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Elena Zattoni
Anna Maria Perdon
Giuseppe Conte *Editors*

Structural Methods in the Study of Complex Systems

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Preface

Complex dynamical systems emerge in a variety of disciplines and domains, ranging from those that deal with physical processes (biology, genetics, environmental sciences, etc.) to those that concern man-made systems (engineering, energy, finance, etc.). Indeed, in these fields, it is becoming more and more common to refer to dynamical structures such as systems of systems, hybrid systems and multimodal systems. In brief, the former ones consist of many interconnected dynamical systems with various topological patterns and hierarchical relations; the second ones are dynamical systems that exhibit dynamics of a different nature, both continuous and discrete; the third ones are dynamical systems whose behaviour may vary during their life cycle owing to different operating conditions or depending on the occurrence of some events. The dynamical structures with these characteristics are currently modelled as multi-agent systems, hybrid impulsive systems, switching systems, implicit switching systems and so on.

Consequently, control design techniques have changed to adapt to the ever-increasing system complexity. In this scenario, structural methodologies (i.e., those methods which have evolved from original graph theories, differential algebraic techniques and geometric approaches) have proven to be particularly powerful for several reasons. Beforehand, the structural approaches privilege the essential features of dynamical systems and their interconnections, thus yielding abstractions that can fit a wide variety of situations. Meanwhile, the geometric perspective, which is often at the basis of the structural approaches, introduces a relevant visual and intuitive component which fosters research advancements. Nevertheless, the formalization of structural and geometric concepts is rendered with algebraic tools, which, in turn, have a direct correspondence with computational algorithms, thus paving the way to actual implementation in engineering applications.

In the latest years, relevant theoretical achievements have been obtained within the scope of each methodology encompassed in the sphere of the structural approaches (i.e., graph-theoretic methods, differential algebraic methods and geometric methods) in relation to fundamental control and observation problems stated for complex systems (e.g., multi-agent systems, hybrid impulsive systems, switching systems, implicit switching systems). Moreover, computational algorithms and case studies have been developed together with the theoretical accomplishments. Thus, the corpus of consolidated results (both theoretical and practical/computational ones) presently available motivates this book, whose primary aim is to illustrate the state of the art on the use of methodological approaches, grounded on structural views, to investigate and solve paradigmatic analysis and synthesis problems formulated for complex dynamical systems. In particular, the different perspectives emerging from the various contributions have the purpose of developing new sensibilities towards the selection of the most suitable tools to handle the specific problems. Furthermore, the thorough discussions of specific topics are expected to outline new directions for solving open problems both in the theory and in the applications.

The book starts with a general description of complexity and structural approaches to it, then it focuses on some fundamental problems and, finally, it dwells on applications. In more detail, an overview on the complex systems arising in the various fields, on the new challenges of engineering design and on how these can be mastered by means of the structural approaches is provided first. A novel geometric view, based on transformations which maintain the invariance of global properties, such as stability or H_∞ norm, is described and shown to provide new tools to investigate stability and to parameterize the set of the stabilizing controllers. A graph-theoretic based approach and the original notion of zero forcing set are the tools used to analyse controllability, fault detectability and identifiability of system networks and, more generally, of systems defined over graphs. How solvability of the output regulation problem in hybrid linear systems with periodic state jumps can be investigated by structural methods is then illustrated. A mixed digraph theory and geometric approach is exploited to introduce the novel concept of subspace arrangement and solve the problem of right-inversion for over-actuated linear switching systems. Furthermore, the synthesis of unknown-input state observers with minimum complexity is tackled by structural tools in the context of linear impulsive systems: necessary and sufficient solvability conditions are derived once a set of essential requirements has been disentangled. The disturbance decoupling problem is investigated for a class of implicit switching systems through geometric considerations inspired to the behavioural approach. In particular, the theoretical results are applied to the synthesis of a Beard–Jones filter. Finally, a structural perspective is adopted to analyse Huygens synchronization over distributed media and it reveals a complex, but structured behaviour behind a seemingly chaotic one.

The book is intended for systems and control scientists interested in developing theoretical and computational tools to solve analysis and synthesis problems involving complex dynamical systems. The different contributions aim at giving a comprehensive picture of the available results together with a stimulating view of possible new directions of investigation in the field. Since the presentations emphasize methodologies supported by a solid computational background and often by specific engineering applications, researchers either focussed on theoretical issues or mainly committed to applications may equally find interesting hints.

The idea of this book has stemmed from the workshop which the editors have organized at the European Control Conference 2018 and its realization has been made possible thanks to the strong and enthusiastic support of the invited speakers and their co-authors, who have contributed their original work and latest achievements in the various chapters.

Bologna, Ancona
March 2019

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Part I
Structure of Complex Dynamical Systems

Chapter 1
Complex Systems and Control: The Paradigms of Structure Evolving Systems and System of Systems

Nicos Karcanias and Maria Livada

Abstract This chapter deals with two rather new notions of complexity emerging in Engineering Systems, reviews existing approaches and results and introduces a number of open problems defining a research agenda in the field. We examine these notions based on the fundamentals of a systemic framework and from the perspective of Systems and Control Theory. The two new major paradigms expressing forms of engineering complexity which have recently emerged are the new paradigms of *Structure Evolving Systems (SES)* and *Systems of Systems (SoS)*. The origin and types of complexity linked to each one of these families are considered, and an effort is made to relate these new types of complexity to engineering problems and link the emerging open issues to problems and techniques from Systems and Control Theory. The engineering areas introducing these new types of complexity are linked to the problems of *Integrated System Design* and *Integrated System Operations*.

1.1 Introduction

Complex Systems is a term that emerges in many disciplines and domains [9] and has many interpretations, implications and problems associated with it. The specific domain provides dominant features and characterizes the nature of problems to be considered. A major classification of such systems is to those linked with *physical processes* (physics, biology, genetics, ecosystems, social, etc.) and the artificial, which are *man-made* (engineering, technology, energy, transport, software, management and finance, etc.). We are dealing with *man-made* systems and we are interested in identifying generic types of system complexity among the different problem domains and then identify the relevant concepts and tools that can handle

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the different types of complexity and then enable the design or redesign of complex systems–processes. There is a need to develop generic methodologies and tools that can be applied across the different problem domains. This research aims to identify Systems and Control concepts and tools which are important in the development of methodologies for the *Management of Complexity* of engineering-type complex systems.

Existing methods in Systems and Control deal predominantly with fixed systems, where components, interconnection topology, measurement–actuation schemes and control structures are specified. Two new major paradigms expressing forms of engineering complexity which have recently emerged are the new paradigms of

- *Structure Evolving Systems (SES)* [32]
- *Systems of Systems (SoS)* [23, 37, 50]

Using the traditional view of the meaning of the system (components, interconnection topology, environment), the common element between the first two new paradigms is that the interconnection topology may vary and evolve in the case of *SES*, whereas in the case of *SoS* the interconnection rule is generalized to a new notion of "systems play" [33] defined on the individual system goals. The paper deals with the fundamentals regarding representation, structure and properties of those two challenging classes, demonstrates the significance of traditional systems and control theory, and introduces a new research agenda for control theory defined by:

Structure Evolving Systems [32]: Such a class of systems emerges in natural processes such as Biology, Genetics, Crystallography [24], etc. The area of man-made processes includes Engineering Design, Power Systems under de-regulation, Integrated Design and Redesign of Engineering Systems (Process Systems, Flexible Space Structures, etc.), Systems Instrumentation, Design over the Life Cycle of processes, Control of Communication Networks, Supply Chain Management, Business Process Re-engineering, etc. This family deviates from the traditional assumption that the system is fixed and its dominant features, introducing types of system complexity related to the following:

- The topology of interconnections is not fixed but may vary through the life cycle of the system (*Variability of Interconnection Topology Complexity*).
- The overall system may evolve through the early–late stages of the design process (*Design Time Evolution*).
- There may be variability and/or uncertainty on the system's environment during the life cycle requiring flexibility in organization and operability (*Life Cycle Complexity*).
- The system may be large scale and multicomponent, and this may impact on methodologies and computations (*Large Scale—Multicomponent Complexity*).
- There may be variability in the Organizational Structures of the information and decision-making (control) in response to changes in goals and operational requirements (*Organizational Complexity Variability*).

The above features characterize a new paradigm in systems theory and introduce major challenges for Control Theory and Design and Systems Engineering. There are different forms of structure evolution. Integrated System Design has been an area that has motivated some of the early studies on *SES*. The integration of traditional design stages [28], such as Process Synthesis *(PS),* Global Instrumentation *(GS)* and finally Control Design *(CD)*, is an evolutionary process as far model system formation and two typical forms of evolution are the s*tructural design evolution*, the *early–late design evolution* and the *interconnection topology evolution* [32]. Methodologies and tools developed for *Fixed Structure Systems (FES)* cannot meet the challenges of the *SES* class and new developments on the level of concepts, modelling, analysis and synthesis methodologies are needed. The research is influenced by the need to address life cycle and redesign issues, and such problems have a strong technological and economic dimension.

System of Systems: The notion of "*System of Systems*" (*SoS*) has emerged in many fields of applications from air traffic control to constellations of satellites, integrated operations of industrial systems in an extended enterprise to future combat systems [23, 50]. Such systems introduce a new systems paradigm with main characteristic the interaction of many independent, autonomous systems, frequently of large dimensions, which are brought together in order to satisfy a global goal and under certain rules of engagement. These complex multisystems are very interdependent, but exhibit features well beyond the standard notion of system composition. They represent a synthesis of systems which themselves have a degree of autonomy, but this composition is subject to a central task and related rules defined as "system plays" [33] expressing the subjection of subsystems to a central task. This generalization of the interconnection topology notion introduces special features and challenging problems, which are different than those linked to the design of traditional systems in engineering. The distinguishing features of this new form of complexity are as follows [32]:

- The role of "*objects*" or "*subsystems*" of the traditional system definition is taken by the notion of the *autonomous agent,* and it is characterized by some form of intelligence. This is linked to the notion of "*integrated intelligent system*" defining an autonomous intelligent agent.
- The notion of "*interconnection topology*" of traditional systems is generalized to that of "*system*s play" which is expressed at the level of goals of autonomous intelligent agents [71].
- Decision-making and control are linked to the nature of the "*system*s play" which among other fields may be linked to cooperative control, game among the subsystems, etc.
- System organization (Hierarchical-Multilevel, Holonic [67], etc.) defines an internal form of system structure and this plays a central role in the characterization of the notion of *emergent properties*.

The problem of *Systems Redesign* has been only partially addressed in engineering as redesign of control structure in response to faults, and it has been an active area in business [65]. This problem may be considered within the framework of Integrated Systems Design and leads to problems in the *SES* area [32]. Understanding the issues linked to *SES* and *SoS* is critical in addressing the problem in its entirety from an engineering perspective. Addressing the issues of *SES* and *SoS* has important implications for the underpinning Control Theory and related Design methodologies. Control Theory and Design has developed considerably in the last 40 years. However, the underlying assumption has always been that the system has been already designed and thus control has been viewed as the final stage of the design process on a system that has been formed. The new paradigms deviating from the "*fixed system structure assumption*" introduce new challenges for Control Theory and Control Design. These force us to reconsider some of the fundamentals (viewing Control as the final design stage on a formed system) and create the need for new developments where Control provides the concept and tools intervening in the overall design process, even at stages where the system is not fixed but may vary, and may be under some evolution. Traditional Control has been capable to deal with uncertainty at the unit process level, but now has to develop to a new stage where it has to handle issues of structural, dynamic evolution of the system as well as control in the context of a "systems play". The paper aims to provide an overview of these new areas, deal with issues of representation, examine different forms of system evolution, define the relevant concepts and tools, provide a systems based characterization of *SoS,* and introduce a research agenda for these new paradigms. Integral part of the effort is the linking of these new challenges to well-defined systems and control concepts and methodologies.

The paper is structured as follows: Section 1.2 reviews the notions of the system and summarizes the emergent forms of complexity. In Sect. 1.3, we review the three major engineering problems which introduce types of complexity, that is, the problems of Integrated Design, Integrated Operations and Re-engineering, and identify the different types of systems complexity which will be the main subject of the subsequent sections. Section 1.4 deals with the evolution of models from the early to late design stages, different types of system evolution are considered and the problems associated with them are specified. We consider external and then internal system representations. We examine the notion of a Progenitor model and the derivation of models for control design. This is linked to a form of evolution where the input and output system dimensions are reduced and considered in Sect. 1.5. An alternative formulation based on internal descriptions, where a process graph is defined with fixed nodal cardinality and subsystem models of variable complexity, and or fixed dynamics of subsystems and variable nodal cardinality. The evolution of systems linked to the cascade design process is considered in Sect. 1.5. We consider an evolution type linked to system composition by design of the interconnection graph, and then additional types of evolution associated with the selection of sets of inputs and outputs, referred to here as "systems instrumentation". Within the latter category, we distinguish two distinct forms of evolution, the introduction of orientation in implicit models and the model projection problems. Section 1.6 deals with multidimensional system view linked to an integrated hierarchical structure and introduces system aspects related to the variable complexity and a different nature of subsystem models. We also provide a characterization of system and emergent properties for the system. The notion of *System of Systems* (*SoS*) is considered in Sect. 1.7. We review first the relative literature which provides an empirical definition of this notion. We then introduce the notion of the *Integrated Autonomous System* which is integral part of the new systemic definition for *SoS*. The crucial element of the new definition is the notion of the "*systems play*" and its characterization in terms of standard systems and control concepts and methods is considered. Finally, Sect. 1.8 provides the conclusions, which are in the form of a research agenda for such new families of complex systems.

1.2 The Notion of the System

The development of a systems framework for general systems is not a new activity [52]. Such developments have been influenced predominantly by the standard engineering paradigm. Addressing the variety of new paradigms emerging in man-made systems requires a further development of the standard notion [31]. We will reconsider existing concepts and notions from the general systems area, detach them from the influences of specific paradigms and generalize them appropriately to make them relevant for the new challenges. We use the following standard systems definition.

Definition 1.1 A *system* is an interconnection and organization of objects that is embedded in a given environment.

This definition is very general and uses as fundamental elements the primitive notions of *objects*, *connectivities–relations* (topology), and *environment,* and for man-made systems involves the notion of *system purpose, goal*. It can be symbolically denoted as in Fig. 1.1.

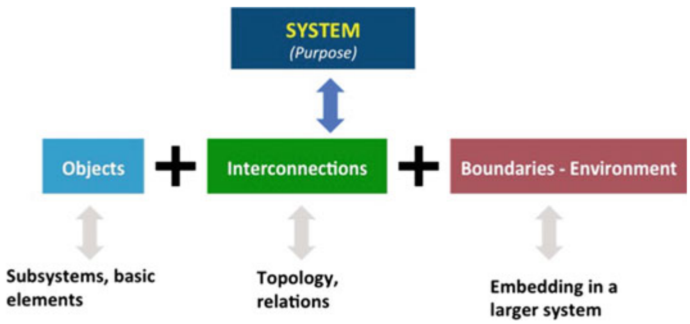


Fig. 1.1 The notion of the system

The notion of the *object* is considered to be the most primitive element, or a system and this allows us to use it in any domain. We define the notion of the object as:

Definition 1.2 An *object*, *B,* is a general unit (abstract, or physical) defined in terms of its attributes and the possible relations between them.

Remark 1.1 This definition of a system is suitable for the study of "soft", as well as "hard" systems and it is based on a variety of paradigms coming from many and diverse disciplines. It refers essentially to *simple systems* since issues of internal organization are reduced only to the interconnection topology. Systems with internal organization will be referred to as *integrated systems* and they will be considered in the following section. These definitions do not make use of notions such as causality, input–output orientation, definition of goal, behaviour, and so on. Quite a few systems do not involve these features, and thus they have to be introduced as additional properties of certain families.

A more explicit description of the notion of the system that involves some form of orientation and which also describes the basic signals is given in Fig. 1.2 where the basic variables are also included. These are the control inputs $\underline{u}$, the outputs $\underline{y}$, the internal variables $\underline{z}$, the input connections $\underline{e}$ and output connections $\underline{w}$. Note that input and output influences are the result of the given system being embedded in a larger system; $\underline{v}$ may also represent disturbances. For composite systems having μ subsystems $S_{a,j}$ we denote by $d_{v,j}, d_{q,j}$ the dimensions of the input and output influences of $S_{a,j}$; then μ will be referred to as the *order* and $\left\{\left(d_{v,j}, d_{q,j}\right), j = 1, ..., \mu\right\}$ as the *cardinality* of the order composite system.

Issues of complexity are naturally connected with the above description and they may be classified in the following categories:

- Objects, Subsystems nature and their variability

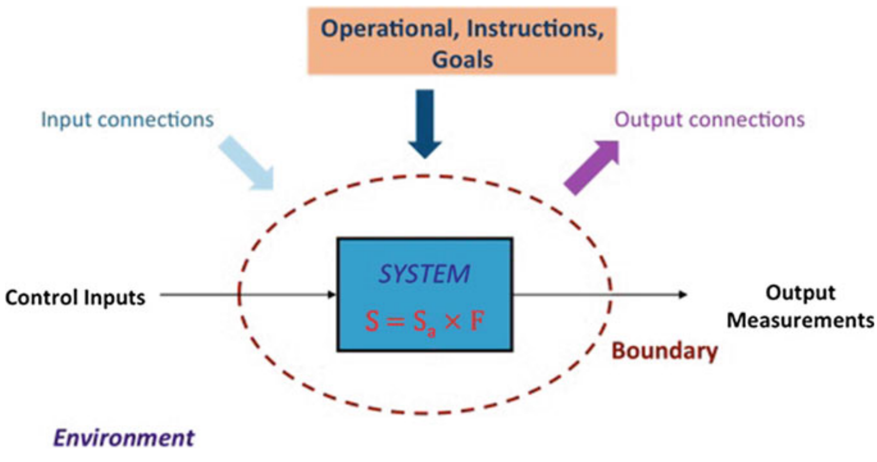


Fig. 1.2 The notion of the system with the basic variables

- Interconnection topology variability (variability of order and cardinality)
- Internal System Organization (non-simple systems)
- Embedding the system to a larger system
- System Design and Redesign
- System Operations
- System Dimensionality
- Support activities related to Data, Information and Computations
- Uncertainty in system description

Central to all above categories of system complexity are issues of system variability due to different types of evolution. The paper is considering the different types of evolutionary processes described above.

1.3 Integrated Design and Operations

The problem of system integration in engineering systems is a technological challenge, and it is perceived by different communities from different viewpoints. Systems Integration means linking the different stages of systems design in the shaping of the system, relating the functions of system operations and establishing a framework where operational targets are translated to design tasks. This problem has been treated mostly as a software problem, and the multidisciplinary nature of the problem (apart from software and data) has been neglected. The significance of integration has created some urgency in working out solutions to difficult problems and this has led to the development of interdisciplinary teams empowered with the task to create such solutions. The key issue here is the lack of methodology that bridges disciplines and provides a framework for studying problems in the interface of particular tasks. The problem of integrating design has been considered in [22, 28, 63]. Recent developments in the area of hybrid systems [5], new developments in the area of organization and overall architectures [67] contribute to the emergence of elements for the integration of system operations. There are, however, many more aspects of the effort to develop a framework of integration which are currently missing. A general view of manufacturing systems involves the following [22]:

1. System Design Issues
2. Operational Issues–Signals and Operations
3. Business Activities
4. Vertical Activities–Data, IT, Software

The diagram indicates a natural nesting of problem areas, where design issues provide the core, linked with the formation of the physical process that realizes production. Production-level activities take place on a given system, they are mostly organized in a hierarchical manner and they realize the higher level strategies decided at the business level. Vertical activities are issues going through the Business–Operations–Design hierarchy and they have different interpretations at the corresponding level. The

Physical Process Dimension deals with issues of design–redesign of the Engineering Process and here the issues are those related to integrated design [8, 22, 28, 49, 57, 58]. The Signals, Operations Dimension is concerned with the study of the different operations, functions based on the Physical Process and it is thus closely related to operations for production. In this area, signals, information extracted from the process are the fundamentals and the problem of integration is concerned with understanding the connectivities between the alternative operations, functionalities and having some means to regulate the overall behaviour. Both design, operations and business generate and rely on data and deploy software tools, and such issues are considered as vertical activities. Compatibility and consistency of the corresponding data structures and software tools express the problem of software integration.

The operation of production of the types frequently found in the Process Industries relies on the functionalities, which are illustrated in Fig. 1.3. Such general activities may be grouped as [22] (i) Enterprise Organization Layers, (ii) Monitoring functions providing information to upper layers and (iii) Control functions setting goals to lower layers. The process unit with its associated Instrumentation are the primary sources of information. However, processing of information can take place at the higher layer. Control actions of different nature are distributed along the different layers of the hierarchy.

The main layer of technical supervisory control functions involves [22, 58]: Quality Analysis and Control; State Assessment, Off Normal Handling and Maintenance; Supervisory control and Optimization; and Identification, Parameter Estimation, Data Reconciliation. These are of supervisory nature activities and refer to the process operator. The automated part of the physical process refers to Process control and involves [22, 58] Regulation, End Point and Sequence Control; Emergency Protection; and Process Instrumentation and Information System.

It is apparent that the complexity of operating the production system is very high. A dominant approach as far as organizing such activities is through a Hierarchical Structuring [53] considered here. However, other forms of organization have emerged [67], but their full potential has not yet been explored. The study of Industrial Processes requires models of different types. The borderlines between the families of Operational Models (OM) and Design Models (DM) are not always very clear and frequently the same model may be used for some functions. Handling the high complexity of the overall system is through aggregation, modularization and hierarchization [8], and this is what characterizes the overall OPPCP structure described in Fig. 1.3. The production system may be viewed as an information system, and thus notions of complexity are naturally associated with it [49].

It is clear that for engineering-type problems the notion of the system emerging is more elaborate than the notion of the *simple system* introduced in the previous section. Systems produced as results of design with operations expressing the functionalities related to the system goal may be referred to as *integrated systems*. Such systems have the design process linked to the physical (engineering) process and an internal organization referred to the different operational functionalities, and all these are supported by signals and data. The *integrated system* has forms of complexity which may be classified as

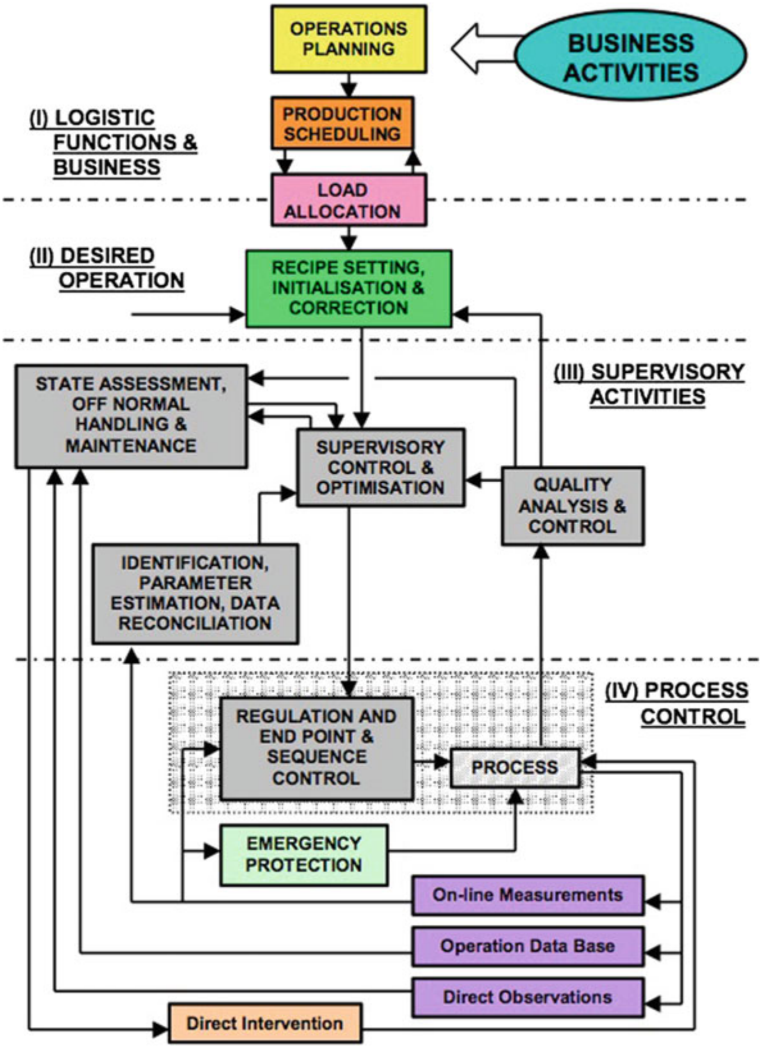


Fig. 1.3 System and its operational hierarchy © [2011] IEEE. Reprinted, with permission, from [22]

1. Integrated Design types of complexity
2. System organization types of complexity
3. System of Systems type of complexity
4. System Re-engineering types of complexity

Note that engineering design is an iterative process and we may distinguish *early stages of design* and *late stages of design* [32]. The transition from early to late design is expressed by models of variable complexity, and this introduces a notion of model embedding with associated complexity. Furthermore, design is a cascade process involving as distinct stages process synthesis, systems instrumentation and finally control design. The transition from one stage to the next expresses a specific form of evolution-type complexity. The large dimensionality (multicomponent nature) of the system refers to the physical system, and this introduces forms of complexity related to design and computations. The system organization is hierarchical or otherwise involves linking functionalities and corresponding models of different nature, and this introduces new forms of complexity with a number of new challenges for Systems and Control theory [22]. The establishing of links between integrated systems at the level of *goals* leads to a new type of complexity referred to as *System of Systems* (*SoS*) [23, 33, 50]. Re-engineering refers to changing the physical system and/or operational processes of an existing system and thus forms of complexity related to both design and operations emerge. These types of complexity are considered subsequently.

1.4 Integrated System Design and Model Complexity Evolution

1.4.1 Integrated Design

The process of overall design of a system is an iterative process [28] (described in Fig. 1.4) which is based on the following design stages:

- Process Synthesis
- Systems (Global) Instrumentation
- Control Design

These three design stages have a cascade nature with feedback loops between the various substages leading to the final structure. Process Synthesis describes the interconnection of the processing units, Systems Instrumentation deals with the problem of selecting the appropriate inputs and outputs and Control System Design is then performed on the final system model. There is an evolutionary process expressed as model shaping during the first two stages which also implies an evolutionary process on the structural properties linked to the final composite system model, which will be used for Control System Design. The Iterative nature of the design process implies that there is an evolutionary process of moving from simple to more complex system descriptions characterized by models of variable complexity. Assuming that the interconnection topology is fixed throughout the design the evolution process is characterized by Early and Late stages. At the *early stages*, simple modelling is required for subprocesses and physical interconnections, and at *late stages* of design, there is need for more detailed, full dynamic models for both subprocesses and physical interconnection structures. Describing the transition from simple modelling to full dynamic models that would enable the study of Systems and control proper-

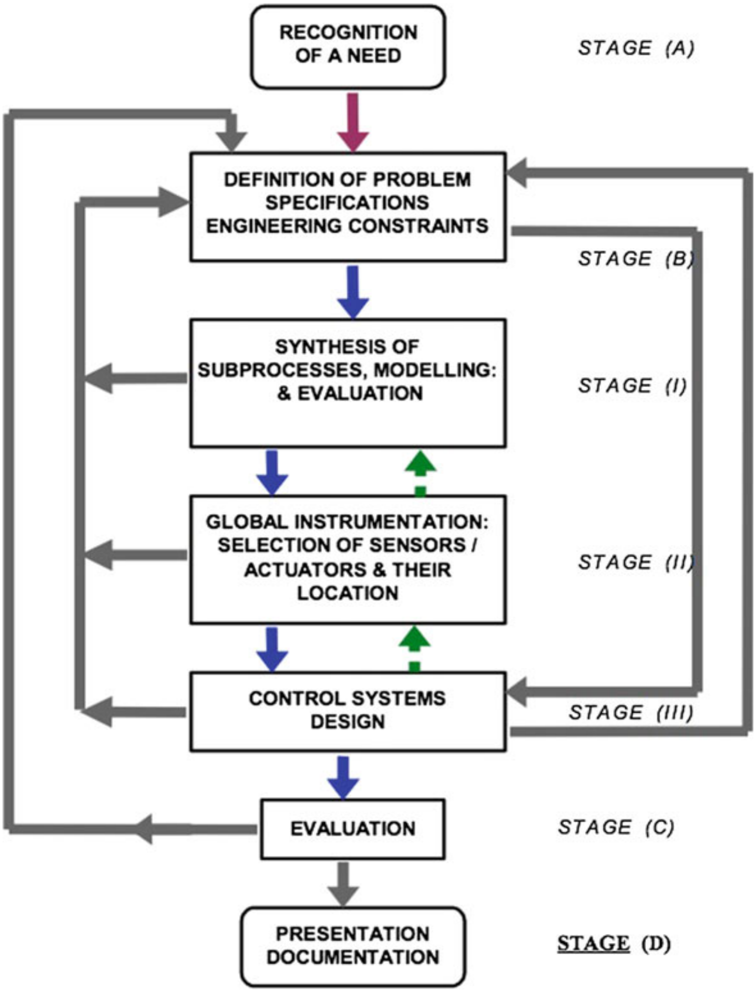


Fig. 1.4 Engineering design process

ties with regard to evolution is challenging. By keeping the same generic topology, the evolution process will develop models of increasing complexity for the subprocesses expressing the embedding of sequential processes and leading to a nesting of system models. An alternative assumption is to consider variations in the topology expressed by dimensional variability of system nodes and/or of the corresponding system cardinality. We distinguish the following types of complexity:

(i) Evolution of models from Early to Late Design stages, referred to as *Design Time Evolution* with *Fixed order* and *Cardinality.*
(ii) Evolution of models from Early to Late Design stages, referred to as *Design Time.*

(iii) Evolution that is naturally linked to the cascade nature of the design process and referred to as *Cascade Design Evolution.*
(iv) Evolution associated with growth and partial death of parts of the system, expressed as *Variable Order* and referred to as *Life Cycle Evolution.*

The study of such processes provides the basis for studying the evolution of system structure and related properties under the different evolutionary mechanisms. Assuming that the interconnection topology is fixed, there are two fundamental aspects of evolution in design, namely, the case of *Fixed Order* and *Fixed Cardinality* linked to the notion of *Dynamic Complexity Evolution* and the case of *Fixed Order,* but *Variable Cardinality*, linked to *Dimensional Complexity Variability* in "early–late" design. These evolution types are considered next.

1.4.2 Early–Late Design Models: The Family of Fixed-Order Models

For a fixed-order composite system at the Early Stages, all the subprocesses and the physical interconnections are represented with simple models, whereas at the Late Stages of design more detailed and may be full dynamic models are required for both subprocesses and physical interconnection structures. This process leads to the notion of *Dynamic Complexity Variability* in the design process. Modelling requires a framework that permits the transition from simple graphs to full dynamic models and allows study of Systems and Control properties in a unifying way. The process that generates families of models has as the simplest element the *Conceptual Model* of the process [15]. *Conceptual Modelling* is being used during the very first steps to translate all the Requirements and Objectives into sets of *Preliminary Designs* leading to the notion of *Conceptual Process Model*.

For composite systems with a fixed order and given cardinality (fixed or variable), the most elementary conceptual model is denoted by $\mathscr{M}_0^c$ [58] and acts as the generator for subsequent models of variable complexity [32]. Every stage of evolution defines a new model which is the successor of the previous one, and it has higher complexity from the previous stage model (for Linear Systems we use the McMillan degree as a measure of complexity). The overall set that contains all such models will be denoted by $\mathscr{M} = \left\{\mathscr{M}_i^c, i = 0, 1, ..., k\right\}$, where k represents the k-th stage of evolution and it is referred to as the *early–late design model set.* This evolutionary process expresses a nesting of models and the simplest model in the chain $\mathscr{M}_0^c$ is referred to as the *basic kernel model* of the chain. There is a need to develop a framework that enables the transition from simple modelling to full dynamic models and allows the study of Systems and Control properties under this form of evolution. Every model $\mathscr{M}_i^c$ in the chain defines a graph which is affected by the cardinality of the subsystems and the description of the physical interconnection streams. The notion of a graph associated with a composite system is defined as the *kernel graph* *model*, and it is the simplest representation of systems of a given order and with cardinality that may be also fixed, or variable. This is defined below.

Definition 1.3 Let us consider a composite system S_C of order μ and correspond to every subsystem S_i a pair of vertices $(\underline{e}_i, \underline{w}_i)$, denoting *input connections* and *output connections*, respectively, and denote by g_i an edge providing an $\underline{e}_i$input-$\underline{w}_i$output description of S_i. If f_{ik} denotes the physical/ information streams connecting the $\underline{w}_i$ and $\underline{e}_k$ vectors, then the set } will be called the *kernel graph* $\mathscr{J}_0$ of the system.

For fixed-order composite systems, we use the kernel model as a starting point in the effort to develop models of increasing complexity, generated from the same $\mathscr{M}_0$ model. There are two cases to distinguish:

(i) fixed cardinality;
(ii) variable cardinality.

In both cases, we use the kernel graphindexgraph!kernel, succeeded by models with increasing complexity for the subprocesses. The chain $\mathscr{M}$ is generated by the *basic kernel model* $\mathscr{M}_0$ which in turn generates a nested sequence of models where $\mathscr{M}_1$ evolves from $\mathscr{M}_0$, $\mathscr{M}_1$ generates $\mathscr{M}_2$ and this procedure goes on, where $\mathscr{M}_{0,nl}$ denotes the simplest nonlinear model.

The derivation of nesting chain is not a simple process, and there is no unique generic procedure for its construction. Specific applications define the chain of models from simple- to complex-based system models based on the knowledge of the particulars of the application. Developing, however, a generic framework requires techniques for generating such chains of models. There are two alternative approaches for such derivations. Both start with a full dynamic model. The first is using alternative methods for model reduction, and the second deploys the theory of partial realization [4, 25] to generate the chain of models. Chains based on model reduction depend on the specific technique used. A technique based on the partial realization for the derivation of chains is generic process and independent from the particulars of model reduction methodologies.

All methods used to generate chains preserve the *kernel model*. Such approaches generate the following sequences of models:

$$\mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \ldots \subset \mathscr{M}_{k+1} \subset \mathscr{M}_{k+2} \subset \ldots \tag{1.1}$$

where $\mathscr{M}_0$ is the kernel model, $\mathscr{M}_1$ is the linear steady-state model, $\mathscr{M}_2$ corresponds to first-order dynamics, and subsequently we increase the complexity up to the most complex linear. The process may continue to the nonlinear case with the use of different complexity Volterra models. $\mathscr{M}_k$ may denote the first Volterra model, $\mathscr{M}_{k+1}$ the second Volterra description, etc. It is very important to note that there is a reversibility between Model Complexity Evolution and Model Simplification Approach. Model Evolution and Model Reduction may become completely reverse processes, if we use systems fixed order and cardinality. The generation of the chains defined in Definition 1.3 leads to three different basic types of model evolution based on the assumptions:

(i) Fixed order and cardinality and variability of complexity of subsystem models.
(ii) Fixed order, variable cardinality and fixed complexity of subsystem models.
(iii) Fixed order, variable cardinality and variability of complexity of subsystem models.

Clearly, the above cases may be also extended to the case of variable order, which is an issue considered later on. The study of system properties under such forms of evolution requires a description of the composite system at each design stage under the assumptions made for the order and cardinality. Thus let us assume that

$$\mathscr{M}_a^j = \left\{ \mathscr{M}_i^j, i = 1, 2, ..., \mu \right\}$$

is the aggregate of the models of the subprocesses of the j-design stage and let $\mathscr{J}^j$ be the interconnection graph defined under the given order and cardinality assumption. The composite system is then defined by

$$\mathscr{M}^j = \mathscr{J}^j * diag \left\{ \mathscr{M}_i^j, i = 1, ..., \mu \right\}.$$

The study of system properties requires a representation of the composite system model $\mathscr{M}^j$ which is a issue that will be considered in a subsequent section.

1.4.3 Early–Late Design: Model Complexity Evolution

1.4.3.1 Fixed-Order and Cardinality Systems

We consider a single system (order 1) and with fixed cardinality. The description of a linear system in terms of the infinite Laurent expansion provides a natural way of deriving approximations of variable complexity by truncation of the infinite series. This natural way of introducing models of variable complexity is linked to the classical problem of partial realization [4, 25]. It is assumed that the information available about a system S is an infinite sequence $(H_k)_{k=1}^{\infty} = \left(CA^{k-1}B\right)_{k=1}^{\infty}$, where $CA^{k-1}B$ are the *Markov parameters*. This input–output information is being used for the realization of a system (A, B, C) that would match only the first v terms of the infinite sequence. This realization is called partial realization. The partial realization establishes families of linear systems of variable dynamic complexity, and this is why our attention is now focused on looking at this classical problem from a different perspective, that is, the evolution in the family of models established by the partial realization [21].

We consider a rational transfer function $G(s) \in \mathbb{R}^{m \times p}(s)$ with a Laurent expansion:

$$G(s) = \sum_{i=1}^{\infty} H_i s^{-i} = H_1 s^{-1} + H_2 s^{-2} + ... \tag{1.2}$$

which defines an infinite sequence $\mathscr{H} = (H_1, H_2, ...)$ where the H_i's are real matrices. Taking the first $\nu\,(\nu > 0)$ terms of Eq. (1.2), we have a finite sequence $\{H\}_\nu \equiv (H_1, H_2, ..., H_\nu)$. A natural way to approximate the rational matrix $G(s)$ is to define a new infinite power series:

$$G^{'}(s) = \sum_{i=1}^{\infty} H_i s^{-i} = H_1 s^{-1} + H_2 s^{-2} + ... + H_\nu s^{-\nu} + H_{\nu+1} s^{-\nu-1} + ... \qquad (1.3)$$

with the first ν coefficients of the above power series being the corresponding H_i of $\{H\}_\nu \equiv (H_1, H_2, ..., H_\nu)$ of the original sequence and the remaining infinite number of terms $\left(H^{'}_{\nu+1}, H^{'}_{\nu+2}, ...\right)$ being dependent on the finite sequence $(H_1, H_2, \ldots H_\nu)$ in some appropriate way that will be defined later on [66]. Such an extension of the finite sequence will be referred to as proper extension and the mechanisms of achieving this are based on the principle of not increasing the McMillan degree of the sequence. This type of approximation is linked to the problem of partial realization [2, 66] and provides a natural way to define models of variable complexity for rational transfer function. In the following, we shall refer to $\mathscr{H} = (H_1, H_2, \ldots)$ as the parent series, the finite sequence $\{H\}_\nu \equiv (H_1, H_2, \ldots H_\nu)$ as the generator set and the infinite sequence based on $\{H\}_\nu$ which has been appropriately extended to $\mathscr{H}_\nu{}' \equiv (H_1, H_2, \ldots H_\nu, H'_{\nu+1}, H'_{\nu+2}, ...)$ with the $(H'_{\nu+1}, H'_{\nu+2}, \ldots)$ as linear functions of $\{H\}_\nu$ as a proper extension of $\{H\}_\nu$.

There always exist triplets of matrices $S \triangleq (A, B, C)$ with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ [25] such that

$$CA^{i-1}B = H_{i,}\ i = 1, 2, \ldots, \nu \text{ and } CA^{i-1}B = H_i\text{'}\ i = \nu + 1, ... \qquad (1.4)$$

Such a triplet S will be called a realization of $\{H\}_\nu$, and the number n defines its *dimension*. Every finite sequence $\{H\}_\nu$ has a realization in the sense defined by (1.4), and the construction of such realizations is based on the rank properties of Hankel matrices [6]. Of all possible realizations of a finite sequence $\{H\}_\nu \equiv (H_1, H_2, \ldots H_\nu)$, there is a family with minimal dimension δ_ν defined by the maximal rank value of the sequence of Hankel matrices constructed from $\{H\}_\nu$, called the *McMillan degree* of $\{H\}_\nu$. A realization of dimension n equal to δ_ν is called *minimal* [4].

Definition 1.4 ([4]) A *realization* $S_\nu \triangleq (A_\nu, B_\nu, C_\nu)$ of $\{H\}_\nu \equiv (H_1, H_2, \ldots H_\nu)$ based on the proper extension, i.e., the infinite sequence

$$\mathscr{H}_\nu{}' \equiv (H_1, H_2, \ldots H_\nu, H'_{\nu+1}, H'_{\nu+2}, ...)$$

with dimension n and McMillan degree δ_ν is called a *partial realization* of the finite sequence $\{H\}_\nu$. If $n = \delta_\nu$, then is a *minimal partial realization* (MPR) $\{H\}_\nu$.

The process of considering $\{H\}_\nu$ finite sequences of the infinite sequence $\mathscr{H}_\nu{}'$ for varying values of ν gives rise to a family of systems

$$\{S\} = \{S_\nu : S_\nu \triangleq (A_\nu, B_\nu, C_\nu), \nu = 1, 2, ...\}$$

with corresponding transfer functions

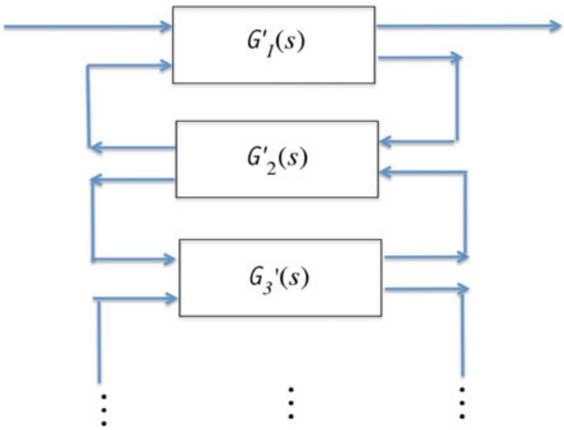
$$\{G\} = \{G_\nu(s) : G_\nu(s) \triangleq C_\nu(sI - A_\nu)^{-1}B_\nu, \nu = 1, 2, ...\}.$$

The study of the properties of such sets is central to the effort to understand evolution of complexity in these chains. The construction of such systems linked to the finite sequences $\{H\}_\nu$ is considered in [2, 10, 66].

The parametrization of the minimal partial realizations is crucial in characterizing the structure evolution. The family of MPRs is characterized by the non-decreasing property of their McMillan degrees [25]. In fact, as we progressively include more and more terms into $\{H\}_\nu \equiv (H_1, H_2, \dots H_\nu)$, the following cases may occur: $\{H\}_\nu \equiv (H_1, H_2, \dots H_\nu)$ gives rise to $S_\nu \triangleq (A_\nu, B_\nu, C_\nu)$, $\{H\}_{\nu+1} \equiv (H_1, H_2, \dots H_{\nu+1})$ gives rise to $S_{\nu+1} \triangleq (A_{\nu+1}, B_{\nu+1}, C_{\nu+1})$ and $\delta_\nu \leq \delta_{\nu+1}$. If an inequality holds at a given position i of the infinite sequence, then this is called *jump point* [4, 25]. In the case, where the McMillan degree of $\{H\}_\nu$ and $\{H\}_{\nu+1}$ remains the same, there are again two different types of $H_{\nu+1}$ extensions: First, the case where the inclusion of $H_{\nu+1}$ results to a different (than the previous) realization of the same McMillan degree and second the case where the inclusion of $H_{\nu+1}$ produces exactly the same realization of the same McMillan degree [4, 25]. An alternative representation of the MPR family is provided by a continued fraction decomposition of rational transfer functions introduced in [2], where a formal power series $G(s) = H_1 s^{-1} + H_2 s^{-2} + \cdots$ a decomposition is introduced having an interpretation as a feedback interconnection of linear systems as shown in Fig. 1.5:

The construction and properties of the above family of linear systems are described in [2] and summary of their properties are as follows:

Fig. 1.5 Partial Realization as feedback interconnection of linear systems. Reprinted from [3], Copyright 1987, with permission from Elsevier



(i) There is a one-to-one correspondence between $\{H\}_\nu$ and the ν-th subsystem of the decomposition (some subsystems might be trivial; the interconnection of the first blocks defines a partial realization of $\{H\}_\nu$.
(ii) The decomposition defines a partitioning of the MacMillan degree of the j-th block, as well as the reachability and observability indices of the j-th subsystem.

The properties of the family of MPRs of a given rational transfer function are central in the study of evolution of structure for these families and it is a challenging issue. Results in this area such as those in [10] are linked to input–output canonical form, and they introduce a parametrization of the set of MPRs establishing links to the row, column Kronecker invariants. The stability properties of MPRs are considered in [11] and demonstrate that not all MPRs preserve important system properties. Results presented for the family of MPRs for a single rational matrix may be transferred to the case of composite systems [29], and this will be elaborated in the following section.

An alternative process for generating families of models with variable complexity has been introduced in [32, 39] and relates to the handling of classes of small numbers on system properties. This classification introduces a numerical form of model nesting and removing small numbers is a form of Robust Structural Simplification, and the derived family of models is referred to as numerical nesting. The derivation of this family is driven by the need that the structural properties of the original system have to be close to those of the reduced system.

Open Issues: The Development of the family of MPRs introduces a nesting of approximate models ordered by their degree of complexity (MacMillan degree). Among the issues related to the characterization of evolution of structure concern,

(i) Investigate whether key system properties of the original system (stability–instability, minimum–non-minimum phase) are preserved within the chain.
(ii) Determine the degree of complexity required to preserve key system properties.
(iii) Characterize the process of evolution of the Kronecker invariants (column, row minimal Indices, and finite and infinite elementary divisors) in the chain of models (the results in [10] deal only with controllability and observability indices).
(iv) Development of the properties of the family of numerically nested models.

1.4.3.2 Fixed-Order and Variable Cardinality Systems

For many engineering processes, the interconnection topology represents "natural flows" or "information flows", referred to as *flow streams,* between the subsystems. The assumption that the dimensionality of these flow streams may vary as we move from early to late design is quite natural. In fact, in the case of a process system, a connection between two subprocesses may be defined in terms of a liquid flow; at early stages, this flow may be expressed in terms of liquid flow, but at later stages other properties such as temperature, pressure, etc. may be included. The number

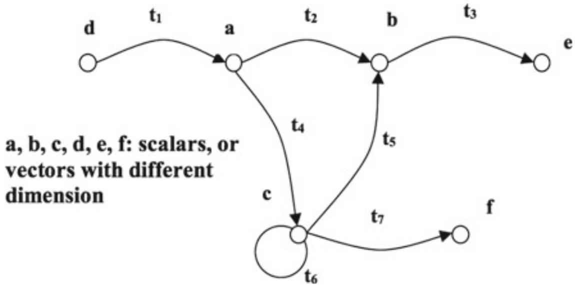


Fig. 1.6 Example of graph dimensional variability

of variables used to represent a system "flow" in natural streams may vary, and this implies a dimensional expansion of them with a subsequent expansion of the related models for the subprocess. This represents a *Dimensional Complexity Variability* from early to late design and corresponds to the case where the order of the composite system is fixed, but we have a variability in the cardinality of subsystems. Thus, although the interconnection rule (kernel graph) may remain the same, the local flows (edges in the graph) may change from scalars to vectors. This expresses an evolution from scalar to vector graphs with a respective evolution of scalar to matrix transfer functions (for the linear case). The nature of the corresponding graph will depend on the stage of the design (Early, Late), and this affects the models of local processes and the description of the physical interconnection streams. In the early stage, the graph consists of the fundamental variables linked to the physical interconnection and contains the minimal interconnection information (Fig. 1.6).

Example 1.1 Consider the directed graph where the corresponding nodes are vectors which according to degree of modelling may have an increasing, or shrinking dimension and thus the corresponding transmittances are also vector transmittances.

The kernel graph model defines a primitive form of the structure that derives from the conceptual modelling of the system, contains the basic information regarding subsystems and physical streams, and provides the minimal information about the physical interconnection topology. At later stages, as the requirements for modelling are increasing, more than one variable is associated with the physical streams and consequently the dimensionality of physical interconnection streams is changing. This defines the variability from one-dimensional to many-dimensional vertices and edges referred to as *Dimensional Complexity Variability of Graphs*.

Open Issues: Fundamental issues related to the dimensional variability of the graph related to the classification of the properties of the directed graph, which are independent of the dimensionality of nodes and those which depend on their dimensionality. Extending the Dimensional Complexity Variability by including the dynamic complexity of the subsystems raises additional issues related to structure evolution and related properties.

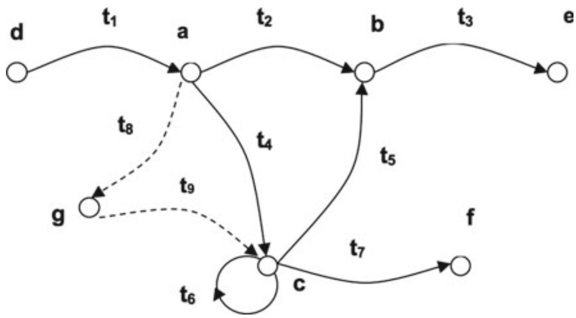


Fig. 1.7 Example of Structural Graph Growth problem. Reprinted from [32], Copyright 2008, with permission from Elsevier

1.4.3.3 Variable Order and Fixed/Variable Cardinality Systems

In engineering problems, the case of system growth or reduction of the system may occur. This involves expansion of the existing system by addition of parts (growth), or removal of parts (death) of the system. These cases lead to a variability of the original system order, emerge in problems of system redesign and are referred to as *graph structural evolution problems*. For such problems, the main interest is the study of evolution of structural and non-structural properties under such transformations. An illustration of the above is provided by the following example (Fig. 1.7).

Example 1.2 Consider the directed graph below represented by the a, b, c, d, e, f nodes and the solid edges. This is modified by adding a new node g and the new dotted line edges and produces an evolution of the previous graph.

The *Structural Graph Growth Problem* introduced here may be combined with the signal, dimensional growth, or *Dimensional Graph Evolution* discussed before. Clearly, combinations of the two may be considered and this may be referred to as the *General Graph Growth Problem*.

Open Issues: The above represent open areas for research on the fundamentals of *Graph Growth–Death*. Some important problems in this area involve the following:

(i) *The representation problem*: Define an appropriate modelling framework for describing the graph augmentation and graph reduction/death problems.

(ii) *The graph structural growth problem*: Investigate how the properties of a directed graph and algebraic invariants (based on given complexity models) are changing by addition of new nodes, or elimination of existing ones.

1.5 Cascade Design System Evolution

The technological stages of the overall system design have been described in Fig. 1.4. The Design is a cascade and complex process which is reminiscent of an evolution process that involves many different forms of system structure evolution. This evolution has many different features. A natural evolution of the system structure is that shaped through the design stages from conceptualization, to process synthesis, global instrumentation and finally control design, and this is referred to as cascade structural evolution. The system used for control design is the evolution of earlier forms shaped through the process synthesis and the systems instrumentation and its different aspects are considered here.

1.5.1 Systems Composition and Complexity

Process or System Synthesis is an act of determining the optimal interconnection of subsystems, as well as the optimal type and design of the units, subsystems within the overall system. The development of a generic synthesis framework that transcends the different application areas is a significant challenge. The modelling of composite systems using energy considerations is examined in [62, 70], and the traditional network synthesis is examined in [64]. The case of process systems is considered in [55]. Here, we consider the linear case and aim to present the evolution from the aggregate to the composite as a function of the interconnection topology. A crucial problem in system synthesis is the *Representation Problem* which is crucial for the study of structure evolution. The results here are based on the reduction of system synthesis to an equivalent feedback design problem using the standard composite system description [14, 28] and its particular characteristics based on the nature of the physical interconnection streams and the selection of the local input and output structure which leads to the notion of *completeness* [28] and providing a representation of the synthesis as generalized feedback design problem.

Let us consider a set of systems $\{S\} = \{S_j, j = 1, 2, .., \mu\}$ of order μ where every subsystem S_i has a pair of vertices $(\underline{e}_i, \underline{w}_i)$ and cardinalities $\{(d_{v,j}, d_{q,j}),$ $j = 1, ..., \mu\}$. We will assume that the subsystems have models $\{M\} = \{M_j, j = 1, 2, .., \mu\}$ of a certain type and if $\mathscr{F}$ is the interconnection rule (described by a graph and the subsystem cardinalities), then $S_a = S_1 \oplus S_2 \oplus \ldots \oplus S_\mu$ denotes the *aggregate system* with a model $\{M_a\} = block - diag\{M_j, j = 1, 2, .., \mu\}$. The *Composite System* is denoted by $S_c = \mathscr{F} * S_a$, where $*$ denotes the action of the interconnection topology $\mathscr{F}$ on S_a. The definition of Composite Systems involves the specification of the physical input and output streams and the selection of inputs and outputs at the subsystem level. Subsystems enter the composite structure, by interconnecting local variables (subsystem connecting inputs, outputs), and this affects drastically the overall properties of the composite system. A first attempt to link model composition to feedback was made in [14] and subsequently developed in [29]. The definition of the composite from the aggregate by the action of the interconnection topology raises important questions, which are linked to

(i) The representation of the composite system;

(ii) The relationships between the structure and properties of the aggregate and the composite in terms of the characteristics of the interconnection topology.

The general scheme that is considered satisfies certain assumptions which are described below [29, 32].

(a) Local Well-Connectedness Assumption (LWCA): The physical linking of a subsystem S_k to the rest of the subsystems implies that there is a connecting input vector $\underline{e}_k$ having as coordinates all variables connected directly to at least one subsystem output, or external variable (manipulated, or disturbance) and having as outputs the vector $\underline{z}_k$ of all possible measurements and connecting variables to at least another subsystem. The pair of vectors $\left(\underline{e}_k, \underline{z}_k\right)$ defines the natural inputs and outputs of the system S_k. A sub-vector of $\underline{z}_k$ is the connecting output vector $\underline{w}_k$ with coordinates all variables which feed to at least one of the subsystems or measured variables. We assume that the transfer functions $H_k(s) : \underline{e}_k \to \underline{z}_k$ are well defined and they are proper. These assumptions are referred to as *Local Well-Connectedness (LWC)* and $H_k(s)$ is the k-th connecting transfer function; furthermore, if $H_k(s)$ represents a minimal system, then the system satisfies the *Minimal LWC (MLWC)* assumption. The aggregate system S_a is represented by the transfer function matrix $H(s) = block - diag\,\{H_k(s), k = 1, 2, ..., \mu\}$.

(b) Local Well-Structured Assumption (LWSA): For every subsystem with $\underline{e}_k$, $\underline{z}_k$ physical inputs and outputs, we shall denote by $\underline{v}_k$, $\underline{y}_k$ the effective input, output vectors. We shall assume that $\underline{y}_k$ is a sub-vector of $\underline{z}_k$ in the sense that $\underline{y}_k = K_k \underline{z}_k,\ K_k \in \mathbb{R}^{p_k \times q_k}$, $p_k \leq q_k$ and that $\underline{e}_k$ is expressed as

$$\underline{e}_k = \underline{f}_k + L_k \underline{u}_k = \underline{f}_k + \underline{v}_k, \tag{1.5}$$

where $\underline{f}_k$ is some vector of variables defined by the interconnections and $\underline{v}_k = L_k \underline{u}_k$ has independently assignable (control or disturbance) variables, defined as a combination of a larger potential vector $\underline{u}_k$; thus $\underline{u}_k$, $\underline{z}_k$ emerge as potential inputs and outputs. This assumption is referred to as *Local Well-Structured (LWS)* assumption.

(c) Global Well-Formedness Assumption: [14] Let $S_a = \{S_k, k = 1, 2, .., \mu\}$ be the system aggregate under the LWC and LWS assumptions. The composite system will be called *Globally Well-Formed (GWF)*, if the interconnection rule $\mathscr{F}$: $\underline{e}_1 \times ... \times \underline{e}_\mu \to \underline{z}_1 \times ... \times \underline{z}_\mu$ represented by the diagram of Fig. 1.8 satisfies the following:

(i) Its output is $\left[\underline{z}_1^t, \ldots, \underline{z}_\mu^t\right]^t = \underline{z}$ and if $\underline{v}_k$ are external input vectors (assignable or disturbances), its inputs $\underline{e}_k$ are expressed as $\underline{e}_k = \sum_{j=1}^{\mu} F_{kj} \underline{z}_j + \underline{v}_k$, F_{kj} real.

(ii) The transfer function from $\underline{v} = [\underline{v}_1^t, \ldots, \underline{v}_\mu^t]^t \to \underline{e} = [\underline{e}_1^t, \ldots, \underline{e}_\mu^t]^t$ is defined.

If $F = [F_{k,j}]_{k,j\in\mu}$, $K = block - diag\{K_i,\ i \in \mu\}$, $L = block - diag\{L_i,\ i \in \mu\}$, $\underline{u} = [.., \underline{u}_i{}^t, ..]^t$, $\underline{y} = [.., \underline{y}_i{}^t, ..]^t$, then

$$\underline{e} = \underline{v} + F\underline{z}, \quad \underline{z} = H_a(s)\underline{e}, \quad \underline{v} = L\underline{u}, \quad \underline{y} = K\underline{z}, \tag{1.6}$$

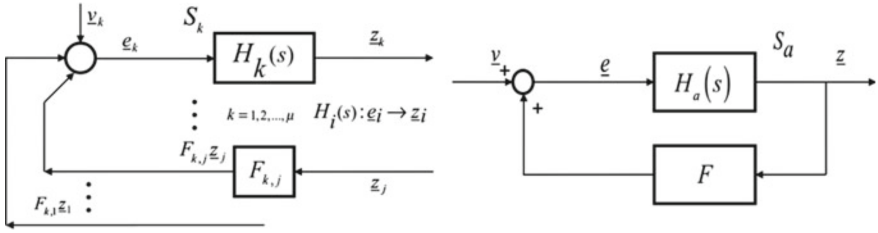


Fig. 1.8 Globally well-formed composite system

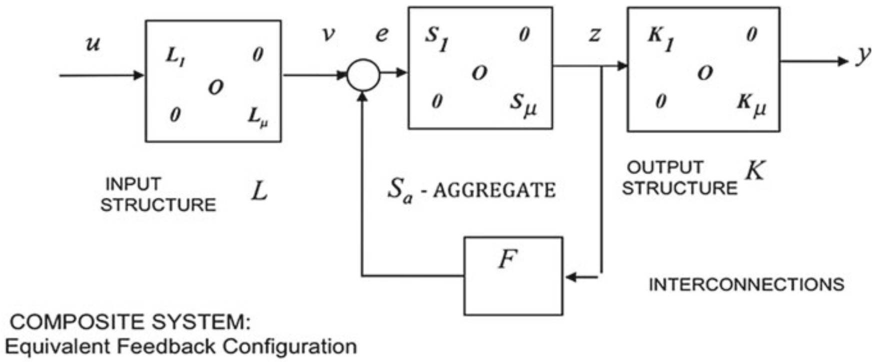


Fig. 1.9 Effective and progenitor system model

and the composite configuration is represented as the feedback configuration of Fig. 1.8.

Note that condition (c.ii) implies that $I - FH_a(s)$ is an invertible matrix. Clearly, the interconnection graph acts as feedback and the selection of effective inputs and outputs is represented as input and output constant compensators and the transfer function of the composite system S_c is

$$G(s) = K \cdot \hat{H}(s), \cdot L \text{ where } \hat{H}(s) = H_a(s) \cdot (I - F \cdot H_a(s))^{-1} \tag{1.7}$$

and it is represented in Fig. 1.9. This expresses the composite system as the action of decentralized input and output reduction (squaring down operation), represented by the input and output transformations K, L, respectively, and of an internal feedback F, representing the topology of the interconnections. The matrix $\hat{H}(s)$ is referred to as *progenitor model* of the composite system and $G(s)$ is the *effective transfer function* of the composite system. The actions of K, L are referred to as *Model Projection* (MP) operations [29, 32] and are forms of "squaring down" [35]. We shall refer to (K, L) pair as the *input–output normalizers*.

The representation of the interconnecting topology as internal feedback provides the means to link the properties of aggregate and the composite system, explains the basic form of evolution in terms of the interconnection graph and allows our intervention in the synthesis problem using results from the feedback theory. The above description of the composite system is general and can lead to family of graphs depending on the cardinality of subsystems and their respective dynamic complexity (see Sect. 1.4.3). A special case of the above general configuration is described below.

(d) Completeness Assumption: The well-formed composite system of Fig. 1.9 will be said to be *complete*, if the following two further conditions hold true:

(i) Every effective subsystem output $\underline{y}_k$ satisfies the condition $\underline{y}_k = K_k \underline{z}_k$, K_k square invertible.
(ii) Every external subsystem vector $\underline{u}_k$ has as many independent coordinates as the dimension of $\underline{e}_k$ input vector, i.e., $\underline{e}_k = L_k \underline{u}_k$ with L_k square and invertible. In this case the matrices K, L are also invertible.

Remark 1.2 The completeness implies that the composite and the aggregate are output feedback and input and output coordinate transformation equivalent, and thus they have the same structural characteristics. Guaranteeing the validity of the above assumptions is both a matter of modelling and selection of input and output schemes.

Open Issues: The representation of the composite linear system given in this section opens up a number of issues for further research. For different interconnection topologies and subsystem cardinalities investigate,

(i) Extending the variable complexity modelling from a single process to a composite system.
(ii) Define the stability and structural properties (McMillan degree, Kronecker invariants, etc.) of the composite system as a function of the corresponding subsystem properties.

1.5.2 Systems Instrumentation and Forms of Evolution

The problem of selection of inputs and outputs sets has a systems dimension, and it is referred to as systems or *global instrumentation* [27, 29]. This problem is different from traditional instrumentation dealing with measuring physical variables, or ways of acting on physical variables. The systems instrumentation involves a number of structure evolution processes linked to the study of four fundamental problems which are

(i) Model Orientation Problems;
(ii) Model Projection Problems;
(iii) Model Expansion Problems;
(iv) Local–Global Structure Evolution Problems.

These problems have a clear model shaping role; each one of them expresses a form of system evolution and their study is reduced to problems of Control Theory and Design. The distinguishing feature of systems instrumentation as far as model shaping is that it acts on the shaping of the input–output structure of the system, rather than the interconnection graph, as described previously. An overview of the overall instrumentation that includes traditional (macro) and systems aspects is given in [27].

1.5.2.1 Model Orientation Problem

A natural system description that makes no distinction as far as the role of process variables and their dependence, or independence is for the linear case the matrix pencil model (first-order differential descriptions) [60], or the general polynomial, or autoregressive model and these characterize the behaviour of an *implicit vector* the coordinates of which are not necessarily independent [7, 36, 69]. For control purposes, there is a need to classify the coordinates of the implicit vector into inputs, outputs and internal variables. This is referred to as *Model Orientation Problem* (MOP) [41]. In many systems, the orientation is not known or, depending on the use of the system, the orientation changes. Questions, such as when is a set of variables implied, or not anticipated by another, or when is it free, have to be answered. The solutions are systems of the standard state-space type and polynomial matrix case. The partitioning of the implicit vector results in a form of evolution of the resulted system from the generator matrix pencil, or autoregressive model with a corresponding evolution of the algebraic structure [41]. If all important variables are included in the physical modelling without a classification into inputs, outputs and internal variables is made, the emerging descriptions based on the implicit vector $\underline{\xi}$ are referred to as *implicit* and in the case of first-order differential descriptions they correspond to the matrix pencil, or generalized autonomous description [36]:

$$S(F, G) : Fp\,\underline{\xi} = G\,\underline{\xi}, \quad F, G \in \mathbb{R}^{\tau \times \upsilon}, \quad p \stackrel{\Delta}{=} d/dt. \tag{1.8}$$

The natural operator associated with such descriptions is the matrix pencil $sF - G$, and the study of such descriptions relies on the structure of $sF - G$ [19]. The classification of the variables in $\underline{\xi}$ into internal variables, or states $\underline{x}$, assignable, or control variables $\underline{u}$, and measurement, or dependent variables $\underline{y}$ is expressed in terms of transformation $\underline{\xi} = Q\,\underline{\tilde{\xi}}$, where $\underline{\tilde{\xi}} = \left[\underline{x}^t, \underline{u}^t, \underline{y}^t\right]^t$ and $Q \in \mathbb{R}^{\upsilon \times \upsilon}$, $|Q| \neq 0$. Q is the *orientation transformation* (OT) and if the original variables in $\underline{\xi}$ are physical and it is desired to preserve them, then Q has to be of the permutation type and it is a *physical* OT. For first-order linear descriptions, the most general form of oriented models is the *general singular* (GS) description [46]:

$$\begin{aligned} &S(E, A, B, C, D) : E\,p\,\underline{x} = A\,\underline{x} + B\,\underline{u}, \;\; \underline{y} = C\,\underline{x} + D\,\underline{u} \\ &E, A \in \mathbb{R}^{\sigma \times n}, \;\; B \in \mathbb{R}^{\sigma \times p}, \;\; C \in \mathbb{R}^{m \times n}, \;\; D \in \mathbb{R}^{m \times p}, \end{aligned} \tag{1.9}$$

where $\tau = m + \sigma$, $\upsilon = n + p + m$ and in general $\sigma \geq n$. In the case where $\sigma = n$, S is called *singular* and if $\sigma = n$ and $|E| \neq 0$, then the description will be called *regular* and it is equivalent to the standard state-space description $S(A, B, C, D) : p\,\underline{x} = A\,\underline{x}$

$+B\,\underline{u}$, $\underline{y} = C\,\underline{x} + D\,\underline{u}$. The *model orientation* problem (MOP) is then expressed as defining a transformation Q (free, or constrained by physical considerations) such that $S(F, G)$ is reduced to $S(E, A, B, C, D)$ or $S(A, B, C, D)$. The study of MOP for matrix pencil models in the unconstrained case has been considered in [41], and it is equivalent to a partitioning of the Kronecker invariants of $sF - G$. The nature of Kronecker invariants determines the type of the resulting oriented system.

A more general implicit description is the polynomial, or the *autoregressive representation* (AR) [69]. It is a more general implicit description, defined by a polynomial matrix $R(p)$, associated with the implicit vector $\underline{w}$ and represents the behaviour of all external variable trajectories $\underline{w}$ satisfying

$$R(p)\underline{w} = 0. \tag{1.10}$$

We may introduce orientation for such descriptions by introducing some internal variables, expressed by a vector $\underline{\xi}$ and this leads to the AR/MA representation which is specified by two polynomial matrices $H(p)$ and $Q(p)$. The external behaviour consisting of all trajectories $\underline{w}$ of the external variables is related to the trajectory $\underline{\xi}$ of the internal variables by

$$H(p)\underline{\xi} = 0, \underline{w} = Q(p)\underline{\xi}. \tag{1.11}$$

For systems with an explicit input/output structure (splitting of $\underline{w}$ into $\underline{u}$ and $\underline{y}$), the Rosenbrock's system matrix [60] in the s-domain provides a natural model, i.e.,

$$T(s)\underline{\xi} = U(s)\underline{u}, \underline{y} = V(s)\underline{\xi} + W(s)\underline{u}, \tag{1.12}$$

where all matrices are polynomial, with $T(s)$ square and invertible. The corresponding transfer function matrix $G(s)$ is represented as $V(s)T^{-1}(s)U(s) + W(s)$. The orientation problem may now be addressed in a more general setup where first we model the system behaviour, in terms of outputs, then we introduce the internal variables and then we consider the orientation. This procedure involves a realization of $R(p)$, which may be in a matrix pencil form that retains $\underline{w}$ as an output vector and includes physical variables that may act as inputs. In this context, it leads to a family of transfer functions $G(s)$ with properties that evolve from those of $R(p)$. This research is linked to the theory of strict equivalence [26].

Open Issues:There are a number of open research issues for the problem of model orientation. These involve

(i) Development of solutions for the case of matrix pencil descriptions when there are physical constraints on the partitioning of the implicit vector.
(ii) Development of solutions to model orientation for autoregressive models and its link to the theory of strict equivalence.

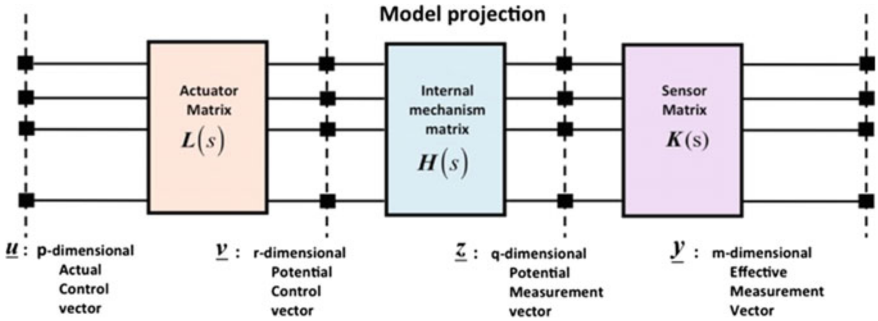


Fig. 1.10 Model projection problems

1.5.2.2 Model Projection Problems

The number of potential control and measurement variables for systems, which may be used, can become very large. For the purpose of control design, we are frequently forced to select a subset of the potential inputs and outputs as effective, operational inputs and outputs. Such a requirement implies reductions in the input and output system maps, variations in the cardinality of the system and results in an evolution process of the resulting system structure from the original one. Developing criteria and techniques for selection of an effective input–output scheme, as projections of the extended input and output vectors, respectively, is what we call *Model Projection Problems* (MPP) described in Fig. 1.10. For linear systems, where orientation has already been decided, and represented by a qxr *progenitor* rational matrix $H(s)$, the MPP is equivalent to selecting the sensor, actuator maps $\mathscr{K}, \mathscr{L}, m \leq q, p \leq r$, with representation the rational matrices $K(s), L(s)$ such that the transfer function $W(s) = K(s)H(s)L(s)$ has certain desirable properties. Clearly, the problem as stated above is in the form of a generalized two-parameter *Model Matching*. The designs of the matrices $K(s), L(s)$ are the instruments defining the evolution and may be assumed in the first instance to be constant. Note that the $\mathscr{K}, \mathscr{L}$ maps are not completely free, but they are constrained by the nature of the specific problem and the need to use certain physical variables. The evolution defined by the MPP family is linked to the process of obtaining new models by reducing the original larger input or output sets. In this sense, projection tends to aggregate and reduce an original model to a smaller dimension with desirable properties. A special problem in this area is the zero assignment by squaring down [35, 43, 59, 61]. A number of key problems related to this form of structure evolution are considered next.

Desired Generic Dimensions Problem: Defining desirable general characteristics, such as number of inputs and outputs on a system model, with some assumed internal structure, is referred to as the *generic dimensionality problem* [32]. We can use conditions, for generic solvability, or generic system properties (such as the Segre index [30]) to define the least required numbers of effective inputs and outputs needed for certain structural properties such as controllability and observability.

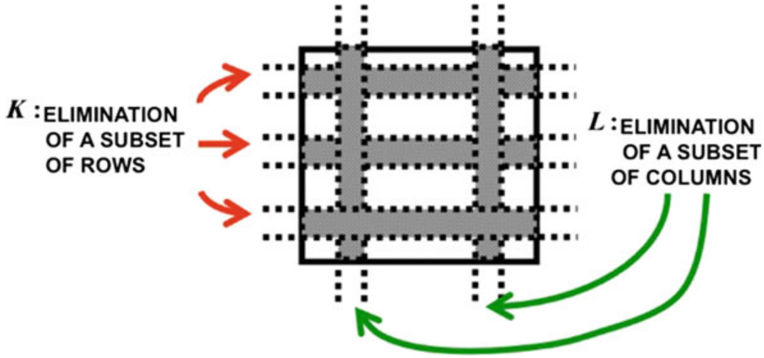


Fig. 1.11 Input–output problems reduction

Generic solvability conditions of control problems, such as pole and zero assignment under different compensation schemes, may lead to constrained integer optimization problems aiming to define generic families of systems for which a range of problems may be solved. The solvability of control problems involves the McMillan degree and/or the generic infinite zero structure of the progenitor transfer function $H(s)$. Use of such invariants requires their robust computation and this is what leads to the study of *structural identification* [42] on early models.

Input–Output Reduction and Well Conditioning of Progenitor Model Problems: The progenitor model in many applications has a large number of physical input and output variables. It is desirable to preserve, such physical variables, but their number may be too large and the progenitor model may not be well conditioned as far as its properties. A special form of model projection may be defined, where only an α subset of inputs and a β subset of outputs is used, leading to an $S_{\alpha,\beta} = S\left(A, B_\alpha, C_\beta, D_{\alpha,\beta}\right)$ subsystem with transfer function $H_{\alpha,\beta}(s)$. The objective is to select the α and β sets such that the resulting $S_{\alpha,\beta}$, $H_{\alpha,\beta}(s)$ is well structured as far as certain properties, which may include input and output regularity, non-degeneracy, minimality, output function controllability, etc. This problem is referred to as *well conditioning by input–output reduction* and it is illustrated in Fig. 1.11. Note that in a transfer function matrix setup, this problem is equivalent to defining sub-matrices of $H(s)$ by eliminating certain columns and rows leading to $W(s) = KH(s)L$ and which have desirable properties [34, 40].

Invariant Structure Transformation/Assignment Problems: The selection of given dimension and structure constant matrices K, L lead to a transfer function $W(s) = KH(s)L$ where the invariant structure of $W(s)$ is obtained by transformation of the set of invariants of $H(s)$. Apart from the study of generic properties and their link to discrete types of invariants, there is also the need to investigate the effects of input and output reduction transformation on well-defined models with fixed parameters. The transformation of one set of invariants to another is a problem not fully understood; certain results in relationship to decoupling have appeared in [13]. The special case of the general model projection is the zero assignment by squaring down leading to a square transfer function [35, 43, 44, 59, 61]. This problem involves transformation of discrete invariants (Forney indices [18]); it is well developed and belongs to the family Determinantal Assignment Problems (DAP) [34] which are studied using exterior algebra and algebraic geometry methods. The two-parameter version of squaring down aims for a transfer function $W(s)$ which is square and has a given zero structure; however, now we can also use the resulting cardinality $(m = p)$ as a design parameter. For all such problems, the overall philosophy is to design K, L such that the resulting model has a given desirable invariant structure or avoids having undesirable structural characteristics. The study of the Morgan's problem may be seen within this class as a transformation of the input structure.

Open Issues: There is a number of open research issues in the area of MPPs which involve

(i) Use of criteria for the selection of numbers of inputs p and outputs m using the McMillan degree n of $H(s)$ based on generic solvability conditions of control problems and conditions for preservation of system properties.

(ii) Characterization of evolution of the structural invariants under the action of constant pre- and post-compensation (L, K) and various input and output dimensions (p, m).

(iii) Selection of suitable (p, m) pair such that zero assignment under (L, K) pair can be achieved.

1.5.2.3 Model Expansion Problems

Defining input–output schemes with the aim to identify (or improve) a system model, or reconstruct an unmeasured internal variable, characterizes the family of *Model Expansion Problems* (MEP). This problem may be seen as prediction of an expanded model from which the current model has evolved. Questions related to the nature of test signals or properties of the measured signals are also important, on top of the more general questions related to the structure of the i/o scheme; the latter gives a distinct signal processing flavour to MEP. Model expansion expresses a form of model structure evolution, where additional inputs and outputs enable a system model to grow to a more full representation of the existing system. This expresses an alternative form of evolution of structure of the model by manipulation of the input–output, external structure. Some distinct problem areas are as follows.

Additional Measurements for Estimation of Variables: In systems, some important variables are not available for measurement. It is then that secondary measurements have to be selected and used in conjunction with estimators to infer the value of unmeasurable variables. The selection of secondary measurements is important for the synthesis of control schemes. The various aspects of the problem are discussed within the well-developed area of state estimation [45].

Input and Output Schemes for System Identification: The selection of input test signals and output measurements is an integral part of the setting up of model identification experiments [17]. In fact, the identified model is always a function of the way the system is excited and observed, i.e., of the way the system is embedded in its experimental environmental. The study of effect of location of the excitation signals and corresponding group of extracted measurements on the identification problem has received less attention. Issues such as how and whether additional excitation signals and extracted measurements may enhance the scope and accuracy of identifiable models are important problems. This area is closely related to the problem of identifiability of models [20, 47].

Model Completion Problems: This class of problems deals with the problem of augmentation of a system operator, like the matrix pencil and has a dual nature to that of model projection, since now we deal with dimensional expansion of the relevant operator. Let $sE - H$ be an $r \times q$ pencil, which is a sub-pencil of the $(r + t) \times (q + v)$ pencil $sE^{'} - H^{'}$, where

$$sE^{'} - H^{'} = \begin{bmatrix} sE - H & X \\ X & X \end{bmatrix}, \tag{1.13}$$

and the X's stand for unspecified pencils of compatible dimensions. Studying the relationships between the sets of invariants of $sE - H$ and $sE^{'} - H^{'}$ pencils and in particular examining the conditions under which we may assign arbitrarily the structure of $sE^{'} - H^{'}$ are known as *Matrix Pencil Completion Problem* [48]. The above formulation may be also extended to that of expansion of polynomial or rational models. It is worth pointing out that such formulations make sense as long as the implicit vector corresponding to the expanded system has new variables which makes sense.

Open Issues: Model expansion requires additional research in areas such as follows:

(i) Detailed system modelling to define family of models which contain the given model as a projection.

(ii) Development of matrix pencil completion problem to the case of polynomial models and rational transfer functions.

1.5.2.4 Local–Global Structure Evolution

When we consider composite systems, then all issues and problems of model orientation, model projection and model expansion may be transferred to the composite system. Note that now the subsystem cardinality may be varying with respective implications on the graph-dependent structure and properties. Guaranteeing structural properties such as controllability, observability, rejection of disturbances, etc. requires the definition of subsets of inputs and outputs at the local, or global level, or appropriate structural combinations of them. In terms of the two-parameter scheme associated with MPP, there is need to define the required Boolean structure of K, L transformations on an internal model, or the modification to the internal graph that can guarantee such properties. This involves examining the Kronecker structure of graph-structured pencil models, which are linked to the presence or the absence of certain system properties.

Open Issues: The role of the interconnection graph is now crucial in defining structure evolution problems which include

(i) Study of Kronecker invariants for structured matrix pencils.
(ii) Development of non-structural (stability, minimum phase) and structural system properties for alternative interconnection graphs and possible variable subsystem cardinality.
(iii) Investigate the role of interconnection graph on system properties, when subprocesses are described by variable complexity models.

1.6 Integrated Operations and Emergent Properties

Figures 1.3 and 1.4 describe the integrated system represented by both operations and designs. Production-level activities take place on a given system; they are mostly organized in a hierarchical manner and they realize the higher level strategies decided at the business level. Vertical activities are issues going through the Business–Operations–Design hierarchy, and they have different interpretations at the corresponding level. The Physical Process Dimension deals with issues of design–redesign of the Engineering Process, and here the issues are those related to integrated design [28]. The Signals, Operations Dimension is concerned with the study of the different operations, functions based on the Physical Process and it is thus closely related to operations for production [58]. In this area, signals and information extracted from the process are the fundamentals and the problem of integration is concerned with understanding the connectivities between the alternative operations and functionalities and having some means to regulate the overall behaviour. Both design, operations and business generate and rely on data and deploy software tools and such issues are considered as vertical activities.

1.6.1 The Multi-modelling and Hierarchical Structure of Integrated Operations

The study of Industrial Processes requires models of different types. The borderlines between the families of Operational Models (OM) and Design Models (DM) are not always very clear. Models linked to design are "off-line", whereas those used for operations are either off-line or "on-line" [58]. For process-type applications, models are classified into two families referred to as "line" and "support" models [58]. Line models are used for determining desired process conditions for the immediate future, whereas support models provide information to control models, or they are used for simulation purposes. A major classification of models is into those referred to as "black" and "white" models [58]. White models are based on understanding the system (physics, chemistry, etc.), and their development requires a lot of process insight and knowledge of physical/chemical relationships. Such models can be applied to a wide range of conditions, contain a small number of parameters and are especially useful in the process design, when experimental data are not available. Black models are of the input–output type and contain many parameters, but require little knowledge of the process and are easy to formulate; such models require appropriate process data and they are only valid for the range, where data are available.

Handling the high complexity of the overall system is through aggregation, modularization and hierarchization [8], and this is what characterizes the overall structure described in Fig. 1.3. To be able to lump a set of subsystems together and treat the composite structure as a single object with a specific function, the subsystems must effectively interact. Modularization refers to the composition of specific function units to achieve a composite function task. Aggregation and modularization refer to physical composition of subsystems through coupling, and it is motivated by the needs of design of systems. Hierarchization is related to the stratification of alternative behavioural aspects of the entire system and it is motivated by the need to manage the overall information complexity. The production system may be viewed as an information system, and thus notions of complexity are naturally associated with it [49].

Hierarchization has to do with identification of design and operational tasks, as well as reduction of externally perceived complexity to manageable levels of the higher layers. At the top of the hierarchy, we perceive and describe the overall production process as an economic activity; at this level we have the lowest complexity, as far as description of the process behaviour. At the next level down, we perceive the process as a set of interacting plant sections, each performing production functions interacting to produce the economic activity of the higher level. At the next level down in the hierarchy, we are concerned with specification of desired operational functions for each unit in a plant section and so on, and we can move down to operation of units with quality, safety, etc., criteria and further down to dynamic performance, etc. In an effectively functioning hierarchy, the interaction between subsystems at lower level is such as to create a reduced level of complexity at the level perceived above [49]. The hierarchization implies a reduction of externally perceived complexity successfully, as we proceed up the hierarchy till the top level. A simpler representation of the overall operational hierarchy of Fig. 1.3 is as shown in Fig. 1.12 [22] having blocks with the following modelling requirements:

0-level: (Signals, Data Level). Physical variables, Instrumentation, Signal processing, Data Structures.

1, 2-levels: (Primary Process Control). Time responses, Simple linear SISO / MIMO models.

3-level: (Supervisory Control Level). Process Optimization Models, Statistical Quality Models (SPC, Multivariate, Filtering, Estimation), Fault Diagnosis, Overall Process State Assessment Models.

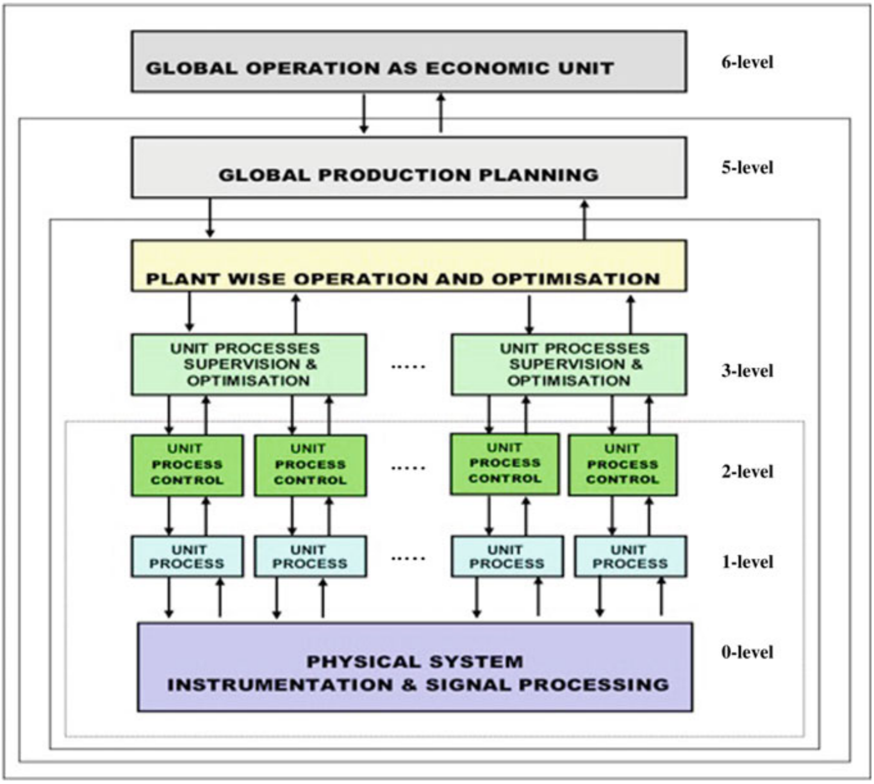


Fig. 1.12 Hierarchy of system operations

4-level: (Plant Operation and Logistics). Nonlinear Static or Dynamic Models for Overall Plant, Operational Research Models, Discrete Event Models (Petri Nets, Languages, Automata).
5-level: (Global Production Planning Level). Production Models, Planning, Forecasting, Economic Models, Operational Research, Game Theory Models.
6-level: (Business Level). Enterprise, Business Modelling, System Dynamics, Forecasting, Graph Models, Economic Models, etc.

A functional representation of the overall system represented by families of relevant models which provide an alternative description of Fig. 1.3 is given in Fig. 1.13. The different types of models in the above groupings are interrelated. Each of the model families on the unit level is simplified and aggregated to models on the plant level and then on the production site, business unit and possibly the enterprise level. Model composition is accompanied by simplification. The latter classification is of functional type, and the Process Control Hierarchy implies a nesting of models to a layered hierarchy with variable complexity as shown in Fig. 1.13. This diagram indicates that at the level of the process we have the richest possible model in terms of signals, data, full dynamic models. Then, as we move up in the hierarchy, the corresponding models become simpler, but also more general since they then refer not to a unit but to a section of the plant. The use of functional-type models for the Process Control Hierarchy implies a nesting of models to a layered hierarchy with variable complexity as shown in Fig. 1.13. The diagram of Fig. 1.13 indicates that at the level of the process we have the richest possible model in terms of signals, data and full dynamic models. Then, as we move up in the hierarchy, the corresponding models become simpler, but also more general since they then refer not to a unit but to a section of the plant. The operation of extraction of the simpler models is some form of projection, whereas wider scale models are obtained by using plant topology and aggregations. These models, although of different nature and scope, are related, since they describe aspects of the same process. Dynamic properties of subsystems are reflected on simpler, but wider area models, although this is what we may refer to as Embedding of Function Models [22].

1.6.1.1 System and Emergent Properties

The notion of emergence is intimately linked with complex systems and has its origins in philosophy [38]. With complex processes such as the Integrated Design and Operations, there is a number of emergent properties appearing which require a systems-based characterization. Emergence refers to understanding how collective properties arise from the properties of parts. More generally, it refers to how behaviour at the collective level of the system arises from the detailed structure, behaviour and relationships on a finer scale. System properties may be classified as intrinsic and extrinsic. An intrinsic property relates to the class of features and characteristics which is inherent and contained wholly within a physical or virtual object. The extrinsic properties are those which are not part of the essential nature of things and have their origin outside the object under scrutiny. Within the context of a system, an emergent property is an extrinsic one that is not an intrinsic property of any constituent of that system, but is manifested by the system as a whole.

A description of system properties in the operational system hierarchy of Fig. 1.3 is given by the diagram in Fig. 1.14. The above diagram provides an overview of the integrated system and the system properties associated with it. The family of system properties is classified into the family of *intrinsic system properties* and the *emergent properties*. Intrinsic system properties such as systems safety, reliability, quality, etc. have the common feature that can be directly assessed from signals and data associated with the physical system (physical, communications, operations layers). Emergent properties on the other hand, such as risk, assurance and sustainability, can be assessed using the inputs provided from indicators associated with the intrinsic system properties. We may define:

Definition 1.5 An *emergent property* is an extrinsic system property, which is assessed using an *emergent property* function with a domain, the set of values of intrinsic system properties.

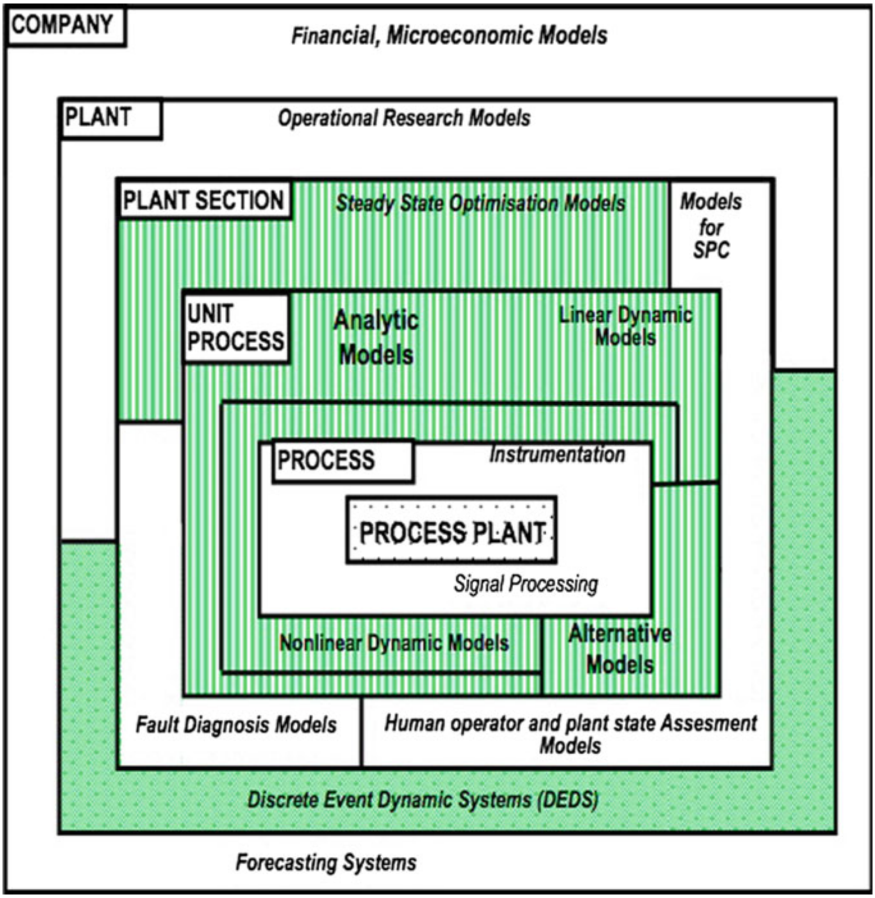


Fig. 1.13 Functional representation of the integrated system

The development of the emergent property function is a study subject in the area of overall assessment of industrial/manufacturing systems.

1.6.1.2 Control in the Hierarchical Structure

The hierarchical model of the Overall Process Operations involves processes of different nature expressing functionalities of the problem. Such processes are interlinked, and each one of them is characterized by a different nature model. We can use input–output descriptions for each of the subprocesses, with an internal state expressing the variables involved in the particular process and inputs and outputs expressing the linking with other processes [22]. Such a model is generic and can be used for all functionalities described in Fig. 1.3. We may adopt a generic description

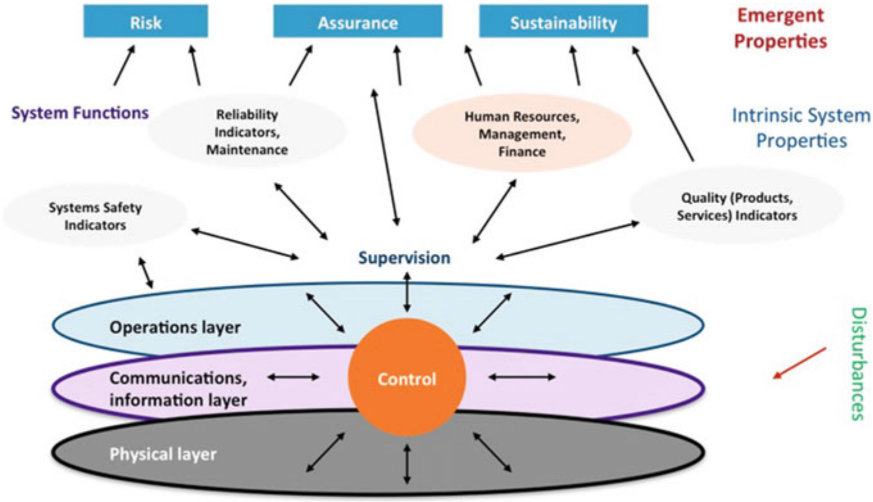


Fig. 1.14 System and emergent properties

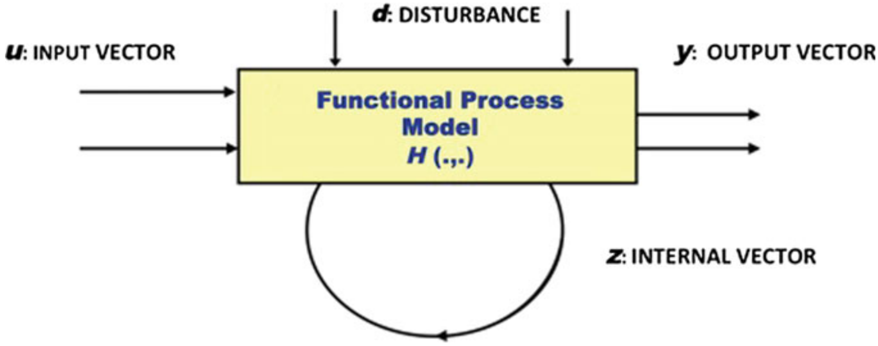


Fig. 1.15 A functional model for a general process

for the various functions as shown in Fig. 1.15, where u_i denote independent manipulated variables of the function model, called system inputs; y_j are the independent controlled variables that can be measured and they are called the system outputs; d_k are the exogenous variables, the disturbances. A description for a functional model of a general process expresses the relationships between the vectors $\underline{u}$, $\underline{d}$, $\underline{y}$ defined by $\underline{y} = H(\underline{u}; \underline{d})$, where H expresses relationships between the relevant variables, and described in Fig. 1.15.

$$M\,(\underline{u},\ \underline{y},\ \underline{d};\ \underline{z})\ \begin{cases} F\,(\underline{z},\ \underline{u},\ \underline{d})\ =\ 0 \\ \underline{y}\ =\ G\ \left(\underline{z},\ \underline{u},\ \underline{d}.\right) \end{cases}$$

The development of such models involves the following:

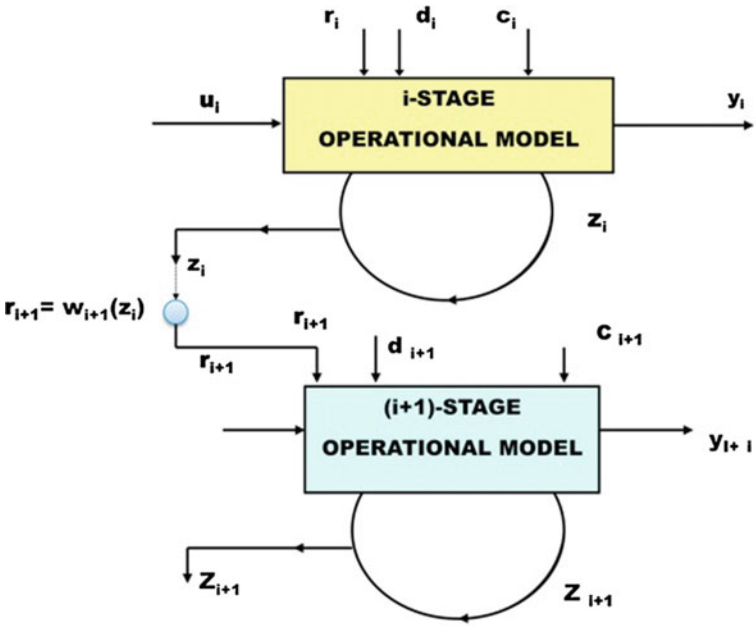


Fig. 1.16 Nesting of models in the hierarchy

(i) For the given function establish a conceptual model based on its role in the operational hierarchy.

(ii) Define the vector of internal variables z and determine its relationships to input and output vectors using any physical insight that we may possess about the functioning of the internal mechanism.

(iii) Establish the relationships between the alternative vectors z associated with problems of the operational hierarchy.

(iv) Define the appropriate formal model to provide an adequate description for the H functional model.

These generic steps provide an approach, which involves many detailed modelling tasks. The above generic steps are providing an approach, which, however, involves many detailed modelling tasks involving issues such as classification of variables to inputs, outputs, disturbances, internal variables, specification of formal description for H, definition of performance indices, etc. The nature of variables and the type of problem under consideration determines the nature of the F, G functions. The implicit model $M(\underline{u}, \underline{y}, \underline{d}, \underline{z})$ is referred to as a $\underline{z}$-stage model.

The selection of the operational stage determines the nature of the internal vector $\underline{z}$ and thus also the corresponding $\underline{z}$-stage. Describing the relationship between different stages, internal vectors are closely related to the problem that is referred to as *Hierarchical Nesting* or *Embedding of Function Models*. The fundamental shell of this hierarchical nesting architecture is described in Fig. 1.16. The reference vector

$\underline{r}_{i+1}$ of operational objectives of the $(i+1)$-stage is defined as a function of the i-th-stage internal vector $\underline{z}_i$, that is, $\underline{r}_{i+1} = w_i(\underline{z}_i)$. A scheme such as the one described above is general and can be used to describe the essence of the hierarchical nesting. This scheme can be extended to describe relations between models associated with functions at the same level of the hierarchy, extend upwards to business-level activities and downwards to the area of the physical process. It is worth pointing out that as going down the hierarchy the complexity and granularity of the subprocess models increase, whereas their nature changes. Note that all subsystems are linked and they express an alternative form of nesting of subsystems of variable complexity and nature, which may be referred to as hierarchical nesting of the operationally integrated system.

The hierarchical nesting introduces new control and measurement issues for the operationally integrated system. If the vector of internal variables in the j-th subsystem in the hierarchy is a state vector x_j, then its state space $\mathscr{X}_j$ is linked to the final subsystem state space $\mathscr{X}_k$ (Physical subsystem state space) in terms of projections/aggregation. The final subsystem state space $\mathscr{X}_k$ of the system corresponds to the physical subsystem of the integrated system. The nesting of state spaces implied by the hierarchical structure is described in Fig. 1.17, and hierarchy-depended system issues related to the implied coupling of subsystems and the different nature of their models. The fact that each stage model in the hierarchy is of different nature than the others makes the overall system of hybrid nature [5].

The nesting of systems implies a multilayer hybrid structure and some new system issues related to the notions of

(i) Global Controllability;
(ii) Global Observability.

Global Controllability Problem: This refers to the crucial issue of whether a high-level objective (possibly generated as the solution of a decision problem at a high level) can be realized within the existing constraints at each of the levels in the hierarchy and finally at lowest level, where we have the physical process (production stage). This is a problem of *Global Controllability*, which may be seen as a problem of *Realization of High-Level Objectives* throughout subprocess in the hierarchy. This new problem requires development of a multilevel hybrid theory, and it can take different forms, according to the nature of the particular stage model. The Global Controllability problem is central to the development of top-down approaches in the study of hierarchical organizations.

Global Observability Problem: This is of dual nature to global controllability and refers to the property of being able to observe aspects of behaviour at the different layers in the hierarchy by appropriate measurements, or estimation processes which are built in the overall scheme. *Global observability* expresses the ability to define *model-based diagnostics* that can predict and evaluate certain aspects of the overall behaviour. It is assumed that the observer has access to the information contained at all stages of the hierarchy, where only external measurement provides the available information. This problem is linked to the development of *bottom-up* approach in the study of hierarchical organizations. The measurements and diagnostics defined

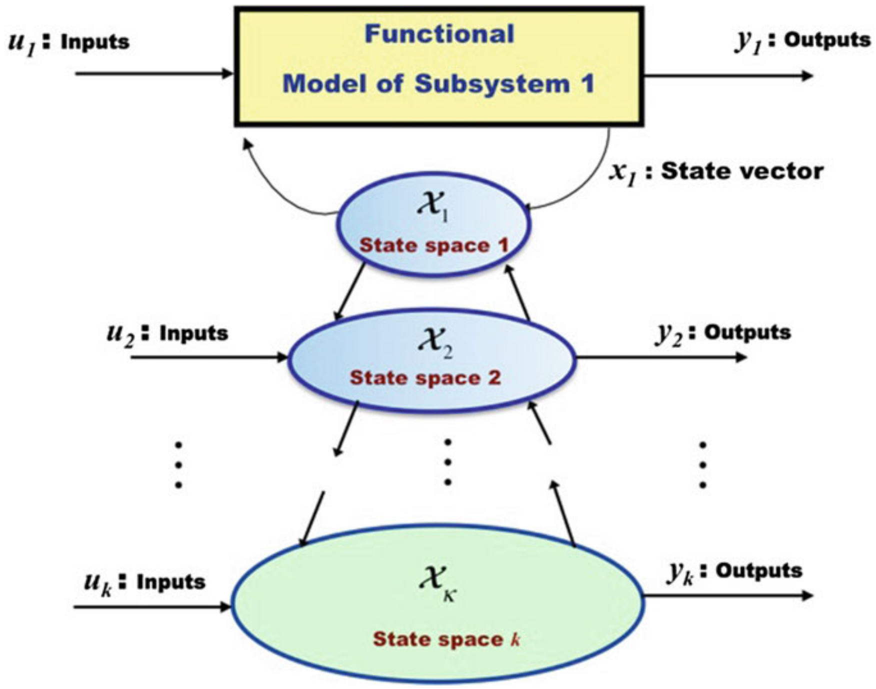


Fig. 1.17 Dynamical nesting in the hierarchy

on the physical process are used to construct the specific property functional models, and thus global observability indicates the quality of the respective functional model.

Open Issues: Integration of Operations requires study of fundamental problems and development of new research areas which include the following:

(i) Understanding the derivation of the different functional models and how they are interfaced, referred to as f*unctional model derivation and interfacing*.
(ii) Defining models for the characterization of emergent properties.
(iii) Development of multilevel hybrid system theory.
(iv) Understanding the different aspects of global controllability and global observability and defining criteria for their characterization.

1.7 The Notion of System of Systems

The concept of "System of Systems" (*SoS*) has emerged in many and diverse fields of applications and describes the integration of many independent, autonomous systems, frequently of large dimensions, which are brought together in order to satisfy a global goal and under certain rules of engagement [23, 50]. These complex multisystems exhibit features well beyond the standard notion of system composition, and represent a synthesis of systems which themselves have a degree of autonomy, but this composition is subject to a central task and related rules. So far, the term *SoS* has been used in a very loose way, by different communities and defined in terms of their characteristics with no special effort to give it a precise definition based on rigorous methodologies and concepts of the Mathematical System Theory [52]. Establishing the links with the traditional system theory approaches is essential, if we are to transfer and appropriately develop powerful and established analytical tools required for their design/redesign. Within this new challenging paradigm, the notion of emergence is also frequently used in a rather loose way. The need for a structured definition of the *SoS* notion has been raised in [33, 37, 52] and will be further developed in this section. A central part of our effort is to explain the difference of *SoS* from that of *Composite Systems* (CoS) which leads to the generalization of the standard notion of *interconnection topology* (linked to composite systems) to the new notion of "*systems play*" [52].

1.7.1 The Empirical Definition of System of Systems

An aggregate of systems leads to the creation of new forms of systems which may be either described within the framework of composite systems, or demonstrate additional features which add complexity to the description and may be referred to as system of systems. The term system of systems (*SoS*) has been used in the literature in different ways and a good treatment of the topic is given in [23]. Most definitions (see references in [23]) describe features or properties of complex systems linked to specific examples. The class of systems exhibiting behaviour of *SoS* typically exhibits aspects of the behaviour met in complex systems; however, not all complex problems fall in the realm of *SoS*. Problem areas characterized as *SoS* exhibit features such as follows [51]:

System of Systems Features: Operational independence of elements; Managerial independence of elements; Evolutionary development; Emergent behaviour; Geographical distribution of elements; Interdisciplinary study; Heterogeneity of systems; Network of systems.

The definitions that have been given so far contain elements of what the abstract notion should have, but they are more linked to specific features linked to areas of applications. A summary of different definitions is given in [37] (Part 1) where the different sources are also listed.

Summary of descriptive definitions for *SoS*

(i) Systems of systems exist when there is a presence of a majority of the following five characteristics: operational and managerial independence, geographic distribution, emergent behaviour and evolutionary development.

(ii) Systems of systems are large-scale concurrent and distributed systems that are comprised of complex systems.

(iii) Enterprise Systems of Systems Engineering is focused on coupling traditional systems engineering activities with enterprise activities of strategic planning and investment analysis.

(iv) System of Systems Integration is a method to pursue development, integration, interoperability and optimization of systems to enhance performance in future battlefield scenarios.

(v) In relation to joint war-fighting, system of systems is concerned with interoperability and synergism of Command, Control, Computers, Communications, and Information and Intelligence, Surveillance, and Reconnaissance Systems.

(vi) System of Systems is a collection of task-oriented or dedicated systems that pool their resources and capabilities together to obtain a new, more complex, "meta-system" which offers more functionality and performance than simply the sum of the constituent systems.

(vii) Systems of Systems are large-scale integrated systems which are heterogeneous and independently operable on their own, but are networked together for a common goal. The goal, as mentioned before, may be cost, performance, robustness, etc.

(viii) A System of Systems is a "super system" comprised of other elements which themselves are independent complex operational systems and interact among themselves to achieve a common goal. Each element of an *SoS* achieves well-substantiated goals even if they are detached from the rest of the *SoS*.

The above definitions are mostly descriptive, but they capture crucial features of what a generic definition should involve; however, they do not answer the question, why is this new notion different than that of composite systems. The last two definitions [23] are more generic and capture the key features of the notion, but they still do not provide a systems working tool for design and redesign of *SoS*. A major task in providing a systems definition for *SoS* is to demonstrate the differences between *SoS* and *Composite Systems* (CoS) and explain why *SoS* is an evolution of *CoS*.

1.7.2 Composite Systems and SoS: The Integrated Autonomous and Intelligent System

Developing the transition from CoS to *SoS*, we need to identify the commonalities and differences between the two notions. We note [37]:

(a) Both CoS and *SoS* are compositions of subsystems and they are embedded in the environment of a larger system.

(b) The subsystems in CoS do not have their independent goal; they are not autonomous and their behaviour is subject to the rules of the interconnection topology.

(c) The interconnection rule in CoS is expressed as a graph topology.

Fig. 1.18 Simplified description of the system

(d) The subsystems in *SoS* may have their own goals, and some of them may be autonomous, semi-autonomous, or organized as autonomous groupings of composite systems.
(e) There may be a connection topology rule expressed as graph topology for the information structures of the subsystems in an *SoS*.
(f) The interconnection rule is described in a more general form than that of the graph topology, named as "*systems play*" and every subsystem enters as an agent with their individual operational sets and goals.

The development of an abstract, systems-based characterization of *SoS* requires the following:

(i) Consider the notion of the system in a more general setup suitable for *SoS*.
(ii) Specify the special features which define the notion of "intelligent autonomous agent".
(iii) Provide a characterization of the generalized notion of relationships defined as "systems play".

First, we revisit the definition of a system as given in Sect. 1.2 and illustrated in Fig. 1.2 and define the notion of autonomy.

Definition 1.6 A simple or composite system is referred to as autonomous, if its relations with other systems in the environment are expressed only through the operational goals.

The notion of autonomy implies that as far as the other systems in the environment are related only through its goals and not through some interconnection topology. This is described in Fig. 1.18.

We are also referring to the notion of intelligent system, and this requires some appropriate interpretation in terms of control capabilities. We may define this notion as follows.

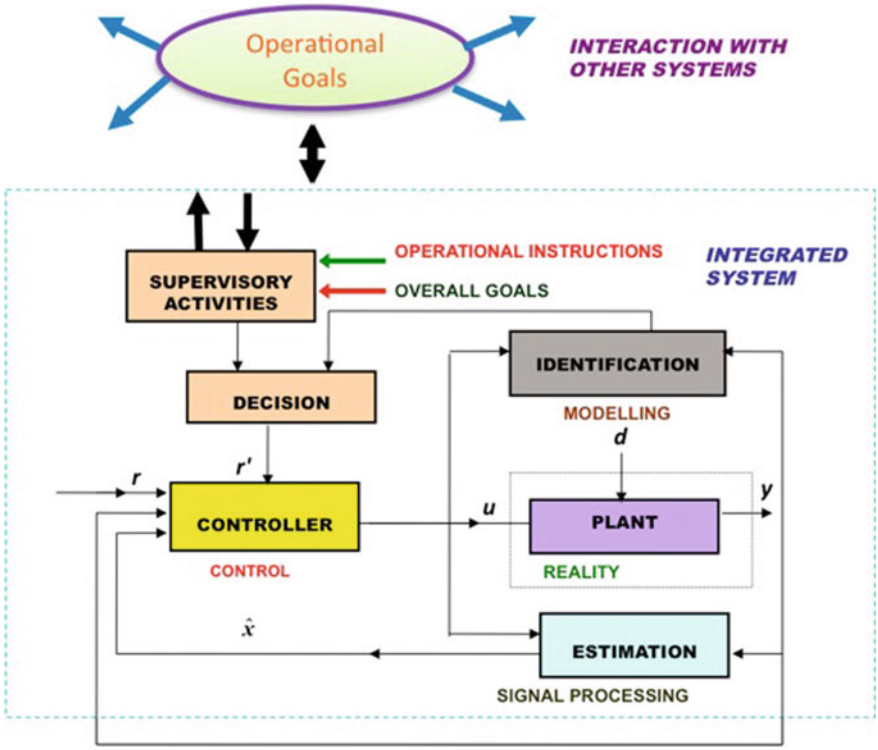


Fig. 1.19 Integrated and autonomous system © [2013] IEEE. Reprinted, with permission, from [33]

Definition 1.7 A system that has the ability to develop measurements for its behaviour, define models for different operational conditions, self-adapt itself to changes in the environment and control its behaviour subject to a set of objectives and constraints will be called *intelligent* or *intelligent agent*.

The intelligent system has the ability to interpret the high-level operational goals to appropriate measurement and control strategies which will enable to implement their realization at all aspects of its behaviour. A scheme that will enable the functioning of a system as an intelligent agent is represented in Fig. 1.19.

In the system representation of Fig. 1.19, the system appears as an autonomous agent (internal system structure together with its inputs and outputs), where its kernel is the physical system (simple or composite), having its operational instructions, and provided with control, modelling, measurement and supervisory capabilities; the supervisory activities allow the realization of the higher level operational goals.

Such a system is an intelligent agent and will be referred to as *integrated system* [33]. As far as its behaviour within the general systems' environment, this type of system is engaged only in terms of its operational goals. The *integrated system* term is used to distinguish it from systems which have no integrated control and information processing capabilities and which may be referred to as basic systems. The integrated structure implies that such goals may be realized and produce the desirable behaviours. Interactions with other systems take place only at the operational goals level.

1.7.3 The Systemic Definition of System of Systems

The distinguishing feature of *SoS* is that the subsystems participate in the composition as intelligent agents with a relative autonomy and behave as actors in a play. The latter property requires that the systems entering the composition, expressed as rules, are of the integrated type, since this requires capabilities for control, estimation, modelling and supervisory capabilities. Features, such as large dimensionality, heterogeneity, network structure, Operational, Adaptability, Emergent Behaviour, etc. may be also present in *SoS* as well as in the case of CoS. We define:

Definition 1.8 Consider a set of systems $\{S\} = \{S_i : i = 1, 2, \ldots, \mu\}$ and let $\mathscr{F}$ be an interconnection rule defined on the information structures of S_i systems. The action of $\mathscr{F}$ on $\{S\}$ defines a *Composite System*, $S_{cos} = \mathscr{F} * \{S_i\}$ or the composition of $\{S\}$ under $\mathscr{F}$.

In the above definition, the subsystems S_i can be basic or integrated. The information structure of each system is defined by the pair of the input and output influence vectors, and the interconnection rule may be represented by a graph topology [32]. The resulted system is embedded in a larger system and it is treated as new system with its own system boundary. This definition may now be extended as follows.

Definition 1.9 Consider a set of integrated and autonomous systems $\{S\} = \{S_i, P_i : i = 1, 2, \ldots, \mu\}$, where P_i are the operational goals of S_i. If $\mathscr{G}$ is a general rule defined on the operational goals P_i, of S_i systems such that $\mathscr{G} : P_1 \times \ldots \times P_\mu \to \mathscr{G}(P_1 \times \ldots \times P_\mu) = \mathscr{G}(P_i)_1^\mu$, then the system

$$S_{SoS} = \mathscr{G} * \{S_i\} = \{S_i, P_i : \mathrm{i} = 1, \ldots, \mu; \mathscr{G}(P_i)_1^\mu\} \tag{1.14}$$

will be called a *System of Systems* associated with a systems play $\mathscr{G}$.

In the above definition, the notion of *SoS* emerges as an evolution of CoS since the systems are assumed to be integrated and autonomous. The notion of the interconnection topology $\mathscr{F}$ is now extended to that of the *systems play* $\mathscr{G}$. The subsystems in *SoS* now act as autonomous intelligent agents which can enter into relations with other systems as defined by the systems play $\mathscr{G}$. The integrated nature of the subsystems implies that the results of the system play can be realized within each subsystem. The nature of the applications defines the *systems play*, which frequently may be expressed as a game defined on intelligent agents.

An *SoS* emerges as a multi-agent system composed of multiple interacting intelligent agents (the subsystems). This multi-agent systems view allows *SoS* to act as vehicle to solve problems which are difficult or impossible for an individual agent. The multi-agent dimension of *SoS* has important characteristics such as follows [1]:

(i) Autonomy: the agents are at least partially autonomous;
(ii) Local views: no agent has a full global view of the system, or the system is too complex for an agent to make practical use of such knowledge;
(iii) Decentralization: there is no designated single controlling agent, but decision and information gathering is distributed.

It is these properties that allow *SoS* to develop "self-organization" capabilities and find the best solution to the problems defined on them.

A major challenge in the development of a unifying approach to the study of *SoS* is the quantitative characterization of the new notion of the *systems play*. Taking into account that *SoS* problems emerge in many and diverse domains, it is clear that some classification of the general *SoS* family into subfamilies with common characteristics is essential before we embark to the characterization of notions such as *systems play*. There is frequently the use of the term *SoS* for *Physical or Natural* systems. Such systems are related to the natural world and social–economic phenomena and are the results of evolution of physical, or socio-economic processes and typical examples are those of the "ecosystem" of a geographical region, or issues related to "social phenomena". Our current approach based on *Artificial or Engineered SoS* requires further development to handle issues of lack of autonomy, or uncertainty in the links between subsystems. Of course, there are grey areas between the two classes of *Artificial and Engineered SoS (E-SoS)* and some further classifications are given in [33].

Note that in *E-SoS* the "goal" is linked to some coordination effort. This leads to another way of classifying *SoS* based on structural and operational characteristics. This classification refers to the mechanisms defining the relations between the subsystems. We may distinguish [33]

- Goal Driven and Unstructured (GU-*SoS*)
- Goal Driven with Central Coordination (GC-*SoS*)

In *GU-SoS* class, the central goal for the system operation is set, as well as the environment within which the system operations will take place. In this case, the nature of the system play is entirely defined by the set goal, which may be in the form of a game where the intelligent agents may participate. A further classification for this class is into

- Pure Goal Driven (P-GU-*SoS*)
- Goal and Scenario Driven (S-GU-*SoS*)

In the *P-GU-SoS* class, the subsystems, as intelligent agents, interpret the central goal, may assign to themselves sub-goals and they then develop actions and self-organization to achieve the central goal, which may be expressed as optimization of a performance index, subject to satisfaction of their individual goals. In *S-GU-SoS*, a scenario linked to the goal is given, the subsystems as intelligent agents undertake roles which aim to optimize a central performance index and satisfy their own particular goals. The *GC-SoS* class on the other hand has the same features as the P-GU-*SoS* and similar subclasses with the additional feature of the existence of coordination. The existence of coordination introduces a structure to the interpretation of the goal by the subsystem and the development of appropriate scenarios to achieve the central goal and partial goal. Coordination is common to E-*SoS* and may be viewed as an interpreter for the development of operational activities. The nature of coordination also introduces special features to *SoS* characterization since it introduces a structure to the resulted *systems play*. Coordination is a form of organization, and there may be different types such as "Hierarchical", "Heterarchical" and "Holonic" [67]. Such forms of organization structure affect the systems play and the development of scenarios. Types of *SoS* where the subsystems are of the engineering type without human action involvement are referred to as "*hard*". Systems involving human presence and behaviour will be referred to as "*soft*" and those involving a mixture of the two types will be called "*hybrid*".

1.7.4 Methods for the Characterization of Systems Play

The development of a description for the systems play depends on the nature of the particular *SoS*. An effort to review the relevant methodologies from system and control which may be used in describing the systems play has been given in [33] and is outlined next. These different methodologies may provide the required framework for the characterization of the systems play and include methods such as Cooperative Control, Market-Based Coordination Techniques, Population Control methodologies and Coalition Games. Each of these methodologies provides formal descriptions of the notion of *systems play*, and they are outlined next.

Cooperative Control: A typical case describing a class of *SoS* is the Vehicle Formation Problem [16] defined as the control of the formation of v vehicles that are performing a shared task; the task depends on the relationship between the locations of the individual vehicles and the task defines the scenario that has to be realized. It is assumed that the vehicles are able to communicate with the other vehicles in carrying out the task and they have capabilities to control their position in the effort to perform the task. Each vehicle is described as a rigid body moving in space and a state vector x_i may be associated with each one; by $x = (x_1, .., x_v)$, we may represent the complete state for the set of v vehicles. The collection of all individual states defines the state of the system, and the execution of assigned task requires the assignment of additional states that can make the system an *SoS*. The development of the scenario and task is handled by introducing for each vehicle an additional discrete state, α_i, which defines the role of the vehicle in the task and which is represented as an element of a discrete set A, the nature of which depends on the specific cooperative control problem. Such problems may be formulated as constrained optimization problem. For *SoS*, the problems of interest are those involving cooperative tasks that can be solved using a decentralized strategy.

Market-Economics Based Coordination Techniques: The distinguishing feature of *SoS* is that there are autonomous units with their own management and control functions that are coupled by resource flows which need to be balanced, over appropriate periods of time depending on local or global storage capacities. The performance of the subsystem consumption and production is influenced by availability of these resources. To perform an arbitration of these flows requires economic balancing mechanisms [12, 68]. The management of the resource flows may be expressed as a network management problem, given that the resource flows define some generic network structure within which we define the flows. Clearly, the overall system performance and behaviour is influenced by discrete decisions taken. Two different approaches that can be used for the management of such flow-coupled *SoS* are *economics-driven coordination* and *market-based mechanisms*. In both cases, the coordinator has only limited information about the behaviour and the constraints of the local units which perform a local optimization of their operational policies. In the economics-driven coordination, it is assumed that the control of *SoS* involves the setting of production / consumption constraints or references between the global *SoS* coordinator and the controllers of individual systems. The *SoS* coordinator utilizes simplified models of the subsystems, and a model of the connecting networks to compute references or constraints on the exchanged flows. The resulting optimization is based on the dynamic price profiles for the resources that are consumed or produced by the subsystems over the planning horizon. An alternative approach is to use mechanisms employing the concepts of *economic markets* to distribute limited resources between subsystems. The market is defined as a population of agents consisting of producers selling goods and consumers buying these goods [12], where the consumers' demand depends on the usefulness or *utility* of a good for the completion of its task. Market-based mechanisms are inherently decentralized and can thus be mapped directly to systems with autonomous subsystems.

Population Control Methods: Population control refers to systems that comprise a large number of semi-independent subsystems, which macroscopically are viewed in terms of their emergent behaviour. Such systems are used in ecology to capture the fluctuations in the populations of interacting species and the relevant models use continuous variables to capture populations and differential equations to capture their evolution. Of special interest is the class of mixed-effect models [54], which address the evolution of a heterogeneous population of individuals, which deploy ordinary differential equations, but with parameters linked to appropriate probability distributions. Population systems dynamics are gaining in importance, as man-made systems become increasingly complex and larger scale and control of the emergent behaviour of large collections of semi-autonomous subsystems becomes an issue. Such methods are primarily motivated by biological applications but have potential for the engineering field of *SoS*.

Coalition Games: The basic idea of *SoS* is to consider the overall system as a set of subsystems that are controlled by local controllers or agents which may exchange information and cooperate. This feature demonstrates the link of *SoS* to distributed and decentralized control schemes with the additional property that the interaction between the subsystems may indicate a time-varying coupling. It is this special feature that indicates the links to a rather new category of management and control schemes referred to as *coalitional management schemes* [54]. In this paradigm, different agents cooperate when there is enough interaction between the controlled systems and they work in a decentralized fashion when there is little interaction. A coalition is a temporary alliance or partnering of groups in order to achieve a common purpose or to engage in a joint activity [56]. A coalition of systems is a temporary system of systems built to achieve a common objective. Forming coalitions requires that the groups have similar values, interests and goals which may allow members to combine their resources and become more powerful than when they each acted alone.

Open Issues: The development of a systems theoretic approach for *SoS* is still in its early stages of development. Broad areas where development is required involve

(i) Characterization of the family of *SoS* according to their origin (engineered, natural, social systems);

(ii) Identifying the methodologies that can contribute to the characterization of *systems play*.

1.8 Conclusions and Future Research

Control Theory and Design have developed around the classical servomechanism paradigm. The area of Systems Integration for large Complex Systems, involving both design and operations, introduces many new challenges and a number of new paradigms generating new requirements and needs for future developments of the Systems and Control Theory beyond the classical paradigm. The identified new paradigms of *Structure Evolving Systems (SES)* and *System of Systems (SoS)* are new areas of complex systems relevant to integrated design and operations for the family of *Engineered Systems*. For the design problem, the challenges in the *SES* area come from the cascade and design-time-dependent evolutionary nature of the process, whereas for system operations the challenges come from the *SoS* type of complexity and specifically the need to characterize the notion of the *systems play*. Additional issues that introduce new dimensions of complexity come from the large dimensions of the processes which have not being considered here. The paper outlines the problem areas and the new challenges posed by *SES* and *SoS* paradigms and identifies some relevant methodologies for their development. The long-term objectives of the proposed research have been the management of complexity in engineering systems and the areas considered here were

(i) Explaining aspects of structure evolution in design and redesign of systems and developing methodologies for controlling the development of the evolutionary design processes.
(ii) Understanding complexity of integrated operations, addressing issues stemming from their organization and characterizing the nature of emergent properties.
(iii) Developing a systems-based characterization of the *SoS* family, characterize its essential features and develop links with concepts and tools of Control Theory.

In the first area, the dominant notions have been time evolution and the cascade design evolution. The overall philosophy for the time evolution has been that there is an evolution from early to late design which is accompanied by an evolution of model structure and associated system properties. The approach followed in cascade design is that each particular design stage shapes a local model; the structure of this local model has important implications on what can be achieved at the next design stage, and it thus determines overall cost, operability, safety and performance of the final process. Structural properties and thus performance, operability, etc. characteristics evolve, but not in a simple manner. This evolution of structure and related potential for delivering certain level of performance is only partially understood for the linear case. We would like to drive the model evolution along paths avoiding the formation of undesirable structural features and where possible to assign desirable characteristics and values. In the effort to formulate a generic system/control-based framework for both aspects of structure evolution, it is essential to address issues such as follows:

General Control Problems

(P.1) Characterization of desirable/undesirable performance characteristics and the limits of what can be achieved.
(P.2) Relate the best achievable performance characteristics to desirable system model structure.
(P.3) Define structure design/redesign problems for the different aspects of system structure.

These are traditional Control Theory tasks which are essential for intervening in the evolution problems related to design. Further aspects related to control of the design evolution process are as follows:

Nests of Variable Complexity Models and Design

(P.4) Development and parametrization of the family of minimal partial realizations (MPRs) according to complexity (MacMillan degree). Extension to nonlinear input–output descriptions.

(P.5) Investigate whether key system properties of the original system (stability–instability, minimum–non-minimum phase) are preserved within the chain and determine the degree of complexity required to preserve key system properties.
(P.6) Characterize the process of evolution of the Kronecker invariants in the MPR chain of models.
(P.7) Development of the structure and properties of the family of numerically nested models.

These problems are linked to the design time evolution family. For cascade design evolution, the emphasis is on the development of composite systems and the evolution of structure under systems instrumentation. Important issues are as follows:

Interconnected Systems and Complexity

(P.8) Explore the role of interconnection graph on structural and non-structural properties of composite systems. Specifically, examine the role of the "completeness" and "lack of completeness" assumptions on the composite system properties.
(P.9) Investigate the structural and non-structural system properties when the cardinality (input and output dimensions) of subsystems changes.
(P.10) Examine interconnected system properties when subprocesses are described by variable complexity models.
(P.11) Study interconnected system properties when the interconnection graph is augmented or loses part.

Structure Evolution in Systems Instrumentation

(P.12) Develop solutions to model orientation for matrix pencil and autoregressive models and study the structure evolution.
(P.13) Examine the structural properties under general input and output projection compensation.
(P.14) Evolution of Structure and Properties under Model Expansion.

The first of the above families of problems refer to generalized process synthesis, and the interconnection graph is central in the characterization of structure and related properties. The second deals with *systems instrumentation* and refers to the model shaping role of the selection of inputs and outputs and the shaping of the evolved system structure, as this is expressed in terms of structural invariants. Each one of the above areas has also a design dimension linked to the shaping of the system model structure. Ideally, we would like to assign desirable properties, but in reality it would be more relevant to avoid the formation of undesirable characteristics.

The area of Integration of Operations requires study of fundamental problems and development of new research areas which include study of emergent properties, system organization, multilevel modelling and control problems in complex hierarchies. Specific areas of interest involve the following:

Complexity in Integrated System Operations

(P.15) Understanding the derivation of the different functional models and how they are interfaced within the framework of different forms of system organization (hierarchical, holonic, etc.).
(P.16) Defining metrics and models for the characterization of emergent properties.
(P.17) Development of multilevel hybrid system theory by exploring the notions of global controllability and observability.

The new definition for the *SoS* is the starting point for the development of methodology that may lead to systematic design. Examining the rules of composition of the subsystems and their coordination as agents in a larger system defines a challenging new area for research and requires links across many disciplines. Examining in detail the special features of the different classes of *SoS* is crucial in the effort to provide a quantitative formulation of the notion of "*systems play*" which may take different forms in the different classes. This is also crucial in quantifying the notion of emergence in the *SoS* context. Key problems in the development of this field are as follows:

System of Systems

(P.18) Characterization of the family of *SoS* according to their origin (engineered, natural, social systems).
(P.19) Identifying the methodologies that can contribute to the characterization of *systems play*.
(P.20) Study the special aspects of emergence in the context of *SoS*.

The chapter has provided an identification of the challenges emerging within the two new classes of complex systems, the *SES* and *SoS*. By introducing the additional dimension of large dimensionality, the above classes take new dimensions. It is then that issues of organization, problem decomposition, decentralization and computational aspects take an additional significance. Design has been central to our study, but for many systems already designed in the past, redesign becomes a crucial issue. So far little effort has been spent in addressing this problem. We note, however, that versions of the above families of problems may be formulated when the redesign problem is considered, where either we want to modify the graph, the input, output structure, or the controller to achieve new requirements and objectives.

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Chapter 2
Stability and the Kleinian View of Geometry

Zoltán Szabó and József Bokor

Abstract Youla parametrization of stabilizing controllers is a fundamental result of control theory: starting from a special, double coprime, factorization of the plant provides a formula for the stabilizing controllers as a function of the elements of the set of stable systems. In this case the set of parameters is universal, i.e., does not depend on the plant but only the dimension of the signal spaces. Based on the geometric techniques introduced in our previous work this paper provides an alternative, geometry based parametrization. In contrast to the Youla case, this parametrization is coordinate free: it is based only on the knowledge of the plant and a single stabilizing controller. While the parameter set itself is not universal, its elements can be generated by a universal algorithm. Moreover, it is shown that on the parameters of the strongly stabilizing controllers a simple group structure can be defined. Besides its theoretical and educative value the presentation also provides a possible tool for the algorithmic development.

2.1 Introduction and Motivation

In many of Euclid's theorems, he moves parts of figures on top of other figures. Felix Klein, in the late 1800s, developed an axiomatic basis for Euclidean geometry that started with the notion of an existing set of transformations and he proposed that geometry should be defined as the study of transformations (symmetries) and of the objects that transformations leave unchanged, or invariant. This view has come to be known as the Erlanger Program. The set of symmetries of an object has a very nice algebraic structure: they form a group. By studying this algebraic structure, we can

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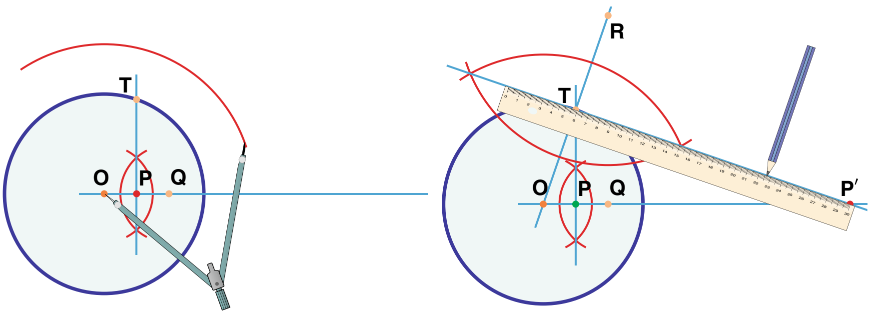
gain deeper insight into the geometry of the figures under consideration. Another advantage of Klein's approach is that it allows us to relate different geometries.

Klein proposed group theory as a mean of formulating and understanding geometrical constructions. In [36] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate various control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The observation that any geometric property of a configuration, which is invariant under an euclidean or hyperbolic motion, may be reliably investigated after the data has been moved into a convenient position in the model, facilitates considerably the solution of the problems. In this work we put an emphasize on this concept of the geometry and its direct applicability to control problems.

The branches of mathematics that are useful in dealing with engineering problems are analysis, algebra, and geometry. Although engineers favour graphic representations, geometry seems to have been applied to a limited extent and elementary geometrical treatment is often considered difficult to understand. Thus, in order to put geometry and geometrical thought in a position to become a reliable engineering tool, a certain mechanism is needed that translates geometrical facts into a more accessible form for everyday algorithms. The compass and ruler should be changed to something else, possibly some series of numbers that can be manipulated more easily and the results can be interpreted more directly in terms of the given engineering problem (Fig. 2.1).



(a) Compass (b) Straightedge

Fig. 2.1 Euclidean constructions Klein proposed group theory as a mean of formulating and understanding geometrical constructions. The idea of constructions comes from a need to create certain objects in the proofs. Geometric constructions were restricted to the use of only a straightedge and compass and are related to Euclid's first three axioms: to draw a straight line from any point to any point, to produce a finite straight line continuously in a straight line and to draw a circle with any center and radius. The idealized ruler, known as a straightedge, is assumed to be infinite in length, and has no markings on it because none of the postulates provides us with the ability to measure lengths. While modern geometry has advanced well beyond the graphical constructions that can be performed with ruler and compass, it is important to stress that visualization might facilitate our understanding and might open the door for our intuition even on fields where, due to an increased complexity, a direct approach would be less appropriate

The link between algebra and geometry goes back to the introduction of real coordinates in the Euclidean plane by Descartes. By fixing a unit and defining the product of two line segments as another segment, Descartes gave a geometric justification of algebraic manipulations of symbols. The axiomatic approach to the Euclidean plane is seldom used because a truly rigorous development is very demanding while the Cartesian product of the reals provides an easy-to-use model. Descartes has managed to solve a lot of ancient problems by algebrizing geometry, and thus by finding a way to express geometrical facts in terms of other entities, in this case, numbers. Note that being a one-to-one mapping, this "naming" preserves information, so that we can study the corresponding group operations simply by looking at these operations' effect on the coordinates ("names"), even though the group elements themselves might be any kind of weird creatures.

The invention of Cartesian coordinates revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra, and provides enlightening geometric interpretations for many other branches of mathematics. Thus, coordinates, in general, are the most essential tools for the applied disciplines that deal with geometry. Descartes justifies algebra by interpreting it in geometry, but this is not the only choice: Hilbert will go the other way, using algebra to produce models of his geometric axioms. Actually this interplay between geometry, its group theoretical manifestation, algebra and control theory is what we are interested in.

The standard way to define the Euclidean plane is a two-dimensional real vector space equipped with an inner product: vectors correspond to the points of the Euclidean plane, the addition operation corresponds to translation, and the inner product implies notions of angle and distance. Since there is no canonical choice of where the origin should go in the space, technically an Euclidean space is not a vector space but rather an affine space on which a vector space acts by translations.

2.1.1 Invariants

In contrast to traditional geometric control theory, see, e.g., [2, 9, 39] for the linear and [1, 18, 19, 25] for the nonlinear theory, which is centered on a local view, our approach revolves around a global view. While the former uses tools from differential geometry, Lie algebra, algebraic geometry, and treats system concepts like controllability, as geometric properties of the state space or its subspaces the latter focuses on an input-output—coordinate free—framework where different transformation groups which leave a given global property invariant play a fundamental role.

In the first case the invariants are the so-called invariant or controlled invariant subspaces, and the suitable change of coordinates and system transforms (diffeomorphisms), see, e.g., the Kalman decomposition, reveal these properties. In contrast, our interest is in the transformation groups that leave a given global property, e.g., stability or $\mathscr{H}_\infty$ norm, invariant. One of the most important consequences of the approach is that through the analogous of the classical geometric constructions it not only might give hints for efficient algorithms but the underlaying algebraic structure, i.e., the given group operation, also provides tools for controller manipulations that preserves the property at hand, called controller blending.

There are a lot of applications for controller blending: both in the LTI system framework, [26, 32] and in the framework using gain-scheduling, LPV techniques, see [8, 15, 16, 31]. While these approaches exploit the so called Youla parametrization of stabilizing controllers, they do not provide an exhaustive characterization of the topic. The approach presented in this book does not only provide a general approach to this problem but, as an interesting side effect of these investigations, also shows that the proposed operation leaves invariant the strongly stabilizing controllers and defines a group structure on them. Moreover, one can define a blending that preserves stability and it is defined directly in terms of the plant and controller, without the necessity to use any factorization.

2.1.2 A Projective View

As a starting point of Euclidean and non-Euclidean worlds the most fundamental geometries are the projective and affine-ones. Perhaps it is not very surprising that feedback stability is related to such geometries. Following the Kleinian project we have to identify the proper mathematical objects and the groups associated to these objects that are related to the concept of stability and stabilizing controllers.

The determination of the stability of dynamical feedback systems from open loop characteristics is of crucial importance in control system design, and its study has attracted considerable research effort during the past fifty years. Until the early 1960s almost all these methods were for scalar input-output feedback systems; however, the rapid developments in the state-space representation of dynamical systems and their realizations from transfer functions led to an equally important development in stability criteria for multivariable feedback systems.

Much of the early work attempted to establish generalizations of the Nyquist, Popov and circle criteria by utilizing an extended version of the mathematical structures used for establishing scalar results. Later it became clear that such system representations are inadequate for the analysis of generalized multivariable operators in feedback systems. It turns out that an approach based upon the systems input-output spaces is required: the only systems representation admissible a priori is the input-output map which defines the system while the existence of every other representations are deduced from these properties. Thus the concept of input-output stability is essentially based upon the theory of operators defined on Hilbert (Banach) spaces.

Control theory should study also stability of feedback systems in which the open-loop operator is unstable or at least oscillatory. Such maps are clearly not contained in Banach spaces and some mathematical description is necessary if feedback stability is to be interpreted from open loop system descriptions. This is achieved by ruling out from the model class those unbounded operators that might "explode" and establishing the stability problem in an extended space which contains well-behaved as well as asymptotically unbounded functions, see [12]. The generalized extended space contains all functions which are integrable or summable over finite intervals. A disadvantage of the method is that the resulting space is a Banach space while we would prefer to work in a Hilbert space context for signals, and the set of stable operators for plants.

Since unbounded operators on a given space do not form an algebra—nor even a linear space, because each one is defined on its own domain—the association of the operator with a linear space, its graph subspace, turns to be fruitful. This leads us to the study of the generalized projective geometries that copy the constructions of the projective plane into a more complex mathematical setting while maintaining the original relations between the main entities and the original ideas. In doing this our main tools are algebraic: group theory, see [33], and the framework of the so called Jordan pairs will help us to obtain the proper interpretations and to achieve new results, see [34].

All these topics involves an advanced mathematical machinery in which often the underlying geometrical ideas remain hidden. Our aim is to highlight some of these geometric governing principles that facilitate the solution of these problems. We try to avoid, wherever is possible, the technical details which can be found in the cited references. We assume, however, some background knowledge from the reader concerning basic mathematical constructions and control theory. Therefore the style of the book is informal where the statements are rather meta-mathematical than mathematical. Throughout the presentation we always assume a reasonable algebraic structure in which our plants and controllers reside: as an example, the set of matrices, MIMO plants form $\mathscr{RL}_\infty$ ($\mathscr{RH}_\infty$), the set of finite dimensional LTV (LPV) plants. In a strictly formal presentation the details would be overwhelming that would distract the reader from the main message of the book. Concerning the possible details that one should complement to the statements of the work in order to construct a formal framework for robust LTV stability see, e.g., [22].

The main concern of this work is to highlight the deep relation that exists between the seemingly different fields of geometry, algebra and control, see Fig. 2.2. While the Kleinian view makes the link between geometry and group theory, through dif-

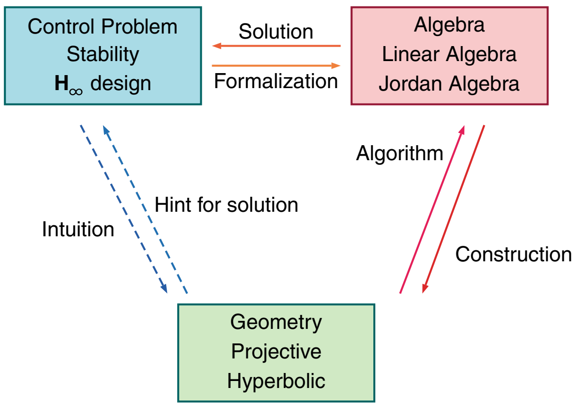


Fig. 2.2 Interplay: geometry, algebra and control

ferent representations and homomorphism the abstract group theoretical facts obtain an algebraic (linear algebraic) formulation that opens the way to engineering applications. We would like to stress that it is a very fruitful strategy to try to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction; finally the solution of the original control problem can be formulated in an algorithmic way by transposing the geometric ideas into the proper algebraic terms.

The main contribution of this work relative to the previous efforts is the following: it is shown that, in contrast to the classical Youla approach, there is a parametrisation of the entire controller set which can be described entirely in a coordinate free way, i.e., just by using the knowledge of the plant P and of the given stabilizing controller K_0. The corresponding parameter set is given in geometric terms, i.e., by providing an associated algebraic (semigroup, group) structure. It turns out that the geometry of stable controllers is surprisingly simple.

2.2 A Glimpse on Modern Geometry—The Kleinian View

Geometry ranges from the very concrete and visual to the very abstract and fundamental: it deals and studies the interrelations between very concrete objects such as points, lines, circles, and planes while on the other side, geometry is a benchmark for logical rigour. Algebraic structures form a parallel world, in which each geometric object and relation has an algebraic manifestation. In this algebraic world the considerations may be also very concrete and algorithmic or very abstract and fundamental.

While it is relatively easy to transform geometric objects into algebraic ones the "naive" approaches to representing geometric objects are very often not the right ones. Introducing more sophisticated algebraic methods often proves to be ultimately more powerful and elegant. Finding the right algebraic structure may open new perspectives on and deep insights into matters that seemed to be elementary at first sight and help to generalize, interpret and understand.

There is a rich interplay of geometric structures and their algebraic counterparts. In this section we will study very simple objects, such as points, lines, circles, conics, angles, distances, and their relations. Also the operations will be quite elementary, e.g., intersecting two lines, intersecting a line and a conic, etc. The emphasis are on structures: the algebraic representation of an object is always related to the operations that should be performed with the object. These advanced representations may lead to new insights and broaden our understanding of the seemingly well-known objects. Moreover these findings will be also useful in our control oriented investigations.

In the plane very elementary operations such as computing the line through two points and computing the intersection of two lines can be very elegantly expressed if lines as well as points are represented by three-dimensional homogeneous coordinates (where nonzero scalar multiples are identified). Taking a closer look at the relation of planar points and their three-dimensional representing vectors, it is apparent that certain vectors do not represent points in the real Euclidean plane. These nonexistent points may be interpreted as points that are infinitely far away; extending the usual plane by these new points at infinity a richer geometric system can be obtained: the system of projective geometry, which turned out to be one of the most fundamental structures having the most elegant algebraic representation.

Projective geometry was viewed as a relatively insignificant area within the domain of Euclidean geometry until in 1859 Cayley demonstrated that projective geometry was actually the most general and that Euclidean geometry was merely a specialization. Later, Klein demonstrated how non-Euclidean geometries could be included. In the spirit of the Erlangen program projective geometry is characterized by invariants under transformations of the projective group. It turns out that the incidence structure and the cross-ratio are the fundamental invariants under projective transformations.

Projective geometry become a fundamental area of modern mathematics with far reaching applications both in the mathematical theory, as algebraic geometry, and also in different applications fields, such as art, computer vision or even control theory, see, e.g., [11]. For a thorough treatment of the subject the interested reader might consult [10] or [4, 6]. In elaborating this chapter we mostly follow the approach of the more recent enlightening account of [30] to the topic.

2.2.1 Elements of Projective Geometry

Following Hilbert's approach a projective plane is a triple $(\mathscr{P}, L, I)$ where $\mathscr{P}$ is a set, called the set of points, L is a set called the set of lines, and I is a subset of $\mathscr{P} \times L$, called the incidence relation ($(P, l) \in I$ means: P is contained in l). The axioms of this geometry are: every two distinct points are contained in a unique line, every two distinct lines contain a unique point and there are four distinct points of which no three are collinear, i.e., lie on a single line. We will denote by $l = A \vee B$ the line passing through two points and by $L = a \wedge b$ the intersection of two lines.

A complete quadrangle is a set of four points A, B, C and D, no three collinear, and the six lines determined by these four points: AB and CD, AC and BD, and AD and BC are said to be pairs of opposite sides. The points at which pairs of opposite sides intersect are called diagonal points of the quadrangle.

A fourth axiom for a projective plane is Fano's Axiom: the three diagonal points of a complete quadrangle are never collinear. A projective plane that does not satisfy this axiom is the Fano plane determined by the seven-point and seven-line geometry.

In the ordinary plane parallel lines do not meet. In contrast, projective geometry formalizes one of the central principles of perspective, i.e., parallel lines meet at infinity. In essence it may be thought of as an extension of Euclidean geometry in which the direction of each line is subsumed within the line as an extra point, and in which a horizon of directions corresponding to coplanar lines is regarded as a line. Thus, two parallel lines meet on a horizon line in virtue of their possessing the same direction (Fig. 2.3).

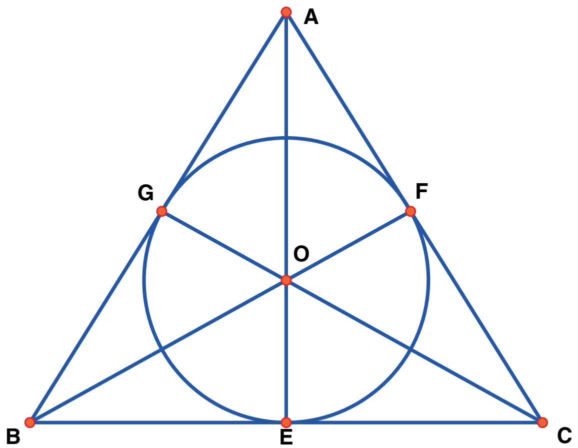


Fig. 2.3 Fano plane: the corresponding projective geometry consists of exactly seven points and seven lines with the incidence relation described by the attached figure. The circle together with the six segments represent the seven lines

Thus we can introduce a special hyperplane, the hyperplane at infinity or ideal hyperplane, and the points at infinity will be those on this hyperplane. Idealized directions are referred to as points at infinity, while idealized horizons are referred to as lines at infinity.

We say that two subspaces are parallel if they have the same intersection with this special hyperplane. Parallelism is an equivalence relation, however, infinity is a metric concept. A purely projective geometry does not single out any points, lines or plane and in this regard parallel and nonparallel lines are not treated as separate cases. In contrast, an affine space can be regarded as a projective space with a distinguished hyperplane.

Coordinates are important in the analytical development of projective geometry as an essential tool for calculations which may be used to verify and illustrate relations unambiguously. However, coordinates are typically based upon metrical considerations and an important question arose: how could such coordinates be logically applied to projective relations? Klein supplied an answer to this by suggesting the use of von Staudt's projective constructions which are employed to define the algebra of points. It is important to emphasize that in projective geometry coordinates are not understand in the ordinary metrical sense; they are a set of numbers, arbitrarily but systematically assigned to different points.

In order to assign coordinates to points on a line m it is required to select three distinct points P_0, P_1 and P_∞ which, by the special nature of the constructions, are endowed with the properties of 0, 1 and ∞.

As an illustration the addition of points on a line is defined using two special projective constructions, see Fig. 2.4. It can be shown that this algebra of points is isomorphic to the field of real numbers and can be extended to include the concept of infinity: a unique real number is associated with each point on the line with the exception of a single point which assumes a correspondence with infinity. The unique real number associated with each point is the non-homogeneous coordinate of the

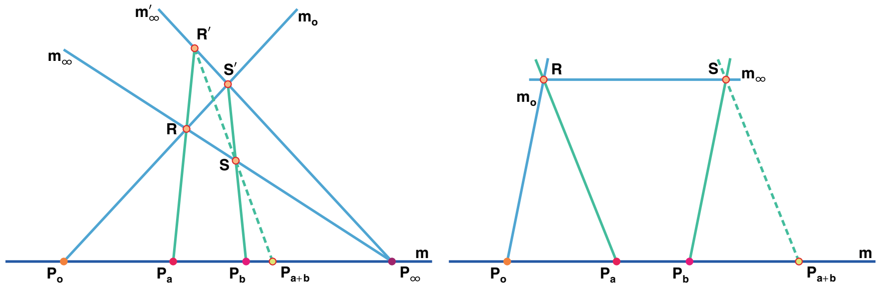


Fig. 2.4 Projective addition: for the addition of two points let us fix the points P_0 and P_∞. Then a fixed line m_0 through P_0 meets the two distinct fixed lines m_∞ and m'_∞ in the points R and S', respectively, while the lines $P_a R$ and $P_b S'$ meet m'_∞ and m_∞ at R' and S. The line $R'S$ meets m at $P_a + P_b = P_{a+b}$. By reversing the latter steps, subtraction can be analogously constructed, e.g., $P_a = P_{a+b} - P_b$. Observe that by sending point P_∞ to infinity we obtain the special configuration based on the "Euclidean" parallels and the common addition on the real line

point on the line. The exceptional role of the point associated with infinity can be removed upon the introduction of homogeneous coordinates.

The cross-ratio plays a fundamental role in the development of projective geometry. It was already known to Pappus of Alexandria and was used by Karl von Staudt to present the first entirely synthetic treatment of the projective geometry by introducing the notion of a throw a pair of ordered pairs of points on a line. Throws are separated into equivalence classes by the projectivities of the line, relative to its situation in a plane.

As a synthetic definition consider a line m embedded in a projective plane and use complete quadrilaterals to define addition and multiplication. Given any throw $\{[A, B], [C, D]\}$ and any fifth point E, there exist many complete quadrilaterals for which each of the pairs of the throws lie on the intersections of opposing lines of the quadrilateral, and such that one of the other lines passes through E. However, for each of these complete quadrilaterals the remaining line cuts m at the same point.

This defines a quinary operator cr on the points of m. One fixes three distinct points of m, calling them 0, 1 and ∞ and then places them in a certain way in three of the arguments of cr to obtain a binary operator. One of these ways defines addition, and another way defines multiplication such that the complement of ∞ in m becomes a field.

In order to obtain coordinates for the points of the projective plane $\mathbb{P}^2(\mathbb{R})$ we should chose a projective basis consisting of four distinct points 0, ∞_x, ∞_y and 1, i.e., the origin, an infinite point on the x-axis m_x, an infinite point on the y-axis m_y and a point with coordinates $(1, 1)^T$, respectively. We can also define points $1_x = (0 \vee \infty_x) \wedge (1 \vee \infty_y)$ and $1_y = (0 \vee \infty_y) \wedge (1 \vee \infty_x)$. A point X on m_x is uniquely determined by the cross-ratio $\mathrm{cr}(0, \infty_x, X, 1_x) = x$ and analogously for a point Y on m_y we have $\mathrm{cr}(0, \infty_y, Y, 1_y) = y$. Any point P of $\mathbb{P}^2(\mathbb{R})$ that does not lie on the line $m_\infty = \infty_x \vee \infty_y$ defines uniquely two points $P_x = m_x \wedge (P \vee \infty_y)$ and $P_y = m_y \wedge (P \vee \infty_x)$ from which it can be reconstructed according to $P = (P_x \vee \infty_y) \wedge (P_y \vee \infty_x)$. For an illustration of this construction see Fig. 2.5.

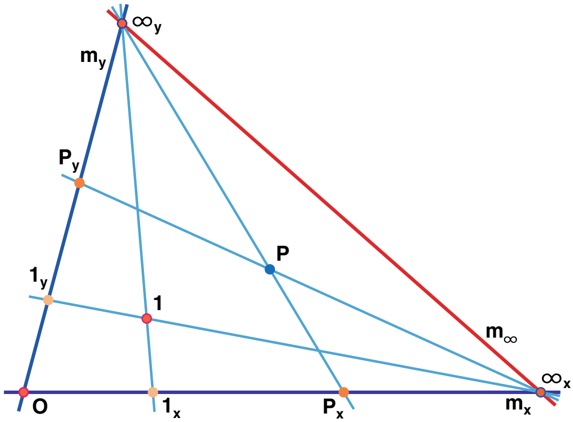


Fig. 2.5 Projective coordinates

Although the point triple (P_0, P_1, P_∞) (called scale) is selected arbitrarily, the addition and multiplication constructions impart them with the special properties associated with (0, 1, ∞). From a projective point of view, however, all points have identical properties. Three distinct new points may be chosen as another scale and all other points relabeled in terms of it. By way of projective transformations, all scales and subsequently all coordinates, are projectively equivalent.

An algebraic model for doing projective geometry in the style of analytic geometry is given by homogeneous coordinates. When the vector space V is coordinatized by fixing a basis, a projective point is a 1-space $\{\lambda(x_0, x_1, \dots, x_n) \mid \lambda \in \mathbb{F}\}$, i.e., an equivalence class $X \sim [x]$ of all vectors that differ by a nonzero multiple, and we can say that this point has coordinates $(x_0, x_1, \dots, x_n)$. Note that $(x_0, x_1, \dots, x_n)$ and $\lambda(x_0, x_1, \dots, x_n)$ denotes the same point for $\lambda \neq 0$. Such coordinates are called homogeneous coordinates. By using homogeneous coordinates we can introduce a special hyperplane, e.g., the one defined by $x_n = 0$, the so called finite points will be the ones with $x_n \neq 0$, while the points at infinity will be those on the hyperplane.

A central concept in projective geometry is that of duality. The simplest illustration of duality is in the projective plane, where the statements "two distinct points determine a unique line" and "two distinct lines determine a unique point" show the same structure as propositions.

A line l passing through two points A and B may be described as the join of the two points, i.e., $l = A \vee B$ and dually, the intersection L point of two lines a and b may be described as the meet of the two lines, i.e., $L = a \wedge b$.

The principle of duality in the plane is that incidence relations remain valid when the roles of points and lines are interchanged, where the point P and line p are (projectively) dual objects.

The dualistic properties of projective geometry may be elegantly expressed in an analytic manner by employing homogeneous coordinates: the condition for a point $X \sim [x]$ with $x = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}^T$ and a line $m \sim [M]$ with $M = \begin{pmatrix} M_0 & M_1 & M_2 \end{pmatrix}$ to be incident may be expressed as the linear relation

$$M_0x_0 + M_1x_1 + M_2x_2 = 0, \text{ i.e., } Mx = 0.$$

Since condition $x_2 \neq 0$ selects out the finite points the line at infinity will corresponds to $m_\infty \sim [(0, 0, 1)]$. Here we assume that all homogeneous coordinates of a point are represented by column vectors while those that corresponds to lines are row vectors. However, it is more convenient to identify the lines with column vectors, too. This can be done through the pairing $\langle \cdot, \cdot \rangle$ as $m \sim \langle m, \cdot \rangle$. Thus the set of all points on the line r through the given points P, Q can be expressed with the condition $\langle r, \lambda p + \mu q \rangle = 0$ for all $\lambda, \mu \in \mathbb{R}$.

Assuming that the coordinates M are fixed while the coordinates x are free to vary, then this equation ($x \in \mathrm{Ker}(m^T)$) represents the locus of points which are incident to the line m. Dually, if the coordinates x are fixed and m is free to vary, then the equation ($m \in \mathrm{Im}(x)^\perp$) represents the pencil of lines which are incident to the point x.

Thus we extend the Euclidean plane by introducing elements at infinity: one point at infinity for each direction and one global line at infinity that contains all these points. We also have a coordinate representation of these objects. Actually the incidence relation $(X, m) \in I$ is expressed as $([x], [m]) \in I_{\mathbb{R}^2}$ defined by the condition $x \perp m$. Thus, by the identification determined by the homogeneous coordinates of the points and lines with equivalence classes of vectors, we have that $(\mathscr{P}_{\mathbb{R}^2}, L_{\mathbb{R}^2}, I_{\mathbb{R}^2})$ is a projective plane: $\mathbb{P}^2(\mathbb{R})$. While this is a simple observation it has an important consequence: it consists the link between geometry and algebra.

From the projective viewpoint the distinction of infinite and finite elements is completely unnatural: it is only a kind of artefact that arises when we interpret the Euclidean plane in a projective setup. Often it is fruitful to interpret Euclidean theorems in a projective framework and vice-versa. To do it we have to model the drawing of a parallel to a line through a point on the projective plane: set the line at infinity (m_∞) and define the operator parallel $(P, m) = P \vee (m \wedge m_\infty)$.

2.2.2 Projective Transformations

Klein stated that a geometry is defined as the properties of a space which remain invariant under all transformations of space (or the coordinate system) by a group of transformations. Thus Euclidean geometry is the theory of objects invariant with respect to Euclidean congruence transformations. For projective geometry, the group of transformations is characterized by those which preserve relations of incidence. An analysis of projective transformations not only identifies important invariant relations but also forms a foundation for developing metrical geometries.

The group of automorphisms of n-dimensional projective space $\mathbb{P}^n(\mathbb{R})$ are induced by the linear automorphisms of $\mathbb{R}^{n+1}$. These can be projective automorphisms, projective collineations or regular projective maps. The group of projective automorphisms of $\mathbb{P}^n(\mathbb{R})$ is denoted by $\mathrm{PGL}(n)$, and is called the projective linear group. Thus the action of projective automorphisms on points can be expressed as $[Ax]$ and, accordingly, on the hyperplanes $[A^{-T}m]$.

The fixed points of the projective automorphisms are given by the (right) eigenvector of the matrix A. It follows that every projective transformation has at least one invariant point and one invariant line. Moreover there is exactly one projective isomorphism which transforms a given fundamental set into another one.

The restriction of a projective mapping in $\mathbb{P}^n(\mathbb{R})$ to a line l is called a projectivity, which is uniquely defined by the images of three distinct points of the line. A projective automorphism of a line, if it is not the identity mapping, has 0, 1, or 2 fixed points. Then the corresponding projective automorphism is called elliptic, parabolic or hyperbolic, respectively. In the complex projective plane there are no elliptic projectivities.

A collineation is a one–to–one linear transformation preserving the incidence relation in which each element is mapped into a corresponding element of the same type (e.g., point to point) whereas a correlation differs in that each element is mapped into a corresponding dual element (e.g., point to line).

It is often useful to consider singular linear mappings, whose domain is a projective space of dimension n and whose image space has a different dimension. Singular projective mapping means a linear mapping which is not quadratic and regular, i.e., it is not a projective isomorphism. Such mappings are generalizations of the concept of central projection from projective three-space onto a plane. A central projection from $\mathbb{P}^n(\mathbb{R})$ onto a subspace V via a center W is given by $\pi(P) = (O \vee P) \wedge V$, where it is required that W and V are complementary subspaces. For all linear mappings $\lambda : \mathbb{P}^n \mapsto \mathbb{P}^m$ there is a central projection π from onto a subspace V and a projective isomorphism α of V onto $\mathbb{P}^m$ such that $\lambda = \pi\alpha$. A linear mapping has a kernel (center or exceptional subspace) Z which is independent of the decomposition. The points $Q \in P \vee Z$ have the property that $\pi(P) = \pi(Q)$.

2.2.3 A Trapezoidal Addition

We conclude this section by reviewing a specific configuration of the projective plane, and its associated special addition law, which bears relevance to the study of feedback stability from a projective point of view.

First, let us list some facts important to us concerning the case $d = 1$, i.e., the projective line $\mathbb{P}^1(\mathbb{R})$. If V is a one dimensional subspaces (line) of a vector space, by choosing a basis of V gives an identification of V with $\mathbb{P}^1(\mathbb{R})$. But another choice of basis of V gives another identification of V with $\mathbb{P}^1(\mathbb{R})$, leading to the group of projective transformations of $\mathbb{P}^1(\mathbb{R})$. As it is shown by this case, groups (isomorphisms) occur in the description of the differences between parametrizations that preserve a certain structure.

The projective line $\mathbb{P}^1(\mathbb{R})$ is the set of lines through 0 in $\mathbb{R}^2$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(\mathbb{R}^2)$ we have the map: $\mathbb{R}^2 \mapsto \mathbb{R}^2,\ x \mapsto Mx$ that sends lines through 0 to lines through 0, and hence gives us a map from $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{P}^1(\mathbb{R})$. Written out in detail

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} ax_0 + bx_1 \\ cx_0 + dx_1 \end{pmatrix},$$

i.e., in inhomogeneous coordinates

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} ax + b \\ cx + d \end{pmatrix}, \quad x \mapsto \frac{ax + b}{cx + d},$$

if $cx + d \neq 0$. Thus the fractional linear transformations from $\mathbb{R}$ to $\mathbb{R}$ is linear in homogeneous coordinates.

It is obvious that M and M' in $\mathrm{GL}(\mathbb{R}^2)$ give the same projective transformation on $\mathbb{P}^1(\mathbb{R})$ precisely when there is a k in $\mathbb{R}^*$ with $M' = kM$. Thus the group of projective transformations (projectivities) of $\mathbb{P}^1(\mathbb{R})$ is the quotient group $\mathrm{PGL}(\mathbb{R}^2)$ of $\mathrm{GL}(\mathbb{R}^2)$ by the subgroup of scalar matrices.

In other words, Möbius transformations can be seen as the restriction of the projective transformation to the set of finite points. Note that while the projective transformation M is linear, and it is defined everywhere, the Möbius transformation is nonlinear (rational) and it is defined only on the domain $cx + d \neq 0$. If $(x, 1)^T$ and $(y = \frac{ax+b}{cx+d}, 1)^T$ are considered as specific (normalized) homogeneous coordinates of the finite points, then we can say that the Möbius transformation acts on a coordinate level while the projectivity M acts on the geometric, projective level.

Projective transformations leave the cross ratio

$$\mathrm{cr}(p,\ q,\ r,\ s) = \frac{(r - p)/(r - q)}{(s - p)/(s - q)}, \quad p,\ q,\ r,\ s \in \mathbb{P}^1(\mathbb{R})$$

invariant, i.e., if g is a projective transformation then

$$\mathrm{cr}(g(p),\ g(q),\ g(r),\ g(s)) = \mathrm{cr}(p,\ q,\ r,\ s).$$

Since Möbius transformations are only a restriction of projective transformations on finite points, invariance holds.

We have already seen that in terms of homogeneous coordinates the Euclidean plane $\mathbb{R}^2$ can be embedded into $\mathbb{P}^2(\mathbb{R})$ by taking its finite points, i.e., by the map

$$\mathbb{R}^2 \mapsto \mathbb{P}^2(\mathbb{R})\ ,\ (x,\ y)^T \mapsto (x, y, 1)^T.$$

The points of $\mathbb{P}^2(\mathbb{R})$ that are not in the image of this map are the ideal points $(x, y, 0)^T$. Thus the set of ideal points is in bijection with the set of points $(x, y)^T$ of the projective line $\mathbb{P}^1(\mathbb{R})$. This is the set of directions in $\mathbb{R}^2$, that correspond to the points on the horizon in $\mathbb{P}^2(\mathbb{R})$.

An example for this embedding in terms of projective coordinates is depicted on Fig. 2.5. Recall that in order to obtain coordinates we should choose a projective basis consisting of four distinct points 0, ∞_x, ∞_y and 1. In an obvious way the construction defines an addition operation on the plane, see Fig. 2.6.

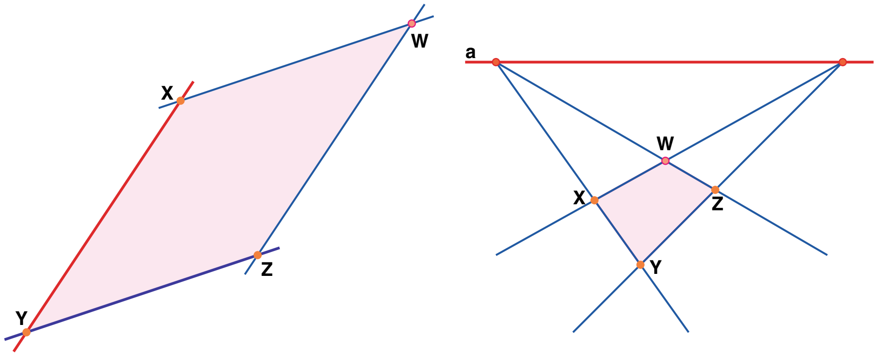


Fig. 2.6 Parallel addition: set the origin to the point Y and let m_x and m_z the directions determined by the points X and Z. If the points ∞_x, ∞_y are set to infinity we obtain a usual setting for parallel vector addition: the coordinates of the point W are constructed by taking parallels to m_x and m_z through W. For a "projective" vector addition we can set the points ∞_x, ∞_y on a given line a of $\mathbb{R}^2$ intersecting m_x and m_z. The point W is provided as $W = \Big[\big((X \vee Y) \wedge a\big) \vee Z\Big] \wedge \Big[\big((Z \vee Y) \wedge a\big) \vee X\Big]$

In [5] these constructions were generalized in order to provide a friendly introduction to Jordan triplets and to illustrate algebraic concepts through elementary constructions performed in the plain geometry. We reproduce here those constructions from [5] that are relevant for our control oriented view.

Parallel addition: recall that we imagine a point "infinitely far" on the line l, given by intersecting l with an ideal line i; then the parallel to X is the line joining this infinitely far point $l \wedge i$ with X: $k = X \vee (l \wedge i)$ is the unique parallel of l through the point X.

In the usual, parallel, view given three non-collinear points X, Y, Z we can construct a fourth point according to

$$W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge i\big) \vee X\Big),$$

which is the intersection of the parallel of $X \vee Y$ through Z with the one of $Z \vee Y$ through X.

Note that the initial construction works well if we assume that X, Y, Z are not collinear, but it is not defined if X, Y, Z are on a common line l. Nevertheless, the map associating to the triple X, Y, Z the fourth point W admits a continuous extension from its initial domain of definition (non-collinear triples) to the bigger set of all triples.

If we choose a line a in the plane and three non-collinear points X, Y, Z in the plane such that the lines $X \vee Y$ and $Z \vee Y$ are not parallel to a then it is possible to construct the point

$$W = \Big(\big((X \vee Y) \wedge a\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big).$$

One can imagine this drawing to be a perspective view onto a plane in 3-dimensional space, where line a represents the horizon. Dragging the line a further and further away from X, Y, Z the perspective view looks more and more like a usual parallelogram construction.

The fourth vertex W is a function of X, Y, Z, therefore we introduce the notation $W = X +_Y Z$, and write $W = \{XYZ\}$. We write O instead of Y if it is fixed as origin, i.e., let $X + Z = X +_O Z$.

Since the operations $\vee$ and $\wedge$ are symmetric in both arguments the law $(X, Z) \mapsto X + Z$ is commutative. But the choice of the origin O is completely arbitrary, thus, the free change of the origin should be facilitated by a more general version of the associative law (called the *para-associative law*):

$$X +_O (U +_P V) = (X +_O U) +_P V$$

where O and P may be different points. Thus, to cope with the problem of collinear points we can use the para-associative law:

$$(X +_O P) +_P V = X +_O (P +_P V) = X +_O V.$$

It turns out that for any fixed origin O the plane $\mathbb{R}^2$ with $X + Z = X +_O Z$ is a commutative group with neutral element O.

Trapezoidal addition: in order to obtain a more general scheme we can introduce two special lines—as if they played the role of the ideal lines—and to define the point addition as:

$$W = \Big(\big((X \vee Y) \wedge a\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge b\big) \vee X\Big),$$

see Fig. 2.7. Note that when the lines a, b and the point Y are kept fixed, the law given by $(X, Z) \mapsto W$ depends nicely on the parameters Y, a, b.

If instead of "parallelograms" we use trapezoids, i.e., $b = i$, the constructions will depend on the choice of some line a in the plane and the underlying set of our constructions will be the set $\mathbb{G} = \mathbb{R}^2 \setminus a$ of all points of the plane $\mathbb{R}^2$ not on a.

Fixing a point Y not on a, and two other points X, Z such that the line $Y \wedge Z$ is not parallel to a we can construct the point

$$W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big).$$

Observe that the map $W = \{XYZ\}$ is not symmetric in X and Z, therefore the law $(X, Z) \mapsto W = \{XYZ\}$ for fixed Y is not commutative. However, the operation $\{XYZ\}$ is associative, moreover, the following generalized associativity law holds:

$$\{XO\{UPV\}\} = \{\{XOU\}PV\}.$$

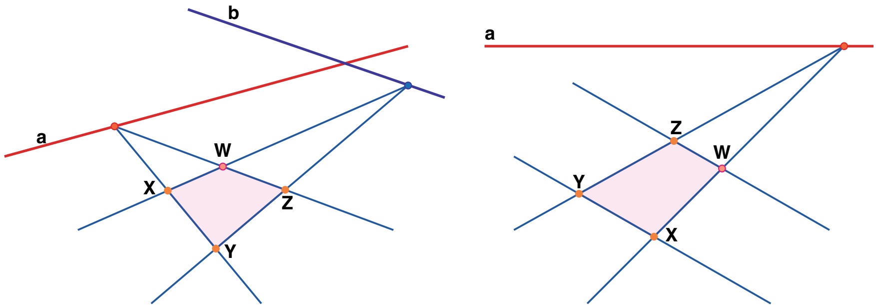


Fig. 2.7 While the quadrangle $XYZW$ is not a parallelogram, its construction has something in common with the one of a parallelogram: the picture illustrates the fundamental process of passing from a commutative, associative law—vector addition, corresponding to usual parallelograms—to a non-commutative law: $W = \Big(\big((X \vee Y) \wedge b\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big)$. Trapezoidal addition, i.e., $b = i$, the point W is provided as $W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big)$

If we fix some element $E \in \mathbb{G}$, then E is a unit element for the binary product $XZ = \{XEZ\}$. Thus for three points X, E, V on a line, we can define a fourth point $W = \{XEV\}$ on the same line.

As a conclusion: for any choice of origin $E \in \mathbb{G}$, the set $\mathbb{G} = \mathbb{R}^2 \setminus a$ is a group with product $XZ = \{XEZ\}$. By using the generalized associativity law follows that $U = (EXE)$ is the inverse of X. The converse is also true: the ternary law $\{XYZ\}$ can be recovered from the binary product in the group $(\mathbb{G}, e)$ with neutral element e as $\{xyz\} = xy^{-1}z$.

We can translate these geometrical facts into analytic formulas by using coordinates of the real vector space $\mathbb{R}^2$. Then, vectors are written as $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ while their sum is defined by $x + y = (x_1 + y_1, x_2 + y_2)^T$. Recall that for two distinct points the affine line spanned by x and y is

$$x \vee y = \big\{tx + (1-t)y \mid t \in \mathbb{R}\big\}.$$

Note that the quadrangle with vertices $xyzw$ is a parallelogram if and only if $w = x - y + z$. Thus, for a fixed element $y \in \mathbb{R}^2$, the law $x +_y z = x - y + z$ defines a commutative group with neutral element y. For $y = 0$, we get back the usual vector addition.

The linear algebra of trapezoid geometry can be obtained by fixing a line a given by $a = \{x \in \mathbb{R}^2 \mid \alpha(x) = 0\}$ for some non-zero linear form $\alpha : \mathbb{R}^2 \to \mathbb{R}$. Since lines $y \vee x$ and $z \vee w$ are parallel, we have that $z + t(x - y)$ for some $t \in \mathbb{R}$. The point $u = a \wedge (y \vee z)$ should be of the form $t_u y + (1 - t_u)z$—since u is on $y \vee z$.

Thus, from $\alpha(u) = 0$ follows that

$$t_u = \frac{1}{1 - \alpha(y)\alpha(z)^{-1}}.$$

Note that $|w-z|/|x-y| = |u-z|/|u-y|$, i.e., $|t| = |t_u|/|1-t_u|$, from which follows that $t = \alpha(z)\alpha(y)^{-1}$.

Thus, on the set $\mathbb{G}$ defined by

$$\mathbb{G} = \{x \in \mathbb{R}^2 \mid \alpha(x) \neq 0\}$$

the point w is defined by

$$\{xyz\} = w = \alpha(z)\alpha(y)^{-1}(x-y) + z. \tag{2.1}$$

Observe that $\alpha(w) = \alpha(x)\alpha(y)^{-1}\alpha(z)$. For a fixed point $e \in \mathbb{G}$ such that $\alpha(e) = 1$ we have that $\mathbb{G}$ is a group with neutral element e and product

$$x \cdot z = (xez) = \alpha(z)(x-e) + z. \tag{2.2}$$

The corresponding group inverse of x is given by

$$x^{-1} = \alpha(x)^{-1}(e-x) + x. \tag{2.3}$$

The set $\mathbb{G}$ is open dense in $\mathbb{R}^2$. Moreover, the group law with the corresponding inversion map are smooth of class C^∞. It can be shown that $\mathbb{G}$ is isomorphic to $(\mathbb{R}, +) \times (\mathbb{R}^\times, \cdot)$, i.e., it is isomorphic to the affine group of the real line:

$$\mathrm{GA}(1, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

To bring this example closer to the feedback setting let us consider a special configuration: take the line a such that $\alpha = (-p,\ 1)$, i.e., the points of this line are $\lambda(1, p)^T$. Set $e = y = (0, 1)$ as the unit element and note that $\alpha(y) = 1$. Consider the set

$$\mathbb{G}_p = \{(k, 1)^T \in \mathbb{R}^2 \mid 1 - pk \neq 0\}$$

and the points $z = (k_z, 1)$ and $x = (k_x, 1)$ from $\mathbb{G}_p$.

Then, we have that

$$x \cdot_p z = \alpha(z)(x-e) + z = (1 - pk_z)\begin{pmatrix} k_x \\ 0 \end{pmatrix} + \begin{pmatrix} k_z \\ 1 \end{pmatrix} = \begin{pmatrix} k_x + k_z - pk_z k_x \\ 1 \end{pmatrix},$$

and

$$x^{-_p} = \alpha(x)^{-1}(e-x) + e = \begin{pmatrix} -(1-pk_x)^{-1}k_x \\ 1 \end{pmatrix}.$$

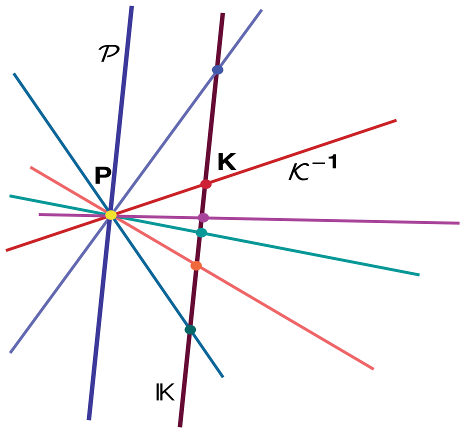


Fig. 2.8 Affine parametrization

In other words, if we fix p and consider all those k for which the matrix

$$F_{p,k} = \begin{pmatrix} 1 & k \\ p & 1 \end{pmatrix} \quad \text{or} \quad F_{-p,-k} = \begin{pmatrix} 1 & -k \\ -p & 1 \end{pmatrix} \quad \text{is nonsingular,}$$

then we obtain exactly the set $\mathbb{G}_p$. Moreover, on this set we have managed to define a group structure, $(\mathbb{G}_p, +_p)$ with unit element 0 defined by

$$k_1 +_p k_2 = k_1 + k_2 - pk_1k_2, \quad k^{-_p} = -(1 - pk)^{-1}k. \tag{2.4}$$

Observe that for $p = 0$ we obtain the usual addition on the real line.

The significance of the result for control is straightforward: take p as a plant and k as a controller. Then condition $1 - pk \neq 0$ selects exactly the controllers that renders the loop well-defined. By taking an arbitrary parallel line with p, its intersection with any other non-parallel line will work as a parametrization of these controllers.

Thus, we have the affine picture sketched on Fig. 2.8. In what follows we are going to provide further explanations in the context of the stable feedback loop.

2.3 The Standard Feedback Loop

A central concept of control theory is that of the feedback and the stability of the feedback loop. For practical reasons our basic objects, the systems, i.e., plants and controllers, are causal. Stability is actually a continuity property of a certain map, more precisely a property of boundedness and causality of the corresponding map. Boundedness here involves some topology. In what follows we consider linear systems, i.e., the signals are elements of some normed linear spaces and an operator means a linear map that acts between signals. Thus, boundedness of the systems is regarded as boundedness in the induced operator norm.

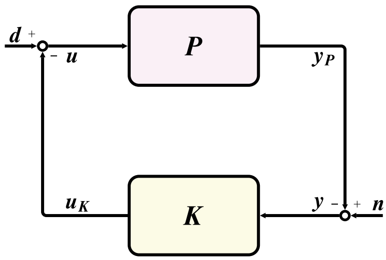


Fig. 2.9 Feedback connection

To fix the ideas let us consider the feedback-connection depicted on Fig. 2.9. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad p = \begin{pmatrix} u \\ y_P \end{pmatrix}, \quad k = \begin{pmatrix} u_K \\ y \end{pmatrix}, \quad z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathscr{H},$$

where $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and we suppose that the signals are elements of the Hilbert space $\mathscr{H}_1, \mathscr{H}_2$ (e.g., $\mathscr{H}_i = \mathscr{L}^{n_i}[0, \infty)$) endowed by a resolution structure which determines the causality concept on these spaces. In this model the plant P and the controller K are linear causal maps. For more details on this general setting, see [12].

The feedback connection is called well-posed if for every $w \in \mathscr{H}$ there is a unique p and k such that $w = p + k$ (causal invertibility) and the pair (P, K) is called stable if the map $w \to z$ is a bounded causal map, i.e., the pair (P, K) is called well-posed if the inverse

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} S_u & S_c \\ S_p & S_y \end{pmatrix} = \begin{pmatrix} (I - KP)^{-1} & -K(I - PK)^{-1} \\ -P(I - KP)^{-1} & (I - PK)^{-1} \end{pmatrix} \tag{2.5}$$

exists (causal invertibility), and it is called stable if all the block elements are stable.

2.3.1 *Youla Parametrization*

A fundamental result concerning feedback stabilization is the description of the set of the stabilizing controllers. A standard assumption is that among the stable factorizations there exists a special one, called double coprime factorization, i.e., $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and there are causal bounded systems $U, V, \tilde{U}$ and $\tilde{V}$, with invertible V and $\tilde{V}$, such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \tilde{\Sigma}_P \Sigma_P = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.6}$$

an assumption which is often made when setting the stabilization problem, [12, 38]. The existence of a double coprime factorization implies feedback stabilizability, actually $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ is a stabilizing controller. In most of the usual model classes actually there is an equivalence.

For a fixed plant P let us denote by $\mathbb{W}_P$ the set of well-posed controllers, while $\mathbb{G}_P \subset \mathbb{W}_P$ denotes the set of stabilizing controllers.

Given a double coprime factorization the set of the stabilizing controllers is provided through the well-known Youla parametrization, [23, 41]:

$$\mathbb{G}_P = \{K = \mathfrak{M}_{\Sigma_P}(Q) \mid Q \in \mathbb{Q}, (V + NQ)^{-1} \text{ exists}\},$$

where $\mathbb{Q} = \{Q \mid Q \text{ stable }\}$ and

$$\mathfrak{M}_{\Sigma_P}(Q) = (U + MQ)(V + NQ)^{-1}. \tag{2.7}$$

For a recent work that covers most of the known control system methodologies using a unified approach based on the Youla parameterization, see [20].

Here $\mathfrak{M}_T(Z)$ is the Möbius transformation corresponding to the symbol T defined by

$$\mathfrak{M}_T(Z) = (B + AZ)(D + CZ)^{-1}, \text{ with } T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

on the domain $\mathrm{dom}_{\mathfrak{M}_T} = \{Z \mid (D + CZ)^{-1} \text{ exists}\}$. Note that

$$Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) = (\tilde{V}K - \tilde{U})(\tilde{M} - \tilde{N}K)^{-1}, \tag{2.8}$$

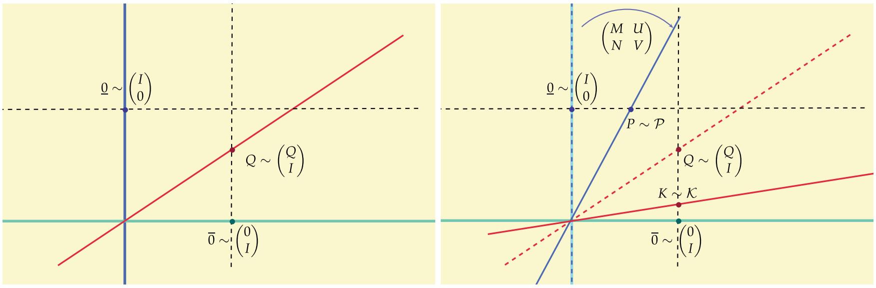
and thus $Q = 0_K$ corresponds to $K_0 = UV^{-1}$.

Since the dimensions of the controller and plant are different, it is convenient to distinguish the zero controller and zero plant by an index, i.e., 0_K and 0_P, respectively.

Observe that the domain of (2.8) is exactly $\mathbb{W}_P$; thus we can introduce the corresponding extended parameter set $\mathbb{Q}_P^{wp} = \{Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) \mid K \in \mathbb{W}_P\}$. Note, that Q_0, i.e., $\mathfrak{M}_{\tilde{\Sigma}_P}(0_K) = -\tilde{U}\tilde{M}^{-1} = -M^{-1}U$, is not in $\mathbb{Q}$, in general. The content of the Youla parametrization is that K is stabilizing exactly when $Q_K \in \mathbb{Q}$, see Fig. 2.10.

2.4 Group of Controllers

In order to design efficient algorithms that operate on the set of controllers that fulfil a given property, e.g., stability or a prescribed norm bound, it is important to have an operation that preserves that property, i.e., a suitable blending method. Available approaches use the Youla parameters in order to define this operation for stability in a trivial way. As these approaches ignore the well-posedness problem by assuming strictly proper plants, they do not provide a general answer to the problem.



(a) Trivial case (b) Transformed case

Fig. 2.10 Youla parametrization

In the particular case when $P = 0_P$ we have $\mathbb{G}_P = \mathbb{Q}$, i.e., mere addition preserves well-posedness and stability. Moreover, the set of these controllers forms the usual additive group $(\mathbb{Q}, +)$ with neutral element 0_K and inverse element $Q \to -Q$. In the general case, however, addition of controllers neither ensure well-posedness nor stability.

2.4.1 *Indirect Blending*

The most straightforward approach to obtain a stability preserving operation is to find a suitable parametrization of the stabilizing controllers, where the parameter space possesses a blending operation. As an example for this indirect (Youla based) blending is provided by the Youla parametrization. However, this mere addition on the Youla parameter level does not lead, in general, to a "simple" operation on the level of controllers:

$$K = \mathfrak{M}_{\Sigma_P}((\mathfrak{M}_{\tilde{\Sigma}_P}(K_1) + \mathfrak{M}_{\tilde{\Sigma}_P}(K_2))). \tag{2.9}$$

The unit element of this operation is the controller K_0 which defines Σ_P, see Fig. 2.11. Its implementation involves three nontrivial transformations.

Note that an obstruction might appear if the sum of the Youla parameters are not in the domain of $\mathfrak{M}_{\Sigma_P}$, e.g., for non strictly proper plants where some of the non strictly proper parameters are out-ruled.

We can formulate this process as a group homomorphism between the usual addition of parameters $\mathbb{Q}$ and the group of automorphisms $Q \mapsto \tau_Q$ associated to the space formed by simple translations, i.e.,

$$\tau_Q = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}, \quad \tau_{Q_1}\tau_{Q_2} = \tau_{Q_1+Q_2}.$$

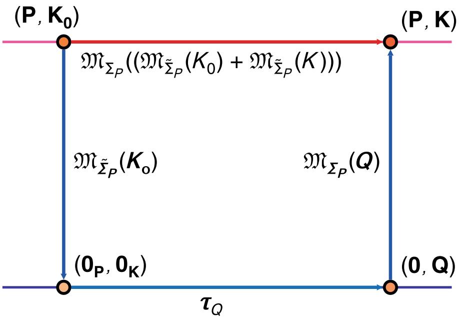


Fig. 2.11 Youla based blending

2.4.2 Direct Blending

The observation that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - PK_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix} \tag{2.10}$$

leads to operation

$$K = K_1(I - PK_2) + K_2 = K_1 \boxdot_P K_2, \tag{2.11}$$

under which well-posed controllers form a group $(\mathbb{W}_P, \boxdot_P)$. The unit of this group is the zero controller $K = 0_K$ and the corresponding inverse elements are given by

$$K^{\boxminus_P} = -K(I - PK)^{-1}. \tag{2.12}$$

Note that

$$I - PK^{\boxminus_P} = (I - PK)^{-1}. \tag{2.13}$$

Clearly not all elements of $\mathbb{W}_P$ are stabilizing, e.g., 0_K is not stabilizing for an unstable plant.

Theorem 2.1 $(\mathbb{G}_P, \boxdot_P)$ *with the operation (blending) defined in* (2.11) *is a semi-group.*

Note, that

$$(I - PK)^{-1} = (I - PK_2)^{-1}(I - PK_1)^{-1}. \tag{2.14}$$

By using the notation

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix} = R_P T_K^{(P)}$$

we have the group homomorphism $T^{(P)}_{K_1} T^{(P)}_{K_2} = T^{(P)}_{K_1 \boxdot_P K_2}$ and $K = \mathfrak{M}_{R_P T^{(P)}_K R_P^{-1}}(0_K)$.

On the level of Youla parameters the corresponding operation is more complex:

$$
\begin{aligned}
Q_{K_2} \odot_P Q_{K_1} &= \tilde{V} U + \tilde{V} M Q_{K_1} + Q_{K_2} \tilde{M} V + Q_{K_2} \tilde{N} M Q_{K_1} = \\
&= (Q_{K_2} - Q_0) \tilde{M} (V + N Q_{K_1}) + Q_{K_1} = \\
&= Q_{K_2} + (\tilde{V} + Q_{K_2} \tilde{N}) M (Q_{K_1} - Q_0), \qquad (2.15)\\
Q_{K^{\boxminus_P}} &= Q_0 - M^{-1} K \tilde{M}^{-1} = \\
&= Q_0 - (Q_K - Q_0)(I + V^{-1} N Q_K)^{-1} V^{-1} \tilde{M}^{-1}. \qquad (2.16)
\end{aligned}
$$

Note that $(\mathbb{G}_P, \boxdot_P)$ and $(\mathbb{Q}, \odot_P)$ are related by only a semigroup homomorphism, while $(\mathbb{W}_P, \boxdot_P)$ and $(\mathbb{Q}_P^{wp}, \odot_P)$ are related, however, through a group homomorphism.

2.4.3 *Strong Stability*

If a plant is stabilizable in general it is not obvious whether there exists a stable controller as a stabilizing one. If such a controller exists, then we call it a strongly stabilizing controller. While their synthesis is non-trivial, in practical applications strongly stabilizing controllers are preferred, see [13, 14].

The semigroup $(\mathbb{G}_P, \boxdot_P)$ does not have a unit, in general. However, if there is a stabilizing controller K_0 such that

$$K_0^{\boxminus_P} = -K_0 (I - P K_0)^{-1}$$

is also a stabilizing controller, i.e., K_0 is stable, then $(\mathbb{G}_P, \boxtimes_P)$ with

$$K_1 \boxtimes_P K_2 = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2$$

is a semigroup with a unit (K_0). This may happen only if the plant is strongly stabilizable.

If we denote by $\mathbb{S}_P$ the set of strongly stabilising controllers, then if this set is not empty, then.

Theorem 2.2 $(\mathbb{S}_P, \boxtimes_P)$ *with the operation (blending) defined as*

$$
\begin{aligned}
K = K_1 \boxtimes_P K_2 &= K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \\
&= K_2 + (K_1 - K_0)(I - P K_0)^{-1}(I - P K_2) \qquad (2.17)
\end{aligned}
$$

is the group of strongly stable controllers, where $K_0 \in \mathbb{S}_P$ is arbitrary. The corresponding inverse is given by

$$K^{\boxtimes_P^{-1}} = K_0 - (K - K_0)(I - PK)^{-1}(I - PK_0). \tag{2.18}$$

Opposed to the possible expectations, we not only have simple expressions for these operations in the Youla parameter space, but the formulae also resemble (2.11) and (2.12):

$$Q_K = Q_{K_2} \otimes Q_{K_1} = Q_{K_2} + Q_{K_1} + Q_{K_2} V^{-1} N Q_{K_1}, \tag{2.19}$$

$$Q_K^{\otimes^{-1}} = -Q(I + V^{-1}NQ)^{-1}. \tag{2.20}$$

It is important to note that while (2.19) keeps the strong stabilizability, as a property, invariant it does not guarantee that the property is fulfilled. This means that the formula also makes sense for parameters that does not correspond to stable controllers.

2.4.4 Example: State Feedback

In this section we provide some examples in order to illustrate the blending properties of these newly defined operators. To do this, let us consider first the state feedback case, i.e., fix the plant $P = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$ parametrized by its state space description and consider the stabilizing (state feedback) controllers given by:

$$K_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_1 \end{array}\right], \quad \text{and} \quad K_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_2 \end{array}\right],$$

respectively.

In order to applying operation (2.11) we need to compute the terms PK_2, K_1PK_2, $K_1 + K_2$, etc. To do this we apply the formulae (2.41), (2.42) and (2.43), respectively. Thus, we obtain first

$$PK_2 = \left[\begin{array}{cc|c} A & 0 & BF_2 \\ 0 & 0 & 0 \\ \hline I & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & BF_2 \\ \hline I & 0 \end{array}\right],$$

where the last equality is obtained by eliminating the uncontrollable and unobservable modes.

Analogously follows

$$K_1PK_2 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & A & BF_2 \\ \hline 0 & F_1 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & BF_2 \\ \hline F_1 & 0 \end{array}\right],$$

and $K_1 + K_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_1 + F_2 \end{array}\right]$. Finally we obtain that the dynamic controller as

$$K = K_1 + K_2 - K_1PK_2 = \left[\begin{array}{c|c} A & BF_2 \\ \hline -F_1 & F_1 + F_2 \end{array}\right].$$

In general this controller is not stable, however, it is stabilizing.

Indeed, one can assemble the closed loop system

$$\begin{aligned} \dot{x} &= Ax - BF_1x_c + B(F_1 + F_2)x, \\ \dot{x}_c &= Ax_c + BF_2x. \end{aligned}$$

Then, by applying the usual change of state variables $[x \; x - x_c]$, the corresponding closed loop matrix is

$$A_{cl} = \begin{pmatrix} A + BF_2 & BF_1 \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected. Moreover, one can also observe the blending of the assigned spectrum.

Taking a stabilizing feedback $K_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_0 \end{array}\right]$ by (2.42) and (2.43) we have

$$(I - PK_0)^{-1} = \left[\begin{array}{c|c} A + BF_0 & BF_0 \\ \hline I & I \end{array}\right],$$

i.e., computing $K_0^{\boxminus_P}$ according to (2.12) gives

$$K_0^{\boxminus_P} = -K_0(I - PK_0)^{-1} = \left[\begin{array}{c|c} A + BF_0 & BF_0 \\ \hline -F_0 & -F_0 \end{array}\right].$$

This is a stable plant, as it was expected. However, it is not a stabilizing controller, in general: by taking the basis $[x \; x + x_c]$, the corresponding closed loop system can be expressed as $A_{cl} = \begin{pmatrix} A & BF_0 \\ 0 & A \end{pmatrix}$, which is not stable, in general.

It is obvious, that every static stabilizing state feedback controller is a strongly stabilizing one. Thus, by fixing a given stabilizing state feedback controller, say K_0, we are going to computing a blending controller according to (2.17). Proceeding as before, we have

$$(I - PK_0)^{-1}(I - PK_2) =$$
$$= \left[\begin{array}{cc|c} A + BF_0 & -BF_0 & BF_0 \\ 0 & A & BF_2 \\ \hline I & -I & I \end{array}\right] = \left[\begin{array}{c|c} A + BF_0 & B(F_0 - F_2) \\ \hline I & I \end{array}\right].$$

This leads to

$$K = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \left[\begin{array}{c|c} A + BF_0 & B(F_2 - F_0) \\ \hline -(F_1 - F_0) & F_1 + F_2 - F_0 \end{array}\right],$$

which is clearly stable. Note that the degree of the controller (n) is less than the expected one ($2n$). We can also verify, that this controller is stabilizing: the matrix of the closed loop system in the usual basis ($[x\ x - x_c]$) can be expressed as

$$A_{cl} = \begin{pmatrix} A + BF_2 & B(F_1 - F_0) \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected. Again, we can also observe the blending of the assigned spectrum.

Taking a stabilizing feedback $K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]$ and computing $K^{\boxtimes_P^{-1}}$ according to (2.18) leads to

$$K^{\boxtimes_P^{-1}} = \left[\begin{array}{c|c} A + BF & B(F - F_0) \\ \hline -(F - F_0) & -F + 2F_0 \end{array}\right],$$

which is clearly stable.

The corresponding closed loop matrix in the basis $[x\ x + x_c]$ can be written as

$$A_{cl} = \begin{pmatrix} A + BF_0 & -B(F - F_0) \\ 0 & A + BF_0 \end{pmatrix},$$

i.e., the closed loop system is stable, as we have already expected.

2.5 A Geometry Based Controller Parametrization

In what follows we fix a stabilizing controller, say K_0, and in the formulae we associate, according to (2.5), the corresponding sensitivities to this controller. Considering

$$\hat{\Sigma}_{P,K_0} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix}$$

we obtain the lower LFT representation of the Youla parametrization, i.e.,

$$K = \mathfrak{M}_{\Sigma_{P,K_0}}(Q) = \mathfrak{F}_l(\hat{\Sigma}_{P,K_0}, Q), \tag{2.21}$$

see, e.g., [42]. Rearranging the terms one has

$$K = \mathfrak{F}_l(\Psi_{K_0,P}, R), \quad \text{with} \quad \Psi_{K_0,P} = \begin{pmatrix} K_0 & I \\ I & S_p \end{pmatrix} \tag{2.22}$$

and

$$R \in \mathbb{R}^Y_{K_0} = \{ \tilde{V}^{-1} Q V^{-1} \mid Q \in \mathbb{Q} \}. \tag{2.23}$$

This fact was already observed for a while, e.g., [24] or [3], where it was used as a starting point for a Youla parametrization based gain scheduling scheme of rational LTI systems. We have recalled this result with the intention to demonstrate how our previous ideas on the geometry of stabilizing controllers can be applied in order to find significantly new information on an already known configuration.

2.5.1 A Coordinate Free Parametrization

In order to relate a Möbius transform to an LFT we prefer to use the formalism presented in [36]. Thus, recall that $\hat{\Sigma}_P$ is the Potapov–Ginsburg transform of Σ_P and formulae like (2.22) can be easily obtained by using the group property of the Möbius transform. Accordingly, we have that

$$K = \mathfrak{M}_{\Gamma_{P,K_0}}(R) = \mathfrak{F}_l(\Psi_{P,K_0}, R), \tag{2.24}$$

$$R = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}}(K) = \mathfrak{F}_l(\Phi_{P,K_0}, K), \tag{2.25}$$

where

$$\Gamma_{P,K_0} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix}, \quad \Psi_{P,K_0} = \hat{\Gamma}_{P,K_0}, \tag{2.26}$$

$$\Gamma^{-1}_{P,K_0} = \begin{pmatrix} I & -K_0 \\ S_p & S_y \end{pmatrix}, \quad \Phi_{P,K_0} = \begin{pmatrix} -K_0 S_y^{-1} & S_u^{-1} \\ S_y^{-1} & P \end{pmatrix}. \tag{2.27}$$

Observe that (2.25) is defined exactly on $\mathbb{W}_P$ and let the restriction on the stabilizing controllers be denoted by $\mathbb{R}_{K_0} = \{\mathfrak{F}_l(\Phi_{P,K_0}, K) \mid K \in \mathbb{G}_P\}$. Apparently, apart the structure of the set $\mathbb{R}^Y_{K_0}$ these formulae do not depend on any special factorization. Moreover, they can be also obtained directly, i.e., without any reference to some factorization of the plant or of the controller, starting from

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} (K - K_0) \begin{pmatrix} 0 & I \end{pmatrix}$$

and applying two times the matrix inversion lemma to obtain first

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix}^{-1} - \begin{pmatrix} S_u \\ S_p \end{pmatrix} R \begin{pmatrix} S_p & S_y \end{pmatrix},$$

with $R = (K - K_0)(I + S_p(K - K_0))^{-1}$ and then

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} R(I - S_p R)^{-1} \begin{pmatrix} 0 & I \end{pmatrix}. \tag{2.28}$$

Thus, it would be desirable to provide, if it exists, a coordinate free description of $\mathbb{R}_{K_0}$. Exactly this is the point where the geometric view and the coordinate free results of Sect. 2.4 can be applied.

As a starting point observe that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix} = \tag{2.29}$$

$$= \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R)^{-1} \end{pmatrix}. \tag{2.30}$$

Analogous to (2.10) we have the factorization

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1} K \\ 0 & I - PK \end{pmatrix}. \tag{2.31}$$

By using the notations

$$\begin{aligned} R_{(P,K_0)} &= \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix} \\ T_K^{(P,K_0)} &= \begin{pmatrix} S_u & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1} K \\ 0 & I - PK \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

we have

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = R_{(P,K_0)} T_K^{(P,K_0)}$$

and

$$T_{K_1}^{(P,K_0)} T_{K_2}^{(P,K_0)} = T_{K_1 \boxdot_P K_2}^{(P,K_0)},$$

moreover

$$K = \mathfrak{M}_{R_{(P,K_0)} T_K^{(P,K_0)} R_{(P,K_0)}^{-1}}(0_K) = \mathfrak{M}_{\Gamma_{P,K_0}}(R),$$

see (2.30) for the last equality. Thus, it is immediate that the operation (2.11) is a natural choice for this new configuration, too.

2.5.2 *Geometric Description of the Parameters*

Considering (2.15) and keeping in mind that $R = \tilde{V}^{-1} Q V^{-1}$ we have the blending rule on $\mathbb{R}_{K_0}^{Y}$:

$$R_2 \odot_{P,K_0} R_1 = K_0 + S_u R_1 + R_2 S_y - R_2 S_y S_p R_1. \tag{2.32}$$

For the stable controllers the parameter blending is more simple:

$$R_2 \otimes_{P,K_0} R_1 = R_2 + R_1 - R_2 S_p R_1, \tag{2.33}$$

$$R^{\otimes_{P,K_0}^{-1}} = -R(I - S_p R)^{-1}, \tag{2.34}$$

see (2.19) and (2.20).

Observe that $K_0 = \tilde{V}^{-1} \tilde{V} U V^{-1} \in \mathbb{R}_{K_0}^{Y}$ and that the corresponding controller is

$$K = K_0 \boxdot_P K_0 = [2K_0]_{\boxdot_P}.$$

Based on (2.32) it is easy to show that to the controller $K = [nK_0]_{\boxdot_P}$ corresponds the parameter $R = (I + \cdots + S_u^{n-1})K_0 \in \mathbb{R}_{K_0}^{Y}$. Thus, if K_0 is stable, then all these parameters are stable. However, the corresponding controllers are not necessarily stable.

Theorem 2.3 *The algebraic structures defined by* (2.32) *and* (2.33) *holds also on* $\mathbb{R}_{K_0}$, *i.e., they can be introduced in a complete coordinate free way.*

Due to lack of space, we do not continue to deduce all the formulae, e.g., inverse, shifted blending, etc., for the parameters. Instead we show, in what follows, that the operation (2.32) can be obtained directly, without the Youla parametrization. To do so, observe that

$$I - PK = (I - PK_0)(I - S_p R_1)^{-1},$$

thus

$$\begin{pmatrix} I & S_u^{-1} K \\ 0 & I - PK \end{pmatrix} = \begin{pmatrix} I & S_u^{-1}(K_0 + S_u R) \\ 0 & S_o^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R)^{-1} \end{pmatrix}.$$

Then, according to (2.30) we have

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_1)^{-1} \end{pmatrix}$$
$$\begin{pmatrix} I & S_u^{-1}(K_0 + S_u R_2) \\ 0 & S_o^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_2)^{-1} \end{pmatrix}. \tag{2.35}$$

Now, keeping in mind that

$$R = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}}(K) = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}\begin{pmatrix} I & K \\ P & I \end{pmatrix}}(0_K) = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix}}(0_K),$$

the assertion follows after evaluating (2.35).

We have already seen that $\{0, K_0\} \subset \mathbb{R}_{K_0}$. Moreover, we have seen that $\mathbb{Q} \subset \mathbb{R}^Y_{K_0}$ by an identification of $Q \to \tilde{V} Q V$, i.e., $R \to Q$. It turns out that this inclusion is also a coordinate free property, i.e., the inclusion holds regardless the existence of any coprime factorization.

Theorem 2.4 *The inclusion* $\mathbb{Q} \subset \mathbb{R}_{K_0}$ *holds.*

Indeed, by taking a controller $K \in \mathscr{K}_{K_0}$, where

$$\mathscr{K}_{K_0} = \{ K = \mathfrak{F}_l(\Psi_{K_0,P}, Q) = K_0 + Q(I - S_p Q)^{-1} \mid Q \in \mathbb{Q}\}, \tag{2.36}$$

after some standard computations, that are left out for brevity, we obtain

$$(I - PK)^{-1} = (I - S_p Q)(I - PK_0)^{-1} \tag{2.37}$$
$$(I - KP)^{-1} = (I - K_0 P)^{-1}(I - Q S_p) \tag{2.38}$$
$$(I - PK)P^{-1} = -(I - S_p Q) S_p \tag{2.39}$$
$$\begin{aligned} K(I - PK)^{-1} &= -S_c + (I - K_0 P)^{-1} Q (I - K_0 P)^{-1} - \\ &\quad - (I - K_0 P)^{-1} Q + Q. \end{aligned} \tag{2.40}$$

Thus $\mathscr{K}_{K_0} \subset \mathbb{G}_P$, as desired.

From a mathematical point of view, there is a small missing here. When a double coprime factorization exists, we should also prove that the set defined by (2.23), and the set defined by (2.28) are equal, i.e., $\mathbb{R}^Y_{K_0} = \mathbb{R}_{K_0}$. But this is equivalent to the fact that the Youla characterization of stabilizing controllers is exhaustive. This is a highly nontrivial issue and it is beyond the scope of this paper to address this topic in general. We should mention, however, that this property holds for discrete time systems, see, e.g., [12].

2.6 From Geometry to Control

As it was already pointed out in the introduction of this paper we have found very useful to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction. In the previous sections some examples were presented to illustrate this point. Now, it is time to demonstrate the way that starts from the abstract level and ends into a directly control relevant result.

The reader customised with system classes, like LTI, LPV (linear parameter varying), nonlinear, switching, etc. might find our presentation of the geometric ideas quite informal. We stress that this is a “feature” of the method. Recall that geometry—and also group theory—does not deal with the existence and the actual nature of the objects that are the primitives of the given geometry but rather captures the “rules” they obeys to. It gives the abstract structures that can be, for a given application, associated with actual objects, i.e., responds to the question “what can be done with these objects” rather than “how to synthesise the object having a given property (e.g., stability)”.

We illustrate this fact by the example of the Youla parametrization. A basic knowledge is to place the topic in the context of finite rank LTI systems, i.e., those associated with rational transfer functions $\mathscr{R}$, and to interpret the result only in this context. However, we should not confine ourselves to this class: it is clear that an LTI plant can be also stabilized by more “complex” controllers, e.g., nonlinear ones, see, e.g., the IQC approach of [35]. This is also clear from the geometry: nothing prevents the Youla parameter to be any stable plant (not necessarily linear) in order to generate the stabilizing controller. Moreover, the nature of the parameter (e.g., nonlinear) is inherited by the controller through the Möbius transform.

We stress that the geometric picture behind the Youla parametrization has been applied under the hood even in the cases when the classes at hand do not have a sound input-output description, e.g., the class of switching systems or even the LPV systems. For the difficulties around these systems when we want to cast them exclusively into in input-output framework see, e.g., [7]. These difficulties does not prevent engineers to reduce the design of the switching controllers to switching between the corresponding values of the parameters, see, e.g., [3, 26, 37]. Moreover, the idea can be extended also for plants that are switching systems themselves, [7, 17], or LPV plants, [40].

Observe that in all these examples the authors spend a considerable amount of effort to solve the existential problem, i.e., how to obtain K_0. In all these cases this problem is cast in a state space framework and the taxonomy of the methods revolves around the type of the Lyapunov function (quadratic vs. polyhedral norm, constant Lyapunov matrix vs. parameter varying) involved that is used as stability certifier.

The motivation behind the increased complexity of the controller is that some additional performance demand is imposed either for the closed loop or for the controller, which cannot be fulfilled in the LTI setting. Concerning closed loop performances, the advantage of the Youla based approaches is that the performance transfer function is affine in the design parameter.

As an example consider the strong stabilizability problem. It is a standard knowledge that in $\mathscr{R}$ the problem does not always have a solution. However, it is less known that if one considers time variant (LTV) controllers, the problem is always solvable, see [21]. Moreover, for the discrete time case the problem is solvable in the disc algebra $\mathscr{A}$ or even in $\mathscr{H}_\infty$, see [28, 29].

To conclude this section we point out some additional properties of the parametrization presented in Sect. 2.5. As a consequence of (2.22) and (2.36), for every controller K_0 there is a stable perturbation ball Δ, contained in the image of the ball with radius $\frac{1}{\|S_p\|}$ under the map $x(1-x)^{-1}$, such that the pair $(P, K_0+\delta)$ is stable for all $\delta \in \Delta$. In particular, if the controller K_0 is strongly stabilizing, then all the controllers from $K_0 + \Delta$ are strongly stabilizing. This fact reveals the role of the sensitivity S_p in relation to the robustness of the stabilizing property of K_0. Due to the symmetry, analogous role is played by S_c for P.

This knowledge, together with (2.32) can be exploited to generate a hole branch of strongly stabilizing controllers starting from an initial one, K_0, with this property; e.g., one has to choose arbitrarily a stable R with a sufficiently small norm (less than $\frac{1}{\|S_p\|}$) and then apply (2.32) iteratively.

2.7 Conclusions

In this work we have shown that based on the direct blending operation the set $\mathbb{R}_{K_0}$ of stabilizing controllers can be defined and characterized in a completely coordinate free way, without any reference to a coprime factorization. For practical purposes it is also interesting to know that $K_0 \in \mathbb{R}_{K_0}$, moreover $\mathbb{Q} \subset \mathbb{R}_{K_0}$ holds as a coordinate free property, too. We emphasize, that a fairly large set of stabilizing controllers can be constructed (parametrized) just starting from the knowledge of a single stabilizing controller, without any additional knowledge (e.g., factorization). This underlines an important property of the geometric (and also input-output) view: describes the structure of the given set—in our case those of the stabilizing controllers – but does not provide a direct method to find any of the actual objects at hand. To do so, we need to ensure (e.g., by a construction algorithm) the existence at least of a single element with the given property.

Up to this point only projective geometric structures were considered. In order to qualify a given controller K as a stabilizing one (validation problem) metric aspects should be also considered, i.e., euclidean, hyperbolic, etc. geometries; e.g., concerning the Youla parametrization $Q_K \in \mathbb{Q}$, or in the geometric parametrization $R_K \in \mathbb{R}_{K_0}$, should be decided. It is subject of further research how these geometries find their way to control theory and vice versa.

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Appendix

A state space realization for the sum of systems is given by

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right], \tag{2.41}$$

while the product of the systems can be expressed as:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]. \tag{2.42}$$

Note that these realizations are not necessarily minimal. If D is invertible then a realization of the inverse system is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]. \tag{2.43}$$

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Chapter 3
Strong Structural Controllability and Zero Forcing

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Abstract In this chapter, we study controllability and output controllability of systems defined over graphs. Specifically, we consider a family of state-space systems, where the state matrix of each system has a zero/non-zero structure that is determined by a given directed graph. Within this setup, we investigate under which conditions all systems in this family are controllable, a property referred to as strong structural controllability. Moreover, we are interested in conditions for strong structural output controllability. We will show that the graph-theoretic concept of zero forcing is instrumental in these problems. In particular, as our first contribution, we prove necessary and sufficient conditions for strong structural controllability in terms of so-called zero forcing sets. Second, we show that zero forcing sets can also be used to state both a necessary and a sufficient condition for strong structural output controllability. In addition to these main results, we include interesting results on the controllability of subfamilies of systems and on the problem of leader selection.

3.1 Introduction

Structural controllability has been an active research area ever since its introduction in the early seventies of the previous century by Lin in [9]. Originally, the concept of structural controllability was introduced in order to deal with uncertainty in the

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state and input matrix representing a given linear input-state system. Instead of being known exactly, some of the entries in these matrices are assumed to be unknown, while the remaining entries are equal to zero. The unknown entries can take arbitrary (zero or non-zero) values. The system is then called weakly structurally controllable if there exists a choice of values for the unknown entries such that the corresponding numerical realization is a controllable pair in the classical sense of Kalman. In the context of structural controllability, the system matrices are no longer matrices with real entries, but are *pattern matrices*, i.e., matrices with some entries equal to zero, and the remaining ones free. With the given pattern matrices defining the system, or rather, the system structure, Lin [9] associated a *directed graph*, and subsequently established conditions for weak structural controllability in terms of topological properties of this graph. Later on, this work was extended to the multi-input case in [6, 14]. In all of the above references, the pattern matrices consist of entries equal to zero and entries whose values are unknown. These unknown entries are allowed to be either zero or non-zero. If we require the non-zero entries to take non-zero values only, then we can also ask the question: under what conditions are *all* numerical realizations controllable in the sense of Kalman. In this case, we call the system defined by the pattern matrices *strongly structurally controllable*. This notion was introduced in [10].

The last decade has witnessed a revival of research related to structural controllability. For a major part, this has been caused by the outburst of research on networks of systems, also referred to as multi-agent systems. The interaction between the agents in such a network is usually represented by a directed graph, where the vertices are identified with the agents, and the arcs correspond to the communication between these agents. In the context of controllability of networks, two types of vertices are distinguished: those that are influenced by inputs from outside the network, called *leaders*, and those that are influenced only by their neighbours, called *followers*. Controllability of such leader–follower networks deals with the issue whether it is possible to steer the states of all vertices in the network to any desired state by applying inputs to the leaders. Clearly, the interaction between the agents is represented by the network graph, including the weights of the arcs.

If one ignores the exact values of the arc weights, and instead focusses on the graph topology only, one arrives at the issue of structural controllability of such leader–follower network: it is called strongly structurally controllable if, roughly speaking, for all (non-zero) values of the arc weights the corresponding numerical realization of the network is controllable. It should be noted that a special role is played by the diagonal entries of the state matrix, which are allowed to take arbitrary (zero or non-zero) values, depending on whether or not the graph contains self-loops. The system is called weakly structurally controllable if there exists a choice of arc weights such that the corresponding network is controllable. The input matrix is determined by the choice of leader vertices. Thus, with a given network graph, a whole family of linear input-state systems is associated, and strong structural controllability deals with the question whether all members of this family are controllable in the classical sense. Weak structural controllability requires that at least one member of the family is controllable. Like in the classical literature on structural controllability, conditions for strong and weak structural controllability of networks are formulated in terms of properties of the underlying network graph, see, e.g., [3–5, 12, 15]. A topological condition for weak structural controllability of networks in terms of maximum matchings was established in [5]. Strong structural controllability was characterized in terms of constrained matchings in [4]. In [12, 15], necessary and sufficient conditions for strong structural controllability were given in terms of *zero forcing sets*.

More recently, research on the topic of strong structural controllability has been extended to strong *output* controllability, also called *strong targeted controllability*. Here, in addition to the subset of leader vertices, a subset of target vertices is given. Then, targeted controllability deals with the question whether the states of the target vertices can be steered to arbitrary desired states by applying appropriate inputs to the leader vertices. For a given network graph with leader set and target set, we have a family of *input-state-output* systems, and the network is called strongly targeted controllable if all members of this family are output controllable, see [16]. Also for strong targeted controllability, graph topological conditions have been obtained. In particular, in [17], such conditions were obtained for the subfamily of all input-state-output systems with distance-information preserving state matrices.

In the present paper, we will give an introduction to the concept of zero forcing and its application to strong structural controllability and strong targeted controllability. The outline of the paper is as follows. In Sect. 3.2, we will introduce preliminaries on the so-called colour change rule, and introduce the concept of zero forcing set. In Sect. 3.3, we introduce the notions of qualitative class associated with a given graph, and give a definition of strong structural controllability. We study the strong structurally reachable subspace associated with a given qualitative class and leader set. This leads to a necessary and sufficient condition for strong structural controllability. Subsequently, we discuss subclasses of the given qualitative class and give a definition of the notion of sufficient richness. Finally, in this section, we address the issue of leader selection, which is concerned with determining the minimal number of leaders required for strong structural controllability. It is shown that this number is equal to the zero forcing number of the graph. Our final section, Sect. 3.4, deals with strong targeted controllability. We define this concept starting from classical output controllability for linear input-state-output systems. Then, we give sufficient conditions for strong targeted controllability. Also, we strengthen these conditions by restricting ourselves to the important subclass of distance-information preserving state matrices. Finally, the paper closes with conclusions in Sect. 3.5.

3.2 Zero Forcing

In this section, we review the notion of zero forcing. Let $G = (V, E)$ be a simple directed graph with vertex (or node) set V and edge set $E \subseteq V \times V$. We say that $v \in V$ is an *out-neighbour* of vertex $u \in V$ if $(u, v) \in E$. Now suppose that the vertices in V are coloured either black or white. The *colour change rule* is defined in the following way. If $u \in V$ is a black vertex and *exactly one* out-neighbour $v \in V$ of u is white, then change the colour of v to black [8]. When the colour change rule is applied to node u to change the colour of v, we say *u forces v*, which we denote by $u \to v$.

Suppose that we have a colouring of G, that is, a set $C \subseteq V$ of only black vertices, and a set $V \setminus C$ consisting of only white vertices. Then the *derived set* $D(C)$ is the set of black vertices obtained by applying the colour change rule until no more changes are possible [8]. It can be shown that for a given graph G and set C, the derived set $D(C)$ is unique [1]. However, note that the order in which forces occur in the colouring process is in general not unique.

The set C is called a *zero forcing set* of G if $D(C) = V$. Let $|C|$ denote the cardinality of C. Then, the *zero forcing number* $Z(G)$ of the graph G is the minimum of $|C|$ over all zero forcing sets C of G. Moreover, a zero forcing set $C \subseteq V$ is called a *minimum zero forcing set* if $|C|$ equals $Z(G)$.

3.3 Zero Forcing and Structural Controllability

A linear time-invariant input/state system of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if and only if the well-known Kalman rank condition holds:

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \text{ is of full row rank.}$$

In this chapter, we are interested in *structural controllability* properties that depend on the (zero) *structure* of the matrices A and B, rather than their numerical values.

To be more specific, let $G = (V, E)$ be a simple directed graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E \subseteq V \times V$. Define the *qualitative class* of G, denoted by $Q(G)$, as

$$Q(G) := \{X \in \mathbb{R}^{n \times n} \mid \text{ for } i \neq j,\ X_{ij} \neq 0 \iff (j, i) \in E\}. \tag{3.1}$$

For $V' = \{v_1, v_2, \ldots, v_k\} \subseteq V$, let $P(V; V')$ denote the $n \times k$ matrix whose ij-th entry is given by

$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Consider the family of linear time-invariant systems

$$\dot{x}(t) = Xx(t) + Uu(t), \tag{3.3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $X \in Q(G)$, and $U = P(V; V_L)$ where $V_L \subseteq V$ is the so-called *leader set*.

Examples of systems of the form (3.3) are encountered in [7] where X is the adjacency matrix, in [2] the normalized matrix, and in [11] the (in-degree or out-degree) Laplacian.

We call the system (3.3) *strongly structurally controllable* if the pair (X, U) is controllable for all $X \in Q(G)$. In that case, we write $(Q(G); V_L)$ is controllable. The term "strong" is used to distinguish with the case of "weak structural controllability" which amounts to the existence of a controllable pair (X, U) with $X \in Q(G)$.

3.3.1 Strong Structural Controllability

In this subsection, we aim at a topological characterization of strong structural controllability and investigate how the graph structure can determine the controllability of $(Q(G); V_L)$, given a leader set V_L. A related problem is *minimal leader selection*, where the goal is to choose a leader set V_L with minimum cardinality such that $(Q(G); V_L)$ is controllable. We will see that both strong structural controllability and minimal leader selections problems are intimately related to the colour change rule and zero forcing sets discussed in Sect. 3.2.

Throughout this chapter, we denote the image (range) of the matrix $P(V; V_L)$ by $\mathscr{V}_L$, and the image of $P(V; D(V_L))$ by $\mathscr{D}(V_L)$. The *reachable subspace* associated with the pair (X, U), denoted by $\langle X \mid \mathrm{im} U\rangle$, is defined as

$$\langle X \mid \mathrm{im} U\rangle = \mathrm{im} U + X \mathrm{im} U + \cdots + X^{n-1} \mathrm{im} U.$$

The subspace $\langle X \mid \mathrm{im} U\rangle$ is the smallest X-invariant subspace containing $\mathrm{im} U$. It is well known (see e.g., [16]) that (X, U) is controllable if and only if $\langle X \mid \mathrm{im} U\rangle = \mathbb{R}^n$.

In the following lemma, we state that the reachable subspace is not affected by the colour change rule.

Lemma 3.1 ([13]) *For any given $X \in Q(G)$ and leader set $V_L \subseteq V$, we have $\langle X \mid \mathscr{V}_L\rangle = \langle X \mid \mathscr{D}(V_L)\rangle$.*

Proof First, we prove that $\langle X \mid \mathscr{D}(V_L)\rangle \subseteq \langle X \mid \mathscr{V}_L\rangle$. This trivially holds in case $D(V_L) = V_L$, and thus $\mathscr{D}(V_L) = \mathscr{V}_L$. Now, suppose that $D(V_L) \neq V_L$, and vertex $v \in V_L$ forces vertex $w \notin V_L$. Then, we *claim* that

$$\mathrm{im} P(V; V_L \cup \{w\}) \subseteq \langle X \mid \mathscr{V}_L\rangle, \tag{3.4}$$

where P is given by (3.2). Clearly, the subspace inclusion (3.4) holds if and only if

$$\langle X \mid \mathscr{V}_L\rangle^\perp \subseteq P(V; V_L \cup \{w\})^\perp. \tag{3.5}$$

Without loss of generality, assume that $V_L = \{1, 2, \ldots, m\}$, $v = m$ and $w = m + 1$. Then, the matrix X can be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}, \tag{3.6}$$

where $X_{11} \in \mathbb{R}^{(m-1)\times(m-1)}$, $X_{22} \in \mathbb{R}$, $X_{33} \in \mathbb{R}$, $X_{44} \in \mathbb{R}^{(n-m-1)\times(n-m-1)}$, and the rest of the matrices involved have compatible dimensions. Notice that the matrix $P(V; V_L \cup \{w\})$ now reads as

$$P(V; V_L \cup \{w\}) = \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\xi \in \mathbb{R}^n$ be a vector in $\langle X \mid \mathscr{V}_L \rangle^{\perp}$. Clearly, we have $\xi^T X^{k-1} P(V; V_L) = 0$ for each $k \in \mathbb{N}$. We write $\xi = \mathrm{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ compatible with the partitioning of X. Setting $k = 1$ yields the equality $\xi^T P(V_L; V) = 0$, and hence $\xi_1 = 0$ and $\xi_2 = 0$. In addition, by setting $k = 2$, we obtain the equality $\xi^T X P(V_L; V) = 0$, which results in

$$\begin{bmatrix} \xi_3^T & \xi_4^T \end{bmatrix} \begin{bmatrix} X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{bmatrix} = 0. \tag{3.7}$$

Since $v \to w$, the vertex v has exactly one out-neighbour in $V \setminus V_L$, and thus we have $X_{32} \neq 0$ and $X_{42} = 0$. Therefore, noting (3.7), the scalar ξ_3 must be equal to zero. As $\xi = \mathrm{col}(0, 0, 0, \xi_4)$ is orthogonal to the subspace $P(V; V_L \cup \{w\})$, the subspace inclusion (3.5) and consequently (3.4) holds. By repeating the argument above, we conclude that $\mathrm{im}\, P(V; D(V_L)) = \mathscr{D}(V_L) \subseteq \langle X \mid \mathscr{V}_L \rangle$, which results in $\langle X \mid \mathscr{D}(V_L) \rangle \subseteq \langle X \mid \mathscr{V}_L \rangle$.

Now, to prove the statement of the lemma, it remains to show that $\langle X \mid \mathscr{V}_L \rangle \subseteq \langle X \mid \mathscr{D}(V_L) \rangle$. The latter holds since $\mathscr{V}_L \subseteq \mathscr{D}(V_L)$, and the proof is complete.

Now that we have established the result of Lemma 3.1, an intriguing question is to characterize the subspace containing all the states that can be reached by applying appropriate input signals to the nodes in the leader set V_L, *independent* of the particular choice of $X \in Q(G)$. Geometrically, this subspace is given by $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle$, and provides a strong structural counterpart of the reachability subspace. Hence, we will refer to it as the *strongly structurally reachable subspace*. Consistent with the previous treatment, here we are after a topological characterization of this subspace.

From Lemma 3.1, we obtain that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle. \tag{3.8}$$

The subspace on the left is the strongly structurally reachable subspace. Interestingly, the one on the right simplifies to $\mathscr{D}(V_L)$, as stated in the theorem below.

Theorem 3.1 ([13]) *For any given leader set $V_L \subseteq V$, we have*

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathscr{D}(V_L). \tag{3.9}$$

Proof Given equality (3.8), and noting that

$$\mathscr{D}(V_L) \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle,$$

it suffices to show that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle \subseteq \mathscr{D}(V_L). \tag{3.10}$$

To this end, we define the set S as

$$S = \{s \in \mathbb{R}^n : s_i = 0 \Leftrightarrow i \in D(V_L)\}. \tag{3.11}$$

Let s be a vector in S. Without loss of generality, let $D(V_L) = \{1, 2, \dots, d\}$. Then, s can be written as $\mathrm{col}(0_d, s_2)$ where each element of $s_2 \in \mathbb{R}^{n-d}$ is non-zero. Let the matrix X be partitioned accordingly as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Clearly, we have

$$s^T X = s_2^T \begin{bmatrix} X_{21} & X_{22} \end{bmatrix}.$$

Note that X_{21} corresponds to the arcs from a vertex $v \in D(V_L)$ to a vertex $w \notin D(V_L)$. Hence, by the colour change rule, each column of X_{21} is either identically zero or contains at least two non-zero elements. We choose these non-zero elements, if any, such that $s_2^T X_{21} = 0$. Since the diagonal elements of X_{22} are free parameters, we conclude that, for any vector $s \in S$, there exists a matrix $X \in Q(G)$ such that $s^T X = 0$. Therefore, we obtain that

$$s \in \langle X \mid \mathscr{D}(V_L) \rangle^{\perp}$$

for some matrix $X \in Q(G)$. Now, let $\xi \in \mathbb{R}^n$ be a vector in $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle$. Hence, by definition, $\xi \in \langle X \mid \mathscr{D}(V_L) \rangle$ for all $X \in Q(G)$. Therefore, we have $s^T \xi = 0$ which yields $s_2^T \xi_2 = 0$, noting $\xi = \mathrm{col}(\xi_1, \xi_2)$. As this conclusion holds for any arbitrary choice of $s \in S$, we conclude that $\xi_2 = 0$. Consequently, we obtain $\xi \in \mathscr{D}(V_L)$, which results in (3.10), and completes the proof.

The most notable consequence of Theorem 3.1 is obtained by looking at the scenario where $D(V_L) = V$, which, by definition, is the case in which V_L is a zero forcing set. In this case, the result of Theorem 3.1 can be used to state necessary and sufficient conditions for strong structural controllability, as stated below.

Theorem 3.2 ([12]) *The system* (3.3) *is strongly structurally controllable, i.e.,* $(Q(G); V_L)$ *is controllable, if and only if* V_L *is a zero forcing set in* G*.*

Proof Suppose that $(Q(G); V_L)$ is controllable. This means that $\langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ for all $X \in Q(G)$. Therefore, $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$. By Theorem 3.1, $\mathscr{D}(V_L) = \mathbb{R}^n$. We conclude that V_L is a zero forcing set. Conversely, suppose that V_L is a zero forcing set. Hence, $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ by Theorem 3.1. We conclude that $\langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ for all $X \in Q(G)$, that is, $(Q(G); V_L)$ is controllable.

3.3.2 Leader Selection

Next, we discuss the *minimal leader selection* problem. In the context of strong structural controllability this amounts to selecting a leader set with minimum cardinality such that (3.3) is strongly structurally controllable. To make this more precise, we define $\ell_{\min}(Q(G))$ as follows:

$$\ell_{\min}(Q(G)) = \min_{V_L \subseteq V(G)} \{|V_L| : (Q(G); V_L) \text{ is controllable}\}. \tag{3.12}$$

An immediate consequence of Theorem 3.2 is

$$\ell_{\min}(Q(G)) = Z(G).$$

The equality above relates the minimal leader selection problem for strong structural controllability of networks to minimal zero forcing sets and the zero forcing number in graph theory [8]. While finding a minimal zero forcing set in general is a difficult combinatorial problem, such sets can be efficiently computed for several types of graphs including path, cycle, acyclic graphs and complete graphs [12].

3.3.3 Qualitative Subclasses

So far, we have investigated controllability of systems given by (3.3), where the matrix X belongs to the family of matrices given by $Q(G)$ in (3.1). In many examples, the state matrix may have more structure than the one captured by $Q(G)$. For instance, in the case that the graph $G = (V, E)$ is *symmetric* (i.e., $(i, j) \in E \iff (j, i) \in E$)

one might be interested in the class of symmetric state matrices only. This gives rise to a subclass of $Q(G)$, namely

$$Q_{\text{sym}}(G) = \{X \in Q(G) \mid X = X^T\}. \tag{3.13}$$

Given a leader set $V_L \subseteq V$ and a subclass of $Q_s(G) \subseteq Q(G)$, we say that $(Q_s(G); V_L)$ is controllable if (X, U) is controllable for all $X \in Q_s(G)$ and $U = P(V; V_L)$.

Obviously, $(Q_s(G); V_L)$ is controllable if $(Q(G); V_L)$ is controllable. The natural question arises here is whether the converse result holds for certain subclasses of $Q(G)$. This motivates the following definition.

Definition 3.1 A subclass $Q_s(G) \subseteq Q(G)$ is called *sufficiently rich* if for any $V_L \subseteq V$ such that $(Q_s(G); V_L)$ is controllable we have that $(Q(G); V_L)$ is controllable as well.

A useful sufficient algebraic condition for a subclass $Q_s(G)$ to be sufficiently rich is provided next.

Lemma 3.2 ([12]) *Let $Q_s(G) \subseteq Q(G)$. Then, $Q_s(G)$ is sufficiently rich if the following implication holds:*

$$z \in \mathbb{R}^n,\ z^T X = 0 \textit{ for some } X \in Q(G) \implies \exists\, X_s \in Q_s(G) \textit{ such that } z^T X_s = 0.$$

Notably, for any symmetric graph G, the algebraic condition in Lemma 3.2 holds for the subclass $Q_{\text{sym}}(G)$ (see Proposition IV.9 of [12]). This implies that $Q_{\text{sym}}(G)$ is a sufficiently rich subclass of $Q(G)$. Hence, Theorem 3.2 can be used to characterize controllability of $(Q_{\text{sym}}(G); V_L)$. Specifically, $(Q_{\text{sym}}(G); V_L)$ is controllable if and only if V_L is a zero forcing set.

Another important subclass of matrices is the class of so-called *distance-information preserving* matrices. To define this class of matrices, we need some terminology first. We define the *distance* $d(u, v)$ between two vertices $u, v \in V$ as the length of the shortest path from u to v. If there does not exist a path from vertex u to v, the distance $d(u, v)$ is defined as infinite. Moreover, the distance from a vertex to itself is equal to zero. For a non-empty subset $S \subseteq V$ and a vertex $j \in V$, the distance from S to j is defined as

$$d(S, j) := \min_{i \in S} d(i, j). \tag{3.14}$$

With this in mind, we state the following definition.

Definition 3.2 Consider a directed graph $G = (V, E)$. A matrix $X \in Q(G)$ is called *distance-information preserving* if for any two distinct vertices $i, j \in V$ we have that $d(j, i) = k$ implies $(X^k)_{ij} \neq 0$.

Although the distance-information preserving property does not hold for all matrices $X \in Q(G)$, it does hold for the adjacency and Laplacian matrices. Because these matrices are often used to describe network dynamics, distance-information preserving matrices form an important subclass of $Q(G)$. We will denote the subclass of distance-information preserving matrices by $Q_d(G)$. It turns out the $Q_d(G)$ is sufficiently rich, as asserted in the following lemma.

Lemma 3.3 ([17]) *The subclass $Q_d(G) \subseteq Q(G)$ is sufficiently rich.*

3.4 Targeted Controllability

In case (3.3) fails to be structurally controllable, it is worthwhile to investigate whether it is "partially" controllable. To elaborate, let $G = (V, E)$ be a simple directed graph where $V = \{1, 2, \ldots, n\}$ is the vertex set and $E \subseteq V \times V$ is the edge set. Consider the following linear time-invariant input/state/output system:

$$\dot{x}(t) = Xx(t) + Uu(t) \tag{3.15a}$$
$$y(t) = Hx(t), \tag{3.15b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. Here, $X \in Q(G)$, $U = P(V; V_L)$ for some given *leader set* $V_L \subseteq V$, and $H = P^T(V; V_T)$ for some given $V_T \subseteq V$ called the *target set*.

In what follows we will investigate the structural output controllability problem for systems of the form (3.15). We therefore first review output controllability for linear systems.

3.4.1 *Output Controllability*

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.16}$$
$$y(t) = Cx(t) \tag{3.17}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Denote the output trajectory corresponding to the initial state x_0 and input u by $y_u(t, x_0)$. The system (3.16) is then called *output controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input u and a $T > 0$ such that $y_u(T, x_0) = y_1$. We sometimes say that the triple (A, B, C) is output controllable meaning that the system (3.16) is output controllable.

It is well known (see, e.g., [16, Exc. 3.22]) that (A, B, C) is output controllable if and only if

$$\operatorname{rank}\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} = p.$$

In turn, this is equivalent to the condition

$$C\langle A \mid \operatorname{im} B\rangle = \mathbb{R}^p.$$

If C has full row rank, the condition above is equivalent to

$$\ker C + \langle A \mid \operatorname{im} B \rangle = \mathbb{R}^n.$$

Finally, by taking orthogonal complements, the latter holds if and only if

$$\operatorname{im} C^T \cap \langle A \mid \operatorname{im} B \rangle^{\perp} = \{0\}.$$

3.4.2 Problem Formulation

Let $G = (V, E)$ be a simple directed graph. Also, let $V_L \subseteq V$ be a leader set and let $V_T \subseteq V$ be a target set. We say that the system (3.15) is *strongly targeted controllable with respect to* $Q' \subseteq Q(G)$ if the system (3.15) is output controllable for all $X \in Q'$. For brevity, we will say $(Q'; V_L; V_T)$ is targeted controllable meaning that (3.15) is strongly targeted controllable with respect to $Q' \subseteq Q(G)$.

The following proposition translates the output controllability results mentioned in Sect. 3.4.1 to targeted controllability.

Proposition 3.1 *The following statements are equivalent:*

(a) $(Q'; V_L; V_T)$ *is targeted controllable.*
(b) $\operatorname{rank} \begin{bmatrix} HU & HXU & HX^2U & \cdots & HX^{n-1}U \end{bmatrix} = p$ *for all* $X \in Q'$.
(c) $H \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^p$ *for all* $X \in Q'$.
(d) $\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L \rangle^{\perp} = \{0\}$ *for all* $X \in Q'$.
(e) $\ker H + \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ *for all* $X \in Q'$.

The main goal of this section is to use Proposition 3.1 to establish conditions for targeted controllability of $(Q'; V_L; V_T)$ in terms of *zero forcing sets*. We will first focus on the case that $Q' = Q(G)$ in Sect. 3.4.3. Subsequently, we discuss the case that $Q' = Q_d(G)$ in Sect. 3.4.4.

3.4.3 Targeted Controllability for $Q(G)$

In this section, we discuss conditions for targeted controllability with respect to the entire qualitative class. That is, we let $Q' = Q(G)$ and investigate under which conditions $(Q(G); V_L; V_T)$ is targeted controllable. We start with a sufficient condition from [13].

Theorem 3.3 *Let* $G = (V, E)$ *be a directed graph with leader set* $V_L \subseteq V$ *and target set* $V_T \subseteq V$. *Then* $(Q(G); V_L; V_T)$ *is targeted controllable if* $V_T \subseteq D(V_L)$.

Proof Assume that $V_T \subseteq D(V_L)$, and thus $\mathscr{V}_T \subseteq \mathscr{D}(V_L)$. By Theorem 3.1, this is equivalent to

$$\mathscr{V}_T \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle. \tag{3.18}$$

Therefore, it is easy to observe that

$$\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L \rangle^{\perp} = \{0\} \tag{3.19}$$

for all $X \in Q(G)$, which results in targeted controllability of $(Q(G); V_L; V_T)$ by Proposition 3.1(d).

Theorem 3.3 provides a sufficient condition for targeted controllability. In particular, targeted controllability is guaranteed provided that the target nodes belong to the derived set of V_L.

As an example, consider the graph depicted in Fig. 3.1, and let $V_L = \{1, 2\}$. It is easy to observe that the derived set of V_L is obtained as $D(V_L) = \{1, 2, 3, 4\}$. By Theorem 3.3, we have that $(Q(G); V_L; V_T)$ is targeted controllable for any

$$V_T \subseteq \{1, 2, 3, 4\}. \tag{3.20}$$

However, this is not necessary as one can show that $(Q(G); V_L; V_T)$ is also targeted controllable with

$$V_T = \{1, 2, 3, 4, 5, 6, 7\}. \tag{3.21}$$

Next, we show that the sufficient condition provided by Theorem 3.3 can be sharpened by extending the derived set of V_L. To this end, we define the subgraph $G' = (V, E')$, where E' is defined as

$$E' := \{(i, j) \mid i \in D(V_L) \text{ and } j \in V_T\}. \tag{3.22}$$

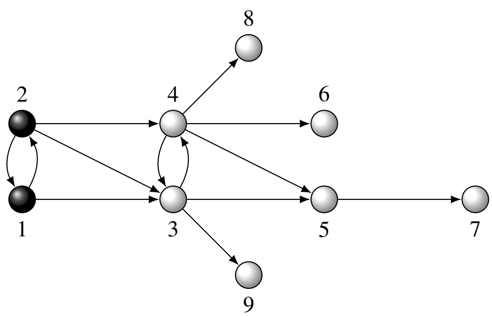

Fig. 3.1 The graph $G = (V, E)$

We define $D'(V_L)$ as the derived set of $D(V_L)$ in the subgraph G'. The following theorem extends the result of Theorem 3.3. In particular, it states that $(Q(G); V_L; V_T)$ is targeted controllable if $V_T \subseteq D'(V_L)$.

Theorem 3.4 *Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Then $(Q(G); V_L; V_T)$ is targeted controllable if $V_T \subseteq D'(V_L)$.*

Proof Assume that $V_T \subseteq D'(V_L)$. If $D'(V_L) = D(V_L)$ the claim follows immediately from Theorem 3.3. It remains to be shown that the theorem holds if $V_E := D'(V_L) \setminus D(V_L) \neq \emptyset$. In this case, we write

$$\mathscr{V}_T \subseteq \mathscr{D}'(V_L) = \mathscr{D}(V_L) \oplus \mathscr{V}_E, \tag{3.23}$$

where $\mathscr{D}'(V_L) = \operatorname{im} P(V; D'(V_L))$, $\mathscr{V}_E = \operatorname{im} P(V; V_E)$ and $\oplus$ denotes direct sum. Without loss of generality, assume that

$$D(V_L) = \{1, 2, \dots, d\}$$

and

$$V_E = \{d + 1, d + 2, \dots, d + e\}.$$

Note that $V_E \subseteq V_T$ and therefore the nodes in V can be relabeled such that

$$V_T = \{d - t, d - t + 1, \dots, d, d + 1, \dots, d + e\}$$

for some $t < d$. Consider the fourth statement in Proposition 3.1. Let $X \in Q(G)$ and ξ be a vector in the subspace $\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L\rangle^{\perp}$. Hence, $\xi \in \mathscr{V}_T \cap \langle X \mid \mathscr{D}(V_L)\rangle^{\perp}$ by Lemma 3.1. We write $\xi \in \mathbb{R}^n$ as $\xi = \operatorname{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ by partitioning the vertices into the subsets $D(V_L) \setminus V_T$, $D(V_L) \cap V_T$, V_E, and $V \setminus D'(V_L)$, respectively. Now, compatible with ξ, let the matrix X be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}. \tag{3.24}$$

By (3.23) we have $\xi \in \mathscr{D}'(V_L)$. This implies that $\xi_4 = 0$. Moreover, we have

$$\xi^T X^{k-1} P(V; D(V_L) = 0 \tag{3.25}$$

for each $k \in \mathbb{N}$. The equality $\xi^T P(V; D(V_L)) = 0$ yields $\xi_1 = \xi_2 = 0$. Then, from $\xi^T X P(V; D(V_L)) = 0$, we obtain that

$$\xi_3^T \begin{bmatrix} X_{31} & X_{32} \end{bmatrix} = 0. \tag{3.26}$$

Note that X_{21}, X_{22}, X_{31} and X_{32} correspond to the arcs from the vertices in the derived set to those in the target set V_T. Therefore, the matrix

$$X' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & X_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

belongs to the qualitative class $Q(G')$. By Lemma 3.1, we have

$$\mathscr{D}'(V_L) \subseteq \langle X' \mid \mathscr{D}(V_L) \rangle. \tag{3.27}$$

It is not difficult to see that the subspace in the right-hand side of (3.27) is computed as

$$\langle X' \mid \mathscr{D}(V_L) \rangle = \operatorname{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, (3.27) yields

$$\mathscr{D}'(V_L) = \operatorname{im} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that $\begin{bmatrix} X_{31} & X_{32} \end{bmatrix}$ is full row rank. Consequently, (3.26) results in $\xi_3 = 0$, that is, $\xi = 0$. We conclude that $(Q(G); V_L; V_T)$ is targeted controllability by the fourth statement of Proposition 3.1.

As an example, consider the graph in Fig. 3.1 with $V_L = \{1, 2\}$. Recall that the derived set of V_L is given by $D(V_L) = \{1, 2, 3, 4\}$. Suppose that V_T is given by

$$V_T = \{1, 2, 3, 4, 5, 6\}. \tag{3.28}$$

Then, Fig. 3.2 shows the subgraph $G' = (V, E')$ with E' given by (3.22). It is straightforward to show that the derived set of $D(V_L)$ in G' is equal to $D'(V_L) = \{1, 2, 3, 4, 5, 6\}$. Therefore, noting that $V_T = D'(V_L)$, we conclude that $(Q(G); V_L; V_T)$ is targeted controllable by Theorem 3.4. Observe that Theorem 3.4 extends Theorem 3.3. Indeed, the condition $V_T \subseteq D(V_L)$ has been replaced by a less conservative condition $V_T \subseteq D'(V_L)$. However, the sufficient condition provided by Theorem 3.4 is not necessary. Indeed, as previously mentioned, $(Q(G); V_L; V_T)$ is targeted controllable for V_T given in (3.21). However, note that node 7 is not contained in the set $D'(V_L)$. Consequently, the conditions of Theorem 3.4 are not satisfied in this case.

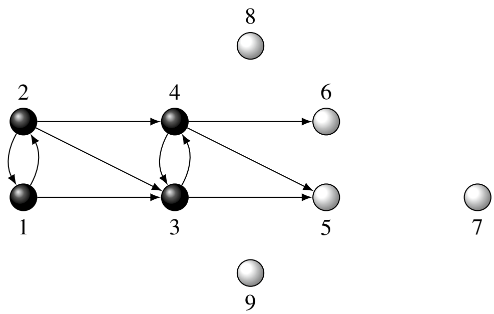

Fig. 3.2 The subgraph $G' = (V, E')$

3.4.4 Targeted Controllability for $Q_d(G)$

In the previous section, we have established sufficient conditions for targeted controllability in the case that state matrices are contained in the qualitative class $Q(G)$. We saw that it is possible to assess targeted controllability even if target nodes are not contained in $D(V_L)$ (but are incident to nodes in $D(V_L)$). However, it turns out to be difficult to assess targeted controllability if target nodes have distance larger than 1 with respect to $D(V_L)$. In this section, we restrict the state matrices to be *distance-information preserving*. We will show that for the class $Q_d(G)$ it is possible to assess targeted controllability even if the distance from $D(V_L)$ to nodes in V_T is arbitrary.

Before we start, we introduce some terminology that will become useful in the rest of this section. A directed graph $G = (V, E)$ is called *bipartite* if there exist disjoint sets of vertices V^- and V^+ such that $V = V^- \cup V^+$ and $(u, v) \in E$ only if $u \in V^-$ and $v \in V^+$. We denote such a bipartite graph by $G = (V^-, V^+, E)$, to indicate the partition of the vertex set. Suppose that the vertex sets V^- and V^+ are given by

$$\begin{aligned} V^- &= \{r_1, r_2, ..., r_s\} \\ V^+ &= \{q_1, q_2, ..., q_t\}. \end{aligned} \tag{3.29}$$

Then, the *pattern class* $\mathscr{P}(G)$ of the bipartite graph G is defined as

$$\mathscr{P}(G) = \{M \in \mathbb{R}^{t \times s} \mid M_{ij} \neq 0 \iff (r_j, q_i) \in E\}. \tag{3.30}$$

Note that the cardinalities of V^- and V^+ can differ; hence, the matrices in the pattern class $\mathscr{P}(G)$ are not necessarily square.

With these definitions in place, we continue our discussion on targeted controllability. Consider any directed graph $G = (V, E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. We assume that all target nodes have finite distance with respect to V_L. This assumption is necessary, as it can be easily shown that $(Q_d(G); V_L; V_T)$ is not targeted controllable if a target node $v \in V_T$ cannot be reached from any leader.

Let $V_S \subseteq V \setminus D(V_L)$ be a subset. We partition the set V_S according to the distance of its nodes with respect to $D(V_L)$, that is,

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{3.31}$$

where for each $i = 1, 2, ..., d$ and $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$. With each of the sets $V_1, V_2, ..., V_d$ we associate a bipartite graph $G_i = (D(V_L), V_i, E_i)$, where for $j \in D(V_L)$ and $k \in V_i$ we have $(j, k) \in E_i$ if and only if $d(j, k) = i$ in the network graph G.

Example 3.1 We consider the network graph $G = (V, E)$ as depicted in Fig. 3.3. The set of leaders is $V_L = \{1, 2\}$, which implies that $D(V_L) = \{1, 2, 3\}$, see Fig. 3.4.

In this example, we define the subset $V_S \subseteq V \setminus D(V_L)$ as $V_S := \{4, 5, 6, 7, 8\}$. Note that V_S can be partitioned according to the distance of its nodes with respect to $D(V_L)$ as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. Then, the bipartite graphs G_1, G_2 and G_3 are given in Figs. 3.5, 3.6 and 3.7, respectively.

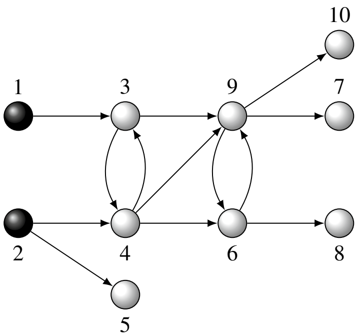

Fig. 3.3 Graph G with $V_L = \{1, 2\}$

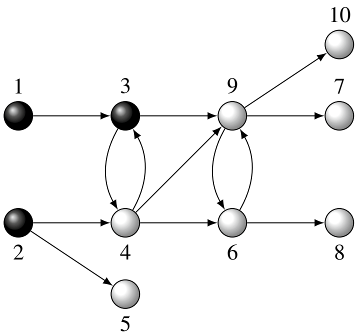

Fig. 3.4 $D(V_L) = \{1, 2, 3\}$

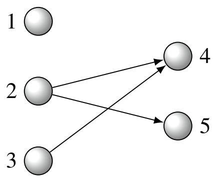

Fig. 3.5 Graph G_1

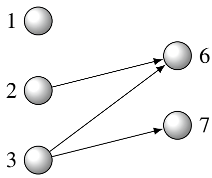

Fig. 3.6 Graph G_2

The next result provides a sufficient graph-theoretic condition for targeted controllability of $(Q_d(G); V_L; V_T)$.

Theorem 3.5 *Consider a directed graph $G = (V, E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $V_T \setminus D(V_L)$ be partitioned as in (3.31). Then $(Q_d(G); V_L; V_T)$ is targeted controllable if $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$.*

In the special case of a single leader, i.e., $|V_L| = 1$, the condition of Theorem 3.5 can be simplified. In this case, $(Q_d(G); V_L; V_T)$ is targeted controllable if no pair of target nodes has the same distance with respect to the leader. This is formulated in the following corollary.

Corollary 3.1 *Consider a directed graph $G = (V, E)$ with singleton leader set $V_L = \{v\} \subseteq V$ and target set $V_T \subseteq V$. Then $(Q_d(G); V_L; V_T)$ is targeted controllable if $d(v, i) \neq d(v, j)$ for all distinct $i, j \in V_T$.*

Note that the condition of Corollary 3.1 is similar to the "k-walk theory" for (weak) targeted controllability established in Theorem 2 of [5]. However, it is worth mentioning that k-walk theory [5] was only proven for directed tree networks with a single leader. On the other hand, Theorem 3.5 establishes a condition for strong targeted controllability that is applicable to general directed networks with multiple leaders.

It is interesting to note that the conditions of Theorem 3.5 are the same as the conditions of Theorem 3.4 in the case that all target nodes have a distance of at most one from $D(V_L)$. However, the advantage of Theorem 3.5 lies in the fact that it can be applied to target nodes that have arbitrary distance with respect to $D(V_L)$.

Example 3.2 Once again, consider the network graph depicted in Fig. 3.3, with leader set $V_L = \{1, 2\}$ and assume the target set is given by $V_T = \{1, 2, ..., 8\}$. The

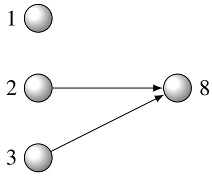

Fig. 3.7 Graph G_3

goal of this example is to prove that $(Q_d(G); V_L; V_T)$. Note that $V_S := V_T \setminus D(V_L)$ is given by $V_S = \{4, 5, 6, 7, 8\}$, which is partitioned as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. The graphs G_1, G_2 and G_3 have been computed in Example 3.1. Note that $D(V_L) = \{1, 2, 3\}$ is a zero forcing set in all three graphs (see Figs. 3.5, 3.6 and 3.7). We conclude by Theorem 3.5 that $(Q_d(G); V_L; V_T)$ is targeted controllable.

Before proving Theorem 3.5, we need two auxiliary lemmas. The following lemma states that $(X^k)_{ij} = 0$ if the distance from j to i is greater than k.

Lemma 3.4 *Consider a directed graph $G = (V, E)$ and two distinct vertices $i, j \in V$. Moreover, let k be a positive integer and $X \in Q(G)$. If $d(j, i) > k$ then $(X^k)_{ij} = 0$.*

The proof of Lemma 3.4 follows simply from induction on k, and is therefore omitted. The next lemma gives conditions under which all matrices in the pattern class $\mathscr{P}(G)$ of a bipartite graph G have full row rank.

Lemma 3.5 *Let $G = (V^-, V^+, E)$ be a bipartite graph and assume V^- is a zero forcing set in G. Then all matrices in $\mathscr{P}(G)$ have full row rank.*

For the proof of Lemma 3.5, we refer to Lemma 7 of [17]. With these results in place, we can now prove Theorem 3.5.

Proof of Theorem 3.5 Suppose that $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$. We want to prove that $(Q_d(G); V_L; V_T)$ is targeted controllable. By Proposition 3.1(c) and Lemma 3.1, $(Q_d(G); V_L; V_T)$ is targeted controllable if and only if $(Q_d(G); D(V_L); V_T)$ is targeted controllable. Therefore, our goal is to prove that $(Q_d(G); D(V_L); V_T)$ is targeted controllable. Relabel the nodes in V such that $D(V_L) = \{1, 2, ..., m\}$, and let the matrix $U = P(V; D(V_L))$ be given by $U = \begin{pmatrix} I & 0 \end{pmatrix}^T$. Furthermore, we let $V_S := V_T \setminus D(V_L)$ be given by $\{m + 1, m + 2, ..., p\}$, where the vertices are ordered in non-decreasing distance with respect to $D(V_L)$. Partition V_S according to the distance of its nodes with respect to $D(V_L)$ as

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{3.32}$$

where for $i = 1, 2, ..., d$ and $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$. We define $\check{V}_i$ and $\hat{V}_i$ to be the sets of vertices in V_S that have distance less than i (respectively, greater than i) from $D(V_L)$. More precisely,

$$\begin{aligned} \check{V}_i &:= V_1 \cup \cdots \cup V_{i-1} \text{ for } i = 2, \ldots, d \\ \hat{V}_i &:= V_{i+1} \cup \cdots \cup V_d \text{ for } i = 1, \ldots, d-1. \end{aligned} \tag{3.33}$$

By convention $\check{V}_1 := \varnothing$ and $\hat{V}_d := \varnothing$. In addition, we assume without loss of generality that the target set V_T contains all nodes in the derived set $D(V_L)$. This implies that the matrix $H = P(V; V_T)^T$ is of the form $H = \begin{pmatrix} I & 0 \end{pmatrix}$. Note that by the structure of H and U, the matrix HX^iU is simply the $p \times m$ upper left corner submatrix of X^i. We now claim that HX^iU can be written as

$$HX^iU = \begin{pmatrix} * \\ M_i \\ 0 \end{pmatrix}, \tag{3.34}$$

where $M_i \in P(G_i)$ is a $|V_i| \times m$ matrix in the pattern class of G_i, the $\left(m + |\check{V}_i|\right) \times m$ matrix $*$ contains elements of less interest and 0 denotes a zero matrix of dimension $|\hat{V}_i| \times m$.

We proceed as follows: first, we prove that the bottom submatrix of (3.34) contains zeros only. Second, we prove that $M_i \in P(G_i)$. From this, we will conclude that Eq. (3.34) holds.

Note that for $k \in D(V_L)$ and $j \in \hat{V}_i$, we have $d(k, j) > i$ and by Lemma 3.4 it follows that $(X^i)_{jk} = 0$. This means that the bottom $|\hat{V}_i| \times m$ submatrix of HX^iU is a zero matrix.

Subsequently, we want to prove that M_i, the middle block of (3.34), is an element of the pattern class $P(G_i)$. Note that the j-th row of M_i corresponds to the element $l := m + |\check{V}_i| + j \in V_i$.

Suppose $(M_i)_{jk} \neq 0$ for a $k \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., |V_i|\}$. As M_i is a submatrix of HX^iU, this implies $(HX^iU)_{lk} \neq 0$. Recall that HX^iU is the $p \times m$ upper left corner submatrix of X^i; therefore, it holds that $(X^i)_{lk} \neq 0$. Note that for the vertices $k \in D(V_L)$ and $l \in V_i$ we have $d(k, l) \geq i$ by the partition of V_S. However, as $(X^i)_{lk} \neq 0$ it follows from Lemma 3.4 that $d(k, l) = i$. Therefore, by the definition of G_i, there is an arc $(k, l) \in E_i$.

Conversely, suppose there is an arc $(k, l) \in E_i$ for $l \in V_i$ and $k \in D(V_L)$. This implies $d(k, l) = i$ in the network graph G. By the distance-information preserving property of X, we consequently have $(X^i)_{lk} \neq 0$. We conclude that $(M_i)_{jk} \neq 0$ and hence $M_i \in P(G_i)$. This implies that Eq. (3.34) holds.

The previous discussion shows that we can write

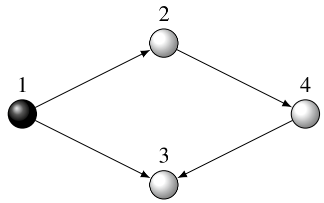

Fig. 3.8 Example showing that Theorem 3.5 not necessary

$$\begin{pmatrix} HU & HXU & HX^2U & \cdots & HX^dU \end{pmatrix} = \begin{pmatrix} I & * & * & \dots & * & * \\ 0 & M_1 & * & \dots & * & * \\ 0 & 0 & M_2 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & * & * \\ \vdots & \vdots & \vdots & \ddots & M_{d-1} & * \\ 0 & 0 & 0 & \dots & 0 & M_d \end{pmatrix}, \tag{3.35}$$

where asterisks denote matrices of less interest. As $D(V_L)$ is a zero forcing set in G_i for $i = 1, 2, ..., d$, the matrices $M_1, M_2, ..., M_d$ have full row rank by Lemma 3.5. We see that the matrix (3.35) has full row rank, and consequently $(Q_d(G); D(V_L); V_T)$ is targeted controllable by Proposition 3.1. We conclude that $(Q_d(G); V_L; V_T)$ is targeted controllable, which proves the theorem.

Note that the condition given in Theorem 3.5 is sufficient, but not necessary. Indeed, one can verify that the graph in Fig. 3.8 with leader set $V_L = \{1\}$ and target set $V_T = \{2, 3\}$ is such that $(Q_d(G); V_L; V_T)$ is targeted controllable. However, this graph does not satisfy the conditions of Theorem 3.5.

In addition to the previously established sufficient condition for targeted controllability, we also give a *necessary* condition in terms of zero forcing sets.

Theorem 3.6 *Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. If $(Q_d(G); V_L; V_T)$ is targeted controllable, then $V_L \cup (V \setminus V_T)$ is a zero forcing set in G.*

Proof Assume without loss of generality that $V_L \cap V_T = \varnothing$. Hence, $V_L \cup (V \setminus V_T) = V \setminus V_T$. We partition the vertex set V into the sets V_L, $V \setminus (V_L \cup V_T)$ and V_T. We label the vertices in V such that $V_L = \{1, 2, \dots, m\}$ and $V_T = \{n - p + 1, n - p + 2, \dots, n\}$. Accordingly, the input and output matrices $U = P(V; V_L)$ and $H = P^T(V; V_T)$ satisfy

$$U = \begin{pmatrix} I & 0 & 0 \end{pmatrix}^T, \tag{3.36}$$

and

$$H = \begin{pmatrix} 0 & 0 & I \end{pmatrix}. \tag{3.37}$$

Note that $\ker H = \operatorname{im} R$, where $R := P(V; (V \setminus V_T))$ is given by

$$R = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}^T . \tag{3.38}$$

By hypothesis, $(Q_d(G); V_L; V_T)$ is targeted controllable. By Proposition 3.1(e), we have

$$\ker H + \langle X \mid \mathrm{im} U \rangle = \mathbb{R}^n \tag{3.39}$$

for all $X \in Q_d(G)$. Equivalently,

$$\mathrm{im} R + \langle X \mid \mathrm{im} U \rangle = \mathbb{R}^n . \tag{3.40}$$

We therefore see that

$$\langle X \mid \mathrm{im} \left(U \;\; R \right) \rangle = \mathbb{R}^n . \tag{3.41}$$

As $\mathrm{im} U \subseteq \mathrm{im} R$, Eq. (3.41) implies $\langle X \mid \mathrm{im} R \rangle = \mathbb{R}^n$ for all $X \in Q_d(G)$. In other words, the pair (X, R) is controllable for all $X \in Q_d(G)$. Furthermore, by sufficient richness of $Q_d(G)$, it follows that (X, R) is controllable for all $X \in Q(G)$ (see Lemma 3.3). We conclude from Theorem 3.2 that $V \setminus V_T$ is a zero forcing set.

3.5 Conclusions

In this chapter, we have studied controllability and output controllability of systems defined over graphs. We have considered a family of state-space systems, where the state matrix of each system has a zero/non-zero structure that is determined by a given directed graph. In this context, we have investigated under which conditions all systems in the family are controllable, in other words, conditions under which the graph is strongly structurally controllable. We have shown that the strongly structurally reachable subspace can be obtained by a graph colouring rule called zero forcing. This yields neat necessary and sufficient conditions for strong structural controllability in terms of zero forcing sets. In addition, we have investigated controllability of certain subfamilies of systems via the notion of sufficient richness. For specific graph structures, we have developed leader selection strategies to find input sets of minimum cardinality that guarantee strong structural controllability. In addition, we have discussed sufficient conditions for strong structural output controllability in terms of zero forcing. We have shown that these results can be strengthened if we restrict the class of state matrices to be distance-information preserving. In the latter case, we have also shown that zero forcing sets can be used to establish a necessary condition for output controllability. Finding necessary and sufficient graph-theoretic conditions for strong structural output controllability is still an open problem that can be considered for future work.

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Part II
Control and Observation of Complex Dynamical Systems

Chapter 4
Output Regulation of Hybrid Linear Systems: Solvability Conditions and Structural Implications

Sergio Galeani and Mario Sassano

Abstract Hybrid output regulation aims at ensuring that the output of a hybrid system under control converges to a desired reference, despite the influence of unknown initial conditions and unmeasured disturbances, assuming that both references and disturbances are described as the output of an exosystem with a known model and unknown initial state. The main objective of this paper is to show how structural methods, based on tools from the geometric approach to control theory, allow a deep understanding of a plethora of surprising and unprecedented phenomena arising in output regulation problems for linear hybrid systems in the presence of time-driven periodic jumps, which do not possess any counterpart in classical output regulation.

4.1 Introduction

Output Regulation is a fundamental problem in control theory. It consists in the problem of steering an output of interest towards a desired reference despite the presence of possibly unmeasured disturbances, where both references and disturbances are assumed to be produced by a known external generator (the *exosystem*). This fundamental problem has received extensive attention in the literature since its formal definition in the linear time-invariant case [17, 19, 38], with extensions to the nonlin-

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ear context [5, 28, 29, 34], in the presence of uncertain parameters in the exosystem [36] or of switching exosystems [24].

The increasing interest in hybrid systems [25] has encouraged the study of the output regulation problem in this novel context, despite the difficulties arising from the interplay between continuous-time (*flow*) dynamics and discrete-time (*jump*) dynamics. For general hybrid systems, key obstructions are related to the fact that jumps occur depending on the state evolution, with the consequence that the plant's and exosystem's jumps are not simultaneous (unless at steady state), thus leading to sudden peaks in the regulation error and nonlinear behaviour even for systems described by linear flow and jump dynamics; such phenomena are well known in the stabilization and tracking problems in billiard systems or mechanical systems subject to impacts, such as juggling and walking robots, and the related state estimation problems [3, 18, 21, 22, 32, 33].

In order to preserve both linearity and the rich phenomena due to the interplay between flow and jump dynamics, [30] focused on a scenario where jumps are *time-driven* (periodic) and *simultaneous* for the exosystem and the plant. The same class of systems has been further investigated by the same authors in [15, 16, 31], mostly considering single-input single-output plants, and under additional restrictions of minimum phaseness (and sometimes relative degree one); the proposed results leverage on the use of a hybrid, time-varying steady-state description. It is important to stress that while the assumption of simultaneous jumps is quite restrictive, the results derived under such assumptions, especially in terms of *necessary* conditions for output regulation, immediately bear on the case of general hybrid systems, since *at steady state the jumps of the plant and of the exosystem need to be synchronized*. At the same time, preserving linearity allows to expose the peculiar features of hybrid regulation uncluttered by technicalities related to nonlinear stabilization (see the discussion in [7, 9]). Due to the above reasons, [31] can be considered as a key reference on the topic of this paper.

Considering the same class of systems with periodic jump times, but in the multi-input multi-output case (possibly with more inputs than outputs) and without any assumption of minimum phaseness or on the relative degree, [6, 7, 10] proposed a hybrid linear time-invariant solution to the output regulation problem, partially exploiting the previous results in [20, 22] (where jump times were not restricted, and only local stability results were provided). Physical examples in the mechanical and electrical domains were considered in [8, 13], which explicitly show (in terms of physically meaningful variables) the internal geometric structure of the hybrid system exploited in [6, 7]. An elegant structural analysis of a closely related problem (under slightly different conditions) is also performed in [39], where the geometric nature of the proposed solution is stressed.

The main objective of this paper is to revisit the results reported in [13] by skipping the details of the technical proofs but highlighting the role of a structural approach in achieving a deeper understanding of the complex phenomena arising from the combination of flows and jumps, and that have no correspondence in classic output regulation. These phenomena include, but are not limited to, the existence of the *heart of the hybrid regulator* (a classic, linear time-invariant regulator hidden inside any hybrid regulator), the *flow zero dynamics internal model principle* (the regulator must contain a copy of the zeros of the flow dynamics, even in the full information case), the fact that for square plants the problem is *generically unsolvable* (no regulator exists for almost all plant parameter values) and even *not well-posed* (solutions exist only in a set of measure zero in the parameter space, so they are the exception rather than the rule), so that the study of robustness in the square case is questionable (unless very specially structured parameter dependence is considered). On the other hand, the same structural tools allow to see that the presence of more inputs than outputs allows to recover generic solvability and the possibility of designing a robust regulator for reasonably large classes of uncertainties.

4.2 Notations, Preliminaries and Assumptions

This paper addresses the same output regulation problem proposed in [31] for a certain class of linear hybrid systems, but for the slightly more general class considered in [6, 7, 13]; as it will turn out, considering such an extended class is crucial to ensure well-posedness of the problem. The class of linear hybrid systems of interest is characterized by the fact that the instants when the state jumps are *time-driven* (that is, jumps occur according to a predetermined periodic sequence of jump times) instead of *state-driven* (that is, determined by the fact that the state touches or belongs to a given set, as is the case for general hybrid systems, see [25]). As a consequence, for the considered class of systems, all solutions are defined in the same *hybrid time domain*, depicted in Fig. 4.1 and described by

$$\mathscr{T} := \{(t,k) : t \in [k\tau_M, (k+1)\tau_M], k \in \mathbb{Z}\}, \tag{4.1}$$

where $\tau_M > 0$ is a given constant, t is the current value of continuous time and k denotes the number of jumps already occurred. Clearly, whenever $t = (k+1)\tau_M$, both (t,k) and $(t,k+1)$ belong to $\mathscr{T}$. The following shorthand notation will be used:

$$\dot{x}(t,k) = \frac{d}{dt}x(t,k), \qquad x^+(t,k) = x(t,k+1).$$

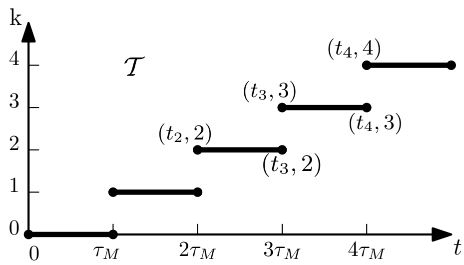


Fig. 4.1 The hybrid time domain $\mathscr{T}$

Since the hybrid time domain $\mathscr{T}$ is the same for all signals, there is no ambiguity in using a compact notation where (t, k) is omitted; actually, the structure of $\mathscr{T}$ implies that, for each $k \in \mathbb{Z}$, the *flow equation* $\dot{x} = f(x, u)$ applies almost everywhere in the intervals $[k\tau_M, (k+1)\tau_M]$, whereas the *jump equation* $x^+ = g(x, u)$ applies if and only if $t = (k+1)\tau_M$ (so that both (t, k) and $(t, k+1)$ belong to $\mathscr{T}$).

For brevity, the notation

$$x_{[k]} := x(k\tau_M, k)$$

will be used. For any function $v(\cdot, \cdot)$ defined on $\mathscr{T}$, $v \equiv 0$ means $v(t, k) = 0$, for all $(t, k) \in \mathscr{T}$. The variable $\sigma := t - k\tau_M$, $\sigma \in [0, \tau_M]$, is used to measure the time elapsed since last jump; clearly, $\dot{v}$ may be used to denote the derivative with respect to both t and σ. Let $\mathbb{C}_g := \{s \in \mathbb{C} : |s| < 1\}$ with $\partial\mathbb{C}_g = \{s \in \mathbb{C} : |s| = 1\}$. The spectrum of a square matrix M is denoted as $\Lambda(M)$.

The following equations describe the dynamics of the LTI hybrid system of interest, having solutions flowing and jumping according to the hybrid time domain $\mathscr{T}$ in (4.1):

$$\dot{x} = Ax + Bu + Pw\,, \tag{4.2a}$$

$$x^+ = Ex + Fu_J + Rw\,, \tag{4.2b}$$

$$e = Cx + Du + Qw\,, \tag{4.2c}$$

with $x(t, k) \in \mathbb{R}^n$, $u(t, k) \in \mathbb{R}^m$, $u_J(t, k) \in \mathbb{R}^{m_J}$, $e(t, k) \in \mathbb{R}^p$ and $w(t, k) \in \mathbb{R}^q$ denotes the state of the LTI hybrid *exosystem*, also flowing and jumping according to the hybrid time domain $\mathscr{T}$ in (4.1). The *exosystem* is described by the equations:

$$\dot{w} = Sw\,, \tag{4.3a}$$

$$w^+ = Jw\,. \tag{4.3b}$$

Clearly, the jumps of the plant (4.2) and of the exosystem (4.3) occur simultaneously since the hybrid time domain $\mathscr{T}$ is the same. As usual in output regulation, the role of the exosystem is to generate disturbances to be rejected and references to be tracked, so that the output e of the plant (defined in (4.2c)) can actually be thought of as a tracking error signal, such that tracking of references and disturbance rejection are simultaneously achieved provided that e is driven to zero (for this reason e is often called the *regulation error*).

As in the classic non-hybrid setting, the output regulation problem will be considered for the two cases of *full information* (i.e., when the states x and w can be measured) and *error feedback* (i.e., when only the regulation error e can be measured); while the second case appears to be (and actually is) the more relevant for practical applications, dealing with both problems allows to separate two key aspects of the overall problem, namely, the first case highlights the *structure of steady-state responses* of the plant (both in the state and in the input) associated with zero regulation error, whereas the second case highlights the need for an *internal model* of the exosystem in order to generate the steady-state input when the measurements of w are not available.

Problem 4.1 *(Full Information, Hybrid Output Regulation Problem (FIHORP))* Find, if possible, a *full information, static hybrid regulator*

$$u = Kx + Gw\,, \tag{4.4a}$$
$$u_J = K_J x + G_J w\,, \tag{4.4b}$$

which satisfies the following two requirements:

(R1) The interconnected system formed by (4.2) and (4.4) with $w \equiv 0$, namely,

$$\dot{x} = A_K x, \qquad A_K := A + BK, \tag{4.5a}$$
$$x^+ = E_K x, \qquad E_K := E + FK_J, \tag{4.5b}$$

is globally asymptotically stable;

(R2) The interconnected system formed by (4.2), (4.3) and (4.4) satisfies

$$\lim_{t+k\to\infty} e(t,k) = 0,$$

for all initial conditions $x_{[0]} \in \mathbb{R}^n$, $w_{[0]} \in \mathbb{R}^q$.

Problem 4.2 *(Error Feedback, Hybrid Output Regulation Problem (EFHORP))* Find, if possible, an *error feedback, dynamic hybrid regulator*

$$\dot{\xi} = G\xi + He\,, \tag{4.6a}$$
$$\xi^+ = G_J\xi + H_J e\,, \tag{4.6b}$$
$$u = K\xi\,, \tag{4.6c}$$
$$u_J = K_J\xi\,, \tag{4.6d}$$

with $\xi(t,k) \in \mathbb{R}^\nu$, which satisfies the following two requirements:

(R1) The interconnected system formed by (4.2) and (4.6) with $w \equiv 0$, namely,

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = A_r \begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad A_r = \begin{bmatrix} A & BK \\ HC & G + HDK \end{bmatrix}, \tag{4.7a}$$
$$\begin{bmatrix} x^+ \\ \xi^+ \end{bmatrix} = E_r \begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad E_r = \begin{bmatrix} E & FK_J \\ H_J C & G_J + H_J DK \end{bmatrix}, \tag{4.7b}$$

is globally asymptotically stable;

(R2) The interconnected system formed by (4.2), (4.3) and (4.6) satisfies

$$\lim_{t+k\to\infty} e(t,k) = 0,$$

for all initial conditions $x_{[0]} \in \mathbb{R}^n$, $\xi_{[0]} \in \mathbb{R}^\nu$, $w_{[0]} \in \mathbb{R}^q$.

Remark 4.1 (*On the need of more control inputs than regulated outputs*) A classic result in **non-hybrid output regulation** shows that, in order for the problem of output regulation to admit a solution (apart from trivial cases), **it is required that** $m \geq p$, that is, at least one distinct (scalar) control input is needed for each to be regulated (scalar) output; such a requirement is *formally* explained as an easy consequence of the *non-resonance condition*,[1] but it should also be *intuitively* clear since each independent (scalar) output might require, in order to be regulated, a suitable control action to be generated by an independent degree of freedom in the input.

One of the most interesting contributions of the structural analysis in this paper is that in **hybrid output regulation**, not only the same requirement $m \geq p$ (quite obviously) still holds, but in fact the problem can be solved with $m = p$ only if some very special relations hold among the elements of the matrices in (4.2) and (4.3), whereas such relations are not required to hold when $m > p$; hence, in practice **it is required that** $m > p$!

Remark 4.2 (*On the generality of the considered class of plants*) The class of plants considered in this paper (see [13]), namely, the one described by (4.2), is more general than the class considered in [30, 31], due to

(i) the presence of the feedthrough term D;
(ii) the presence of the input u_J on the jump dynamics;
(iii) no minimum phase requirements;
(iv) no relative degree requirements;
(v) the possibility of having a "non-square" (*i.e.,* $m > p$) plant.

Note that (i) is usually allowed in non-hybrid output regulation theory, and inputs as in (ii) are considered in the literature (see e.g., [2]); on the other hand, minimum phase and relative degree requirements are used in [30, 31] to achieve stabilization by high gain techniques, which, however, are not really required in the present linear scenario, as shown in [7, 11, 12, 14]; finally, the discussion in Remark 4.1 motivates the interest for (v), which is an enabling factor in developing a theory which is actually applicable to a large class of systems.

Remark 4.3 (*On simultaneous jumps for the plant and the exosystem*) It is apparent that the request that the plant and the exosystem jump simultaneously is quite restrictive. However, there are at least two very important reasons to deal with this case thoroughly. First, such a scenario preserves the linearity of the underlying flow and jump dynamics, while still providing a plethora of extremely interesting phenomena not present in the non-hybrid case (as will be shown later), in a context where their understanding and discussion is unhindered by excessive technicalities. Second, several results derived in this restricted scenario, and especially necessary conditions directly hold for hybrid systems with state-driven jumps along periodic trajectories (see e.g., [21, 22]), since at steady state the exosystem and the plant *must*

[1] The non-resonance condition will play a role later in this paper, where it will be introduced and discussed; hence, the reader unfamiliar with such a condition should not worry at this time, and just trust the mentioned fact for the moment being!

be synchronized, so any obstruction to achieve regulation under the assumption of simultaneous jumps directly extends to the general case.

We close this section with some final comments about the requirements in the two considered problems.

As for requirement (R1), leveraging on the periodicity of the time domain and the fact that the dynamics are linear and time-invariant, it is possible to show that *(R1)* is actually equivalent to stronger properties and that it is easy to check if *(R1)* is achieved by resorting to its associated *monodromy system* (a tool reminiscent of Floquet theory and abundantly used in the literature about analysis and control of periodic systems, see [4] and references therein). In particular, the *monodromy system* associated with (4.2) is a discrete-time *linear time-invariant* system that, at the discrete-time instant k, has

- as state $\tilde{x}(k)$, the state of (4.2) "sampled" at hybrid time $(k\tau_M, k)$ (that is, $x_{[k]}$);
- as input $\tilde{u}(k)$, the lifted input u and u_J of (4.2) during the hybrid time interval $[k\tau_M, (k+1)\tau_M] \times \{k\})$ (that is, the time history of u on $[k\tau_M, (k+1)\tau_M] \times \{k\})$ plus the value of u_J at $((k+1)\tau_M, k))$;
- as output $\tilde{y}(k)$, the output e of (4.2) during the hybrid time interval $[k\tau_M, (k+1)\tau_M] \times \{k\})$.

Formally, the monodromy system can be written as

$$\tilde{x}^+ = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} + \tilde{P}\tilde{w}\,, \tag{4.8a}$$

$$\tilde{e} = \tilde{C}\tilde{x} + \tilde{D}\tilde{u} + \tilde{Q}\tilde{w}\,, \tag{4.8b}$$

where $\tilde{A} = Ee^{A\tau_M}$ is a matrix whereas all other $\tilde{B}$, $\tilde{P}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{Q}$ are suitable (linear time-invariant) operators. Matrix $\tilde{A}$ is usually called the *monodromy matrix*. Since the monodromy system is an equivalent representation of the original system, the following facts are easily shown to hold.

Fact 1 *All motions of (4.2) have the same stability and attractivity properties (hence, there is no confusion in talking about stability of the system).*

Fact 2 *(4.2) is asymptotically stable if and only if* $\Lambda(\tilde{A}) = \Lambda(Ee^{A\tau_M}) \subset \mathbb{C}_g$.

The following fact is the key tool for guaranteeing that a well-defined steady-state solution exists, whenever requirement *(R1)* is satisfied.

Fact 3 *If (4.2) is asymptotically stable then it is also globally exponentially incrementally stable (that is, the difference between any pair of solutions obtained by applying the same inputs converges to zero exponentially).*

Fact 4 *The boundedness and convergence properties of* x *in (4.2) are equivalent to boundedness and convergence of* $\tilde{x}$ *in (4.8).*

As a consequence of the above facts, stability for a system in the class of (4.2) can be easily checked by computing the eigenvalues of its monodromy matrix; and a number of strategies devised for stabilizing a periodic system can be brought to bear on stabilization of (4.2), see [7, 12] for more details. A deeper consequence is the fact that the non-trivial part in solving 4.1 and 4.2 consists exactly in achieving *(R2)*.

As for requirement (R2), note that since in requirement *(R1)* of both Problem 4.1 and Problem 4.2 it is imposed that the closed-loop system between the plant and the regulator (with the exosystem disconnected) is asymptotically stable, it follows that regulation is achieved for free whenever the exosystem only generates signals decaying to zero (because the output of the asymptotically stable system with converging exogenous input will be convergent as well). Hence, since all asymptotically stable dynamics in the exosystem are automatically taken care of as a consequence of requirement *(R1)*, it is not restrictive to consider only exosystems satisfying the following assumption, stating that no eigenvalue of the exosystem's monodromy matrix $\tilde{S} := Je^{S\tau_M}$ has modulus less than 1; such assumption will be tacitly adopted in the rest of the paper.

Assumption 1 $\Lambda(\tilde{S}) \cap \mathbb{C}_g = \emptyset$.

4.3 Solvability Conditions, Without a Structural Approach

Having set the stage for the discussion of hybrid output regulation in the considered class of systems, it is possible now to develop a solution in a form that is not based on any structural property; such solution was essentially given in [30, 31] and later generalized in [6, 13].

4.3.1 Full Information Case

In the case of full information feedback, Proposition 4.1 provides necessary and sufficient conditions for the solution of Problem 4.1. As in the non-hybrid output regulation context, a key role is played by the *hybrid regulator equations:*

$$\dot{\Pi}(\sigma) + \Pi(\sigma)S = A\Pi(\sigma) + B\Gamma(\sigma) + P\,, \tag{4.9a}$$

$$\Pi(0)J = E\Pi(\tau_M) + F\Gamma_J + R\,, \tag{4.9b}$$

$$0 = C\Pi(\sigma) + D\Gamma(\sigma) + Q\,, \tag{4.9c}$$

$$\Gamma(\sigma) = K\Pi(\sigma) + G\,, \tag{4.9d}$$

$$\Gamma_J = K_J\Pi(\tau_M) + G_J\,, \tag{4.9e}$$

in the unknowns $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$ and $\Gamma_J \in \mathbb{R}^{m_J\times q}$, where Π and Γ describe the steady-state evolution of the state and the input of (4.2) (provided that requirement *(R1)* has been achieved) according to the relations

$$x_{ss}(t, k) = \Pi(\sigma)w(t, k), \quad u_{ss}(t, k) = \Gamma(\sigma)w(t, k).$$

Proposition 4.1 *Consider the linear hybrid system (4.2) driven by the exosystem (4.3). The static hybrid regulator (4.4) solves Problem 4.1 if and only if*

(C1) $\Lambda(E_K e^{A_K\tau_M}) \subset \mathbb{C}_g$;
(C2) *there exist* $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$ *and* $\Gamma_J \in \mathbb{R}^{m_J\times q}$ *solving (4.9).*

While the previous result is existential, the following result provides an explicit formula for the computation of the solutions Π, Γ and Γ_J of (4.9).

Proposition 4.2 *Suppose that* $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$ *and* $\Gamma_J \in \mathbb{R}^{m_J\times q}$ *solve the hybrid regulator equations (4.9). Then, necessarily*

$$\Pi(\sigma) = e^{A\sigma}\left(\bar{\Pi} + \bar{\Gamma}(\sigma)\right)e^{-S\sigma}, \tag{4.10a}$$

$$\bar{\Gamma}(\sigma) = \int_0^\sigma e^{-A\tau}(P + B\Gamma(\tau))e^{S\tau}d\tau, \tag{4.10b}$$

where $\bar{\Pi} \in \mathbb{R}^{n\times q}$ *and* $\bar{\Gamma}(\sigma)$*, for all* $\sigma \in [0, \tau_M]$*, satisfy*

$$\bar{\Pi}\tilde{S} = \tilde{E}\bar{\Pi} + \tilde{E}\bar{\Gamma}(\tau_M) + F\tilde{\Gamma}_J + \tilde{R}, \tag{4.11a}$$

$$0 = Ce^{A\sigma}(\bar{\Pi} + \bar{\Gamma}(\sigma)) + (D\Gamma(\sigma) + Q)e^{S\sigma}, \tag{4.11b}$$

with $\tilde{E} = Ee^{A\tau_M}$, $\tilde{R} = Re^{S\tau_M}$, $\tilde{S} = Je^{S\tau_M}$ *and* $\tilde{\Gamma}_J = \Gamma_J e^{S\tau_M}$.

Remark 4.4 As in the non-hybrid output regulation problem, the solution is defined by means of a specific Sylvester equation (4.11a), enforcing invariance of a desired steady-state manifold, together with a linear matrix equation (4.11b), imposing zero error on the manifold. In fact, once $\bar{\Pi}$ and $\bar{\Gamma}$ are computed by solving (4.11a)–(4.11b), the mapping $\Pi(\sigma)$ is easily computed by (4.10). However, solving (4.11) is a non-trivial task, due to the infinite number of constraints present in (4.11b).

Remark 4.5 When $D = 0$, the satisfiability of (4.11) does not depend on the *instantaneous* value of the matrix-valued function $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$, since in such a case Γ only appears inside integrals, and what matters is just its cumulative behaviour; but even when $D \neq 0$ there can be degrees of freedom left in the choice of Γ, as can be seen by the structural analysis in Sect. 4.5. Clearly, different choices of Γ yield different Π's, so that the steady-state evolution of the state can actually be shaped through the choice of Γ. The freedom in choosing Γ can be exploited in order to simplify the control law, as discussed in Sect. 4.7, where semi-classical solutions are introduced.

Remark 4.6 The presence of the term $e^{-S\sigma}$ in (4.10a) has a particularly interesting interpretation, related to an intrinsic *sample-and-hold structure of the solution of the hybrid regulator equations* (4.9). In fact, noting that $w_{[k]} = e^{-S\sigma} w(t,k)$ and rearranging the terms in (4.10), it is easy to see that the steady-state $x_{ss}(t,k) = \Pi(\sigma)w(t,k)$ can be rewritten as

$$\begin{aligned} x_{ss}(t,k) &= e^{A\sigma}\bar{\Pi}w_{[k]} + e^{A\sigma}\int_0^{\sigma} e^{-A\tau}(P + B\Gamma(\tau))e^{S\tau}d\tau w_{[k]} \\ &= e^{A\sigma}\bar{\Pi}w_{[k]} + \int_{k\tau_M}^{t} e^{A(t-\tau)}(P + B\Gamma(\tau - k\tau_M))w(\tau,k)d\tau. \end{aligned}$$

The term $e^{A\sigma}\bar{\Pi}w_{[k]}$ is in fact a *free response* of (4.2a) to the *sampled* value $w_{[k]}$; the other term represents the forced response of (4.2a) to a forcing input depending on the current value of w, through P and Γ. Due to (4.9d), a similar structure is present in the steady-state input.

From the point of view of regulator design, neither conditions (4.9) nor conditions (4.10), (4.11) are easy to solve in order to find the required regulator parameters. As extensively discussed in [6, 7, 9], in order to address the issue of generating input-state pairs enforcing zero output e, it is useful to perform a structural analysis of the plant that makes it possible to reveal a special structure in the flow dynamics, as will be discussed in Sect. 4.4.1.

4.3.2 Error Feedback Case

Considering the more common case when only error measurements are available, necessary and sufficient conditions for the solution of Problem 4.2 are given in Proposition 4.3. Again, as in the non-hybrid output regulation context, a key role is played by the *hybrid regulator equations:*

$$\begin{aligned} \dot{\Pi}(\sigma) + \Pi(\sigma)S &= A\Pi(\sigma) + B\Gamma(\sigma) + P\,, &\quad (4.12a)\\ \Pi(0)J &= E\Pi(\tau_M) + F\Gamma_J + R\,, &\quad (4.12b)\\ 0 &= C\Pi(\sigma) + D\Gamma(\sigma) + Q\,, &\quad (4.12c)\\ \dot{\Sigma}(\sigma) + \Sigma(\sigma)S &= G\Sigma(\sigma)\,, &\quad (4.12d)\\ \Sigma(0)J &= G_J\Sigma(\tau_M)\,, &\quad (4.12e)\\ \Gamma(\sigma) &= K\Sigma(\sigma)\,, &\quad (4.12f)\\ \Gamma_J &= K_J\Sigma(\tau_M)\,. &\quad (4.12g) \end{aligned}$$

Proposition 4.3 *Consider the linear hybrid system (4.2) driven by the exosystem (4.3). The dynamic hybrid regulator (4.6) solves the error feedback output regulation problem if and only if*

(C1) $\Lambda(E_r e^{A_r \tau_M}) \subset \mathbb{C}_g$;

(C2) *there exist* $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$, $\Gamma_J : \mathbb{R}^{m_J\times q}$ *and* $\Sigma : [0, \tau_M] \to \mathbb{R}^{\nu\times q}$ *solving (4.12).*

Remark 4.7 Note that (4.12a), (4.12b) and (4.12c) coincide with (4.9a), (4.9b) and (4.9c), which means that solvability of the full information problem is indeed a necessary condition for solvability of the error feedback problem. Moreover, (4.12d), (4.12e) and (4.12f) are an expression of the classic *internal model principle* in the present hybrid context; such principle essentially tells us that regulator (4.6) is able to internally generate the input $\Gamma(\sigma)w(t,k)$ when $e \equiv 0$, that is, for each $w_{[0]}$ there is an initial state $\xi_{[0]}$ such that the ensuing free motion of (4.6) generates exactly the input $\Gamma(\sigma)w(t,k)$ required for regulation for all $(t,k) \in \mathscr{T}$. It will be shown in Sect. 4.5.3 that, somewhat surprisingly, in the hybrid case (unlike in the non-hybrid case) there is also a sort of internal model principle *related to the plant flow dynamics* embedded in (4.12a), (4.12b) and (4.12c) (equivalently, (4.9a), (4.9b) and (4.9c)).

As was done before (see Proposition 4.2), it is again possible to provide explicit expressions for the solutions of (4.12), as in Proposition 4.4.

Proposition 4.4 *Suppose that* $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Sigma : [0, \tau_M] \to \mathbb{R}^{\nu\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$ *and* $\Gamma_J \in \mathbb{R}^{m_J\times q}$ *solve the hybrid regulator equations (4.12). Then, necessarily* Π *and* Σ *are of the form*

$$\Pi(\sigma) = e^{A\sigma}\left(\bar{\Pi} + \bar{\Gamma}(\sigma)\right)e^{-S\sigma}\,, \tag{4.13a}$$

$$\bar{\Gamma}(\sigma) = \int_0^\sigma e^{-A\tau}(P + B\Gamma(\tau))e^{S\tau}d\tau\,, \tag{4.13b}$$

$$\Sigma(\sigma) = e^{G\sigma}\bar{\Sigma}e^{-S\sigma}\,, \tag{4.13c}$$

where $\bar{\Pi}$, $\bar{\Sigma}$ *and* $\bar{\Gamma}(\sigma)$*, for all* $\sigma \in [0, \tau_M]$*, satisfy*

$$\bar{\Pi}\tilde{S} = \tilde{E}\bar{\Pi} + \tilde{E}\bar{\Gamma}(\tau_M) + F\tilde{\Gamma}_J + \tilde{R}\,, \tag{4.14a}$$

$$0 = Ce^{A\sigma}(\bar{\Pi} + \bar{\Gamma}(\sigma)) + (D\Gamma(\sigma) + Q)e^{S\sigma}\,, \tag{4.14b}$$

$$\bar{\Sigma}\tilde{S} = \tilde{G}_J\bar{\Sigma}\,, \tag{4.14c}$$

with $\tilde{E} = Ee^{A\tau_M}$, $\tilde{\Gamma}_J = \Gamma_J e^{S\tau_M}$ *(as in Proposition 4.2), and* $\tilde{G}_J = G_J e^{G\tau_M}$.

Remark 4.8 The comments in Remark 4.6 hold here too, possibly in a stronger fashion. Again exploiting the presence of the term $e^{-S\sigma}$ in (4.13a) and the fact that $w_{[k]} = e^{-S\sigma}w(t,k)$, the steady-state $x_{ss}(t,k) = \Pi(\sigma)w(t,k)$ and $\xi_{ss}(t,k) = \Sigma(\sigma)w(t,k)$ can be rewritten as

$$\begin{aligned}\xi_{ss}(t,k) &= e^{G(t-k\tau_M)}\bar{\Sigma}w_{[k]},\\ x_{ss}(t,k) &= e^{A(t-k\tau_M)}\bar{\Pi}w_{[k]} + \int_{k\tau_M}^t e^{A(t-\tau)}BKe^{G\tau}\bar{\Sigma}w_{[k]}d\tau + \int_{k\tau_M}^t e^{A(t-\tau)}Pw(\tau,k),\end{aligned}$$

which shows that, apart from the last term which represents the forced response of (4.2a) to the current value of w entering through P, the steady-state response in the state is actually the sum of a *free response* and a *forced response* of (4.2a) to a (suitably processed) *sampled* value $w_{[k]}$ of w. Due to (4.12f), a similar comment holds for the steady-state input as well.

Looking back at the provided solutions to Problems 4.1 and 4.2, there are several reasons not to be completely satisfied with the answers found thus far. Among the natural questions to ask at this point, the following ones are probably the most relevant:

- what is the source of the time dependence (via σ) of Π and Γ?
- How is such time dependence related to the plant or exosystem data?
- During an interval of flow (between two consecutive jumps), why is the solution different from the solution of the corresponding classic (flow-only) output regulation problem?

As already mentioned, addressing these questions will require the introduction of suitable structural tools.

4.4 Some Structural Tools

Understanding the structure and inner workings of the solutions of the hybrid regulator equations, as well as the nature of the steady-state solutions in an hybrid output regulation problem and the regulator inducing them, requires a fine decomposition of the hybrid dynamics based on isolating certain controlled invariant subspaces of the flow dynamics and considering how the jump dynamics interacts with them. While a full appreciation of the steps and results involved in such a plan can only be based on a sufficient knowledge of the geometric approach [1], for brevity in the following only the description based on coordinate transformations will be given.

4.4.1 A Decomposition of the Flow Dynamics

A deep understanding of the structure of the solution set of the hybrid regulator equations (4.9) can be achieved by leveraging on the following proposition (proven in [23]), which will be used as a powerful *analysis* tool in the sequel.

Proposition 4.5 *Modulo a coordinate change in the input, state and output spaces, matrices A, B, C and D in (4.2) are in the form*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} + \begin{bmatrix} B_{12} \\ B_{22} \\ B_{32} \end{bmatrix} \begin{bmatrix} \bar{A}_{31} & \bar{A}_{32} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}, \tag{4.15a}$$

$$C = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & D_2 \end{bmatrix}, \tag{4.15b}$$

where $B_{11} \in \mathbb{R}^{n_1 \times (m-p)}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $A_{33} \in \mathbb{R}^{n_3 \times n_3}$, *the pair* (A_{11}, B_{11}) *is reachable, and the subsystem* $(A_{33}, B_{32}, C_3, D_2)$ *is square, reachable, observable, without finite invariant zeros and invertible.*

From (4.15), the preliminary state feedback

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\bar{A}_{31} & -\bar{A}_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \tag{4.16}$$

where $\hat{u}_1$ and $\hat{u}_2$ are new input signals, yields the even simpler structure:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}, \tag{4.17a}$$

$$C = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & D_2 \end{bmatrix}. \tag{4.17b}$$

Based on Proposition 4.5, there is clearly no loss of generality in assuming that (4.2) has already been given in the form (4.15). Similarly, it is possible to compute the steady-state evolution of x and $\hat{u}$ using the form (4.17), and then to obtain the steady-state evolution of u for the original system from (4.16).

Remark 4.9 Invertibility of $(A_{33}, B_{32}, C_3, D_2)$ implies that for each possible output of such subsystem, there is exactly one compatible evolution for the input and the state. It is immediate to see from (4.17) that the subspace $\{x : x_3 = 0\}$ corresponds, when u_1 is absent, to the *flow zero dynamics* of (4.2) (namely, the zero dynamics of the flow equations of (4.2)), and due to reachability of the pair (A_{11}, B_{11}), the subspace $\{x : x_3 = 0, x_2 = 0\}$ corresponds to the "reachable subset" of the flow zero dynamics. More formally (see [37, Chap. 7]), the above subspaces correspond, respectively, to the *weakly unobservable subspace* and to the *controllable weakly unobservable subspace* of the flow dynamics; in the case when $D = 0$, those subspaces reduce to the *largest controlled invariant subspace contained in* $\ker(C)$, usually denoted as $\mathscr{V}^*$, and to the *largest controllability subspace contained in* $\ker(C)$, usually denoted as $\mathscr{R}^*$. Using the block partition highlighted in (4.17) allows to explicitly show the actual degrees of freedom that can be exploited in order to generate the steady-state input signal (as shown in [6]), and in fact particularizing the hybrid regulator equations (4.12) to the case in which system (4.2) is in the form of (4.17) yields simple design equations.

For completeness, it is mentioned that if (4.16) is modified as

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} K_1 & 0 & 0 \\ -\bar{A}_{31} & -\bar{A}_{32} & K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \tag{4.18}$$

with K_1, K_3 such that the matrices $\bar{A}_{11} + \bar{B}_{11}K_3$, $\bar{A}_{22}$, $\bar{A}_{33} + \bar{B}_{32}K_3$ have pairwise disjoint spectra (this is always possible because, by Proposition 4.5, the pairs (A_{11}, B_{11}) and (A_{33}, B_{32}) are reachable) then, possibly under an additional change of coordinates, matrix A can be made block diagonal (preserving the block structure of the remaining matrices); this is possible since disjoint spectra imply complementability of the relevant subspaces [1].

4.4.2 On the Solution of an Integral Equation

The decomposition in (4.17) shows that, using the input $\hat{u}_1$, it is possible to drive the states corresponding to the pair (A_{11}, B_{11}) without affecting the plant output e. Hence, these states turn into a kind of *extra input* for the jump dynamics, which is present even when u_J is absent; clearly such inputs become an extra degree of freedom that helps in solving the second hybrid regulator equation (the one associated with jumps, and involving $\Pi(0)$ and $\Pi(\tau_M)$). The analysis in this subsection is essentially devoted to explaining how to determine the input $\hat{u}_1$ acting during a flow in order to achieve a certain desired effect at the following jump. Such choice is usually based on the use of time-varying gains, which would make the resulting regulator time-varying as well (whereas only time-invariant regulators are considered in Problems 4.1 and 4.2); for this reason, the subsection is closed by showing how the same result can be achieved by replacing the time-varying gain with a dynamic LTI filter.

In order to show how the hybrid Francis equations (4.9) can be actually solved for a plant in the form (4.17), two instrumental results are introduced. In both results, $\mathscr{H}$ is the set of matrix functions $H : [0, \tau_M] \mapsto \mathbb{R}^{(m-p)\times n_1}$ with piecewise continuous elements having at most k discontinuities, for some fixed $k \in \mathbb{N}$, equipped with the uniform norm, and $\mathscr{H}_g$ is the subset of $\mathscr{H}$ such that rank $(\Xi) = n_1$, with

$$\Xi := \int_0^{\tau_M} e^{A_{11}(\tau_M - \tau)} B_{11} H(\tau) d\tau. \tag{4.19}$$

The next lemma shows that there is an abundance of choices of functions H such that Ξ in (4.19) has full row rank, so that a good H can always be selected.

Lemma 4.1 *Let (A_{11}, B_{11}) be reachable. Then $\mathscr{H}_g$ is open and everywhere dense in $\mathscr{H}$.*

The next proposition shows how to compute a solution of an equation of the form (4.19) in the unknown function H.

Proposition 4.6 *Let $H \in \mathscr{H}_g$ and Ξ as in (4.19). Then*

$$\hat{\Gamma}_{1\nu}(\tau) = H(\tau)\Xi^{-1}\tilde{\Gamma}_1 e^{-S\tau} \tag{4.20}$$

satisfies $\tilde{\Gamma}_1 = \int_0^{\tau_M} e^{A_{11}(\tau_M-\tau)} B_{11}\hat{\Gamma}_{1\nu}(\tau)e^{S\tau}d\tau$.

Specific choices of H adopted in earlier papers [6, 7, 9, 12] include *gramian-based* solutions of the form

$$H(\tau) := B_{11}' e^{A_{11}'(\tau_M-\tau)}, \tag{4.21}$$

and *piecewise constant* solutions of the form

$$H(\tau) = H_h, \quad \tau \in [(h-1)\tau_\nu, h\tau_\nu),\ h = 1, \ldots, \nu,$$

where $\tau_\nu := \frac{\tau_M}{\nu}$, and the integer $\nu > 0$ and matrices $H_h \in \mathbb{R}^{(m-p)\times n_1}$, $h = 1, \ldots, \nu$ are chosen such that rank $(R_\nu) =$ rank $(R_\nu H^\circ) = n_1$, with

$$\begin{aligned}
A_\nu &:= e^{A\tau_\nu}, \quad B_\nu = \int_0^{\tau_\nu} e^{A_{11}(\tau_\nu-\tau)}d\tau B_{11},\\
R_\nu &:= \left[B_\nu\ A_\nu B_\nu\ \cdots\ A_\nu^{\nu-1}B_\nu\right],\\
H^\circ &:= \left[H_\nu\ H_{\nu-1}\ \cdots\ H_1\right].
\end{aligned}$$

More generally, generalizing the *gramian-based* solution, $H(\tau)$ in (4.19), (4.20), can be chosen as $H(\tau) = C_\Gamma e^{A_\Gamma\tau}$ for an (almost arbitrary) pair (C_Γ, A_Γ) with A_Γ square and having dimension at least n_1; this is a key ingredient if in the end an LTI regulator is desired for implementation (see the discussion around (4.35) in Sect. 4.5.3).

4.5 Solvability Conditions, Untangled: A Structural Interpretation

Using the decomposition of the flow dynamics and the properties reported in Sect. 4.4, it is possible to achieve a deeper understanding of the solution given in Sect. 4.3 and to give answers to the questions that were posed at the end of that section, about the source of the time dependence (via σ) in Π and Γ, and about how such time dependence is related to the plant or exosystem data.

It was stressed in Remark 4.7 that (4.12a), (4.12b) and (4.12c) coincide with (4.9a), (4.9b) and (4.9c), and provide the (necessary and sufficient) solvability condition of the full information problem and a necessary solvability condition of the error feedback problem. This section is devoted to an in-depth analysis of such conditions, which will result in several far-reaching consequences, by showing that

(P1) *at the heart of any hybrid regulator* (which, as it has been shown, necessarily satisfies the hybrid regulator equations (4.9)) *lies a classic regulator*, that is, a solution of the regulation problem defined for the flow-only data (4.3a), (4.2a), (4.2c) (without jumps);

(P2) *in order to achieve regulation, the steady-state input must contain both a copy of the exosystem's natural modes* (as in the classic, non-hybrid case) *and a suitable copy of the natural modes of the flow zero dynamics of the system.*

Property (P2) (whose precise formulation for the general case will be given later) extends the classic *internal model principle* by adding the need for a suitable duplication of the zero dynamics of the system, and then it is reasonably named the *flow zero dynamics internal model principle*. Property (P1) is also very interesting, in our opinion, since it shows an (expected, but yet to be proven) fact that the solution of the hybrid case should arise as an extension of the underlying flow problem (which is indeed solved in between each pair of jumps). Clearly, a preliminary step in order to show the above properties in Sect. 4.5.3 consists in reformulating the hybrid regulator equations in the coordinates of Sect. 4.4.1, which were chosen to highlight the relevant underlying structural properties of the flow dynamics.

4.5.1 Structural Formulation and Solution of the hybrid regulator equations

Although, as noted in Remark 4.4, the hybrid regulator equations (4.9) are infinite-dimensional and hence difficult to solve directly, it is possible to reduce their solution to the solution of two uncoupled, finite-dimensional Francis equations.

As is apparent from the structural decomposition in (4.15), the square subsystem with state x_3 is the only relevant part to ensure regulation during flows, and in fact the first reduced equation (known as ***reduced flow*** *Francis equation*) essentially picks a solution for the flow regulation problem for such subsystem (and is a controlled invariance condition for the steady-state subspace over which zero regulation error is achieved during flows). On the other hand, the subsystem with state $[x_1' \ x_2']'$ is irrelevant during flows (as far as its effect on the state x_3 is cancelled by u_2, again thanks to the structure of the last block row of matrices A and B in (4.15)) but becomes crucial at jumps (after one period of flow), since then it determines if the state after the jump lands again on the controlled invariant subspace associated with zero regulation error which is characterized by the reduced flow Francis equation; as a consequence, the second reduced equation (known as ***monodromy*** *Francis equation*) is written in terms of the dynamics for the $[x_1' \ x_2']'$ subsystem over one period (hence, the *monodromy* name, where one period means τ_M time units of flow followed by a single jump) and has as output the difference between the value of x_3 after the jump, and the value that x_3 should have during flows according to the solution of the reduced flow Francis equation.

The *reduced flow Francis equation* has the form

$$\Pi S = A_{33}\Pi + B_{32}\Gamma + P_3\,, \tag{4.22a}$$
$$0 = C_3\Pi + D_2\Gamma + Q\,. \tag{4.22b}$$

Note that since $(A_{33}, B_{32}, C_3, D_2)$ is square, invertible and has no invariant zeros (see Proposition 4.5), the reduced flow Francis equation (4.22) *always* admits a *unique solution* (Π_{3c}, Γ_{2c}), and then its solvability is never an issue; its relevance consists in the fact that, as will be shown next, its solution (Π_{3c}, Γ_{2c}) is the core of any hybrid regulator (the so-called *heart of the regulator*). It is remarked that, perhaps somewhat surprisingly, *the reduced flow Francis equation will be solvable even if the standard Francis equation is not solvable for the flow dynamics of the plant and the exosystem.*

The *monodromy Francis equation* has the form:

$$\tilde{\Pi}\tilde{S} = \tilde{A}\tilde{\Pi} + \tilde{B}\tilde{\Gamma} + \tilde{P}, \tag{4.23a}$$
$$0 = \tilde{C}\tilde{\Pi} + \tilde{D}\tilde{\Gamma} + \tilde{Q}, \tag{4.23b}$$

where $\tilde{\Pi} = \begin{bmatrix}\tilde{\Pi}_1' & \tilde{\Pi}_2'\end{bmatrix}'$, $\tilde{\Gamma} = \begin{bmatrix}\tilde{\Gamma}_1' & \tilde{\Gamma}_2'\end{bmatrix}'$, $\tilde{\Pi}_1 \in \mathbb{R}^{n_1\times q}$, $\tilde{\Pi}_2 \in \mathbb{R}^{n_2\times q}$, $\tilde{\Gamma}_1 \in \mathbb{R}^{n_1\times q}$, $\tilde{\Gamma}_2 \in \mathbb{R}^{m_J\times q}$ and

$$A_0 = \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}, \qquad \tilde{S} = Je^{S\tau_M} \tag{4.24a}$$
$$\tilde{A} = \begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix} e^{A_0\tau_M}, \qquad \tilde{B} = \begin{bmatrix}E_{11} & F_1\\ E_{21} & F_2\end{bmatrix}, \tag{4.24b}$$
$$\tilde{C} = \begin{bmatrix}E_{31} & E_{32}\end{bmatrix} e^{A_0\tau_M}, \qquad \tilde{D} = \begin{bmatrix}E_{31} & F_3\end{bmatrix}, \tag{4.24c}$$
$$\tilde{P} = \begin{bmatrix}R_1 + E_{13}\Pi_{3c}\\ R_2 + E_{23}\Pi_{3c}\end{bmatrix} e^{S\tau_M} + \begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}\int_0^{\tau_M} e^{A_0(\tau_M-\tau)}\hat{P}e^{S\tau}d\tau, \tag{4.24d}$$
$$\tilde{Q} = (R_3 + E_{33}\Pi_{3c} - \Pi_{3c}J)e^{S\tau_M} + \begin{bmatrix}E_{31} & E_{32}\end{bmatrix}\int_0^{\tau_M} e^{A_0(\tau_M-\tau)}\hat{P}e^{S\tau}d\tau, \tag{4.24e}$$
$$\hat{P} = \begin{bmatrix}\hat{P}_1\\ \hat{P}_2\end{bmatrix} = \begin{bmatrix}P_1\\ P_2\end{bmatrix} + \begin{bmatrix}A_{13}\\ A_{23}\end{bmatrix}\Pi_{3c} + \begin{bmatrix}B_{12}\\ B_{22}\end{bmatrix}\Gamma_{2c}. \tag{4.24f}$$

While the *reduced flow Francis equation* poses no limits to the solvability of the hybrid output regulation problem, the *monodromy Francis equation* is the really crucial object that determines if a given instance of the problem is solvable or not. This fact is formally proven as the first item in the next proposition, which also shows the explicit relation between the solutions of the *hybrid Francis equations* (4.9) (with no structure highlighted) and the solutions of the *reduced* and *monodromy Francis equations*.

Proposition 4.7 *Let (4.2) be as in (4.17).*

(i) *The* hybrid Francis equations *(4.9a), (4.9b) and (4.9c) are solvable if and only if the* monodromy Francis equation *(4.23) is solvable.*

(ii) *If $(\Pi(\sigma), \hat{\Gamma}(\sigma), \Gamma_J)$ solves (4.9) for a plant in the form (4.17), then*

$$\Pi(\sigma) = \begin{bmatrix} \Pi_{1v}(\sigma) \\ \Pi_{2v}(\sigma) \\ \Pi_{3c} \end{bmatrix}, \qquad \hat{\Gamma}(\sigma) = \begin{bmatrix} \hat{\Gamma}_{1v}(\sigma) \\ \Gamma_{2c} \end{bmatrix}, \tag{4.25}$$

with (Π_{3c}, Γ_{2c}) solving (4.22), and a solution $(\tilde{\Pi}, \tilde{\Gamma})$ of (4.23) is given by

$$\tilde{\Pi}_1 = \Pi_{1v}(0), \qquad \tilde{\Pi}_2 = \Pi_{2v}(0), \tag{4.26a}$$

$$\tilde{\Gamma}_1 = \int_0^{\tau_M} e^{A_{11}(\tau_M - \tau)} B_{11} \hat{\Gamma}_{1v}(\tau) e^{S\tau} d\tau, \tag{4.26b}$$

$$\tilde{\Gamma}_2 = \Gamma_J e^{S\tau_M}. \tag{4.26c}$$

(iii.a) *If $(\tilde{\Pi}, \tilde{\Gamma})$ solves (4.23) then a solution of (4.9a), (4.9b) and (4.9c) is given by (4.25) with (Π_{3c}, Γ_{2c}) solving (4.22), $\hat{\Gamma}_{1v} : [0, \tau_M] \to \mathbb{R}^{(m-p)\times q}$ any function satisfying (4.26b), $\Pi_{1v}(\sigma)$, $\Pi_{2v}(\sigma)$ such that*

$$\begin{bmatrix} \Pi_{1v}(\sigma) \\ \Pi_{2v}(\sigma) \end{bmatrix} = e^{A_0\sigma} \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} e^{-S\sigma} + \int_0^{\sigma} e^{A_0(\sigma-\tau)} \begin{bmatrix} B_{11}\hat{\Gamma}_{1v}(\tau) + \hat{P}_1 \\ \hat{P}_2 \end{bmatrix} e^{S(\tau-\sigma)} d\tau, \tag{4.27a}$$

$$\Gamma_J = \tilde{\Gamma}_2 e^{-S\tau_M}. \tag{4.27b}$$

(iii.b) *Moreover, if (4.9d) is also satisfied, then a LTI static hybrid regulator as in (4.4) is obtained.*

The previous proposition essentially answers the first two questions asked at the end of Sect. 4.3, about the origin, and the relation with the problem data, of the time dependence in the solution of (4.9). By additional analysis in the subsequent pages, further insight into the above questions will also be gained. Note that if a time-varying G is allowed in (4.9d), then (4.9d) can be always be satisfied via the time-varying choice $G(\sigma) := \Gamma(\sigma) - K\Pi(\sigma)$, and then it does not really restrict the possibility of solving the output regulation problem; however, item (iii.b) in Proposition 4.7 stresses that if $\Gamma(\sigma) - K\Pi(\sigma)$ is constant then it is possible to build an LTI regulator (with constant G).

Corollary 4.1 *The triple $(\Pi(\sigma), \Gamma(\sigma), \Gamma_J)$ solves (4.9) for system (4.2) as in (4.15) if and only if*

$$\Pi(\sigma) = \begin{bmatrix} \Pi'_{1v}(\sigma) \ \Pi'_{2v}(\sigma) \ \Pi'_{3c} \end{bmatrix}',$$

and the triple $(\Pi(\sigma), \hat{\Gamma}(\sigma), \Gamma_J)$, *where*

$$\hat{\Gamma}(\sigma) = \Gamma(\sigma) + \begin{bmatrix} 0 & 0 \\ \bar{A}_{31} & \bar{A}_{32} \end{bmatrix} \begin{bmatrix} \Pi_{1v}(\sigma) \\ \Pi_{2v}(\sigma) \end{bmatrix}, \tag{4.28}$$

solves (4.9) for (4.2) as in (4.17).

The *construction of a solution* for (4.9) based upon Proposition 4.6 and item (iii) of Proposition 4.7 proceeds as follows: first, the fixed part of any solution of the considered hybrid output regulation problem which ensures tracking during flow intervals (and which we already named the *heart of the regulator*) is computed by (4.22); then, (4.23) is solved (whenever a solution of the regulation problem at hand exists, by item (i) of Proposition 4.7); finally, (4.26) is applied with $\hat{\Gamma}_{1v}$ given by (4.20) for an arbitrary choice of $H \in \mathscr{H}_g$. Putting together the several pieces thus computed provides the required regulator.

Having addressed the problem of computing a solution (whenever it exists), and having pointed out how solvability is essentially equivalent to the possibility of solving (4.23), it is of interest to further study the solvability of this last equation. A well-known sufficient condition for solvability of (4.23) is given by the *non-resonance* condition

$$\operatorname{rank} \begin{bmatrix} \tilde{A} - sI & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = n, \quad \forall s \in \Lambda(\tilde{S}), \tag{4.29}$$

which clearly requires that $n_1 + m_J \geq n_3$ (which, in turn, implies that $m > p$ when $m_J = 0$, because otherwise also $n_1 = 0$). In the case $m > p$, condition (4.29) can be further refined by exploiting the relation between the first block columns of $\begin{bmatrix} \tilde{A}' & \tilde{C}' \end{bmatrix}'$ and $\begin{bmatrix} \tilde{B}' & \tilde{D}' \end{bmatrix}'$.

Proposition 4.8 *Equation (4.23) is solvable if*

$$\operatorname{rank} \begin{bmatrix} E_{22} e^{A_{22}\tau_M} - sI & E_{21} & F_2 \\ E_{32} e^{A_{22}\tau_M} & E_{31} & F_3 \end{bmatrix} = n_2 + n_3,$$

$\forall s \in \Lambda(\tilde{S})$.

As a corollary, in the case $n_2 = 0$, the following sufficient condition holds.

Corollary 4.2 *Equation (4.23) is solvable if* $n_2 = 0$ *and* $\operatorname{rank} \begin{bmatrix} E_{31} & F_3 \end{bmatrix} = n_3$.

It is worth mentioning that the case $n_2 = 0$ is particularly relevant when $m > p$ since in this case it is *generic*[2]; in fact, the decomposition of the subspace $\{x : x_3 = 0\}$ in (4.15) into the components x_1 and x_2 is essentially a *reachability decomposition*

[2]The reader unfamiliar with the concept of *generic property* will find the relevant basic definitions in Sect. 4.6.

with respect to u_1 of the residual dynamics on $\{x : x_3 = 0\}$, and it is well known that a generic pair (A, B) is reachable. On the other hand, clearly if $m = p$ there is no input u_1 (since its dimension should be zero) and then the subspace $\{x : x_3 = 0\}$ is completely unreachable (namely, there is no x_1). It follows that, at least generically, the only cases of interest are those with $n_1 = 0$ (when $m = p$) and $n_2 = 0$ (when $m > p$); clearly, if additional structure on the problem data is assumed, the mentioned generic properties are affected by the considered structure and the above discussion for $m > p$ does not apply.

Remark 4.10 The advantage of the formulation in Proposition 4.7, and in particular of item (iii), when compared to Propositions 4.1 and 4.2, consists in showing that solutions of the hybrid regulation problem can be computed by simply solving two finite-dimensional, linear Francis equations (the *reduced flow* and the *monodromy* Francis equations), and then applying the formulas provided in Propositions 4.7 and 4.6. The finite-dimensional nature of the provided conditions also makes it straightforward to check if a specific instance of the problem is solvable. Even more interestingly, such finite-dimensional conditions provide a basis for discussing the universal solvability and well-posedness of the hybrid Francis equations, a topic addressed in Sect. 4.6.

4.5.2 The Heart of the Hybrid Regulator

The analysis in Proposition 4.7 and Corollary 4.1 shows that a unique pair (Π_{3c}, Γ_{2c}) solving the reduced flow Francis equation (4.22) is always present, albeit possibly hidden, in any solution of the hybrid Francis equations (4.9a)–(4.9d). The pair (Π_{3c}, Γ_{2c}) solves a classic, purely continuous-time output regulation problem associated with the system $(A_{33}, B_{32}, C_3, D_2)$ with exogenous input w entering through matrices P_3 and Q. Due to the ubiquitous presence of the pair (Π_{3c}, Γ_{2c}), it seems appropriate to call such a pair (or the classic, flow-only full information regulator associated with such a pair) *the heart of the hybrid regulator*.

The inevitable presence of (Π_{3c}, Γ_{2c}) can be associated with the fact that, during each of the intervals when $t \in (k\tau_M, (k+1)\tau_M)$ for $(t, k) \in \mathcal{T}$, the hybrid output regulation problem is actually a classic, purely continuous-time output regulation problem. A similar result holds in the error feedback case, in which case the heart of the regulator becomes

$$\Pi S = A_{33}\Pi + B_{32}\Gamma + P_3\,, \tag{4.30a}$$

$$0 = C_3\Pi + D_2\Gamma + Q\,, \tag{4.30b}$$

$$G_0\Sigma_0 = \Sigma_0 S\,, \tag{4.30c}$$

$$\Gamma = K_{20}\Sigma_0\,, \tag{4.30d}$$

where (4.30c) and (4.30d) express a reduced form of the internal model principle for the purely continuous-time output regulation problem associated with the system $(A_{33}, B_{32}, C_3, D_2)$ with exogenous input w entering through matrices P_3 and Q (the exact expression of K_{20} can be derived but is not crucial here, the main point being that it relates the steady-state state response, parameterized via Σ_0, with the steady-state input, parameterized by Γ). The above discussion can then be summarized by saying that

> at the heart of any hybrid output regulator lies a classic regulator for the invertible square subsystem of (4.2) *relating u_2 to e*.

The above discussion also answers the last question asked at the end of Sect. 4.3, namely, why during flows the solution of the hybrid Francis equations has apparently nothing in common with the solution of a classic (flow-only) regulator. Our structural analysis has shown that:

- the overall control input u can be decomposed into two components, u_1 and u_2;
- u_1 is irrelevant to regulation during flows, and then the related part in the solution of the hybrid Francis equations (that is, the part of $\Gamma(\sigma)$ relative to u_1) can be an arbitrary function of time;
- u_2 is *uniquely determined by the regulation requirement* (since the x_3 subsystem is square and invertible), and it is given by two terms: a first term $\Gamma_{2c}w$ that is *exactly the solution of the associated flow-only output regulation problem*, plus a second term that is in charge of cancelling the influence of the states x_1 and x_2 on the dynamics of x_3.
- as a consequence, if the matrices $\bar{A}_{31}$ and $\bar{A}_{32}$ in (4.15) are zero, u_2 reduces to the output of a classic regulator, and the only possibly time-varying part in a matrix $\Gamma(\sigma)$ solving the hybrid Francis equations is related to u_1 and used to solve the monodromy Francis equation (4.23).

4.5.3 The Flow Zero Dynamics Internal Model Principle

As pointed out in Remark 4.7, Eqs. (4.12d), (4.12e) and (4.12f) extend the classic internal model principle to the present hybrid context; such equations express the fact that the regulator's dynamics must contain a suitable "copy" of the exosystem's dynamics, so that at steady-state the regulator produces as a free response a steady-state input to the plant including the exosystem's natural modes. However, differently from what happens in the classic setting, in the hybrid case also, Eqs. (4.12a), (4.12b) and (4.12c) (equivalently, (4.9a), (4.9b), (4.9c)) entail an internal model principle, although essentially related to the plant; in particular, it is shown in this section that

> in order to be able to generate $u_{ss}(t, k) = \Gamma(\sigma)w(t, k)$*, the hybrid regulator must contain a suitable* ***copy of the flow zero dynamics*** *of the plant.*

Note that $u_{ss}(t, k)$ above is the steady-state control input to be implemented in the full information (and nominal parameter values) case, so that the required "internal model" is to be implemented even in the full information case; additionally, in the error feedback case, since w is not available, the classic internal model of the exosystem dynamics must also be provided.

In order to better understand the mentioned phenomenon, it is useful to consider the *feedforward output regulation problem*, which is the special case of the full information/error feedback problem in which the plant (4.2) is already asymptotically stable, so that no feedback is needed in order to achieve stabilization (requirement (R1) in Problems 4.1 and 4.2), and w is measured (or estimated), so that a purely feedforward solution is viable. Note that, by the discussion in Remark 4.7, the conclusions derived for the *full information* case remain valid also for the more general case of *error feedback*. Moreover, the (explanatory) scenario of asymptotically stable plants may be considered without loss of generality, since, should this simplifying assumption not be satisfied, the stabilizing component of the control law (4.4a) will not affect the conclusions of this section, which are concerned with the steady-state behaviour of the control input. The characterization of the solution to the *feedforward output regulation problem* is as follows.

Proposition 4.9 *Consider the linear hybrid system (4.2) driven by the exosystem (4.3), with (4.2) asymptotically stable. The feedforward output regulation problem is solved by a static hybrid feedforward regulator*

$$u(t,k) = G(\sigma)w(t,k)\,, \quad \sigma = t - k\tau_M\,, \tag{4.31a}$$

$$u_J = G_J w\,, \tag{4.31b}$$

if and only if there exist $\Pi : [0, \tau_M] \to \mathbb{R}^{n\times q}$, $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times q}$ *and* $\Gamma_J \in \mathbb{R}^{m_J\times q}$ *solutions of (4.9a), (4.9b), (4.9c) and* $G(\sigma) = \Gamma(\sigma)$, $G_J = \Gamma_J$.

Proposition 4.9 shows that the steady-state input required for achieving regulation has a "time-varying nature", which is somewhat hidden in (4.4a), (4.9d). The real meaning of this nature is investigated now.

Time-varying components in $\Gamma(\sigma)$ are due to (i) $\Gamma_{1v}(\sigma)$ and (ii) $\hat{\Gamma}_2(\sigma) - \Gamma_2(\sigma) = \bar{A}_{31}\Pi_{1v}(\sigma) + \bar{A}_{32}\Pi_{2v}(\sigma)$ (see Corollary 4.1, and recall that $\hat{\Gamma}_2(\sigma) = \Gamma_{2c}$ is constant). As already mentioned in the previous section, $\Gamma_{1v}(\sigma)$ can be chosen in a rather arbitrary fashion (as long as the monodromy Francis equation is satisfied), and then is not so interesting in terms of deriving structural information about the steady-state input. On the other hand, an insightful analysis can be performed on the term $\hat{\Gamma}_2(\sigma) - \Gamma_2(\sigma) = \bar{A}_{31}\Pi_{1v}(\sigma) + \bar{A}_{32}\Pi_{2v}(\sigma)$. Using (4.27), it is possible to write

$$\begin{bmatrix} \Pi_{1v}(\sigma) \\ \Pi_{2v}(\sigma) \end{bmatrix} w(t,k) = e^{A_0\sigma} \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} w_{[k]} + \int_0^{\sigma} e^{A_0(\sigma-\tau)} \begin{bmatrix} B_{11}\hat{\Gamma}_{1v}(\tau) + \hat{P}_1 \\ \hat{P}_2 \end{bmatrix} e^{S\tau} d\tau w_{[k]}, \tag{4.32}$$

where the fact that $e^{-S\sigma}w(t,k) = e^{-S\sigma}w(\sigma + k\tau_M, k) = w_{[k]}$ has been used.

Focusing on the case $m = p$ (where the input u_1 and the state x_1 disappear) yields the special but insightful case:

$$\dot{\zeta} = A_{22}\zeta + \hat{P}_2 w, \qquad \zeta_{[0]} = \tilde{\Pi}_2 w_{[0]} \tag{4.33a}$$

$$\zeta^+ = \tilde{\Pi}_2 J w, \tag{4.33b}$$

$$u_{ss} = \bar{A}_{32}\zeta + \Gamma_{2c} w, \tag{4.33c}$$

$$u_{J,ss} = \Gamma_J w. \tag{4.33d}$$

It is easy to see that, during flows, the steady-state input u_{ss} contains not only the modes of w (corresponding to the eigenvalues of S) but also the modes of the free response of the subsystem ζ, corresponding to the eigenvalues of A_{22}, that is, to the zeros of the flow dynamics of (4.2). It is worth stressing that, even if the flow zero dynamics only contains stable modes (that is, even if the flow dynamics is *minimum phase*) such modes will not disappear, because at the beginning of each period of flow there is a new initialization of such modes due to the jump dynamics of ζ^+, which is forced by the persistent signal $w_{[k]}$. *So, provided that the pair* $(\bar{A}_{22}, \bar{A}_{32})$ *is observable (a generically true property), in order to produce the input* u_{ss} *it is necessary that the regulator contains a model of the complete flow zero dynamics.*

More generally, it follows from (4.32) that the steady-state input u_{ss}, $u_{J,ss}$ is generated as output of the following system:

$$\dot{\zeta} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \zeta + \begin{bmatrix} B_{11}\hat{\Gamma}_{1v}(\sigma) + \hat{P}_1 \\ \hat{P}_2 \end{bmatrix} w, \tag{4.34a}$$

$$\zeta^+ = \begin{bmatrix} \tilde{\Pi}_1 J \\ \tilde{\Pi}_2 J \end{bmatrix} w, \tag{4.34b}$$

$$u_{ss} = \begin{bmatrix} 0 & 0 \\ -\bar{A}_{31} & -\bar{A}_{32} \end{bmatrix} \zeta + \begin{bmatrix} \hat{\Gamma}_{1v}(\sigma) \\ \Gamma_{2c} \end{bmatrix} w, \tag{4.34c}$$

$$u_{J,ss} = \Gamma_J w, \tag{4.34d}$$

$$\zeta_{[0]} = \left[\tilde{\Pi}_1' \ \tilde{\Pi}_2'\right]' w_{[0]}. \tag{4.34e}$$

Note that, as already mentioned, the time dependence in $u_{ss,1} = \Gamma_{1v}(\sigma) w$ is not essential, since if the problem is solvable at all, it is also solvable by a $\Gamma_{1v}(\sigma)$ chosen as in (4.20) with $H(\sigma) = C_\Gamma e^{A_\Gamma \sigma}$, so that the linear time-varying system (4.34) can be replaced by the LTI system:

$$\dot{\zeta} = \begin{bmatrix} A_{11} & A_{12} & B_{11}C_\Gamma \\ 0 & A_{22} & 0 \\ 0 & 0 & A_\Gamma \end{bmatrix} \zeta + \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ 0 \end{bmatrix} w, \tag{4.35a}$$

$$\zeta^+ = \begin{bmatrix} \tilde{\Pi}_1 J \\ \tilde{\Pi}_2 J \\ \Xi^{-1}\tilde{\Gamma}_1 J \end{bmatrix} w, \tag{4.35b}$$

$$u_{ss} = \begin{bmatrix} 0 & 0 & C_\Gamma \\ -\bar{A}_{31} & -\bar{A}_{32} & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ \Gamma_{2c} \end{bmatrix} w, \tag{4.35c}$$

$$u_{J,ss} = \Gamma_J w, \tag{4.35d}$$

$$\zeta_{[0]} = \left[\tilde{\Pi}_1' \; \tilde{\Pi}_2' \; (\Xi^{-1}\tilde{\Gamma}_1)'\right]' w_{[0]}. \tag{4.35e}$$

As for the term $u_{ss,2}(t,k) = \Gamma_2(\sigma)w(t,k)$, a more in-depth analysis can be performed which extends the considerations already made for the case when $m = p$, yielding the following statement which clarifies exactly the need for a suitable internal model of the flow zero dynamics inside the regulator (even in the full information case), and the exact meaning of "suitable".

Theorem 4.1 (Flow zero dynamics internal model principle) *The steady-state input $u_{ss,2}(t,k) = \Gamma_2(\sigma)w(t,k)$ includes all the observable and reachable natural modes of the continuous-time system:*

$$\dot{\chi} = A_\chi \chi + B_\chi u_\chi, \quad y_\chi = C_\chi \chi + D_\chi u_\chi, \tag{4.36a}$$

$$\left[\begin{array}{c|c} A_\chi & B_\chi \\ \hline C_\chi & D_\chi \end{array}\right] = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & (\tilde{\Pi}_1 + \hat{P}_1)J \\ 0 & A_{22} & 0 & (\tilde{\Pi}_2 + \hat{P}_2)J \\ \hline -\bar{A}_{31} & -\bar{A}_{32} & 0 & 0 \end{array}\right], \tag{4.36b}$$

whose dynamics are associated with the zero dynamics of the flow equations of system (4.2) (see Remark 4.9).

The flow zero dynamics internal model principles have far-reaching consequences, for example, in terms of the possibility of achieving *robust output regulation*, that is, output regulation in the presence of plant parameter variations. Since the regulator needs to be able to generate the modes of the flow zero dynamics, in order for a fixed regulator to be able to cope with parameter variations, one should assume that either such variations do not modify the zeros of the plant (this choice is made in [31], where it is assumed that the required time-varying components in $\Gamma(\sigma)$ can be expressed as linear combinations of known functions; but the structural analysis has revealed that such functions are exactly the modes of the flow zero dynamics, which then must be fixed, which implies that only a rather benign kind of parameter variations are allowed) or that such modes are naturally decoupled by the x_3 dynamics during flow (this choice is made in [14], where physically motivated examples are provided; see also [8]. In this case, there is no problem if the flow zero dynamics changes, as long as it remains decoupled during flows). At the present time, a general solution to the hybrid robust output regulation problem is not available.

4.6 On Well-Posedness, Universal and Generic Solvability

In this section, some remarkable (and classic, see [17, 35, 37]) properties about the solvability of the output regulation problem are first reviewed for the LTI non-hybrid case, and then considered in the hybrid setting.

All such properties are essentially derived by leveraging on two facts, namely, that if a matrix is full rank, such a property is preserved when its entries are changed by any arbitrary but sufficiently small quantity, and that a set of $k \leq n$ randomly chosen vectors in $\mathbb{R}^n$ are linearly independent with probability 1.

Another notion of interest in the following analysis is the fact that a property defined for objects in a certain set $\mathscr{S}$ (such that a specific element of the set may or may not enjoy the property) is said to be *generic* if it holds for all elements in the set $\mathscr{S}$, possibly with the exception of a set of measure zero; for example, if the set $\mathscr{S}$ is the closed unit ball in $\mathbb{R}^3$, the property of having all non-zero coordinates is generic, whereas the property of having at least one positive coordinate is not generic (nor its negation is generic). An often encountered, equivalent way of expressing the same concept is in terms of randomly chosen elements from $\mathscr{S}$: a property is generic on $\mathscr{S}$ if, choosing a random element of $\mathscr{S}$ according to a uniform distribution, the considered property is true with probability 1 (hence, unsolvable instances of the problem are a "rarity").

4.6.1 The Classic (Non-hybrid) Case

Consider the classic Francis equation

$$\Pi S = A\Pi + B\Gamma + P\,, \tag{4.37a}$$

$$0 = C\Pi + D\Gamma + Q\,, \tag{4.37b}$$

for an output regulation problem in the non-hybrid setting, where (A, B, C, D, S, P, Q) are the data, with $m \geq p$. Since in practice the data (A, B, C, D, S, P, Q) are never exactly known (provided the minimal polynomial of S is exactly known), it is relevant to know if the conclusions drawn using the nominal parameter values still hold for (slightly) different actual parameter values; hence, assuming that a solution of (4.37) exists for the nominal values of the data, (4.37) is said to be ***universally solvable*** if a solution exists for any (P, Q), and ***well-posed*** if a solution exists also for arbitrary, sufficiently small perturbations of the nominal parameters (including P and Q). Moreover, it is also interesting to know if solvable output regulation problems are a rule or an exception; hence, (4.37) is said to be ***generically solvable*** if solutions fail to exist only on a set of measure zero in the space of problem data (namely, if the elements of the matrices in the data are randomly chosen with uniform distribution over a full-dimensional compact set, the problem instance is solvable with probability 1).

It turns out that the key condition for all these issues is given by the *non-resonance condition*

$$\operatorname{rank}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + p, \quad \forall s \in \Lambda(S), \tag{4.38}$$

which, clearly, has to be checked just at the nominal values of (A, B, C, D). In addition to being a sufficient condition for solvability of the classic Francis equation, becomes a **necessary and sufficient condition for both well-posedness and universal solvability**. It is worth noting that satisfaction of the *non-resonance condition* implies $m \geq p$.

In turn, the simple *size constraint* $m \geq p$ guarantees that (4.38) generically holds, and then that (4.37) is *generically solvable*.

Note that *generic solvability* implies, among other properties, that for a plant with $m \geq p$ the output regulation problem is *solvable for almost any choice of the exosystem's dynamics*.

The above conditions are actually *necessary unless structural assumptions are introduced about the admissible parameter variations*, like dependence on a small number of *physical parameters*, see [26, 27].

4.6.2 The Hybrid Case

Using the same tools and reasoning exploited in the classic case for proving the properties discussed in the previous subsection and exploiting the fact that the solvability conditions in this paper (see item (i) in Proposition 4.7) formally correspond to the Francis equation (4.23), it is possible to derive the corresponding properties for the hybrid case.

Proposition 4.10 *If the plant (4.2) and the exosystem (4.3) satisfy the non-resonance condition (4.29), namely*

$$\operatorname{rank}\begin{bmatrix}\tilde{A}-sI & \tilde{B}\\ \tilde{C} & \tilde{D}\end{bmatrix}=n,\quad \forall s\in\Lambda(\tilde{S}),$$

then the hybrid output regulation Problems 4.1 and 4.2 are well-posed *and* universally solvable.

It turns out that if the polynomial matrix appearing in the rank condition (4.29) has full row rank for some $s_0 \in \mathbb{C}$, then it fails to have the same rank only on a finite set of values in $\mathbb{C}$; hence, the following corollary holds.

Corollary 4.3 *Under the non-degeneracy condition*

$$\exists s_0\in\mathbb{C}:\quad \operatorname{rank}\begin{bmatrix}\tilde{A}-s_0I & \tilde{B}\\ \tilde{C} & \tilde{D}\end{bmatrix}=n, \tag{4.39}$$

the hybrid output regulation Problems 4.1 and 4.2 for plant (4.2) are solvable for almost all exosystems (4.3).

A condition for generic solvability based on size constraints only can also be given. Since the condition $m \geq p$ is already imposed by the heart of the regulator, namely, the need of solving (4.22), if a generic $D \neq 0$ is allowed, it will follow that generically rank $(D) = p$ and then regulation is trivially achieved by the steady-state input $u_{ss} = -D_0 Q w$ (for any D_0 such that $D D_0 = I$). Hence, only the non-trivial (and more practically relevant) case in which $D = 0$ structurally holds is considered.

Proposition 4.11 *Let $D = 0$ structurally and $m \geq p$. The hybrid output regulation Problems 4.1 and 4.2 are generically solvable if and only if either $m > p$, $n + m_J \geq 2p$ or $m = p$, $m_J \geq p$.*

The following simple (but perhaps somewhat striking) corollary is obtained by focusing on the case without jump inputs ($m_J = 0$). The first corollary implies that the hybrid output regulation problem is essentially not well-posed in the "square plant" case (with $D = 0$), so that robust regulation can be obtained in such a case only with respect to *(very) structured parameter variations* (and in fact, even the nominal parameters of the problem must happen to belong to a "lucky" set of measure zero in the space of parameters). Intuitively, this is due to the fact that in the considered case $n_1 = 0$ (structurally) and then $\tilde{B}$ and $\tilde{D}$ are absent in (4.23), which becomes an overdetermined system of equations. One of the consequences is that in the framework of [31], the admissible parametric uncertainties have to ensure both that the zeros of the flow dynamics are not affected, and that the mentioned overdetermined system of equations remains solvable (that is, such parametric uncertainties must be very structured, and, for example, small random errors on system parameters due to measurement noise during identification are not allowed).

Corollary 4.4 *If $D = 0$ structurally, $m_J = 0$ and $m = p$, the hybrid output regulation problem for (4.2) is generically not solvable.*

On the positive side, even for $m_J = 0$ and $D = 0$, generic solvability is ensured by Proposition 4.11 already by having just a single additional flow input and a state dimension at least twice as large as the output dimension. Such generic solvability properties seem to suggest that the natural framework to address robust hybrid output regulation problems should be the one considering $m \geq p$ as in [13] (with the relevant case being, in fact, when $m > p$).

4.7 Semi-classical Solution to Hybrid Output Regulation

The solution of the *nominal* output regulation problem (namely, the case when the plant parameters are exactly known) is completely characterized by the Francis equations. Obviously, the most interesting problem in applications is the *robust* output regulation problem (namely, the case when the plant parameters are only approximately known, and a regulator has to be designed ensuring regulation despite possible plant parameter deviations). A number of obstructions to achieve actual robust hybrid output regulation have been provided in the previous sections, highlighting how currently available solutions (essentially, the two approaches in [14, 31]) are valid only under more or less restrictive hypotheses. Without going into the details of the robust regulator design in [14], this section provides a description of the conceptual tool that enables such design.

4.7.1 The Classic (Non-hybrid) Case

When parameters are perturbed, it is no longer possible to use the steady-state input $u_{ss} = \Gamma w$ directly, since the correct Γ is unknown (as it critically depends on the *actual* unknown plant parameters via the Francis equations). In the classic (non-hybrid) theory, considering the case $p = 1$ for brevity, such problem is circumvented by noting that choosing any observable pair (S^*, Γ^*) such that S^* and S have the same minimal polynomial, and letting $\Psi^*(t) = \Gamma^* e^{S^* t}$, for any Γ and $w(0)$ it is possible to find a vector $\zeta(0)$ such that

$$u_{ss}(t) = \Gamma e^{St} w(0) = \Psi^*(t)\zeta(0), \tag{4.40}$$

thus replacing the need to know the *plant parameter-dependent coefficient matrix* Γ with the need to have a *suitable initial state for the plant parameter-independent system* $\dot{\zeta} = S^*\zeta$, $y_\zeta = \Gamma^*\zeta$, which provides an internal model for the exosystem. Once this key step is performed, the mere *existence* of the "good" initial state for the internal model implies that the output regulation problem is solved by interconnecting the plant, the internal model and a suitable error feedback stabilizer; in fact, even if the initial state of the internal model is not the "good" one, global asymptotic (in fact, incremental) stability of the LTI closed-loop system implies that all responses will converge to the ideal response corresponding to the "good" initial state $\zeta(0)$. Note that, in order for the above idea to work, the only crucial requirement is that any steady-state input $u_{ss}(\cdot)$ can be expressed (see (4.40)) as the output-free response of a *known* linear system not dependent on the plant parameters; and this is possible since the steady-state solution x_{ss}, u_{ss} only contains the exosystem's natural modes (and not the plant's ones).

In the attempt to mimic the above reasoning in the considered hybrid setting, it is immediately clear that the flow zero dynamics internal model principle is a structural obstruction towards expressing the steady-state input $u_{ss}(\cdot)$ as the output-free response of a known linear system not depending on the plant parameters, since the natural modes of the flow zero dynamics appear in $\Pi(\cdot)$ and $\Gamma(\cdot)$. It is then interesting to provide conditions under which it is possible to have the desired form for u_{ss}; still, this leaves the possibility of having x_{ss} in the more complex form including the natural modes of the flow zero dynamics. Such steady-state solutions are called *semi-classical* in [10], since they exhibit a steady-state input u_{ss} similar to the one in classic output regulation, and a steady-state state response x_{ss} in the general form for hybrid output regulation.

4.7.2 The Hybrid Case: Periodic Semi-classical Solutions

Periodic semi-classical solutions are obtained if u_{ss} is only required to be expressed as the free output response of a possibly LTV system (not designed on the exact knowledge of the plant parameters, as is the case for the piecewise constant approach right after (4.21)). From (4.25), (4.28) and the flow zero dynamics internal model principle, *periodic semi-classical solutions* are characterized by the fact that, partitioning $\varGamma(\sigma) = \left[\varGamma_1(\sigma)' \ \varGamma_2(\sigma)'\right]'$, the only time-varying term is $\varGamma_1(\sigma) = \varGamma_{1v}(\sigma)$, whereas $\varGamma_2(\sigma) = \varGamma_{2c}$ is constant and $\bar{A}_{31}\varPi_{1v}(\sigma) + \bar{A}_{32}\varPi_{2v}(\sigma) = 0$. Note that this still leaves the (periodically) time-varying term $\varGamma_{1v}(\sigma)$, but it only contains known functions, which can be chosen quite arbitrarily in light of Lemma 4.1. By the flow zero dynamics internal model principle, the following proposition follows immediately.

Proposition 4.12 *The hybrid output regulation problem for (4.2), (4.3) is solvable by periodic semi-classical solutions if and only if $W_\chi(s) = 0$.*

A sufficient condition guaranteeing $W_\chi(s) = 0$ is given by $\left[\bar{A}_{31} \ \bar{A}_{32}\right] = 0$, imposing that (4.2) has the form (4.17). While at first this might appear too strong to be satisfied in practice, several systems satisfy such a condition; a simple physical example is the spinning and bouncing disk considered in [10]. More generally, such a structure appears in those systems having several components that interact only at jumps (i.e., during flows they evolve independently from each other) with only a subset of them affecting the regulated output. While satisfaction of $W_\chi(s) = 0$ in general would depend on the exosystem (via $\tilde{\varPi}_1$, $\tilde{\varPi}_2$ and J in (4.36), obviously it holds for any exosystem in case of plants in the form (4.17).

4.7.3 The Hybrid Case: Constant Semi-classical Solutions

Constant semi-classical solutions, characterized by a constant $\varGamma(\sigma) = \left[\varGamma_1' \ \varGamma_{2c}'\right]'$, can be obtained by strengthening the condition in Proposition 4.12. Considering such solutions, and denoting by θ the (yet unknown) value of $\varGamma_1$, the key step is to parametrize the variable $\tilde{\varGamma}$ in the monodromy Francis equation (4.23) according to

$$\tilde{\varGamma}_1(\theta) = \sum_{i=1}^{m-p} \sum_{j=1}^{q} \tilde{\varGamma}_{1,i,j}\theta_{i,j},$$

where $\tilde{\varGamma}_{1,i,j}$ is obtained applying (4.26b) for $\hat{\varGamma}_{1v}(\tau)$ equal to the constant matrix with all elements equal to zero, except the element in position (i, j) equal to one. Although still linear, the equation obtained substituting $\tilde{\varGamma}_1(\theta)$ in (4.23) is not in the form of a Francis equation with respect to the unknowns $(\tilde{\varPi}, \theta, \varGamma_J)$ where $\varGamma_J = \tilde{\varGamma}_2 e^{-S\tau_M}$ (as for the notation in Sect. 4.5.2); hence, in order to give solvability conditions, it must be expressed in vectorized form. Let $\mathrm{Vec}(\cdot)$ denote the vectorization operator (which applied to a matrix M with columns $M_1, \ldots, M_r$ results in $\mathrm{Vec}(M) = \begin{bmatrix} M_1' & \cdots & M_r' \end{bmatrix}'$), let $\otimes$ denote the Kronecker product of two matrices (such that $A \otimes B$ is a block matrix whose (i, j)-th block is $a_{i,j}B$, with $a_{i,j}$ being the (i, j)-th element of A), and recall that the linear equation $AXB = C$ can be rewritten as $(B' \otimes A)\mathrm{Vec}(X) = \mathrm{Vec}(C)$ [28, Appendix A]. For $i = 1, \ldots, m - p$ and $j = 1, \ldots, q$, define the h-th column of matrices V and W according to

$$\begin{aligned} V_h &= \mathrm{Vec}(\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} \tilde{\Gamma}_{1,i,j}), \quad W_h = \mathrm{Vec}(E_{31}\tilde{\Gamma}_{1,i,j}), \\ h &= (j-1)(m-p) + i. \end{aligned}$$

Let also

$$\begin{aligned} \Psi &:= \begin{bmatrix} (\tilde{S}' \otimes I - I \otimes \tilde{A}) & V & e^{S'\tau_M} \otimes \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ -I \otimes \tilde{C} & W & e^{S'\tau_M} \otimes F_3 \end{bmatrix}, \\ \Phi &:= \begin{bmatrix} \mathrm{Vec}(\tilde{P}) \\ \mathrm{Vec}(\tilde{Q}) \end{bmatrix}, \quad \xi = \begin{bmatrix} \mathrm{Vec}(\tilde{\Pi}) \\ \mathrm{Vec}(\Gamma_1) \\ \mathrm{Vec}(\Gamma_J) \end{bmatrix}. \end{aligned}$$

The following result provides conditions for the existence and design of *constant semi-classical solutions*; it involves the pseudoinverse[3] of Ψ, denoted as $\Psi^\sharp$.

Proposition 4.13 *A* constant semi-classical solution *exists if and only if* $W_\chi(s) = 0$ *and* $\Phi \in \mathrm{Im}\,(\Psi)$. *In such a case, the constant matrices* Γ_1 *and* Γ_J *can be computed by solving the linear equation* $\Psi\xi = \Phi$, *that is, by choosing* $\xi = \Psi^\sharp\Phi$.

Moreover, since the results in [12] show that an (possibly dynamic) LTI hybrid stabilizer for (4.2) can always be designed, the above discussion has an interesting consequence for all three hybrid output regulation problems considered in this paper (full information, error feedback, feedforward), namely, the fact that whenever output regulation can be achieved by a linear time-varying regulator, it can also be achieved by an LTI regulator, possibly at the price of introducing extra dynamics. The preceding discussion proves the following result.

Theorem 4.2 *If the hybrid output regulation problem is solvable, then there exists an (possibly dynamic) LTI regulator solving it.*

Finally, it is worth mentioning that it is possible to give generic conditions for the existence of periodic and constant semi-classical solutions, similar to what was done in the general case. One condition of this kind is given by the following proposition.

[3]If $M = USV'$ is the singular value decomposition (SVD) of M, then $M^\sharp = VS^\sharp U'$ with $S^\sharp$ computed from S by inverting the non-zero elements on its diagonal.

Proposition 4.14 *Let $D = 0$ and $\left[\bar{A}_{31}\ \bar{A}_{32}\right] = 0$ structurally, and $m \geq p$. The hybrid output regulation Problems 4.1 and 4.2 generically admit a periodic semi-classical solution if either $m > p$, $n + m_J \geq 2p$ or $m = p$, $m_J \geq p$. Moreover, if $m_J + m \geq n$, the same problems generically admit a constant semi-classical solution.*

4.8 Example

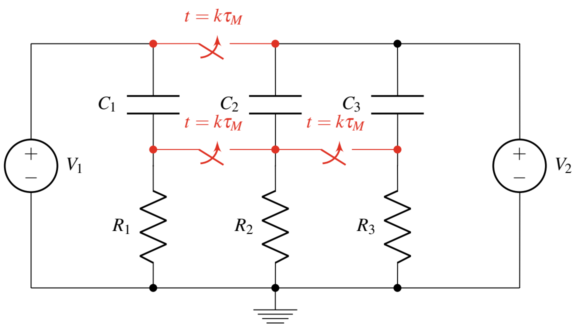
Several concepts and considerations reported in the previous sections can be appreciated in the following example, taken from [13]. Consider the RC circuit depicted in Fig. 4.2, which comprises three capacitors, described by the capacitances C_i, $i = 1, 2, 3$, each in series with a corresponding resistor, characterized by the resistances R_i, $i = 1, 2, 3$. The tension in the circuit is regulated by two independent voltage generators. The capacitors are interconnected periodically every τ_M seconds by means of switches and the objective consists in regulating the voltage across the capacitor C_3 in such a way that it tracks a desired reference profile.

Such task may be formulated within the framework defined in this paper by considering the continuous-time equations of the circuit in Fig. 4.2, defined as

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{R_1 C_1} x_1 + \frac{1}{R_1 C_1} u_1 , \\ \dot{x}_2 &= -\frac{1}{R_2 C_2} x_2 + \frac{1}{R_2 C_2} u_2 , \\ \dot{x}_3 &= -\frac{1}{R_3 C_3} x_3 + \frac{1}{R_3 C_3} u_2 , \end{aligned} \tag{4.41}$$

where $x_i \in \mathbb{R}$ denotes the voltage across the i-th capacitor and $u_i \in \mathbb{R}$ denotes the (controlled) tension of the i-th voltage generator. The discrete-time behaviour is described instead by equations identical to (4.2b) without F and R and with

Fig. 4.2 Schematics of the RC circuit; the switches are closed periodically at integer multiples of τ_M



$$E = \frac{1}{C_1 + C_2 + C_3} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \end{bmatrix}. \tag{4.42}$$

The control task consists in regulating the voltage of the capacitor C_3, namely, x_3, to track the first component of the state of the exosystem described by equations of the form (4.3) with

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.43}$$

The above system is such that $m = 2$, $p = 1$ and $q = 2$ and is already in the form (4.17), with $A_{ii} = -1/R_iC_i$, $i = 1, 2, 3$, $B_{11} = 1/R_1C_1$, $B_{12} = 0$, $B_{22} = 1/R_2C_2$ and $B_{32} = 1/R_3C_3$. Due to the passive nature of the components and the dissipation introduced by the R elements, such a system is asymptotically stable and then it is possible to focus directly on the steady-state generation. Since $A_{13} = A_{23} = 0$, the condition $W_\chi(s) = 0$ in Proposition 4.12 is satisfied and it is possible to find a semi-classical steady-state solution; the approach in Sect. 4.7.2 yields in fact the constant semi-classical solution

$$\varPi = \begin{bmatrix} -1.9501 & 3.4105 \\ 1.6 & 1.2 \\ 1 & 0 \end{bmatrix}, \quad \varGamma = \begin{bmatrix} -5.3606 & 1.4605 \\ 1 & 2 \end{bmatrix},$$

where the third row of $\varPi$ and the second row of $\varGamma$ are the heart of the regulator for the case under consideration.

In the following simulation, we let $C_1 = C_2 = C_3 = 1$, $R_1 = 1$, $R_2 = 0.5$, $R_3 = 2$ and $\tau_M = 1$. Figure 4.3 displays the time history of the output regulation error $e(t)$ when the steady-state input $u_{ss} = \varGamma w$ is implemented and the system is initial-

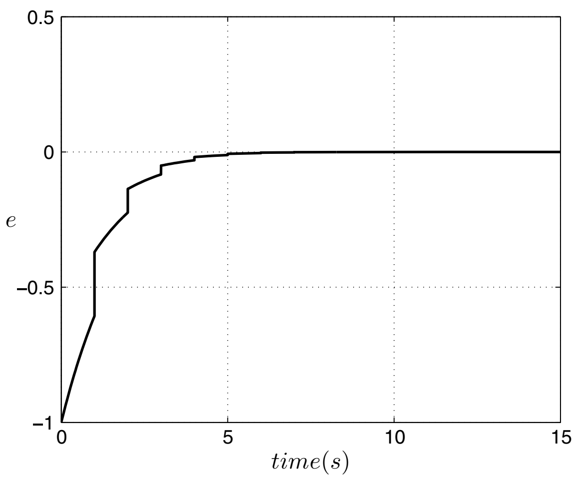


Fig. 4.3 Time history of the output regulation error $e(t)$

ized at $x(0,0)) = 0$. Since the plant is LTI and asymptotically stable, the convergence of the regulation error e to zero depends solely on the application of the appropriate steady-state input u_{ss}. It is stressed that the considered solution is semi-classical, that is, although $u_{ss} = \Gamma w$, the correct characterization of x_{ss} is not given by the classical $x_{ss,c} = \Pi w$ but requires the hybrid characterization $x_{ss}(t,k) = \Pi(\sigma)w(t,k)$ in Proposition 4.2 with $\Pi(0) = \Pi$. This aspect can be visualized in the top and middle graphs of Figs. 4.5 and 4.6. In Fig. 4.5, the response to the semi-classic steady-state input $u_{ss} = \Gamma w$ converges to the hybrid steady-state response $x_{ss}(t,k) = \Pi(\sigma)w(t,k)$. On the other hand, the top and middle graphs of Fig. 4.6 compare the same response in Fig. 4.5 to the classic steady-state state

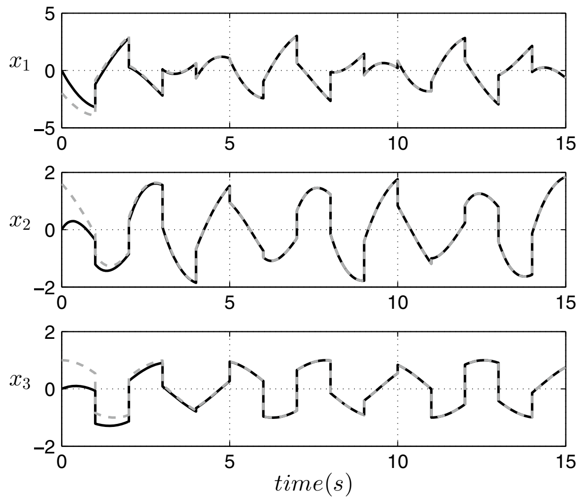


Fig. 4.4 Input evolution for the subvectors u_1 (solid line) and u_2 (dashed line)

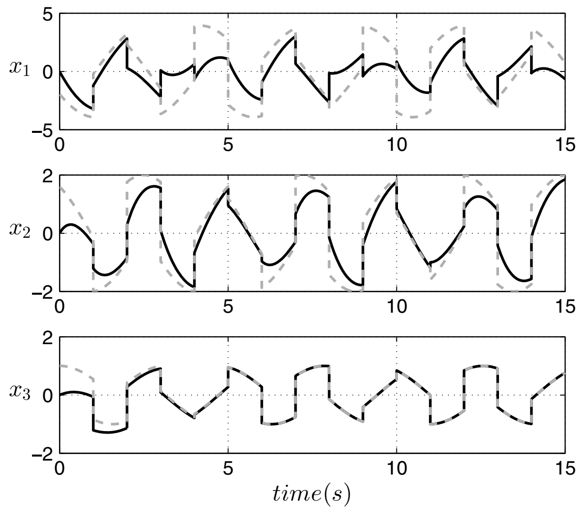


Fig. 4.5 Time histories of the state components: hybrid steady-state response (dashed gray) and state response to the semi-classical steady-state input $u_{ss} = \Gamma w$ from $x(0,0) = 0$ (solid black)

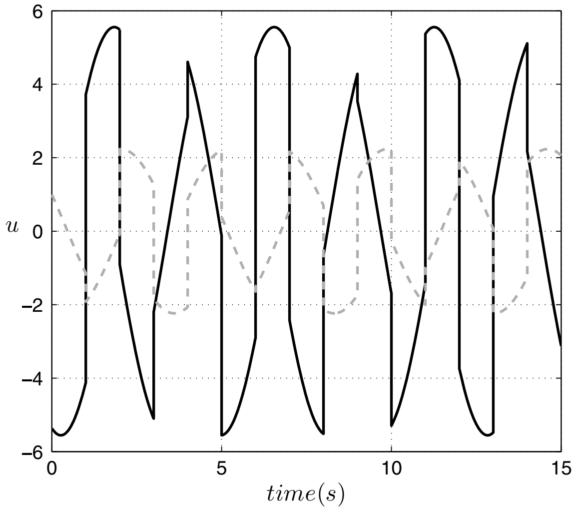


Fig. 4.6 Time histories of the state components: classic steady-state response $x_{ss,c} = \Pi w$ (dashed) and state response to the semi-classical steady-state input $u_{ss} = \Gamma w$ from $x(0, 0) = 0$ (solid)

response $x_{ss,c} = \Pi w$, which is clearly not describing the real steady-state behaviour of the system. It is worth noting that the bottom graphs of Figs. 4.5 and 4.6 actually coincide (implying that the third component of $x_{ss,c}$ is in fact the steady-state behaviour), as is to be expected by the fact that the last row of Π corresponds to the heart of the regulator (cfr Sect. 4.5.2) that has a classic nature. Finally, Fig. 4.4 shows the time histories of the control inputs u_1 and u_2.

4.9 Conclusions

In this paper, the problem of output regulation proposed in [31] has been revisited for general fat systems (having more inputs than outputs), with special emphasis on necessary conditions for achieving regulation both in the full information and in the error feedback case. A discussion about *well-posedness*, *universal* and *generic solvability* of the necessary and sufficient conditions for output regulation has been provided. Although the issue of robust regulation is not directly addressed in this paper, knowing that the solvability of the related conditions is not destroyed by arbitrarily small errors in the data, as well as knowing that under certain conditions (e.g., having more inputs than outputs) unsolvable instances of the problem lie in a zero measure set in the space of parameters, are a sort of *a priori* guarantee of a solid formulation (as discussed in classical references as [17, 38]). It is shown that a *flow zero dynamics internal model principle* holds, which particularized to the case of square plants imposes that any hybrid error feedback regulator must contain a suitable copy of the zero dynamics of the system, in addition to the usual copy of the exosystem dynamics. As another consequence, the flow zero dynamics internal model principle shows that while in non-hybrid output regulation the steady-state input can always be linearly parameterized in terms of known functions (in fact, just the exosystem's modes) even in the presence of plant parameter variations, assuming a similar parameterization in the hybrid case is tantamount to assume that *the natural modes of the flow zero dynamics are known and not affected by any variation of the plant parameters*; it follows that robust output regulation results based on this assumption (e.g., [31]) can be robust only for *structured* (or *physical parameter*) *uncertainties* as in [26, 27], and not for more general *unstructured uncertainties* as in classic results [17, 38]. Several additional constructive aspects have also been discussed. Elaborating on the LTI framework in [6, 7], constant and periodic semi-classical solutions have been proposed. Such solutions are characterized by the property that, contrary to the general case in the papers above, the required steady-state input can be generated as a simple linear function of the state of the exosystem, with either constant or piecewise constant gains, although the associated steady-state response has a genuinely hybrid structure. Apart from being interesting *per se* (being sort of a combination between a classic and a fully hybrid solution), semi-classical solutions are easier to implement and can be used for the design of robust regulators along classic ideas; related constructions (exploiting the results presented in this paper) are proposed in [11], where the problem of robust output regulation (not solved here) is considered.

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Chapter 5
A Stratified Geometric Approach to the Disturbance Decoupling Problem with Stability for Switched Systems Over Digraphs

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Abstract This chapter considers the disturbance decoupling problem with stability (more precisely, eigenvalue assignment) for switched discrete-time linear systems, where switching occurs within a set of admissible transitions defined via a weighted directed graph. The concept of subspace arrangement, as a collection of linear subspaces, is employed as the main tool for the definition of appropriate geometric properties tailored to the considered setup. Appropriate forms of feedback laws that achieve disturbance decoupling with spectral assignability under certain classes of admissible switching signals are characterized in terms of two distinct notions of controlled invariance. The result specializes more general approaches based on robust controlled invariance by exploiting the graph structure of the switching topology.

5.1 Introduction

The geometric approach pioneered by Basile and Marro [2] and by Morse and Wonham [16] has been recognized as the most fundamental tool for studying the structure of linear dynamical systems and solving a large class of control problems such as disturbance decoupling, tracking and regulation, and noninteraction. The geometric approach was then generalized to nonlinear systems [9, 11], infinite-dimensional linear systems [8] and linear time-varying (LTV) systems [10].

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More recently, geometric concepts have been applied to linear parameter-varying (LPV) systems [1], switched linear systems [7, 22] and piecewise linear systems [21]. This body of work is largely based on the concept of robust controlled invariance [3], where a subspace is found to be invariant under all possible values of a parameter vector on a bounded set for LPV systems, or under all possible switching modes for switched linear systems. Such an approach bypasses intrinsic dependence of controlled invariant subspaces on time-varying parameters or switching modes, which may be available information, as opposed to parametric uncertainty. Consequently, methodologies based on robust controlled invariance may lead to conservative results in the presence of a known structure for the switching topology.

This paper employs stratified geometric concepts as natural extensions of familiar ones in linear systems theory to the framework of switched discrete-time linear systems, where a weighted directed graph is employed to characterize the switching rule [23]. Following the introduction of a subspace arrangement as a finite set of linear subspaces corresponding to each switched subsystem, invariance concepts are characterized when the system is switched from one mode to the next. The consideration of two types of feedback control, namely, a mode-dependent piecewise constant control and a graph-based time-varying control, induces different notions of controlled invariance, leading to strong and weak controlled invariant subspace arrangements, respectively. Sufficient conditions for solvability of the Disturbance Decoupling Problem with Stability (DDPS) under certain classes of admissible switching signals are expressed in terms of appropriate extensions of controllability subspaces to the framework of subspace arrangements. These results specialize previous approaches based on robust controlled invariance to the considered setup, providing a constructive solution in cases where those methodologies fail.

The concept of subspace arrangement was first introduced in [12] to compute the maximal controlled invariant set of switched linear systems. However, the definition in [12, Prop.3], based on the union of subspaces in the arrangement, does not classify the subspaces in the arrangement with respect to the switching modes and ignores the role of switching logic in the invariance property. As a consequence, it requires one to actively control the switching mode in order to guarantee the invariance of the trajectory within the subspace arrangement. Conversely, the approach presented here leaves the switching signal unspecified, apart from requirements imposed on dwell time to ensure stabilizability.

The chapter is organized as follows: In Sect. 5.2, notation from graph theory is recalled, and the concept of subspace arrangement is defined. Section 5.3 provides background material on switched discrete-time linear systems needed in the sequel. The disturbance decoupling for switched systems over digraphs is stated in Sect. 5.4, followed by the development of invariance concepts for subspace arrangements in Sect. 5.5. Finally, solutions to disturbance decoupling problem with stability are given in Sect. 5.6, and concluding remarks are offered in Sect. 5.7.

5.2 Background and Notation

A *directed graph* (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = \{v_1, ..., v_N\}$ and an edge set $\mathcal{E} = \{(v_i, v_j),\ i, j \in \{1, ..., N\}\}$ with ordered pairs (v_i, v_j) as two-element subsets of $\mathcal{V}$, namely, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A loop (also called a self-loop) is an edge that connects a vertex to itself. A *weighted directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph with weights w_{ij} assigned to every edge $(v_i, v_j) \in \mathcal{E}$ in the graph. Node v_j is said *out-neighbour* or *successor* of v_i if $(v_i, v_j) \in \mathcal{E}$. Given a node v_i, the set

$$\Omega_i = \{v_j \in \mathcal{V} | (v_i, v_j) \in \mathcal{E}\}$$

defines the *out-neighbour set* of v_i, and the *out degree* of node i is equal to the cardinality of the set Ω_i. A *directed path* in a digraph is a sequence of distinct vertices $(v_0, ..., v_r)$ in which there is a directed edge pointing from each vertex in the sequence to its successor in the sequence, i.e., $(v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, ..., r$. A digraph is said to be *strongly connected* if there exists a *directed path* joining any two vertices in $\mathcal{V}$.

A *subspace arrangement* in $\mathbb{R}^n$ is a finite set of subspaces $\mathcal{V} := \{\mathscr{V}_1, ..., \mathscr{V}_N\}$, where $\mathscr{V}_i$ is a linear subspace of $\mathbb{R}^n$ for every $i \in \mathcal{I} := \{1, ..., N\}$ (see [4]). A subspace arrangement $\mathcal{V}$ is called *pure* if $\dim(\mathscr{V}_i) = \dim(\mathscr{V}_j)$ for all $i, j \in \mathcal{I}$. For $x \in \mathbb{R}^n$, we say $x \in \mathcal{V}$ if there exists $i \in \mathcal{I}$ such that $x \in \mathscr{V}_i$. For a subspace arrangement $\mathcal{V} := \{\mathscr{V}_1, ..., \mathscr{V}_N\}$, denote $\mathcal{I} = \{1, ..., N\}$ the ordered index set. For subspace arrangements $\mathcal{V}_a = \{\mathscr{V}_{a1}, ..., \mathscr{V}_{aN}\}$ and $\mathcal{V}_b = \{\mathscr{V}_{b1}, ..., \mathscr{V}_{bN}\}$ in $\mathbb{R}^n$ of same ordered index set, we say $\mathcal{V}_a \subseteq \mathcal{V}_b$ if $\mathscr{V}_{ai} \subseteq \mathscr{V}_{bi}$, $\forall i \in \mathcal{I}$. Addition (intersection) of subspace arrangements is defined as the sum (intersection) of the corresponding elements, i.e.,

$$\mathcal{V}_a + \mathcal{V}_b := \{\mathscr{V}_i : \mathscr{V}_i = \mathscr{V}_{ai} + \mathscr{V}_{bi},\ i \in \mathcal{I}\} \tag{5.1a}$$

$$\mathcal{V}_a \cap \mathcal{V}_b := \{\mathscr{V}_i : \mathscr{V}_i = \mathscr{V}_{ai} \cap \mathscr{V}_{bi},\ i \in \mathcal{I}\}, \tag{5.1b}$$

where, by definition, $\mathcal{V}_a + \mathcal{V}_b$ and $\mathcal{V}_a \cap \mathcal{V}_b$ are also subspace arrangements in $\mathbb{R}^n$ of the same ordered index set $\mathcal{I}$. The family of all subspace arrangements in $\mathbb{R}^n$ under the operations $+$ and $\cap$ forms a lattice[1]: $\mathcal{V}_a + \mathcal{V}_b$ is the smallest subspace arrangement containing both $\mathcal{V}_a$ and $\mathcal{V}_b$, while $\mathcal{V}_a \cap \mathcal{V}_b$ is the largest subspace arrangement contained in both $\mathcal{V}_a$ and $\mathcal{V}_b$. For a subspace arrangement $\mathcal{V} = \{\mathscr{V}_1, ..., \mathscr{V}_N\}$ in $\mathscr{X} \cong \mathbb{R}^n$, we define the factor space arrangement $\mathscr{X}/\mathcal{V} := \{\mathscr{X}/\mathscr{V}_1, ..., \mathscr{X}/\mathscr{V}_N\}$.

[1]Unlike linear vector subspaces, the family of all subspace arrangements in $\mathbb{R}^n$ is not partially ordered by inclusion.

5.3 Switched Discrete-time Linear Systems

A *switched discrete-time linear system* is a collection of a finite number of subsystems modelled by difference equations

$$\Sigma: \quad \begin{cases} x(t+1) = A_i x(t) + B_i u(t) + D_i d(t) \\ y(t) = C_i x(t) \qquad i \in \mathcal{I}, \end{cases} \tag{5.2}$$

where $t \in \mathbb{Z} := \{0, 1, ...\}$ is the time sequence, $x \in \mathscr{X} \cong \mathbb{R}^n$ is the state, $u \in \mathscr{U} \cong \mathbb{R}^m$ is the control input, $d \in \mathscr{D} \cong \mathbb{R}^q$ is the disturbance input and $y \in \mathscr{Y} \cong \mathbb{R}^p$ is the output. For each $i \in \mathcal{I}$, the triplet (C_i, A_i, B_i) describes a linear time-invariant (LTI) system. We let $\mathcal{A} := \{A_i\}_{i\in\mathcal{I}}$, $\mathcal{B} := \{B_i\}_{i\in\mathcal{I}}$ and $\mathcal{C} := \{C_i\}_{i\in\mathcal{I}}$ be the collection of the switched subsystem matrices. The logic rule that generates switching signals is described in terms of piecewise constant signals that are assumed to depend on time only, that is, $\sigma : \mathbb{Z} \to \mathcal{I}$, where $\mathcal{I} := \{1, ..., N\}$ represents the finite index set of discrete switching modes. The value of the switching signal $\sigma(t) = i$, $i \in \mathcal{I}$ specifies the active mode at the time instant t. A standard requirement on σ is that the switching events only occur at the sampling instants, such that the issue of synchronization between sampling time and switching time need not be considered.

A weighted digraph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is associated with the switched system (5.2), where the vertex set $\mathcal{V} := \mathcal{I}$ is the index set of switched subsystems, while the edge set $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{I}$ consists of a directed edge (i, j) whenever it is permissible to switch from subsystem i to subsystem j for $i, j \in \mathcal{I}$. Switching signals are assumed to dwell at each mode for at least one sampling period, and thus self-loops always exist for the switched systems at each vertex, that is, $(i, i) \in \mathcal{E}$ for all $i \in \mathcal{I}$. A pictorial representation is given in Fig. 5.1.

Definition 5.1 A switching signal σ is said to be *admissible* if and only if $(\sigma(t), \sigma(t+1)) \in \mathcal{E}$ for all $t \in \mathbb{Z}$.

Let $\varphi(\mathcal{G})$ denote the class of all admissible switching signals for (5.2) corresponding to the underlying digraph $\mathcal{G}$. We further classify the switching signals by defining by $\varphi_{\tau_D}(\mathcal{G})$ the set of switching signals where each σ is restricted to a dwell

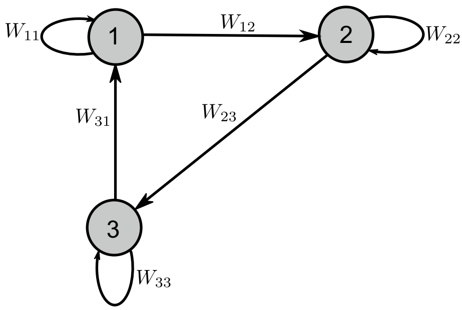

Fig. 5.1 The digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a switched discrete-time linear system: The vertex set $\mathcal{V} = \{1, 2, 3\}$ defines three switched modes; only transitions in the directed edge set $\mathcal{E} = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 1)\}$ are admissible

time bounded away from $0 < \tau_D < \infty$, and by $\varphi_{\tau_a}(\mathcal{G})$ the set of arbitrary switching signals $\sigma \in \varphi(\mathcal{G})$.

Moreover, for $i \in \mathcal{I}$, denote the set

$$\Omega_i = \{j \in \mathcal{I} \mid (i, j) \in \mathcal{E}\},\ i \in \mathcal{I}$$

as the out-neighbour set of node i that represents all admissible switches from i-th subsystem. The weight in each directed edge of the associated digraph defines the transition matrix of the switched system that transfers a state at node i to one at node j. Specifically, $W_{ij} = A_i \in \mathbb{R}^{n\times n}$ for autonomous systems and $W_{ij} = (A_i, B_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ for systems with control inputs.

5.4 Problem Statement

The disturbance decoupling problem considered in this study amounts in finding a feedback law that eliminates the effect of disturbances on the output under (possibly) arbitrary switching signals $\sigma \in \varphi(\mathcal{G})$. Let $F_{\sigma(t)} : \mathscr{X} \to \mathscr{U}$ denote a switched state feedback control designed for the switched system (5.2) associated with the switching signal $\sigma(t)$. Then, the problem of disturbance decoupling with stability of the closed-loop system is formally stated as follows.

Problem 5.1 Given the switched discrete-time linear system (5.2), find a switched state feedback $F_{\sigma(t)}$ such that the following requirements are satisfied:

I-1 The forced response of the controlled output $y(\cdot) : \mathbb{Z} \to \mathscr{Y}$ is equal to zero for all $t \in \mathbb{Z}$ when $x(t_0) = 0$ and any disturbance $d(\cdot)$, that is,

$$y(t) = C_{\sigma(t)} \sum_{j=t_0}^{t-1} \Phi_{\mathcal{A}_F}(t; j+1, \sigma) D_{\sigma(t)} d(t) = 0,\ \forall t \in \mathbb{Z}.$$

I-2 The closed-loop system is internally stable.

It is noteworthy that solvability conditions for the disturbance decoupling problem for switched linear systems have been given in [7, 17] on the basis of the geometric concept of robust controlled invariant subspace [3]. Letting

$$\mathscr{K} := \bigcap_{i\in\mathcal{I}} \ker C_i$$

and $\mathscr{V}^\star(\mathcal{A}, \mathcal{B}, \mathscr{K})$ be the maximal robust $(\mathcal{A}, \mathcal{B})$-controlled invariant subspace contained in $\mathscr{K}$, the results in [7, 17] state that the disturbance decoupling problem (without internal stability) is solvable for general switching systems if and only if

$$\sum_{i\in\mathcal{I}} \operatorname{im} D_i \subseteq \mathscr{V}^\star(\mathcal{A}, \mathcal{B}, \mathscr{K}). \tag{5.3}$$

This study aims at finding less stringent solvability condition than [7, 17] for the considered problem by exploiting the digraph structure of the switching topology. The proposed approach is based on the stratified geometric characterization proposed in [23], which will be recalled in the forthcoming section.

5.5 Invariant Subspace Arrangements

In this section, the concepts of invariant and controlled invariant subspace arrangements are introduced as suitable generalizations of well-known concepts. It is found that these invariance concepts do not depend on the switching signals but only on the structure of the digraph that governs the switching mechanisms.

5.5.1 Invariant Subspace Arrangements

Definition 5.2 For the system (5.2) with switching signal $\sigma \in \varphi(\mathcal{G})$, a subspace arrangement $\mathcal{V} := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ is said to be $\mathcal{A}$-invariant over digraph $\mathcal{G}$ if

$$A_i\mathscr{V}_i \subseteq \mathscr{V}_j,\ \forall(i, j) \in \mathcal{E}.$$

The definition of invariant subspace arrangement is motivated by inspecting the state trajectory under arbitrary switching signals over a digraph $\mathcal{G}$. Namely, $\forall x \in \mathscr{V}_i$, a self-loop maintains the state x within the i-th subspace, that is, $A_i x \subseteq \mathscr{V}_i, i \in \mathcal{I}$, which agrees with the traditional concept of subspace invariance at each switching node. This property is termed *loop invariance*. Moreover, $\forall x \in \mathscr{V}_i$, a transition edge $(i, j) \in \mathcal{E}$ and $i \neq j$ transfers the state x into the out-neighbouring subspace $\mathscr{V}_j$, that is, $A_i x \subseteq \mathscr{V}_j$, which is defined under the topology of the graph that governs the switching mechanism, a property that is termed *transition invariance*.

5.5.2 Controlled Invariant Subspace Arrangements

Two different types of feedback control are introduced in this section, leading to the definition of strong and weak controlled invariant subspace arrangements. Geometric conditions are pursued each definition to obtain an equivalent formulation that does not depend on the specific form of feedback control law.

5.5.2.1 Strong Controlled Invariant Subspace Arrangements

Definition 5.3 For system (5.2) with switching signal $\sigma \in \varphi(\mathcal{G})$, a subspace arrangement $\mathcal{V} := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ is said to be a *strong* $(\mathcal{A}, \mathcal{B})$*-invariant subspace arrangement* over the digraph $\mathcal{G}$ if for all $i \in \mathcal{I}$, there exists $F_i \in \mathbb{R}^{m\times n}$ such that

$$(A_i + B_i F_i)\mathscr{V}_i \subseteq \mathscr{V}_j, \quad \forall (i, j) \in \mathcal{E}. \tag{5.4}$$

Following the standard definition, the collection of such feedback control matrices $\mathcal{F} := \{F_i\}_{i\in\mathcal{I}}$ is called a friend of the strong $(\mathcal{A}, \mathcal{B})$-invariant subspace arrangement $\mathcal{V}$.

Lemma 5.1 ([23]) *A subspace arrangement* $\mathcal{V} := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ *is a strong* $(\mathcal{A}, \mathcal{B})$*-invariant over* $\mathcal{G}$ *for system* (5.2) *if and only if*

$$A_i\mathscr{V}_i \subseteq \bigcap_{j\in\Omega_i} \mathscr{V}_j + \operatorname{im} B_i. \tag{5.5}$$

5.5.2.2 Weak Controlled Invariant Subspace Arrangements

A strong controlled invariant subspace arrangement is induced by a mode-dependent feedback control. A different concept of controlled invariance, termed *weak controlled invariance*, can be realized by enlarging the class of feedback laws to include time-varying laws as well.

Definition 5.4 For system (5.2) with switching signal $\sigma \in \varphi(\mathcal{G})$, a subspace arrangement $\mathcal{V} := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ is said to be a *weak* $(\mathcal{A}, \mathcal{B})$*-invariant subspace arrangement* over $\mathcal{G}$ if for any $(i, j) \in \mathcal{E}$, there exists $F_{ij} \in \mathbb{R}^{m\times n}$ such that

$$(A_i + B_i F_{ij})\mathscr{V}_i \subseteq \mathscr{V}_j, \quad \forall (i, j) \in \mathcal{E}. \tag{5.6}$$

Lemma 5.2 ([23]) *A subspace arrangement* $\mathcal{V} := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ *is a weak* $(\mathcal{A}, \mathcal{B})$*-invariant over* $\mathcal{G}$ *for system* (5.2) *if and only if*

$$A_i\mathscr{V}_i \subseteq \mathscr{V}_j + \operatorname{im} B_i \tag{5.7}$$

holds for any $(i, j) \in \mathcal{E}$.

Finally, it has been shown in [23] that there exists a unique maximal weak (strong) controlled invariant subspace arrangement contained in a given arrangement. The existence of a maximal element among the class of weak (strong) controlled invariant subspace arrangements is instrumental to the following result, proved in [23].

Theorem 5.1 *Let $\mathcal{V}^\star := \{\mathscr{V}_i\}_{i\in\mathcal{I}}$ be the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace arrangement over the digraph $\mathcal{G}$ contained in $\mathcal{K}$ where $\mathscr{K}_i = \ker C_i$, $\forall\, i \in \mathcal{I}$. The DDP for switched discrete-time linear system (5.2) with switching signal $\sigma \in \varphi(\mathcal{G})$ is solvable if* $\operatorname{im} D_i \subseteq \Psi_{\mathscr{V}_i}$, $\forall\, i \in \mathcal{I}$, *where* $\Psi_{\mathscr{V}_i} = \bigcap_{i\in\Omega_i} \mathscr{V}_j$.

5.6 Disturbance Decoupling Problem with Stability

In this section, we introduce the concept of reachability subspace arrangement, which generalizes the familiar concept of controllability subspace. Controllability subspaces have been originally introduced in [20]: Given a system pair (A, B), the controllable subspace of $\langle A + BF, BG\rangle$ is called a controllability subspace, which is a subfamily of the (A, B)-invariant subspaces. The importance of controllability subspace stems from the fact that the restriction of $A + BF$ to an $(A + BF)$-invariant controllability subspace can be assigned an arbitrary spectrum by a suitable choice of the friend F. Therefore, disturbance decoupling with stability can be achieved through eigenvalue assignment if appropriate conditions involving the reachable space associated with the disturbance input hold.

Definition 5.5 *(Reachability subspace arrangement)* For the switched discrete-time linear system (5.2), a subspace arrangement $\mathcal{R} := \{\mathscr{R}_i\}_{i\in\mathcal{I}}$ is called a *reachability subspace arrangement* if for all $(i, j) \in \mathcal{E}$ and $i \in \mathcal{I}$ there exist $F_{ij} \in \mathbb{R}^{m\times n}$ and $G_i \in \mathbb{R}^{m\times m}$ such that

$$(A_i + B_i F_{ij})\mathscr{R}_i \subseteq \mathscr{R}_j, \qquad \forall (i, j) \in \mathcal{E} \tag{5.8}$$

$$\mathscr{R}_i = \langle A_i + B_i F_{ii} | \operatorname{im}(B_i G_i)\rangle, \qquad \forall i \in \mathcal{I}. \tag{5.9}$$

Remark 5.1 Equation (5.8) guarantees that the defined reachability subspace arrangement $\mathcal{R}$ is a weak $(\mathcal{A}, \mathcal{B})$-invariant over digraph $\mathcal{G}$. Equation (5.9) states that each element $\mathscr{R}_i$ in the reachability subspace arrangement $\mathcal{R}$ is a controllability subspace for the corresponding i-th subsystem.

Useful results similar to those holding for controllability subspaces are also valid for reachability subspace arrangements, namely,

- $\mathcal{R}$ is a reachability subspace arrangement if and only if there exists $F_{ij} \in \mathbb{R}^{m\times n}$ such that

$$(A_i + B_i F_{ij})\mathscr{R}_i \subseteq \mathscr{R}_j, \qquad \forall (i, j) \in \mathcal{E}$$
$$\mathscr{R}_i = \langle A_i + B_i F_{ii} | \operatorname{im} B_i \cap \mathscr{R}_i\rangle, \qquad \forall i \in \mathcal{I}$$

- Suppose $\mathcal{R}$ is a reachability subspace arrangement. If matrices $F_{ij} \in \mathbb{R}^{m\times n}$ exist that satisfy $(A_i + B_i F_{ij})\mathscr{R}_i \subseteq \mathscr{R}_j$ for all $(i, j) \in \mathcal{E}$, then

$$\mathscr{R}_i = \langle A_i + B_i F_{ii} | \operatorname{im} B_i \cap \mathscr{R}_i\rangle, \qquad \forall i \in \mathcal{I}$$

The above results provide a method for checking if a controlled invariant subspace arrangement $\mathcal{V}$ is a reachability subspace arrangement: Construct any $F_{ij} \in \mathbb{R}^{m\times n}$ such that $(A_i + B_i F_{ij})\mathscr{V}_i \subseteq \mathscr{V}_j, \forall(i, j) \in \mathcal{E}$, and then check that

$$\mathscr{V}_i = \langle A_i + B_i F_{ii} | \mathrm{im}\, B_i \cap \mathscr{V}_i \rangle, \qquad \forall i \in \mathcal{I}.$$

The closure property of reachability subspace arrangements under the operation of arrangement addition is also not hard to verify, and thus there exists a unique maximal reachability subspace arrangement $\mathcal{R}^\star := \{\mathscr{R}_i^\star\}_{i\in\mathcal{I}}$ contained in $\mathcal{K}$ such that $\mathscr{R}_i^\star \subseteq \mathscr{K}_i$ for all $i \in \mathcal{I}$.

5.6.1 Stabilization of Switched Discrete-Time Linear Systems

Stabilization[2] of switched systems have attracted considerable attention [13–15, 19]. A thorough survey of stability analysis was shown in [15], which is largely based on different forms of Lyapunov-based analysis. The relationship between controllability and stabilizability has been studied in relation to eigenstructures of the closed-loop systems [18], or overshoot estimation of the transition matrices via pole placement [6]. In particular, it has been shown that if for a switched continuous-time system each pair (A_i, B_i), $i \in \mathcal{I}$ is controllable, then for any given scalar $\tau_d > 0$, there exists a feedback control $u(t) = K_{\sigma(t)}x(t)$ such that the closed-loop system is exponentially stable under any switching signal subject to dwell time τ_d. However, it is also known that controllability of each individual subsystem does not necessarily guarantee that the switched system can be stabilized under arbitrary switching signal (i.e., zero dwell time) [19].

The following theorem, presented without proof, is a simple extension of the stabilizability results of [6, Theorem 2.1].

Theorem 5.2 *Consider a switched discrete-time linear system described by a finite set of matrices $\{A_i\}_{i\in\mathcal{I}}$ and $\{B_i\}_{i\in\mathcal{I}}$ such that every pair (A_i, B_i), $i \in \mathcal{I}$ is controllable. Then, there exists a set of control matrices $\{K_i\}_{i\in\mathcal{I}}$ such that for any given positive dwell time τ_D and any switching law $\sigma \in \varphi_{\tau_D}$, the switched linear system under the switched feedback law $u(t) = K_{\sigma(t)}x(t)$ is internally stable.*

Since this study also considers a graph-based time-varying control (for weak controlled invariance), the following theorem is considered as a proper extension of the above results.

Theorem 5.3 *Consider a switched discrete-time linear system described by a finite set of matrices $\{A_i\}_{i\in\mathcal{I}}$ and $\{B_i\}_{i\in\mathcal{I}}$ such that every pair (A_i, B_i), $i \in \mathcal{I}$ is controllable. Given a set of transition feedback laws $\{K_{ij}\}_{(i,j)\in\mathcal{E}\wedge i\neq j}$, there exists a set*

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[2]This study only focuses on stabilization through feedback control laws, as the switching signal is assumed to be assigned.

of feedback laws $\{K_{ii}\}_{i\in\mathcal{I}}$ *such that for any given positive dwell time* τ_D *and any switching law* $\sigma \in \varphi_{\tau_D}$, *the switched linear system under the switched feedback law* $u(t) = K_{\sigma(t)\sigma(t+1)}x(t)$ *is internally stable.*

Remark 5.2 If the pairs $\{A_i\}_{i\in\mathcal{I}}$ and $\{B_i\}_{i\in\mathcal{I}}$ are only stabilizable, it is generally not sufficient to guarantee the stabilizability of the switched system under any given dwell time. On the other hand, analysis based on multiple Lyapunov functions [5] reveals that the closed-loop system is asymptotically stable under slow switching with sufficiently large dwell time [13].

5.6.2 Solution to DDP with Stability

The proposed solution to DDP with stability in this section is built on structural decomposition via controlled invariance, and exponential stabilizability under dwell-time switching signal via eigenvalue assignment. The first result is concerned with DDPS under arbitrary dwell-time switching. Consider the switched discrete-time linear system given by the triplets $\Sigma_i = \{C_i, A_i, B_i\}$ and digraph $\mathcal{G}$ that defines the switching mechanism. We make the following assumption.

Assumption 1 (A_i, B_i) is controllable for all $i \in \mathcal{I}$.

Using well-known results [20], the following lemma summarizes some important properties of controlled invariant arrangements and reachability arrangements:

Lemma 5.3 *Let* $\mathcal{V}^\star := \{\mathscr{V}_i^\star\}_{i\in\mathcal{I}}$ *be the maximal* $(\mathcal{A}, \mathcal{B})$*-invariant subspace arrangement over digraph* $\mathcal{G}$ *contained in arrangement* $\mathcal{K}$ *where* $\mathcal{K}_i = \ker C_i$, $i \in \mathcal{I}$. *Let* $\mathcal{R}^\star := \{\mathscr{R}_i^\star\}_{i\in\mathcal{I}}$ *be the maximal reachability subspace arrangement over digraph* $\mathcal{G}$ *contained in* $\mathcal{K}$, *and let* $\{F_{ij}\}_{(i,j)\in\mathcal{E}} \in \mathbb{F}(\mathcal{V}^\star)$ *be a friend of* $\mathcal{V}^\star$. *Then the following results hold:*

- *For any* $i \in \mathcal{I}$, $\operatorname{spec}(A_i + B_iF_{ii})|\mathscr{R}_i^\star$ *is assignable;*
- *For any* $i \in \mathcal{I}$, $\operatorname{spec}(A_i + B_iF_{ii})|\mathscr{V}_i^\star/\mathscr{R}_i^\star$ *is fixed;*
- *For any* $i \in \mathcal{I}$, $\operatorname{spec}(A_i + B_iF_{ii}|\mathscr{X}/\mathscr{V}_i^\star)$ *is assignable.*

With the above lemma at hand, the solution to DDPS for switched discrete-time linear systems subject to arbitrary dwell time is readily obtained.

Theorem 5.4 *Consider the switched discrete-time linear system over digraph* $\mathcal{G}$ *comprised of the triplets* $\Sigma_i = \{C_i, A_i, B_i\}$. *Let* $\mathcal{R}^\star := \{\mathscr{R}_i\}_{i\in\mathcal{I}}$ *be the maximal reachability subspace arrangement contained in* $\mathcal{K}$ *where* $\mathcal{K}_i = \ker C_i$, $i \in \mathcal{I}$. *Given any positive (integer) dwell time* τ_D, *the disturbance decoupling problem is solvable with exponentially stable closed-loop system for switching signals* $\sigma \in \varphi_{\tau_D}(\mathcal{G})$ *if*

$$\operatorname{im} D_i \subseteq \Psi_{\mathscr{R}_i}\ \forall i \in \mathcal{I}, \quad \textit{where} \quad \Psi_{\mathscr{R}_i} = \bigcap_{j\in\Omega_i} \mathscr{R}_j.$$

In some cases, the pair $\{A_i, B_i\}_{i\in\mathcal{I}}$ may not be controllable but only stabilizable. Furthermore, closed-loop stability needs not be satisfied for arbitrarily small dwell time but only for sufficiently large ones. Consequently, it is of interest to relax the previous assumption and consider the following one in the subsequent analysis.

Assumption 2 (A_i, B_i) is stabilizable for all $i \in \mathcal{I}$.

To solve DDPS under this weaker assumption, it is natural to introduce the family of maximal internally stabilizable $(\mathcal{A}, \mathcal{B})$-controlled invariant subspace arrangement $\mathcal{V}_g^\star$ contained in $\mathcal{K}$. By definition, $\mathcal{V}_g^\star$ is a controlled invariant arrangement, thus $\mathcal{V}_g^\star \subseteq \mathcal{V}^\star$, and satisfies the following property:

$$\mathcal{V}_g^\star = \left\{\mathscr{V}_{g,i}^\star \subseteq \mathscr{V}_i^\star, i \in \mathcal{I} : \exists \mathcal{F} \in \mathbb{F}(\mathcal{V}_g^\star), \text{ s.t. } \operatorname{spec}(A_i + B_i F_{ii})|\mathscr{V}_{g,i}^\star \subset \mathbb{C}^-\right\}. \tag{5.10}$$

Since $\operatorname{spec}(A_i + B_i F_{ii})|\mathscr{R}_i^\star$ can be assigned arbitrarily, $\mathcal{R}^\star \subseteq \mathcal{V}_g^\star$. While nothing can be said about $\operatorname{spec}(A_i + B_i F_i)|(\mathscr{V}_i^\star/\mathscr{R}_i^\star)$, the subspace arrangement $\mathcal{V}^\star/\mathcal{R}^\star$ can be factorized as

$$\mathcal{V}^\star/\mathcal{R}^\star = \{\mathscr{V}_i^\star/\mathscr{R}_i^\star\}_{i\in\mathcal{I}} = \{\mathscr{X}_{g,i}^\star \oplus \mathscr{X}_{b,i}^\star\}_{i\in\mathcal{I}},$$

where $\operatorname{spec}(A_i + B_i F_{ii})|\mathscr{X}_{g,i}^\star \subset \mathbb{C}^-$ and $\operatorname{spec}(A_i + B_i F_{ii})|\mathscr{X}_{b,i}^\star \subset \bar{\mathbb{C}}^+$. Then, the subspace of "good" modes can be made to lie in $\mathcal{V}_g^\star$, that is

$$\operatorname{spec}(A_i + B_i F_i)|\mathscr{V}_{g,i}^\star = \operatorname{spec}(A_i + B_i F_i)|\mathscr{R}_i^\star \biguplus \operatorname{spec}(A_i + B_i F_i)|\mathscr{X}_{g,i}^\star. \tag{5.11}$$

Clearly, the above definition defines the internal stabilizability of $A_i + B_i F_{ii}$ restricted to $\mathscr{V}_{g,i}^\star$ for each individual subsystem $i \in \mathcal{I}$. Moreover, since (A_i, B_i) are stabilizable, it is possible to select F_{ii} such that $\operatorname{spec}(A_i + B_i F_{ii}|\mathscr{X}/\mathcal{V}_{g,i}^\star) \subset \mathbb{C}^-$. Then, the solution to DDPS for switched discrete-time linear system for sufficiently large dwell time is cast in the following theorem.

Theorem 5.5 *Let $\mathcal{V}_g^\star := \{\mathscr{V}_{g,i}^\star\}_{i\in\mathcal{I}}$ be the maximal internally stabilizable $(\mathcal{A}, \mathcal{B})$-invariant subspace arrangement over the digraph $\mathcal{G}$ contained in $\mathcal{K}$ where $\mathcal{K}_i = \ker C_i, \forall i \in \mathcal{I}$. For sufficiently large dwell time τ_D, the disturbance decoupling problem is solvable with asymptotically stable closed-loop system for switching signals $\sigma \in \varphi_{\tau_D}(\mathcal{G})$ if*

$$\operatorname{im} D_i \subseteq \Psi_{\mathscr{V}_{g,i}^\star}\ \forall i \in \mathcal{I}, \quad \textit{where} \quad \Psi_{\mathscr{V}_{g,i}^\star} = \bigcap_{j\in\Omega_i} \mathscr{V}_{g,j}^\star.$$

5.7 Conclusions

Invariant subspace arrangements have been considered in this study to characterize invariance properties of state trajectories of switched systems within a set of linear subspaces when arbitrary switching events occur. The proposed invariance notions are closely related to the structure of the underlying digraph that governs the switching rules. Two different notions of controlled invariance were defined and connected to the solvability conditions of DDP that specializes previous results to exploit the graph structure of the switching topology. A natural extension of the concept of controllability subspaces to the stratified geometric setup considered here leads to simple sufficient conditions (reminiscent of classical ones) for solvability of DDPS under different classes of admissible switching signals.

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Chapter 6
Unknown-Input State Observers for Hybrid Dynamical Structures

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Abstract This chapter investigates the unknown-input state observation problem for hybrid dynamical systems with state jumps. The problem considered is that of deriving an asymptotic estimate of a linear function of the state of a given system in the presence of unknown inputs. The systems addressed are hybrid dynamical systems which exhibit a continuous-time linear behaviour except at isolated points of the time axis, where their state shows abrupt changes ruled by an algebraic linear equation. The systems belonging to this class are also known as linear impulsive systems. It will be assumed that the time interval between two consecutive state discontinuities is lower bounded by a positive real constant. The presence of unknown inputs precludes, in general, the possibility of asymptotically estimating the whole system state, but may permit the asymptotic estimation of a linear function of the state. Thus, the objective of this chapter is to state and prove necessary and sufficient conditions for the existence of solutions to this problem, in the context of linear impulsive systems. The methodology adopted is structural, based on the use of properly defined geometric objects and properties. A general necessary and sufficient condition is proven first. Then, under more restrictive, yet acceptable in contexts of practical interest, assumptions, a constructive necessary and sufficient condition is shown. The latter condition can be checked by an algorithmic procedure, since it is based on subspaces which can be easily computed and on properties which can be systematically ascertained. Special attention is paid to the synthesis of asymptotic observers whose state has the minimal possible dimension, briefly referred to as minimal-order observers.

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6.1 Introduction

The unknown-input state observation problem—namely, the problem of asymptotically estimating a linear function of the state of a given system in the presence of unknown inputs—is a classical problem of system and control theory, originally stated for continuous-time linear time-invariant systems [1, 2, 4]. Since this problem lends itself to manifold solutions, it has been widely investigated in the last decades, either in the context of continuous-time linear systems [12, 13, 19, 22, 40, 42] or in the discrete-time case [14, 16, 23, 30, 35, 38].

Owing to its close connections with issues of high impact on applications—like, in particular, fault detection (see, e.g., [10], but also more recent papers such as [17, 28, 39], just to mention some)—in the past years, the unknown-input state observation problem has been reformulated and solved for several different classes of dynamical systems: i.e., descriptor systems [24, 36], affine systems [21], bilinear systems [27], 2D systems [5], time-delay systems [47, 48], and many others. Nonetheless, this problem attracts the interest of the scientific community still today and this is especially true for its most recent formulations, which mainly concern nonlinear complex dynamical systems [8, 9, 43], linear parameter varying systems [31, 41], switched systems [15].

In this work, the unknown-input state observation problem is investigated for hybrid linear systems with state jumps, or linear impulsive systems, as these systems are also called in literature [18, 20, 25, 29, 44]. The dynamical systems of this class exhibit a continuous-time linear dynamics, called *flow dynamics*, except at isolated points of the time axis, where their state shows discontinuities, or *jumps*. In particular, it is assumed herein that the length of the time interval between consecutive jump instants has a positive lower bound. The study of this kind of dynamical systems is motivated by their effectiveness in modeling physical systems whose state drastically changes due to the occurrence of some abrupt phenomena, as is the case, for instance, of impacts in mechanical systems or of the action of switches in electrical circuits. Although some classic control problems like output regulation and disturbance decoupling have been investigated in depth in the context of linear impulsive systems (see, e.g., [6, 7, 37, 45, 46]), this is not yet the case of unknown-input state observation. In fact, only few contributions are presently available and several issues are still open [11, 26, 33, 34]. Hence, the goal of this chapter is to provide an exhaustive discussion of this topic.

The geometric methods, which have originally fostered the solution of the unknown-input state observation problem for linear time-invariant systems (see the cited works [1, 2, 4]), have recently proven to be valid and effective also in dealing with linear impulsive systems: see, e.g., [26, 33, 34]. More precisely, in [34], necessary and sufficient conditions for the existence of asymptotic observers were given by using both geometric and LMI conditions. The state estimation problem was solved essentially by finding suitable geometric conditions in [33]. However, the systems considered therein were characterized by measurements only available at the jump times. Instead, in [26], exact observers, which needed to be initialized with the correct value of the state variable (modulo its projection over a certain subspace which depends on the unknown input distribution matrix), were considered and geometric conditions for their existence were given.

In this chapter, the approach to the unknown-input state observation problem is similar to that of the earlier work [11], where sufficient geometric conditions for the solvability of the full order unknown-input state observer problem were demonstrated. However, in this work, a general necessary and sufficient condition for the existence of an asymptotic unknown-input observer is provided. Morever, conditions for the synthesis of asymptotic observers whose state has the minimal possible dimension are investigated. Finally, under slightly more restrictive conditions, generally satisfied in circumstances of practical interest, necessary and sufficient conditions which can be straightforwardly checked by an algorithmic procedure are proven. The relevant feature of the proposed solution is that the synthesis of the minimal order observer is based on the largest *hybrid* conditioned invariant subspace in a specific class and that the minimal order observer, if it exists, turns out to be globally asymptotically stable also if the largest conditioned hybrid conditioned invariant subspace is not internally stabilizable. This property, along with order minimality, facilitates achieving of an asymptotic estimation of the component of the system state not affected by the unknown inputs.

This chapter is organized as follows. In Sect. 6.2, the notion of asymptotic observer of a linear function of the system state in the presence of unknown inputs is defined for linear impulsive systems whose state jumps are subject to a minimal dwell time constraint. The minimal order unknown-input state observer problem is then stated. In Sect. 6.3, the structural notions needed to investigate the observer problem are presented. The main ones are those of hybrid conditioned invariant subspace and hybrid controlled invariant subspace, along with that of hybrid dynamics induced by the output injection performed by a friend of the considered conditioned invariant subspace. The stability properties of the observer and, consequently, the convergence of the estimate are shown to depend on the internal stabilizability property of certain hybrid conditioned invariant subspaces. In Sect. 6.4, a necessary and sufficient condition for the existence of an asymptotic observer of a linear function of the state of a given system in the presence of unknown inputs is stated first. Conditions for deriving an asymptotic observer whose state has the minimal possible dimension are then discussed. In Sect. 6.5, a necessary and sufficient condition for the existence of an asymptotic observer is stated under less general assumptions. The new condition, compared with the former, has the advantage of being based on subspaces that can be directly computed through algorithmic procedures and on properties that can be straightforwardly ascertained. Finally, Sect. 6.6 contains the conclusions.

Notation: The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ are used to denote the sets of natural numbers, real numbers, and non-negative real numbers, respectively. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathscr{V}$. The quotient space of a vector space $\mathscr{X}$ over a subspace $\mathscr{V} \subseteq \mathscr{X}$ is denoted by $\mathscr{X}/\mathscr{V}$. The subspace of $\mathscr{X}$ orthogonal to $\mathscr{V}$ is denoted by $\mathscr{V}^\perp$. The dimension of a subspace $\mathscr{V}$ is denoted by dim $\mathscr{V}$. Matrices and associated linear maps between vector spaces are denoted by slanted capital letters, like A. The image and the kernel of A are denoted by Im A and Ker A, respectively. The transpose of A is denoted by $A^\top$. The restriction of a linear map A to an A-invariant subspace $\mathscr{V}$ is denoted by $A|_{\mathscr{V}}$. The symbols I_n and $0_{m\times n}$ are respectively used to denote the $n\times n$ identity matrix and the $m\times n$ zero matrix.

6.2 Preliminaries and Problem Statement

Let $\mathcal{S}$ be the set of all maps $\sigma:\mathbb{N}\to\mathbb{R}^+$ such that $\tau_\sigma>0$, where

$$\tau_\sigma = \inf\{\sigma(0), \sigma(i+1)-\sigma(i);\ i\in\mathbb{N}\}.$$

Hence, $\sigma\in\mathcal{S}$ implies that $\sigma(i+1)$ is greater than $\sigma(i)$ for all $i\in\mathbb{N}$ and that

$$\operatorname{Im}\sigma = \{t\in\mathbb{R}^+,\ t=\sigma(i) \text{ for some } i\in\mathbb{N}\}$$

is a discrete, countably infinite subset of $\mathbb{R}^+$, whose subsets (including $\operatorname{Im}\sigma$ itself) have no accumulation points. For any given $\sigma\in\mathcal{S}$, the positive real constant τ_σ is said to be the *dwell time* of σ. Moreover, given a positive real constant τ, the set of all $\sigma\in\mathcal{S}$ whose dwell time τ_σ is no smaller than τ is denoted by $\mathcal{S}_\tau$: i.e.,

$$\mathcal{S}_\tau = \{\sigma\in\mathcal{S},\ \tau_\sigma\geq\tau\}.$$

Let Σ_σ be a linear impulsive system described by

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) = A_C\,x(t) + B_C\,u(t) + D_C\,q(t), & \\ y(t) = C_C\,x(t), & \text{with } t\neq\sigma(i),\ i\in\mathbb{N}, \\ x(\sigma(i)) = A_J\,x^-(\sigma(i)) + B_J\,u(\sigma(i)) + D_J\,q(\sigma(i)), & \\ y(\sigma(i)) = C_J\,x(\sigma(i)), & \text{with } i\in\mathbb{N}, \end{cases} \tag{6.1}$$

where $t\in\mathbb{R}^+$ is the time variable; $\sigma\in\mathcal{S}$ is the map of the jump times; $x\in\mathscr{X}=\mathbb{R}^n$, $u\in\mathscr{U}=\mathbb{R}^m$, $q\in\mathscr{Q}=\mathbb{R}^s$ and $y\in\mathscr{Y}=\mathbb{R}^p$ are the state, the known input, the unknown input and the output, respectively; $x^-(\sigma(i))$ denotes the limit of $x(t)$ when t goes to $\sigma(i)$ from the left: i.e.,

$$x^-(\sigma(i)) = \lim_{t\to\sigma(i)^-} x(t).$$

A_C, B_C, D_C, C_C, A_J, B_J, D_J and C_J are real matrices of suitable dimensions. The matrices B_C, D_C, B_J and D_J are assumed to be full column rank, while C_C and C_J are assumed to be full row rank. The jumps are assumed to be instantaneously detectable or, in other words, the map of the jump times σ is assumed to be measurable.

According to the dynamics described by the first block of equations of (6.1), the state $x(t)$ of Σ_σ evolves continuously on the time interval $[0, \sigma(0))$, starting from initial condition $x(0) = x_0$ at time $t = 0$. Then, as specified in the second block of equations of (6.1), at time $t = \sigma(0)$, the state jumps to $A_J\, x^-(\sigma(0))$, instead of taking the value $x^-(\sigma(0))$. The same behaviour repeats on each subsequent time interval $[\sigma(i), \sigma(i+1))$, with initial condition $x(\sigma(i))$. The first block of equations of (6.1) describes the so-called *flow dynamics* of Σ_σ, while the second block of equations describes the *jump behaviour* of Σ_σ. Jumps occur at all points $\sigma(i) \in \operatorname{Im}\sigma$ and the dwell time τ_σ represents the lower bound of the distance between consecutive, distinct jump times.

It is worth noting that the time interval between two consecutive jumps is not assumed to be upper bounded. It is also worth pointing out that the input and output matrices of the flow dynamics are assumed to be different from the corresponding ones in the jump behaviour, in general. However, if some matrices describing the continuous-time behaviour are equal to their homologous for the jump behaviour, the results that will be derived in this work apply straightforwardly.

Concerning the unknown-input state observation problem, it is worth just mentioning that an asymptotic estimate of the whole state $x(t)$ of Σ_σ, based on the knowledge of the input $u(t)$, of the output $y(t)$ and of the map $\sigma(i)$, is precluded by the presence of the unknown input $q(t)$. Nevertheless, given a linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, the problem of finding an asymptotic estimate of the linear function $L\, x(t)$ of the state of Σ_σ may have a solution. Henceforth, conditions for the existence of solutions to this latter problem are investigated assuming as the candidate observer a linear impulsive system subject to the same jump time sequence as that of the given system Σ_σ, fed by the known input $u(t)$ and by the output $y(t)$ of Σ_σ and providing the desired estimate of $L\, x(t)$ as output.

Hence, let the candidate observer, from now on denoted by $\Sigma_{O\sigma}$, be described by

$$\Sigma_{O\sigma} \equiv \begin{cases} \dot z(t) = A_{Co}\, z(t) + B_{Co}\, u(t) - G_{Co}\, y(t), & \\ w(t) = H_{Co}\, z(t) + K_{Co}\, y(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ z(\sigma(i)) = A_{Jo}\, z^-(\sigma(i)) + B_{Jo}\, u(\sigma(i)) - G_{Jo}\, y^-(\sigma(i)), & \\ w(\sigma(i)) = H_{Jo}\, z(\sigma(i)) + K_{Jo}\, y(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases} \tag{6.2}$$

where $z \in \mathscr{Z} = \mathbb{R}^r$, $w \in \mathscr{W} = \mathbb{R}^l$ and A_{Co}, B_{Co}, G_{Co}, H_{Co}, K_{Co}, A_{Jo}, B_{Jo}, G_{Jo}, H_{Jo}, K_{Jo} are real matrices of suitable dimensions.

Consequently, the estimation error is defined by

$$e(t) = w(t) - L\, x(t), \quad \text{with } t \in \mathbb{R}^+. \tag{6.3}$$

The following statement highlights the features that a candidate observer like $\Sigma_{O\sigma}$ should exhibit in order to be defined an asymptotic observer, in the context of linear impulsive systems whose dwell time is lower bounded away from zero.

Definition 6.1 Given the linear impulsive system Σ_σ, the positive real constant τ and the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, the linear impulsive system $\Sigma_{O\sigma}$ is said to be an *asymptotic observer for* $L\,x(t)$ *over* $\mathcal{S}_\tau$ if

(i) $\Sigma_{O\sigma}$ is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$;
(ii) $\lim_{t\to\infty} e(t) = 0$ for all the initial states $x(0) = x_0$ of Σ_σ and $z(0) = z_0$ of $\Sigma_{O\sigma}$ and the time evolution of $e(t)$ is independent of the input signals $u(\cdot)$ and $q(\cdot)$ of Σ_σ for all $\sigma \in \mathcal{S}_\tau$;
(iii) there exists a linear map $P : \mathscr{X} \to \mathscr{Z}$, with $\mathrm{Im}\,P = \mathscr{Z}$, such that $e(t) = 0$ for all $t \in \mathbb{R}^+$, for all the input signals $u(\cdot)$ and $q(\cdot)$ of Σ_σ, and for all $\sigma \in \mathcal{S}_\tau$, provided that, given the initial state $x(0) = x_0$ of Σ_σ, the initial state of $\Sigma_{O\sigma}$ is set to $z(0) = P\,x_0$ (any such P is said to be an *exact initializing linear map*).

With reference to Definition 6.1 it is worth explicitly pointing out the following aspects.

Remark 6.1 In particular, the state evolution and the output of the observer $\Sigma_{O\sigma}$ depend on the specific map σ, which is assumed to be measurable. Nevertheless, Conditions (i) to (iii) of Definition 6.1 are expressed *over* $\mathcal{S}_\tau$: i.e., they are required to hold *for all* the maps σ whose dwell time is no smaller than the given τ.

Remark 6.2 Condition (i) in Definition 6.1 (i.e., global asymptotic stability of $\Sigma_{O\sigma}$ over $\mathcal{S}_\tau$) is required to make the observer usable in practice. Instead, Condition (iii) (i.e., the existence of an exact initializing linear map) is useful to detect the action of the unknown input when the initial state of the given system is known.

In a sligthly less general case, sufficient conditions for the existence of an asymptotic observer whose state space has the same dimension as that of the given system were proven in [11], where a synthesis procedure was also presented. In the following sections, observers whose state space has smaller dimension will be considered, so as to relax the existing condition and reduce the implementation complexity. More precisely, the focus will be on the problem of finding an asymptotic observer whose state space has the minimal possible dimension. This latter problem can be formally stated as follows.

Problem 6.1 Let the linear impulsive system Σ_σ, the positive real constant τ and the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, be given. The *Minimal Order Unknown Input Observer (MOUIO) Problem* consists in finding an asymptotic observer $\Sigma_{O\sigma}$ for $L\,x(t)$ over $\mathcal{S}_\tau$ such that the dimension r of its state space is minimal.

In light of the problem statement above, it is interesting to highlight the features of this work with respect to [26], which is one of the contributions more similar to this one in the available literature, not only for the topic dealt with but also for the methodology adopted. In [26], the author introduces an observer which differs from the observer $\Sigma_{O\sigma}$ described by (6.2) for the presence of an additional term in the jump equation. Such term represents an estimate of $x(\sigma(i))_{\mathrm{mod}\,\mathscr{Q}}$, where $\mathscr{Q} \subseteq \mathscr{X}$ is a suitably defined subspace, and its value is available by virtue of the exact initialization of the observer state to $z(t_0) = x(t_0)_{\mathrm{mod}\,\mathscr{Q}}$ at some $t_0 \in \mathbb{R}^+$. Indeed, the problem considered by Lawrence [26], in his own words, is that of *maintaining the state estimate*, modulo a specific subspace, in the presence of the unknown input. Instead, the problem considered herein consists in *providing an asymptotic state estimate*, modulo a specific subspace, in the presence of the unknown input, for any initialization of both the system and the observer. In other words, the approach developed in this work prescinds from any exact initialization of the observer state, since this information is not assumed to be available. For this reason, the observer structure proposed in [26] is not a viable option herein, while, under appropriate hypotheses, the observer $\Sigma_{O\sigma}$ described by (6.2) provides an estimate of the linear function of the state which converges to $L\,x(t)$, in the presence of the disturbance, for any initial value of $x(0)$ and $z(0)$.

6.3 A Structural Approach to the MOUIO Problem

In this section, the structural notions and properties functional to the solution of the MOUIO problem for linear impulsive systems are presented.

6.3.1 Hybrid Conditioned Invariance

The most relevant structural notion in the formulation of the solvability conditions for the MOUIO problem is that of hybrid conditioned invariance. Nevertheless, the basic concept of hybrid invariance must be set in order to put the key notion in context. To avoid lengthy phrasing, the symbol $\mathscr{H}$ replaces the word *hybrid* from now on. Moreover, for immediate reference, the structural notions of $\mathscr{H}$-invariance and $\mathscr{H}$-conditioned invariance are introduced herein with respect to the linear impulsive system Σ_σ defined in (6.1).

A subspace $\mathscr{S} \subseteq \mathscr{X}$ is said to be an $\mathscr{H}$-*invariant subspace* for Σ_σ if $A_C\,\mathscr{S} \subseteq \mathscr{S}$ and $A_J\,\mathscr{S} \subseteq \mathscr{S}$. From the dynamic point of view, $\mathscr{H}$-invariance is characterized by the fact that the trajectory of any free motion which originates inside an $\mathscr{H}$-invariant subspace $\mathscr{S}$ is contained in $\mathscr{S}$.

A subspace $\mathscr{S} \subseteq \mathscr{X}$ is said to be an $\mathscr{H}$-conditioned *invariant subspace* for Σ_σ if

$$A_C\,(\mathscr{S} \cap \operatorname{Ker} C_C) \subseteq \mathscr{S}, \tag{6.4}$$

$$A_J\,(\mathscr{S} \cap \operatorname{Ker} C_J) \subseteq \mathscr{S}. \tag{6.5}$$

As is known [3, Theorem 4.1-3], the inclusions (6.4), (6.5) are respectively equivalent to the existence of linear maps $G_C,\ G_J : \mathscr{Y} \to \mathscr{X}$ such that

$$(A_C + G_C\, C_C)\, \mathscr{S} \subseteq \mathscr{S}, \tag{6.6}$$

$$(A_J + G_J\, C_J)\, \mathscr{S} \subseteq \mathscr{S}. \tag{6.7}$$

Hence, in light of the two equivalences mentioned above, a subspace $\mathscr{S} \subseteq \mathscr{X}$ is an $\mathscr{H}$-conditioned invariant subspace for Σ_σ *if and only if* there exists a pair (G_C, G_J) that satisfies (6.6), (6.7). Any such pair is called *a friend of* $\mathscr{S}$.

It is worth noting that the characterization discussed above relates the $\mathscr{H}$-conditioned invariance of a subspace $\mathscr{S}$ with respect to Σ_σ to its $\mathscr{H}$-invariance with respect to a linear impulsive system derived from Σ_σ by respectively modifying its flow and jump dynamics through the output injections of a pair enjoying the property of being a friend of $\mathscr{S}$.

The notion of conditioned invariance has been previously considered in the context of linear impulsive systems and in relation to the unknown-input observation problem by several authors [26, 32, 34]. In [32, Property 3.5], the conditioned invariance of a subspace with respect to a linear impulsive system is characterized through its invariance with respect to the flow dynamics and its conditioned invariance with respect to the jump behaviour, according to the fact that, in the linear impulsive systems given therein, the measurements are only available at the jump times. Instead, in [34, Definition 3], the conditioned invariance of a subspace with respect to a linear impulsive system is characterized through its conditioned invariance with respect to both the flow dynamics and the jump behaviour, consistently with the fact that [34] is focused on the more general circumstance where the measurements are available both in the continuous-time intervals and at the jump times (although the output-distribution matrix is not same). The same assumption on measurement availability both in the continuous-time intervals and at the jump times is also made in [26]. However, in [26, Definition 3.1], the conditioned invariance of a subspace with respect to a linear impulsive system is defined in a peculiar fashion which involves the structure postulated for the linear impulsive observer. In that context, the conditioned invariance of a subspace with respect to both the flow and the jump dynamics turns out to be a sufficient-only condition for the conditioned invariance with respect to the overall linear impulsive system. As is clear from the previous discussion, the present work shares the setting and the definition of [34], even if the objective is different (in fact, in [34] the focus is on fault detection and isolation).

As can be shown by simple arguments of linear algebra, given a subspace $\mathscr{E} \subseteq \mathscr{X}$, the set of all $\mathscr{H}$-conditioned invariant subspaces containing $\mathscr{E}$ is closed with respect to the subspace intersection. Hence, there exists a minimal $\mathscr{H}$-conditioned invariant subspace containing $\mathscr{E}$, henceforth denoted by $\mathscr{S}^*_{\mathscr{H}}(\mathscr{E})$.

Moreover, by an appropriate generalization of the proof of [3, Algorithm 4.1-1], it can be shown that the sequence of subspaces $\{\mathscr{S}_k,\ k \in \mathbb{N}\}$, defined by

$$\begin{cases} \mathscr{S}_0 = \mathscr{E}, \\ \mathscr{S}_k = \mathscr{E} + A_C\,(\mathscr{S}_{k-1} \cap \operatorname{Ker} C_C) + A_J\,(\mathscr{S}_{k-1} \cap \operatorname{Ker} C_J), \quad k \in \mathbb{N}, \end{cases} \tag{6.8}$$

is a non-decreasing nested sequence of subspaces of $\mathscr{X}$ converging to $\mathscr{S}^*_{\mathscr{H}}(\mathscr{E})$ in a finite number of steps.

6.3.2 Synthesis of the Observer

As mentioned in the previous section, $\mathscr{H}$-conditioned invariant subspaces and their friends are strictly related to the synthesis of unknown-input observers for linear impulsive systems. To illustrate this fact, the relationship between an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ and a friend (G_C, G_J) is first characterized through the following construction.

Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ. First, note that the inclusion (6.4) is equivalent to $A_C^\top\, \mathscr{S}^\perp \subseteq \mathscr{S}^\perp + \operatorname{Im} C_C^\top$. Let $P^\top$ be a matrix whose columns form a basis of $\mathscr{S}^\perp$, so that $\mathscr{S} = \operatorname{Ker} P$ and $P : \mathscr{X} \to \mathscr{X}/\mathscr{S}$ is the projection from $\mathscr{X}$ onto $\mathscr{X}/\mathscr{S}$. Then, there exists a pair of matrices of suitable dimensions, say $(\bar{A}_C, M_C)$, such that $A_C^\top\, P^\top = P^\top\, \bar{A}_C^\top + C_C^\top\, M_C^\top$. By transposing the latter equation and taking a matrix G_C such that $M_C = -\,P\, G_C$, one gets

$$P\,(A_C + G_C\, C_C) = \bar{A}_C\, P. \tag{6.9}$$

Likewise, (6.5) is equivalent to the existence of a pair of matrices $(\bar{A}_J, M_J)$ such that $A_J^\top\, P^\top = P^\top\, \bar{A}_J^\top + C_J^\top\, M_J^\top$. By transposing the latter equation and picking a matrix G_J such that $M_J = -\,P\, G_J$, one gets

$$P\,(A_J + G_J\, C_J) = \bar{A}_J\, P. \tag{6.10}$$

In light of $\mathscr{S} = \operatorname{Ker} P$, (6.9) and (6.10) respectively imply

$$P\,(A_C + G_C\, C_C)\,\mathscr{S} = \bar{A}_C\, P\,\mathscr{S} = \{0\}, \tag{6.11}$$

$$P\,(A_J + G_J\, C_J)\,\mathscr{S} = \bar{A}_J\, P\,\mathscr{S} = \{0\}. \tag{6.12}$$

In turn, (6.11), (6.12) are respectively equivalent to (6.6), (6.7) and the latter show that the pair (G_C, G_J) obtained as shown above is a friend of $\mathscr{S}$.

Conversely, let (G_C, G_J) be a friend of an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ and let $P^\top$ be a matrix whose columns form a basis of $\mathscr{S}^\perp$, the following arguments show how to derive two matrices $\bar{A}_C$ and $\bar{A}_J$ which respectively satisfy (6.9) and (6.10). Let $T = [P^\top\;\; S]^\top$, where S denotes a matrix whose columns form a basis of $\mathscr{S}$. Then, (6.9) can be written as $\bar{A}_C\,[I_r\;\; 0_{r\times(n-r)}]\,T = P\,(A_C + G_C\, C_C)$, where $r = \dim\, \mathscr{S}^\perp$. Therefore, $\bar{A}_C$ can be expressed as

$$\bar{A}_C = P\,(A_C + G_C\, C_C)\, T^{-1} \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix}. \tag{6.13}$$

Similarly, from (6.10) it ensues that

$$\bar{A}_J = P\,(A_J + G_J\,C_J)\,T^{-1}\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}. \tag{6.14}$$

Hence, it will be shown how, under appropriate assumptions, the matrices $\bar{A}_C$ and $\bar{A}_J$ introduced above can be used to synthesize a candidate asymptotic observer for a linear function of the state of the linear impulsive system Σ_σ. Let the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \le n$, and the positive real constant τ be given. Let $\mathscr{J} \subseteq \mathscr{X}$ denote the subspace of all the states x of Σ_σ which can be expressed as $x = x(\sigma(0))$, for some initial state $x(0) \in \mathscr{X}$ and some input signals $u(\cdot)$ and $q(\cdot)$. If the $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ satisfies the conditions

$$\mathscr{S} \cap \operatorname{Ker} C_C \subseteq \operatorname{Ker} L, \tag{6.15}$$

$$\mathscr{S} \cap \operatorname{Ker} C_J \cap \mathscr{J} \subseteq \operatorname{Ker} L, \tag{6.16}$$

a candidate asymptotic observer $\Sigma_{O\sigma}$ for $L\,x(t)$ over $\mathcal{S}_\tau$ can be obtained as follows.

First, note that (6.15) is equivalent to $\mathscr{S}^\perp + (\operatorname{Ker} C_C)^\perp \supseteq (\operatorname{Ker} L)^\perp$, which can also be written as $\operatorname{Im} P^\top + \operatorname{Im} C_C^\top \supseteq \operatorname{Im} L^\top$. Then, by the latter inclusion, there exist two matrices, H_C and K_C, such that $L^\top = P^\top H_C^\top + C_C^\top K_C^\top$ or, equivalently,

$$L = H_C\,P + K_C\,C_C. \tag{6.17}$$

Similarly, (6.16) implies the existence of two matrices, H_J and K_J, such that

$$L\,x = H_J\,Px + K_J\,C_J\,x, \tag{6.18}$$

for all $x \in \mathscr{J}$ (it is worth noting that $L = H_J\,P + K_J\,C_J$ does not necessarily hold). Then, let the matrices of the candidate observer $\Sigma_{O\sigma}$, described by (6.2), be defined by

$$A_{Co} = \bar{A}_C,\; B_{Co} = P\,B_C,\; G_{Co} = P\,G_C, \tag{6.19a}$$

$$H_{Co} = H_C,\; K_{Co} = K_C, \tag{6.19b}$$

$$A_{Jo} = \bar{A}_J,\; B_{Jo} = P\,B_J,\; G_{Jo} = P\,G_J, \tag{6.19c}$$

$$H_{Jo} = H_J,\; K_{Jo} = K_J, \tag{6.19d}$$

so that

$$\Sigma_{O\sigma} \equiv \begin{cases} \dot{z}(t) = \bar{A}_C\,z(t) + P\,B_C\,u(t) - P\,G_C\,y(t), & \\ w(t) = H_C\,z(t) + K_C\,y(t), & \text{with } t \neq \sigma(i),\; i \in \mathbb{N}, \\ z(\sigma(i)) = \bar{A}_J\,z^-(\sigma(i)) + P\,B_J\,u(\sigma(i)) - P\,G_J\,y^-(\sigma(i)), & \\ w(\sigma(i)) = H_J\,z(\sigma(i)) + K_J\,y(\sigma(i)), & \text{with } i \in \mathbb{N}. \end{cases} \tag{6.20}$$

It is worthwhile noting that the matrices $\bar{A}_C$ and $\bar{A}_J$ respectively define the flow component and the jump component of the dynamics of the observer and, therefore, one can say that the dynamics of the observer is *induced by the choice of $\mathscr{S}$ and its friend* (G_C, G_J).

To investigate the behaviour of the estimation error $e(t)$, generated by $\Sigma_{O\sigma}$ according to (6.3), it is convenient to define the auxiliary variable $e_{aux}(t)$ as

$$e_{aux}(t) = z(t) - P\,x(t). \tag{6.21}$$

Thus, $e(t)$ can be regarded as the output of the system $\Sigma_{E\sigma}$ defined by

$$\Sigma_{E\sigma} \equiv \begin{cases} \dot{e}_{aux}(t) = \bar{A}_C\, e_{aux}(t) - P\,D_C\,q(t), & \\ e(t) = H_C\,e_{aux}(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ e_{aux}(\sigma(i)) = \bar{A}_J\, e^-_{aux}(\sigma(i)) - P\,D_J\,q(\sigma(i)), & \\ e(\sigma(i)) = H_J\,e_{aux}(\sigma(i)), & \text{with } i \in \mathbb{N}. \end{cases} \tag{6.22}$$

In fact, for the flow dynamics of $\Sigma_{E\sigma}$, one gets

$$\begin{aligned} \dot{e}_{aux}(t) &= \dot{z}(t) - P\,\dot{x}(t) \\ &= \bar{A}_C\,z(t) + P\,B_C\,u(t) - P\,G_C\,y(t) - P\,A_C\,x(t) - P\,B_C\,u(t) - P\,D_C\,q(t) \\ &= \bar{A}_C\,z(t) - P\,(A_C + G_C\,C_C)\,x(t) - P\,D_C\,q(t) \\ &= \bar{A}_C\,z(t) - \bar{A}_C\,P\,x(t) - P\,D_C\,q(t) \\ &= \bar{A}_C\,e_{aux}(t) - P\,D_C\,q(t), \end{aligned}$$

where (6.21), (6.20), (6.1), and (6.9) have been taken into account. Similarly, for the jump behaviour, one has

$$\begin{aligned} e_{aux}(\sigma(i)) &= z(\sigma(i)) - P\,x(\sigma(i)) \\ &= \bar{A}_J\,z^-(\sigma(i)) + P\,B_J\,u(\sigma(i)) - P\,G_J\,y^-(\sigma(i)) - P\,A_J\,x^-(\sigma(i)) \\ &\quad - P\,B_J\,u(\sigma(i)) - P\,D_J\,q(\sigma(i)) \\ &= \bar{A}_J\,z^-(\sigma(i)) - P\,(A_J + G_J\,C_J)\,x^-(\sigma(i)) - P\,D_J\,q(\sigma(i)) \\ &= \bar{A}_J\,z^-(\sigma(i)) - \bar{A}_J\,P\,x^-(\sigma(i)) - P\,D_J\,q(\sigma(i)) \\ &= \bar{A}_J\,e^-_{aux}(\sigma(i)) - P\,D_J\,q(\sigma(i)), \end{aligned}$$

where (6.21), (6.20), (6.1), and (6.10) have been considered. Moreover, for the output equation in the continuous-time intervals, one gets

$$\begin{aligned} e(t) &= w(t) - L\,x(t) \\ &= H_C\,z(t) + K_C\,y(t) - L\,x(t) \\ &= H_C\,z(t) + (K_C\,C_C - L)\,x(t) \\ &= H_C\,(z(t) - P\,x(t)) \\ &= H_C\,e_{aux}(t), \end{aligned}$$

where (6.3), (6.20), (6.1), (6.17) and (6.21) have been considered.

Furthermore, for the output equation at the jump times, one obtains

$$\begin{aligned} e(\sigma(i)) &= w(\sigma(i)) - L\,x(\sigma(i)) \\ &= H_J\, z(\sigma(i)) + K_J\, y(\sigma(i)) - L\,x(\sigma(i)) \\ &= H_J\, z(\sigma(i)) + (K_J\, C_J - L)\,x(\sigma(i)) \\ &= H_J\,(z(\sigma(i)) - P\,x(\sigma(i))) \\ &= H_J\, e_{aux}(\sigma(i)), \end{aligned}$$

as a consequence of (6.3), (6.20), (6.1), (6.18) and (6.21).

The linear impulsive system $\Sigma_{E\sigma}$, defined by (6.22), shows two important facts. The first is that the estimation error $e(t)$ depends on the initialization error $e_{aux}(0)$ on the state of the observer with respect to the projection of the state of the given system. The second is that the effect of the unknown input $q(t)$ on the evolution of the estimation error $e(t)$ is conditioned by the matrix P.

In light of the previous arguments, it ensues that the observer $\Sigma_{O\sigma}$ in (6.20) meets Definition 6.1 if the considered $\mathscr{H}$-conditioned invariant susbspace $\mathscr{S}$ satisfies the inclusion

$$\mathscr{D} \subseteq \mathscr{S}, \tag{6.23}$$

where

$$\mathscr{D} = \mathrm{Im}\, D_C + \mathrm{Im}\, D_J,$$

in addition to (6.15) and (6.16), and, furthermore, it is such that, for some friend (G_C, G_J), the matrices $\bar{A}_C$ and $\bar{A}_J$ which satisfy (6.9) and (6.10) define an asymptotically stable linear impulsive dynamics for all $\sigma \in \mathcal{S}_\tau$.

The conditions expressed by (6.15), (6.16) and (6.23) are structural conditions and they are easy to check. The qualitative condition about asymptotic stability is more difficult to ascertain and its analysis requires the parametrization of the set of matrices $\bar{A}_C$ and $\bar{A}_J$ which respectively satisfy (6.9) and (6.10) for some friend (G_C, G_J) and which are used to construct $\Sigma_{O\sigma}$ as well as to describe the dynamics of the auxiliary variable $e_{aux}(t)$. This will be done in the following subsection.

6.3.3 Parametrization of the Induced Dynamics

Concerning the relationships between $\mathscr{H}$-conditioned invariant subspaces, friends and induced dynamics, the following results are established.

Proposition 6.1 *Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ and let $P^\top$ be a matrix whose columns form a basis of $\mathscr{S}^\perp$. Let $\bar{A}_C$, G_C, $\bar{A}_J$, G_J be matrices which respectively satisfy (6.9) and (6.10). Then, the matrices $\bar{A}'_C$, G'_C, $\bar{A}'_J$, G'_J respectively satisfy the homologous relations*

$$P\,(A_C + G'_C\,C_C) = \bar{A}'_C\,P \tag{6.24}$$

$$P\,(A_J + G'_J\,C_J) = \bar{A}'_J\,P \tag{6.25}$$

if and only if (up to a change of basis in $\mathscr{X}/\mathscr{S}$*)*

$$\bar{A}'_C = \bar{A}_C + Q_C\,N_1^\top, \tag{6.26a}$$

$$P\,G'_C = P\,G_C - Q_C\,N_2^\top, \tag{6.26b}$$

and

$$\bar{A}'_J = \bar{A}_J + Q_J\,M_1^\top, \tag{6.27a}$$

$$P\,G'_J = P\,G_J - Q_J\,M_2^\top, \tag{6.27b}$$

where $\left[N_1^\top\ N_2^\top\right]^\top$ *and* $\left[M_1^\top\ M_2^\top\right]^\top$ *are matrices whose columns form respective bases of* $\operatorname{Ker}\,[P^\top\ C_C^\top]$ *and* $\operatorname{Ker}\,[P^\top\ C_J^\top]$*, while* Q_C *and* Q_J *are arbitrary matrices of suitable dimensions.*

Proof The equivalence between (6.24) and (6.26) is considered first. Note that $[P^\top\ C_C^\top]\,[N_1^\top\ N_2^\top]^\top = P^\top\,N_1 + C_C^\top\,N_2 = 0$, by definition of the matrix $\left[N_1^\top\ N_2^\top\right]^\top$. Then, $P^\top\,N_1 = -\,C_C^\top\,N_2$ holds, which can also be written as

$$N_1^\top\,P = -\,N_2^\top\,C_C. \tag{6.28}$$

Hence, it can be shown that (6.26) implies (6.24). In fact, by exploiting (6.26a), (6.9), (6.28) and (6.26b), one gets

$$\begin{aligned}
\bar{A}'_C\,P &= (\bar{A}_C + Q_C\,N_1^\top)\,P \\
&= \bar{A}_C\,P + Q_C\,N_1^\top\,P \\
&= P\,(A_C + G_C\,C_C) - Q_C\,N_2^\top C_C \\
&= P\,A_C + P\,G_C\,C_C - Q_C\,N_2^\top C_C \\
&= P\,A_C + (P\,G_C - Q_C\,N_2^\top)\,C_C \\
&= P\,A_C + P\,G'_C\,C_C \\
&= P\,(A_C + P\,G'_C\,C_C).
\end{aligned}$$

The reverse implication can be shown as follows. From (6.9) and (6.24), it ensues that $(\bar{A}'_C - \bar{A}_C)\,P = P\,(G'_C - G_C)\,C_C$, which can also be written as $\left[\bar{A}'_C - \bar{A}_C - P\,(G'_C - G_C)\right]\left[P^\top\ C_C^\top\right]^\top = 0$. By transposing both members of the latter, one gets

$$\left[P^\top\ C_C^\top\right]\begin{bmatrix} (\bar{A}'_C - \bar{A}_C)^\top \\ -\,(G'_C - G_C)^\top\,P^\top \end{bmatrix} = 0,$$

which shows that there exists a matrix Q_C such that

$$\begin{bmatrix} (\bar{A}'_C - \bar{A}_C)^\top \\ -(G'_C - G_C)^\top P^\top \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} Q_C^\top.$$

Again, by transposing both members, the latter becomes $\left[\bar{A}'_C - \bar{A}_C \quad -P\,(G'_C - G_C)\right] = Q_C \left[N_1^\top \; N_2^\top\right]$, which, in turn, is equivalent to (6.26). The equivalence between (6.25) and (6.27) can be shown likewise. □

Proposition 6.2 *Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ and let $P^\top$ be a matrix whose columns form a basis of $\mathscr{S}^\perp$. Let (G_C, G_J) be a friend of $\mathscr{S}$. Then, the matrices $\bar{A}_C$ and $\bar{A}_J$ which respectively satisfy (6.9) and (6.10) are unique (up to a change of basis in $\mathscr{X}/\mathscr{S}$), independently of the choice of $P^\top$.*

Proof Let $P^\top$ and $P_1^\top$ be two matrices whose columns form respective bases of $\mathscr{S}^\perp$. Then, $P_1^\top = P^\top Q$ for some invertible matrix Q of suitable dimension. Morever, let $\bar{A}_C$ and $\bar{A}'_C$ be matrices that respectively satisfy (6.9) and the like of (6.9) with P replaced by P_1: i.e., $\bar{A}'_C\,P_1 = P_1\,(A_C + G_C\,C_C)$. From the latter equality, by replacing P_1 with $Q^\top P$, pre-multiplying by the inverse of $Q^\top$ and taking (6.9) into account, one gets $(Q^\top)^{-1}\bar{A}'_C\,Q^\top P = \bar{A}_C\,P$. Since P is full row rank and $Q^\top$, as a square and nonsingular matrix, can be regarded as a change of basis in $\mathscr{X}/\mathscr{S}$, the latter equation proves the statement in relation to $\bar{A}_C$. Concerning $\bar{A}_J$, the proof can be obtained by similar arguments. □

Proposition 6.3 *Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ and let $P^\top$ be a matrix whose columns form a basis of $\mathscr{S}^\perp$. Let (G_C, G_J) be a friend of $\mathscr{S}$. Let the following conditions hold:*

$$\mathscr{S} + \operatorname{Ker} C_C = \mathscr{X}, \tag{6.29}$$

$$\mathscr{S} + \operatorname{Ker} C_J = \mathscr{X}. \tag{6.30}$$

Then, the matrices $\bar{A}_C$ and $\bar{A}_J$ which respectively satisfy (6.9) and (6.10) are unique (up to a change of basis in $\mathscr{X}/\mathscr{S}$), independently of the choice of (G_C, G_J).

Proof The uniqueness of $\bar{A}_C$ is proven first. Considering the orthogonal subspaces, (6.29) is equivalent to $\operatorname{Im} P^\top \cap \operatorname{Im} C_C^\top = \{0\}$, which, in turn, is equivalent to $\operatorname{Ker}\,[P^\top \; C_C^\top] = \{0\}$. Hence, the latter equality proves the uniqueness of $\bar{A}_C$ by Proposition 6.1. The uniqueness of $\bar{A}_J$ can be shown by similar arguments. □

Remark 6.3 Proposition 6.3 points out that if (6.29) and (6.30) hold, the dynamics of the observer $\Sigma_{O\sigma}$ synthesized in Sect. 6.3.2 by means of the $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ does not depend on the choice of the friend (G_C, G_J) of $\mathscr{S}$.

Remark 6.4 The matrices $\bar{A}_C$ and $\bar{A}_J$ that respectively satisfy (6.9) and (6.10) and that have been used for the synthesis of $\Sigma_{O\sigma}$ (and, consequently, for describing the dynamics of the auxiliary variable $e_{aux}(t)$) can also be obtained by a different method, as is shown below. Given an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ and a friend (G_C, G_J) of $\mathscr{S}$, the linear impulsive dynamics obtained from that of Σ_σ by applying the output injections G_C and G_J to the flow dynamics and the jump behaviour, respectively, is considered. Namely, the linear impulsive system

$$\Sigma_\sigma^G \equiv \begin{cases} \dot{\xi}(t) = (A_C + G_C\, C_C)\, \xi(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ \xi(\sigma(i)) = (A_J + G_J\, C_J)\, \xi^-(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases} \tag{6.31}$$

is considered, where $\xi \in \mathbb{R}^n = \mathscr{X}$. Let the change of basis $\xi = T\,\zeta$ with $T = [P^\top\ S]$, where $P^\top$ and S are matrices whose columns form respective bases of $\mathscr{S}^\perp$ and $\mathscr{S}$, be applied in the state space of Σ_σ^G. Then, with respect to the new coordinates, Σ_σ^G is described by

$$\Sigma_\sigma^G \equiv \begin{cases} \dot{\zeta}_1(t) = A_{C11}^G\, \zeta_1(t), & \\ \dot{\zeta}_2(t) = A_{C21}^G\, \zeta_1(t) + A_{C22}^G\, \zeta_2(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ \zeta_1(\sigma(i)) = A_{J11}^G\, \zeta_1^-(\sigma(i)), & \\ \zeta_2(\sigma(i)) = A_{J21}^G\, \zeta_1^-(\sigma(i)) + A_{J22}^G\, \zeta_2^-(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases} \tag{6.32}$$

where the partition $\zeta = [\zeta_1^\top\ \zeta_2^\top]^\top$ is consistent with that of T. Equation (6.32) show that the matrices A_{C11}^G and A_{J11}^G respectively describe the flow dynamics and the jump behaviour of the linear impulsive dynamics induced by Σ_σ^G on the quotient space $\mathscr{X}/\mathscr{S}$. Let $\Pi : \mathscr{X} \to \mathscr{X}/\mathscr{S}$ denote the canonical projection in the new coordinates: i.e., $\Pi\,\zeta = \Pi\,[\zeta_1^\top\ \zeta_2^\top]^\top = \zeta_1$. Then, $A_{C11}^G\,\Pi = \Pi\,T^{-1}(A_C + G_C\,C_C)\,T$. Since $\Pi\,T^{-1} = P$, by post-multiplying by T^{-1} both members of the previous equality, it ensues that $A_{C11}^G\,P = P\,(A_C + G_C\,C_C)$. The comparison of the latter equality with (6.9) shows that $A_{C11}^G\,P = \bar{A}_C\,P$ and, since P is full row rank, $A_{C11}^G = \bar{A}_C$. By similar arguments, one can also show that $A_{J11}^G = \bar{A}_J$.

In light of the previous reasonings, one can say that $\bar{A}_C$ and $\bar{A}_J$ define the linear impulsive *quotient dynamics* induced by that of Σ_σ^G on the quotient space $\mathscr{X}/\mathscr{S}$.

6.3.4 Stabilizability of Conditioned Invariant Subspaces

The aim of this section is to introduce the notions needed to investigate the existence of friends of a given $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$, such that the observer $\Sigma_{O\sigma}$ obtained with the procedure described in Sect. 6.3.2 exhibits a linear impulsive dynamics with the property of being globally asymptotically stable over a set of jump time sequences characterized by a minimum dwell time. The key notion is that of external stabilizability of an $\mathscr{H}$-conditioned invariant subspace. The complementary notion of internal stabilizability will also be defined, so as an in-depth comparison with the results of [11] can be made. Moreover, a necessary and sufficient condition for an $\mathscr{H}$-conditioned invariant subspace to be externally stabilizable will be established. Furthermore, a sufficient only, easy-to-check condition will be given.

Definition 6.2 Let the positive real constant τ be given. An $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ is said to be

- *internally stabilizable over* $\mathcal{S}_\tau$ if there exists a friend (G_C, G_J) of $\mathscr{S}$ such that the linear impulsive dynamics induced by Σ_σ^G on $\mathscr{S}$ is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$;
- *externally stabilizable over* $\mathcal{S}_\tau$ if there exists a friend (G_C, G_J) of $\mathscr{S}$ such that the linear impulsive dynamics induced by Σ_σ^G on the quotient space $\mathscr{X}/\mathscr{S}$ is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$.

In light of Remark 6.4, external stabilizability of $\mathscr{S}$ over $\mathcal{S}_\tau$ is equivalently characterized by the existence of a friend (G_C, G_J) of $\mathscr{S}$ such that the dynamics of the observer $\Sigma_{O\sigma}$ obtained as described in Sect. 6.3.2, with state space dimension $r = \dim \mathscr{S}^\perp$ and dynamic matrices $A_{Co} = \bar{A}_C$ and $A_{Jo} = \bar{A}_J$, where $\bar{A}_C$ and $\bar{A}_J$ are respectively given by (6.13) and (6.14), is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$.

Proposition 6.4 *Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ and let $P^\top$ and S be matrices whose columns form respective bases of $\mathscr{S}^\perp$ and $\mathscr{S}$. Let (G_C, G_J) be a friend of $\mathscr{S}$. Let $\bar{A}_C$ and $\bar{A}_J$ be defined as in (6.13) and (6.14). Then, $\mathscr{S}$ is externally stabilizable over $\mathcal{S}_\tau$ for a given positive real constant τ if and only if there exist two matrices Q_C and Q_J such that the linear impulsive dynamics*

$$\Sigma_\sigma^G|_{\mathscr{X}/\mathscr{S}} \equiv \begin{cases} \dot{\zeta}_1(t) = (\bar{A}_C + Q_C N_1^\top)\, \zeta_1(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ \zeta_1(\sigma(i)) = (\bar{A}_J + Q_J M_1^\top)\, \zeta_1^-(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases} \tag{6.33}$$

where $[N_1^\top\ N_2^\top]^\top$ and $[M_1^\top\ M_2^\top]^\top$ are matrices whose columns form respective bases of $\operatorname{Ker}[P^\top\ C_C^\top]$ *and* $\operatorname{Ker}[P^\top\ C_J^\top]$*, is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$.*

Proof It directly follows from Proposition 6.1. □

Proposition 6.5 *Under the same hypotheses of Proposition 6.4 and with the same notations, $\mathscr{S}$ is externally stabilizable over $\mathcal{S}_\tau$, for a given positive real constant τ, if $(\bar{A}_C, N_1^\top)$ is an observable pair.*

Proof By [46, Proposition 3], on the assumption of observability of the pair $(\bar{A}_C, N_1^\top)$, it is possible to find a matrix Q_C such that the linear impulsive system $\Sigma_\sigma^G|_{\mathscr{X}/\mathscr{S}}$, defined by (6.33) with any given Q_J, is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$. □

6.3.5 Hybrid Controlled Invariance

The aim of this section is to introduce the notion of hybrid controlled invariance, which, like that of hybrid conditioned invariance, is needed to state the solvability conditions of the MOUIO problem. It is worth premising that in the definition of hybrid controlled invariance that will be given herein, only the unknown-input distribution matrices, D_C and D_J, of the linear impulsive system Σ_σ are considered. In fact, since the problem dealt with is a state observation problem, the effect on the estimation error of the known input, which is applied to the given linear impulsive system Σ_σ through the matrices B_C and B_J, can be annihilated by simply applying the same input to the observer $\Sigma_{O\sigma}$ through matrices suitably related to the same B_C and B_J—see (6.20) and (6.22). In brief, since the known input can be compensated straightforwardly, there is no need to consider its distribution matrices in the definition of the hybrid controlled invariant subspaces functional to the solution of the observer problem.

That being said, the main definitions and properties concerning the notion of hybrid controlled invariance in the context of the unknown-input state observation are stated as follows.

A subspace $\mathscr{V} \subseteq \mathscr{X}$ is said to be an *$\mathscr{H}$-controlled invariant subspace* for Σ_σ if

$$A_C\,\mathscr{V} \subseteq \mathscr{V} + \operatorname{Im} D_C, \tag{6.34}$$

$$A_J\,\mathscr{V} \subseteq \mathscr{V} + \operatorname{Im} D_J. \tag{6.35}$$

As is known [3, Theorem 4.1-2], the inclusions (6.34), (6.35) are respectively equivalent to the existence of linear maps $F_C, F_J : \mathscr{Q} \to \mathscr{X}$ such that

$$(A_C + D_C\,F_C)\,\mathscr{V} \subseteq \mathscr{V}, \tag{6.36}$$

$$(A_J + D_J\,F_J)\,\mathscr{V} \subseteq \mathscr{V}. \tag{6.37}$$

Hence, a subspace $\mathscr{V} \subseteq \mathscr{X}$ is an $\mathscr{H}$-controlled invariant subspace for Σ_σ *if and only if* there exists a pair (F_C, F_J) which satisfies (6.36), (6.37). Any such pair is called *a friend of* $\mathscr{V}$.

As can be proven by basic arguments of linear algebra, given a subspace $\mathscr{W} \subseteq \mathscr{X}$, the set of all $\mathscr{H}$-controlled invariant subspaces contained in $\mathscr{W}$ is closed with respect to the sum of subspaces. Therefore, there exists a maximum $\mathscr{H}$-controlled invariant subspace contained in $\mathscr{W}$, which will be denoted by $\mathscr{V}^*_{\mathscr{H}}(\mathscr{W})$.

Moreover, by properly extending the proof of [3, Algorithm 4.1-2], one can show that the sequence of subspaces $\{\mathscr{V}_k,\ k \in \mathbb{N}\}$, defined by

$$\begin{cases} \mathscr{V}_0 = \mathscr{W}, \\ \mathscr{V}_k = \mathscr{W} \cap A_C^{-1}\,(\mathscr{V}_{k-1} + \operatorname{Im} D_C) \cap A_J^{-1}\,(\mathscr{V}_{k-1} \cap \operatorname{Im} D_J), \quad k \in \mathbb{N}, \end{cases} \tag{6.38}$$

is a non-increasing nested sequence of subspaces of $\mathscr{X}$ converging to $\mathscr{V}^*_{\mathscr{H}}(\mathscr{W})$ in a finite number of steps.

Remark 6.5 The notions of $\mathscr{H}$-conditioned invariance and $\mathscr{H}$-controlled invariance are dual to each other. In fact, a subspace $\mathscr{S} \subseteq \mathscr{X}$ satisfies (6.4) and (6.5) or, equivalently, $\mathscr{S} \cap \operatorname{Ker} C_C \subseteq A_C^{-1}\mathscr{S}$ and $\mathscr{S} \cap \operatorname{Ker} C_J \subseteq A_J^{-1}\mathscr{S}$ if and only if $A_C^\top \mathscr{S}^\perp \subseteq \mathscr{S}^\perp + \operatorname{Im} C_C^\top$ and $A_J^\top \mathscr{S}^\perp \subseteq \mathscr{S}^\perp + \operatorname{Im} C_J^\top$. The latter inclusions show that the subspace $\mathscr{V} = \mathscr{S}^\perp$ is an $\mathscr{H}$-controlled invariant subspace for the linear impulsive system $\Sigma_\sigma^\top$ given by

$$\Sigma_\sigma^\top \equiv \begin{cases} \dot{x}(t) = A_C^\top x(t) + C_C^\top y(t), & \\ q(t) = D_C^\top x(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ x(\sigma(i)) = A_J^\top x^-(\sigma(i)) + C_J^\top y(\sigma(i)), & \\ q(\sigma(i)) = D_J^\top x(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases}$$

and, apart from the action of the known input, $\Sigma_\sigma^\top$ is the dual counterpart of Σ_σ given by (6.1).

6.3.6 *The Maximal Hybrid Conditioned Invariant Subspace*

The aim of this section is to show that, under certain assumptions that will be specified, the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing a given subspace exhibits some special properties, instrumental in synthesizing the minimal order unknown-input observer.

Henceforth, $\mathscr{D}$ and $\mathscr{C}$ respectively denote the sum and the intersection of subspaces defined by

$$\mathscr{D} = \operatorname{Im} D_C + \operatorname{Im} D_J, \tag{6.39}$$

$$\mathscr{C} = \operatorname{Ker} C_C \cap \operatorname{Ker} C_J. \tag{6.40}$$

With this notation, it can be stated that, if the condition

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) = \mathscr{X} \tag{6.41}$$

holds, the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing $\mathscr{D}$ enjoys the properties presented below.

Proposition 6.6 *Let (6.41) hold. Then, any friend (G_C, G_J) of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ is also a friend of any other $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ containing $\mathscr{D}$. More generally, any friend (G_C, G_J) of an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}_1$ such that $\mathscr{D} \subseteq \mathscr{S}_1 \subseteq \mathscr{S}$ is also a friend of $\mathscr{S}$.*

Proof Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ containing the subspace $\mathscr{D}$ and let s be a vector of $\mathscr{S}$. By virtue of (6.41), s can be written as $s = \bar{s} + k$, for some $\bar{s} \in \mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and some $k \in \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})$. Since the minimality of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ as an $\mathscr{H}$-conditioned invariant subspace for Σ_σ containing $\mathscr{D}$ implies $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \subseteq \mathscr{S}$ and since the definition of $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C})$ implies $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) \subseteq \mathscr{C}$, it ensues that $k = s - \bar{s}$ belongs to $\mathscr{S} \cap \mathscr{C}$. Let (G_C, G_J) be a friend of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$. Then,

$$(A_C + G_C\, C_C)\, s = (A_C + G_C\, C_C)\, (\bar{s} + k) = (A_C + G_C\, C_C)\, \bar{s} + A_C\, k = s_1 + s_2, \tag{6.42}$$

with $s_1 = (A_C + G_C\, C_C)\, \bar{s}$ and $s_2 = A_C\, k$. Note that $s_1 \in \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \subseteq \mathscr{S}$, since (G_C, G_J) is a friend of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$. Morever, $s_2 \in \mathscr{S}$, since, as it has been shown, $k \in \mathscr{S} \cap \mathscr{C}$. Consequently, (6.42) proves that (6.6) holds. The inclusion (6.7) can be proven likewise. Thus, it has been shown that (G_C, G_J) is also a friend of $\mathscr{S}$. The more general statement can be shown along the same lines, since (6.41) and minimality of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ imply

$$\mathscr{S}_1 + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) = \mathscr{X}.$$

□

Proposition 6.7 *Let (6.41) hold. Then, the sum of any two $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing $\mathscr{D}$ is an $\mathscr{H}$-conditioned invariant subspace for Σ_σ containing $\mathscr{D}$.*

Proof Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be any two $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing $\mathscr{D}$. Hence, the subspace $\mathscr{D}$ is contained in $\mathscr{S}_1 + \mathscr{S}_2$, since it is contained in both $\mathscr{S}_1$ and $\mathscr{S}_2$. Let (G_C, G_J) be a friend of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$. By Proposition 6.6, (G_C, G_J) is also a friend of both $\mathscr{S}_1$ and $\mathscr{S}_2$. Consequently,

$$(A_C + G_C\, C_C)(\mathscr{S}_1 + \mathscr{S}_2) = (A_C + G_C\, C_C)\, \mathscr{S}_1 + (A_C + G_C\, C_C)\, \mathscr{S}_2 \subseteq \mathscr{S}_1 + \mathscr{S}_2$$

and, similarly,

$$(A_J + G_J\, C_J)(\mathscr{S}_1 + \mathscr{S}_2) = (A_J + G_J\, C_J)\, \mathscr{S}_1 + (A_J + G_J\, C_J)\, \mathscr{S}_2 \subseteq \mathscr{S}_1 + \mathscr{S}_2.$$

The proof is thus complete, since the latter inclusions are equivalent to $\mathscr{H}$-conditioned invariance of $\mathscr{S}_1 + \mathscr{S}_2$. □

As was pointed out at the end of Sect. 6.3.1, the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing a given subspace is closed with respect to the intersection of subspaces. On assumption (6.41), the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing the subspace $\mathscr{D}$ is closed with respect to the sum of subspaces, by virtue of Proposition 6.7. Consequently, on assumption (6.41), the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ containing $\mathscr{D}$ and contained in a given subspace $\mathscr{W} \subseteq \mathscr{X}$ is a lattice with respect to the intersection and the sum of subspaces. Hence, in particular, that set has a maximum, which will be called the maximal $\mathscr{H}$-conditioned invariant subspace contained in $\mathscr{W}$ and containing $\mathscr{D}$ and which will be denoted by $\mathscr{S}_{\mathscr{H}*}(\mathscr{W})$.

As will be shown in Sect. 6.5, the subspace $\mathscr{S}_{\mathscr{H}*}(\mathscr{W})$ plays a key role in the statement of directly checkable, necessary and sufficient solvability conditions for the MOUIO problem.

The next proposition leads to an algorithmic procedure for computing $\mathscr{S}_{\mathscr{H}*}(\mathscr{W})$.

Proposition 6.8 *Given a subspace $\mathscr{W} \subseteq \mathscr{X}$, consider the subspace $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$, i.e., the maximal $\mathscr{H}$-controlled invariant subspace contained in $\mathscr{C} \cap \mathscr{W}$. Let (6.41) hold. Moreover, let*

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \subseteq \mathscr{W}. \tag{6.43}$$

Then,

$$\mathscr{S}_{\mathscr{H}*}(\mathscr{W}) = \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}). \tag{6.44}$$

Proof The statement will be proven by showing that $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ is the maximal $\mathscr{H}$-conditioned invariant subspace contained in $\mathscr{W}$ and containing $\mathscr{D}$. Firstly, note that $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ contains $\mathscr{D}$, since $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ contains $\mathscr{D}$. Secondly, note that $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ is contained in $\mathscr{W}$ since both $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ are contained in $\mathscr{W}$.

To prove the $\mathscr{H}$-conditioned invariance of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$, note that $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \subseteq (\mathscr{C} \cap \mathscr{W})$ implies

$$(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})) \cap \operatorname{Ker} C_C = (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \cap \operatorname{Ker} C_C) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}).$$

Consequently, it ensues that

$$\begin{aligned}
&A_C \left((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})) \cap \operatorname{Ker} C_C\right) \\
&\quad = A_C \left((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \cap \operatorname{Ker} C_C) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})\right) \\
&\quad = A_C \left(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \cap \operatorname{Ker} C_C\right) + A_C\, \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \\
&\quad \subseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) + \operatorname{Im} D_C \\
&\quad = \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}).
\end{aligned}$$

The inclusion

$$A_J \left((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})) \cap \operatorname{Ker} C_J\right) \subseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$$

can be derived similarly. Therefore, the $\mathscr{H}$-conditioned invariance of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ has been proven.

To prove the maximality of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$, it will be shown that

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \supseteq \mathscr{S}, \tag{6.45}$$

where $\mathscr{S}$ denotes any $\mathscr{H}$-conditioned invariant subspace containing $\mathscr{D}$ and contained in $\mathscr{W}$. To prove (6.45), it will be shown that

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \supseteq \mathscr{S}_1, \tag{6.46}$$

where the subspace $\mathscr{S}_1$ is defined by

$$\mathscr{S}_1 = \mathscr{S} + (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})) \tag{6.47}$$

and, thus, $\mathscr{S}_1 \supseteq \mathscr{S}$. Since (6.41) holds, $\mathscr{S}_1$ is an $\mathscr{H}$-conditioned invariant subspace containing $\mathscr{D}$, as it is the sum of two $\mathscr{H}$-conditioned invariant subspaces containing $\mathscr{D}$ (Proposition 6.7). Moreover, $\mathscr{S}_1 \subseteq \mathscr{W}$ because $\mathscr{S}$, $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ are contained in $\mathscr{W}$. Then, in order to prove (6.46), it will be shown that

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \supseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}_1, \tag{6.48}$$

where the subspace $\mathscr{V}_1$ is defined by

$$\mathscr{V}_1 = \mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C}), \tag{6.49}$$

since, as it will also be shown,

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}_1 = \mathscr{S}_1. \tag{6.50}$$

To this aim, note that $\mathscr{V}_1$ is an $\mathscr{H}$-controlled invariant subspace. In fact,

$$\begin{aligned}
A_C\,\mathscr{V}_1 &= A_C\,(\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C}))\\
&= A_C\,\big((\mathscr{S}_1 \cap \mathrm{Ker}\,C_C) \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})\big)\\
&\subseteq A_C\,(\mathscr{S}_1 \cap \mathrm{Ker}\,C_C) \cap A_C\,\mathscr{V}^*_{\mathscr{H}}(\mathscr{C})\\
&\subseteq \mathscr{S}_1 \cap (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) + \mathrm{Im}\,D_C)\\
&= (\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})) + \mathrm{Im}\,D_C\\
&= \mathscr{V}_1 + \mathrm{Im}\,D_C
\end{aligned}$$

and $A_J\,\mathscr{V}_1 \subseteq \mathscr{V}_1 + \mathrm{Im}\,D_J$ can be shown likewise. Moreover, $\mathscr{V}_1 \subseteq \mathscr{C} \cap \mathscr{W}$ because $\mathscr{S}_1 \subseteq \mathscr{W}$ and $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) \subseteq \mathscr{C}$. Therefore,

$$\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W}) \supseteq \mathscr{V}_1, \tag{6.51}$$

by maximality of $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{W})$ as an $\mathscr{H}$-controlled invariant subspace contained in $\mathscr{C} \cap \mathscr{W}$. Hence, (6.51) implies (6.48). Finally, it remains to show (6.50) or, equivalently,

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + (\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})) = \mathscr{S}_1, \tag{6.52}$$

by (6.49). To this aim, let s_1 be a vector of $\mathscr{S}_1$. By virtue of (6.41), s_1 can be expressed as $s_1 = s + k$ for some $s \in \mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and some $k \in \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})$. Since $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \subseteq \mathscr{S}_1$ by (6.47), the vector $k = s_1 - s$ belongs to $\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})$, which implies

$$\mathscr{S}_1 \subseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + (\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})). \tag{6.53}$$

Conversely,

$$\mathscr{S}_1 \supseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + (\mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})), \tag{6.54}$$

since $\mathscr{S}_1 \supseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and, obviously, $\mathscr{S}_1 \supseteq \mathscr{S}_1 \cap \mathscr{V}^*_{\mathscr{H}}(\mathscr{C})$. Thus, (6.53) and (6.54) imply (6.52) and this completes the proof. □

6.4 Problem Solution

The aim of this section is to present necessary and sufficient conditions for the existence of observers which provide an asymptotic estimate of the linear function $L\,x(t)$ of the state of Σ_σ, for a given L, and, in particular, to show conditions for the synthesis of observers with minimal order.

6.4.1 Necessary and Sufficient Conditions for the Existence of Asymptotic Observers

The first result consists of the following theorem, which establishes a necessary and sufficient condition for the existence of an asymptotic observer (not necessarily of minimal order) of a linear function of the state of the given system in the presence of the unknown input.

Theorem 6.1 *Let the linear impulsive system Σ_σ, the positive real constant τ and the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, be given. Let $\mathscr{J} \subseteq \mathscr{X}$ denote the subspace of all the states x of Σ_σ which can be expressed as $x = x(\sigma(0))$, for some initial state $x(0) \in \mathscr{X}$ and some input signals $u(\cdot)$ and $q(\cdot)$. Then, there exists an asymptotic observer $\Sigma_{O\sigma}$ for $L\,x(t)$ over $\mathcal{S}_\tau$ if and only if there exists an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ such that*

(i) *$\mathscr{S}$ satisfies the conditions (6.15), (6.16) and (6.23): i.e., $\mathscr{S} \cap \operatorname{Ker} C_C \subseteq \operatorname{Ker} L$, $\mathscr{S} \cap \operatorname{Ker} C_J \cap \mathscr{J} \subseteq \operatorname{Ker} L$, and $\mathscr{D} \subseteq \mathscr{S}$;*
(ii) *$\mathscr{S}$ is externally stabilizable over $\mathcal{S}_\tau$.*

Proof *Sufficiency.* Let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ satisfying conditions (i)–(ii) of the statement. Then, it will be shown that the observer $\Sigma_{O\sigma}$ described by (6.2), with a proper choice of the defining matrices, satisfies Definition 6.1.

To this aim, let $P^\top$ and S be basis matrices of $\mathscr{S}^\perp$ and $\mathscr{S}$, respectively, and let $T=[P^\top\ S]^\top$. By condition (ii) of the statement, there exists a friend (G_C, G_J) of $\mathscr{S}$ such that the dynamics of the observer $\Sigma_{O\sigma}$ described by (6.2), with $r=\dim \mathscr{S}^\perp$ and A_{Co}, B_{Co}, G_{Co}, A_{Jo}, B_{Jo}, G_{Jo} respectively given by (6.19a) and (6.19c), where $\bar{A}_C$ and $\bar{A}_J$ are taken as in (6.13) and (6.14), is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$. Namely, $\Sigma_{O\sigma}$ with the flow and jump equations designed as above satisfies condition (i) of Definition 6.1.

Moreover, let the output distribution matrices H_{Co}, K_{Co}, H_{Jo} and K_{Jo} of $\Sigma_{O\sigma}$ be given by (6.19b) and (6.19d), where the matrices H_C, K_C, H_J and K_J are such that (6.17) and, respectively, (6.18) hold. Note that such matrices H_C, K_C, H_J and K_J exist by virtue of condition (i) of the statement and, in particular, by virtue of (6.15) and (6.16), according to the reasoning developed in Sect. 6.3.2.

With the matrix choice above, the observer $\Sigma_{O\sigma}$ turns out to be described by (6.20) and the estimation error $e(t)$ can be seen as the output of the system $\Sigma_{E\sigma}$ described by (6.22). In particular, note that, since $P^\top$ is defined as a basis matrix of $\mathscr{S}^\perp$ and $\mathscr{D}$ is defined by (6.39), condition (i) of the statement and, in particular, (6.23) implies $P\,D_C = P\,D_J = 0$. Consequently, the evolution of $e(t)$ is not affected by $q(.)$. Since the evolution of $e(t)$ is not even affected by $u(.)$ and since the dynamics of $\Sigma_{E\sigma}$ coincides with that of $\Sigma_{O\sigma}$, it follows that the observer $\Sigma_{O\sigma}$ thus designed also satisfies condition (ii) of Definition 6.1.

Furthermore, by taking $P : \mathscr{X} \to \mathscr{X}/\mathscr{S}$ as exact initializing map, it is easy to see that also condition (iii) of Definition 6.1 is met.

Necessity. Let $\Sigma_{O\sigma}$, described by (6.2), be an asymptotic observer for $L\,x(t)$: i.e., let $\Sigma_{O\sigma}$ satisfy conditions (i)–(iii) of Definition 6.1. Then, it will be shown that there exists an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ satisfying conditions (i)–(ii) of the statement.

First, it will be shown that there is no loss of generality in assuming that $\Sigma_{O\sigma}$ has no unobservable modes.

To this aim, let us denote by $\mathscr{Z}_O$ the subspace of $\mathscr{Z}$ formed by all the unobservable states of $\Sigma_{O\sigma}$: i.e., all the states $z \in \mathscr{Z}$ whose free evolution produces the output $y(t)=0$ for all $t \in \mathbb{R}^+$ and for all $\sigma \in \mathcal{S}_\tau$. The subspace $\mathscr{Z}_O$ is easily seen to be an $\mathscr{H}$-invariant subspace for $\Sigma_{O\sigma}$ and its states are useless to asymptotically estimate $L\,x(t)$. In fact, let the change of basis $z=T\,\xi$, where $T=[T_1\ T_2]$ and T_1 is a basis matrix of $\mathscr{Z}_O$, be applied in the state space $\mathscr{Z}$ of $\Sigma_{O\sigma}$. Then, with respect to the new coordinates, $\Sigma_{O\sigma}$ is described by

$$
\Sigma_{O\sigma} \equiv
\begin{cases}
\dot{\xi}_1(t) = A'_{Co_{11}}\,\xi_1(t) + A'_{Co_{12}}\,\xi_2(t) + B'_{Co_1}\,u(t) - G'_{Co_1}\,y(t), & \\
\dot{\xi}_2(t) = A'_{Co_{22}}\,\xi_2(t) + B'_{Co_2}\,u(t) - G'_{Co_2}\,y(t), & \\
w(t) = H'_{Co_2}\,\xi_2(t) + K_{Co}\,y(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\
\xi_1(\sigma(i)) = A'_{Jo_{11}}\,\xi_1^-(\sigma(i)) + A'_{Jo_{12}}\,\xi_2^-(\sigma(i)) + B'_{Jo_1}\,u(\sigma(i)) - G'_{Jo_1}\,y^-(\sigma(i)), & \\
\xi_2(\sigma(i)) = A'_{Jo_{22}}\,\xi_2^-(\sigma(i)) + B'_{Jo_2}\,u(\sigma(i)) - G'_{Jo_2} y^-(\sigma(i)), & \\
w(\sigma(i)) = H'_{Jo_2}\,\xi_2(\sigma(i)) + K_{Jo}\,y(\sigma(i)), & \text{with } i \in \mathbb{N}.
\end{cases}
\tag{6.55}
$$

Then, the subsystem of $\Sigma_{O\sigma}$ evolving in $\mathscr{Z}_O$ can be factored out by simply disregarding the equations that describe the behaviour of the component ξ_1 in (6.55). The remaining subsystem inherits all the properties of $\Sigma_{O\sigma}$ as far as the estimation of $L\,x(t)$ is concerned. Therefore, it is an asymptotic observer of $L\,x(t)$ with no unobservable states.

In light of the previous reasoning, the asymptotic observer $\Sigma_{O\sigma}$ will be assumed henceforth to have no unobservable states, or, equivalently, to have the unobservable subspace $\mathscr{Z}_O$ coinciding with the origin of the state space $\mathscr{Z}$.

The asymptotic observer $\Sigma_{O\sigma}$, as such, satisfies conditions (i)–(iii) of Definition 6.1. Let $P : \mathscr{X} \to \mathscr{Z}$ be the exact initializing linear map, according to condition (iii). Hence, it will be shown that the subspace $\mathscr{S} = \operatorname{Ker} P$ is an $\mathscr{H}$-conditioned invariant subspace for Σ_σ satisfying conditions (i)–(ii) of the statement.

To this aim, consider the auxiliary variable

$$e_{aux}(t) = z(t) - P\,x(t).$$

By following a reasoning similar to that developed in Sect. 6.3.2, the estimation error $e(t)$ can be regarded as the output of the linear impulsive system $\bar{\Sigma}_{E\sigma}$, whose dynamics describes the time evolution of $e_{aux}(t)$. Namely, from equations (6.2) of $\Sigma_{O\sigma}$ and equations (6.1) of Σ_σ, one derives the following equations for $\bar{\Sigma}_{E\sigma}$:

$$\bar{\Sigma}_{E\sigma} \equiv \begin{cases} \dot{e}_{aux}(t) = A_{Co}\,e_{aux}(t) + (A_{Co}\,P - G_{Co}\,C_C - P\,A_C)\,x(t) + \\ \qquad (B_{Co} - P\,B_C)\,u(t) - P\,D_C\,q(t), \\ e(t) = H_{Co}\,e_{aux}(t) + (H_{Co}\,P + K_{Co}\,C_C - L)\,x(t), \qquad \text{with}\, t \neq \sigma(i),\ i \in \mathbb{N}, \\ e_{aux}(\sigma(i)) = A_{Jo}\,e^-_{aux}(\sigma(i)) + (A_{Jo}\,P - G_{Jo}\,C_J - P\,A_J)\,x^-(\sigma(i)) + \\ \qquad (B_{Jo} - P\,B_J)\,u(\sigma(i)) - P\,D_J\,q(\sigma(i)), \\ e(\sigma(i)) = H_{Jo}\,e_{aux}(\sigma(i)) + (H_{Jo}\,P + K_{Jo}\,C_J - L)\,x(\sigma(i)),\ \text{with}\ i \in \mathbb{N}. \end{cases} \tag{6.56}$$

Note that, since the estimation error vanishes when the state of the observer is initialized exactly (condition (ii) of Definition 6.1), the equations

$$L\,x = (H_{Co}\,P + K_{Co}\,C_C)\,x,$$
$$L\,x = (H_{Jo}\,P + K_{Jo}\,C_J)\,x,$$

hold for all $x \in \mathscr{X}$, which implies

$$L = H_{Co}\,P + K_{Co}\,C_C, \tag{6.57}$$
$$L = H_{Jo}\,P + K_{Jo}\,C_J. \tag{6.58}$$

In light of (6.57) and (6.58), equations (6.56) of $\bar{\Sigma}_{E\sigma}$ become

$$
\bar{\Sigma}_{E\sigma} \equiv \begin{cases} \dot{e}_{aux}(t) = A_{Co}\, e_{aux}(t) + (A_{Co}\, P - G_{Co}\, C_C - P\, A_C)\, x(t) + \\ \qquad (B_{Co} - P\, B_C)\, u(t) - P\, D_C\, q(t), \\ e(t) = H_{Co}\, e_{aux}(t), \qquad \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ e_{aux}(\sigma(i)) = A_{Jo}\, e_{aux}^{-}(\sigma(i)) + (A_{Jo}\, P - G_{Jo}\, C_J - P\, A_J)\, x^{-}(\sigma(i)) + \\ \qquad (B_{Jo} - P\, B_J)\, u(\sigma(i)) - P\, D_J\, q(\sigma(i)), \\ e(\sigma(i)) = H_{Jo}\, e_{aux}(\sigma(i)), \qquad \text{with } i \in \mathbb{N}. \end{cases} \tag{6.59}
$$

Since the dynamic matrices and the output distribution matrices of $\bar{\Sigma}_{E\sigma}$ are the same as those of Σ_O, the system $\bar{\Sigma}_{E\sigma}$ does not have unobservable states either. Therefore, by virtue of condition (ii) of Definition 6.1, it ensues that

$$A_{Co}\, P - G_{Co}\, C_C - P\, A_C = 0, \tag{6.60}$$

$$A_{Jo}\, P - G_{Jo}\, C_J - P\, A_J = 0, \tag{6.61}$$

$$B_{Co} - P\, B_C = 0, \tag{6.62}$$

$$B_{Jo} - P\, B_J = 0, \tag{6.63}$$

$$P\, D_C = 0, \tag{6.64}$$

$$P\, D_J = 0, \tag{6.65}$$

Equalities (6.60), (6.61) are equivalent to

$$P\, A_C\, (\text{Ker}\, P \cap \text{Ker}\, C_C) = \{0\},$$

$$P\, A_J\, (\text{Ker}\, P \cap \text{Ker}\, C_J) = \{0\},$$

which, in turn, are equivalent to (6.4) and (6.5), since $\mathscr{S} = \text{Ker}\, P$. Therefore, the subspace $\mathscr{S}$ is an $\mathscr{H}$-conditioned invariant subspace for Σ_σ.

Equalities (6.57) and (6.58) show that $\mathscr{S}$ satisfies (6.15) and (6.16). Moreover, (6.64) and (6.65), in light of (6.39), show that $\mathscr{S}$ also satisfies (6.23). Hence, the subspace $\mathscr{S}$ satisfies condition (i) of the statement.

In order to prove that the $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ is externally stabilizable, note that, since $P : \mathscr{X} \to \mathscr{Z}$ is such that $\text{Im}\, P = \mathscr{Z}$, the matrices G_{Co} and G_{Jo} of $\Sigma_{O\sigma}$ can be written as $G_{Co} = P\, G_C$ and $G_{Jo} = P\, G_J$, for suitable matrices G_C and G_J. Then, (6.60) and (6.61) can be re-written as

$$P\, (A_C + G_C\, C_C) = A_{Co}\, P \tag{6.66}$$

$$P\, (A_J + G_J\, C_J) = A_{Jo}\, P. \tag{6.67}$$

Equations (6.66) and (6.67) show that (G_C, G_J) is a friend of $\mathscr{S} = \text{Ker}\, P$ and that the dynamics induced on $\mathscr{X}/\mathscr{S}$ by that of Σ_σ^G has a flow component and a jump component that are respectively described by A_{Co} and A_{Jo}. Since A_{Co} and A_{Jo} are the flow and jump dynamic matrices of $\Sigma_{O\sigma}$ and since, by property (i) of Definition 6.1, $\Sigma_{O\sigma}$ is globally asymptotically stable for all $\sigma \in \mathcal{S}_\tau$, it ensues that $\mathscr{S}$, as an $\mathscr{H}$-conditioned invariant subspace of Σ_σ is externally stabilizable. $\square$

Remark 6.6 In [11], a procedure for the synthesis of an asymptotic observer with the same dynamic order of the given system was shown. It is therefore worthwhile to point out the differences between the construction illustrated therein and the one developed in the proof of Theorem 6.1. To this aim, let the linear impulsive system Σ_σ, the linear function $L : \mathscr{X} \to \mathbb{R}^l$ and the positive real constant τ be given. Moreover, let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ, satisfying property (ii) of Theorem 6.1 and, as far as property (i) is concerned, satisfying (6.23) and

$$\mathscr{S} \subseteq \operatorname{Ker} L \tag{6.68}$$

in place of (6.15) and (6.16) – by the way, it is clear that (6.68) is more demanding than the pair of (6.15) and (6.16). Let (G_C, G_J) be a friend of $\mathscr{S}$ such that the dynamics induced on $\mathscr{X}/\mathscr{S}$ by that of system (6.31) be globally asymptotically stable over $\mathcal{S}_\tau$ and let the state equations of the observer $\Sigma_{O\sigma}$, described by (6.2), be defined by picking

$$A_{Co} = A_C + G_C\, C_C, \quad B_{Co} = B_C, \quad G_{Co} = G_C, \tag{6.69}$$

$$A_{Jo} = A_J + G_J\, C_J, \quad B_{Jo} = B_J, \quad G_{Jo} = G_J. \tag{6.70}$$

As to the output equations, from (6.68) it ensues that there exists a matrix H such that $L = H\, P$, where $P^\top$ denotes a basis matrix of $\mathscr{S}^\perp$. Then, let

$$H_{Co} = H\, P, \quad K_{Co} = 0, \tag{6.71}$$

$$H_{Jo} = H\, P, \quad K_{Jo} = 0. \tag{6.72}$$

Consequently, the estimation error $e(t) = w(t) - L\, x(t)$ can be written as $e(t) = H\, P\, z(t) - L\, x(t)$, where $z(t)$ denotes the state of the observer thus devised, with $t \neq \sigma(i)$, $i \in \mathbb{N}$, and it can also be written as $e(\sigma(i)) = H\, P\, z^-(\sigma(i)) - L\, x^-(\sigma(i))$, with $i \in \mathbb{N}$. Next, let $\eta(t) = P\, z(t) - P\, x(t)$, so that $e(t) = H\, \eta(t)$. Therefore, the time evolution of $\eta(t)$ is described by

$$\begin{cases} \dot\eta(t) = P\,(A_C + G_C\, C_C)\,\eta(t), & \text{with } t \neq \sigma(i),\ i \in \mathbb{N}, \\ \eta(\sigma(i)) = P\,(A_J + G_J\, C_J)\,\eta^-(\sigma(i)), & \text{with } i \in \mathbb{N}, \end{cases} \tag{6.73}$$

where (6.23) has been taken into account. Equations (6.73) show that $\lim_{t\to\infty} \eta(t) = 0$, independently of the known input signal $u(\cdot)$ and of the unknown input signal $q(\cdot)$, for any initial state $z(0)$ of $\Sigma_{O\sigma}$ and any initial state $x(0)$ of Σ_σ, and for all $\sigma \in \mathcal{S}_\tau$. In other words, $P\, z(t)$ is an asymptotic estimate of $P\, x(t)$ for all $\sigma \in \mathcal{S}_\tau$, independently of the inputs $u(\cdot)$ and $q(\cdot)$ and for all initial conditions $z(0)$ and $x(0)$. By virtue of the linear relation existing between $\eta(t)$ and the estimation error $e(t)$, the latter enjoys the same properties of the former or, in other words, $w(t)$ is an asymptotic estimate of $L\, x(t)$ over $\mathcal{S}_\tau$. However, it is important to highlight that the dynamics of the observer thus devised – i.e., the dynamics described by (6.31) – is globally asymptotically stable over $\mathcal{S}_\tau$ only if $\mathscr{S}$ is also *internally stabilizable* over $\mathcal{S}_\tau$ and the friend (G_C, G_J) is chosen in such a way that also the dynamic induced on $\mathscr{S}$ by that of (6.31) is globally asymptotically stable over $\mathcal{S}_\tau$. Furthermore, the initializing map P is *not surjective*, as the subsystem of (6.31) which evolves on $\mathscr{S}$ plays no role in the estimation of $L\,x(t)$.

6.4.2 Order Minimization

The observer derived in the proof of sufficiency of Theorem 6.1 is not necessarily of minimal order or, in other words, its state space does not necessarily have the minimal possible dimension. Considerations on how to derive an observer with this property are, in fact, the object of this section, which paves the way to the necessary and sufficient condition for solvability of the MOUIO problem.

On the assumption that the set of all $\mathscr{H}$-conditioned invariant subspaces for Σ_σ satisfying conditions (i)–(ii) of Theorem 6.1 be non-empty, an element $\mathscr{S}$ of that set is said to be *of maximal dimension* if $\dim \mathscr{S} \geq \dim \mathscr{S}'$ for all $\mathscr{S}'$ in the same set. By exploiting this definition, the following theorem can be stated.

Theorem 6.2 *On the same assumptions of Theorem 6.1, let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ satisfying conditions (i)–(ii) of Theorem 6.1 and being of maximal dimension. Then, the observer $\Sigma_{O\sigma}$ devised as in the proof of Theorem 6.1 by using $\mathscr{S}$ has the minimal order among all those devised in the same way by using any other externally stabilizable $\mathscr{H}$-conditioned invariant subspace satisfying conditions (i)–(ii) of Theorem 6.1.*

Proof It follows from Theorem 6.1 and from $\mathscr{S}$ being of maximal dimension. □

Let $\mathscr{C}_{\Sigma_{O\sigma}}$ denote the *class* of all the linear impulsive observers devised as in the *if*-part of the proof of Theorem 6.1, by using an externally stabilizable $\mathscr{H}$-conditioned invariant subspace for Σ_σ satisfying conditions (i)-(ii) of the same Theorem 6.1. The following theorem provides a sufficient condition for solvability of the MOUIO Problem for that class of observers.

Theorem 6.3 *Let the linear impulsive system Σ_σ, the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, and the positive real constant τ be given. Then, the MOUIO Problem for the observers in the class $\mathscr{C}_{\Sigma_{O\sigma}}$ is solvable if there exists an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ satisfying conditions (i)-(ii) of Theorem 6.1 and of maximal dimension.*

Proof It follows from Theorem 6.2 and from $\mathscr{S}$ being of maximal dimension. □

The existence of an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ satisfying conditions (i)-(ii) of Theorem 6.1, either of maximal dimension or not, may be difficult to ascertain. A necessary condition for the existence of a subspace with these properties is provided by the following proposition. The main feature of that necessary condition is that it can be directly checked by using the algorithm for the computation of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$, the minimal $\mathscr{H}$-conditioned invariant subspace for Σ_σ containing $\mathscr{D}$ – i.e., the recursive algorithm described by (6.8).

Proposition 6.9 *On the same assumptions of Theorem 6.1, let $\mathscr{S}$ be an $\mathscr{H}$-conditioned invariant subspace for Σ_σ, satisfying condition (i) of Theorem 6.1. Then, the following conditions hold:*

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \cap \operatorname{Ker} C_C \subseteq \operatorname{Ker} L, \tag{6.74}$$

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \cap \operatorname{Ker} C_J \cap \mathscr{J} \subseteq \operatorname{Ker} L. \tag{6.75}$$

Proof It follows from the minimality of $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ in the set of the $\mathscr{H}$-conditioned invariant subspaces containing $\mathscr{D}$, as can be easily proven by contradiction. □

Remark 6.7 The necessary condition established by Proposition 6.9 is a structural one and, as such, it only deals with the existence of an $\mathscr{H}$-conditioned invariant subspace for Σ_σ, satisfying condition (i) of Theorem 6.1. Namely, Proposition 6.9 does not take into account condition (ii), which refers to external stabilizability of the $\mathscr{H}$-conditioned invariant subspace. In other words, if the necessary condition is satisfied, an externally stabilizable $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ may either exist or not.

6.5 A Checkable Necessary and Sufficient Condition

In this section, the solvability of the MOUIO Problem is characterized by a necessary and sufficient condition that can be directly checked by an algorithmic procedure. To this aim, it is assumed that condition (6.41)—namely,

$$\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) = \mathscr{X}$$

and

$$\mathscr{J} = \mathscr{X} \tag{6.76}$$

hold. Thus, in addition to the results of Sect. 6.3.6, the following can be proven.

Proposition 6.10 *Let (6.41) and (6.76) hold together with (6.74) and (6.75). Then, the set of all $\mathscr{H}$-conditioned invariant subspaces $\mathscr{S}$ for Σ_σ satisfying (6.15), (6.16) and (6.23) has a maximal element, henceforth denoted by $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$.*

Proof The statement will be proven by showing that the set of all $\mathscr{H}$-conditioned invariant subspaces $\mathscr{S}$ for Σ_σ satisfying (6.15), (6.16) and (6.23) is closed with respect to the sum. Hence, let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two $\mathscr{H}$-conditioned invariant subspaces for Σ_σ satisfying (6.15), (6.16) and (6.23). Since (6.41) holds, the sum $\mathscr{S}_1 + \mathscr{S}_2$ is an $\mathscr{H}$-conditioned invariant subspace satisfying (6.23), by Proposition 6.7. Then, since dim $\mathscr{X}$ is finite, showing that $\mathscr{S}_1 + \mathscr{S}_2$ satisfies (6.15) and (6.16) leads to the conclusion. To this aim, first note that (6.41) is equivalent to

$$(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp \cap (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}))^\perp = \{0\}. \tag{6.77}$$

Moreover, note that conditions (6.15), (6.16) and (6.23) satisfied by $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, are equivalent to

$$\mathscr{S}_i^\perp + (\operatorname{Ker} C_C)^\perp \supseteq (\operatorname{Ker} L)^\perp, \qquad \text{with } i = 1, 2, \tag{6.78}$$
$$\mathscr{S}_i^\perp + (\operatorname{Ker} C_J)^\perp \supseteq (\operatorname{Ker} L)^\perp, \qquad \text{with } i = 1, 2, \tag{6.79}$$
$$\mathscr{D}^\perp \supseteq \mathscr{S}_i^\perp, \qquad \text{with } i = 1, 2, \tag{6.80}$$

where (6.76) has also been considered. Then, in order to show that

$$(\mathscr{S}_1 + \mathscr{S}_2) \cap \operatorname{Ker} C_C \subseteq \operatorname{Ker} L \tag{6.81}$$

holds, note that any $x \in (\operatorname{Ker} L)^\perp$ can be written as $x = s_1 + k_1 = s_2 + k_2$ for some $s_i \in \mathscr{S}_i^\perp$ and $k_i \in (\operatorname{Ker} C_C)^\perp$, with $i = 1, 2$, by (6.78). Since $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \subseteq \mathscr{S}_i$ is equivalent to $\mathscr{S}_i^\perp \subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp$ and since $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}) \subseteq \operatorname{Ker} C_C$ is equivalent to $(\operatorname{Ker} C_C)^\perp \subseteq (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C}))^\perp$, from (6.77) it follows that $\mathscr{S}_i^\perp \cap (\operatorname{Ker} C_C)^\perp = \{0\}$, with $i = 1, 2$. Consequently, $k_1 = k_2$ and $s_1 = s_2 \in (S_1^\perp \cap S_2^\perp)$. Hence, the relation $(S_1^\perp \cap S_2^\perp) + (\operatorname{Ker} C_C)^\perp \supseteq (\operatorname{Ker} L)^\perp$, or, equivalently, (6.81) holds. The inclusion

$$(\mathscr{S}_1 + \mathscr{S}_2) \cap \operatorname{Ker} C_J \subseteq \operatorname{Ker} L, \tag{6.82}$$

where (6.76) has been taken into account, can be shown likewise. □

Hence, on the assumptions of Proposition 6.10, $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is the maximal $\mathscr{H}$-conditioned invariant subspace satisfying (6.15), (6.16) and (6.23). With this notation, the following theorem can be proven.

Theorem 6.4 *Let the linear impulsive system Σ_σ, the linear map $L : \mathscr{X} \to \mathbb{R}^l$, with $l \leq n$, and the positive real constant τ be given. Let (6.41) and (6.76) hold together with (6.74) and (6.75). Then, there exists an $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ with a friend (G_C, G_J) such that the observer $\Sigma_{O\sigma}$ devised as in the proof of Theorem 6.1 by means of $\mathscr{S}$ and (G_C, G_J) is an asymptotic observer for $L\,x(t)$ over $\mathcal{S}_\tau$ if and only if $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is externally stabilizable.*

Proof Sufficiency. The statement follows from Theorem 6.1 with $\mathscr{S} = \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ and, in light of Proposition 6.3 and Remark 6.3, with any friend (G_C, G_J) of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$.

Necessity. Let $\mathscr{S} \subseteq \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ be an $\mathscr{H}$-conditioned invariant subspace with a friend (G_C, G_J) such that the observer $\Sigma_{O\sigma}$ devised as in the proof of sufficiency of Theorem 6.1, by means of $\mathscr{S}$ and (G_C, G_J), is an asymptotic observer for $L\,x(t)$ over $\mathcal{S}_\tau$. By Proposition 6.6, (G_C, G_J) is also a friend of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$. Hence, in particular, $(A_C + G_C\, C_C)\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D}) \subseteq \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ holds. Let $\bar{A}_C$ be such that $\bar{A}_C\, P = P\,(A_C + G_C\, C_C)$, where $P^\top$ is a matrix whose columns are a basis of $S^\perp$. Thus, from $P\,(A_C + G_C\, C_C)\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D}) \subseteq P\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$, in light of the relation characterizing $\bar{A}_C$, it ensues that $P\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is $\bar{A}_C$-invariant. Likewise, by picking $\bar{A}_J$ such that $\bar{A}_J\, P = P\,(A_J + G_J\, C_J)$, one can show that $P\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is $\bar{A}_J$-invariant. Hence, in a suitable basis, the matrices $\bar{A}_C$ and $\bar{A}_J$ exhibit the upper block-triangular form

$$\bar{A}_C = \begin{bmatrix} \bar{A}_{C11} & \bar{A}_{C12} \\ 0 & \bar{A}_{C22} \end{bmatrix}, \quad \bar{A}_J = \begin{bmatrix} \bar{A}_{J11} & \bar{A}_{J12} \\ 0 & \bar{A}_{J22} \end{bmatrix},$$

where the dimensions of the matrices $\bar{A}_{C11}$ and $\bar{A}_{J11}$ are equal to

$$\dim\,(P\, \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})) = \dim \bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D}) - \dim \mathscr{S}.$$

The matrices $\bar{A}_{C22}$ and $\bar{A}_{J22}$ define the dynamics of the observer devised by means of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$, while the matrices $(\bar{A}_C, \bar{A}_J)$ define the dynamics of the observer devised by means of $\mathscr{S}$. Since the latter is globally asymptotically stable over $\mathcal{S}_\tau$, so is the first one and, therefore, $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is externally stabilizable over $\mathcal{S}_\tau$. □

Theorem 6.4 provides a necessary and sufficient condition for the existence of asymptotic observers and, by maximality of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$, a way to find a solution to the MOUIO Problem, if any exists. Namely, if $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is externally stabilizable, a solution is given by the observer devised as in the proof of sufficiency of Theorem 6.1 by means of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ and any of its friends (G_C, G_J).

In order to provide a practical test and to complete the procedure for the observer synthesis, an algorithm for computing $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ is still needed. To this aim, let (6.41) and (6.76) hold, together with (6.74) and (6.75). As seen in the proof of Proposition 6.10, (6.41) is equivalent to (6.77). Moreover, $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) \supseteq \mathscr{D}$ and the inclusions (6.74) and (6.75), in light of (6.76), are respectively equivalent to

$$\mathscr{D}^\perp \supseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp, \tag{6.83}$$

$$(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp + (\mathrm{Ker}\, C_C)^\perp \supseteq (\mathrm{Ker}\, L)^\perp, \tag{6.84}$$

$$(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp + (\mathrm{Ker}\, C_J)^\perp \supseteq (\mathrm{Ker}\, L)^\perp. \tag{6.85}$$

Let the subspace $\mathscr{M}$ be defined by

$$\mathscr{M} = (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp \cap (\mathscr{C}^\perp + (\mathrm{Ker}\, L)^\perp), \tag{6.86}$$

where $\mathscr{C}$ was defined in (6.40). Hence, in light of (6.83), (6.84) and (6.85), it can be shown that the subspace $\mathscr{M}$, which is obviously contained in $(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp$, allows us to state the following property.

Proposition 6.11 *Let (6.41) and (6.76) hold, together with (6.74) and (6.75). Then, the maximal $\mathscr{H}$-conditioned invariant subspace satisfying (6.15), (6.16) and (6.23), namely $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$, coincides with the maximal $\mathscr{H}$-conditioned invariant subspace containing $\mathscr{D}$ and contained in $\mathscr{M}^\perp$, namely $\mathscr{S}_{\mathscr{H}*}(\mathscr{M}^\perp)$: i.e.,*

$$\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D}) = \mathscr{S}_{\mathscr{H}*}(\mathscr{M}^\perp) = \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^\perp). \tag{6.87}$$

Proof First, it will be shown that any $\mathscr{H}$-conditioned invariant subspace $\mathscr{S}$ for Σ_σ which satisfies (6.15), (6.16) and (6.23) is contained in $\mathscr{M}^\perp$. To this aim, let us consider the subspace $\mathscr{S}^\perp$ and note that $\mathscr{S} \supseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ is equivalent to

$$\mathscr{S}^\perp \subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp. \tag{6.88}$$

Moreover, as already observed, (6.15) and (6.16) are equivalent to

$$\mathscr{S}^\perp + (\mathrm{Ker}\, C_C)^\perp \supseteq (\mathrm{Ker}\, L)^\perp, \tag{6.89}$$
$$\mathscr{S}^\perp + (\mathrm{Ker}\, C_J)^\perp \supseteq (\mathrm{Ker}\, L)^\perp, \tag{6.90}$$

where (6.76) has been taken into account. The inclusions (6.89) and (6.90) imply

$$\begin{aligned}(\mathrm{Ker}\, L)^\perp &\subseteq \mathscr{S}^\perp + (\mathrm{Ker}\, C_C)^\perp + (\mathrm{Ker}\, C_J)^\perp \\ &= \mathscr{S}^\perp + (\mathrm{Ker}\, C_C \cap \mathrm{Ker}\, C_J)^\perp \\ &= \mathscr{S}^\perp + \mathscr{C}^\perp.\end{aligned} \tag{6.91}$$

Let s be an element of $\mathscr{M}$. Since $\mathscr{M} \subseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp$, the vector s can be written as

$$s = (c + l) \in (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp \cap (\mathscr{C}^\perp + (\mathrm{Ker}\, L)^\perp), \tag{6.92}$$

for some $c \in \mathscr{C}^\perp$ and some $l \in (\mathrm{Ker}\, L)^\perp$. By (6.91), the vector l, which is equal to the difference between s and c can also be written as

$$s - c = l = s' + c', \tag{6.93}$$

for some $s' \in \mathscr{S}^\perp$ and some $c' \in \mathscr{C}^\perp$. From (6.41), in the equivalent formulation (6.77), it ensues that

$$(\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^\perp \cap C^\perp = \{0\}.$$

Therefore, in light of (6.88), it follows that $-c = c'$ and $s = s'$. Since s' belongs to $\mathscr{S}^\perp$, the same is true for s and, therefore, $\mathscr{M} \subseteq \mathscr{S}^\perp$, or equivalently $\mathscr{S} \subseteq \mathscr{M}^\perp$.

Then, it will be shown that the maximal $\mathscr{H}$-conditioned invariant subspace containing $\mathscr{D}$ and contained in $\mathscr{M}^\perp$, namely

$$\mathscr{S}_{\mathscr{H}*}(\mathscr{M}^\perp) = \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^\perp),$$

satisfies (6.15) and (6.16). To this aim, note that, by the definitions of the involved subspaces, the following relations hold

$$\mathscr{D} \subseteq \mathscr{S}^*_{\mathscr{H}}(\mathscr{D}) + \mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp}), \tag{6.94}$$
$$\mathscr{M} \subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp}, \tag{6.95}$$
$$\mathscr{M} \subseteq (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp}))^{\perp}. \tag{6.96}$$

From (6.95) and (6.96), it follows that

$$\mathscr{M} \subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} \cap (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp}))^{\perp}$$

Then, by (6.84) and (6.85) and the definition of the involved subspaces, one gets

$$\begin{aligned}(\mathrm{Ker}\, L)^{\perp} &\subseteq ((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} + (\mathrm{Ker}\, C_C)^{\perp}) \cap ((\mathrm{Ker}\, C_C)^{\perp} + (\mathrm{Ker}\, L)^{\perp})\\ &= ((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} \cap ((\mathrm{Ker}\, C_C)^{\perp} + (\mathrm{Ker}\, L)^{\perp})) + (\mathrm{Ker}\, C_C)^{\perp}\\ &\subseteq ((\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} \cap (\mathscr{C}^{\perp} + (\mathrm{Ker}\, L)^{\perp})) + (\mathrm{Ker}\, C_C)^{\perp}\\ &\subseteq \mathscr{M} + (\mathrm{Ker}\, C_C)^{\perp}\\ &\subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} \cap (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp}))^{\perp} + (\mathrm{Ker}\, C_C)^{\perp}\end{aligned} \tag{6.97}$$

In a similar way, one gets

$$(\mathrm{Ker}\, L)^{\perp} \subseteq (\mathscr{S}^*_{\mathscr{H}}(\mathscr{D}))^{\perp} \cap (\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp}))^{\perp} + (\mathrm{Ker}\, C_J)^{\perp}. \tag{6.98}$$

Considering the orthogonal subspaces, from (6.97) and (6.98), one derives that $\mathscr{S}_{\mathscr{H}*}(\mathscr{M}^{\perp})$ satisfies (6.15) and (6.16), which completes the proof. □

Remark 6.8 Proposition 6.11 shows that the subspace $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$, which plays a central role in the solution of the MOUIO problem, can be algorithmically computed as the maximal $\mathscr{H}$-conditioned invariant subspace for Σ_σ that contains $\mathscr{D}$ and that is contained in $\mathscr{M}^{\perp}$, by using the procedures illustrated in Sect. 6.3.1 and in Sect. 6.3.5 for $\mathscr{S}^*_{\mathscr{H}}(\mathscr{D})$ and $\mathscr{V}^*_{\mathscr{H}}(\mathscr{C} \cap \mathscr{M}^{\perp})$, respectively. Then, one can analyze external stabilizability of $\bar{\mathscr{S}}_{\mathscr{H}*}(\mathscr{D})$ by means of any friend (G_C, G_J), by virtue of Proposition 6.3. Thus, the condition of Theorem 6.4 can be ascertained through an algorithmic procedure.

6.6 Conclusions

In this chapter, a general, necessary and sufficient condition for the existence of an asymptotic observer of a linear function of the state of a given system in the presence of unknown inputs has been stated. Conditions for deriving an asymptotic observer of minimal order—or, more precisely, whose state space has the minimal possible dimension—have then been discussed. Moreover, a new necessary and sufficient condition for the existence of an asymptotic observer is stated under slightly more restrictive assumptions. The new condition, in comparison with the first, is more interesting from the practical point of view, since it is based on subspaces that can be computed by algorithms and on properties that can be directly ascertained.

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Chapter 7
Advances of Implicit Description Techniques in Modelling and Control of Switched Systems

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Abstract Our contribution is devoted to a constructive overview of the implicit system approach in modern control of switched dynamic models. We study a class of non-stationary autonomous switched systems and formally establish the existence of solution. We next incorporate the implicit systems approach into our consideration. At the beginning of the contribution, we also develop a specific system example that is used for illustrations of various system aspects that we consider. Our research involves among others a deep examination of the reachability property in the framework of the implicit system framework that we propose. Based on this methodology, we finally propose a resulting robust control design for the switched systems under consideration and the proposed control strategy is implemented in the context of the illustrative example.

Notation
Let us first introduce the necessary notation used in this manuscript.

- Script capitals $\mathscr{V}$, $\mathscr{W}$, ... denote finite-dimensional linear spaces with elements v, w, ... The dimension of a space $\mathscr{V}$ is denoted by $\dim(\mathscr{V})$, $\mathscr{V} \approx \mathscr{W}$ stands for $\dim(\mathscr{V}) = \dim(\mathscr{W})$. Moreover, in the case $\mathscr{V} \subset \mathscr{W}$, $\frac{\mathscr{W}}{\mathscr{V}}$ or $\mathscr{W}/\mathscr{V}$ stands for the quotient space $\mathscr{W}$ modulo $\mathscr{V}$. The direct sum of independent spaces is written as $\oplus$. $X^{-1}\mathscr{V}$ stands for the inverse image of the subspace $\mathscr{V}$ by the linear transformation X. Given a linear transformation $X : \mathscr{V} \to \mathscr{W}$, the expression $\mathrm{Im}\, X = X\mathscr{V}$ denotes

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its image and Ker$X$ denotes the corresponding kernel. In the case $\mathscr{V} \approx \mathscr{W}$, we write $X : \mathscr{V} \leftrightarrow \mathscr{W}$. Given a space $\mathscr{X} = \mathscr{S} \oplus \mathscr{T}$, the natural projection on $\mathscr{S}$ along $\mathscr{T}$ is denoted as $P : \mathscr{X} \to \mathscr{S}//\mathscr{T}$. A zero-dimensional subspace is denoted by $\{0\}$, and the identity operator is I. $e_i \in \mathbb{R}^n$ stands for the vector whose i-th entry is equal to 1 and the other ones are equal to 0. $\{e_k, \ldots, e_\ell\}$ stands for the subspace generated by the vectors $e_k, \ldots, e_\ell$.

- Additionally $\mathbb{R}^+$, $\mathbb{R}^{+*}$, $\mathbb{Z}^+$ and $\mathbb{N}$ stand for sets of non-negative real numbers, positive real numbers, non-negative integers and correspondingly for positive integers (the natural numbers), respectively. The notations $\mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{V}\right)$ and $\mathcal{L}^\infty\left(\mathbb{R}^+, \mathscr{V}\right)$ are used for the space of infinitely differentiable functions and space of bounded functions from $\mathbb{R}^+$ to $\mathscr{V}$.

7.1 Introduction

We review recent contributions related to the implicit linear systems and to the corresponding modelling approaches. We mainly analyse here the effective control design schemes for some specific classes of complex systems, namely, for dynamic systems with switches. First, let us mention the celebrated *implicit systems representation* proposed by Rosenbrock [39]. It was developed in the context of a specific generalization of proper linear systems (see also [28]).

Recall that an *implicit representation* $\Sigma^{imp}(E, A, B, C)$ is a set of differential and algebraic equations of the generic form[1]:

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu \;\text{ and }\; y = Cx, \quad \forall\, t \geq 0, \tag{7.1}$$

where $E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$ and $C : \mathscr{X}_d \to \mathscr{Y}$ are linear maps. The spaces $\mathscr{X}_d \approx \mathbb{R}^n$, $\underline{\mathscr{X}}_{eq} \approx \mathbb{R}^{n_{eq}}$, $\mathscr{U} \approx \mathbb{R}^m$ and $\mathscr{Y} \approx \mathbb{R}^p$ are usually called the "*descriptor*","*equation*" and the "*input*" and the "*output*" spaces, respectively. In [5], it was shown that under the condition $n_{eq} \leq n$ one can constructively describe a linear system with an internal Variable Structure. However, in case $n_{eq} < n$, when the system under consideration is solvable, solutions are generally non-unique. In some sense, there is an additional degree of freedom in (7.1) that can finally incorporate (by an implicit way) a *structure variation*. In [8], a non-square implicit description was effectively used for modelling and control of various classes of linear systems. This effective control approach also includes systems with internal switches. Moreover, the necessary and sufficient conditions for a unique dynamic system behaviour (expressed in terms of the overall implicit model) are developed. These conditions imply existence of the system parts which are associated with the common internal dynamic equation and also with the algebraic constraints. The last one are "*controlled*" (in an hidden way) by the degree of freedom. It was also shown how to include the variable internal structure representation into the common square

[1] In the case where there is no output equation $y = Cx$, we simply write $\Sigma^{imp}(E, A, B)$.

implicit descriptions for an (A, E, B) invariant subspace generated by the kernel of the generic output map. The above-mentioned embedding makes it possible to get an unobservable variable internal structure. As a consequence of this effect, a proper closed-loop system with a controllable pre-specified structure was obtained.

In [13], we have taken advantage of the results obtained in [8] for a particular model (a class) of the so-called "*time-dependent, autonomous switched systems*" [29]. In [10], the authors propose a specific variable structure decoupling control strategy based on the celebrated ideal proportional and derivative (PD) feedback. Moreover, our contribution [14] is dedicated to a proper practical approximation of the ideal PD feedback mentioned above. This control strategy "rejects" in some sense the given variable structure and makes it possible to establish the stability property (stabilization) of both implicit control strategies. In [15], the authors have tackled the descriptor variable observation problem for implicit descriptions having *column minimal indices* blocks. In this paper, two concrete design procedures are considered, namely, the *(i) Linear descriptor observers* approach (based on the fault detection techniques) and the *(ii) Indirect variable descriptor observers* technique. The last one is based on the finite time structure detection methodology. In the first design scheme, the observer is composed of the celebrated Beard–Jones filter which makes it possible to observe the existing degree of freedom in rectangular implicit representations. Since this observation is accomplished by a pole-zero cancellation, this technique is reserved to minimum phase systems. The second idea from the contribution mentioned above is based on an adaptive structure detection. This technique is implemented in finite time and guarantees avoiding of possible stability problems (due to the temporarily unstable closed-loop systems into the detection procedure [11]).

Our contribution is organized as follows. In Sect. 7.2, we review a class of *time-dependent autonomous switched systems* which can be studied in combination with the newly developed approach of the *linear time-invariant implicit systems* [8, 13]. In Sect. 7.3, we review some properties of the *rectangular implicit representations* [8, 12]. Section 7.4 is devoted to the important structural property associated with the system reachability [6, 8, 9, 12, 16]. Section 7.5 includes the control strategy development for the *rectangular implicit representations* when the descriptor variable is available [8, 14]. In Sect. 7.6, we present some numerical simulations. Section 7.7 summarizes our contribution.

7.2 Time-Dependent Autonomous Switched Systems

In [13], taking into consideration the analytic results obtained in [8], an important class of the so-called *time-dependent autonomous switched systems* [29, 40] has been considered. It can be formally represented by the following *state-space representation* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\bar{x} = A_{q_i}\bar{x} + Bu \text{ and } y = C_{q_i}\bar{x}, \tag{7.2}$$

where u and y are the input and output variables. Here, q_i are elements of a finite set of indexes (locations) $\mathscr{Q}$. The system remains in location

$$q_i \in \mathscr{Q} = \{q_1, \ldots, q_\eta |\ q_i \in \mathbb{R}^\mu,\ i \in \{1, \ldots, \eta\}\}$$

for all time instants $t \in [T_{i-1}, T_i)$, and some $i \in \mathbb{N}$,

$$T_{i-1} \in \mathfrak{T} = \left\{ T_i \in \mathbb{R}^+ \,\middle|\, T_0 = 0,\ T_{i-1} < T_i\ \forall i \in \mathbb{N},\ \text{with } \lim_{i\to\infty} T_i = \infty \right\}.$$

The matrices A_{q_i} and C_{q_i} have here a particular structure (cf. [33, 42]):

$$A_{q_i} = \overline{A}_0 + \overline{A}_1\overline{D}(q_i) \text{ and } C_{q_i} = \overline{C}_0 + \overline{C}_1\overline{D}(q_i), \tag{7.3}$$

where $B \in \mathbb{R}^{\bar{n}\times m}$, $\overline{A}_0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\overline{C}_0 \in \mathbb{R}^{p\times\bar{n}}$, $\overline{A}_1 \in \mathbb{R}^{\bar{n}\times\hat{n}}$, $\overline{C}_1 \in \mathbb{R}^{p\times\hat{n}}$ and $\overline{D}(q_i) \in \mathbb{R}^{\hat{n}\times\bar{n}}$, $i \in \{1, \ldots, \eta\}$. Moreover, $\overline{A}_1$ and B are monic, C_1 and $\overline{D}(q_i)$ are epic, and $\overline{D}(0) = 0$.

Let us first introduce an illustrative example which will be used along with this chapter.

7.2.1 Example (Part 1)

Consider (7.2) and (7.3) with the following state-space matrices ($q = (\alpha, \beta)$):

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \overline{A}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \overline{C}_0 = \begin{bmatrix} 0 & 2 \end{bmatrix},\ \overline{C}_1 = \begin{bmatrix} 1 \end{bmatrix},\ \overline{D}(q) = \begin{bmatrix} \alpha & \beta \end{bmatrix},$$
$$q \in \{q_1, q_2, q_3\},\ q_1 = (-1, -1),\ q_2 = (-1, 0),\ q_3 = (-1, -2). \tag{7.4}$$

Here

$$A_{q_i} = \begin{bmatrix} \alpha & (1+\beta) \\ (1+\alpha) & \beta \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_{q_i} = \begin{bmatrix} \alpha & (\beta+2) \end{bmatrix}. \tag{7.5}$$

For a pair (α, β) we have an expression of the transfer function ($i \in \{1, 2, 3\}$)

$$F_{q_i}(\mathrm{s}) = C_{q_i}(\mathrm{sI} - A_{q_i})^{-1}B = \frac{(\beta+2)\mathrm{s} - \alpha}{(\mathrm{s}+1)(\mathrm{s}-(1+\alpha+\beta))}. \tag{7.6}$$

For the possible three index values $q \in \mathscr{Q}$, we obtain the corresponding three dynamic behaviours [37, 44, 45]:

$$
\begin{aligned}
\mathfrak{B}^{\infty}_{q_1} &= \left\{ (u(\cdot),\, y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_i,\, \mathbb{R}^2\right) \middle| \left[-1 \,\middle|\, (\mathrm{d}/\mathrm{dt}+1) \right] \begin{bmatrix} u \\ \hline y \end{bmatrix} = 0 \right\} \\
\mathfrak{B}^{\infty}_{q_2} &= \left\{ (u(\cdot),\, y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_j,\, \mathbb{R}^2\right) \middle| \left[-(2\mathrm{d}/\mathrm{dt}+1) \,\middle|\, (\mathrm{d}/\mathrm{dt}+1)(\mathrm{d}/\mathrm{dt}) \right] \begin{bmatrix} u \\ \hline y \end{bmatrix} = 0 \right\} \\
\mathfrak{B}^{\infty}_{q_3} &= \left\{ (u(\cdot),\, y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_k,\, \mathbb{R}^2\right) \middle| \left[-1 \,\middle|\, (\mathrm{d}/\mathrm{dt}+1)(\mathrm{d}/\mathrm{dt}+2) \right] \begin{bmatrix} u \\ \hline y \end{bmatrix} = 0 \right\} \qquad (7.7)
\end{aligned}
$$

associated with the disjoints $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}_k \in \left\{ \mathcal{I}_\tau = [T_{\tau-1}, T_\tau) \subset \mathbb{R}^+ \mid \tau \in \mathbb{N},\ T_{\tau-1} \in \mathfrak{T} \right\}$, $i, j, k \in \mathbb{N}$.

Taking into consideration the dynamic "behaviour" determined by Eq. (7.7), one can interpret it as a result of formally different state-space representations. However, one can show that it is a consequence of the same system represented by (7.2), (7.3) and by (7.4). The evident change of its internal structure is caused by the pole-zero cancellation. The last one generates uncontrollable and/or unobservable modes, see Fig. 7.1.

Comparing Fig. 7.1 with the dynamic behaviours (7.7), one can conclude that the lack of order of $\mathfrak{B}^{\infty}_{q_1}$ is due to an unobservable mode $\overline{\mathcal{O}}$. In fact, one could carefully handle the unobservable subspace for getting a desired internal structure. Indeed, in [13] we have taken an advantage of the particular structure (7.3) in the sense of handling the unobservable subspace. This approach also includes the consequent changes of structure inside a known set of models. Let us refer in that connection to the so-called *ladder systems* which consider irreducible factors of order 1 over $\mathbb{R}$, irreducible factors of order 2 over $\mathbb{R}$ and lead/lag compensation networks [7, 13].

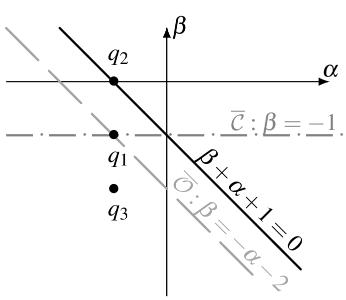


Fig. 7.1 Structural properties of the *state-space representation* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$ with matrices (7.5). The characteristic polynomial is $\det(\mathrm{s}\,\mathrm{I} - A_{q_i}) = (\mathrm{s}+1)\big(\mathrm{s} - (1+\alpha+\beta)\big)$, the uncontrollable and unobservable modes, $\overline{\mathcal{C}}$ and $\overline{\mathcal{O}}$, are $\det \left[\begin{array}{c|c} 0 & (1+\beta) \\ 1 & \beta \end{array} \right] = -(1+\beta)$ and $\det \begin{bmatrix} \alpha & (\beta+2) \\ \hline (\alpha^2 + (\alpha+1)(\beta+2)) & (\alpha(\beta+1) + \beta(\alpha+2)) \end{bmatrix} = -(2+\alpha+\beta)^2$, respectively

As it is shown in the next sections, the particular matrices' structure (7.3) of the *time-dependent autonomous switched systems* (7.2) enables its representation by *time-independent linear rectangular implicit descriptions* (7.1). The corresponding control here is given in the form of static or dynamic descriptor variable feedbacks.

7.3 Implicit Systems

Let us come back to the *time-dependent autonomous switched systems* described by (7.2) and (7.3). We next define the *descriptor variable* $x = \begin{bmatrix} \bar{x}^T & \hat{x}^T \end{bmatrix}^T$, where $\hat{x} = -\overline{D}(q_i)\bar{x}$, and get the following expression:

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} x = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} x. \tag{7.8}$$

From (7.8), we easily deduce that all the possible structure variations of (7.2) and (7.3) share the same dynamics represented by the *rectangular implicit representation*:

$$\begin{bmatrix} \mathrm{I} & 0 \end{bmatrix} \mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \end{bmatrix} x + Bu \quad \text{and} \quad y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} x\,. \tag{7.9}$$

So, if there is a static or dynamic descriptor variable feedback controlling the *rectangular implicit representation* (7.9), it also controls (7.2) and (7.3).

We next review some properties of the *rectangular implicit representations*. These useful properties make it easy to study the *time-dependent autonomous switched systems* in the theoretic framework of *linear time-invariant implicit systems theory*.

7.3.1 Existence of Solution

Let us begin with some formal definitions of the *implicit representation* Σ^{imp} (E, A, B, C) of the generic form (cf. (7.1)).

Definition 7.1 (*Implicit representation*) An *implicit representation*, Σ^{imp} (E, A, B, C), is a *set of differential and algebraic equations* of the form (7.1), where

Hypothesis $\boldsymbol{H}1$. $\operatorname{Ker} B = \{0\}$ and $\operatorname{Im} C = \mathscr{Y}$.
Hypothesis $\boldsymbol{H}2$. $\operatorname{Im} \begin{bmatrix} E & A & B \end{bmatrix} = \underline{\mathscr{X}}_{eq}$.

Definition 7.2 (*Input/descriptor system* [12, 24, 37]) An implicit representation Σ^{imp} (E, A, B),

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu \quad \forall\, t \geq 0, \tag{7.10}$$

is called an *input/descriptor system*, when for all initial condition $x_0 \in \mathscr{X}_d$, there exists at least one solution $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$, such that $x(0) = x_0$. The *input/descriptor system* is completely defined by a triple: $\Sigma_{i/d} = \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d,\right.$ $\left.\mathfrak{B}_{[E,A,B]}\right)$, with behaviour:

$$\mathfrak{B}_{[E,A,B]} = \left\{(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{U} \times \mathscr{X}_d\right) \middle|\ \left[\,(E\mathrm{d}/\mathrm{dt} - A)\ -B\,\right] \begin{bmatrix} x \\ u \end{bmatrix} = 0\right\}. \tag{7.11}$$

At this point, it is important to clarify what exactly means the qualitative expression "*there exists at least one solution*".

Let us first review the existence of solution for two conceptually crucial points:

1. *given any initial condition* and
2. *for all admissible inputs*.

7.3.1.1 Existence of Solution Given Any Initial Condition

A. Behavioural approach

Following Hautus [20] and Hautus and Silverman [21], Geerts [18] generalized the solvability results of [19]. An important advantage of this generalization is the natural way of definition. It is based on the distributional framework [41] and, moreover, considers the usual time domain associated with the ordinary differential equations. This fact constitutes the real starting point of the so-called behavioural approach [37]. Geerts introduced the following definition for the distributional version[2] of the *implicit representation* (7.10)[3]: $pEx = Ax + Bu + Ex_0$.

Definition 7.3 (*C-solvability in the function sense* [18]) Given the *solution set*, $S_C(x_0, u) \stackrel{\text{def}}{=} \left\{x \in \mathscr{C}^{n_d}_{\text{imp}} \middle|\ [pE - A]x = Bu + Ex_0\right\}$, the *implicit representation* (7.10) is *C-solvable in the function sense* if $\forall x_0 \in \mathscr{X}_d\ \exists\ u \in \mathscr{C}^m_{\text{sm}} : S_C(x_0, u) \cap \mathscr{C}^n_{\text{sm}} \neq \emptyset$. Given the *"consistent initial conditions set"*, $\mathcal{I}_C \stackrel{\text{def}}{=} \left\{z_0 \in \mathscr{X}_d |\ \exists u \in \mathscr{C}^m_{\text{sm}}\ \exists x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}} : x(0^+) = z_0\right\}$, and the *"weakly consistent initial conditions set"*, $\mathcal{I}^w \stackrel{\text{def}}{=} \left\{z_0 \in \mathscr{X}_d |\ \exists u \in \mathscr{C}^m_{\text{sm}}\ \exists x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}}\right\}$, a point $x_0 \in \mathscr{X}_d$ is called *C-consistent* if $x_0 \in \mathcal{I}_C$ and *weakly C-consistent* if $x_0 \in \mathcal{I}^w$.

The *C-solvability in the function sense* is concerned with solutions only composed of some arbitrarily often differentiable ordinary functions. The two notions of consistency, *C-consistent* and *weakly C-consistent*, lead to *smooth solutions*, namely,

---

[2]Geerts [18] considered the linear combinations of impulsive and smooth distributions, with μ coordinates, denoted by $\mathscr{C}^\mu_{\text{imp}}$, as the signal sets. The set $\mathscr{C}^\mu_{\text{imp}}$ is a subalgebra and it can be decomposed as $\mathscr{C}^\mu_{\text{p-imp}} \oplus \mathscr{C}^\mu_{\text{sm}}$, where $\mathscr{C}^\mu_{\text{p-imp}}$ and $\mathscr{C}^\mu_{\text{sm}}$ denote the subalgebras of pure impulses and smooth distributions, respectively [41]. The unit element of this subalgebra is the Dirac delta distribution, δ. Any linear combination of δ and its distributional derivatives $\delta^{(\ell)}$, $\ell > 1$, is called impulsive.

[3]Ex_0 stands for $Ex_0\delta$, $x_0 \in \mathscr{X}_d$ being the initial condition, and pEx stands for $\delta^{(1)} * Ex$ ($*$ denotes convolution); if pEx is smooth and $E\dot{x}$ stands for the distribution that can be identified with the ordinary derivative, $E\mathrm{d}x/\mathrm{d}t$, then $pEx = E\dot{x} + Ex_{0^+}$.

with no impulsions, but the *C-consistency* avoids jumps at the origin, namely, the *smooth solutions* are continuous on the left. Note that the *weakly C-consistent* case enables jumps at the origin.

Geerts in [18] characterized the existence of solutions for every initial condition in his main result (see his Corollary 3.6, Proposition 4.2 and Theorem 4.5). Hereafter, we summarize some results concerning *smooth solutions*, together with their geometric equivalences (see [12] for details).

Theorem 7.1 (C-solvability in the function sense [18]) *If **H**2 is fulfilled, then the* implicit representation *(7.10) is* C-solvable in the function sense *if and only if* $\mathcal{I}^w = \mathscr{X}_d$*, namely, if and only if* $\operatorname{Im} E + \mathbf{A}\operatorname{Ker} E + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}$*, i.e., if and only if*

$$E\mathscr{V}^*_{\mathscr{X}_d} = \operatorname{Im} E. \tag{7.12}$$

Moreover, the initial conditions will be C-consistent, $\mathcal{I}_C = \mathscr{X}_d$*, if and only if* $\operatorname{Im} E + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}$*, i.e., if and only if*

$$E\mathscr{V}^*_{\mathscr{X}_d} + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}. \tag{7.13}$$

$\mathscr{V}^*_{\mathscr{X}_d}$ is the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$ [30, 43],

$$\mathscr{V}^*_{\mathscr{X}_d} \stackrel{\text{def}}{=} \sup\{\mathscr{V} \subset \mathscr{X}_d \mid A\mathscr{V} \subset E\mathscr{V} + \operatorname{Im} B\}, \tag{7.14}$$

which is the limit of the following algorithm:

$$\mathscr{V}^0_{\mathscr{X}_d} = \mathscr{X}_d\,, \quad \mathscr{V}^{\mu+1}_{\mathscr{X}_d} = A^{-1}\left(E\mathscr{V}^{\mu}_{\mathscr{X}_d} + \operatorname{Im} B\right). \tag{7.15}$$

B. Viability approach

In order to study the reachability problem for implicit systems, Frankowska in [16] introduced the specific set-valued map (the set of all admissible velocities) $\mathbf{F} : \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}\big(Ax + \operatorname{Im} B\big) = \{v \in \mathscr{X} \,|\, Ev \in Ax + \operatorname{Im} B\}$, and considered the generic differential inclusion:

$$\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x), \quad \text{where } \; x(0) = x_0. \tag{7.16}$$

Frankowska [16] showed that the solutions of (7.10) and (7.16) are the same. Additionally, the meaning of a viable solution was constructively clarified. The largest subspace of such viable solutions is given as follows.

Definition 7.4 (*Viability kernel* [2, 16]) An absolutely continuous function, $x : \mathbb{R}^+ \to \mathscr{X}_d$, is called a *trajectory* of (7.16), if $x(0) = x_0$ and $\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x)$ for almost every $t \in \mathbb{R}^+$, that is to say, if there exists a measurable function, $u : \mathbb{R}^+ \to \mathscr{U}$, such that $x(0) = x_0$ and $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$ for almost every $t \in \mathbb{R}^+$.

Let $\mathcal{K}$ be a subspace[4] of $\mathcal{X}_d$. A trajectory x of (7.16) is called *viable in* $\mathcal{K}$, if $x(t) \in \mathcal{K}$ for all $t \geq 0$. The set of such trajectories is called *the set of viable solutions in* $\mathcal{K}$. The subspace $\mathcal{K}$ is called *a viability domain of* $\mathbf{F}$, if for all $x \in \mathcal{K}$: $\mathbf{F}(x) \cap \mathcal{K} \neq \emptyset$. The subspace $\mathcal{K}$ is called *the viability kernel of* (7.16) when it is the largest viability domain of $\mathbf{F}$.

Theorem 7.2 (Viability kernel [2]) *The supremal* (A, E, B)*-invariant subspace contained in* $\mathcal{X}_d$*,* $\mathcal{V}^*_{\mathcal{X}_d}$*, is the viability kernel of* $\mathcal{X}_d$ *for the set-valued map,* $\mathbf{F} : \mathcal{X}_d \rightsquigarrow \mathcal{X}_d$*,* $\mathbf{F}(x) = E^{-1}(Ax + \operatorname{Im} B)$*. Moreover, for all* $x_0 \in \mathcal{V}^*_{\mathcal{X}_d}$*, there exists a trajectory,* $x \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathcal{V}^*_{\mathcal{X}_d}\right)$*, solution of (7.10), satisfying* $x(0) = x_0$*.*

A singular system is "strict" *when the viability kernel coincides with the whole descriptor space,* $\mathcal{X}_d$*, namely,*

$$\mathcal{V}^*_{\mathcal{X}_d} = \mathcal{X}_d. \tag{7.17}$$

As we have shown in Theorem 7.1, the specific condition (7.13) implies that for any initial condition $\lim_{t \to 0^+} x(t) = x_0 \in \mathcal{X}_d$, there exists at least one solution pair $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathcal{U} \times \mathcal{X}_d\right)$ of (7.10).

7.3.1.2 Existence of Solution for All Admissible Inputs

When an implicit representation $\Sigma^{imp}\ (E,\ A,\ B)$ has a solution for all admissible inputs, it is simply called *solvable*.

Definition 7.5 (*Solvable representation* [4]) The *implicit representation* (7.10) is called *solvable*, if for any $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathcal{U}\right)$, there exists at least one trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+,\ \mathcal{X}_d)$ solution of $\left[E\mathrm{d}/\mathrm{d}t - A\right]x(t) = Bu(t), \forall\, t \geq 0$.

Lemma 7.1 (Existence of solution [25, 27, 28]) *The* implicit representation *(7.10) admits at least one solution for all* $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathcal{U}\right)$
if and only if

$$\operatorname{rang}\left[\,\lambda E - A \;\; B\,\right] = \operatorname{rang}\left[\,\lambda E - A\,\right],\ \ \textit{for almost all}\ \ \lambda \in \mathbb{C} \tag{7.18}$$

if and only if

$$\operatorname{Im} B \subset \operatorname{Im}\,(\lambda E - A),\ \ \textit{for almost any}\ \ \lambda \in \mathbb{C} \tag{7.19}$$

if and only if

$$\operatorname{Im} B \subset \mathcal{B}^*_1 + \mathcal{B}^*_2, \tag{7.20}$$

[4]We restrict our discussion to subspaces of finite-dimensional vector spaces. In [16] and in [2], these definitions are stated in the more general framework of closed sets of normed vector spaces.

where $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ are the limits of the following geometric algorithms:

$$\mathcal{B}_1^0 = \underline{\mathscr{X}}_{eq} \ , \quad \mathcal{B}_1^{\mu+1} = EA^{-1}\mathcal{B}_1^{\mu} \tag{7.21}$$

$$\mathcal{B}_2^0 = \{0\} \ , \quad \mathcal{B}_2^{\mu+1} = AE^{-1}\mathcal{B}_2^{\mu} . \tag{7.22}$$

Corollary 7.1 (Existence of solution [8]) *The following statements hold true:*

1. *If the geometric condition*

$$\mathrm{Im}\, A + \mathrm{Im}\, B \subset \mathrm{Im}\, E \tag{7.23}$$

   holds, then the implicit representation *(7.10) admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty \left(\mathbb{R}^+, \mathscr{U}\right)$, and for any initial condition $\lim_{t\to 0^+} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^\infty \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$ solution of (7.10).*
2. *If the geometric condition*

$$\mathrm{Im}\, E + \mathrm{Im}\, A = \underline{\mathscr{X}}_{eq} \tag{7.24}$$

   holds, then the implicit representation *(7.10) admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty \left(\mathbb{R}^+, \mathscr{U}\right)$.*

Indeed, (7.23) implies $\mathcal{B}_1^* = \mathrm{Im}\, E$ and $\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d$, and (7.24) implies (7.19).

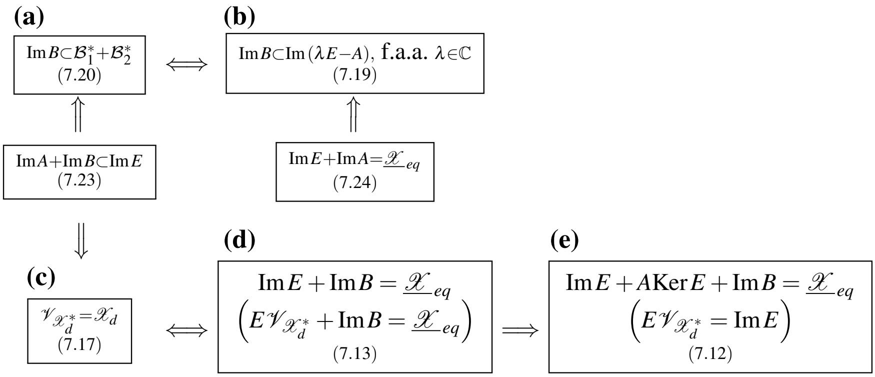


Fig. 7.2 Connexions between the notions of existence of solution. **a**, **b** Conditions of existence of at least one solution for all admissible inputs of Lebret [25]. **c** Condition of viable solution of Aubin and Frankowska [2] or smooth solution (without any jump) of Özçaldiran and Haliločlu [36]. **d** Condition of Geerts [18] guaranteeing that the set of consistent initial conditions equals the whole space. **e** Condition of *C-solvability in the function sense* of Geerts [18] or the condition of Przyluski and Sosnowski [38] guaranteeing that the set of initial conditions of smooth solutions (with possible jumps) equals the whole space, or the impulse controllability condition of Ishihara and Terra [23], or the impulse-mode controllability with arbitrary initial conditions of Hou [22]

In Fig. 7.2, we compare the solvability conditions of Geerts (7.12) and (7.13), based on a distributional framework, with the solvability condition of Frankowska (7.17), based on a viability approach; as well as the solvability conditions of Lebret [25], (7.19) and (7.20).

7.3.2 Proper Implicit Representations

We now are interested in the proper linear systems in the presence of internal switches, which can be represented and controlled by means of *implicit representations*. Let us first introduce some basic definitions and present the necessary analytic results which naturally lead to *implicit representations* given by a proper linear system with internal structure variations.

Definition 7.6 (*Regularity* [17]) A pencil $[\lambda E - A]$, with $\lambda \in \mathbb{C}$, is called *regular* if it is square and its determinant is not the zero polynomial. An *implicit representation* $\Sigma^{imp}\ (E, A, B, C)$ is called *regular* if its pencil $[\lambda E - A]$ is regular.

Definition 7.7 (*Internal properness* [1, 3]) An *implicit representation* $\Sigma^{imp}\ (E, A, B, C)$ is called *internally proper* if its pencil $[\lambda E - A]$ is proper, namely, if its pencil is regular and has no infinite elementary divisor greater than 1. In other words, there is no derivative action in the system dynamics.

It is common knowledge that an *implicit representation* is completely characterized by the canonical Kronecker form of its pencil $[\lambda E - A]$, with $\lambda \in \mathbb{C}$. Usually, there are four possible types of suitable blocks [17]:

1. *Finite elementary divisors (fed)*, as for example, $[\lambda E_{fed} - A_{fed}] = \begin{bmatrix} (\lambda - \alpha) & 1 \\ 0 & (\lambda - \alpha) \end{bmatrix}$. The *fed* corresponds to the proper part of the system (integral actions), and it was geometrically characterized by Wong [46] and Bernhard [3].
2. *Infinite elementary divisors (ied)*, as for example, $[\lambda E_{ied} - A_{ied}] = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. The *ied* corresponds to the non-proper part of the system (time-derivative actions), and it was geometrically characterized by Armentano [1].
3. *Minimal column indices (mci)*, as for example, $[\lambda E_{mci} - A_{mci}] = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix}$. The *mci* corresponds to the existence of a certain degree of freedom (more variables than equations), and it was geometrically characterized by Armentano [1].
4. *Minimal row indices (mri)*, as for example, $[\lambda E_{mri} - A_{mri}] = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \\ 0 & 1 \end{bmatrix}$. The *mri* is related with the existence of algebraic and differential constraints on the external signals. For example, an admissible input has to satisfy some given algebraic and differential equations. Clearly, [1] geometrically characterized the *mri*.

Example 7.1 Let us consider the following *implicit representation*:

$$[E\mathrm{d}/\mathrm{dt} - A]x - Bu = \left[\begin{array}{c|cc} \boxed{1}_{ied} & 0 & 0 \\ \hline 0 & \multicolumn{2}{|c}{\boxed{\left[\, \mathrm{d}/\mathrm{dt} \;\, 1 \,\right]}_{mci}} \end{array}\right] \left[\begin{array}{c} x_{ied} \\ \hline \bar{x}_{mci} \\ \hat{x}_{mci} \end{array}\right] - \left[\begin{array}{c} -1 \\ \hline 1 \end{array}\right] u = 0,$$

$$y - Cx = y - \left[\begin{array}{c|cc} 1 & 2 & -1 \end{array}\right] \left[\begin{array}{c} x_{ied} \\ \hline \bar{x}_{mci} \\ \hat{x}_{mci} \end{array}\right] = 0, \tag{7.25}$$

and let us suppose that the degree of freedom satisfies the algebraic equation:

$$Dx = \left[\begin{array}{c|cc} a & b & c \end{array}\right] \left[\begin{array}{c} x_{ied} \\ \hline \bar{x}_{mci} \\ \hat{x}_{mci} \end{array}\right] = 0, \tag{7.26}$$

1. if $\left[\begin{array}{c|cc} a & b & c \end{array}\right] = \left[\begin{array}{c|cc} 1 & 1 & 0 \end{array}\right]$, we then get the non-proper external behaviour:

$$y(t) = \mathrm{d}u(t)/\mathrm{d}t, \tag{7.27}$$

2. if $\left[\begin{array}{c|cc} a & b & c \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & -1 \end{array}\right]$, we then get the proper external behaviour:

$$y(t) = \mathrm{e}^{-t}\bar{x}_c(0) + \int_0^t \mathrm{e}^{-(t-\tau)}u(\tau)\mathrm{d}\tau - u(t). \tag{7.28}$$

As we can see from the analysis realized above the existence of the degree of freedom can lead to a non-proper solution. This fact implies the necessity to add some specific geometric conditions on the degree of freedom in order that proper solutions are guaranteed.

Let us consider an implicit representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, where $\mathbb{E} : \mathscr{X}_d \to \underline{\mathscr{X}}_g$, $\mathbb{A} : \mathscr{X}_d \to \underline{\mathscr{X}}_g$, $\mathbb{B} : \mathscr{U} \to \underline{\mathscr{X}}_g$, $C : \mathscr{X}_d \to \mathscr{Y}$, such that the following hypotheses are satisfied:

Hypothesis ***H*3**. $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ satisfies the standard assumptions **H**1 and **H**2, namely,

$$\mathrm{Ker}\mathbb{B} = \{0\}, \quad \mathrm{Im}\, C = \mathscr{Y} \text{ and } \mathrm{Im}\left[\, \mathbb{E}\ \mathbb{A}\ \mathbb{B} \,\right] = \underline{\mathscr{X}}_g. \tag{7.29}$$

Hypothesis ***H*4**. $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, which is implied by (cf. Corollary 7.1 (7.24)):

$$\mathrm{Im}\,\mathbb{E} + \mathrm{Im}\,\mathbb{A} = \underline{\mathscr{X}}_g. \tag{7.30}$$

Hypothesis ***H*5**. The differential part of $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, say $\Sigma^{ir}(E, A, B)$: $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$ ($E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$, $\underline{\mathscr{X}}_{eq} \subset$

$\underline{\mathscr{X}}_g$), admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, and for any initial condition $\lim_{t\to 0^+} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$ solution of $\Sigma^{ir}(E, A, B)$, which is implied by (cf. Corollary 7.1: (7.23))

$$\operatorname{Im} A + \operatorname{Im} B \subset \operatorname{Im} E. \tag{7.31}$$

Keeping in mind assumptions (7.29), (7.30) and (7.31), the implicit representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ can be expressed as follows:

$$\underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{\mathbb{E}} \frac{\mathrm{d}}{\mathrm{dt}} x = \underbrace{\begin{bmatrix} A \\ D \end{bmatrix}}_{\mathbb{A}} x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathbb{B}} u \;\text{ and }\; y = Cx. \tag{7.32}$$

Lemma 7.2 (Σ^{ig} internally proper [8]) *Let us consider the implicit global representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, (7.32), satisfying the standard assumptions (7.29), and the solvability assumptions (7.30) and (7.31). Then, (7.32) is internally proper if and only if*

$$\operatorname{Ker} D \oplus \operatorname{Ker} E = \mathscr{X}_d. \tag{7.33}$$

Let us note that assumptions (7.29.c) and (7.30) are equivalent to

$$\underline{\mathscr{X}}_g = \operatorname{Im} E \oplus \operatorname{Im} D. \tag{7.34}$$

Let us introduce the following implicit representations definitions.

Definition 7.8 (*Rectangular implicit representation*) An implicit representation $\Sigma^{ir}(E, A, B, C)$,

$$E \frac{\mathrm{d}}{\mathrm{dt}} x = Ax + Bu \quad \text{and} \quad y = Cx, \tag{7.35}$$

where the matrices E and A have more columns than rows, and the solvability condition (7.31) is satisfied, is called *implicit rectangular representation*.

Definition 7.9 (*Algebraic constraint*) An *algebraic constraint* is a set of algebraic equations independent of the input variable, $\Sigma^{alc}(0, D, 0)$:

$$0 = Dx, \tag{7.36}$$

where $D : \mathscr{X}_d \to \underline{\mathscr{X}}_{alc}$ is a linear map and the finite-dimensional space, $\underline{\mathscr{X}}_{alc}$, is called the algebraic constraint space.

Definition 7.10 (*Global implicit representation*) If we gather the *implicit rectangular representation* (7.35) with the *algebraic constraint* (7.36), which describes the degree of freedom, we get the following *global implicit representation*, $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$:

$$\mathbb{E}\frac{\mathrm{d}}{\mathrm{dt}}x = \mathbb{A}x + \mathbb{B}u \quad \text{and} \quad y = Cx$$
$$\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ \mathbb{A} = \begin{bmatrix} A \\ D \end{bmatrix}, \ \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{7.37}$$

The Cartesian product, $\underline{\mathscr{X}}_g \stackrel{\text{def}}{=} \underline{\mathscr{X}}_{eq} \times \underline{\mathscr{X}}_{cal}$, is the *space of global equations*. We shall assume that (7.34) is satisfied.

7.3.3 Switched Systems

Let us note that the above-mentioned conditions (7.33) and (7.34) imply the following important statement: there exist bases in $\mathscr{X}_d$ and also in $\underline{\mathscr{X}}_g$ such that (7.32) takes the specific form ($\overline{D}(q_i)$ is a variable matrix with respect to the location $q_i \in \mathscr{Q}$):

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \text{ and } y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}, \tag{7.38}$$

and defining: $\begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}$, we get

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} (\overline{A}_0 + \overline{A}_1 \overline{D}(q_i)) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} (\overline{C}_0 + \overline{C}_1 \overline{D}(q_i)) & -\overline{C}_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix}, \tag{7.39}$$

which coincides with *time-dependent autonomous switched systems* (7.2) with the particular structure (7.3).

Remark 7.1 (Implicit representation of switched systems) The *time-dependent autonomous switched systems* (7.2), with the particular structure (7.3), are described by the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ (7.37), where the linear maps $E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{X}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$, $C : \mathscr{X}_d \to \mathscr{Y}$ and $D_i : \mathscr{X}_d \to \underline{\mathscr{X}}_{alc}$ are equal to

$$E = \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix}, \ A = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \end{bmatrix}, \ C = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix}, \ D_i = \begin{bmatrix} \overline{D}(q_i) & \mathrm{I} \end{bmatrix}. \tag{7.40}$$

- The fixed structure of all $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$, which is active for a particular D_i, is described by the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ (7.1).
- The degree of freedom is characterized by the *algebraic constraints* $\Sigma^{alc}(0, D_i, 0)$ (7.36).
- Since dim $\underline{\mathscr{X}}_{eq}$ < dim $\mathscr{X}_d$, there then exists a degree of freedom.

- Since $\operatorname{Im} A + \operatorname{Im} B \subset \operatorname{Im} E = \underline{\mathscr{X}}_{eq}$, $\Sigma^{ir}(E, A, B,\ C)$ admits at least one solution for all $u(\cdot) \in \mathcal{C}^{\infty}\left(\mathbb{R}^{+},\ \mathscr{U}\right)$, and for any initial condition $\lim_{t \to 0^{+}} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^{\infty}\left(\mathbb{R}^{+},\ \mathscr{U} \times \mathscr{X}_d\right)$ solution of $\Sigma^{ir}(E, A, B)$.
- Since $\operatorname{Im} D_i = \underline{\mathscr{X}}_{alc}$ and $\underline{\mathscr{X}}_{eq} \times \underline{\mathscr{X}}_{alc} \approx \mathscr{X}_d$ for all $i \in \{1, \ldots, \eta\}$ then the $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ have unique solutions for any $i \in \{1, \ldots, \eta\}$.
- Since $\mathrm{Ker} D_i \oplus \mathrm{Ker} E = \mathscr{X}_d$ for all $i \in \{1, \ldots, \eta\}$ then the *implicit global representations* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ are proper.

7.3.4 Example (Part 2)

Let us continue Example 7.2.1

A. Global implicit representation
The *state-space representation* $\Sigma^{state}(A_{q_i}, B,\ C_{q_i})$, (7.2) and (7.5), is also represented by the following *global implicit representation* (cf. (7.39)):

$$
\begin{array}{r} E\,\mathrm{Ker}D \\ \\ D\,\mathrm{Ker}E \end{array}
\overset{\mathrm{Ker}E}{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]}
\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} \bar{x} \\ \hline \tilde{x} \end{bmatrix}
=
\overset{\mathrm{Ker}D}{\left[\begin{array}{cc|c} \alpha & (1+\beta) & 0 \\ (1+\alpha) & \beta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]}
\begin{bmatrix} \bar{x} \\ \hline \tilde{x} \end{bmatrix}
+
\left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] u
$$

$$
y = \left[\begin{array}{cc|c} \alpha & (\beta+2) & -1 \end{array}\right] \begin{bmatrix} \bar{x} \\ \hline \tilde{x} \end{bmatrix}
\tag{7.41}
$$

Let us note that

- $\operatorname{Im} A + \operatorname{Im} B \subset \operatorname{Im} E$, $\underline{\mathscr{X}}_g = E\mathrm{Ker}D \oplus D\mathrm{Ker}E = \operatorname{Im} E \oplus \operatorname{Im} D$ and $\mathscr{X}_d = \mathrm{Ker}D \oplus \mathrm{Ker}E$ then the *implicit global representation* (7.41) is externally proper (cf. Lemma 7.2).
- The part limited to $\mathrm{Ker}E$ and $D\mathrm{Ker}E$ is algebraically redundant.
- The part of the *implicit global representation* (7.41) limited to $\mathrm{Ker}D$, in the domain, and to $E\mathrm{Ker}D$, in the co-domain, which matrices are depicted with continuous lines, coincides with the state representations (7.2) and (7.5).
- The upper part of the *implicit global representation* (7.41) is an *implicit rectangular representation*, in which matrices are depicted with dashed lines. This part explicitly contains the changes in the behaviour which are due to the switches in the α and β parameters.
- The lower part of the *implicit global representation* (7.41) is an *algebraic constraint* which includes the components of the descriptor variable which are always zero.

B. Fixed structure

Premultiplying (7.41) by $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right]$, and defining: $x = \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline -\alpha & -\beta & 1 \end{array}\right] \begin{bmatrix} \bar{x} \\ \bar{\bar{x}} \end{bmatrix}$, we get the *implicit global representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ (cf. (7.37)):

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \frac{\mathrm{d}}{\mathrm{dt}} x = \left[\begin{array}{ccc} 0 & 1 & -1 \\ 1 & 0 & -1 \\ \hline \alpha & \beta & 1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] u \ \text{ and } \ y = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} x. \tag{7.42}$$

In the upper part of the *implicit global representation* (7.42), we get the *rectangular implicit representation* $\Sigma^{ir}(E, A, B, C)$ (cf. (7.35)):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} x = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ \text{ and } \ y = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} x. \tag{7.43}$$

In the lower part of the *implicit global representation* (7.42), we get the *algebraic constraint* $\Sigma^{alc}(0, D_i, 0)$ (cf. (7.36)):

$$0 = \begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x. \tag{7.44}$$

C. Kronecker normal form

In order to better understand how the internal structure variation is acting in the implicit representations $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ and $\Sigma^{ir}(E, A, B, C)$, let us obtain their respective Kronecker normal forms.

C.1 Kronecker normal forms of $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$:

- If $\beta = -1$:

$$G_{ig1}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig1} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda - \alpha)}_{fed} & 0 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}. \tag{7.45}$$

- If $\beta \neq -1$ and $\alpha + \beta = -2$:

$$G_{ig2}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig2} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda + 1)}_{fed} & 1 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}. \tag{7.46}$$

- If $\beta \neq -1$ and $\alpha + \beta \neq -2$:

$$G_{ig3}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig3} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda - 1 - \alpha - \beta)}_{fed} & 0 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}, \quad (7.47)$$

where $G_{ig1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, $D_{ig1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & (1-\alpha) & 1 \end{bmatrix}$, $G_{ig2} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ (1+\beta) & -(1+\beta) & 0 \end{bmatrix}$, $D_{ig2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{1+\beta} \\ 1 & 2 & \frac{\beta}{1+\beta} \end{bmatrix}$, $G_{ig3} = \begin{bmatrix} 0 & 0 & -1 \\ \left(1 - \frac{1+\beta}{2+\alpha+\beta}\right) & \frac{1+\beta}{2+\alpha+\beta} & 1 \\ (1+\beta) & -(1+\beta) & 0 \end{bmatrix}$ and $D_{ig3} = \begin{bmatrix} 0 & 1 & \frac{1}{2+\alpha+\beta} \\ 0 & 1 & \left(\frac{1}{2+\alpha+\beta} - \frac{1}{(1+\beta)}\right) \\ 1 & -(\alpha+\beta) & \left(\frac{\beta}{1+\beta} - \frac{\alpha+\beta}{2+\alpha+\beta}\right) \end{bmatrix}$.

C.2 Kronecker normal form of $\Sigma^{ir}(E, A, B, C)$:

$$G_{ir}[\lambda E - A]D_{ir} = \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \end{matrix}}_{mci} & 0 \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{(\lambda + 1)}_{fed} \end{bmatrix} \quad (7.48)$$

where $G_{ir} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $D_{ir} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Remark 7.2 (Internal structure variation) When we split the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, (7.41), *via* the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$, (7.42), into the *rectangular implicit representation* $\Sigma^{ir}(E, A, B, C)$, (7.43), and the *algebraic constraint* $\Sigma^{alc}(0, D_k, 0)$, (7.44), we get the common structure of the system which is described by $\Sigma^{ir}(E, A, B, C)$ (7.43).

When comparing the Kronecker normal forms, (7.45), (7.46) and (7.47), of the pencils associated with (7.42), with the Kronecker normal form, (7.48), of the pencil associated with (7.43), we realize that the **variable internal structure** of the *global implicit representation* (7.42) is taken into account by the **fixed block** *minimal column index* of the Kronecker normal form (7.48), $\boxed{\lambda \ 1}_{mci}$, associated with the *rectangular implicit representation* (7.43).

7.4 Reachability

Reachability is the most important concept studied in System Theory, since it characterizes the set of vectors which can be reached from the origin, in a finite time, following trajectories, solutions of the system. For *state-space representations* Σ^{state} (A, B), $\mathrm{d}x/\mathrm{d}t = Ax + Bu$, this set of vectors is geometrically characterized by the reachability subspace (see, for example, [47]):

$$\mathscr{R}^* = \langle A \mid \mathrm{Im}\, B \rangle \stackrel{\mathrm{def}}{=} \mathrm{Im}\, B + A\mathrm{Im}\, B + \cdots + A^{n-1}\mathrm{Im}\, B, \tag{7.49}$$

and the trajectories are generated by the external control input, u.

For the case of *implicit representations* $\Sigma^{imp}(E, A, B)$, $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$, where E and A are square but $[\lambda E - A]$ is not necessarily invertible, Özçaldiran extended his geometric characterization of reachability by considering the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$ [34, 35]:

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}_{\mathscr{X}^*_d} \cap \mathscr{S}^*_{\mathscr{X}_d}, \tag{7.50}$$

where $\mathscr{V}^*_{\mathscr{X}_d}$ is the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$ (7.14), computed by (7.15), and $\mathscr{S}^*_{\mathscr{X}_d}$ is the infimal (E, A, B)-invariant subspace associated with $\mathrm{Im}\, B$,

$$\mathscr{S}^*_{\mathscr{X}_d} = \inf \left\{ \mathscr{S} \subset \mathscr{X}_d \;\middle|\; \mathscr{S} = E^{-1}\left(A\mathscr{S} + \mathrm{Im}\, B\right) \right\}, \tag{7.51}$$

which is the limit of the algorithm:

$$\mathscr{S}^0_{\mathscr{X}_d} = \mathrm{Ker} E\ , \quad \mathscr{S}^{\mu+1}_{\mathscr{X}_d} = E^{-1}\left(A\mathscr{S}^{\mu}_{\mathscr{X}_d} + \mathrm{Im}\, B\right). \tag{7.52}$$

The geometric characterization of $\mathscr{R}^*_{\mathscr{X}_d}$, given by (7.50), (7.14) and (7.51), is a nice generalization of the classical state-space characterization (7.49). Indeed, for $\Sigma^{state}(A, B) = \Sigma^{imp}(\mathrm{I}, A, B)$: $\mathscr{V}^*_{\mathscr{X}} = \mathscr{X}$ and $\mathscr{S}^*_{\mathscr{X}} = \langle A \mid \mathrm{Im}\, B \rangle$. Thus, it would appear quite natural that for the more general representations $\Sigma^{imp}(E, A, B)$, with E and A not necessarily square, the reachability would be also characterized by $\mathscr{R}^*_{\mathscr{X}_d}$.

The trueness of this conjecture was established by Frankowska [16] using tools of differential inclusions. But, as enhanced later on, this reachability concept needed to be further determined in order to discriminate the action of an effective external control input from an internal degree of freedom.

Indeed, the trajectories generated by $\Sigma^{imp}(E, A, B)$ depend on the initial conditions, $x(0)$, and not only on the external control input, but also possibly on internal degrees of freedom, which are completely free and unknown. Since the system $\Sigma^{imp}(E, A, B)$ represented by (7.10) has more unknowns than equations, when a solution does exist, this is, in general, non-unique. The possible resulting trajectories can be studied within so-called *viability domains*, see Frankowska [16].

7.4.1 $\mathscr{R}^*_{\mathscr{X}_d}$: Reachable Subspace

Frankowska formally defined reachability as follows:

Definition 7.11 (*Reachability* [16]) The implicit representation (7.10) is called *reachable* if for any possible $x_0, x_1 \in \mathscr{X}_d$ and for any time $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot)$, solution of (7.10), such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

And using tools of differential inclusions, she proved for the more general case:

Theorem 7.3 (Reachability [2, 16]) *For any $t_1 > t_0 \geq 0$, the reachable subspace of (7.10) at time t_1, starting from any initial value $x(t_0)$, is equal to $\mathscr{R}^*_{\mathscr{X}_d}$. Moreover, $\mathscr{R}^*_{\mathscr{X}_d}$ is the largest subspace such that for any $x_0, x_1 \in \mathscr{R}^*_{\mathscr{X}_d}$ and any $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{R}^*_{\mathscr{X}_d}\right)$, solution of (7.10), with $x(t_0) = x_0$ and $x(t_1) = x_1$.*

Note that the reachability Definition 7.11 requires no explicit control action!

In order to have a better understanding of Frankowska's reachability concept, let us decompose the *descriptor* and *equation* spaces in function of the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$, $\mathscr{V}^*_{\mathscr{X}_d}$, and of the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$, $\mathscr{R}^*_{\mathscr{X}_d}$.

In the third lemma of [12], it is proved that there exist some complementary subspaces, $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{B}_\mathcal{C}$ and $\mathscr{R}_\mathcal{C}$, such that

$$\begin{aligned} \mathscr{X}_d &= \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{X}_1, \\ \mathscr{V}^*_{\mathscr{X}_d} &= \mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{X}_2, \\ \mathscr{R}^*_{\mathscr{X}_d} &= \mathscr{R}_\mathcal{C} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \operatorname{Ker} E), \end{aligned} \tag{7.53}$$

$$\begin{aligned} \underline{\mathscr{X}}_{eq} &= (E\mathscr{V}^*_{\mathscr{X}_d} + \operatorname{Im} B) \oplus A\mathscr{X}_1, \\ E\mathscr{V}^*_{\mathscr{X}_d} + \operatorname{Im} B &= (A\mathscr{R}^*_{\mathscr{X}_d} + \operatorname{Im} B) \oplus E\mathscr{X}_2, \\ A\mathscr{R}^*_{\mathscr{X}_d} + \operatorname{Im} B &= E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_\mathcal{C}, \end{aligned} \tag{7.54}$$

$$\mathscr{U} = B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \oplus B^{-1}\mathscr{B}_\mathcal{C}, \tag{7.55}$$

satisfying

$$\begin{gathered} \mathscr{R}_\mathcal{C} \approx E\mathscr{R}^*_{\mathscr{X}_d}, \quad \mathscr{X}_2 \approx E\mathscr{X}_2, \quad \mathscr{X}_1 \approx A\mathscr{X}_1, \\ \mathscr{V}^*_{\mathscr{X}_d} \cap \operatorname{Ker} E = \mathscr{R}^*_{\mathscr{X}_d} \cap \operatorname{Ker} E, \quad \operatorname{Im} B \cap E\mathscr{V}^*_{\mathscr{X}_d} = \operatorname{Im} B \cap E\mathscr{R}^*_{\mathscr{X}_d}. \end{gathered} \tag{7.56}$$

Given the geometric decompositions (7.53), (7.54) and (7.55), the implicit representation (7.10) takes the following form (recall (7.56)):

$$\begin{array}{cc} & \begin{array}{cccc} \mathcal{R}_{\mathcal{C}} & \mathcal{R}^*_{\mathcal{X}_d}\cap\operatorname{Ker}E & \mathcal{X}_2 & \end{array} \\ \begin{array}{c} E\mathcal{R}^*_{\mathcal{X}_d} \\ \mathcal{B}_{\mathcal{C}} \\ E\mathcal{X}_2 \\ \\ \end{array} & \left[\begin{array}{cc|c|c} \mathrm{I}_{\mathcal{C}} & 0 & 0 & * \\ 0 & 0 & & \\ \hline 0 & & \mathrm{I}_2 & \\ \hline 0 & & & 0 \end{array}\right] \end{array} \frac{\mathrm{d}}{\mathrm{dt}}x = \begin{array}{cc} & \begin{array}{c} \mathcal{R}^*_{\mathcal{X}_d} \end{array} \\ \begin{array}{c} A\mathcal{R}^*_{\mathcal{X}_d}+\operatorname{Im}B \\ \\ \\ A\mathcal{X}_1 \end{array} & \left[\begin{array}{cc|c|c} \overline{A}_{1,1} & \overline{A}_{1,2} & \widehat{A}_1 & 0 \\ \overline{A}_{2,1} & \overline{A}_{2,2} & \widehat{A}_2 & \\ \hline 0 & & \widehat{A}_3 & \\ \hline 0 & & & \mathrm{I}_1 \end{array}\right] \\ & \begin{array}{cc} \mathcal{V}^*_{\mathcal{X}_d} & \mathcal{X}_1 \end{array} \end{array} x + \begin{array}{c} \begin{array}{cc} B^{-1}E\mathcal{R}^*_{\mathcal{X}_d} & B^{-1}\mathcal{B}_{\mathcal{C}} \end{array} \\ \left[\begin{array}{c|c} \overline{B}_1 & 0 \\ 0 & \mathrm{I}_{\mathcal{B}_{\mathcal{C}}} \\ \hline 0 & \\ \hline 0 & \end{array}\right] \end{array} u \tag{7.57}$$

In the third lemma of [12], it is also proved that

$$E\mathcal{R}^*_{\mathcal{X}_d} = \left\langle \overline{A}_{1,1} \;\middle|\; \operatorname{Im}\left[\,\overline{A}_{1,2}\;\;\overline{B}_1\,\right]\right\rangle. \tag{7.58}$$

- $E\mathcal{R}^*_{\mathcal{X}_d}$ has the form of the classical state reachable subspace (7.49).
- $E\mathcal{R}^*_{\mathcal{X}_d}$ is handled by two actions: *(i)* the input action, *via* $\operatorname{Im}\overline{B}_1$, and *(ii)* the internal degree of freedom action, *via* $\operatorname{Im}\overline{A}_{1,2}$.
- The pair $\left(\overline{A}_{1,1},\left[\overline{A}_{1,2}\;\overline{B}_1\right]\right)$ is reachable in the classical state sense.

Example 7.2 Let us consider the following implicit representation, which is constituted by a *minimal column index* and has no input actions:

$$\left[\,1\;0\,\right]\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}\bar{x}\\ \hat{x}\end{bmatrix} = \left[\,0\;1\,\right]\begin{bmatrix}\bar{x}\\ \hat{x}\end{bmatrix} + \left[\,0\,\right]u. \tag{7.59}$$

Let us compute its reachability subspace $\mathcal{R}^*_{\mathcal{X}_d}$: From algorithms (7.15) and (7.52), we get

$$\mathcal{R}^*_{\mathcal{X}_d} = \mathcal{V}^*_{\mathcal{X}_d} = \{e_1,\; e_2\} = \mathcal{X}_d,$$

and also (cf. (7.53), (7.54) and (7.55)):

$$\begin{gathered}\operatorname{Im}E = E\mathcal{R}^*_{\mathcal{X}_d} = \mathcal{R}_{\mathcal{C}} = \{e_1\},\quad \mathcal{R}^*_{\mathcal{X}_d}\cap\operatorname{Ker}E = \{e_2\},\\ \mathcal{B}_{\mathcal{C}} = \operatorname{Im}B = B^{-1}E\mathcal{R}^*_{\mathcal{X}_d} = B^{-1}\mathcal{B}_{\mathcal{C}} = \{0\}.\end{gathered}$$

The matrices involved in (7.57) are

$$\overline{A}_{1,1} = 0,\quad \overline{A}_{1,2} = 1,\quad \overline{A}_{2,1} = \overline{A}_{2,2} = \emptyset,\quad \overline{B}_1 = 0,\quad \mathrm{I}_{\mathcal{C}} = 1,\quad \mathrm{I}_{\mathcal{B}_{\mathcal{C}}} = \emptyset.$$

- (7.59) is reachable: $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d}$.
- (7.59) has no inputs actions: $\operatorname{Im} B = \{0\}$.
- (7.59) is handled by its internal degree of freedom: $E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \;\middle|\; \operatorname{Im}\left[\, \overline{A}_{1,2} \;\; \overline{B}_1 \,\right]\right\rangle = \langle 0 \mid \operatorname{Im}[\, 1 \;\; 0 \,]\rangle$.

7.4.2 External Reachability

In order to avoid the pathologies illustrated in the previous example, in [6] we have introduced the concept of *external reachability*.

Definition 7.12 (*External reachability* [6]) The implicit representation (7.10) is called *externally reachable* (by $P.D.$ feedback) if

- It is *reachable*.
- The spectrum of $\lambda(E - BF_d) - (A + BF_p)$ can be freely assigned by the selection of $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t$.

Theorem 7.4 (External reachability [6]) *(7.10) is* externally reachable *(by $P.D.$ feedback) if and only if*

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d \tag{7.60}$$

$$\dim\left(\frac{\operatorname{Im} B}{E\mathscr{V}^*_{\mathscr{X}_d} \cap \operatorname{Im} B}\right) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d} \cap \operatorname{Ker} E\right). \tag{7.61}$$

- To prove this Theorem, Bonilla, and Malabre [6] have used tools from Kronecker theory.
- Theorem 7.4 is the combination of the notion of *reachability* by Frankowska [16] and the notion of *unicity of the descriptor variable solution* by Lebret [25]
- Indeed, if there exists a proportional and derivative feedback of the descriptor variable which insures the unicity of the descriptor variable, no internal degree of freedom will be present. This implies that the trajectory of the descriptor variable is compulsorily due to an action of the external control input.

Example 7.3 Let us consider again the implicit representation (7.59) of Example 7.2. For that example, we have computed $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, $\mathscr{B}_{\mathcal{C}} = \{0\}$ and $\mathscr{R}^*_{\mathscr{X}_d} \cap \operatorname{Ker} E = \{e_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.4 is not satisfied.

This means that there exists no external control input $u \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{U}\right)$, to control the whole descriptor variable, x of system (7.59).

Example 7.4 If we add an effective input action to (7.59), say

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u, \tag{7.62}$$

we get from algorithms (7.15) and (7.52): $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} = \{e_1,\ e_2\} = \mathscr{X}_d$, which imply $\mathrm{Im}\, B = \mathrm{Im}\, E = E\mathscr{R}^*_{\mathscr{X}_d} = \mathrm{Im}\, B \cap E\mathscr{R}^*_{\mathscr{X}_d} = \{e_1\} \approx \mathbb{R}^1$ and $\mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker} E = \{e_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.4 is still not satisfied. This means that there exists no external control input $u \in \mathcal{C}^\infty \left(\mathbb{R}^+,\ \mathscr{U}\right)$ able to control the whole descriptor variable, x, of system (7.62).

However, we would like to control, at least partly, systems with representations like (7.62).

7.4.3 *Externally Assignable Output Dynamics*

In order to partly control implicit representations with an internal degree of freedom, like (7.62), we have introduced in [9] the concept of *external output dynamics assignment*.

Definition 7.13 (*External output dynamics assignment* [9]) The implicit representation (7.10) has an *assignable external output dynamics* when there exists a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$ such that the closed-loop system is *externally reachable*.

Theorem 7.5 (External output dynamics assignment [9]) *The implicit representation* $\Sigma^{imp}(E, A, B, C)$*, (7.1), has an* assignable external output dynamics *if and only if*

$$\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{X}_d \tag{7.63}$$

$$\dim \left(\frac{\mathrm{Im}\, B}{E\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Im}\, B} \right) \geq \dim \left(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Ker} E \right) - \dim \left(\mathscr{V}^* \cap E^{-1} \mathrm{Im}\, B \right), \tag{7.64}$$

$\mathscr{V}^*$ is the supremal (A, E, B) invariant subspace contained in $\mathrm{Ker} C$ [30, 31],

$$\mathscr{V}^* \overset{\mathrm{def}}{=} \sup \{ \mathscr{V} \subset \mathrm{Ker} C \mid A\mathscr{V} \subset E\mathscr{V} + \mathrm{Im}\, B \}, \tag{7.65}$$

which is the limit of the following algorithm:

$$\mathscr{V}^0 = \mathscr{X}_d\ ,\quad \mathscr{V}^{\mu+1} = \mathrm{Ker} C \cap A^{-1} (E\mathscr{V}^\mu + \mathrm{Im}\, B). \tag{7.66}$$

$\mathscr{V}^*$ characterizes the supremal part of the implicit representation $\Sigma^{imp}\,(E, A, B, C)$ which can be made unobservable when using a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, namely, for all derivative feedback $F_d : \mathscr{X}_d \to \mathscr{U}$ there exists a proportional feedback $F_p : \mathscr{X}_d \to \mathscr{U}$, such that $(A + BF_p)\mathscr{V}^* \subset (E - BF_d)\mathscr{V}^*$. The set feedback pairs (F_p, F_d) satisfying this geometric inclusion is noted as $\mathbf{F}(\mathscr{V}^*)$.

Condition (7.64) has been established by Lebret [25] to guarantee unicity of the output.

Example 7.5 For the implicit representation (7.62) of Example 7.4, let us add an output equation:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u \;\text{ and }\; y = \begin{bmatrix} a & b \end{bmatrix} x, \tag{7.67}$$

with $a^2 + b^2 \neq 0$.

From (7.67), (7.15) and (7.52), we get $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, $\mathrm{Im}\, B = \mathrm{Im}\, E = E\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Im}\, B = \{e_1\} \approx \mathbb{R}^1$ and $\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Ker} E = \{e_2\} \approx \mathbb{R}^1$, then $E^{-1}\mathrm{Im}\, B = \mathscr{X}_d$.

From (7.67) and (7.66), follows that $\mathscr{V}^* = \mathrm{Ker} C = \{be_1 - ae_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.5 is satisfied and there exists a $P.D.$ feedback, $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, such that the output dynamics of the closed-loop system is externally reachable, like, for example,

$$u = \begin{bmatrix} (1-a) & -b \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} a & (b-1) \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + u_r$$

obtaining in this way:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u_r \;\text{ and }\; y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix},$$

where $\begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}$.

7.4.4 Example (Part 3)

Let us come back to the *rectangular implicit representation* (7.43), which comes from the *implicit global representation* (7.42) of the *switched system* of Sects. 7.2.1 and 7.3.4, described by (7.2) and (7.5), or by (7.41).

From (7.43), (7.15) and (7.52), we get $\mathrm{Im}\, A + \mathrm{Im}\, B \subset \mathrm{Im}\, E = \underline{\mathscr{X}}_{eq}$, $\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{R}^*_d = \mathscr{X}_d$, $E\mathscr{V}^*_{\mathscr{X}_d} = \mathrm{Im}\, E$ and $E\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Im}\, B = \mathrm{Im}\, B$. Also, $\mathrm{Ker} E = \{e_3\} \approx \mathbb{R}^1$ and $E^{-1}\mathrm{Im}\, B = \{e_2, e_3\}$.

From (7.43) and (7.66), follows that $\mathscr{V}^* = \mathrm{Ker} C = \{e_1, e_2\}$, then $\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B = \{e_2\} \approx \mathbb{R}^1$.

Hence, Theorem 7.5 is satisfied and there then exists a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, such that the output dynamics of the closed-loop system is externally reachable.

7.5 Control

Lebret and Loiseau [26] have extended the famous Morse Canonical Form [32] to the general case of *implicit descriptions*. In that paper, Lebret and Loiseau have completely characterized the internal structure of the *implicit descriptions*. With respect to the *minimal column indices*, which are responsible for the variation of the internal structure, they have distinguished two kinds of blocks, namely,

Blocks L_{q_i}: These blocks characterize the *degree of freedom* which are observable at the output. For reachable representations, $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, their number is characterized as follows:

$$\text{card}\left\{L_{q(i)};\, q_i \geq 1\right\} = \dim\left(\frac{\text{Ker}\,E}{\mathscr{V}^* \cap \text{Ker}\,E}\right). \tag{7.68}$$

Blocks L_{σ_i}: These blocks characterize the *degree of freedom* which are unobservable at the output. For reachable representations, $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, their number is characterized as follows:

$$\text{card}\left\{L_{\sigma(i)};\, \sigma_i \geq 1\right\} = \dim\left(\mathscr{V}^* \cap \text{Ker}\,E\right). \tag{7.69}$$

The internal structure variation will then be unobservable at the output if there exists a pair $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, such that

$$\mathscr{V}^* \supset \text{Ker}(E - BF_d^*). \tag{7.70}$$

7.5.1 *Decoupling of the Variable Structure*

In [8], we have introduced the *variable structure decoupling problem*.

Problem 7.1 (*Variable structure decoupling* [8]) Let us consider the *global implicit representation* $\Sigma^{ig}(\mathbf{E}, \mathbf{A}, \mathbf{B}, C)$, (7.37), such that the solvability assumptions, (7.31) and (7.34), and the internal properness condition, (7.33), are satisfied.

Under which geometric conditions does there exist a $P.D.$ feedback, $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$, for the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$, (7.35), such that the external behaviour of the closed-loop system is time-invariant with prespecified dynamics?

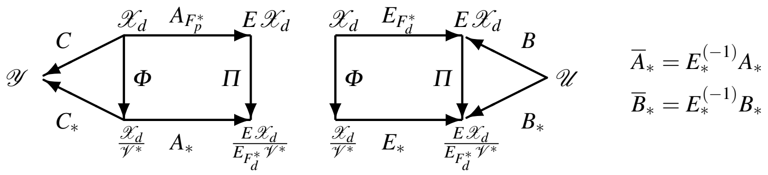


Fig. 7.3 Maps induced by $A_{F_p^*}$ and $E_{F_d^*}$. Φ and Π are canonical projections. The map E_* is invertible and $E_*^{(-1)}$ is its inverse

Theorem 7.6 (Variable structure decoupling [8]) *If the* implicit rectangular representation $\Sigma^{ir}(E, A, B, C)$, *(7.35), satisfies (7.31), (7.34), (7.33) and* [5]

$$\dim(\mathrm{Ker} E) \leq \dim\left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B\right), \tag{7.71}$$

there then exists a P.D. feedback, $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$, *such that the internal variable structure of the closed-loop system* implicit rectangular representation $\Sigma^{ir}(E_{F_d^*}, A_{F_p^*}, B, C)$ *is made unobservable, namely,*

$$\mathscr{V}^* \supset \mathrm{Ker}(E - BF_d^*),$$

where $(F_p^*, F_d^*) \in \boldsymbol{F}(\mathscr{V}^*)$.

Moreover, $\Sigma^{ir}(E_{F_d^*}, A_{F_p^*}, B, C)$ *is externally equivalent*[6] *to the* state-space representation $\Sigma^{state}(\overline{A}_*, \overline{B}_*, C_*)$, *where* $E_{F_d^*} = E - BF_d^*$ *and* $A_{F_p^*} = A + BF_p^*$, *and* $\overline{A}_*$, $\overline{B}_*$ *and* C_*, *are induced by* $A_{F_p^*}$ *and* $E_{F_d^*}$ *as it is shown in Fig. 7.3.*

Furthermore, if $\Sigma^{ir}(E, A, B, C)$, *(7.35), satisfies*

$$\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{X}_d,$$

then Σ^{state} $(\overline{A}_*, \overline{B}_*, C_*)$ *is controllable (reachable), namely,* $\left\langle \overline{A}_* \,\middle|\, \mathrm{Im}\, \overline{B}_* \right\rangle = \mathscr{X}_d \big/ \mathscr{V}^*$.

For proving Theorem 7.6, in [8] we have done the following geometric decompositions:

$$\begin{aligned} \mathscr{X}_d &= \left(\mathscr{V}^* + E^{-1}\mathrm{Im}\, B\right) \oplus \mathscr{X}_0, \\ \mathscr{V}^* &= \mathscr{X}_{\mathscr{V}^*} \oplus \left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B\right), \\ E^{-1}\mathrm{Im}\, B &= \left(\left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B\right) + \mathrm{Ker} E\right) \oplus \mathscr{X}_3, \\ \mathrm{Ker} E &= (\mathscr{V}^* \cap \mathrm{Ker} E) \oplus \mathscr{X}_E, \end{aligned} \tag{7.72}$$

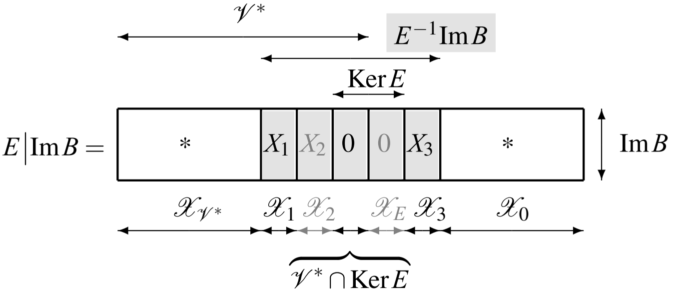
[5] Let us note that (7.31) implies that (cf. Fig. 7.2): $E\mathscr{V}_{\mathscr{X}_d^*} + \mathrm{Im}\, B = \underline{\mathscr{X}}_{eq}$ and $\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d$, hence (7.64) takes the form (7.71).

[6] Recall that two representations are externally equivalent when the sets of all possible trajectories for their external signals (here u and y) are identical (see [37, 44, 45]).

and we have shown that (7.71) implies that there exist two complementary subspaces, $\mathscr{X}_1$ and $\mathscr{X}_2$, such that

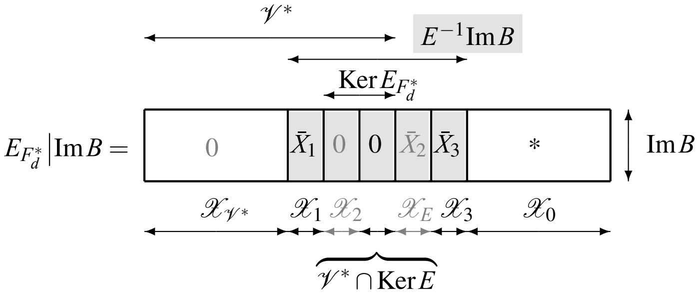
$$\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus (\mathscr{V}^* \cap \mathrm{Ker}\, E) \;\text{ and }\; \mathscr{X}_2 \approx \mathscr{X}_E. \tag{7.73}$$

Hence, under the bases (7.72) and (7.73), the map E restricted to Im B takes the following form:



(7.74)

Now, in view of the isomorphism $\mathscr{X}_2 \approx \mathscr{X}_E$, for satisfying (7.70) we only have to move the zero block of $\mathscr{X}_E$ to $\mathscr{X}_2$ by means of the derivative action F_d^*, namely,

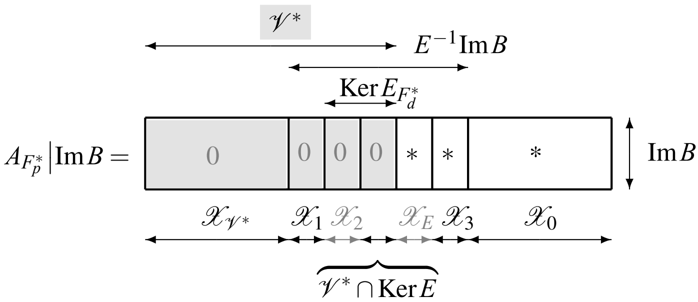


(7.75)

After having chosen F_d^*, we shall select F_p^*, such that $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, namely,

$$(A + BF_p)\mathscr{V}^* \subset (E - BF_d)\mathscr{V}^*. \tag{7.76}$$

Hence,



(7.77)

7.5.2 Example (Part 4)

Let us now verify if the *rectangular implicit representation* (7.43) satisfies the geometric conditions of Theorem 7.6: $\mathscr{R}_d^* = \mathscr{X}_d$, $\mathscr{V}^* = \{e_1, -e_2 - 2e_3\}$, $\mathrm{Ker}\,E = \{e_3\} \approx \mathbb{R}^1$ and $E^{-1}\mathrm{Im}\,B = \{e_2, e_3\}$; hence, $\mathscr{V}^* \cap E^{-1}\mathrm{Im}\,B = \{-e_2 - 2e_3\} \approx \mathbb{R}^1$, which implies (7.71). There then exists $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$ making unobservable the structure variation.

In order to satisfy (7.70), the derivative part of the control law has to contain the term $\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$. Indeed, $\mathrm{Ker}(E - BF_d^*) = \{-e_2 - 2e_3\} \subset \mathscr{V}^*$.

In order to satisfy (7.76), the proportional part of the control law has to contain the term[7] $\begin{bmatrix} -1 & -2/\tau & (1 + 1/\tau) \end{bmatrix}$, where τ is a positive real number. Indeed, $(A + BF_p^*)\mathscr{V}^* = \{e_1\} = (E - BF_d^*)\mathscr{V}^*$.

Thus, the proportional and derivative feedback is

$$u^* = \begin{bmatrix} -1 & -2/\tau & (1 + 1/\tau) \end{bmatrix} x + \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \mathrm{d}x/\mathrm{d}t + \begin{bmatrix} 1/\tau \end{bmatrix} u_r. \qquad (7.78)$$

Applying the control law (7.78) to system (7.42), we get the closed-loop system:

$$\begin{gathered} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \boxed{1} \\ \hline 0 & 0 & 0 \end{array}\right] \frac{\mathrm{d}}{\mathrm{d}t}\xi = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & \boxed{-1/\tau} \\ \hline \alpha & -(\beta+2) & -1 \end{array}\right] \xi + \left[\begin{array}{c} 0 \\ \boxed{1/\tau} \\ \hline 0 \end{array}\right] u_r, \\ y^* = \begin{bmatrix} 0 & 0 & \boxed{1} \end{bmatrix} \xi \end{gathered}$$

(7.79)

[7] Since Theorem 7.5 is satisfied, one can also assign the output dynamics.

where $\xi = T^{-1}x$, $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$. In [14], we show that the necessary stability region is[8]

$$\mathcal{R}^*_{\mathcal{NSC}}(\alpha, \beta) = \left\{(\alpha, \beta) \middle| \, \alpha \cdot (\beta + 2) < 0\right\} \cup \left\{(\alpha, \beta) \middle| \, \beta = -2 \,\&\, \alpha \neq 0\right\}, \quad (7.80)$$

and that the sufficient stability region is[9]

$$\begin{aligned} \mathcal{R}^*_{\mathcal{SSC}}(\alpha, \beta) = &\left\{(\alpha, \beta) \middle| \, \alpha \cdot (\beta + 2) < 0, \ \underline{\beta} \leq |\, \beta + 2 \,| \leq \overline{\beta}, \ \underline{\alpha} \leq |\, \alpha \,| \leq \overline{\alpha}\right\} \\ &\cup \left\{(\alpha, \beta) \middle| \, \beta = -2, \ \underline{\alpha} \leq |\, \alpha \,| \leq \overline{\alpha}\right\}, \end{aligned} \quad (7.81)$$

where $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$ and $\overline{\beta}$ are some given real numbers $0 < \underline{\alpha} \leq \overline{\alpha}$ and $0 < \underline{\beta} \leq \overline{\beta}$.

7.5.3 Rejection of the Variable Structure

Since the implementation of the "pure" derivative-based actions is not practically feasible, we have to generate a proper filter with the aim to approximate the external behaviour of the ideal non-proper controller.

In [14], we have considered the following problem.

Problem 7.2 (*Variable structure rejection* [14]) Let us consider a *global implicit representation* $\Sigma^{ig}(\mathbf{E}, \mathbf{A}, \mathbf{B}, C)$, (7.37), such that the solvability assumptions, (7.31) and (7.34), and the internal properness condition, (7.33), are satisfied, and $Ex(t)$ is continuous for all $t \geq 0$. Let us consider the $P.D.$ feedback

$$u^* = F^*_p x + F^*_d \mathrm{d}x/\mathrm{d}t + u_r \quad (7.82)$$

which constitutes a solution of Problem 7.1, where the feedback pair $(F^*_p, F^*_d) \in \mathbf{F}(\mathscr{V}^*)$ was chosen as it is indicated in Theorem 7.6.

Find a proper approximation of the ideal control law (7.82) such that the closed-loop system is BIBO-stable and moreover, for a given $\delta > 0$

$$\left|\, y(t) - y^*(t) \,\right| \leq \delta \qquad \forall\, t \geq t^*(\delta), \quad (7.83)$$

where $t^*(\delta)$ is a fixed transient time, y^* is the output for the ideal control law (7.82) and y is the output associated with the proper approximation of (7.82).

---

[8]This region is obtained from $\det \begin{bmatrix} \mathrm{s} & -1 & -1 \\ 0 & 0 & (\mathrm{s} + 1/\tau) \\ \hline -\alpha & (\beta + 2) & 1 \end{bmatrix} = -((\beta + 2)\mathrm{s} - \alpha)(\mathrm{s} + 1/\tau)$.

[9]This region is obtained following the methodology of [42], namely, we solve two Lyapunov equations for the two cases: *(i)* $\beta \neq -2$ and *(ii)* $\alpha \neq 0$ (with $\beta = -2$), with a common positive definite matrix P.

Theorem 7.7 (Variable structure rejection [14]) *Under the same conditions like in Theorem 7.6 and the additional assumptions*

Hypothesis ***H6***. $u_r \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathbb{R}^m\right)$ *and* $\mathrm{d}^2u_r/\mathrm{d}t^2 \in \mathcal{L}^\infty\left(\mathbb{R}^+,\ \mathbb{R}^m\right)$,
Hypothesis ***H7***. *The matrix* $\overline{A}_*$, *defined in Theorem 7.6 is Hurwitz,*
Hypothesis ***H8***. *Given* $\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_\ell \in \mathcal{Q}$, $g = \left[\, g_1 \cdots g_\ell \,\right]^T$, $g_1, \ldots, g_\ell \in \mathbb{R}^+$, *the locations* $q \in \mathcal{Q}$ *belong to the convex set*

$$\overline{\mathcal{Q}}_{\bar{q}_0}(g) = \left\{ q \in \mathcal{Q} \,\middle|\, q = \bar{q}_0 + \sum_{j=1}^{\ell} \gamma_{(i,j)} g_j \bar{q}_j \right\},$$

where for each $\left[T_{i-1}, T_i\right)$, *the value of* $\gamma_{(i,j)}$ *takes constant values in the closed subset of* $\mathbb{R}$: $\left[0, 1\right]$,
we now consider the following proper approximation of the ideal control law (7.82):

$$\begin{aligned} \mathrm{d}\bar{x}/\mathrm{d}t &= -(1/\varepsilon)\bar{x} + (1/\varepsilon)F_d^* x, \\ u &= -(1/\varepsilon)\bar{x} + \left((1/\varepsilon)F_d^* + F_p^*\right) x + u_r, \end{aligned} \tag{7.84}$$

where $\varepsilon > 0$. *If for a given pair* $(\varepsilon, \overline{A}_*)$, *there exists a nonempty convex* sufficient stability condition region $\mathcal{R}^{\#}_{\mathcal{SSC}}(q; \varepsilon)$ *contained in the stability region of the ideal solution,* $\mathcal{R}^{*}_{\mathcal{SSC}}(q)$, *for which the linear combination* $\overline{X} + \Gamma\overline{\Delta}_0$, *of the matrices coming from*

$$\left[\mathrm{s} \left[\begin{array}{cc} E & 0 \\ -BF_d^* & \varepsilon \mathrm{I} \\ \hline 0 & 0 \end{array} \right] - \left[\begin{array}{cc} A_{F_p^*} & \mathrm{I} \\ 0 & -\mathrm{I} \\ \hline D_i & 0 \end{array} \right] \right] \approx \left[\begin{array}{cc} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{array} \right], \tag{7.85}$$

is a Hurwitz matrix and moreover, there exist constant positive definite matrices $\overline{P}$, $\overline{Q}_0$ *such that*

$$\begin{aligned} \left(\overline{X} + \Gamma\overline{\Delta}_0\right)^T \overline{P} + \overline{P}\left(\overline{X} + \Gamma\overline{\Delta}_0\right) &= -\overline{Q}_0, \\ \lambda_{\min}\left(\overline{Q}_0\right) + \sum_{j=1}^{\ell} g_j \lambda_{\min}\left(\overline{Q}_j\right) &> 0, \end{aligned} \tag{7.86}$$

then Problem 7.2 has a solution. Here we denote $\overline{Q}_j = \left(\Gamma\overline{\Delta}_j\right)^T \overline{P} + \overline{P}\left(\Gamma\overline{\Delta}_j\right)$, $j \in \{1, \ldots, \ell\}$.

7.5.4 Example (Part 5)

The proper approximation of the ideal control law (7.78) is (cf. (7.84))

$$\begin{aligned} \mathrm{d}\bar{x}/\mathrm{d}t &= -(1/\varepsilon)\bar{x} + \begin{bmatrix} 0 & -1/\varepsilon & 1/\varepsilon \end{bmatrix} x \\ u &= -(1/\varepsilon)\bar{x} + \begin{bmatrix} -1 & -(2/\tau + 1/\varepsilon) & (1 + 1/\tau + 1/\varepsilon) \end{bmatrix} x + \begin{bmatrix} 1/\tau \end{bmatrix} u_r. \end{aligned} \tag{7.87}$$

ε is a positive number which tunes the precision of the approximation.

Equation (7.85) takes the form (cf. (7.42) and (7.78), with $x = T\xi$)

$$\left[\mathrm{s} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 1 & 1 & 0 & \varepsilon \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right] - \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1/\tau & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline \alpha & -(\beta+2) & 1 & 0 & 0 \end{array} \right] \right] \approx \begin{bmatrix} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{bmatrix}$$

$$\overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/\varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ (1/\varepsilon + 1/\tau) \\ 0 \\ 1/(\varepsilon\tau) \end{bmatrix}, \quad \overline{\Delta}_i = \begin{bmatrix} \alpha & -(\beta+2) & 0 & 0 \end{bmatrix}.$$

If in our example we put $\tau = 4$ and $\varepsilon = 1/4$, we then get

$$\overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 17/4 \\ 0 \\ 1 \end{bmatrix},$$

and $\phi_q^{\#} = \det \begin{bmatrix} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{bmatrix} = -(\mathrm{s}+4)\big(\mathrm{s}^3 + (17\beta/4 - \alpha + 17/2)\mathrm{s}^2 + (\beta - 17\alpha/4 + 2)\mathrm{s} - \alpha\big)$. Thus the necessary stability region is

$$\begin{aligned} \mathcal{R}^{\#}_{\mathcal{NSC}}(\alpha, \beta; \varepsilon = 1/4) = \Big\{ (\alpha, \beta) \Big| \; & \alpha < 0, \quad 17\beta/4 - \alpha + 17/2 > 0 \\ & \beta^2 + (4 - 305\alpha/68)\,\beta + (4\alpha/17 + 4\,(17\alpha/4 - 2)\,(\alpha - 17/2)/17) > 0 \Big\}, \end{aligned}$$

thus $q_1, q_2, q_3 \in \mathcal{R}^{\#}_{\mathcal{NSC}}(-1, \beta; \varepsilon = 1/4) = \big\{ (\alpha, \beta) \big| -(\beta+2) < 0.1775 \big\}$ (see (7.4) and Fig. 7.1). Then $(\underline{\alpha}, \underline{\beta}) = (1, 0)$ and $(\overline{\alpha}, \overline{\beta}) = (1, 2)$. We assume that (*c.f.* **H**8)

$$\big(\alpha, -(\beta+2)\big) \in \Big\{(-1,\beta) \in \mathscr{Q} \big|\ (-1,-(\beta+2)) = (-1,0) + 0\gamma_1(-1,0) + 2\gamma_2(0,-1),\quad \gamma_1, \gamma_2 \in \big[0,1\big]\Big\}.$$

Then,

$$\overline{\Delta}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix},\ \overline{\Delta}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ \overline{\Delta}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\overline{X} + \Gamma\overline{\Delta}_0 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -17/4 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma\overline{\Delta}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 17/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\quad \Gamma\overline{\Delta}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 17/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Choosing $\overline{Q}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we get

$$\overline{P} = \begin{bmatrix} 3.7404 & -0.5000 & 0.7480 & -1.1154 \\ -0.5000 & 1.1154 & 0.0620 & -0.5000 \\ 0.7480 & 0.0620 & 0.3120 & -0.2633 \\ -1.1154 & -0.5000 & -0.2633 & 2.7404 \end{bmatrix},\ \sigma\{\overline{P}\} = \begin{cases} 0.1237 \\ 0.7815 \\ 2.3950 \\ 4.6079 \end{cases},$$

$$\overline{Q}_1 = \begin{bmatrix} 1.0000 & 3.7404 & 0.7482 & -0.5000 \\ 3.7404 & 0 & 0 & 0 \\ 0.7482 & 0 & 0 & 0 \\ -0.5000 & 0 & 0 & 0 \end{bmatrix},\quad \sigma\{\overline{Q}_1\} = \begin{cases} 4.3795 \\ -3.3795 \\ 0 \\ 0 \end{cases},$$

$$\overline{Q}_2 = \begin{bmatrix} 0 & 0.5000 & 0 & 0 \\ 0.5000 & 7.4808 & 0.7482 & -0.5000 \\ 0 & 0.7482 & 0 & 0 \\ 0 & -0.5000 & 0 & 0 \end{bmatrix},\quad \sigma\{\overline{Q}_2\} = \begin{cases} 7.6199 \\ 0.0000 \\ -0.1391 \\ 0 \end{cases}.$$

Then

$$\lambda_{\min}\left(\overline{Q}_0\right) + (\overline{\alpha} - \underline{\alpha})\lambda_{\min}\left(\overline{Q}_1\right) + (\overline{\beta} - \underline{\beta})\lambda_{\min}\left(\overline{Q}_2\right) = 1 + 0(-3.3795) + 2(-0.1391) = 0.7218 > 0.$$

The last condition implies that the stability condition (7.86) is satisfied.

7.6 Numerical Simulation

We made a MATLAB® numerical simulation:

> *"Start time" = 0.0, "Stop time" = 150, "Type" = "Variable–Step", "Solver" = "ode45 Domand–Prince", "Max step size" = "auto", "Relative tolerance" = 1e^{-4}, "Min step size" = "auto", "Absolute tolerance" = "auto", "Initial step size" = "auto", "Consecutive min step size violations allowed" = 1, "States shape preservation" = "Disable all", et "Zero crossing control" = "Disable all".*

The behaviours, $\mathfrak{B}^{\infty}_{q_i}$, take place as follows (recall (7.7)): In the time interval $[0, 50)$ takes place $\mathfrak{B}^{\infty}_{q_1}$. In the time interval $[50, 100)$ takes place $\mathfrak{B}^{\infty}_{q_2}$. In the time interval $[100, 150]$ takes place $\mathfrak{B}^{\infty}_{q_3}$.

We apply the proper approximation (7.87) of the ideal control law (7.78), with the choice: $\tau = 4$ and $\varepsilon = 0.25$. We assume that we do not have access to the descriptor variable x, so we use the following descriptor variable observer synthesized in [15] (see equations (3.35) and (3.19) in [15]):

$$\begin{aligned} \mathrm{d}\hat{x}_c/\mathrm{d}t &= \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ \hat{x}_\ell &= \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} -1 \end{bmatrix} y, \end{aligned} \tag{7.88}$$

where

$$\begin{bmatrix} \bar{x}_c \\ \bar{x}_\ell \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]^{-1} x. \tag{7.89}$$

The reference u_r has been chosen as follows (see Definition 2.4.5–[37]):

$$\begin{aligned} \phi(t) &= \begin{cases} \mathrm{e}^{-\frac{1}{1-(t')^2}}, & t \in A = (\frac{1}{6}, \frac{2}{6}),\ t' = 12t - 3 \\ -\mathrm{e}^{-\frac{1}{1-(t'')^2}}, & t \in B = (\frac{4}{6}, \frac{5}{6}),\ t'' = -12t + 9 \\ 0 \quad , & t \in \mathbb{R} \backslash (A \cup B) \end{cases} \\ r(t) &= \int_0^t \left(\sum_{i=0}^{3} (-1)^i \phi(\frac{2}{75}\sigma - i) \right) \mathrm{d}\sigma, \quad t \in [0, 150]. \end{aligned}$$

The model matching error is computed as follows:

$$\left| y(t) - y^*(t) \right| = \left| y(t) - \int_0^t \mathrm{e}^{-\frac{1}{\tau}(t-\sigma)} u_r(\sigma) \mathrm{d}(\sigma) \right|.$$

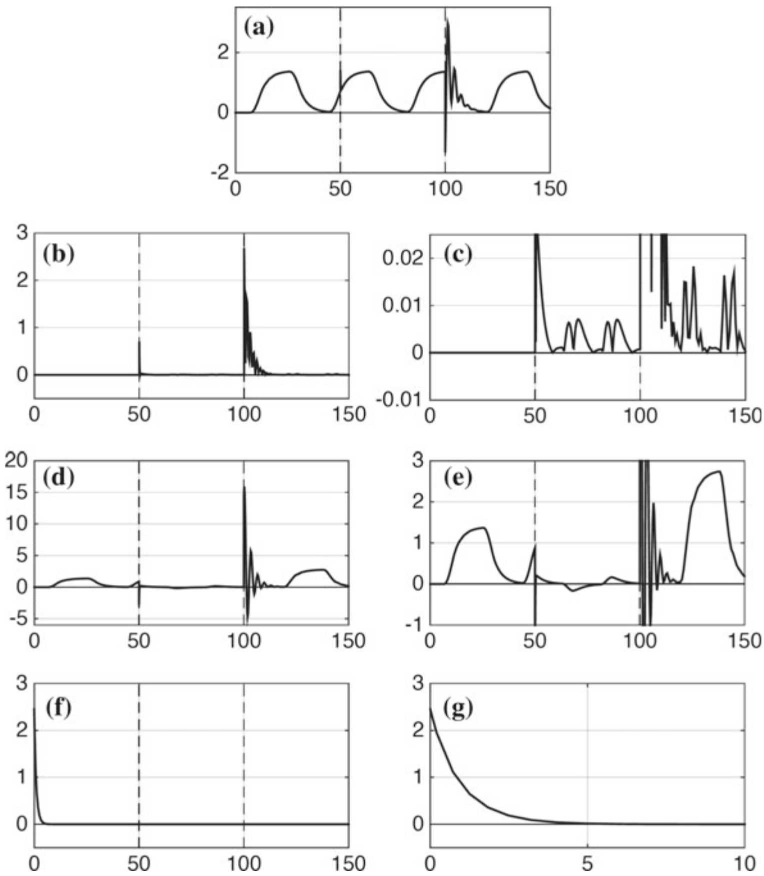


Fig. 7.4 Simulation results of Control with observation of the descriptor variable. **a** Output, y. **b** and **c** Model matching error, $| y(t) - y^*(t) |$. **d** and **e** Control input, u. **f** and **g** Observation error, $\| \hat{x} - x \|_2$

In Fig. 7.4, we show the numerical simulations for this minimum phase case. In order to appreciate the performance of the remainder generator, in this simulation, we have set the initial condition: $\bar{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

7.7 Summary

In Sect. 7.2, we have shown how to use the *linear implicit systems theory* in order to model and control, in an efficient way, a class of complex systems, namely, *time-varying, autonomous, switched systems*. Thanks to a simple example, we have shown that the internal structure variations can take into account wide situations with varying parameters, such as, among others: (1) the relative degree, (2) the system gain and (3) the values of the finite zeros.

In Sect. 7.3, we have shown how linear time-invariant implicit systems theory can be efficiently used to model a certain class of switched systems with autonomous location transitions.

For the equivalent *state-space representations* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$ (7.2) and (7.3), we have determined the common fixed structure. The general systems structure is represented by the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ in (7.35). We also have derived a *linear time-invariant implicit representation* for the initial linear switched system with autonomous location transitions, $\Sigma^{state}(A_{q_i}, B, C_{q_i})$. Note that the *implicit global representations* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ in (7.37) and (7.40) are time-dependent. Alternatively, the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ in (7.35) is time-invariant.

As shown in some simple examples, the corresponding structure variation has a wide structure. For instance, it includes variable relative degree, variable gain and variable finite zeros.

In the particular structure (7.3) studied above, only the matrices A_{q_i} and C_{q_i} have a generic "switched" structure and additionally depend on index q_i. The independence (assumed above) of matrix B on the switchings $q_i \in \mathscr{Q}$ does not involve any restriction into the used formulation. The zeros and the unobservable subspace of the systems under consideration are indeed only characterized by the structure of the matrices A_{q_i} and C_{q_i}.

The main advantage of these implicit representations is reflected in the structural concept of solvability. Indeed, condition (7.30) for having at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$ and condition (7.31) for the existence of solution (for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$ and for any initial condition) naturally lead to the *global implicit representation* (7.37).

When we restrict to proper systems, the matrices of the *implicit rectangular representation* (7.35), the *algebraic constraint* (7.36) and the *implicit rectangular representation* (7.37) have the particular form (7.40), which are precisely the structure of the switched system, (7.2), and (7.3) here considered.

In Sect. 7.4, we have tackled the most important concept studied in System Theory, *the reachability*. For the general case of implicit systems, represented by (7.10), with E and A not necessarily square, Frankowska [16] has been the first to give a functional interpretation of *reachability*. For this, she has used the Viability Theory [2]. More precisely, she has shown that *reachability* is equivalent to finding a smooth trajectory $x(\cdot)$, solution of (7.10), starting from the initial condition x_0 and reaching the desired x_1 in a given finite time t_1, namely, $x(0) = x_0$ and $x(t_1) = x_1$. Frankowska [16] has shown that *reachability* is geometrically characterized by the well-known *reachable space*, $\mathscr{R}^*_{\mathscr{X}_d}$. Of course, $\mathscr{R}^*_{\mathscr{X}_d}$ is contained in the viability kernel $\mathscr{V}^*_{\mathscr{X}_d}$, since this last guaranties the existence of at least one trajectory, solution of (7.10), leaving from x_0. This is also clear from $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d}$.

It has to be pointed out that the fundamental *reachability* Definition 7.11 requires no explicit control actions, and in general, the trajectories inside the *reachability* *subspace* are handled by input actions and by internal degree of freedom actions; it might exist *reachable systems* without any effective external input (see, for instance, Example 7.2). In order to guarantee that the trajectories are caused by control inputs, we have introduced the *external reachability* concept; for this, we have combined the Frankowska's *reachability* notion [16] with the notion of *unicity of the descriptor variable solution* of Lebret [25].

In the case of *implicit descriptions* constituted by *minimal column indices*, there exists no external input for controlling the whole descriptor variable: this is due to the existence of completely free variables. In order to partly control implicit representations having an internal degree of freedom, we have introduced the *external output dynamics assignment* concept; for this we have used the characterization of Lebret and Loiseau [26], which enables us to make unobservable the degree of freedom by means of a $P.D.$ feedback, and insure that the closed-loop system gets the *external reachability* property.

In Sect. 7.5, we have proposed a control scheme based on *proportional and derivative feedbacks* of the descriptor variable, in order to obtain a *closed-loop system* which is *proper, linear and time-invariant*, whatever be the positions of the internal switches.

The Canonical Form of Lebret and Loiseau [26] has enabled us to characterize the internal structure of the *implicit descriptions*.

Following the typical geometric procedure of the *disturbance decoupling problem* [47], we have decoupled the variable structure by means of an ideal $P.D.$ feedback [8].

In [10], we have proposed an effective procedure to approximate the ideal static $P.D.$ feedback by means of a dynamic $P.$ feedback. Following the ideas of [33, 42], in [14] we have studied the stability aspects.

In Sect. 7.6, we have presented a MATLAB® numerical simulation. We have used a descriptor variable observer based on fault detection techniques [15]. This observer is composed of a Beard–Jones filter, which aim is to observe the existing degree of freedom in rectangular implicit representations. Notice that after the initial transient, this observer remains insensitive to the switchings events (see Fig. 7.4f, g); this is the case, because the observer is based on the fault detection of a continuous linear system. Since this observation is accomplished by a pole-zero cancellation, this technique is reserved to minimum phase systems, with respect to the output-degree-of-freedom transfer, namely, to *implicit rectangular representations* having Hurwitz output decoupling zeros. When unstable zeros are present, alternatives exist (see [15]).

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Part III
Applications of Complex Dynamical Systems

Chapter 8
Huygens Synchronization Over Distributed Media—Structure Versus Complex Behavior

Vladimir Răsvan

Abstract This analysis is concerned with Huygens synchronization over distributed media—vibrating strings or LC transmission lines. If e.g., two oscillators with lumped parameters i.e., described by ordinary differential equations, which display self sustained oscillations, are coupled to some distributed medium, they interact in function of the structural properties of the resulting system. If this medium has infinite length then, according to the structural properties of the difference operator describing propagation, either synchronization with the external frequency or some “complex behavior” can occur. If the coupling has a finite length, again the qualitative properties are determined by the structure of the aforementioned difference operator: either the self sustained oscillations are quenched, the system approaching asymptotically a stable equilibrium (opposite of the Turing coupling of two “cells” that is lumped oscillators) or again the aforementioned “complex behavior” can occur. We state finally the *conjecture that this “complex behavior” is in fact some almost periodic oscillation and not a chaotic behavior*. It is worth mentioning that the two types of qualitative behavior are in connection with the physical nature of the considered systems. Specifically, the difference operator is strongly stable for electrical systems and critically stable for mechanical systems. It is this aspect that explains proneness to standard or “complex” behavior. However, if the aforementioned conjecture will not be disproved, the difference between the corresponding oscillatory behaviors will consist only in the contents of harmonics given by the Fourier series attached.

8.1 Introduction and Overview

One of the “contemporary classics” in the theory of dynamical systems introduced in his monograph [37] the following far going dichotomies: stability/instability and synchronization/stochasticity. The first dichotomy appears to have an obvious sig-

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nificance, especially if one is thinking to the properties of equilibria: while the Lyapunov stability theory is meant for "motions" i.e., not for equilibria only (as it had been conceived e.g., by Thomson and Tait [55]), most of its results, especially the applicable ones are obtained for equilibria. Concerning the stability of motion i.e., of system's trajectories, the Lyapunov theory operates a difference between basic and perturbed motions. Usually a basic motion is a solution not defined by initial conditions but on the whole real ("time") axis: bounded, periodic, almost periodic. A standard example of almost periodic motion is the quasi-periodic motion e.g., $\Psi(\omega_1 t + \phi_1, \ldots, \omega_m t + \phi_m)$ where $\Psi(\xi_1, \ldots, \xi_m)$ is 2π-periodic in all its arguments and ω_k, $k = \overline{1, m}$ are rationally independent. As pointed out in [37], this function does not describe a stochastic process but it might appear as periodic with sufficiently large period. It has approximate repeatability over sufficiently large time intervals $T(\varepsilon)$ (ε being the precision of the repeatability). But, again citing [37], on intervals shorter than $T(\varepsilon)$, *it is similar to a stochastic process*.

In this paper we aim to clarify this confusion which falsifies in some sense the idea of complex processes. However, in order to make this purpose more clear, we have to refer to the second dichotomy—synchronization/stochasticity. Unlike stability (where Lyapunov stability is dominant among all other concepts), there are several concepts and models of synchronization. They rely on various structures of interacting oscillators.

A classical version goes back to Lagrange [37] and starts from the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial q_i} + Q_i = 0,\ i = 1, \ldots, n \tag{8.1}$$

with q_i—the generalized coordinates, T—the kinetic energy, U—the potential energy and Q_i—the generalized forces. Assuming that $q_1 = q_2 = \ldots = q_n = 0$ is an equilibrium, linearization around it will give

$$\sum_1^n (a_{ij}\ddot{q}_j + b_{ij}\dot{q}_j + c_{ij}q_j) = 0,\ i = 1, \ldots, n \tag{8.2}$$

and this suggests n linear oscillators described by

$$a_{ii}\ddot{q}_i + b_{ii}\dot{q}_i + c_{ii}q_i = 0,\ i = 1, \ldots, n \tag{8.3}$$

suitably connected among them.

The immediate generalization of the aforementioned representation is to consider some local oscillators connected through a structure that can be described by a graph. An overview of these representations can be found in [24, 25] and other.

A separate discussion has to be done on synchronization. In [5] there are given several definitions of synchronization. Among them there is mentioned the *Huygens synchronization*, viewed as frequency synchronization. It is defined for periodic motions with frequencies $\omega_1, \ldots, \omega_k$ and is understood as rational dependence of

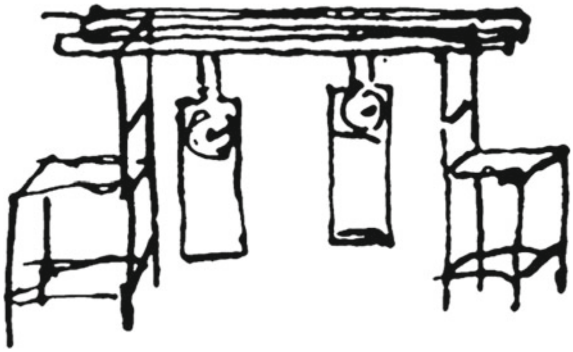


Fig. 8.1 The original setting of Huygens

the frequencies $\omega_j = n_j\omega$, $j = \overline{1, k}$. Huygens synchronization is considered mainly by the physicists [40]; in this reference there are presented many applications at the physical level of rigor.

In our analysis we shall start from the Huygens basic experiment (Fig. 8.1 see [40]) and keep in mind the coupling of the two pendula (displaying periodic motions) through a medium described possibly by distributed parameters. This setting of the synchronization is to be found also in [38] (a "toy" example), [17, 31]. An interesting survey is [39]. We describe in brief the physical setting. One (in the "toy" example) or several mechanical oscillators are "hanging" on a rope (like in the Huygens experiment) or are connected through an elastic rod—both (the rope and the rod) with distributed parameters—being thus possibly synchronized through the aforementioned medium.

We have still to mention two cases: (i) the rope is infinite and, as it will appear in the following, can act as a synchronizing external agent; (ii) the rope or the rod are of finite length and the self sustained oscillations can be quenched by an induced damping (which nevertheless occurred in the other case—of the infinite length rope—also).

To these models arising from mechanics we shall add an electronic analogue: two electronic oscillators coupled through a lossless LC transmission line that may be of finite or infinite length.

In what is left of this paper we shall present the mathematical models of the aforementioned physical structures—what is common to all them—and put on a rigorous mathematical basis the various qualitative behaviors displayed. In this way some illustrations of what is called complex behavior will be obtained.

From the mathematical point of view there will be considered boundary value problems for hyperbolic partial differential equations in the plane. These problems will be approached by associating some functional equations resulting from the integration of the Riemann invariants along the characteristics (following [1, 12, 13]).

The functional equations thus obtained will be tackled with the methods of the Theory of Oscillations. The conclusions of the paper will focus both on the interpretation of the results as well as on some perspective/unsolved problems.

8.2 Basic Mathematical Models—The "Toy" Application

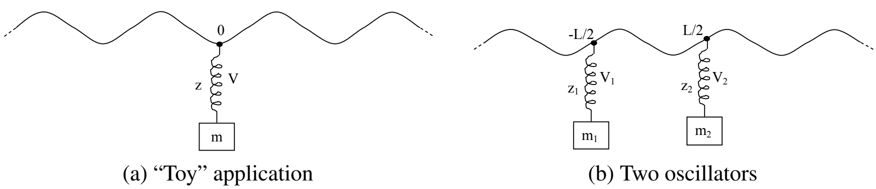
The basis for the entire approach for all cases can be seen on the "toy" model (the single oscillator—Fig. 8.2a) [38]

$$\begin{array}{l} \partial_t^2 y - c^2 \partial_x^2 y = 0;\ c^2 = T/\rho\ ,\ t > 0\ ,\ -\infty < x < \infty \\ m\ddot{z} + Q(z) = T(\partial_x y_r(0,t) - \partial_x y_l(0,t)),\ y_r(0,t) = y_l(0,t) = z(t) \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) \end{array} \tag{8.4}$$

where T is the strain force and ρ—the rope density; therefore $c = \sqrt{T/\rho}$ is the velocity of the wave propagation through the rope; $y_l(x,t)$ and $y_r(x,t)$ are the displacements of the rope for $x < 0$ (at the "left") and $x > 0$ (at the "right") respectively; obviously the oscillator is "hanging" at $x = 0$. The oscillator is clearly a conservative system hence its free oscillations do not admit a limit cycle but are rather "moving" on cycles in the phase plane—defined by the initial conditions only (Figs. 8.2a and 8.3).

Here, as throughout the entire paper, only smooth enough classical solutions will be considered, allowing to introduce

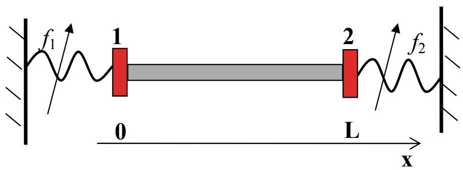
$$v(x,t) := \partial_t y(x,t),\ w(x,t) := \partial_x y(x,t) \tag{8.5}$$



(a) "Toy" application (b) Two oscillators

Fig. 8.2 Mechanical oscillators on a string

Fig. 8.3 Two van der Pol oscillators on the elastic rod



which are subject to a system in the symmetric Friedrichs form

$$
\begin{array}{l}
\partial_t v = c^2 \partial_x w,\ \ \partial_t w = \partial_x v \\
m\ddot{z} + Q(z) = T(w_r(0,t) - w_l(0,t)),\ \ v_r(0,t) = v_l(0,t) = \dot{z}(t) \\
v(x,0) = y_1(x),\ \ w(x,0) = y_0'(x)
\end{array}
\tag{8.6}
$$

Following the standard textbook [56] we observe that (8.6) define in fact two boundary value problems – both forced by the external signal $\dot{z}(t)$; finally the two problems will be "coupled" at $x = 0$ by the dynamic (non-standard) boundary conditions defined by the oscillator. Introducing the Riemann invariants by

$$
u^{\pm}(x,t) = v(x,t) \mp cw(x,t);\ \ v = \frac{1}{2}(u^- + u^+),\ \ w = \frac{1}{2c}(u^- - u^+) \tag{8.7}
$$

the following equations are obtained

$$
\begin{array}{l}
\partial_t u^+ + c\partial_x u^+ = 0,\ \ \partial_t u^- - c\partial_x u^- = 0 \\
m\ddot{z} + Q(z) = \dfrac{T}{2c}(u_r^-(0,t) - u_r^+(0,t) - u_l^-(0,t) + u_l^+(0,t)) \\
u_l^-(0,t) + u_l^+(0,t) = u_r^-(0,t) + u_r^+(0,t) = 2\dot{z}(t) \\
u^{\mp}(x,0) = y_1(x) \pm cy_0'(x)
\end{array}
\tag{8.8}
$$

Equation (8.8) allow to define the two aforementioned boundary value problems

$$
\begin{array}{l}
\partial_t u_r^+ + c\partial_x u_r^+ = 0,\ \ \partial_t u_r^- - c\partial_x u_r^- = 0 \\
u_r^-(0,t) + u_r^+(0,t) = 2\dot{z}(t),\ \ t > 0;\ \ u_r^{\mp}(x,0) = y_1(x) \pm cy_0'(x);\ \ x > 0
\end{array}
\tag{8.9}
$$

and

$$
\begin{array}{l}
\partial_t u_l^+ + c\partial_x u_l^+ = 0,\ \ \partial_t u_l^- - c\partial_x u_l^- = 0 \\
u_l^-(0,t) + u_l^+(0,t) = 2\dot{z}(t),\ \ t > 0;\ \ u_l^{\mp}(x,0) = y_1(x) \pm cy_0'(x);\ \ x < 0
\end{array}
\tag{8.10}
$$

Consider first (8.9) and let $(x,t), x > 0, t > 0$ be a point crossed by two characteristic straight lines

$$
t^{\pm}(\xi; x,t) = t \pm (\xi - x)/c \tag{8.11}
$$

The increasing characteristic $t^+(\xi; x,t)$ can be extended "to the left" to cross either $x = 0$ at $t^+(0; x,t) = t - x/c$ (provided $t > x/c$) or $t = 0$ at $\xi = x - ct$ (provided $t < x/c$ i.e., $x - ct > 0$). Since $u_r^+(\xi; t + (\xi - x)/c) \equiv$ const (with respect to ξ) it follows that

$$
u_r^+(x,t) = \begin{cases} u_r^+(0, t - x/c),\ \ x - ct < 0 \\ y_1(x - ct) - cy_0'(x - ct),\ \ x - ct > 0 \end{cases} \tag{8.12}
$$

Remark that we cannot use the boundary condition at $x = 0$ without the representation of the backward wave $u_r^-(x,t)$ which is constant along the decreasing characteristic $t^-(\xi; x, t)$ which will always cross $t = 0$ at $\xi = x + ct > 0$. Therefore

$$u_r^-(x,t) = u_r^-(x+ct,0) = y_1(x+ct) + cy_0'(x+ct) \tag{8.13}$$

and, for $x = 0$

$$u_r^-(0,t) = 2\dot{z}(t) - u_r^+(0,t) = y_1(ct) + cy_0'(ct) \;\Rightarrow\; u_r^+(0,t) = 2\dot{z}(t) - y_1(ct) - cy_0'(ct)$$

We can thus re-write (8.12) as

$$u_r^+(x,t) = \begin{cases} y_1(x-ct) - cy_0'(x-ct), & x - ct > 0 \\ 2\dot{z}(t - x/c) - y_1(ct - x) - cy_0'(ct - x), & x - ct < 0 \end{cases} \tag{8.14}$$

By direct check we obtain that $u_r^\pm(x,t)$ thus defined by (8.13) and (8.14) are a solution of (8.9). In a similar way we obtain that $u_l^\pm(x,t)$ defined by

$$\begin{aligned} u_l^-(x,t) &= \begin{cases} y_1(x+ct) + cy_0'(x+ct), & 0 < t < -x/c \\ 2\dot{z}(t + x/c) - y_1(-x-ct) + cy_0'(-x-ct), & t > -x/c \end{cases} \\ u_l^+(x,t) &= y_1(x-ct) - cy_0'(x-ct) \end{aligned} \tag{8.15}$$

are a solution of (8.10). Turning to (8.8) and taking into account the representation formulae (8.13), (8.14) and (8.15) we obtain the equation of the oscillator

$$m\ddot{z} + (2T/c)\dot{z} + Q(z) = (T/c)[y_1(ct) + cy_0'(ct) + y_1(-ct) - cy_0'(-ct)] \tag{8.16}$$

We are now in position to make several significant comments. The first ones concern the oscillator equation (8.16). A damping $2(T/c)\dot{z}$ occurred, called by the physicists “radiation dissipation”. Next, the oscillator is forced by a forcing term determined by the initial conditions of the rope equations. Under certain assumptions which will be stated in the following, the oscillator will display a global solution, very much alike to the forcing term, and this solution will result exponentially stable.

The second group of comments concerns solving (including numerically) of the aforementioned initial boundary value problems. Indeed, given the initial conditions $y_0(x)$ and $y_1(x)$—sufficiently smooth e.g., twice differentiable, Eq. (8.16) can be solved on any finite interval $(0, T_1)$. Therefore $u_r^\pm(x,t)$ and $u_l^\pm(x,t)$ will follow from (8.13), (8.14) and (8.15) respectively. Taking into account (8.7), the solution of (8.6) hence of (8.4) is obtained if (8.5) are considered.

We turn now back to the forced oscillator (8.16) and state the following mathematical result

Theorem 8.1 *Consider the system (8.16) under the following assumptions:*
(i) $Q(z)$ is subject to

$$0 < \mu \le \frac{Q(z_1) - Q(z_2)}{z_1 - z_2} \le \mu + L, \;\forall z_1, z_2 \in \mathbb{R} \tag{8.17}$$

(ii) The following inequality holds

$$L < 4(T/c)\left(\sqrt{\frac{\mu}{m\mu}} + \frac{1}{m}(T/c)\right) = 4\left(\frac{T}{c\sqrt{m}}\right)\left(\sqrt{\mu} + \frac{T}{c\sqrt{m}}\right) \tag{8.18}$$

Then system (8.16) has a unique solution defined on $\mathbb{R}$ *which is bounded if* $y_1(\cdot)$ *and* $y_0'(\cdot)$ *are bounded on* $\mathbb{R}$ *and, moreover, it is exponentially stable. If* $y_0'(\cdot)$ *and* $y_1(\cdot)$ *are* Λ*-periodic, then the aforementioned solution is* Λ/c*-periodic and if* $y_0'(\cdot)$ *and* $y_1(\cdot)$ *are almost periodic then the aforementioned global solution is also almost periodic.*

Theorem 8.1 follows by simple application of a theorem from [58] which we reproduce here for the sake of completeness.

Theorem 8.2 (V.A. Yakubovich) *Consider the nonlinear system of ordinary differential equations*

$$\dot{x} = Ax - b\phi(c^*x) + g(t) \tag{8.19}$$

where x, b, c *are* n*-vectors,* A *is a* $n \times n$ *matrix with constant entries and* $g : \mathbb{R} \mapsto \mathbb{R}^n$ *is a* n*-valued vector function. Assume that:*
(i) A *is a Hurwitz matrix;*
(ii) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ *satisfies*

$$0 \le \frac{\phi(\sigma_1) - \phi(\sigma_2)}{\sigma_1 - \sigma_2} \le L,\ \forall \sigma_1, \sigma_2 \in \mathbb{R} \tag{8.20}$$

(iii) $|g(t)| \le M$ *i.e., it is globally bounded on* $\mathbb{R}$*;*
(iv) the following frequency domain inequality holds

$$\frac{1}{L} + \Re e H(\imath\omega) > 0,\ \forall \omega \in \mathbb{R}^+ \cup \{+\infty\} \tag{8.21}$$

where $H(\lambda) = c^*(\lambda I - A)^{-1}b$ *is the transfer function of the linear part of (8.19). Then system (8.19) has a unique, bounded on* $\mathbb{R}$ *solution which is exponentially stable. Moreover, this solution is* T*-periodic if* $g(t)$ *is* T*-periodic and almost periodic if* $g(t)$ *is such.*

Note that (8.18) is the expression of (8.21) in the case of (8.16) where $H(\lambda) = (m\lambda^2 + (2T/c)\lambda + \mu)^{-1}$. Also $\phi(\sigma) = Q(\sigma) - \mu\sigma$ and $\mu > 0$ is required to have the linear part of a transformed (8.16) exponentially stable. If (8.18) is replaced by $L < 4(T/(c\sqrt{m}))^2$ then $\mu > 0$ can be arbitrarily small.

8.3 Two Electronic Oscillators on a *LC* Transmission Line

We continue our analysis by considering an electronic analogue of the mechanical oscillating system of Fig. 8.2b.

By writing down the Kirchhoff “laws” we obtain the following

$$
\begin{aligned}
& L\partial_t I + \partial_x V = 0,\ C\partial_t V + \partial_x I = 0 \\
& V_l(-l/2,t) = R_1'(I_l(-l/2,t) - I_c(-l/2,t)) + V_{D1} = V_c(-l/2,t) \\
& V_c(l/2,t) = R_2'(I_c(l/2,t) - I_r(l/2,t)) + V_{D2} = V_r(l/2,t) \\
& -V_1 + R_1 I_{L1} + L_1 \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} + V_{D1} = 0 \\
& C_1 \frac{\mathrm{d}V_{D1}}{\mathrm{d}t} = I_{L1} - f_1(V_{D1}) + I_l(-l/2,t) - I_c(-l/2,t) \\
& -V_2 + R_2 I_{L2} + L_2 \frac{\mathrm{d}I_{L2}}{\mathrm{d}t} + V_{D2} = 0 \\
& C_2 \frac{\mathrm{d}V_{D2}}{\mathrm{d}t} = I_{L2} - f_2(V_{D2}) + I_c(l/2,t) - I_r(l/2,t)
\end{aligned}
\tag{8.22}
$$

The subscript “l” denotes the variables “at the left” i.e., for $x < -l/2$, the subscript “r”—the variables “at the right” i.e., for $x > l/2$ and the subscript “c”—those of the central strip i.e., for $-l/2 \le x \le l/2$.

It already becomes clear that (8.22) define three boundary value problems. Since the approach has been already sketched in Sect. 8.2, we shall give only what is necessary to understand the synchronization problem and its structural aspects.

A. The right hand boundary value problem is defined on $(l/2, \infty) \times \mathbb{R}^+$ by the following equations

$$
\begin{aligned}
& L\partial_t I_r + \partial_x V_r = 0,\ C\partial_t V_r + \partial_x I_r = 0,\ x > l/2,\ t > 0 \\
& V_r(l/2,t) + R_2' I_r(l/2,t) = V_{D2}(t) + R_2' I_c(l/2,t) := V_r''(t)
\end{aligned}
\tag{8.23}
$$

where $V_r''(t)$ appears as a forcing term “injected” by the central zone dynamics. We follow the way sketched in Sect. 8.2: introduce the Riemann invariants, consider them along the characteristics to obtain finally the representation

$$
V_r(x,t) = \begin{cases}
\frac{1}{2}[V_0(x - t/\sqrt{LC}) + V_0(x + t/\sqrt{LC}) + \sqrt{L/C}(I_0(x - t/\sqrt{LC}) + \\
+ I_0(x + t/\sqrt{LC}))];\ x \ge l/2,\ 0 \le t \le \sqrt{LC}(x - l/2) \\
\frac{1}{2}[V_0(x + t/\sqrt{LC}) - \rho_2 V_0(l - x + t/\sqrt{LC}) - \\
- \sqrt{L/C}(I_0(x + t/\sqrt{LC}) + \rho_2 I_0(l - x + t/\sqrt{LC})) + \\
+ (1 + \rho_2) V_r''(t + \sqrt{LC}(l/2 - x))];\ x \ge l/2,\ t \ge \sqrt{LC}(x - l/2)
\end{cases}
\tag{8.24}
$$

$$
I_r(x,t) = \begin{cases}
\frac{1}{2}[I_0(x - t/\sqrt{LC}) + I_0(x + t/\sqrt{LC}) + \sqrt{C/L}(V_0(x - t/\sqrt{LC}) - \\
- V_0(x + t/\sqrt{LC}))];\ x \ge l/2,\ 0 \le t \le \sqrt{LC}(x - l/2) \\
\frac{1}{2}[I_0(x + t/\sqrt{LC}) - \rho_2 I_0(l - x + t/\sqrt{LC}) - \\
- \sqrt{C/L}(V_0(x + t/\sqrt{LC}) + \rho_2 V_0(l - x + t/\sqrt{LC})) + \\
+ (1 + \rho_2)\sqrt{C/L} V_r''(t + \sqrt{LC}(l/2 - x))]; \\
x \ge l/2,\ t \ge \sqrt{LC}(x - l/2)
\end{cases}
\tag{8.25}
$$

where we denoted $\rho_2 = (1 - R_2'\sqrt{C/L})(1 + R_2'\sqrt{C/L})^{-1}$; obviously $|\rho_2| < 1$.

The left hand boundary value problem is defined on $(-\infty, -l/2) \times \mathbb{R}^+$ by the following equations

$$\begin{array}{l} L\partial_t I_l + \partial_x V_l = 0\ ,\ \ C\partial_t V_l + \partial_x I_l = 0,\ \ x < -l/2,\ \ t > 0 \\ V_l(-l/2, t) - R_1' I_l(l/2, t) = V_{D1}(t) - R_1' I_c(-l/2, t) := V_l'(t) \end{array} \tag{8.26}$$

where $V_l'(t)$ is again a forcing term "injected by the central strip dynamics. As in the case of the right hand boundary value problem, we introduce the Riemann invariants, consider them along the characteristics and recombine them to obtain finally the representation

$$V_l(x,t) = \begin{cases} \frac{1}{2}[V_0(x - t/\sqrt{LC}) + V_0(x + t/\sqrt{LC}) + \sqrt{L/C}(I_0(x - t/\sqrt{LC}) + \\ -I_0(x + t/\sqrt{LC}))];\ x \le -l/2,\ 0 \le t \le -\sqrt{LC}(x + l/2) \\ \frac{1}{2}[V_0(x - t/\sqrt{LC}) - \rho_1 V_0(-l - x - t/\sqrt{LC}) + \\ +\sqrt{L/C}(I_0(x - t/\sqrt{LC}) - \rho_1 I_0(-l - x - t/\sqrt{LC})) + \\ +(1 + \rho_1)V_l'(t + \sqrt{LC}(l/2 + x))];\ x \le -l/2,\ t \ge -\sqrt{LC}(x + l/2) \end{cases} \tag{8.27}$$

$$I_l(x,t) = \begin{cases} \frac{1}{2}[I_0(x - t/\sqrt{LC}) - I_0(x + t/\sqrt{LC}) + \sqrt{C/L}(V_0(x - t/\sqrt{LC}) + \\ +V_0(x + t/\sqrt{LC}))];\ x \le -l/2,\ 0 \le t \le -\sqrt{LC}(x + l/2) \\ \frac{1}{2}[I_0(x - t/\sqrt{LC}) + \rho_1 I_0(-l - x - t/\sqrt{LC}) + \\ +\sqrt{C/L}(V_0(x - t/\sqrt{LC}) + \rho_1 V_0(l - x + t/\sqrt{LC})) - \\ -(1 + \rho_2)\sqrt{C/L}V_l'(t + \sqrt{LC}(l/2 + x))]; \\ x \le -l/2,\ t \ge -\sqrt{LC}(x + l/2) \end{cases} \tag{8.28}$$

where we denoted $\rho_1 = (1 - R_1'\sqrt{C/L})(1 + R_1'\sqrt{C/L})^{-1}$; obviously $|\rho_1| < 1$.

To conclude this subsection, we obtained the representation of the solutions for the two side boundary value problems on $(l/2, \infty)$ and $(-\infty, l/2)$ using the solutions' behavior along the characteristics. Such boundary value problems, defined on semi-infinite intervals, are single point boundary value problems and were pointed out and analyzed in [56]. The forcing terms are "injected" exactly at the aforementioned boundaries and they contain signals from the "stuck" boundaries of the central boundary value problem superposed with the signals from the local (lumped) oscillators which are located also on the boundaries. Moreover, the effect of these forcing signals appear for "large t"—see formulae (8.24), (8.25) and (8.27), (8.28)—superposed on the initial conditions $(V_0(x), I_0(x))$ which are defined on $(-\infty, \infty)$ hence on $(-l/2, l/2)$ also.

B. We shall consider now the two point boundary value problem defined on $(-l/2, l/2) \times \mathbb{R}$ by the Eq. (8.29) that follow. Observe there, from the beginning that the boundary conditions coincide at $x = \pm l/2$ with those of the side boundary value problems but now they are "viewed from inside the central boundary value problem". The central boundary value problem is now "forced" by the signals of the side ones. Since the side boundary value problems had been shown as forced by the central one, we discover here a structure with *internal feedback*. Moreover, the boundary conditions of (8.29) are fed by signals arising from the local (lumped) oscillators. Since the aforementioned local oscillators are also fed with signals from the boundary conditions, we can recognize here another *internal feedback*. On the other hand it has been pointed out that feedback can be related either to oscillatory behavior (see Chap. IV of [57]) or to instabilities [37]. Such aspects may turn essential for our paper. They also show the importance of the boundary value problem defined by (8.29), which will be under study.

$$
\begin{aligned}
& L\partial_t I_c + \partial_x V_c = 0,\ C\partial_t V_c + \partial_x I_c = 0 \\
& V_c(-l/2,t) - R_1' I_c(-l/2,t) = V_{D1}(t) + R_1' I_l(-l/2,t) := V_l''(t) \\
& V_c(l/2,t) - R_2' I_c(l/2,t) = V_{D2}(t) - R_2' I_r(l/2,t) := V_r'(t) \\
& L_1 \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} = -R_1 I_{L1} - V_{D1} + E_1 \\
& C_1 \frac{\mathrm{d}V_{D1}}{\mathrm{d}t} = I_{L1} - f_1(V_{D1}) - I_c(-l/2,t) + I_l(-l/2,t) \\
& L_2 \frac{\mathrm{d}I_{L2}}{\mathrm{d}t} = -R_2 I_{L2} - V_{D2} + E_2 \\
& C_2 \frac{\mathrm{d}V_{D2}}{\mathrm{d}t} = I_{L2} - f_2(V_{D2}) + I_c(l/2,t) - I_r(l/2,t)
\end{aligned}
\tag{8.29}
$$

We start by expressing the forcing terms in (8.29): at $\pm l/2$ we shall have $t \geq 0$ hence only the second form of the representation is valid and only in (8.25) and (8.28) since only currents are needed

$$
\begin{aligned}
I_l(-l/2,t) = & \frac{1+\rho_1}{2}[I_0(-l/2 - t/\sqrt{LC}) + \sqrt{C/L}(V_0(-l/2 - t/\sqrt{LC}) - \\
& - V_{D1}(t) + R_1' I_c(-l/2,t))] \\
I_r(l/2,t) = & \frac{1+\rho_2}{2}[I_0(l/2 + t/\sqrt{LC}) - \sqrt{C/L}(V_0(l/2 + t/\sqrt{LC}) - \\
& - V_{D2}(t) - R_2' I_c(l/2,t))]
\end{aligned}
\tag{8.30}
$$

Next we substitute (8.30) in (8.29) to obtain, after some simple and straightforward manipulation

$$
\begin{aligned}
& L\partial_t I_c + \partial_x V_c = 0\ ,\ C\partial_t V_c + \partial_x I_c = 0 \\
& V_c(-l/2,t) + \frac{1+\rho_1}{2} R_1' I_c(-l/2,t) = \frac{1+\rho_1}{2} V_{D1}(t) + \frac{1-\rho_1}{2}\psi_1(t) \\
& V_c(l/2,t) - \frac{1+\rho_2}{2} R_2' I_c(l/2,t) = \frac{1+\rho_2}{2} V_{D2}(t) + \frac{1-\rho_2}{2}\psi_2(t) \\
& L_1 \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} = -R_1 I_{L1} - V_{D1} + E_1 \\
& C_1 \frac{\mathrm{d}V_{D1}}{\mathrm{d}t} = I_{L1} - (\frac{1+\rho_1}{2}\sqrt{C/L}V_{D1} + f_1(V_{D1})) - \frac{1+\rho_1}{2}(I_c(-l/2,t) - \sqrt{C/L}\psi_1(t))
\end{aligned}
\tag{8.31}
$$

$$L_2\frac{\mathrm{d}I_{L2}}{\mathrm{d}t} = -R_2 I_{L2} - V_{D2} + E_2$$
$$C_2\frac{\mathrm{d}V_{D2}}{\mathrm{d}t} = I_{L2} - \left(\frac{1+\rho_2}{2}\sqrt{C/L}V_{D2} + f_2(V_{D2})\right) + \frac{1+\rho_2}{2}(I_c(l/2,t) + \sqrt{C/L}\psi_2(t))$$

where we denoted

$$\begin{aligned}\psi_1(t) &:= V_0(-l/2 - t/\sqrt{LC}) + \sqrt{L/C}I_0(-l/2 - t/\sqrt{LC})\\ \psi_2(t) &:= V_0(l/2 + t/\sqrt{LC}) - \sqrt{L/C}I_0(l/2 + t/\sqrt{LC})\end{aligned} \tag{8.32}$$

8.4 The Single Oscillator

We shall consider here the electronic oscillator of the structure presented in Fig. 8.4

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = -RI_L - V_D + E,\; C\frac{\mathrm{d}V_D}{\mathrm{d}t} = I_L - f(V_D) \tag{8.33}$$

where $f(V_D)$ is the nonlinear characteristic of the tunnel diode (worth mentioning that most of the contemporary electronic oscillators strongly rely on the nonlinear characteristic of the tunnel diode).

The d.c. voltage source E ensures the "bias" of the operating point as in Fig. 8.5—the point denoted by Q_2, where the resistance is negative; the equations of the operating point—the equilibrium—are

$$R\bar{I}_L + \bar{V}_D = E,\; \bar{I}_L = f(\bar{V}_D) \tag{8.34}$$

The equilibrium $(\bar{V}_D, \bar{I}_L)$ is the crossing point of the nonlinear tunnel diode characteristic with the straight line $\bar{I}_L = (E - \bar{V}_D)/R$. We can consider the deviations

$$i_L = I_L - \bar{I}_L,\; v_D = V_D - \bar{V}_D \tag{8.35}$$

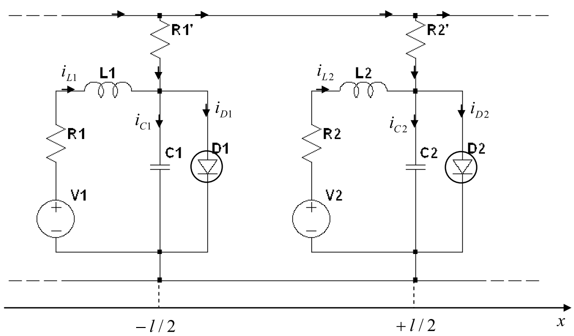


Fig. 8.4 Two electronic oscillators on the infinite LC line

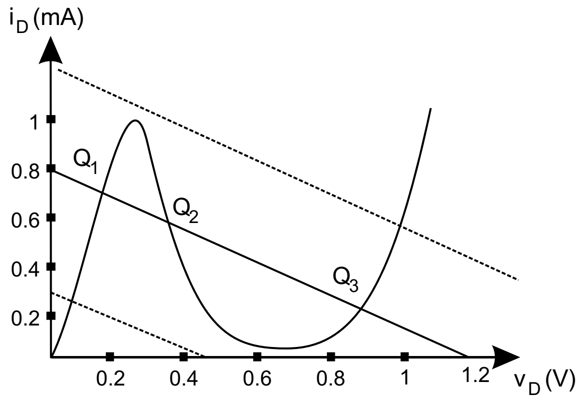


Fig. 8.5 Nonlinear characteristic of the tunnel diode

and write down the system in deviations

$$L\frac{\mathrm{d}i_L}{\mathrm{d}t} = -Ri_L - v_D,\ C\frac{\mathrm{d}v_D}{\mathrm{d}t} = i_L - (f(v_D + \bar{V}_D) - f(\bar{V}_D)) \tag{8.36}$$

System (8.36) is equivalent to the following second order differential equation

$$LC\frac{\mathrm{d}^2 v_D}{\mathrm{d}t^2} + (RC + L\varphi'(v_D))\frac{\mathrm{d}v_D}{\mathrm{d}t} + v_D + R\varphi(v_D) = 0 \tag{8.37}$$

where we denoted $\varphi(x) := f(x + \bar{V}_D) - f(\bar{V}_D)$; obviously $\varphi(0) = 0$.

Equation (8.37) belongs to the class of Liénard equations [33] having the general form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + h(y)\frac{\mathrm{d}y}{\mathrm{d}t} + g(y) = 0 \tag{8.38}$$

Worth mentioning that almost all types of electronic and mechanical oscillators are described by Liénard type equations—see e.g., [3]. Among them, the most famous and the most studied is the equation of van der Pol—a special case of (8.38) with

$$g(y) \equiv y\ ,\ h(y) = \varepsilon(y^2 - 1) \tag{8.39}$$

(Observe that (8.37) does not allow to deduce the equation of van der Pol as a special case).

Equation (8.38) was considered a very suitable application for the Poincaré–Bendixson theory of the limit cycles. Results on this subject can be found in such renowned classical monographs as [9, 27, 53]. Additionally there exist quite a lot of interesting papers containing effective criteria for existence and uniqueness of an orbitally limit cycle of (8.38). For the aforementioned reasons it is but common to assume that the oscillators of (8.22)—considered independently, see (8.33)—have a unique orbitally stable limit cycle. There exist also computational approaches allowing to estimate with good precision the period of the periodic solution defined by the limit cycle—see e.g., [7, 34, 36]. All aforementioned estimates have in common the dependence of the period on system's parameters and *structure*: the structure is given here by the circuit structure ensuring generation of a stable limit cycle with a stable period: in Fig. 8.4 we had one type of oscillator. The van der Pol oscillator based on a tunnel diode is described by the following equations

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = U_c,\ C\frac{\mathrm{d}U_c}{\mathrm{d}t} = -I_L - f(U_c + E_0) \tag{8.40}$$

where $f(\cdot)$ is the nonlinear characteristic of the tunnel diode. The steady state is given by $\bar{U}_c = 0$, $\bar{I}_L = f(E_0)$ hence the equations in deviations are

$$L\frac{\mathrm{d}i_L}{\mathrm{d}t} = u_c,\ C\frac{\mathrm{d}u_c}{\mathrm{d}t} = -i_L - [f(u_c + E_0) - f(E_0)] \tag{8.41}$$

with $i_L = I_L - f(E_0)$, $u_c = U_c$. In realizing a van der Pol oscillator based on the tunnel diode, the nonlinear characteristic of the diode i.e., $f(z) = az^3 + bz^2 + cz$ is biased by E_0 in order to eliminate the quadratic term. A rather elementary manipulation shows that by choosing $E_0 = -b/(3a)$ it follows that

$$f(u_c + E_0) - f(E_0) = u_c(au_c^2 + c - b^2/(3a)),\ c - b^2/(3a) < 0 \tag{8.42}$$

the last inequality ensuring three real roots for $f(u_c + E_0) - f(E_0) = 0$. Next, starting from the second order differential equation deduced from (8.41),

$$LC\frac{\mathrm{d}^2u_c}{\mathrm{d}t^2} - L(b^2/(3a) - c - 3au_c^2)\frac{\mathrm{d}u_c}{\mathrm{d}t} + u_c = 0 \tag{8.43}$$

which is a Liénard type equation, the following change of variables is made

$$y(\tau) := \sqrt{\frac{3a}{b^2/(3a) - c}}u_c(\tau\sqrt{LC}) = \frac{1}{\sqrt{\left(\frac{b^2}{3a}\right)^2 - \frac{c}{3a}}}u_c(\tau\sqrt{LC}) \tag{8.44}$$

to obtain the standard van der Pol equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \varepsilon(1 - y^2)\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0;\ \varepsilon := \sqrt{\frac{L}{C}}\left(\frac{b}{3a} - c\right) \tag{8.45}$$

We can consider also other types of electronic oscillators having a unique stable limit cycle but their analysis is outside the aim of this paper. The inquisitive reader can consult [3, 7, 8, 34] and, possibly, adapt the found structures to the actual electronic technology.

In what follows we shall focus on system (8.31)—its theoretical background.

8.5 The Functional Differential Equations of the Coupled Oscillators

We start from system (8.31) where the forcing terms $\psi_k(t)$ are given by (8.32) which are voltages; these terms are induced by the semi-infinite side boundary value problems, $V_0(x)$ and $I_0(x)$ being the initial conditions of the LC line. In order to apply the standard approaches of the oscillation theory, we shall introduce new state variables—the deviations with respect to the constant steady states of the unforced oscillators. These steady states are obtained as solutions of the system

$$
\begin{gathered}
\bar{V}_c + \frac{1+\rho_1}{2} R_1' \bar{I}_c = \frac{1+\rho_1}{2} \bar{V}_{D1},\ \ \bar{V}_c - \frac{1+\rho_2}{2} R_2' \bar{I}_c = \frac{1+\rho_2}{2} \bar{V}_{D2} \\
\bar{I}_{L1} = \frac{E_1 - \bar{V}_{D1}}{R_1},\ \ \bar{I}_{L1} - \left(\frac{1+\rho_1}{2} \sqrt{C/L} \bar{V}_{D1} + f_1(\bar{V}_{D1}) \right) - \frac{1+\rho_1}{2} \bar{I}_c = 0 \\
\bar{I}_{L2} = \frac{E_2 - \bar{V}_{D2}}{R_2},\ \ \bar{I}_{L2} - \left(\frac{1+\rho_2}{2} \sqrt{C/L} \bar{V}_{D2} + f_2(\bar{V}_{D2}) \right) - \frac{1+\rho_2}{2} \bar{I}_c = 0
\end{gathered} \tag{8.46}
$$

Under some standard and quite mild assumptions on the functions $f_k(\cdot)$, $k = 1, 2$, system (8.46) has a unique solution. We now introduce the deviations

$$
\begin{array}{l}
i_c(x,t) = I_c(x,t) - \bar{I}_c,\ \ v_c(x,t) = V_c(x,t) - \bar{V}_c; \\
i_{Lk}(t) = I_{Lk}(t) - \bar{I}_{Lk},\ \ v_{Dk}(t) = V_{Dk}(t) - \bar{V}_{Dk},\ \ k = 1, 2
\end{array} \tag{8.47}
$$

Consequently the following system in deviations is obtained

$$
\begin{array}{l}
L\partial_t i_c + \partial_x v_c = 0,\ \ C\partial_t v_c + \partial_x i_c = 0 \\
v_c(-l/2, t) + \dfrac{1+\rho_1}{2} R_1' i_c(-l/2, t) = \dfrac{1+\rho_1}{2} V_{D1}(t) + \dfrac{1-\rho_1}{2} \psi_1(t) \\
v_c(l/2, t) - \dfrac{1+\rho_2}{2} R_2' i_c(l/2, t) = \dfrac{1+\rho_2}{2} V_{D2}(t) + \dfrac{1-\rho_2}{2} \psi_2(t) \\
L_1 \dfrac{\mathrm{d} i_{L1}}{\mathrm{d}t} = -R_1 i_{L1} - v_{D1};\ \ L_2 \dfrac{\mathrm{d} i_{L2}}{\mathrm{d}t} = -R_2 i_{L2} - v_{D2} \\
C_1 \dfrac{\mathrm{d} v_{D1}}{\mathrm{d}t} = i_{L1} - \tilde{\varphi}_1(v_{D1}) - \dfrac{1+\rho_1}{2} i_c(-l/2, t) + \dfrac{1+\rho_1}{2} \sqrt{C/L} \psi_1(t) \\
C_2 \dfrac{\mathrm{d} v_{D2}}{\mathrm{d}t} = i_{L2} - \tilde{\varphi}_2(v_{D2}) + \dfrac{1+\rho_2}{2} i_c(l/2, t) + \dfrac{1+\rho_2}{2} \sqrt{C/L} \psi_2(t)
\end{array} \tag{8.48}
$$

where we introduced the new functions $\tilde{\varphi}_k : \mathbb{R} \mapsto \mathbb{R}$ by

$$
\tilde{\varphi}_k(\sigma) = \frac{1+\rho_k}{2} \sqrt{C/L} \sigma + f_k(\sigma + \bar{V}_{Dk}) - f_k(\bar{V}_{Dk}) \tag{8.49}
$$

with $\tilde{\varphi}_k(0) = 0$. We consider from now on system (8.48) and apply to it the approach of the previous sections by considering the Riemann invariants along the characteristics.

A. Introduce first the Riemann invariants as in Sect. 8.2, here

$$\begin{aligned} &u_c^{\pm}(x,t) = v_c(x,t) \pm \sqrt{L/C}\, i_c(x,t);\\ &v_c(x,t) = \frac{1}{2}(u_c^+(x,t) + u_c^-(x,t)),\ i_c(x,t) = \frac{1}{2}\sqrt{C/L}(u_c^+(x,t) - u_c^-(x,t)) \end{aligned} \tag{8.50}$$

where $u_c^{\pm}(x,t)$ signify forward and backward voltage waves respectively. After some quite straightforward but tedious manipulation the boundary value problem (8.48) is expressed in the terms of the Riemann invariants

$$\begin{aligned} &\sqrt{LC}\partial_t u_c^{\pm} \pm \partial_x u_c^{\pm} = 0\\ &u_c^+(-l/2,t) + \frac{1+\rho_1'}{2} u_c^-(-l/2,t) = (1+\rho_1')v_{D1}(t) + (1-\rho_1')\sqrt{C/L}\psi_1(t)\\ &u_c^-(l/2,t) + \frac{1+\rho_2'}{2} u_c^+(l/2,t) = (1+\rho_2')v_{D2}(t) + (1-\rho_2')\sqrt{C/L}\psi_2(t)\\ &L_k\frac{\mathrm{d}i_{Lk}}{\mathrm{d}t} = R_k i_{Lk} - v_{Dk},\ k=1,2\\ &C_1\frac{\mathrm{d}v_{D1}}{\mathrm{d}t} = i_{L1} - \tilde{\varphi}_1(v_{D1}) - \frac{1+\rho_1}{2}\frac{1}{2}\sqrt{C/L}(u_c^+(-l/2,t) - u_c^-(-l/2,t)) +\\ &\qquad + \frac{1+\rho_1}{2}\sqrt{C/L}\psi_1(t)\\ &C_2\frac{\mathrm{d}v_{D2}}{\mathrm{d}t} = i_{L2} - \tilde{\varphi}_2(v_{D2}) + \frac{1+\rho_2}{2}\frac{1}{2}\sqrt{C/L}(u_c^+(-l/2,t) - u_c^-(-l/2,t)) +\\ &\qquad + \frac{1+\rho_2}{2}\sqrt{C/L}\psi_2(t) \end{aligned} \tag{8.51}$$

where we denoted

$$\rho_k' = (1 - 2R_k'\sqrt{C/L})(1 + 2R_k'\sqrt{C/L})^{-1},\ k=1,2 \tag{8.52}$$

Consider first the increasing characteristic line crossing $(x,t) \in (-l/2, l/2) \times \mathbb{R}^+$

$$t^+(\xi; x, t) = t + \sqrt{LC}(\xi - x)$$

which can be extended "to the left" in two ways: (a) if $t + \sqrt{LC}(-l/2 - x) < 0$ i.e., $t < \sqrt{LC}(l/2 + x)$ ("small" times) then the extension can be made up to $\underline{\xi}_c$ deduced from $t + \sqrt{LC}(\underline{\xi}_c - x) = 0$. Since $u_c^+(x,t)$ is constant along the characteristic, it follows that

$$u_c^+(x,t) = u_c^+(x - t/\sqrt{LC}, 0) = \frac{1}{2}(v_0(x - t/\sqrt{LC}) + \sqrt{L/C}\, i_0(x - t/\sqrt{LC})) \tag{8.53}$$

for $t < \sqrt{LC}(l/2 + x)$ hence $t \in (0, l\sqrt{LC})$. In particular

$$u_c^+(l/2,t) = \frac{1}{2}(v_0(l/2 - t/\sqrt{LC}) + \sqrt{L/C}\, i_0(l/2 - t/\sqrt{LC}))$$

with the argument of v_0 and i_0 ranging on $(-l/2, l/2)$. We denoted by v_0 and i_0 the deviations of the initial conditions with respect to the aforementioned steady state values $\bar{V}_c$, $\bar{I}_c$—see (8.46); (b) if $t + \sqrt{LC}(-l/2 - x) > 0$ i.e., $t > \sqrt{LC}(l/2 + x)$ ("large" times) then

$$u_c^+(x, t) = u_c^+(l/2, t + \sqrt{LC}(l/2 - x)) \tag{8.54}$$

and, in particular $u_c^+(-l/2, t) = u_c^+(l/2, t + l\sqrt{LC})$ showing that $u_c^+(x, t)$ is indeed a forward wave. Consider now the decreasing characteristic line crossing $(x, t) \in (-l/2, l/2) \times \mathbb{R}^+$

$$t^-(\xi; x, t) = t - \sqrt{LC}(\xi - x)$$

and the backward wave $u_c^-(x, t)$ which is constant along it. The characteristic line can be extended "to the right" in two ways: (a) if $t - \sqrt{LC}(l/2 - x) < 0$ i.e., $t < \sqrt{LC}(l/2 - x)$ ("small" times that is $t \in (0, l\sqrt{LC})$) then the extension can be done up to $\bar{\xi}_c$ deduced from $t - \sqrt{LC}(\bar{\xi}_c - x) = 0$ i.e., $\bar{\xi}_c = x + t/\sqrt{LC} > x > -l/2$ but also $\bar{\xi}_c < x + l/2 - x = l/2$. We deduce

$$u_c^-(x, t) = u_c^-(x + t/\sqrt{LC}, 0) = \frac{1}{2}(v_0(x + t/\sqrt{LC}) + \sqrt{L/C}i_0(x + t/\sqrt{LC})) \tag{8.55}$$

for $t < \sqrt{LC}(l/2 - x)$ hence $t \in (0, l\sqrt{LC})$. In particular

$$u_c^-(-l/2, t) = \frac{1}{2}(v_0(-l/2 + t/\sqrt{LC}) + \sqrt{L/C}i_0(-l/2 + t/\sqrt{LC}))$$

with the argument of v_0 and i_0 ranging on $(-l/2, l/2)$; (b) if $t - \sqrt{LC}(l/2 - x) > 0$ i.e., $t > \sqrt{LC}(l/2 - x)$ ("large" times that is $t > l\sqrt{LC}$)) then the extension can be done up to $\xi = l/2$

$$u_c^-(x, t) = u_c^-(-l/2, t - \sqrt{LC}(-l/2 - x)) \tag{8.56}$$

and, in particular $u_c^-(l/2, t) = u_c^-(-l/2, t + l\sqrt{LC})$ showing that $u_c^-(x, t)$ is indeed a backward wave. Denoting

$$y_c^\pm(t) := u_c^\pm(p \pm l/2, t) \;\Rightarrow\; u_c^\pm(\mp l/2, t) = y_c^\pm(t + l\sqrt{LC}),\;\; t > l\sqrt{LC} \tag{8.57}$$

we have also

$$y_c^\pm(t) = \frac{1}{2}(v_0(\pm l/2 \mp t/\sqrt{LC}) \pm \sqrt{L/C}i_0(\pm l/2 \mp t/\sqrt{LC})) \tag{8.58}$$

for $0 < t < l\sqrt{LC}$. Introducing also $\eta_c^\pm(t) := y_c^\pm(t + l\sqrt{LC})$ it follows that

$$\eta_c^{\pm}(t) = \frac{1}{2}(v_0(\mp l/2 \mp t/\sqrt{LC}) \pm \sqrt{L/C}i_0(\mp l/2 \mp t/\sqrt{LC})) \tag{8.59}$$

for $-l\sqrt{LC} < t < 0$. Substituting $\eta_c^{\pm}(t)$ thus defined in the boundary conditions and the differential equations of (8.51) we obtain, after some manipulation, the following system of functional differential equations composed of coupled delay differential and difference equations

$$\begin{aligned}
&L_k \frac{\mathrm{d}i_{Lk}}{\mathrm{d}t} = R_k i_{Lk} - v_{Dk},\ k = 1, 2\\
&C_1 \frac{\mathrm{d}v_{D1}}{\mathrm{d}t} = i_{L1} - \varphi_1(v_{D1}) - \frac{1+\rho_1}{2}\frac{3+\rho_1'}{4}\eta_c^-(t - l\sqrt{LC}) + \frac{1+\rho_1}{2}\frac{3+\rho_1'}{4}\sqrt{C/L}\psi_1(t)\\
&C_2 \frac{\mathrm{d}v_{D2}}{\mathrm{d}t} = i_{L2} - \varphi_2(v_{D2}) - \frac{1+\rho_2}{2}\frac{3+\rho_2'}{4}\eta_c^+(t - l\sqrt{LC}) + \frac{1+\rho_2}{2}\frac{3+\rho_2'}{4}\sqrt{C/L}\psi_2(t)
\end{aligned}$$

$$\begin{aligned}
\eta_c^+(t) + \frac{1+\rho_1'}{2}\eta_c^-(t - l\sqrt{LC}) &= (1+\rho_1')v_{D1}(t) + \frac{1-\rho_1'}{2}\psi_1(t)\\
\eta_c^-(t) + \frac{1+\rho_2'}{2}\eta_c^+(t - l\sqrt{LC}) &= (1+\rho_2')v_{D2}(t) + \frac{1-\rho_2'}{2}\psi_2(t)
\end{aligned} \tag{8.60}$$

where we denoted

$$\varphi_k(\sigma) := \frac{1+\rho_k}{2}\frac{1+\rho_k'}{2}\sqrt{C/L}\sigma + \tilde{\varphi}_k(\sigma),\ k = 1, 2 \tag{8.61}$$

Summarizing the development of this Section we have in fact proven the following mathematical result

Theorem 8.3 *Consider system (8.48) and let($v_c(x,t)$, $i_c(x,t)$, $i_{Lk}(t)$, $v_{Dk}(t)$) be a classical solution of it, defined by the initial conditions ($v_0(x)$, $i_0(x)$, $-l/2 \le x \le l/2$; $i_{Lk}(0)$, $v_{Dk}(0)$, $k = 1, 2$) and with given $\psi_k(t)$, $k = 1, 2$. Then ($i_{Lk}(t)$, $v_{Dk}(t)$; $\eta_c^{\pm}(t)$) is a solution of (8.60) on $t > 0$ with $\eta_c^{\pm}(t) := y_c^{\pm}(t + l\sqrt{LC})$, $y_c^{\pm}(t)$ defined by (8.58) and with the initial conditions ($i_{Lk}(0)$, $v_{Dk}(0)$; $\eta_c^{\pm}(t)$, $-l\sqrt{LC} \le t \le 0$), $\eta_c^{\pm}(t)$ being defined on $(-l\sqrt{LC}, 0)$ by (8.59).*

Conversely, let ($i_{Lk}(t)$, $v_{Dk}(t)$; $\eta_c^{\pm}(t)$) be some solution of (8.60) corresponding to the initial conditions ($i_{Lk}(0)$, $v_{Dk}(0)$; $\eta_c^{\pm}(t)$, $-l\sqrt{LC} \le t \le 0$). Then ($v_c(x,t)$, $i_c(x,t)$, $i_{Lk}(t)$, $v_{Dk}(t)$) is a (possibly discontinuous) classical solution of (8.48), where $v_c(x,t)$ and $i_c(x,t)$ are obtained from (8.50) and

$$u_c^+(x,t) = \eta_c^+(t - \sqrt{LC}(l/2 + x)),\ u_c^-(t + \sqrt{LC}(x - l/2)) \tag{8.62}$$

(deduced from the representation formulae (8.54)–(8.55), the notations (8.57) and the definition of $\eta_c^{\pm}(t)$).

This theorem is a result of the aforementioned (in Sect. 8.1) methodology of [1, 12, 13], clarified and completed in [44]. Theorem 8.3 establishes in a rigorous way a one to one correspondence between the solutions of two mathematical objects: the boundary value problem (8.48) for hyperbolic partial differential equations and the initial value (Cauchy) problem (8.60) for a system of coupled delay differential and difference equations. In this way all mathematical results obtained for one object are projected back onto the other one.

We shall illustrate this assertion in the following. Consider first the basic theory i.e., existence, uniqueness and smooth data dependence of the solutions: for partial differential equations this is called well posedness in the sense of J. Hadamard. In the case of (8.48) we observe that the boundary problem is non standard: the boundary conditions are coupled (in some kind of internal feedback) to a system of ordinary differential equations. It is thus useful to focus on system (8.60). Its solution can be constructed by steps on intervals $(kl\sqrt{LC}, (k+1)l\sqrt{LC})$ with $k > 0$ a positive integer; the construction ensures uniqueness. However the solution is not smoothed along the construction: $\eta_c^{\pm}(t)$ have the smoothness of their initial conditions on $(-l\sqrt{LC}, 0)$ and are discontinuous in $kl\sqrt{LC}$, k being an integer; $i_{Lk}(t)$ and $v_{Dk}(t)$, $k = 1, 2$ are differentiable and their derivatives have the smoothness of $\eta_c^{\pm}(t)$ as well as discontinuities in $kl\sqrt{LC}$.

These aforementioned properties suggest that system (8.60) as system of functional differential equations with deviated argument may be considered of neutral type. This is true in general: consider the system

$$\begin{aligned} \dot{x} &= A_0x(t) + A_1y(t-\tau) - b_0\phi(c^*x(t)) + f_1(t) \\ y(t) &= A_2x(t) + A_3y(t-\tau) + f_2(t) \end{aligned} \tag{8.63}$$

with $\dim\ x = n$, $\dim\ y = m$ and A_0, A_1, A_2, A_3, b_0 of appropriate dimensions. For $y(t)$, $f_2(t)$ smooth enough, (8.63) can be reduced to

$$\begin{aligned} &\dot{x} = A_0x(t) + A_1y(t-\tau) - b_0\phi(c^*x(t)) + f_1(t) \\ &\frac{\mathrm{d}}{\mathrm{d}t}(y(t) - A_3y(t-\tau)) = A_2A_0x(t) + A_2A_1y(t-\tau) + A_2f_1(t) + \dot{f}_2(t) \end{aligned} \tag{8.64}$$

which is clearly of neutral type—see [26] for details. Conversely, if

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - Dx(t-\tau)) = A_0x(t) + A_1x(t-\tau) - b\phi(c^*x(t)) + f(t) \tag{8.65}$$

is a functional differential equation of neutral type, then it can be given the form

$$\begin{aligned} \dot{z}(t) &= A_0x(t) + A_1x(t-\tau) - b\phi(c^*x(t)) + f(t) \\ x(t) &= z(t) + Dx(t-\tau) \end{aligned} \tag{8.66}$$

which is alike (8.63). Also the discontinuities of the solutions are specific to neutral functional differential equations. But in our case they have an additional significance, being a consequence of the mismatch between the initial and the boundary conditions of the boundary value problem defined by (8.48). If one takes into account that the initial conditions of a dynamical system “integrate” the effect of short period perturbations—which are unknown—while the boundary conditions are a part of system's description, the aforementioned mismatch is the rule and the matching—the exception.

To end this section and to open the way to the next one, we recall here the so called *Stability Postulate* of Četaev [10, 11]—see also [49]—"only stable trajectories are physically observable and measurable". We shall thus consider stability of (8.60) and, therefore of (8.48) in order to check the Postulate; for its importance we included its checking within the so called augmented validation of the mathematical models for physical systems [44, 49].

8.6 Stability and Forced Oscillations of the System of Functional Differential Equations

A. Consider again system (8.60) with $\psi_1(t) \equiv \psi_2(t) \equiv 0$; considering (8.59) it appears that we thus took $i_0(x) \equiv 0, v_0(x) \equiv 0$ for $x < -l/2, x > l/2$. Consequently (8.60) becomes an autonomous system with $\{0\}$ as equilibrium. Moreover this system belongs to the class of coupled delay differential and difference equations described by

$$\begin{aligned} \dot{x} &= A_0 x(t) + A_1 y(t-\tau) - b_0\phi(c^* x(t)) \\ y(t) &= A_2 x(t) + A_3 y(t-\tau) - b_1\phi(c^* x(t)) \end{aligned} \tag{8.67}$$

This class of systems was introduced in [50, 51] precisely in connection with boundary value problems for hyperbolic partial differential equations arising from electrical and control engineering—see [45, 52]. For it the following stability result holds

Theorem 8.4 ([50, 51]) *Consider system (8.67) under the following assumptions: (i) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is subject to*

$$0 \leq \phi(\sigma)\sigma \leq k\sigma^2,\ \phi(0) = 0 \tag{8.68}$$

(ii) the characteristic equation of the linear subsystem has its roots in the left half plane of $\mathbb{C}$ defined by $\Re e(s) \leq -\alpha < 0$ for some $\alpha > 0$:

$$\det\ G(s) = \det \begin{pmatrix} sI - A_0 & -A_1 e^{-s\tau} \\ -A_2 & I - A_3 e^{-s\tau} \end{pmatrix} \neq 0,\ \Re e(s) > -\alpha \tag{8.69}$$

and the matrix A_3 has all its roots inside the unit disk $\mathbb{D}_1 \subset \mathbb{C}$.

If there exists a nonnegative $q \geq 0$ such that the following frequency domain inequality holds

$$\frac{1}{k} + \Re e(1 + \imath\omega q)H(\imath\omega) > 0,\ \omega \in \bar{\mathbb{R}}^+ \tag{8.70}$$

where $H(s)$ is the transfer function of the linear part of (8.67) defined by

$$H(s) = (c^* \quad 0)G(s)^{-1}\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \tag{8.71}$$

then the zero solution of (8.67) is globally asymptotically stable with respect to all nonlinear functions subject to (8.68).

This is a standard result on absolute stability: we point out some specific features for application to (8.60) in the autonomous case. In the case of (8.60) matrix A_3 has the form

$$A_3 = \begin{pmatrix} 0 & -(1+\rho_1')/2 \\ -(1+\rho_2')/2 & 0 \end{pmatrix}$$

and the eigenvalues $\pm(1/2)\sqrt{(1+\rho_1')(1+\rho_2')}$, located inside $\mathbb{D}_1$ since $|\rho_k'| < 1$—see (8.52). This condition—called strong stability of the difference operator [23]—is rather important in the study of the functional differential equations of neutral type. Among other, it is necessary for (8.69) and also for the corresponding result of the type Barbashin–Krasovskii–LaSalle invariance principle [23] (Theorem 8.2 of Chap. 9).

We have still two more remarks to add here. The first one concerns the sector condition (8.68) which is sometimes viewed as a dissipativeness/passivity condition on the corresponding nonlinear block. If we examine (8.29), the functions $f_i(\cdot)$ are nonlinearities in the structure of the local oscillators: as illustrated in Sect. 8.4, they are such to obey the conditions of the existence, uniqueness and stability of the limit cycle and *do not* satisfy (8.68). If however (8.48) is considered, one can see from (8.49) that the new nonlinear functions can possibly satisfy a condition like (8.68) because of the additional local damping—the so called *radiation dissipation*—mentioned at Sect. 8.2.

The next remark concerns the fact that (8.48) and, therefore, (8.60), contain two nonlinear functions. Consequently the linear subsystem of (8.60) has two inputs and two outputs; its transfer function becomes a 2×2 matrix transfer function and the frequency domain inequality (8.70) becomes a matrix one. We reached here the main *drawback* of the frequency domain approach of V. M. Popov for nonlinear systems: the *dimension of the frequency domain inequality—as matrix inequality—equals the number of the nonlinear functions*.

The (older) counterpart of the frequency domain approach is the method of the Lyapunov function(al). If we turn back to the Stability Postulate and follow the original setting of Četaev, it is explained there that "the physical laws should be subject to the Stability Postulate". We analyzed this idea along several physics, chemistry, engineering and non-engineering applications throughout two monographs [20, 46]: the underlying unifying factor was existence of some "natural" Lyapunov function which in some cases could be deduced from the right hand side of model's equations. For mechanical and electrical engineering structures this had been particularly true since the Lyapunov function had turned to be the stored energy. As already shown in [41, 48], the energy stored in the electromagnetic structure of (8.48)—see also Fig. 8.4 reads

$$\mathscr{E}(v_{Dk}(t), i_{Lk}(t), i_c(\cdot,t), v_c(\cdot,t)) = \frac{1}{2}[C_1 v_{D1}^2(t) + C_2 v_{D2}^2(t) + L_1 i_{L1}^2(t) + L_2 i_{L2}^2(t)] + \int_{-l/2}^{l/2} (L i_c(x,t)^2 + C v_c(x,t)^2)\mathrm{d}x \tag{8.72}$$

As known, when a system is structured in discretely space distributed elements, the stored energy is a diagonal quadratic function of the state variables. Moreover, this function(al) is positive definite (the presence of a distributed parameter LC line gives to $\mathscr{E}$ the character of a functional).

We differentiate (8.72) along the solutions of the autonomous system (8.48) ($\psi_1(t) \equiv \psi_2(t) \equiv 0$) to obtain

$$-\mathscr{W}(v_{Dk}(t), i_{Lk}(t), i_c(-l/2,t), i_c(l/2,t)) = R_1 i_{L1}^2(t) + \tilde{\varphi}(v_{D1}(t))v_{D1}(t) + R_2 i_{L2}^2(t) + \tilde{\varphi}(v_{D2}(t))v_{D2}(t) + R_1' i_c^2(-l/2,t) + R_2' i_c^2(l/2,t) \tag{8.73}$$

Since $\mathscr{W}$ is negative definite in its arguments and non-positive definite with respect to the state space of (8.48), the energy Lyapunov functional (8.72) is non-increasing along the solutions of (8.48). More precisely we can write

$$\mathscr{E}(v_{Dk}(t), i_{Lk}(t), i_c(\cdot,t), v_c(\cdot,t)) \le \mathscr{E}(v_{Dk}(0), i_{Lk}(0), i_0(\cdot), v_0(\cdot)) \tag{8.74}$$

in this way we obtained Lyapunov stability *in the sense of the norm induced by the Lyapunov functional* (8.72) *itself*. Worth mentioning that the aforementioned Lyapunov stability holds provided the sector condition $\tilde{\varphi}_k(\sigma)\sigma > 0, k = 1, 2$, holds. As already mentioned, $f_k(\sigma + \bar{V}_{Dk}) - f_k(\bar{V}_{Dk})$ do not satisfy a sector condition since they are subject to the conditions ensuring existence and orbital stability of a limit cycle. The inequality $\tilde{\varphi}_k(\sigma)\sigma > 0$ displays the so called *sector rotation*—the mathematical expression of the *radiation dissipation* (a notion arising from Physics). It is in fact an additional local damping arising from the coupling of the oscillators to the LC line.

Now, since $\mathscr{W}$ is only non-positive definite, asymptotic stability can be obtained from the invariance principle Barbashin–Krasovskii–LaSalle. Its validity for (8.48) is not proven (up to our knowledge) but it is proven for the associated system of functional differential equations (8.60) with $\varphi_k(\cdot)$ defined by (8.61).

Making use of the representation formulae (8.54), (8.56) and (8.57) as well as of the definition (8.50) of the Riemann invariants, we obtain after some simple manipulation

$$\int_{-l/2}^{l/2} (L i_c(x,t)^2 + C v_c(x,t)^2)\mathrm{d}x = \sqrt{\frac{C}{L}} \int_{-l\sqrt{LC}}^{0} (\eta_c^+(t+\theta)^2 + \eta_c^-(t+\theta)^2)\mathrm{d}\theta \tag{8.75}$$

We express then $\mathscr{E}$ in the language of the state variables of (8.60) using again formulae (8.54), (8.56) and (8.57) as well as the definition (8.50) of the Riemann invariants. The associated quadratic Lyapunov functional is thus obtained

$$\begin{aligned}\mathscr{V}(v_{Dk}, i_{Lk}, \eta_c^{\pm}(\cdot)) = \frac{1}{2}[C_1 v_{D1}^2 + C_2 v_{D2}^2 + L_1 i_{L1}^2 + L_2 i_{L2}^2]+ \\ + \int_{-l\sqrt{LC}}^{0} (\eta_c^+(\theta)^2 + \eta_c^-(\theta)^2)\mathrm{d}\theta\end{aligned} \tag{8.76}$$

($\mathscr{V}$ is written as a state functional. *not* along system's solutions). At its turn the derivative functional of (8.73) becomes

$$\begin{aligned}-\mathscr{W}(v_{Dk}, i_{Lk}, \eta_c^{\pm}(\cdot)) = R_1 i_{L1}^2 + R_2 i_{L2}^2 + \tilde{\varphi}(v_{D1})v_{D1} + \tilde{\varphi}(v_{D2})v_{D2}+ \\ +\frac{1+\rho_1'}{2} R_1'(\eta_c^+(0) - \eta_c^-(-l\sqrt{LC}))^2 + \frac{1+\rho_2'}{2} R_2'(\eta_c^-(0) - \eta_c^+(-l\sqrt{LC}))^2\end{aligned} \tag{8.77}$$

This function is also written as a state function. It vanishes on the set defined by

$$i_{L1} = i_{L2} = v_{D1} = v_{D2} = 0;\ \ \eta_c^+(0) = \eta_c^-(-l\sqrt{LC}),\ \ \eta_c^-(0) = \eta_c^+(-l\sqrt{LC}) \tag{8.78}$$

Considering (8.60), it is easily seen that the only invariant set of it, contained in the set defined by (8.78), is the singleton $\{0\}$. Application of the Barbashin–Krasovskii–LaSalle invariance principle—in our case Theorem 9.8.1 of [23], p. 293—gives global asymptotic stability of the zero solution of (8.60). Moreover this property holds for the zero solution of (8.48) —with $\psi_k(t) \equiv 0$, $k = 1, 2$—from the representation formulae (8.54), (8.56) and (8.57). The final result reads as follows

Theorem 8.5 *Consider system (8.60) and assume that the nonlinear functions $f_k(\sigma)$ are such that the transformed functions $\tilde{\varphi}(\sigma)$ defined by (8.49) are subject to the sector conditions $\tilde{\varphi}_k(\sigma)\sigma > 0$. Then the zero equilibrium of (8.60) and, therefore, of (8.48), is globally asymptotically stable.*

The physical/engineering significance of this result is a rather simple one: *"synchronization" by oscillation quenching*. More precisely, if each local oscillator—taken separately—displays a stable limit cycle whose period can be computed at least approximately—see e.g., [7, 36]—the coupling to the LC line will introduce an additional damping ensuring fulfilment of the aforementioned sector conditions and oscillation quenching.

An additional remark appears as necessary: Theorem 9.8.2 of [23] holds under the assumption that the difference operator of the neutral system is strongly stable. It has been shown from the beginning of this Section that system (8.60) – with $\psi_k(t) \equiv 0$, $k = 1, 2$—belongs to the class defined by (8.67). Its difference operator being defined by matrix A_3 must have its roots inside the unit disk $\mathbb{D}_1$. We showed that this is indeed the case since the eigenvalues of A_3 are $\pm(1/2)\sqrt{(1+\rho_1')(1+\rho_2')}$. The numbers ρ_k' are given by (8.52) and satisfy $0 < |\rho_k'| < 1$. In fact this important structural restriction is due to local dissipation induced by the resistors $R_k' > 0$.

The case of the oscillators coupled to a finite length LC line can be obtained from (8.22) by taking in the boundary conditions $I_l(-l/2,t) \equiv 0$, $I_r(l/2,t) \equiv 0$ and considering the central boundary value problem only. The approach is exactly the aforementioned one and only the autonomous systems leading to oscillation quenching are obtained—see [48].

B. We shall consider now the case of the forced oscillations imposed to the coupled oscillators: this means $\psi_k(t) \not\equiv 0$ in (8.48) and (8.60). The forcing terms are in fact defined by the voltage and current initial conditions on the lateral semi-infinite segments of the LC line i.e., $x < -l/2$, $x > l/2$. Obviously we can imagine them being periodic or almost periodic.

System (8.60) belongs to the class defined by

$$\begin{aligned} \dot{x} &= A_0 x(t) + A_1 y(t-\tau) - \sum_1^r b_k^0 \phi_k(\sigma_k(t)) + f(t) \\ y(t) &= A_2 x(t) + A_3 y(t-\tau) - \sum_1^r b_k^1 \phi_k(\sigma_k(t)) + g(t) \\ \sigma_k &= c_k^* x \ , \ k = 1, \dots, r \end{aligned} \tag{8.79}$$

with dim $x = n$, dim $y = m$, $|f(t)| + |g(t)| \le M$—possibly periodic or almost periodic. For this system two basic results are known, based either on a matrix frequency domain inequality [22] or on the properties of a suitably found Lyapunov functional [43]. Since we already discussed the drawbacks of the matrix frequency domain inequality face to the Lyapunov functional approach and, moreover, we have at our disposal the energy functional, we shall reproduce here the basic result of [43]

Theorem 8.6 ([43]) *Consider system (8.79) under the following assumptions: (i) there exist positive definite matrices P and R such that the following matrix inequalities hold*

$$\begin{gathered} \begin{pmatrix} A_0^*P + PA_0 + A_2^*RA_2 & PA_1 + A_2^*RA_3 & -(PB_0 + A_2^*RB_1) + \frac{1}{2}C \\ * & A_3^*RA_3 - R & -A_3^*RB_1 \\ * & * & B_1^*RB_1 - K^{-1} \end{pmatrix} \le 0 \\ \begin{pmatrix} A_0^*P + PA_0 + A_2^*RA_2 & PA_1 + A_2^*RA_3 \\ * & A_3^*RA_3 - R \end{pmatrix} < 0 \end{gathered} \tag{8.80}$$

where $C = (c_1 \ c_2 \dots c_r)$, $B_0 = (b_1^0 \dots b_r^0)$, $B_1 = (b_1^1 \dots b_r^1)$, $K^{-1} = diag\{k_1^{-1} \dots k_r^{-1}\}$; (ii) the nonlinear functions are continuous and such that

$$0 \le \frac{\phi_i(\sigma_1) - \phi_i(\sigma_2)}{\sigma_1 - \sigma_2} \le k_i \ , \ i = 1, \dots, r \ , \ \forall \sigma_1 \ne \sigma_2 \tag{8.81}$$

(iii) $|f(t)| + |g(t)| \le M$. Then there exists a bounded on the entire real axis solution of (8.79) which is exponentially stable. If $f(t)$, $g(t)$ are T-periodic, then this solution is also T-periodic and if they are almost periodic then this solution is also almost periodic.

As shown in [43], this theorem is obtained based on the Lyapunov–Krasovskii functional of the form

$$\mathscr{V}(x, y) = x^* P x + \int_{-\tau}^{0} y^*(\theta) R y(\theta) \mathrm{d}\theta \tag{8.82}$$

where $P > 0$ and $R > 0$ are those of (8.80). In our case the matrices P and R, as resulting from (8.76) are diagonal

$$P = \frac{1}{2} diag\{L_1, L_2, C_1, C_2\}, \ R = \sqrt{C/L} I_2 \tag{8.83}$$

with I_2 the 2×2 identity matrix.

From the application of the aforementioned general theorem of [43] there is obtained an existence theorem for a periodic or almost periodic limit regime of (8.60) hence of (8.48). This regime is such according to the properties of the forcing terms $\psi_i(t)$, $i = 1, 2$. At their turn these properties are imposed by the properties of the initial conditions (voltage and current) of the LC line—$V_0(x)$, $I_0(x)$, $-\infty < x < \infty$.

Two final remarks concern this case. The first one concerns the physical/ engineering significance of the result: it is *an obvious synchronization phenomenon* achieved by the coupling through the LC line: regardless the period of the individual (local) oscillators, the common oscillation period is "forced" by the forcing periodic signal provided it is such. If the forcing signal is almost periodic, the common oscillation is almost periodic; however we can state nothing about its frequency content.

The second remark concerns the sense of the periodicity or of the almost periodicity. Here we have to say something about obtaining the oscillation existence and the stability result. The properties of the Lyapunov–Krasovskii functional allow to obtain some estimates of system's solutions. At their turn these estimates imply fulfilment of the assumptions needed for the Kurzweil–Halanay theorem on invariant manifolds for flows on Banach spaces [21, 29]. According to this theorem, periodicity and almost periodicity must be viewed in the sense of the norm of the Banach space. In our case the Banach space is the Hilbert space $\mathbb{R}^4 \times \mathscr{L}^2(-l\sqrt{LC}, 0; \mathbb{R}^2)$. Therefore, if periodicity/almost periodicity of the local oscillators is the standard (real axis) one, the periodicity/almost periodicity of the state variables of the LC line must be viewed in a generalized sense e.g., in the sense of V. V. Stepanov—see [22]; additional details can be obtained from the classical monographs [2, 14, 15].

8.7 Two Mechanical Oscillators on the String

We shall refer here to the settings of Figs. 8.2b and 8.3 which describe the systems considered in [17, 31]. For the former system of two mechanical oscillators hanging on an infinite rope, the equations are as follows

$$
\begin{aligned}
& y_{tt} - c^2 y_{xx} = 0,\ -\infty < x < \infty;\ c^2 = T/\rho \\
& m_1 \ddot{z}_1 + V_1(z_1) = T(y_x(-l/2+0,t) - y_x(-l/2-0,t)) \\
& m_2 \ddot{z}_2 + V_2(z_2) = T(y_x(l/2+0,t) - y_x(l/2-0,t))
\end{aligned}
\tag{8.84}
$$

For the latter system of two mechanical oscillators on a finite length rod the equations are as follows

$$
\begin{aligned}
& y_{tt} - c^2 y_{xx} = 0,\ -\infty < x < \infty;\ c^2 = E/\rho \\
& m_1 \ddot{z}_1 + f_1(z_1, \dot{z}_1) = ESy_x(0,t) \\
& m_2 \ddot{z}_2 + f_2(z_2, \dot{z}_2) = ESy_x(l,t)
\end{aligned}
\tag{8.85}
$$

with the following significance of the physical parameters: T—the string tension; ρ—linear string density; $c = \sqrt{T/\rho}$—sound velocity through the string; S—rod cross section area; E—the Young modulus of the rod; ρ—rod density (volume rated); $c = \sqrt{E/\rho}$ sound velocity through the rod.

Concerning the description of the two local oscillators, the following can be added. In (8.84) the two oscillators are in fact conservative systems displaying cycles but not a single stable limit cycle—see also the "toy" application of Sect. 8.2. This type of oscillator was chosen in [31, 38] in order to illustrate the occurrence of the additional damping introduced by the coupling to the string (rope/rod)—the aforementioned *radiation dissipation*. Unlikely, the oscillators of (8.85) are viewed as generalizing the van der Pol oscillators which display a unique orbitally stable limit cycle; in fact this description generalizes also the Liénard equations—also displaying a unique orbitally stable limit cycle (under definite assumptions).

In the following we shall consider system (8.84) for a more detailed analysis, but we shall nevertheless point out the results that can migrate to (8.85). Examination of both Fig. 8.2b and Eq. (8.84) shows that, as in the case of the electronic oscillators, the boundary value problem (8.84) can be split up in three sub-problems: left hand side for $x < -l/2$, right hand side for $x > l/2$ and central problem, for $x \in (-l/2, l/2)$. Analogous problems on semi-infinite intervals [56] and finite interval have already been considered in Sects. 8.2 and 8.3; therefore we shall only sketch the approach for (8.84).

We write down (8.84) in the Friedrichs form by introducing the new variables

$$
v(x,t) := y_t(x,t),\ w(x,t) := y_x(x,t)
\tag{8.86}
$$

thus obtaining

$$
\begin{aligned}
& v_t - c^2 w_x = 0,\ w_t = v_x \\
& m_1 \ddot{z}_1 + V_1(z_1) = T(w(-l/2+0,t) - w(-l/2-0,t)) \\
& m_2 \ddot{z}_2 + V_2(z_2) = T(w(l/2+0,t) - w(l/2-0,t))
\end{aligned}
\tag{8.87}
$$

If the initial conditions for (8.84) had been $y(x,0) = y_0(x)$, $y_t(x,0) = y_1(x)$ now we shall have

$$
v(x,0) = y_1(x),\ w(x,0) = y_0'(x) := \frac{\mathrm{d}}{\mathrm{d}x} y_0(x)
\tag{8.88}
$$

The aforementioned considerations show that we considered only classical solutions thus having sufficient smoothness. Moreover we write down some boundary conditions accounting for the continuity of the string

$$z_1(t) = y(-l/2, t)\ ,\ z_2(t) = y(l/2, t) \tag{8.89}$$

These conditions are explicitly mentioned in [17] and assumed implicitly in [31].

The equations of the three boundary value problems associated to (8.87) are as follows

$$\begin{array}{l} \partial_t v_r - c^2 \partial_x w_r = 0,\ \partial_t w_r - \partial_x v_r = 0;\ x > l/2 \\ v_r(l/2, t) = \dot{z}_2(t);\ v_r(x, 0) = y_1(x),\ w_r(x, 0) = y_0'(x),\ x > l/2 \end{array} \tag{8.90}$$

next

$$\begin{array}{l} \partial_t v_l - c^2 \partial_x w_l = 0,\ \partial_t w_l - \partial_x v_l = 0;\ x < -l/2 \\ v_l(-l/2, t) = \dot{z}_1(t);\ v_l(x, 0) = y_1(x),\ w_l(x, 0) = y_0'(x),\ x < -l/2 \end{array} \tag{8.91}$$

and, finally

$$\begin{array}{l} \partial_t v_c - c^2 \partial_x w_c = 0,\ \partial_t w_c - \partial_x v_c = 0;\ -l/2 \le x \le l/2 \\ v_c(-l/2, t) = \dot{z}_1(t),\ v_c(l/2, t) = \dot{z}_2(t) \\ v_c(x, 0) = y_1(x),\ w_c(x, 0) = y_0'(x),\ -l/2 \le x \le l/2 \end{array} \tag{8.92}$$

Here, as in the previous sections, the subscripts r, l, c account for the right hand side, left hand side and central boundary value problem respectively. Observe that all these boundary value problems have standard boundary conditions with forcing terms supplied by the local oscillators. Since the local oscillators are also supplied by output signals from the boundary value problems dynamics, some internal feedback is to be pointed out. However, consideration of the "independent" boundary value problems described by (8.90)–(8.92) displays "feedback breaking" (for a while).

In dealing with (8.90) and (8.91) we can take the same approach as in Sect. 8.3 to obtain the following representation formulae

$$\begin{array}{l} v_r(x, t) = \begin{cases} \frac{1}{2}[y_1(x + ct) + c y_0'(x + ct) + y_1(x - ct) - c y_0'(x - ct)], \\ 0 < t < (x - l/2)/c,\ x > l/2 \\ \frac{1}{2}[y_1(x + ct) + c y_0'(x + ct) - y_1(l + ct - x) - c y_0'(l + ct - x) + \\ 2\dot{z}_2(t + (l/2 - x)/c)],\ t > (x - l/2)/c,\ x > l/2 \end{cases} \\ w_r(x, t) = \begin{cases} \frac{1}{2c}[y_1(x + ct) + c y_0'(x + ct) - y_1(x - ct) + c y_0'(x - ct)], \\ 0 < t < (x - l/2)/c,\ x > l/2 \\ \frac{1}{2c}[y_1(x + ct) + c y_0'(x + ct) + y_1(l + ct - x) - c y_0'(l + ct - x) - \\ 2\dot{z}_2(t + (l/2 - x)/c)],\ t > (x - l/2)/c,\ x > l/2 \end{cases} \end{array} \tag{8.93}$$

for the right hand side boundary value problem and

$$
\begin{aligned}
v_l(x,t) &= \begin{cases} \frac{1}{2}[y_1(x+ct)+cy_0'(x+ct)+y_1(x-ct)-cy_0'(x-ct)], \\ 0<t<-(x+l/2)/c,\ x<-l/2 \\ \frac{1}{2}[-y_1(-l-x-ct)+cy_0'(-l-x-ct)+y_1(x-ct)-cy_0'(x-ct)+ \\ 2\dot{z}_1(t+(l/2+x)/c)],\ t>-(x+l/2)/c,\ x<-l/2 \end{cases} \\
w_l(x,t) &= \begin{cases} \frac{1}{2c}[y_1(x+ct)+cy_0'(x+ct)-y_1(x-ct)+cy_0'(x-ct)], \\ 0<t<-(x+l/2)/c,\ x<-l/2 \\ \frac{1}{2c}[-y_1(-l-x-ct)+cy_0'(-l-x-ct)-y_1(x-ct)+cy_0'(x-ct)+ \\ 2\dot{z}_2(t+(l/2+x)/c)],\ t>(x+l/2)/c,\ x<-l/2 \end{cases}
\end{aligned}
\tag{8.94}
$$

for the left hand side boundary value problem.

Observe that the terms $\dot{z}_i(\cdot), i=1,2$ in (8.93) and (8.94) already suggest an additional damping in the local oscillators—the "radiation dissipation". Equation (8.92) together with the ordinary differential equations of (8.87) will give the following equations of the central boundary value problem

$$
\begin{aligned}
&\partial_t v_c - c^2\partial_x w_c = 0,\ \partial_t w_c = \partial_x v_c \\
&v_c(-l/2,t) = \dot{z}_1(t),\ v_c(l/2,t) = \dot{z}_2(t) \\
&m_1\ddot{z}_1 + V_1(z_1) = T(w_c(-l/2,t) - w_l(-l/2,t)) \\
&m_2\ddot{z}_2 + V_2(z_2) = T(w_r(l/2,t) - w_c(l/2,t))
\end{aligned}
\tag{8.95}
$$

For this problem we observe the forcing terms which are exactly the boundary values of the side boundary value problems. Using (8.93) and (8.94) for $t>0$ we obtain

$$
\begin{aligned}
&\partial_t v_c - c^2\partial_x w_c = 0,\ \partial_t w_c = \partial_x v_c \\
&v_c(-l/2,t) = \dot{z}_1(t),\ v_c(l/2,t) = \dot{z}_2(t) \\
&m_1\ddot{z}_1 + (T/c)\dot{z}_1 + V_1(z_1) = Tw_c(-l/2,t)+ \\
&+(T/c)(y_1(-l/2-ct) - cy_0'(-l/2-ct)) \\
&m_2\ddot{z}_2 + (T/c)\dot{z}_2 + V_2(z_2) = -Tw_c(l/2,t)+ \\
&+(T/c)(y_1(l/2+ct) + cy_0'(l+ct))
\end{aligned}
\tag{8.96}
$$

For (8.96) we apply the same approach as in the case of (8.48). Introduce first the Riemann invariants by

$$
u_c^{\pm} = v_c \mp cw_c;\ v_c = \frac{1}{2}(u_c^- + u_c^+),\ w_c = \frac{1}{2c}(u_c^- - u_c^+) \tag{8.97}
$$

and let (x,t) some point within the strip $\{(x,t) \mid -l/2 < x < l/2, t>0\}$ crossed by the characteristics $t^{\pm}(\zeta;x,t) = t \pm (\zeta - x)/c$. The forward wave $u_c^+(x,t)$ is constant along the increasing characteristic $t^+(\cdot;x,t)$ while the backward wave $u_c^-(x,t)$ is constant along the decreasing characteristic $t^-(\cdot;x,t)$.

Denoting $\xi_c^{\pm}(t) := u_c^{\pm}(\pm l/2,t)$ we obtain $u_c^{\pm}(\mp l/2,t) = \xi_c^{\pm}(t+l/c)$. With the final notation $\eta_c^{\pm}(t) := \xi_c^{\pm}(t+l/c)$, the following system of forced coupled delay differential and difference equations is obtained

$$\begin{aligned}
&m_1\ddot{z}_1 + (2T/c)\dot{z}_1 + V_1(z_1) = (T/c)\xi_c^-(t - l/c) + \\
&+ (T/c)(y_1(-l/2 - ct) - cy_0'(-l/2 - ct)) \\
&m_2\ddot{z}_2 + (2T/c)\dot{z}_2 + V_2(z_2) = (T/c)\xi_c^+(t - l/c) + \\
&+ (T/c)(y_1(l/2 + ct) + cy_0'(l + ct)) \\
&\xi_c^+(t) = -\xi_c^-(t - l/c) + 2\dot{z}_1(t);\ \xi_c^-(t) = -\xi_c^+(t - l/c) + 2\dot{z}_2(t)
\end{aligned} \tag{8.98}$$

System (8.98) belongs to the class defined by (8.79). If the initial conditions of the string on $(-\infty, -l/2) \cup (l/2, \infty)$ are identically zero, system (8.98) has no forcing term i.e., it has the form (8.79) with $f(t) \equiv 0$, $g(t) \equiv 0$. The same type of system could have been obtained provided we started from the models of the oscillators coupled through a finite length rod [17]—see also our Refs. [41, 47]. The specific feature of this model is that the matrix A_3 (if we refer to (8.79) as general representation) has the form

$$A_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{8.99}$$

and has its eigenvalues ± 1 thus being in a critical case of the difference operator stability. This critical case is not considered within the (largely popular) line open by the assumption of J. K. Hale on the strong stability of the difference operator. Therefore no recent reference on the subject exists. However the aforementioned models occurring in synchronization and arising from Mechanics [17, 31] as well as other ones from Mechanical Engineering and Mechanical vibration control and quenching (including oil drillstring vibrations) [42, 44] belong to this class. Considering this critical case appears as an urgent task.

8.8 Challenges and Existing Results

Consider (8.98) with zero forcing term: $y_1(x) \equiv y_0'(x) \equiv 0$ on $(-\infty, -l/2) \cup (l/2, \infty)$ and with linear elastic springs i.e., $V_i(z_i) = \mu_i z_i$. The linear autonomous system thus obtained has the characteristic equation

$$\begin{aligned}
\Delta(\lambda) = &(m_1\lambda^2 + (2T/c)\lambda + \mu_1)(m_2\lambda^2 + (2T/c)\lambda + \mu_2) \\
&-(m_1\lambda^2 + \mu_1)(m_2\lambda^2 + \mu_2)\mathrm{e}^{-2(l/c)\lambda}
\end{aligned} \tag{8.100}$$

Its zeros coincide with the zeros of

$$\Phi(\lambda) = 1 - \frac{m_1\lambda^2 + \mu_1}{m_1\lambda^2 + (2T/c)\lambda + \mu_1} \cdot \frac{m_2\lambda^2 + \mu_2}{m_2\lambda^2 + (2T/c)\lambda + \mu_2}\mathrm{e}^{-2(l/c)\lambda} \tag{8.101}$$

Taking $\lambda = \sigma + \imath\mu$, $\sigma > 0$, it is easily shown that (8.101) has no zeros for $\sigma \geq 0$ i.e., in $\mathbb{C}^+ \cup \imath\mathbb{R}$ but it is impossible to find some $\sigma_0 < 0$ such that the roots of (8.101) i.e., of (8.100) satisfy $\Re e(\lambda) \leq \sigma_0 < 0$. This condition is required for linear neutral functional differential equations in order to be exponentially stable.

Since the difference operator is in critical case, it is possible for some chains of roots of (8.100) to accumulate towards the imaginary axis $\imath\mathbb{R}$ while "arising" from the left half plane $\mathbb{C}^-$. Results concerning the asymptotics of the roots of the quasi-polynomials may be found in the (by now) classical monographs [4, 6, 32] and also in such memoirs as [16, 30, 59, 60]. Two papers of specific interest in this problem might be [18, 19] where solutions of neutral functional differential equations in critical cases are described.

8.9 Conclusions and Perspective

We discussed throughout this paper some models of Huygens synchronization in Mechanics and Electronics. Consideration of the connecting media with distributed parameters introduced what one might define as "synchronization by propagation". Besides the aforementioned motivating applications, it is useful to "know better the history" by mentioning here such almost forgotten pioneering papers as [35, 54].

Among the several existing concepts of synchronization, we discussed here Huygens synchronization viewed as a problem of stability ("synchronization to 0" i.e., oscillation quenching and synchronization *via* an external signal viewed as a problem of forced oscillations). As follows from the presented results, there exist two basic cases which in fact are *structurally different*: the strongly stable case—of the strongly stable difference operator in the associated system of neutral functional differential equations—and the critically stable case—of the critically stable difference operator. The strongly stable case can be considered, as in the general theory of stability, the fundamental (basic) case. For such systems the stability problems, including those connected to Huygens synchronization, can be solved using either the Lyapunov method or the methods of the complex domain (Laplace transform, root distribution of the characteristic equations, Popov-like frequency domain inequalities); the contents of Sects. 8.3–8.6 illustrate this assertion.

In the critical case however the aforementioned methods fail. One cannot rely on the Barbashin–Krasovskii–LaSalle invariance principle since it is not proven in this case. The frequency domain methods also require a strongly stable difference operator [22, 50, 51]. On the other hand, the systems displaying a structure with critically stable difference operator e.g., those in [17, 31] are supposed to illustrate the so called "complex" (if not even *chaotic*) behavior. In [37] two basic dichotomies are considered in dynamics: stability/instability and synchronization/chaotic behavior. We send also to the comprehensive survey [28]. Our *conjecture* is however that the aforementioned complex behavior relies on *almost periodic processes*. The subsequent Physics/Engineering problem is and will be a most important one supplying a rather urgent task: *how to assess in practice almost periodicity*?

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