香港中文大學(深圳)
The Chinese University of Hong Kong

(Materials of this lecture are NOT included in the midterm and final exams)

CSC3100 Data Structures
Lecture 16: red-black tree

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Red-black tree

- A "balanced" binary search tree
  - It guarantees an O(logn) running time for many operations, such as search, insertion, and deletion

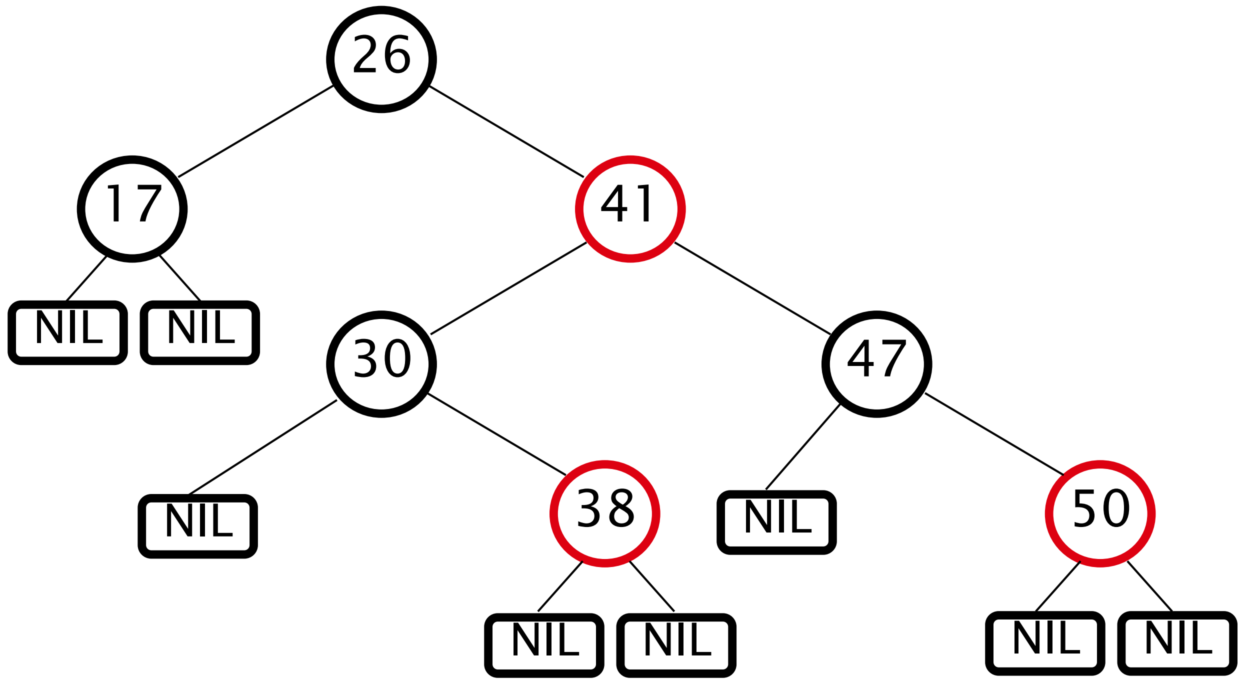
- Red-black tree
  - A binary search tree has an additional attribute for its nodes: color which can be **red** or **black**
  - Restrict the way that nodes can be colored on any path from the root to a leaf
  - Ensures that no path is more than twice as long as any other path

Red-black tree properties

1. Every node is either <u>red</u> or <u>**black**</u>
2. The root is <u>**black**</u>
3. Every leaf (NIL) is <u>**black**</u>
4. If a node is <u>**red**</u>, then both its children are <u>**black**</u>

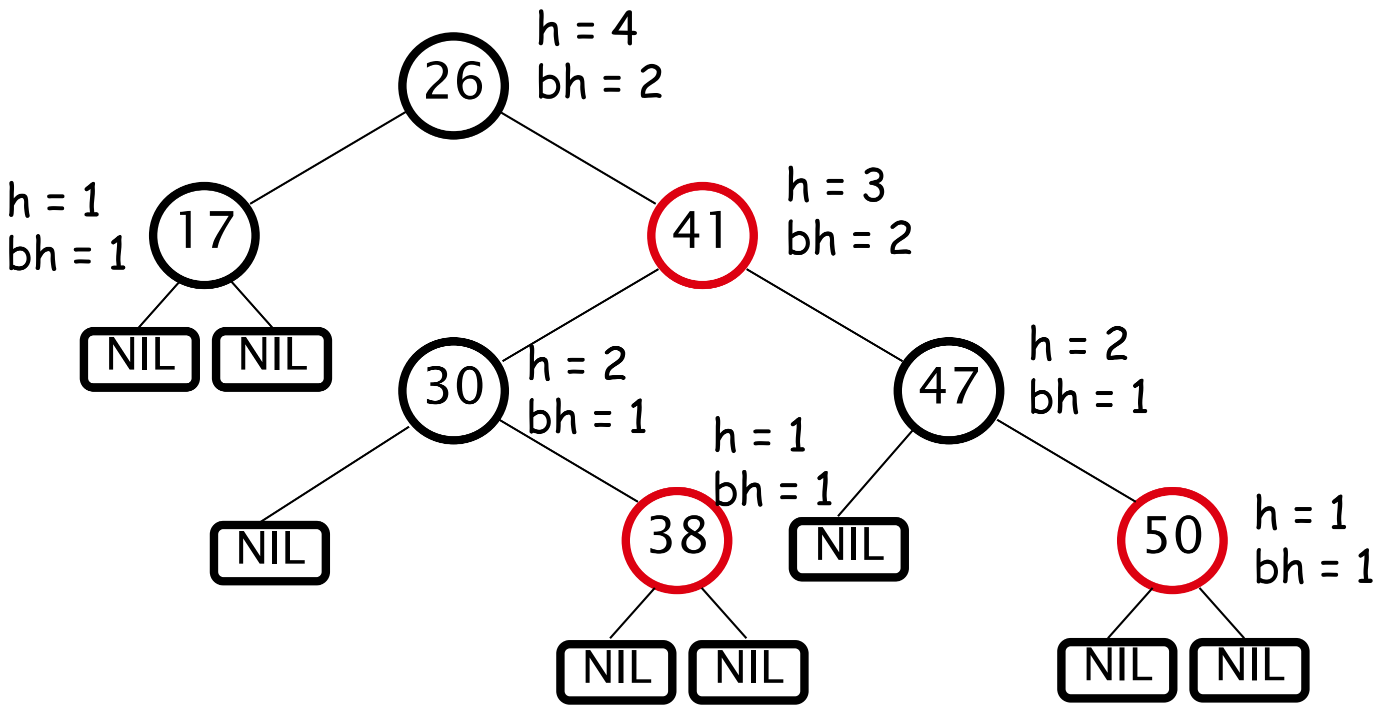
   No two consecutive red nodes on a simple path from the root to a leaf
5. For each node, all paths from that node to descendant leaves contain the same number of <u>**black**</u> nodes

Example



- For convenience we use a sentinel NIL[T] to represent all the NIL nodes at the leaves
  - NIL[T] has the same fields as an ordinary node
  - Color[NIL[T]] = BLACK
  - The other fields may be set to arbitrary values

Black height of a node



- **Height of a node:**
  - The number of edges in the longest path to a leaf
- **Black-height of a node x:**
  - bh(x) is the number of black nodes (including NIL) on the path from x to a leaf, <u>not counting x</u>

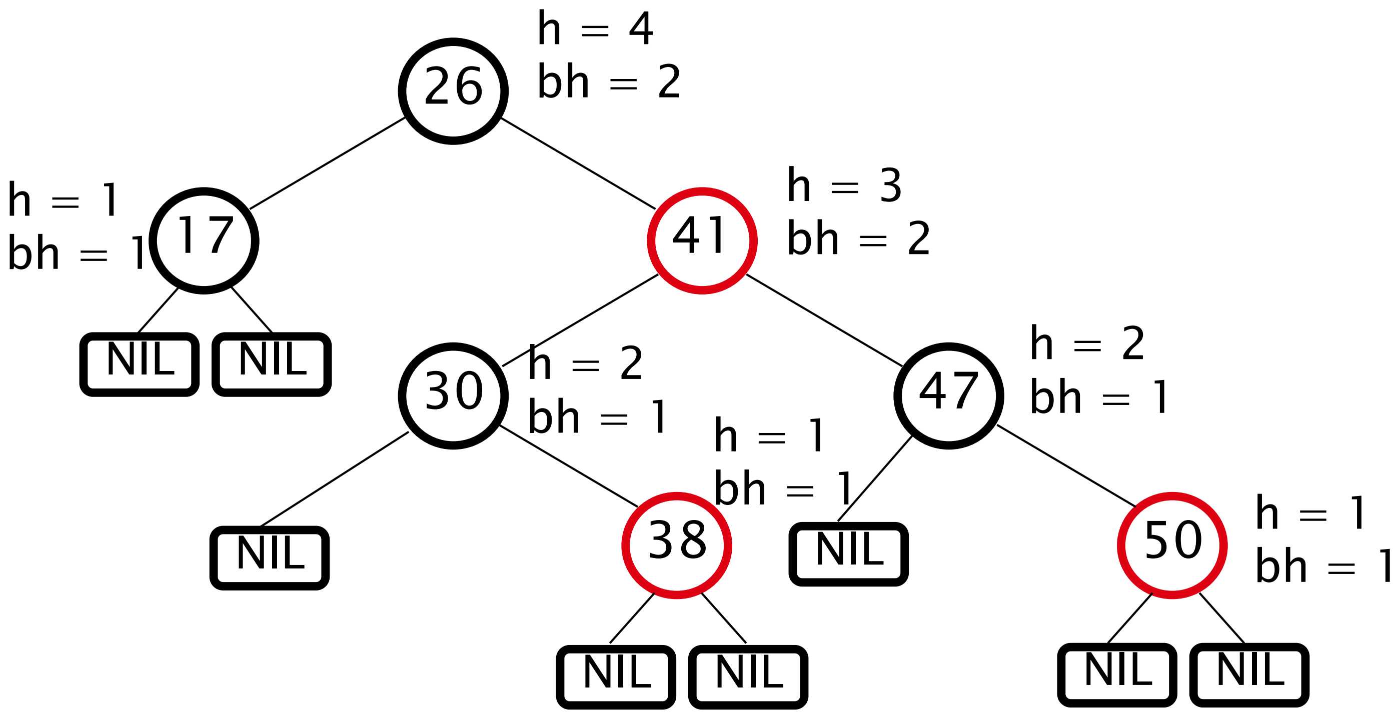
Important property of red-black tree

A red-black tree with n internal nodes has height <u>at most</u> $2\log(n + 1)$

- Need to prove two claims first …

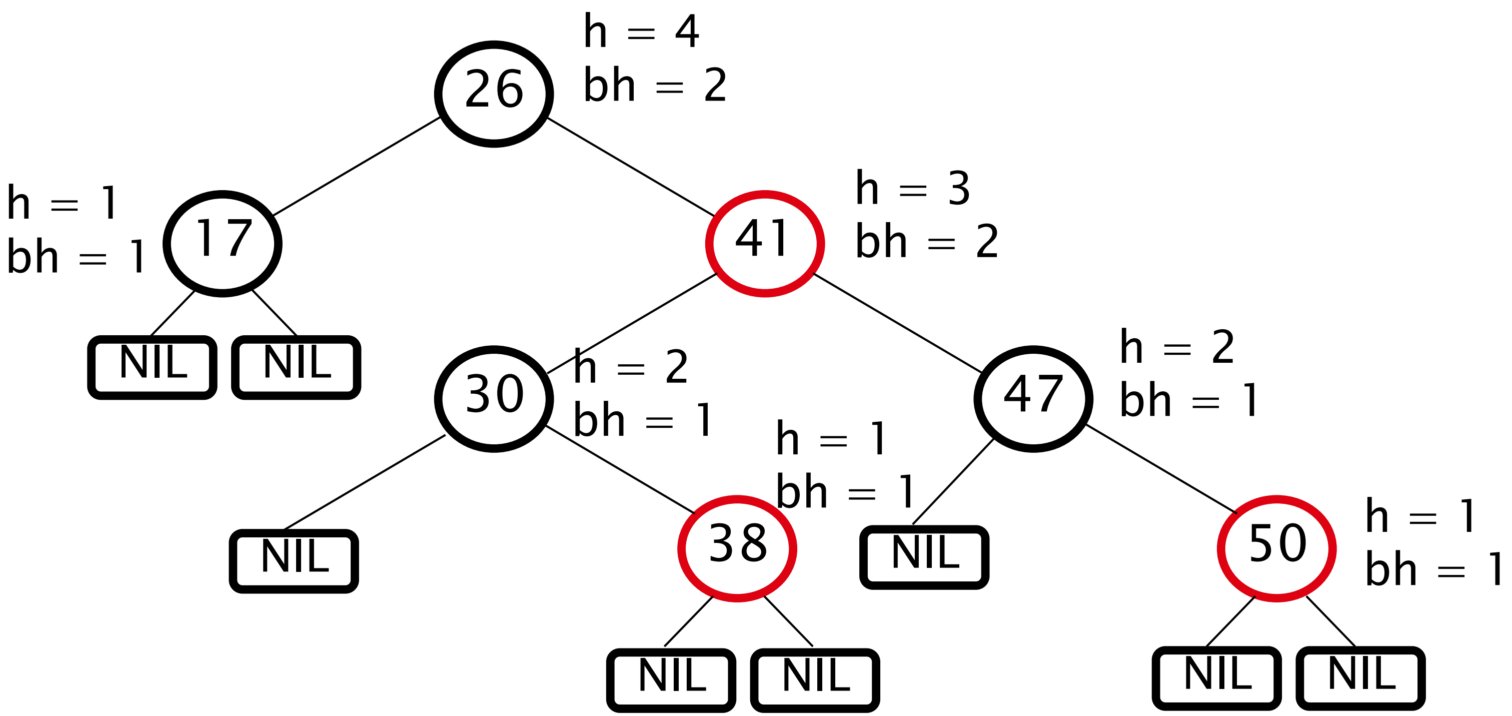
Claim 1

- Any node x with height h(x) has bh(x) ≥ h(x)/2
- **Proof**
  - By property 4, at most h/2 <u>red</u> nodes on the path from the node to a leaf
  - Hence at least h/2 are <u>**black**</u>



Claim 2

- The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes



Claim 2 (Cont'd)

Proof: By induction on **h[x]**

Basis: h[x] = 0 $\Rightarrow$

x is a leaf (NIL[T]) $\Rightarrow$

bh(x) = 0 $\Rightarrow$

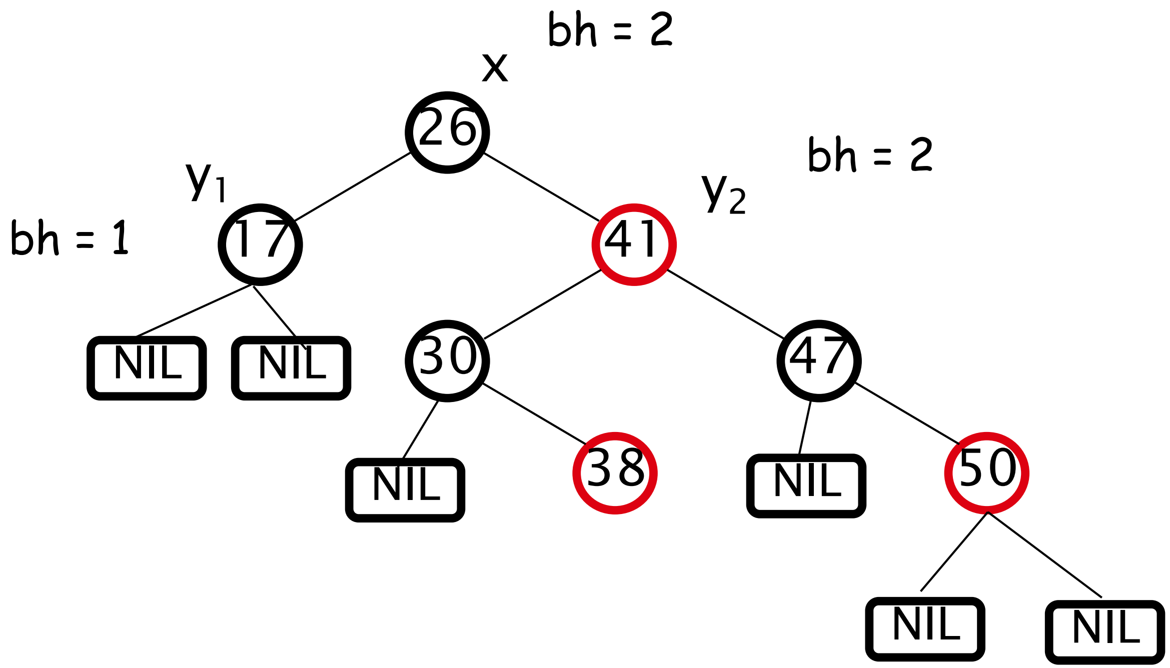
# of internal nodes: $2^0 - 1 = 0$

Inductive Hypothesis: assume it is true for h[x]=h-1

Claim 2 (Cont'd)

Inductive step:

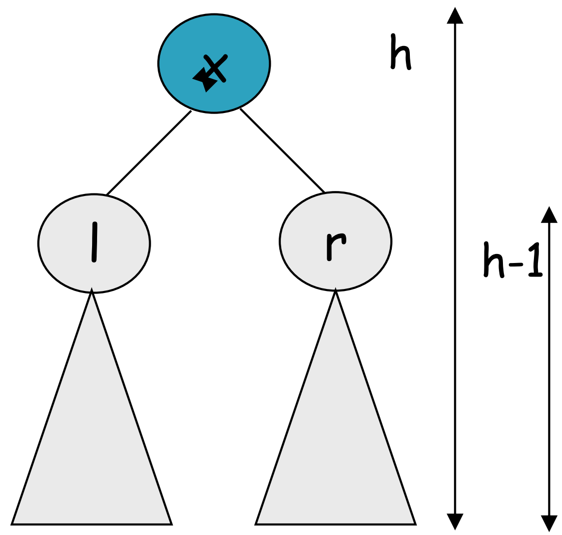
- Prove it for h[x]=h
- Let bh(x) = b. Then, any child y of x has:
  - bh (y) = b (if the child is **red**), or
  - bh (y) = b - 1 (if the child is **black**)



Claim 2 (Cont'd)

- Using inductive hypothesis, the number of internal nodes for each child of x is at least (if it is black):

$$2^{bh(x)-1} - 1$$



- The subtree rooted at x has at least:

$(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1$
$= 2 \cdot (2^{bh(x)-1} - 1) + 1$
$= 2^{bh(x)} - 1$ internal nodes

$bh(l) \geq bh(x)-1$

$bh(r) \geq bh(x)-1$

Important property of red-black tree

A red-black tree with n internal nodes has height <u>at most</u> $2\log(n + 1)$

Proof in the next slides.

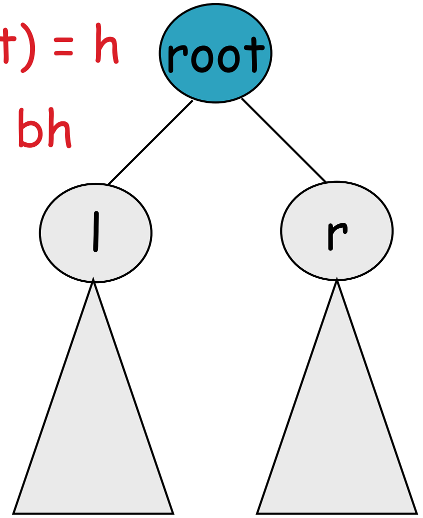
- Claim 1: Any node x with height $h(x)$ has $bh(x) \geq h(x)/2$
- Claim 2: The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes

Height of red-black tree

Lemma: A red-black tree with n internal nodes has height at most $2\log(n + 1)$.

height(root) = h

bh(root) = bh



Proof:

$$n \quad \geq 2^{bh} - 1 \quad \geq 2^{h/2} - 1$$

number n of internal nodes

since bh $\geq h/2$

- Add 1 to both sides and then take logs:

$$n + 1 \geq 2^{bh} \geq 2^{h/2}$$

$$\log(n + 1) \geq h/2$$

$$\Rightarrow h \leq 2\log(n + 1)$$

Operations of red-black tree

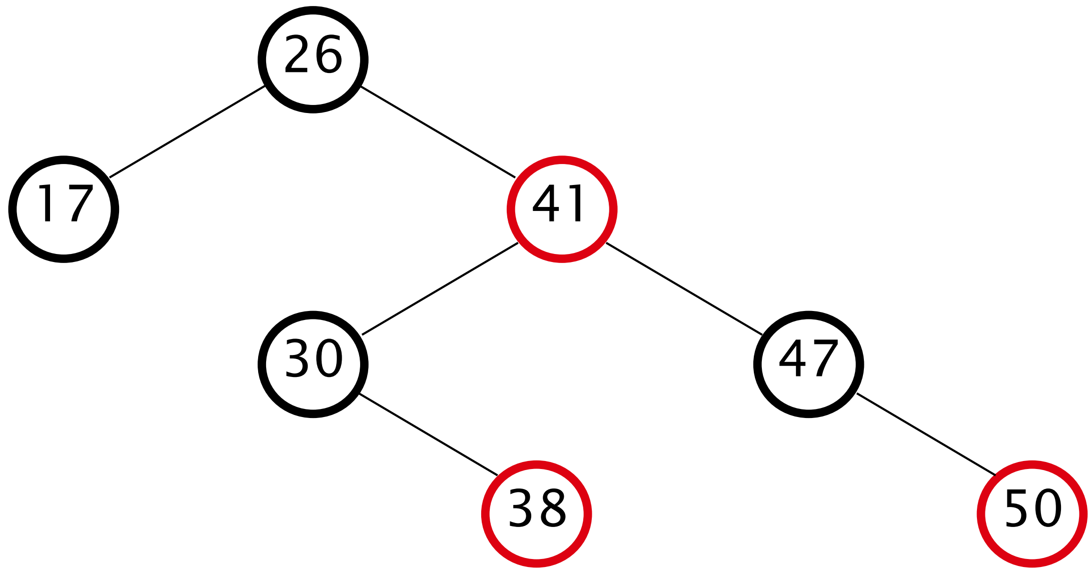
- The non-modifying operations: MINIMUM, MAXIMUM, and SEARCH run in O(h) time
  - They take O(logn) time on red-black trees
  - SEARCH is similar to the search on binary search tree

- What about INSERT and DELETE?
  - They will still run in O(logn) time
  - We have to guarantee that the modified tree will still be a red-black tree

INSERT operation

INSERT: Suppose we want to insert 35. What color to make the new node?

- Red?
  - Property 4 is violated: if a node is red, then its children are black
- Black?
  - Property 5 is violated: all paths from a node to its leaves contain the same number of black nodes

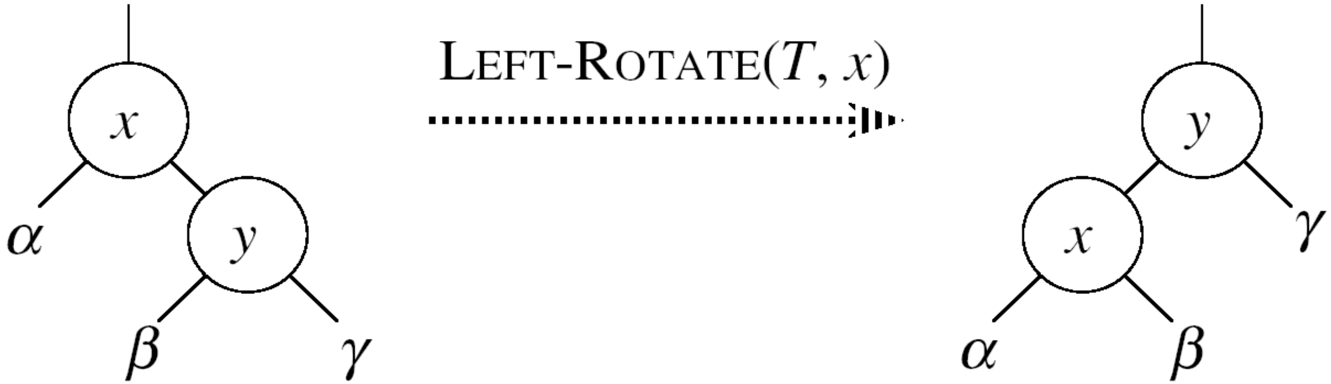


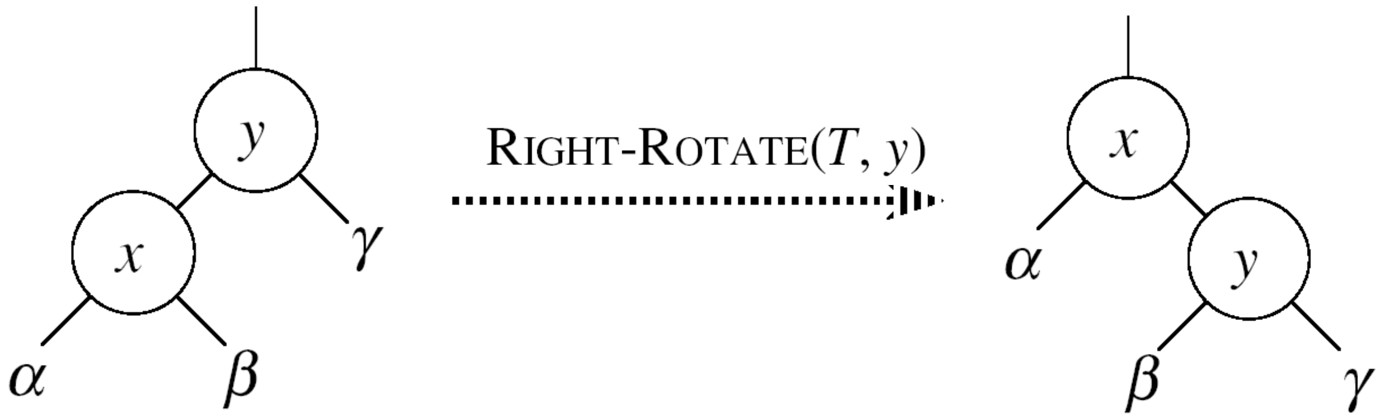
Rotations

- After insertion and deletion on red-black trees, we need to restore the red-black tree properties

- Rotations take a red-black tree and a node within the tree and:
  - Two types of rotations: Left & right rotations
  - Together with some node re-coloring they help restore the red-black-tree property
  - Change some of the pointer structure
  - **Do not** change the binary search tree property

Left rotation and right rotation





INSERT

- Goal:
  - Insert a new node z into a red-black-tree

- Idea:
  - Insert node z into the tree as for an ordinary binary search tree
  - Color the node **red**
  - Restore the red-black-tree properties
    - Use an auxiliary procedure RB-INSERT-FIXUP

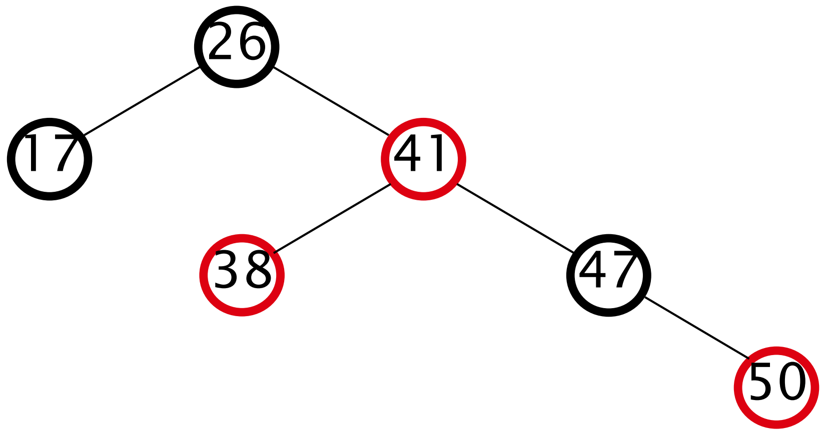
Properties affected by INSERT

1. Every node is either <u>red</u> or <u>black</u> OK!
2. The root is <u>black</u>

   If the root is changed
   ⇒ May not OK
3. Every leaf (NIL) is <u>black</u> OK!
4. If a node is <u>red</u>, then both its children are <u>black</u>

   If p(z) is red ⇒ not OK
   z and p(z) are both red

   OK!
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes



INSERT(T,z)

```
y ← NIL
x ← root[T]
```

- Initialize nodes x and y
- Throughout the algorithm y points to the parent of x

```
while x ≠ NIL
        do y ← x
              if key[z] < key[x]
                 then x ← left[x]
                 else x ← right[x]
```

- Go down the tree until reaching a leaf
- At that point y is the parent of the node to be inserted

```
p[z] ← y
```

Sets the parent of z to be y

```
if y = NIL
   then root[T] ← z
```

The tree was empty: set the new node to be the root

```
   else if key[z] < key[y]
           then left[y] ← z
           else right[y] ← z
```

Otherwise, set z to be the left or right child of y, depending on whether the inserted node is smaller or larger than y's key

```
left[z] ← NIL
right[z] ← NIL
color[z] ← RED
```

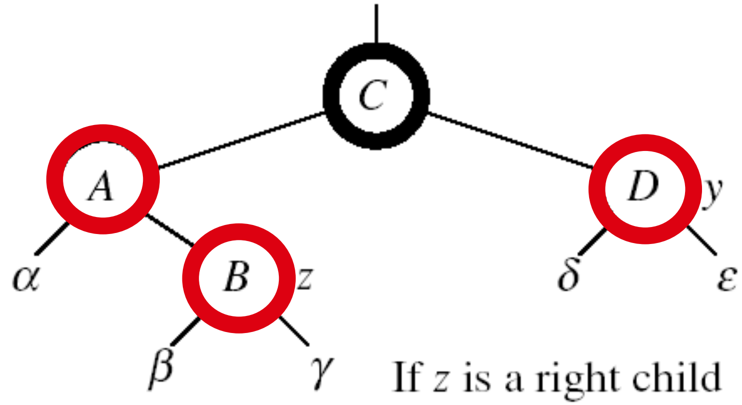
Set the fields of the newly added node

```
RB-INSERT-FIXUP(T, z)
```

Fix any inconsistencies that could have been introduced by adding this new red node

RB-Insert-Fixup(T, z)

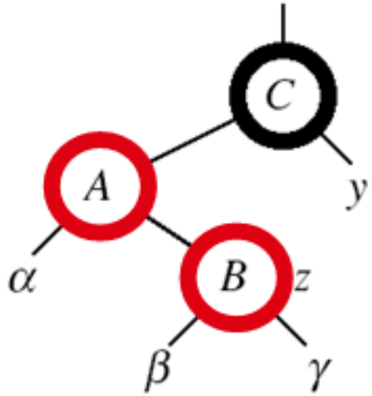
- Case 1: z's uncle y is red
  - Solution: recolor



If z is a right child

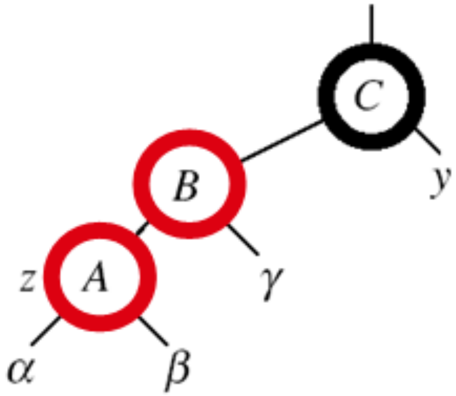
- Case 2: z's uncle y is black and z is a right child
  - Solution: double rotation
  - Can be transferred to Case 3

Case 2



- Case 3: z's uncle y is black and z is a left child
  - Solution: single rotation

Case 3



RB-Insert-Fixup(T, z)

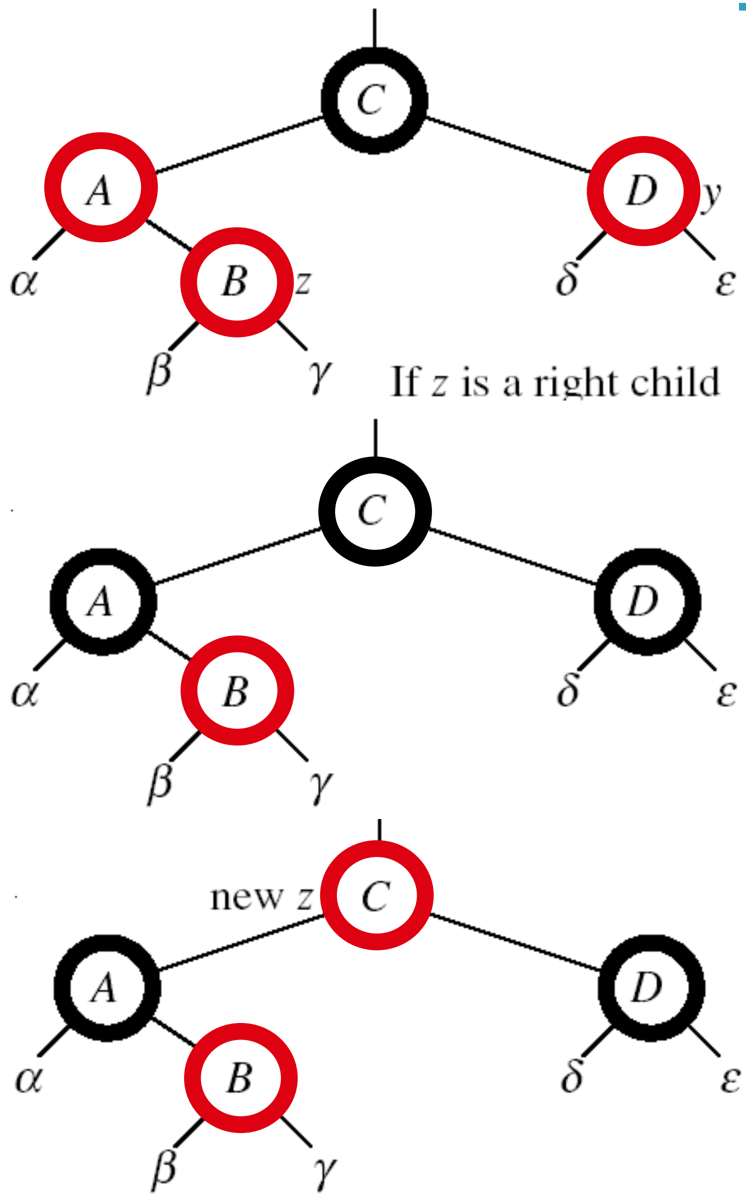
```
while z.p.color == red        ← The while loop repeats only when
                                Case 1 is executed: O(lgn) times
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == red
            z.p.color = black            // case 1
            y.color = black              // case 1
            z.p.p.color = red            // case 1
            z = z.p.p                    // case 1
        else if z == z.p.right
                z = z.p                  // case 2
                Left-rotation (T, z)     // case 2
            z.p.color = black            // case 3
            z.p.p.color = red            // case 3
            Right-rotation (T, z.p.p)    // case 3
    else (same as then clause with "right" and "left" exchanged)
T.root.color = black    ← may just insert the root or the red violation reach root
```

INSERT - Case 1

z's "uncle" (y) is red

Idea: (<u>z is a right</u>)

- p[p[z]] (z's grandparent) must be black: p[z] is red
- Color p[z] black
- Color y black
- Color p[p[z]] red
- z = p[p[z]]
  - Push the "red" violation up the tree

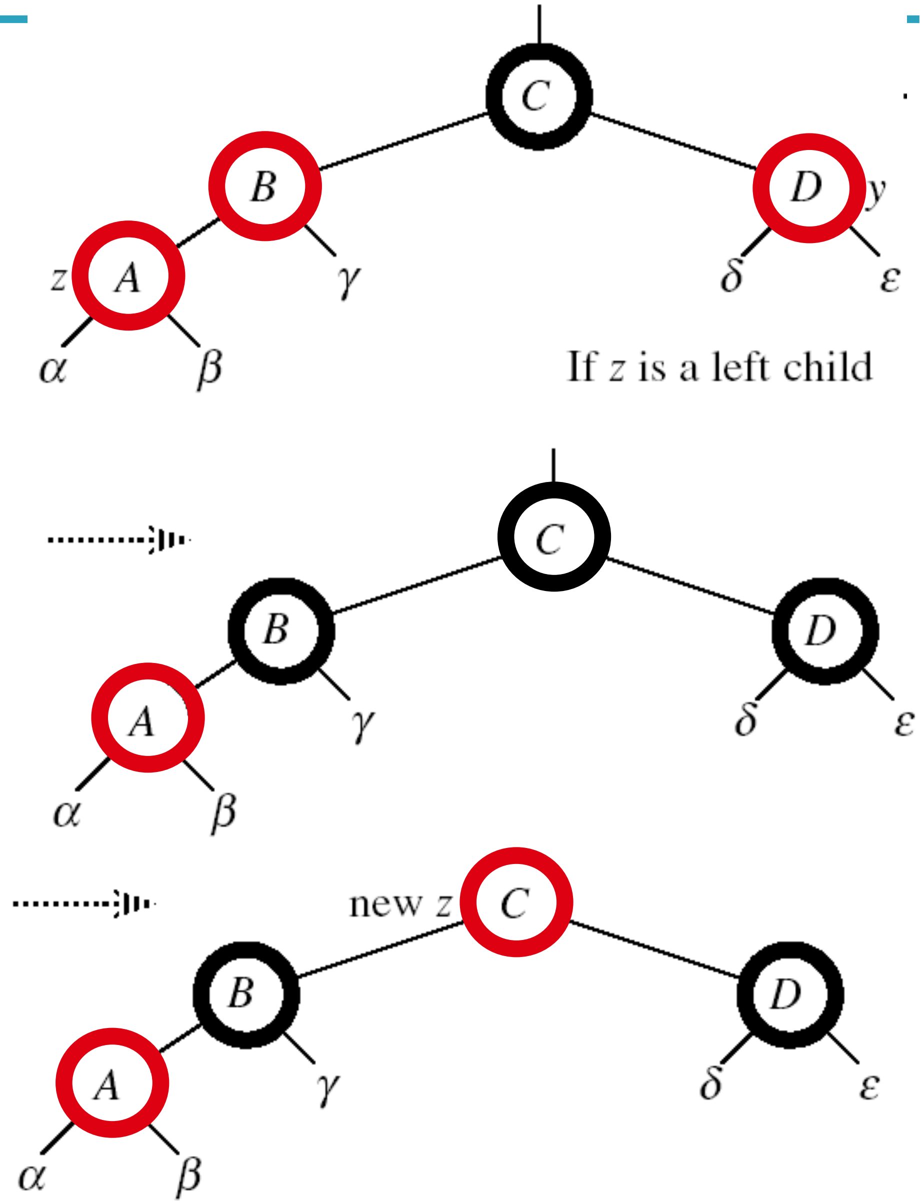


INSERT - Case 1

z's "uncle" (y) is red

Idea: (<u>z is a left child</u>)

- p[p[z]] (z's grandparent) must be black: p[z] is red
- Color p[z] ← black
- Color y ← black
- Color p[p[z]] ← red
- z = p[p[z]]
  - Push the "red" violation up the tree



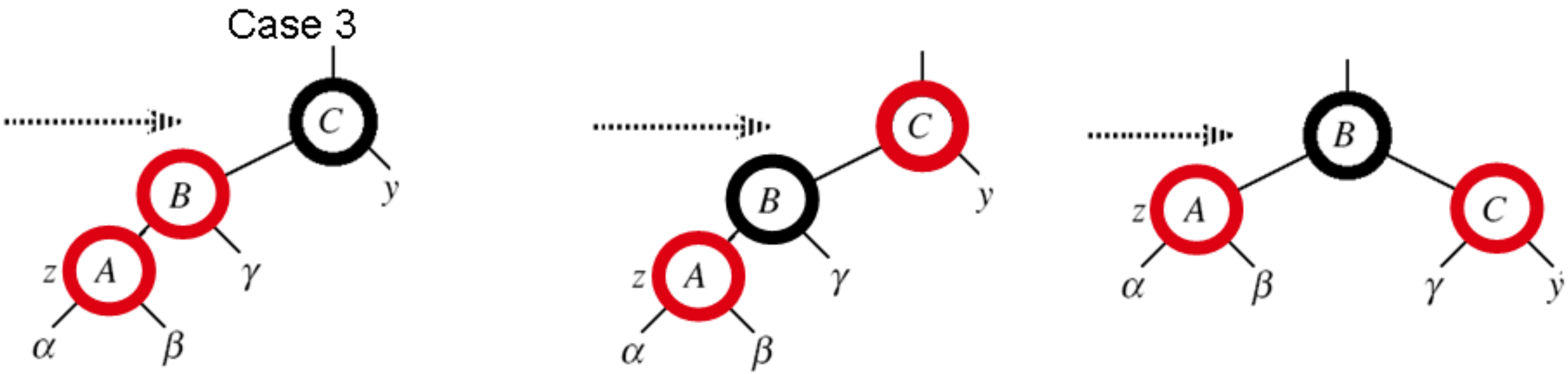
INSERT - Case 3

Case 3:

- z's "uncle" (y) is black
- z is a left child

Idea:

- Color p[z] ← black
- Color p[p[z]] ← red
- RIGHT-ROTATE(T, p[p[z]])
- No longer have 2 reds in a row
- p[z] is now black



INSERT - Case 2

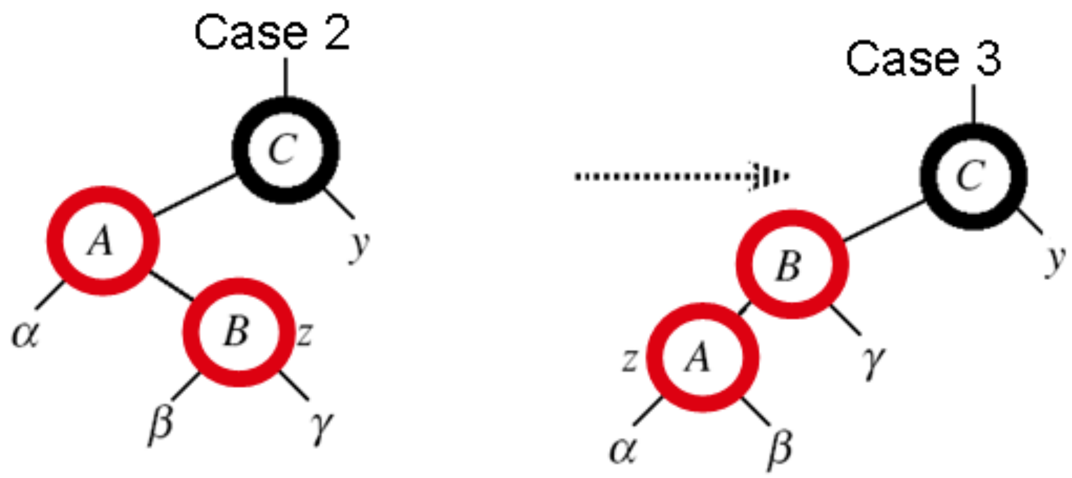
Case 2:

- z's "uncle" (y) is black
- z is a right child

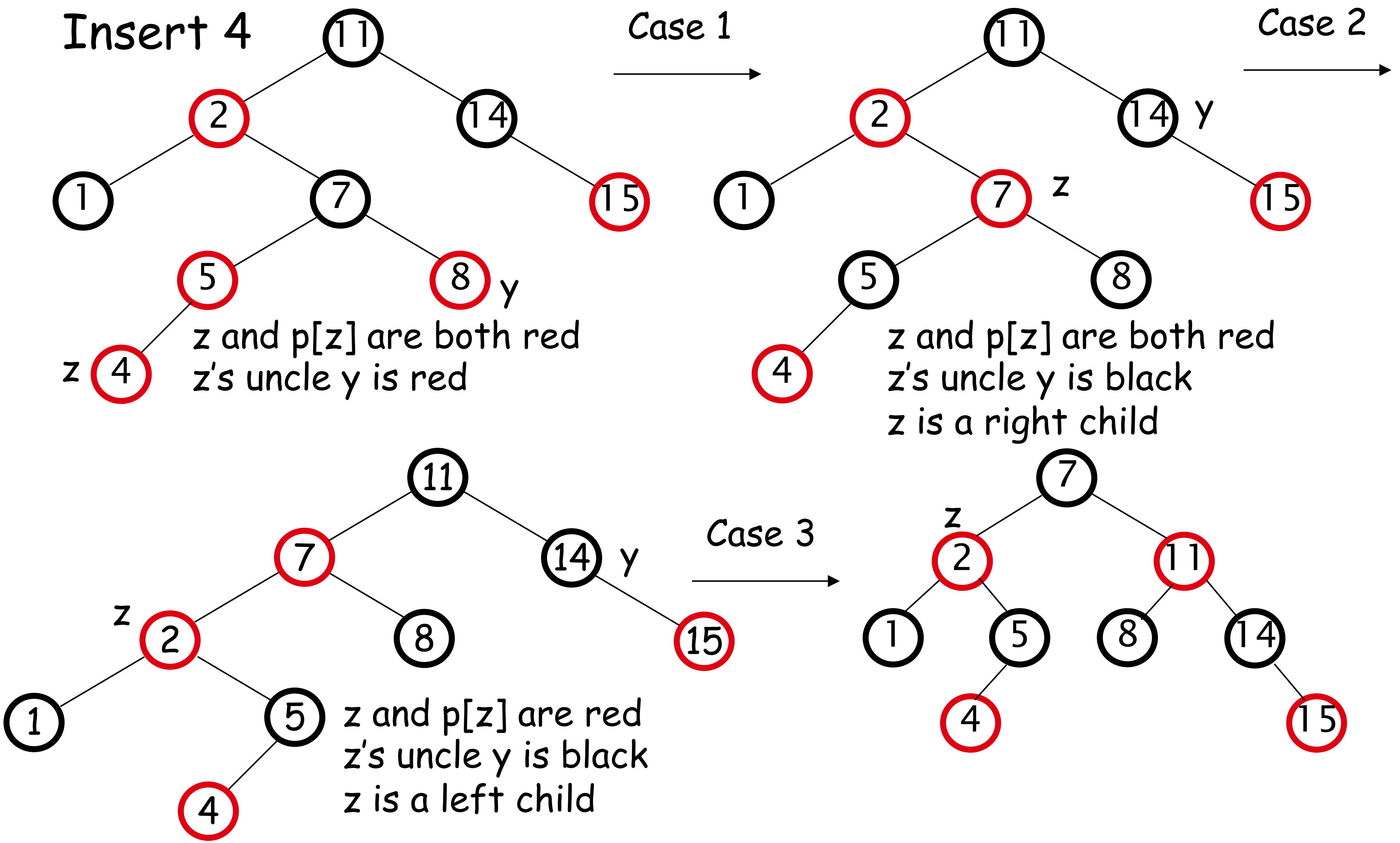
Idea:

- z ← p[z]
- LEFT-ROTATE(T, z)

⇒ now z is a left child, and both z and p[z] are red ⇒ case 3



Example



Insert 4
Case 1
11
2
14
1
7
15
5
8
y
z
4
z and p[z] are both red
z's uncle y is red
Case 2
11
2
14
y
1
7
z
15
5
8
4
z and p[z] are both red
z's uncle y is black
z is a right child
11
7
14
y
z
2
8
15
1
5
4
z and p[z] are red
z's uncle y is black
z is a left child
Case 3
7
z
2
11
1
5
8
14
4
15

Complexity analysis

- **Time complexity of detailed steps**
  - A red-black tree has $O(\log n)$ height
  - Search for insertion location takes $O(\log n)$ time
  - Addition to the node takes $O(1)$ time

  - The while loop will be executed at most $O(\log n)$ time
    - Each recoloring and each rotation take $O(1)$ time
    - Never performs more than two rotations, since the loop terminates if case 2 or case 3 is executed

  - An insertion in a red-black tree takes $O(\log n)$ time

What are the advantages of red-black tree over AVL tree?

Exercises

- What is the ratio between the longest path and the shortest path in a red-black tree?
  - The shortest path is at least bh(root)
  - The longest path is equal to h(root)
  - Since h(root)≤2bh(root), the ratio is ≤2

- When we insert a node into a red-black tree, we initially set the color of the new node to red. Why didn't we choose to set the color to black?

Red-black trees: summary

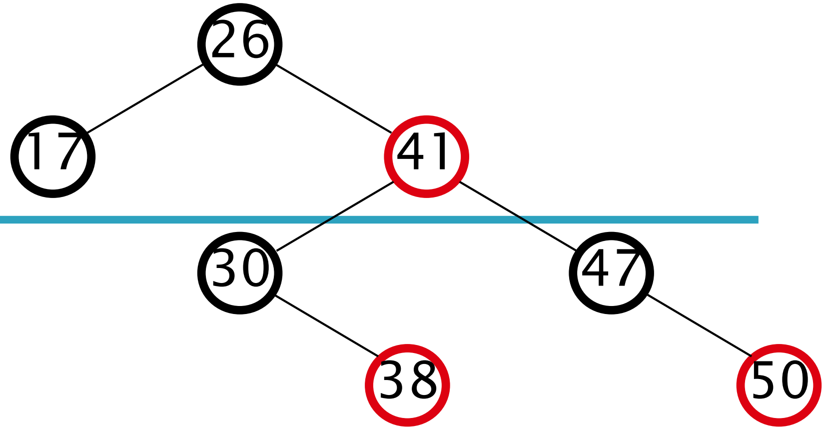
- Red-black-trees guarantee that the height of the tree will be O(logn)

- Operations on red-black-trees:
  - SEARCH O(h)
  - PREDECESSOR O(h)
  - SUCCESOR O(h)
  - MINIMUM O(h)
  - MAXIMUM O(h)
  - INSERT O(h)
  - DELETE O(h)

Recommended reading

- Reading
  - Chapter 13, textbook
- Next lectures
  - Heap, chapter 6&12, textbook

DELETE operation

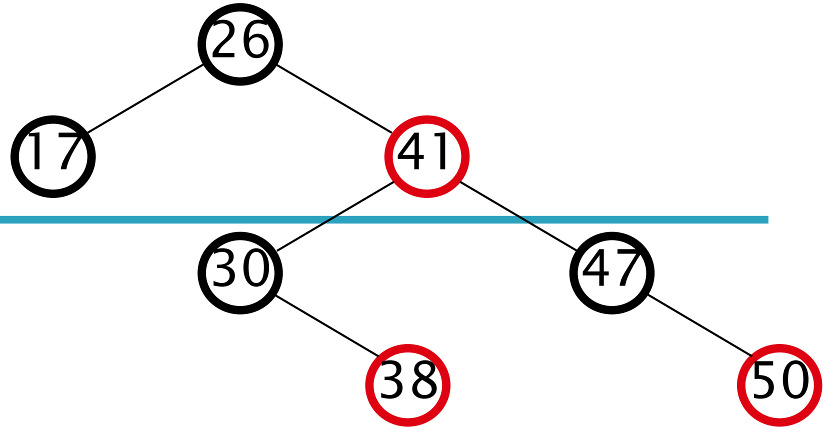


DELETE: the color of the node to be removed -- red

1. Every node is either red or **black** OK!
2. The root is **black** OK!
3. Every leaf (NIL) is **black** OK!
4. If a node is red, then both its children are **black** OK!
5. For each node, all paths from the node to descendant leaves contain the same number of **black** nodes OK!

Note: the deletion of a red node is the same as the deletion of a node in BST

DELETE operation



DELETE: the color of the node to be removed -- **Black**

1. Every node is either red or black OK!
2. The root is black
3. Every leaf (NIL) is black OK!
4. If a node is red, then both its children are black

Not OK! If removing the root and the child that replaces it is **red**

Not OK! Could create two red nodes in a row

Not OK! Could change the black heights of some nodes

5. For each node, all paths from the node to descendant leaves contain the same number of black nodes

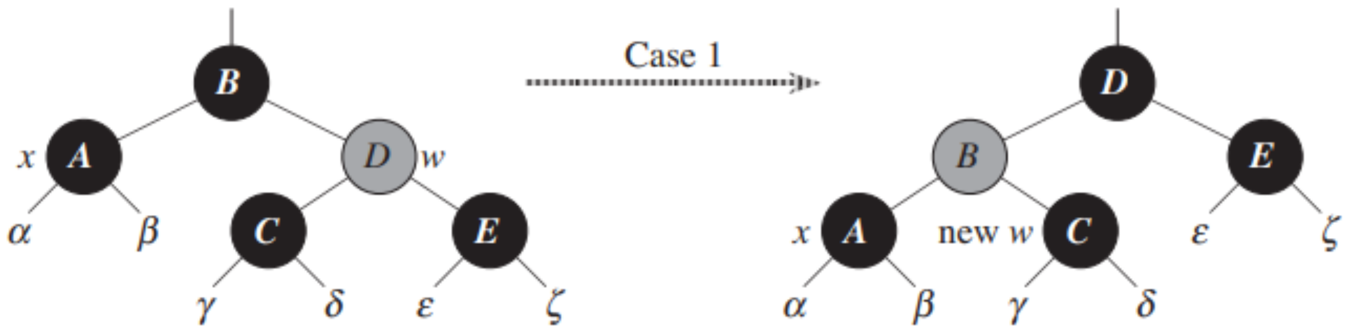
Deletion on red-black tree

- Similar to the deletion on BST, but need to use an auxiliary procedure RB-Delete-Fixup to restore the red-black tree properties

- Four different cases of RB-Delete-Fixup
  - Case 1: x's sibling w is red
  - Case 2:x's sibling w is black, and both of w's children are black
  - Case 3:x's sibling w is black, w's left child is red, and w's right child is black
  - Case 4: x's sibling w is black, and w's right child is red (left child either color)

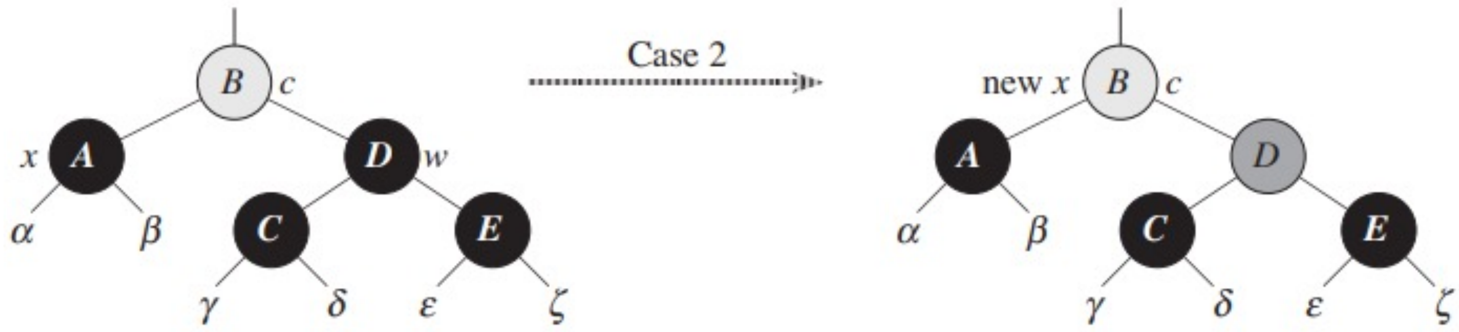
Case 1

- Case 1: x's sibling w is red
  - Solution: rotate and recolor



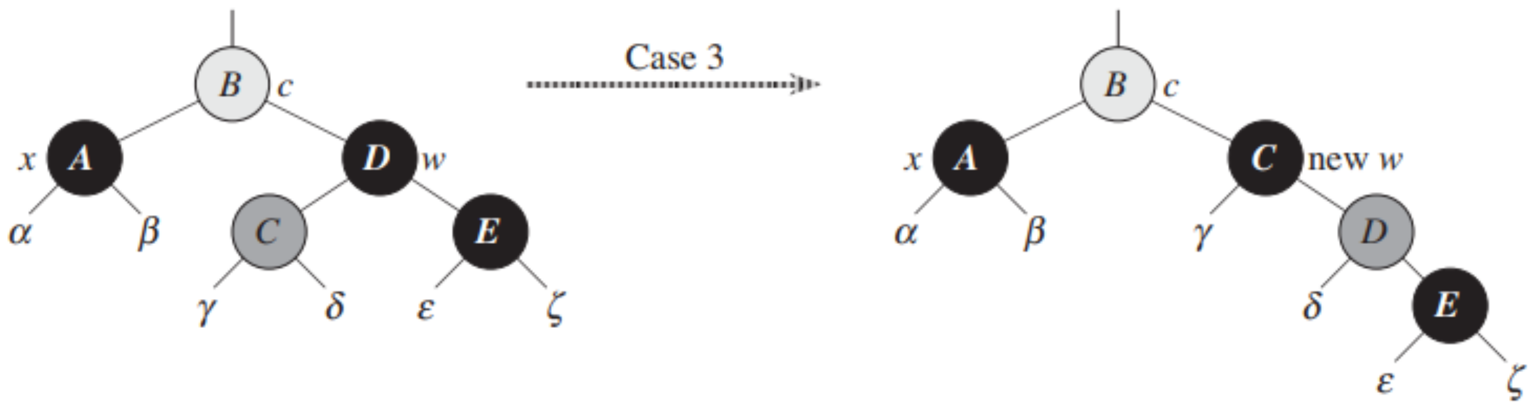
Case 2

- Case 2: x's sibling w is **black**, and both of w's children are **black**
  - Solution: recolor



Case 3

- Case 3: x's sibling w is <u>black</u>, w's left child is <u>red</u>, and w's right child is <u>black</u>



Case 4

- Case 4: x's sibling w is **<u>black</u>**, and w's right child is <u>red</u> (left child either color)