The Math Behind Paper

Investigating Geometric Sequences Behind the ISO 216 Series-A Paper Format

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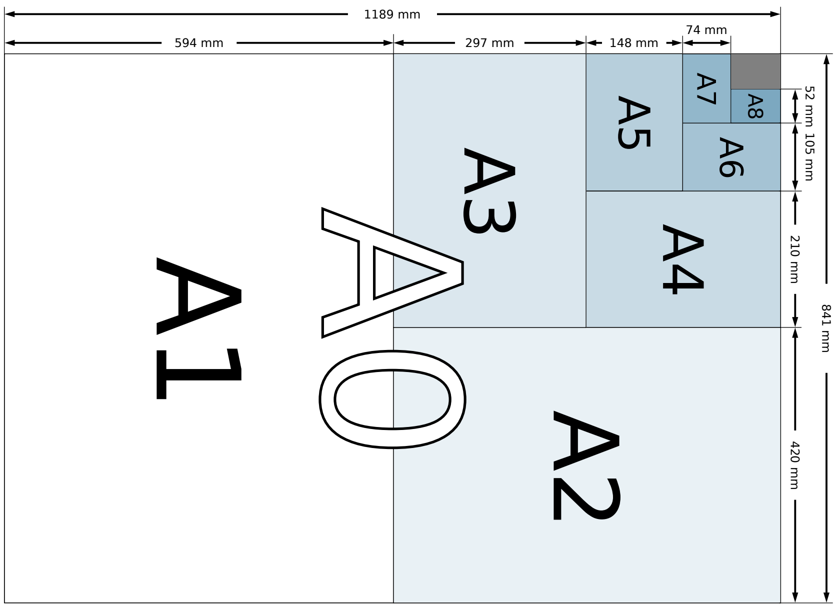
June 4, 2019



Introduction

Understanding the properties of arithmetic and geometric sequences are essential for quantifying the world, and for computing in general. In this paper, I will investigate the different sizes of ISO 216 standard A-series paper, and quantify their properties using my knowledge of sequences.

The Properties of ISO 216 A-Series Paper

Let us begin by quantifying the properties of ISO 216 A-series paper (henceforth referred to as A-paper). It is given that the largest A-paper size, A_0, has a total area of 1 m². Likewise, we know that each successive smaller paper is the previous paper folded in half.

Formalising the Area

We can formalise this property as the following geometric sequence:

$$
\begin{aligned}
a_n &= a \times r^{n-1} \\
A_n &= A \times \left(\frac{1}{2}\right)^{n-1} \\
&= 2^{-n}
\end{aligned}
$$

By plugging in the the numbers, it is trivial to generate a table of area for each successive A-series paper:

n	Area (fractional m²)	Area (decimal m²)
0	1	1
1	$\frac{1}{2}$	0.5
2	$\frac{1}{4}$	0.25
3	$\frac{1}{8}$	0.125
4	$\frac{1}{16}$	0.0625
5	$\frac{1}{32}$	0.03125
6	$\frac{1}{64}$	0.015625

Table 1: List of A-series paper areas for $0 \geq n \leq 6$

Formalizing the Length and Width

The geometric sequence behind the area of the A-series paper is trivial to discover, but what about the length and width of the paper for a given n in A_n? Recall that every time an A-series paper is folded in half, we make the fold at the largest side (the length) of the paper:

$$A_n = L_n \times W_n$$
$$A_{n+1} = \frac{L_n}{2} \times W_n$$

Where essentially the new "length" of A_{n+1} is actually the width of the previous larger paper, namely A_n:

$$L_{n+1} = W_n$$
$$W_{n+1} = \frac{L_n}{2}$$

This relation is apparent even if we just list out the sequences:

$$L_n = L_0,\ \tfrac{L_0}{2},\ \tfrac{L_0}{2},\ \tfrac{L_0}{4},\ \tfrac{L_0}{4},\ \tfrac{L_0}{8},\ \tfrac{L_0}{8},$$
$$W_n = W_0, W_0, \tfrac{W_0}{2}, \tfrac{W_0}{2}, \tfrac{W_0}{4}, \tfrac{W_0}{4}, \tfrac{W_0}{8},$$

This property is an important one, because it allows us to discover the ratio, or *aspect ratio* between the two sides of the paper. We must do this prior to the next section on the scaling factor, because without knowing the aspect ratio of the sides, there is no way for us to do proper scaling conversions. Hence, let R_n be the aspect ratio of A_n

$$
\begin{aligned}
R_n &= \frac{L_n}{W_n} \\
R_{n+1} &= \frac{L_{n+1}}{W_{n+1}} \\
&= \frac{W_n}{L_n \div 2} \\
&= \frac{2W_n}{L_n} \\
&= \frac{2}{L_n \div W_n}
\end{aligned}
$$

Originally, I hoped that somehow the terms would cancel out and I would be able to have a numerical solution, but that didn't happen. However, not all was lost, because in the process of working out this math I came to another realisation. Now recall that we defined R_n as $\frac{L_n}{W_n}$. Therefore:

$$
\begin{aligned}
R_{n+1} &= \frac{2}{L_n \div W_n} \\
&= \frac{2}{R_n}
\end{aligned}
$$

Do you recall how in the Lecture, we were told that one of the properties of the A-series paper, was how each time you fold it in half it preserves the proportions of the previous paper? This is a property that is unique to the ISO 216 paper, if you take a U.S. Letter paper and fold it in half, every time you fold it you end up with a slightly different rectangle. Now, there is only one way that this property can happen — namely that the aspect ratio of each subsequent halve must be the same, e.g $R_n = R_{n+1} = R_{n+2}$ for $n \to \infty$ where the aspect ratio is $1 : R_n$ Therefore:

$$
R_{n+1} = \frac{2}{R_n} = R_n
$$

Now we have the simple numerical solution that I was looking for! We can find the value of R_n simply by solving for it:

$$\begin{aligned} R_n &= \frac{2}{R_n} \\ R_n \times R_n &= 2 \\ (R_n)^2 &= 2 \\ R_n &= \sqrt{2} \end{aligned}$$

Therefore, the aspect ratio of the ISO 216 A-series paper is $1 : \sqrt{2}$. What does this mean? It means the ratio of the length to the width of the paper is that of $1 : \sqrt{2}$, or expressed in Euclidean terms:[1]

$$L_n : W_n :: 1 : \sqrt{2}$$

Exact Algebraic Solutions to Lengths and Widths

Now that we have laid the required groundwork in place, we can find the exact (and numerical approximations) solutions of the length and width for any arbitary A_n. This is done using the aspect ratio we discovered in the previous section.

Exact Algebraic Solution for L_0, W_0

First, in order to create an expression that will find the L_n, W_n for any A_n, we must find the L_0, W_0 of A_0. Given $A_0 = L_0 \times W_0$, and that $A_0 = 1$ (as by definition), a system of simulatneous equations of the 2nd order can be used to find the exact algebraic solutions to L_0, W_0:

[1] The Euclidean continuous proportion reads: "As L_n is to W_n, in the manner of 1 to $\sqrt{2}$".

$$
\begin{cases}
L_0 \times W_0 & = 1 \\
W_0 \times \sqrt{2} & = L_0
\end{cases}
$$

Note that $W_0 \times \sqrt{2} = L_0$ holds true due to the aspect ratio, as we discussed in the previous section. Hence, to solve for W_0, we substitute L_0 from the 2nd equation in the system:

$$
\begin{aligned}
W_0 \times \sqrt{2} \times W_0 &= 1 \\
(W_0)^2 \times \sqrt{2} &= 1 \\
(W_0)^2 &= \frac{1}{\sqrt{2}} \\
W_0 &= \sqrt{\frac{1}{\sqrt{2}}} \\
&= \left(2^{-\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= 2^{-\frac{1}{4}} \\
&= \frac{1}{\sqrt[4]{2}}
\end{aligned}
$$

With $W_0 = \frac{1}{\sqrt[4]{2}}$, finding the solution for L_0 is trivial:

$$
\begin{aligned}
L_0 \times \frac{1}{\sqrt[4]{2}} &= 1 \\
\frac{L_0}{1} \times \frac{1}{\sqrt[4]{2}} &= 1 \\
\frac{\sqrt[4]{2}}{\sqrt[4]{2}} &= 1 \\
L_0 &= \sqrt[4]{2}
\end{aligned}
$$

Therefore, we now know the length and width for A_0, namely:

$$L_0 = \sqrt[4]{2}$$
$$W_0 = \frac{1}{\sqrt[4]{2}}$$

And to check if our work is correct, all we have to do is to multiply the length and the width together, and check if they give us an exact integer solution of 1m^2 for A_0's area:

$$L_0 \times W_0 = 1$$
$$\sqrt[4]{2} \times \frac{1}{\sqrt[4]{2}} = 1$$

Generalised Exact Algebraic Solutions

Now that we defined the length and width of A_0, creating an expression or function for the length and width of any subsequent A_n is trivial. The solutions for L_n and W_n can be represented as geometric sequences of the form $a_n = a \times r^{n-1}$, where $r = \frac{1}{\sqrt{2}}$ as each subsequent smaller paper size is effectively divided by the aspect ratio:

$$L_n = \sqrt[4]{2} \times \left(\frac{1}{\sqrt{2}}\right)^n$$
$$W_n = \frac{1}{\sqrt[4]{2}} \times \left(\frac{1}{\sqrt{2}}\right)^n$$

For the purpose of generating the required table of lengths and widths, we will represent those expressions as functions instead:

$$L(n) = \sqrt[4]{2} \times \left(\frac{1}{\sqrt{2}}\right)^n$$
$$W(n) = \frac{1}{\sqrt[4]{2}} \times \left(\frac{1}{\sqrt{2}}\right)^n$$

Generated Table of Values for $A_0 - A_4$

Now that we have the functional form of the sequence notation, we can fulfill part 1 and 2 of the assignment by generating a table of the exact and approximative magnitudes of the length and width.

Note that, in order to save space in the table, we will not use the pretty radical notation of the terms, but rather the more compact (and arguablly, less pretty) exponent notation:

$$L_n = \sqrt[4]{2} \times \left(\frac{1}{\sqrt{2}}\right)^n = 2^{\frac{1}{4} - \frac{n}{2}}$$
$$W_n = \frac{1}{\sqrt[4]{2}} \times \left(\frac{1}{\sqrt{2}}\right)^n = 2^{-\frac{1}{4} - \frac{n}{2}}$$

Table of Exact Values[2]

Paper	Length (m)	Width (m)	Area (m²)
A_0	$2^{\frac{1}{4}}$	$2^{-\frac{1}{4}}$	1
A_1	$2^{-\frac{1}{4}}$	$2^{-\frac{3}{4}}$	$\frac{1}{2}$
A_2	$2^{-\frac{3}{4}}$	$2^{-\frac{5}{4}}$	$\frac{1}{4}$
A_3	$2^{-\frac{5}{4}}$	$2^{-\frac{7}{4}}$	$\frac{1}{8}$
A_4	$2^{-\frac{7}{4}}$	$2^{-\frac{9}{4}}$	$\frac{1}{16}$
A_n	$2^{\frac{1}{4}-\frac{n}{2}}$	$2^{-\frac{1}{4}-\frac{n}{2}}$	$\frac{1}{2^n}$

Table 2: Exact (algebraic) values for any given A_n

Note that I have simplified the values to their simplest exponent form.

Table of Approximative Values

Paper	Length (m)	Width (m)	Area (m²)
A_0	1.1892	0.8408	1
A_1	0.8408	0.5946	0.5
A_2	0.5946	0.4204	0.25
A_3	0.4204	0.2973	0.125
A_4	0.2973	0.2102	0.0625

Table 3: Approximate (numerical) values for any given A_n

Note that the figures are rounded to 3 decimal places.

[2]Note that the numbers following the 2 are exponents, although the lack of space in the table can make them look like regular terms. For example, under the length column for A_2, the entry $2^{-\frac{3}{4}}$ reads "two to the power of negative three over four". Please don't confuse the small numbering with regular terms!

Research credit

Special thanks to Giuseppe Stelluto for his demonstration of an alternative method for deriving the formula for the sequences L_n and W_n. His approach was much more technically rigorous, but I felt that my approach was more accessible to those without a formal education in mathematics.

Image credit

Cover page illustration is a diagram illustrating ISO 216 A-series paper sizes, sourced from Wikipedia under Creative Commons (CC BY-SA 3.0) license.

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