrix P is said to be *negative semidefinite* (respectively, *negative definite*) if $-P$ is *positive semidefinite* (respectively, *positive definite*). Usually, to express the property (of a matrix P) of being positive definite (respectively, positive semidefinite) the notation $P > 0$ (respectively, $P \geq 0$) is used.[3] Likewise, the notation $P < 0$ (respectively $P \leq 0$) is used to express the property that P is negative definite (respectively, negative semidefinite). If P and R are symmetric matrices, the notations

$$P \geq R \quad \text{and} \quad P > R$$

stand for "*the matrix* $P - R$ *is positive semidefinite*" and, respectively, for "*the matrix* $P - R$ *is positive definite*".

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[2]The matrix $Q - SR^{-1}S^{\mathrm{T}}$ is called the *Schur's complement* of R in (A.1).

[3]Note that this *is not* the same as $p_{ij} > 0$ for all i, j. For example, the matrix

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

in which the off-diagonal elements are negative, is positive definite (use Sylvester's criterion above).

Any matrix P that can be written in the form $P = M^{\mathrm{T}}M$, in which M is a possibly non-square matrix, is positive semidefinite. In fact $x^{\mathrm{T}}Px = x^{\mathrm{T}}M^{\mathrm{T}}Mx = \|Mx\|^2 \geq 0$.

Conversely, any matrix P which is positive semidefinite can always be expressed as $P = M^{\mathrm{T}}M$. In fact, if P is positive semidefinite all its eigenvalues are nonnegative. Let r denote the number of nonzero eigenvalues and let the eigenvalues be sorted so that $\lambda_{r+1} = \cdots = \lambda_n = 0$. Then

$$P = Q\Lambda Q^{\mathrm{T}} = Q\begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathrm{T}}$$

in which Λ_1 is an $r \times r$ diagonal matrix, whose diagonal elements are all positive. For $i = 1, \ldots, r$, let σ_i denote the positive square root of λ_i, let Q_1 be the $n \times r$ matrix whose columns coincide with the first r columns of Q and set

$$M = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix} Q_1^{\mathrm{T}}$$

Then, the previous identity yields

$$P = M^{\mathrm{T}}M$$

in which M is a $n \times r$ matrix of rank r.

Finally, recall that, if P is symmetric and invertible, the eigenvalues of P^{-1} are the inverse of the eigenvalues of P. Thus, in particular, if P is positive definite, so is also P^{-1}.

A.2 Linear Matrix Equations

In this section, we discuss the existence of solutions of two relevant linear matrix equations that arise in the analysis of linear systems. One of such equation is the so-called *Sylvester's equation*

$$AX - XS = R \tag{A.3}$$

in which $A \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{d\times d}$, in the unknown $X \in \mathbb{R}^{n\times d}$. An equation of this kind arises, for instance, when it is desired to transform a given block-triangular matrix into a (purely) block-diagonal one, by means of a similarity transformation, as in

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & R \\ 0 & S \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Another instance in which an equation of this kind arises is the analysis of the stability of a linear system, where this equation assumes the special form $AX + XA^{\mathrm{T}} = Q$, known as *Lyapunov's equation*.

Another relevant linear matrix equation is the so-called *regulator* or *Francis's equation*

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C\Pi + Q \end{aligned} \tag{A.4}$$

in which $A \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, in the unknowns $\Pi \in \mathbb{R}^{n\times d}$ and $\Psi \in \mathbb{R}^{m\times d}$. This equation arises in the study of the problem of output regulation of linear systems.

The two equations considered above are special cases of an equation of the form

$$A_1 X q_1(S) + \cdots + A_k X q_k(S) = R \tag{A.5}$$

in which, for $i = 1, \ldots, k$, $A_i \in \mathbb{R}^{\bar n\times \bar m}$ and $q_i(\lambda)$ is a polynomial in the indeterminate λ, $S \in \mathbb{R}^{\bar d\times \bar d}$, $R \in \mathbb{R}^{\bar n\times \bar d}$, in the *unknown* $X \in \mathbb{R}^{\bar m\times \bar d}$. In fact, the Sylvester's equation corresponds to the case in which $k = 2$ and

$$A_1 = A, \quad q_1(\lambda) = 1, \quad A_2 = I, \quad q_2(\lambda) = -\lambda,$$

while the Francis' equation corresponds to the case in which $k = 2$ and

$$A_1 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad q_1(\lambda) = 1, \quad A_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad q_2(\lambda) = -\lambda, \quad R = \begin{pmatrix} -P \\ -Q \end{pmatrix}.$$

Equations of the form (A.5) are also known as *Hautus' equations*.[4] Noting that the left-hand side of (A.5) can be seen as a linear map

$$\begin{aligned} \mathscr{H} : \mathbb{R}^{\bar m\times \bar d} &\to \mathbb{R}^{\bar n\times \bar d} \\ : \quad X \quad &\mapsto \mathscr{H}(X) := A_1 X q_1(S) + \cdots + A_k X q_k(S), \end{aligned}$$

to say that (A.5) has a solution is to say that $R \in \mathrm{Im}(\mathscr{H})$.

In what follows, we are interested in the case in which (A.5) has solutions for all R, i.e., in the case in which the map $\mathscr{H}$ is *surjective*.[5]

Theorem A.1 *The map $\mathscr{H}$ is surjective if and only if the $\bar n$ rows of the matrix*

$$A(\lambda) = A_1 q_1(\lambda) + \cdots + A_k q_k(\lambda)$$

are linearly independent for each λ which is an eigenvalue of S. If this is the case and $\bar n = \bar m$, the solution X of (A.5) is unique.

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[4] See [3].

[5] Note that, if this is the case and $\bar n = \bar m$, the map is also *injective*, i.e., it is an invertible linear map. In this case the solution X of (A.5) is *unique*.

From this, it is immediate to deduce the following Corollaries.

Corollary A.1 *The Sylvester's equation (A.3) has a solution for each R if and only if $\sigma(A) \cap \sigma(S) = \emptyset$. If this is the case, the solution X is unique.*

Corollary A.2 *The Francis' equation (A.4) has a solution for each pair (P, Q) if and only if the rows of the matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

are linearly independent for each λ which is an eigenvalue of S. If this is the case and $m = p$, the solution pair (Π, Ψ) is unique.

A.3 The Theorems of Lyapunov for Linear Systems

In this section we describe a powerful criterion useful to determine when a $n \times n$ matrix of real numbers has all eigenvalues with negative real part.[6]

Theorem A.2 (Direct Theorem) *Let $A \in \mathbb{R}^{n\times n}$ be a matrix of real numbers. Let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix of real numbers and suppose that the matrix*

$$PA + A^{\mathrm{T}}P$$

is negative definite. Then, all eigenvalues of the matrix A have negative real part.

Proof Let λ be an eigenvalue of A and x an associated eigenvector. Let x_R and x_I denote the real and, respectively, imaginary part of x, i.e., set $x = x_R + jx_I$ and let $x^* = x_R^{\mathrm{T}} - jx_I^{\mathrm{T}}$. Then

$$x^*Px = (x_R)^{\mathrm{T}}Px_R + (x_I)^{\mathrm{T}}Px_I.$$

Since P is positive definite and x_R and x_I cannot be both zero (because $x \neq 0$), we deduce that

$$x^*Px > 0. \tag{A.6}$$

With similar arguments, since $A^{\mathrm{T}}P + PA$ is negative definite, we deduce that

$$x^*(A^{\mathrm{T}}P + PA)x < 0. \tag{A.7}$$

Using the definition of x and λ (i.e., $Ax = x\lambda$, that implies $x^*A^{\mathrm{T}} = \lambda^*x^*$), obtain

$$x^*(A^{\mathrm{T}}P + PA)x = \lambda^*x^*Px + x^*Px\lambda = (\lambda + \lambda^*)x^*Px$$

[6] For further reading, see e.g., [1].

from which, using (A.6) and (A.7) we conclude

$$\lambda + \lambda^* = 2\mathrm{Re}[\lambda] < 0. \qquad \triangleleft$$

Remark A.1 The criterion described in the previous theorem provides a sufficient condition under which all the eigenvalues of a matrix A have negative real part. In the analysis of linear systems, this criterion is used as a sufficient condition to determine whether the equilibrium $x = 0$ of the autonomous system

$$\dot{x} = Ax \tag{A.8}$$

is (globally) asymptotically stable. In this context, the previous proof—which only uses algebraic arguments—can be replaced by the linear version of the proof of Theorem B.1, which can be summarized as follows. Let $V(x) = x^{\mathrm{T}}Px$ denote the positive-definite quadratic function associated with P, let $x(t)$ denote a generic trajectory of system (A.8) and consider the composite function $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$. Observe that

$$\frac{\partial V}{\partial x} = 2x^{\mathrm{T}}P.$$

Therefore, using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\partial V}{\partial x}\bigg|_{x=x(t)} \frac{\mathrm{d}x}{\mathrm{d}t} = 2x^{\mathrm{T}}(t)PAx(t) = x^{\mathrm{T}}(t)(PA + A^{\mathrm{T}}P)x(t).$$

If P is positive definite and $PA+A^{\mathrm{T}}P$ is negative definite, there exist positive numbers a_1, a_2, a_3 such that

$$a_1\|x\|^2 \le V(x) \le a_2\|x\|^2 \quad \text{and} \quad x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x \le -a_3\|x\|^2.$$

From this, it is seen that $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le -\lambda V(x(t))$$

with $\lambda = a_3/a_2 > 0$ and therefore

$$a_1\|x(t)\|^2 \le V(x(t)) \le e^{-\lambda t}V(x(0)) \le e^{-\lambda t}a_2\|x(0)\|^2.$$

Thus, for any initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$. This proves that all eigenvalues of A have negative real part. $\triangleleft$

Theorem A.3 (Converse Theorem) *Let $A \in \mathbb{R}^{n\times n}$ be a matrix of real numbers. Suppose all eigenvalues of A have negative real part. Then, for any choice of a symmetric positive-definite matrix Q, there exists a unique symmetric positive-definite matrix P such that*

$$PA + A^{\mathrm{T}}P = -Q. \tag{A.9}$$

Proof Consider (A.9). This is a Sylvester equation, and—since the spectra of A and $-A^{\mathrm{T}}$ are disjoint—a unique solution P exists. We compute it explicitly. Define

$$M(t) = e^{A^{\mathrm{T}}t}Qe^{At}$$

and observe that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At} + e^{A^{\mathrm{T}}t}Qe^{At}A = A^{\mathrm{T}}M(t) + M(t)A.$$

Integrating over $[0, T]$ yields

$$M(T) - M(0) = A^{\mathrm{T}}\int_0^T M(t)dt + \int_0^T M(t)dt\; A.$$

Since the eigenvalues of A have negative real part,

$$\lim_{T\to\infty} M(T) = 0$$

and

$$P := \lim_{T\to\infty}\int_0^T M(t)dt < \infty.$$

We have shown in this way that P satisfies (A.9). It is the unique solution of this equation.

To complete the proof it remains to show that P is positive definite, if so is Q. By contradiction, suppose is not. Then there exists $x_0 \neq 0$ such that

$$x_0^{\mathrm{T}}Px_0 \leq 0\,,$$

which, in view of the expression found for P, yields

$$\int_0^\infty x_0^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}x_0 dt \leq 0.$$

Setting

$$x(t) = e^{At}x_0$$

this is equivalent to

$$\int_0^\infty x^{\mathrm{T}}(t)Qx(t)dt \leq 0\,,$$

which, using the estimate $x^{\mathrm{T}}Qx \geq \lambda_{\min}(Q)\|x\|^2$ in turn yields

$$\int_0^\infty \|x(t)\|^2 dt \le 0$$

which then yields

$$x(t) = 0, \qquad \text{for all } t \in [0, \infty).$$

Bearing in mind the expression of $x(t)$, this implies $x_0 = 0$ and completes the proof.

A.4 Stabilizability, Detectability and Separation Principle

In this section, a few fundamental facts about the stabilization of linear systems are reviewed.[7] Consider a linear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{A.10}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$ and we summarize the properties that determine the existence of a (dynamic) output feedback controller of the form

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y, \end{aligned} \tag{A.11}$$

with state $x_c \in \mathbb{R}^{n_c}$, that stabilizes the resulting closed-loop system

$$\begin{aligned} \dot{x} &= (A + BD_c C)x + BC_c x_c \\ \dot{x}_c &= B_c C x + A_c x_c. \end{aligned} \tag{A.12}$$

Definition A.1 The pair (A, B) is *stabilizable* if there exists a matrix F such that $(A + BF)$ has all the eigenvalues in $\mathbb{C}^-$.

Definition A.2 The pair (A, C) is *detectable* if there exists a matrix G such that $(A - GC)$ has all the eigenvalues in $\mathbb{C}^-$.

Noting that the closed-loop system (A.12) can be written as $\dot{x}_{c\ell} = A_{c\ell} x_{c\ell}$, with $x_{c\ell} = \text{col}(x, x_c)$ and

$$A_{c\ell} = \begin{pmatrix} (A + BD_c C) & BC_c \\ B_c C & A_c \end{pmatrix}, \tag{A.13}$$

we have the following fundamental result.

Theorem A.4 *There exist matrices A_c, B_c, C_c, D_c such that (A.13) has all the eigenvalues in $\mathbb{C}^-$ if and only if the pair (A, B) is stabilizable* and *pair (A, C) is detectable.*

[7] For further reading, see e.g., [1].

Proof (Necessity) Suppose all eigenvalues of (A.13) have negative real part. Then, by the converse Lyapunov's Theorem, there exists a unique, symmetric, and positive-definite solution $P_{c\ell}$ of the matrix equation

$$P_{c\ell}A_{c\ell} + A_{c\ell}^{T}P_{c\ell} = -I. \tag{A.14}$$

Let $P_{c\ell}$ be partitioned as in

$$P_{c\ell} = \begin{pmatrix} P & S \\ S^{T} & P_{c} \end{pmatrix}$$

consistently with the partition of $A_{c\ell}$ (note, in this respect, that the two diagonal blocks may have different dimensions n and n_c). Note also that P and P_c are necessarily positive definite (and hence also nonsingular) because so is $P_{c\ell}$. Consider the matrix

$$T = \begin{pmatrix} I & 0 \\ -P_{c}^{-1}S^{T} & I \end{pmatrix}.$$

Define $\tilde{P} := T^{T}P_{c\ell}T$ and note that

$$\tilde{P} = \begin{pmatrix} P - SP_{c}^{-1}S^{T} & 0 \\ 0 & P_{c} \end{pmatrix}. \tag{A.15}$$

Define $\tilde{A} := T^{-1}A_{c\ell}T$ and, by means of a simple computation, observe that

$$\tilde{A} = \begin{pmatrix} A + B(D_{c}C - C_{c}P_{c}^{-1}S^{T}) & * \\ * & * \end{pmatrix}$$

in which we have denoted by an asterisk blocks whose expression is not relevant in the sequel.

From (A.14) it is seen that

$$\begin{aligned} \tilde{P}\tilde{A} + \tilde{A}^{T}\tilde{P} &= (T^{T}P_{c\ell}T)(T^{-1}A_{c\ell}T) + (T^{T}A_{c\ell}^{T}(T^{-1})^{T})(T^{T}P_{c\ell}T) \\ &= T^{T}(P_{c\ell}A_{c\ell} + A_{c\ell}^{T}P_{c\ell})T = -T^{T}T. \end{aligned} \tag{A.16}$$

This shows that the matrix $\tilde{P}\tilde{A} + \tilde{A}^{T}\tilde{P}$ is negative definite (because T is nonsingular) and hence so is its upper-left block. The latter, if we set

$$P_0 = P - SP_{c}^{-1}S^{T}, \qquad F = D_{c}C - C_{c}P_{c}^{-1}S^{T}$$

can be written in the form

$$P_0(A + BF) + (A + BF)^{T}P_0. \tag{A.17}$$

The matrix P_0 is positive definite, because it is the upper-left block of the positive-definite matrix (A.15). The matrix (A.17) is negative definite, because it is the upper-left block of the negative-definite matrix (A.16). Thus, by the direct criterion of Lyapunov, it follows that the eigenvalues of $A + BF$ have negative real part. This completes the proof that, if $A_{c\ell}$ has all eigenvalues in $\mathbb{C}^-$, there exists a matrix F such that $A + BF$ has all eigenvalues in $\mathbb{C}^-$, i.e., the pair (A, B) is stabilizable. In a similar way it is proven that the pair (A, C) is detectable.

(Sufficiency) Assuming that (A, B) is stabilizable and that (A, C) is detectable, pick F and G so that $(A+BF)$ has all eigenvalues in $\mathbb{C}^-$ and $(A-GC)$ has all eigenvalues in $\mathbb{C}^-$. Consider the controller

$$\begin{aligned} \dot{x}_c &= (A + BF - GC)x_c + Gy \\ u &= Fx_c, \end{aligned} \tag{A.18}$$

i.e., set

$$A_c = A + BF - GC, \quad B_c = G, \quad C_c = F, \quad D_c = 0.$$

This yields a closed-loop system

$$\begin{aligned} \dot{x} \ &= Ax + BFx_c \\ \dot{x}_c &= GCx + (A + BF - GC)x_c. \end{aligned}$$

The change of variables $z = x - x_c$ changes the latter into the system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

This system is in block-triangular form and both diagonal blocks have all eigenvalues in $\mathbb{C}^-$. Thus the controller (A.18) guarantees that the matrix $A_{c\ell}$ has all eigenvalues in $\mathbb{C}^-$.[8]

To check whether the two fundamental properties in question hold, the following tests are useful.

Lemma A.3 *The pair* (A, B) *is stabilizable if and only if*

$$\operatorname{rank} \begin{pmatrix} A - \lambda I & B \end{pmatrix} = n \tag{A.19}$$

for all $\lambda \in \sigma(A)$ *having nonnegative real part.*

[8]Observe that the choices of F and G are *independent* of each other, i.e., F is only required to place the eigenvalues of $(A + BF)$ in $\mathbb{C}^-$ and G is only required to place the eigenvalues of $(A - GC)$ in $\mathbb{C}^-$. For this reason, the controller (A.18) is said to be a controller inspired by a *separation principle*.

Lemma A.4 *The pair* (A, C) *is detectable if and only if*

$$\operatorname{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n \tag{A.20}$$

for all $\lambda \in \sigma(A)$ *having nonnegative real part.*

Remark A.2 For the sake of completeness, we recall how the two properties of stabilizability and detectability invoked above compare with the properties of reachability and observability. To this end, we recall that the linear system (A.25) is *reachable* if and only if

$$\operatorname{rank}\begin{pmatrix} B \; AB \; \dots \; A^{n-1}B \end{pmatrix} = n \tag{A.21}$$

or, what is the same, if and only if the condition (A.19) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having nonnegative real part). The linear system (A.25) is *observable* if and only if

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} = n \tag{A.22}$$

or, what is the same, if and only if the condition (A.20) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having nonnegative real part).

It is seen from this that, in general, reachability is a property stronger than stabilizability and observability is a property stronger than detectability. The two (pairs of) properties coincide when all eigenvalues of A have nonnegative real part. If the rank of the matrix on the left-hand side of (A.21) is $n_1 < n$, the system is *not* reachable and there exists a nonsingular matrix T such that[9]

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

in which $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{11}, B_1) is reachable. This being the case, it is easy to check that the pair (A, B) is stabilizable if and only if all eigenvalues of A_{22} have negative real part. A similar criterion determines the relation between detectability and observability. If the rank of the matrix on the left-hand side of (A.22) is $n_1 < n$, the system is *not* observable and there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

in which $A_{22} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{22}, C_2) is observable. This being the case, it is easy to check that the pair (A, C) is detectable if and only if all eigenvalues of A_{11} have negative real part.

[9] This is the well-know *Kalman's* decomposition of a system into reachable/unreachable parts.

A.5 Steady-State Response to Harmonic Inputs

Invariant subspaces. Let A be a fixed $n \times n$ matrix. A subspace $\mathscr{V}$ of $\mathbb{R}^n$ is *invariant* under A if

$$v \in \mathscr{V} \quad \Rightarrow \quad Av \in \mathscr{V}.$$

Let $d < n$ denote the dimension of $\mathscr{V}$ and let $\{v_1, v_2, \ldots, v_d\}$ be a basis of $\mathscr{V}$, that is a set of d linearly independent vectors $v_i \in \mathbb{R}^n$ such that

$$\mathscr{V} = \mathrm{Im}(V)$$

where V is the $n \times d$ matrix

$$V = \begin{pmatrix} v_1 \; v_2 \; \ldots \; v_d \end{pmatrix}.$$

Then, it is an easy matter to check that $\mathscr{V}$ is invariant under A if and only if there exists a $d \times d$ matrix $A_{\mathscr{V}}$ such that

$$AV = VA_{\mathscr{V}}.$$

The map $z \mapsto A_{\mathscr{V}} z$ characterizes the *restriction* to $\mathscr{V}$ of the map $x \mapsto Ax$. This being the case, observe that if (λ_0, z_0) is a pair eigenvalue-eigenvector for $A_{\mathscr{V}}$ (i.e., a pair satisfying $A_{\mathscr{V}} z_0 = \lambda_0 z_0$), then $(\lambda_0, V z_0)$ is a pair eigenvalue-eigenvector for A.

Let the matrix A have n_s eigenvalues in $\mathbb{C}^-$, n_a eigenvalues in $\mathbb{C}^+$ and n_c eigenvalues in $\mathbb{C}^0 = \{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] = 0\}$, with (obviously) $n_\mathrm{s} + n_\mathrm{a} + n_\mathrm{c} = n$. Then (passing for instance through the Jordan form of A) it is easy to check that there exist three invariant subspaces of A, denoted $\mathscr{V}_\mathrm{s}$, $\mathscr{V}_\mathrm{a}$, $\mathscr{V}_\mathrm{c}$, of dimension $n_\mathrm{s}, n_\mathrm{a}, n_\mathrm{c}$ that are complementary in $\mathbb{R}^n$, i.e., satisfy

$$\mathscr{V}_\mathrm{s} \oplus \mathscr{V}_\mathrm{a} \oplus \mathscr{V}_\mathrm{c} = \mathbb{R}^n, \tag{A.23}$$

with the property that the restriction of A to $\mathscr{V}_\mathrm{s}$ is characterized by a $n_\mathrm{s} \times n_\mathrm{s}$ matrix A_s whose eigenvalues are precisely the n_s eigenvalues of A that are in $\mathbb{C}^-$, the restriction of A to $\mathscr{V}_\mathrm{a}$ is characterized by a $n_\mathrm{a} \times n_\mathrm{a}$ matrix A_a whose eigenvalues are precisely the n_a eigenvalues of A that are in $\mathbb{C}^+$ and the restriction of A to $\mathscr{V}_\mathrm{c}$ is characterized by a $n_\mathrm{c} \times n_\mathrm{c}$ matrix A_c whose eigenvalues are precisely the n_c eigenvalues of A that are in $\mathbb{C}^0$. These three subspaces are called the *stable eigenspace*, the *antistable eigenspace* and the *center eigenspace*.

Consider now the autonomous linear system

$$\dot{x} = Ax \tag{A.24}$$

with $x \in \mathbb{R}^n$. It is easy to check that if subspace $\mathscr{V}$ is invariant under A, then for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ is such that $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$.[10] Because of (A.23), any trajectory $x(t)$ of (A.24) can be uniquely decomposed as

$$x(t) = x_{\mathrm{s}}(t) + x_{\mathrm{a}}(t) + x_{\mathrm{c}}(t)$$

with $x_{\mathrm{s}}(t) \in \mathscr{V}_{\mathrm{s}}$, $x_{\mathrm{a}}(t) \in \mathscr{V}_{\mathrm{a}}$, $x_{\mathrm{c}}(t) \in \mathscr{V}_{\mathrm{c}}$. Moreover, since the restriction of A to $\mathscr{V}_{\mathrm{s}}$ is characterized by a matrix A_{s} whose eigenvalues are all in $\mathbb{C}^-$,

$$\lim_{t \to \infty} x_{\mathrm{s}}(t) = 0$$

and, since the restriction of A to $\mathscr{V}_{\mathrm{a}}$ is characterized by a matrix A_{a} whose eigenvalues are all in $\mathbb{C}^+$,

$$\lim_{t \to -\infty} x_{\mathrm{a}}(t) = 0.$$

A geometric characterization of the steady-state response. It is well known that a stable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

subject to a (harmonic) input of the form

$$u(t) = u_0 \cos(\omega_0 t) \tag{A.25}$$

exhibits a well-defined *steady-state response*, which is itself a harmonic function of time. The response in question can be easily characterized by means of a simple geometric construction. Observe that the input defined above can be viewed as generated by an autonomous system of the form

$$\begin{aligned} \dot{w} &= Sw \\ u &= Qw \end{aligned} \tag{A.26}$$

in which $w \in \mathbb{R}^2$ and

$$S = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}, \qquad Q = u_0 \begin{pmatrix} 1 & 0 \end{pmatrix},$$

[10]Note that also the converse of such implication holds. If $\mathscr{V}$ is a subspace with the property that, for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ satisfies $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$, then $\mathscr{V}$ is invariant under A. The property in question is sometimes referred to as the *integral version* of the notion of invariance, while the property indicated in the text above is referred to as the *infinitesimal version* on the notion of invariance.

set in the initial state[11]

$$w(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.27}$$

In this way, the *forced* response of the given linear system, from any initial state $x(0)$, to the input (A.25) can be identified with the *free* response of the composite system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} \tag{A.28}$$

from the initial state $(x(0), w(0))$ with $w(0)$ given by (A.27).

Since A has all eigenvalues with negative real part and S has eigenvalues on the imaginary axis, the Sylvester equation

$$\Pi S = A\Pi + BQ \tag{A.29}$$

has a unique solution Π. The composite system (A.28) possesses two complementary invariant eigenspaces: a *stable eigenspace* and a *center eigenspace*, which can be respectively expressed as

$$\mathscr{V}^{\mathrm{s}} = \operatorname{span} \begin{pmatrix} 0 \\ I \end{pmatrix}, \qquad \mathscr{V}^{\mathrm{c}} = \operatorname{span} \begin{pmatrix} I \\ \Pi \end{pmatrix}.$$

The latter, in particular, shows that the center eigenspace is the set of all pairs (w, x) such that $x = \Pi w$.

Consider now the change of variables $\tilde{x} = x - \Pi w$ which, after a simple calculation which uses (A.29), yields

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{x}} &= A\tilde{x}. \end{aligned}$$

Since the matrix A is Hurwitz, for any initial condition

$$\lim_{t\to\infty} \tilde{x}(t) = 0\,,$$

which shows that the (unique) projection of the trajectory along the stable eigenspace asymptotically tends to zero. In the original coordinates, this reads as

$$\lim_{t\to\infty} [x(t) - \Pi w(t)] = 0\,,$$

[11] To check that this is the case, simply bear in mind that the solution $w(t)$ of (A.26) is given by

$$w(t) = e^{St} w(0) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

.

from which we see that the *steady-state response* of the system to any input generated by (A.26) can be expressed as

$$x_{ss}(t) = \Pi w(t). \tag{A.30}$$

It is worth observing that the steady-state response $x_{ss}(t)$ thus defined can also be identified with an *actual* forced response of the system to the input (A.25), provided that the initial state $x(0)$ is appropriately chosen. In fact, since the center eigenspace is invariant for the composite system (A.28), if the initial condition of the latter is taken on $\mathscr{V}^c$, i.e., if $x(0) = \Pi w(0)$, the motion of such system remains confined to $\mathscr{V}^c$ for all t, i.e., $x(t) = \Pi w(t)$ for all t. Thus, if $x(0) = \Pi w(0)$, the actual forced response $x(t)$ of the system to the input (A.25) coincides with the steady-state response $x_{ss}(t)$. Note that, in view of the definition of $w(0)$ given by (A.27), this initial state $x(0)$ is nothing else than the first column of the matrix Π.

The calculation of the solution Π of the Sylvester equation (A.29) is straightforward. Set

$$\Pi = \begin{pmatrix} \Pi_1 & \Pi_2 \end{pmatrix}$$

and observe that the equation in question reduces to

$$\Pi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} = A\Pi + Bu_0 \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

An elementary calculation (multiply first both sides on the right by the vector $\begin{pmatrix} 1 & j \end{pmatrix}^{\mathrm{T}}$) yields

$$\Pi_1 + j\Pi_2 = (j\omega_0 I - A)^{-1} B u_0,$$

i.e.,

$$\Pi = \begin{pmatrix} \mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 & \mathrm{Im}[(j\omega_0 I - A)^{-1}B]u_0 \end{pmatrix}.$$

As shown above, the steady-state response has the form (A.30). Hence, in particular, the periodic input

$$u(t) = u_0 \cos(\omega_0 t)$$

produces the periodic state response

$$x_{ss}(t) = \Pi w(t) = \Pi_1 \cos(\omega_0 t) - \Pi_2 \sin(\omega_0 t), \tag{A.31}$$

and the periodic output response

$$\begin{aligned} y_{ss}(t) &= Cx_{ss}(t) + Du_0 \cos(\omega_0 t) \\ &= \mathrm{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \mathrm{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t), \end{aligned} \tag{A.32}$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D.$$

A.6 Hamiltonian Matrices and Algebraic Riccati Equations

In this section, a few fundamental facts about algebraic Riccati equations are reviewed.[12] An algebraic Riccati equation is an equation of the form

$$A^{\mathrm{T}}X + XA + Q + XRX = 0 \tag{A.33}$$

in which all matrices involved are $n \times n$ matrices and R, Q are *symmetric* matrices. Such equation can also be rewritten the equivalent form as

$$\begin{pmatrix} X & -I \end{pmatrix} \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = 0.$$

From either one of these expressions, it is easy to deduce the following identity

$$\begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} (A + RX) \tag{A.34}$$

and to conclude that X is a solution of the Riccati equation (A.33) if and only if the subspace

$$\mathscr{V} = \mathrm{Im} \begin{pmatrix} I \\ X \end{pmatrix} \tag{A.35}$$

is an (n-dimensional) invariant subspace of the matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix}. \tag{A.36}$$

In particular, (A.34) also shows that if X is a solution of (A.33), the matrix $A + RX$ characterizes the restriction of H to its invariant subspace (A.35). A matrix of the form (A.36), with real entries and in which R and Q are symmetric matrices, is called a Hamiltonian matrix. Some relevant features of the Hamiltonian matrix (A.36) and their relationships with the Riccati equation (A.33) are reviewed in what follows.

Lemma A.5 *The spectrum of the Hamiltonian matrix (A.36) is symmetric with respect to the imaginary axis.*

---

[12] For further reading, see e.g., [2].

Proof Set

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and note that

$$J^{-1}HJ = \begin{pmatrix} -A^{\mathrm{T}} & Q \\ -R & A \end{pmatrix} = -H^{\mathrm{T}}.$$

Hence H and $-H^{\mathrm{T}}$ are similar. As a consequence, if λ is an eigenvalue of H so is also $-\lambda$. Since the entries of H are real numbers, and therefore the spectrum of this matrix is symmetric with respect to the real axis, the result follows. ◁

Suppose now that the matrix (A.36) has no eigenvalues on the imaginary axis. Then, the matrix in question has exactly n eigenvalues in $\mathbb{C}^-$ and n eigenvalues in $\mathbb{C}^+$. As a consequence, there exist two complementary n-dimensional invariant subspaces of H: a subspace $\mathscr{V}^{\mathrm{s}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{s}}$ has all eigenvalues in $\mathbb{C}^-$, the *stable eigenspace*, and a subspace $\mathscr{V}^{\mathrm{a}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{a}}$ has all eigenvalues in $\mathbb{C}^+$, the *antistable eigenspace*. A situation of special interest in the subsequent analysis is the one in which the stable eigenspace (respectively, the antistable eigenspace) of the matrix (A.36) can be expressed in the form (A.35); in this case in fact, as observed before, it is possible to associate with this subspace a particular solution of the Riccati equation (A.33).

If there exists a matrix X^- such that

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^- + X^-A + Q + X^-RX^- = 0$$

and the matrix $A+RX^-$ has all eigenvalues in $\mathbb{C}^-$. This matrix is the *unique*[13] solution of the Riccati equation (A.33) having the property that $A + RX$ has all eigenvalues in $\mathbb{C}^-$ and for this reason is called *the stabilizing* solution of the Riccati equation (A.33).

Similarly, if there exists a matrix X^+ such that

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^+ + X^-A + Q + X^+RX^+ = 0$$

[13] If $\mathscr{V}$ is an n-dimensional subspace of $\mathbb{R}^{2n}$, and $\mathscr{V}$ can be expressed in the form (A.35), the matrix X is necessarily unique.

and the matrix $A+RX^+$ has all eigenvalues in $\mathbb{C}^+$. This matrix is the unique solution of the Riccati equation (A.33) having the property that $A+RX$ has all eigenvalues in $\mathbb{C}^+$ and for this reason is called *the antistabilizing* solution of the Riccati equation (A.33).

The existence of such matrices X^- and X^+ is discussed in the following statement.

Proposition A.1 *Suppose the Hamiltonian matrix (A.36) has no eigenvalues on the imaginary axis and R is a (either positive or negative) semidefinite matrix.*

If the pair (A, R) is stabilizable, the stable eigenspace $\mathscr{V}^{\mathrm{s}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix}$$

in which X^- is a symmetric matrix, the (unique) stabilizing solution of the Riccati equation (A.33).

If the pair (A, R) is antistabilizable, the antistable eigenspace $\mathscr{V}^{\mathrm{a}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix}$$

in which X^- is a symmetric matrix, the (unique) antistabilizing solution of the Riccati equation (A.33).

The following proposition describes the relation between solutions of the algebraic Riccati equation (A.33) and of the algebraic Riccati *inequality*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0. \tag{A.37}$$

Proposition A.2 *Suppose R is negative semidefinite. Let X^- (respectively X^+) be a solution of the Riccati equation (A.33) having the property that $\sigma(A+RX^-) \in \mathbb{C}^-$ (respectively, $\sigma(A+RX^+) \in \mathbb{C}^+$). Then, the set of solutions of*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0\,,$$

is not empty and any X in this set satisfies $X < X^-$ (respectively, $X > X^+$).

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Appendix B
Stability and Asymptotic Behavior of Nonlinear Systems

B.1 The Theorems of Lyapunov for Nonlinear Systems

We assume in what follows that the reader is familiar with basic concepts concerning the stability of equilibrium in a nonlinear system. In this section we provide a sketchy summary of some fundamental results, mainly to the purpose of introducing notations and results that are currently used throughout the book.[14]

Comparison functions. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathscr{K}_\infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for each fixed t, the function

$$\begin{aligned} \alpha : [0, a) &\to [0, \infty) \\ r &\mapsto \beta(r, t) \end{aligned}$$

belongs to class $\mathscr{K}$ and, for each fixed r, the function

$$\begin{aligned} \varphi : [0, \infty) &\to [0, \infty) \\ t &\mapsto \beta(r, t) \end{aligned}$$

is decreasing and $\lim_{t\to\infty} \varphi(t) = 0$.

The composition of two class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathscr{K}$ function, defined on $[0, a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0, b) \to [0, a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \text{ for all } r \in [0, a) \\ \alpha(\alpha^{-1}(r)) &= r, \text{ for all } r \in [0, b). \end{aligned}$$

[14] For further reading, see [2, 4, 5, 6].

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathscr{K}$ function. If $\alpha(\cdot)$ is a class $\mathscr{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot,\cdot)$ is a class $\mathscr{K}\mathscr{L}$ function and $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions, the function thus defined

$$\begin{aligned} \gamma : [0,a) \times [0,\infty) &\rightarrow [0,\infty) \\ (r,t) \quad\quad &\mapsto \alpha_1(\beta(\alpha_2(r),t)) \end{aligned}$$

is a class $\mathscr{K}\mathscr{L}$ function.

The Theorems of Lyapunov. Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{B.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well-known criterion of Lyapunov, which, using comparison functions, can be expressed as follows. Let B_d denote the open ball of radius d in $\mathbb{R}^n$, i.e.,

$$B_d = \{x \in \mathbb{R}^n : \|x\| < d\}.$$

Theorem B.1 (Direct Theorem) *Let $V : B_d \rightarrow \mathbb{R}$ be a C^1 function such that, for some class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0,d)$,*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \textit{for all } x \in B_d. \tag{B.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \textit{for all } x \in B_d, \tag{B.3}$$

the equilibrium $x = 0$ of (B.1) is stable.

If, for some class $\mathscr{K}$ function $\alpha(\cdot)$, defined on $[0,d)$,

$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \textit{for all } x \in B_d, \tag{B.4}$$

the equilibrium $x = 0$ of (B.1) is locally asymptotically stable.

If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathscr{K}_\infty$ functions, the equilibrium $x = 0$ of (B.1) is globally asymptotically stable.

Remark B.1 The usefulness of the comparison functions, in the statement of the theorem, is motivated by the following simple arguments. Suppose (B.3) holds. Then, so long as $x(t) \in B_d$, $V(x(t))$ is non-increasing, i.e., $V(x(t)) \le V(x(0))$. Pick $\varepsilon < d$ and define $\delta = \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$. Then, using (B.2), it is seen that, if $\|x(0)\| < \delta$,

$$\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \overline{\alpha}(\|x(0)\|) \le \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon)$$

which implies $\|x(t)\| \le \varepsilon$. This shows that $x(t)$ exists for all t and the equilibrium $x = 0$ is stable.

Suppose now that (B.4) holds. Define $\gamma(r) = \alpha(\overline{\alpha}^{-1}(r))$, which is a class $\mathscr{K}$ function. Using the estimate on the right of (B.2), it is seen that $\alpha(\|x\|) \ge \gamma(V(x))$ and hence

$$\frac{\partial V}{\partial x} f(x) \le -\gamma(V(x)).$$

Since $V(x(t))$ is a continuous function of t, non-increasing and nonnegative for each t, there exists a number $V^* \ge 0$ such that $\lim_{t\to\infty} V(x(t)) = V^*$. Suppose V^* is strictly positive. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \le -\gamma(V(x(t)) \le -\gamma(V^*) < 0.$$

Integration with respect to time yields

$$V(x(t)) \le V(x(0)) - \gamma(V^*)t$$

for all t. This cannot be the case, because for large t the right-hand side is negative, while the left-hand side is nonnegative. From this it follows that $V^* = 0$ and therefore, using the fact that $V(x)$ vanishes only at $x = 0$, it is concluded that $\lim_{t\to\infty} x(t) = 0$. Note also that identical arguments hold for the analysis of the asymptotic properties of a time-dependent system

$$\dot{x} = f(x, t)$$

so long $f(0, t) = 0$ for all $t \ge 0$ and $V(x)$ is independent of t. ◁

Sometimes, in the design of feedback laws, while it is difficult to obtain a system whose equilibrium $x = 0$ is globally asymptotically stable, it is relatively easier to obtain a system in which trajectories are bounded (maybe for a specific set of initial conditions) and have suitable decay properties. Instrumental, in such context, is the notion of *sublevel set* of a Lyapunov function $V(x)$ which, for a fixed nonnegative real number c, is defined as

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}.$$

The function $V(x)$, which is *positive definite* (i.e., is positive for all nonzero x and zero at $x = 0$) is said to be *proper* if, for each $c \in \mathbb{R}$, the sublevel set Ω_c is a *compact* set. Now, it is easy to check that the function $V(x)$ is proper if and only the inequality on the left-hand side of (B.2) holds for all $x \in \mathbb{R}^n$, with a function $\underline{\alpha}(\cdot)$ which is of class $\mathscr{K}_\infty$. Note also that, if $V(x)$ is proper, for any $c > 0$ it is possible to find a numbers $c_1 > 0$ and $c_2 > 0$ such that

$$B_{c_1} \subset \Omega_c \subset B_{c_2}.$$

A typical example of how sublevel sets can be used to analyze boundedness and decay of trajectories is the following one. Let r_1 and r_2 be two positive numbers, with $r_2 > r_1$. Suppose $V(x)$ is a function satisfying (B.2), with $\underline{\alpha}(\cdot)$ a class $\mathscr{K}_\infty$ function. Pick any pair of positive numbers c_1, c_2, such that

$$\Omega_{c_1} \subset B_{r_1} \subset B_{r_2} \subset \Omega_{c_2}.$$

and let $S_{c_1}^{c_2}$ denote the "annular" compact set

$$S_{c_1}^{c_2} = \{x \in \mathbb{R}^n : c_1 \le V(x) \le c_2\}.$$

Suppose that, for some $a > 0$,

$$\frac{\partial V}{\partial x} f(x) \le -a \qquad \text{for all}\, x \in S_{c_1}^{c_2}.$$

Then, for each initial condition $x(0) \in B_{r_2}$, the trajectory $x(t)$ of (B.1) is defined for all t and there exists a finite time T such that $x(t) \in B_{r_1}$ for all $t \ge T$. In fact, take any $x(0) \in B_{r_2} \setminus \Omega_{c_1}$. Such $x(0)$ is in $S_{c_1}^{c_2}$. So long as $x(t) \in S_{c_1}^{c_2}$, the function $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t) \le -a$$

and hence

$$V(x(t)) \le V(x(0)) - at \le c_2 - at.$$

Thus, at a time $T \le (c_2 - c_1)/a$, $x(T)$ is on the boundary of the set Ω_{c_1}. On the boundary of Ω_{c_1} the derivative of $V(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_{c_1} and remains there for all $t \ge T$.

It is well known that the criterion for asymptotic stability provided by the previous theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem B.1 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (B.1). In particular, the following result holds.

Theorem B.2 (Converse Theorem) *Suppose the equilibrium $x = 0$ of (B.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold. If the equilibrium $x = 0$ of (B.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold with $d = \infty$.*

It is well known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (B.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \le M e^{-\lambda t} \|x(0)\| \quad \text{for all}\, t \ge 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In this context, the following criterion is useful.

Lemma B.1 *The equilibrium $x = 0$ of nonlinear system (B.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, three class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and three real numbers $\delta > 0$, $\underline{a} > 0$, $a > 0$, such that*

$$\begin{aligned} \underline{\alpha}(\|x\|) \le V(x) &\le \overline{\alpha}(\|x\|) \\ \frac{\partial V}{\partial x} f(x) &\le -\alpha(\|x\|) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = \underline{a}s^2, \qquad \alpha(s) = as^2$$

for all $s \in B_\delta$.

B.2 Input-to-State Stability and the Theorems of Sontag

In the analysis of *forced* nonlinear systems, the property of *input-to-state stability*, introduced and thoroughly studied by E.D. Sontag, plays a role of paramount importance.[15] Consider a forced nonlinear system

$$\dot{x} = f(x, u) \tag{B.5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0, 0) = 0$ and $f(x, u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (B.5) can be any piecewise continuous bounded function. The space of all such functions is endowed with the so-called supremum norm $\|u(\cdot)\|_\infty$, which is defined as

$$\|u(\cdot)\|_\infty = \sup_{t \ge 0} \|u(t)\|.$$

Definition B.1 System (B.5) is said to be input-to-state stable if there exist a class $\mathscr{K}\mathscr{L}$ function $\beta(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$, called a *gain function*, such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \tag{B.6}$$

for all $t \ge 0$.

[15] The concept of input-to-state stability, its properties and applications have been introduced in the sequence of papers [10, 11, 14]. A summary of the most relevant aspect of the theory can also be found in [5, p. 17–31].

Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta, \gamma\} \le \beta + \gamma \le \max\{2\beta, 2\gamma\}$, an alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_\infty)\} \tag{B.7}$$

for all $t \ge 0$. Note also that, letting $\|u(\cdot)\|_{[0,t]}$ denote the supremum norm of the restriction of $u(\cdot)$ to the interval $[0, t]$, namely

$$\|u(\cdot)\|_{[0,t]} = \sup_{s \in [0,t]} \|u(s)\| \,,$$

the bound (B.6) can be also expressed in the alternative form[16]

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_{[0,t]}) \,, \tag{B.8}$$

and the bound (B.7) in the alternative form

$$\|x(t)\| \le \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_{[0,t]})\} \,,$$

both holding for all $t \ge 0$.

The property of input-to-state stability can be given a characterization which extends the well-known criterion of Lyapunov for asymptotic stability. The key tool for such characterization is the notion of *ISS-Lyapunov function*.

Definition B.2 A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (B.5) if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{B.9}$$

and

$$\frac{\partial V}{\partial x} f(x, u) \le -\alpha(\|x\|) \quad \text{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfying } \|x\| \ge \chi(\|u\|). \tag{B.10}$$

An equivalent form in which the notion of an ISS-Lyapunov function can be described is the following one.

[16] In fact, since $\|u(\cdot)\|_{[0,t]} \le \|u(\cdot)\|_\infty$ and $\gamma(\cdot)$ is increasing, (B.8) implies (B.6). On the other hand, since $x(t)$ depends only on the restriction of $u(\cdot)$ to the interval $[0, t]$, one could in (B.6) replace $u(\cdot)$ with an input $\bar{u}(\cdot)$ defined as $\bar{u}(s) = u(s)$ for $0 \le s \le t$ and $\bar{u}(s) = 0$ for $s > t$, in which case $\|\bar{u}(\cdot)\|_\infty = \|u(\cdot)\|_{[0,t]}$, and observe that (B.6) implies (B.8).

Lemma B.2 *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (B.5) if and only if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\sigma(\cdot)$ such that (B.9) holds and*

$$\frac{\partial V}{\partial x} f(x, u) \le -\alpha(\|x\|) + \sigma(\|u\|) \qquad \textit{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{B.11}$$

The existence of an ISS-Lyapunov function turns out to be a necessary and sufficient condition for input-to-state stability.

Theorem B.3 *System (B.5) is input-to-state stable if and only if there exists an ISS-Lyapunov function. In particular, if such function exists, then an estimate of the form (B.6) holds with $\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)))$.*

The following elementary examples describe how the property of input-to-state stability can be checked and an estimate of the gain function can be evaluated.

Example B.1 A stable linear system

$$\dot{x} = Ax + Bu$$

is input-to-state stable, with a linear gain function. In fact, let P denote the unique positive-definite solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$ and observe that $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\frac{\partial V}{\partial x}(Ax + Bu) \le -\|x\|^2 + c\|x\|\,\|u\|$$

for some $c > 0$. Pick $0 < \varepsilon < 1$ and set $\ell = c/(1 - \varepsilon)$. Then, it is easy to see that

$$\|x\| \ge \ell\|u\| \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(Ax + Bu) \le -\varepsilon\|x\|^2.$$

The system is input-to-state, with $\chi(r) = \ell r$. Since $\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2$, we obtain following the estimate for the (linear) gain function

$$\gamma(r) = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\ell r.$$

◁

Example B.2 Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p u,$$

in which $k \in \mathbb{N}$ is *odd*, $a > 0$ and $p \in \mathbb{N}$ is such that $p < k$. Pick $V(x) = \frac{1}{2}x^2$ and note that, since $k + 1$ is even,

$$\frac{\partial V}{\partial x}(-ax^k + bx^p u) \le -a|x|^{k+1} + |b||x|^{p+1}|u|$$

Pick $0 < \varepsilon < a$ and define (recall that $k > p$)

$$\chi(r) = \Big(\frac{|b|r}{a-\varepsilon}\Big)^{\frac{1}{k-p}}.$$

Then, it is easy to see that

$$\|x\| \ge \chi(\|u\|) \qquad \Rightarrow \qquad \frac{\partial V}{\partial x}(-ax^k + bx^p u) \le -\varepsilon|x|^{k+1}.$$

The system is input-to-state stable, with $\gamma(r) = \chi(r)$.

Note that the condition $k > p$ is essential. In fact, the following system, in which $k = p = 1$,

$$\dot{x} = -x + xu$$

is not input-to-state stable. Under the bounded (constant) input $u(t) = 2$ the state $x(t)$ evolves as a solution of $\dot{x} = x$ and hence diverges to infinity. ◁

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which one of the most useful can be expressed as follows.

Theorem B.4 *System (B.5) is input-to-state stable if and only if there exist class $\mathscr{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any bounded input and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x(0)\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\le \gamma(\limsup_{t\to\infty} \|u(t)\|). \end{aligned}$$

B.3 Cascade-Connected Systems

In this section we investigate the *asymptotic* stability of the equilibrium $(z, \xi) = (0, 0)$ of a pair of cascade-connected subsystems of the form

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi), \end{aligned} \tag{B.12}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $f(0, 0) = 0$, $g(0) = 0$, and $f(z, \xi)$, $g(\xi)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. B.1).

Since similar cascade connections occur quite often in the analysis (and feedback design) of nonlinear systems, it is important to understand under what conditions

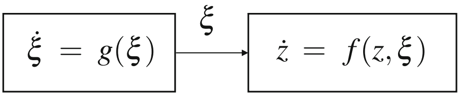

Fig. B.1 A cascade connection of systems

the stability properties of the two components subsystems determine the stability of the cascade. If both systems were linear systems, the cascade would be a system modeled as

$$\begin{aligned}\dot{z} &= Fz + G\xi \\ \dot{\xi} &= A\xi,\end{aligned}$$

and it is trivially seen that if both F and G have all eigenvalues in $\mathbb{C}^-$, the cascade is an asymptotically stable system. The nonlinear counterpart of such property, though, requires some extra care.

The simplest scenario, in this respect, is one in which one is interested in seeking only local stability. In this case, the following result holds.[17]

Lemma B.3 *Suppose the equilibrium* $z = 0$ *of*

$$\dot{z} = f(z, 0) \tag{B.13}$$

is locally asymptotically stable and the equilibrium $\xi = 0$ *of* $\dot{\xi} = g(\xi)$ *is stable. Then the equilibrium* $(z, \xi) = (0, 0)$ *of (B.12) is stable. If the equilibrium* $\xi = 0$ *of* $\dot{\xi} = g(\xi)$ *is locally asymptotically stable, then the equilibrium* $(z, \xi) = (0, 0)$ *of (B.12) is locally asymptotically stable.*

It must be stressed, though, that in this lemma only the property of *local* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ is considered. In fact, by means of a simple counterexample, it can be shown that the *global* asymptotic stability of $z = 0$ as an equilibrium of (B.13) and the *global* asymptotic stability of $\xi = 0$ as an equilibrium of $\dot{\xi} = g(\xi)$ *do not* imply, in general, *global* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ of the cascade. As a matter of fact, the cascade connection of two such systems may even have finite escape times. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

Example B.3 Consider the case in which

$$\begin{aligned}f(z, \xi) &= -z + z^2\xi \\ g(\xi) &= -\xi.\end{aligned}$$

Clearly $z = 0$ is a globally asymptotically equilibrium of $\dot{z} = f(z, 0)$ and $\xi = 0$ is a globally asymptotically equilibrium of $\dot{\xi} = g(\xi)$. However, this system has finite escape times. To show that this is the case, consider the differential equation

[17] More details and proofs of the results stated in this section can be found in [5, p. 11–17 and 31–36].

$$\dot{\tilde{z}} = -\tilde{z} + \tilde{z}^2 \tag{B.14}$$

with initial condition $\tilde{z}(0) = z_0$. Its solution is

$$\tilde{z}(t) = \frac{-z_0}{z_0 - 1 - z_0 \exp(-t)} \exp(-t).$$

Suppose $z_0 > 1$. Then, the $\tilde{z}(t)$ escapes to infinity in finite time. In particular, the maximal (positive) time interval on which $\tilde{z}(t)$ is defined is the interval $[0, t_{\max}(z_0))$ with

$$t_{\max}(z_0) = \ln\left(\frac{z_0}{z_0 - 1}\right).$$

Now, return to system (B.12), with initial condition (z_0, ξ_0) and let ξ_0 be such that

$$\xi(t) = \exp(-t)\xi_0 \geq 1 \qquad \text{for all } t \in [0, t_{\max}(z_0)).$$

Clearly, on the time interval $[0, t_{\max}(z_0))$, we have

$$\dot{z} = -z + z^2\xi \geq -z + z^2.$$

By comparison with (B.14), it follows that

$$z(t) \geq \tilde{z}(t).$$

Hence $z(t)$ escapes to infinity, at a time $t^* \leq t_{\max}(z_0)$. The lesson learned from this example is that, even if $\xi(t)$ exponentially decreases to 0, this may not suffice to prevent finite escape time in the upper system. The state $z(t)$ escapes to infinity at a time in which the effect of $\xi(t)$ on the upper equation is still not negligible. $\triangleleft$

The following results provide the extra condition needed to ensure global asymptotic stability in the cascade.

Lemma B.4 *Suppose the equilibrium $z = 0$ of (B.13) is asymptotically stable, and let S be a subset of the domain of attraction of such equilibrium. Consider the system*

$$\dot{z} = f(z, \xi(t)). \tag{B.15}$$

in which $\xi(t)$ is a continuous function, defined for all $t \geq 0$ and suppose that $\lim_{t\to\infty} \xi(t) = 0$. *Pick $z_0 \in S$, and suppose that the integral curve $z(t)$ of (B.15) satisfying $z(0) = z_0$ is defined for all $t \geq 0$, bounded, and such that $z(t) \in S$ for all $t \geq 0$. Then* $\lim_{t\to\infty} z(t) = 0$.

This last result implies, in conjunction with Lemma B.3, that if the equilibrium $z = 0$ of (B.13) is globally asymptotically stable, if the equilibrium $\xi = 0$ of the lower subsystem of (B.12) is globally asymptotically stable, and all trajectories of the composite system (B.12) *are bounded*, the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is globally asymptotically stable.

To be in a position to use this result in practice, one needs to determine conditions under which the boundedness property holds. This is indeed the case if the upper subsystem of the cascade, viewed as a system with state z and input ξ, is input-to-state stable. In view of this, it can be claimed that if the upper subsystem of the cascade is input-to-state stable and the lower subsystem is globally asymptotically stable (at the equilibrium $\xi = 0$), the cascade is globally asymptotically stable (at the equilibrium $(z, \xi) = (0, 0)$).

As a matter fact, a more general result holds, which is stated as follows.

Theorem B.5 *Suppose that system*

$$\dot{z} = f(z, \xi), \tag{B.16}$$

viewed as a system with input ξ and state z is input-to-state stable and that system

$$\dot{\xi} = g(\xi, u), \tag{B.17}$$

viewed as a system with input u and state ξ is input-to-state stable as well. Then, system

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi, u) \end{aligned}$$

is input-to-state stable.

Example B.4 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2\xi_1\xi_2 \\ \dot{x}_2 &= -x_2 + \xi_1^2 - x_1\xi_1\xi_2 \\ \dot{\xi}_1 &= -\xi_1^3 + \xi_1 u_1 \\ \dot{\xi}_2 &= -\xi_2 + u_2. \end{aligned}$$

The subsystem consisting of the two top equations, seen as a system with state $x = (x_1, x_2)$ and input $\xi = (\xi_1, \xi_2)$ is input-to-state stable. In fact, let this system be written as

$$\dot{x} = f(x, \xi)$$

and consider the candidate ISS-Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

for which we have

$$\frac{\partial V}{\partial x} f(x, \xi) = -(x_1^2 + x_2^2) + x_2\xi_1^2 \leq -x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{2}\xi_1^4 \leq -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\xi\|^4.$$

Thus, the function $V(x)$ satisfies the condition indicated in Lemma B.2, with

$$\alpha(r) = \frac{1}{2}r^2\,, \qquad \sigma(r) = \frac{1}{2}r^4.$$

The subsystem consisting of the two bottom equations is composed of two separate subsystems, both of which are input-to-state stable, as seen in Examples B.1 and B.2. Thus, the overall system is input-to-state stable. $\triangleleft$

B.4 Limit Sets

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{B.18}$$

with $x \in \mathbb{R}^n$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for all $x_0 \in \mathbb{R}^n$ the solution of (B.18) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Definition B.3 Let $x_0 \in \mathbb{R}^n$ be fixed. Suppose that $x(t, x_0)$ is defined for all $t \geq 0$. A point x is said to be an ω-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is the union of all ω-limit points of the motion $x(t, x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t, x_0)$ as $t \to \infty$, because the solution in question may not admit any limit as $t \to \infty$ (see for instance Fig. B.2).[18]

However, it is known that, if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches the set $\omega(x_0)$, as specified in the lemma that follows.[19] In this respect, recall that a set $S \subset \mathbb{R}^n$ is said to be *invariant* under (B.18) if for all initial conditions $x_0 \in S$ the solution $x(t, x_0)$ of (B.18) exists for all $t \in (-\infty, +\infty)$ and

[18] Figures B.2, B.3, B.4 are reprinted from *Annual Reviews in Control*, Vol. 32, A. Isidori and C.I. Byrnes, Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection, pp. 1–16, Copyright (2008), with permission from Elsevier.

[19] See [1, p. 198].

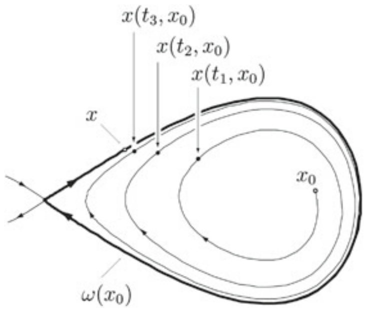

Fig. B.2 The ω-limit set of the point x_0

$x(t, x_0) \in S$ for all such t.[20] Moreover, the *distance* of a point $x \in \mathbb{R}$ from a set $S \subset \mathbb{R}^n$, denoted $\mathrm{dist}(x, S)$, is the nonnegative real number defined as

$$\mathrm{dist}(x, S) = \inf_{z \in S} \|x - z\|.$$

Lemma B.5 *Suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty connected compact set, invariant under (B.18). Moreover,*

$$\lim_{t \to \infty} \mathrm{dist}(x(t, x_0), \omega(x_0)) = 0.$$

Example B.5 Consider the classical (stable) Van der Pol oscillator, written in state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 \end{aligned} \tag{B.19}$$

in which, as it is well known, the damping term $\mu(1 - x_1^2)y$ can be seen as a model of a nonlinear resistor, negative for small x_1 and positive for large x_1 (see [6]). From the phase portrait of this system (depicted in Fig. B.3 for $\mu = 1$) it is seen that all motions except the trivial motion occurring for $x_0 = 0$ are bounded in positive time and approach, as $t \to \infty$, the limit cycle $\mathscr{L}$. As consequence, $\omega(x_0) = \mathscr{L}$ for any $x_0 \neq 0$, while $\omega(0) = \{0\}$. $\lhd$

An important useful application of the notion of ω-limit set of a point is found in the proof of the following result, commonly known as LaSalle's invariance principle.[21]

---

[20] We recall, for the sake of completeness, that a set S is said to be *positively invariant*, or *invariant in positive time* (respectively, *negatively invariant* or *invariant in negative time*) if for all initial conditions $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$ (respectively exists for all $t \leq 0$ and $x(t, x_0) \in X$ for all $t \leq 0$). Thus, a set is invariant if it is both positively invariant and negatively invariant.

[21] See e.g., [6].

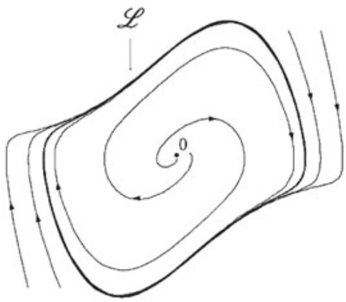

Fig. B.3 The phase portrait of the Van der Pol oscillator

Theorem B.6 *Consider system (B.18). Suppose there exists a* C^1 *function* $V : \mathbb{R}^n \to \mathbb{R}$ *satisfying*

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n,$$

for some pair of class $\mathscr{K}_\infty$ *functions* $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ *and such that*

$$\frac{\partial V}{\partial x} f(x) \le 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{B.20}$$

Let $\mathscr{E}$ *denote the set*

$$\mathscr{E} = \{x \in \mathbb{R}^n : \frac{\partial V}{\partial x} f(x) = 0\}. \tag{B.21}$$

Then, for each x_0*, the integral curve* $x(t, x_0)$ *of (B.18) passing through* x_0 *at time* $t = 0$ *is bounded, and*

$$\omega(x_0) \subset \mathscr{E}.$$

Proof A direct consequence of (B.20) is that, for any x_0, the motion $x(t, x_0)$ is bounded in positive time. In fact, this property yields $V(x(t, x_0)) \le V(x_0)$ for all $t \ge 0$ and this in turn implies (see Remark B.1)

$$\|x(t, x_0)\| \le \underline{\alpha}^{-1}(\overline{\alpha}(\|x_0\|)).$$

Thus, the limit set $\omega(x_0)$ is nonempty, compact and invariant. The nonnegative-valued function $V(x(t, x_0))$ is non-increasing for $t \ge 0$. Thus, there is a number $V_0 \ge 0$, possibly dependent on x_0, such that

$$\lim_{t \to \infty} V(x(t, x_0)) = V_0.$$

By definition of limit set, for each point $x \in \omega(x_0)$, there exists a sequence of times $\{t_k\}$, with $\lim_{k \to \infty} t_k = \infty$, such that $\lim_{k \to \infty} x(t_k, x_0) = x$. Thus, since $V(x)$ is continuous,

$$V(x) = \lim_{k \to \infty} V(x(t_k, x_0)) = V_0.$$

In other words, the function $V(x)$ takes the same value V_0 at any point $x \in \omega(x_0)$. Now, pick any initial condition $\bar{x}_0 \in \omega(x_0)$. Since the latter is invariant, we have $x(t, \bar{x}_0) \in \omega(x_0)$ for all $t \in \mathbb{R}$. Thus, along this particular motion, $V(x(t, \bar{x}_0)) = V_0$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} V(x(t, \bar{x}_0)) = \frac{\partial V}{\partial x} f(x)\Big|_{x=x(t,\bar{x}_0)}.$$

This, implies

$$x(t, \bar{x}_0) \in \mathscr{E}, \qquad \text{for all } t \in \mathbb{R}$$

and, since $\bar{x}_0$ is any point in $\omega(x_0)$, proves the theorem. ◁

This theorem is often used to determine the asymptotic properties of the integral curves of (B.18). In fact, in view of Lemma B.5, it is seen that if a function $V(x)$ can be found such that (B.20) holds, any trajectory of (B.18) is bounded and converges, asymptotically, to an invariant set that is entirely contained in the set $\mathscr{E}$ defined by (B.21). In particular, if system (B.18) has an equilibrium at $x = 0$ and it can be determined that, in the set $\mathscr{E}$, the only possible invariant set is the point $x = 0$, then the equilibrium in question is globally asymptotically stable.

Example B.6 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1(1 + x_1x_2)\,,\end{aligned}$$

pick $V(x) = x_1^2 + x_2^2$, and observe that

$$\frac{\partial V}{\partial x} f(x) = -2(x_1x_2)^2.$$

The function on the right-hand side is not negative definite, but it is negative semi-definite, i.e., satisfies (B.20). Thus, trajectories converge to bounded sets that are invariant and contained in the set

$$\mathscr{E} = \{x \in \mathbb{R}^2 : x_1x_2 = 0\}.$$

Now, it is easy to see that no invariant set may exist, other than the equilibrium, entirely contained in the set $\mathscr{E}$. In fact, if a trajectory of the system is contained in $\mathscr{E}$ for all $t \in \mathbb{R}$, this trajectory must be a solution of

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1\,.\end{aligned}$$

This system, a harmonic oscillator, has only one trajectory entirely contained in $\mathscr{E}$, the trivial trajectory $x(t) = 0$. Thus, the equilibrium point $x = 0$ is the only possible invariant set contained in $\mathscr{E}$ and therefore this equilibrium is globally asymptotically stable. ◁

Returning to the analysis of the properties of limit sets, let $B \subset \mathbb{R}^n$ be a fixed bounded set and suppose that *all* motions with initial condition $x_0 \in B$ are bounded in positive time. Since any motion $x(t, x_0)$ asymptotically approaches the limit set $\omega(x_0)$ as $t \to \infty$, it seems reasonable to look at the set

$$\Omega = \bigcup_{x_0 \in B} \omega(x_0)\,, \tag{B.22}$$

as to a "target" set that is asymptotically approached by motions of (B.18) with initial conditions in B. However, while it is true that the distance of $x(t, x_0)$ from the set (B.22) tends to 0 as $t \to \infty$ for any x_0, the convergence to such set may fail be *uniform* in x_0, even if the set B is compact. In this respect, recall that—by definition—the distance of $x(t, x_0)$ from a set S tends to 0 as $t \to \infty$ if for every ε there exists T such that

$$\mathrm{dist}(x(t, x_0), S) \leq \varepsilon\,, \qquad \text{for all } t \geq T. \tag{B.23}$$

The number T in this expression depends on ε but also on x_0.[22] The distance of $x(t, x_0)$ from S is said to tend to 0, as $t \to \infty$, *uniformly* in x_0 on B, if for every ε there exists T, which depends on ε *but not* on x_0, such that (B.23) holds for all $x_0 \in B$.

Example B.7 Consider again the Example B.5, in which the set Ω defined by (B.22) consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $\mathscr{L}$ and let B be a compact set satisfying $B \supset \mathscr{L}$. All $x_0 \in B$ are such that $\mathrm{dist}(x(t, x_0), \Omega) \to 0$ as $t \to \infty$. However, the convergence is not uniform in x_0. In fact, observe that, if $x_0 \neq 0$ is inside $\mathscr{L}$, the motion $x(t, x_0)$ is bounded in negative time and remains inside $\mathscr{L}$ for all $t \leq 0$ (as a matter of fact, it converges to 0 as $t \to -\infty$). Pick any $x_1 \neq 0$ inside $\mathscr{L}$ such that $\mathrm{dist}(x_1, \mathscr{L}) > \varepsilon$ and let T_1 be the minimal time needed to have $\mathrm{dist}(x(t, x_1), \mathscr{L}) \leq \varepsilon$ for all $t \geq T_1$. Let $T_0 > 0$ be fixed and define $x_0 = x(-T_0, x_1)$. If T_0 is large, x_0 is close to 0, and the minimal time T needed to have $\mathrm{dist}(x(t, x_0), \Omega) \leq \varepsilon$ for all $t \geq T$ is $T = T_0 + T_1$. Since the time T_0 can be taken arbitrarily large, it follows that the time $T > 0$ needed to have $\mathrm{dist}(x(t, x_0), \Omega) \leq \varepsilon$ for all $t \geq T$ can be made arbitrarily large, even if x_0 is taken within a compact set. ◁

Uniform convergence to the target set is important for various reasons. On the one hand, for practical purposes it is important to have a fixed bound on the time needed to get within an ε-distance of that set. On the other hand, uniform convergence plays a relevant role in the existence Lyapunov functions, an indispensable tool in analysis and design of feedback systems. While convergence to the set (B.22) is not guaranteed to be uniform, there is a larger set—though—for which such property holds.

[22] In fact it is likely that, the more is x_0 distant from S, the longer one has to wait until $x(t, x_0)$ becomes ε-distant from S.

Definition B.4 Let B be a bounded subset of $\mathbb{R}^n$ and suppose $x(t, x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_k) = x.$$

It is clear from the definition that, if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question returns the definition of ω-limit set of a point. It is also clear that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, with all the x_k's being taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B).$$

However, the converse inclusion is not true in general.

Example B.8 Consider again the system in Example B.5, and let B be a compact set satisfying $B \supset \mathscr{L}$. We know that $\{0\}$ and $\mathscr{L}$, being ω-limit sets of points of B, are in $\omega(B)$. But it is also easy to see that any other point inside $\mathscr{L}$ is a point of $\omega(B)$. In fact, let $\bar{x}$ be any of such points and pick any sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$. Since $x(t, \bar{x})$ remains inside $\mathscr{L}$ (and hence in B) for all negative values of t, it is seen that $x_k := x(-t_k, \bar{x})$ is a point in B for all k. The sequence $\{x_k, t_k\}$ is such that $x(t_k, x_k) = \bar{x}$ and therefore the property required for $\bar{x}$ to be in $\omega(B)$ is trivially satisfied. This shows that $\omega(B)$ includes not just $\{0\}$ and $\mathscr{L}$, but also all points of the open region surrounded by $\mathscr{L}$. $\triangleleft$

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma B.5, can be summarized as follows.[23]

Lemma B.6 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

Thus, as it is the case for the ω-limit set of a point, the ω-limit set *of a bounded set B* is compact and invariant. Being invariant, the set $\omega(B)$ is filled with motions which exist for all $t \in (-\infty, +\infty)$ and all such motions, since this set is compact, are bounded in positive and in negative time. Moreover, this set is uniformly approached by motions with initial conditions $x_0 \in B$. We conclude the section with another property, that will be used later to define the concept of *steady-state behavior* of a system.[24]

Lemma B.7 *If B is a compact set invariant for (B.18), then $\omega(B) = B$.*

[23] For a proof see, e.g., [3, 7, 8].

[24] For a proof, see [17].

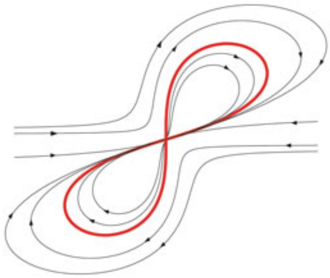

Fig. B.4 The phase portrait of system (B.24)

B.5 Limit Sets and Stability

It is well known that, in a nonlinear system, an equilibrium point which attracts all motions with initial conditions in some open neighborhood of this point is not necessarily stable in the sense of Lyapunov. A classical example showing that convergence to an equilibrium does not imply stability is provided by the following 2-dimensional system.[25]

Example B.9 Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix} \tag{B.24}$$

in which $f(0,0) = g(0,0) = 0$ and

$$\begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix} = \frac{1}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)} \begin{pmatrix} x_1^2(x_2 - x_1) + x_2^5 \\ x_2^2(x_2 - 2x_1) \end{pmatrix}$$

for $(x_1, x_2) \neq (0, 0)$. The phase portrait of this system is the one depicted in Fig. B.4. This system has only one equilibrium at $(x, y) = (0, 0)$ and any initial condition $(x_1(0), x_2(0))$ in the plane produces a motion that asymptotically tends to this point. However, it is not possible to find, for every $\varepsilon > 0$, a number $\delta > 0$ such that every initial condition in a disc of radius δ produces a motion which remains in a disc of radius ε for all $t \geq 0$. $\lhd$

It is also known—though—that if the convergence to the equilibrium is *uniform*, then the equilibrium in question *is stable*, in the sense of Lyapunov. This property is a consequence of the fact that $x(t, x_0)$ depends continuously on x_0 (see for example [2, p. 181]).

We have seen before that bounded motions of (B.18) with initial conditions in a bounded set B asymptotically approach the compact invariant set $\omega(B)$. Thus, the question naturally arises to determine whether or not this set is also stable in the sense of Lyapunov. In this respect, we recall that the notion of *asymptotic stability*

[25] See [2, pp. 191–194] and [9].

of a closed invariant set $\mathscr{A}$ is defined as follows. The set $\mathscr{A}$ is asymptotically stable if:

(i) for every $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{dist}(x_0, \mathscr{A}) \le \delta \quad \Rightarrow \quad \text{dist}(x(t, x_0), \mathscr{A}) \le \varepsilon \quad \text{for all } t \ge 0.$$

(ii) there exists a number $d > 0$ such that

$$\text{dist}(x_0, \mathscr{A}) \le d \quad \Rightarrow \quad \lim_{t\to\infty} \text{dist}(x(t, x_0), \mathscr{A}) = 0.$$

It is not difficult to show (see [12] or [15]) that if the set $\mathscr{A}$ is also bounded and hence compact, and the convergence in (ii) is *uniform* in x_0, then property (ii) implies property (i). This yields the following important property of the set $\omega(B)$.

Lemma B.8 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \le M$ for all $t \ge 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Suppose also that $\omega(B)$ is contained in the interior of B. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains B.*

B.6 The Steady-State Behavior of a Nonlinear System

We use the concepts introduced in the previous section to define a notion of *steady state* for a nonlinear system.

Definition B.5 Consider system (B.18) with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose that X is positively invariant. The motions of this system are said to be *ultimately bounded* if there is a bounded subset $B \subset X$ with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \ge T$ and all $x_0 \in X_0$.

Motions with initial conditions in a set B having the property indicated in the previous definition are indeed bounded and hence it makes sense to consider the limit set $\omega(B)$, which—according to Lemma B.6—is nonempty and has all the properties indicated in that lemma. What it is more interesting, though, is that—while a set B having the property indicated in the previous definition is clearly not unique—the set $\omega(B)$ is a unique well-defined set.

Lemma B.9 [26]*Let the motions of (B.18) be ultimately bounded and let B' be any other bounded subset of X with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \ge T$ and all $x_0 \in X_0$. Then, $\omega(B) = \omega(B')$.*

[26]See [17] for a proof.

It is seen from this that, in any system whose motions are ultimately bounded, all motions asymptotically converge to a well-defined compact invariant set, which is filled with trajectories that are bounded in positive and negative time. This motivates the following definition.

Definition B.6 Suppose the motions of system (B.18), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady-state motion* is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady-state locus* of (B.18) and the restriction of (B.18) to $\omega(B)$ is the *steady-state behavior* of (B.18). ◁

This definition characterizes the steady-state *behavior* of a nonlinear *autonomous* system, such as system (B.18). It can be used to characterized the steady-state *response* of a *forced* nonlinear system

$$\dot{z} = f(z, u) \tag{B.25}$$

so long as the input u can be seen as the output of an *autonomous* "input generator"

$$\begin{aligned} \dot{w} &= s(w) \\ u &= q(w). \end{aligned} \tag{B.26}$$

In this way, the concept of steady-state response (to specific classes of inputs) can be extended to nonlinear systems.

The idea of seeing the steady-state response of a forced system as a particular response of an augmented autonomous system has been already exploited in Sect. A.5, in the analysis of the steady-state response of a stable linear system to harmonic inputs. In the present setting, the results of such analysis can be recast as follows. Let (B.25) be a stable linear system, written as

$$\dot{z} = Az + Bu\,, \tag{B.27}$$

in which $z \in \mathbb{R}^n$, and let (B.26) be the "input generator" defined in (A.26). The composition of (B.25) and (B.26) is the autonomous linear system (compare with (A.28))

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{B.28}$$

Pick a set W_c defined as

$$W_c = \{w \in \mathbb{R}^2 : \|w\| \leq c\}$$

in which c is a fixed number, and consider the set $X = W_c \times \mathbb{R}^n$. The set X is a closed set, positively invariant for the motion of (B.28). Moreover, since the lower subsystem of (B.28) is a linear asymptotically stable system driven by a bounded input, the motions of system (B.28), with initial conditions taken in X, are ultimately bounded. In fact, let Π be the solution of the Sylvester equation (A.29) and recall that the difference $z(t) - \Pi w(t)$ tends to zero as $t \to \infty$. Then, any bounded set B of the form

$$B = \{(w, z) \in W_c \times \mathbb{R}^n : \|z - \Pi w\| \leq d\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. It is easy to check that

$$\omega(B) = \{(w, z) \in W_c \times \mathbb{R}^n : z = \Pi w\},$$

that is, $\omega(B)$ is the graph of the restriction of the linear map $x = \Pi w$ to the set W_c. The set $\omega(B)$ is invariant for (B.28), and the restriction of (B.28) to the set $\omega(B)$ characterizes the steady-state response of (B.27) to harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c.

A totally similar result holds if the input generator is a nonlinear system of the form (B.26), whose initial conditions are chosen in a *compact invariant* set W. The fact that W is invariant for the dynamics of (B.26) implies, as a consequence of Lemma B.8, that the steady-state locus of (B.26) is the set W itself, i.e., that the input generator is "in steady state".[27] The composition of (B.26) and (B.27) yields an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Az + Bq(w), \end{aligned} \tag{B.29}$$

in which $(w, z) \in X := W \times \mathbb{R}^n$. Note that, since W is invariant for (B.26), the set X is invariant for (B.29).

Since the inputs generated by (B.26) are bounded and the lower subsystem of (B.29) is input-to-state stable, the motions of system (B.29), with initial conditions taken in X, are ultimately bounded. In fact, since W is compact and invariant, there exists a number U such that $\|q(w(t))\| \leq U$ for all $t \in \mathbb{R}$ and all $w(0) \in W$. Therefore, standard arguments can be invoked to deduce the existence of positive numbers K, λ and M such that

$$\|z(t)\| \leq Ke^{-\lambda t}\|z(0)\| + MU$$

for all $t \geq 0$. From this, it is immediate to check that any bounded set B of the form

$$B = \{(w, z) \in W \times \mathbb{R}^n : \|z\| \leq (1 + d)MU\}$$

[27] Note that the set W_c considered in the previous example had exactly this property.

in which d is any positive number, has the property requested in the definition of ultimate boundedness. This being the case, it can be shown that the steady-state locus of (B.29) is the graph of the (nonlinear) map[28]

$$\begin{aligned} \pi : W &\to \mathbb{R}^n \\ w &\mapsto \pi(w) = \int_\infty^0 e^{-A\tau} Bq(\bar{w}(\tau, w))d\tau \,, \end{aligned} \tag{B.30}$$

i.e.,

$$\omega(B) = \{(w, z) \in W \times \mathbb{R}^n : z = \pi(w)\} \,,$$

To check that this is the case, observe first of all that—since $q(\bar{w}(t, w))$ is by hypothesis a bounded function of t and all eigenvalues of A have negative real part—the integral on the right-hand side of (B.30) is finite for every $w \in W$. Then, observe that the graph of the map $z = \pi(w)$ is invariant for (B.29). In fact, pick an initial state for (B.29) on the graph of this map, i.e., a pair (w_0, z_0) satisfying $z_0 = \pi(w_0)$ and compute the solution $z(t)$ of the lower equation of (B.29), via the classical variation of constants formula, to obtain

$$\begin{aligned} z(t) &= e^{At} \int_{-\infty}^0 e^{-A\tau} Bq(\bar{w}(\tau, w_0))d\tau + \int_0^t e^{A(t-\tau)} Bq(\bar{w}(\tau, w_0))d\tau \\ &= \int_\infty^0 e^{-A\theta} Bq(\bar{w}(\theta + t, w_0))d\theta = \int_\infty^0 e^{-A\theta} Bq(\bar{w}(\theta, w(t, w_0)))d\theta. \end{aligned}$$

This shows that $z(t) = \pi(w(t))$ and proves the invariance of the graph of $\pi(\cdot)$ for (B.29). Since the graph of $\pi(\cdot)$ is a compact set invariant for (B.29), this set is necessarily a subset of the steady-state locus of (B.29). Finally, observe that, since the eigenvalues of A have negative real part, all motions of (B.29) whose initial conditions are not on the graph of $\pi(\cdot)$ are unbounded in negative time and therefore cannot be contained in the steady-state locus, which by definition is a bounded invariant set. Thus, the only points in the steady-state locus are precisely the points of the graph of $\pi(\cdot)$.

This result shows that the steady-state response of a stable linear system to an input generated by a nonlinear system of the form (B.26), with initial conditions $w(0)$ taken in a compact invariant set W, can be expressed in the form

$$z_{ss}(t) = \pi(w(t))$$

in which $\pi(\cdot)$ is the map defined in (B.30).

Remark B.2 Note that the motions of the autonomous input generator (B.26) are not necessarily periodic motions, as it was the case for the input generator (A.26). For

[28] In the following formula, $\bar{w}(t, w)$ denotes the integral curve of $\dot{w} = s(w)$ passing through w at time $t = 0$. Note that, as a consequence of the fact that W is closed and invariant, $\bar{w}(t, w)$ is defined for all $(t, w) \in \mathbb{R} \times W$.

instance, the system in question could be a stable Van der Pol oscillator, with W defined as the set of all points inside and on the boundary of the limit cycle. In this case, it is possible to think of the steady-state response of (B.27) not just as of the (single) periodic input obtained when the initial condition of (B.26) is taken on the limit cycle, but also as of all (non periodic) inputs obtained when the initial condition is taken in the interior of W. $\triangleleft$

Consider now the case of a general nonlinear system of the form (B.25), in which $z \in \mathbb{R}^n$, with input u supplied by a nonlinear input generator of the form (B.26). Suppose that system (B.25) is input-to-state stable and that the initial conditions of the input generator are taken in compact invariant set W. It is easy to see that the motions of the augmented system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, q(w)), \end{aligned} \tag{B.31}$$

with initial conditions in the set $X = W \times \mathbb{R}^n$, are ultimately bounded. In fact, since W is a compact set, there exists a number $U > 0$ such that

$$\|u(\cdot)\|_\infty = \|q(w(\cdot))\|_\infty \leq U$$

for all $w(0) \in W$. Since (B.25) is input-to-state stable, there exist a class $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \leq \beta(\|z(0)\|, t) + \gamma(U)$$

for all $t \geq 0$. Since $\beta(\cdot, \cdot)$ is a class $\mathscr{KL}$ function, for any compact set Z and any number $d > 0$, there exists a time T such that $\beta(\|z(0)\|, t) \leq d\gamma(U)$ for all $z(0) \in Z$ and all $t \geq T$. Thus, it follows that the set

$$B = \{(w, z) \in W \times \mathbb{R}^n : \|z\| \leq (1 + d)\gamma(U)\}$$

has the property requested in the definition of ultimate boundedness.

Since the motions of the augmented system (B.31) are ultimately bounded, its steady-state locus $\omega(B)$ is well defined. As a matter of fact, it is possible to prove that also in this case the set in question is the *graph of a map* defined on W.

Lemma B.10 *Consider a system of the form (B.31) with $(w, z) \in W \times \mathbb{R}^n$. Suppose its motions are ultimately bounded. If W is a compact set invariant for $\dot{w} = s(w)$, the steady-state locus of (B.31) is the graph of a (possibly set-valued) map defined on W.*

Proof Since W is compact and invariant for (B.26), $\omega(W) = W$. As a consequence, for all $\bar{w} \in W$ there is a sequence $\{w_k, t_k\}$ with w_k in W for all k such that $\bar{w} = \lim_{k\to\infty} w(t_k, w_k)$. Set $x = \mathrm{col}(w, z)$ and let $x(t, x_0)$ denote the integral curve of (B.31) passing through x_0 at time $t = 0$. Pick any point $z_0 \in \mathbb{R}^n$ and let $x_k =$

$\text{col}(w_k, z_0)$. All such x_k's are in a compact set. Hence, by definition of ultimate boundedness, there is a bounded set B and a integer $k^* > 0$ such that $x(t_{k^*+\ell}, x_k) \in B$ for all $\ell \geq 0$ and all k. Set $\bar{x}_\ell = x(t_{k^*}, x_\ell)$ and $\tau_\ell = t_{k^*+\ell} - t_{k^*}$, for $\ell \geq 0$, and observe that, by construction, $x(\tau_\ell, \bar{x}_\ell) = x(t_{k^*+\ell}, x_\ell)$, which shows that all $x(\tau_\ell, \bar{x}_\ell)$'s are in B, a bounded set. Hence, there exists a subsequence $\{x(\tau_h, \bar{x}_h)\}$ converging to a point $\hat{x} = \text{col}(\hat{w}, \hat{z})$, which is a point of $\omega(B)$ because all $\bar{x}_h$'s are in B. Since system (B.31) is upper triangular, necessarily $\hat{w} = \bar{w}$. This shows that, for any point $\bar{w} \in W$, there is at least one point $\hat{z} \in \mathbb{R}^n$ such that $(\bar{w}, \hat{z}) \in \omega(B)$. $\triangleleft$

It should be stressed that the map whose graph characterizes the steady-state locus of (B.31) may fail to be single-valued and, also, may fail to be continuously differentiable, as shown in the examples below.

Example B.10 Consider the system

$$\dot{z} = -z^3 + zu\,, \tag{B.32}$$

which is input-to-state stable, with input u provided by the input generator

$$\begin{aligned} \dot{w} &= 0 \\ u &= w \end{aligned}$$

for which we take $W = \{w \in \mathbb{R} : |w| \leq 1\}$. Thus, $u(t) = w(t) = w(0) := w_0$. If $w_0 \leq 0$, system (B.32) has a globally asymptotically stable equilibrium at $z = 0$. If $w_0 > 0$, system (B.32) has one unstable equilibrium at $z = 0$ and two locally asymptotically stable equilibria at $z = \pm\sqrt{w_0}$. For every fixed $w_0 > 0$, trajectories of (B.32) with initial conditions satisfying $|z_0| > \sqrt{w_0}$ asymptotically converge to either one of the two asymptotically stable equilibria, while the compact set

$$\{(w, z);\, w = w_0,\, |z| \leq \sqrt{w_0}\}$$

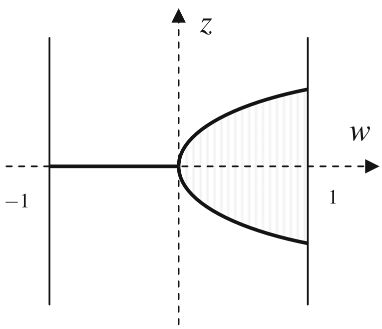

Fig. B.5 The steady-state locus of system (B.32) is invariant. As a consequence, the steady-state locus of the augmented system

$$\begin{aligned}\dot{z} &= -z^3 + zw\\ \dot{w} &= 0\end{aligned}$$

is the graph of the set-valued map

$$\pi \ : \ w \in W \quad \mapsto \quad \pi(w) \subset \mathbb{R}$$

defined as (Fig. B.5)

$$\begin{aligned}-1 \le w \le 0 \quad &\Rightarrow \quad \pi(w) = \{0\}\\ 0 < w \le 1 \quad &\Rightarrow \quad \pi(w) = \{z \in \mathbb{R} : |z| \le \sqrt{w}\}.\end{aligned}$$ ◁

Example B.11 Consider the system

$$\dot{z} = -z^3 + u$$

which is input to state stable, with input u provided by the harmonic oscillator

$$\begin{aligned}\dot{w}_1 &= w_2\\ \dot{w}_2 &= -w_1\\ u &= w_1\end{aligned}$$

for which we take $W = \{w \in \mathbb{R}^2 : \|w\| \le 1\}$. It can be shown[29] that, for each $w(0) \in W$, there is one and only one value $z(0) \in \mathbb{R}$ from which the motion of the resulting augmented system (B.31) is bounded both in positive and negative time. The set of all such pairs identifies a single-valued map $\pi : W \to \mathbb{R}$, whose graph characterizes the steady-state locus of the system. The map in question is continuously differentiable at any nonzero w, but it is only continuous at $w = 0$. ◁

If the map whose graph characterizes the steady-state locus of (B.31) *is single-valued*, the steady-state response of an input-to-state stable system of the form (B.25) to an input generated by a system of the form (B.26) can be expressed as

$$z_{\mathrm{ss}}(t) = \pi(w(t)),$$

in which $\pi(\cdot)$ is a map defined on W. In general, it is not easy to give explicit expressions of this map (such as the one considered earlier in (B.30)). However, if $\pi(\cdot)$ is continuously differentiable, a very expressive implicit characterization is possible. In fact, recall that the steady-state locus of (B.31) is by definition an invariant set, i.e., $z(t) = \pi(w(t))$ for all $t \in \mathbb{R}$ along any trajectory of (B.31) with initial condition satisfying $z(0) = \pi(w(0))$. Along all such trajectories,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = f(z(t), q(w(t))) = f(\pi(w(t)), q(w(t))).$$

[29] See [16].

If $\pi(w)$ is continuously differentiable, then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \left.\frac{\partial\pi}{\partial w}\right|_{w=w(t)} \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \left.\frac{\partial\pi}{\partial w}\right|_{w=w(t)} s(w(t))$$

and hence it is seen that $\pi(w)$ satisfies the partial differential equation

$$\frac{\partial\pi}{\partial w} s(w) = f(\pi(w), q(w)) \qquad \text{for all } w \in W. \tag{B.33}$$

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Index

A. Isidori, *Lectures in Feedback Design for Multivariable Systems*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8