Nets and Filters

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Overview

This is a personal note on nets and filters, derived from a study on functional analysis. The main purpose of this document is to provide a short introduction to the study of well-known topological concepts in the language of nets and filter, rather than sequences and neighborhoods.

1 is a preparatory section, where basic concepts and results, such as maximal filter (net), its characterization, and derivation, etc., are introduced. It soon turns out that derivation is a convenient tool to investigate the properties of nets and filters at the same time. For instance, in 1.5 we see existence and characterization of maximal net immediately follow from the corresponding results on filter through derivation.

2 is our main part, which includes the characterization of various topological concepts, such as continuity, compactness, and Hausdorff separation axiom, in the language of nets and filters. The section closes with an application to compact space, in which filter-based proofs are provided to well-known statements. The highlight is a simple proof of Tychonoff's Thereom.

More extensive treatment on the subject can be found in [1], [2].

Notations

currently editing. the list below is just a sample.

- $\neg$: negation.
- $\vee$: disjunction.
- $\wedge$: conjunction.

1 Net and Filter as Set-Theoretic Objects

1.1 Overview

This section collects basic results on net and filter that can be defined and discussed without any underlying topological structure. The main results are mostly included in 1.2.1, 1.2.4, 1.3 and 1.4, where we investigate the properties of (maximal) filter, and then establish the way to deduce the corresponding results on net from those on filter through derivation. The argument are independent of 1.2.2 and 1.2.3, which are not needed until the subsequent section 2.

1.2 Filter

Filter is a generalization of neighborhood. Although neighborhood is a topological concept, filter is defined purely in a set-theoretic language, as follows.

1.2.1 Definition

Definition 1.1. *(Filter) A nonempty collection $\mathscr{F}$ of subsets of a set X is called a filter of X if it satisfies*

- $\emptyset \notin \mathscr{F}$.
- $X \supset B \supset A \in \mathscr{F} \implies B \in \mathscr{F}$.
- $A, B \in \mathscr{F} \implies A \cap B \in \mathscr{F}$.

Given two filters $\mathscr{F}$ and $\mathscr{G}$ of the same set, $\mathscr{G}$ is called finer than $\mathscr{F}$, or $\mathscr{F}$ is called coarser than $\mathscr{G}$, if $\mathscr{G} \supset \mathscr{F}$, and said to be strictly so if the inclusion is strict.

Remark 1.2. Every filter $\mathscr{F}$ has the following finite intersection property; for every finitely many member $F_1, \ldots, F_n$ of $\mathscr{F}$ there holds $\cap F_i \neq \emptyset$.

As is often the case in mathematics, a filter can be generated by its smaller subset.

Definition 1.3. *(Filter basis) A nonempty collection $\mathscr{B}$ of subset of a set X is called a filter basis if it has the following property;*

- $\emptyset \notin \mathscr{B}$.
- *For $B_1, B_2 \in \mathscr{B}$, there is $B_3 \in \mathscr{B}$ such that $B_3 \subset B_1 \cap B_2$.*

A filter basis $\mathscr{B}$ is called a basis of a filter $\mathscr{F}$ if for every $F \in \mathscr{F}$ there is $B \in \mathscr{B}$ such that $B \subset F$.

Remark 1.4. Given a filter basis $\mathscr{B}$, the filter $\mathscr{F}$ defined by

$$\mathscr{F} = \{F \mid F \supset B, \exists B \in \mathscr{B}\}$$

contains $\mathscr{B}$ as it subcollection. It is easy to see that $\mathscr{B}$ is a basis of $\mathscr{F}$. Whence $\mathscr{B}$ is said to generate $\mathscr{F}$.

A typical set theoretic argument establishes the following minimality property.

Proposition 1.5. *The filter generated by a filter basis is the coarsest filter containing the basis.*

Example 1.6. (Filter generated by a set) $\mathscr{B} := \{A\}$ with $A \neq \emptyset$ is the simplest filter basis. The filter generated by $\mathscr{B}$ is called the filter generated by A, and denoted by $\langle A \rangle$.

Definition 1.7. *(Trace of filter basis) A filter basis is said to have trace on a set A if A intersects every member of the filter basis.*

Definition 1.8. *(Compatibility) Two filter bases $\mathscr{A}$ and $\mathscr{B}$ are said to be compatible if $A \cap B \neq \emptyset$ for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$.*

Proposition 1.9. *(Characterization of compatibility) Two filter bases $\mathscr{F}$ and $\mathscr{G}$ are compatible if and only if they admits a common finer filter.*

Proof. If they are compatible, then

$$\{F \cap G \mid F \in \mathscr{F}, G \in \mathscr{G}\}$$

is a common finer filter.

Conversely, suppose a common finer filter exists. Then every $F \in \mathscr{F}$ and $G \in \mathscr{G}$ is a member of the filter, and thus $F \cap G \neq \emptyset$ necessarily follows from definition of filter. □

Remark 1.10. If a filter $\mathscr{F}$ has trace on A, then we can construct a finer filter since the filter $\langle A \rangle$ generated by A is compatible with $\mathscr{F}$. This procedure yields a strictly finer filter if and only if $A \notin \mathscr{F}$. This fact is frequently exploited.

1.2.2 Transformation of Filter

Definition 1.11. *(Image of filter) Let $f : X \to Y$ be a map. The image of a filter $\mathscr{F}$ of X under f is the filter generated by the filter basis*

$$I_f(\mathscr{F}) := \{f(F) \mid F \in \mathscr{F}\},$$

and is denoted by $f(\mathscr{F})$. The image of a filter basis $\mathscr{B}$ is defined to be $I_f(\mathscr{B})$ itself, and is also denoted by $f(\mathscr{B})$.

Exercise 1.12. Let $f : X \to Y$ be a map, and let $\mathscr{F}$ be a filter of X. If f is surjective, then $f(\mathscr{F}) = I_f(\mathscr{F}) = \{G \subset Y \mid f^{-1}(G) \in \mathscr{F}\}$.

Exercise 1.13. Let $f : X \to Y$ be a map, and let $\mathscr{A}$ and $\mathscr{B}$ be filter bases of X. Then $\mathscr{A} \subset \mathscr{B}$ implies $f(\mathscr{A}) \subset f(\mathscr{B})$, and conversely if f is injective. (Use the fact that $f^{-1}(f(A)) = A$ if f is injective.)

Exercise 1.14. (Inverse image of filter basis) Let $f : X \to Y$ be a map, and let $\mathscr{B}$ be a filter basis of Y. Then $f^{-1}(\mathscr{B}) := \{f^{-1}(B) \mid B \in \mathscr{B}\}$ is a filter basis of X if and only if $\mathscr{B}$ has trace on $f(X)$.

When one of these conditions are fulfilled, $f^{-1}(\mathscr{B})$ is called the inverse image of $\mathscr{B}$ under f. We use the same notation for the filter generated by the filter basis $f^{-1}(\mathscr{B})$.

Exercise 1.15. Let $f : X \to Y$ be a map. Suppose $\mathscr{B}_X$ and $\mathscr{B}_Y$ are filter bases of X and Y, respectively. Then the following relation holds, provided that each of the inverse image is properly defined as a filter basis.

(1) $f(f^{-1}(\mathscr{B}_Y)) \supset \mathscr{B}_Y$.

(2) $f^{-1}(f(\mathscr{B}_X)) \subset \mathscr{B}_X$, with equality if f is injective.

Exercise 1.16. Let $f : X \to Y$ be a surjective map, and let $\mathscr{A}$ and $\mathscr{B}$ be filter bases of Y. Then $\mathscr{A} \subset \mathscr{B}$ if and only if $f^{-1}(\mathscr{A}) \subset f^{-1}(\mathscr{B})$, where the inverse images are considered as filter bases. (Use the fact that $f(f^{-1}(A)) = A$ if f is surjective.)

1.2.3 Induced Filter

Definition 1.17. *(Induced filter) Suppose X is a set, and suppose $(X_i)_{i\in I}$ is an indexed set. Let $\mathscr{F}_i$ be a filter on X_i and $\pi_i : X \to X_i$ a map such that $\pi_i^{-1}(\mathscr{F}_i)$ is a filter basis of X for every $i \in I$. Then the filter $\mathscr{F}$ induced on X by $(\mathscr{F}_i, \pi_i)_{i\in I}$ is the coarsest one satisfying*

$$\pi_i(\mathscr{F}) = \mathscr{F}_i$$

for every $i \in I$.

Example 1.18. (Product filter) In the above definition, take $\pi_i : X \ni x = (x_i) \mapsto x_i \in X_i$ as the natural projection. The resulting filter is called the *product filter* of $\mathscr{F}_i$, and is denoted by $\prod \mathscr{F}_i$. Note that each π_i is surjective, and therefore the inverse image of filter is automatically well-defined.

Example 1.19. (Restriction of filter onto trace) Suppose a filter $\mathscr{F}$ of X has trace on A. Then the pair of $\mathscr{F}$ and an inclusion map $i_A : A \to X$ fulfills the condition of Exercise 1.14. The resulting induced filter of A given by $i_A^{-1}(\mathscr{F}) = \{F \cap A \mid F \in \mathscr{F}\}$ is called the *restriction* of $\mathscr{F}$ onto A.

Proposition 1.20. *(Characterization of induced filter) Let $\mathscr{F}$ be a filter of X induced by the pair $(\mathscr{F}_i, \pi_i)_{i\in I}$ as in the above definition. Then $\mathscr{F}$ is generated by a filter basis of X consisting of all subsets of the form $\cap_{i\in J}\pi_i^{-1}(F_i)$, where J is a finite subset of I and $F_i \in \mathscr{F}_i$.*

Proof. Let $\mathscr{B}$ be the filter basis indicated in the claim, and let $\mathscr{G}$ be the filter generated by $\mathscr{B}$. It suffice to show that $\mathscr{F} = \mathscr{G}$.

Suppose $A \in \mathscr{F}$. It follows from definition that, for each i, there holds $\pi_i(A) = F_i$ for some $F_i \in \mathscr{F}_i$, and hence $A \supset \pi_i^{-1}(F_i)$, in which the last term is a member of the basis of $\mathscr{G}$. Thus, $\mathscr{F} \subset \mathscr{G}$. To prove the opposite inclusion, we claim $\pi_i(\mathscr{B}) \subset \mathscr{F}_i$ for every index i, from which we deduce that $\pi_i(\mathscr{G}) \subset \mathscr{F}_i$ for every index i, and thus $\mathscr{G} \subset \mathscr{F}$ by minimality of $\mathscr{G}$. Suppose $A \in \mathscr{B}$, that is, suppose A is of the form

$$A = \bigcap_{j \in J} \pi_j^{-1}(F_j),$$

where J is a finite subset of I, and $F_j \in \mathscr{F}_j$. Then we have

$$\pi_j(A) \subset \bigcap_{j \in J} F_j \subset F_j \in \mathscr{F}_j$$

for every $j \in J$, from which our claim follows. $\square$

1.2.4 Maximal Filter

Definition 1.21. *(Maximal, ultra-filter) A filter $\mathscr{F}$ of X is called a maximal (or ultra) filter if it admits no strictly finer filter of X.*

Proposition 1.22. *(Existence of maximal filter) If a collection of sets has the finite intersection property, then there exists a maximal filter containing it.*

Proof. Let $\mathscr{A}$ be the collection of sets. Let $\mathscr{C}$ be the collection of all collections of sets such that each $\mathscr{F}_i \in \mathscr{C}$ has the finite intersection property and $\mathscr{F}_i \supset \mathscr{A}$. Introduce an order on $\mathscr{C}$ by the rule $\mathscr{F}_1 \succeq \mathscr{F}_2 \iff \mathscr{F}_1 \supset \mathscr{F}_2$. It is easy to see that $\mathscr{C}$ is partially ordered, and that every totally ordered subset admits a supremum. We then have a maximal element $\mathscr{F}$ of $\mathscr{C}$ by Zorn's Lemma. It remains to show that $\mathscr{F}$ so derived is a filter.

$\emptyset \notin \mathscr{F}$ is obvious. For $B \supset A \in \mathscr{F}$, we see that $\{B\} \cap \mathscr{F} \in \mathscr{C}$ and hence $B \in \mathscr{F}$ by maximality. For $A, B \in \mathscr{F}$, we similarly have $\{A \cap B\} \cup \mathscr{F} \in \mathscr{C}$ by the finite intersection property, and thus $A \cap B \in \mathscr{F}$ again by maximality. $\square$

Theorem 1.23. *(Characterization of maximal filter) Let $\mathscr{F}$ be a filter in a set X. The following conditions are equivalent.*

(1) $\mathscr{F}$ is maximal.

(2) Every set intersecting every member of $\mathscr{F}$ belongs to $\mathscr{F}$.

(3) For any subset A of X, either A or $X \setminus A$ belongs to $\mathscr{F}$.

Proof. (1) $\implies$ (2): Let A be such intersecting subset and let $\mathscr{F}$ be maximal. Let $\mathscr{C}$ be the collection of sets consisting of the sets C such that $C \supset A \cap B$ for some $B \in \mathscr{F}$. It is easy to check that $\mathscr{C}$ is a filter, and that there holds $\mathscr{F} \subset \mathscr{C}$ and $A \in \mathscr{C}$. It then follows from maximality that $\mathscr{F} = \mathscr{C}$, and hence $A \in \mathscr{F}$.

(2) $\implies$ (3): Suppose (3) fails, that is, suppose $A \notin \mathscr{F}$ and $X \setminus A \in \mathscr{F}$ for some $A \subset X$. It necessarily follows that $F \nsubseteq A$ and $F \nsubseteq X \setminus A$ for every $F \in \mathscr{F}$. Then $A \cap F \neq \emptyset$ and $(X \setminus A) \cap F \neq \emptyset$ for every $F \in \mathscr{F}$. Thus, (2) fails.

(3) $\implies$ (1): Suppose $A \in \mathscr{G} \supset \mathscr{F}$, where $\mathscr{G}$ is a filter. It necessarily follows from the definition of filter that $X \setminus A \notin \mathscr{G}$, and hence $X \setminus A \notin \mathscr{F}$. Thus, $A \in \mathscr{F}$ by Assumption. $\square$

Proposition 1.24. *(Invariance of filter maximality under surjection) Let $f : X \to Y$ be a surjective map. If a filter $\mathscr{F}$ of X is maximal, so is $f(\mathscr{F})$.*

Proof. Let $G \subset Y$. By maximality, there holds either $f^{-1}(G) \in \mathscr{F}$ or not. If this is the case, it follows that $f^{-1}(G) = F$ for some $F \in \mathscr{F}$, and hence $G \supset f(f^{-1}(G)) = f(F)$, implying $G \in f(\mathscr{F})$. If not, we have $X \setminus f^{-1}(G) = f^{-1}(X \setminus G) \in \mathscr{F}$. Thus, $X \setminus G \in f(\mathscr{F})$. $\square$

Example 1.25. (Counterexample to the converse) Here we see how the converse of Proposition 1.24 fails. Let $\mathscr{N}$ be the set of open intervals over 0 in real line:

$$\mathscr{N} := \{(a, b) \subset \mathbb{R} \mid a < 0 < b\}.$$

Observe that, by Proposition 1.23, $\mathscr{N}$ is not maximal while $\mathscr{N} \cup \{0\}$ is. Define $f : \mathbb{R} \to \mathbb{R}_+$ via

$$f(x) := \max\{|x| - 1, 0\}.$$

Then we have

$$f(\mathscr{N}) = \{[0, c) \subset \mathbb{R}_+ \mid c > 0\} \cup \{0\} = f(\mathscr{N} \cup \{0\}).$$

Proposition 1.24 shows that $f(\mathscr{N}) = f(\mathscr{N} \cup \{0\})$ is maximal.

Proposition 1.26. *(Representation of filter) Every filter is the intersection of all finer maximal filters.*

Proof. Suppose a filter $\mathscr{F}$ is given. It is obvious that the intersection is finer than $\mathscr{F}$. Conversely, suppose $A \notin \mathscr{F}$. Then $\mathscr{F}$ has trance on $X \setminus A$ since A contains no members of $\mathscr{F}$, from which it follows that $\mathscr{F}$ admits a finer filter that contains $X \setminus A$. Thus, every maximal filter finer than $\mathscr{F}$ necessarily contains $X \setminus A$. $\square$

1.3 Net

Net is a generalization of sequence, whose index set is (possibly uncountable) a directed set.

Definition 1.27. *(Net) Let Λ be a directed set and let X be a set. A net (on Λ) is a mapping $x : \Lambda \ni \lambda \mapsto x_\lambda \in X$, and is denoted by $(x_\lambda)_{\lambda \in \Lambda}$ or x_λ, or sometimes simply by x if there is no fear of confusion.*

Definition 1.28. *(Eventually and frequently) Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net of a directed set Λ into a set X. Suppose A is a subset of X.*

- *x is said to be eventually in A or residual in A if there is $\lambda_0 \in \Lambda$ such that $x_\lambda \in A$ for all $\lambda > \lambda_0$.*
- *x is said to be frequently in A or cofinal in A if for every $\lambda_0 \in \Lambda$ there is $\lambda > \lambda_0$ such that $x_\lambda \in A$.*

- *x_λ is called a maximal net or an ultra-net if x is eventually either in A or $X \setminus A$ for every $A \subset X$.*

Definition 1.29. *(Subnet, $\succeq$) Let x_λ be a net in a set X. We say that a net $(s_d)_{d \in D}$ is a subnet of x_λ, and write $s \succeq x$, if for every $\lambda \in \Lambda$ there is $d \in D$ such that*

$$s(D_d) \subset x(\Lambda_\lambda),$$

where $D_d := \{e \in D \mid e > d\}$ and Λ_λ is defined similarly.

If s is a subnet of a net x but not the converse, then s is called a proper subnet of x.

Lemma 1.30. *A net g is a subnet of a net f if and only if g is eventually in A whenever f is eventually in A.*

Definition 1.31. *(Relation of subnet: $\sim$) Two nets x and y are called equivalent, and are denoted by $x \sim y$, if they are subnets of each other, that is, if $x \succeq y$ and $y \succeq x$.*

Remark 1.32. It is obvious that the relation $\succeq$ and the equivalence relation $\sim$ fulfill the axiom of partial order and the axiom of equivalence relation, respectively.

Definition 1.33. *(Maximal net) A net is called maximal if it admits no proper subnet.*

1.4 Derivation

In this subsection we see that every filter and net has its essentially unique twin. Each of individual filter and net turns into the corresponding twin by the conversion called *derivation*, and then comes back to the original one by another derivation. Very importantly, derivation preserves the orders that nets and filters naturally induce, and consequently interesting properties such as maximality and convergence are also preserved.

Definition 1.34. *(Derived net) Let $\mathscr{B}$ be a filter basis, and let $\mathscr{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ be the generated filter. A net x_λ derived from the filter basis $\mathscr{B}$ is a mapping*

$$x : \Lambda \ni \lambda \mapsto x_\lambda \in F_\lambda \subset X,$$

where Λ is directed by the partial order $>$ defined by $\lambda_1 > \lambda_2 \iff F_{\lambda_1} \subset F_{\lambda_2}$. We also say that $\mathscr{F}$ (or $\mathscr{B}$) generates x.

Definition 1.35. *(Derived filter) The filter derived from a net x is the collection of sets in which x eventually lies, and is denoted by $\mathscr{F}_x$. We also say that x generates $\mathscr{F}_x$.*

Remark 1.36. Note that derived filter is uniquely specified as a filter once a net is designated. In contrast, derived net no way specifies unique filter except in trivial cases. In fact, derived nets in general consist of a collection of projection nets of indexing set of a filter into a member of the filter.

You can safely skip the next result. Actually, we will shortly establish several results that make it trivial (see Remark 1.45). The statement is placed here just for an exercise.

Exercise 1.37. (Invariance of maximality under derivation)

- Every net derived from a maximal filter is maximal.
- Conversely, the filter derived from a maximal net is maximal.

Proof. Suppose $\mathscr{F} = \{F_\lambda\}$ is a maximal filter. For every $A \subset X$, there holds either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$. That is, every F_λ is of the form either A or $X \setminus A$. Thus, every derived net is eventually in either A or $X \setminus A$.

Conversely, suppose a net x is maximal, that is, x is eventually in either A or $X \setminus A$ for every $A \subset X$. Then, by definition, the derived filter $\mathscr{F}$ consists of A or $X \setminus A$. Thus, $\mathscr{F}$ is maximal. $\square$

The next Lemma immediately follows from definition.

Lemma 1.38. *(relation induced by eventual behavior) Two nets are equivalent if they share the same derived filter, that is, if they eventually lie in the same set.*

Lemma 1.39. *(Idempotent property of derivation)*

(1) If x is a net derived from a filter $\mathscr{F}$, then $\mathscr{F} = \mathscr{F}_x$.

(2) If $\mathscr{F}_x$ is the filter derived from a net x, then every net f derived from $\mathscr{F}_x$ is equivalent to x.

Proof. (1): Write $x = (x_\lambda)$. Observe $A \in \mathscr{F}_x$ if and only if x is eventually in A. This is the case if and only if for every λ_0 we have $x_\lambda \in A$ for all $\lambda > \lambda_0$. This holds if and only if there exists a subset $F_\lambda \in \mathscr{F}$ such that $x_\lambda \in F_\lambda \subset A$, from which it follows that $A \in \mathscr{F}$, and conversely (consider contraposition).

(2): Note that the filter $\mathscr{F}$ derived from f coincides with $\mathscr{F}_x$ by (1). Thus x and f are equivalent by Lemma 1.38. $\square$

The following theorem and its corollary are the highlight of this section. The proof is almost obvious since it is just a rephrase of the above Lemma.

Theorem 1.40. *(Fundamental theorem of derivation)*

(1) For every filter there exists a net that generates the filter. Moreover, the net is unique up to the subnet-equivalence modulo.

(2) For every net there exists unique filter that generates a net equivalent to the given net.

Proof. Lemma 1.39 has establishes the existence part. It remains to show the uniqueness. To prove the first claim, let $\mathscr{F}$ be a filter, and suppose two nets x and y generates the identical filter $\mathscr{F}$. But this is just rephrase of $x \sim y$ by definition. Similarly, the second claim immediately follows from Lemma 1.39. $\square$

Remark 1.41. (Implication of fundamental theorem of derivation) Theorem 1.40 says that derivation works like an idempotent bijection between the set of filters and that of equivalence classes of nets, as we have suggested at the beginning of the section. It is thus natural to introduce a notation that represents the paired relationship, as follows.

Definition 1.42. *(Equivalence of net and filter: $\simeq$) We say that a net x and a filter $\mathscr{F}$ are equivalent, and write $x \simeq \mathscr{F}$, if $\mathscr{F}_x = \mathscr{F}$.*

Corollary 1.43. *(Invariance of order under derivation) Suppose filters $\mathscr{F}$, $\mathscr{G}$ and nets f, g have the following equivalence relations:*

$$\mathscr{F} \simeq f, \quad \mathscr{G} \simeq g.$$

Then we have the following equivalence:

$$\mathscr{G} \supset \mathscr{F} \iff g \succeq f.$$

Proof. By Theorem 1.40, we may assume that $\mathscr{F}$ and $\mathscr{G}$ are derived from f and g, respectively. g is a subnet of f if and only if, by Lemma 1.30, g is eventually in A whenever f is eventually in A. It then follows from Theorem 1.40 and definition of derived filter that $\mathscr{G} = \mathscr{F}_g \supset \mathscr{F}_f = \mathscr{F}$, and conversely. $\square$

Proposition 1.44. *(Invariance of equivalence under mapping) Let $f : X \to Y$ be a map, and let x be a net and $\mathscr{F}$ a filter. Then*

$$x \simeq \mathscr{F} \implies f(x) \simeq f(\mathscr{F}).$$

Proof. By Theorem 1.40, we may assume that the filter is derived by the net, that is, $\mathscr{F} = \mathscr{F}_x$. It suffices to show that

$$f^{-1}(A) \in \mathscr{F}_x \iff A \in f(\mathscr{F}_x).$$

($\implies$) is obvious. The converse is also obvious for a filter basis $\mathscr{B} = I_f(\mathscr{F}_x)$ that generates $f(\mathscr{F}_x)$. It therefore follows that for every $A \in f(\mathscr{F}_x)$ we can choose $B \in \mathscr{B}$ such that $B \subset A$ and $f^{-1}(B) \in \mathscr{F}_x$, which gives $f^{-1}(A) \in \mathscr{F}_x$. $\square$

1.5 Application of Derivation: Maximal Net

Derivation makes it easy to translate results on filter into those on net, and vice versa. We demonstrate it with several results regarding maximal net here.

Remark 1.45. Look at Exercise 1.37 you might have skipped, which claims that maximality is invariant under derivation. Now the claim is an immediate consequence of Corollary 1.43 and Proposition 1.46.

Theorem 1.46. *(Characterization of maximal net) Let x be a net. The following conditions are equivalent.*

(1) x is eventually either in A or $X \setminus A$ for every $A \subset X$.

(2) x admits no proper subsets.

(3) The derived filter $\mathscr{F}_x$ is maximal.

Proof. (1) $\iff$ (3) by Exercise 1.37.

(2) $\iff$ (3): By Corollary 1.43, x admits no proper subnets if and only if x is equivalent to a maximal filter. Thus $\mathscr{F}_x$ is maximal, and conversely. ☐

The existence of a maximal subnet has become almost trivial.

Corollary 1.47. *(Existence of maximal subnet) Every net admits a subnet which has no proper subnet (or equivalently, is maximal).*

Proof. For every net x, the derived filter $\mathscr{F}_x$ admits a maximal filter $\mathscr{F}^\star$ by Proposition 1.22, which in turn admits a corresponding maximal net y by Exercise 1.37. Whence we have

$$x \simeq \mathscr{F}_x \subset \mathscr{F}^\star \simeq y,$$

and hence $x \succeq y$ by Corollary 1.43. This means y is a subnet of x. ☐

2 Net and Filter in Topology

In this section we always assume that an underlying universal set is equipped with a topological structure.

2.1 Overview

In this section we see that various topological concepts are re-captured and sometimes even characterized by the convergence and the cluster point of net and filter. After basic concepts and associated results are introduced in 2.2, a series of the characterizations is shown in 2.3. The closing content 2.4 is an application to compact space, where filter-based proofs are provided to well-known statements.

2.2 Convergence and Cluster Point

2.2.1 Convergence and Cluster Point of Filter

Definition 2.1. *(Convergence of filter) We say that filter $\mathscr{F}$ converges to a point p, and write $\mathscr{F} \to p$, if every neighborhood N of p belongs to $\mathscr{F}$.*

Remark 2.2. Note (verify!) that $\mathscr{F} \to p$ if and only if for every neighborhood N of p there is $F \in \mathscr{F}$ such that $F \subset N$.

Definition 2.3. *(Cluster point of filter) A point p is called a cluster point of a filter $\mathscr{F}$ if p lies in the closure of every member of $\mathscr{F}$.*

Definition 2.4. *(Convergence and cluster point of filter basis) We say that a filter basis $\mathscr{B}$ converges to a point p, and write $\mathscr{B} \to p$, if for every neighborhood N of p there is $B \in \mathscr{B}$ such that $B \subset N$.*

A point p is called a cluster point of a filter basis $\mathscr{B}$ if $p \in \overline{B}$ for every $B \in \mathscr{B}$.

Remark 2.5. Here we adopt, as definition of convergence of filter basis, the necessary and sufficient condition of convergence of filter. See remark 2.2.

Remark 2.6. Clearly, p is a cluster point of a filter basis $\mathscr{B}$ if and only if $\mathscr{B}$ has trace on every neighborhood of p.

Example 2.7. (basis of neighborhoods as a filter) Suppose $\mathscr{N}(p)$ is a basis of neighborhoods of a point p of a topological space. $\{p\} \cup \mathscr{N}(p)$ is a filter basis, but not necessarily a filter. On the other hand, $\mathscr{N}(p)$ is a filter. Both of them converge to p.

In general, $\mathscr{F} \to p$ implies $\mathscr{G} \to p$ for a filter $\mathscr{G} \supset \mathscr{F}$, and not conversely. The following proposition gives a special example in which the converse holds.

Proposition 2.8. *Suppose a filter basis $\mathscr{B}$ generates a filter $\mathscr{F}$. Then $\mathscr{B} \to p$ if and only if $\mathscr{F} \to p$.*

Proof. Suppose $\mathscr{F} \to p$. By definition, for every neighborhood N of p there is $F \in \mathscr{F}$ such that $F \subset N$. For this F there is $B \in \mathscr{B}$ such that $B \subset N$ since $\mathscr{B}$ generates $\mathscr{F}$. Thus, $\mathscr{B} \to p$. The converse is trivial. □

Lemma 2.9. *(Relation of convergence and cluster point) Let $\mathscr{F}$ be a filter and p be a point. If $\mathscr{F} \to p$, then p is a cluster point of $\mathscr{F}$. Moreover, the converse is also true if $\mathscr{F}$ is a maximal filter. Thus, if $\mathscr{F}^\star$ is a maximal filter, p is a cluster point of $\mathscr{F}^\star$ precisely when $\mathscr{F}^\star \to p$.*

Proof. Suppose $\mathscr{F} \to p$. Let $A \in \mathscr{F}$. Since every neighborhood N of p belongs to $\mathscr{F}$, we have, by definition, $A \cap N \neq \emptyset$, and hence $p \in \overline{A}$. Thus, p is a cluster point of A.

Conversely, suppose p is a cluster point of a maximal filter $\mathscr{F}$. Then, by definition of cluster point, every neighborhood N of p intersects every member of $\mathscr{F}$. By Theorem 1.23, we have $N \in \mathscr{F}$, and therefore $\mathscr{F} \to p$. □

2.2.2 Convergence and Cluster Point of Net

Definition 2.10. *(Convergence and cluster point of net)*

- *A net x_λ is said to converge to a point p, and is denoted by $x_\lambda \to p$, if x_λ is eventually in every neighborhood of p.*
- *A point p is called a cluster point of a net x_λ if x is frequently in every neighborhood of p.*

Exercise 2.11. (Convergence of subnet) If a net converges to a point, so is every subnet of it.

Theorem 2.12. *(Invariance of convergence under derivation) Let $\mathscr{F}$ be a filter and let x be a net. Suppose p is a point. If $\mathscr{F} \simeq x$, then we have the following equivalence:*

$$\mathscr{F} \to p \iff x \to p.$$

Proof. By Theorem 1.40, we may assume that $\mathscr{F}$ is derived from x. For every neighborhood N of p, we see that $x \to p$ implies x is eventually in N, and hence $N \in \mathscr{F}$, and thus $\mathscr{F} \to p$, and conversely. □

An analogous argument to the sequence case shows the following.

Exercise 2.13. (Frequent-eventual relation) A net x is frequently in a set A if and only if x admits a subnet that is eventually in A.

2.3 Characterization of Topological Properties

Proposition 2.14. *(Characterization of cluster point) Suppose p is a point of a topological space X. Let x be a net of X, and let $\mathscr{F}$ be the filter of X equivalent to x. The following conditions are equivalent.*

(1) p is a cluster point of x.

(2) p is a cluster point of $\mathscr{F}$.

(3) x admits a subnet converging to p.

(4) There is a filter $\mathscr{G}$ such that $\mathscr{G} \supset \mathscr{F}$ and $\mathscr{G} \to p$.

Proof. (1) $\iff$ (3): By Proposition 2.13.

(3) $\iff$ (4): By invariance of order (Corollary 1.43) and invariance of convergence (Theorem 2.12).

(2) $\iff$ (4): p is a cluster point of $\mathscr{F}$ precisely when $\mathscr{F}$ has trace on every $N \in \mathscr{N}$, where $\mathscr{N}$ is a basis of neighborhood of p. This is the case precisely when $\mathscr{F}$ and $\mathscr{N}$ are compatible, which is true precisely when (4) holds by Proposition 1.9. □

Proposition 2.15. *(Characterization of neighborhood) Suppose p is a point of a topological space X, and suppose U is a subset of X. The following conditions are equivalent.*

(1) U is a neighborhood of p.

(2) $U \cap \{x\} \neq \emptyset$ for every net x converging to p.

(3) Every filter converging to p contains U as its member.

Proof. (1) $\implies$ (2): Obvious.

(2) $\implies$ (3): Let $\mathscr{F}$ be a filter with $U \notin \mathscr{F} \to p$. Every net derived x from F converges to p but $U \cap \{x\} = \emptyset$.

(3) $\implies$ (1): Let $\mathscr{N} := \mathscr{N}(p)$ be the set of all neighborhoods of p. Clearly, $\mathscr{N} \to p$, and thus $U \in \mathscr{N}$ by assumption. □

Proposition 2.16. *(Characterization of closure) Suppose A is a subset of a topological space X. Let p be a point in X. The following conditions are equivalent.*

(1) $p \in \overline{A}$.

(2) There is a net of A converging to p.

(3) p is a cluster point of a filter of A.

(4) There is a filter basis of A converging to p.

Proof. Let $\mathscr{N}(p)$ be the set of all neighborhoods of p. (1) is true if and only if $\mathscr{N}(p)$ has trace on A. This is the case if and only if (4) holds. Other equivalences are easy to deduce. □

Exercise 2.17. (Characterization of closed sets) For a set A in a topological space X, the following conditions are equivalent.

(1) A is closed.

(2) If a filter $\mathscr{F}$ of A converges to a point p, then $p \in A$.

Exercise 2.18. (Continuous mapping of cluster point) Suppose $f : X \to Y$ is continuous, and suppose a filter $\mathscr{F}$ has a cluster point p. Then the filter $f(\mathscr{F})$ has $f(p)$ as one of its cluster points.

Proposition 2.19. *(Characterization of continuity) Let $f : X \to Y$ be a map. The following conditions are equivalent.*

(1) f is continuous at a point p.

(2) For any filter basis $\mathscr{B}$ of X, if $\mathscr{B} \to p$, then $f(\mathscr{B}) \to f(p)$.

(3) For any filter $\mathscr{F}$ of X, if $\mathscr{F} \to p$, then $f(\mathscr{F}) \to f(p)$.

(4) For any net x of X, if $x \to p$, then $f(x) \to f(p)$.

Proof. (1) $\implies$ (2): Suppose N is a neighborhood of $f(p)$. By assumption, there is a neighborhood U of p such that $f(U) \subset N$. But by continuity, there is $B \in \mathscr{B}$ such that $B \subset U$, and thus $f(B) \subset f(U) \subset N$.

(2) $\implies$ (3): Obvious.

(3) $\implies$ (4): Since $\mathscr{F}_x \to p$ by assumption and Proposition 2.12, it follows from (3) and Exercise 1.44 that $f(\mathscr{F}_x) \simeq f(x) \to p$.

(4) $\implies$ (1): Take net as an usual sequence. □

Proposition 2.20. *(Characterization of compactness) Let X be a topological space. The following conditions are equivalent.*

(1) X is compact.

(2) Every closed collection of subsets of X with the finite intersection property has a nonempty intersection.

(3) Every filter of X has a cluster point.

(4) Every maximal filter is convergent.

Proof. (1) $\iff$ (2): Every elementary course on topology should contain this result, so the proof is left to the reader.

(2) $\implies$ (3): For every filter $\mathscr{F}$, define

$$\overline{\mathscr{F}} := \{\overline{F} \mid F \in \mathscr{F}\}.$$

$\overline{\mathscr{F}}$ is obviously a closed filter with the finite intersection property. Then (2) implies that there is a point p such that $p \in \overline{F}$ for all $F \in \mathscr{F}$. Thus, p is a cluster point of $\mathscr{F}$.

(3) $\implies$ (4): Apply Lemma 2.9.

(4) $\implies$ (2): Let $\mathscr{C}$ be the collection of closed sets with the finite intersection property. By Proposition 1.22, there is a finer maximal filter, which is convergent by assumption. Suppose p is a limit point of it. Then p is a cluster point of the filter basis $\mathscr{C}$. Thus, $p \in \overline{C} = C$ for every $C \in \mathscr{C}$. □

Exercise 2.21. Show (4) $\implies$ (1) directly in the Proposition.

Proof. Suppose (4) holds. We prove that *every covering $\mathscr{U}$ of X that admits no finite subcovering is not open.* Let $\mathscr{U}$ be such a covering. Observe that

$$\mathscr{B} := \{X \setminus U \mid U \in \mathscr{U}\}$$

has the finite intersection property. We can therefore, by Proposition 1.22, construct a maximal filter $\mathscr{F} \supset \mathscr{B}$. By (4), we have $\mathscr{F} \to p$ for some $p \in X$, which is a cluster point of $\mathscr{F}$ by Proposition 2.14. Then, by definition of cluster point, we have $p \in \overline{F}$ for every $F \in \mathscr{F}$, in particular, $p \in \overline{X \setminus U}$ for every $U \in \mathscr{U}$. On the other hand, we have $p \in U$, that is, $p \notin X \setminus U$ for some $U \in \mathscr{U}$ since $\mathscr{U}$ is a covering of the whole space. This means $\overline{X \setminus U} \supsetneq X \setminus U$ for some $U \in \mathscr{U}$. Thus, $\mathscr{U}$ is not open. □

Proposition 2.22. *(Characterization of Hausdorff separation axiom) The following conditions about a topological space X are equivalent;*

(1) X is a Hausdorff space.

(2) Every filter basis of X converges at most one point.

(3) Every filter basis of X has at most one cluster point.

Proof. Let $\mathscr{N}(p)$ be a basis of neighborhoods of p. (1) $\implies$ (2): Suppose a filter basis $\mathscr{B}$ converges to a point p. Pick a point q with $q \neq p$. Then there exist $U \in \mathscr{N}(p)$ and $V \in \mathscr{N}(q)$ such that $U \cap V = \emptyset$. Convergence of $\mathscr{F}$ implies that there is $B \in \mathscr{B}$ such that $B \subset U$ and there is no $B \in \mathscr{B}$ such that $B \subset V$. Thus, $\mathscr{B}$ converges to no point but p.

(2) $\implies$ (3): Obvious by Proposition 2.14.

(3) $\implies$ (1): In a non-Hausdorff space, there are two distinct point p and q such that $\mathscr{N}(p)$ and $\mathscr{N}(q)$ are compatible. Then, p and q are cluster points. □

Theorem 2.23. *(Characterization of convergence of induced filter in initial topology) Suppose X is the topological space equipped with the initial topology induced by topological spaces $(X_\lambda)_{\lambda\in\Lambda}$ and maps $(\pi_\lambda : X \to X_\lambda)_{\lambda\in\Lambda}$, that is, the coarsest topology with respect to which every π_λ is continuous. Let $(\mathscr{F}_\lambda)_{\lambda\in\Lambda}$ be filters such that each $\pi_\lambda^{-1}(\mathscr{F}_\lambda)$ is a filter basis. Let $\mathscr{F}$ be the filter of X induced by $(\mathscr{F}_\lambda)$ and (π_λ).*

(1) Each $\mathscr{F}_\lambda$ converges to a point p_λ if and only if $\mathscr{F}$ converges to $p := (p_\lambda)$.

(2) Each $\mathscr{F}_\lambda$ admits a cluster point p_λ if and only if $\mathscr{F}$ admits a cluster point p.

Proof. (1): Sufficiency follows from continuity of π_λ. Conversely, suppose each $\mathscr{F}_\lambda$ converges to p_λ. Let N be a neighborhood of $p := (p_\lambda)_{\lambda\in\Lambda}$ in X. By construction of initial topology, there are finite subset J of I such that

$$N \supset \bigcap_{j\in J} \pi_j^{-1}(N_j),$$

where each N_j is a neighborhood of p_j. There also exists $F_j \in \mathscr{F}_j$ such that $F_j \subset N_j$ for each $j \in J$. This implies a basis of $\mathscr{F}$ converges to p by Proposition 1.20.

(2): Sufficiency follows again from continuity of π. Conversely, if p_λ is a cluster point of $\mathscr{F}_\lambda$, then, by Proposition 2.14, there is a filter $\mathscr{G}_\lambda$ that is finer than $\mathscr{F}_\lambda$ and converges to p_λ for every $\lambda \in \Lambda$. Then (1) implies the filter $\mathscr{G}$ induced by $(\mathscr{G}_\lambda, \pi_\lambda)$ converges to (p_λ). Since $\mathscr{G}$ is obviously finer than $\mathscr{F}$, (p_λ) is a cluster point of $\mathscr{F}$ by Proposition 2.14. □

2.4 Compact Space with Filter

Proposition 2.24. *(Invariance of compactness under continuous transformation) Let X be a compact space and Y be a topological space. If $f : X \to Y$ is a surjective and continuous map, then Y is compact.*

Proof. For any filter $\mathscr{G}$ of Y, we see that $f^{-1}(\mathscr{G})$ is a filter of X. By assumption, $f^{-1}(\mathscr{G})$ admits a cluster point p in X, which is, by Exercise 2.18, also a cluster point of $f(f^{-1}(\mathscr{G})) \supset \mathscr{G}$. This implies that $\mathscr{G}$ also admits a cluster point, and hence Y is compact. □

Proposition 2.25. *(invariance of compactness under closed intersection) Every closed set of a compact space is compact.*

Proof. Let $\mathscr{F}$ be a filter of a closed subset of B of a compact space. By assumption, $\mathscr{F}$ admits a cluster point belonging to B. Thus, B is compact. □

Proposition 2.26. *Every compact set of a Hausdorff space is closed.*

Proof. Suppose a filter $\mathscr{F}$ of a compact set B of a Hausdorff space X converges to a point p in X. By compactness, $\mathscr{F}$ has a cluster point q in B. By proposition

2.14, there is a finer filter $\mathscr{G}$ than $\mathscr{F}$ such that $\mathscr{G} \to q$ in B, as well as in X as a filter basis. Then Proposition 2.22 implies $p = q \in B$. Thus, B is closed by Exercise 2.17. $\square$

Proposition 2.27. *Let X be a compact space and Y a Hausdorff space. Then every continuous bijection of X into Y is homeomorphism.*

Proof. It suffice to show that f is a closed mapping. Suppose F is a closed set of X. Then, by by Proposition 2.25, it is compact, whose image under f is compact by Proposition 2.24, and thus closed by Proposition 2.26. $\square$

Theorem 2.28. *(Tychonoff) Suppose a set X is equipped with the initial topology induced by topological spaces $(X_\lambda)_{\lambda\in\Lambda}$ and surjective maps $(\pi_\lambda : X \to X_\lambda)_{\lambda\in\Lambda}$. Then X is compact if and only if each X_λ is compact.*

Proof. Necessity is obvious. Conversely, suppose each X_λ is compact. Let $\mathscr{F}$ be a maximal filter of X. By Proposition 1.24, $\mathscr{F}_\lambda := \pi_\lambda(\mathscr{F})$ is also maximal. Proposition 2.20 implies each $\mathscr{F}_\lambda$ is convergent; suppose $\mathscr{F}_\lambda \to p_\lambda$. It then follows from Proposition 2.23 that $\prod \mathscr{F}_\lambda \to p := (p_\lambda)$, and thus $\mathscr{F} \to p$. $\square$

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