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Cooperative Control of Nonlinear Networked Systems

Infinite-time and Finite-time Design Methods

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To my family for all the love, support and understanding.

Yongduan Song

To my family, for all the support, love and kindness.

Yujuan Wang

Preface

Motivation. Multiple networked dynamical systems, involving non-parametric or non-vanishing uncertainties under certain communication topologies, have the potential for many important applications, making it important and interesting to control such systems. Due to the requirement for a powerful central station with global information flow, the traditional centralized control has not been the ideal control framework for the multiple networked systems under limited communication condition. In this regard, the distributed control, dedicated to achieving a global common objective by using only local communication in the absence of central coordination, is more promising due to the advantages such as limited communication/sensing range, cooperative fashion, flexibility, and scalability. On the other hand, adaptive control is known as one of the most effective ways to deal with parametric uncertainties by employing online parameter estimators. As such, it is necessary to investigate the cooperative adaptive control of multiple networked nonlinear systems involving non-parametric or non-vanishing uncertainties.

Technical challenge in cooperative control of multiple networked nonlinear systems with non-parametric/non-vanishing uncertainties lies in two aspects: (1) how to realize the global cooperative objective in a distributive manner and (2) how to effectively compensate the non-parametric/non-vanishing uncertainties and meanwhile achieve the cooperative control objective with high precision and high rate of error convergence. Although a rich collection of cooperative control results are available in the literature for nonlinear multi-agent systems (MASs), most existing works appear to only for the nonlinear MAS with linearly parameterized nonlinearities or achieve the so-called cooperative uniformly ultimately bounded (CUUB) stability result. This is partly due to the constraint of the local information transformation allowed in the communication network and partly due to the extra residual terms produced by the non-vanishing uncertainties that cannot be canceled in the Lyapunov stability analysis (for the MAS with non-vanishing uncertainties).

Hence, it is crucial to develop control tools in the continuum domain, for the distributed cooperative control of multiple networked nonlinear systems subject to non-parametric/non-vanishing uncertainties to achieve a global cooperative objective with high precision and high rate of error convergence under relatively weak communication topology condition.

What does the book cover? This book is concerned with the distributed cooperative control of multiple networked nonlinear systems in the presence of unknown non-parametric nonlinearities and non-vanishing uncertainties under a certain communication condition. The technical coverage extends to

- Stability analysis such as input-to-state stability (ISS), cooperative uniformly ultimately bounded (CUUB), globally exponentially stability, finite-time stability, and finite-time cooperative uniformly ultimately bounded (FT-CUUB), etc., to analyze and synthesize the multiple networked nonlinear systems;
- Control design techniques and tools including matrix analysis tool (to develop the distributed controller), core function technique and virtual parameter estimation error technique (to construct the adaptive part of the Lyapunov function candidate to deal with the non-parametric uncertainties), and Young's inequality (to establish various inequalities for the stability analysis), etc.;
- Various control schemes ranging from regular state feedback-based cooperative adaptive control (to achieve CUUB), fractional state feedback-based distributed finite-time control (to achieve finite-time stability), fractional state feedback-based distributed finite-time adaptive control (to achieve FT-CUUB), and so on.

The infinite-time stability theory and finite-time stability theory allow for the establishment of both qualitative and quantitative properties of the closed-loop dynamical systems, and the adaptive control theory plays an important role in coping with system uncertainties by employing online parameter estimators in numerous dynamical systems. The required communication topologies can be either undirected or directed, and the convergence time can be endowed with either infinite time or finite time.

Technical contents of the book are divided into three parts. Part I consists of Chaps. 1 and 2. Chapter 1 provides an overview of the existing results on cooperative control of multiple networked systems from the two aspects: infinite-time control and finite-time control. Chapter 2 contains a review of algebraic graph theory, certain matrix analysis theory associated with the graph, and stability analysis theory and design tools.

Part II, composed of Chaps. 3–5, deals with the infinite-time cooperative leaderless consensus control of multiple networked nonlinear systems subject to non-parametric/non-vanishing uncertainties under some fixed and local communication conditions. In Chap. 3, we present some key technique methods, including the virtual parameter estimation error method and the core function technique, to handle the inherent unknown time-varying control gain and the unknown non-parametric uncertainties, and also introduce the construction of the distributed part of the Lyapunov function (it is related to the graph topology) that is important for the Lyapunov stability analysis and distributed controller design. Chapter 4 shows how to design the distributed adaptive controller for the networked MAS with non-parametric/non-vanishing uncertainties from first-order to high-order dynamics under the fixed and undirected topology, and also gives the Lyapunov stability analysis. In Chap. 5, we study the cooperative leaderless consensus control problem for networked MAS with first-order and second-order dynamics under the directed topology condition.

In Part III of the book, which is comprised of Chaps. 6–9, we study the finite-time adaptive consensus control for multiple networked nonlinear systems subject to non-parametric/non-vanishing uncertainties linked to each other by a local communication graph. The idea of fractional state feedback-based distributed finite-time adaptive control is introduced in Chap. 6. We present some key inequalities related to the fractional adaptive updated law and some useful lemmas related to the directed topology condition, all of which are important for the finite-time stability analysis and finite-time leaderless consensus controller design. Chapter 7 shows the finite-time controller design and stability analysis for the networked MAS with non-parametric/non-vanishing uncertainties with second-order dynamics under the fixed and undirected topology. In Chap. 8, we extend the results obtained in Chap. 7 to the directed topology. Chapter 8 studies the finite-time cooperative leaderless consensus control problem for networked MAS with high-order dynamics under the directed topology condition.

Whom is the book for? The book presents various solutions to cooperative control problems of multiple networked nonlinear systems on graphs. The researcher, engineer, or student in cooperative adaptive control of nonlinear MAS will find a wealth of examples that are solved and presented in full analytical and numerical detail, graphically illustrated, and with the intuition that helps understand the results. It should be noted that although the overall system with linearly parameterized nonlinearities is nonlinear, such nonlinearity is sufficiently benign and globally tractable. This is unlike the nonlinearities one encounters in the nonlinear MAS with non-parametric or non-vanishing uncertainties. This book is intended primarily as a bridge between the distributed control of MAS with linearly parameterized nonlinearities and the one with non-parametric uncertainties. The researchers and scientists in research institutes as well as academics in universities working on nonlinear systems, adaptive control, and distributed control will find the book filled with problems of a new kind.

We propose to use mathematical tools such as algebraic graph theory and certain matrix analysis theory, and potential readers who already had some exposure to basic systems theory and algebra graph theory can obtain a deeper understanding of the roles of matrix operators as mathematical machineries for cooperative control of MAS. This book can be used as an introductory material to fresh graduates (at bachelor or master's level) joining the control workforce to work on distributed control of multiple networked systems. The book can also be used as reference textbooks for new postgraduate students who have the intention to work on this arena of research. At the same time, many universities have established programs and courses in this new field, with many cross-faculty and interdiscipline research going on in this arena as well.

We believe that there are a significant number of potential readers for our proposed book among the professionals working in the engineering fields, which include control engineers, mechanical engineers, and aerospace engineers who deal with various mechanics problems with spatially varying coefficients even if control is not their focus. This is because the methods employed are not from the generic control catalog but are very structure-specific. Physicists would find the book of interest for the same reason.

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Chongqing, China
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Contents

Abbreviations and Notation

Abbreviations

CUUB	Cooperative uniformly ultimately bounded
LMI	Linear matrix inequality
LTI	Linear time invariant
MIMO	Multiple input, multiple output
NMAS	Networked multi-agent systems
SISO	Single input, single output
TSM	Terminal sliding mode

Notation

$\mathbb{R}$	Field of real numbers
Σ	Summation
$\lvert a \rvert$	The absolute of a scalar a
$\lVert X \rVert$	The norm of a vector X
max	Maximum
min	Minimum
sup	Supremum, the least upper bound
inf	Infimum, the greatest lower bound
$\forall$	For all
$\in$	Belongs to
$\rightarrow$	Tends to
$<(>)$	Less (greater) than
$\leq(\geq)$	Less (greater) than or equal to
$\ll(\gg)$	Much less (greater) than
$\otimes$	Kronecker product
$\mathbb{R}^N$	The N-dimensional Euclidean space
I_N	The N-dimensional identity matrix
$1_N\ (0_N)$	The N-dimensional vector with each entry being 1(0)
$\dot{y}$	The first derivative of y with respect to time

$\ddot{y}$	The second derivative of y with respect to time
$y^{(i)}$	The ith derivative of y with respect to time
w.r.t.	With respect to
$\mathrm{sat}(\cdot)$	The saturation function
$\mathrm{sgn}(\cdot)$	The signum function
$\mathrm{diag}\{a_1, \cdots, a_N\}$	A diagonal matrix with diagonal elements a_1 to a_N
$\mathrm{diag}\{A_1, \cdots, A_N\}$	A diagonal matrix with diagonal elements A_1 to A_N, here A_i ($i = 1, \cdots, N$) denotes matrix
$P > 0$	A positive definite matrix P
$P \geq 0$	A positive semi-definite matrix P
$A^T(X^T)$	The transpose of matrix A (of vector X)
$\bar{\sigma}(A)(\underline{\sigma}(A))$	The maximum (minimum) singular value of matrix A
$\lambda_{\max}(P)(\lambda_{\min}(P))$	The maximum (minimum) eigenvalue of symmetric matrix P
$f^{-1}(\cdot)$	The inverse of function f

Part I
Overview and Preliminary

Chapter 1
Introduction

This chapter introduces the cooperative control of multi-agent systems (MASs) and overviews recent research results in distributed cooperative control of networked MAS. The recent research results in distributed cooperative control of MAS are roughly categorized as leaderless consensus, leader–follower consensus, distributed formation control, and distributed containment control. A short discussion is given to propose several future research directions and problems that deserve further investigation.

1.1 Cooperative Control of Multi-agent Systems

In the past two decades, the control of multiple dynamical systems interconnected by a communication network has attracted extensive attention among scientific communities due to the benefits obtained by replacing a solo complicated system with several simple systems. For the control of multiple networked dynamical systems, two methods are commonly used: the centralized control method and the distributed control method. As shown in Fig. 1.1, the centralized control method relies on the assumption that each subsystem of the group has the ability to communicate to a central station or share information via a fully connected network. Essentially, the centralized control method is a direct extension of the traditional single-system control methodology. As a result, the centralized control method may result in a catastrophic failure of the overall system due to its single point of failure. Also, real-world communication topologies are not usually fully connected. Therefore, cooperative control in the presence of real-world communication constraints becomes a significant challenge. Different from the traditional centralized control method, the distributed control method, as shown in Fig. 1.2, does not require the existence of a central station. The distributed control is dedicated to achieving a global common objective by using only local communication in the absence of central coordination and thus is more promising due to the advantages such as limited communication/sensing range,

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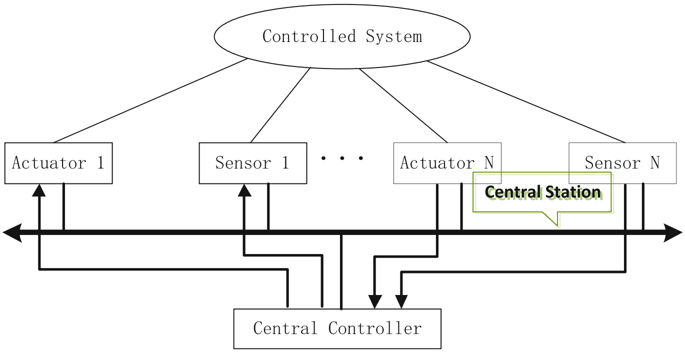

Fig. 1.1 Centralized control

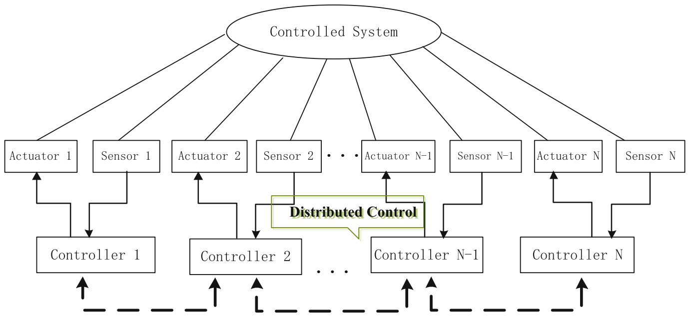

Fig. 1.2 Distributed control

cooperative fashion, flexibility, and scalability, although it is far more complex than the centralized one.

By cooperative control system on networks, it is referred to a system that consists of a group of autonomous subsystems with sensing or communication capabilities, for example, vehicles with communication devices and robots with camera sensors. Of fundamental concern for the networked cooperative system is the study of the interactions and collective behaviors of each subsystem under the influence of the information flow allowed in the communication network. This communication network can be modeled as a graph with directed edges or links corresponding to the allowed flow of information among the subsystems, and the subsystems are modeled as the nodes (sometimes called agents) in the graph. The goal of the networked cooperative system is to achieve a common group behavior by using only local information available to each agent from sensing or communication devices. Such local information may include relative configuration and motion obtained from sensing or communication among agents, agents' sensor measurements, and so on. Thus, a cooperative system has four basic elements: group objective, agents, information topology, and control protocols governing the motion of agents. Specifically, the fundamental problem in the networked cooperative control system is the design of distributed protocols in the sense that the control law of each agent is only allowed to depend on local information from its neighbors in the networks to guarantee that all the agents achieve a common group objective or that all the states of agents achieve a common value.

Cooperative control of networked multiple dynamical systems poses significant theoretical and practical challenges. First, the research objective is to develop a system of subsystems rather than a single system. Second, the communication bandwidth and connectivity of the team are often limited, and the information exchange among subsystems may be unreliable. It is also difficult to decide what to communicate and when and with whom the communication takes place. Third, arbitration between team goals and individual goals needs to be negotiated. Fourth, the computational resources of each individual subsystem will often be limited.

In general, the control problems of the cooperative systems can be roughly categorized based on the following directions:

1. Consensus/agreement/synchronization/rendezvous. Distributed consensus is a study dedicated to ensuring an agreement between states or output variables among networked systems, in which distributed controller is designed for each agent such that the states or output of all agents is eventually driven to an unprescribed common value. This common value may be a constant or may be a time-varying function which is generally a function of all the agents' initial states [1]. In many cases, the four words can be used without discrimination.
2. Leader–follower consensus. For the leader–follower consensus problem, a leader agent acting as a command generator is involved with all the other agents being follower agents. The leader agent generates the desired reference trajectory and does not accept information from follower agents, and all the follower agents attempt to follow the leader agent's trajectory. The control objective in leader–follower consensus is to design distributed controller for each follower such that each follower can synchronize to the leader agent. This problem is also known as coordinated tracking, synchronization to a leader [2], model reference consensus [3], leader-following control [4], or pinning control [5].
3. Distributed formation control. Distributed formation control problem refers to the scenario that all the agents in the cooperative system are required to operate according to the prescribed trajectory to form a certain geometrical configuration through local interaction with or without a group reference. The control objective in the formation control is to stabilize the relative distances/position between the agents to achieve a pre-specified desired value.
4. Distributed containment control. The distributed containment control problem arises when there exist multiple leaders. It refers to the scenario that all the follower agents are forced to enter the convex hull formed by the leader agents through local interaction among the neighbor follower agents only.

How to achieve cooperative consensus with the least possible topological requirement and how to achieve it in a timely and distributive manner are two common challenges in distributed control of networked cooperative systems. In this sense, the convergence rate is one of the most important performance indicators for the cooperative control of networked multi-agent systems. So far, most of the existing cooperative control results are achieving asymptotical convergence, meaning that the convergence rate is at best exponential with infinite settling time. That is, it needs infinite time to achieve cooperative consensus or formation or containment or other cooperative objectives. It is indicated in [6] that the second smallest eigenvalue of the Laplacian matrix quantifies the convergence rate of the networked multi-agent systems. Several researchers have tried to find better interaction graph to get a larger second smallest eigenvalue such that a better convergence rate can be obtained. To maximize the second smallest eigenvalue of the interaction graph, the researchers in [7] try to find the best vertex positional configuration. By utilizing the semi-definite convex programming, the researchers in [8] design the weights among the agents to increase the convergence rate. It is worth noting that all these efforts are to find proper interaction graphs, but not to find proper control schemes to achieve higher performance. In practice, it is often required that the cooperative control objective can be reached in finite time. In addition to bearing the benefit of faster rate of convergence, finite-time control systems exhibit the feature of better disturbance rejection and robustness against uncertainties as shown in [9], making the finite-time cooperative control problem one of the most active research topics. Thus, several researchers invoke finite-time control schemes for networked multi-agent systems to guarantee that the cooperative control objective is reached within finite time.

1.2 Overview of Recent Research Results in Cooperative Control of Multi-agent Systems

In this section, we overview recent research results in distributed cooperative control of MAS. According to different convergence rates of the cooperative control systems, we roughly categorize the recent research results into the following three classes:

1. Non-finite time convergence: The non-finite time convergence means that the convergence rate of the cooperative control systems is at best exponential with infinite settling time. That is, it needs infinite time to achieve cooperative consensus or formation or containment or other cooperative objectives. The control results corresponding to the non-finite time convergence include the asymptotical stability, exponential stability, ultimately uniformly boundedness (UUB).
2. Increased convergence speed: This is corresponding to the case to choose proper interaction graphs to increase the convergence rate but not to find proper control schemes to achieve high performance.
3. Finite-time convergence: The finite-time convergence means that the control objectives of the cooperative control systems are realized in finite time. The finite-time control systems not only possess faster convergence rate but also possess better disturbance rejection and robustness against uncertainties.

1.2.1 Non-finite Time Convergence

In this subsection, we overview recent research results in distributed cooperative control of networked MAS with non-finite time convergence.

The research on cooperative control of MAS has gained considerable attention from scientific communities. This is partly due to the broad potential applications of the cooperative control of MAS in various areas, such as rendezvous [10], flocking [11], synchronization [12, 13], and formation control [14–16]. Consensus is known as one of the most important issues in cooperative control of MAS, which refers to all agents reaching an agreement on certain variables of interest. Vicsek et al. proposed in [17] in 1995 a simple but compelling discrete-time model of n autonomous agents (points or particles) all moving in the plane with the same speed but with different headings, in which each agent's heading is updated by using a local rule based on the average of its own heading plus the headings of its "neighbors." This work provided study base for the cooperative consensus studies. Vicsek et al. in [17] provided simulation results to demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves, on basis of which Jadbabaie et al. further provided a theoretical explanation for the observed behavior in [18]. In [6], Olfati-Saber and Murray addressed consensus problems for networks with directed information flow and provided analytical tools relying on control theory, algebraic graph theory, and matrix theory, which provides a theoretical framework to analyze and study the consensus problem of multi-agent systems. In [12], Moreau studied a simple but compelling model of networked agents interacting via time-dependent communication links and represented necessary and/or sufficient conditions for the convergence of the individual agents' states to a common value. Ren et al. showed in [19, 20] that the consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs has a spanning tree frequently enough as the system evolves, which extends sufficient conditions for the cooperative control systems to achieve consensus to more general case. In [21], Cortes addressed the problem of designing (continuous-time) coordination algorithms that make a networked system asymptotically agree upon the value of a desired function of the initial state of the individual agents. Ren and Atkins proposed second-order consensus protocols that take motions of the information states and their derivatives into account, extending first-order protocols from the literature, and also showed that unlike the first-order case, having a (directed) spanning tree is a necessary rather than a sufficient condition for consensus seeking with second-order dynamics in [22, 23]. Yu et al. studied some necessary and sufficient conditions for second-order consensus: Both the real and imaginary parts of the eigenvalues of the Laplacian matrix associated with the interaction graph topology play key roles in reaching consensus in [24]. In [25], Abdessameud et al. dealt with consensus strategy design for double-integrator dynamics without velocity measurements and with input constraints. The works in [26–29] studied the leaderless consensus problem for high-order linear time-invariant

(LTI) multi-agent systems with asymptotical convergence. Yu et al. addressed the leaderless consensus problem for second-order multi-agent systems with nonlinear dynamics under directed topologies in [30]. In [31], Wang and Song investigated the leaderless consensus for a class of second-order multi-agent systems with unknown time-varying control gains and non-parametric uncertainties as well as the unknown actuator failures.

It is noted that the aforementioned works are all about the leaderless consensus, and the final consensus value is some functions that is related to the initial states of all agents in cooperative system, i.e., χ-consensus [21], including the average consensus [32–35] as a special case. For more detailed results on consensus control, the authors can be referred to [36, 37].

In some practical applications, there exists one or even multiple leaders that provide/provides group reference states or external control instructions for networked multi-agent systems. Consensus with one leader is often named as consensus tracking or coordinated tracking. In [38], Ren studied the multi-vehicle consensus with a time-varying reference state, in which the cooperative tracking problem for the first-order multi-agent systems with one time-varying leader has been solved. Qin et al. considered the consensus problem of multiple second-order vehicles with a time-varying reference signal under directed communication topology in [39]. In [40], Cao et al. addressed the distributed discrete-time consensus tracking with a time-varying reference trajectory and limited communication. Gu et al. investigated a leader–follower flocking system that can track the desired trajectory led by group leaders in [41]. By using the polynomial filtering algorithms, a control method to accelerate the convergence rate for given network matrices was proposed in [42]. The work in [43] addressed the stochastic bounded consensus tracking problems of leader–follower multi-agent systems, in which the information measured at the sampling instants from its neighbors or a virtual leader with a time-varying reference trajectory is available, the measurements are corrupted by random noises, and the signal sampling process induces a small sampling delay. The second-order consensus of leader–follower multi-agent systems with jointly connected topologies and time-varying delays was studied in [44]. A distributed consensus protocol was developed for a class of homogeneous time-varying nonlinear multi-agent systems in the work [45], in which the agent dynamics were supposed to be in the strict feedback form and satisfy Lipschitz conditions with time-varying gains. In the work [46], a coordinated consensus problem with an active leader under variable interconnection topology was considered, in which the leader's state not only keeps changing but also may not be measured. The work in [4] considered a leader-following problem for multi-agent systems with a switching interconnection topology, in which distributed observers were designed for second-order follower agents under a common assumption that the active leader's velocity cannot be measured in real time. The consensus tracking problems for high-order LTI multi-agent systems under both fixed and switching interaction topologies were coped with by Ni and Cheng in [47]. The work [48] solved a distributed coordinated tracking problem via a variable structure approach. The work [49] investigated the coordinated tracking problem of multiple networked dynamic systems with both physical connections and virtual connections in which unknown unidentical nonlinearities and time-varying yet undetectable actuation faults and varying actuation authorities are involved.

If there exist multiple leaders, containment problems requiring that the states of followers converge to the convex hull formed by the states of leaders arise. Cao and Ren studied the containment control problem of a group of mobile autonomous agents with single-integrator dynamics in the presence of both multiple stationary and dynamic leaders under fixed and switching directed network topologies in [50], and studied containment control problem for double-integrator dynamics in the presence of both stationary and dynamic leaders in [51], respectively. Mei et al. [52] investigated the distributed containment problem for multiple Lagrangian systems in the presence of parametric uncertainties with multiple stationary leaders under a directed graph without using the neighbors' velocity information in the absence of communication. The containment problems for low-order multi-agent systems have also been considered in [54–59]. Li et al. [53] discussed the containment control problems for both continuous-time and discrete-time high-order LTI multi-agent systems under directed communication topologies by using a dynamic output protocol. Dong et al. [60] investigated the containment control problem for high-order LTI singular multi-agent systems with time delays. By classifying the agents into boundary agents and internal agents, Liu et al. [61] proposed a criterion for the states of internal agents to converge to a convex combination of the states of the boundary agents.

1.2.2 Increased Convergence Speed

To achieve a faster convergence rate in consensus control of networked MAS, two main methods have been proposed: The first one is to use the physical communication network as an information flow graph but with the optimized weights for all information flow edges, and the second one is to introduce a few shortcut (non-local) multi-hop edges to the information flow network without physically changing or adding any edges in the underlying communication network. The most interesting and famous one in the later category is small-world networks that can dramatically enhance the algebraic connectivity of regular complex networks. As shown in [6], the second smallest eigenvalue of the Laplacian matrix quantifies the convergence rate. Several researchers have tried to find better interaction graph to get a larger second smallest eigenvalue such as to get better convergence rate. On the basis of the work [6], Olfati-Saber further demonstrated in [62] that without adding new links or nodes to the network it was still possible to dramatically increase the algebraic connectivity of a regular complex network by 1000 times or more. However, Nosrati et al. showed the fact in [63] that, "the small-world network construction has a negligible effect on the time-delay robustness of the consensus protocol over a initial regular network," is not true. Xiao et al. [64] showed how problem structure can be exploited to speed up interior-point methods for solving the fastest distributed linear iteration problem for networks with up to a thousand or so edges. Kim et al. [65] posed the problem of finding the best vertex positional configuration to maximize the second smallest eigenvalue of the associated Laplacian. In order to achieve a fast consensus seeking, Jin and Murray [66] proposed the multi-hop relay protocols, where each agent can expand its knowledge by employing multi-hop paths in the network, and demonstrated that multi-hop relay protocols can enlarge the algebraic connectivity without physically changing the network topology. Yang et al. [67] analyzed the relationship between the second smallest eigenvalue of the associated Laplacian and the convergence rate of the consensus control algorithm, and found that the convergence rate of the consensus control algorithm on a completely regular lattice can be greatly enhanced by just randomly rewiring a very small number of links in the network.

1.2.3 Finite-Time Convergence

It is worth noting that all these solutions mentioned in Sect. 1.2.2 are to choose proper interaction graphs to achieve a faster convergence, but not to find proper control schemes to achieve a higher performance. It is often required in practice that the cooperative control objective is reached in finite time. In addition to bearing the benefit of faster rate of convergence, finite-time control systems exhibit the feature of better disturbance rejection and robustness against uncertainties as shown in [9], enticing increasing research attention from control community. Thus, several researchers seek finite-time control algorithms to guarantee that the cooperative control objective is reached within finite time.

Weiss and Infante [68] gave the definition of finite-time stability of systems in 1965 and extended the finite-time stability theory to the non-autonomous systems under the influence of external forces in [69], which were the embryo of finite-time stability theory. However, these results obtained in [68, 69] are only about the analysis on the finite-time stability, and there are not the analysis on how to design the finite-time controller for systems. In general, according to the continuousness of the control input signals, the finite-time control methods in the existing literatures are classified into three categories: discontinuous finite-time control, continuous but non-smooth finite-time control, and continuous and smooth finite-time control. The discontinuous finite-time control methods mainly include signum function feedback-based control and terminal sliding mode (TSM)-based control; the continuous but non-smooth finite-time control methods mainly include single fractional state feedback-based control, homogeneous finite-time control, and adding power integrator-based control; the continuous and smooth finite-time control methods mainly include the time-varying gain-based regular state feedback control proposed by Song and Wang in [70, 71].

Signum function feedback-based control: The signum function feedback-based control is a class of discontinuous control method based on the signs of system states. Take the simple first-order linear system as an example,

$$\dot{x} = u, \tag{1.1}$$

where x and u denote the system state and control input, respectively. The signum function feedback-based control is of the basic form,

$$u = -c\text{sgn}(x) \tag{1.2}$$

where $c > 0$ is a freely chosen finite constant parameter. This control is based on the signum function, in which infinite bandwidth is required and the control action is discontinuous. Under such discontinuous control, the system states converge to the origin in a finite time T, satisfying

$$T \leq x(t_0)/c. \tag{1.3}$$

The signum function-based control can ensure the finite-time stability of systems and also possesses better disturbance rejection; however, the discontinuous terms in control input might cause the notorious chatting phenomenon during system operation. Cortes [72] discussed the application of the discontinuous control method based on signum function to consensus problems in first-order multi-agent systems and showed how the proposed non-smooth gradient flows achieve consensus in finite time. Chen et al. [73] investigated the finite-time consensus problem for first-order linear multi-agent systems under specific directed topology using the signum protocol. Li and Qu [74] extended the signum function-based control to first-order nonlinear multi-agent systems under the general setting of directed and switching topologies to achieve finite-time consensus. Upon using the signum-based control method, Zhang et al. [75] studied the finite-time consensus tracking problem of networked nonlinear multi-agent systems subject to unknown external disturbances under the directed topology that satisfies detailed balanced condition. Franceschelli et al. [76] proposed a decentralized consensus algorithm for a network of continuous-time integrators subject to unknown-but-bounded time-varying disturbances by using the discontinuous local interaction rule, in which it was proven that the agents achieve an approximated consensus condition by attenuating the destabilizing effect of the disturbances after a finite transient time. Wen et al. [77] proposed a class of discontinuous control protocols that only utilize the position information of each agent by using the signum-based control method and solved the finite-time coordinated tracking control problem. Meng and Lin [78] studied the finite-time consensus tracking control problem for multiple double-integrator networked systems involving a time-varying velocity leader and unknown bounded external disturbances, in which a discontinuous consensus tracking control protocol upon using a distributed finite-time observer is proposed. Based on signum function control method, Fu and Wang [79] addressed the finite-time consensus tracking problem for high-order integrator networked systems with bounded matched uncertainties.

Terminal sliding mode (TSM) control: Sliding mode control is a special kind of variable structure control proposed by Emelyanov and Taran in [80]. Sliding mode control is known to drive the system states to the sliding surface in finite time, and further, it is indicated that the sliding mode control has strong robustness and thus provides an effective control method for the nonlinear control systems. Yu and Man introduced the terminal sliding mode (TSM) control method in [81, 82], in which the terminal sliding mode (TSM) was defined by the first-order dynamics

$$\dot{x} + \beta x^{q/p} = 0 \tag{1.4}$$

where x is a scalar variable and $\beta > 0$ and $p > q > 0$ are odd integers such that, for any real number x, $x^{q/p}$ returns a real number. For system (1.4), the trajectories converge to an equilibrium state in a finite time T, satisfying

$$T = \frac{p}{\beta(p-q)}|x(0)|^{(p-q)/p}. \tag{1.5}$$

Further, the dynamics (1.4) can be extended to higher-dimensional terminal sliding mode case by introducing the hierarchical sliding mode structure as

$$\begin{aligned} s_1 &= \dot{s}_0 + \beta_1 s_0^{q_1/p_1} \\ s_2 &= \dot{s}_1 + \beta_2 s_1^{q_2/p_2} \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ s_{n-1} &= \dot{s}_{n-2} + \beta_{n-1} s_{n-2}^{q_{n-1}/p_{n-1}} \end{aligned} \tag{1.6}$$

where $s_0 = x$, $\beta_i > 0$, and $p_i > q_i$ are positive odd integers, and the finite convergence time T^s for system (1.6) satisfies

$$T^s = t_1^s + \sum_{i=1}^{n-1} \frac{p_{n-i}}{\beta_{n-i}(p_{n-i} - q_{n-i})} |s_{n-i-1}(t_i^s)|^{(p_{n-i}-q_{n-i})/p_{n-i}} \tag{1.7}$$

where t_i^s is the reaching time of the terminal sliding mode $s_{n-i} = 0$ for $i = 1, \ldots, n-1$. The TSM control can ensure the finite-time convergence of systems and is robust to input disturbances [83]. However, this method may cause the singularity problem around the equilibrium. Khoo et al. [84] studied the integral terminal sliding mode control of networked multi-robot systems, in which an integral terminal sliding mode surface for a class of first-order systems was proposed and the finite-time coordinated consensus tracking of multi-robot networked systems can be achieved on this integral terminal sliding mode surface. Khoo et al. [85] further studied the finite-time consensus tracking control for leader–follower multi-agent systems dominated by second-order dynamics and proved that the global finite-time consensus tracking can be achieved on the terminal sliding mode surface by switching control law. Zou and Kumar [86] applied the terminal sliding mode control to the attitude coordination of a group of spacecraft in the presence of external disturbances, in which a fast terminal sliding manifold was presented and a robust control term based on the hyperbolic tangent function was employed to suppress the bounded external disturbances. By using sliding mode control theory, Chen et al. [87] investigated the finite-time consensus tracking problem for a class of networked

Euler–Lagrange systems with a leader–follower structure, where the leader has an active dynamics and only a subset of followers have access to the leader system. Ghasemi and Nersesov [88] considered the finite-time coordination for a group of agents described by fully actuated Euler–Lagrange dynamics along with a leader agent by using a sliding mode control approach. With the construction of a nonlinear tracking control protocol via non-singular terminal sliding mode scheme, He et al. [89] investigated the finite-time cooperative attitude synchronization and tracking control for multiple rigid spacecraft with a time-varying leader whose attitude is represented by modified Rodriguez parameters. By introducing decentralized sliding mode estimators, Cao et al. [90] proposed a framework to achieve finite-time decentralized formation tracking of multiple autonomous vehicles. Upon using the second-order twisting sliding mode control system, Kamble et al. studied the leader–follower finite-time formation control of networked multi-agent systems with input disturbances in [91]. Li et al. [92] investigated the robust finite-time coordinated tracking problem for a group of second-order multi-agent systems with nonlinear dynamics under directed topology by using the fast terminal sliding mode control method. Zhao and Hua constructed a nonlinear distributed consensus tracking protocol via non-singular terminal sliding mode scheme in [93] to address the finite-time consensus tracking problem for second-order multi-agent systems composed of a leader with bounded input signal and n followers with bounded disturbances. Yang et al. [94] proposed a distributed robust control method for synchronized tracking of networked Euler–Lagrange systems, in which a disturbance observer was introduced to compensate for the low-passed-coupled uncertainties and a sliding mode control term was employed to handle the uncertainties that the disturbance observer cannot compensate for sufficiently.

The above-mentioned two control methods are discontinuous, and the discontinuous control terms would lead to chatting in control systems, which is not desired in the actual control. Thus, it is highly desired to develop continuous finite-time control methods for systems to avoid chatting.

Most of the existing works on continuous finite-time control are based on the theory of finite-time stability proposed in [9, 95, 96]. Following the terminology of [9, 95, 96], the global finite-time stability can be introduced in mathematical terms as follows.

Definition 1.1 Consider the system

$$\dot{x} = f(x, t), \quad f(0, t) = 0, \quad x \in \mathbb{R}^n, \tag{1.8}$$

where $f : U_0 \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous w. r. t. x on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is said to be (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \subseteq U_0$ of the origin. By "finite-time convergence," we mean that if there is a finite settling time function $T : U/\{0\} \to (0, \infty)$, such that $\forall x_0 \in U$, every solution $x(t; 0, x_0)$ of system (1.8) with x_0 as the initial condition is defined, and $x(t; 0, x_0) \in U/\{0\}$ for

$t \in [0, T(x_0))$; $\lim_{t \to T(x_0)} x(t; 0, x_0) = 0$. The origin is said to be a globally finite-time stable equilibrium if $U = \mathbb{R}^n$.

Lemma 1.1 *Consider system (1.8). Suppose that there exists a continuously differentiable function $V(x) : \hat{U} \to \mathbb{R}$ ($\hat{U} \subset U_0 \subset \mathbb{R}^n$), finite real number $c > 0$ and $\alpha \in (0, 1)$, such that*

(i) $V(x)$ is positive and definite on $\hat{U}$.

(ii) $\dot{V}(x) + cV(x)^\alpha$ is negative semi-definite (along the trajectory) on $\hat{U}$.

Then, the origin of system (1.8) is finite-time stable. Moreover, the finite settling time T satisfies

$$T \leq \frac{1}{c(1-\alpha)} V(x)^{1-\alpha} \tag{1.9}$$

for all x in some open neighborhood of the origin.

Single fractional-order state feedback-based control. By employing the theory of finite-time stability proposed in [95, 96], Wang and Xiao [97] discussed finite-time state consensus problems for multi-agent systems with single-integrator dynamics under undirected topology and directed topology that contains a spanning tree, respectively, and presented one framework for constructing effective distributed finite-time control algorithms based on fractional state feedback. Xiao et al. [98] extended the finite-time consensus results obtained in [97] to finite-time formation control for single-integrator multi-agent systems. On the basis of these two works mentioned above, Xiao et al. [99] further proposed a general class of nonlinear protocols in the form of continuous state feedbacks for finite-time consensus in integrator-like networked dynamic agents with directional link failure, which, with adjustable terms, were applicable in a wide range of situations, such as with input saturation restrictions and with convergence rate constraints. Upon using the single fractional state feedback-based method, Cao and Ren [100] analyzed the finite-time convergence of a nonlinear but continuous consensus algorithm for first-order multi-agent systems with unknown inherent nonlinear dynamics under directed switching topology graph that had a directed spanning tree at each time interval. Zuo and Lin [101, 102] constructed a class of global continuous time-invariant consensus protocols based on single fractional-order state feedback for each single-integrator agent in networked multi-agent systems under undirected topology and directed topology, respectively, with the settling time being upper bounded for arbitrary initial conditions.

Homogeneous finite-time control. The finite-time stability of homogeneous systems was proposed by Bhat in [103]. It is stated in [103] that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity, and a finite-time stable homogeneous system has a smooth homogeneous Lyapunov function that satisfies a finite-time differential inequality. Hong et al. [104–107] further deepened this theory and applied it to finite-time control of homogeneous systems. This approach is mainly based on the properties of homogeneous systems. In the following, we introduce some concepts about homogeneous systems which can be found, for example, in [103, 104, 108, 109].

Definition 1.2 Dilation $\Delta_\varepsilon^{(r_1,\ldots,r_n)}$ is mapping dependent on dilation coefficients $(r_1, \ldots, r_n)$, which assigns to every real number $\varepsilon > 0$ a diffeomorphism

$$\Delta_\varepsilon^{(r_1,\ldots,r_n)}(x_1, \ldots, x_n) = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) \tag{1.10}$$

where $x_1, \ldots, x_n$ are suitable coordinates on $\mathbb{R}^n$ and $r_1, \ldots, r_n$ are positive real numbers.

A scalar function $\varphi(x_1, \ldots, x_n)$ is called homogeneous of degree $\sigma \in \mathbb{R}$ w. r. t. the dilation $\Delta_\varepsilon^{(r_1,\ldots,r_n)}$ if there exists $\sigma \in \mathbb{R}$ such that

$$\varphi(\Delta_\varepsilon^{(r_1,\ldots,r_n)}(x_1, \ldots, x_n)) = \varepsilon^\sigma \varphi(x_1, \ldots, x_n). \tag{1.11}$$

A vector field $f(x) = [f_1(x), \ldots, f_n(x)]^T$ is called homogenous of degree $k \in \mathbb{R}$ w. r. t. the dilation $\Delta_\varepsilon^{(r_1,\ldots,r_n)}$ if there exists $k \in \mathbb{R}$ such that

$$f_i(\Delta_\varepsilon^{(r_1,\ldots,r_n)}(x_1, \ldots, x_n)) = \varepsilon^{k+r_i} f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{1.12}$$

System $\dot{x} = f(x)$ is called homogeneous if its vector field f is homogeneous.

Lemma 1.2 *Suppose that system $\dot{x} = f(x)$ is homogeneous of degree $k < 0$ w. r. t. dilation $\Delta_\varepsilon^{(r_1,\ldots,r_n)}$, in which f is continuous and $x = 0$ is its asymptotically stable equilibrium. Then, the equilibrium of this system $\dot{x} = f(x)$ is globally finite-time stable.*

Lemma 1.3 *Suppose that system (1.8) is homogeneous of degree $k < 0$ w. r. t. dilation $\Delta_\varepsilon^{(r_1,\ldots,r_n)}$, in which f is continuous and $x = 0$ is its asymptotically stable equilibrium. Then, the equilibrium of system (1.8) is globally finite-time stable. Moreover, if $\tilde{f}(x)$ is a continuous vector field defined on $\mathbb{R}^n$ such that*

$$\lim_{\varepsilon \to 0} \frac{f_i(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)}{\varepsilon^{k+r_i}} = 0, \quad i = 1, \ldots, n, \quad \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)^T \tag{1.13}$$

uniformly for $\|x\| = 1$, then the zero solution of

$$\dot{x} = f(x) + \tilde{f}(x), \quad \tilde{f}(0) = 0 \tag{1.14}$$

is (locally) finite-time stable.

By merging the theories of semi-stability and finite-time stability, Hui et al. [110] developed a rigorous framework for finite-time semi-stability, in which sufficient and necessary conditions for finite-time semi-stability of homogeneous systems were addressed by exploiting the fact that a homogeneous system is finite-time semi-stable if and only if it is semi-stable and has a negative degree of homogeneity. Upon using the finite-time control technique and homogenous systems theory, Du et al. [111] addressed the finite-time synchronization problem for a class of nonlinear leader–follower multi-agent systems with second-order dynamics by constructing a finite-time state feedback controller and a finite-time observer. With the aid of homogeneous theory, Zhang and Yang [112] studied the finite-time consensus control problems of second-order multi-agent systems with one and multiple leaders under a directed graph and established some sufficient conditions for the achievement of the finite-time consensus tracking. Based on the matrix theory, graph theory, homogeneity with dilation, and LaSalle's invariance principle, Guan et al. [113] investigated the finite-time consensus tracking control problem for second-order networked multi-agent systems under fixed and switched topologies, and designed finite-time consensus protocols by the pinning control technique without the assumption that the topology graph is connected or the leader is globally reachable. Based on homogeneous theory, Lu et al. [114] considered finite-time consensus tracking control problems for multi-agent systems with a virtual leader under fixed and switching topologies, respectively, in which an improved consensus tracking protocol was proposed in that the chattering phenomenon occurred in non-Lipschitz dynamical systems can be eliminated by introducing a saturation function to replace the original sign function. By using tools from homogeneous theory, Zhao et al. [115] addressed the finite-time consensus tracking problem of networked multiple agents with Euler–Lagrange dynamics. Based on the homogeneous control method, finite-time position consensus and collision avoidance problems were investigated for multiple autonomous underwater vehicle (multi-AUV) systems by Li and Wang in [116]. Lu et al. [117] considered the finite-time consensus tracking problems for double-integrator multi-agent systems with bounded control input by using homogeneous theory. By applying the homogeneous theory to stability analysis, the work in [118] by Zhang et al. was devoted to the distributed finite-time observers for multi-agent systems with input saturations and without velocity measurements. The work in [119] by Yu et al. was concerned with the finite-time consensus problem of distributed agents having nonidentical unknown nonlinear dynamics, to a leader agent that also has unknown nonlinear control input signal, in which homogeneous Lyapunov functions and homogeneous vector fields were introduced in the stability analysis although the whole system is not homogeneous.

Adding power integrator-based control. Adding a power integrator, as a feedback design tool, was first introduced by Caron in [120] for the stabilization problem and then was applied extensively by Lin and Qian [121–123] to solve the problem of stabilization for uncertain nonlinear systems with unmatched uncertainties. Based on adding a power integrator method, Li et al. [124] addressed the finite-time consensus control problems for linear multi-agent systems with second-order dynamics and unknown bounded disturbances in two cases: without a leader and with one leader, and showed that the disturbance rejection property of the closed-loop system can be enhanced by adjusting the fractional power in the continuous non-smooth finite-time consensus controller. However, the communication topology considered in [124] is restricted to be an undirected connected graph, and the effect of communication delays is not considered. By integrating the non-smooth finite-time control technique and adding a power integrator method, Du et al. [125] investigated the robust consensus control problem for second-order multi-agent systems subject to external disturbances for more general cases: (i) The communication topology is a directed graph which is more general than that of [124]; (ii) there exists communication delay between the agents which is not considered in [124]. On the basis of the work in [124], Du, Li, and Lin, by applying the adding a power integrator technique and a finite-time observer, further studied the finite-time formation control of multiple second-order agents via dynamic output feedback under the assumption that the velocities of all agents cannot be measured. Wang et al. [127] further extended the finite-time consensus tracking method proposed in [124] to the finite-time containment control for second-order linear multi-agent systems with multiple leaders under undirected topology. With the adding a power integrator method, Huang et al. [128] considered the finite-time leaderless consensus for a group of nonlinear mechanical systems under undirected topology with a backstepping-like adaptive control scheme, in which the nonlinearities in the system are linearly parameterized. Upon using the non-smooth finite-time control method and adding a power integrator technique, Ou, Du, and Li further addressed the finite-time consensus tracking control problem for multiple non-holonomic wheeled mobile robots in dynamic model in [129]. With the aid of the adding a power integrator technique, Wang et al. [130] investigated the problem of achieving rotating formation and containment simultaneously via finite-time control schemes for second-order linear multi-agent systems, in which the desired rotating formation structure is time-varying. By integrating the fractional state feedback method and adding a power integrator technique, Wang et al. [131] considered the finite-time leaderless consensus problem for a group of nonlinear mechanical systems under single-way directed interaction topology and unknown actuation failures, in which the resultant control gain of the system is unknown and time-varying, making the control impact on the system uncertain and the finite-time control synthesis non-trivial. In [132], Wang et al. further extended the finite-time leaderless consensus control methods for second-order nonlinear multi-agent systems with non-parametric uncertainties to high-order case under directed topology by using the adding a power integrator technique associated with high-order case. Based on fractional power feedback of local error and the adding a power integrator technique, Wang et al. proposed a finite-time distributed adaptive control solution to the formation–containment problem for multiple networked nonlinear systems with uncertainties and directed communication constraints in [133].

Time-varying gain-based regular state feedback control. The time-varying gain-based regular state feedback finite-time control was first introduced by Song et al. in [70] for the robust regulation of normal-form nonlinear systems in prescribed finite time and then was applied extensively in [134] to solve the pre-specified finite-time leader-following control problem of high-order linear multi-agent systems under directed graphs. This finite-time control method is based on the time-varying control gain together with regular state feedback, and thus the control input signal is not only continuous but also C^1 smooth. Particularly, the convergence time is not only finite, but can be pre-specified by the designer due to the independence of any other design parameters and initial conditions. For more detailed results on the prescribed finite-time control, the authors can be referred to [71].

1.3 Notes

As finite-time convergence is of special interest to many important applications that involve multiple agent systems (MASs) [95], the past few years have witnessed sustained growing interest in finite-time control of MAS, leading to fruitful results on finite-time consensus and/or containment of MAS in the literatures; see [68–133]. It is noted that, however, most existing finite-time control methods cannot guarantee the convergence within pre-specified finite time in that the actual converging time period is not uniform nor pre-specifiable. This is because the finite time T^* is determined by $T^* \leq \frac{V(t_0)^{1-\alpha}}{\gamma(1-\alpha)}$ [96], with $\gamma > 0$ and $0 < \alpha < 1$ being design parameters, and $V(t_0)$ being the Lyapunov function of system initial states, from which it is seen that the finite time T^* depends on both the initial conditions and a set of design parameters, and thus cannot be pre-specified arbitrarily. In other words, for a prescribed T^*, it is non-trivial to determine the corresponding value for the crucial parameters α, γ, etc.

Furthermore, several other constraints are imposed in the existing finite-time control methods. For instance, in [72–79], signum function-based discontinuous control has to be used to achieve finite-time consensus of MAS under certain topology conditions, which makes the control action discontinuous. The methods suggested in [124–133] involve the fractional power state feedback, rendering non-smooth control action. So far, as summarized in the above analysis, most existing finite-time control schemes for MAS are either discontinuous or non-smooth, and the problem of smooth distributed control for networked multiple systems (especially for the nonlinear multi-agent systems with unknown uncertainties) featured with uniform pre-specified finite settling time has remained almost untouched, which represents an interesting topic for future research.

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Chapter 2
Preliminaries

This chapter introduces algebraic graph theory, matrix analysis theory on graphs, stability analysis theory on cooperative control systems, and the theory of finite-time stability analysis.

2.1 Algebraic Graph Theory

For a group of agents interacting with each other through communication network, it is natural to model the interaction among agents by a directed weighted graph. Suppose that there are N agents in the group. A directed graph with N nodes (agents) is represented by $\mathscr{G} = (\iota, \varepsilon)$, in which $\iota = \{\iota_1, \iota_2, \ldots, \iota_N\}$ denotes a nonempty finite set of nodes (agents) and $\varepsilon \subseteq \iota \times \iota$ denotes the set of edges between two distinct nodes. A weighted graph denotes that a graph associates a weight with every edge in the graph. A directed edge is represented by an ordered pair of distinct nodes $\varepsilon_{ij} = (\iota_i, \iota_j)$, where ι_i is called the parent (terminal) node and ι_j is called the child (initial) node, meaning that node ι_i can receive information from node ι_j, but not necessarily *vice versa*. We assume the graph is simple, i.e., no self loops ($(\iota_i, \iota_i) \notin \varepsilon$, $\forall i \in \{1, \ldots, N\}$) and no multiple edges between the same pairs of nodes. Node ι_j is an in-neighbor of node ι_i if ι_i can obtain information from ι_j, i.e., $(\iota_j, \iota_i) \in \varepsilon$. The set of in-neighbors of node ι_i is defined as $\mathscr{N}_i = \{\iota_j \in \iota | (\iota_j, \iota_i) \in \varepsilon\}$. In an undirected graph, the pairs of nodes are unordered, in which the edge (ι_i, ι_j) corresponds to the edges (ι_i, ι_j) and (ι_j, ι_i) in the directed graph. An undirected graph is actually a special case of a directed graph. A directed path is denoted by a sequence of ordered edges $(\iota_{i_1}, \iota_{i_2})$, $(\iota_{i_2}, \iota_{i_3})$,..., where $\iota_{i_k} \in \iota$. A cycle refers to a directed path starting and ending at the same node. A directed graph is *strongly connected* if there is a directed path from every node to any other distinct node. For an undirected graph, it is said to be *connected* if there is an undirected path between any pair of distinct nodes, and further, if there exists an edge between any pair of distinct nodes, this undirected graph is said to be *fully connected*. A directed graph is *complete* if there is an edge from every node to any other node. A *root* is a node that has no parent and

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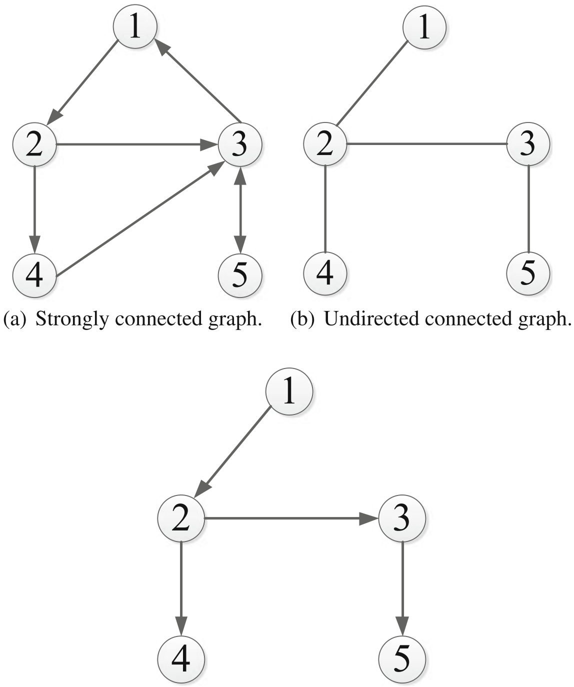

(a) Strongly connected graph. (b) Undirected connected graph.

(c) Directed graph containing a spanning tree.

Fig. 2.1 Different network graph topologies. Notice that graph **c** contains a directed spanning tree with 1 being the root node but is not strongly connected because nodes 2, 3, 4, and 5 do not have directed paths to all other nodes

has directed paths to all other nodes in a directed graph. A *directed tree* is a directed graph where every node has exactly one parent except for the root. A directed tree has no cycle because every edge is oriented away from the root. In the undirected graph, a *tree* refers to a graph where each pair of nodes is connected by exactly one undirected path. A directed spanning tree is a directed tree formed by graph edges that connects all the nodes of the graph. In a graph if a subset of the edges forms a directed tree, the graph is said to have a spanning tree. A graph may have multiple spanning trees. If a graph is strongly connected (Fig. 2.1a, b), then it contains at least one spanning tree (Fig. 2.1c). The in-degree of ι_i is the number of nodes ι_i can obtain information from whom. The out-degree of ι_i is the number of nodes who can obtain information from ι_i. If the in-degree of a node ι_i $(i = 1, \ldots, N)$ is equal to its out-degree, the node is balanced. If all nodes ι_i in a graph are balanced, the graph is balanced.

2.2 Matrix Analysis on Graphs

The graph structure and properties can be studied by examining the properties of the graph topology of the graph. This is known as algebraic graph theory [1, 2].

Denote by a_{ij} the weight for the edge $\varepsilon_{ij} \in \varepsilon$, then the graph topology of the network $\mathscr{G}$ can be represented by a weighted adjacency matrix $\mathscr{A} = [a_{ij}]$, where $\varepsilon_{ij} \in \varepsilon \Leftrightarrow a_{ij} > 0$, and otherwise, $a_{ij} = 0$. Self-edges are not allowed, that is, $a_{ii} = 0$ for all i, unless otherwise indicated. If the weight is not relevant, then a_{ij} is set equal to 1 if $\varepsilon_{ij} \in \varepsilon$. In an undirected graph, the adjacency matrix $\mathscr{A}$ is symmetric, i.e., $\mathscr{A} = \mathscr{A}^T$.

The weighted in-degree of node i is denoted by the i-th row sum of $\mathscr{A}$, i.e., $d_i = \sum_{j=1}^{N} a_{ij}$, and the weighted out-degree of node ι_i is denoted by the i-th column sum of A, i.e., $d_i^0 = \sum_{j=1}^{N} a_{ji}$. Note that $\sum_{j=1}^{N} a_{ij} = \sum_{j \in \mathscr{N}_i} a_{ij}$. For a graph, if $d_i = d_i^0$ for all i, the graph is weight balanced. For brevity, we sometimes simply refer to node ι_i as node i, and refer simply to in-degree, out-degree, and the balanced property, without the qualifier 'weight'.

Definition 2.1 ([3]) The weighted Laplacian graph matrix is denoted by

$$\mathscr{L} = [l_{ij}] = \mathscr{D} - \mathscr{A} \in \mathbb{R}^{N \times N},$$

where $\mathscr{D}$ is the diagonal in-degree matrix defined as

$$\mathscr{D} = diag(d_1, \ldots, d_N) \in \mathbb{R}^{N \times N}.$$

The Laplacian matrix is of extreme importance in the study of dynamical multi-agent systems on graphs. For an undirected graph, $\mathscr{L}$ is symmetric and is called the Laplacian matrix. For a directed graph, $\mathscr{L}$ is not necessarily symmetric and is sometimes called the nonsymmetric Laplacian matrix [4] or directed Laplacian matrix [5].

Lemma 2.1 ([6]) *Let $\mathscr{L}$ be the nonsymmetric Laplacian matrix (Laplacian matrix) of the directed graph $\mathscr{G}$ (undirected graph $\mathscr{G}$). Then $\mathscr{L}$ has at least one zero eigenvalue and all its nonzero eigenvalues have positive real parts (are positive). Furthermore, $\mathscr{L}$ has a simple zero eigenvalue and all its other eigenvalues have positive real parts (are positive) if and only if $\mathscr{G}$ has a directed spanning tree (is connected). In addition, $\mathscr{L} 1_N = 0_N$ and there exists a nonnegative vector $p \in \mathbb{R}^N$ satisfying $p^T \mathscr{L} = 0_{1 \times N}$ and $p^T 1_N = 1$.*

For an undirected graph $\mathscr{G}$, it holds that $x^T \mathscr{L} x = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(x_i - x_j)^2$, where $x = [x_1, \ldots, x_N]^T$.

Lemma 2.2 ([6]) *Let $\mathscr{L}$ be the nonsymmetric Laplacian matrix of the directed graph $\mathscr{G}$. If $\mathscr{G}$ is balanced, then $x^T \mathscr{L} x \geq 0$. If $\mathscr{G}$ is strongly connected and balanced (containing undirected and connected graph as a special case), then $\mathscr{L} x = 0_N$ or $x^T \mathscr{L} x = 0$ if and only if $x_i = x_j$ for all $i, j \in \{1, \ldots, N\}$.*

Let $\lambda_1(\mathscr{L}) \leq \lambda_2(\mathscr{L}) \leq \cdots \leq \lambda_N(\mathscr{L})$ be the N eigenvalues of $\mathscr{L}$ associated an undirected graph $\mathscr{G}$, so that $\lambda_1(\mathscr{L}) = 0$ and $\lambda_2(\mathscr{L})$ denotes the algebraic connectivity, which is positive if and only if $\mathscr{G}$ is connected according to Lemma 2.1. The algebraic connectivity quantifies the convergence rate of consensus algorithms [6].

Lemma 2.3 ([7]) *Suppose that* $z = [z_1^T, \ldots, z_N^T]^T$ *with* $z_i \in \mathbb{R}^m$. *Let* $\mathscr{A}$ *and* $\mathscr{L}$ *be the adjacency matrix and nonsymmetric Laplacian matrix associated with the directed graph* $\mathscr{G}$ *of order* N*, respectively. Then the following five conditions are equivalent:*

(i) The directed graph $\mathscr{G}$ *has a directed spanning tree;*
(ii) The rank of $\mathscr{L}$ *is* $N - 1$*;*
(iii) $\mathscr{L}$ *has a simple zero eigenvalue with an associated eigenvector* 1_N *and all other eigenvalues have positive real parts;*
(iv) $(\mathscr{L} \otimes I_m)z = 0_{Nm}$ *if and only if* $z_1 = \cdots = z_N$*;*
(v) Consensus is reached for the closed-loop system $\dot{z} = -(\mathscr{L} \otimes I_m)z$.

For the leader–follower MAS containing 1 leader and N followers, where a leader is an agent without in-neighbors and a follower is an agent that has at least one in-neighbor, let $\bar{\mathscr{G}}$ be the corresponding directed graph between the 1 leader, labeled as agent 0, and N followers, labeled as agents or followers 1 to N. Let $\bar{\mathscr{A}}$ and $\bar{\mathscr{L}}$ be the adjacency matrix and the nonsymmetric Laplacian matrix (Laplacian matrix) associated with $\bar{\mathscr{G}}$, respectively. In addition, let $\mathscr{G}$, $\mathscr{A}$, and $\mathscr{L}$ be, respectively, the directed graph between the N followers, the adjacency matrix, and the nonsymmetric Laplacian matrix (Laplacian matrix) associated with $\mathscr{G}$. In such case, $\bar{\mathscr{A}} = \begin{bmatrix} 0 & 0_{1\times N} \\ \bar{a} & \mathscr{A} \end{bmatrix}$, where $\bar{a} = [a_{10}, \ldots, a_{N0}]^T$ and $a_{i0} > 0$ $(i = 1, \ldots, N)$ if agent 0 is an in-neighbor of agent i and $a_{i0} = 0$ otherwise, and $\bar{\mathscr{L}} = \begin{bmatrix} 0 & 0_{1\times N} \\ \bar{a} & \mathscr{H} \end{bmatrix}$, where $\mathscr{H} = \mathscr{L} + \mathrm{diag}\{a_{10}, \ldots, a_{N0}\}$.

Lemma 2.4 ([6]) *Let* $\mathscr{G}$ *and* $\mathscr{H}$ *be defined the same as in the above, if* $\mathscr{G}$ *is directed (undirected), all eigenvalues of* $\mathscr{H}$ *have positive real parts (*$\mathscr{H}$ *is symmetric positive definite) if and only if the leader in the directed graph* $\bar{\mathscr{G}}$ *has directed paths to all followers.*

2.3 Matrix and Algebra Theory Background

Definition 2.2 ([8]) A Hurwitz matrix or a stability matrix is a matrix $A \in \mathbb{R}^{N\times N}$ satisfying that all its eigenvalues have negative real parts.

Definition 2.3 ([9]) The M-matrix is a matrix of the form

$$B = \alpha I_N - C,$$

where $\alpha \in \mathbb{R}_+$, $C \in \mathbb{R}^{N\times N}$ is semi-positive definite, satisfying $\rho(C) \leq \alpha$ ($\rho(C)$ is the modulus of C). Further, if $\rho(C) < \alpha$, the matrix B is a nonsingular M-matrix.

Lemma 2.5 ([9]) *Let* $\mathbb{Z}^{N\times N} = \{B = [b_{ij}] \in \mathbb{R}^{N\times N} | b_{ij} \leq 0, i \neq j\}$. *A matrix* $B \in \mathbb{Z}^{N\times N}$ *is a nonsingular* M*-matrix if and only if* B^{-1} *exists and* $B^{-1} \geq 0$.

Lemma 2.6 ([10] Gershgorin's disk theorem) *Let* $A = [a_{ij}] \in \mathbb{R}^{N\times N}$*, and*

$$R_i^{'}(A) = \sum_{j=1, j\neq i}^{N} |a_{ij}|, \quad i = 1, \ldots, N,$$

be the deleted absolute row sums of A*. Then all eigenvalues of* A *are located in the union of* N *disks*

$$\cup_i^N \{z \in C : |z - a_i i| \leq R_i^{'}(A)\}.$$

Furthermore, if a union of k *of these* N *disks forms a connected region that is disjoint from all of the remaining* $N - k$ *disks, then in this region there are precisely* k *eigenvalues of* A*.*

Definition 2.4 ([11]) Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B is

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Lemma 2.7 ([11]) *Let* A*,* B*,* C*, and* D *be matrices with appropriate dimensions and* $k \in \mathbb{R}$*, then the following equalities hold.*

(1) $k(A \otimes B) = (kA) \otimes B = A \otimes (kB)$*;*
(2) $A \otimes (B + C) = A \otimes B + A \otimes C$*;* $(B + C) \otimes A = B \otimes A + C \otimes A$*;*
(3) $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$*;*
(4) $(A \otimes B) \otimes C = A \otimes (B \otimes C) = A \otimes B \otimes C$*;*
(5) $(A \otimes B)^T = A^T \otimes B^T$*;*
(6) $(A \otimes B)(C \otimes D) = AC \otimes BD$*.*

Lemma 2.8 ([6] Holder inequality) *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for all vectors* $f \in \mathbb{R}^k$ *and* $g \in \mathbb{R}^k$*, it holds that*

$$|f^T g| \leq \|f\|_p \|g\|_q.$$

Lemma 2.9 ([12]) *For any* $x, y \in \mathbb{R}^N$ *and any symmetric positive-definite matrix* $A \in \mathbb{R}^{N\times N}$*, it holds that*

$$2x^T y \leq x^T A^{-1} x + yAy.$$

Lemma 2.10 ([10] Rayleigh-Ritz theorem) *Let* $A \in \mathbb{R}^{N\times N}$ *be a symmetric matrix. Then for all* $x \in \mathbb{R}^N$*, it holds that*

$$\lambda_{\min}(A)x^T x \le x^T Ax \le \lambda_{\max}(A)x^T x,$$

where $\lambda_{\min}(A) = \min_{x\neq 0_N} \frac{x^T Ax}{x^T x} = \min_{x^T x=1} \frac{x^T Ax}{x^T x}$, $\lambda_{\max}(A) = \max_{x\neq 0_N} \frac{x^T Ax}{x^T x} = \max_{x^T x=1} \frac{x^T Ax}{x^T x}$.

Lemma 2.11 ([13] Young's inequality) *Let* $a, b \ge 0$ *and* $c > 0$*, then it holds that*

$$ab \le \frac{a^{1+c}}{1+c} + \frac{cb^{1+\frac{1}{c}}}{1+c}.$$

Lemma 2.12 ([14]) *Let* $a_1, a_2 > 0$ *and* $0 < c < 1$*, then it holds that*

$$(a_1 + a_2)^c \le a_1^c + a_2^c.$$

Lemma 2.13 ([14]) *Let* $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$ *and* $0 < p < 2$*, then the following inequality holds,*

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^p \le (a_1^p + a_2^p + \cdots + a_n^p)^2.$$

Lemma 2.14 ([15]) *For any* $\varepsilon \in \mathbb{R}_+$ *and* $x \in \mathbb{R}$*, it holds that*

$$0 \le |x| - x\tanh(\frac{x}{\varepsilon}) \le k\varepsilon, \quad k = 0.2785.$$

Lemma 2.15 ([16]) *If* $h = h_2/h_1 \ge 1$*, where* h_1, h_2 *are positive odd integers, then it holds that* $|x - y|^h \le 2^{h-1}|x^h - y^h|$.

Lemma 2.16 ([16]) *If* $0 < h = h_1/h_2 \le 1$*, where* $h_1, h_2 > 0$ *are odd integers, then* $|x^h - y^h| \le 2^{1-h}|x - y|^h$.

Lemma 2.17 ([16]) *If* $c, d > 0$*, then for any* $x, y \in \mathbb{R}$, $|x|^c|y|^d \le c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.

Lemma 2.18 ([17]) *If* $0 < h \le 1$*, then for* $x_i \in \mathbb{R}$ $(i = 1, 2, \ldots, N)$*, it holds that* $(\sum_{i=1}^N |x_i|)^h \le \sum_{i=1}^N |x_i|^h \le N^{1-h}(\sum_{i=1}^N |x_i|)^h$.

2.4 Stability Analysis Theory on Cooperative Control Systems

In the study of dynamical systems, different kinds of stability problems arise. This section is mainly concerned with the stability theory of networked cooperative control systems.

To investigate the stability of networked cooperative systems, some basic definitions and concepts are introduced as follows.

Consider a networked cooperative system consisting of N subsystems,

$$\dot{x}_i = f_i(x, t), \quad i = 1, \ldots, N \tag{2.1}$$

where $x_i \in \mathbb{R}^m$ denotes the state of subsystem i ($i = 1, \ldots, N$), $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^{mN}$, $f_i : \mathbb{D} \times \mathbb{R}_+ \to \mathbb{R}^m$ ($\mathbb{D} \subset \mathbb{R}^{mN}$ is a domain contains the origin) is locally Lipschitz in x and piecewise continuous in t on $\mathbb{D} \times \mathbb{R}_+$.

For the networked cooperative systems without a leader, the neighborhood error is defined as

$$e_i = \sum_{i \in \mathcal{N}_i} a_{ij}(x_i - x_j), \quad i = 1, \ldots, N. \tag{2.2}$$

For the networked cooperative system with N followers and 1 leader, labeled 0, the neighborhood error is defined as

$$e_i = \sum_{i \in \mathcal{N}_i} a_{ij}(x_i - x_j) + a_{i0}(x_i - x_0), \quad i = 1, \ldots, N, \tag{2.3}$$

and the tracking error (or disagreement variable) is defined as

$$\varpi_i = x_i - x_0, \quad i = 1, \ldots, N. \tag{2.4}$$

Denote by $e = [e_1, \ldots, e_N]^T$ and $\varpi = [\varpi_1, \ldots, \varpi_N]^T$.

In the following, we define the concepts of stability, asymptotic stability, exponential stability, and uniform ultimately boundedness for networked cooperative system (2.1), which extend the standard concepts of stability for nonautonomous system in [8].

Definition 2.5 The networked cooperative system (2.1) with (without) a leader is
• stable (Lyapunov stable) if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that

$$\|e(t_0)\| < \delta \Rightarrow \|e(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0. \tag{2.5}$$

• uniformly stable if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ (independent of t_0), such that (2.5) is satisfied.
• unstable if it is not stable.
• asymptotically stable if it is stable and there exists a finite constant $c = c(t_0) > 0$ such that

$$\lim_{t \to \infty} e(t) = 0, \quad \forall\, \|e(t_0)\| < c. \tag{2.6}$$

• uniformly asymptotically stable if it is uniformly stable and there exists a finite constant $c > 0$ (independent of t_0), such that

$$\lim_{t\to\infty} e(t) = 0, \quad \forall \, \|e(t_0)\| < c \tag{2.7}$$

• uniformly in t_0. That is, for each $\eta > 0$, there exists $T = T(\eta) > 0$ such that

$$\|e(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \quad \forall \, \|e(t_0)\| < c. \tag{2.8}$$

• globally uniformly asymptotically stable if it is uniformly stable, and $\delta(\varepsilon)$ can be chosen such that $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, and, for every finite $\eta > 0$ and $c > 0$, there exists $T = T(\eta, c) > 0$ such that

$$\|e(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \quad \forall \, \|e(t_0)\| < c. \tag{2.9}$$

Definition 2.6 The networked cooperative system (2.1) with (without) a leader is
• exponentially stable, if there exist finite constants $c, k, \lambda > 0$ such that

$$\|e(t)\| \leq k\|e(t_0)\| \exp^{-\lambda(t-t_0)}, \quad \forall \, \|e(t_0)\| < c. \tag{2.10}$$

• globally exponentially stable if (2.10) is satisfied for any given initial state $x(t_0)$.

Definition 2.7 The networked cooperative system (2.1) with (without) a leader is
• uniformly bounded if there exists a finite constant $c > 0$ (independent of t_0), and for any $a \in (0, c)$, there exists $\beta = \beta(a) > 0$ (independent of t_0), such that

$$\|e(t_0)\| < a \Rightarrow \|e(t)\| < \beta, \quad \forall \, t \geq t_0 \geq 0. \tag{2.11}$$

• globally uniformly bounded if (2.11) is satisfied for arbitrarily large a.
• uniformly ultimately bounded (UUB) with ultimate bound b if there exist finite constants $b, c > 0$ (independent of t_0), and for any $a \in (0, c)$, there exists $T = T(a, b) \geq 0$ (independent of t_0), such that

$$\|e(t_0)\| < a \Rightarrow \|e(t)\| < b, \quad \forall \, t \geq t_0 + T. \tag{2.12}$$

• globally UUB if (2.12) is satisfied for arbitrarily large a.

Definition 2.8 The networked cooperative system (2.1) with (without) a leader is
• locally finite-time stable if it is Lyapunov stable, and for any $x(t_0) \in \mathbb{D}_0$ ($\mathbb{D}_0 \subset \mathbb{D}$) there exists a settling time $t_0 < T < \infty$ such that

$$\begin{aligned} \lim_{t\to T} e(t) &= 0, \\ e(t) &= 0, \quad \forall t \geq T. \end{aligned} \tag{2.13}$$

• globally finite-time stable if (2.13) holds for any given initial state $x(t_0)$.

In the above, we have introduced the concepts of stability of networked cooperative systems. Our task now is to introduce the Lyapunov analysis method to determine stability.

Definition 2.9 ([8]) A continuously differentiable (C^1) function $V : \mathbb{D} \to \mathbb{R}$ is said to be a Lyapunov function if it satisfies

$$V(0) = 0 \ \text{ and } \ V(x) > 0 \ \text{ in } \ \mathbb{D} - \{0\} \tag{2.14}$$

$$\dot{V}(x) \leq 0. \tag{2.15}$$

Lemma 2.19 ([8]) *Consider system (2.1). If there exists a C^1 Lyapunov function $V(x) : \mathbb{D} \to \mathbb{R}_+$ such that*

$$\dot{V}(x) < 0 \tag{2.16}$$

for $x \in \mathbb{D} - \{0\}$, where $\dot{V}(x)$ is the derivative along the system trajectory, i.e., $\dot{V}(x) = \frac{\partial V}{\partial x}(x) f(x)$, then $V(x) \to 0$ as $t \to \infty$, and system (2.1) is asymptotically stable.

Lemma 2.20 ([8]) *Consider system (2.1). If there exist a C^1 Lyapunov function $V(x) : \mathbb{D} \to \mathbb{R}_+$, and some real constants $0 < c, d < \infty$, such that*

$$\dot{V}(x) \leq -cV(x) + d \tag{2.17}$$

then system (2.1) is uniformly ultimately bounded (UUB).

Lemma 2.21 ([18]) *Consider system (2.1). If there exists a C^1 Lyapunov function $V(x) : \mathbb{D} \to \mathbb{R}_+$, and some constants $0 < c < \infty$ and $0 < \alpha < 1$, such that*

$$\dot{V}(x) \leq -cV(x)^{\alpha} \tag{2.18}$$

then system (2.1) is finite-time stable for any given $x(t_0) \in D_0 \subset D$, in which the finite settling time T^* satisfies

$$T^* \leq \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)}. \tag{2.19}$$

Lemma 2.22 ([19]) *Consider system (2.1). If there exist a C^1 Lyapunov function $V(x) : \mathbb{D} \to \mathbb{R}_+$, and also some positive constants c, d, and $0 < \alpha < 1$, such that*

$$\dot{V}(x) \leq -cV(x)^{\alpha} + d, \tag{2.20}$$

then system (2.1) is stable within a finite settling time T^* satisfying

$$T^* \leq \frac{V(x_0)^{1-\alpha}}{c\theta_0(1-\alpha)}, \tag{2.21}$$

for any given $x(t_0) \in \mathbb{D}_0 \subset \mathbb{D}$*, where* $0 < \theta_0 < 1$ *is a real constant. Moreover, when* $t \geq T^*$*, the state trajectory convergent to a compact set,*

$$\Theta = \left\{ x \middle| V(x) \leq \frac{d}{c(1-\theta_0)^{\frac{1}{\alpha}}} \right\}. \tag{2.22}$$

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Part II
Infinite-Time Cooperative Adaptive Control of Multiple Networked Nonlinear Systems

Chapter 3
Lyapunov Analysis for Cooperative Adaptive Consensus Under Undirected Graph

For the distributed adaptive cooperative control of nonlinear multi-agent systems subject to unknown time-varying gain and non-parametric uncertainties, it requires the use of special Lyapunov functions that depend on the graph Laplacian and parametric estimation error in a certain way. The technical difficulties in the cooperative adaptive controller design and stability analysis for such uncertain systems mainly arise from: (1) the time-varying yet unknown control gain; (2) the non-parametric uncertainties; and (3) the local communication constraints. One of the key approaches to solve these difficulties is to select special Lyapunov functions for the cooperative adaptive control design that depends on the graph topology and parameter estimation error in special ways. In this chapter, we will show how to construct the Lyapunov function that is related to the graph topology and parameter estimation error. In addition, some key concepts needed for the distributed adaptive controller design and stability analysis on graph are introduced, such as the virtual parameter estimation error technique; the core function concept; and distributed Lyapunov function design dependent of the graph topology.

In Sect. 3.1, we formulate the leaderless consensus problem on a group of networked dynamical systems subject to unknown time-varying gain and non-parametric/non-vanishing uncertainties. Section 3.2 introduces the concepts of the core function and the virtual parameter estimation error. Section 3.3 addresses the problem of how to construct the Lyapunov function for the stability analysis of cooperative MAS in the presence of unknown time-varying gain and non-parametric/non-vanishing uncertainties under fixed and undirected connected graph topology. The constructed Lyapunov function mainly depends on the graph topology and the

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parameter estimation errors, which consists of two parts: a distributed Lyapunov function part constituted by the local neighborhood error and graph Laplacian matrix, and an adaptive Lyapunov function part dependent of the parameter estimation errors. Section 3.4 presents the basic results and key ideas of the adaptive distributed leaderless consensus controller design and stability analysis for the continuous-time MAS subject to unknown and non-parametric uncertainties under the undirected topology condition. The importance of the key techniques described earlier, including the core function, the virtual parameter estimation error, and the Lyapunov function design is shown in Sect. 3.4. These studies open the way for many well-known results from systems theory to be extended to the case of multi-agent systems on graphs.

3.1 Leaderless Consensus Formulation

Autonomous control of networked MAS has become an interesting topic for research due to the potential for many important applications. Cooperative consensus is known as the basic issue in the cooperative control of MAS, which is a study dedicated to ensuring an agreement between states or output variables among networked MAS. Consider a group of N networked agents subject to non-parametric uncertainties modeled by

$$\dot{x}_i(t) = g_i(\cdot)u_i(t) + f_i(z_i(t), t) + f_{di}(z_i(t), t) \tag{3.1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i is the control gain, f_i denotes the system nonlinearities, f_{di} denotes the non-vanishing uncertainties including the external disturbance, and $z_i = \bigcup_{j \in \mathcal{N}_i \bigcup i} x_j$.

For ease of notation, it is assumed that the agents' states are scalars ($x_i \in \mathbb{R}$), and the case for the vector states ($x_i \in \mathbb{R}^m$, $m > 1$) can be derived similarly by using standard methods involving the Kronecker products employed in various chapters in the book.

It is assumed that the nonlinear dynamics $f_i(\cdot)$ ($i = 1, 2, \ldots, N$) are unknown and thus cannot be available for the controller design. Furthermore, $f_i(\cdot)$ ($i = 1, 2, \ldots, N$) considered in this chapter are non-parametric uncertainties; that is, $f_i(\cdot)$ ($i = 1, 2, \ldots, N$) cannot be linearly parametric decomposition or cannot be approximated by neural networks. For the control gain $g_i(\cdot)$, it is assumed that the control gain is unknown and possibly time-varying.

Remark 3.1 Note that each node has its own distinct dynamics, and both the node nonlinearities $f_i(\cdot)$ and the node non-vanishing uncertainties $f_{di}(\cdot)$ are unknown and non-identical. In addition, $f_i(\cdot)$ and $f_d i(\cdot)$ associated with the ith agent are not only related to the state of the ith agent itself but also influenced by its neighbors; that is, the coupling effects among the neighboring agents are considered in this model. Furthermore, the impact of the parameter variation of the agent is explicitly reflected in the model. Such variation literally leads to unknown and time-varying control gains in the model, thus making the control design and stability analysis much more involved as compared to the case of known and constant control gains.

The focus on this chapter is the controller design and stability analysis for the cooperative leaderless consensus problem of nonlinear MAS (3.1), in which both the unknown time-varying control gain and the non-parametric/non-vanishing modeling uncertainties are simultaneously addressed.

For each agent in the networked system, only local state information relative to its neighbors is obtainable, which means that, for node i, only local neighborhood synchronization error is available for controller design. Thus, we introduce the local neighborhood state error for node i as

$$e_i = \sum_{j \in \mathscr{N}_i} a_{ij}(x_i - x_j), \quad i = 1, \ldots, N. \tag{3.2}$$

Let $E = [e_1, \ldots, e_N]^T$, $X = [x_1, \ldots, x_N]^T$ such that $E = LX$.

3.2 Core Function and Virtual Parameter Estimation Error

In this section, we introduce the concepts of core function and virtual parameter estimation error, both of which are crucial for the controller design and stability analysis of the cooperative control of MAS with unknown time-varying control gain and non-parametric uncertainties.

Note that for the system with constant control gain and with nonlinearities that can be linearly parameterized or can be approximated by neural networks, there exists some unknown finite constant, p_i, and some known scalar function, $\phi_i(x_i)$, related to the states variables such that the nonlinear term can be linearly parametric decomposed as $f_i(\cdot) = p_i\phi_i(\cdot)$, or there exists some "optimal" vector $W_i \in \mathbb{R}^l$, activation function vector $\phi(x_i) \in \mathbb{R}^l$, and bounded error $\varepsilon(x_i) \in \mathbb{R}$ (l denotes the number of neural nodes), such that for x_i in some compact set, $f_i(\cdot)$ can be approximated by neural networks as $f_i(\cdot) = W_i^T\phi(\cdot) + \varepsilon(\cdot)$. For such nonlinear systems, one can perform the standard adaptive control method by constructing the parameter estimation error of the normal form $\tilde{\bullet} = \bullet - \hat{\bullet}$, where $\hat{\bullet}$ denotes the estimation of the unknown constant parameter $\bullet$, to obtain the system stability results. However, for the system that involves time-varying control gain as well as non-parametric uncertainties, the control problem becomes more challenging.

In the following, we introduce the concepts of core function and virtual parameter estimation error, with aid of both of which the obstacles caused by the unknown time-varying control gain and non-parametric uncertainties in the controller design and stability analysis for cooperative control of MAS can be circumvented without the need for linearization or approximation to the original systems.

To proceed, we should specify the assumptions on the unknown time-varying control gain and the system non-parametric/non-vanishing uncertainties. The following assumptions on system (3.1) are imposed.

Assumption 3.1 For system (3.1), it is assumed that the control gain g_i ($i = 1, 2, \dots, N$) is unknown, time-varying, and bounded away from zero; that is, there exist unknown finite constants $\underline{g}_i$ and $\bar{g}_i$ such that $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$, and g_i is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$), which means that the control direction is definite.

Assumption 3.2 For the non-parametric uncertainty $f_i(\cdot)$ ($i = 1, 2, \dots, N$), certain crude structural information on non-parametric nonlinearity $f_i(\cdot)$ is available to allow an unknown constant $c_{fi} \geq 0$ and a known scalar function $\varphi_i(z_i)$ to be extracted, such that $|f_i(\cdot)| \leq c_{fi}\varphi_i(\cdot)$ for $t \in [t_0, \infty)$, where $\varphi_i(\cdot)$ is bounded for any x_i ($i = 1, 2, \dots, N$). The non-vanishing uncertainty $f_{di}(\cdot)$ ($i = 1, 2, \dots, N$) is bounded by some unknown finite constants $\theta_i \geq 0$ such that $|f_{di}(\cdot)| \leq \theta_i < \infty$.

Remark 3.2 (1) Assumption 3.2 is related to the extraction of the deep-rooted information from the non-parametric nonlinearities $f_i(\cdot)$, where the constant c_{fi}, although associated with the system parameters, bears no physical meaning, thus named as virtual parameter. (2) We call $\varphi_i(z_i)$ as "core function" [1] for it contains the deep-rooted information of the system that is easily computable. For any practical system, the core function (which is non-unique) can be readily obtained with certain crude structural information on $f_i(\cdot)$, as seen from the later part. (3) To extract the core function $\varphi_i(z_i)$ from the $f_i(\cdot)$, one can perform upper normalization on $f_i(\cdot)$ [2]. Although this might render the virtual parameter overestimated, excessive control effort is avoided because the corresponding control algorithm does not use such parameter directly, as seen later.

Remark 3.3 By using the concept of deep-rooted (core) function, non-parametric uncertainties and time-varying parameters in the system can be gracefully handled. For instance, for the lumped uncertain function of the form

$$f_i(\cdot) = \rho_i(t)\cos(ax_i) + x_i\sin(bt + dx_i^2) + (x_i\zeta_i(t) + e_i^{-|hx_i|})^2 \tag{3.3}$$

where $|\rho_i(t)| \leq \bar{\rho} < \infty$ and $|\zeta_i(t)| \leq \bar{\zeta} < \infty$ are two unknown and time-varying parameters, and a, b, d, and h are unknown constants. Clearly, it is impossible to carry out parametric decomposition to separate the parameters $\rho_i(t), a, b, d, \zeta_i(t)$, and h from $f_i(\cdot)$. However, it is effortless to obtain the deep-rooted function of the form $\varphi_i(z_i) = 1 + |x_i| + x_i^2$ independent of those parameters, such that $f_i(\cdot) \leq c_{fi}\varphi_i(z_i)$, with c_{fi} being the virtual parameter defined by $c_{fi} = \max\{\bar{\rho} + 1, 2\bar{\zeta} + 1, \bar{\zeta}^2\}$. Note that $\varphi_i(z_i)$ is readily computable and non-unique.

It is interesting to note that the virtual parametric decomposition of the lumped uncertainty $f_i(\cdot)$ as imposed in Assumption 3.2 does exist if $f_i(\cdot)$ can be dealt with by NN-based method as stated in the following fact.

Fact 1: If the lumped uncertainty $f_i(\cdot)$ can be approximated by neural networks, then Assumption 3.2 holds naturally.

Proof of Fact 1: According to the universal approximation capability property [3], there exist some "optimal" vector $W_i \in \mathbb{R}^l$, activation function vector $\phi_i(z_i) \in \mathbb{R}^l$, and bounded error function $\varepsilon_i(z_i) \in \mathbb{R}$ (l denotes the number of neural nodes), such that for z_i in some compact set,

$$f_i(z_i) = W_i^T \phi_i(z_i) + \varepsilon_i(z_i) \tag{3.4}$$

it then follows that

$$|f_i(z_i)| \leq \|W_i\| \|\phi_i(z_i)\| + |\varepsilon_i(z_i)| \leq \|W_i\| \|\phi_i(z_i)\| + \bar{\varepsilon}_i = c_{fi} \varphi_i(z_i) \tag{3.5}$$

where $\bar{\varepsilon}_i$ is an unknown finite constant satisfying $|\varepsilon_i(z_i)| \leq \bar{\varepsilon}_i$, $c_{fi} = \max\{\|W_i\|, \bar{\varepsilon}_i\}$ is the virtual parameter needed to be estimated and $\varphi_i(z_i) = 1 + \|\phi_i(z_i)\|$ is the core function which is computable.

The significant implications of Fact 1 are threefold: (1) Assumption 3.2 is reasonable and justifiable; (2) any nonlinear systems with uncertainties that can be dealt with by NN-based technique can also be handled by the proposed method; (3) if little information on $f_i(\cdot)$ is available to extract the deep-rooted function from the lumped uncertainties, one can use the combination of the well-known basis functions as the core function as defined above.

Also, there exist certain situations that might pose challenges to NN-based approximation technique, as noted in the following fact. However, such case can be handled by the core function method gracefully.

Fact 2: If $f_i(z_i)$ is unbounded, it is logically impossible to globally approximate such unbounded function with a bounded function of the form $W_i^T \phi_i(z_i) + \varepsilon_i(z_i)$ anymore. In fact, any unbounded function cannot be represented (approximated) by a bounded function over the entire domain of interest. Instead, it is only possible to approximate such unbounded function with $W_i^T \phi_i(z_i) + \varepsilon_i(z_i)$ in which W_i is unknown bounded but $\phi_i(z_i)$ and $\varepsilon_i(z_i)$ are bounded only if z_i is bounded.

In the following, we introduce the concept of the virtual parameter estimation error [4]. Upon using $\underline{g}$ (the lower bound of the control gain g_i), the virtual parameter estimation error is constructed as

$$\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}. \tag{3.6}$$

Note that such defined parameter estimation error is in contrast to the regular error of the form $\tilde{\bullet} = \bullet - \hat{\bullet}$. This error is named as virtual parameter estimation error. By introducing it into a novel chosen Lyapunov function, the difficulty associated with the unknown and time-varying control gain can be handled effectively.

3.3 Lyapunov Function Design Under Undirected Topology

The cooperative adaptive control design for nonlinear networked multi-agent systems with unknown uncertainties requires the use of special Lyapunov function that depends on the graph topology and the parameter estimation error in a certain way. In the design of admissible adaptive controllers, only distributed control protocols and distributed adaptive laws are permitted. It is not straightforward to develop adaptive laws for cooperative MAS on graphs that only require information from that agent and its neighbors. The key to solving this problem is to select special Lyapunov function for adaptive control design dependent of the graph topology and parameter estimation errors in specific ways.

It is noted that the Lyapunov function is essential for the Lyapunov stability analysis of the adaptive leaderless consensus control of the nonlinear MAS with unknown time-varying gain and non-parametric uncertainties. In this section, we show how to construct the Lyapunov function based on the graph Laplacian, neighborhood error, and parametric estimation error. The Lyapunov function in the adaptive cooperative control of nonlinear MAS usually consists of two parts: the distributed part related to the neighborhood error and the Laplacian matrix, and the adaptive part dependent of the parameter estimation error.

3.3.1 The Distributed Part of Lyapunov Function

In this subsection, we show how to construct the distributed part of the Lyapunov function.

We first introduce the definition related to uniform ultimate boundedness (UUB) for cooperative control systems.

Definition 3.1 ([5, 6]) For the error $E(t) \in \mathbb{R}^N$, if there exists a compact set $\Omega \in \mathbb{R}^N$ with the property that $\{0_N\} \subset \Omega$, so that $\forall E(t_0) \in \Omega$, there exist a finite positive constant C and a finite time $t_f(C, X(t_0))$, both independent of $t_0 > 0$, such that for any $t \geq t_0 + t_f$, $\|E(t)\| \leq C$, then we say $E(t)$ is cooperative uniform ultimate boundedness (CUUB).

Suppose that the network communication among the N agents is modeled by a directed graph $\mathscr{G}$, with the weighted adjacency matrix $\mathscr{A} = [a_{ij}]$ and the Laplacian matrix $\mathscr{L}$ (see Chap. 2 for the details on graph theory).

The following results of graph theory are useful for establishing leaderless consensus stability results of the MAS in this chapter.

Lemma 3.1 ([7]) *For an undirected and connected graph $\mathscr{G}$, the Laplacian $\mathscr{L}$ of $\mathscr{G}$ has only one zero eigenvalue associated with the eigenvector 1_N, i.e., $\mathscr{L}1_N = 0$, and all the other eigenvalues of $\mathscr{L}$ are positive real constants. If $1_N^T X = 0$, then $X^T \mathscr{L} X \geq \lambda_2(\mathscr{L}) X^T X$, where $\lambda_2(\mathscr{L})$ denotes the second minimum eigenvalue of $\mathscr{L}$.*

Lemma 3.2 ([7]) *The vector $\mathscr{L}X$, with $X = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$, is a column stack vector of $\sum_{j=1}^N a_{ij}(x_i - x_j)$, $i = 1, \ldots, N$. If graph $\mathscr{G}$ is undirected (and hence $\mathscr{L}$ is symmetric), then $X^T\mathscr{L}X = \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N a_{ij}(x_i - x_j)^2$. If the undirected graph $\mathscr{G}$ is connected, then $\mathscr{L}X = 0_N$ or $X\mathscr{L}X = 0$ if and only if $x_i = x_j$, for all $i, j = 1, \ldots, N$.*

Lemma 3.3 ([8]) *The Laplacian matrix $\mathscr{L}$ associated with an undirected connected graph $\mathscr{G}$ is symmetric and semi-positive definite, that is, $\mathscr{L} = \mathscr{L}^T \in \mathbb{R}^{N\times N}$ and $\mathscr{L} \geq 0$. Furthermore, the null space $N(\mathscr{L}) = \{X|X^T\mathscr{L}X = 0\} = span\{1_N\}$.*

Proof We first show that $\mathscr{L}$ is symmetric and semi-positive definite. Since the graph $\mathscr{G}$ is undirected, it is straightforward that the adjacency matrix $\mathscr{A}$ is symmetric, implying that $\mathscr{L}$ is symmetric. According to Lemma 3.1, the Laplacian matrix $\mathscr{L}$ associated with an undirected and connected graph has only one zero eigenvalue with all its other eigenvalues being positive real constants; thus, $\mathscr{L}$ is semi-positive definite, and we also derive that the null space of $\mathscr{L}$ is $N(\mathscr{L}) = \text{span}\{1_N\}$. According to Fact 8.15.2 in [9], if $\mathscr{L} = \mathscr{L}^T \in \mathbb{R}^{N\times N}$ satisfying $\mathscr{L} \geq 0$ or $\mathscr{L} \leq 0$, then the null space is $N(\mathscr{L}) = \{X|X^T\mathscr{L}X = 0\}$. Therefore, the null space is equal to $N(\mathscr{L}) = \{X|X^T\mathscr{L}X = 0\} = \text{span}\{1_N\}$. □

According to Lemma 3.2 or 3.3, the leaderless consensus (CUUB consensus) is achieved if and only if $E = \mathscr{L}X = 0$ ($\|E\| = \|\mathscr{L}X\| \leq C$) or $X^T\mathscr{L}X = 0$ ($\|X^T\mathscr{L}X\| \leq C$) according to Definition 3.1. Thus, the objective of the leaderless consensus (CUUB consensus) is to make $E = 0$ ($\|E\| \leq C$) or $X^T\mathscr{L}X = 0$ ($\|X^T\mathscr{L}X\| \leq C$).

According to the above analysis, we next construct the first part of Lyapunov function candidate.

Before moving on, let us first introduce two matrices: $\tilde{\Lambda} = \text{diag}\{a, \lambda_2, \ldots, \lambda_N\}$ and $\Lambda = U_N^T\tilde{\Lambda}^{-1}U_N$, where a is an arbitrarily positive constant, and U_N is an orthogonal matrix such that $\mathscr{L} = U_N^T\text{diag}\{0, \lambda_2, \ldots, \lambda_N\}U_N = U_N^T\Lambda_0U_N$ with $\Lambda_0 = \text{diag}\{0, \lambda_2, \ldots, \lambda_N\}$ provided by Lemma 3.1.

Such matrix Λ is a symmetric and positive definite matrix, with which we construct the first part of Lyapunov function as

$$V_1 = E^T\Lambda E. \tag{3.7}$$

By recalling that $E = \mathscr{L}X$, it then holds that

$$\begin{aligned} V_1 &= X^T\mathscr{L}^T(U_N^T\tilde{\Lambda}^{-1}U_N)\mathscr{L}X \\ &= X^T(U_N^T\Lambda_0U_N)^T(U_N^T\tilde{\Lambda}^{-1}U_N)(U_N^T\Lambda_0U_N)X \\ &= X^TU_N^T\Lambda_0\tilde{\Lambda}^{-1}\Lambda_0U_NX \\ &= X^TU_N^T\Lambda_0U_NX \\ &= X^T\mathscr{L}X. \end{aligned} \tag{3.8}$$

It is seen from (3.7) that if $V_1 \to 0$ ($V_1 \in L_\infty$), then $E \to 0$ ($E \in L_\infty$). The distributed controller can be designed to make the distributed Lyapunov function constructed as in (3.7) to converge to zero (be bounded) such that the neighborhood error E converges to zero (is bounded), which further implies that the leaderless consensus (CUUB consensus) is achieved by Lemma 3.2 or 3.3 according to Definition 3.1.

3.3.2 *The Adaptive Part of Lyapunov Function*

In this subsection, we show how to construct the adaptive part of Lyapunov function available for the cooperative adaptive control of nonlinear MAS with unknown time-varying control gain and non-parametric uncertainties.

With the aid of the virtual parameter estimation error introduced in Sect. 3.2, we construct the adaptive part of the Lyapunov function candidate as

$$V_2 = \frac{1}{2\underline{g}}\tilde{\bullet}^T \Gamma^{-1} \tilde{\bullet} = \frac{1}{2\underline{g}}(\bullet - \underline{g}\hat{\bullet})^T \Gamma^{-1}(\bullet - \underline{g}\hat{\bullet}) \tag{3.9}$$

in which $\Gamma = \text{diag}\{\sigma_1, \ldots, \sigma_N\} > 0$ is a design parameter matrix. The derivative of the adaptive part of the Lyapunov function candidate is taken as

$$\dot{V}_2 = (\bullet - \underline{g}\hat{\bullet})^T \Gamma^{-1}(-\dot{\hat{\bullet}}). \tag{3.10}$$

With such defined adaptive part of Lyapunov function, the unknown time-varying gain and non-parametric uncertainties can be handled gracefully without the need for linearization or approximation to the original systems.

The next section shows how the graph Laplacian matrix and virtual parameter estimation error works in the Lyapunov stability analysis for the distributed leaderless consensus control of nonlinear MAS with unknown non-parameter uncertainties.

3.4 Lyapunov Analysis for Adaptive Leaderless Consensus

In this section, we take the Lyapunov stability analysis for adaptive leaderless consensus of system (3.1) subject to unknown time-varying control gain and non-parametric uncertainties as an example to show how the Lyapunov function works in the Lyapunov stability analysis.

The distributed adaptive controller based on the signum function is designed as follows:

$$u_i = -k_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) + u_{ci}, \quad i = 1, \ldots, N, \tag{3.11}$$

where u_{ci} is a compensation control term

$$u_{ci} = -\text{sgn}(e_i)\hat{c}_{fi}\varphi_i - \text{sgn}(e_i)\hat{\theta}_i, \tag{3.12}$$

with the updated law

$$\dot{\hat{c}}_{fi} = \gamma_{1i}\varphi_i|e_i|, \dot{\hat{\theta}}_i = \gamma_{2i}|e_i|, \tag{3.13}$$

and $k_i, \gamma_{1i}, \gamma_{2i} > 0$ are finite design constant parameters, $\hat{c}_{fi}$ and $\hat{\theta}_i$ are the estimations of the virtual parameters c_{fi} and θ_i (c_{fi} and θ_i are defined as in Assumption 3.2), respectively, and $\varphi_i(\cdot)$ is the readily computable scalar function as given in Assumption 3.2.

Next, we provide the Lyapunov stability analysis to show how the distributed part of Lyapunov function and adaptive part of Lyapunov function work, respectively.

Theorem 3.1 *With the distributed control law given in (3.11) together with the compensation control term (3.12) and the adaptive law (3.13), system (3.1) under Assumptions 3.1–3.2 and an undirected connected graph topology $\mathscr{G}$ achieves asymptotically stable consensus. Further, the internal signals including the control input signal, the neighborhood error, and parameter estimation error in the system remain uniformly bounded.*

Proof The proof consists of three steps.

Step 1. Introducing the Laplacian matrix into the distributed part of Lyapunov function.

We first construct the following distributed Lyapunov function as

$$V_1 = \frac{1}{2}E^T \Lambda E \tag{3.14}$$

where $\Lambda = U_N^T \tilde{\Lambda}^{-1} U_N$ is given in Sect. 3.3.1. By recalling that $E^T \Lambda E = X^T \mathscr{L} X$ derived in (3.8), we take the time derivative of V_1 along (3.1) and get

$$\dot{V}_1 = X^T \mathscr{L} \dot{X} = E^T \dot{X} = \sum_{i=1}^{N} e_i(g_i u_i + f_i + f_{di}). \tag{3.15}$$

By performing the control law proposed in (3.11) and the compensation term (3.12), (3.15) then becomes

$$\begin{aligned}
\dot{V}_1 &= \sum_{i=1}^{N} e_i[g_i(-k_i e_i + u_{ci}) + f_i + f_{di}] \\
&= -\sum_{i=1}^{N} k_i g_i e_i^2 + e_i\left[g_i(-\text{sgn}(e_i)\hat{c}_{fi}\varphi_i - \text{sgn}(e_i)\hat{\theta}_i) + f_i + f_{di}\right]
\end{aligned}$$

$$\le -\sum_{i=1}^{N} k_i g_i e_i^2 + \sum_{i=1}^{N} [-\underline{g}\hat{c}_{fi}\varphi_i|e_i| - \underline{g}\hat{\theta}_i|e_i| + c_{fi}\varphi_i|e_i| + \theta_i|e_i|]$$
$$= -\sum_{i=1}^{N} k_i g_i e_i^2 + \sum_{i=1}^{N} \Big[(c_{fi} - \underline{g}\hat{c}_{fi})\varphi_i|e_i| + (\theta_i - \underline{g}\hat{\theta}_i)|e_i|\Big]. \tag{3.16}$$

Step 2. Introducing the virtual parameter estimation error into the adaptive part of Lyapunov function.

Note that (3.16) involves the parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$; we then introduce such virtual parameter estimation error into the adaptive part of Lyapunov function candidate as

$$V_2 = \frac{1}{2\underline{g}}\tilde{C}_f^T \Gamma_1^{-1}\tilde{C}_f + \frac{1}{2\underline{g}}\tilde{\theta}^T \Gamma_2^{-1}\tilde{\theta} \tag{3.17}$$

with

$$\tilde{C}_f = C_f - \underline{g}\hat{C}_f, \tilde{\theta} = \theta - \underline{g}\hat{\theta} \tag{3.18}$$

where $\tilde{C}_f = [\tilde{c}_{f1} \ \cdots \ \tilde{c}_{fN}]^T \in \mathbb{R}^N$, $C_f = [c_{f1} \ \cdots \ c_{fN}]^T \in \mathbb{R}^N$, $\hat{C}_f = [\hat{c}_{f1} \ \cdots \ \hat{c}_{fN}]^T \in \mathbb{R}^N, \tilde{\theta} = [\tilde{\theta}_1 \ \cdots \ \tilde{\theta}_N]^T \in \mathbb{R}^N, \theta = [\theta_1 \ \cdots \ \theta_N]^T \in \mathbb{R}^N, \hat{\theta} = [\hat{\theta}_1 \ \cdots \ \hat{\theta}_N]^T \in \mathbb{R}^N$, $\Gamma_1 = \mathrm{diag}\{\gamma_{1i}\} \in \mathbb{R}^{N\times N}$, and $\Gamma_2 = \mathrm{diag}\{\gamma_{2i}\} \in \mathbb{R}^{N\times N}$. Such treatment allows the unknown time-varying parameters involved in the system to be processed gracefully, as can be seen shortly.

Upon using the updated law (3.13), the time derivative of V_2 is taken as

$$\dot{V}_2 = \tilde{C}_f^T \Gamma_1^{-1}(-\dot{\hat{C}}_f) + \tilde{\theta}^T \Gamma_2^{-1}(-\dot{\hat{\theta}})$$
$$= \sum_{i=1}^{N} \tilde{c}_{fi}\left(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}}\right) + \sum_{i=1}^{N} \tilde{\theta}_i\left(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}\right)$$
$$= \sum_{i=1}^{N} (c_{fi} - \underline{g}\hat{c}_{fi})\varphi_i|e_i| + \sum_{i=1}^{N} (\theta_i - \underline{g}\hat{\theta}_i)|e_i|. \tag{3.19}$$

Step 3. Constructing the Lyapunov function candidate.

We choose the Lyapunov function candidate as

$$V = V_1 + V_2. \tag{3.20}$$

By combining (3.16) and (3.19), we then get the derivative of V as

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \le -\sum_{i=1}^{N} k_i g_i e_i^2 \le -\underline{k}\underline{g}\|E\|^2 \le 0 \tag{3.21}$$

where $\underline{k} = \min\{k_1, \ldots, k_N\}$. From (3.21), we see that $\|E\|^2 \leq -\dot{V}(t)/\underline{k}g$, which then implies that

$$\int_0^{\infty} \|E(t)\|^2 dt \leq \int_0^{\infty} -\frac{\dot{V}(t)}{\underline{k}g} dt = -\frac{1}{\underline{k}g}\left(V(\infty) - V(0)\right) \leq \frac{1}{\underline{k}g} V(0) < \infty \tag{3.22}$$

By recalling the fact that E is uniformly continuous, which, together with (3.22), implies that $\lim_{t\to\infty} E(t) = \lim_{t\to\infty} \mathscr{L}X(t) = 0$ upon using the Barbalat lemma. According to Lemma 3.2, we see that $\mathscr{L}X = 0$ if and only if $x_i = x_j$ for all $i, j = 1, \ldots, N$. Therefore, $x_i = x_j$ for all $i, j = 1, \ldots, N$ as $t \to \infty$, which means that the asymptotically stable leaderless consensus is achieved under the proposed control scheme (3.11)–(3.13). In addition, from (3.21), we can deduce that $V \in L_\infty$, which further ensures that $E \in L_\infty$, $\tilde{C} \in L_\infty$, and $\tilde{\theta} \in L_\infty$. From the definition of $\tilde{C}$ and $\tilde{\theta}$, it is straightforward that $\hat{C} \in L_\infty$ and $\hat{\theta} \in L_\infty$, which further implies that $U \in L_\infty$. It is then concluded that the control input, the neighborhood errors, and the parameter estimation errors in the system remain uniformly bounded under the proposed control. □

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Chapter 4
Cooperative Adaptive Consensus Control for Uncertain Multi-agent Systems with *n*th-Order Dynamics Under Undirected Graph

In Chap. 3 we showed how to construct the Lyapunov function that is related to the graph topology and parameter estimation error, and also introduced some key concepts needed for the distributed adaptive controller design and stability analysis on graph such as the virtual parameter estimation error technique, the core function concept, and the distributed Lyapunov function design dependent of the graph topology. In this chapter we address the controller design and stability analysis for the distributed leaderless consensus control of networked multi-agent systems, including first-order systems, second-order systems and high-order systems, all with unknown time-varying gain and non-parametric/non-vanishing uncertainties under the undirected communication topology condition. Each agent is mathematically modeled by a continuous nonlinear system, and the communication network among agents is modeled by a graph. Of particular interest is the development of the distributed adaptive consensus control protocol capable of simultaneously compensating time-varying yet unknown control gain and non-parametric/non-vanishing uncertainties under the local communication constraints.

Section 4.1 addresses the adaptive leaderless consensus problem for first-order continuous-time MAS with unknown time-varying gain and non-parametric/non-vanishing uncertainties under undirected topology based on continuous control method. The basic structure and key ideas of the distributed adaptive control protocols to confront the leaderless consensus problems emerge from this study. To understand the relationships between the distributed adaptive controller design for first-order systems and for general high-order systems, it was natural to study the second-order systems and then high-order systems. Sections 4.2 and 4.3 study the adaptive leaderless cooperative consensus for the second-order position–velocity continuous-time uncertain systems and general high-order continuous-time uncertain

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Part of Sects. 4.2 and 4.4 have been reproduced from Wang, Y. J., Song, Y. D., Lewis, F. L.: Robust adaptive fault-tolerant control of multi-agent systems with uncertain non-identical dynamics and undetectable actuation failures. IEEE Transactions on Industrial Electronics, vol. 62, no. 6, pp. 3978–3988, 2015 © 2015, IEEE, reprinted with permission.

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dynamical systems on graph, respectively. The control schemes proposed in this chapter are user-friendly in that there is no need for detailed dynamic information of the agent in controller design and implementation, resulting in structurally simple and computationally inexpensive solutions for the leaderless consensus problem of networked uncertain multi-agent systems. Section 4.4 gives a discussion on the cooperative adaptive consensus control of MAS with unknown time-varying gain and non-parametric/non-vanishing uncertainties.

4.1 Cooperative Adaptive Control of First-Order Uncertain Systems Under Undirected Topology

In this section, we address the leaderless consensus problem for first-order MAS with unknown time-varying gain and non-parametric uncertainties on graph based on the principles of core function, virtual parameter estimation error, and Lyapunov stability theory introduced in Chap. 3. To help with the understanding of the fundamental idea and technical development of the proposed method, we start with controller design for the first-order MAS, followed by the extension to the second-order and high-order case in later sections. This section formulates the basic structure of the distributed control schemes to confront the leaderless consensus problems and show how to learn the unknown non-parametric uncertainties online to ensure the leaderless consensus by using distributed adaptive control technique on graph.

4.1.1 System Dynamics

We consider a group of N networked agents with the same dynamics as in (3.1)

$$\dot{x}_i(t) = g_i(\cdot)u_i(t) + f_i(z_i(t), t) + f_{di}(z_i(t), t) \tag{4.1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i is the control gain, f_i and f_{di} are the system nonlinearities and non-vanishing uncertainties, respectively; $z_i = \bigcup_{j \in \mathscr{N}_i \bigcup i} x_j$.

Assumption 4.1

a. g_i ($i = 1, 2, \ldots, N$) is unknown, time-varying, and bounded away from zero; that is, there exist unknown finite positive constants $\underline{g}_i$ and $\bar{g}_i$ such that $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$, and g_i is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$).
b. Certain crude structural information on $f_i(z_i(t), t)$ is available to allow an unknown constant $c_{fi} \geq 0$ and a known scalar function $\varphi_i(z_i)$ to be extracted [2], such that $|f_i(z_i(t), t)| \leq c_{fi}\varphi_i(z_i)$ for $t \in [t_0, \infty)$, where $\varphi_i(z_i)$ is bounded for any x_i ($i = 1, 2, \ldots, N$). There exists unknown finite positive constant θ_i such that $|f_{di}(z_i(t), t)| \leq \theta_i < \infty$.

The local neighborhood error for node i is

$$e_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j), \quad i = 1, \ldots, N. \tag{4.2}$$

4.1.2 Controller Design and Stability Analysis

We consider the controller design and stability analysis for the first-order uncertain MAS (4.1). It should be stressed that the control scheme proposed in Sect. 3.4 for the same system model (4.1), although capable of dealing with the unknown time-varying gain and non-parametric/non-vanishing uncertainties, might cause the undesired chattering phenomenon when the consensus error crosses zero due to the utilization of the sign function. In this subsection, we propose a continuous control solution not only to compensate the unknown time-varying gain and non-parametric/non-vanishing uncertainties but also to avoid the undesirable chatting phenomenon.

Before moving on, we first give a useful lemma as follows.

Lemma 4.1 ([1]) *For a real diagonal matrix M with appropriate dimension, if vectors X and Y satisfy that each component of their product X^TY is nonnegative, it then holds that*

$$\lambda_{\min}(M)X^TY \le X^TMY \le \lambda_{\max}(M)X^TY$$

with $\lambda_{max}(M)$ and $\lambda_{min}(M)$ denoting the maximum and minimum eigenvalues of M, respectively.

The continuous distributed adaptive control scheme is proposed as follows,

$$u_i = -k_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) + u_{ci}, \quad i = 1, \ldots, N \tag{4.3}$$

where u_{ci} is a compensation control term given as

$$u_{ci} = -\frac{\hat{c}_{fi}\varphi_i^2 e_i}{\varphi_i|e_i| + \varepsilon} - \frac{\hat{\theta}_i e_i}{|e_i| + \varepsilon}, \tag{4.4}$$

with the updated law

$$\dot{\hat{c}}_{fi} = -\gamma_{1i}\sigma_{1i}\hat{c}_{fi} + \frac{\gamma_{1i}\varphi_i^2 e_i^2}{\varphi_i|e_i| + \varepsilon}, \quad \dot{\hat{\theta}}_i = -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \frac{\gamma_{2i}e_i^2}{|e_i| + \varepsilon}, \tag{4.5}$$

$k_i, \gamma_{1i}, \gamma_{2i}, \sigma_{1i}, \sigma_{2i} > 0$ are finite design constant parameters, ε is a small positive constant, $\hat{c}_{fi}$ and $\hat{\theta}_i$ are the estimations of the virtual parameters c_{fi} and θ_i (c_{fi} and

θ_i are defined as in Assumption 4.1), respectively, and $\varphi_i(\cdot)$ is the readily computable scalar function as given in Assumption 4.1.

The main result in this subsection is given as follows.

Theorem 4.1 *Suppose that the graph topology $\mathscr{G}$ is undirected and connected. The MAS* (4.1) *under Assumption 4.1 with the distributed control law proposed in* (4.3) *and* (4.5) *achieves the CUUB leaderless consensus. In addition, the internal system signals including the control input signal, the position error, and the parameter estimation error remain uniformly bounded.*

Proof Choose the Lyapunov function candidate as

$$V = V_1 + V_2 \tag{4.6}$$

with

$$V_1 = \frac{1}{2} E^T \Lambda E, \quad V_2 = \frac{1}{2\underline{g}} \sum_{i=1}^{N} \frac{\tilde{c}_{fi}^2}{\gamma_{1i}} + \frac{1}{2\underline{g}} \sum_{i=1}^{N} \frac{\tilde{\theta}_i^2}{\gamma_{2i}} \tag{4.7}$$

where $\Lambda = U_N^T \tilde{\Lambda}^{-1} U_N$ is given the same as in Sect. 3.3.1, $\tilde{c}_{fi} = c_{fi} - \underline{g}\hat{c}_{fi}$ and $\tilde{\theta}_i = \theta_i - \underline{g}\hat{\theta}_i$ denote the virtual parameter estimation error.

Taking the time derivative of V_1 along (4.1), one gets

$$\dot{V}_1 = X^T \mathscr{L} \dot{X} = E^T \dot{X} = \sum_{i=1}^{N} e_i (g_i u_i + f_i + f_{di}). \tag{4.8}$$

By performing the control law u_i proposed in (4.3) and the compensation unit u_{ci} given in (4.4), the time derivative of V_1 then becomes

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^{N} e_i \left[g_i \left(-k_i e_i + u_{ci} \right) + f_i + f_{di} \right] \\ &\leq -\sum_{i=1}^{N} k_i g_i e_i^2 + \sum_{i=1}^{N} \left[-g_i \frac{\hat{c}_{fi} \varphi_i^2 e_i^2}{\varphi_i |e_i| + \varepsilon} - g_i \frac{\hat{\theta}_i e_i^2}{|e_i| + \varepsilon} + |e_i| c_{fi} \varphi_i + |e_i| \theta_i \right] \\ &\leq -\underline{k}\underline{g} \sum_{i=1}^{N} e_i^2 + \sum_{i=1}^{N} \left[-\underline{g} \frac{\hat{c}_{fi} \varphi_i^2 e_i^2}{\varphi_i |e_i| + \varepsilon} - \underline{g} \frac{\hat{\theta}_i e_i^2}{|e_i| + \varepsilon} \right. \\ &\quad \left. + \frac{c_{fi} \varphi_i^2 e_i^2 + c_{fi} \varphi_i |e_i| \varepsilon}{\varphi_i |e_i| + \varepsilon} + \frac{\theta_i e_i^2 + \theta_i |e_i| \varepsilon}{|e_i| + \varepsilon} \right] \end{aligned} \tag{4.9}$$

By noting that $\frac{\varphi_i |e_i|}{\varphi_i |e_i| + \varepsilon} \leq 1$ and $\frac{|e_i|}{|e_i| + \varepsilon} \leq 1$, we then arrive at

$$\dot{V}_1 \leq -\underline{k}\underline{g}\sum_{i=1}^{N} e_i^2 + \sum_{i=1}^{N}\left[\left(c_{fi} - \underline{g}\hat{c}_{fi}\right)\frac{\varphi_i^2 e_i^2}{\varphi_i|e_i| + \varepsilon} + \left(\theta_i - \underline{g}\hat{\theta}_i\right)\frac{e_i^2}{|e_i| + \varepsilon} + c_{fi}\varepsilon + \theta_i\varepsilon\right] \tag{4.10}$$

On the other hand, upon using the updated laws for $\hat{c}_{fi}$ and $\hat{\theta}_i$ designed in (4.5), one gets the derivative of V_2 as

$$\begin{aligned}\dot{V}_2 =& \sum_{i=1}^{N}(c_{fi} - \underline{g}\hat{c}_{fi})(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}}) + \sum_{i=1}^{N}(\theta_i - \underline{g}\hat{\theta}_i)(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}) \\ =& \sum_{i=1}^{N}(c_{fi} - \underline{g}\hat{c}_{fi})(\sigma_{1i}\hat{c}_{fi} - \frac{\varphi_i^2 e_i^2}{\varphi_i|e_i| + \varepsilon}) + \sum_{i=1}^{N}(\theta_i - \underline{g}\hat{\theta}_i)(\sigma_{2i}\hat{\theta}_i - \frac{e_i^2}{|e_i| + \varepsilon}).\end{aligned} \tag{4.11}$$

By combining (4.10) and (4.11), we then get the derivative of the Lyapunov function candidate V as

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \leq -\underline{k}\underline{g}\sum_{i=1}^{N} e_i^2 + \sum_{i=1}^{N}(c_{fi}\varepsilon + \theta_i\varepsilon) + \sum_{i=1}^{N}\sigma_{1i}\tilde{c}_{fi}\hat{c}_{fi} + \sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i\hat{\theta}_i. \tag{4.12}$$

By using the relation that $ab = \frac{1}{2}\left[(a+b)^2 - a^2 - b^2\right]$, we then have

$$\begin{aligned}\tilde{c}_{fi}\hat{c}_{fi} &= \tilde{c}_{fi}\frac{1}{\underline{g}}(c_{fi} - \tilde{c}_{fi}) = \frac{1}{2\underline{g}}\left[c_{fi}^2 - \tilde{c}_{fi}^2 - (c_{fi} - \tilde{c}_{fi})^2\right], \\ \tilde{\theta}_i\hat{\theta}_i &= \tilde{\theta}_i\frac{1}{\underline{g}}(\theta_i - \tilde{\theta}_i) = \frac{1}{2\underline{g}}\left[\theta_i^2 - \tilde{\theta}_i^2 - (\theta_i - \tilde{\theta}_i)^2\right].\end{aligned} \tag{4.13}$$

Inserting (4.13) into (4.12), we arrive at

$$\begin{aligned}\dot{V} \leq& -\underline{k}\underline{g}E^T E + \sum_{i=1}^{N}(c_{fi} + \theta_i)\varepsilon \\ &- \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}\tilde{c}_{fi}^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}c_{fi}^2 - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2.\end{aligned} \tag{4.14}$$

By recalling that matrix Λ is positive and definite, it then holds

$$\lambda_{\min}(\Lambda)E^T E \leq E^T \Lambda E \leq \lambda_{\max}(\Lambda)E^T E. \tag{4.15}$$

Denote by

$$\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}, \quad \underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}, \qquad (4.16)$$
$$d = \frac{1}{2\underline{g}} \sum_{i=1}^{N} \left(\sigma_{1i} c_{fi}^2 + \sigma_{2i}\theta_i^2\right) \in L_\infty.$$

Upon using (4.14), (4.15) and (4.16) is represented as

$$\dot{V} \leq -\underline{k}\underline{g} \frac{1}{\lambda_{\max}(\Lambda)} E^T \Lambda E - \frac{\underline{\sigma}_1}{2\underline{g}} \sum_{i=1}^{N} \tilde{c}_{fi}^2 - \frac{\underline{\sigma}_2}{2\underline{g}} \sum_{i=1}^{N} \tilde{\theta}_i^2 + d \qquad (4.17)$$

Denote by

$$\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}, \quad \underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}. \qquad (4.18)$$

It then follows from (4.17) that

$$\dot{V} \leq -\frac{2\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)} \cdot \frac{1}{2} E^T \Lambda E - \frac{\underline{\sigma}_1 \underline{\gamma}_1}{2\underline{g}} \sum_{i=1}^{N} \frac{\tilde{c}_{fi}^2}{\gamma_{1i}} - \frac{\underline{\sigma}_2 \underline{\gamma}_2}{2\underline{g}} \sum_{i=1}^{N} \frac{\tilde{\theta}_i^2}{\gamma_{2i}} + d. \qquad (4.19)$$

By letting

$$c = \min\left\{\frac{2\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)}, \underline{\sigma}_1 \underline{\gamma}_1, \underline{\sigma}_2 \underline{\gamma}_2\right\}, \qquad (4.20)$$

we then represent (4.19) as

$$\dot{V} \leq -cV + d. \qquad (4.21)$$

From (4.21), we can conclude that the set $\Omega = \{(e_i, \tilde{c}_{fi}, \tilde{\theta}_i) | V \leq d/c\}$ is globally attractive. Once $(e_i, \tilde{c}_{fi}, \tilde{\theta}_i) \notin \Omega$, then $\dot{V} < 0$. Therefore, there exists a finite time t^* such that $(e_i, \tilde{c}_{fi}, \tilde{\theta}_i) \in \Omega$ for $\forall t > t^*$. This further implies, together with (4.15), that

$$\|E\| = \sqrt{E^T E} \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)} V_1} \leq \sqrt{\frac{2d}{\lambda_{\min}(\Lambda) c}} \qquad (4.22)$$

when $t \geq t^*$, meaning that the leaderless error E is CUUB.

In the sequel, we prove all the internal signals in the system are uniformly bounded. By solving (4.21), it follows that

$$V(t) \leq \exp^{-ct} V(0) + \frac{d}{c} \in L_\infty \tag{4.23}$$

for all $t \geq 0$, from which we can deduce that $E \in L_\infty$, $\hat{c}_{fi} \in L_\infty$ and $\hat{\theta}_i \in L_\infty$ for all $i = 1, \ldots, N$. Therefore, $U \in L_\infty$ i.e., the control input remains bounded. This completes the proof. □

4.1.3 Simulation Example

This section gives a simulation example to illustrate the effectiveness of the distributed adaptive control scheme proposed in (4.3)–(4.5). It is shown that under the proposed control a group of agents subject to unknown time-varying control gain, unknown non-parametric uncertainties, and unknown bounded non-vanishing disturbances achieves leaderless consensus successfully.

We use four agents to demonstrate the effectiveness of our proposed control scheme. The communication graph among the four agents is presented in Fig. 4.1, which satisfies the assumption that the graph topology is undirected and connected. Each edge weight is set to be 1.

The dynamics of the four agents is of the following form,

$$\begin{aligned}
\dot{x}_1 =& (5 + 0.5\exp^{-|x_1|})u_1 + 0.1\exp^{-x_1 - x_2} x_1 + f_{d1}, \\
\dot{x}_2 =& (5 + 0.5\exp^{-|x_2|})u_2 + 0.1\exp^{-x_1 - x_2^2} x_2^2 + f_{d2}, \\
\dot{x}_3 =& (5 + 0.5\exp^{-|x_3|})u_3 + 0.1\exp^{-x_2 - x_3^2} x_3^2 + f_{d3}, \\
\dot{x}_4 =& (5 + 0.5\exp^{-|x_4|})u_4 + 0.1\exp^{-x_3 - x_4} x_4 + f_{d4},
\end{aligned} \tag{4.24}$$

in which the non-vanishing uncertainties are of the form: $f_{d1} = 0.02\sin(0.05t)$, $f_{d2} = 0.01\cos(0.06t)$, $f_{d3} = 0.02\cos(0.06t)$, and $f_{d4} = 0.01\sin(0.05t)$.

The earlier theoretical analysis has declared that the nonlinear system (4.24) can achieve CUUB leaderless consensus under the control scheme proposed in (4.3)–(4.5). To set up the proposed controller, we only need to select the design parameters

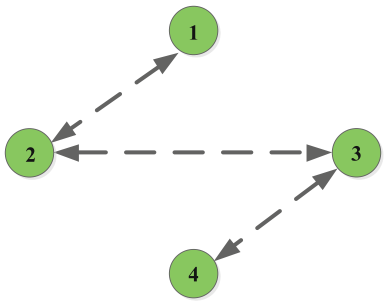


Fig. 4.1 Communication graph topology among the four agents

Fig. 4.2 The position trajectories x_i of all the four agents

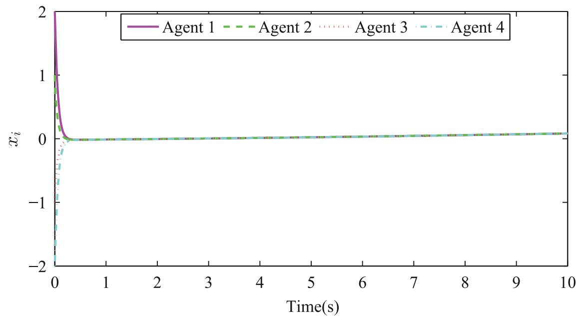


Fig. 4.3 The neighborhood errors e_i of the four agents

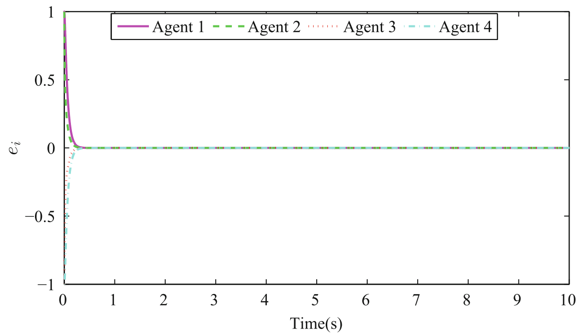
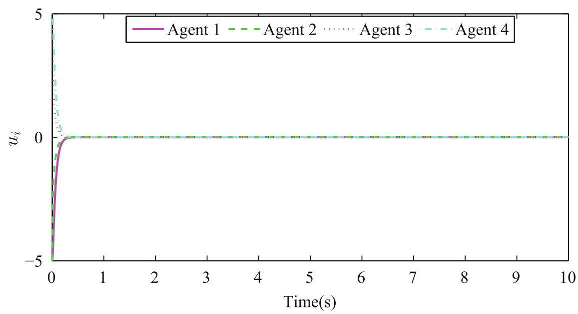


Fig. 4.4 The control inputs u_i of the four agents



k_i, γ_i, and σ_i $(i = 1, 2, 3, 4)$ and does not need to take any time-consuming analytical derivation for choosing design parameters. The design parameters in this simulation are taken as: $k_i = 5$ $(i = 1, 2, 3, 4)$, $\gamma_i = 0.01$ and $\sigma_i = 0.01$ $(i = 1, 2)$, $\varepsilon = 0.01$. In addition, the core functions $\varphi_i(\cdot)$ $(i = 1, 2, 3, 4)$ is easily taken as $\varphi_1 = |x_1|$, $\varphi_2 = x_2^2$, $\varphi_3 = x_3^2$, and $\varphi_4 = |x_4|$ according to the core information of the system.

Figure 4.2 presents the position trajectories of all the four agents, and Fig. 4.3 shows the neighborhood position synchronization errors of all the four agents, both of which verify that the position synchronization is achieved with small residual errors in some finite time. Figure 4.4 shows the control input signals of the four agents u_i $(i = 1, 2, 3, 4)$, from which we observe that the control input signals of all

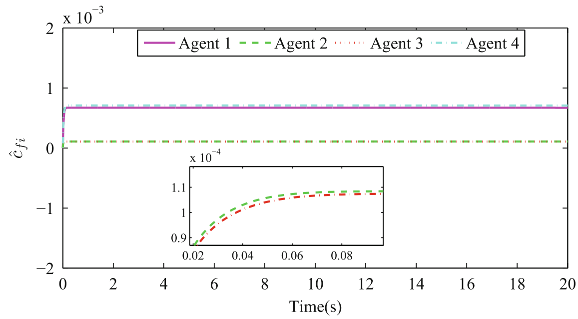


Fig. 4.5 The parameter estimation $\hat{c}_{fi}$ of the four agents

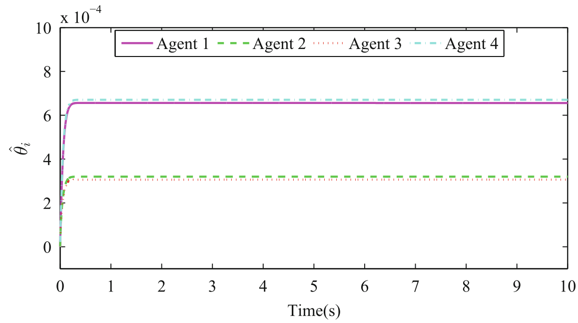


Fig. 4.6 The parameter estimation $\hat{\theta}_i$ of the four agents

the four agents are smooth and bounded. Figures 4.5 and 4.6 portray the parameter estimation values of the virtual estimation parameters c_{fi} and θ_i $(i = 1, 2, 3, 4)$, respectively, from both of which we see that all the parameter estimation values, $\hat{c}_{fi}$ and $\hat{\theta}_i$ $(i = 1, 2, 3, 4)$, are bounded under the proposed control algorithm.

4.2 Second-Order Systems

In this section, we address the leaderless consensus problem for the second-order nonlinear multi-agent systems subject to non-parametric nonlinearities and unknown bounded non-vanishing uncertainties.

Consider a group of N agents with scalar second-order dynamics

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= g_i(z_i, t)u_i(t) + f_i(z_i, t) + f_{di}(z_i, t) \quad i = 1, \ldots, N \end{aligned} \tag{4.25}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ denote the position state, velocity state, and control input of the ith agent, respectively, g_i is the control gain, possibly time-varying and unavailable for controller design, f_i denotes the system nonlinearity, and f_{di} denotes the non-vanishing uncertainty including the external disturbance acting on the ith agent. In addition, $z_i = \bigcup_{j \in \mathcal{N}_i \bigcup i} \{x_j, v_j\}$.

Suppose that g_i, f_i, and f_{di} satisfy Assumption 4.1. Of particular interest in this section is to design distributed adaptive controller for the second-order nonlinear multi-agent system (4.25) to ensure the leaderless consensus and to give a strict stability analysis to demonstrate the stability of the system.

4.2.1 Consensus Error Dynamics

Let the local neighborhood state error still be defined as in (4.2). To solve the leaderless consensus problem for the second-order multi-agent systems, we need to introduce two filtered variables ξ_i and s_i $(i = 1, \ldots, N)$ as

$$\xi_i = \dot{x}_i + \beta x_i, \quad s_i = \dot{e}_i + \beta e_i, \tag{4.26}$$

where β is a positive constant chosen by the designer. Let $\xi = [\xi_1, \ldots, \xi_N]^T \in \mathbb{R}^N$ and $S = [s_1, \ldots, s_N]^T \mathbb{R}^N$ such that $S = L\xi$.

With the filtered variable ξ, the consensus error dynamics for system (4.25) can be rewritten as a compact form,

$$\dot{\xi} = \ddot{X} + \beta \dot{X} = GU + F + F_d + \beta v \tag{4.27}$$

where $X = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$, $v = [v_1, \ldots, v_N]^T \in \mathbb{R}^N$, $G = \mathrm{diag}\{g_i\} \in \mathbb{R}^N$, $U = [u_1, \ldots, u_N]^T \in \mathbb{R}^N$, $F = [f_1, \ldots, f_N]^T \in \mathbb{R}^N$, and $F_d = [f_{d1}, \ldots, f_{dN}]^T \in \mathbb{R}^N$.

4.2.2 Controller Design and Stability Analysis

In this subsection, we address the controller design and stability analysis for system (4.25).

The controller for the second-order multi-agent system (4.25) is designed as

$$u_i = -k_i s_i + u_{ci}, \quad i = 1, \ldots, N \tag{4.28}$$

where the compensation term u_{ci} is given as

$$u_{ci} = -\frac{\hat{c}_i \psi_i^2 s_i}{\psi_i |s_i| + \varepsilon} - \frac{\hat{\theta}_i s_i}{|s_i| + \varepsilon}, \tag{4.29}$$

the updated law is designed as

$$\dot{\hat{c}}_i = -\gamma_{1i}\sigma_{1i}\hat{c}_i + \frac{\gamma_{1i}\varphi_i^2 s_i^2}{\psi_i|s_i| + \varepsilon}, \quad \dot{\hat{\theta}}_i = -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \frac{\gamma_{2i}s_i^2}{|s_i| + \varepsilon}, \tag{4.30}$$

and k_i, γ_{1i}, γ_{2i}, σ_{1i}, and σ_{2i} are finite positive design parameters, $\hat{c}_i$ and $\hat{\theta}_i$ are the estimation values of the virtual parameters c_i and θ_i ($c_i = \max\{c_{fi}, \beta\}$ and θ_i is defined by $|f_{di}| \leq \theta_i$ given as in Assumption 4.1), respectively, $\psi_i(\cdot) = \varphi_i(\cdot) + |v_i(t)|$ with $\varphi_i(\cdot)$ given as in Assumption 4.1, and $\varepsilon > 0$ is a small finite constant.

The main result in this subsection is given as follows.

Theorem 4.2 *Suppose that the graph topology $\mathcal{G}$ is undirected and connected, and g_i, f_i, and f_{di} satisfy Assumption 4.1. The second-order nonlinear multi-agent system (4.25) with the distributed control scheme (4.28), the compensation control term (4.29), and the updated control law (4.30) achieves CUUB leaderless consensus. In addition, all the internal system signals remain bounded.*

Proof With the positive definite matrix $\Lambda = U_N^T \tilde{\Lambda}^{-1} U_N$ given in Sect. 3.3.1, we construct the distributed Lyapunov function as

$$V_1 = \frac{1}{2}S^T \Lambda S. \tag{4.31}$$

By following the similar procedure as in the proof in (3.8), we readily obtain that

$$V_1 = \frac{1}{2}S^T \Lambda S = \frac{1}{2}\xi^T L \xi. \tag{4.32}$$

Taking the derivative of V_1 along the error dynamic (4.27) yields

$$\dot{V}_1 = S^T \dot{\xi} = S^T(GU + F + F_d + \beta v) = \sum_{i=1}^{N}(g_i u_i + f_i + f_{di} + \beta v_i) \tag{4.33}$$

From Assumptions 4.1, we see that $|f_i| \leq c_{fi}\varphi_i$ and $|f_{di}| \leq \theta_i$, and we further have that $|f_i + \beta v_i| \leq c_{fi}\varphi_i + \beta|v_i| \leq c_i\psi_i$ with $c_i = \max\{c_{fi}, \beta\}$ and $\psi_i = \varphi_i + |v_i|$. Upon using the control law (4.28) and the compensating unit u_{ci} (4.29), one then gets the time derivative of V_1 as

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{N} s_i(g_i u_i + f_i + f_{di} + \beta v_i)\\ &= \sum_{i=1}^{N} s_i[g_i(-k_i s_i + u_{ci}) + f_i + f_{di} + \beta v_i]\end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{i=1}^{N} k_i g_i s_i^2 + \sum_{i=1}^{N}[-g_i \frac{\hat{c}_i \psi_i^2 s_i^2}{\psi_i |s_i| + \varepsilon} - g_i \frac{\hat{\theta}_i s_i^2}{|s_i| + \varepsilon} + |s_i| c_i \psi_i + |s_i| \theta_i] \\
&\leq -\underline{k}\underline{g} \sum_{i=1}^{N} s_i^2 + \sum_{i=1}^{N}[(c_i - \underline{g}\hat{c}_i) \frac{\psi_i^2 s_i^2}{\psi_i |s_i| + \varepsilon} \\
&\quad + (\theta_i - \underline{g}\hat{\theta}_i) \frac{s_i^2}{|s_i| + \varepsilon} + c_i \varepsilon + \theta_i \varepsilon].
\end{aligned} \tag{4.34}
$$

Next, we construct the adaptive part of the Lyapunov function as

$$
V_2 = \frac{1}{2\underline{g}} \sum_{i=1}^{N} \frac{c_i^2}{\gamma_{1i}} + \frac{1}{2\underline{g}} \sum_{i=1}^{N} \frac{\theta_i^2}{\gamma_{2i}}. \tag{4.35}
$$

By applying the updated law $\dot{\hat{c}}_i$ and $\dot{\hat{\theta}}_i$ given in (4.30), one gets the derivative of V_2 as

$$
\begin{aligned}
\dot{V}_2 &= \sum_{i=1}^{N} (c_i - \underline{g}\hat{c}_i)(-\frac{\dot{\hat{c}}_i}{\gamma_{1i}}) + \sum_{i=1}^{N} (\theta_i - \underline{g}\hat{\theta}_i)(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}) \\
&= \sum_{i=1}^{N} (c_i - \underline{g}\hat{c}_i)(\sigma_{1i}\hat{c}_i - \frac{\varphi_i^2 s_i^2}{\varphi_i |s_i| + \varepsilon}) + \sum_{i=1}^{N} (\theta_i - \underline{g}\hat{\theta}_i)(\sigma_{2i}\hat{\theta}_i - \frac{s_i^2}{|s_i| + \varepsilon}).
\end{aligned} \tag{4.36}
$$

Now, we choose the Lyapunov function candidate as

$$
V = V_1 + V_2
$$

with V_1 and V_2 being given in (4.32) and (4.35), respectively. By combining (4.34) and (4.36), we thus have the derivative of V as

$$
\dot{V} = \dot{V}_1 + \dot{V}_2 \leq -\underline{k}\underline{g} \sum_{i=1}^{N} s_i^2 + \sum_{i=1}^{N} (c_i \varepsilon + \theta_i \varepsilon) + \sum_{i=1}^{N} \sigma_{1i} \tilde{c}_i \hat{c}_i + \sum_{i=1}^{N} \sigma_{2i} \tilde{\theta}_i \hat{\theta}_i. \tag{4.37}
$$

Upon using the inequality $ab = \frac{1}{2}\left[(a+b)^2 - a^2 - b^2\right]$, we have

$$
\tilde{c}_i \hat{c}_i = \frac{1}{2\underline{g}} \left[c_i^2 - \tilde{c}_i^2 - (c_i - \tilde{c}_i)^2\right], \quad \tilde{\theta}_i \hat{\theta}_i = \frac{1}{2\underline{g}} \left[\theta_i^2 - \tilde{\theta}_i^2 - (\theta_i - \tilde{\theta}_i)^2\right]. \tag{4.38}
$$

By inserting (4.38) into (4.37), we arrive at

$$
\begin{aligned}
\dot{V} \leq & -\underline{k}\underline{g}S^T S + \sum_{i=1}^{N}(c_i + \theta_i)\varepsilon \\
& - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}\tilde{c}_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}c_i^2 - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2 \\
\leq & -\frac{\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)}S^T \Lambda S - \frac{\underline{\sigma}_1}{2\underline{g}}\sum_{i=1}^{N}\tilde{c}_i^2 - \frac{\underline{\sigma}_2}{2\underline{g}}\sum_{i=1}^{N}\tilde{\theta}_i^2 \\
& + \sum_{i=1}^{N}(c_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}c_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2 \\
\leq & -\frac{2\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)} \cdot \frac{1}{2}S^T \Lambda S - \frac{\underline{\sigma}_1\underline{\gamma}_1}{2\underline{g}}\sum_{i=1}^{N}\frac{\tilde{c}_i^2}{\gamma_{1i}} - \frac{\underline{\sigma}_2\underline{\gamma}_2}{2\underline{g}}\sum_{i=1}^{N}\frac{\tilde{\theta}_i^2}{\gamma_{2i}} \\
& + \sum_{i=1}^{N}(c_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}c_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2,
\end{aligned}
\tag{4.39}
$$

where $\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}$, $\underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}$, $\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$, and $\underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. Let $c = \min\{\frac{2\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)}, \underline{\sigma}_1\underline{\gamma}_1, \underline{\sigma}_2\underline{\gamma}_2\}$, $d = \sum_{i=1}^{N}(c_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}(\sigma_{1i}c_i^2 + \sigma_{2i}\theta_i^2)$. We then have from (4.39) that

$$
\dot{V} \leq -cV + d. \tag{4.40}
$$

Similar to the analysis given in the proof of Theorem 4.1, we readily derive that

$$
\|S\| = \sqrt{S^T S} \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}V_1} \leq \sqrt{\frac{2d}{\lambda_{\min}(\Lambda)c}} \tag{4.41}
$$

after some finite time t^*, which further implies that E and $\dot{E}$ are bounded after some finite time t^*. Therefore, E and $\dot{E}$ are CUUB.

In the sequel, we prove that all the internal system signals remain bounded. By solving (4.40), it is deduced that

$$
V(t) \leq \exp^{-ct} V(0) + \frac{d}{c} \in L_\infty \tag{4.42}
$$

for all $t \geq 0$, which then implies that $S \in L_\infty$, $E \in L_\infty$, $\dot{E} \in L_\infty$, $\hat{c}_i \in L_\infty$, and $\hat{\theta}_i \in L_\infty$ for all $i = 1, \ldots, N$. Therefore, $U \in L_\infty$; that is, the control input remains bounded. □

4.2.3 Simulation Example

In this section, we demonstrate the effectiveness of the distributed adaptive control algorithm proposed in (4.28)–(4.30) by a simulation example. For convenience, in the simulation example, we still use four agents with the communication topology among the agents being the same as in Sect. 4.1.3, as shown in Fig. 4.7. Each edge weight is set to be 1.

The four agents are modeled by the following dynamics,

$$\begin{aligned} \dot{x}_i =& v_i, \\ \dot{v}_i =& g_i u_i + f_i + f_{di}, \quad i = 1, 2, 3, 4 \end{aligned} \tag{4.43}$$

with

$$\begin{aligned} \dot{v}_1 =&(5 + 0.5 \exp^{-|x_1|})u_1 + l \sin(v_1) + l \exp^{-v_1 - x_2} v_1 + f_{d1}, \\ \dot{v}_2 =&(5 + 0.5 \exp^{-|x_2|})u_2 + l \sin(v_2) + l \exp^{-x_1 - v_2^2} v_2^2 + f_{d2}, \\ \dot{v}_3 =&(5 + 0.5 \exp^{-|x_3|})u_3 + l \sin(v_3) + l \exp^{-x_2 - v_3^2} v_3^2 + f_{d3}, \\ \dot{v}_4 =&(5 + 0.5 \exp^{-|x_4|})u_4 + l \sin(v_4) + l \exp^{-x_3 - v_4} v_4 + f_{d4} \end{aligned}$$

where the non-vanishing uncertainties are: $f_{d1} = 0.02 \sin(0.05t)$, $f_{d2} = 0.01 \cos(0.06t)$, $f_{d3} = 0.02 \cos(0.06t)$, and $f_{d4} = 0.01 \sin(0.05t)$.

In the simulation, only the design parameters k_i, γ_i, and σ_i $(i = 1, 2, 3, 4)$ are needed to be selected. They are chosen as: $k_i = 5$ $(i = 1, 2, 3, 4)$, $\gamma_i = 0.1$ and $\sigma_i = 0.1$ $(i = 1, 2)$, $\varepsilon = 0.01$ and $\beta = 1$. In addition, according to the core information of the system, the core functions $\varphi_i(\cdot)$ $(i = 1, 2, 3, 4)$ are easily selected as $\varphi_1 = 1 + |v_1|$, $\varphi_2 = 1 + v_2^2$, $\varphi_3 = 1 + v_3^2$, and $\varphi_4 = 1 + |v_4|$.

Figures 4.8 and 4.9 show the position and velocity trajectories of all the four agents, respectively. Figures 4.10 and 4.11 present the neighborhood position errors and velocity synchronization errors of all the four agents, which verify that system (4.43) achieves both the position synchronization and velocity synchronization with small residual errors under the control algorithm proposed in (4.28). Figure 4.12

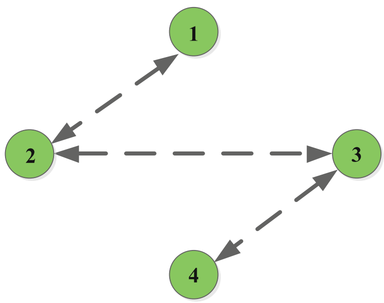


Fig. 4.7 Communication graph topology between the four agents

Fig. 4.8 The position trajectories x_i of the four agents

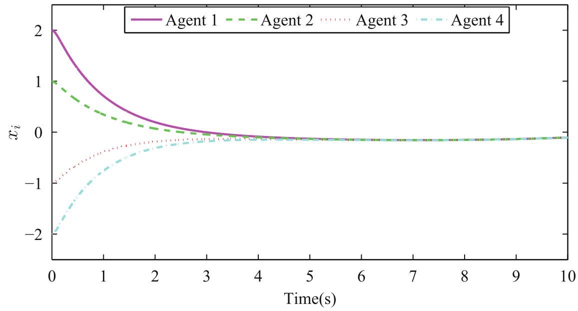


Fig. 4.9 The velocity trajectories v_i of the four agents

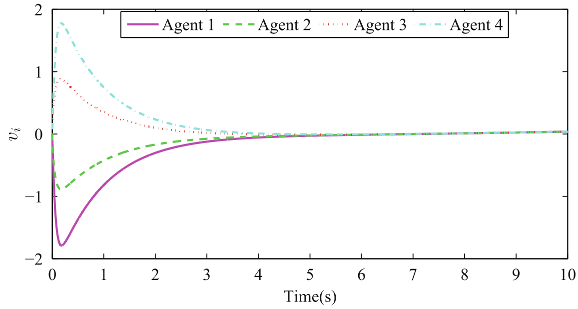


Fig. 4.10 The position neighborhood errors e_i of the four agents

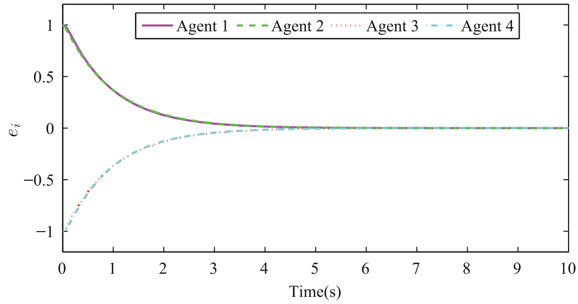
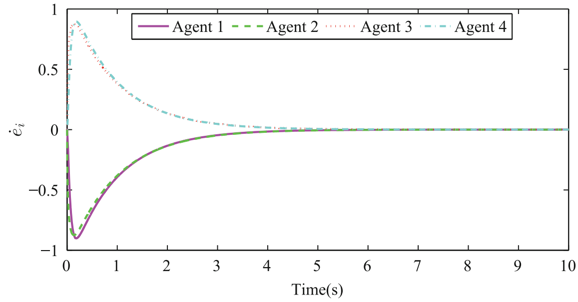


Fig. 4.11 The velocity neighborhood errors $\dot{e}_i$ of the four agents



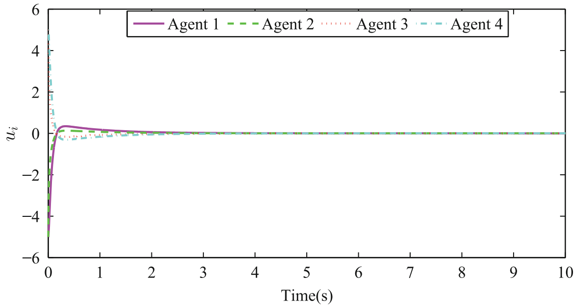


Fig. 4.12 The control inputs u_i of the four agents

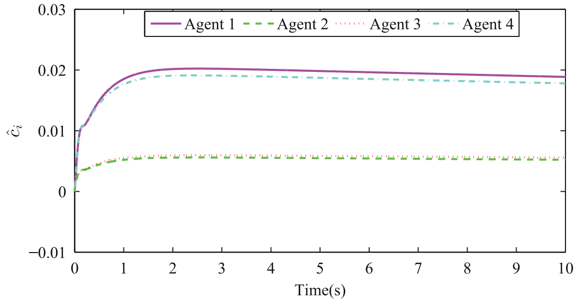


Fig. 4.13 The parameter estimation $\hat{c}_i$ of the four agents

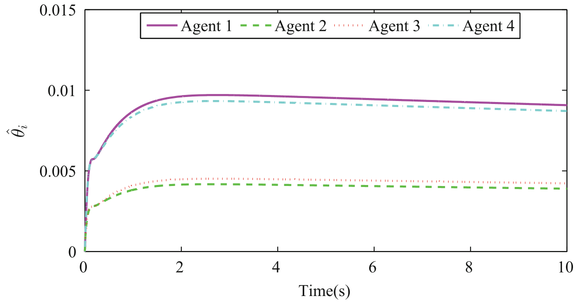


Fig. 4.14 The parameter estimation $\hat{\theta}_i$ of the four agents

shows the control input signals u_i $(i = 1, 2, 3, 4)$ of the four agents, from which we observe that all the control input signals of the four agents remain continuous and bounded. Figures 4.13 and 4.14 represent the parameter estimation values of the virtual parameters c_i and θ_i $(i = 1, 2, 3, 4)$, respectively, from both of which we see that all the parameter estimation values $\hat{c}_i$ and $\hat{\theta}_i$ $(i = 1, 2, 3, 4)$ are bounded under the proposed control scheme.

4.3 High-Order Systems

In this section, we address the leaderless consensus problem for high-order nonlinear multi-agent systems subject to non-parametric uncertainties and unknown bounded external disturbances.

We consider a group of N agents with scalar high-order dynamics

$$\begin{aligned}\dot{x}_{q,i}(t) &= x_{q+1,i}(t), \quad q = 1, \ldots, n+1,\\ \dot{x}_n(t) &= g_i(\cdot)u_i(t) + f_i(z_i(t), t) + f_{di}(z_i(t), t), \quad i = 1, \ldots, N,\end{aligned} \tag{4.44}$$

where $x_{q,i} \in \mathbb{R}\,(q = 1, \ldots, n)$ and $u_i \in \mathbb{R}$ are the qth state and control input of the ith agent, respectively, g_i is the control gain, f_i denotes the system nonlinearity, and f_{di} denotes the non-vanishing uncertainty. In addition, $z_i = \bigcup_{j \in \mathscr{N}_i \bigcup i} \{x_{1,j}, \ldots, x_{n,j}\}$.

Suppose that g_i, f_i and f_{di} satisfy Assumption 4.1. Of particular interest in this section is to design distributed adaptive controller for the high-order nonlinear multi-agent system (4.44) with unknown time-varying gain and non-parametric uncertainties to make the system achieve consensus. We also give strict theoretical analysis and numerical simulation to demonstrate and verify the effectiveness of the proposed control algorithm.

4.3.1 Consensus Error Dynamics

Let the local neighborhood state error be defined as in (4.2). To solve the leaderless consensus problem for high-order multi-agent systems, we introduce two filtered variables ζ_i and ω_i $(i = 1, \ldots, N)$ as

$$\begin{aligned}\zeta_i &= \beta_1 x_{1,i} + \beta_2 x_{2,i} + \cdots + \beta_{n-1} x_{n-1,i} + x_{n,i}, \quad i = 1, \ldots, N,\\ \omega_i &= \beta_1 e_{1,i} + \beta_2 e_{2,i} + \cdots + \beta_{n-1} e_{n-1,i} + e_{n,i}, \quad i = 1, \ldots, N,\end{aligned} \tag{4.45}$$

where β_q $(q = 1, \ldots, n-1)$ are some positive constants chosen by the designer such that the polynomial $r^{n-1} + \beta_{n-1} r^{n-2} + \cdots + \beta_1$ is Hurwitz. Let $\zeta = [\zeta_1, \ldots, \zeta_N]^T \in \mathbb{R}^N$ and $\omega = [\omega_1, \ldots, \omega_N]^T \in \mathbb{R}^N$ such that $\omega = \mathscr{L}\zeta$.

With the filtered variable ζ, the consensus error dynamics for system (4.44) is rewritten as the following compact form

$$\begin{aligned}\dot{\zeta} &= \beta_1 X_2 + \beta_2 X_3 + \cdots + \beta_{n-1} X_n + \dot{X}_n\\ &= GU + F + F_d + \beta_1 X_2 + \beta_2 X_3 + \cdots + \beta_{n-1} X_n\end{aligned} \tag{4.46}$$

with $X_q = [x_{q,1}, \ldots, x_{q,N}]^T \in \mathbb{R}^N$ $(q = 1, \ldots, n)$, $G = \text{diag}\{g_i\} \in \mathbb{R}^{N \times N}$, $U = [u_1, \ldots, u_N]^T \in \mathbb{R}^N$, $F = [f_1, \ldots, f_N]^T \in \mathbb{R}^N$, and $F_d = [f_{d1}, \ldots, f_{dN}]^T \in \mathbb{R}^N$.

4.3.2 Controller Design and Stability Analysis

In this subsection, we conduct the controller design and stability analysis for system (4.44).

The controller for the n-order ($n \geq 2$) multi-agent system (4.44) is designed as

$$u_i = -k_i\omega_i + u_{ci}, \quad i = 1, \ldots, N \tag{4.47}$$

where the compensation term u_{ci} is given as

$$u_{ci} = -\frac{\hat{b}_i\phi_i^2\omega_i}{\phi_i|\omega_i| + \varepsilon} - \frac{\hat{\theta}_i\omega_i}{|\omega_i| + \varepsilon}, \tag{4.48}$$

and the updated law is designed as

$$\dot{\hat{b}}_i = -\gamma_{1i}\sigma_{1i}\hat{b}_i + \frac{\gamma_{1i}\phi_i^2\omega_i^2}{\phi_i|\omega_i| + \varepsilon}, \quad \dot{\hat{\theta}}_i = -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \frac{\gamma_{2i}\omega_i^2}{|\omega_i| + \varepsilon}, \tag{4.49}$$

where k_i, γ_{1i}, γ_{2i}, σ_{1i}, and σ_{2i} are finite positive design parameters, $\varepsilon > 0$ is a small finite constant, $\hat{b}_i$ and $\hat{\theta}_i$ are the estimation values of the virtual parameters b_i and θ_i with

$$b_i = \max\{c_{fi}, \beta_1, \ldots, \beta_{n-1}\}, \quad |f_{di}| \leq \theta_i \tag{4.50}$$

and

$$\phi_i(\cdot) = \varphi_i(\cdot) + |x_{2,i}(t)| + \cdots + |x_{n,i}(t)| \tag{4.51}$$

with $\varphi_i(\cdot)$ being a scalar and readily computable function given in Assumption 4.1.

The main result in this subsection is given as follows.

Theorem 4.3 *Suppose that the graph topology $\mathscr{G}$ is undirected and connected, and g_i, f_i, and f_{di} satisfy Assumption 4.1. The n-order ($n \geq 2$) nonlinear multi-agent system* (4.44) *with the distributed control scheme* (4.47)*, the compensation control term* (4.48)*, and the updated control law* (4.49) *achieves CUUB leaderless consensus. In addition, all the internal system signals remain bounded.*

Proof With the positive definite matrix $\Lambda = U_N^T\tilde{\Lambda}^{-1}U_N$ given in Sect. 3.3.1, we construct the distributed Lyapunov function as

$$V_1 = \frac{1}{2}\omega^T\Lambda\omega \tag{4.52}$$

By following the similar procedure as in the proof in (3.8), we can get

$$V_1 = \frac{1}{2}\omega^T \Lambda \omega = \frac{1}{2}\zeta^T \mathscr{L} \zeta. \tag{4.53}$$

Taking the derivative of V_1 along the error dynamic (4.46) yields

$$\begin{aligned}\dot{V}_1 &= \omega^T \dot{\zeta} = \omega^T (GU + F + F_d + \beta_1 X_2 + \beta_2 X_3 + \cdots + \beta_{n-1} X_n) \\ &= \sum_{i=1}^{N} \omega_i (g_i u_i + f_i + f_{di} + \beta_1 x_{2,i} + \beta_2 x_{3,i} + \cdots + \beta_{n-1} x_{n,i})\end{aligned} \tag{4.54}$$

Denote by $b_i = \max\{c_{fi}, \beta_1, \ldots, \beta_{n-1}\}$ and $\phi_i = \varphi_i + |x_{2,i}| + \cdots + |x_{n,i}|$, we then get from Assumptions 4.1 that

$$\begin{aligned}&|f_i + \beta_1 x_{2,i} + \beta_2 x_{3,i} + \cdots + \beta_{n-1} x_{n,i}| \\ \le &c_{fi}\varphi_i + \beta_1 |x_{2,i}| + \beta_2 |x_{3,i}| + \cdots + \beta_{n-1} |x_{n,i}| \\ \le &b_i \phi_i\end{aligned} \tag{4.55}$$

Upon using the control law u_i given in (4.47) and the compensating unit u_{ci} given in (4.48), one gets the time derivative of V_1 as

$$\begin{aligned}\dot{V}_1 =&\sum_{i=1}^{N} \omega_i \left(g_i u_i + f_i + f_{di} + \beta_1 x_{2,i} + \beta_2 x_{3,i} + \cdots + \beta_{n-1} x_{n,i}\right) \\ =&\sum_{i=1}^{N} \omega_i \left[g_i(-k_i \omega_i + u_{ci}) + f_i + f_{di} + \beta_1 x_{2,i} + \beta_2 x_{3,i} + \cdots + \beta_{n-1} x_{n,i}\right] \\ \le &-\sum_{i=1}^{N} k_i g_i \omega_i^2 + \sum_{i=1}^{N} \left[-g_i \frac{\hat{b}_i \phi_i^2 \omega_i^2}{\phi_i |\omega_i| + \varepsilon} - g_i \frac{\hat{\theta}_i \omega_i^2}{|\omega_i| + \varepsilon} + |\omega_i| b_i \phi_i + |\omega_i| \theta_i\right] \\ \le &-\underline{k}\underline{g} \sum_{i=1}^{N} \omega_i^2 + \sum_{i=1}^{N} \left[\left(b_i - \underline{g}\hat{b}_i\right) \frac{\phi_i^2 \omega_i^2}{\phi_i |\omega_i| + \varepsilon} + \left(\theta_i - \underline{g}\hat{\theta}_i\right) \frac{\omega_i^2}{|\omega_i| + \varepsilon} + b_i \varepsilon + \theta_i \varepsilon\right].\end{aligned} \tag{4.56}$$

Next, we construct the adaptive part of the Lyapunov function as

$$V_2 = \frac{1}{2\underline{g}} \tilde{B}^T \Gamma_1^{-1} \tilde{B} + \frac{1}{2\underline{g}} \tilde{\theta}^T \Gamma_2^{-1} \tilde{\theta} \tag{4.57}$$

with $\tilde{B} = [\tilde{b}_1, \ldots, \tilde{b}_N]^T, \tilde{\theta} = [\tilde{\theta}_1, \ldots, \tilde{\theta}_N]^T$, $\Gamma_1 = \mathrm{diag}\{\gamma_{1i}\}$, and $\Gamma_2 = \mathrm{diag}\{\gamma_{2i}\}$. By applying the updated laws $\dot{\hat{b}}_i$ and $\dot{\hat{\theta}}_i$ given in (4.49), one then gets the derivative of V_2 as

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^{N}\left(b_i - \underline{g}\hat{b}_i\right)\left(-\frac{\dot{\hat{b}}_i}{\gamma_{1i}}\right) + \sum_{i=1}^{N}\left(\theta_i - \underline{g}\hat{\theta}_i\right)\left(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}\right)\\ &= \sum_{i=1}^{N}\left(b_i - \underline{g}\hat{b}_i\right)\left(\sigma_{1i}\hat{b}_i - \frac{\phi_i^2\omega_i^2}{\phi_i|\omega_i| + \varepsilon}\right) + \sum_{i=1}^{N}\left(\theta_i - \underline{g}\hat{\theta}_i\right)\left(\sigma_{2i}\hat{\theta}_i - \frac{\omega_i^2}{|\omega_i| + \varepsilon}\right).\end{aligned} \tag{4.58}$$

Now, we construct the Lyapunov function candidate as

$$V = V_1 + V_2$$

with V_1 and V_2 being given in (4.53) and (4.57), respectively. The derivative of V thus follows from (4.56) and (4.58) that

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \leq -\underline{k}\underline{g}\sum_{i=1}^{N}\omega_i^2 + \sum_{i=1}^{N}(b_i\varepsilon + \theta_i\varepsilon) + \sum_{i=1}^{N}\sigma_{1i}\tilde{b}_i\hat{b}_i + \sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i\hat{\theta}_i. \tag{4.59}$$

Upon using the inequality, $ab = \frac{1}{2}\left[(a+b)^2 - a^2 - b^2\right]$, we have

$$\tilde{b}_i\hat{b}_i = \frac{1}{2\underline{g}}\left[b_i^2 - \tilde{b}_i^2 - (b_i - \tilde{b}_i)^2\right], \quad \tilde{\theta}_i\hat{\theta}_i = \frac{1}{2\underline{g}}\left[\theta_i^2 - \tilde{\theta}_i^2 - (\theta_i - \tilde{\theta}_i)^2\right]. \tag{4.60}$$

By inserting (4.60) into (4.59), we arrive at,

$$\begin{aligned}\dot{V} \leq& -\underline{k}\underline{g}\omega^T\omega + \sum_{i=1}^{N}(b_i + \theta_i)\varepsilon\\ &- \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}\tilde{b}_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}b_i^2 - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2\\ \leq& -\underline{k}\underline{g}\frac{1}{\lambda_{\max}(\Lambda)}\omega^T\Lambda\omega - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}\tilde{b}_i^2 - \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\tilde{\theta}_i^2\\ &+ \sum_{i=1}^{N}(b_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}b_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2\\ \leq& -\frac{2\underline{k}\underline{g}}{\lambda_{\max}(\Lambda)}\cdot\frac{1}{2}\omega^T\Lambda\omega - \frac{\underline{\sigma}_1\underline{\gamma}_1}{2\underline{g}}\sum_{i=1}^{N}\frac{\tilde{b}_i^2}{\gamma_{1i}} - \frac{\underline{\sigma}_2\underline{\gamma}_2}{2\underline{g}}\sum_{i=1}^{N}\frac{\tilde{\theta}_i^2}{\gamma_{2i}}\\ &+ \sum_{i=1}^{N}(b_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{1i}b_i^2 + \frac{1}{2\underline{g}}\sum_{i=1}^{N}\sigma_{2i}\theta_i^2,\end{aligned} \tag{4.61}$$

where $\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}$, $\underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}$, $\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$, and $\underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. Let $c = \min\{\frac{2k\underline{g}}{\lambda_{\max}(\Lambda)}, \underline{\sigma}_1\underline{\gamma}_1, \underline{\sigma}_2\underline{\gamma}_2\}$, $d = \sum_{i=1}^{N}(b_i + \theta_i)\varepsilon + \frac{1}{2\underline{g}}\sum_{i=1}^{N}(\sigma_{1i}b_i^2 + \sigma_{2i}\theta_i^2)$. We then have from (4.61) that

$$\dot{V} \leq -cV + d. \tag{4.62}$$

Similar to the analysis given in the proof of Theorem 4.1, we readily derive that

$$\|\omega\| = \sqrt{\omega^T\omega} \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}V_1} \leq \sqrt{\frac{2d}{\lambda_{\min}(\Lambda)c}} \tag{4.63}$$

after some finite time t^*, which further implies that all the qth ($q = 1, \ldots, n$) neighborhood error E_q are bounded after some finite time t^* by the definition of ω according to Lemma 3 in [3]. Therefore, all E_q ($q = 1, \ldots, n$) are CUUB.

In the sequel, we prove all the internal system signals are bounded. By solving (4.62), it is deduced that $V(t) \leq \exp^{-ct} V(0) + \frac{d}{c} \in L_\infty$ for all $t \geq 0$, which then implies that $\omega \in L_\infty$, $E_q \in L_\infty$ ($q = 1, \ldots, n$), $\hat{b}_i \in L_\infty$ and $\hat{\theta}_i \in L_\infty$ for all $i = 1, \ldots, N$. Hence, $U \in L_\infty$; i.e., the control input is bounded. This completes the proof. ∎

4.4 Discussion on Cooperative Adaptive Consensus of MAS with Non-parametric Uncertainties

1. Note that each node in systems (4.1), (4.25), and (4.44) has its own distinct dynamics, and both the node nonlinearities $f_i(\cdot)$ and the node non-vanishing uncertainties (including the external disturbances) $f_{di}(\cdot)$ are unknown and non-identical. In addition, $f_i(\cdot)$ and $f_{di}(\cdot)$ associated with the ith agent are not only related to the state of the ith agent itself but also influenced by its neighbors; this is in contrast to most existing works that ignore the coupling effects among the neighboring agents.
2. The impact of the parameter variation of the agent is explicitly reflected in the models (4.1), (4.25), and (4.44). Note that such variation literally leads to unknown and time-varying control gains in the model, thus making the control design and stability analysis much more involved as compared to the case of known and constant control gains [3–11], and unknown yet constant gains [12].
3. It should be stressed that in practice it would be very difficult, if not impossible, to obtain the exact values of those bounds involved in Assumptions 4.1. In this work, the control scheme is developed without the need for analytical estimation of such bounds, although the fact that those bounds do exist is used for stability analysis.

4. The control scheme as given in (4.3) and (4.4) consists of a robust control and an adaptive control; the later contains two crucial parameters that are adaptively adjusted during the system operation. The updating algorithm (4.5) has a constant decay term to prevent the estimated parameter from drifting. The role of the small constant ε in the denominator is to ensure the continuity of the control action. The reason for including the other term in the algorithm is to accommodate the lumped uncertainties in the model to guarantee CUUB consensus, as seen in the stability analysis.
5. In building the compensation control u_{ci}, the adaptively adjusted parameters $\hat{c}_{fi}$ and $\hat{\theta}_i$ rather than c_{fi} and θ_i are used, thus avoiding the conservative and excessive control effort. In particular, during the start-of stage, the initial control effort from u_{ci} can be made as small as zero by setting $\hat{c}_i(0) = 0$ and $\hat{\theta}_i(0) = 0$, which can be chosen freely by the designer.
6. In developing the control schemes, a number of virtual parameters such as c_{fi} and θ_i are defined and used in the stability analysis, but those parameters are not involved in the control algorithms, thus analytical estimation of those parameters (a nontrivial task) is not needed in setting up and implementing the proposed control strategies.

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Chapter 5
Cooperative Adaptive Consensus for Multi-agent Systems Under Directed Topology

In Chap. 4, we studied the controller design and stability analysis for the cooperative adaptive leaderless consensus control problem of networked multi-agent systems, including first-order systems, second-order systems, and high-order systems, all with unknown time-varying gain and non-parametric/non-vanishing uncertainties under the undirected communication topology condition. Considering that the networked communication among the subsystems is not always bidirectional in practice, it is necessary to take the one-way directed communication condition into consideration. In comparison with the undirected topology condition, the one-way directed communication topology obviously complicates the underlying problem significantly. The technical difficulties mainly arise from the asymmetry of the Laplacian matrix associated with the directed topology graph. In this chapter, we attempt to address the distributed leaderless consensus control problem of networked multi-agent systems under local and one-way directed communication conditions.

The key to tackle the technical difficulty arising from the directed communication constraints depends on how to construct the Lyapunov function on graph. In Sect. 5.1, we derive some results related to the Laplacian matrix, which is crucial to construct the Lyapunov function suitable for the leaderless consensus stability analysis, and show how to construct the special Lyapunov function on graph. In Sect. 5.2, we address the leaderless cooperative consensus control problem for first-order continuous-time multi-agent systems subject to uncertainties under directed topology. The basic results and key ideas of cooperative control under directed topology emerge from this study. Section 5.3 investigates the leaderless consensus control problem of nonlinear multi-agent systems with second-order dynamics in the presence of unknown time-varying gain and non-parametric/non-vanishing uncertainties. The focus of this section is the controller design and stability analysis for the cooperative consensus of second-order uncertain multi-agent systems, in which both unknown time-varying gain and non-parametric/non-vanishing modeling uncertainties should be compensated and the directed topology condition should be addressed

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simultaneously. Section 5.4 gives a simulation example to illustrate the effectiveness of the distributed control scheme proposed in Sect. 5.3 for the second-order uncertain multi-agent systems under directed topology.

5.1 Graph Laplacian Potential Under Directed Topology

Graph Laplacian potential was introduced in [1] for undirected graph with 0–1 adjacency elements. Later, it was extended to weighted undirected graphs [2]. In this section, we extend the concept of Laplacian potential to directed graphs and show how the Lyapunov function is related to the directed graph Laplacian and the neighborhood error. In the next section, we show how to apply such Laplacian potential in the Lyapunov analysis of the leaderless consensus problems for nonlinear MAS under directed topology.

Before moving on, we first present several useful lemmas. First, it is important to note that for directed communication topology the Laplacian matrix $\mathscr{L}$ is no longer symmetric in contrast to the case associated with undirected graph where the Laplacian matrix is symmetric, and semi-positive definite with one simple zero eigenvalue. This imposes technical challenge in constructing Lyapunov function. To circumvent the technical difficulty in constructing Lyapunov function arising from asymmetric property of the original Laplacian matrix $\mathscr{L}$ under directed graph, a new matrix Q is introduced that exhibits the property of symmetry and semi-positive definiteness with one simple zero eigenvalue, as stated in the following Lemma.

Lemma 5.1 *Suppose that the digragh $\mathscr{G}$ is strongly connected. Let $p = [p_1, p_2, \ldots, p_n]^T$ be the left eigenvector of $\mathscr{L}$ associated with the zero eigenvalue. Define*

$$Q = (diag(p)\mathscr{L} + \mathscr{L}^T diag(p)), \tag{5.1}$$

where $diag(p) = diag\{p_1, \ldots, p_N\} \in \mathbb{R}^{N\times N} > 0$, then Q is a Laplacian matrix corresponding to a connected undirected graph topology. More specifically, Q is symmetric, positive semi-definite and has a simple zero eigenvalue with all the other eigenvalues positive and real.

Proof It is worth noting that in the work by [3] it has been shown that Q is symmetric and positive semi-definite. Here we further reveal that such Q is a Laplacian corresponding to a connected undirected graph and thus has only one simple zero eigenvalue.

First, note that $p^T\mathscr{L} = 0_{1\times N}$, which implies that $\sum_{j=1}^{N} a_{ij} p_i = \sum_{j=1}^{N} a_{ji} p_j$ for all $i, j \in \{1, \ldots, N\}$, and then it is derived that

$$
\begin{aligned}
Q &= \mathrm{diag}(p)\mathscr{L} + \mathscr{L}^T \mathrm{diag}(p) \\
&= \begin{bmatrix}
\sum_{j=1}^{N}(a_{1j}p_1 + a_{j1}p_j) & \cdots & -a_{1N}p_1 - a_{N1}p_N \\
-a_{21}p_2 - a_{12}p_1 & \cdots & -a_{2N}p_2 - a_{N2}p_N \\
\vdots & \vdots & \vdots \\
-a_{N1}p_N - a_{1N}p_1 & \cdots & \sum_{j=1}^{N}(a_{Nj}p_N + a_{jN}p_j)
\end{bmatrix}
\end{aligned} \tag{5.2}
$$

Let $z_{ij} = a_{ij}p_i + a_{ji}p_j$ ($i, j \in \{1, \ldots, N\}$) and $\mathscr{Z} = [z_{ij}]$. It is interesting to note that since $p_i > 0$ ([4]), we have $z_{ij} > 0 \Leftrightarrow a_{ij} > 0$ or $a_{ji} > 0$, and $z_{ii} = 0 \Leftrightarrow a_{ii} = 0$. Note that $\mathscr{A} = [a_{ij}]$ is the weighted adjacency matrix of the strongly connected digraph $\mathscr{G}$, and moreover, $z_{ij} = z_{ji}$. It is thus seen that $\mathscr{Z}$ so defined is symmetric and is a weighted adjacency matrix associated with a connected undirected graph according to the definition of the adjacency matrix in graph theory. Define $b_i = \sum_{j=1}^{N} z_{ij}$ and $\mathscr{B} = \mathrm{diag}(b_1, \ldots, b_N) \in \mathbb{R}^{N\times N}$. It is straightforward from (5.2) that $Q = \mathscr{B} - \mathscr{Z}$, and it thus holds that Q is a Laplacian of a connected undirected graph topology by the Laplacian definition, that is, Q is symmetric, positive semi-definite and has a simple zero eigenvalue with all the other eigenvalues positive and real. □

The significance of the feature on Q (defined in (5.1)) as revealed in Lemma 5.1 is that the original directed graph problem (where the Laplacian matrix $\mathscr{L}$ is non-symmetric, thus imposing significant challenge for consensus design because one cannot use such $\mathscr{L}$ to construct Lyapunov function directly) can be treated as an undirected graph problem (where the newly defined matrix Q is equivalent to the Laplacian with the feature as stated in Lemma 5.1). Furthermore, such feature allows for the development of the following two lemmas that are crucial for the construction of the distributed part of Lyapunov function subject to directed graph.

Lemma 5.2 *For a strongly connected directed network with Laplacian $\mathscr{L}$, let Q be defined as in Lemma 5.1, then $\forall x \neq 0_N$,*

$$x^T Q x = 0 \tag{5.3}$$

if and only if

$$x = c1_N \tag{5.4}$$

where $c \neq 0$ is a constant. Moreover, $\min\limits_{x\neq c1_N} \frac{x^T Qx}{x^T x}$ exists and

$$0 < \min_{x\neq c1_N} \frac{x^T Qx}{x^T x} \leq \sum_{i=2}^{N} \lambda_i(Q). \tag{5.5}$$

Proof According to Lemma 5.1, Q is a Laplacian matrix of a connected undirected graph. Let Ξ be the diagonal matrix associated with Q, that is, there exists an orthogonal matrix $R = (r_1, r_2, \ldots, r_N)$ such that $Q = R\Xi R^T$. Let $y = R^T x = [y_1, \ldots, y_N]^T$. Then

$$\begin{aligned} x^T Qx = x^T R\Xi R^T x = y^T \Xi y = \sum_{i=1}^{N} \lambda_i(Q) y_i^2 \\ = 0 \cdot y_1^2 + \lambda_2(Q) y_2^2 + \cdots + \lambda_N(Q) y_N^2, \end{aligned} \tag{5.6}$$

where $\lambda_i(Q)$ $(i = 1, \ldots, N)$ denotes the eigenvalue of Q. From (5.6), it can be concluded that for $\forall x \neq 0_N$ (i.e., $\forall y \neq 0_N$),

$$x^T Qx = y^T \Lambda y = 0 \tag{5.7}$$

if and only if

$$\begin{cases} y_1 \neq 0, \\ y_i = 0, \quad i = 2, \ldots, N \end{cases} \tag{5.8}$$

Note that $r_1, r_2, \ldots, r_N$ are the N eigenvectors of Q associated with the N different eigenvalues $\lambda_1(Q)$, $\lambda_2(Q)$, $\cdots$, $\lambda_N(Q)$, respectively. Thus, $\mathbb{R}^N = \text{span}\{r_1, r_2, \ldots, r_N\}$, and moreover, $r_i \perp r_j$ $(i \neq j, i, j = 1, \ldots, N)$. There exist some constants a_i $(i = 1, \ldots, N)$ such that

$$x = a_1 r_1 + a_2 r_2 + \cdots + a_N r_N \tag{5.9}$$

and then

$$y_i = r_i^T x = r_i^T (a_1 r_1 + a_2 r_2 + \cdots + a_N r_N) = a_i r_i^T r_i. \tag{5.10}$$

Thus, the condition in (5.8) is equal to

$$\begin{cases} a_1 r_1^T r_1 \neq 0, \\ a_i r_i^T r_i = 0, \ i = 2, \ldots, N \end{cases} \Longleftrightarrow \begin{cases} a_1 \neq 0, \\ a_i = 0, \ i = 2, \ldots, N \end{cases}$$

which implies that

$$x = a_1 r_1. \tag{5.11}$$

Note that 1_N is the eigenvector of Q associated with the simple zero eigenvalue, i.e., $\lambda_1(Q)$, therefore $r_1 = k 1_N$, with k being a nonzero constant, and then

$$x = a_1 k 1_N = c 1_N \tag{5.12}$$

where $c = a_1 k$. Thus, $\forall x \neq 0_N$,

$$\begin{aligned}\min_{x \neq c1_N} \frac{x^T Q x}{x^T x} &\leq \min_{x \neq c1_N, x^T x = 1} x^T Q x = \min_{x \neq c1_N, y^T y = 1} y^T \Xi y \\ &\leq \min_{x \neq c1_N, y^T y = 1, y_3 = \cdots = y_N = 0} \sum_{i=1}^{N} \lambda_i(Q) y_i^2 \leq \lambda_2(Q). \end{aligned} \tag{5.13}$$

On the other hand, $\forall x \neq 0_N$, since $x \neq c1_N$, then $x^T Q x \neq 0$, which, together with the fact that Q is a positive semi-definite, implies $x^T Q x > 0$, and therefore (5.5) holds. □

Upon using Lemma 5.2, the following result can be established.

Lemma 5.3 *$\forall E \neq 0_N$, there exists a constant $k_m > 0$ such that*

$$\frac{E^T Q E}{E^T E} \geq k_m. \tag{5.14}$$

Proof Note that $p^T E = p^T \mathscr{L} X = 0$, i.e., $\sum_{i=1}^{N} p_i e_i = 0$, and $p_i > 0$ according to [4], from which we know that for $E \neq 0$ and $i = 1, \ldots, N$, it is impossible that $\text{sgn}(e_i) = 1$ (or $\text{sgn}(e_i) = -1$). Therefore, $E \neq c1_N$ with c being a nonzero constant. According to Lemma 5.2, there exists a constant $k_\text{m} = \min\limits_{E \neq 0_N} \frac{E^T QE}{E^T E} > 0$ such that $\frac{E^T QE}{E^T E} \geq k_\text{m} > 0$. □

According to the above analysis, we next construct the first part of Lyapunov function candidate as

$$V_1 = E^T \text{diag}(p) E, \tag{5.15}$$

from which we see that if $V_1 \to 0$ ($V_1 \in L_\infty$), then $E \to 0$ ($E \in L_\infty$), implying that the leaderless consensus (CUUB leaderless consensus) is achieved.

The next section shows how the graph Laplacian matrix works in the Lyapunov stability analysis of leaderless consensus of nonlinear MAS under the local directed topology.

5.2 Leaderless Consensus of First-Order Systems Under Directed Topology

5.2.1 *Problem Formulation*

The focus of this section is to show how to tackle the directed topology in the leaderless consensus control problem, so here we consider the following normal-form continuous-time system with first-order dynamics,

$$\dot{x}_i(t) = g_i(z_i(t), t)u_i(t) + f_i(z_i(t), t) + f_{di}(z_i(t), t) \tag{5.16}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i is the control gain, f_i denotes the system nonlinearities, and f_{di} denotes the non-vanishing uncertainties; $z_i = \bigcup_{j\in\mathcal{N}_i \bigcup i} x_j$.

Assumption 5.1 The control gain g_i is known and nonzero, $f_i(z_i, t)$ and $f_{di}(z_i, t)$ are known, either continuously differentiable or Lipschitz.

5.2.2 *Controller Design and Stability Analysis for First-Order Systems*

The control objective of this section is to design a control strategy under the directed topology condition such that the leaderless consensus errors uniformly converge to zero asymptotically, and meanwhile, to give the strict stability analysis.

To this end, the control input for each ith ($i = 1, \ldots, N$) agent is designed as

$$u_i = -\frac{1}{g_i}\left[ke_i + f_i + f_{di}\right], \tag{5.17}$$

where $k > 0$ is design parameter chosen by the designer.

Theorem 5.1 ***Distributed Leaderless Consensus Control under Directed Topology*** *Consider the nonlinear MAS as described by (5.16) under Assumption 5.1. Suppose that the communication topology $\mathcal{G}$ is directed and strongly connected. If the distributed control law (5.17) is applied, then asymptotically leaderless consensus is achieved in that the leaderless consensus errors e_i ($i = 1, \ldots, N$) converge to zero as $t \to \infty$.*

Proof By substituting (5.17) into (5.16), one gets

$$\dot{x}_i = -ke_i. \tag{5.18}$$

We then choose the Lyapunov function candidate as

$$V = \frac{1}{2}E^T \mathrm{diag}(p)E, \tag{5.19}$$

whose derivative along (5.18) is

$$\dot{V}(t) = E^T \mathrm{diag}(p)\dot{E} = E^T(\mathrm{diag}(p)\mathcal{L})\dot{X} = -kE^T(\mathrm{diag}(p)\mathcal{L})E. \tag{5.20}$$

Upon using Lemma 5.3, we have

$$\dot{V}(t) = -\,kE^T\,(\mathrm{diag}(p)\mathscr{L})\,E \leq -\frac{kk_{\mathrm{m}}}{2}E^T E = -\frac{kk_{\mathrm{m}}}{2}\sum_{i=1}^{N} e_i^2. \tag{5.21}$$

Note that

$$V(t) = \frac{1}{2}E^T \mathrm{diag}(p)E \leq \frac{\bar{p}}{2}\sum_{i=1}^{N} e_i^2, \tag{5.22}$$

where $\bar{p} = \max\{p_1, \ldots, p_N\}$, and $p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue. It thus follows from (5.21) and (5.22) that

$$\dot{V}(t) \leq -\frac{kk_{\mathrm{m}}}{\bar{p}}V(t). \tag{5.23}$$

By solving the differential inequality (5.23), we get

$$V(t) \leq \exp^{-\frac{kk_{\mathrm{m}}}{\bar{p}}(t-t_0)}\,V(t_0). \tag{5.24}$$

From (5.24) we conclude that, for any given initial state $X(t_0) \in \mathbb{R}^N$,

$$V(t) \to 0 \ \text{ as } \ t \to +\infty, \tag{5.25}$$

which further implies

$$E \to 0 \ \text{ as } \ t \to +\infty, \tag{5.26}$$

meaning that the leaderless consensus is achieved asymptotically under the proposed control (5.17). □

5.3 Leaderless Consensus of Second-Order Uncertain Systems Under Directed Topology

The focus of this section is to address the leaderless consensus problem of second-order dynamic multi-agent systems with unknown time-varying gain and non-parametric uncertainties under directed topology.

5.3.1 Problem Formulation—Second-Order Uncertain Systems Under Directed Topology

Consider a group of subsystems with second-order uncertain dynamics described by

$$\begin{aligned} \dot{x}_i(t) =& v_i(t), \\ \dot{v}_i(t) =& g_i(z_i, t)u_i(t) + f_i(z_i, t) + f_{di}(z_i, t), \end{aligned} \tag{5.27}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position state, velocity state, and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i denotes the control gain, possibly time-varying and unavailable for controller design, f_i denotes the system nonlinearities, and f_{di} denotes the non-vanishing uncertainties (f_i and f_{di} are either continuously differentiable or Lipschitz). In addition, $z_i = \bigcup_{j \in \mathscr{N}_i \bigcup i} \{x_j, v_j\}$.

The following standard assumptions are in order.

Assumption 5.2

a. The unknown and time-varying gain g_i is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$) and $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$ with $\underline{g}_i$, $\bar{g}_i$ being unknown finite constants, which ensures the global controllability of the system.
b. It is assumed that $f_i(z_i, t)$ is unknown and non-parameterized. Thus, they are not available in the finite-time control design in this section. Certain crude structural information is available to allow an unknown finite positive constant c_{fi} and a known scalar function $\varphi_i(z_i)$ such that $|f_i(z_i, t)| \leq c_{fi}\varphi_i(z_i)$ for all $t \geq t_0$, where $\varphi_i(z_i)$ is bounded for all z_i $(i = 1, \ldots, N)$.
c. The unknown and non-vanishing $f_{di}(z_i, t)$ is bounded by some finite positive constant θ_i, that is, $|f_{di}(z_i, t)| \leq \theta_i < \infty$.

In addition, it is assumed for ease of notation that the agents' states are scalars, i.e., $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$. If the states are vectors, i.e., $x_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$, the results of this chapter can be extended by using the standard method involving the Kronecker products.

The local neighborhood state error for node i is

$$e_i = \sum_{j \in \mathscr{N}_i} a_{ij}(x_i - x_j) \tag{5.28}$$

Let $E = [e_1, \ldots, e_N]^T$, $X = [x_1, \ldots, x_N]^T$ such that $E = \mathscr{L}X$.

The distributed leaderless consensus control design problem confronted herein is as follows: (1) Design control scheme for all agents i in $\mathscr{G}$ under the directed communication topology condition to make an agreement. The control scheme must be distributed in the sense that they can only depend on local information about the agent itself and its neighbors in graph. (2) Overcome the technical difficulty in constructing the Lyapunov function arising from asymmetric property of the original Laplacian matrix under the directed communication topology condition. (3) Tackle unknown and time-varying control gains and accommodate unknown non-parametric uncertainties and unknown non-vanishing nonlinearities.

5.3.2 Leaderless Consensus Control Design and Stability Analysis

The control objective of this section is to design a distributed adaptive controller under the directed topology condition such that the impacts arising from the asymmetric property of the original Laplacian matrix under directed topology can be removed, and meanwhile the unknown time-varying control gain, unmodeling nonlinearities, and non-vanishing uncertainties can be compensated, and finally, the leaderless consensus of the networked multi-agent systems with second-order uncertain dynamics is ensured.

The control input for each ith ($i = 1, \ldots, N$) agent is designed to consist of two parts: (1) the negative feedback control term u_{0i} and (2) the compensation control term u_{ci}, which is of the following form,

$$u_i = u_{0i} + u_{ci}, \tag{5.29}$$

in which the negative feedback control term u_{0i} is generated by

$$u_{0i} = -c_1\delta_i, \tag{5.30}$$

and (2) the compensation control term u_{ci} is generated by

$$u_{ci} = -\hat{c}_{fi}\varphi_i \tanh(\delta_i\varphi_i/\tau_i) - \hat{\theta}_i \tanh(\delta_i/\tau_i) \tag{5.31}$$

with the updated laws

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi} + \gamma_{1i}\varphi_i \tanh(\delta_i\varphi_i/\tau_i)\delta_i, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \gamma_{2i}\tanh(\delta_i/\tau_i)\delta_i, \end{aligned} \tag{5.32}$$

where $\delta_i = v_i - v_i^*$ with $v_i^* = -c_2 e_i$, $c_1, c_2 > 0$ are design parameters, $\hat{c}_{fi}$ and $\hat{\theta}_i$ are, respectively, the estimations of c_{fi} and θ_i (the parameters to be estimated), $\varphi_i(\cdot)$ is the scalar and readily computable function as given in Assumption 5.2, and τ_i, γ_{1i}, γ_{2i}, σ_{1i}, and σ_{2i} are positive design parameters chosen by the designer.

Theorem 5.2 ***Distributed Adaptive Control for Leaderless Consensus under Directed Topology*** *Consider the second-order nonlinear MAS as described by (5.27) under Assumption 5.2. Let the communication topology be directed and strongly* *connected. If the distributed control laws (5.29)–(5.32) are applied, then leaderless consensus is achieved in that*

(1) the leaderless consensus positive errors e_i ($i = 1, \ldots, N$) and velocity errors converge to a small residual set Ω_1 defined by

$$\Omega_1 = \left\{ |e_i| \leq \sqrt{\frac{2dk_v}{\underline{p}k_d}}, \quad |v_i - v_j| \leq 2\left(\sqrt{\frac{2c_2^2 dk_v}{k_d}} + c_2\sqrt{\frac{2dk_v}{\underline{p}k_d}}\right), \right.$$
$$\left. \forall i, j \in \{1, \ldots, N\} \right\}, \tag{5.33}$$

where k_d and k_v are given, respectively, in (5.58) and (5.61), which are explicitly computable;

(2) the generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a small set Ω_{1p} defined by

$$\Omega_{1p} = \left\{ \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_1 dk_v}, |\tilde{\theta}_i| \leq \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_2 dk_v}, \ \forall i \in \{1, \ldots, N\} \right\}; \tag{5.34}$$

(3) all signals in the closed-loop system remain uniformly bounded.

Proof The proof of the result can be done by the following six steps.

Step 1. Construct the distributed part of the Lyapunov function candidate as

$$V_1(t) = \frac{1}{2} E^T \text{diag}(p) E. \tag{5.35}$$

Taking the time derivative of $V_1(t)$ along (5.27) yields that

$$\dot{V}_1(t) = E^T \text{diag}(p) \dot{E} = E^T (\text{diag}(p) \mathscr{L}) v \tag{5.36}$$

where $v = \dot{X}$. Let $v^* = -c_2 E$ be the virtual control of v, with c_2 being a design constant. Upon using Lemma 5.3, we have

$$\begin{aligned}
\dot{V}_1(t) =& - c_2 E^T (\text{diag}(p)\mathscr{L}) E + E^T (\text{diag}(p)\mathscr{L})(v - v^*) \\
\leq& - \frac{c_2 k_{\mathrm{m}}}{2} E^T E + E^T (\text{diag}(p)\mathscr{L})(v - v^*) \\
=& - \frac{c_2 k_{\mathrm{m}}}{2} \sum_{i=1}^{N} e_i^2 + \sum_{i=1}^{N} \left(v_i - v_i^*\right) \sum_{j=1}^{N} \ell_{ji} e_j,
\end{aligned} \tag{5.37}$$

where ℓ_{ji} is the (j, i)th element of $\text{diag}(p)L$. By recalling that $\delta_i = v_i - v_i^*$, we get that

$$\begin{aligned}
&\sum_{i=1}^{N}\left(v_i - v_i^*\right)\sum_{j=1}^{N}\ell_{ji}e_j \leq \sum_{i=1}^{N}|\delta_i|\sum_{j=1}^{N}|\ell_{ji}||e_j| \\
\leq&\ell_{\max}\sum_{i=1}^{N}|\delta_i|\sum_{j=1}^{N}|e_i| \leq \ell_{\max}\frac{1}{2}\left[\left(\sum_{i=1}^{N}|\delta_i|\right)^2 + \left(\sum_{j=1}^{N}|e_j|\right)^2\right] \\
\leq&\frac{1}{2}\ell_{\max}\left[N\sum_{i=1}^{N}|\delta_i|^2 + N\sum_{i=1}^{N}|e_i|^2\right] = \frac{N\ell_{\max}}{2}\sum_{i=1}^{N}\left[\delta_i^2 + e_i^2\right],
\end{aligned} \tag{5.38}$$

where $\ell_{\max} = \max_{j,i\in\{1,\ldots,N\}}|\ell_{ji}|$, and the fact that $\left(\sum_{i=1}^{N}x\right)^2 \leq N\sum_{i=1}^{N}x^2$ has been used. By substituting (5.38) into (5.37), it thus follows that

$$\dot{V}_1(t) \leq -\frac{c_2k_{\mathrm{m}}}{2}\sum_{i=1}^{N}e_i^2 + \frac{N\ell_{\max}}{2}\sum_{i=1}^{N}\left(\delta_i^2 + e_i^2\right). \tag{5.39}$$

Step 2. Define the second part of the Lyapunov function candidate as

$$V_2(t) = \frac{1}{2c_2^2}\sum_{i=1}^{N}(v_i - v_i^*)^2. \tag{5.40}$$

Taking the derivative of $V_2(t)$ yields

$$\begin{aligned}
\dot{V}_2(t) =&\frac{1}{c_2^2}\sum_{i=1}^{N}(v_i - v_i^*)(\dot{v}_i - \dot{v}_i^*) \\
=&\frac{1}{c_2^2}\sum_{i=1}^{N}\delta_i(\dot{v}_i - \dot{v}_i^*) = \frac{1}{c_2^2}\sum_{i=1}^{N}\delta_i(\dot{v}_i - c_2\dot{e}_i) \\
=&\frac{1}{c_2^2}\sum_{i=1}^{N}\left[\delta_i\dot{v}_i - c_2\delta_i\sum_{j\in\mathcal{N}_i}a_{ij}(v_i - v_j)\right].
\end{aligned} \tag{5.41}$$

Let $\bar{a} = \max_{\forall i\in\{1,\ldots,N\}}\{\sum_{j\in\mathcal{N}_i}a_{ij}\}$ and $\bar{b} = \max_{\forall i,j\{1,\ldots,N\}}\{a_{ij}\}$, we then have from the second term of the right side of (5.41) that

$$
\begin{aligned}
& -\frac{1}{c_2^2}\sum_{i=1}^{N} c_2\delta_i \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) \\
\leq & \frac{1}{c_2}\sum_{i=1}^{N} |\delta_i| \left| \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) \right| \\
= & \frac{1}{c_2}\sum_{i=1}^{N} |\delta_i|^q \left| \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) \right| \\
\leq & \frac{1}{c_2}\sum_{i=1}^{N} |\delta_i| \left(\bar{a}|v_i| + \bar{b}\sum_{j\in\mathcal{N}_i} |v_j| \right). \qquad (5.42)
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\delta_i||v_j| \leq & |\delta_i|\left|v_j - v_j^*\right| + |\delta_i|\left|v_j^*\right| \leq |\delta_i||\delta_j| + c_2|\delta_i||e_j| \\
\leq & \frac{1}{2}\left(|\delta_i|^2 + |\delta_j|^2\right) + \frac{c_2}{2}\left(|\delta_i|^2 + |e_j|^2\right), \qquad (5.43)
\end{aligned}
$$

we then have from (5.42) that

$$
\begin{aligned}
& \frac{1}{c_2}\sum_{i=1}^{N} |\delta_i| \left(\bar{a}|v_i| + \bar{b}\sum_{j\in\mathcal{N}_i} |v_j| \right) \\
\leq & \frac{1}{c_2}\sum_{i=1}^{N} \Bigg[\bar{a}|\delta_i|^2 + \frac{\bar{a}c_2}{2}\left(|\delta_i|^2 + |e_i|^2\right) \\
& + \bar{b}\frac{1}{2}\sum_{j\in\mathcal{N}_i}\left(|\delta_i|^2 + |\delta_j|^2\right) + \frac{\bar{b}c_2}{2}\sum_{j\in\mathcal{N}_i}\left(|\delta_i|^2 + |e_j|^2\right) \Bigg] \\
\leq & \frac{1}{c_2}\sum_{i=1}^{N} \left[\bar{a}|\delta_i|^2 + \frac{\bar{a}c_2}{2}\left(|\delta_i|^2 + |e_i|^2\right) + \bar{b}\bar{N}|\delta_i|^2 + \frac{\bar{b}c_2\bar{N}}{2}\left(|\delta_i|^2 + |e_i|^2\right) \right] \\
= & \frac{\bar{a} + \bar{b}\bar{N}}{c_2}\sum_{i=1}^{N} \left[\left(1 + \frac{c_2}{2}\right)|\delta_i|^2 + \frac{c_2}{2}|e_i|^2 \right], \qquad (5.44)
\end{aligned}
$$

where $\bar{N}$ denoting the maximum number of the in-degree and out-degree of each ith agent for all $i \in \{1, \ldots, N\}$. By combining (5.42) and (5.44), the second term of the right side of (5.41) can be further written as

$$
\begin{aligned}
&- \frac{1}{c_2^2} \sum_{i=1}^{N} c_2 \delta_i \sum_{j \in \mathcal{N}_i} a_{ij}(v_i - v_j) \\
&\leq \frac{\bar{a} + \bar{b}\bar{N}}{c_2} \sum_{i=1}^{N} \left[\left(1 + \frac{c_2}{2}\right) |\delta_i|^2 + \frac{c_2}{2} |e_i|^2 \right]. \qquad (5.45)
\end{aligned}
$$

By applying the control laws given in (5.29)–(5.31) to the first term of the right hand of (5.41), we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \delta_i \dot{v}_i = \sum_{i=1}^{N} \delta_i \left(g_i u_i + f_i + f_{di}\right) \\
&\leq \sum_{i=1}^{N} \Bigg[- c_1 g_i \delta_i^2 - \delta_i g_i \hat{c}_{fi} \varphi_i \tanh(\delta_i \varphi_i / \tau_i) - \delta_i g_i \hat{\theta}_i \tanh(\delta_i / \tau_i) \\
&\quad + |\delta_i| c_{fi} \varphi_i + |\delta_i| \theta_i \Bigg] \\
&\leq \sum_{i=1}^{N} \Bigg[- c_1 \underline{g} \delta_i^2 - \underline{g} \hat{c}_{fi} \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) - \underline{g} \hat{\theta}_i \delta_i \tanh(\delta_i / \tau_i) \\
&\quad + c_{fi} \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) + 0.2785 \tau_i c_{fi} + \theta_i \delta_i \tanh(\delta_i / \tau_i) + 0.2785 \tau_i \theta_i \Bigg] \\
&= \sum_{i=1}^{N} \Bigg[- c_1 \underline{g} \delta_i^2 + 0.2785 \tau_i (c_{fi} + \theta_i) + (c_{fi} - \underline{g} \hat{c}_{fi}) \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) \\
&\quad + (\theta_i - \underline{g} \hat{\theta}_i) \delta_i \tanh(\delta_i / \tau_i) \Bigg] \qquad (5.46)
\end{aligned}
$$

where we have used the fact that $0 \leq |s| - s \cdot \tanh(s/k) \leq 0.2785k$ [5].

By substituting (5.45) and (5.46) into (5.41), we arrive at

$$
\begin{aligned}
\dot{V}_2(t) \leq & k_1 \sum_{i=1}^{N} \delta_i^2 + k_2 \sum_{i=1}^{N} e_i^2 + \sum_{i=1}^{N} \frac{1}{c_2^2} \Big[0.2785 \tau_i \left(c_{fi} + \theta_i\right) \\
& + (c_{fi} - \underline{g} \hat{c}_{fi}) \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) + (\theta_i - \underline{g} \hat{\theta}_i) \delta_i \tanh(\delta_i / \tau_i) \Big] \qquad (5.47)
\end{aligned}
$$

where

$$
k_1 = -\frac{c_1 \underline{g}}{c_2^2} + \frac{\bar{a} + \bar{b}\bar{N}}{c_2} \left(1 + \frac{c_2}{2}\right), \quad k_2 = \frac{\bar{a} + \bar{b}\bar{N}}{2}. \qquad (5.48)
$$

Step 3. Note that in (5.47) the parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ is involved, which motivates us to introduce the generalized weight parameter estimation errors $\tilde{c}_{fi}$ ($i \in \{1, \ldots, N\}$) and $\tilde{\theta}_i$ as follows:

$$\tilde{c}_{fi} = c_{fi} - \underline{g}\hat{c}_{fi}, \quad \tilde{\theta}_i = \theta_i - \underline{g}\hat{\theta}_i, \tag{5.49}$$

with which we introduce the third part of the Lyapunov function candidate as

$$V_3(t) = \sum_{i=1}^{N} \frac{\tilde{c}_{fi}^2}{2c_2^2 \underline{g}\gamma_{1i}} + \sum_{i=1}^{N} \frac{\tilde{\theta}_i^2}{2c_2^2 \underline{g}\gamma_{2i}}. \tag{5.50}$$

By applying the adaptive laws for $\hat{c}_{fi}$ and $\hat{\theta}_i$ given in (5.32), we get the derivative of V_3 as

$$\begin{aligned}
\dot{V}_3(t) =& \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{c_2^2}\left(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}}\right) + \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{c_2^2}\left(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}\right) \\
=& \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{c_2^2}\left[\sigma_{1i}\hat{c}_{fi} - \delta_i\varphi_i \tanh(\delta_i\varphi_i/\tau_i)\right] \\
&+ \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{c_2^2}\left[\sigma_{2i}\hat{\theta}_i - \delta_i \tanh(\delta_i/\tau_i)\right] \\
=& \sum_{i=1}^{N} \frac{\sigma_{1i}}{c_2^2}\tilde{c}_{fi}\hat{c}_{fi} + \sum_{i=1}^{N} \frac{\sigma_{2i}}{c_2^2}\tilde{\theta}_i\hat{\theta}_i \\
&- \sum_{i=1}^{N} \frac{1}{c_2^2}(c_{fi} - \underline{g}\hat{c}_{fi})\delta_i\varphi_i \tanh(\delta_i\varphi_i/\tau_i) \\
&- \sum_{i=1}^{N} \frac{1}{c_2^2}(\theta_i - \underline{g}\hat{\theta}_i)\delta_i \tanh(\delta_i/\tau_i).
\end{aligned} \tag{5.51}$$

Step 4. Define the Lyapunov function candidate as

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{5.52}$$

where $V_1(t)$, $V_2(t)$, and $V_3(t)$ are given in (5.35), (5.40), and (5.50), respectively.
By combining (5.39), (5.47), and (5.51), we then arrive at

$$\begin{aligned}
\dot{V}(t) \leq& -k_3\sum_{i=1}^{N}\delta_i^2 - k_4\sum_{i=1}^{N}e_i^2 + \sum_{i=1}^{N}\frac{\sigma_{1i}}{c_2^2}\tilde{c}_{fi}\hat{c}_{fi} \\
&+ \sum_{i=1}^{N}\frac{\sigma_{2i}}{c_2^2}\tilde{\theta}_i\hat{\theta}_i + \sum_{i=1}^{N}\frac{0.2785\tau_i(c_{fi} + \theta_i)}{c_2^2}
\end{aligned} \tag{5.53}$$

where

$$k_3 = -\frac{N\ell_{\max}}{2} + \frac{c_1\underline{g}}{c_2^2} - \frac{\bar{a}+\bar{b}\bar{N}}{c_2}\left(1+\frac{c_2}{2}\right),$$
$$k_4 = \frac{c_2 k_{\mathrm{m}}}{2} - \frac{N\ell_{\max}}{2} - \frac{\bar{a}+\bar{b}\bar{N}}{2}. \tag{5.54}$$

Thus, c_1 and c_2 can be chosen as $c_1 > c_2^2\underline{g}^{-1}\left[\frac{N\ell_{\max}}{2} + \frac{\bar{a}+\bar{b}\bar{N}}{c_2}\left(1+\frac{c_2}{2}\right)\right]$ and $c_2 > \frac{1}{k_{\mathrm{m}}}\left(N\ell_{\max} + \bar{a} + \bar{b}\bar{N}\right)$ such that k_3 and $k_4 > 0$.

Upon using the inequality, $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$, we have

$$\tilde{c}_{fi}\hat{c}_{fi} = \frac{1}{2\underline{g}}\left[c_{fi}^2 - \tilde{c}_{fi}^2 - (c_{fi} - \tilde{c}_{fi})^2\right], \quad \tilde{\theta}_i\hat{\theta}_i = \frac{1}{2\underline{g}}\left[\theta_i^2 - \tilde{\theta}_i^2 - (\theta_i - \tilde{\theta}_i)^2\right]. \tag{5.55}$$

By substituting (5.55) into (5.53), one gets

$$\dot{V}(t) \le -k_3\sum_{i=1}^N \delta_i^2 - k_4\sum_{i=1}^N e_i^2 - \frac{\underline{\sigma}_1}{2\underline{g}c_2^2}\sum_{i=1}^N \tilde{c}_{fi}^2 - \frac{\underline{\sigma}_2}{2\underline{g}c_2^2}\sum_{i=1}^N \tilde{\theta}_i^2 + d \tag{5.56}$$

where $\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}$, $\underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}$, and

$$d = \frac{1}{c_2^2}\sum_{i=1}^N\left[\frac{\sigma_{1i}c_{fi}^2 + \sigma_{2i}\theta_i^2}{2\underline{g}} + 0.2785\tau_i\left(c_{fi} + \theta_i\right)\right] < \infty. \tag{5.57}$$

By introducing k_d as

$$k_d = \min\left\{k_3, k_4, \frac{\underline{\sigma}_1}{2\underline{g}c_2^2}, \frac{\underline{\sigma}_2}{2\underline{g}c_2^2}\right\}, \tag{5.58}$$

we further represent (5.56) as

$$\dot{V}(t) \le -k_d\sum_{i=1}^N\left(\delta_i^2 + e_i^2 + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2\right) + d. \tag{5.59}$$

Step 5. We prove that there exists some finite time $t^* > 0$ such that the leaderless error and the parameter estimation error will enter into a compact after t^*.

Note that

$$
\begin{aligned}
V_1(t) &= \frac{1}{2}E^T \mathrm{diag}(p) E \le \frac{\bar{p}}{2}\sum_{i=1}^{N} e_i^2, \\
V_2(t) &= \frac{1}{2c_2^2}\sum_{i=1}^{N}(v_i - v_i^*)^2 = \frac{1}{2c_2^2}\sum_{i=1}^{N}\delta_i^2, \\
V_3(t) &\le \frac{1}{2c_2^2 g \underline{\gamma}_1}\sum_{i=1}^{N}\tilde{c}_{fi}^2 + \frac{1}{2c_2^2 g \underline{\gamma}_2}\sum_{i=1}^{N}\tilde{\theta}_i^2
\end{aligned} \tag{5.60}
$$

where $\bar{p} = \max\{p_1, \ldots, p_N\}$ ($p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue), $\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$ and $\underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. Let

$$
k_v = \max\left\{\frac{\bar{p}}{2}, \frac{1}{2c_2^2}, \frac{1}{2c_2^2 g \underline{\gamma}_1}, \frac{1}{2c_2^2 g \underline{\gamma}_2}\right\}, \tag{5.61}
$$

we then have from (5.60) that

$$
V(t) \le k_v \sum_{i=1}^{N}\left(\delta_i^2 + e_i^2 + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2\right). \tag{5.62}
$$

By combining (5.59) and (5.62), we then arrive at

$$
\dot{V}(t) \le -\frac{k_d}{k_v}V(t) + d, \tag{5.63}
$$

from which we can conclude that the set $\Theta = \{(x_i, v_i) : V(t) < \frac{dk_v}{k_d}\}$ is globally attractive. Once $(x_i, v_i) \notin \Theta$, then $\dot{V} < 0$. Therefore, there exists a finite time t^* such that $(x_i, v_i) \in \Theta$ for $\forall t > t^*$. This further implies that

$$
V(t) < \frac{dk_v}{k_d} \tag{5.64}
$$

when $t > t^*$.

Step 6. Derive the estimation for steady-state errors of all agents.

Note that for all $i = 1, \ldots, N$, we have

$$
|e_i| = \sqrt{e_i^2} \le \sqrt{\sum_{i=1}^{N} e_i^2} \le \sqrt{\frac{2}{\underline{p}}V_1(t)} \le \sqrt{\frac{2}{\underline{p}}V(t)} \le \sqrt{\frac{2dk_v}{\underline{p}k_d}}, \tag{5.65}
$$

with $\underline{p} = \min\{p_1, \ldots, p_N\}$ ($p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue). Note that

$$|v_i - v_i^*| \le \sqrt{\sum_{i=1}^{N} \delta_i^2} \le \sqrt{2c_2^2 V_2(t)} \le \sqrt{2c_2^2 V(t)} \le \sqrt{\frac{2c_2^2 d k_v}{k_d}}. \tag{5.66}$$

On the other hand,

$$|v_i^*| = |-c_2 e_i| \le c_2 \sqrt{\frac{2dk_v}{\underline{p} k_d}}. \tag{5.67}$$

It thus follows from (5.66) and (5.67) that

$$|v_i| \le |v_i - v_i^*| + |v_i^*| \le \sqrt{\frac{2c_2^2 d k_v}{k_d}} + c_2 \sqrt{\frac{2dk_v}{\underline{p} k_d}}, \tag{5.68}$$

which then implies, for $\forall i, j \in \{1, \ldots, N\}$, that

$$|v_i - v_j| \le |v_i| + |v_j| \le 2\left(\sqrt{\frac{2c_2^2 d k_v}{k_d}} + c_2 \sqrt{\frac{2dk_v}{\underline{p} k_d}}\right). \tag{5.69}$$

In addition, for all $\forall i \in \{1, \ldots, N\}$,

$$|\tilde{c}_{fi}| \le \sqrt{\sum_{i=1}^{N} \tilde{c}_{fi}^2} \le \sqrt{2c_2^2 \underline{g} \bar{\gamma}_1 V_3(t)} \le \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_1 d k_v}, \tag{5.70}$$

and

$$|\tilde{\theta}_i| \le \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_2 d k_v}, \tag{5.71}$$

with $\bar{\gamma}_1 = \max\{\gamma_{11}, \ldots, \gamma_{1N}\}$, and $\bar{\gamma}_2 = \max\{\gamma_{21}, \ldots, \gamma_{2N}\}$.

From the above analysis, we conclude that under the proposed control scheme (5.29)–(5.32), the position and velocity errors between neighbor agents will converge to a small region Ω_1, defined by

$$\Omega_1 = \left\{ |e_i| \le \sqrt{\frac{2dk_v}{\underline{p} k_d}},\ \ |v_i - v_j| \le 2\left(\sqrt{\frac{2c_2^2 d k_v}{k_d}} + c_2 \sqrt{\frac{2dk_v}{\underline{p} k_d}}\right),\ \forall i, j \in \{1, \ldots, N\} \right\}, \tag{5.72}$$

and the generalized parameter estimation converges to the region Ω_{1p} given as

$$\Omega_{1p} = \left\{ |\tilde{c}_{fi}| \le \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_1 d k_v}, |\tilde{\theta}_i| \le \sqrt{\frac{2}{k_d} c_2^2 \underline{g} \bar{\gamma}_2 d k_v},\ \forall i \in \{1, \ldots, N\} \right\} \quad (5.73)$$

when $t > t^*$. □

5.4 Simulation Examples

This section gives a simulation example on a group of networked autonomous surface vessels (ASVs) [6] with second-order uncertain dynamics to illustrate the effectiveness of the distributed adaptive control algorithm proposed in (5.29)–(5.32) under the local and directed communication topology condition. It is shown that under the proposed control scheme, the group of agents subject to unknown time-varying control gain, unknown non-parametric nonlinearities, and unknown bounded non-vanishing uncertainties achieves consensus successfully under the directed topology condition.

We use four ASVs to demonstrate the effectiveness of our proposed control scheme. The communication topology among the four ASVs is presented in Fig. 5.1, which satisfies the assumption that the graph topology is directed and strongly connected. Each edge weight is taken as 0.1. The left eigenvector of $\mathscr{L}$ associated with eigenvalue 0 is $[2, 1, 1, 1]^T$.

The dynamics of the kth ($k = 1, 2, 3, 4$) ASV is modeled by

$$\begin{bmatrix} m_{x,k} & 0 & 0 \\ 0 & m_{y,k} & 0 \\ 0 & 0 & m_{z,k} \end{bmatrix} \cdot \begin{bmatrix} \dot{v}_{x,k} \\ \dot{v}_{y,k} \\ \dot{v}_{z,k} \end{bmatrix} = \begin{bmatrix} f_{x,k} \\ f_{y,k} \\ f_{z,k} \end{bmatrix} + \begin{bmatrix} u_{x,k} \\ u_{y,k} \\ d_{x,k} \end{bmatrix} + \begin{bmatrix} u_{x,k} \\ d_{y,k} \\ d_{z,k} \end{bmatrix} \quad (5.74)$$

in which $M_k = \text{diag}\{m_{x,k}, m_{y,k}, m_{z,k}\}$ denotes the mass matrix; $r_k = [x_k, y_k, z_k]^T$, $v_k = \dot{r}_k = [v_{x,k}, v_{y,k}, v_{z,k}]^T$, and $u_k = [u_{x,k}, u_{y,k}, u_{z,k}]^T$ denote the position, velocity,

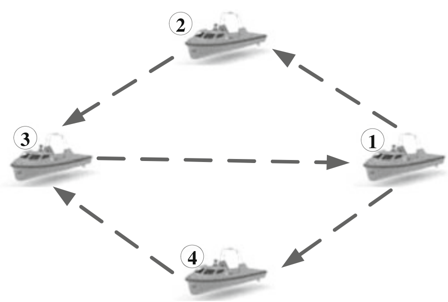
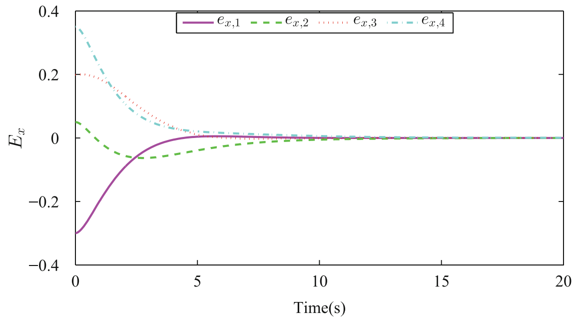


Fig. 5.1 Directed communication topology among the four ASVs

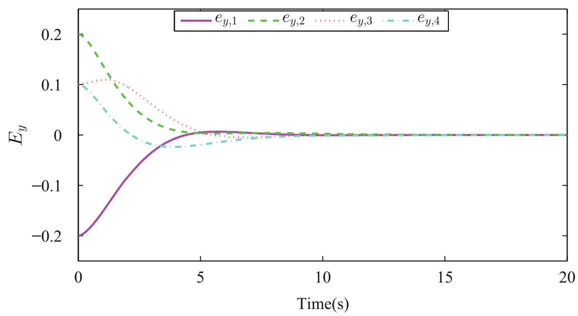
and control input vector, respectively; $D_k = [d_{x,k}, d_{y,k}, d_{z,k}]^T$ is environment disturbance, and $f_k = [f_{x,k}, f_{y,k}, f_{z,k}]^T$ represents coriolis, centripetal, and hydrodynamic damping forces and torques acting on the body, where

$$f_k = \begin{bmatrix} A_{x,k} + A_{|x,k|}|v_{x,k}| & -m_{y,k}v_{z,k} & 0 \\ m_{x,k}v_{z,k} & B_{y,k} + B_{|y,k|}|v_{y,k}| & 0 \\ 0 & 0 & C_{z,k} + C_{|z,k|}|v_{z,k}| \end{bmatrix} \cdot \begin{bmatrix} v_{x,k} \\ v_{y,k} \\ v_{z,k} \end{bmatrix} \tag{5.75}$$

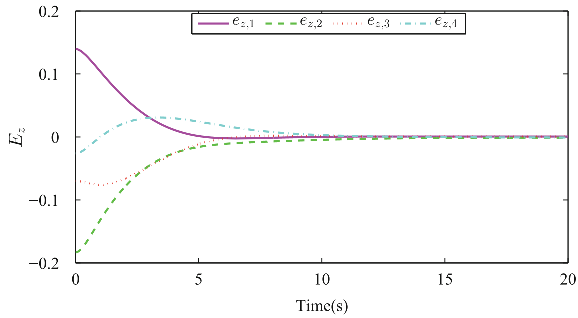
Fig. 5.2 Position error convergence of the four ASVs under the proposed finite-time control scheme



(a) E in x-direction.



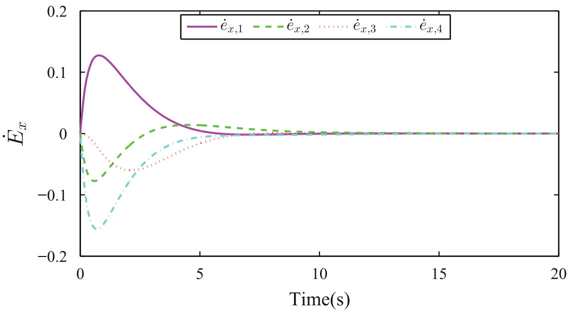
(b) E in y-direction.



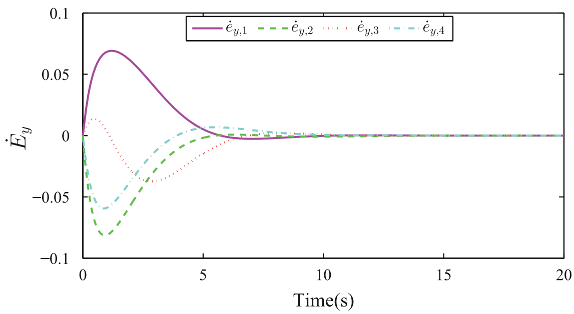
(c) E in z-direction.

In the simulation, the physical parameters are taken as: $M_k = \text{diag}\{600 + 6(-1)^k + 6\Delta m(t), 1000 + 10(-1)^k + 10\Delta m(t), 800 + 8(-1)^k + 8\Delta m(t)\}$ with $\Delta m(t) = \sin(\pi t/50 - \pi)$, $A_{x,k} = -1 + 0.1(-1)^k$, $A_{|x,k|} = -25 + 2.5(-1)^k$, $B_{y,k} = -10 + (-1)^k$, $B_{|y,k|} = -200 + 20(-1)^k$, $C_{z,k} = -0.5 + 0.05(-1)^k$, and $C_{|z,k|} = -1500 + 150(-1)^k$ for $k = 1, 2, 3, 4$. The external disturbance is taken as $D_k = [3 + 3(-1)^k \sin(t/50) + 2\sin(t/10), -1 + 3(-1)^k \sin(t/20 - \pi/6) + 2\sin(t), -5(-1)^k \sin(0.1t) - \sin(t + \pi/3)]^T$.

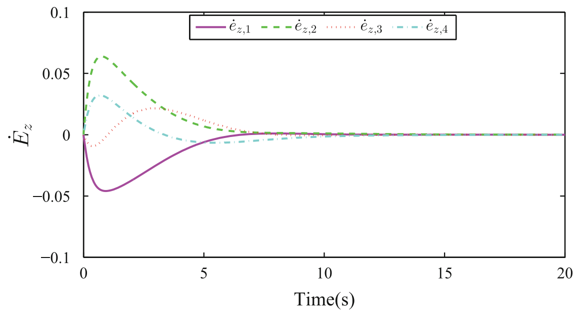
Fig. 5.3 Velocity error convergence of the four ASVs under the proposed finite-time control scheme



(a) $\dot{E}$ in x-direction.



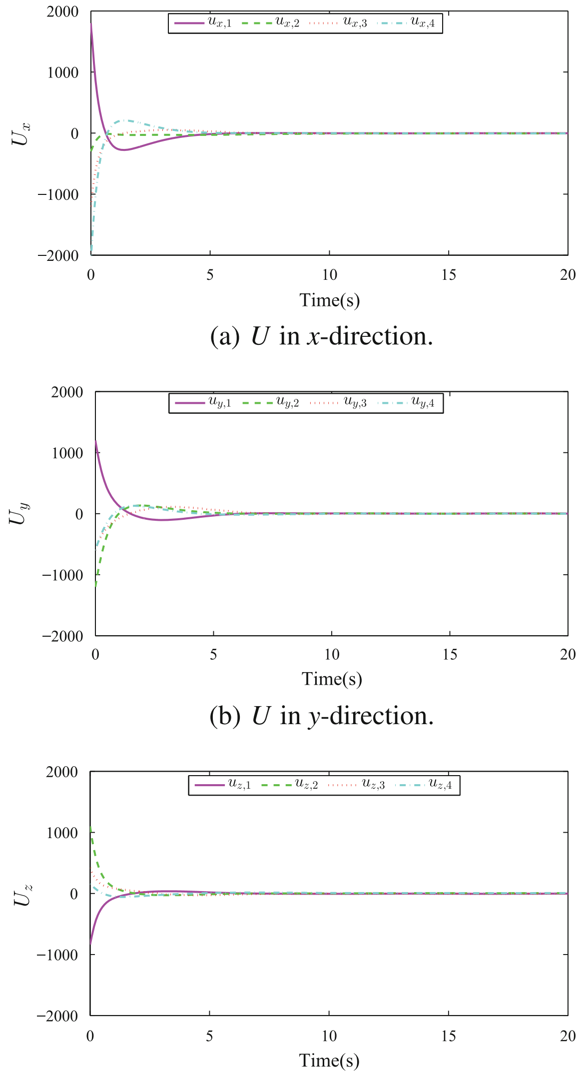
(b) $\dot{E}$ in y-direction.



(c) $\dot{E}$ in z-direction.

The simulation objective is that the four ASVs are required to achieve consensus by using the proposed control law given in (5.29)–(5.32) under directed communication topology condition. The initial conditions of the vessels are $r_1(0) = (-1.5\text{m}, 0\text{m}, \pi/3\text{rad})$, $r_2(0) = (-1\text{m}, 2\text{m}, -\pi/4\text{rad})$, $r_3(0) = (1.5\text{m}, 2\text{m}, -\pi/9\text{rad})$, $r_4(0) = (2\text{m}, 1\text{m}, \pi/4\text{rad})$, $v_k(0) = (0, 0, 0)\,(k = 1, 2, 3, 4)$, respectively. The control parameters are taken as: $s = 2$, $c_1 = 2000$, and $c_2 = 3$. In addition, the initial values of the estimates are chosen as $\hat{c}_{fx,k} = \hat{c}_{fy,k} = \hat{c}_{fz,k} = 0$ and $\hat{\theta}_{x,k} = \hat{\theta}_{y,k} =$

Fig. 5.4 Control inputs of the four ASVs under the proposed finite-time control scheme



(a) U in x-direction.

(b) U in y-direction.

(c) U in z-direction.

$\hat{\theta}_{z,k} = 0$ for $k = 1, 2, 3, 4$. The simulation runs for 15 s. The position error convergence results in x-direction, y-direction, and z-direction under the proposed distributed adaptive control scheme are represented in Fig. 5.2. In addition, the velocity error convergence results in x-direction, y-direction, and z-direction under the proposed finite-time control scheme are given in Fig. 5.3. From both Figs. 5.2 and 5.3, we see that the consensus is achieved under the proposed control scheme. Figure 5.4 shows the control input signals of the four ASVs under the proposed finite-time control scheme, from which we see the control input signal is smooth and bounded under the local and directed communication condition.

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Part III
Finite-Time Cooperative Adaptive Control for Multiple Networked Nonlinear Systems

Chapter 6
Finite-Time Leaderless Consensus Control for Systems with First-Order Uncertain Dynamics

This chapter investigates the problem of finite-time leaderless consensus of networked multi-agent systems with first-order uncertain dynamics under local communication topology condition. Finite-time convergence behavior is of special importance in cooperative control of MAS, but the vast majority research on finite-time control of MAS has been focused on linear systems or nonlinear systems with nonlinearities that can be linearly parameterized, that is, the nonlinearities in that systems are assumed to exhibit the linear parametric property. The control results on finite-time distributed control of nonlinear MAS with unknown non-parametric and non-vanishing uncertainties are scarce. Extending the existing finite-time control methods for linear systems or nonlinear systems with linearly parameterized nonlinearities to MAS subject to non-parametric and non-vanishing uncertainties encounters significant technical challenge. The main hindrance stems from the fact that, in the presence of the non-parametric uncertainties, the commonly used adaptive control law cannot be used to derive the finite-time convergence because it can not ensure an important relation that is crucial to derive the finite-time convergence result.

This chapter presents a solution to this problem by focusing on achieving leaderless consensus of MAS with first-order non-parametric uncertain dynamics in finite-time with sufficient accuracy. To circumvent the aforementioned technical obstacles, we introduce the locally defined neighborhood error and fraction power-based adaptive updated law for the parameter estimation to embed into the control scheme, which renders the crucial relation $\dot{V} \leq cV^{\alpha} + d$ $(0 < \alpha < 1)$ needed for the finite-time convergence to be satisfied.

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Section 6.1 represents some useful lemmas that is related to the finite-time stability analysis for the multi-agent systems. Section 6.2 addresses the leaderless consensus control problem for the first-order multi-agent systems with unknown time-varying gain and non-parametric uncertainties under undirected topology. A distributed finite-time leaderless consensus control scheme is derived such that the consensus error uniformly converges to a small residual set in finite-time, and all the internal signals are ensured to be uniformly bounded. In addition, the finite convergence time for each agent to reach the required consensus configuration is explicitly established and recipes for control parameter selection to make the residual errors as small as desired are provided. Section 6.3 addresses the leaderless consensus control problem for the first-order multi-agent systems with known control gain and bounded non-parametric uncertainties under the directed topology condition, in which the technical difficulty arising from the non-symmetric property of the Laplacian matrix is circumvented in the finite-time stability analysis. Section 6.4 gives a simulation example to confirm the effectiveness and benefits of the proposed finite-time leaderless consensus control scheme for the multi-agent systems with first-order non-parametric uncertain dynamics.

6.1 Preliminaries for the Finite-Time Control

In this section, we represent some useful lemmas for the finite-time stability analysis. We first introduce the definition related to finite-time stability.

Lemma 6.1 ([1]) *Consider the system*

$$\dot{x} = f(x,t),\ f(0,t) = 0,\ x \in \mathbb{U} \subset \mathbb{R}^N \tag{6.1}$$

where $f : \mathbb{U} \times \mathbb{R}_+ \to \mathbb{R}^N$ is continuous on an open neighborhood $\mathbb{U}$ of the origin $x = 0$. Suppose there exists a continuously differentiable function $V(x,t) : \mathbb{U} \times \mathbb{R}_+ \to \mathbb{R}$, a real number $c > 0$ and $0 < \alpha < 1$, and a neighborhood $\mathbb{U}_0 \subset \mathbb{U}$ of the origin such that $V(\cdot)$ is positive definite and $\dot{V}(x,t) + cV^{\alpha}(x,t) \leq 0$ on $\mathbb{U}_0$, where $\dot{V}(x) = \frac{\partial V}{\partial x}(x) f(x)$. Then the origin is a finite-time-stable equilibrium of (6.1). Moreover, there exists a finite settling time T^* satisfying

$$T^* \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)} \tag{6.2}$$

such that for any given initial state $x(t_0) \in \mathbb{U}_0/\{0\}$, $lim_{t\to T^*} V(x,t) = 0$ and $V(x,t) = 0$ for $t \geq T^*$.

Lemma 6.2 ([2]) *If $h = h_2/h_1 \geq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x - y|^h \leq 2^{h-1}|x^h - y^h|$.*

Lemma 6.3 ([2]) *If* $0 < h = h_1/h_2 \le 1$, *where* $h_1, h_2 > 0$ *are odd integers, then* $|x^h - y^h| \le 2^{1-h}|x - y|^h$.

Lemma 6.4 ([2]) *For* $x, y \in \mathbb{R}$, *if* $c, d > 0$, *then* $|x|^c|y|^d \le c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.

Lemma 6.5 ([3]) *For* $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, $0 < h \le 1$, *then* $(\sum_{i=1}^{N} |x_i|)^h \le \sum_{i=1}^{N} |x_i|^h \le N^{1-h}(\sum_{i=1}^{N} |x_i|)^h$.

Lemma 6.6 ([4]) *For* $x \in \mathbb{R}$, $y \in \mathbb{R}$, $0 < h = h_1/h_2 \le 1$, *where* $h_1, h_2 > 0$ *are positive odd integers, then* $x^h(y - x) \le \frac{1}{1+h}\left(y^{1+h} - x^{1+h}\right)$.

Proof Let $g(x) = x^h(c - x) - \frac{1}{1+h}\left(c^{1+h} - x^{1+h}\right)$. Taking the derivation of $g(x)$ with respect to x yields

$$\frac{d}{dx}g(x) = hx^{h-1}(c - x) - x^h + x^h = hx^{h-1}(c - x) \tag{6.3}$$

It is straightforward to derive that $g(x)$ arrives its maximum value at $x = c$, which implies

$$g(x) \le g(c) = 0, \quad \forall x \in \mathbb{R}. \tag{6.4}$$

Thus, $x^h(y - x) \le \frac{1}{1+h}(y^{1+h} - x^{1+h})$ holds. ■

6.2 Finite-Time Leaderless Consensus Control Under Undirected Topology

In this section, we address the leaderless consensus control problem for networked multi-agent systems with first-order uncertain dynamics. The agent dynamics are subject to unknown time-varying gain and unknown non-parametric uncertainties, meaning that the parameters are unknown and time-varying and the dynamics are not allowed to appear in the control protocols. This section formulates the basic structure of the distributed finite-time control protocols to confront these problems, allowing the leaderless consensus to be achieved in finite-time with sufficient accuracy in spite of the unknown time-varying gain and unknown non-parametric uncertainties. We introduce the locally defined neighborhood error of fractional-order and fraction power-based adaptive updated law for the parameter estimation to embed into the control scheme and show how to achieve the finite-time convergence by the negative feedback of the fractional-order local error and how to compensate the unknown non-parametric uncertainties by using the adaptive control technique based on fraction power-based adaptive updated law.

6.2.1 Problem Formulation of Finite-Time Leaderless Consensus for First-Order Uncertain Systems

Consider a group of N agents with scalar first-order dynamics

$$\dot{x}_i(t) = g_i(x_i, t)u_i(t) + f_i(z_i, t) + f_{di}(z_i, t) \tag{6.5}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i denotes the control gain, f_i denotes the system nonlinearities and f_{di} denotes the non-vanishing uncertainties (f_i and f_{di} are either continuously differentiable or Lipschitz); $z_i = \bigcup_{j \in \mathcal{N}_i \bigcup i} x_j$.

Standard assumptions for the existence of unique solutions and global controllability of system are made.

Assumption 6.1

a. The control effectiveness gain g_i (unknown and time-varying) is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$) and $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$ with $\underline{g}_i$, $\bar{g}_i$ being unknown finite constants, which ensures the global controllability of the system.
b. It is assumed that $f_i(z_i, t)$ is unknown and non-parameterized. Thus, they are not available in the finite-time control design. Moreover, certain crude structural information is available to allow that $|f_i(z_i, t)| \leq c_{fi}\varphi_i(z_i)$ for all $t \geq t_0$, where $c_{fi} \geq 0$ is an unknown finite constant and $\varphi_i(z_i)$ is a known scalar function, and $\varphi_i(z_i)$ is bounded for any z_i (x_i) ($i = 1, 2, \ldots, N$).
c. It is assumed that $f_{di}(z_i, t)$ is unknown and non-vanishing, and $f_{di}(z_i, t)$ is bounded by some unknown finite constant θ_i, i.e., $|f_{di}(z_i, t)| \leq \theta_i < \infty$.

For ease of notation, the agents' states are scalars, i.e., $x_i \in \mathbb{R}$. If the states are vectors, i.e., $x_i \in \mathbb{R}^n$, the results of this chapter can be extended by using the standard method involving the Kronecker products. In addition, it should be stressed that although the bounds mentioned in Assumption 6.1 are assumed to exist, they are not used in the controller design and do not have to be known. They appear in the error bounds in the control results derived in Theorem 6.1 as can be seen later.

The local neighborhood state error for node i is denote by

$$e_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) \tag{6.6}$$

Let $E = [e_1, \ldots, e_N]^T$, $X = [x_1, \ldots, x_N]^T$ such that $E = LX$.

The distributed finite-time leaderless consensus control design problem confronted herein is as follows: (1) Design control schemes for all agents i in $\mathcal{G}$ to make an agreement in finite time. The control scheme must be distributed in the sense that they can only depend on local information about the agent itself and its neighbors in graph, and meanwhile, the control scheme should be endowed with the finite-time quality; (2) Tackle the unknown and time-varying control effectiveness gain explicitly; (3) Accommodate the unknown non-parametric nonlinearities and unknown non-vanishing uncertainties upon using the fraction power-based adaptive updated law.

6.2.2 Finite-Time Leaderless Consensus Control Design

The control objective of this section is to design a control strategy such that the leaderless consensus error uniformly converges to a small residual set in finite time, in spite of the unknown time-varying gain and unknown non-parametric uncertainties. In addition, the finite convergence time for each agent to reach the required consensus configuration is explicitly established and recipes for control parameter selection to make the residual errors as small as desired are provided.

To this end, the control input for each ith ($i = 1, \ldots, N$) agent is designed to consist of two parts: (1) the negative feedback control term u_{0i}, and (2) the compensation control term u_{ci}, which is of the following form,

$$u_i = u_{0i} + u_{ci} \tag{6.7}$$

in which the negative feedback control term based on the fractional-order local error is generated by

$$u_{0i} = -k_i e_i^q \tag{6.8}$$

and (2) the compensation control term is generated by

$$u_{ci} = -\hat{c}_{fi}\varphi_i \tanh(e_i\varphi_i/\tau_i) - \hat{\theta}_i \tanh(e_i/\tau_i) \tag{6.9}$$

with the fraction power-based updated law

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi}^q + \gamma_{1i}\varphi_i \tanh(e_i\varphi_i/\tau_i)e_i, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i^q + \gamma_{2i}\tanh(e_i/\tau_i)e_i, \end{aligned} \tag{6.10}$$

where $0 < q = q_1/q_2 < 1$ (q_1, q_2 are positive odd integers), $\hat{c}_{fi}$ and $\hat{\theta}_i$ are the estimations of c_{fi} and θ_i (the parameters to be estimated), respectively, $\varphi_i(\cdot)$ is the scalar and readily computable function as given in Assumption 6.1, k_i, τ_i, γ_{1i}, γ_{2i}, σ_{1i}, and σ_{2i} are positive finite design parameters chosen arbitrarily by the designer.

Remark 6.1 It is seen from (6.8) to (6.10) that the proposed control structure is of "sublinear" form in that the control scheme is built upon fraction state feedback control and the adaptation rule (i.e., the leakage term) is also of fraction form, rather than just simply linear state feedback. It is such sublinear control with fraction leakage adaptation that ensures the multi-agent systems to have the attractive performance as stated in the next theorem.

6.2.3 Finite-Time Leaderless Consensus Stability Analysis

We need to prove a relation firstly, which is crucial to derive the main results.

Lemma 6.7 *For the generalized weight parameter estimation error* $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ *and the constant* $0 < q = q_1/q_2 < 1$ *(*q_1 *and* q_2 *are positive odd integrators), it holds that*

$$\tilde{\bullet}\hat{\bullet}^q \le \underline{g}^{-q}\frac{1}{1+q}\Bigg[\left(2^{(q-1)(1-q)} - 2^{q-1}\right)\tilde{\bullet}^{1+q} + \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q}2^{-(q-1)^2(1+q)}\right)\bullet^{1+q}\Bigg]. \tag{6.11}$$

Proof Firstly, by noting that $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$, we get

$$\tilde{\bullet}\hat{\bullet}^q = \tilde{\bullet}\left[\underline{g}^{-1}\left(\bullet - \tilde{\bullet}\right)\right]^q = \underline{g}^{-q}\tilde{\bullet}\left(\bullet - \tilde{\bullet}\right)^q. \tag{6.12}$$

According to Lemma 6.6, we let $x = \bullet - \tilde{\bullet}$, $y = \bullet$, and then $y - x = \tilde{\bullet}$, thus we have

$$\tilde{\bullet}\hat{\bullet}^q \le \underline{g}^{-q}\frac{1}{1+q}\left[\bullet^{1+q} - \left(\bullet - \tilde{\bullet}\right)^{1+q}\right]. \tag{6.13}$$

By noting that $1 + q = 1 + \frac{q_1}{q_2} = \frac{q_1+q_2}{q_2}$, and $q_1 + q_2$ is an even number, we then get from (6.13) that

$$\tilde{\bullet}\hat{\bullet}^q = \underline{g}^{-q}\frac{1}{1+q}\left[\bullet^{1+q} - |\bullet - \tilde{\bullet}|\,|\bullet - \tilde{\bullet}|^q\right]. \tag{6.14}$$

Upon using Lemma 6.3, we arrive at

$$\tilde{\bullet}\hat{\bullet}^q \le \underline{g}^{-q}\frac{1}{1+q}\left[\bullet^{1+q} - 2^{q-1}\,|\bullet - \tilde{\bullet}|\,\left|\bullet^q - \tilde{\bullet}^q\right|\right]. \tag{6.15}$$

Note that both the function $g(s) = s$ $(s \in \mathbb{R})$ and function $f(s) = s^q$ $(s \in \mathbb{R})$ are non-decreasing, we then have $|\bullet - \tilde{\bullet}|\,\left|\bullet^q - \tilde{\bullet}^q\right| = (\bullet - \tilde{\bullet})\left(\bullet^q - \tilde{\bullet}^q\right)$, and (6.15) becomes

$$\tilde{\bullet}\hat{\bullet}^q \le \underline{g}^{-q}\frac{1}{1+q}\left[\bullet^{1+q} - 2^{q-1}\,\bullet^{1+q} + 2^{q-1}\,\bullet\,\tilde{\bullet}^q + 2^{q-1}\tilde{\bullet}\,\bullet^q - 2^{q-1}\tilde{\bullet}^{1+q}\right]. \tag{6.16}$$

Upon using Lemma 6.4, we get that

$$2^{q-1}\tilde{\bullet}\bullet^q = \left(2^{q-1}\tilde{\bullet}\right)\bullet^q \le \frac{1}{1+q}\left(2^{q-1}\tilde{\bullet}\right)^{1+q} + \frac{q}{1+q}\bullet^{1+q}, \tag{6.17}$$

and

$$
\begin{aligned}
2^{q-1} \bullet \tilde{\bullet}^{q} &= 2^{q-1} \bullet \left(2^{q-1}\tilde{\bullet}\right)^{q} 2^{-(q-1)q} = 2^{(q-1)(1-q)} \bullet \left(2^{q-1}\tilde{\bullet}\right)^{q} \\
&\le \frac{q}{1+q}\left(2^{q-1}\tilde{\bullet}\right)^{1+q} + \frac{1}{1+q}\left(2^{(q-1)(1-q)}\bullet\right)^{1+q}.
\end{aligned}
\tag{6.18}
$$

By inserting (6.17) and (6.18) into (6.16), we then arrive at

$$
\begin{aligned}
\tilde{\bullet}\hat{\bullet}^{q} \le{}& \underline{g}^{-q}\frac{1}{1+q}\Bigg[\bullet^{1+q} - 2^{q-1}\bullet^{1+q}\frac{q}{1+q}\left(2^{q-1}\tilde{\bullet}\right)^{1+q} + \frac{1}{1+q}\left(2^{(q-1)(1-q)}\bullet\right)^{1+q} \\
&+ \frac{1}{1+q}\left(2^{q-1}\tilde{\bullet}\right)^{1+q} + \frac{q}{1+q}\bullet^{1+q} - 2^{q-1}\tilde{\bullet}^{1+q}\Bigg] \\
={}& \underline{g}^{-q}\frac{1}{1+q}\Bigg[\left(2^{(q-1)(1-q)} - 2^{q-1}\right)\tilde{\bullet}^{1+q} \\
&+ \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q}2^{-(q-1)^2(1+q)}\right)\bullet^{1+q}\Bigg],
\end{aligned}
\tag{6.19}
$$

which then implies that (6.11) holds. □

Theorem 6.1 (Distributed Adaptive Finite-Time Control for Leaderless Consensus) *Suppose that the communication topology is undirected and connected, and Assumption 6.1 is satisfied. The nonlinear MAS subject to unknown time-varying gain and non-parametric/non-vanishing uncertainties as described by* (6.5) *with the proposed control scheme* (6.7)–(6.10) *achieves the finite-time leaderless consensus in that*

(1) the leaderless consensus errors e_i $(i = 1, \ldots, N)$ converge to a small residual set Ω_1 defined by

$$
\Omega_1 = \left\{ |e_i| \le \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}}\left(\frac{k_v^{\frac{1+q}{2}} d}{\eta_2\eta_1 k_d}\right)^{\frac{1}{1+q}},\ \forall i = 1, \ldots, N \right\}
\tag{6.20}
$$

in a finite time T^* satisfying

$$
T^* \le \frac{V(t_0)^{\frac{1-q}{2}} k_v^{\frac{1+q}{2}}}{(1-\eta_2)\eta_1 k_d \tilde{c}(\frac{1-q}{2})}
\tag{6.21}
$$

where $0 < \eta_1 \le 1$, $0 < \eta_2 < 1$, $V(t_0)$ is known, k_d and k_v are given in (6.37) *and* (6.41)*, respectively, which are explicitly computable;*

(2) the generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a small set Ω_{1p} defined by

$$\Omega_{1p} = \left\{ |\tilde{c}_{fi}| \leq \sqrt{2\underline{g}\bar{\gamma}_1} \left(\frac{k_v^{\frac{1+q}{2}} d}{\eta_2 \eta_1 k_d} \right)^{\frac{1}{1+q}}, \right.$$
$$\left. |\tilde{\theta}_i| \leq \sqrt{2\underline{g}\bar{\gamma}_2} \left(\frac{k_v^{\frac{1+q}{2}} d}{\eta_2 \eta_1 k_d} \right)^{\frac{1}{1+q}}, \ \forall i = 1, \ldots, N \right\} \tag{6.22}$$

in the finite time T^*;

(3) all signals in the closed-loop system remain uniformly bounded.

Proof The proof of the result can be done by the following five steps.
Step 1. Construct the distributed part of the Lyapunov function candidate as

$$V_1(t) = \frac{1}{2} E^T \Lambda E = \frac{1}{2} X^T \mathscr{L} X \tag{6.23}$$

where $\Lambda = U_N^T \tilde{\Lambda}^{-1} U_N$, $\tilde{\Lambda} = \mathrm{diag}\{a, \lambda_2, \ldots, \lambda_N\}$ (a is an arbitrarily finite positive constant), U_N is an orthogonal matrix such that $\mathscr{L} = U_N^T \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\} U_N = U_N^T \Lambda_0 U_N$, with $\Lambda_0 = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$ provided by Lemma 3.1. Such matrix Λ is symmetric and positive definite, implying that the distributed part of Lyapunov function $V_1(t)$ is well defined. Taking the time derivative of $V_1(t)$ along (6.5) yields that

$$\dot{V}_1(t) = E^T \dot{X} = \sum_{i=1}^{N} e_i \dot{x}_i = \sum_{i=1}^{N} e_i (g_i u_i + f_i + f_{di}). \tag{6.24}$$

Inserting the control input u_i given in (6.7) and u_{0i} given in (6.8) into (6.24) yields

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{N} \left(-k_i g_i e_i^{1+q} + g_i e_i u_{ci} + e_i f_i + e_i f_{di} \right) \\ &\leq \sum_{i=1}^{N} \left(-k_i g_i e_i^{1+q} + g_i e_i u_{ci} + |e_i| c_{fi} \varphi_i + |e_i| \theta_i \right) \end{aligned} \tag{6.25}$$

By using the fact that $0 \leq |s| - s \cdot \tanh(s/k) \leq 0.2785k$ [5], and by inserting the compensation control term u_{ci} given in (6.9) into (6.25), we arrive at

$$\begin{aligned}\dot{V}_1(t) \leq & \sum_{i=1}^{N}\big[-k_i g_i e_i^{1+q} - g_i e_i \varphi_i \hat{c}_{fi}\tanh(e_i\varphi_i/\tau_i) - g_i e_i \hat{\theta}_i \tanh(e_i/\tau_i) \\ & + c_{fi}e_i\varphi_i \tanh(e_i\varphi_i/\tau_i) + \theta_i e_i \tanh(e_i/\tau_i) + 0.2785\tau_i(c_{fi}+\theta_i)\big] \\ \leq & \sum_{i=1}^{N}\big[-k_i g_i e_i^{1+q} + (c_{fi} - \underline{g}\hat{c}_{fi})e_i\varphi_i \tanh(e_i\varphi_i/\tau_i) \\ & + (\theta_i - \underline{g}\hat{\theta}_i)e_i \tanh(e_i/\tau_i) + 0.2785\tau_i(c_{fi}+\theta_i)\big]. \end{aligned} \tag{6.26}$$

Step 2. Note that in (6.26) the parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ is involved, which motivates us to introduce the generalized weight parameter estimation errors $\tilde{c}_f$ and $\tilde{\theta}$ as follows,

$$\tilde{c}_f = c_f - \underline{g}\hat{c}_f, \quad \tilde{\theta} = \theta - \underline{g}\hat{\theta} \tag{6.27}$$

where $\tilde{c}_f = [\tilde{c}_{f1}, \ldots, \tilde{c}_{fN}]^T \in \mathbb{R}^N$, and $\tilde{\theta} = [\tilde{\theta}_1, \ldots, \tilde{\theta}_N]^T \in \mathbb{R}^N$.

Upon using the generalized weight parameter estimation error, we define the distributed part of the Lyapunov function candidate as

$$V_2(t) = \frac{1}{2\underline{g}}\tilde{c}_f^T \Gamma_1^{-1}\tilde{c}_f + \frac{1}{2\underline{g}}\tilde{\theta}^T \Gamma_2^{-1}\tilde{\theta} \tag{6.28}$$

where $\Gamma_1 = \mathrm{diag}\{\gamma_{11}, \ldots, \gamma_{1N}\} \in \mathbb{R}^{N\times N}$ and $\Gamma_2 = \mathrm{diag}\{\gamma_{21}, \ldots, \gamma_{2N}\} \in \mathbb{R}^{N\times N}$.

Taking the derivative of V_2 yields

$$\dot{V}_2(t) = \sum_{i=1}^{N}\frac{1}{\underline{g}}\tilde{c}_{fi}\left(-\frac{\underline{g}\dot{\hat{c}}_{fi}}{\gamma_{1i}}\right) + \sum_{i=1}^{N}\frac{1}{\underline{g}}\tilde{\theta}_i\left(-\frac{\underline{g}\dot{\hat{\theta}}_i}{\gamma_{2i}}\right) = \sum_{i=1}^{N}\tilde{c}_{fi}\left(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}}\right) + \sum_{i=1}^{N}\tilde{\theta}_i\left(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}\right). \tag{6.29}$$

Now applying the adaptive laws for $\hat{c}_{fi}$ and $\hat{\theta}_i$ given in (6.10)–(6.29), we get

$$\dot{V}_2(t) = \sum_{i=1}^{N}\big[\sigma_{1i}\tilde{c}_{fi}\hat{c}_{fi}^q - \tilde{c}_{fi}\varphi_i e_i \tanh(e_i\varphi_i/\tau_i) + \sigma_{2i}\tilde{\theta}_i\hat{\theta}_i^q - \tilde{\theta}_i e_i \tanh(e_i/\tau_i)\big]. \tag{6.30}$$

Step 3. Define the Lyapunov function candidate as

$$V(t) = V_1(t) + V_2(t) \tag{6.31}$$

where $V_1(t)$ and $V_2(t)$ are given in (6.23) and (6.28), respectively.

By combining the derivative of $V_1(t)$ given in (6.26) and the derivative of $V_2(t)$ given in (6.30), and recalling that $\tilde{c}_f = c_f - \underline{g}\hat{c}_f$ and $\tilde{\theta} = \theta - \underline{g}\hat{\theta}$, we then arrive at

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) \\ &\leq \sum_{i=1}^{N} \big[-k_i g_i e_i^{1+q} + 0.2785\tau_i(c_{fi} + \theta_i) + \sigma_{1i}\tilde{c}_{fi}\hat{c}_{fi}^q + \sigma_{2i}\tilde{\theta}_i\hat{\theta}_i^q \big]. \end{aligned} \tag{6.32}$$

Upon using Lemma 6.7, it is straightforward that

$$\begin{aligned}\tilde{c}_{fi}\hat{c}_{fi}^q \leq& \frac{1}{\underline{g}^q(1+q)}\Bigg[\left(2^{(q-1)(1+q)} - 2^{q-1}\right)\tilde{c}_{fi}^{1+q} \\ &+ \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q}2^{-(q-1)^2(1+q)}\right) c_{fi}^{1+q}\Bigg], \end{aligned} \tag{6.33}$$

$$\begin{aligned}\tilde{\theta}_i\hat{\theta}_i^q \leq& \frac{1}{\underline{g}^q(1+q)}\Bigg[\left(2^{(q-1)(1+q)} - 2^{q-1}\right)\tilde{\theta}_i^{1+q} \\ &+ \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q}2^{-(q-1)^2(1+q)}\right) \theta_i^{1+q}\Bigg]. \end{aligned} \tag{6.34}$$

By substituting (6.33) and (6.34) into (6.32), one gets

$$\dot{V}(t) \leq -\sum_{i=1}^{N}\left(k_{d1}e_i^{1+q} + k_{d2}\tilde{c}_{fi}^{1+q} + k_{d3}\tilde{\theta}_i^{1+q}\right) + d \tag{6.35}$$

where

$$\begin{aligned} k_{d1} =& \underline{kg} \\ k_{d2} =& \frac{\underline{\sigma}_1}{\underline{g}^q(1+q)}\left(2^{q-1} - 2^{(q-1)(1+q)}\right) \\ k_{d3} =& \frac{\underline{\sigma}_2}{\underline{g}^q(1+q)}\left(2^{q-1} - 2^{(q-1)(1+q)}\right) \\ d =& \sum_{i=1}^{N}\Bigg[0.2785\tau_i(c_{fi} + \theta_i) + \frac{1}{\underline{g}^q(1+q)}\left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q}2^{-(q-1)^2(1+q)}\right) \\ &\times\left(\sigma_{1i}c_{fi}^{1+q} + \sigma_{2i}\theta_i^{1+q}\right)\Bigg] < \infty, \end{aligned} \tag{6.36}$$

with $\underline{k} = \min\{k_1, \ldots, k_N\}$, $\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}$, and $\underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}$. Note that $2^{q-1} - 2^{(q-1)(1+q)} > 0$, which ensures that $k_{d2} > 0$ and $k_{d3} > 0$. By introducing k_d as

$$k_d = \min\{k_{d1}, k_{d2}, k_{d3}\}, \tag{6.37}$$

we can further represent (6.35) as

$$\dot{V}(t) \leq -k_d \sum_{i=1}^{N} \left(e_i^{1+q} + \tilde{c}_{fi}^{1+q} + \tilde{\theta}_i^{1+q} \right) + d. \tag{6.38}$$

Step 4. We prove that there exists a finite time $T^* > 0$ that can be explicitly expressed and a bounded constant $0 < \zeta < \infty$ such that $V(t) < \zeta$ when $t \geq T^*$ in the sequel.

Note that

$$V_1(t) = \frac{1}{2} E^T \Lambda E \leq \frac{\lambda_{\max}(\Lambda)}{2} \sum_{i=1}^{N} e_i^2, \tag{6.39}$$

$$V_2(t) \leq \frac{1}{2\underline{\gamma}_1 \underline{g}} \sum_{i=1}^{N} \tilde{c}_{fi}^2 + \frac{1}{2\underline{\gamma}_2 \underline{g}} \sum_{i=1}^{N} \tilde{\theta}_i^2, \tag{6.40}$$

where $\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$ and $\underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. By introducing k_v as

$$k_v = \max\{\frac{\lambda_{\max}(\Lambda)}{2}, \frac{1}{2\underline{\gamma}_1 \underline{g}}, \frac{1}{2\underline{\gamma}_2 \underline{g}}\}, \tag{6.41}$$

we then get from (6.39) and (6.40) that

$$V(t) \leq k_v \sum_{i=1}^{N} \left(e_i^2 + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2 \right). \tag{6.42}$$

According to Lemma 6.5, we get from (6.42) that

$$V(t)^{\frac{1+q}{2}} \leq k_v^{\frac{1+q}{2}} \left[\sum_{i=1}^{N} \left(e_i^2 + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2 \right) \right]^{\frac{1+q}{2}} \leq k_v^{\frac{1+q}{2}} \sum_{i=1}^{N} \left(e_i^{1+q} + \tilde{c}_{fi}^{1+q} + \tilde{\theta}_i^{1+q} \right). \tag{6.43}$$

Let $\tilde{c} = \frac{\eta_1 k_d}{k_v^{\frac{1+q}{2}}}$ (k_d and k_v are given in (6.37) and (6.41), respectively, and $0 < \eta_1 \leq 1$). By combining (6.38) and (6.43), it thus follows that

$$\dot{V}(t) \leq -\tilde{c} V(t)^{\frac{1+q}{2}} + d. \tag{6.44}$$

Let $\Theta = \{X : V(t) < (\frac{d}{\eta_2 \tilde{c}})^{\frac{2}{1+q}}, 0 < \eta_2 < 1\}$. According to Theorem 5.2 in [6], for any $X \notin \Theta$ for all $t \in [0, t_x]$, it holds that $V(t) \geq (d/\eta_2 \tilde{c})^{\frac{2}{1+q}}$ i.e., $d \leq \eta_2 \tilde{c} V(t)^{\frac{1+q}{2}}$, for all $t \in [0, t_x]$. This fact, together with (6.44), implies that

$$\dot{V}(t) \leq -\tilde{c} V(t)^{\frac{1+q}{2}} + \eta_2 \tilde{c} V(t)^{\frac{1+q}{2}} = -(1 - \eta_2) \tilde{c} V(t)^{\frac{1+q}{2}} \tag{6.45}$$

for all $t \in [0, t_x]$. Note that $V(t) \geq (d/\eta_2\tilde{c})^{\frac{2}{1+q}} > 0$ for $t \in [0, t_x]$, it thus follows from (6.45) that $t_x < \frac{V(t_0)^{1-\frac{1+q}{2}}}{(1-\eta_2)\tilde{c}(1-\frac{1+q}{2})}$ according to Lemma 6.1. Therefore for $\forall t \geq T^*$, with T^* satisfying

$$T^* \leq \frac{V(t_0)^{1-\frac{1+q}{2}}}{(1-\eta_2)\tilde{c}(1-\frac{1+q}{2})}, \tag{6.46}$$

we have

$$V(t) < (\frac{d}{\eta_2\tilde{c}})^{\frac{2}{1+q}} = \zeta. \tag{6.47}$$

Step 5. Derive the estimation for steady-state errors of all agents.

Note that for all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} |e_i| = \sqrt{e_i^2} \leq \sqrt{\sum_{i=1}^{N} e_i^2} \leq \sqrt{\frac{2V_1(t)}{\lambda_{\min}(\Lambda)}} \\ \leq \sqrt{\frac{2V(t)}{\lambda_{\min}(\Lambda)}} \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}} \left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}}. \end{aligned} \tag{6.48}$$

In addition, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned} |\tilde{c}_{fi}| = \sqrt{\tilde{c}_{fi}^2} \leq \sqrt{\sum_{i=1}^{N} \tilde{c}_{fi}^2} \leq \sqrt{2\underline{g}\bar{\gamma}_1 V_2(t)} \\ \leq \sqrt{2\underline{g}\bar{\gamma}_1 V(t)} \leq \sqrt{2\underline{g}\bar{\gamma}_1} \left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}} \end{aligned} \tag{6.49}$$

and

$$|\tilde{\theta}_i| \leq \sqrt{2\underline{g}\bar{\gamma}_2} \left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}} \tag{6.50}$$

with $\bar{\gamma}_1 = \max\{\gamma_{11}, \ldots, \gamma_{1N}\}$ and $\bar{\gamma}_2 = \max\{\gamma_{21}, \ldots, \gamma_{2N}\}$.

Thus under the proposed finite-time control scheme (6.7)–(6.10), the leaderless consensus errors between neighbor agents will converge to a small region Ω_1, defined by

$$\Omega_1 = \left\{ |e_i| \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}} \left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}},\ \forall i = 1, \ldots, N \right\}, \tag{6.51}$$

and the generalized parameter estimation errors will converge to the region Ω_{1p},

$$\Omega_{1p} = \left\{ |\tilde{c}_{fi}| \le \sqrt{2\underline{g}\bar{\gamma}_1} \left(\frac{d}{\eta_2 \tilde{c}} \right)^{\frac{1}{1+q}}, |\tilde{\theta}_i| \le \sqrt{2\underline{g}\bar{\gamma}_2} \left(\frac{d}{\eta_2 \tilde{c}} \right)^{\frac{1}{1+q}}, \ \forall i = 1, \ldots, N \right\} \tag{6.52}$$

in the finite time T^*. □

6.2.4 Comparison with Regular State Feedback-Based Infinite-Time Method

Note that if we take the value of the fraction power q as 1, the control scheme (6.7)–(6.10) reduces to the following regular state feedback-based infinite-time control

$$u_i = -k_i e_i - \hat{c}_{fi}\varphi_i \tanh(e_i\varphi_i/\tau_i) - \hat{\theta}_i \tanh(e_i/\tau_i), \tag{6.53}$$

with the updated laws

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi} + \gamma_{1i}\varphi_i \tanh(e_i\varphi_i/\tau_i)e_i, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \gamma_{2i}\tanh(e_i/\tau_i)e_i. \end{aligned} \tag{6.54}$$

With such regular state feedback-based control, the following result is derived.

Theorem 6.2 *Consider system* (6.5) *under Assumption 6.1 and undirected connected topology condition. If controlled by the adaptive control law given in* (6.53)–(6.54)*, the cooperative uniformly ultimately bounded (UUB) leaderless consensus is achieved in that the leaderless consensus errors converge to a set Ω_2 defined by*

$$\Omega_2 = \left\{ |e_i| \le \sqrt{\frac{2k_v d}{\lambda_{\min}(\Lambda)\eta_2\eta_1 k_d}}, \ \forall i = 1, \ldots, N \right\}, \tag{6.55}$$

and the generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a set Ω_{2p} given as

$$\Omega_{2p} = \left\{ \left|\tilde{c}_{fi}\right| \le \sqrt{\frac{2\underline{g}\bar{\gamma}_1 k_v d}{\eta_2\eta_1 k_d}}, \ \left|\tilde{\theta}_i\right| \le \sqrt{\frac{2\underline{g}\bar{\gamma}_2 k_v d}{\eta_2\eta_1 k_d}}, \ \forall i = 1, \ldots, N \right\}, \tag{6.56}$$

respectively, where $0 < \eta_1 \le 1$, $0 < \eta_2 < 1$, $V(t_0)$ is known, k_d and k_v are given, respectively, in (6.37) and (6.41) with $q = 1$, which are explicitly computable.

Proof Following the same line as in the proof of Theorem 6.1 and by setting $q = 1$, it can be derived that

$$\dot{V}(t) \le -\frac{k_d}{k_v} V(t) + d \tag{6.57}$$

where k_d, k_v, and d are defined the same as in Theorem 6.1 associated with $q = 1$. Define

$$\bar{\Theta} = \left\{ X : V(t) < \frac{k_v d}{\eta_2 k_d}, 0 < \eta_2 < 1 \right\} \tag{6.58}$$

Since $d \le \frac{\eta_2 k_d V(t)}{k_v}$ when $X \notin \bar{\Theta}$, it follows from (6.57) that

$$\dot{V}(t) \le -\frac{k_d}{k_v} V(t) + \frac{\eta_2 k_d}{k_v} V(t) < 0. \tag{6.59}$$

Hence, $\dot{V}(t) < 0$ once $X \notin \bar{\Theta}$. Note that $\bar{\Theta}$ is a global attractive region due to the fact that $\bar{\Theta}$ is a level set of the Lyapunov function, that is, the states would not escape from $\bar{\Theta}$ after entering it.

In addition, it can be readily derived that under the control laws (6.53)–(6.54) the leaderless consensus errors converge to the set Ω_2 specified by (6.55) and the generalized estimation errors converge to Ω_{2p} given in (6.56), respectively. □

6.2.5 Notes

It is worth noting that several parameters in the proposed control scheme (6.7)–(6.10) can be adjusted, which are roughly identified into three groups: one is in the definition of k_d, another one is in the definition of k_v, and the third one is in the definition of d. The finite-time T^* defined in (6.21) can be made smaller by letting $k_v^{\frac{1+q}{2}}/k_d$ smaller, i.e., making k_v smaller and k_d larger; the error convergence bounds in region Ω_1 can be made smaller by letting $dk_v^{\frac{1+q}{2}}/k_d$ smaller, i.e., making k_v and d smaller and k_d larger. Upon examining the definition of k_v, k_d, and d, it is seen that increasing k_i lead to an increasing k_d; increasing k_i, γ_{1i} and γ_{2i} results in a decreasing k_v; increasing k_i and decreasing σ_{1i}, σ_{2i} and τ_i lead to a smaller d. In addition, by increasing the parameters σ_{1i}, σ_{2i}, γ_{1i}, and γ_{2i}, the adaptation rate for $\hat{c}_{fi}$ and $\hat{\theta}_i$ becomes faster. However, certain tradeoff between transient performance and control precision need to be made in practice. This recipe for parameter selection makes it clear and straightforward to choose the suitable parameters to affect the settling time T^* as defined in (6.21) and the control precision as reflected in Ω_1 and Ω_2, enabling the proposed control scheme with the abilities to make both the residual errors and the convergence time as small as desired.

It is seen that by choosing the control gains k_i large enough, all convergence regions, i.e., Ω_1 and Ω_{1p} in the proposed finite-time control scheme, Ω_2 and Ω_{2p} in the regular state feedback-based control scheme, can be reduced as small as desired. Nevertheless, due to control saturation constraint, k_i are not allowed to be set too large. While in the proposed method one can choose the fractional power q properly to enhance the disturbance rejection performance and increase the control precision without the need for excessively large k_i.

The main contributions of this section are summarized as follows. First, the distributed adaptive finite-time control scheme is established based upon a fractional-order local state negative feedback and fractional power-based adaptive updated law, allowing the leaderless consensus to be achieved in finite time with sufficient accuracy. Second, by introducing the generalized parameter estimation error, the unknown and time-varying control gain inherent in the systems is dealt with gracefully. Third, with the norm-bounding skill the lumped uncertainties of each agent are converted into a simplified relation that only involves a virtual parameter to be estimated and a computable scalar function with core information, thus facilitating the online computations and control design, and the unknown and non-parametric uncertainties are explicitly accommodated.

6.3 Finite-Time Leaderless Consensus Control Under Directed Topology

In this section, we address the finite-time leaderless consensus problem for networked multi-agent systems with first-order dynamics under directed graph topology. We first introduce the Lyapunov function design for the stability analysis under directed topology, and then we start with controller design and stability analysis to explain the fundamental idea and the technical development of the proposed method.

6.3.1 Problem Formulation

The focus of this section is to show how to tackle the technical difficulty arising from the asymmetric Laplacian matrix under the directed communication topology condition in the finite-time leaderless consensus problem, so here we consider the first-order normal-form system (6.5) under the directed communication topology condition in the case that the control gain g_i is known and nonzero, and $f_i(z_i, t)$, $f_{di}(z_i, t)$ are known.

Standard assumptions for the existence of unique solutions and global controllability of system are made, e.g., f_i and f_{di} are either continuously differentiable or Lipschitz.

The finite-time leaderless consensus control design problem confronted herein is as follows: (1) Design control schemes for all agents i in $\mathscr{G}$ to make an agreement in finite-time under directed topology. The control schemes must be distributed in the sense that they can only depend on local information about the agent itself and its neighbors in graph, and meanwhile, the control schemes should be endowed with the finite-time quality; (2) Overcome the technical difficulty in constructing the Lyapunov function arising from asymmetric property of the original Laplacian matrix under the directed communication topology condition.

6.3.2 Lyapunov Function Design Under Directed Topology

In this subsection, we introduce the Lyapunov function design for the finite-time stability analysis under the strongly connected directed topology. In the next subsection, we will show how to apply such Lyapunov function in the Lyapunov analysis of the finite-time leaderless consensus problem for multi-agent systems under the directed topology condition.

Before moving on, we first present several useful lemmas. It is important to note that the Laplacian matrix $\mathscr{L}$ is no longer symmetric under the one-way directed topology, which imposes significant challenge for the finite-time leaderless consensus control design because one cannot use such $\mathscr{L}$ to construct Lyapunov function directly. To circumvent the technical difficulty in constructing Lyapunov function arising from asymmetric property of the original Laplacian matrix $\mathscr{L}$ under directed graph, we introduce a new matrix Q [7],

$$Q = (\mathrm{diag}(p)\mathscr{L} + \mathscr{L}^T \mathrm{diag}(p)), \tag{6.60}$$

where $\mathrm{diag}(p) = \mathrm{diag}\{p_1, \ldots, p_N\} \in \mathbb{R}^{N\times N} > 0$ with $p = [p_1, \ldots, p_N]^T$ being the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue. Such defined matrix Q is actually a Laplacian corresponding to a connected undirected graph [4], which exhibits the property of symmetry and semi-positive definiteness with one simple zero eigenvalue.

The significance of the feature on Q is that the original directed graph problem can be treated as an undirected graph problem. Furthermore, such feature allows for the development of the following two lemmas that are crucial for the construction of the Lyapunov function design subject to directed graph.

Lemma 6.8 *For a strongly connected digraph $\mathscr{G}$ with Laplacian $\mathscr{L}$, let Q be defined as in (6.60), then $\forall x \neq 0_N$, $x^T Qx = 0$ if and only if $x = c1_N$, where $c \neq 0$ is a constant. Moreover, $\min\limits_{x\neq c1_N} \frac{x^T Qx}{x^T x}$ exists and*

$$0 < \min_{x\neq c1_n} \frac{x^T Qx}{x^T x} \leq \sum_{i=2}^{n} \lambda_i(Q). \tag{6.61}$$

Upon using Lemma 6.8, the following result can be established.

Lemma 6.9 $\forall E^q \neq 0_N$ *(*$E^q = \left[e_1^q, \ldots, e_N^q\right]^T$*), there exists a constant* $k_m > 0$ *such that*

$$\frac{(E^q)^T Q E^q}{(E^q)^T E^q} \geq k_m. \tag{6.62}$$

Proof Note that for $p^T E = \mathscr{L} X = 0$, i.e., $\sum_{i=1}^N p_i e_i = 0$, and $p_i > 0$ according to [8], from which we know that for $E \neq 0$ and $i = 1, \ldots, N$, it is impossible that $\text{sgn}(e_i) = 1$ (or $\text{sgn}(e_i) = -1$). Therefore, $E \neq c1_N$ with c being a nonzero constant. Note that $\text{sgn}(e_i) = \text{sgn}(e_i^q)$ $(i = 1, \ldots, N)$, it is straightforward that $E^q \neq c1_N$. According to Lemma 6.8, there exists a constant $k_\text{m} = \min\limits_{E^q \neq 0_N} \frac{(E^q)^T Q E^q}{(E^q)^T E^q} > 0$ such that $\frac{(E^q)^T Q E^q}{(E^q)^T E^q} \geq k_\text{m} > 0$. □

According to the above analysis, we construct the Lyapunov function candidate as

$$V = \frac{1}{1+q} \left(E^{\frac{1+q}{2}}\right)^T \text{diag}(p) E^{\frac{1+q}{2}}, \tag{6.63}$$

where $E^{\frac{1+q}{2}} = \left[e_1^{\frac{1+q}{2}}, \ldots, e_N^{\frac{1+q}{2}}\right]^T$. Hereafter, we let $E^h = \left[e_1^h, \ldots, e_N^h\right]^T$ with $h \in \mathbb{R}$.

6.3.3 Finite-Time Leaderless Consensus Control Design and Stability Analysis Under Directed Topology

The control objective of this section is to design a finite-time control strategy under the directed topology condition such that the leaderless consensus errors uniformly converge to zero in finite-time, and meanwhile, to give the strict stability analysis.

To this end, the control input for each ith $(i = 1, \ldots, N)$ agent is designed as

$$u_i = -\frac{1}{g_i} \left[k e_i^q + f_i + f_{di}\right], \tag{6.64}$$

where $0 < q = q_1/q_2 < 1$, q_1, q_2 are positive odd integers, and $k > 0$ is a design parameter chosen by the designer.

Theorem 6.3 (Distributed Finite-Time Control for Leaderless Consensus under Directed Topology) *Consider the nonlinear MAS as described by* (6.5) *under the assumptions that* g_i *is known and nonzero and* $f_i(z_i, t)$*,* $f_{di}(z_i, t)$ *are known. Let the communication topology be directed and strongly connected. If the distributed control law* (6.64) *is applied, then finite-time leaderless consensus is achieved in that*

(1) *the leaderless consensus errors e_i $(i = 1, \ldots, N)$ converge to zero in a finite time T^*, which satisfies*

$$T^* \leq \frac{2\bar{p}^{\frac{2q}{1+q}}\left[(1+q)\,V(t_0)\right]^{\frac{1-q}{1+q}}}{\eta_1 k k_m\,(1-q)}, \tag{6.65}$$

where $\bar{p} = \max\{p_1, \ldots, p_N\}$ $(p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with it zero eigenvalue), $0 < \eta_1 \leq 1$, $V(t_0)$ is known, which is explicitly computable.

Proof By applying the control scheme (6.64) into (6.5), we get

$$\dot{x}_i = -ke_i^q. \tag{6.66}$$

We then choose the Lyapunov function candidate $V = \frac{1}{1+q}(E^{\frac{1+q}{2}})^T \mathrm{diag}(p) E^{\frac{1+q}{2}}$, whose derivative along (6.66) is

$$\begin{aligned}\dot{V}(t) &= \left(E^q\right)^T \mathrm{diag}(p)\dot{E} = \left(E^q\right)^T (\mathrm{diag}(p)\mathscr{L})\dot{X}\\ &= -k\left(E^q\right)^T (\mathrm{diag}(p)\mathscr{L})E^q,\end{aligned} \tag{6.67}$$

with $E^q = \left[e_1^q, \ldots, e_N^q\right]^T$. Upon using Lemma 6.9, we have

$$\begin{aligned}\dot{V}(t) =& -k(E^q)^T\,(\mathrm{diag}(p)\mathscr{L})\,E^q\\ \leq& -\frac{kk_\mathrm{m}}{2}\left(E^q\right)^T E^q = -\frac{kk_\mathrm{m}}{2}\sum_{i=1}^{N}(e_i)^{2q}.\end{aligned} \tag{6.68}$$

Note that

$$V(t) = \frac{1}{1+q}\left(E^{\frac{1+q}{2}}\right)^T \mathrm{diag}(p)E^{\frac{1+q}{2}} \leq \frac{\bar{p}}{1+q}\sum_{i=1}^{N}e_i^{1+q}. \tag{6.69}$$

According to Lemma 6.5, we then get from (6.69) that,

$$V(t)^{\frac{2q}{1+q}} \leq \left(\frac{\bar{p}}{1+q}\right)^{\frac{2q}{1+q}}\left(\sum_{i=1}^{N}e_i^{1+q}\right)^{\frac{2q}{1+q}} \leq \left(\frac{\bar{p}}{1+q}\right)^{\frac{2q}{1+q}}\sum_{i=1}^{N}e_i^{2q} \tag{6.70}$$

Let $\tilde{c} = \frac{1}{2}\eta_1 kk_\mathrm{m}(\frac{1+q}{\bar{p}})^{\frac{2q}{1+q}}$ $(0 < \eta_1 \leq 1)$. It thus follows from (6.68) and (6.70) that

$$\dot{V}(t) \leq -\tilde{c}V(t)^{\frac{2q}{1+q}}. \tag{6.71}$$

Then according to Lemma 6.1, we get from (6.71) that, there exists a finite-time constant T^*, which satisfies,

$$T^* \leq \frac{V(t_0)^{1-\frac{2q}{1+q}}}{\tilde{c}\left(1-\frac{2q}{1+q}\right)}, \tag{6.72}$$

such that for any given initial state $X(t_0) \in \mathbb{R}^N/\{0\}$, $\lim_{t\to T^*} V(X,t) = 0$ and $V(X,t) = 0$ for $t \geq T^*$, which further implies that

$$\begin{aligned} &E \to 0 \ \text{ as } \ t \to T^{*-}, \\ &E = 0 \ \text{ when } \ t \geq T^*. \end{aligned} \tag{6.73}$$

This implies that the leaderless consensus is achieved in the finite time T^* under the proposed control (6.64). In addition, by substituting $\tilde{c}$ into (6.72), we further get that,

$$T^* \leq \frac{2\bar{p}^{\frac{2q}{1+q}}\left[(1+q)V(t_0)\right]^{\frac{1-q}{1+q}}}{\eta_1 k k_{\mathrm{m}}(1-q)}, \tag{6.74}$$

which then yields (6.65). □

6.4 Simulation Example

This section gives a simulation example on a first-order nonlinear multi-agent system with unknown time-varying gain and non-parametric uncertainties to illustrate the feasibility of the distributed adaptive finite-time control algorithm proposed in (6.7)–(6.10).

We use four agents to demonstrate the effectiveness of our proposed control scheme. The communication graph among the four agents is presented in Fig. 6.1, which satisfies the assumption that the graph topology is undirected and connected. Each edge weight is set to be 1.

The dynamics of the four agents is of the following form,

$$\begin{aligned} \dot{x}_1 &=(5+\exp^{-x_1^2})u_1 + 0.1\exp^{-x_1^2-x_2^2}x_1 + f_{d1}, \\ \dot{x}_2 &=(5+\exp^{-x_2^2})u_2 + 0.1\exp^{-x_2^2-x_3^2}x_2^2 + f_{d2}, \\ \dot{x}_3 &=(5+\exp^{-x_3^2})u_3 + 0.1\exp^{-x_2^2-x_4^2}x_3^2 + f_{d3}, \\ \dot{x}_4 &=(5+\exp^{-x_4^2})u_4 + 0.1\exp^{-x_1^2-x_4^2}x_4 + f_{d4}, \end{aligned} \tag{6.75}$$

in which the non-vanishing uncertainties are of the form: $f_{d1} = 0.02\sin(0.05t)$, $f_{d2} = 0.01\cos(0.06t)$, $f_{d3} = 0.02\cos(0.05t)$, and $f_{d4} = 0.01\sin(0.06t)$.

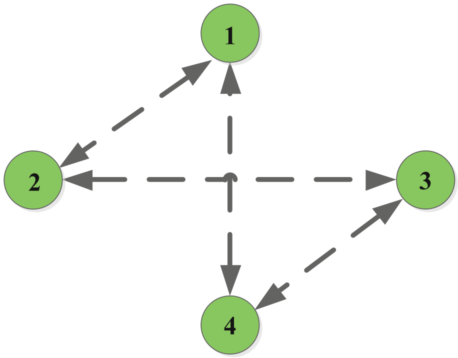

Fig. 6.1 Communication graph topology among the four agents

The earlier theoretical analysis has declared that the nonlinear system (6.75) can achieve finite-time leaderless consensus under the control scheme proposed in (6.7)–(6.10). To set up the proposed controller, we only need to select the design parameters k_i, γ_i, and σ_i $(i = 1, 2, 3, 4)$ and does not need to take any time-consuming analytical derivation for choosing design parameters. The simulation runs for $0.6s$. The initial condition are $X(0) = [2, 1, -1, -2]^T$. In addition, the initial values of the parameter estimates are chosen as $\hat{c}_{fi}(0) = \hat{\theta}_i(0) = 0$ for $i = 1, 2, 3, 4$. The simulation is conducted by applying the control laws given in (6.7)–(6.10), where the design parameters are taken as: $k_i = 1$ $(i = 1, 2, 3, 4)$, $\gamma_i = 0.01$, and $\sigma_i = 0.01$ $(i = 1, 2)$, $\varepsilon = 0.01$. In addition, the core functions $\varphi_i(\cdot)$ $(i = 1, 2, 3, 4)$ are easily taken as $\varphi_1 = |x_1|$, $\varphi_2 = x_2^2$, $\varphi_3 = x_3^2$, and $\varphi_4 = |x_4|$ according to the core information of the system.

Figure 6.2 presents the position trajectories of all the four agents, and Fig. 6.3 shows the neighborhood position synchronization errors of all the four agents, both

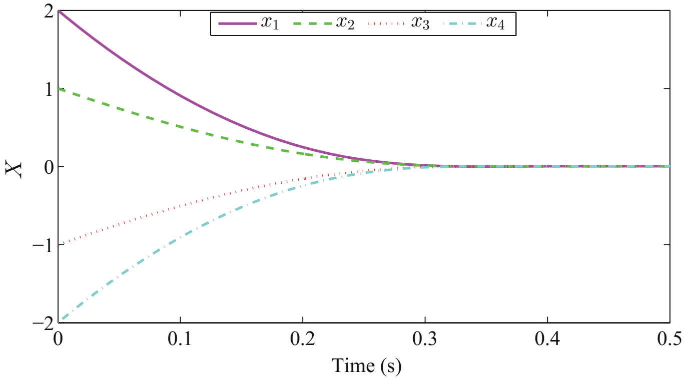

Fig. 6.2 Position trajectories x_i $(i = 1, 2, 3, 4)$ of all the four agents

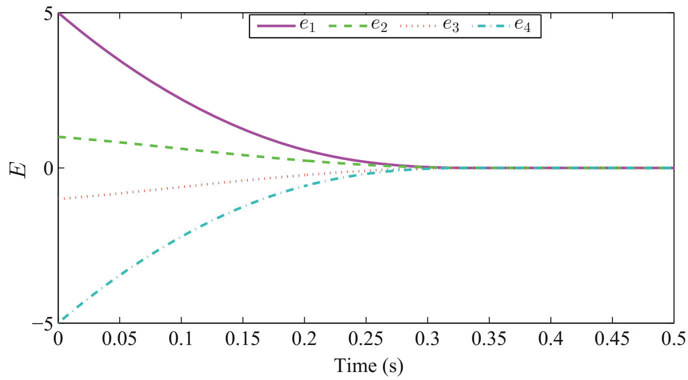

Fig. 6.3 Neighborhood errors e_i $(i = 1, 2, 3, 4)$ of the four agents under the proposed finite-time control scheme (6.7)

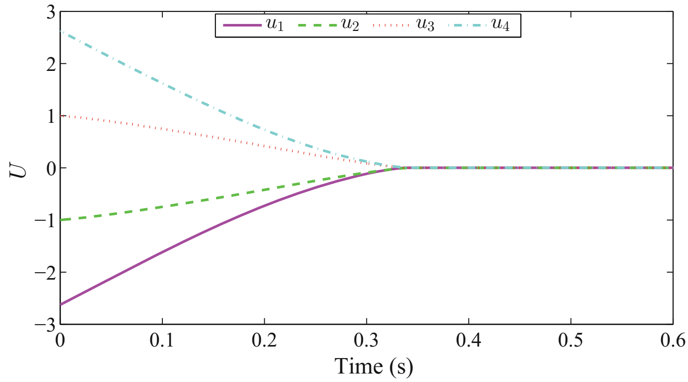

Fig. 6.4 Control inputs u_i $(i = 1, 2, 3, 4)$ of the four agents

of which verify that the position synchronization is achieved in finite time. Figure 6.4 shows the control input signals of the four agents u_i $(i = 1, 2, 3, 4)$, from which we observe that the control input signals of all the four agents are continuous and bounded.

To show that better performance is achieved with the proposed finite-time control scheme, we also tested the convergence property between the proposed finite-time control scheme (6.7)–(6.10) and the typical infinite-time-based adaptive scheme (6.53)–(6.54). Both the two control laws are applied to system (6.75) by using the same design parameters. It is observed from Figs. 6.3, 6.5, and 6.6 that the convergence rate is faster and the error precision is better with the finite-time controller compared with the infinite-time controller.

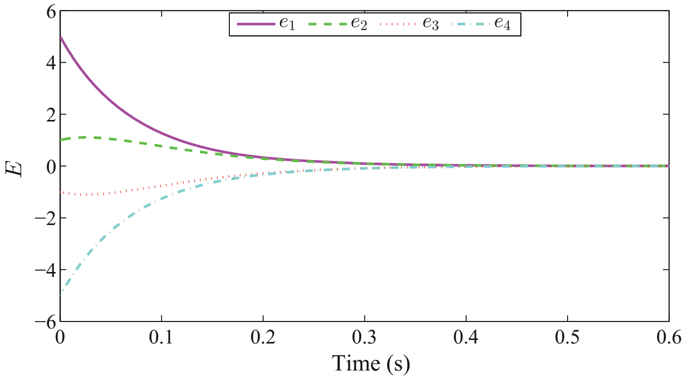

Fig. 6.5 Neighborhood errors e_i ($i = 1, 2, 3, 4$) of the four agents under the infinite-time control scheme (6.53)

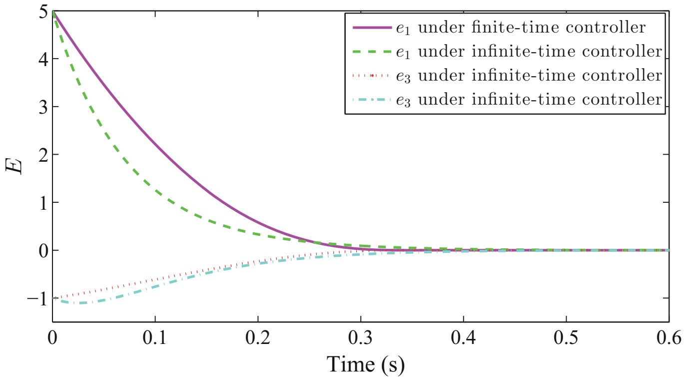

Fig. 6.6 Neighborhood error convergence comparison of agent 1 and 3 under the two different control schemes

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Chapter 7
Finite-Time Consensus for Systems with Second-Order Uncertain Dynamics Under Undirected Topology

In Chap. 6, we have addressed the finite-time leaderless consensus control problem for networked multi-agent systems with first-order uncertain dynamics. In this chapter, we investigate the distributed adaptive finite-time consensus control for cooperative multi-agent systems with second-order dynamics where the unknown time-varying effectiveness gain and non-parametric uncertainties are involved. By considering the requirement in most practical applications that both position and velocity must be controlled such as in formation control and coordination among multiple unmanned aerial vehicles, and also considering the fact that during system operation, mass property of the system might change due to payload releasing, reloading, and/or fuel burning, thus leading to unknown and time-varying parameters in the system, a MAS model with second-order dynamics that explicitly accounts for non-parametric uncertainties, non-vanishing external disturbances as well as unknown time-varying effectiveness gain is necessary to be established and studied. In particular, to study the finite-time control of such networked multi-agent systems with second-order uncertain dynamics is highly requirable.

However, extending the finite-time control methods for first-order systems to second-order case encounters significant technical challenge. The main hindrance stems from the fact that the filtering technique commonly used for system with second-order dynamics cannot be used here to derive the finite-time result upon utilizing Babarlat lemma or UUB theory because in the finite-time stability analysis, the convergence of the filtered error to zero or to a small value cannot ensure the finite time convergence of the original error. To circumvent this technical difficulty, we introduce a virtual controller and resort to adding a power integrator technique, which allows the networked multi-agent systems with second-order uncertain dynamics to achieve the leaderless consensus in finite time with sufficient accuracy.

Section 7.1 formulates the leaderless consensus control problem for multi-agent systems with second-order dynamics in the presence of unknown time-varying gain and non-parametric uncertainties under undirected topology. Section 7.2 represents some useful lemmas that is related to the finite-time stability analysis for the multi-agent systems. Section 7.3 addresses the leaderless consensus controller design and

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stability analysis for the second-order uncertain multi-agent systems. A distributed finite-time leaderless consensus control scheme is derived such that the consensus error uniformly converges to a small residual set in finite time and all the internal signals are ensured to be uniformly bounded. In addition, the finite convergence time for each agent to reach the required consensus configuration is explicitly established and recipes for control parameter selection to make the residual errors as small as desired are provided. Section 7.4 conducts numerical simulation to confirm the effectiveness of the theoretical results obtained in the previous sections. Section 7.5 gives some notes on the finite-time control of the second-order uncertain multi-agent systems.

7.1 Formulation of Finite-Time Consensus for Second-Order Uncertain Systems

In this section, we formulate the leaderless consensus control problem for networked multi-agent systems with second-order dynamics. The agent dynamics are subject to unknown time-varying gain and unknown non-parametric uncertainties, meaning that the parameters are unknown and time-varying and the dynamics are not allowed to appear in the control protocols.

We consider a group of N agents with scalar second-order dynamics

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= g_i(z_i, t)u_i(t) + f_i(z_i, t) + f_{di}(z_i, t), \end{aligned} \tag{7.1}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position state, velocity state, and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i denotes the control effectiveness gain, possibly time-varying and unavailable for controller design, f_i denotes the system nonlinearities, and f_{di} denotes the non-vanishing uncertainties (f_i and f_{di} are either continuously differentiable or Lipschitz). In addition, $z_i = \bigcup_{j\in\mathcal{N}_i \bigcup i}\{x_j, v_j\}$.

Standard assumptions for existence of unique solutions and global controllability of system are made.

Assumption 7.1 a. The control gain g_i is unknown and time-varying and is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$). In addition, $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$ with $\underline{g}_i$, $\bar{g}_i$ being unknown finite constants, which ensures the global controllability of the system.

b. Certain crude structural information is available to allow the unknown and non-parametric uncertainty $f_i(z_i, t)$ satisfy $|f_i(z_i, t)| \leq c_{fi}\varphi_i(z_i)$ for all $t \geq t_0$, where $c_{fi} \geq 0$ is an unknown finite constant and $\varphi_i(z_i)$ is a known scalar function, and $\varphi_i(z_i)$ is bounded for all z_i ($i = 1, \ldots, N$).

c. The unknown and non-vanishing $f_{di}(z_i, t)$ is bounded by some unknown finite constant θ_i, that is, $|f_{di}(z_i, t)| \leq \theta_i < \infty$.

It is assumed for ease of notation that the agents' states are scalars, i.e., $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$. If the states are vectors, i.e., $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$, the results of this chapter can be extended by using the standard method involving the Kronecker products.

The local neighborhood state error for node i is

$$e_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) \tag{7.2}$$

Let $E = [e_1, \ldots, e_N]^T$, $X = [x_1, \ldots, x_N]^T$ such that $E = \mathcal{L}X$.

The distributed finite-time leaderless consensus control design problem confronted herein is as follows: (1) Design control schemes for all agents i with second-order uncertain dynamics in $\mathcal{G}$ to make an agreement in finite time. The control schemes must be distributed in the sense that they can only depend on local information about the agent itself and its neighbors in graph, and meanwhile, the control schemes should be endowed with the finite-time quality; (2) Tackle unknown and time-varying control gains explicitly; (3) Accommodate the unknown non-parametric uncertainties and unknown non-vanishing nonlinearities by using the control method based on fractional power-based adaptive laws.

7.2 Useful Lemmas

In this section, we represent some useful lemmas for the finite-time stability analysis.

Lemma 7.1 ([1]) *Suppose there exists a continuously differentiable function $V(x,t) : \mathbb{U} \times \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{U} \subset \mathbb{R}^N$ is an open neighborhood of the origin), a real number $c > 0$ and $0 < \alpha < 1$, such that $V(x,t)$ is positive definite and $\dot{V}(x,t) + cV(x,t)^\alpha \leq 0$ on $\mathbb{U}_0$ ($\mathbb{U}_0 \subset \mathbb{U}$), then $V(x,t)$ is locally in finite-time convergent with a finite settling time satisfying $T^* \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$, such that for any given initial state $x(t_0) \in \mathbb{U}_0/\{0\}$, $\lim_{t\to T^*} V(x,t) = 0$ and $V(x,t) = 0$ for $t \geq T^*$.*

Lemma 7.2 ([2]) *If $h = h_2/h_1 \geq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x - y|^h \leq 2^{h-1}|x^h - y^h|$.*

Lemma 7.3 ([2]) *If $0 < h = h_1/h_2 \leq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x^h - y^h| \leq 2^{1-h}|x - y|^h$.*

Lemma 7.4 ([2]) *For $x, y \in \mathbb{R}$, if $c, d > 0$, then $|x|^c|y|^d \leq c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.*

Lemma 7.5 ([3]) *For $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, $0 < h \leq 1$, then $(\sum_{i=1}^N |x_i|)^h \leq \sum_{i=1}^N |x_i|^h \leq N^{1-h}(\sum_{i=1}^N |x_i|)^h$.*

Lemma 7.6 ([4]) *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $0 < h = h_1/h_2 \leq 1$, where $h_1, h_2 > 0$ are positive odd integers, then $x^h(y - x) \leq \frac{1}{1+h}(y^{1+h} - x^{1+h})$.*

Lemma 7.7 ([5]) *For a connected undirected graph $\mathscr{G}$, the Laplacian matrix L has the following properties: (1) L is semi-definite. (2) 0 is a simple eigenvalue of L and 1 is the associated eigenvector. (3) Assuming the eigenvalue of L is denoted as $0, \lambda_2(L), \ldots, \lambda_N(L)$ satisfying $0 \leq \lambda_2(L) \leq \cdots \leq \lambda_N(L)$, then the second smallest eigenvalue $\lambda_2(L) > 0$. Furthermore, if $1_N^T X = 0$, then $X^T L X \geq \lambda_2(L) X^T X$.*

Lemma 7.8 *For a connected undirected graph $\mathscr{G}$, it holds that*

$$E^T E \geq \lambda_2(L) X^T L X \tag{7.3}$$

where $E = [e_1, \ldots, e_N]^T$ is the leaderless consensus error vector defined by (7.2) and $X = [x_1, \ldots, x_N]^T$ is the position state vector.

Proof Note that $E = LX$. Thus, $E^T E = X^T L^T \mathscr{L} X = X^T L^2 X$. Since L is a diagonalizable symmetric semi-positive definite matrix, then it is easy to prove that $L^{1/2}$ is also a symmetric semi-positive definite matrix, and $L^2 = L^{1/2} L L^{1/2}$. Let $W = L^{1/2} 1_N$, we get $W^T W = \left(L^{1/2} 1_N\right)^T \left(L^{1/2} 1_N\right) = 1_N^T L 1_N = 0$. Thus, $W = 0_N$, which yields $W^T X = 0$, i.e., $1_N^T L^{1/2} X = 0$. Then according to Lemma 7.7, it holds that

$$E^T E = \left(L^{1/2} X\right)^T L \left(L^{1/2} X\right) \geq \lambda_2(L) X^T L X. \tag{7.4}$$

$\square$

Lemma 7.9 *For the generalized weight parameter estimation error $\tilde{\bullet} = \bullet - \underline{g} \hat{\bullet}$ and the constant $0 < q = q_1/q_2 < 1$ (q_1 and q_2 are positive odd integrators), it holds that*

$$\begin{aligned} \tilde{\bullet} \hat{\bullet}^q \leq & \underline{g}^{-q} \frac{1}{1+q} \Bigg[\left(2^{(q-1)(1-q)} - 2^{q-1}\right) \tilde{\bullet}^{1+q} \\ & + \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q} 2^{-(q-1)^2(1+q)}\right) \bullet^{1+q} \Bigg]. \end{aligned} \tag{7.5}$$

Proof The proof can be seen in Lemma 6.7, so it is omitted here.

7.3 Distributed Adaptive Finite-Time Control for Second-Order Uncertain Systems

This section represents the basic structure of the distributed adaptive finite-time control protocols for the second-order uncertain multi-agent systems, allowing the leaderless consensus to be achieved in finite time in spite of the unknown time-varying gain and unknown non-parametric uncertainties. We show how to realize the finite-time convergence by fractional-order state feedback and how to compensate the unknown non-parametric uncertainties by using adaptive control techniques based on the fractional-order adaptive updated law to guarantee finite-time leaderless consensus of second-order uncertain multi-agent systems.

7.3.1 Finite-Time Consensus Control Design

The control objective of this section is to design distributed adaptive finite-time controller such that the impacts arisen from the time-varying control gain, unmodeling nonlinearities, and non-vanishing uncertainties can be compensated and meanwhile the leaderless consensus of the networked multi-agent system is ensured in finite time.

Before design the controller, we first introduce the virtual control of v_i as

$$v_i^* = -c_2 e_i^q, \tag{7.6}$$

where $0 < q = q_1/q_2 < 1, q_1, q_2$ are positive odd integers, and $c_2 > 0$ is an arbitrarily finite constant. We also introduce the virtual error as

$$\delta_i = v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}}. \tag{7.7}$$

The control input for each ith ($i = 1, \ldots, N$) agent is designed to consist of two parts: (1) the negative feedback control term u_{0i} and (2) the compensation control term u_{ci}, which is of the following form

$$u_i = u_{0i} + u_{ci}, \tag{7.8}$$

in which the negative feedback control term u_{0i} is generated by

$$u_{0i} = -c_1 \delta_i^{2q-1}, \tag{7.9}$$

and (2) the compensation control term u_{ci} is generated by

$$u_{ci} = -\hat{c}_{fi}\varphi_i \tanh(\delta_i^{2-q}\varphi_i/\tau_i) - \hat{\theta}_i \tanh(\delta_i^{2-q}/\tau_i) \tag{7.10}$$

with the updated laws

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi}^q + \gamma_{1i}\varphi_i \tanh(\delta_i^{2-q}\varphi_i/\tau_i)\delta_i^{2-q}, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i^q + \gamma_{2i}\tanh(\delta_i^{2-q}/\tau_i)\delta_i^{2-q}, \end{aligned} \tag{7.11}$$

where δ_i is defined by (7.7), $0 < q = q_1/q_2 < 1$, q_1, q_2 are positive odd integers, $c_1, c_2 > 0$ are design parameters, $\hat{c}_{fi}$ and $\hat{\theta}_i$ are, respectively, the estimations of c_{fi} and θ_i (the parameters to be estimated), $\varphi_i(\cdot)$ is the scalar and readily computable function as given in Assumption 7.1, and $k_i, \tau_i, \gamma_{1i}, \gamma_{2i}, \sigma_{1i}$, and σ_{2i} are positive finite design parameters chosen by the designer.

7.3.2 Finite-Time Consensus Stability Analysis

Theorem 7.1 (Distributed Adaptive Finite-Time Consensus of Second-Order Uncertain Systems) *Suppose that Assumption 7.1 holds and the communication topology is undirected and connected. The uncertain MAS (7.1) with the distributed adaptive control law (7.8)–(7.11) achieves finite-time leaderless consensus in that*

(1) the leaderless consensus positive errors e_i $(i = 1, \ldots, N)$ and velocity errors converge to a small residual set Ω_1 defined by

$$\Omega_1 = \Bigg\{ |e_i| \leq \sqrt{\frac{2k_v}{\lambda_{\min}(\Lambda)}} \left(\frac{d}{\eta_2\eta_1 k_d}\right)^{\frac{1}{1+q}}, \quad |v_i - v_j| \leq 2\Bigg[\frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}} 2^{\frac{q}{2}-1}} + c_2\left(\frac{2}{\lambda_{\min}(\Lambda)}\right)^{\frac{q}{2}}\Bigg] \left(\frac{d k_v^{\frac{1+q}{2}}}{\eta_2\eta_1 k_d}\right)^{\frac{q}{1+q}}, \ \forall i, j \in \{1, \ldots, N\}\Bigg\}, \tag{7.12}$$

in a finite time T^* satisfying

$$T^* = \frac{2V(t_0)^{\frac{1-q}{2}} k_v^{\frac{1+q}{2}}}{(1-\eta_2)(1-q)\eta_1 k_d}, \tag{7.13}$$

where $0 < \eta_1 \leq 1$, $0 < \eta_2 < 1$, $V(t_0)$ is known, k_d and k_v are given respectively in (7.42) and (7.46), which are explicitly computable;

(2) the generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a small set Ω_{1p} defined by

$$\Omega_{1p} = \Bigg\{ |\tilde{c}_{fi}| \leq \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1} \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1}{1+q}}, \quad |\tilde{\theta}_i| \leq \sqrt{2k_\gamma \underline{g}\bar{\gamma}_2} \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1}{1+q}}, \ \forall i \in \{1, \ldots, N\}\Bigg\} \tag{7.14}$$

in the finite time T^*;

(3) all signals in the closed-loop system remain uniformly bounded.

Proof The proof of the result can be done by the following six steps.

Step 1. Construct the distributed part of the Lyapunov function candidate as

$$V_1(t) = \frac{1}{2}E^T \varLambda E = \frac{1}{2}X^T \mathscr{L} X \tag{7.15}$$

where $\varLambda = U_N^T \tilde{\varLambda}^{-1} U_N$, $\tilde{\varLambda} = diag\{a, \lambda_2, \ldots, \lambda_N\}$ (a is an arbitrarily positive constant), U_N is an orthogonal matrix such that $\mathscr{L} = U_N^T \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\} U_N = U_N^T \varLambda_0 U_N$, with $\varLambda_0 = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$ provided by Lemma 3.1. Such matrix $\varLambda$ is symmetric and positive definite, implying that the distributed part of Lyapunov function $V_1(t)$ is well defined. Taking the time derivative of $V_1(t)$ along (7.1) yields that

$$\dot{V}_1(t) = E^T \dot{X} = \sum_{i=1}^{N} e_i v_i. \tag{7.16}$$

Let $v_i^* = -c_2 e_i^q$ be the virtual control of v_i, then we have

$$\dot{V}_1(t) = \sum_{i=1}^{N} e_i v_i^* + \sum_{i=1}^{N} e_i (v_i - v_i^*) = -c_2 \sum_{i=1}^{N} e_i^{1+q} + \sum_{i=1}^{N} e_i (v_i - v_i^*). \tag{7.17}$$

By recalling that $\delta_i = v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}}$, we get from Lemma 7.3 that

$$\begin{aligned} \sum_{i=1}^{N} e_i (v_i - v_i^*) &\le \sum_{i=1}^{N} |e_i||v_i - v_i^*| \\ &= \sum_{i=1}^{N} |e_i| \left| \left(v_i^{\frac{1}{q}}\right)^q - \left((v_i^*)^{\frac{1}{q}}\right)^q \right| \le 2^{1-q} \sum_{i=1}^{N} |e_i||\delta_i|^q. \end{aligned} \tag{7.18}$$

Upon using Lemma 7.4, it follows from (7.18) that

$$\sum_{i=1}^{N} e_i (v_i - v_i^*) \le \frac{2^{1-q}}{1+q} \sum_{i=1}^{N} \left(|e_i|^{1+q} + q|\delta_i|^{1+q}\right) = \frac{2^{1-q}}{1+q} \sum_{i=1}^{N} \left(e_i^{1+q} + q\delta_i^{1+q}\right). \tag{7.19}$$

By substituting (7.19) into (7.17), we arrive at

$$\dot{V}_1(t) \le -c_2 \sum_{i=1}^{N} e_i^{1+q} + \frac{2^{1-q}}{1+q} \sum_{i=1}^{N} \left(e_i^{1+q} + q\delta_i^{1+q}\right). \tag{7.20}$$

Step 2. Define the second part of the Lyapunov function candidate by adding a power integrator [2]

$$V_2(t) = \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{2-q} d\varsigma \tag{7.21}$$

which is positive semi-definite and C^1 [2]. Taking the derivative of $V_2(t)$ yields

$$\begin{aligned}
\dot{V}_2(t) &= \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \Bigg[\left(v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{2-q} \dot{v}_i \\
&\quad + (2-q) \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} d\varsigma \cdot \frac{d(-(v_i^*)^{\frac{1}{q}})}{dt} \Bigg] \\
&= \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \left[\delta_i^{2-q} \dot{v}_i + (2-q) \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} d\varsigma \cdot c_2^{\frac{1}{q}} \dot{e}_i \right] \\
&= \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \Bigg[\delta_i^{2-q} \dot{v}_i + (2-q) \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} d\varsigma \\
&\quad \times c_2^{\frac{1}{q}} \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \Bigg].
\end{aligned} \tag{7.22}$$

Since

$$\begin{aligned}
\left| \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} d\varsigma \right| &\leq \left| \left(v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} \right| \left| v_i - v_i^* \right| = |\delta_i|^{1-q} \left| v_i - v_i^* \right| \\
&\leq |\delta_i|^{1-q} 2^{1-q} |\delta_i|^q = 2^{1-q} |\delta_i|
\end{aligned} \tag{7.23}$$

then the second term of the right side of (7.22) becomes

$$\begin{aligned}
&\frac{2-q}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-q} d\varsigma \cdot c_2^{\frac{1}{q}} \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \\
\leq &\frac{2-q}{c_2} \sum_{i=1}^{N} |\delta_i| \left| \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \right|.
\end{aligned} \tag{7.24}$$

Let $\bar{a} = \max_{\forall i \in \{1,\ldots,N\}} \{ \sum_{j \in \mathcal{N}_i} a_{ij} \}$, $\bar{b} = \max_{\forall i, j \{1,\ldots,N\}} \{a_{ij}\}$, and $\bar{N}$ denoting the maximum number of the out-degree of all the ith ($i \in \{1, \ldots, N\}$) agents in the graph $\mathcal{G}$. We then see that

$$
\begin{aligned}
\frac{2-q}{c_2}\sum_{i=1}^{N}|\delta_i|\left|\sum_{j\in\mathcal{N}_i}a_{ij}(v_i-v_j)\right| \le & \frac{2-q}{c_2}\sum_{i=1}^{N}|\delta_i|\left(\bar{a}|v_i|+\bar{b}\sum_{j\in\mathcal{N}_i}|v_j|\right)\\
\le & \frac{2-q}{c_2}\sum_{i=1}^{N}|\delta_i|\left(\bar{a}|v_i|+\bar{b}\bar{N}|v_i|\right). \qquad (7.25)
\end{aligned}
$$

Upon using Lemmas 7.3 and 7.4, we have

$$
\begin{aligned}
|\delta_i||v_i| \le |\delta_i|\left|v_i-v_i^*\right|+|\delta_i|\left|v_i^*\right| \le & 2^{1-q}|\delta_i||\delta_i|^q+c_2|\delta_i||e_i|^q\\
\le & 2^{1-q}|\delta_i|^{1+q}+\frac{c_2}{1+q}\left(|\delta_i|^{1+q}+q|e_i|^{1+q}\right). \qquad (7.26)
\end{aligned}
$$

Upon using (7.25) and (7.26), (7.24) can be further written as

$$
\begin{aligned}
&\frac{2-q}{2^{1-q}c_2^{1+1/q}}\sum_{i=1}^{N}\int_{v_i^*}^{v_i}\left(\varsigma^{\frac{1}{q}}-(v_i^*)^{\frac{1}{q}}\right)^{1-q}d\varsigma\cdot c_2^{\frac{1}{q}}\sum_{j\in\mathcal{N}_i}a_{ij}(v_i-v_j)\\
&\le \frac{2-q}{c_2}(\bar{a}+\bar{b}\bar{N})\sum_{i=1}^{N}\left[\left(2^{1-q}+\frac{c_2}{1+q}\right)\delta_i^{1+q}+\frac{c_2}{1+q}e_i^{1+q}\right]. \qquad (7.27)
\end{aligned}
$$

By applying the control laws given in (7.8)–(7.10) to the first term of the right hand of (7.22), we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\delta_i^{2-q}\dot{v}_i=\sum_{i=1}^{N}\delta_i^{2-q}\left(g_iu_i+f_i+f_{di}\right)\\
&\le\sum_{i=1}^{N}\Big[-c_1g_i\delta_i^{1+q}-\delta_i^{2-q}g_i\hat{c}_{fi}\varphi_i\tanh\left(\delta_i^{2-q}\varphi_i/\tau_i\right)-\delta_i^{2-q}g_i\hat{\theta}_i\tanh\left(\delta_i^{2-q}/\tau_i\right)\\
&\quad+|\delta_i^{2-q}|c_{fi}\varphi_i+|\delta_i^{2-q}|\theta_i\Big]\\
&\le\sum_{i=1}^{N}\Big[-c_1\underline{g}\delta_i^{1+q}-\underline{g}\hat{c}_{fi}\delta_i^{2-q}\varphi_i\tanh\left(\delta_i^{2-q}\varphi_i/\tau_i\right)-\underline{g}\hat{\theta}_i\delta_i^{2-q}\tanh\left(\delta_i^{2-q}/\tau_i\right)\\
&\quad+c_{fi}\delta_i^{2-q}\varphi_i\tanh\left(\delta_i^{2-q}\varphi_i/\tau_i\right)+0.2785\tau_ic_{fi}+\theta_i\delta_i^{2-q}\tanh\left(\delta_i^{2-q}/\tau_i\right)+0.2785\tau_i\theta_i\Big]\\
&=\sum_{i=1}^{N}\Big[-c_1\underline{g}\delta_i^{1+q}+0.2785\tau_i(c_{fi}+\theta_i)+\left(c_{fi}-\underline{g}\hat{c}_{fi}\right)\delta_i^{2-q}\varphi_i\tanh\left(\delta_i^{2-q}\varphi_i/\tau_i\right)\\
&\quad+\left(\theta_i-\underline{g}\hat{\theta}_i\right)\delta_i^{2-q}\tanh\left(\delta_i^{2-q}/\tau_i\right)\Big] \qquad (7.28)
\end{aligned}
$$

where we have used the fact that $0\le|s|-s\cdot\tanh(s/k)\le0.2785k$ [6].

By substituting (7.27)–(7.28) into (7.22), we arrive at

$$\dot{V}_2(t) \leq k_1 \sum_{i=1}^{N} \delta_i^{1+q} + k_2 \sum_{i=1}^{N} e_i^{1+q} + \sum_{i=1}^{N} \frac{1}{2^{1-q} c_2^{1+1/q}} \Big[0.2785\tau_i \left(c_{fi} + \theta_i \right) \\ + \left(c_{fi} - \underline{g}\hat{c}_{fi} \right) \delta_i^{2-q} \varphi_i \tanh\left(\delta_i^{2-q} \varphi_i / \tau_i \right) + \left(\theta_i - \underline{g}\hat{\theta}_i \right) \delta_i^{2-q} \tanh\left(\delta_i^{2-q} / \tau_i \right) \Big] \tag{7.29}$$

where

$$k_1 = -\frac{c_1 \underline{g}}{2^{1-q} c_2^{1+1/q}} + \frac{2-q}{c_2} (\bar{a} + \bar{b}\bar{N}) \left(2^{1-q} + \frac{c_2}{1+q} \right),$$
$$k_2 = \frac{2-q}{1+q} (\bar{a} + \bar{b}\bar{N}). \tag{7.30}$$

Step 3. Note that in (7.29) the parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ is involved, which motivates us to introduce the generalized weight parameter estimation errors $\tilde{c}_{fi}$ $(i \in \{1, \ldots, N\})$ and $\tilde{\theta}_i$ as follows:

$$\tilde{c}_{fi} = c_{fi} - \underline{g}\hat{c}_{fi}, \quad \tilde{\theta}_i = \theta_i - \underline{g}\hat{\theta}_i, \tag{7.31}$$

with which we introduce the third part of the Lyapunov function candidate as

$$V_3(t) = \sum_{i=1}^{N} \frac{\tilde{c}_{fi}^2}{2k_\gamma \underline{g} \gamma_{1i}} + \sum_{i=1}^{N} \frac{\tilde{\theta}_i^2}{2k_\gamma \underline{g} \gamma_{2i}}. \tag{7.32}$$

where $k_\gamma = 2^{1-q} c_2^{1+1/q}$.

By applying the adaptive laws for $\hat{c}_{fi}$ and $\hat{\theta}_i$ given in (7.11), we get the derivative of V_3 as

$$\begin{aligned} \dot{V}_3(t) &= \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{k_\gamma} \left(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}} \right) + \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{k_\gamma} \left(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}} \right) \\ &= \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{k_\gamma} \left[\sigma_{1i} \hat{c}_{fi}^q - \delta_i^{2-q} \varphi_i \tanh(\delta_i^{2-q} \varphi_i / \tau_i) \right] \\ &\quad + \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{k_\gamma} \left[\sigma_{2i} \hat{\theta}_i^q - \delta_i^{2-q} \tanh(\delta_i^{2-q} / \tau_i) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \frac{\sigma_{1i}}{k_\gamma} \tilde{c}_{fi} \hat{c}_{fi}^{q} + \sum_{i=1}^{N} \frac{\sigma_{2i}}{k_\gamma} \tilde{\theta}_i \hat{\theta}_i^{q} \\
&\quad - \sum_{i=1}^{N} \frac{1}{k_\gamma} \left(c_{fi} - \underline{g} \hat{c}_{fi} \right) \delta_i^{2-q} \varphi_i \tanh\left(\delta_i^{2-q} \varphi_i / \tau_i \right) \\
&\quad - \sum_{i=1}^{N} \frac{1}{k_\gamma} \left(\theta_i - \underline{g} \hat{\theta}_i \right) \delta_i^{2-q} \tanh\left(\delta_i^{2-q} / \tau_i \right). \qquad (7.33)
\end{aligned}
$$

Step 4. Define the Lyapunov function candidate as

$$
V(t) = V_1(t) + V_2(t) + V_3(t) \qquad (7.34)
$$

where $V_1(t)$, $V_2(t)$, and $V_3(t)$ are given in (7.15), (7.21), and (7.32), respectively. By combining (7.20), (7.29), and (7.33), we then arrive at

$$
\begin{aligned}
\dot{V}(t) \leq &- k_3 \sum_{i=1}^{N} \delta_i^{1+q} - k_4 \sum_{i=1}^{N} e_i^{1+q} + \sum_{i=1}^{N} \frac{\sigma_{1i}}{k_\gamma} \tilde{c}_{fi} \hat{c}_{fi}^{q} \\
&+ \sum_{i=1}^{N} \frac{\sigma_{2i}}{k_\gamma} \tilde{\theta}_i \hat{\theta}_i^{q} + \sum_{i=1}^{N} \frac{0.2785 \tau_i (c_{fi} + \theta_i)}{k_\gamma} \qquad (7.35)
\end{aligned}
$$

where

$$
\begin{aligned}
k_3 &= -\frac{2^{1-q} q}{1+q} + \frac{c_1 \underline{g}}{2^{1-q} c_2^{1+1/q}} - \frac{2-q}{c_2} (\bar{a} + \bar{b} \bar{N}) \left(2^{1-q} + \frac{c_2}{1+q} \right), \\
k_4 &= c_2 - \frac{2^{1-q}}{1+q} - \frac{2-q}{1+q} (\bar{a} + \bar{b} \bar{N}). \qquad (7.36)
\end{aligned}
$$

Thus, c_1 and c_2 can be chosen as $c_1 > 2^{1-q} c_2^{1+1/q} \underline{g}^{-1} \left[\frac{2^{1-q} q}{1+q} + \frac{2-q}{c_2} (\bar{a} + \bar{b} \bar{N}) \left(2^{1-q} + \frac{c_2}{1+q} \right) \right]$ and $c_2 > \frac{1}{1+q} \left[2^{1-q} + (2-q)(\bar{a} + \bar{b} \bar{N}) \right]$ such that k_3 and $k_4 > 0$.

Upon using Lemma 7.9, it is straightforward that

$$
\begin{aligned}
\tilde{c}_{fi} \hat{c}_{fi}^{q} \leq &\frac{1}{\underline{g}^{q} (1+q)} \Bigg[\left(2^{(q-1)(1-q)} - 2^{q-1} \right) \tilde{c}_{fi}^{1+q} \\
&+ \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q} 2^{-(q-1)^2 (1+q)} \right) c_{fi}^{1+q} \Bigg], \qquad (7.37) \\
\tilde{\theta}_i \hat{\theta}_i^{q} \leq &\frac{1}{\underline{g}^{q} (1+q)} \Bigg[\left(2^{(q-1)(1-q)} - 2^{q-1} \right) \tilde{\theta}_i^{1+q} \\
&+ \left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q} 2^{-(q-1)^2 (1+q)} \right) \theta_i^{1+q} \Bigg]. \qquad (7.38)
\end{aligned}
$$

By substituting (7.37) and (7.38) into (7.35), one gets

$$\dot{V}(t) \leq -k_3 \sum_{i=1}^{N} \delta_i^{1+q} - k_4 \sum_{i=1}^{N} e_i^{1+q} - k_5 \sum_{i=1}^{N} \tilde{c}_{fi}^{1+q} - k_6 \sum_{i=1}^{N} \tilde{\theta}_i^{1+q} + d \tag{7.39}$$

where

$$k_5 = \frac{\underline{\sigma}_1 \left(2^{(q-1)(1-q)} - 2^{q-1}\right)}{2^{1-q} c_2^{1+1/q} \underline{g}^q (1+q)},$$

$$k_6 = \frac{\underline{\sigma}_2 \left(2^{(q-1)(1-q)} - 2^{q-1}\right)}{2^{1-q} c_2^{1+1/q} \underline{g}^q (1+q)},$$

$$d = \sum_{i=1}^{N} \Bigg[\left(1 - 2^{q-1} + \frac{q}{1+q} + \frac{1}{1+q} 2^{-(q-1)^2(1+q)}\right) \cdot \frac{(\sigma_{1i} c_{fi}^{1+q} + \sigma_{2i} \theta_i^{1+q})}{2^{1-q} c_2^{1+1/q} \underline{g}^q (1+q)} + \frac{0.2785 \tau_i \left(c_{fi} + \theta_i\right)}{2^{1-q} c_2^{1+1/q}} \Bigg] < \infty, \tag{7.40}$$

with $\underline{\sigma}_1 = \min\{\sigma_{11}, \ldots, \sigma_{1N}\}$ and $\underline{\sigma}_2 = \min\{\sigma_{21}, \ldots, \sigma_{2N}\}$. Note that $2^{(q-1)(1-q)} - 2^{q-1} = 2^{q-1}\left(2^{1-q} - 1\right) > 0$, and thus, $k_5 > 0$ and $k_6 > 0$. We further represent (7.39) as

$$\dot{V}(t) \leq -k_d \sum_{i=1}^{N} \left(\delta_i^{1+q} + e_i^{1+q} + \tilde{c}_{fi}^{1+q} + \tilde{\theta}_i^{1+q}\right) + d \tag{7.41}$$

where

$$k_d = \min\{k_3, k_4, k_5, k_6\}. \tag{7.42}$$

Step 5. We prove that there exists a finite time $T^* > 0$ and a bounded constant $0 < \zeta < \infty$ such that $V(t) < \zeta$ when $t \geq T^*$ in the sequel.

According to Lemma 7.8, we get

$$V_1(t) = \frac{1}{2} X^T \mathscr{L} X \leq \frac{1}{2\lambda_2(\mathscr{L})} E^T E = \frac{1}{2\lambda_2(\mathscr{L})} \sum_{i=1}^{N} e_i^2. \tag{7.43}$$

Upon using Lemma 7.3, we have

$$
\begin{aligned}
V_2(t) &= \frac{1}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{2-q} d\varsigma \\
&\leq \frac{1}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} \left| \left(v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{2-q} \right| \left| v_i - v_i^* \right| \\
&\leq \frac{1}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} |\delta_i|^{2-q} \cdot 2^{1-q} |\delta_i|^q = \frac{1}{c_2^{1+1/q}} \sum_{i=1}^{N} \delta_i^2. \qquad (7.44)
\end{aligned}
$$

From the definition of $V_3(t)$ given in (7.32), it is not hard to see that

$$
V_3(t) \leq \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g}\gamma_{1\mathrm{m}}} \sum_{i=1}^{N} \tilde{c}_{fi}^2 + \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g}\gamma_{2\mathrm{m}}} \sum_{i=1}^{N} \tilde{\theta}_i^2, \qquad (7.45)
$$

with $\gamma_{1\mathrm{m}} = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$ and $\gamma_{2\mathrm{m}} = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. Let

$$
k_v = \max\left\{ \frac{1}{2\lambda_2(\mathscr{L})}, \frac{1}{c_2^{1+1/q}}, \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g}\gamma_{1\mathrm{m}}}, \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g}\gamma_{2\mathrm{m}}} \right\}, \qquad (7.46)
$$

we then have from (7.43)–(7.44) that

$$
V(t) \leq k_v \sum_{i=1}^{N} \left(\delta_i^2 + e_i^2 + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2 \right). \qquad (7.47)
$$

Upon using Lemma 7.5, we then arrive at

$$
V(t)^{\frac{1+q}{2}} \leq k_v^{\frac{1+q}{2}} \sum_{i=1}^{N} \left(e_i^{1+q} + \delta_i^{1+q} + \tilde{c}_{fi}^{1+q} + \tilde{\theta}_i^{1+q} \right). \qquad (7.48)
$$

Let $\tilde{c} = \frac{\eta_1 k_d}{k_v^{\frac{1+q}{2}}}$ (k_d and k_v are given in (7.42) and (7.46), respectively, and $0 < \eta_1 \leq 1$). It thus follows from (7.41) and (7.48) that

$$
\dot{V}(t) \leq -\tilde{c}V(t)^{\frac{1+q}{2}} + d. \qquad (7.49)
$$

Let $\Theta = \left\{ (x_i, v_i) : V(t) < \left(\frac{d}{\eta_2 \tilde{c}} \right)^{\frac{2}{1+q}}, 0 < \eta_2 < 1 \right\}$. According to Theorem 5.2 in [7], for any $(x_i, v_i) \notin \Theta$ for all $t \in [0, t_x]$, it holds that $V(t) \geq (d/\eta_2\tilde{c})^{\frac{2}{1+q}}$, i.e., $d \leq \eta_2 \tilde{c} V(t)^{\frac{1+q}{2}}$, for all $t \in [0, t_x]$. This fact, together with (7.49), implies that

$$
\dot{V}(t) \leq -\tilde{c}V(t)^{\frac{1+q}{2}} + \eta_2\tilde{c}V(t)^{\frac{1+q}{2}} = -(1-\eta_2)\tilde{c}V(t)^{\frac{1+q}{2}} \qquad (7.50)
$$

for all $t \in [0, t_x]$. Note that $V(t) \geq (d/\eta_2\tilde{c})^{\frac{2}{1+q}} > 0$ for $t \in [0, t_x]$, it thus follows from (7.50) that $t_x < \frac{V(t_0)^{1-\frac{1+q}{2}}}{(1-\eta_2)\tilde{c}\left(1-\frac{1+q}{2}\right)}$ according to Lemma 7.1. Therefore for $\forall t \geq T^*$, with T^* satisfying

$$T^* = \frac{V(t_0)^{1-\frac{1+q}{2}}}{(1-\eta_2)\tilde{c}\left(1-\frac{1+q}{2}\right)}, \tag{7.51}$$

we have

$$V(t) < \left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{2}{1+q}} = \zeta. \tag{7.52}$$

Step 6. Derive the estimation for steady-state errors of all agents.
Note that for all $i \in \{1, \ldots, N\}$, we have

$$\begin{aligned} |e_i| =& \sqrt{e_i^2} \leq \sqrt{\sum_{i=1}^{N} e_i^2} \leq \sqrt{\frac{2V_1(t)}{\lambda_{\min}(\Lambda)}} \\ \leq& \sqrt{\frac{2V(t)}{\lambda_{\min}(\Lambda)}} \leq \sqrt{\frac{2}{\lambda_{\min}(\Lambda)}}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}}. \end{aligned} \tag{7.53}$$

According to Lemma 7.2, we have $|\varsigma^{1/q} - (v_i^*)^{1/q}| \geq 2^{1-1/q}|\varsigma - v_i^*|^{1/q}$, and thus, if $v_i \geq v_i^*$, it holds that

$$\begin{aligned} V_2(t) \geq& \frac{1}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} \left[2^{1-1/q}\left(\varsigma - v_i^*\right)^{1/q}\right]^{2-q} d\varsigma \\ =& \frac{1}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} 2^{(1-1/q)(2-q)} \left(\varsigma - v_i^*\right)^{\frac{2}{q}-1} d\varsigma \\ =& \frac{1}{\frac{2}{q}-1+1} \frac{2^{(1-1/q)(2-q)}}{2^{1-q}c_2^{1+1/q}} \sum_{i=1}^{N} \left(\varsigma - v_i^*\right)^{\frac{2}{q}-1+1} \Big|_{v_i^*}^{v_i} \\ =& \frac{q2^{1-\frac{2}{q}}}{c_2^{1+1/q}} \sum_{i=1}^{N} \left(v_i - v_i^*\right)^{\frac{2}{q}}. \end{aligned} \tag{7.54}$$

If $v_i < v_i^*$, the proof of (7.54) is similar. Then we have

$$
\begin{aligned}
|v_i - v_i^*| &= \left[\left(v_i - v_i^*\right)^{\frac{2}{q}}\right]^{\frac{q}{2}} \le \left[\sum_{i=1}^{N}(v_i - v_i^*)^{\frac{2}{q}}\right]^{\frac{q}{2}} \\
&\le \left[\frac{c_2^{1+1/q}}{q2^{1-2/q}}V_2(t)\right]^{\frac{q}{2}} = \frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}}2^{\frac{q}{2}-1}}V_2(t)^{\frac{q}{2}} \le \frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}}2^{\frac{q}{2}-1}}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{q}{1+q}}.
\end{aligned}
\tag{7.55}
$$

On the other hand, from (7.53), we see that

$$
|v_i^*| = |-c_2e_i^q| \le c_2\left(\frac{2}{\lambda_{\min}(\Lambda)}\right)^{\frac{q}{2}}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{q}{1+q}}.
\tag{7.56}
$$

It thus follows from (7.55) and (7.56) that

$$
|v_i| \le |v_i - v_i^*| + |v_i^*| \le \left[\frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}}2^{\frac{q}{2}-1}} + c_2\left(\frac{2}{\lambda_{\min}(\Lambda)}\right)^{\frac{q}{2}}\right]\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{q}{1+q}}
\tag{7.57}
$$

which then implies, for $\forall i,\ j \in \{1,\ldots,N\}$, that

$$
|v_i - v_j| \le |v_i| + |v_j| \le 2\left[\frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}}2^{\frac{q}{2}-1}} + c_2\left(\frac{2}{\lambda_{\min}(\Lambda)}\right)^{\frac{q}{2}}\right]\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{q}{1+q}}.
\tag{7.58}
$$

In addition, for all $\forall i \in \{1,\ldots,N\}$,

$$
\begin{aligned}
|\tilde{c}_{fi}| &= \sqrt{\tilde{c}_{fi}^2} \le \sqrt{\sum_{i=1}^{N}\tilde{c}_{fi}^2} \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1 V_3(t)} \\
&\le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1 V(t)} \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}}.
\end{aligned}
\tag{7.59}
$$

and

$$
|\tilde{\theta}_i| \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_2}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{1+q}},
\tag{7.60}
$$

with $\bar{\gamma}_1 = \max\{\gamma_{11},\ldots,\gamma_{1N}\}$, and $\bar{\gamma}_2 = \max\{\gamma_{21},\ldots,\gamma_{2N}\}$.

From the above analysis, we conclude that under the proposed finite-time control scheme (7.8)–(7.11), the position and velocity errors between neighbor agents will converge to a small region Ω_1, defined by

$$\Omega_1 = \Bigg\{ |e_i| \le \sqrt{\frac{2k_v}{\lambda_{\min}(\Lambda)}} \left(\frac{d}{\eta_2\eta_1 k_d}\right)^{\frac{1}{1+q}}, \quad |v_i - v_j| \le 2\Bigg[\frac{c_2^{\frac{1}{2}(q+1)}}{q^{\frac{q}{2}}2^{\frac{q}{2}-1}}$$
$$+ c_2 \left(\frac{2}{\lambda_{\min}(\Lambda)}\right)^{\frac{q}{2}}\Bigg] \left(\frac{dk_v^{\frac{1+q}{2}}}{\eta_2\eta_1 k_d}\right)^{\frac{q}{1+q}}, \ \forall i, j \in \{1, \ldots, N\}\Bigg\}, \tag{7.61}$$

and the generalized parameter estimation converges to the region Ω_{1p} given as

$$\Omega_{1p} = \Bigg\{ |\tilde{c}_{fi}| \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1} \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1}{1+q}},$$
$$|\tilde{\theta}_i| \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_2} \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1}{1+q}}, \ \forall i \in \{1, \ldots, N\}\Bigg\} \tag{7.62}$$

in the finite time T^*. □

7.4 Numerical Simulations

To verify the effectiveness of the proposed finite-time control algorithms, we perform the numerical simulation on a group of four networked autonomous surface vessels (ASVs) [8] with nonlinear dynamics.

The dynamics of the kth ($k = 1, 2, 3, 4$) ASV is modeled by

$$M_k \dot{v}_k = f_k(r_k, v_k) + u_k + D_k \tag{7.63}$$

where $M_k = \mathrm{diag}\{m_{x,k}, m_{y,k}, m_{z,k}\}$ denotes the mass matrix; $r_k = [x_k, y_k, z_k]^T$, $v_k = [v_{x,k}, v_{y,k}, v_{z,k}]^T$, and $u_k = [u_{x,k}, u_{y,k}, u_{z,k}]^T$ denote the position, velocity, and control input vector, respectively; D_k is environment disturbance, and $f_k(r_k, v_k)$ represents coriolis, centripetal, and hydrodynamic damping forces and torques acting on the body, where

$$f_k(r_k, v_k) = \begin{bmatrix} A_{x,k} + A_{|x,k|}|v_{x,k}| & -m_{y,k}v_{z,k} & 0 \\ m_{x,k}v_{z,k} & B_{y,k} + B_{|y,k|}|v_{y,k}| & 0 \\ 0 & 0 & C_{z,k} + C_{|z,k|}|v_{z,k}| \end{bmatrix} \cdot \begin{bmatrix} v_{x,k} \\ v_{y,k} \\ v_{z,k} \end{bmatrix} \tag{7.64}$$

In the simulation, the physical parameters are taken as: $M_k = \mathrm{diag}\{600 + 6(-1)^k + 6\Delta m(t), 1000 + 10(-1)^k + 10\Delta m(t), 800 + 8(-1)^k + 8\Delta m(t)\}$ ($\Delta m(t) = \sin(\pi t/50 - \pi)$), $A_{x,k} = -1 + 0.1(-1)^k$, $A_{|x,k|} = -25 + 2.5(-1)^k$, $B_{y,k} = -10 + (-1)^k$, $B_{|y,k|} = -200 + 20(-1)^k$, $C_{z,k} = -0.5 + 0.05(-1)^k$, and $C_{|z,k|} = -1500 + 150(-1)^k$ for $k = 1, 2, 3, 4$. The external disturbance is taken as $D_k = [3 +$

$3(-1)^k \sin(t/50) + 2\sin(t/10), -1 + 3(-1)^k \sin(t/20 - \pi/6) + 2\sin(t), -5(-1)^k \sin(0.1t) - \sin(t + \pi/3)]^T$. The networked communication topology is undirected and connected as shown in Fig. 7.1. Each edge weight is taken as 0.1.

The simulation objective is that the four ASVs are required to achieve consensus in a finite time by using the proposed control law given in (7.8)–(7.11) under the undirected and connected topology. The initial conditions of the vessels are $r_1(0) = (-1\text{m}, 0\text{m}, \pi/3\text{rad})$, $r_2(0) = (-0.5\text{m}, 1.5\text{m}, -\pi/4\text{rad})$, $r_3(0) = (1\text{m}, 1.5\text{m}, -\pi/9\text{rad})$, $r_4(0) = (1.5\text{m}, 0.5\text{m}, \pi/4\text{rad})$, $v_k(0) = (0, 0, 0)$ $(k = 1, 2, 3, 4)$, respectively. The control parameters are chosen as: $s = 4$, $c_1 = 2000$, and $c_2 = 2$. In addition, the initial values of the estimates are chosen as $\hat{\theta}_{x,k} = \hat{\theta}_{y,k} = \hat{\theta}_{z,k} = 0$ and $\hat{\zeta}_{x,k} = \hat{\zeta}_{y,k} = \hat{\zeta}_{z,k} = 0$ for $k = 1, 2, 3, 4$. The simulation runs for $15s$. The trajectories of the four ASVs are represented in Fig. 7.2, which shows the trajectory of each ASV from the initial position to the finial position. The position error convergence results in x-direction, y-direction, and z-direction under the finite-time control scheme proposed in (7.8)–(7.11) are represented in Fig. 7.3. In addition, the

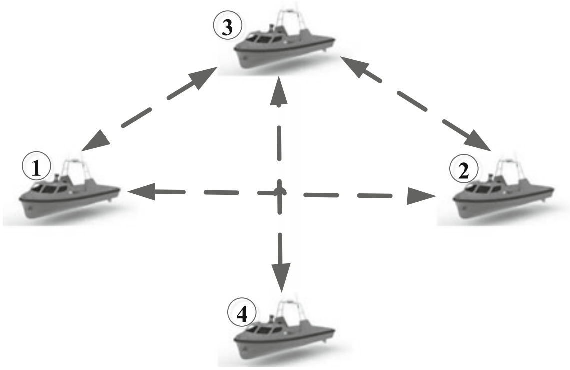


Fig. 7.1 Undirected connected topology among the four ASVs

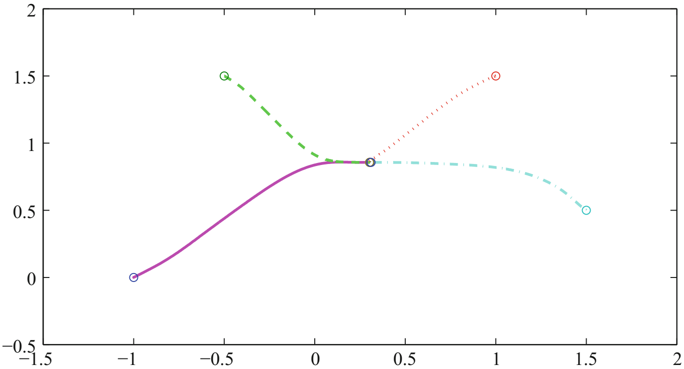
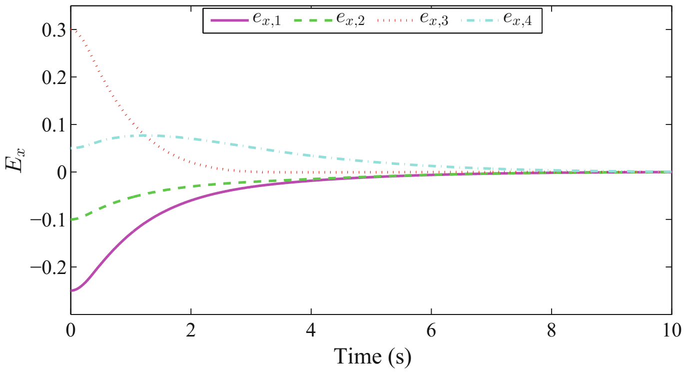
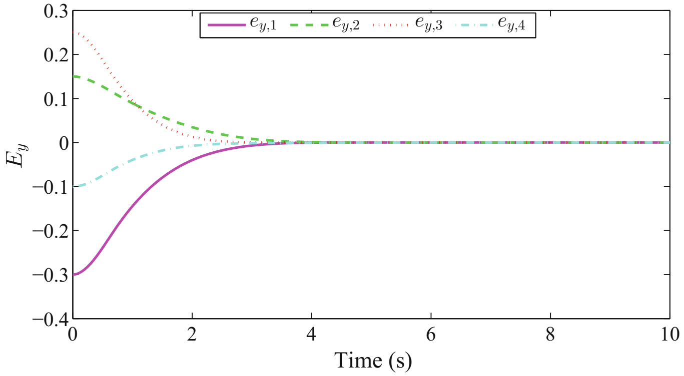


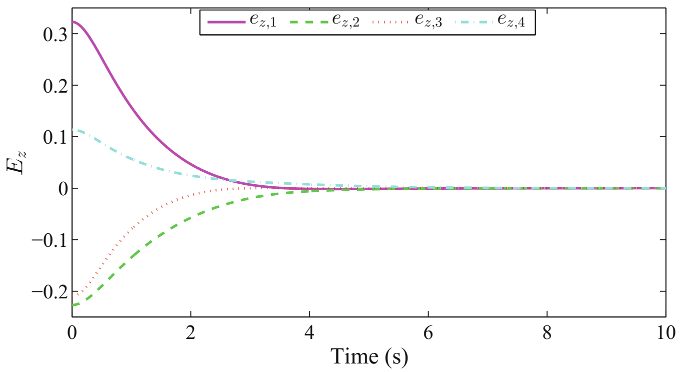
Fig. 7.2 Trajectory of each ASV from the initial position to the final position



(a) E in x-direction.

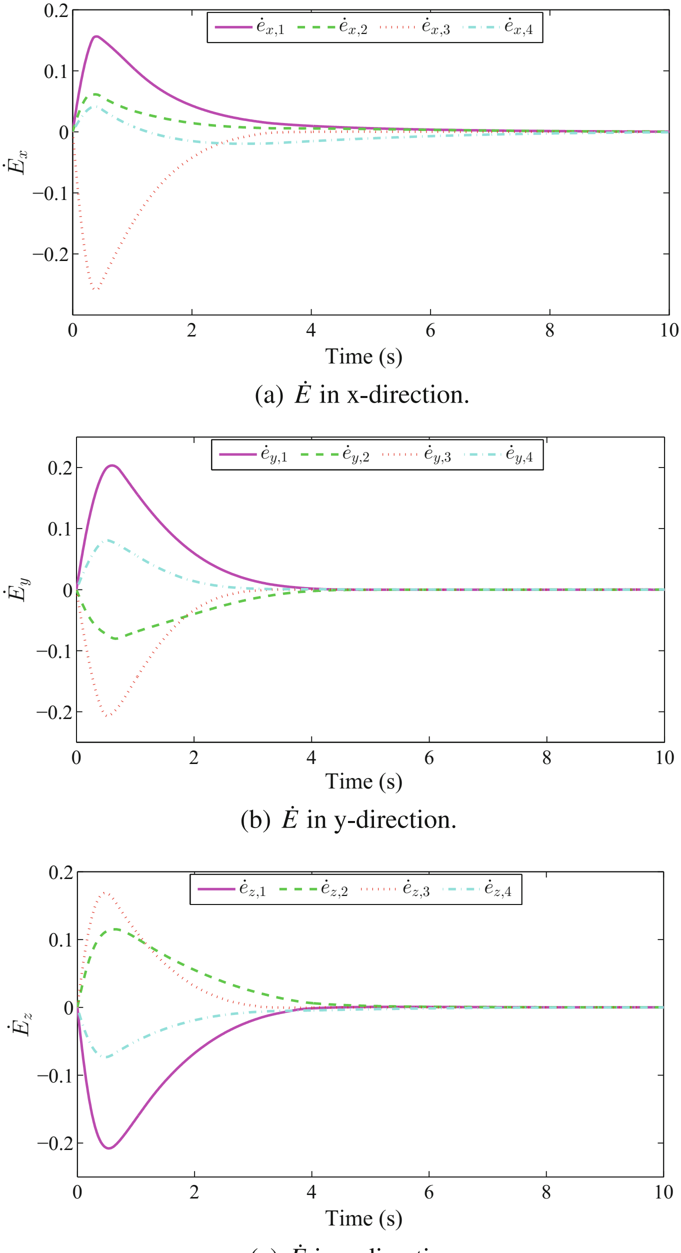


(b) E in y-direction.



(c) E in z-direction.

Fig. 7.3 Position error convergence of the four ASVs under the proposed finite-time control scheme

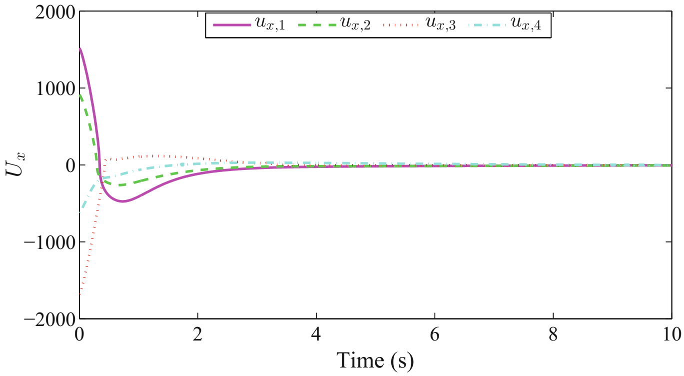


(a) $\dot{E}$ in x-direction.

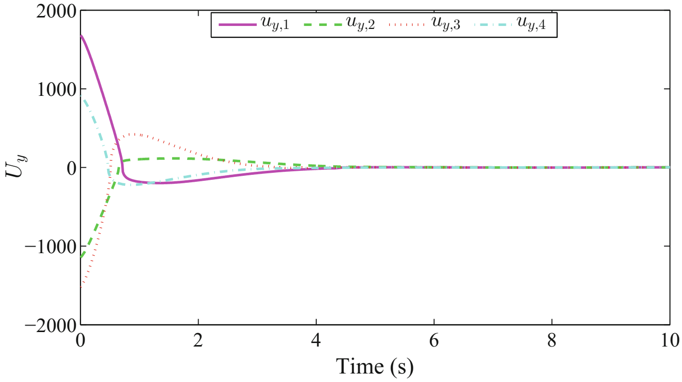
(b) $\dot{E}$ in y-direction.

(c) $\dot{E}$ in z-direction.

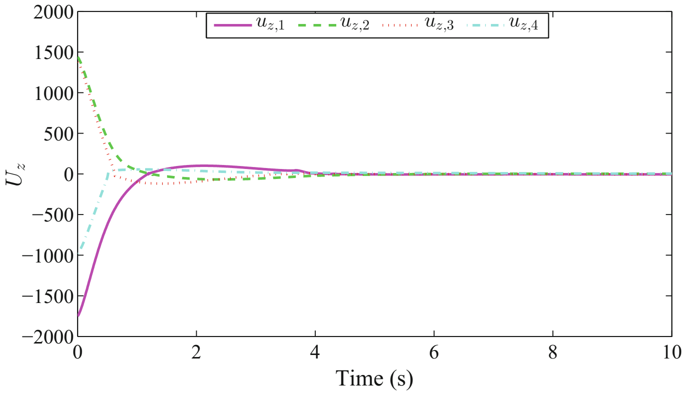
Fig. 7.4 Velocity error convergence of the four ASVs under the proposed finite-time control scheme



(a) U in x-direction.

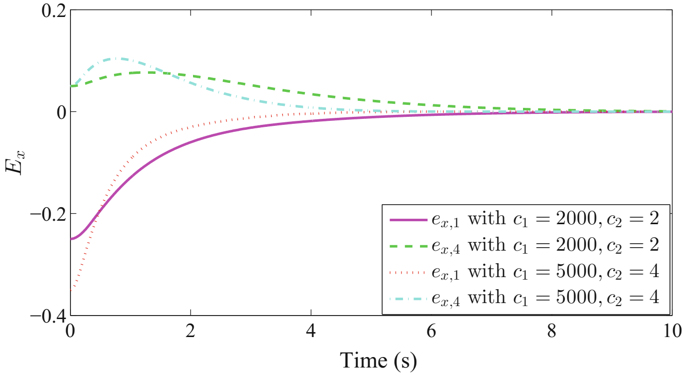


(b) U in y-direction.

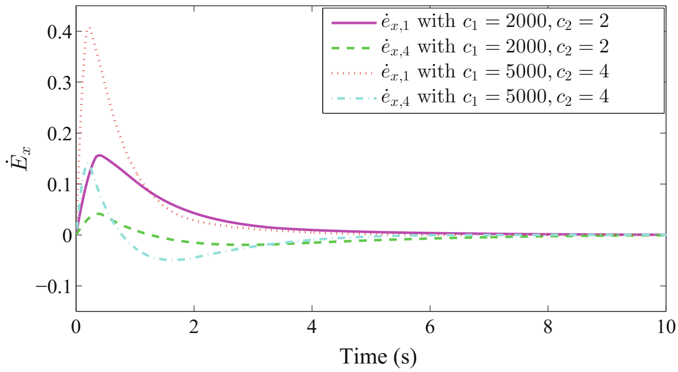


(c) U in z-direction.

Fig. 7.5 Control inputs of the four ASVs under the proposed finite-time control scheme



(a) E x-direction .



(b) $\dot{E}$ in x-direction.

Fig. 7.6 Error convergence comparison with x-direction under the finite-time control scheme with different control parameters

velocity error convergence results in x-direction, y-direction, and z-direction under the proposed finite-time control scheme are given in Fig. 7.4. It is observed from both Figs. 7.3 and 7.4 that the proposed finite-time controller (7.8)–(7.11) leads to fast convergence rate and high control accuracy. Figure 7.5 shows the control input signals of the four ASVs in x-direction, y-direction and z-direction, respectively, under the proposed finite-time control scheme, from which we see that the control input signals of all the four ASVs in the three directions are continuous and bounded.

To see the effects of the control parameters on the error convergence time, we consider two group of control parameters: $s = 4$, $c_1 = 2000$ and $c_2 = 2$; $s = 4$, $c_1 = 5000$ and $c_2 = 4$, respectively. The position and velocity error convergence comparison results are depicted in Fig. 7.6. It is observed from the comparison results that the finite time T^* specified in (7.13) can be adjusted smaller by choosing larger controller parameters c_1 and c_2, but it will result in larger initial control effort.

Remark 7.1 It should be mentioned that the control parameter c_1 is set relatively large as compared with c_2 in the simulation. This is because $\max\{m_{x,k}, m_{y,k}, m_{z,k}\} = 1010$, leading to the equivalent control gain $1/1010$, thus in order to have enough power to steer the system, the designed control parameter c_1 has to be chosen large enough to counteract the small control gain $1/1010$.

7.5 Notes

The majority of the research on leaderless consensus control of nonlinear MAS has been focused on the systems with nonlinearities that can be linearly parameterized, and the convergence time is not finite. The work in this chapter explicitly addressed the finite-time leaderless consensus control of nonlinear MAS with second-order dynamics in the presence of unknown time-varying gain and non-parametric uncertainties. It is shown that under the proposed control scheme, not only the finite-time convergence is realized but also the unknown time-varying gain and the non-parametric uncertainties are compensated successfully.

Not considered in this chapter is the local directed communication topology condition, which obviously further complicates the underlying problem significantly. For agents with second-order dynamics in the presence of non-parametric uncertainties under local and one-way directed communication topology, the finite-time leaderless consensus control is still a challenging problem, thus worthy of further studying. We give a solution to this problem in the next chapter.

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Chapter 8
Finite-Time Consensus for Systems with Second-Order Uncertain Dynamics Under Directed Topology

In Chap. 7, we have explicitly addressed the finite-time leaderless consensus control problem of nonlinear MAS with second-order dynamics in the presence of unknown time-varying gain and non-parametric uncertainties under the local and undirected communication condition. By considering that in most practical applications, the networked communication among the subsystems is not only local but also not bidirectional, and it is highly desirable to consider the local and one-way directed communication condition. For agents with second-order dynamics in the presence of non-parametric uncertainties under local and one-way directed communication topology, the finite-time leaderless consensus control problem is interesting and also challenging, worthy of further studying. Compared with the control results derived in Chap. 7, the finite-time leaderless consensus of second-order nonlinear MAS with non-parametric uncertainties under one-way directed communication topology is much more challenging, where the one-way directed communication topology obviously complicates the underlying problem significantly. This is mainly because the symmetric property does not hold for the directed Laplacian, which imposes a significant technique challenge to extend the existing finite-time adaptive consensus control methods derived from undirected graph to directed graph. In this chapter, we attempt to provide a solution to this problem under directed graph topology. Such a solution is made possible by deriving some crucial algebraic connectivity properties established associated with the newly defined Laplacian matrix.

Section 8.1 formulates the leaderless consensus control problem for multi-agent systems with second-order dynamics in the presence of unknown time-varying gain and non-parametric uncertainties under directed topology. Section 8.2 represents some useful results that are helpful to tackle the technical difficulties arising from the directed communication topology in the finite-time stability analysis for distributed control. Section 8.3 addresses the leaderless consensus controller design and stability

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Part of Sect. 8.2 has been reproduced from Wang, Y. J., Song, Y. D., Krstic, M., and Wen, C. Y.: Fault-tolerant finite time consensus for multiple uncertain nonlinear mechanical systems under single-way directed communication interactions and actuation failures. Automatica, vol. 63, pp. 374–383, 2016 © 2016 Elsevier Ltd., reprinted with permission.

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analysis for the second-order uncertain multi-agent systems under directed topology, in which the technical difficulties arising from the non-symmetric property of the Laplacian matrix are circumvented in the finite-time stability analysis. Section 8.4 conducts numerical simulation to confirm the effectiveness of the theoretical results obtained in the previous sections. Section 8.5 gives some notes on the finite-time control of the second-order uncertain multi-agent systems.

8.1 Formulation of Finite-Time Consensus for Second-Order Uncertain Systems Under Directed Topology

In this section, we formulate the leaderless consensus control problem for networked multi-agent systems with second-order uncertain dynamics under directed topology, where the agent dynamics are subjected to unknown time-varying gain and unknown non-parametric uncertainties. In the next section, we will introduce the Lyapunov function design for the finite-time stability analysis under directed topology, and then we start with controller design and stability analysis to explain the fundamental idea and the technical development of the proposed method.

8.1.1 Control Problem Formulation Under Directed Topology

The focus of this section is to address the finite-time leaderless consensus problem of second-order dynamic multi-agent systems with unknown time-varying gain and non-parametric uncertainties under the directed topology.

We consider the system described by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= g_i(z_i, t)u_i(t) + f_i(z_i, t) + f_{di}(z_i, t), \end{aligned} \tag{8.1}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position state, velocity state, and control input of the ith ($i = 1, 2, \ldots, N$) agent, respectively, g_i denotes the control gain, possibly time-varying and unavailable for controller design, f_i denotes the system nonlinearities, and f_{di} denotes the non-vanishing uncertainties. In addition, $z_i = \bigcup_{j \in \mathcal{N}_i \bigcup i} \{x_j, v_j\}$.

The following standard assumptions are in order.

Assumption 8.1 a. Standard assumptions for the existence of unique solutions and global controllability of system are made; e.g., f_i and f_{di} are either continuously differentiable or Lipschitz, and g_i (unknown and time-varying) is sign-definite (w. l. o. g. here $\mathrm{sgn}(g_i) = +1$) and $0 < \underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i < \infty$ with $\underline{g}_i$, $\bar{g}_i$ being unknown finite constants, which ensures the global controllability of the system.

b. It is assumed that $f_i(z_i, t)$ is unknown and non-parameterized. Thus, they are not available in the finite-time control design in this section. Moreover, certain crude structural information is available to allow that $|f_i(z_i, t)| \leq c_{fi}\varphi_i(z_i)$ for all $t \geq t_0$, where $c_{fi} \geq 0$ is an unknown finite constant and $\varphi_i(z_i)$ is a known scalar function, and $\varphi_i(z_i)$ is bounded if z_i is bounded.
c. It is assumed that $f_{di}(z_i, t)$ is unknown and non-vanishing, and $f_{di}(z_i, t)$ is bounded by $|f_{di}(z_i, t)| \leq \theta_i < \infty$ with θ_i being some unknown finite constant.

In addition, it is assumed for ease of notation that the agents' states are scalars, i.e., $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$. If the states are vectors, i.e., $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$, the results of this chapter can be extended by using the standard method involving the Kronecker products.

The local neighborhood state error for node i is

$$e_i = \sum_{j \in \mathscr{N}_i} a_{ij}(x_i - x_j) \tag{8.2}$$

Let $E = [e_1, \ldots, e_N]^T$, $X = [x_1, \ldots, x_N]^T$ such that $E = \mathscr{L}X$.

The distributed finite-time leaderless consensus control design problem confronted herein is as follows: (1) design control schemes for all agents i in $\mathscr{G}$ under the directed communication topology condition to make an agreement in finite time. The control schemes must be distributed in the sense that they can only depend on local information about the agent itself and its neighbors in graph, and meanwhile, the control schemes should be endowed with the finite-time quality; (2) overcome the technical difficulty in constructing the Lyapunov function arising from asymmetric property of the original Laplacian matrix under the directed communication topology condition; and (3) tackle unknown and time-varying control gains, and accommodate unknown non-parametric uncertainties and unknown non-vanishing nonlinearities by using the control method based on fractional-order adaptive laws.

8.2 Preliminaries

8.2.1 Useful Lemmas Related to Finite-Time Control

The following lemmas are useful for establishing the finite-time stability results of the MAS.

Lemma 8.1 ([1]) *Suppose there exists a continuously differentiable function $V(x, t) : \mathbb{U} \times \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{U} \subset \mathbb{R}^N$ is an open neighborhood of the origin), a real number $c > 0$ and $0 < \alpha < 1$, such that $V(x, t)$ is positive definite and $\dot{V}(x, t) + cV(x, t)^\alpha \leq 0$ on $\mathbb{U}_0$ ($\mathbb{U}_0 \subset \mathbb{U}$), then $V(x, t)$ is locally in finite-time convergent with a finite settling time satisfying $T^* \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$, such that for any given initial state $x(t_0) \in \mathbb{U}_0/\{0\}$, $\lim_{t \to T^*} V(x, t) = 0$ and $V(x, t) = 0$ for $t \geq T^*$.*

Lemma 8.2 ([2]) *If $h = h_2/h_1 \geq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x - y|^h \leq 2^{h-1}|x^h - y^h|$.*

Lemma 8.3 ([2]) *If $0 < h = h_1/h_2 \leq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x^h - y^h| \leq 2^{1-h}|x - y|^h$.*

Lemma 8.4 ([2]) *For $x, y \in \mathbb{R}$, if $c, d > 0$, then $|x|^c|y|^d \leq c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.*

Lemma 8.5 ([3]) *For $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, $0 < h \leq 1$, then $(\sum_{i=1}^{N} |x_i|)^h \leq \sum_{i=1}^{N} |x_i|^h \leq N^{1-h}(\sum_{i=1}^{N} |x_i|)^h$.*

Lemma 8.6 ([4]) *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $0 < h = h_1/h_2 \leq 1$, where $h_1, h_2 > 0$ are positive odd integers, then $x^h(y - x) \leq \frac{1}{1+h}(y^{1+h} - x^{1+h})$.*

Lemma 8.7 *For the generalized weight parameter estimation error $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ and the constant $q = \frac{4h-1}{4h+1}$ ($h \in \mathbb{Z}_+$), it holds that*

$$\begin{aligned} \tilde{\bullet}\hat{\bullet}^{\frac{3q-1}{1+q}} \leq & \frac{1+q}{4q}\underline{g}^{\frac{1-3q}{1+q}}\Bigg[\left(2^{\frac{8q(q-1)}{(1+q)^2}} - 2^{\frac{2(q-1)}{1+q}}\right)\tilde{\bullet}^{\frac{4q}{1+q}} \\ & + \left(1 - 2^{\frac{2q-2}{1+q}} + \frac{3q-1}{4q} + \frac{(1+q)2^{\frac{-16q(q-1)^2}{(1+q)^3}}}{4q}\right)\bullet^{\frac{4q}{1+q}}\Bigg]. \end{aligned} \tag{8.3}$$

Proof Firstly, by noting that $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$, we get

$$\tilde{\bullet}\hat{\bullet}^{\frac{3q-1}{1+q}} = \tilde{\bullet}\left[\underline{g}^{-1}(\bullet - \tilde{\bullet})\right]^{\frac{3q-1}{1+q}} = \underline{g}^{\frac{1-3q}{1+q}}\tilde{\bullet}(\bullet - \tilde{\bullet})^{\frac{3q-1}{1+q}}. \tag{8.4}$$

According to Lemma 8.6, we let $x = \bullet - \tilde{\bullet}$, $y = \bullet$, and then $y - x = \tilde{\bullet}$, and we then have

$$\tilde{\bullet}(\bullet - \tilde{\bullet})^{\frac{3q-1}{1+q}} \leq \frac{1+q}{4q}\left[\bullet^{\frac{4q}{1+q}} - (\bullet - \tilde{\bullet})^{\frac{4q}{1+q}}\right]. \tag{8.5}$$

Note that $\frac{4q}{1+q} = 1 + \frac{3q-1}{1+q}$ and $\frac{3q-1}{1+q} = \frac{3\times\frac{4h-1}{4h+1}-1}{1+\frac{4h-1}{4h+1}} = \frac{2h-1}{2h}$ ($h \in \mathbb{Z}_+$), we thus see that both the function $g(s) = s$ ($s \in \mathbb{R}$) and function $f(s) = s^{\frac{3q-1}{1+q}}$ ($s \in \mathbb{R}$) are non-decreasing, and then we have

$$(\bullet - \tilde{\bullet})^{\frac{4q}{1+q}} = (\bullet - \tilde{\bullet})(\bullet - \tilde{\bullet})^{\frac{3q-1}{1+q}} = |\bullet - \tilde{\bullet}|\,|\bullet - \tilde{\bullet}|^{\frac{3q-1}{1+q}}, \tag{8.6}$$

from which, (8.5) becomes

$$\tilde{\bullet}(\bullet - \tilde{\bullet})^{\frac{3q-1}{1+q}} \leq \frac{1+q}{4q}\left[\bullet^{\frac{4q}{1+q}} - |\bullet - \tilde{\bullet}|\,|\bullet - \tilde{\bullet}|^{\frac{3q-1}{1+q}}\right]. \tag{8.7}$$

Upon using Lemma 8.3, we arrive at

$$\tilde{\bullet}\,(\bullet-\tilde{\bullet})^{\frac{3q-1}{1+q}} \le \frac{1+q}{4q}\left[\bullet^{\frac{4q}{1+q}} - 2^{\frac{2q-2}{1+q}}\,|\bullet-\tilde{\bullet}|\left|\bullet^{\frac{3q-1}{1+q}}-\tilde{\bullet}^{\frac{3q-1}{1+q}}\right|\right]. \tag{8.8}$$

Similarly, by noting that the function $g(s)=s$ $(s\in\mathbb{R})$ and function $f(s)=s^{\frac{3q-1}{1+q}}$ $(s\in\mathbb{R})$ are non-decreasing, we then have $|\bullet-\tilde{\bullet}|\left|\bullet^{\frac{3q-1}{1+q}}-\tilde{\bullet}^{\frac{3q-1}{1+q}}\right|$ $=(\bullet-\tilde{\bullet})\left(\bullet^{\frac{3q-1}{1+q}}-\tilde{\bullet}^{\frac{3q-1}{1+q}}\right)$, and (8.8) becomes

$$\begin{aligned}\tilde{\bullet}\,(\bullet-\tilde{\bullet})^{\frac{3q-1}{1+q}} \le &\frac{1+q}{4q}\Big[\bullet^{\frac{4q}{1+q}} - 2^{\frac{2q-2}{1+q}}\bullet^{\frac{4q}{1+q}} + 2^{\frac{2q-2}{1+q}}\bullet^{\frac{3q-1}{1+q}}\tilde{\bullet}\\ &+2^{\frac{2q-2}{1+q}}\bullet\tilde{\bullet}^{\frac{3q-1}{1+q}} - 2^{\frac{2q-2}{1+q}}\tilde{\bullet}^{\frac{4q}{1+q}}\Big].\end{aligned} \tag{8.9}$$

Upon using Lemma 8.4, we get that

$$2^{\frac{2q-2}{1+q}}\tilde{\bullet}\bullet^{\frac{3q-1}{1+q}} = \left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)\bullet^{\frac{3q-1}{1+q}} \le \frac{1+q}{4q}\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{4q}{1+q}} + \frac{3q-1}{4q}\bullet^{\frac{4q}{1+q}}, \tag{8.10}$$

and

$$\begin{aligned}2^{\frac{2q-2}{1+q}}\bullet\tilde{\bullet}^{\frac{3q-1}{1+q}} =&2^{\frac{2q-2}{1+q}}\bullet\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{3q-1}{1+q}} + 2^{\frac{2q-2}{1+q}\cdot\frac{1-3q}{1+q}}\\ =&2^{\frac{2q-2}{1+q}\cdot\frac{2-2q}{1+q}}\bullet\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{3q-1}{1+q}}\\ \le&\frac{1+q}{4q}\left(2^{-4\left(\frac{q-1}{1+q}\right)^2}\bullet\right)^{\frac{4q}{1+q}} + \frac{3q-1}{4q}\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{4q}{1+q}}.\end{aligned} \tag{8.11}$$

By inserting (8.10)–(8.11) into (8.9), we then arrive at

$$\begin{aligned}\tilde{\bullet}\,(\bullet-\tilde{\bullet})^{\frac{3q-1}{1+q}} \le&\frac{1+q}{4q}\Big[\bullet^{\frac{4q}{1+q}} - 2^{\frac{2q-2}{1+q}}\bullet^{\frac{4q}{1+q}} + \frac{1+q}{4q}\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{4q}{1+q}} + \frac{3q-1}{4q}\bullet^{\frac{4q}{1+q}}\\ &+\frac{1+q}{4q}\left(2^{-4\left(\frac{q-1}{1+q}\right)^2}\bullet\right)^{\frac{4q}{1+q}} + \frac{3q-1}{4q}\left(2^{\frac{2q-2}{1+q}}\tilde{\bullet}\right)^{\frac{4q}{1+q}} - 2^{\frac{2q-2}{1+q}}\tilde{\bullet}^{\frac{4q}{1+q}}\Big]\\ \le&\frac{1+q}{4q}\Big[\left(2^{\frac{8q(q-1)}{(1+q)^2}} - 2^{\frac{2(q-1)}{1+q}}\right)\tilde{\bullet}^{\frac{4q}{1+q}}\\ &+\left(1-2^{\frac{2q-2}{1+q}} + \frac{3q-1}{4q} + \frac{(1+q)2^{\frac{-16q(q-1)^2}{(1+q)^3}}}{4q}\right)\bullet^{\frac{4q}{1+q}}\Big],\end{aligned} \tag{8.12}$$

which then implies (8.3). □

8.2.2 Lyapunov Function Design Under Directed Topology

In this subsection, we introduce the Lyapunov function design for the finite-time stability analysis under strongly connected directed topology. In the next subsection, we will show how to apply such Lyapunov function in the Lyapunov analysis of the finite-time leaderless consensus problem for multi-agent systems under directed topology.

Before moving on, we first present several useful lemmas. It is important to note that the Laplacian matrix $\mathscr{L}$ is no longer symmetric under the one-way directed topology, which imposes a significant challenge for the finite-time leaderless consensus control design because one cannot use such $\mathscr{L}$ to construct Lyapunov function directly. To circumvent the technical difficulty in constructing Lyapunov function arising from asymmetric property of the original Laplacian matrix $\mathscr{L}$ under directed graph, we introduce a new matrix Q [5],

$$Q = (\mathrm{diag}(p)\mathscr{L} + \mathscr{L}^T \mathrm{diag}(p)), \tag{8.13}$$

where $\mathrm{diag}(p) = \mathrm{diag}\{p_1, \ldots, p_N\} \in \mathbb{R}^{N \times N} > 0$ with $p = [p_1, \ldots, p_N]^T$ being the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue. Such defined matrix Q is actually a Laplacian corresponding to a connected undirected graph [4], which exhibits the property of symmetry and semi-positive definiteness with one simple zero eigenvalue.

The significance of the feature on Q is that the original directed graph problem can be treated as an undirected graph problem. Furthermore, such feature allows for the development of the following two lemmas that are crucial for the construction of the Lyapunov function design subject to directed graph.

Lemma 8.8 ([4]) *For a strongly connected digraph $\mathscr{G}$ with Laplacian $\mathscr{L}$, let Q be defined as in (8.13), then $\forall x \neq 0_N$, $x^T Qx = 0$ if and only if $x = c1_N$, where $c \neq 0$ is a constant. Moreover, $\min\limits_{x \neq c1_N} \frac{x^T Qx}{x^T x}$ exists and*

$$0 < \min_{x \neq c1_N} \frac{x^T Qx}{x^T x} \leq \sum_{i=2}^{N} \lambda_i(Q). \tag{8.14}$$

Upon using Lemma 8.8, the following result can be established.

Lemma 8.9 ([4]) $\forall E^q \neq 0_N$ *($E^q = \left[e_1^q, \ldots, e_N^q\right]^T$, $q > 0$), there exists a finite constant $k_m > 0$ such that*

$$\frac{(E^q)^T QE^q}{(E^q)^T E^q} \geq k_m. \tag{8.15}$$

According to the above analysis, we construct the Lyapunov function related to the directed graph as

$$V = \frac{1}{1+q}\left(E^{\frac{1+q}{2}}\right)^T \mathrm{diag}(p) E^{\frac{1+q}{2}}, \tag{8.16}$$

where $E^{\frac{1+q}{2}} = \left[e_1^{\frac{1+q}{2}}, \ldots, e_N^{\frac{1+q}{2}}\right]^T$. Hereafter, we let $E^h = \left[e_1^h, \ldots, e_N^h\right]^T$ with $h \in \mathbb{R}_+$.

8.3 Distributed Adaptive Finite-Time Control Design and Stability Analysis Under Directed Topology

The control objective in this section is to design a distributed adaptive finite-time controller under the directed topology condition such that the impacts arisen from the asymmetric property of the original Laplacian matrix under directed topology can be removed, meanwhile the unknown time-varying control gain, unmodeling nonlinearities, and non-vanishing uncertainties can be compensated, and finally the leaderless consensus of the networked uncertain multi-agent systems with second-order uncertain dynamics is ensured in finite time.

The control input for each ith ($i = 1, \ldots, N$) agent is designed to consist of two parts: (1) the negative feedback control term u_{0i} and (2) the compensation control term u_{ci}, which is of the following form

$$u_i = u_{0i} + u_{ci}, \tag{8.17}$$

in which the negative feedback control term u_{0i} is generated by

$$u_{0i} = -c_1 \delta_i^{2q-1}, \tag{8.18}$$

and the compensation control term u_{ci} is generated by

$$u_{ci} = -\hat{c}_{fi}\varphi_i \tanh(\delta_i\varphi_i/\tau_i) - \hat{\theta}_i \tanh(\delta_i/\tau_i) \tag{8.19}$$

with the updated laws

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi}^{\frac{3q-1}{1+q}} + \gamma_{1i}\varphi_i \tanh(\delta_i\varphi_i/\tau_i)\delta_i, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i^{\frac{3q-1}{1+q}} + \gamma_{2i}\tanh(\delta_i/\tau_i)\delta_i, \end{aligned} \tag{8.20}$$

where $\delta_i = v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}}$ with $v_i^* = -c_2 e_i^q$ and $q = \frac{4h-1}{4h+1}$ ($h \in \mathbb{Z}_+$), $c_1, c_2 > 0$ are design parameters, $\hat{c}_{fi}$ and $\hat{\theta}_i$ are, respectively, the estimations of c_{fi} and θ_i (the parameters to be estimated), $\varphi_i(\cdot)$ is the scalar and readily computable function as given in Assumption 8.1, and τ_i, γ_{1i}, γ_{2i}, σ_{1i}, and σ_{2i} are positive design parameters chosen by the designer.

Theorem 8.1 (Distributed Adaptive Finite-Time Control for Leaderless Consensus under Directed Topology) *Consider the nonlinear MAS as described by (8.1) under Assumption 8.1. Let the communication topology be directed and strongly connected. If the distributed control laws (8.17)–(8.20) are applied, then finite-time leaderless consensus is achieved in that*

(1) The leaderless consensus positive errors e_i ($i = 1, \ldots, N$) and velocity errors converge to a small residual set Ω_1 defined by

$$\Omega_1 = \left\{ |e_i| \le \left(\frac{1+q}{\underline{p}} k_v\right)^{\frac{1}{1+q}} \left(\frac{d}{\eta_2 \eta_1 k_d}\right)^{\frac{1}{2q}}, \quad |v_i - v_j| \le 2\left[\left(\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q} q}\right)^{\frac{q}{1+q}} \right.\right.$$
$$\left.\left. + c_2 \left(\frac{1+q}{\underline{p}}\right)^{\frac{q}{1+q}} \right] \sqrt{\frac{d k_v^{\frac{2q}{1+q}}}{\eta_2 \eta_1 k_d}}, \ \forall i, j \in \{1, \ldots, N\} \right\}, \tag{8.21}$$

in a finite time T^* satisfying

$$T^* = \frac{V(t_0)^{\frac{1-q}{1+q}} k_v^{\frac{2q}{1+q}} (1+q)}{(1-\eta_2)\eta_1 k_d (1-q)}, \tag{8.22}$$

where $0 < \eta_1 \le 1$, $0 < \eta_2 < 1$, $V(t_0)$ is known, and k_d and k_v are given, respectively, in (8.49) and (8.54), which are explicitly computable.

(2) The generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a small set Ω_{1p} defined by

$$\Omega_{1p} = \left\{ |\tilde{c}_{fi}| \le \sqrt{2^{2-q} c_2^{1+1/q} \underline{g} \bar{\gamma}_1 k_v} \left(\frac{d}{\eta_2 \eta_1 k_d}\right)^{\frac{1+q}{4q}}, \right.$$
$$\left. |\tilde{\theta}_i| \le \sqrt{2^{2-q} c_2^{1+1/q} \underline{g} \bar{\gamma}_2 k_v} \left(\frac{d}{\eta_2 \eta_1 k_d}\right)^{\frac{1+q}{4q}}, \ \forall i \in \{1, \ldots, N\} \right\} \tag{8.23}$$

in the finite time T^*.

(3) All signals in the closed-loop system remain uniformly bounded.

Proof The proof of the result can be done by the following six steps.

Step 1. Construct the distributed part of the Lyapunov function candidate as

$$V_1(t) = \frac{1}{1+q}\left(E^{\frac{1+q}{2}}\right)^T \operatorname{diag}(p) E^{\frac{1+q}{2}}. \tag{8.24}$$

Taking the time derivative of $V_1(t)$ yields that

$$\dot{V}_1(t) = \left(E^q\right)^T \operatorname{diag}(p)\dot{E} = \left(E^q\right)^T (\operatorname{diag}(p)\mathscr{L})v \tag{8.25}$$

where $v = \dot{X}$. Let $v^* = -c_2 E^q$ be the virtual control of v, with c_2 being a finite design constant. Upon using Lemma 8.9, we have

$$\begin{aligned}
\dot{V}_1(t) =& - c_2\left(E^q\right)^T (\operatorname{diag}(p)\mathscr{L})E^q + \left(E^q\right)^T (\operatorname{diag}(p)\mathscr{L})(v - v^*) \\
\leq& - \frac{c_2 k_{\mathrm{m}}}{2}\left(E^q\right)^T E^q + \left(E^q\right)^T (\operatorname{diag}(p)\mathscr{L})(v - v^*) \\
=& - \frac{c_2 k_{\mathrm{m}}}{2}\sum_{i=1}^{N} e_i^{2q} + \sum_{i=1}^{N}\left(v_i - v_i^*\right)\sum_{j=1}^{N}\ell_{ji}e_j^q,
\end{aligned} \tag{8.26}$$

where ℓ_{ji} is the (j, i)th element of $\operatorname{diag}(p)L$. By recalling that $\delta_i = (v_i)^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}}$, we get from Lemmas 8.3 and 8.4 that

$$\begin{aligned}
&\sum_{i=1}^{N}\left(v_i - v_i^*\right)\sum_{j=1}^{N}\ell_{ji}e_j^q \leq \sum_{i=1}^{N} 2^{1-q}|\delta_i|^q \sum_{j=1}^{N}|\ell_{ji}||e_j|^q \\
\leq& 2^{1-q}\ell_{\max}\sum_{i=1}^{N}|\delta_i|^q\sum_{j=1}^{N}|e_i|^q \leq 2^{1-q}\ell_{\max}\frac{1}{2}\left[\left(\sum_{i=1}^{N}|\delta_i|^q\right)^2 + \left(\sum_{j=1}^{N}|e_j|^q\right)^2\right] \\
\leq& 2^{1-q}\times\frac{1}{2}\ell_{\max}\left[N\sum_{i=1}^{N}|\delta_i|^{2q} + N\sum_{i=1}^{N}|e_i|^{2q}\right] = 2^{-q}N\ell_{\max}\sum_{i=1}^{N}\left[\delta_i^{2q} + e_i^{2q}\right],
\end{aligned} \tag{8.27}$$

where $\ell_{\max} = \max_{j,i\in\{1,\ldots,N\}}|\ell_{ji}|$, and the fact that $\left(\sum_{i=1}^{N} x\right)^2 \leq N\sum_{i=1}^{N} x^2$ has been used. By substituting (8.27) into (8.26), it thus follows that

$$\dot{V}_1(t) \leq -\frac{c_2 k_{\mathrm{m}}}{2}\sum_{i=1}^{N} e_i^{2q} + 2^{-q}N\ell_{\max}\sum_{i=1}^{N}\left(\delta_i^{2q} + e_i^{2q}\right). \tag{8.28}$$

Step 2. Define the second part of the Lyapunov function candidate by adding a power integrator

$$V_2(t) = \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right) d\varsigma \tag{8.29}$$

which is positive semi-definite and C^1 [2]. Taking the derivative of $V_2(t)$ yields

$$\begin{aligned}
\dot{V}_2(t) =& \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \Bigg[\left(v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right) \dot{v}_i \\
&+ \int_{v_i^*}^{v_i} \left(\varsigma^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}} \right)^{1-1} d\varsigma \cdot \frac{d(-(v_i^*)^{\frac{1}{q}})}{dt} \Bigg] \\
=& \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \left[\delta_i \dot{v}_i + \left(v_i - v_i^* \right) \cdot c_2^{\frac{1}{q}} \dot{e}_i \right] \\
=& \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \left[\delta_i \dot{v}_i + \left(v_i - v_i^* \right) \cdot c_2^{\frac{1}{q}} \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \right].
\end{aligned} \tag{8.30}$$

Upon using Lemma 8.3, the second term of the right side of (8.30) becomes

$$\begin{aligned}
& \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \left(v_i - v_i^* \right) \cdot c_2^{\frac{1}{q}} \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \\
\leq & \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} 2^{1-q} \, |\delta_i|^q \cdot c_2^{\frac{1}{q}} \left| \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \right| \\
= & \frac{1}{c_2} \sum_{i=1}^{N} |\delta_i|^q \left| \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \right|.
\end{aligned} \tag{8.31}$$

Let $\bar{a} = \max_{\forall i \in \{1,\ldots,N\}} \{ \sum_{j \in \mathcal{N}_i} a_{ij} \}$ and $\bar{b} = \max_{\forall i, j \{1,\ldots,N\}} \{a_{ij}\}$. We then see that

$$\frac{1}{c_2} \sum_{i=1}^{N} |\delta_i|^q \left| \sum_{j \in \mathcal{N}_i} a_{ij} (v_i - v_j) \right| \leq \frac{1}{c_2} \sum_{i=1}^{N} |\delta_i|^q \left(\bar{a} |v_i| + \bar{b} \sum_{j \in \mathcal{N}_i} |v_j| \right). \tag{8.32}$$

Upon using Lemmas 8.3 and 8.4, we have

$$\begin{aligned}|\delta_i|^q|v_j| \leq&|\delta_i|^q\left|v_j-v_j^*\right|+|\delta_i|^q\left|v_j^*\right| \leq 2^{1-q}|\delta_i|^q|\delta_j|^q+c_2|\delta_i|^q|e_j|^q\\ \leq&2^{-q}\left(|\delta_i|^{2q}+|\delta_j|^{2q}\right)+\frac{c_2}{2}\left(|\delta_i|^{2q}+|e_j|^{2q}\right).\end{aligned} \tag{8.33}$$

Upon using (8.33) to (8.32), we have

$$\begin{aligned}&\frac{1}{c_2}\sum_{i=1}^N|\delta_i|^q\left(\bar{a}|v_i|+\bar{b}\sum_{j\in\mathcal{N}_i}|v_j|\right)\\ \leq&\frac{1}{c_2}\sum_{i=1}^N\Bigg[\bar{a}2^{1-q}|\delta_i|^{2q}+\frac{\bar{a}c_2}{2}\left(|\delta_i|^{2q}+|e_i|^{2q}\right)\\ &+\bar{b}2^{-q}\sum_{j\in\mathcal{N}_i}\left(|\delta_i|^{2q}+|\delta_j|^{2q}\right)+\frac{\bar{b}c_2}{2}\sum_{j\in\mathcal{N}_i}\left(|\delta_i|^{2q}+|e_j|^{2q}\right)\Bigg]\\ \leq&\frac{1}{c_2}\sum_{i=1}^N\left[\bar{a}2^{1-q}|\delta_i|^{2q}+\frac{\bar{a}c_2}{2}\left(|\delta_i|^{2q}+|e_i|^{2q}\right)+2^{1-q}\bar{b}\bar{N}|\delta_i|^{2q}+\frac{\bar{b}c_2\bar{N}}{2}\left(|\delta_i|^{2q}+|e_i|^{2q}\right)\right]\\ =&\frac{\bar{a}+\bar{b}\bar{N}}{c_2}\sum_{i=1}^N\left[\left(2^{1-q}+\frac{c_2}{2}\right)|\delta_i|^{2q}+\frac{c_2}{2}|e_i|^{2q}\right],\end{aligned} \tag{8.34}$$

where $\bar{N}$ denotes the maximum number of the in-degree and out-degree of each ith agent for all $i \in \{1, \ldots, N\}$. By combining (8.31), (8.32), and (8.34), the second term of the right side of (8.30) can be further written as

$$\begin{aligned}&\frac{1}{2^{1-q}c_2^{1+1/q}}\sum_{i=1}^N\left(v_i-v_i^*\right)\cdot c_2^{\frac{1}{q}}\sum_{j\in\mathcal{N}_i}a_{ij}(v_i-v_j)\\ \leq&\frac{\bar{a}+\bar{b}\bar{N}}{c_2}\sum_{i=1}^N\left[\left(2^{1-q}+\frac{c_2}{2}\right)|\delta_i|^{2q}+\frac{c_2}{2}|e_i|^{2q}\right].\end{aligned} \tag{8.35}$$

By applying the control laws given in (8.17)–(8.19) to the first term of the right hand of (8.30), we have

$$\begin{aligned}&\sum_{i=1}^N\delta_i\dot{v}_i=\sum_{i=1}^N\delta_i\left(g_iu_i+f_i+f_{di}\right)\\ \leq&\sum_{i=1}^N\Bigg[-c_1g_i\delta_i^{2q}-\delta_ig_i\hat{c}_{fi}\varphi_i\tanh(\delta_i\varphi_i/\tau_i)-\delta_ig_i\hat{\theta}_i\tanh(\delta_i/\tau_i)\\ &+|\delta_i|c_{fi}\varphi_i+|\delta_i|\theta_i\Bigg]\\ \leq&\sum_{i=1}^N\Bigg[-c_1\underline{g}\delta_i^{2q}-\underline{g}\hat{c}_{fi}\delta_i\varphi_i\tanh(\delta_i\varphi_i/\tau_i)-\underline{g}\hat{\theta}_i\delta_i\tanh(\delta_i/\tau_i)\end{aligned}$$

$$
\begin{aligned}
&+ c_{fi}\delta_i\varphi_i \tanh(\delta_i\varphi_i/\tau_i) + 0.2785\tau_i c_{fi} + \theta_i\delta_i \tanh(\delta_i/\tau_i) + 0.2785\tau_i\theta_i \Big] \\
= &\sum_{i=1}^{N} \Big[- c_1 \underline{g}\delta_i^{2q} + 0.2785\tau_i(c_{fi} + \theta_i) + \left(c_{fi} - \underline{g}\hat{c}_{fi}\right) \delta_i\varphi_i \tanh(\delta_i\varphi_i/\tau_i) \\
&+ \left(\theta_i - \underline{g}\hat{\theta}_i\right) \delta_i \tanh(\delta_i/\tau_i) \Big]
\end{aligned} \tag{8.36}
$$

where we have used the fact that $0 \le |s| - s \cdot \tanh(s/k) \le 0.2785k$ [6].

By substituting (8.35) and (8.36) into (8.30), we arrive at

$$
\begin{aligned}
\dot{V}_2(t) \le &k_1 \sum_{i=1}^{N} \delta_i^{2q} + k_2 \sum_{i=1}^{N} e_i^{2q} + \sum_{i=1}^{N} \frac{1}{2^{1-q}c_2^{1+1/q}} \Big[0.2785\tau_i \left(c_{fi} + \theta_i\right) \\
&+ \left(c_{fi} - \underline{g}\hat{c}_{fi}\right) \delta_i\varphi_i \tanh(\delta_i\varphi_i/\tau_i) + \left(\theta_i - \underline{g}\hat{\theta}_i\right) \delta_i \tanh(\delta_i/\tau_i) \Big]
\end{aligned} \tag{8.37}
$$

where

$$
\begin{aligned}
k_1 &= -\frac{c_1 \underline{g}}{2^{1-q}c_2^{1+1/q}} + \frac{\bar{a} + \bar{b}\bar{N}}{c_2} \left(2^{1-q} + \frac{c_2}{2}\right), \\
k_2 &= \frac{\bar{a} + \bar{b}\bar{N}}{2}.
\end{aligned} \tag{8.38}
$$

Step 3. Note that in (8.37) the parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\hat{\bullet}$ is involved, which motivates us to introduce the generalized weight parameter estimation errors $\tilde{c}_{fi}$ ($i \in \{1, \dots, N\}$) and $\tilde{\theta}_i$ as follows

$$
\tilde{c}_{fi} = c_{fi} - \underline{g}\hat{c}_{fi}, \quad \tilde{\theta}_i = \theta_i - \underline{g}\hat{\theta}_i \tag{8.39}
$$

with which we introduce the third part of the Lyapunov function candidate as

$$
V_3(t) = \sum_{i=1}^{N} \frac{\tilde{c}_{fi}^2}{2k_\gamma \underline{g}\gamma_{1i}} + \sum_{i=1}^{N} \frac{\tilde{\theta}_i^2}{2k_\gamma \underline{g}\gamma_{2i}} \tag{8.40}
$$

where $k_\gamma = 2^{1-q}c_2^{1+1/q}$.

By applying the adaptive laws for $\hat{c}_{fi}$ and $\hat{\theta}_i$ given in (8.20), we get the derivative of V_3 as

$$
\dot{V}_3(t) = \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{k_\gamma} \Big(-\frac{\dot{\hat{c}}_{fi}}{\gamma_{1i}}\Big) + \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{k_\gamma} \Big(-\frac{\dot{\hat{\theta}}_i}{\gamma_{2i}}\Big)
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \frac{\tilde{c}_{fi}}{k_\gamma} \left[\sigma_{1i} \hat{c}_{fi}^{\frac{3q-1}{1+q}} - \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) \right] \\
&\quad + \sum_{i=1}^{N} \frac{\tilde{\theta}_i}{k_\gamma} \left[\sigma_{2i} \hat{\theta}_i^{\frac{3q-1}{1+q}} - \delta_i \tanh(\delta_i / \tau_i) \right] \\
&= \sum_{i=1}^{N} \frac{\sigma_{1i}}{k_\gamma} \tilde{c}_{fi} \hat{c}_{fi}^{\frac{3q-1}{1+q}} + \sum_{i=1}^{N} \frac{\sigma_{2i}}{k_\gamma} \tilde{\theta}_i \hat{\theta}_i^{\frac{3q-1}{1+q}} \\
&\quad - \sum_{i=1}^{N} \frac{1}{k_\gamma} \left(c_{fi} - \underline{g} \hat{c}_{fi} \right) \delta_i \varphi_i \tanh(\delta_i \varphi_i / \tau_i) \\
&\quad - \sum_{i=1}^{N} \frac{1}{k_\gamma} \left(\theta_i - \underline{g} \hat{\theta}_i \right) \delta_i \tanh(\delta_i / \tau_i).
\end{aligned} \tag{8.41}
$$

Sep 4. Define the Lyapunov function candidate as

$$
V(t) = V_1(t) + V_2(t) + V_3(t) \tag{8.42}
$$

where $V_1(t)$, $V_2(t)$, and $V_3(t)$ are given in (8.24), (8.29), and (8.40), respectively.
By combining (8.28), (8.37), and (8.41), we then arrive at

$$
\begin{aligned}
\dot{V}(t) \leq & - k_3 \sum_{i=1}^{N} \delta_i^{2q} - k_4 \sum_{i=1}^{N} e_i^{2q} + \sum_{i=1}^{N} \frac{\sigma_{1i}}{k_\gamma} \tilde{c}_{fi} \hat{c}_{fi}^{\frac{3q-1}{1+q}} \\
& + \sum_{i=1}^{N} \frac{\sigma_{2i}}{k_\gamma} \tilde{\theta}_i \hat{\theta}_i^{\frac{3q-1}{1+q}} + \sum_{i=1}^{N} \frac{0.2785 \tau_i (c_{fi} + \theta_i)}{k_\gamma}
\end{aligned} \tag{8.43}
$$

where

$$
\begin{aligned}
k_3 &= -2^{-q} N \ell_{\max} + \frac{c_1 \underline{g}}{2^{1-q} c_2^{1+1/q}} - \frac{\bar{a} + \bar{b} \bar{N}}{c_2} \left(2^{1-q} + \frac{c_2}{2} \right), \\
k_4 &= \frac{c_2 k_{\mathrm{m}}}{2} - 2^{-q} N \ell_{\max} - \frac{\bar{a} + \bar{b} \bar{N}}{2}.
\end{aligned} \tag{8.44}
$$

Thus, c_1 and c_2 can be chosen as $c_1 > 2^{1-q} c_2^{1+1/q} \underline{g}^{-1} \left[2^{-q} N \ell_{\max} + \frac{\bar{a}+\bar{b}\bar{N}}{c_2} \left(2^{1-q} + \frac{c_2}{2} \right) \right]$ and $c_2 > \frac{2}{k_{\mathrm{m}}} \left(2^{-q} N \ell_{\max} + \frac{\bar{a}+\bar{b}\bar{N}}{2} \right)$ such that k_3 and $k_4 > 0$.
Upon using Lemma 8.7, it is straightforward that

$$
\tilde{c}_{fi} \hat{c}_{fi}^{\frac{3q-1}{1+q}} \leq \frac{1+q}{4q} \underline{g}^{\frac{1-3q}{1+q}} \left[\left(2^{\frac{8q(q-1)}{(1+q)^2}} - 2^{\frac{2(q-1)}{1+q}} \right) \tilde{c}_{fi}^{\frac{4q}{1+q}} \right.
$$

$$+\left(1-2^{\frac{2q-2}{1+q}}+\frac{3q-1}{4q}+\frac{(1+q)2^{\frac{-16q(q-1)^2}{(1+q)^3}}}{4q}\right)c_{fi}^{\frac{4q}{1+q}}\Bigg], \tag{8.45}$$

$$\begin{aligned}\tilde{\theta}_i\hat{\theta}_i^{\frac{3q-1}{1+q}} \leq & \frac{1+q}{4q}\underline{g}^{\frac{1-3q}{1+q}}\Bigg[\left(2^{\frac{8q(q-1)}{(1+q)^2}}-2^{\frac{2(q-1)}{1+q}}\right)\tilde{\theta}_i^{\frac{4q}{1+q}} \\ & +\left(1-2^{\frac{2q-2}{1+q}}+\frac{3q-1}{4q}+\frac{(1+q)2^{\frac{-16q(q-1)^2}{(1+q)^3}}}{4q}\right)\theta_i^{\frac{4q}{1+q}}\Bigg].\end{aligned} \tag{8.46}$$

By substituting (8.45) and (8.46) into (8.43), one gets

$$\dot{V}(t) \leq -k_3\sum_{i=1}^{N}\delta_i^{2q}-k_4\sum_{i=1}^{N}e_i^{2q}-k_5\sum_{i=1}^{N}\tilde{c}_{fi}^{\frac{4q}{1+q}}-k_6\sum_{i=1}^{N}\tilde{\theta}_i^{\frac{4q}{1+q}}+d \tag{8.47}$$

where

$$\begin{aligned}k_5 =& \frac{\underline{\sigma}_1(1+q)\underline{g}^{\frac{1-3q}{1+q}}\left(2^{\frac{2(q-1)}{1+q}}-2^{\frac{8q(q-1)}{(1+q)^2}}\right)}{2^{3-q}c_2^{1+1/q}q}, \\ k_6 =& \frac{\underline{\sigma}_2(1+q)\underline{g}^{\frac{1-3q}{1+q}}\left(2^{\frac{2(q-1)}{1+q}}-2^{\frac{8q(q-1)}{(1+q)^2}}\right)}{2^{3-q}c_2^{1+1/q}q}, \\ d =& \sum_{i=1}^{N}\Bigg[\left(1-2^{\frac{2q-2}{1+q}}+\frac{3q-1}{4q}+\frac{(1+q)2^{\frac{-16q(q-1)^2}{(1+q)^3}}}{4q}\right)\underline{g}^{\frac{1-3q}{1+q}} \\ & \times\frac{(1+q)(\sigma_{1i}c_{fi}^{\frac{4q}{1+q}}+\sigma_{2i}\theta_i^{\frac{4q}{1+q}})}{2^{3-q}c_2^{1+1/q}q}+\frac{0.2785\tau_i\left(c_{fi}+\theta_i\right)}{2^{1-q}c_2^{1+1/q}}\Bigg]<\infty,\end{aligned} \tag{8.48}$$

where $\underline{\sigma}_1=\min\{\sigma_{11},\ldots,\sigma_{1N}\}$ and $\underline{\sigma}_2=\min\{\sigma_{21},\ldots,\sigma_{2N}\}$. Note that $\frac{2(q-1)}{1+q}-\frac{8q(q-1)}{(1+q)^2}=\frac{2(q-1)(1-3q)}{(1+q)^2}=\frac{16(2h-1)}{(4h-1)^2}$ and $h\in Z_+$. Thus, $k_5>0$, and then $k_6>0$. By introducing k_d as

$$k_d=\min\{k_3,k_4,k_5,k_6\}, \tag{8.49}$$

we further represent (8.47) as

$$\dot{V}(t) \leq -k_d\sum_{i=1}^{N}\left(\delta_i^{2q}+e_i^{2q}+\tilde{c}_{fi}^{\frac{4q}{1+q}}+\tilde{\theta}_i^{\frac{4q}{1+q}}\right)+d. \tag{8.50}$$

Step 5. We prove that there exists a finite time $T^*>0$ and a bounded constant $0<\zeta<\infty$ such that $V(t)<\zeta$ when $t\geq T^*$ in the sequel.

By recalling the definition of $V_1(t)$ given in (8.24), we have

$$V_1(t) = \frac{1}{1+q}\left(E^{\frac{1+q}{2}}\right)^T \mathrm{diag}(p)E^{\frac{1+q}{2}} \leq \frac{\bar{p}}{1+q}\sum_{i=1}^{N} e_i^{1+q}, \tag{8.51}$$

where $\bar{p} = \max\{p_1, \ldots, p_N\}$, and $p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue. Upon using Lemma 8.3, we have

$$\begin{aligned} V_2(t) \leq & \frac{1}{2^{1-q}c_2^{1+1/q}}\sum_{i=1}^{N}\left|v_i^{\frac{1}{q}} - (v_i^*)^{\frac{1}{q}}\right|\left|v_i - v_i^*\right| \\ \leq & \frac{1}{2^{1-q}c_2^{1+1/q}}\sum_{i=1}^{N}|\delta_i| \cdot 2^{1-q}|\delta_i|^q = \frac{1}{c_2^{1+1/q}}\sum_{i=1}^{N}\delta_i^{1+q}. \end{aligned} \tag{8.52}$$

From the definition of $V_3(t)$ given in (8.40), it is not hard to see that

$$V_3(t) \leq \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g\gamma}_1}\sum_{i=1}^{N}\tilde{c}_{fi}^2 + \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g\gamma}_2}\sum_{i=1}^{N}\tilde{\theta}_i^2, \tag{8.53}$$

with $\underline{\gamma}_1 = \min\{\gamma_{11}, \ldots, \gamma_{1N}\}$ and $\underline{\gamma}_2 = \min\{\gamma_{21}, \ldots, \gamma_{2N}\}$. Let

$$k_v = \max\left\{\frac{\bar{p}}{1+q}, \frac{1}{c_2^{1+1/q}}, \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g\gamma}_1}, \frac{1}{2^{2-q}c_2^{1+1/q}\underline{g\gamma}_2}\right\}, \tag{8.54}$$

we then have from (8.51) to (8.53) that

$$V(t) \leq k_v\sum_{i=1}^{N}\left(\delta_i^{1+q} + e_i^{1+q} + \tilde{c}_{fi}^2 + \tilde{\theta}_i^2\right). \tag{8.55}$$

Upon using Lemma 8.5, we then arrive at

$$V(t)^{\frac{2q}{1+q}} \leq k_v^{\frac{2q}{1+q}}\sum_{i=1}^{N}\left(\delta_i^{2q} + e_i^{2q} + \tilde{c}_{fi}^{\frac{4q}{1+q}} + \tilde{\theta}_i^{\frac{4q}{1+q}}\right). \tag{8.56}$$

Let $\tilde{c} = \frac{\eta_1 k_d}{k_v^{\frac{2q}{1+q}}}$ (k_d and k_v are given in (8.49) and (8.54), respectively, and $0 < \eta_1 \leq 1$). It thus follows from (8.50) and (8.56) that

$$\dot{V}(t) \leq -\tilde{c}V(t)^{\frac{2q}{1+q}} + d. \tag{8.57}$$

Let $\Theta = \{(x_i, v_i) : V(t) < (\frac{d}{\eta_2 \tilde{c}})^{\frac{1+q}{2q}}, 0 < \eta_2 < 1\}$. According to Theorem 5.2 in [7], for any $(x_i, v_i) \notin \Theta$ for all $t \in [0, t_x]$, it holds that $V(t) \geq (d/\eta_2 \tilde{c})^{\frac{1+q}{2q}}$, i.e., $d \leq \eta_2 \tilde{c} V(t)^{\frac{2q}{1+q}}$, for all $t \in [0, t_x]$. This fact, together with (8.57), implies that

$$\dot{V}(t) \leq -\tilde{c} V(t)^{\frac{2q}{1+q}} + \eta_2 \tilde{c} V(t)^{\frac{2q}{1+q}} = -(1-\eta_2)\tilde{c} V(t)^{\frac{2q}{1+q}} \tag{8.58}$$

for all $t \in [0, t_x]$. Note that $V(t) \geq (d/\eta_2 \tilde{c})^{\frac{1+q}{2q}} > 0$ for $t \in [0, t_x]$, and it thus follows from (8.58) that $t_x < \frac{V(t_0)^{1-\frac{2q}{1+q}}}{(1-\eta_2)\tilde{c}\left(1-\frac{2q}{1+q}\right)}$ according to Lemma 8.1. Therefore for $\forall t \geq T^*$, with T^* satisfying

$$T^* = \frac{V(t_0)^{1-\frac{2q}{1+q}}}{(1-\eta_2)\tilde{c}\left(1-\frac{2q}{1+q}\right)}, \tag{8.59}$$

we have

$$V(t) < \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1+q}{2q}} = \zeta. \tag{8.60}$$

Step 6. Derive the estimation for steady-state errors of all agents.
Note that for all $i \in \{1, \ldots, N\}$, we have

$$\begin{aligned} |e_i| = \left(e_i^{1+q}\right)^{\frac{1}{1+q}} \leq \left(\sum_{i=1}^{N} e_i^{1+q}\right)^{\frac{1}{1+q}} \leq \left(\frac{1+q}{\underline{p}} V_1(t)\right)^{\frac{1}{1+q}} \\ \leq \left(\frac{1+q}{\underline{p}} V(t)\right)^{\frac{1}{1+q}} \leq \left(\frac{1+q}{\underline{p}}\right)^{\frac{1}{1+q}} \left(\frac{d}{\eta_2 \tilde{c}}\right)^{\frac{1}{2q}}, \end{aligned} \tag{8.61}$$

with $\underline{p} = \min\{p_1, \ldots, p_N\}$ ($p = [p_1, \ldots, p_N]^T$ is the left eigenvector of $\mathscr{L}$ associated with its zero eigenvalue). According to Lemma 8.2, we have $|\varsigma^{1/q} - (v_i^*)^{1/q}| \geq 2^{1-1/q} |\varsigma - v_i^*|^{1/q}$, and thus, if $v_i \geq v_i^*$,

$$\begin{aligned} V_2(t) \geq & \frac{1}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \int_{v_i^*}^{v_i} 2^{1-1/q} \left(\varsigma - v_i^*\right)^{1/q} d\varsigma \\ = & \frac{1}{\frac{1}{q}+1} \times \frac{2^{1-1/q}}{2^{1-q} c_2^{1+1/q}} \sum_{i=1}^{N} \left(\varsigma - v_i^*\right)^{\frac{1}{q}+1} \Big|_{v_i^*}^{v_i} \\ = & \frac{2^{q-\frac{1}{q}} q}{c_2^{1+1/q}(1+q)} \sum_{i=1}^{N} \left(v_i - v_i^*\right)^{1+\frac{1}{q}}. \end{aligned} \tag{8.62}$$

If $v_i < v_i^*$, the proof of (8.62) is similar. Then, we have

$$|v_i - v_i^*| = \left[\left(v_i - v_i^*\right)^{1+1/q}\right]^{\frac{1}{1+1/q}} \le \left[\sum_{i=1}^{N}(v_i - v_i^*)^{1+1/q}\right]^{\frac{1}{1+1/q}}$$
$$\le \left[\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q}q}V_2(t)\right]^{\frac{1}{1+1/q}} \le \left[\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q}q}\right]^{\frac{q}{1+q}}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{2}}. \tag{8.63}$$

On the other hand, from (8.61), we see that

$$\left|v_i^*\right| = \left|-c_2 e_i^q\right| \le c_2\left(\frac{1+q}{\underline{p}}\right)^{\frac{q}{1+q}}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{2}}. \tag{8.64}$$

It thus follows from (8.63) to (8.64) that

$$|v_i| \le |v_i - v_i^*| + |v_i^*| \le \left[\left(\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q}q}\right)^{\frac{q}{1+q}} + c_2\left(\frac{1+q}{\underline{p}}\right)^{\frac{q}{1+q}}\right]\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{2}} \tag{8.65}$$

which then implies, for $\forall i,\ j \in \{1, \ldots, N\}$, that

$$|v_i - v_j| \le |v_i| + |v_j| \le 2\left[\left(\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q}q}\right)^{\frac{q}{1+q}} + c_2\left(\frac{1+q}{\underline{p}}\right)^{\frac{q}{1+q}}\right]\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1}{2}}. \tag{8.66}$$

In addition, for all $\forall i \in \{1, \ldots, N\}$,

$$|\tilde{c}_{fi}| = \sqrt{\tilde{c}_{fi}^2} \le \sqrt{\sum_{i=1}^{N}\tilde{c}_{fi}^2} \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1 V_3(t)}$$
$$\le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1 V(t)} \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_1}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1+q}{4q}}, \tag{8.67}$$

and

$$|\tilde{\theta}_i| \le \sqrt{2k_\gamma \underline{g}\bar{\gamma}_2}\left(\frac{d}{\eta_2\tilde{c}}\right)^{\frac{1+q}{4q}}, \tag{8.68}$$

with $\bar{\gamma}_1 = \max\{\gamma_{11}, \ldots, \gamma_{1N}\}$, and $\bar{\gamma}_2 = \max\{\gamma_{21}, \ldots, \gamma_{2N}\}$.

From the above analysis, we conclude that under the proposed finite-time control scheme (8.17)–(8.20), the position and velocity errors between neighbor agents will converge to a small region Ω_1, defined by

$$\Omega_1 = \Bigg\{ |e_i| \le \left(\frac{1+q}{\underline{p}} k_v\right)^{\frac{1}{1+q}} \left(\frac{d}{\eta_2\eta_1 k_d}\right)^{\frac{1}{2q}}, \quad |v_i - v_j| \le 2\Bigg[\left(\frac{c_2^{1+1/q}(1+q)}{2^{q-1/q} q}\right)^{\frac{q}{1+q}} + c_2 \left(\frac{1+q}{\underline{p}}\right)^{\frac{q}{1+q}} \Bigg] \sqrt{\frac{d k_v^{\frac{2q}{1+q}}}{\eta_2\eta_1 k_d}}, \ \forall i, j \in \{1, \ldots, N\} \Bigg\}, \tag{8.69}$$

and the generalized parameter estimation converges to the region Ω_{1p} given as

$$\Omega_{1p} = \Bigg\{ |\tilde{c}_{fi}| \le \sqrt{2^{2-q} c_2^{1+1/q} \underline{g} \bar{\gamma}_1 k_v} \left(\frac{d}{\eta_2\eta_1 k_d}\right)^{\frac{1+q}{4q}}, \quad |\tilde{\theta}_i| \le \sqrt{2^{2-q} c_2^{1+1/q} \underline{g} \bar{\gamma}_2 k_v} \left(\frac{d}{\eta_2\eta_1 k_d}\right)^{\frac{1+q}{4q}}, \ \forall i \in \{1, \ldots, N\} \Bigg\} \tag{8.70}$$

in the finite time T^*. □

8.3.1 Comparison with Regular State Feedback-Based Infinite-Time Method Under Directed Topology

Note that if we take the value of the fraction power q as 1, the control scheme (8.17)–(8.20) reduces to the following regular state feedback-based infinite-time control

$$u_i = -c_1\delta_i - \hat{c}_{fi}\varphi_i \tanh(\delta_i\varphi_i/\tau_i) - \hat{\theta}_i \tanh(\delta_i/\tau_i), \tag{8.71}$$

with the updated laws

$$\begin{aligned} \dot{\hat{c}}_{fi} &= -\gamma_{1i}\sigma_{1i}\hat{c}_{fi} + \gamma_{1i}\varphi_i \tanh(\delta_i\varphi_i/\tau_i)\delta_i, \\ \dot{\hat{\theta}}_i &= -\gamma_{2i}\sigma_{2i}\hat{\theta}_i + \gamma_{2i}\tanh(\delta_i/\tau_i)\delta_i, \end{aligned} \tag{8.72}$$

where $\delta_i = v_i - v_i^*$ with $v_i^* = -c_2 e_i$.

With such regular state feedback-based control, the following result is derived.

Theorem 8.2 *The second-order uncertain MAS (8.1) under Assumption 8.1 and directed strongly connected topology with the control scheme (8.71)–(8.72) achieves cooperative uniformly ultimately bounded (UUB) leaderless consensus in that the* *leaderless consensus positive errors e_i ($i = 1, \ldots, N$) and velocity errors converge to a small residual set Ω_2 defined by*

$$\Omega_2 = \Bigg\{ |e_i| \le \sqrt{\frac{2k_v d}{\eta_2 \eta_1 \underline{p} k_d}}, \quad |v_i - v_j| \le 2\left(\sqrt{2c_2^2} + c_2\sqrt{\frac{2}{\underline{p}}}\right) \times \sqrt{\frac{dk_v}{\eta_2\eta_1 k_d}}, \ \forall i, j \in \{1, \ldots, N\} \Bigg\}, \tag{8.73}$$

and the generalized parameter estimate errors $\tilde{c}_{fi}$ and $\tilde{\theta}_i$ converge to a small set Ω_{1p} defined by

$$\Omega_{1p} = \Bigg\{ |\tilde{c}_{fi}| \le \sqrt{\frac{2c_2^2 \underline{g} \bar{\gamma}_1 k_v d}{\eta_2 \eta_1 k_d}}, \quad |\tilde{\theta}_i| \le \sqrt{\frac{2c_2^2 \underline{g} \bar{\gamma}_2 k_v d}{\eta_2 \eta_1 k_d}}, \ \forall i \in \{1, \ldots, N\} \Bigg\} \tag{8.74}$$

where $0 < \eta_1 \le 1$, $0 < \eta_2 < 1$, $V(t_0)$ is known, and k_d and k_v are given, respectively, in (8.49) and (8.54), which are explicitly computable. In addition, all signals in the closed-loop system remain uniformly bounded.

Proof The proof can be done by letting $q = 1$ and following the same line as in the proof of Theorem 8.1. □

8.4 Numerical Simulations

To verify the effectiveness of the proposed finite-time control algorithms, numerical simulation on a group of four networked autonomous surface vessels (ASVs) [8] with nonlinear dynamics is performed.

The dynamics of the kth ($k = 1, 2, 3, 4$) ASV is modeled by

$$\begin{bmatrix} m_{x,k} & 0 & 0 \\ 0 & m_{y,k} & 0 \\ 0 & 0 & m_{z,k} \end{bmatrix} \cdot \begin{bmatrix} \dot{v}_{x,k} \\ \dot{v}_{y,k} \\ \dot{v}_{z,k} \end{bmatrix} = \begin{bmatrix} f_{x,k} \\ f_{y,k} \\ f_{z,k} \end{bmatrix} + \begin{bmatrix} u_{x,k} \\ u_{y,k} \\ d_{x,k} \end{bmatrix} + \begin{bmatrix} u_{x,k} \\ d_{y,k} \\ d_{z,k} \end{bmatrix} \tag{8.75}$$

in which $M_k = \mathrm{diag}\{m_{x,k}, m_{y,k}, m_{z,k}\}$ denotes the mass matrix; $r_k = [x_k, y_k, z_k]^T$, $v_k = \dot{r}_k = [v_{x,k}, v_{y,k}, v_{z,k}]^T$, and $u_k = [u_{x,k}, u_{y,k}, u_{z,k}]^T$ denote the position, velocity, and control input vector, respectively; $D_k = [d_{x,k}, d_{y,k}, d_{z,k}]^T$ is environment disturbance; and $f_k = [f_{x,k}, f_{y,k}, f_{z,k}]^T$ represents coriolis, centripetal, and hydrodynamic damping forces and torques acting on the body, where

$$f_k = \Delta \cdot v_k$$

with

$$\Delta = \begin{bmatrix} A_{x,k} + A_{|x,k|}|v_{x,k}| & -m_{y,k}v_{z,k} & 0 \\ m_{x,k}v_{z,k} & B_{y,k} + B_{|y,k|}|v_{y,k}| & 0 \\ 0 & 0 & C_{z,k} + C_{|z,k|}|v_{z,k}| \end{bmatrix}.$$

In the simulation, the physical parameters are taken as: $M_k = \text{diag}\{600 + 6(-1)^k + 6\Delta m(t), 1000 + 10(-1)^k + 10\Delta m(t), 800 + 8(-1)^k + 8\Delta m(t)\}$ with $\Delta m(t) = \sin(\pi t/50 - \pi)$, $A_{x,k} = -1 + 0.1(-1)^k$, $A_{|x,k|} = -25 + 2.5(-1)^k$, $B_{y,k} = -10 + (-1)^k$, $B_{|y,k|} = -200 + 20(-1)^k$, $C_{z,k} = -0.5 + 0.05(-1)^k$, and $C_{|z,k|} = -1500 + 150(-1)^k$ for $k = 1, 2, 3, 4$. The external disturbance is taken as $D_k = [3 + 3(-1)^k \sin(t/50) + 2\sin(t/10), -1 + 3(-1)^k \sin(t/20 - \pi/6) + 2\sin(t), -5(-1)^k \sin(0.1t) - \sin(t + \pi/3)]^T$. The networked communication topology is directed and strongly connected as shown in Fig. 8.1. Each edge weight is taken as 0.1. The left eigenvector of $\mathscr{L}$ associated with eigenvalue 0 is $[2, 1, 1, 1]^T$.

The simulation objective is that the four ASVs are required to achieve consensus in a finite time by using the proposed control law given in (8.17)–(8.20) under the directed communication topology condition. The initial condition of the vessels is $r_1(0) = (-1.5\text{m}, 0\text{m}, \pi/3\text{rad})$, $r_2(0) = (-1\text{m}, 2\text{m}, -\pi/4\text{rad})$, $r_3(0) = (1.5\text{m}, 2\text{m}, -\pi/9\text{rad})$, $r_4(0) = (2\text{m}, 1\text{m}, \pi/4\text{rad})$, $v_k(0) = (0, 0, 0)$ $(k = 1, 2, 3, 4)$, respectively. The control parameters are taken as: $s = 2$, $c_1 = 2000$ and $c_2 = 3$. In addition, the initial values of the estimates are chosen as $\hat{c}_{fx,k} = \hat{c}_{fy,k} = \hat{c}_{fz,k} = 0$ and $\hat{\theta}_{x,k} = \hat{\theta}_{y,k} = \hat{\theta}_{z,k} = 0$ for $k = 1, 2, 3, 4$. The simulation runs for 15s. The trajectories of the ASVs are represented in Fig. 8.2, which shows the trajectory of each ASV

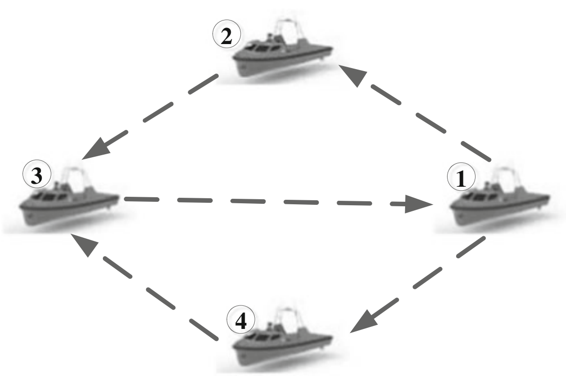


Fig. 8.1 Directed communication topology among the four ASVs

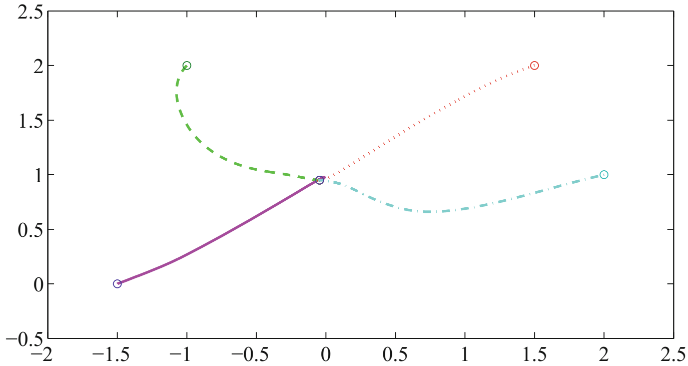
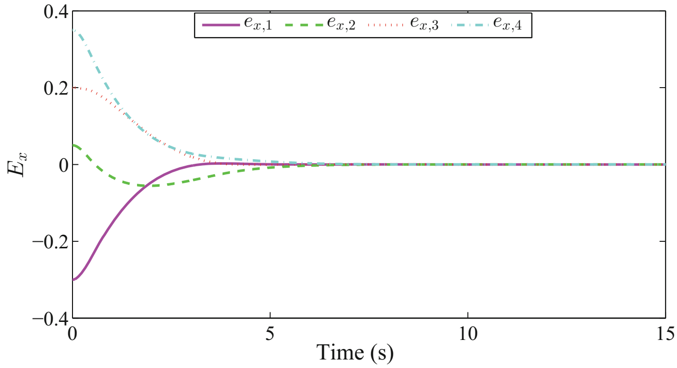


Fig. 8.2 Trajectory of each ASV from the initial position to the final position

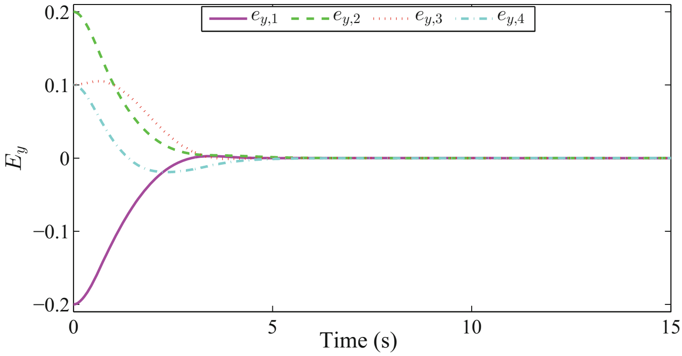
from the initial position to the finial position. The position error convergence results in x-direction, y-direction, and z-direction under the finite-time control scheme proposed in (8.17)–(8.20) are represented in Fig. 8.3. In addition, the velocity error convergence results in x-direction, y-direction, and z-direction under the proposed finite-time control scheme are given in Fig. 8.4. From both Figs. 8.3 and 8.4, we see that the consensus is achieved under the proposed finite-time control scheme with high accuracy and fast convergence rate. Figure 8.5 shows the control input signals of the four ASVs under the proposed finite-time control scheme.

To show the effectiveness of our proposed scheme, we make a comparison on convergence time between the proposed control scheme (8.17)–(8.20) and the typical infinite-time-based adaptive scheme (8.71)–(8.72). The two control schemes are applied to the same group of ASVs (8.75) with the same unknown parameters mentioned above. The convergence comparison results are shown in Fig. 8.6. It is observed from the comparison results that the convergence rates of the four ASVs, including the position error convergence rates and velocity error convergence rates, are faster under the finite-time controller compared with the infinite-time controller.

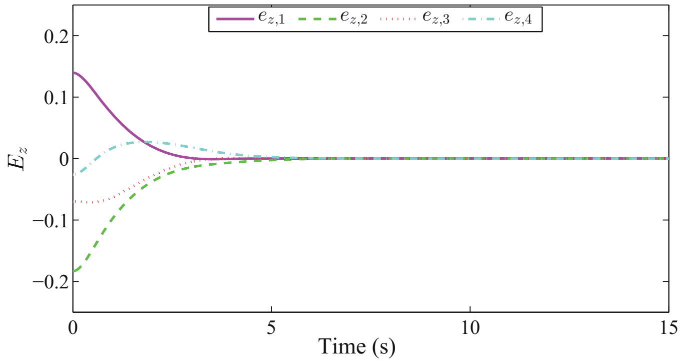
On the other hand, to see the effects of the control parameters on the error convergence time, we consider two groups of control parameters: $s = 2$, $c_1 = 2000$ and $c_2 = 3$; $s = 2$, $c_1 = 5000$ and $c_2 = 4$, respectively. The position and velocity error convergence comparison results are depicted in Fig. 8.7, which indicates that the finite-time T^* specified in (8.22) can be adjusted smaller by choosing larger controller parameters c_1 and c_2, but it will result in larger initial control effort.



(a) E in x-direction.

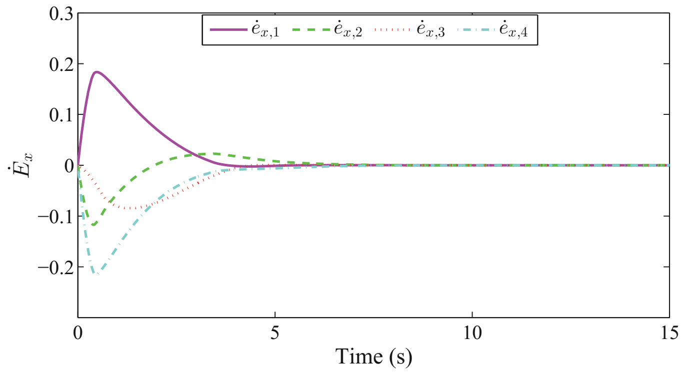


(b) E in y-direction.

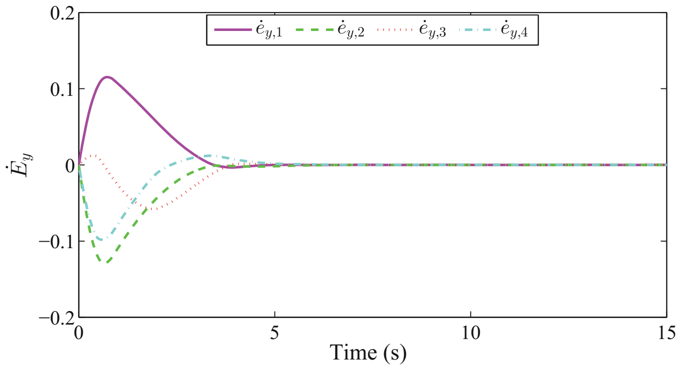


(c) E in z-direction.

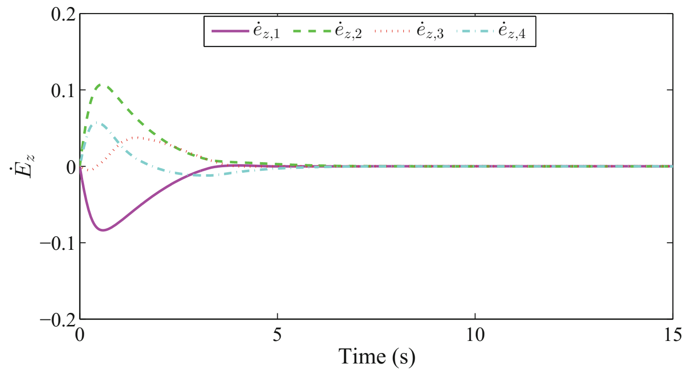
Fig. 8.3 Position error convergence of the four ASVs under the proposed finite-time control scheme



(a) $\dot{E}$ in x-direction.

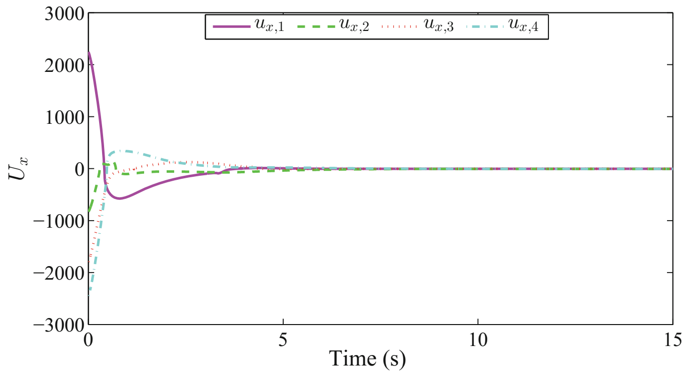


(b) $\dot{E}$ in y-direction.

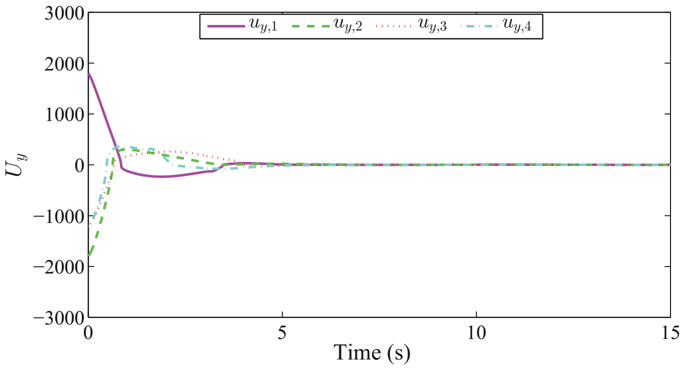


(c) $\dot{E}$ in z-direction.

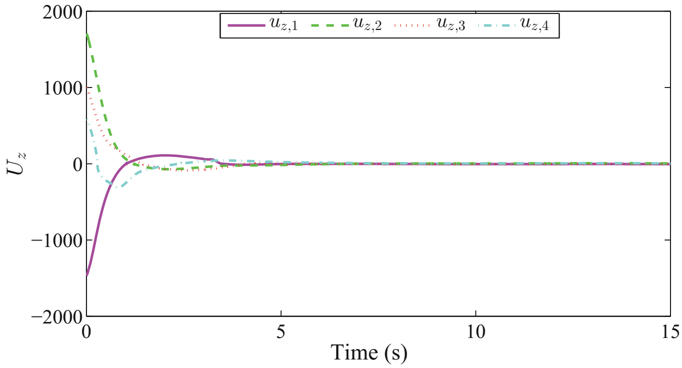
Fig. 8.4 Velocity error convergence of the four ASVs under the proposed finite-time control scheme



(a) U in x-direction.

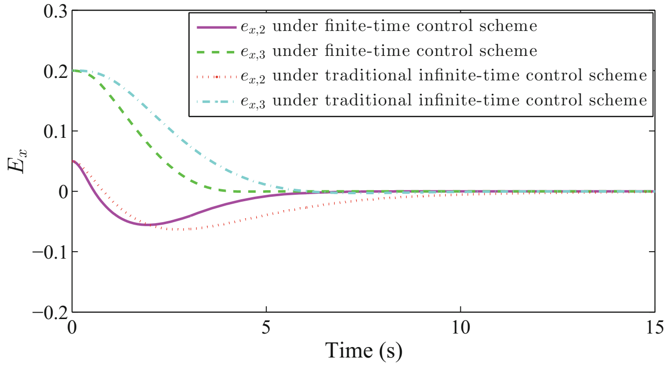


(b) U in y-direction.

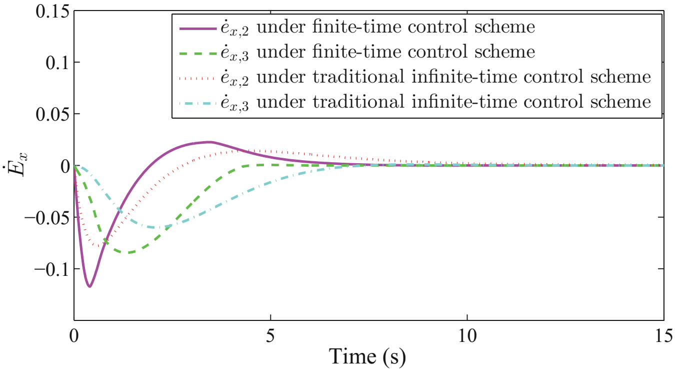


(c) U in z-direction.

Fig. 8.5 Control inputs of the four ASVs under the proposed finite-time control scheme



(a) E in x-direction.



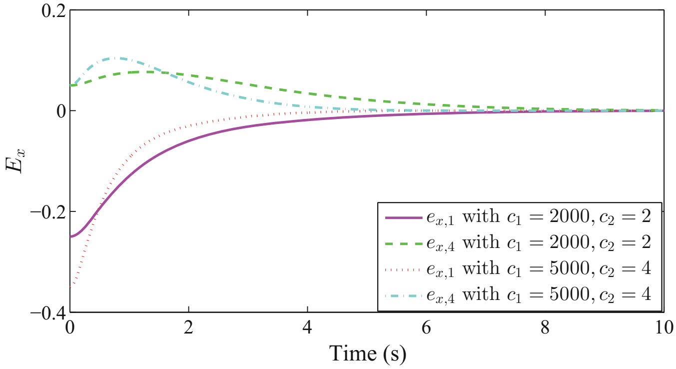
(b) $\dot{E}$ in x-direction.

Fig. 8.6 Error convergence comparison with x-direction under the two control schemes

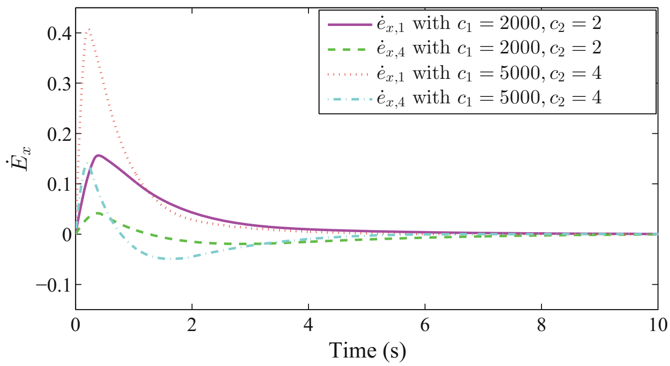
8.5 Notes

The work in this chapter explicitly addressed the finite-time leaderless consensus control of nonlinear MAS with second-order dynamics in the presence of unknown time-varying gain and non-parametric uncertainties under the one-way directed communication topology. It is shown that under the proposed control scheme, not only the finite-time convergence is realized but also the unknown time-varying gain and the non-parametric uncertainties are compensated successfully.

The main contributions of the work in this chapter are summarized as follows: (1) The finite-time consensus control scheme is derived for a group of networked nonlinear systems with second-order uncertain dynamics under the directed communication topology. It is shown that all the internal signals are ensured to be uniformly bounded and the consensus configuration error uniformly converges to a small residual set in finite time. (2) The one-way directed communication condition is taken



(a) E x-direction .



(b) $\dot{E}$ in x-direction.

Fig. 8.7 Error convergence comparison with x-direction under the finite-time control scheme with different control parameters

into consideration. The technical difficulty arising from the directed topology is tackled by deriving the general algebraic connectivity properties established in Lemmas 8.8–8.9 with the newly defined Laplacian matrix Q. (3) By introducing the skillfully defined weighting parameter estimate error and by establishing an important inequality in Lemma 8.7, the lumped uncertainties in the system with unknown and time-varying control gains are compensated gracefully.

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Chapter 9
Finite-Time Leaderless Consensus Control for Systems with High-Order Uncertain Dynamics

The majority of the research on finite-time consensus control has been focused on MAS described by single or double integrator under undirected topology. This work explicitly addressed the problem of finite-time consensus for MAS with high-order nonlinear uncertain dynamics under directed communication constraints. Finite-time leaderless consensus for networked multi-agent systems (MAS) with high-order nonlinear uncertain dynamics under local communication condition is an challenging problem due to the involvement of high-order nonlinear uncertain dynamics, local communication constraints, and unknown time-varying control effectiveness gain. In this chapter, based upon the locally defined consensus error and fractionally composed virtual error, a number of useful intermediate results are derived, with which, the finite-time consensus solutions are established for networked MAS with high-order uncertain dynamics under single-way directed communication topology. By including fraction power integration as part of the Lyapunov function candidate and by using inductive analysis, it is shown that the proposed distributed solution is able to achieve consensus in finite time with sufficient accuracy. The benefits and effectiveness of the developed algorithm are also confirmed by numerical simulations.

Section 9.1 addresses the finite-time leaderless consensus control problem for the high-order multi-agent systems with unknown non-affine dynamics under local directed communication topology. Section 9.2 gives some useful lemmas and also derives a number of useful intermediate results that are related to the finite-time stability analysis for the high-order uncertain multi-agent systems. In Sect. 9.3, the locally defined neighborhood error and the fractionally composed virtual error are introduced into the control scheme, where the fraction power feedback law and fraction power adaptive updated law for the parameter estimation are designed. The main control results are given in Sect. 9.4, in which rigorous proof of the stability is carried out by inductive analysis upon adding the fraction power integration term to

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the Lyapunov function candidate. Section 9.5 shows the comparison results between the infinite-time method and the finite-time method. Section 9.6 gives two simulation examples to verify the effectiveness of the developed algorithm. Section 9.7 gives a discussion on the finite-time consensus control problem of high-order uncertain MAS.

9.1 Finite-Time Consensus Formulation for High-Order Uncertain Systems

In this chapter, we consider a group of networked systems consisting of N subsystems with high-order non-affine dynamics, modeled by

$$\begin{aligned} \dot{x}_{i,m} &= x_{i,m+1}, \quad m = 1, \ldots, n-1 \\ \dot{x}_{i,n} &= f_i(\bar{x}_i, u_i) + f_{di}(\bar{x}_i, t), \\ y_i &= x_{i,1}, \quad i = 1, \ldots, N, \end{aligned} \tag{9.1}$$

where $\bar{x}_i = [x_{i,1}, \ldots, x_{i,n}]^T \in \mathbb{R}^n$, $x_{i,m} \in \mathbb{R}$, $u_i \in \mathbb{R}$, and $y_i \in \mathbb{R}$ are system state, control input, and control output, respectively, $f_i(\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth but unknown non-affine function, and $f_{di}(\cdot)$ denotes all the uncertainties and disturbances acting on the ith subsystem.

Let the communication network among the N subsystems be represented by a directed graph $\mathscr{G} = (\iota, \varepsilon)$ and each subsystem can be seen as a node in such graph [1], where $\iota = \{\iota_1, \iota_2, \ldots, \iota_N\}$ denotes the set of nodes, $\varepsilon \subseteq \iota \times \iota$ is the set of edges between two distinct nodes. The set of neighbors of node ι_i is denoted by $\mathscr{N}_i = \{\iota_j \in \iota | \varepsilon_{ji} \in \varepsilon\}$, in which $\varepsilon_{ji} = (\iota_j, \iota_i) \in \varepsilon$ indicates that node i can obtain information from node j. The graph topology is represented by the weighted adjacency matrix $\mathscr{A} = [a_{ij}]$, where $a_{ij} > 0 \Leftrightarrow \varepsilon_{ji} \in \varepsilon$, and $a_{ij} = 0$, otherwise. Moreover, it is assumed that $a_{ii} = 0$ for all $i = 1, \ldots, N$. The in-degree matrix $\mathscr{B} = \mathrm{diag}(\mathscr{B}_1, \ldots, \mathscr{B}_N) \in \mathbb{R}^{N\times N}$ is defined as $\mathscr{B}_i = \sum_{j\in\mathscr{N}_i} a_{ij}$ such that the Laplacian matrix is defined by $L = [l_{ij}] = \mathscr{B} - \mathscr{A} \in \mathbb{R}^{N\times N}$.

The local position error is defined as

$$e_i = \sum_{j\in\mathscr{N}_i} a_{ij}(y_i - y_j), \quad i, j \in \{1, \ldots, N\} \tag{9.2}$$

The control objective is to establish distributed consensus protocols for general high-order nonlinear MAS in the presence of non-affine dynamics and non-parametric uncertainties as described by (9.1) via finite-time control method such that not only the stable cooperative output consensus is achieved, but also the finite-time convergence is ensured with sufficient accuracy, i.e., there exists a finite time T^* that can be explicitly expressed such that for all $t \geq T^*$ and all $i = 1, \ldots, N$, $|e_i(t)| < \bar{\varepsilon}$, where $e_i(t)$ denotes the local neighborhood error, $\bar{\varepsilon} > 0$ is a constant which can be made sufficiently small, and meanwhile, all the internal signals in the closed-loop systems are bounded.

Note that by non-affine dynamics, the control action enters into the system through an implicit and uncertain way, making the controller design much more challenging. To proceed, the non-affine dynamic model (9.1) is first transformed into an affine form by using the mean value theorem [2], that is, there exists a constant $\xi_i \in (0, u_i)$ $(i \in \{1, \ldots, N\})$ such that

$$f_i(\bar{x}_i, u_i) = f_i(\bar{x}_i, 0) + \frac{\partial f_i(\bar{x}_i, \xi_i)}{\partial u_i} u_i. \tag{9.3}$$

Let $g_i(\bar{x}_i, \xi_i) = \partial f_i(\bar{x}_i, \xi_i)/\partial u_i$ denote the control effectiveness gain, and $L_{fi}(\bar{x}_i, t) = f_i(\bar{x}_i, 0) + f_{di}(\bar{x}_i, t)$ denote the lumped uncertainties. By substituting (9.3) into (9.1), one gets

$$\dot{x}_{i,n} = g_i(\bar{x}_i, \xi_i)u_i + L_{fi}(\bar{x}_i, t). \tag{9.4}$$

The following fundamental assumptions are in order for the controller design.

Assumption 9.1

a. The directed communication network $\mathscr{G}$ is strongly connected.
b. The control effectiveness gain $g_i(\cdot)$ is unknown and time-varying but bounded away from zero, that is, there exist unknown constants g_{li} and g_{ui} such that $0 < g_{li} \leq |g_i(\cdot)| \leq g_{ui} < \infty$ and $g_i(\cdot)$ is sign-definite (we assume $\mathrm{sgn}(g_i) = +1$ in this paper without loss of generality).
c. For the lumped uncertainties $L_{fi}(\cdot)$, there exists an unknown constant $\alpha_i \geq 0$ and a known scalar function $\psi_i(\bar{x}_i) \geq 0$ such that $|L_{fi}(\cdot)| \leq \alpha_i \psi_i(\cdot)$. In addition, if $\bar{x}_i$ is bounded, so is $\psi_i(\bar{x}_i)$.

It should be addressed that in Assumption 9.1-b, although g_{li} is unknown, we can always estimate its lower bound, i.e., $\underline{g} < g_{li}$ for some constant $\underline{g} > 0$. In addition, Assumption 9.1-b is commonly imposed in most existing works in addressing the tracking control problem, for instance [3–8]. Assumption 9.1-c is related to the extraction of the core information from the nonlinearities of the system, which can be readily done for any practical system with only crude model information [9].

9.2 Useful Lemmas and Intermediate Results

In this section, we represent some useful lemmas and derive a number of useful intermediate results that play a crucial role in the finite-time stability analysis for the high-order uncertain multi-agent systems.

Lemma 9.1 ([10]) *Suppose there exists a continuously differentiable function* $V(x, t) : \mathbb{U} \times \mathbb{R}_+ \to \mathbb{R}$ *(*$\mathbb{U} \subset \mathbb{R}^N$ *is an open neighborhood of the origin), a real* *number* $c > 0$ *and* $0 < \alpha < 1$*, such that* $V(x,t)$ *is positive definite and* $\dot{V}(x,t) + cV(x,t)^{\alpha} \le 0$ *on* $\mathbb{U}_0$ *(*$\mathbb{U}_0 \subset \mathbb{U}$*), then* $V(x,t)$ *is locally in finite-time convergent with a finite settling time satisfying* $T^* \le \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$*, such that for any given initial state* $x(t_0) \in \mathbb{U}_0/\{0\}$, $\lim_{t\to T^*} V(x,t) = 0$ *and* $V(x,t) = 0$ *for* $t \ge T^*$*.*

Lemma 9.2 ([11]) *Let c, d be positive real numbers and* $\gamma(x,y) > 0$ *a real-valued function. Then,* $|x|^c|y|^d \le \frac{c\gamma(x,y)|x|^{c+d}}{c+d} + \frac{d\gamma(x,y)^{-c/d}|y|^{c+d}}{c+d}$.

Lemma 9.3 ([11]) *If* $h = h_2/h_1 \ge 1$*, where* $h_1, h_2 > 0$ *are odd integers, then* $|x - y|^h \le 2^{h-1}|x^h - y^h|$.

Lemma 9.4 ([11]) *If* $0 < h = h_1/h_2 \le 1$*, where* $h_1, h_2 > 0$ *are odd integers, then* $|x^h - y^h| \le 2^{1-h}|x-y|^h$.

Lemma 9.5 ([12]) *For* $x_i \in \mathbb{R}$*,* $i = 1, 2, \ldots, N$*,* $0 < h \le 1$*, then* $(\sum_{i=1}^N |x_i|)^h \le \sum_{i=1}^N |x_i|^h \le N^{1-h}(\sum_{i=1}^N |x_i|)^h$.

Lemma 9.6 ([7]) *For* $x \in \mathbb{R}$*,* $y \in \mathbb{R}$*,* $0 < h = h_1/h_2 \le 1$*, where* $h_1, h_2 > 0$ *are positive odd integers, then* $x^h(y-x) \le \frac{1}{1+h}(y^{1+h} - x^{1+h})$.

Lemma 9.7 *Let* $\Xi = 1/2(PL + L^T P)$*, where* $P = diag\{p\}$ *and* $p = [p_1, \ldots, p_N]^T$ *is the left eigenvector of L associated with the zero eigenvalue. With the graph satisfying Assumption 9.1-a,* $\forall E^s \ne 0_N$*, there exists a constant* $k_s > 0$ *such that*

$$(E^s)^T \Xi E^s \ge k_s (E^s)^T E^s, \tag{9.5}$$

where $s = (4ln - 1)/(4ln + 1)$*,* $l \in Z_+$*.*

Proof Before moving on, we first give an important property on Ξ, which has been proved in our previous work [7]: $\forall X \ne 0_N$, $X^T \Xi X = 0$ if and only if $X = c1_N$ with c being a nonzero constant, and moreover, $0 < \min_{X \ne c1_N, X \ne 0_N} \frac{X^T \Xi X}{X^T X} \le \lambda_2(\Xi)$ (we denote by $\lambda_i(\Xi)$ the eigenvalue of Ξ). In the following, we prove that indeed $E^s \ne c1_N$, and then $\min_{E^h \ne 0_N} \frac{(E^h)^T \Xi E^h}{(E^h)^T E^h} > 0$. Let $k_s = \min_{E^h \ne 0_N} \frac{(E^h)^T \Xi E^h}{(E^h)^T E^h}$, then it holds that $(E^h)^T \Xi E^h \ge k_s (E^h)^T E^h$. Note that $p^T E = p^T L(y - \varpi) = 0$, i.e., $p_1 e_{1,1} + \cdots + p_N e_{N,1} = 0$, thus for $E \ne 0_N$, it is impossible that $\mathrm{sgn}(e_{i,1}) = 1$ (or $\mathrm{sgn}(e_{i,1}) = -1$) for all $i = 1, 2, \ldots, N$ due to the fact that $p_i > 0$ ([13]). Hence, $E \ne c1_N$ with c being a nonzero constant. Note that $\mathrm{sgn}(e_{i,1}) = \mathrm{sgn}(e_{i,1}^h)$ for $i = 1, \ldots, N$, it is readily established that $E^h \ne c1_N$. ∎

Lemma 9.8 *Suppose* $V : \mathbb{D} \to \mathbb{R}$ *is a continuous function satisfying:*

(i) V *is positive definite.*
(ii) There exist real numbers $\tilde{c} > 0$*,* $\alpha \in (0, 1)$ *and* $0 < \varsigma < \infty$*, and an open neighborhood* $\mathbb{U} \subseteq \mathbb{D}$ *of the origin such that*

$$\dot{V}(x,t) \le -\tilde{c}V(x,t)^{\alpha} + \varsigma, \;\; x \in \mathbb{U}/\{0\}. \tag{9.6}$$

Then there exists a finite time T^* (called the settling time) satisfying

$$T^* \leq \frac{V(x,0)^{1-\alpha}}{(1-\mu_1)\tilde{c}(1-\alpha)}, \quad 0 < \mu_1 < 1 \tag{9.7}$$

such that for $\forall t \geq T^*$, we have $V(x,t)$ is bounded by

$$V(x,t) < (\varsigma/(\mu_1\tilde{c}))^{\frac{1}{\alpha}}. \tag{9.8}$$

Proof Let $\Delta = \{x : V(x,t) < (\frac{\varsigma}{\mu_1\tilde{c}})^{\frac{1}{\alpha}}, 0 < \mu_1 < 1\}$. According to Theorem 5.2 in [14] for any $x \notin \Delta$ for all $t \in [0, t_x]$, it holds that $V(x,t) \geq (\varsigma/\mu_1\tilde{c})^{\frac{1}{\alpha}}$, i.e., $\varsigma \leq \mu_1\tilde{c}V(x,t)^\alpha$, for all $t \in [0, t_x]$. This fact, together with (9.6), implies that

$$\begin{aligned}\dot{V}(x,t) &\leq -\tilde{c}V(x,t)^\alpha + \mu_1\tilde{c}V(x,t)^\alpha \\ &= -(1-\mu_1)\tilde{c}V(x,t)^\alpha\end{aligned} \tag{9.9}$$

for all $t \in [0, t_x]$. Note that $V(x,t) \geq (\varsigma/\eta_1\tilde{c})^\alpha > 0$ for $t \in [0, t_x]$. According to Lemma 9.1 and (9.9), we know $t_x < \frac{V(x,0)^{1-\alpha}}{(1-\mu_1)\tilde{c}(1-\alpha)} = T^*$. Thus (9.8) holds for $\forall t \geq T^*$. ■

Lemma 9.9 *Consider a continuous function of the form*

$$\dot{\vartheta}(t) = -k_\vartheta\vartheta(t)^h + q\varsigma(t), \tag{9.10}$$

where $k_\vartheta > 0$, $0 < h = h_1/h_2 \leq 1$ ($h_1, h_2 > 0$ are positive odd integers), $q > 0$ is a constant, and $\varsigma(t) \geq 0$ is a scalar function. Then, for any given bounded initial condition $\vartheta(0) \geq 0$, we have $\vartheta(t) \geq 0$, $\forall t \geq 0$.

Proof The assertion is vacuously true if $\varsigma(t) \equiv 0$. In fact, in this case (9.10) becomes

$$\dot{\vartheta}(t) = -k_\vartheta\vartheta(t)^h. \tag{9.11}$$

By direct integration, we obtain the solution to (9.11) is

$$\begin{aligned}&\vartheta(t) = \left[|\vartheta(0)|^{1-h} - k_\vartheta(1-h)t\right]^{\frac{1}{1-h}}, \ t < \bar{T}, \vartheta(0) \neq 0, \\ &\vartheta(t) = 0, \qquad\qquad\qquad\qquad\qquad \text{otherwise}\end{aligned} \tag{9.12}$$

where $\bar{T} = \frac{|\vartheta(0)|^{1-h}}{k_\vartheta(1-h)}$, which implies that $\vartheta(t) \geq 0$ for $\forall t \geq 0$ and any given bounded initial condition $\vartheta(0) \geq 0$.

We now consider the case that $\varsigma(t)$ is not always equal to 0. If there exists some time constant $t_1 > 0$ such that $\varsigma(t_1) = 0$, then by continuity there exists a small neighborhood $[t_1, t_1 + \Delta t_1)$ such that $\vartheta(t) \geq 0$ on $[t_1, t_1 + \Delta t_1)$ ($\Delta t_1 > 0$) from the above analysis; if there exists some time constant $t_2 > 0$ such that $\varsigma(t_2) > 0$

and $\vartheta(t_2) > \left(\frac{q\varsigma(t_2)}{k_\vartheta}\right)^{\frac{1}{h}} > 0$, then by continuity there exists a small neighborhood $[t_2, t_2 + \Delta t_2)$ such that $\vartheta(t) > 0$ on $[t_2, t_2 + \Delta t_2)$ $(\Delta t_2 > 0)$; if there exists some time constant $t_3 > 0$ such that $\varsigma(t_3) > 0$ and $\vartheta(t_3) = \left(\frac{q\varsigma(t_3)}{k_\vartheta}\right)^{\frac{1}{h}} > 0$, then by (9.10) it holds that $\dot{\vartheta}(t_3) = -k_\vartheta \left[\left(\frac{q\varsigma(t_3)}{k_\vartheta}\right)^{\frac{1}{h}}\right]^h + q\varsigma(t_3) = 0$, and by continuity there exists a small neighborhood $[t_3, t_3 + \Delta t_3)$ such that $\vartheta(t) = \left(\frac{q\varsigma(t_3)}{k_\vartheta}\right)^{\frac{1}{h}} > 0$ on $[t_3, t_3 + \Delta t_3)$ $(\Delta t_3 > 0)$; and if there exists some time constant $t_4 > 0$ such that $\varsigma(t_4) > 0$ and $0 < \vartheta(t_4) < \left(\frac{q\varsigma(t_4)}{k_\vartheta}\right)^{\frac{1}{h}}$, it follows from (9.10) that $\dot{\vartheta}(t_4) > -k_\vartheta \left[\left(\frac{q\varsigma(t_4)}{k_\vartheta}\right)^{\frac{1}{h}}\right]^h + q\varsigma(t_4) = 0$, and then there exists a small neighborhood $[t_4, t_4 + \Delta t_4)$ such that $\vartheta(t) > \vartheta(t_4) > 0$ on $[t_4, t_4 + \Delta t_4)$ $(\Delta t_4 > 0)$ by continuity. By global continuity of the function $\vartheta(t)$, we have $\vartheta(t) \geq 0$ $\forall t \geq 0$ for any given bounded initial condition $\vartheta(0) \geq 0$ in this case. ■

Lemma 9.10 *Let $\tilde{\zeta} = \zeta - \hat{\zeta}$. It then holds that*

$$\begin{aligned}\tilde{\zeta}\hat{\zeta}^{\frac{3s-1}{1+s}} \leq & \frac{1+s}{4s}\Bigg[\left(2^{\frac{8s(s-1)}{(1+s)^2}} - 2^{\frac{2(s-1)}{1+s}}\right)\tilde{\zeta}^{\frac{4s}{1+s}} + \Bigg(1 - 2^{\frac{2s-2}{1+s}} \\ & + \frac{3s-1}{4s} + \frac{(1+s)2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s}\Bigg)\zeta^{\frac{4s}{1+s}}\Bigg],\end{aligned} \tag{9.13}$$

where $s = (4ln - 1)/(4ln + 1)$, $l \in Z_+$.

Proof Upon using Lemma 9.6, it is deduced from Lemma 9.4 that

$$\begin{aligned}\tilde{\zeta}\left(\zeta - \tilde{\zeta}\right)^{\frac{3s-1}{1+s}} \leq & \frac{1+s}{4s}\left[\zeta^{\frac{4s}{1+s}} - \left(\zeta - \tilde{\zeta}\right)^{\frac{4s}{1+s}}\right] \\ = & \frac{1+s}{4s}\left[\zeta^{\frac{4s}{1+s}} - \left|\zeta - \tilde{\zeta}\right|\left|\zeta - \tilde{\zeta}\right|^{\frac{3s-1}{1+s}}\right] \\ \leq & \frac{1+s}{4s}\left[\zeta^{\frac{4s}{1+s}} - 2^{\frac{2s-2}{1+s}}\left|\zeta - \tilde{\zeta}\right|\left|\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}}\right|\right].\end{aligned} \tag{9.14}$$

Note that $s = \frac{4ln-1}{4ln+1}$, straightforward derivation yields that $\frac{3s-1}{1+s} = \frac{2ln-1}{2ln}$, and therefore, $\left|\zeta - \tilde{\zeta}\right|\left|\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}}\right| = \left(\zeta - \tilde{\zeta}\right)\left(\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}}\right)$. By using Lemma 9.2, it then holds that

$$
\begin{aligned}
&\tilde{\zeta}\left(\zeta-\tilde{\zeta}\right)^{\frac{3s-1}{1+s}}\\
\leq&\frac{1+s}{4s}\left[\zeta^{\frac{4s}{1+s}}-2^{\frac{2s-2}{1+s}}\zeta^{\frac{4s}{1+s}}+2^{\frac{2s-2}{1+s}}\tilde{\zeta}\zeta^{\frac{3s-1}{1+s}}+2^{\frac{2s-2}{1+s}}\zeta\tilde{\zeta}^{\frac{3s-1}{1+s}}-2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}}\right]\\
\leq&\frac{1+s}{4s}\Bigg[\zeta^{\frac{4s}{1+s}}-2^{\frac{2s-2}{1+s}}\zeta^{\frac{4s}{1+s}}+\frac{(1+s)\left(2^{\frac{2s-2}{1+s}}\tilde{\zeta}\right)^{\frac{4s}{1+s}}}{4s}+\frac{(3s-1)\zeta^{\frac{4s}{1+s}}}{4s}\\
&+\frac{(1+s)\left(2^{-4(\frac{s-1}{1+s})^2}\zeta\right)^{\frac{4s}{1+s}}}{4s}+\frac{(3s-1)\left(2^{\frac{2s-2}{1+s}}\tilde{\zeta}\right)^{\frac{4s}{1+s}}}{4s}-2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}}\Bigg]\\
\leq&\frac{1+s}{4s}\left[\left(2^{\frac{8s(s-1)}{(1+s)^2}}-2^{\frac{2(s-1)}{1+s}}\right)\tilde{\zeta}^{\frac{4s}{1+s}}+\left(1-2^{\frac{2s-2}{1+s}}+\frac{3s-1}{4s}+\frac{(1+s)2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s}\right)\zeta^{\frac{4s}{1+s}}\right].
\end{aligned}
\tag{9.15}
$$

9.3 Inductive Controller Design

The focus in this section is on deriving the finite-time output consensus control scheme for MAS with dynamics (9.1). We first introduce the neighborhood error and local virtual error and then conduct the inductive control design and stability analysis.

To develop the distributed finite-time control scheme for the high-order MAS on graph, we define the following local position error $e_{i,1}$ and local virtual errors $e_{i,m}$ $(m = 2, \ldots, n)$ as

$$
\begin{aligned}
e_{i,1} &= \sum_{j\in\mathcal{N}_i} a_{ij}(y_i-\varpi_i-y_j+\varpi_j),\quad i,j\in\{1,\ldots,N\}\\
e_{i,m} &= x_{i,m}^{1/q_m}-x_{i,m}^{*1/q_m},\quad i\in\{1,\ldots,N\},\quad m=2,\ldots,n
\end{aligned}
\tag{9.16}
$$

where ϖ_i are constants denoting the final consensus configuration such that $y_i - y_j = \varpi_i - \varpi_j$ $(i, j \in \{1, \ldots, N\})$; $q_m = (4ln + 3 - 2m)/(4ln + 1) > 0$ $(l \in Z_+)$; $x^*_{i,m}$ $(m = 2, \ldots, n)$ is the virtual control given by $x^*_{i,m} = -\beta_{i,m-1}e^{q_m}_{i,m-1}$, where $\beta_{i,m-1}$ is positive constant to be designed. Denote $y = [y_1, \ldots, y_N]^T \in \mathbb{R}^N$, $\varpi = [\varpi_1, \ldots, \varpi_N]^T \in \mathbb{R}^N$, and $E = [e_{1,1}, \ldots, e_{N,1}]^T \in \mathbb{R}^N$, then it holds that $E = L(y - \varpi)$.

It is noted that unlike the existing method [15] in defining the fraction power q_m, here we introduce the integer l into q_m. Such treatment leads to the fraction power feedback control capable of adjusting the control precision and finite convergence time, as shown in sequel.

With the intermediate results and lemmas as represented in Sect. 9.2, we are ready to address the finite-time adaptive consensus control design for the high-order non-affine MAS (9.1).

The design procedure consists of n steps. At each of the first m ($m = 1, 2, \ldots, n-1$) step, a virtual control $x^*_{i,m+1}$ is designed, and finally, the true control law u_i is derived at step n. In the following, we denote by $x_k = [x_{1,k}, \ldots, x_{N,k}]$ for $k = 1, \ldots, n$.

Step 1: Construct the first part of Lyapunov function candidate as

$$V_1(x_1) = \frac{1}{(1+s)k_\mathrm{s}}(E^{\frac{1+s}{2}})^T P E^{\frac{1+s}{2}}, \tag{9.17}$$

where s and k_s are defined as in Lemma 9.7. With simple computation, it is obtained that

$$\begin{aligned}\dot{V}_1(x_1) =& \frac{1}{k_\mathrm{s}}(E^s)^T P\dot{E} = \frac{1}{k_\mathrm{s}}(E^s)^T PLx_2 \\ =& \frac{1}{k_\mathrm{s}}(E^s)^T PLx_2^* + \frac{1}{k_\mathrm{s}}(E^s)^T PL(x_2 - x_2^*),\end{aligned} \tag{9.18}$$

where $x_2 = [x_{1,2}, \ldots, x_{N,2}]^T \in \mathbb{R}^N$. By introducing the virtual control $x_2^* = -\beta_1 E^s$ ($\beta_1 > 0$ is design constant) into (9.18), we have

$$\begin{aligned}\dot{V}_1(x_1) =& -\frac{\beta_1}{k_\mathrm{s}}(E^s)^T PLE^s + \frac{1}{k_\mathrm{s}}(E^s)^T PL(x_2 - x_2^*) \\ =& -\frac{\beta_1}{k_\mathrm{s}}(E^s)^T \Xi E^s + \frac{1}{k_\mathrm{s}}\sum_{i=1}^N (x_{i,2} - x_{i,2}^*)\sum_{j=1}^N \ell_{ji}(e_{j,1})^s\end{aligned} \tag{9.19}$$

where $\Xi = 1/2(PL + L^T P)$ defined as in Lemma 9.7, and ℓ_{ji} is the (j, i)th element of (PL). Let $\ell_{\max} = \max_{j,i\in\{1,\ldots,N\}}|\ell_{ji}|$, a straightforward derivation by using Lemmas 9.2 and 9.4 yields

$$\begin{aligned}&\frac{1}{k_\mathrm{s}}\sum_{i=1}^N (x_{i,2} - x_{i,2}^*)\sum_{j=1}^N \ell_{ji}(e_{j,1})^s \le \frac{1}{k_\mathrm{s}}\sum_{i=1}^N 2^{1-s}|e_{i,2}|^s \sum_{j=1}^N |\ell_{ji}||e_{j,1}|^s \\ \le& 2^{1-s}\ell_{\max}\frac{1}{k_\mathrm{s}}\sum_{i=1}^N |e_{i,2}|^s \sum_{j=1}^N |e_{j,1}|^s \le \frac{1}{2N}\left(\sum_{i=1}^N |e_{i,1}|^s\right)^2 + \tilde{C}_2\left(\sum_{i=1}^N |e_{i,2}|^s\right)^2 \\ \le& \frac{1}{2}\sum_{i=1}^N (e_{i,1})^{2s} + C_2\sum_{i=1}^N (e_{i,2})^{2s}\end{aligned} \tag{9.20}$$

in which the fact $\left(\sum\limits_{i=1}^N x_i\right)^2 \le N\sum\limits_{i=1}^N x_i^2$ has been used, and $C_2 = N\tilde{C}_2$ is a fixed constant.

In light of Lemma 9.7 and (9.20), it is deduced from (9.19) that

$$\dot{V}_1(x_1) \leq -\beta_1 \sum_{i=1}^{N} (e_{i,1})^{2s} + \frac{1}{2} \sum_{i=1}^{N} (e_{i,1})^{2s} + C_2 \sum_{i=1}^{N} (e_{i,2})^{2s}. \tag{9.21}$$

Step 2: Construct the second part of Lyapunov function candidate as

$$V_2(x_1, x_2) = V_1(x_1) + W_2(x_1, x_2), \tag{9.22}$$

with

$$W_2(x_1, x_2) = \sum_{i=1}^{N} \int_{x_{i,2}^*}^{x_{i,2}} \left(\tau^{\frac{1}{q_2}} - x_{i,2}^{*\frac{1}{q_2}} \right)^{1+s-q_2} d\tau. \tag{9.23}$$

It is obvious that $W_2(\cdots)$ is C^1, then $V_2(\cdot)$ is C^1, and moreover, $V_2(\cdot)$ is positive definite and $V_2(\cdot) \leq \bar{k}_b \sum_{i=1}^{N} (e_{i,1}^{1+s} + e_{i,2}^{1+s})$, where $\bar{k}_b = \max \left\{ \frac{1}{(1+s)k_s}, 2 \right\}$. Note that

$$\begin{aligned}
\frac{\partial W_2}{\partial x_{i,2}} &= e_{i,2}^{1+s-q_2} = e_{i,2}, \\
\frac{\partial W_2}{\partial x_{j,1}} &= -(1+s-q_2) \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \int_{x_{i,2}^*}^{x_{i,2}} \left(\tau^{\frac{1}{q_2}} - x_{i,2}^{*\frac{1}{q_2}} \right)^{s-q_2} d\tau \\
&= -\frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \left(x_{i,2} - x_{i,2}^* \right), \quad j \in \bar{\mathcal{N}_i} = \mathcal{N}_i \cup i.
\end{aligned} \tag{9.24}$$

The derivative of $W_2(t)$ is thus derived as

$$\begin{aligned}
\dot{W}_2(\cdot) =& \sum_{i=1}^{N} \left[e_{i,2} \dot{x}_{i,2} - \left(x_{i,2} - x_{i,2}^* \right) \sum_{j \in \bar{\mathcal{N}_i}} \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \dot{x}_{j,1} \right] \\
\leq & \sum_{i=1}^{N} \left[e_{i,2}(x_{i,3} - x_{i,3}^*) + e_{i,2} x_{i,3}^* + 2^{1-q_2} \beta_1^{1/q_2} \cdot |e_{i,2}|^{q_2} | \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2})| \right].
\end{aligned} \tag{9.25}$$

According to Lemma 9.5, we have

$$|x_{i,2}| \leq \left| e_{i,2} + x_{i,2}^{*1/q_2} \right|^{q_2} \leq |e_{i,2}|^{q_2} + \beta_1 |e_{i,1}|^{q_2}. \tag{9.26}$$

Denote by $r_1 = \max_{\forall i \in J} \{ \sum_{j \in \mathcal{N}_i} a_{ij} \}$ and $r_2 = \max_{\forall i,j \in J} \{a_{ij}\}$. It then follows from (9.26) that

$$
\begin{aligned}
&2^{1-q_2}\beta_1^{1/q_2}\sum_{i=1}^{N}|e_{i,2}|^{q_2}\left|\sum_{j\in\mathcal{N}_i}a_{ij}(x_{i,2}-x_{j,2})\right| \\
\leq &2^{1-q_2}\beta_1^{1/q_2}\sum_{i=1}^{N}|e_{i,2}|^{q_2}\left(r_1|x_{i,2}|+r_2\sum_{j\in\mathcal{N}_i}|x_{j,2}|\right) \\
\leq &2^{1-q_2}\beta_1^{1/q_2}\sum_{i=1}^{N}\bigg[r_1|e_{i,2}|^{2q_2}+r_1\beta_1|e_{i,2}|^{q_2}|e_{i,1}|^{q_2} \\
&+r_2|e_{i,2}|^{q_2}\sum_{j\in\mathcal{N}_i}(|e_{j,2}|^{q_2}+\beta_1|e_{j,1}|^{q_2})\bigg]. \qquad (9.27)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&2^{1-q_2}\beta_1^{1+1/q_2}r_1|e_{i,2}|^{q_2}|e_{i,1}|^{q_2}\leq 2^{2(1-q_2)}\beta_1^{2(1+1/q_2)}r_1^2|e_{i,2}|^{2q_2}+\frac{1}{4}|e_{i,1}|^{2q_2}, \\
&2^{1-q_2}\beta_1^{1/q_2}r_2|e_{i,2}|^{q_2}\sum_{j\in\mathcal{N}_i}|e_{j,2}|^{q_2}\leq 2^{-q_2}\beta_1^{1/q_2}r_2\sum_{j\in\mathcal{N}_i}(|e_{i,2}|^{2q_2}+|e_{j,2}|^{2q_2}), \\
&2^{1-q_2}\beta_1^{1+1/q_2}r_2|e_{i,2}|^{q_2}\sum_{j\in\mathcal{N}_i}|e_{j,1}|^{q_2} \\
\leq &\sum_{j\in\mathcal{N}_i}(2^{2(1-q_2)}\beta_1^{2(1+1/q_2)}r_2^2c_{\mathcal{N}}|e_{i,2}|^{2q_2}+\frac{1}{4c_{\mathcal{N}}}|e_{j,1}|^{2q_2}), \qquad (9.28)
\end{aligned}
$$

in which $c_{\mathcal{N}}=\max_{i\in\{1,\ldots,N\}}\{\dim\{\mathcal{N}_i\}\}$ with $\dim\{\mathcal{N}_i\}$ being the number of the elements in $\mathcal{N}_i$.

By substituting (9.28) into (9.27) we have

$$
2^{1-q_2}\beta_1^{1/q_2}\sum_{i=1}^{N}|e_{i,2}|^{q_2}\cdot\left|\sum_{j\in\mathcal{N}_i}a_{ij}(x_{i,2}-x_{j,2})\right|\leq\frac{1}{2}\sum_{i=1}^{N}e_{i,1}^{2q_2}+\bar{C}_2\sum_{i=1}^{N}e_{i,2}^{2q_2} \qquad (9.29)
$$

with $\bar{C}_2=2^{1-q_2}\beta_1^{1/q_2}r_1+2^{2(1-q_2)}\beta_1^{2(1+1/q_2)}r_1^2+2^{1-q_2}\beta_1^{1/q_2}r_2c_{\mathcal{N}}+2^{2(1-q_2)}\beta_1^{2(1+1/q_2)}r_2^2c_{\mathcal{N}}^2$.

By combining (9.22), (9.21), (9.25), and (9.29), we see that

$$
\begin{aligned}
\dot{V}_2\leq&-(\beta_1-1)\sum_{i=1}^{N}(e_{i,1})^{2s}+(C_2+\bar{C}_2)\sum_{i=1}^{N}(e_{i,2})^{2s} \\
&+\sum_{i=1}^{N}\left[e_{i,2}(x_{i,3}-x_{i,3}^*)+e_{i,2}x_{i,3}^*\right]. \qquad (9.30)
\end{aligned}
$$

Note that $q_3 = q_2 - 2/(4ln+1)$, and then $1 + q_3 = 2q_2 = 2s$. Thus by choosing $\beta_1 > n - 1 + k_c$ with $k_c > 0$ being a design parameter and the virtual control $x^*_{i,3} = -\beta_{i,2} e_{i,2}^{q_3}$ with $\beta_{i,2} \geq n - 2 + k_c + C_2 + \bar{C}_2$, we get

$$\dot{V}_2 \leq -(n-2+k_c) \sum_{i=1}^{N} \sum_{m=1}^{2} (e_{i,m})^{2s} + \sum_{i=1}^{N} e_{i,2}(x_{i,3} - x^*_{i,3}). \tag{9.31}$$

Step k ($3 \leq k \leq n-1$): To proceed, we conduct the proof by using an inductive argument. Suppose at step $k-1$, there exists a C^1 Lyapunov function V_{k-1} $(x_1, \ldots, x_{k-1})$, which is positive definite and satisfies

$$V_{k-1}(\cdot) \leq \bar{k}_b \sum_{i=1}^{N} (e_{i,1}^{1+s} + \cdots + e_{i,k-1}^{1+s}) \tag{9.32}$$

such that

$$\dot{V}_{k-1}(\cdot) \leq -(n-k+1+k_c) \sum_{i=1}^{N} \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^{N} e_{i,k-1}^{1+s-q_{k-1}} (x_{i,k} - x^*_{i,k}). \tag{9.33}$$

We claim that (9.32) and (9.33) also hold at step k. To prove this, consider

$$V_k(x_1, \ldots, x_k) = V_{k-1}(\cdot) + W_k(x_1, \ldots, x_k), \tag{9.34}$$

with

$$W_k(x_1, \ldots, x_k) = \sum_{i=1}^{N} \int_{x^*_{i,k}}^{x_{i,k}} \left(\tau^{\frac{1}{q_k}} - x_{i,k}^{*\frac{1}{q_k}} \right)^{1+s-q_k} d\tau. \tag{9.35}$$

It is not hard to check that $W_k(\cdot)$ is C^1, and moreover,

$$\begin{aligned} \frac{\partial W_k}{\partial x_{i,k}} &= e_{i,k}^{1+s-q_k}, \\ \frac{\partial W_k}{\partial x_{i,m}} &= -(1+s-q_k) \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \int_{x^*_{i,k}}^{x_{i,k}} \left(\tau^{\frac{1}{q_k}} - x_{i,k}^{*\frac{1}{q_k}} \right)^{s-q_k} d\tau, \quad m = 1, \ldots, k-1, \end{aligned} \tag{9.36}$$

Therefore, $V_k(\cdot)$ is C^1, positive definite and satisfies $V_k(\cdot) \leq \bar{k}_b \sum_{i=1}^{N} (e_{i,1}^{1+s} + \cdots + e_{i,k}^{1+s})$.

By recalling (9.33), we derive the derivative of $V_k(\cdot)$ as

$$
\begin{aligned}
\dot{V}_k(\cdot) \leq & -(n-k+1+k_{\mathrm{c}}) \sum_{i=1}^{N} \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^{N} e_{i,k-1}^{1+s-q_{k-1}} \left(x_{i,k} - x_{i,k}^{*}\right) \\
& + \sum_{i=1}^{N} e_{i,k}^{1+s-q_k} x_{i,k+1} + \sum_{i=1}^{N} \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right).
\end{aligned} \tag{9.37}
$$

To continue, we examine the second term and the fourth term on the right-hand side of (9.37). First, according to Lemma 9.4, it holds that

$$
\left| e_{i,k-1}^{1+s-q_{k-1}} (x_{i,k} - x_{i,k}^{*}) \right| \leq 2^{1-q_k} |e_{i,k-1}|^{1+s-q_{k-1}} |e_{i,k}|^{q_k} \leq \frac{e_{i,k-1}^{2s}}{2} + C_{i,k} e_{i,k}^{2s} \tag{9.38}
$$

with $C_{i,k}$ being a constant. As for the fourth term on the right-hand side of (9.37), we have

$$
\begin{aligned}
& \left| \sum_{i=1}^{N} \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \\
\leq & \sum_{i=1}^{N} (1+s-q_k) 2^{1-q_k} |e_{i,k}|^{s} \cdot \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right|.
\end{aligned} \tag{9.39}
$$

Now we show that

$$
\sum_{i=1}^{N} \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \leq \sum_{i=1}^{N} \left(\sum_{m=1}^{k} \rho_{ki,m} |e_{i,m}|^{s} \right), \tag{9.40}
$$

for a constant $\rho_{ki,m} \geq 0$. This is done by inductive argument. First of all, it is deduced from (9.24)–(9.26) that

$$
\left| \sum_{i=1}^{N} \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \leq \sum_{i=1}^{N} \beta_1^{1/q_2} \left| \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2}) \right| \leq \sum_{i=1}^{N} \left(\sum_{m=1}^{2} \rho_{2i,m} |e_{i,m}|^{s} \right), \tag{9.41}
$$

where $\rho_{2i,m} \geq 0$ is a constant. Assume that for $k-1$, it holds,

$$
\sum_{i=1}^{N} \left| \sum_{m=2}^{k-2} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \leq \sum_{i=1}^{N} \left(\sum_{m=1}^{k-1} \rho_{(k-1)i,m} |e_{i,m}|^{s} \right) \tag{9.42}
$$

with $\rho_{(k-1)i,m} \geq 0$ $(m = 1, \ldots, k-1)$ being constants. By recalling that $e_{i,m} = x_{i,m}^{1/q_m} - x_{i,m}^{*1/q_m}$ and $x_{i,m}^{*} = -\beta_{i,m-1} e_{i,m-1}^{q_m}$, it thus holds that

$$x_{i,m}^{*1/q_m} = -\beta_{i,m-1}^{1/q_m} e_{i,m-1}, \tag{9.43}$$

$$|x_{i,m}| \leq \left| e_{i,m} + x_{i,m}^{*1/q_m} \right|^{q_m} \leq |e_{i,m}|^{q_m} + \beta_{i,m-1} |e_{i,m-1}|^{q_m}, \tag{9.44}$$

which, together with (9.42), yields that

$$\begin{aligned}
&\sum_{i=1}^{N} \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}_i}} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \\
=& \sum_{i=1}^{N} \left| -\beta_{i,k-1}^{1/q_k} \left(\sum_{m=2}^{k-1} \frac{\partial e_{i,k-1}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}_i}} \frac{\partial e_{i,k-1}}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \\
\leq& \sum_{i=1}^{N} \beta_{i,k-1}^{1/q_k} \left| \frac{x_{i,k-1}^{\frac{1}{q_{k-1}}-1}}{q_{k-1}} \dot{x}_{i,k-1} - \sum_{m=2}^{k-2} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{i,m}} \dot{x}_{i,m} - \sum_{j \in \bar{\mathcal{N}_i}} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \\
\leq& \sum_{i=1}^{N} \beta_{i,k-1}^{1/q_k} \left[\frac{1}{q_{k-1}} \cdot \left(e_{i,k-1}^{1-q_{k-1}} + \beta_{i,k-2}^{\frac{1}{q_{k-1}}-1} e_{i,k-2}^{1-q_{k-1}} \right) \right. \\
&\left. \cdot \left(|e_{i,k}|^{q_k} + |e_{i,k-1}|^{q_k} \beta_{i,k-1} \right) + \sum_{m=1}^{k-1} \rho_{(k-1)i,m} |e_{i,m}|^{s} \right] \\
\leq& \sum_{i=1}^{N} \sum_{m=1}^{k} \rho_{ki,m} |e_{i,m}|^{s}.
\end{aligned} \tag{9.45}$$

Therefore,

$$\begin{aligned}
&\left| \sum_{i=1}^{N} \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \bar{\mathcal{N}_i}} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \\
\leq& \sum_{i=1}^{N} (1 + s - q_k) 2^{1-q_k} |e_{i,k}|^{s} \left(\sum_{m=1}^{k} \rho_{ki,m} |e_{i,m}|^{s} \right) \\
\leq& \frac{1}{2} \sum_{i=1}^{N} \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^{N} \bar{C}_{i,k} e_{i,k}^{2s}
\end{aligned} \tag{9.46}$$

with $\bar{C}_{i,k} > 0$ being a constant.

By substituting (9.38) and (9.46) into (9.37), we arrive at

$$\dot{V}_k(\cdot) \leq -(n-k+k_{\mathrm{c}}) \sum_{i=1}^{N} \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^{N} \left(C_{i,k} + \bar{C}_{i,k}\right) e_{i,k}^{2s} + \sum_{i=1}^{N} e_{i,k}^{1+s-q_k} \left(x_{i,k+1} - x_{i,k+1}^{*}\right) + \sum_{i=1}^{N} e_{i,k}^{1+s-q_k} x_{i,k+1}^{*}. \tag{9.47}$$

By introducing the virtual control $x_{i,k+1}^{*} = -\beta_{i,k} e_{i,k}^{q_{k+1}}$ with $\beta_{i,k} \geq n-k+k_{\mathrm{c}} + C_{i,k} + \bar{C}_{i,k} > 0$, we get from (9.47) that

$$\dot{V}_k(\cdot) \leq -(n-k+k_{\mathrm{c}}) \sum_{i=1}^{N} \sum_{m=1}^{k} e_{i,m}^{2s} + \sum_{i=1}^{N} e_{i,k}^{1+s-q_k} \left(x_{i,k+1} - x_{i,k+1}^{*}\right). \tag{9.48}$$

Step n: Consider the nth part of Lyapunov function candidate

$$V_n(x_1, \ldots, x_n) = V_{n-1}(\cdot) + W_n(x_1, \ldots, x_n) \tag{9.49}$$

with

$$W_n(x_1, \ldots, x_n) = \sum_{i=1}^{N} \int_{x_{i,n}^{*}}^{x_{i,n}} \left(\tau^{\frac{1}{q_n}} - x_{i,n}^{*\frac{1}{q_n}}\right)^{1+s-q_n} d\tau. \tag{9.50}$$

Clearly, $V_n(\cdot)$ so defined is C^1, positive definite and satisfies $V_n(\cdot) \leq \bar{k}_b \sum_{i=1}^{N} (e_{i,1}^{1+s} + \cdots + e_{i,n}^{1+s})$.

From the above inductive argument, one can conclude that

$$\dot{V}_n(\cdot) \leq -k_{\mathrm{c}} \sum_{i=1}^{N} \sum_{m=1}^{n-1} e_{i,m}^{2s} + \sum_{i=1}^{N} e_{i,n}^{1+s-q_n} \dot{x}_{i,n} + \sum_{i=1}^{N} \left(C_{i,n} + \bar{C}_{i,n}\right) e_{i,n}^{2s}. \tag{9.51}$$

where $C_{i,n}$ and $\bar{C}_{i,n}$ are finite positive constants.

The actual control u_i is designed as

$$u_i = -\beta_{i,n} e_{i,n}^{q_{n+1}} - \frac{\hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n}^{1+s-q_n} \tag{9.52}$$

with the updated law

$$\dot{\hat{\zeta}}_i = -\sigma_i \gamma_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} + \frac{\sigma_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)}, \tag{9.53}$$

where $\hat{\zeta}_i$ is the estimation of ζ_i (here ζ_i is a virtual parameter to be defined later), $\psi_i(\cdot)$ is a scalar and readily computable function, $\beta_{i,n}$, σ_i, γ_i, and η_i are positive design parameters chosen arbitrarily by the designer. Now we are ready to present the main control result in the next section.

9.4 The Finite-Time Stability Analysis for the Uncertain High-Order MAS

In this section, we give the main control result and conduct the finite-time stability analysis for the uncertain non-affine multi-agent systems (9.1).

Theorem 9.1 *Consider the networked high-order non-affine uncertain MAS (9.1) under Assumption 9.1. If the distributed adaptive control scheme (9.52)–(9.53) with the virtual control $x_{i,m}^*$ ($m = 1, \ldots, n$) is applied, then for any initial conditions satisfying $\hat{\zeta}_i(0) \geq 0$ and $V(0) \leq \mu$ ($\mu > 0$ is a finite constant), the finite-time leaderless consensus for system (9.1) is achieved in that*
(1) the neighborhood error E converges to a small residual set Θ_0, namely

$$\Theta_0 = \left\{ \|E\| \leq \left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}} \right\}, \tag{9.54}$$

and the parameter estimate error $\tilde{\zeta}$ converges to a small region Θ_1, given by

$$\Theta_1 = \left\{ \|\tilde{\zeta}\| \leq \left[\left(\frac{2\sigma_{\max} k_b}{\underline{g}} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}} \right\} \tag{9.55}$$

in a finite time T^* that satisfies

$$T^* \leq \frac{V(0)^{1-\frac{2s}{1+s}}}{(1-\mu_1)\tilde{c}(1-\frac{2s}{1+s})}, \tag{9.56}$$

where

$$\begin{aligned} & k_a = \min\left\{ k_c, \frac{1+s}{4s} \underline{g} \gamma_i \left(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} \right) \right\}, \quad k_b = \max\left\{ \bar{k}_b, \frac{\underline{g}}{2\sigma_i} \right\}, \\ & \varsigma = \sum_{i=1}^N \frac{\eta_i^2}{4} + \sum_{i=1}^N \frac{1+s}{4s} \underline{g} \gamma_i \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta_i^{\frac{4s}{1+s}} \\ & 0 < \mu_1 < 1, \quad 0 < \mu_2 \leq 1, \quad \tilde{\zeta} = [\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N]^T, \quad \sigma_{\max} = \max\{\sigma_1, \ldots, \sigma_N\}. \end{aligned} \tag{9.57}$$

(2) all of the signals in the controlled system are semi-globally uniformly ultimately bounded.

Proof Choose the Lyapunov function candidate as

$$V = V_n + \sum_{i=1}^{N} \frac{\underline{g}}{2\sigma_i} \tilde{\zeta}_i^2. \tag{9.58}$$

Taking the derivative of V along (9.4) yields that

$$\begin{aligned} \dot{V}(\cdot) \leq & -k_c \sum_{i=1}^{N} \sum_{m=1}^{n-1} e_{i,m}^{2s} + \sum_{i=1}^{N} e_{i,n}^{1+s-q_n} \left(g_i u_i + L_{fi} \right) \\ & + \sum_{i=1}^{N} \left(C_{i,n} + \bar{C}_{i,n} \right) e_{i,n}^{2s} + \sum_{i=1}^{N} \tilde{\zeta}_i \left(-\frac{\underline{g}}{\sigma_i} \dot{\hat{\zeta}}_i \right). \end{aligned} \tag{9.59}$$

According to Assumption 9.1 and Young's inequality, we have, for any $\eta_i > 0$, that

$$e_{i,n}^{1+s-q_n} L_{fi} \leq \alpha_i \psi_i |e_{i,n}|^{1+s-q_n} \leq \frac{\alpha_i^2}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} + \frac{\eta_i^2}{4}. \tag{9.60}$$

Let $\zeta_i = \underline{g}^{-1} \alpha_i^2$. By using (9.52) and (9.60), it is deduced from (9.59) that

$$\begin{aligned} \dot{V}(\cdot) \leq & -k_c \sum_{i=1}^{N} \sum_{m=1}^{n-1} e_{i,m}^{2s} - \sum_{i=1}^{N} \left(g_i \beta_{i,n} - C_{i,n} - \bar{C}_{i,n} \right) e_{i,n}^{2s} \\ & + \sum_{i=1}^{N} \left(-\frac{g_i \hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} + \frac{\underline{g} \zeta_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} \right) \\ & + \sum_{i=1}^{N} \frac{\eta_i^2}{4} + \sum_{i=1}^{N} \tilde{\zeta}_i \left(-\frac{\underline{g}}{\sigma_i} \dot{\hat{\zeta}}_i \right). \end{aligned} \tag{9.61}$$

According to Lemma 9.9 and (9.53), we see that $\hat{\zeta}_i(t) \geq 0$ for any given initial estimate $\hat{\zeta}_i(0) \geq 0$. If $\beta_{i,n}$ is chosen such that $\beta_{i,n} \underline{g} - C_{i,n} - \bar{C}_{i,n} > k_c$, it thus follows from (9.61) that

$$\begin{aligned} \dot{V}(\cdot) \leq & -k_c \sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{2s} + \sum_{i=1}^{N} \frac{\underline{g}}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} (\zeta_i - \hat{\zeta}_i) \\ & + \sum_{i=1}^{N} \frac{\eta_i^2}{4} + \sum_{i=1}^{N} \tilde{\zeta}_i (-\frac{\underline{g}}{\sigma_i} \dot{\hat{\zeta}}_i). \end{aligned} \tag{9.62}$$

By applying the adaptive law for $\hat{\zeta}_i$ given in (9.53), one has

$$\dot{V}(\cdot) \leq -k_{\mathrm{c}} \sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{2s} + \sum_{i=1}^{N} \underline{g} \gamma_i \tilde{\zeta}_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} + \sum_{i=1}^{N} \frac{\eta_i^2}{4}. \tag{9.63}$$

According to Lemma 9.10, one gets

$$\begin{aligned} \tilde{\zeta}_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} \leq & -\frac{1+s}{4s} \left(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} \right) \tilde{\zeta}_i^{\frac{4s}{1+s}} \\ & + \frac{1+s}{4s} \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s) 2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta_i^{\frac{4s}{1+s}}, \end{aligned} \tag{9.64}$$

where $2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} > 0$. Let

$$\varsigma = \sum_{i=1}^{N} \frac{\eta_i^2}{4} + \sum_{i=1}^{N} \frac{1+s}{4s} \underline{g} \gamma_i \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s) 2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta_i^{\frac{4s}{1+s}}. \tag{9.65}$$

By inserting (9.64) into (9.63), it then holds that

$$\dot{V}(\cdot) \leq -k_{\mathrm{c}} \sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{2s} - \sum_{i=1}^{N} \frac{1+s}{4s} \underline{g} \gamma_i \left(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} \right) \tilde{\zeta}_i^{\frac{4s}{1+s}} + \varsigma. \tag{9.66}$$

By denoting that

$$k_a = \min_{i \in \{1,\dots,N\}} \left\{ k_{\mathrm{c}}, \frac{1+s}{4s} \underline{g} \gamma_i \left(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} \right) \right\}, \tag{9.67}$$

(9.66) can be expressed as

$$\dot{V}(\cdot) \leq -k_a \left[\sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{2s} + \sum_{i=1}^{N} \left(\tilde{\zeta}_i^{\frac{2}{1+s}} \right)^{2s} \right] + \varsigma. \tag{9.68}$$

In the sequel, we prove that there exists a finite time $T^* > 0$ and a bounded constant c_V such that $V(t) < c_V$ when $t \geq T^*$.

Let $k_b = \max_{i \in \{1,\dots,N\}} \left\{ \bar{k}_b, \frac{g}{2\sigma_i} \right\}$, and thus,

$$V(\cdot) \leq k_b \left[\sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{1+s} + \sum_{i=1}^{N} \left(\tilde{\zeta}_i^{\frac{2}{1+s}} \right)^{1+s} \right] \tag{9.69}$$

which further implies, upon using Lemma 9.5, that

$$V^{\frac{2s}{1+s}}(\cdot) \leq k_b^{\frac{2s}{1+s}} \left(\sum_{i=1}^{N} \sum_{m=1}^{n} e_{i,m}^{2s} + \sum_{i=1}^{N} \tilde{\zeta}_i^{\frac{4s}{1+s}} \right). \tag{9.70}$$

Let $\tilde{c} = \frac{\mu_2 k_a}{k_b^{\frac{2s}{1+s}}}$ $(0 < \mu_2 \leq 1)$. It is straightforward from (9.68) and (9.70) that

$$\dot{V}(\cdot) \leq -\tilde{c} V^{\frac{2s}{1+s}}(\cdot) + \varsigma. \tag{9.71}$$

Upon using Lemma 9.8, it is seen that there exists a finite time T^* satisfying

$$T^* \leq \frac{V(0)^{1-\frac{2s}{1+s}}}{(1-\mu_1)\tilde{c}(1-\frac{2s}{1+s})}, \tag{9.72}$$

such that for $\forall t \geq T^*$,

$$V(\cdot) < \left(\frac{\varsigma}{\mu_1 \tilde{c}} \right)^{\frac{1+s}{2s}} = k_b \left(\frac{\varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1+s}{2s}} = c_V. \tag{9.73}$$

Next, we examine the steady-state errors of all agents. Note that

$$\|E\| \leq [(1+s)k_s V_1(\cdot)]^{\frac{1}{1+s}} \leq [(1+s)k_s V(\cdot)]^{\frac{1}{1+s}} \leq \left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}}. \tag{9.74}$$

Let $\tilde{\zeta} = [\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N]^T \in \mathbb{R}^N$ and $\sigma_{\max} = \max\{\sigma_1, \ldots, \sigma_N\}$. Then we have from (9.58) that

$$\|\tilde{\zeta}\| \leq \sqrt{\frac{2\sigma_{\max}}{\underline{g}} V(\cdot)} \leq \left[\left(\frac{2\sigma_{\max} k_b}{\underline{g}} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}}. \tag{9.75}$$

Thus, it has been established that under the proposed finite-time control scheme, the neighborhood error converges to a small region Θ_0 given as

$$\Theta_0 = \left\{ \|E\| \leq \left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}} \right\}, \tag{9.76}$$

and the parameter estimate errors converge to the region Θ_1,

$$\Theta_1 = \left\{ \|\tilde{\zeta}\| \leq \left[\left(\frac{2\sigma_{\max} k_b}{\underline{g}} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}} \right\} \tag{9.77}$$

in the finite time T^* as given in (9.72).

9.5 Comparison with Regular State Feedback-Based Infinite-Time Method

Note that if we set the fraction power $s = 1$, $q_1 = q_2 = \cdots = q_n = 1$, the proposed finite-time control scheme reduces to the regular infinite-time control and the convergence result is in infinite-time sense. In this case, the local virtual error $e_{i,m}$ ($i \in \{1, \ldots, N\}$, $m = 2, \ldots, n$) becomes $e_{i,m} = x_{i,m} - x^*_{i,m}$, with the virtual control $x^*_{i,m}$ given as

$$x^*_{i,m} = -\beta_{i,m-1} e_{i,m-1}, \tag{9.78}$$

and the actual control u_i becomes

$$u_i = -\beta_{i,n} e_{i,n} - \frac{\hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n} \tag{9.79}$$

with the updated law

$$\dot{\hat{\zeta}}_i = -\sigma_i \gamma_i \hat{\zeta}_i + \frac{\sigma_i}{\eta_i^2} \psi_i^2 e_{i,n} \tag{9.80}$$

then the cooperative uniformly ultimately bounded (CUUB) consensus is achieved. More specifically, under this control scheme, the neighborhood error converges to a set given as

$$\Theta_2 = \left\{ \|E\| \leq \left(\frac{2k_b \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2}} \right\}, \tag{9.81}$$

and the parameter estimate errors converge to a set

$$\Theta_3 = \left\{ \|\tilde{\zeta}\| \leq \left(\frac{2\sigma_{\max} k_b \varsigma}{\underline{g} \mu_1 \mu_2 k_a} \right)^{\frac{1}{2}} \right\} \tag{9.82}$$

with $\tilde{c}$ and ς corresponding to the values with $s = 1$ and $q_1 = \cdots = q_n = 1$.

Remark 9.1 It is seen that in these two control schemes, i.e., (9.52)–(9.53) and (9.79)–(9.80), the steady-state errors are controlled by the convergence regions, i.e., Θ_0, Θ_1, Θ_2, and Θ_3, respectively, all of which can be reduced as small as desired by choosing the control parameters. Nevertheless, due to control saturation constraint, the control parameters are not allowed to be set too large. While in the proposed method, one can choose the fractional power $s = (4ln - 1)/(4ln + 1)$ properly to enhance the disturbance rejection performance and increase the control precision without the need for excessively large control gain. For instance, suppose $\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}}\varsigma}{\mu_1\mu_2 k_a} < 1$ is satisfied (this is allowed because all the parameters in k_a and k_b, i.e., k_c, γ_i, and σ_i are free design parameters), note that $1/s$ is much larger than 1, which makes $\left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}}\varsigma}{\mu_1\mu_2 k_a}\right)^{\frac{1}{2s}} \ll \left(\frac{2k_b k_s \varsigma}{\mu_1\mu_2 k_a}\right)^{\frac{1}{2}}$, thus leading to better control precision as compared with regular state feedback-based infinite-time control method.

Remark 9.2 It is worth noting that the proposed finite-time control method is essentially different from the regular state feedback-based UUB control method, here the feedback error and the adaptive compensation are not used directly in building the control scheme and the adaptive updated law, instead the fractional power of the feedback error of the form $e_{i,m}^{q_m}$ and the fractional power of adaptive term $\hat{\zeta}_i^{\frac{3s-1}{1+s}}$ are embedded into the algorithm, it is such special treatment that makes it possible to achieve sufficient control precision in finite time by the adaptive approach. In other words, for the proposed control method, the convergence precision is adjustable and the convergence time is easily adjustable without the need for high feedback gain to reach the given precision, whereas for regular state-based UUB control, although it can reach a bound in a finite time, both the bound and the finite time cannot be prescribed and one can only enlarge the control gain to get a better control precision.

Remark 9.3 Note that if letting the number of the agents N be equal to 1 and the neighborhood error $e_{i,1} = \sum_{j\in\mathcal{N}_i} a_{ij}(x_{i,1} - x_{j,1})$ be replaced by $x_{i,1}$, and accordingly $r_1 = \max_{\forall i\in\{1,\ldots,N\}}\{\sum\limits_{j\in\mathcal{N}_i} a_{ij}\} = 1$ and $r_2 = \max_{\forall i,j\in\{1,\ldots,N\}}\{a_{ij}\} = 0$, then the proposed control scheme is immediately applicable to single (non-networked) systems with high-order nonlinear dynamics. Compared with the existing works on single nonlinear systems by [15–17], the control scheme presented herein is able to deal with much less restrictive uncertainties including non-parametric uncertainties and external disturbances gracefully. In addition, in defining the fraction power q_m, we introduce the integer l into q_m here, such treatment makes the fraction power feedback control capable of adjusting the control precision and finite convergence time, leading to a more favorable solution for finite-time control of high-order systems.

Remark 9.4 Compared with the finite-time control for the system with second-order dynamics, the work proposed in this chapter extends the result to high-order case, which poses significant technical challenge. In fact, the main hindrance stems from the fact that in the presence of high-order dynamics, the commonly used filtering technique [18] cannot be used with Babarlat lemma or UUB theory to derive finite time convergence because the convergence of the filtered error to zero or to a small value cannot ensure the finite-time convergence of the original error. Compared with the other two works [19, 20] on adaptive finite-time consensus for nonlinear MAS, where (i) the systems are second-order; (ii) the communication is undirected; and (iii) the uncertainties are assumed to exhibit the linear parametric property, the work proposes in this chapter gives an adaptive distributed control solution to the problem of the cooperative consensus of nonlinear MAS with general high-order dynamics involving non-parametric uncertainties and non-vanishing disturbances under directed communication, by focusing on achieving consensus in finite time with sufficient accuracy.

9.6 Numerical Simulations

In order to demonstrate the feasibility of the proposed finite-time control, two numerical examples are given.

Example 9.1 We simulate on a group of third-order MAS with non-affine dynamics in the form of (9.1) with

$$\dot{x}_{i,3} = (0.6 + 0.1\exp(-x_{i,2}^2 - x_{i,3}^2))u_i + 0.1\sin(u_i) + f_{di}(\cdot), \tag{9.83}$$

for $i = 1, 2, 3, 4$, where the system uncertainties f_{di} are taken as $f_{di}(\cdot) = d_i + 0.5(-1)^i x_{i,1}^2 + 0.5(-1)^i(x_{i,2}^2) + 0.5x_{i,3}^2$, and d_i denotes disturbances, taken as random and bounded by $|d_i| \leq 0.1$. The communication topology is directed and connected as shown in Fig. 9.1. Each edge weight is taken as 0.1. The left eigenvector of L associated with eigenvalue 0 is $[1, 1, 1, 1]^T$.

The simulation runs for 20 s. The initial conditions are $x_1(0) = [-0.2, -0.3, 0, 0.2]^T$, $x_2(0) = [0.2, 0.3, 0, -0.2]^T$, and $x_3(0) = \mathbf{0}_4$. The simulation is conducted by applying the control laws given in (9.52)–(9.53), where the control parameters are taken as: $l = 1$, $\beta_1 = 4$, $\beta_{i,2} = 12$, $\beta_{i,3} = 40$, $\sigma_i = 1$, $\gamma_i = 1$, and $\eta_i = 0.8$. The scalar function $\psi_i(\bar{x}_i) = 0.2 + x_{i,1}^2 + x_{i,2}^2 + x_{i,3}^2$. In addition, the initial values of the estimates are chosen as $\hat{\zeta}_i(0) = 0.5$ for $i = 1, 2, 3, 4$. The error convergence result of the four agents is depicted in Fig. 9.2. Figure 9.3 represents the parameter estimation $\hat{\zeta}_i$ $(i = 1, 2, 3, 4)$.

To show that better performance is achieved with the proposed control scheme, we also tested the convergence property between the proposed finite-time adaptive control scheme (9.52)–(9.53) and the typical infinite-time-based adaptive scheme (9.79)–(9.80). Both the two control laws are applied to system (9.83) by using the same design parameters. It is observed from Fig. 9.4 that the convergence rate is faster and the error precision is better with the finite-time controller compared with the infinite-time controller.

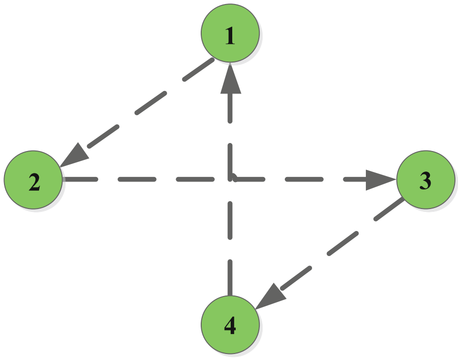

Fig. 9.1 Directed communications among the four agents

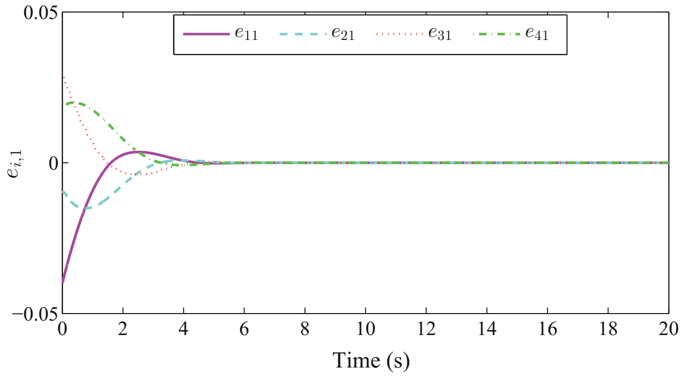

Fig. 9.2 Position neighborhood errors under the finite-time controller

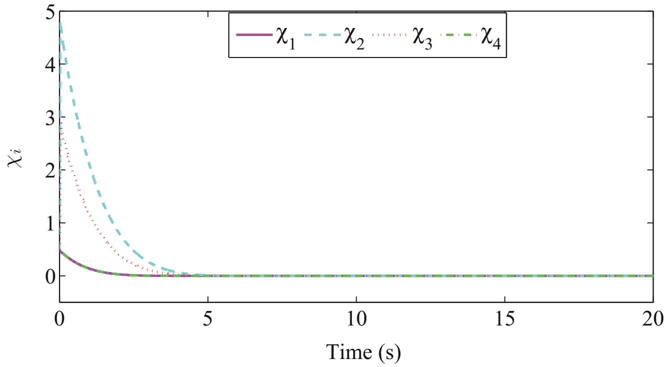

Fig. 9.3 Parameter estimation under the finite time controller

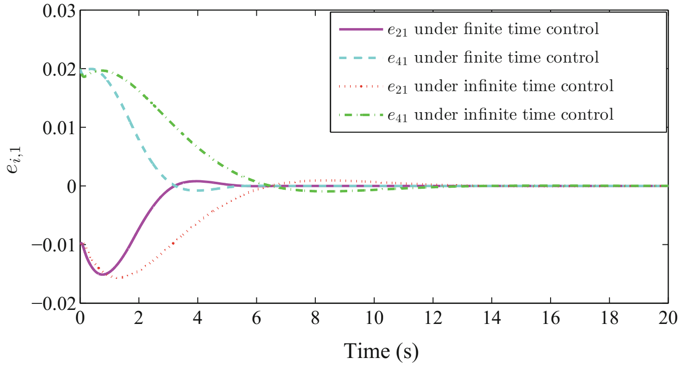

Fig. 9.4 Position neighborhood errors under the two different controllers

Example 9.2 We consider a one-link manipulator [21] with non-affine actuation system,

$$D_i \ddot{q}_i + B_i \dot{q} + N_i \sin(q_i) = f(u_i), \tag{9.84}$$

for $i = 1, 2, 3, 4$, where q_i, $\dot{q}_i$, and $\ddot{q}_i$ denote the link angular position, velocity, and acceleration of the ith robot, respectively; $f(u_i) = u_i + 0.5cos(u_i)$ is the driving torque in which u_i is the motor current, $G_g(t) = -0.1\sin(t)$ is the external disturbance. For convenience, we consider the same communication condition as in Example 1 with each edge weight being 0.1. We set the final consensus configuration ϖ_i as $\varpi = 0.15[\pi/2, 5\pi/8, 3\pi/4, 7\pi/8]^T$. The initial conditions are $x_1(0) = [-0.1, -0.2, 0, 0.1]^T$, $x_2(0) = \mathbf{0}_4$, and $\hat{\zeta}(0) = 0.5\mathbf{1}_4$.

Figure 9.5 is the position trajectories of the four robots, and Fig. 9.6 is the neighborhood errors. For comparison, we also compare the convergence property of the proposed control with the typical infinite-time-based control (9.79)–(9.80). Figure 9.7 demonstrates that the convergence property including the convergence rate and the error precision is better with the finite-time controller as compared with the infinite-time method, as theoretically predicted.

9.7 Notes

Finite-time convergence behavior is of special importance in cooperative control of MAS, but the vast majority research on finite-time control of MAS has been focused on linear single or double integrator ([22–25], to just name a few). Although effort on finite-time consensus of MAS with first- or second-order feedback linearizable nonlinear dynamics has been made by [20, 26–28] by using homogeneous (local) approximation, results on finite-time distributed control of MAS with general

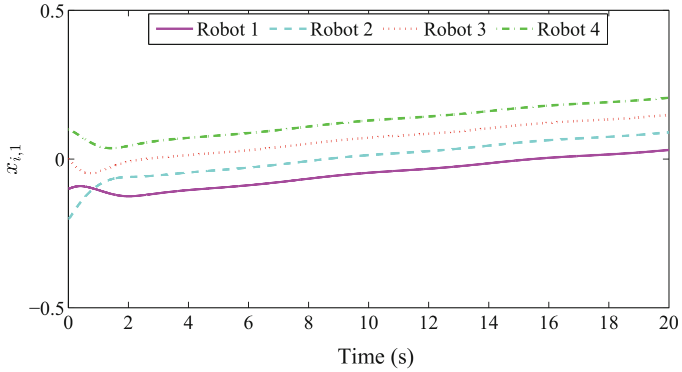

Fig. 9.5 Position trajectories of the four agents under the finite-time controller

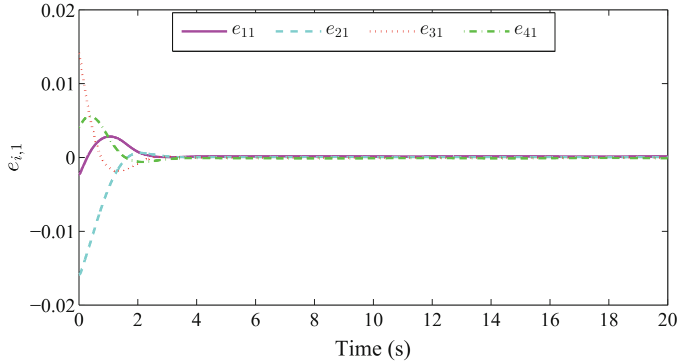

Fig. 9.6 Position neighborhood errors under the finite-time controller

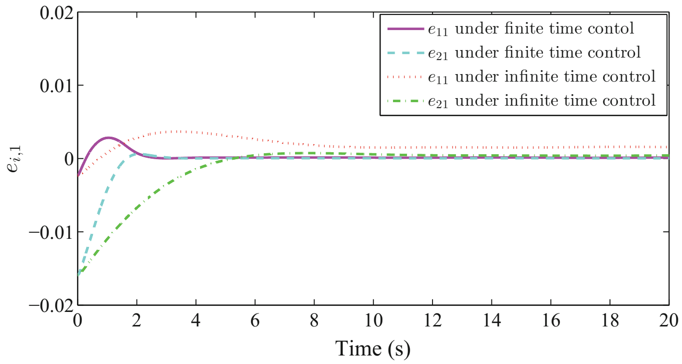

Fig. 9.7 Position neighborhood errors under the two different controllers

high-order nonlinear dynamics are scarce, and extending the finite-time control methods for single or double integrator to higher-order case encounters significant technical challenge. The main hindrance stems from the fact that in the presence of high-order dynamics, the commonly used filtering technique [18] cannot be used with Babarlat lemma or uniformly ultimately bounded (UUB) theory to derive finite-time convergence because the convergence of the filtered error to zero or to a small value cannot ensure the finite-time convergence of the original error.

The underlying problem becomes even more challenging when non-parametric uncertainties and unknown time-varying control effectiveness gains (or non-affine dynamics) are involved. In fact, the existence of non-parametric uncertainties literally makes it impossible to use continuous control to achieve zero steady-state error in finite time even for non-networked (single) system. Although zero error convergence is no longer feasible for such situation, convergence with sufficient steady-state accuracy is acceptable for most practical applications. Motivated by the work of [14], where it is shown that the finite-time convergence systems with nonlinearities possess many good features including faster convergence rate, better disturbance rejection, and robustness against uncertainties, it is thus practically meaningful and plausible to pursue achieving finite-time convergence with sufficient accuracy. However, new issues and challenges arise along with this effort. One is that if the control effectiveness gain is unknown and time-varying, literally any designed protocol will be polluted or disguised, making the control impact on the agent uncertain and unclear such that the widely used negative feedback control method is inapplicable for direct cancelation. Furthermore, to guarantee global finite-time stable control with sufficient accuracy for high-order nonlinear MAS, the derivative of the Lyapunov function $V(x, t)$ must satisfy the relation of $\dot{V}(x, t) \leq -cV(x, t)^{\alpha} + \varsigma$ for suitable constants $c > 0$, $0 < \alpha < 1$ and bounded constant $\varsigma > 0$, which brings about another technical difficulty to adaptive control design because it is non-trivial to synthesize the adaptive compensation to fulfill such relation. The problem under consideration is made further complicated by the local communication topology, especially the local one-way directed communication topology, which imposes significant challenge to extend existing finite-time adaptive control methods to distributed consensus control of networked MAS. As such, the problem of distributed finite-time consensus control of MAS with high-order nonlinear dynamics has not been well addressed; this is particularly true when there involve non-parametric uncertainties and disturbances in the model and the communication is local and especially one-way directed.

In this chapter, we present a solution to such problem by focusing on achieving consensus of MAS in finite time with sufficient accuracy. To circumvent the aforementioned technical obstacles, we first introduce the locally defined neighborhood error and the fractionally composed virtual error, which are embedded into the control scheme; and then, we design the fraction power feedback law and fraction power adaptive updated law for the parameter estimation, which renders the crucial relation $\dot{V}(x, t) \leq -cV(x, t)^{\alpha} + \varsigma$ to be satisfied. To carry out rigorous proof of the stability by inductive analysis, we add the fraction power integration term to the Lyapunov function candidate. By integrating the adjustable fractional power feedback control method with adaptive control method, not only the inherent uncertainties with unknown and time-varying control gains can be effectively compensated but also the cooperative consensus can be achieved in finite time with sufficient accuracy.

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