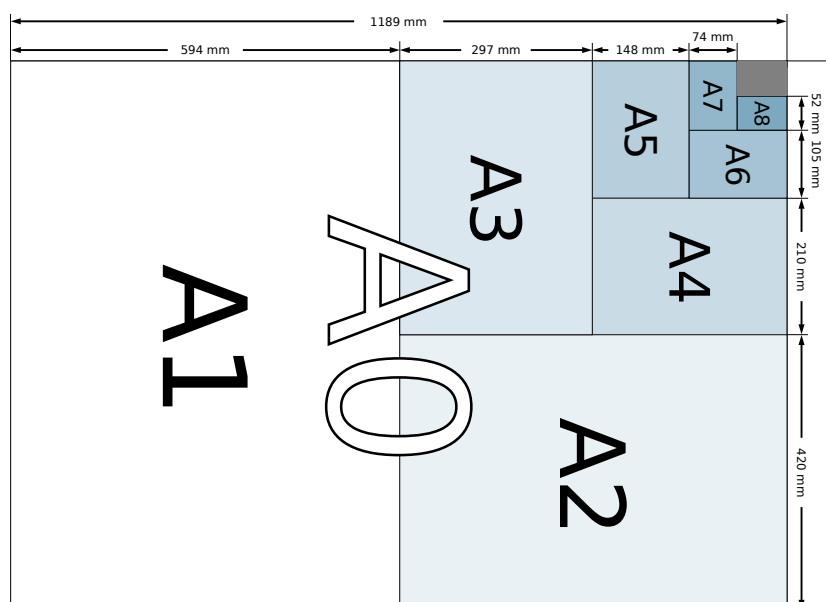


# THE MATH BEHIND PAPER

## Investigating Geometric Sequences Behind the ISO 216 Series-A Paper Format

By Shen Zhou Hong      Goldsmiths, University of London

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## Introduction

Understanding the properties of arithmetic and geometric sequences are essential for quantifying the world, and for computing in general. In this paper, I will investigate the different sizes of ISO 216 standard A-series paper, and quantify their properties using my knowledge of sequences.

# The Properties of ISO 216 A-Series Paper

Let us begin by quantifying the properties of ISO 216 A-series paper (henceforth referred to as A-paper). It is given that the largest A-paper size,  $A_0$ , has a total area of  $1 \text{ m}^2$ . Likewise, we know that each successive smaller paper is the previous paper folded in half.

## Formalising the Area

We can formalise this property as the following geometric sequence:

$$\begin{aligned}a_n &= a \times r^{n-1} \\A_n &= A \times \left(\frac{1}{2}\right)^{n-1} \\&= 2^{-n}\end{aligned}$$

By plugging in the the numbers, it is trivial to generate a table of area for each successive A-series paper:

$n$	Area (fractional $\text{m}^2$ )	Area (decimal $\text{m}^2$ )
0	1	1
1	$\frac{1}{2}$	0.5
2	$\frac{1}{4}$	0.25
3	$\frac{1}{8}$	0.125
4	$\frac{1}{16}$	0.0625
5	$\frac{1}{32}$	0.03125
6	$\frac{1}{64}$	0.015625

Table 1: List of A-series paper areas for  $0 \leq n \leq 6$

## Formalizing the Length and Width

The geometric sequence behind the area of the A-series paper is trivial to discover, but what about the length and width of the paper for a given  $n$  in  $A_n$ ? Recall that every time an A-series paper is folded in half, we make the fold at the largest side (the length) of the paper:

$$\begin{aligned}A_n &= L_n \times W_n \\A_{n+1} &= \frac{L_n}{2} \times W_n\end{aligned}$$

Where essentially the new “length” of  $A_{n+1}$  is actually the width of the previous larger paper, namely  $A_n$ :

$$\begin{aligned}L_{n+1} &= W_n \\W_{n+1} &= \frac{L_n}{2}\end{aligned}$$

This relation is apparent even if we just list out the sequences:

$$\begin{aligned}L_n &= L_0, \frac{L_0}{2}, \frac{L_0}{2}, \frac{L_0}{4}, \frac{L_0}{4}, \frac{L_0}{8}, \frac{L_0}{8}, \\W_n &= W_0, W_0, \frac{W_0}{2}, \frac{W_0}{2}, \frac{W_0}{4}, \frac{W_0}{4}, \frac{W_0}{8},\end{aligned}$$

This property is an important one, because it allows us to discover the ratio, or *aspect ratio* between the two sides of the paper. We must do this prior to the next section on the scaling factor, because without knowing the aspect ratio of the sides, there is no way for us to do proper scaling conversions. Hence, let  $R_n$  be the aspect ratio of  $A_n$

$$\begin{aligned}
R_n &= \frac{L_n}{W_n} \\
R_{n+1} &= \frac{L_{n+1}}{W_{n+1}} \\
&= \frac{W_n}{L_n \div 2} \\
&= \frac{2W_n}{L_n} \\
&= \frac{2}{L_n \div W_n}
\end{aligned}$$

Originally, I hoped that somehow the terms would cancel out and I would be able to have a numerical solution, but that didn't happen. However, not all was lost, because in the process of working out this math I came to another realisation. Now recall that we defined  $R_n$  as  $\frac{L_n}{W_n}$ . Therefore:

$$\begin{aligned}
R_{n+1} &= \frac{2}{L_n \div W_n} \\
&= \frac{2}{R_n}
\end{aligned}$$

Do you recall how in the Lecture, we were told that one of the properties of the A-series paper, was how each time you fold it in half it preserves the proportions of the previous paper? This is a property that is unique to the ISO 216 paper, if you take a U.S. Letter paper and fold it in half, every time you fold it you end up with a slightly different rectangle. Now, there is only one way that this property can happen — namely that the aspect ratio of each subsequent halve must be the same, e.g  $R_n = R_{n+1} = R_{n+2}$  for  $n \rightarrow \infty$  where the aspect ratio is 1 :  $R_n$ . Therefore:

$$R_{n+1} = \frac{2}{R_n} = R_n$$

Now we have the simple numerical solution that I was looking for! We can find the value of  $R_n$  simply by solving for it:

$$\begin{aligned} R_n &= \frac{2}{R_n} \\ R_n \times R_n &= 2 \\ (R_n)^2 &= 2 \\ R_n &= \sqrt{2} \end{aligned}$$

Therefore, the aspect ratio of the ISO 216 A-series paper is  $1 : \sqrt{2}$ . What does this mean? It means the ratio of the length to the width of the paper is that of  $1 : \sqrt{2}$ , or expressed in Euclidean terms:<sup>1</sup>

$$L_n : W_n :: 1 : \sqrt{2}$$

## Exact Algebraic Solutions to Lengths and Widths

Now that we have laid the required groundwork in place, we can find the exact (and numerical approximations) solutions of the length and width for any arbitrary  $A_n$ . This is done using the aspect ratio we discovered in the previous section.

### Exact Algebraic Solution for $L_0, W_0$

First, in order to create an expression that will find the  $L_n, W_n$  for any  $A_n$ , we must find the  $L_0, W_0$  of  $A_0$ . Given  $A_0 = L_0 \times W_0$ , and that  $A_0 = 1$  (as by definition), a system of simultaneous equations of the 2nd order can be used to find the exact algebraic solutions to  $L_0, W_0$ :

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<sup>1</sup>The Euclidean continuous proportion reads: “As  $L_n$  is to  $W_n$ , in the manner of 1 to  $\sqrt{2}$ ”.

$$\begin{cases} L_0 \times W_0 &= 1 \\ W_0 \times \sqrt{2} &= L_0 \end{cases}$$

Note that  $W_0 \times \sqrt{2} = L_0$  holds true due to the aspect ratio, as we discussed in the previous section. Hence, to solve for  $W_0$ , we substitute  $L_0$  from the 2nd equation in the system:

$$\begin{aligned} W_0 \times \sqrt{2} \times W_0 &= 1 \\ (W_0)^2 \times \sqrt{2} &= 1 \\ (W_0)^2 &= \frac{1}{\sqrt{2}} \\ W_0 &= \sqrt{\frac{1}{\sqrt{2}}} \\ &= \left(2^{-\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= 2^{-\frac{1}{4}} \\ &= \frac{1}{\sqrt[4]{2}} \end{aligned}$$

With  $W_0 = \frac{1}{\sqrt[4]{2}}$ , finding the solution for  $L_0$  is trivial:

$$\begin{aligned} L_0 \times \frac{1}{\sqrt[4]{2}} &= 1 \\ \frac{L_0}{1} \times \frac{1}{\sqrt[4]{2}} &= 1 \\ \frac{\sqrt[4]{2}}{\sqrt[4]{2}} &= 1 \\ L_0 &= \sqrt[4]{2} \end{aligned}$$

Therefore, we now know the length and width for  $A_0$ , namely:

$$L_0 = \sqrt[4]{2}$$

$$W_0 = \frac{1}{\sqrt[4]{2}}$$

And to check if our work is correct, all we have to do is to multiply the length and the width together, and check if they give us an exact integer solution of  $1\text{m}^2$  for  $A_0$ 's area:

$$L_0 \times W_0 = 1$$

$$\sqrt[4]{2} \times \frac{1}{\sqrt[4]{2}} = 1$$

## Generalised Exact Algebraic Solutions

Now that we defined the length and width of  $A_0$ , creating an expression or function for the length and width of any subsequent  $A_n$  is trivial. The solutions for  $L_n$  and  $W_n$  can be represented as geometric sequences of the form  $a_n = a \times r^{n-1}$ , where  $r = \frac{1}{\sqrt{2}}$  as each subsequent smaller paper size is effectively divided by the aspect ratio:

$$L_n = \sqrt[4]{2} \times \left( \frac{1}{\sqrt{2}} \right)^n$$

$$W_n = \frac{1}{\sqrt[4]{2}} \times \left( \frac{1}{\sqrt{2}} \right)^n$$

For the purpose of generating the required table of lengths and widths, we will represent those expressions as functions instead:

$$L(n) = \sqrt[4]{2} \times \left( \frac{1}{\sqrt{2}} \right)^n$$

$$W(n) = \frac{1}{\sqrt[4]{2}} \times \left( \frac{1}{\sqrt{2}} \right)^n$$

### **Generated Table of Values for $A_0 - A_4$**

Now that we have the functional form of the sequence notation, we can fulfill part 1 and 2 of the assignment by generating a table of the exact and approximative magnitudes of the length and width.

Note that, in order to save space in the table, we will not use the pretty radical notation of the terms, but rather the more compact (and arguably, less pretty) exponent notation:

$$L_n = \sqrt[4]{2} \times \left( \frac{1}{\sqrt{2}} \right)^n = 2^{\frac{1}{4} - \frac{n}{2}}$$

$$W_n = \frac{1}{\sqrt[4]{2}} \times \left( \frac{1}{\sqrt{2}} \right)^n = 2^{-\frac{1}{4} - \frac{n}{2}}$$



### Table of Exact Values<sup>2</sup>

Paper	Length (m)	Width (m)	Area (m <sup>2</sup> )
$A_0$	$2^{\frac{1}{4}}$	$2^{-\frac{1}{4}}$	1
$A_1$	$2^{-\frac{1}{4}}$	$2^{-\frac{3}{4}}$	$\frac{1}{2}$
$A_2$	$2^{-\frac{3}{4}}$	$2^{-\frac{5}{4}}$	$\frac{1}{4}$
$A_3$	$2^{-\frac{5}{4}}$	$2^{-\frac{7}{4}}$	$\frac{1}{8}$
$A_4$	$2^{-\frac{7}{4}}$	$2^{-\frac{9}{4}}$	$\frac{1}{16}$
$A_n$	$2^{\frac{1}{4}-\frac{n}{2}}$	$2^{-\frac{1}{4}-\frac{n}{2}}$	$\frac{1}{2^n}$

Table 2: Exact (algebraic) values for any given  $A_n$

Note that I have simplified the values to their simplest exponent form.

### Table of Approximative Values

Paper	Length (m)	Width (m)	Area (m <sup>2</sup> )
$A_0$	1.1892	0.8408	1
$A_1$	0.8408	0.5946	0.5
$A_2$	0.5946	0.4204	0.25
$A_3$	0.4204	0.2973	0.125
$A_4$	0.2973	0.2102	0.0625

Table 3: Approximate (numerical) values for any given  $A_n$

Note that the figures are truncated to 3 decimal places.

<sup>2</sup>Note that the numbers following the 2 are exponents, although the lack of space in the table can make them look like regular terms. For example, under the length column for  $A_2$ , the entry  $2^{-\frac{3}{4}}$  reads “two to the power of negative three over four”. Please don’t confuse the small numbering with regular terms!

## Scaling Factor (e.g. Aspect Ratio) In Percentage

As the final part of the assignment, we are tasked to calculate a “scaling factor” that allows us to easily convert between a given  $A_n$  to  $A_m$ . A scaling factor is a number that we multiply the sides of the paper by, in order to reach a bigger or smaller sized paper. In order to do this, we use the aspect ratio  $\sqrt{2}$ , converted to a percentage. I will demonstrate this method as well as generate a table with percentage conversions as required.

### Calculating Scaling Factor for arbitrary $A_n$ to $A_m$

Recall that the aspect ratio is calculated as  $\sqrt{2}$ . Therefore, to move “up” a paper size (e.g. from  $A_n$  to  $A_{n-1}$ ), the given paper size is multiplied by  $\sqrt{2}$ . Likewise, to move “down” a paper size, the given paper size is divided by the same aspect ratio. Generalising towards an expression for finding the scaling factor  $s$  of arbitrary conversions from  $A_n$  to  $A_m$ :

$$s = 1 \times \left(\sqrt{2}\right)^{m-n}$$

This expression does not give us the percentage scaling factor, of course. In order to find that, we simply have to modify it slightly:

$$s = 100 \times \left(\sqrt{2}\right)^{m-n}$$

As you can see, now given an arbitrary  $A_n$  and  $A_m$ , we can easily find the percentage scale factor:

$$\begin{aligned}
A_2 &\rightarrow A_5 \\
n &= 2 \\
m &= 5 \\
s &= 100 \times \left(\sqrt{2}\right)^{5-2} \\
s &\approx 282.84\%
\end{aligned}$$

## Scaling Factor Table

Hence, using the above expression we can easily generate a table for scale factors. Note: I generated the first row of values by hand, but it was extremely tedious and time-consuming, so I ended up filling the rest of the rows with existing data. See endpage for attribution info.

Size	A0	A1	A2	A3	A4	A5	A6	A7
A0	100%	71%	50%	35%	25%	18%	12.50%	8.80%
A1	141%	100%	71%	50%	35%	25%	18%	12.50%
A2	200%	141%	100%	71%	50%	35%	25%	18%
A3	283%	200%	141%	100%	71%	50%	35%	25%
A4	400%	283%	200%	141%	100%	71%	50%	35%
A5	566%	400%	283%	200%	141%	100%	71%	50%
A6	800%	566%	400%	283%	200%	141%	100%	71%
A7	1132%	800%	566%	400%	283%	200%	141%	100%

Table 4: Scale Factors for Conversions between arbitrary  $A_n$  to  $A_m$ .

## Research credit

Special thanks to Giuseppe Stelluto for his demonstration of an alternative method for deriving the formula for the sequences  $L_n$  and  $W_n$ . His approach was much more technically rigorous, but I felt that my approach was more accessible to those without a formal education in mathematics.

Additional thanks to papersizes.org for supplying an CSV (comma separated values) file with the scaling factors. Originally I was making the table by hand in LaTeX, but that literally took forever. With an premade CSV file, I was able to work much faster, and simply convert the CSV format to a LaTeX table.

## Image credit

Cover page illustration is a diagram illustrating ISO 216 A-series paper sizes, sourced from Wikipedia under Creative Commons (CC BY-SA 3.0) license.

## Technical Notes

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