Towards Infinity and Beyond: The Geometric and Convergent Properties of Recursive Pinwheel Triangles

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Contents

1 Introduction

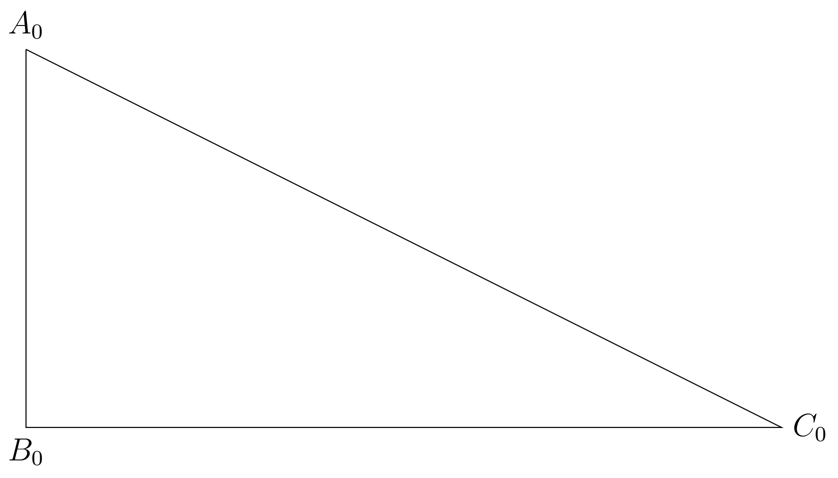
1.1 Rationale

2 Constructions

2.1 Pinwheel tile

1. Draw $\triangle A_0B_0C_0$ where:

$$\begin{aligned} \angle A_0B_0C_0 &= 90^\circ \\ B_0C_0 &= 2A_0B_0 \\ A_0C_0 &= \sqrt{(A_0B_0)^2 + (B_0C_0)^2} \end{aligned} \tag{1}$$



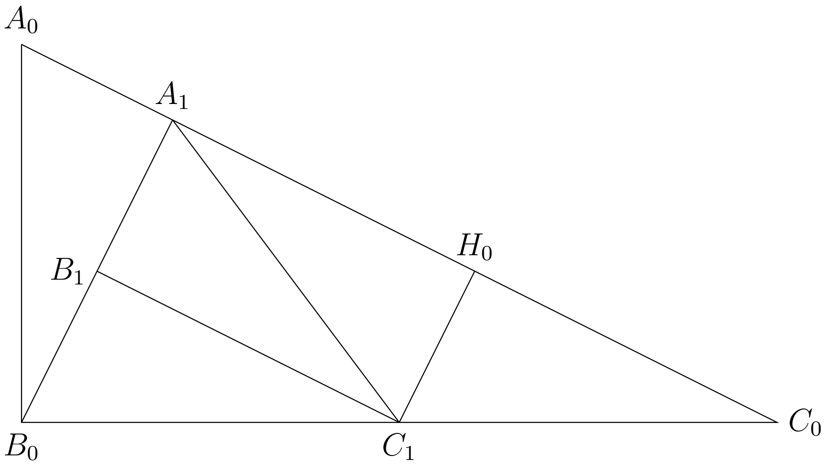
2. Draw line B_0A_1, where $B_0A_1 \perp A_0C_0$.

3. Draw point C_1 at the midpoint of B_0C_0.

4. Draw line C_1H_0, where $C_0H_0 \perp A_0C_0$.

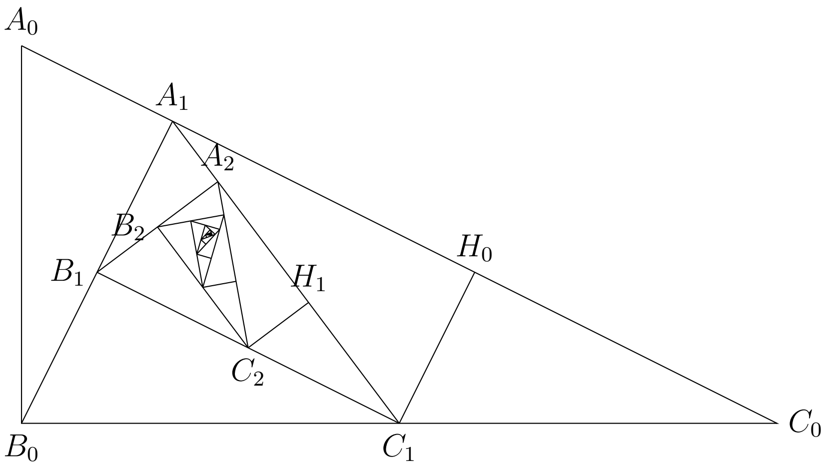
5. Draw line B_1C_1, where $B_1C_1 \perp B_0A_1$.

6. Draw line A_1C_1.



2.2 Recursive pinwheel tiling

Repeat steps 2 through 6, operating on each subsequent $\triangle A_nB_nC_n$.



3 Properties of the recursive pinwheel tiling

3.1 Angles

For each angle in $\triangle A_nB_nC_n$, where the vertices are of the same level n (e.g. $\angle A_6B_6C_6$). This property is apparent becasue each subsequent triangle is similar, therefore all angles are equal.

$$\begin{aligned} \angle A_nB_nC_n &= 90^\circ \\ \angle B_nC_nA_n &= \tan^{-1}\frac{1}{2} \\ \angle B_nA_nC_n &= \tan^{-1} 2 \end{aligned} \tag{2}$$

3.2 Area and dilation factor

Let length $A_0B_0 = x$. Therefore, the area of $\triangle A_0B_0C_0 = x$. Because $\triangle A_0B_0C_0$ contains five isometric subtriangles of the order $\triangle A_1B_1C_1$, the area of each $n = 1$ triangle equals $\frac{x}{5}$. Recall that length $A_1C_1 = A_0C_0$, where: $A_1C_1 = \sqrt{(A_1B_1)^2 + (B_1C_1)^2}$. As a result, it is shown that each shape is shrunk by an dilation factor of $\frac{1}{\sqrt{5}}$, for $\sqrt{5} \times \frac{1}{\sqrt{5}} = 1$:

$$\begin{aligned} A_{n-1}B_{n-1} &= \frac{A_nB_n}{\sqrt{5}} \\ B_{n-1}C_{n-1} &= \frac{B_nC_n}{\sqrt{5}} \\ A_{n-1}C_{n-1} &= \frac{A_nC_n}{\sqrt{5}} \end{aligned} \tag{3}$$

Likewise, the area of each subsequent triangle $\triangle A_nB_nC_n$ is shrunk by the same factor.

3.3 Geometric series, and limit of the area

Define $[\triangle A_nB_nC_n]$ as the notation for the area of $\triangle A_nB_nC_n$. Recall how the area of each triangle decreases by an dilation factor of $\frac{1}{\sqrt{5}}$:

$$\begin{aligned}
\left[\triangle A_0B_0C_0\right] &= 1, \\
\left[\triangle A_1B_1C_1\right] &= \frac{1}{5}, \\
\left[\triangle A_2B_2C_2\right] &= \frac{1}{25}, \\
\left[\triangle A_3B_3C_3\right] &= \frac{1}{125}, \\
&\ldots
\end{aligned} \tag{4}$$

It is apparent that the decreasing area of the pinwheel triangle follow the following geometric series:

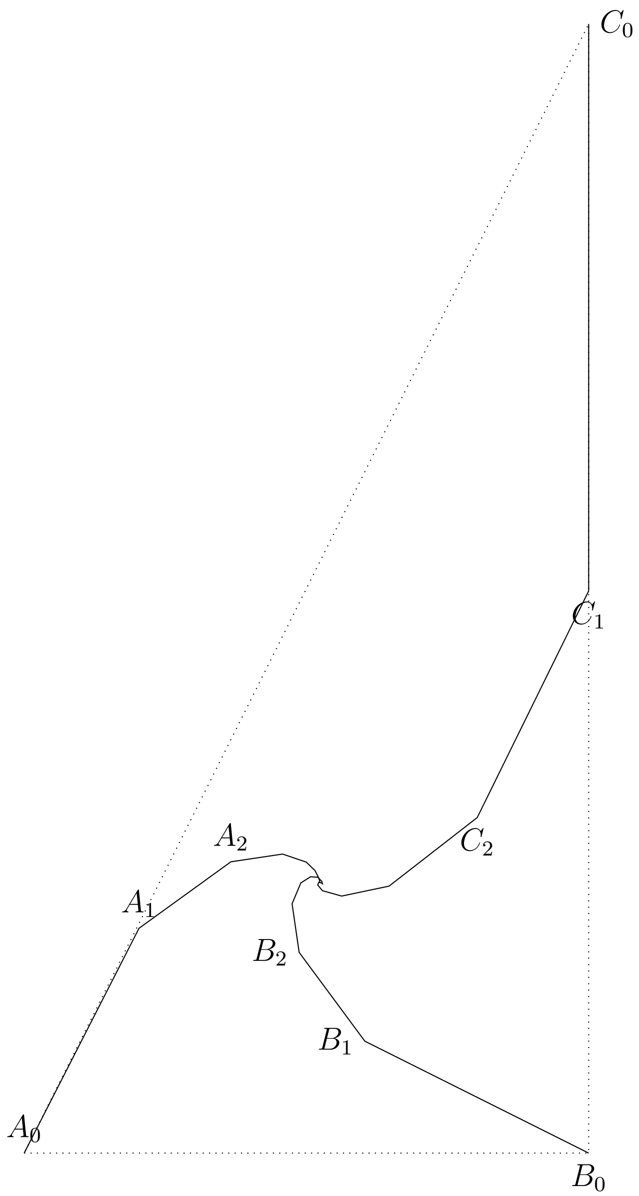
$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \ldots \tag{5}$$

$$1 + 1\left(\frac{1}{5}\right)^1 + 1\left(\frac{1}{5}\right)^2 + 1\left(\frac{1}{5}\right)^3 + 1\left(\frac{1}{5}\right)^4 + \ldots \tag{6}$$

Which generalizes to the form $a + ar^1 + ar^2 + \ldots$, with $a = 1, r = \frac{1}{5}$.

4 Properties of the pinwheel spiral

Note how each self-similar segment in subsequent subtriangles form a spiral with their parent segments. This 'spiral' shall be referred to as the pinwheel spiral:



5 Bibliography

Timm, F. Annette. "Sex with a Purpose: Prostitution, Venereal Disease, and Militarized Masculinity in the Third Reich". *Journal of the History of Sexuality* 11.1/2 (Special Issue: Sexuality and German Fascism Jan. 2002–Apr. 2002): 223–225. Web.