香港中文大學（深圳）
The Chinese University of Hong Kong

CSC3100 Data Structures
Lecture 12: Sorting algorithms

Yixiang Fang
School of Data Science (SDS)
The Chinese University of Hong Kong, Shenzhen

Outline

- Comparison-based sorting algorithms
  - ShellSort

- Non-comparison-based sorting algorithms
  - CountingSort
  - BucketSort
  - RadixSort

- A summary of 10 classic sorting algorithms

ShellSort

- Break the quadratic time barrier by comparing elements that are distant
- The distance between comparisons decreases as the algorithm runs until the last phase, in which adjacent elements are compared (*diminishing increment sort*)
- An increment sequence $h_1, h_2, h_3, \ldots, h_t$, used in reverse order with $h_1=1$

ShellSort

- *After a phase, with an increment h_k, A [i] <= A [i + h_k]*
- All elements spaced h_k apart are sorted (insertion sort)

- Example (1,3,5)

Original	**81**	**94**	**11**	**96**	**12**	**35**	**17**	**95**	**28**	**58**	**41**	**75**	**15**
5-sorted	**35**	**17**	**11**	**28**	**12**	**41**	**75**	**15**	**96**	**58**	**81**	**94**	**95**
3-sorted	**28**	**12**	**11**	**35**	**15**	**41**	**58**	**17**	**94**	**75**	**81**	**96**	**95**
1-sorted	**11**	**12**	**15**	**17**	**28**	**35**	**41**	**58**	**75**	**81**	**94**	**95**	**96**

Standard insertion sort

ShellSort with {1,2,4,8,…,n/2}

```
public static void shellSort(int[ ] a) {
    int j;
    for (int gap = a.length/2; gap > 0; gap /=2)
        for (int i = gap; i < a.length; i++) {
            int tmp = a[i];
            for (j = i; j >= gap && tmp < a[j-gap]; j-= gap)
                a[j] = a[j-gap];
            a[j] = tmp;
        }
}
```

ShellSort

- Analysis of Shellsort
  - Very hard (average-case is a long-standing open problem)

  - Depend on the selection of an increment sequence

  - Theorem: the worst-case running time of Shellsort, using some increment, is $\Theta(N^2)$
    - Put the largest N/2 numbers in the even positions

      e.g., 4,12,1,10,3,11,2,9
    - Use the increments {…, 8,4,2,1}
    - Before the last sort, the N/2 largest numbers are still in the even positions, e.g., 1,9,2,10,3,11,4,12
    - The numbers of inversions is 1+2+…+(N-1)/2 = $\Theta(N^2)$

Non-comparison-based sorting algorithms

CountingSort

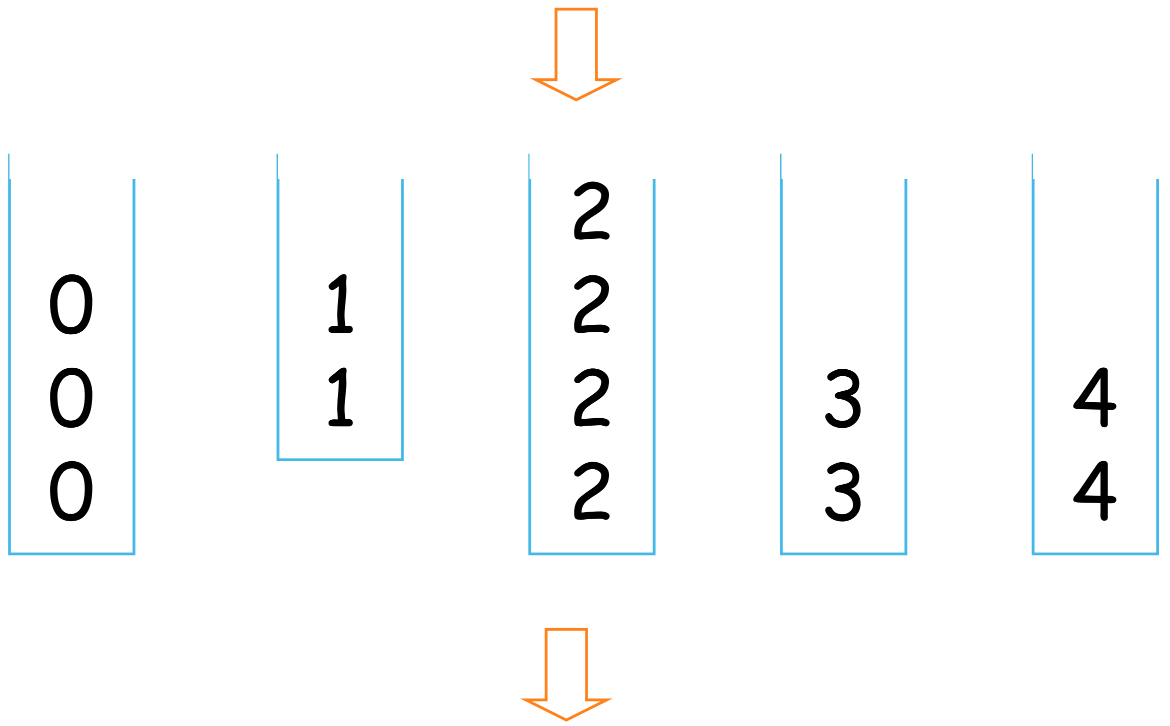
- Idea: suppose the values are integers in [0, m-1]

- Steps
  - Start with m empty *buckets* numbered 0 to m-1
  - Scan the list and place element s[i] in bucket s[i]
  - Output the buckets in order

- Will need an array of buckets, and the values to be sorted will be the indexes to the buckets
  - No comparisons will be necessary

CountingSort

4	2	1	2	0	3	2	1	4	0	2	3	0



0	0	0	1	1	2	2	2	2	3	3	4	4

CountingSort

Algorithm CountingSort(S)
(values in S are between 0 and m-1)

```
for j ← 0 to m-1 do     // initialize m buckets
   b[j] ← 0
for i ← 0 to n-1 do     // place elements in their appropriate buckets
   b[S[i]] ← b[S[i]]  + 1
i ← 0
for j ← 0 to m-1 do     // place elements in buckets
   for r ← 1 to b[j] do // back in S
         S[i] ← j
         i ← i + 1
```

Exercise

- Use CountingSort to sort the following sequence of integer values

  4,3,2,1,5,2,4,1,2,0,4,2,0

- How to process the case that the minimum value in the input sequence of integers is very large?

- How to process the case that the values in the sequence vary greatly (i.e., 1, 10, 101, 1000, 100001) ?

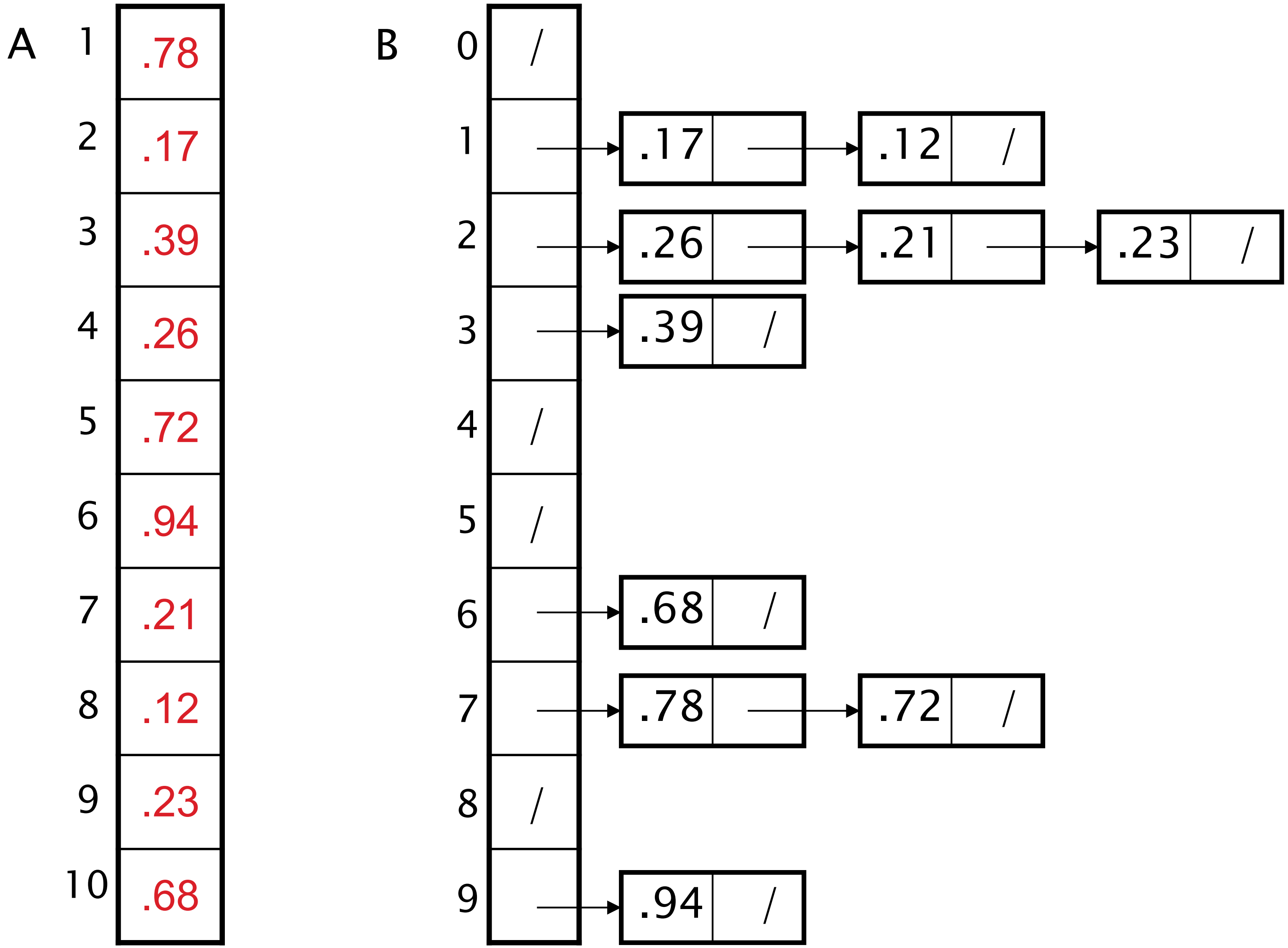
BucketSort

- Assumption:
  - The input is generated by a random process that distributes elements uniformly over [0, 1)

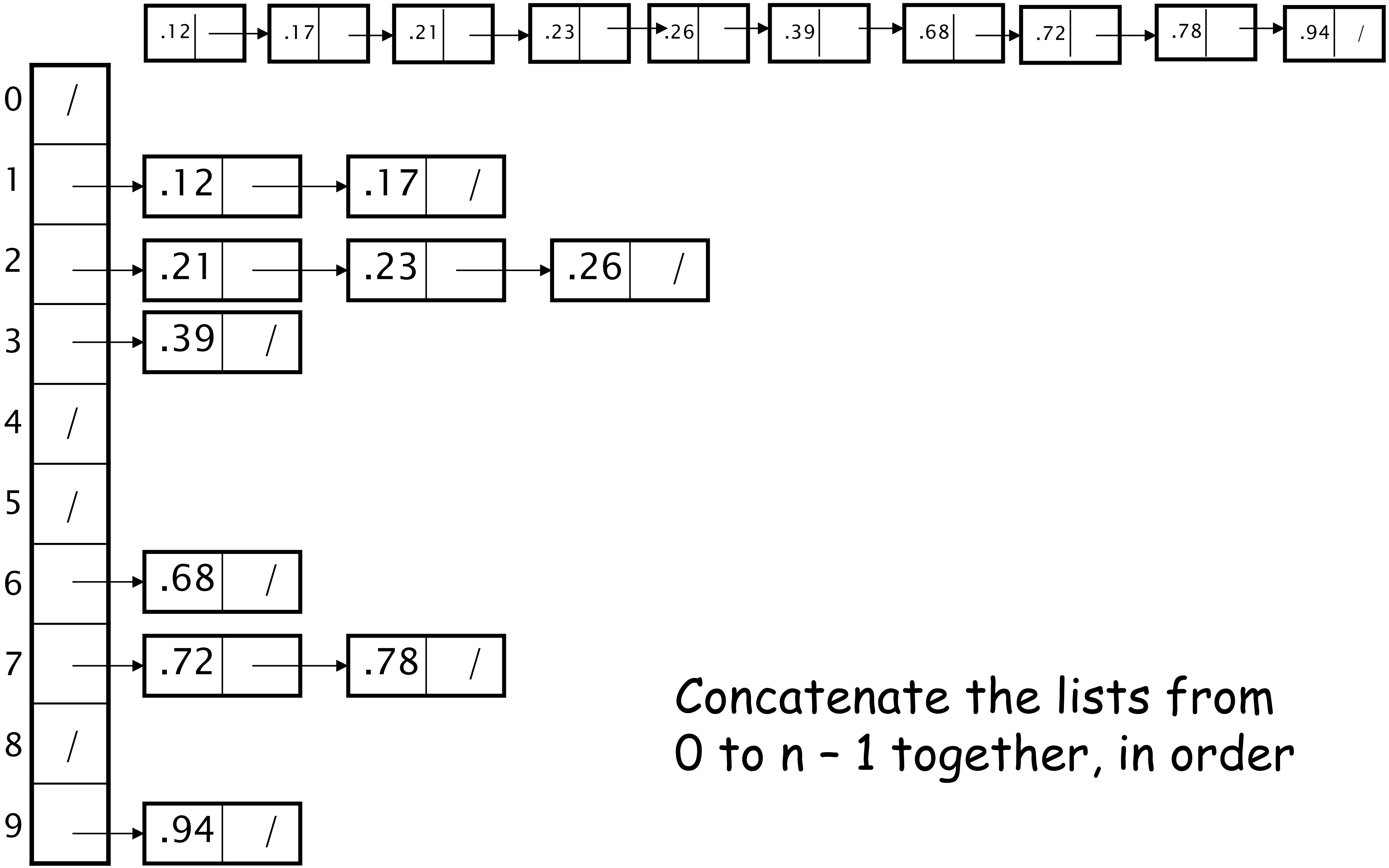
- Idea:
  - Divide [0, 1) into n equal-sized buckets
  - Distribute the n input values into the buckets
  - Sort each bucket (e.g., using QuickSort)
  - Go through the buckets in order, listing elements in each one

- **Input and output**
  - Input: $A[1 \,..\, n]$, where $0 \le A[i] < 1$ for all i
  - Output: elements $A[i]$ sorted
  - Extra array: $B[0 \,..\, n - 1]$ of linked lists, each list initially empty

Bucket Sort


A
1 .78
2 .17
3 .39
4 .26
5 .72
6 .94
7 .21
8 .12
9 .23
10 .68
B
0 /
1 .17 .12 /
2 .26 .21 .23 /
3 .39 /
4 /
5 /
6 .68 /
7 .78 .72 /
8 /
9 .94 /

BucketSort



Concatenate the lists from 0 to n – 1 together, in order

BucketSort

BUCKET-SORT(A, n)

for i ← 1 **to** n

  do insert A[i] into list B[$\lfloor$nA[i]$\rfloor$]

for i ← 0 **to** n - 1

  do sort list B[i] with QuickSort

concatenate lists B[0], B[1], . . . , B[n -1]

together in order

return the concatenated lists

$$\left.\begin{array}{l}\text{for } i \leftarrow 1 \text{ to } n \\ \quad \text{do insert } A[i] \text{ into list } B[\lfloor nA[i]\rfloor]\end{array}\right\} O(n)$$

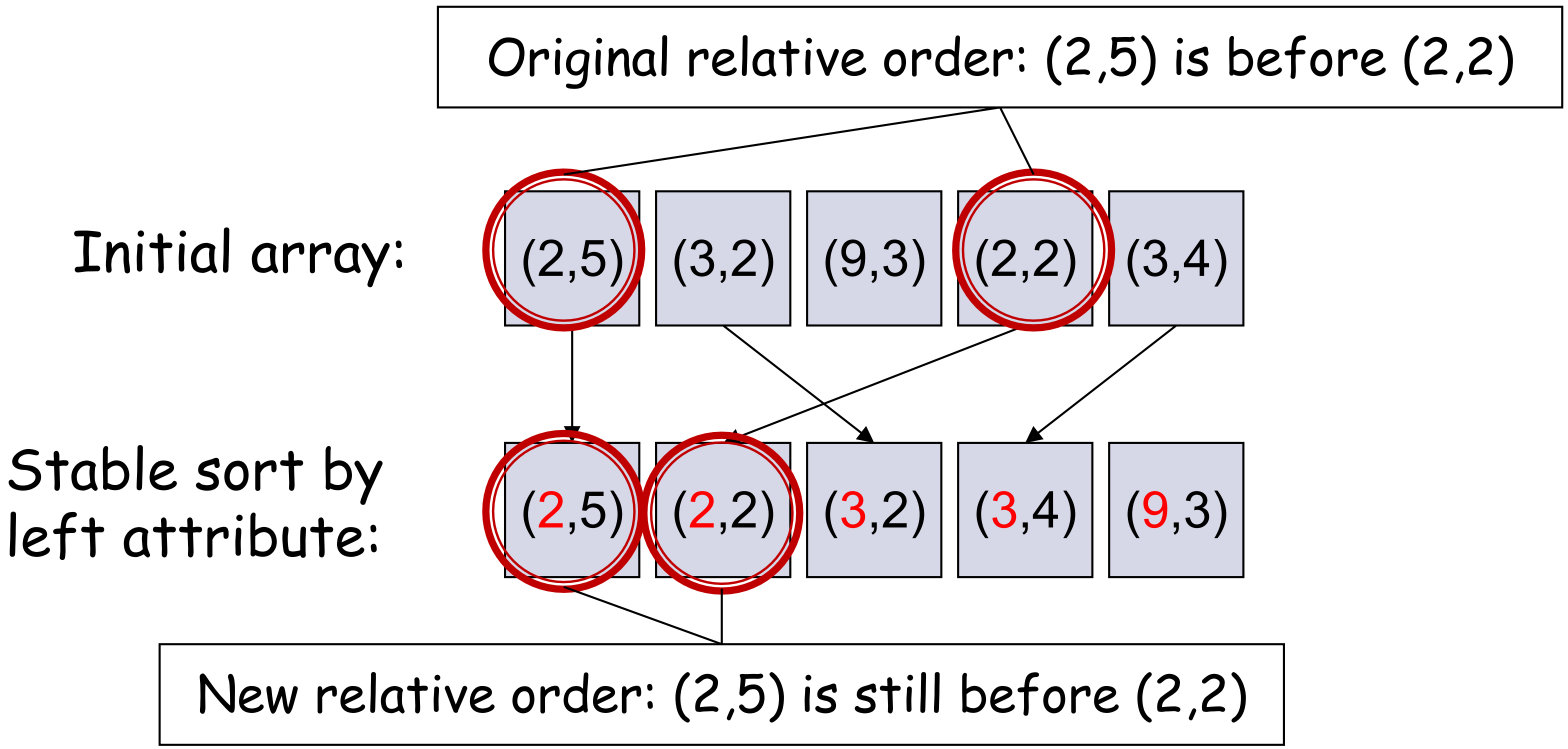
$$\left.\begin{array}{l}\text{for } i \leftarrow 0 \text{ to } n-1 \\ \quad \text{do sort list } B[i] \text{ with QuickSort}\end{array}\right\} \Theta(n)$$

$$\left.\begin{array}{l}\text{concatenate } \ldots \\ \text{return the concatenated lists}\end{array}\right\} O(n)$$

$\Theta(n)$

Concept: stable sorting algorithm

- Definition: A *stable* sorting algorithm is one that preserves the original relative order of elements with equal key
  - E.g., suppose the left attribute is the key attribute



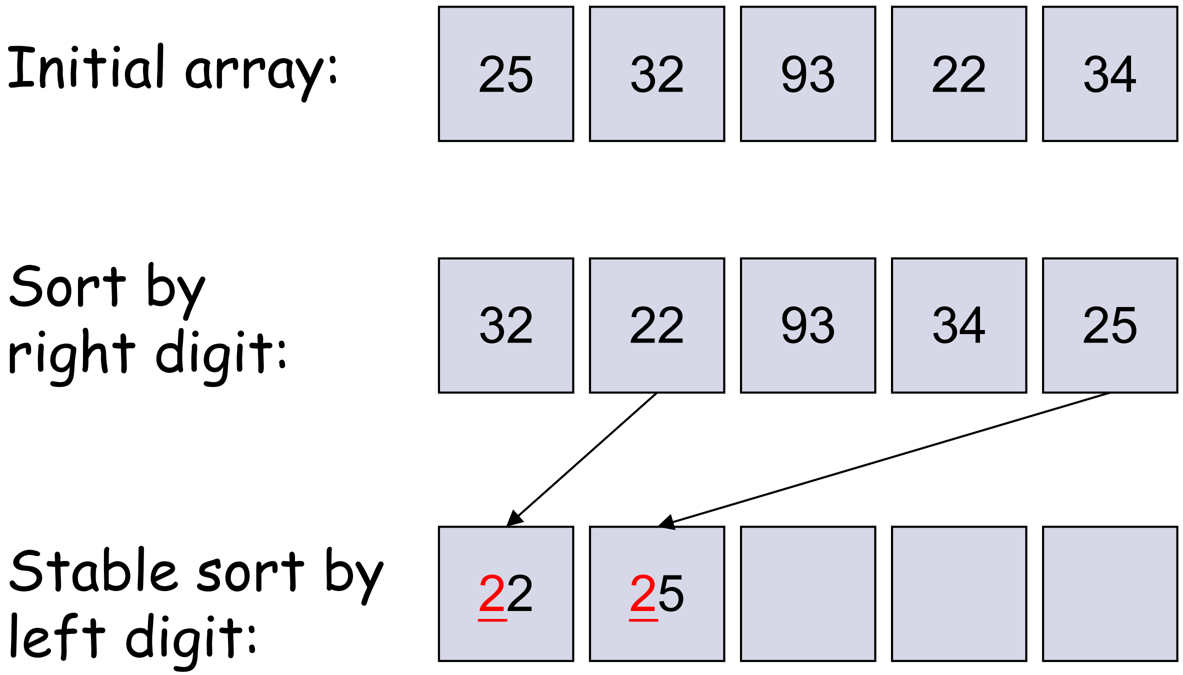
Using stable sort (1/4)

- Suppose we sort some 2-digit integers
- **Phase 1: Stable sort by the right digit (the least significant digit)**

Initial array:	25	32	93	22	34
Sort by right digit:	32	22	93	34	25

Using stable sort (2/4)

- Suppose we sort some 2-digit integers
- **Phase 2: Stable sort by the left digit (the second least significant digit)**



Using stable sort (3/4)

- Suppose we sort some 2-digit integers
- **Phase 2: Stable sort by the left digit (the second least significant digit)**

Initial array:	25	32	93	22	34
Sort by right digit:	32	22	93	34	25
Stable sort by left digit:	22	25	32	34	

Using stable sort (4/4)

- Suppose we sort some 2-digit integers
- **Phase 2: Stable sort by the left digit (the second least significant digit)**

Initial array:	25	32	93	22	34
Sort by right digit:	32	22	93	34	25
Stable sort by left digit:	<u>2</u>2	<u>2</u>5	<u>3</u>2	<u>3</u>4	<u>9</u>3

RadixSort

- CountingSort is not efficient if m is large
- The idea of RadixSort:
  - Apply stable bucket sort on each digit (from Least Significant Digit to Most Significant Digit)
- A complication:
  - Just keeping the count is not enough
  - Need to keep the actual elements
  - Use a queue for each digit

RadixSort: example (1/3)

- Input: 170, 045, 075, 090, 002, 024, 802, 066
- The first pass
  - Consider **the least significant digits** as keys and move the keys into their buckets

0	170, 090
1	
2	002, 802
3	
4	024
5	045, 075
6	066
7	
8	
9	

  - Output: 170, 090, 002, 802, 024, 045, 075, 066

RadixSort: example (2/3)

- The second pass
  - Input: 170, 090, 002, 802, 024, 045, 075, 066
  - Consider **the second least significant digits** as keys and move the keys into their buckets

0	002, 802
1	
2	024
3	
4	045
5	
6	066
7	170, 075
8	
9	090

  - Output: 002, 802, 024, 045, 066, 170, 075, 090

RadixSort: example (3/3)

- The **third** pass
  - Input: 002, 802, 024, 045, 066, 170, 075, 090
  - Consider **the third least significant digits** as keys and move the keys into their buckets

0	002, 024, 045, 066, 075, 090
1	170
2	
3	
4	
5	
6	
7	
8	802
9	

  - Output: 002, 024, 045, 066, 075, 090, 170, 802 (Sorted)

Codes (1/2)

```
// items to be sorted are in {0,…,10^d-1},
// i.e., the type of d-digit integers
void radixsort(int A[], int n, int d)
{
   int i;
   for (i=0; i<d; i++)
      bucketsort(A, n, i);
}

// To extract d-th digit of x
int digit(int x, int d)
{
   int i;
   for (i=0; i<d; i++)
      x /= 10; // integer division
   return x%10;
}
```

Codes (2/2)

```
void bucketsort(int A[], int n, int d)
// stable-sort according to d-th digit
{
   int i, j;
   Queue *C = new Queue[10];
   for (i=0; i<10; i++) C[i].makeEmpty();
   for (i=0; i<n; i++)
      C[digit(A[i],d)].EnQueue(A[i]);
   for (i=0, j=0; i<10; i++)
      while (!C[i].empty())
      { // copy values from queues to A[]
         C[i].DeQueue(A[j]);
         j++;
      }
}
```

Inductive proof that RadixSort works

- Keys: k-digit numbers, base B
  - (that wasn't hard!)

- Claim: after i^{th} BucketSort, the least significant i digits are sorted
  - Base case: $i=0$. 0 digits are sorted.
  - Inductive step: Assume for i, prove for $i+1$

    Consider two numbers: X, Y. Say X_i is i^{th} digit of X:
    - $X_{i+1} < Y_{i+1}$ then $i+1^{th}$ BucketSort will put them in order
    - $X_{i+1} > Y_{i+1}$, same thing
    - $X_{i+1} = Y_{i+1}$, order depends on last i digits. Induction hypothesis says already sorted for these digits because BucketSort is **stable**

Worst-case time complexity

- Assume k digits, each digit comes from {0,…,M-1}

- For each digit,
  - O(M) time to initialize M queues
  - O(n) time to distribute n numbers into M queues

- Total time = O(k(M+n))

- When k is constant and M = O(n), we can make RadixSort run in linear time, i.e., O(n)

Questions

- Can we start from the most significant digit?

Now let sort three 3-digit numbers?
478, 430, 356

1st digit:
4, 4, 3 => 3, 4, 4 => 356, 478, 430
2nd digit:
5, 7, 3 => 3, 5, 7 => 430, 356, 478
3rd digit:
0, 6, 8 => 0, 6, 8 => 430, 356, 478

Questions

- Since RadixSort is faster than QuickSort, why is QuickSort still preferable in many cases?
  - Although RadixSort runs in $\Theta(n)$ while QuickSort $\Theta(n \lg n)$, QuickSort has much smaller constant factor c
  - RadixSort requires extra memory, whereas QuickSort works in place

10 classic sorting algorithms

Sorting algorithm	Stability	Time cost			Extra space cost
		Best	Average	Worst	
Bubble sort	√	O(n)	$O(n^2)$	$O(n^2)$	O(1)
Insertion sort	√	O(n)	$O(n^2)$	$O(n^2)$	O(1)
Selection sort	×	O(n)	$O(n^2)$	$O(n^2)$	O(1)
MergeSort	√	O(nlogn)	O(nlogn)	O(nlogn)	O(n)
HeapSort	×	O(nlogn)	O(nlogn)	O(nlogn)	O(1)
QuickSort	×	O(nlogn)	O(nlogn)	$O(n^2)$	O(1)
ShellSort	×	O(n)	$O(n^{1.3})$	$O(n^2)$	O(1)
CountingSort	√	O(n+k)	O(n+k)	O(n+k)	O(k)
BucketSort	√	O(n)	O(n+k)	$O(n^2)$	O(k)
RadixSort	√	O(nk)	O(nk)	O(nk)	O(n)

Recommended reading

- Reading this week
  - Chapter 8, textbook
- Next lecture
  - Tree data structure: chapter 12