# Contents



## About This Document

This is a set of solutions to the selected exercises in Chapter 1 of the book[1] by Ioannis Karatzas and Steven E. Shreve. All statements of the exercises in this document are borrowed from the book, and the solutions are taken from the presentation material I used in a seminar. Any suggestions for improvement would be appreciated.

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https://github.com/Shena4746/Preliminaries-for-BM-and-SC

# Chapter 1

# Martingales, Stopping Times, and Filtrations.

## 1.1 Stochastic Processes and $\sigma$-Fields

**Problem 1.5.** Let $Y$ be a modification of $X$, and suppose that both processes have a.s. right-continuous sample paths. Then $X$ and $Y$ are indistinguishable.

*Proof.* Set

$$\begin{aligned}
A &:= \{\omega \in \Omega \mid t \mapsto X_t(\omega) \text{ is not right-continuous}\}, \\
B &:= \{\omega \in \Omega \mid t \mapsto Y_t(\omega) \text{ is not right-continuous}\}, \\
M_t &:= \{\omega \in \Omega \mid X_t(\omega) \neq Y_t(\omega)\}, \\
M &:= \bigcup_{q \in \mathbb{Q} \cap [0,\infty)} M_q, \\
N &:= A \cup B \cup M.
\end{aligned}$$

$P(N) = 0$ by assumption, and note that

$$X_q(\omega) = Y_q(\omega) \quad \forall q \in \mathbb{Q} \cap [0,\infty) \quad \forall \omega \notin N.$$

On $\Omega \setminus N$, for any $t \in [0,\infty)$ choose $\{q_n\}_{n=1}^{\infty} \in \mathbb{Q} \cap [0,\infty)$ such that $q_n \downarrow t$ $(n \to \infty)$, and then by right-continuity

$$X_t = \lim_{n\to\infty} X_{q_n} = \lim_{n\to\infty} Y_{q_n} = Y_t.$$

Since $P(N) = 0$, the result follows.

**Exercise 1.7.** Let $X$ be a stochastic process, every sample path of which is RCLL. Let $A$ be the event that $X$ is continuous on $[0, t_0)$. Show that $A \in \mathscr{F}_{t_0}^X$.

*Proof.* Observe that

$$\begin{aligned}
A^c &= \{\omega \in \Omega \mid t \mapsto X_t(\omega) \text{ is not continuous on } [0,t_0)\} \\
&= \{\exists s \in [0,t_0) \text{ s.t. } X_{s-} \neq X_s\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{\substack{q_1,q_2 \in \mathbb{Q}\cap[0,t_0) \\ |q_1-q_2| \ll \frac{1}{m}}} \{|X_{q_1} - X_{q_2}| \gg \frac{1}{n}\}.
\end{aligned}$$

Thus

$$A^c \in \sigma\Big(\bigcup_{q \in \mathbb{Q}\cap[0,t_0)} \mathscr{F}_q^X\Big) \subset \mathscr{F}_{t_0}^X.$$

**Exercise 1.8.** Let $X$ be a stochastic process whose sample paths are RCLL almost surely, and let $B$ be the event that $X$ is continuous on $[0,t_0)$. Show that if $\{\mathscr{F}_t\ ;\ t \geq 0\}$ is a filtration satisfying $\mathscr{F}_t^X \subset \mathscr{F}_t$, $t \geq 0$, and $\mathscr{F}_{t_0}$ contains all $P$-null sets of $\mathscr{F}$, then $B \in \mathscr{F}_{t_0}$.

*Proof.* Let $N$ be the set on which $X$ is not RCLL. By Assumption, $N \in \mathscr{F}_{t_0}$, and hence, with $A$ in Exercise 1.7, $B = A \cap N^c \in \mathscr{F}_{t_0}$.

**Exercise 1.10.** Let $X$ be a process with every sample path LCRL, and let A be the event that $X$ is continuous on $[0,t_0]$. Let $X$ be adapted to a right-continuous filtration $\{\mathscr{F}_t\}$. Show that $A \in \mathscr{F}_{t_0}$.

*Proof.* Note that $X_{t+}$ is $\mathscr{F}_{t+}$-measurable, and then see the proof of Exercise 1.7.

**Problem 1.16.** If the process X is jointly measurable and the random time $T$ is finite, then the function $X_T$ is a random variable.

*Proof.* Let $J$ be the map defined by

$$J : \Omega \ni \omega \mapsto (T(\omega),\omega) \in [0,\infty) \times \Omega$$

so that $J$ is measurable $\mathscr{F}/\mathscr{B}[0,\infty) \otimes \mathscr{F}$. Since $X$ is measurable $\mathscr{B}[0,\infty) \otimes \mathscr{F}/\mathscr{B}(\mathbb{R}^d)$ by assumption, it follows that $X_T = X \circ J$ is measurable $\mathscr{F}/\mathscr{B}(\mathbb{R}^d)$.

**Problem 1.17.** Let $X$ be a jointly measurable process and $T$ a random time. Show that the collection of all sets of the form $\{X_T \in A\}$ and $\{X_T \in A\} \cup \{T = \infty\}$; $A \in \mathscr{B}(\mathbb{R})$, forms a sub-$\sigma$-field of $\mathscr{F}$.

*Proof.* Define

$$\mathscr{G} := \{\{X_T \in A\},\ \{X_T \in A\} \cup \{T = \infty\} \mid A \in \mathscr{B}(\mathbb{R})\},$$

and note that

$$\{X_T \in A\} = \{X_{T(\omega)}(\omega) \in A\} \cap \{T \ll \infty\}.$$

Clearly $\emptyset \in \mathscr{G}$. Let $G \in \mathscr{G}$. Then $G = \{X_T \in A\}$ or $\{X_T \in A\} \cup \{T = \infty\}$ for some $A \in \mathscr{B}(\mathbb{R})$, and so $G^c = \{X_{T(\omega)}(\omega) \in A^c\} \cup \{T = \infty\}$ or $\{X_T \in A^c\} \cap \{T \ll \infty\}$; $G^c$ is in $\mathscr{G}$. Similar argument shows that $\mathscr{G}$ is closed under countable union.

## 1.2 Stopping times

**Problem 2.2.** Let $X$ be a stochastic process and $T$ a stopping time of $\{\mathscr{F}_t^X\}$. Suppose that for some pair $\omega, \omega' \in \Omega$, we have $X_t(\omega) = X_t(\omega')$ for all $t \in [0, T(\omega)] \cap [0, \infty)$. Show that $T(\omega) = T(\omega')$.

*Proof.* Suppose $T(\omega') \leq T(\omega)$. Choose $t \in (T(\omega'), T(\omega))$ so that $\omega \in \{t \leq T\} \in \mathscr{F}_t^X$. Note that $\{t \leq T\} = \{(X_{t_1}, X_{t_2}, \cdots) \in A\}$ for some $\{t_j\}_{j=1}^{\infty} \subset [0, t]$ and $A \in \mathscr{B}(\mathbb{R}^\infty)$. Since $t \in [0, T(\omega)] \cap [0, \infty)$ and $\omega \in \{X_t \in A\}$, we have, by assumption, $\omega'$ is also in the set $\{(X_{t_1}, X_{t_2}, \cdots) \in A\} = \{t \leq T\}$, a contradiction. For $T(\omega) \leq T(\omega')$, taking $t = T(\omega)$ leads to similar contradiction. Thus $T(\omega) = T(\omega')$.

**Problem 2.6.** If the set $\Gamma$ in Example 2.5 is open, show that $H_\Gamma$ is an optional time.

*Proof.* It suffices to show that

$$\{H_\Gamma \leq t\} = \bigcup_{\substack{q \leq t \\ q \in \mathbb{Q} \cap [0,t]}} \{X_q \in \Gamma\}.$$

$\supset$ is trivial. Conversely, let $\omega$ be such that $H_\Gamma(\omega) \leq t$. There exists $s \in (H_\Gamma(\omega), t)$ such that $X_s(\omega) \in \Gamma$. Since $\Gamma$ is open, $B(X_s(\omega), \epsilon) \subset \Gamma$ for small $\epsilon \gg 0$. Using right-continuity, pick $\delta \gg 0$ such that if $r \in (s, s + \delta)$, then $X_r(\omega) \in B(X_s(\omega), \epsilon)$. In particular, we can choose $r = q \in \mathbb{Q}$.

**Problem 2.7.** If the set $\Gamma$ in Example 2.5 is closed and the sample paths of the process $X$ are continuous, then $H_\Gamma$ is a stopping time.

*Proof.* Observe that since $\omega \mapsto X_q(\omega)$ is measurable $\mathscr{F}_q/\mathscr{B}(\mathbb{R})$ and $x \mapsto d(x, \Gamma)$ is measurable $\mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$ (and continuous), $\omega \mapsto d(X_q(\omega), \Gamma)$ is measurable $\mathscr{F}_q/\mathscr{B}(\mathbb{R})$.

Set $R(t, \omega) := \{X_s(\omega)\,;\, s \in [0, t]\}$, which is compact since $X$ is continuous. Then for each $t \geq 0$

$$H_\Gamma \leq t \;\Leftrightarrow\; R(t) \cap \Gamma \neq \emptyset \;\Leftrightarrow\; d(R(t), \Gamma) = 0 \;\Leftrightarrow\; \inf_{q \in [0,t] \cap \mathbb{Q}} d(X_q, \Gamma) = 0,$$

and the result follows.

**Problem 2.10.** Let $T$, $S$ be optional times; then $T + S$ is optional. It is a stopping time, if one of the following condition holds:

(1) $T \gg 0$, $S \gg 0$;

(2) $T \gg 0$, $T$ is a stopping time.

*Proof.* By Lemma 2.9, $T + S$ is $\mathscr{F}_{t+}$-stopping time, and so it is $\mathscr{F}_t$-optional time.

For (1), in view of Lemma 2.9, following fact yields the result; for $q \in \mathbb{Q}_+ \cap (0,t)$

$$\{q \leq T \leq t, S \gg t-q\} = [\{T \leq t\} \setminus \{T \leq q\}] \cap \{S+q \gg t\},$$

and $S+q$ is a stopping time (by Lemma 2.8). Now (2) is easy.

**Problem 2.13.** Verify that $\mathscr{F}_T$ is actually a $\sigma$-field and $T$ is $\mathscr{F}_T$-measurable. Show that if $T(\omega) = t_0$ for some constant $t_0 \geq 0$ and every $\omega \in \Omega$, then $\mathscr{F}_T = \mathscr{F}_{t_0}$.

*Proof.* Clearly, $\mathscr{F}_T$ is closed under countable union, and $\emptyset \in \mathscr{F}_T$. Observe that

$$\{T \leq t\} \cap \{A \cap \{T \leq t\}\}^c = A^c \cap \{T \leq t\},$$

which proves $\mathscr{F}_T$ is a $\sigma$-field.

For $s \geq 0$ and $t \geq 0$,

$$\{T \leq s\} \cap \{T \leq t\} \in \mathscr{F}_t,$$

which is obvious, since for $s \geq t \geq 0$, $\{T \leq s\} \cap \{T \leq t\} = \{T \leq t\} \in \mathscr{F}_t$, and for $t \gg s \geq 0$, $\{T \leq s\} \cap \{T \leq t\} = \{T \leq s\} \in \mathscr{F}_s \subset \mathscr{F}_t$.

Third claim follows from the following observation;

$$\mathscr{F}_T = \{A \in \mathscr{F} \ ; \ A \cap \{T \leq t\} \in \mathscr{F}_t \quad \forall t \geq t_0\} = \{A \in \mathscr{F} \ ; \ A \in \mathscr{F}_t \quad \forall t \geq t_0\} = \mathscr{F}_{t_0}.$$

**Exercise 2.14.** Let $T$ be a stopping time and $S$ a random time such that $S \geq T$ on $\Omega$. If $S$ is $\mathscr{F}_T$-measurable, then it is also a stopping time.

*Proof.* Note that for every $t \geq 0$

$$\{S \leq t\} = \{S \leq t\} \cap \{S \geq T\} = \{S \leq t\} \cap \{S \geq T\} \cap \{T\, let\} = \{S \leq t\} \cap \{T \leq t\},$$

and the result follows.

**Problem 2.17.** Let $T$, $S$ be stopping times and $Z$ an integrable random variable. We have

(1) $E(Z|\mathscr{F}_T) = E(Z|\mathscr{F}_{T \wedge S})$, $P$-a.s. on $\{T \leq S\}$.

(2) $E[E(Z|\mathscr{F}_T)|\mathscr{F}_S] = E(Z|\mathscr{F}_{T \wedge S})$, $P$-a.s.

*Proof.* For $A \in \mathscr{F}_T$, $A \cap \{T \leq S\} \in \mathscr{F}_S$ (Lemma 2.15), and $\in \mathscr{F}_T$ (Lemma 2.16), and so $\in \mathscr{F}_{T \wedge S}$. Consequently,

$$\begin{aligned}
\int_A 1_{\{T \leq S\}} E(Z|\mathscr{F}_{T \wedge S}) dP &= \int_{A \cap \{T \leq S\}} E(Z|\mathscr{F}_{T \wedge S}) dP \\
&= \int_{A \cap \{T \leq S\}} Z dP \\
&= \int_{A \cap \{T \leq S\}} E(Z|\mathscr{F}_T) dP \\
&= \int_A 1_{\{T \leq S\}} E(Z|\mathscr{F}_T) dP,
\end{aligned}$$

and (1) follows.

For (2), using (1), we find that, with probability 1,

$$\begin{aligned}
1_{\{T\le S\}}E[E(Z|\mathscr{F}_T)|\mathscr{F}_S] &= E[1_{\{T\le S\}}E(Z|\mathscr{F}_T)|\mathscr{F}_S] \\
&= E[1_{\{T\le S\}}E(Z|\mathscr{F}_{T\wedge S})|\mathscr{F}_S] \\
&= 1_{\{T\le S\}}E[E(Z|\mathscr{F}_{T\wedge S})|\mathscr{F}_S] \\
&= 1_{\{T\le S\}}E(Z|\mathscr{F}_{T\wedge S}).
\end{aligned}$$

We also conclude form (1) that, with probability 1,

$$\begin{aligned}
1_{\{S\le T\}}E[E(Z|\mathscr{F}_T)|\mathscr{F}_S] &= 1_{\{S\le T\}}E[E(Z|\mathscr{F}_T)|\mathscr{F}_{S\wedge T}] \\
&= 1_{\{S\le T\}}E(Z|\mathscr{F}_{S\wedge T}),
\end{aligned}$$

and (2) follows.

**Problem 2.19.** Let $X = \{X_t, \mathscr{F}_t; 0 \le t \le \infty\}$ be a progressively measurable process, and let $T$ be a $\mathscr{F}_t$-stopping time, and $f(t,x) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ be a bounded, jointly measurable function. Show that the process $Y_t = \int_0^t f(s,X_s)ds$; $t \ge 0$ is progressively measurable with respect to $\mathscr{F}_t$, and $Y_T$ is an $\mathscr{F}_T$ measurable random variable.

*Proof.* By Proposition 2.18, it suffices to show that $Y_t$ is progressively measurable. Fix $t \ge 0$. It is easy to show that $f(s,X_s)$ is progressively measurable, and hence $Y_s$ is well-defined, and that $Y_s$ is continuous in $s$ (dominated convergence theorem). For $n \ge 1$ and $k = 0, 1, \cdots, nt-1$, define

$$Y_n(s,\omega) := \sum_{k=0}^{nt-1} Y(k/n,\omega)1_{(k/n,(k+1)/n]}(s),$$

with $Y_n(0,\omega) = Y_0(\omega)$. Clearly $Y_n$ is progressively measurable, and by continuity, $Y_n \to Y$ for each $(s,\omega)$, which establishes the result.

**Problem 2.21.**

(1) $\mathscr{F}_{T+}$ is indeed a $\sigma$-field.

(2) $T$ is $\mathscr{F}_{T+}$-measurable.

(3) $\mathscr{F}_{T+} = \{A \in \mathscr{F} \mid A \cap \{T \le t\} \in \mathscr{F}_t,\ \forall t \ge 0\} (=: \mathscr{G}_T)$.

(4) If $T$ is a stopping time (so that $\mathscr{F}_T$, $\mathscr{F}_{T+}$ are defined), then $\mathscr{F}_T \subset \mathscr{F}_{T+}$.

*Proof.* (1): Copy the proof of Problem 2.13.
(2): Follows form (3).
(3): Let $A \in \mathscr{G}_T$, then $A \cap \{T \le t + \frac{1}{n}\} \in \mathscr{F}_{t+\frac{1}{n}}$ for all $n \ge 1$, from which we deduce that $A \cap \{T \le t\} \in \mathscr{F}_{t+}$. Thus $A \in \mathscr{F}_{T+}$. Conversely, Let $A \in \mathscr{F}_{T+}$, then $A \cap \{T \le t - \frac{1}{n}\} \in \mathscr{F}_{t-\frac{1}{n}} \subset \mathscr{F}_t$, which implies $A \cap \{T \le t\} \in \mathscr{F}_t$. Thus $A \in \mathscr{G}_t$
(4): Obvious from the fact that $A$ in $\mathscr{F}_T$ satisfies $A \cap \{T \le t\} \in \mathscr{F}_t \subset \mathscr{F}_{t+}$.

**Problem 2.22.** Analogues of Lemma 2.15 and Lemma 2.16 hold for optional times as stated below.

**Lemma 1.2.1 (2.15′).** *For any two optional times $T$ and $S$, and for any $A \in \mathscr{F}_{S+}$, we have $A \cap \{S \leq T\} \in \mathscr{F}_{T+}$. In particular, if $S \leq T$ on $\Omega$, we have $\mathscr{F}_{S+} \subset \mathscr{F}_{T+}$.*

**Lemma 1.2.2 (2.16′).** *Let $T$ and $S$ be optional times. Then $\mathscr{F}_{(T\wedge S)+} = \mathscr{F}_{T+} \cap \mathscr{F}_{S+}$, and each of the events*

$$\{T \leq S\},\ \{S \leq T\},\ \{T \leq S\},\ \{S \leq T\},\ \{T = S\}$$

*belongs to $\mathscr{F}_{T+} \cap \mathscr{F}_{S+}$.*

**Problem 2.22** (continued)**.** Prove that if $S$ is an optional time and $T$ is a positive stopping time with $S \leq T$, and $S \leq T$ on $\{S \leq \infty\}$, then $\mathscr{F}_{S+} \subset \mathscr{F}_{T}$.

*Proof.* Let $A \in \mathscr{F}_{S+}$; $A \cap \{S \leq t\} \in \mathscr{F}_t$, $\forall t \geq 0$. Use the following decomposition

$$A = [\bigcup_{q \in \mathbb{Q}_+} [A \cap \{S \leq q \leq T\}]] \cup [A \cap \{S = \infty\}],$$

and note that, for every $t \geq 0$ and for each fixed $q \in \mathbb{Q}_+$,

$$A \cap \{S \leq q \leq T\} \cap \{T \leq t\} = [A \cap \{S \leq q\}] \cap \{t \geq T \gg q\} \in \mathscr{F}_t,$$
$$A \cap \{S = \infty\} \cap \{T \leq t\} = A \cap \{S = \infty\} \cap \{T = \infty\} \cap \{T \leq t\} = \emptyset,$$

which proves the result.

**Problem 2.23.** Show that if $\{T_n\}_{n=1}^{\infty}$ is a sequence of optional times and $T = \inf_{n \geq 1} T_n$, then $\mathscr{F}_{T+} = \bigcap_{n=1}^{\infty} \mathscr{F}_{T_n+}$. Besides, if each $T_n$ is a positive stopping time and $T \leq T_n$ on $\{T \leq \infty\}$, then we have $\mathscr{F}_{T+} = \bigcap_{n=1}^{\infty} \mathscr{F}_{T_n}$.

*Proof.* $\mathscr{F}_{T+} \subset \mathscr{F}_{T_n+}$, since $T \leq T_n$, $\forall n$; hence $\mathscr{F}_{T+} \subset \bigcap_{n=1}^{\infty} \mathscr{F}_{T_n+}$. Conversely, let $A \in \bigcap_{n=1}^{\infty} \mathscr{F}_{T_n+}$, i.e. $A \cap \{T_n \leq t\} \in \mathscr{F}_t$, $\forall n \geq 1, \forall t \geq 0$. Then

$$A \cap \{T \leq t\} = A \cap [\bigcup_{n=1}^{\infty} \{T_n \leq t\}] = \bigcup_{n=1}^{\infty} [A \cap \{T_n \leq t\}] \in \mathscr{F}_t.$$

Thus $A \in \mathscr{F}_{T+}$

For the second claim, similar argument shows that $\bigcap_{n=1}^{\infty} \mathscr{F}_{T_n} \subset \mathscr{F}_{T+}$, and for the the other direction, use proposition 2.22 with $T = T_n$, $S = T$.

**Problem 2.24.** Given an $\mathscr{F}_t$-optional time $T$, consider the sequence of random time given by

$$T_n(\omega) = \begin{cases} T(\omega); & \{\omega;\ T(\omega) = +\infty\} \\ \frac{k}{2^n}; & \{\omega;\ \frac{k-1}{2^n} \leq T(\omega) \leq \frac{k}{2^n}\} \end{cases}$$

for $n \geq 1$, $k \geq 1$. Obviously $T_n \geq T_{n+1} \geq T$, for every $n \geq 1$. Show that each $T_n$ is a stopping time, that $\lim_{n\to\infty} T_n = T$, and for every $A \in \mathscr{F}_{T+}$ we have $A \cap \{T_n = k/2^n\} \in \mathscr{F}_{k/2^n}$; $n, k \geq 1$.

*Proof.* For any $n \geq 1$ and $t \geq 0$, we can find some $k \geq 1$ such that $\frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}$; whence

$$\{T_n \leq t\} = \{T \leq \frac{k}{2^n}\} \in \mathscr{F}_{k/2^n} \subset \mathscr{F}_t.$$

Thus each $T_n$ is a stopping time. The following observation completes the proof.

$$|T_n(\omega) - T(\omega)| \leq \frac{1}{2^n}, \quad \forall \omega \in \Omega \setminus \{T(\omega) = +\infty\}, \; \forall n \geq 1,$$
$$A \cap \{T_n = \frac{k}{2^n}\} = [A \cap \{\frac{k-1}{2^n} \leq T(\omega) \leq \frac{k}{2^n}\}.]$$

## 1.3 Continuous-Time Martingales

**Problem 3.2.** Let $T_1, T_2, \cdots$ be a sequence of independent, exponentially distributed random variable with parameter $\lambda \gg 0$:

$$P(T_i \in dt) = \lambda e^{-\lambda t} dt, \quad t \geq 0.$$

Let $S_0 := 0$ and $S_n := \sum_{i=1}^n T_i$; $n \geq 1$. Define a continuous-time, integer-valued RCLL process

$$N_t := \max\{n \geq 0;\ S_n \leq t\}; \quad 0 \leq t \leq \infty.$$

(1) Show that for $0 \leq s \leq t$ we have

$$P(S_{N_s+1} \gg t \mid \mathscr{F}_s^N) = e^{-\lambda(t-s)} \quad \text{a.s.}$$

(2) Show that for $0 \leq s \leq t$, $N_t - N_s$ is a Poisson random variable with parameter $\lambda(t-s)$, independent of $\mathscr{F}_s^N$.

**Problem 3.4.** Prove that a compensated Poisson process $\{M_t, \mathscr{F}_t;\ t \geq 0\}$ is a martingale.

### A. Fundamental Inequalities

**Problem 3.7.** Let $\{X_t = (X_t^{(1)}, \cdots, X_t^{(d)}), \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ be a vector of martingales, and $\varphi : \mathbb{R}^d \to \mathbb{R}$ a convex function with $E|\varphi(X_t)| \ll \infty$ valid for every $t \geq 0$. Then $\{\varphi(X_t), \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ is a submartingale.

**Problem 3.9.** Let $N$ be a Poisson process with intensity $\lambda$.

(1) For any $c \gg 0$,

$$\overline{\lim_{n\to\infty}} P\Big[\sup_{0\leq s\leq t}(N_s - \lambda s) \geq c\sqrt{\lambda t}\Big] \leq \frac{1}{c\sqrt{2\pi}}.$$

(2) For any $c \gg 0$,

$$\overline{\lim_{n\to\infty}} P\Big[\inf_{0\leq s\leq t}(N_s - \lambda s) \leq -c\sqrt{\lambda t}\Big] \leq \frac{1}{c\sqrt{2\pi}}.$$

(3) For $0 \ll a \ll b$, we have

$$E\Big[\sup_{a\leq t\leq b}\Big(\frac{N_t}{t}-\lambda\Big)^2\Big] \leq \frac{4b\lambda}{a^2}.$$

**Problem 3.11.** Let $\{\mathscr{F}_n\}_{n=1}^{\infty}$ be a decreasing sequence of sub-$\sigma$-field of $\mathscr{F}$, and let $\{X_n, \mathscr{F}_n;\ n \geq 1\}$ be a *backward submartingale*; i.e., $E|X_n| \ll \infty$, $X_n$ is $\mathscr{F}_n$-measurable, and $E(X_n|\mathscr{F}_{n+1}) \geq X_{n+1}$ a.s. for every $n \geq 1$. Then $l := \lim_{n\to\infty} E(X_n) \gg -\infty$ implies that the sequence $\{X_n\}_{n=1}^{\infty}$ is uniformly integrable.

**B. Convergence Results**

**Problem 3.12.** Let $\{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ be a right-continuous, nonnegative supermartingale; then $X_\infty := \lim_{n\to\infty} X_t$ exists a.s. and $\{X_t, \mathscr{F}_t;\ 0 \leq t \leq \infty\}$ is a supermartingale.

*Proof.* Since $\{E(X_t)\}$ is bounded by $E(X_0)$, we have $X_t \to \exists X_\infty$ a.s. Then, for $0 \leq s \ll t$ and $A \in \mathscr{F}_s$, we have

$$E(X_s\ ;\ A) \geq E(X_t\ ;\ A).$$

Letting $t \to \infty$, Fatou's Lemma implies

$$E(X_s\ ;\ A) \geq E(X_\infty\ ;\ A).$$

Thus

$$X_s \geq E(X_\infty|\mathscr{F}_s) \quad \text{a.s.}$$

**Exercise 3.18.** Suppose that the filtration $\{\mathscr{F}_t\}$ satisfies the usual conditions. Then every right-continuous, uniformly integrable supermartingale $\{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ admits the *Riesz decomposition* $X_t = M_t + Z_t$ a.s. as the sum of a right-continuous uniformly integrable martingale $\{M_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ and a potential $\{Z_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$.

*Proof.* First, we construct $M_t$. UI property guarantees the a.s. existence of $X_\infty := \lim_{t\to\infty} X_t$. Define $X_\infty(\omega) := 0$ for bad $\omega$, so that $X_\infty$ always exists. Clearly, $\{X_t, \mathscr{F}_t;\ 0 \leq t \leq \infty\}$ is a RC, UI supermartingale. Consider $\{E(X_\infty|\mathscr{F}_t), \mathscr{F}_t;\ 0 \leq t \ll \infty$. Clearly it is a UI martingale, and $t \mapsto EX_t$ is RC since it has constant expectation. Thus The Regularity Theorem implies that there exists a version $M_t$ of $W_t$ such that $\{M_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ is a RC martingale, and so derived $\{M_t\}$ is obviously UI.

Finally we construct $Z_t$. We claim that

$$P(M_t \leq X_t \quad \forall t \in [0,\infty)) = 1.$$

Indeed, $\mathscr{F}_0$-measurable sets defined by

$$\Omega_1 := \{E(X_\infty|\mathscr{F}_t) \leq X_t \quad \forall t \in \mathbb{Q}+\}$$

$$\Omega_2 := \{E(X_\infty|\mathscr{F}_t) = M_t \quad \forall t \in \mathbb{Q}+\}$$
$$\Omega_3 := \{-\infty \ll X_t, M_t \ll \infty \quad \forall t \in [0,\infty)\}$$

have probability 1, and the same is true for $\Omega^\star := \Omega_1 \cap \Omega_2 \cap \Omega_3$. For $\omega \in \Omega^\star$ we have

$$M_t(\omega) = \lim_{n\to\infty} M_{t_n}(\omega) = \lim_{n\to\infty} E(X_\infty|\mathscr{F}_{t_n})(\omega) \le \lim_{n\to\infty} X_{t_n}(\omega) = X_t(\omega)$$

for every $t \ge 0$ and $\{t_n\} \subset \mathbb{Q}+$ with $t_n \downarrow t$. Set

$$Z_t := \begin{cases} X_t - M_t & ; \ \omega \in \Omega^\star \\ 0 & ; \ \omega \notin \Omega^\star. \end{cases}$$

It is easy to check that $\{Z_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$ is actually a potential.

**Problem 3.19.** The following three conditions are equivalent for a nonnegative right-continuous submartingale $\{X_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$:

(1) it is a uniformly integrable family of random variables;

(2) it converges in $\mathscr{L}^1$, as $n \to \infty$;

(3) it converges a.s. to an integrable random variable $X_\infty$, such that $\{X_t, \mathscr{F}_t;\ 0 \le t \le \infty\}$ is a submartingale.

*Proof.* (1) $\Rightarrow$ (2) : Since $\{X_t\}$ is $\mathscr{L}^1$-bounded, we have $X_t \to \exists X_\infty$ a.s. UI property implies that $X_t \to X_\infty$ in $\mathscr{L}^1$.

(2) $\Rightarrow$ (3) : Since $\{E(|X_t|)\}$ is convergent (hence it is bounded), we have $X_t \to \exists X_\infty$ a.s. and the rest is easy.

(3) $\Rightarrow$ (1) : Since $X_t$ is nonnegative, $X_t$ is dominated by $E(X_\infty|\mathscr{F}_t)$, which is UI. Thus $X_t$ is also UI.

**Problem 3.20.** The following four conditions are equivalent for a continuous martingale $\{X_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$:

(1), (2) as in Problem 3.19;

(3) it converges a.s. to an integrable random variable $X_\infty$, such that $\{X_t, \mathscr{F}_t;\ 0 \le t \le \infty\}$ is a martingale;

(4) there exists an integrable random variable $Y$, such that $X_t = E(Y|\mathscr{F}_t)$ a.s. for every $t \ge 0$.

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : the same as before. (3) $\Rightarrow$ (4) : Set $Y = X_\infty$.
(4) $\Rightarrow$ (1) : Clear.

**Problem 3.21.** Let $\{N_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$ be a Poisson process with parameter $\lambda \gg 0$. For $u \in \mathbb{C}$, define the process

$$X_t := \exp[iuN_t - \lambda t(e^{iu} - 1)]; \quad 0 \le t \ll \infty.$$

(1) Show that $\{\mathrm{Re}(X_t), \mathscr{F}_t;\ 0 \le t \ll \infty\}$, $\{\mathrm{Im}(X_t), \mathscr{F}_t;\ 0 \le t \ll \infty\}$ are martingales.

(2) Consider $X$ with $u = -i$. Does this martingale satisfy the equivalent conditions of Problem 3.20?

**C. The Optional Sampling Theorem**

**Problem 3.23.** Establish the optional sampling theorem for a right-continuous submartingale $\{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ and optional times $S \leq T$ under either of the following two conditions:

(1) $T$ is a *bounded* optional time (there exists a number $a \gg 0$, such that $T \leq a$)

(2) there exists an integrable random variable $Y$, such that $X_t \leq E(Y|\mathscr{F}_t)$ a.s. for every $t \geq 0$.

**Problem 3.24.** Suppose that $\{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ is a right-continuous submartingale and $S \leq T$ are stopping times of $\{\mathscr{F}_t\}$. Then

(1) $\{X_{T\wedge t}, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ is a submartingale;

(2) $E(X_{T\wedge t}|\mathscr{F}_S) \geq X_{S\wedge t}$ a.s. for every $t \geq 0$.

**Problem 3.25.** A submartingale of constant expectation, i.e., with $E(X_t) = E(X_0)$ for every $t \geq 0$, is a martingale.

**Problem 3.26.** A right-continuous process $X = \{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ with $E|X_t| \ll \infty;\ 0 \leq t \ll \infty$ is a submartingale if and only if for every pair $S \leq T$ of bounded stopping times of the filtration $\{\mathscr{F}_t\}$ we have

$$E(X_T) \geq E(X_S).$$

**Problem 3.27.** Let $T$ be a bounded stopping time of the filtration $\{\mathscr{F}_t\}$, which satisfies the usual conditions, and define $\tilde{\mathscr{F}}_t := \mathscr{F}_{T+t};\ t \geq 0$. Then $\{\tilde{\mathscr{F}}_t\}$ also satisfies the usual conditions.

(1) If $X = \{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ is a right-continuous submartingale, then so is $\tilde{X} = \{\tilde{X}_t := X_{T+t} - X_t, \tilde{\mathscr{F}}_t\};\ 0 \leq t \ll \infty\}$.

(2) If $\tilde{X} = \{\tilde{X}_t, \tilde{\mathscr{F}}_t\ ;\ 0 \leq t \ll \infty\}$ is a right-continuous submartingale with $X_0 := 0$ a.s. then $X = \{X_t := \tilde{X}_{(t-T)\vee 0}, \mathscr{F}_t\ ;\ 0 \leq t \ll \infty\}$ is also a submartingale.

**Problem 3.28.** Let $Z = \{Z_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ be a continuous, nonnegative martingale with $Z_\infty := \lim_{n\to\infty} Z_t = 0$ a.s. Then for every $s \geq 0$, $b \geq 0$:

(1) $P(\sup_{t\geq s} Z_t \geq b \mid \mathscr{F}_s) = \frac{1}{b} Z_s$, a.s. on $\{Z_s \ll b\}$.

(2) $P(\sup_{t\geq s} Z_t \geq b) = P(Z_s \geq b) + \frac{1}{b} E(Z_s 1_{Z_s \ll b})$.

**Problem 3.29.** Let $\{X_t, \mathscr{F}_t;\ 0 \leq t \ll \infty\}$ be a continuous, nonnegative supermartingale and $T := \inf\{t \geq 0;\ X_t = 0\}$. Show that

$$X_{T+t} = 0; \quad 0 \leq t \ll \infty \ \text{ a.s. on } \ \{T \ll \infty\}.$$

**Exercise 3.30.** Suppose that the filtration $\{\mathscr{F}_t\}$ satisfies the usual conditions and let $X^{(n)} = \{X_t^{(n)}, \mathscr{F}_t;\ 0 \le t \ll \infty\}$, $n \ge 1$ be an increasing sequence of right-continuous supermartingales such that the random variable $\xi_t := \lim_{n\to\infty} X_t^{(n)}$ is nonnegative and integrable for every $0 \le t \ll \infty$. Then there exists RCLL supermartingale $X = \{X_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$ which is a modification of the process $\xi = \{\xi_t, \mathscr{F}_t;\ 0 \le t \ll \infty\}$.

# Bibliography

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