1 Reproducing Kernel Hilbert Space

Definition 1.1 (Reproducing Kernel). *Let E be a nonempty set. A function K defined by*

$$K : E \times E \ni (x, y) \mapsto K(x, y) \in \mathbb{F}$$

is called a reproducing kernel of a Hilbert space H of functions on E if it satisfies the following conditions:

(a) $K(\cdot, x) \in H$ *for every* $x \in E$

(b) $\langle f, K(\cdot, x)\rangle_H = f(x)$ *for every* $x \in E$ *and every* $f \in H$.

Such Hilbert space is called a reproducing kernel hilbert space (RKHS, for short), and is denoted by $(H(E), K)$ or (H, K).

Theorem 1.2 (Characterization of RKHS). *A Hilbert space H of functions on a nonempty set E admits a reproducing kernel K if and only if all evaluation functionals $\{\mathrm{ev}_x\}_{x\in E}$ are continuous on H.*

Proof. Suppose (H, K) is a RKHS. For $x \in E$ and for $f \in H$ we have

$$|\mathrm{ev}_x(f)| = |f(x)| = |\langle f, K(\cdot, x)\rangle| \le \|f\| \, \|K(\cdot, x)\| \le \|f\| \, K(x, x)^{1/2} \to 0$$

as $\|f\| \to 0$. Thus, ev_x is continuous linear functional (with norm $K(x, x)^{1/2}$).

Conversely, if $\mathrm{ev}_x : H \ni f \mapsto f(x) \in \mathbb{F}$ is continuous, then, by Riesz's representation theorem, there exists $r_x \in H$ such that

$$\langle f, r_x\rangle = f(x)$$

for every $f \in H$. If this happens for every $x \in E$, then $K(x, y) := r_x(y)$ is a reproducing kernel of H. □

Corollary 1.3. *Every convergent sequence in RKHS converges pointwise to the same limit.*

Proof. $|f_n(x) - f(x)| = |\mathrm{ev}_x(f_n - f)| \to 0$ when $f_n \to f$ in norm by continuity of evaluation functional. □

Definition 1.4. *(Positive definite function) Let E be a nonempty set. A function $K : E \times E \to \mathbb{C}$ is called positive definite if for any $n \in \mathbb{N}$ and for any $a \in \mathbb{C}^n$ and $x \in E^n$ there holds*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} K(x_i, x_j) \ge 0,$$

where $\overline{c}$ is the complex conjugate of c.

Proposition 1.5. *Suppose φ is a mapping of a set E into a Hilbert space H. Then the mapping $K : E \times E \ni (x, y) \mapsto \langle \varphi(x), \varphi(y) \rangle \in \mathbb{C}$ is positive definite.*

Proof. For a and x taken as in the definition, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} K(x_i, x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \langle \varphi(x_i), \varphi(x_j) \rangle = \left\| \sum_{i=1}^{n} a_i \varphi(x_i) \right\|^2 \geq 0.$$

□

Proposition 1.6. *Every reproducing kernel is positive definite.*

Proof. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} K(x_i, x_j) = \|\sum_{i=1}^{n} K(\cdot, x_i)\|^2$. □

Proposition 1.7. *Every positive definite function $K : E \times E \to \mathbb{C}$ satisfies*

(a) $K(x, x) \geq 0$ *for every* $x \in E$

(b) $K(x, y) = \overline{K(y, x)}$ *for every* $x, y \in E$

(c) $\overline{K}$ *is also positive definite*

(d) $|K(x, y)| \leq K(x, x)K(y, y)$ *for every* $x, y \in E$.

Proof. (a) and (c) clearly hold. For $\alpha, \beta \in \mathbb{C}$ and $x, y \in E$, we have

$$g(\alpha, \beta) := |\alpha|^2 K(x, x) + \alpha\overline{\beta}K(x, y) + \overline{\alpha}\beta K(y, x) + |\beta|^2 K(y, y) \geq 0.$$

Choose $\alpha = \beta = 1$ and $\alpha = i$, $\beta = 1$ to get

$$\begin{aligned} K(x, y) + K(y, x) &= g(1, 1) - K(x, x) - K(y, y) =: A \in \mathbb{R} \\ iK(x, y) - iK(y, x) &= g(i, 1) - K(x, x) - K(y, y) =: B \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2K(y, x) &= A + iB \\ 2K(x, y) &= A - iB, \end{aligned}$$

which proves (b). Finally, for $x, y \in E$ with $K(x, y) \neq 0$ and for $r \in \mathbb{R}$, (b) gives

$$0 \geq g(r, K(x, y)) = r^2 K(x, x) + 2r |K(x, y)|^2 + |K(x, y)|^2 K(y, y).$$

As RHS is quadratic in r, it must satisfy

$$|K(x, x)|^4 - |K(x, y)|^2 K(x, x)K(y, y) \leq 0,$$

from which (c) follows. □

Theorem 1.8. *Let H_0 be the subspace of $\mathbb{F}^E$, equipped with an inner product $\langle \cdot, \cdot \rangle_{H_0}$ with norm $\|\cdot\|_{H_0}$. Then there exists unique RKHS (H, K) that extends H_0 in the sense that*

(a) *$H_0 \subset H \subset \mathbb{F}^E$ and the subspace topology of H_0 in H coincides with the topology of $(H_0, \|\cdot\|_{H_0})$*

if and only if H_0 satisfies the following conditions:

(b) *every evaluation functional ev_x is continuous in $(H_0, \|\cdot\|_{H_0})$*

(c) *any Cauchy sequence $\{f_n\} \subset H_0$ converging pointwise to 0 converges to 0 also in H_0-norm.*

Proof. Suppose such extension H exists. H satisfies (b) by Theorem1.2. Since H is complete, Cauchy sequence $\{f_n\} \subset H_0$ tends to some f, for which we have

$$f(x) = \mathrm{ev}_x(f) = \lim_{n\to\infty} \mathrm{ev}_x(f_n) = \lim_{n\to\infty} f_n(x) = 0$$

for every $x \in E$. Therefore, f is identically 0.

Conversely, suppose (b)(c) hold. Let H be the set of all functions $f \in \mathbb{F}^E$ for which there exists a Cauchy sequence $\{f_n\} \subset H_0$ converging pointwise to f. Clearly, $H_0 \subset H \subset \mathbb{F}^E$. The rest of proof consists of the following Lemmas. □

Lemma 1.9. *Let $f, g \in H$ and let $\{f_n\}$ and $\{g_n\}$ be two Cauchy sequences in H_0 that converge pointwise to f and g respectively.*

(A) *The sequence $\langle f_n, g_n \rangle_{H_0}$ is convergent.*

(B) *The limit $\lim_{n\to\infty} \langle f_n, g_n \rangle_{H_0}$ depends solely on f and g, independent of the choice of f_n and g_n.*

(C) *$\langle f, g \rangle_H := \lim_{n\to\infty} \langle f_n, g_n \rangle_{H_0}$ is an inner product on H.*

Proof. It follows from the definition of f_n and g_n that

$$\begin{aligned} \left| \langle f_n, g_n \rangle_{H_0} - \langle f_m, g_m \rangle_{H_0} \right| &= |\langle f_n - f_m, g_n \rangle - \langle f_m, g_n - g_m \rangle| \\ &\geq \|g_n\| \, \|f_n - f_m\| + \|f_m\| \, \|g_n - g_m\| \to 0, \end{aligned}$$

which proves (A). In order to verify (B), suppose $\{f'_n\}$ and $\{g'_n\}$ are also such approximating sequences. We then similarly deduce that

$$|\langle f_n, g_n \rangle - \langle f'_n, g'_n \rangle| \leq \|g_n\| \, \|f_n - f'_n\| + \|f'_n\| \, \|g_n - g'_n\| .$$

$\{f_n - f'_n\}$ and $\{g_n - g'_n\}$ are Cauchy sequences tending pointwise to 0. Thus, assumption (c) gives $\|f_n - f'_n\| \to 0$ and $\|g_n - g'_n\| \to 0$. So, (A) and (B) show that $\langle f, g\rangle_H$ is well-defined. Note that if $\langle f, f\rangle_H = 0$, then for every $x \in E$

$$f(x) = \mathrm{ev}_x(f) = \lim_{n\to\infty} \mathrm{ev}_x(f_n) = \lim_{n\to\infty} f_n(x) = 0,$$

and hence $f \equiv 0$. As the symmetry, positivity, linearity are quite obvious, we conclude that (C) is true. $\square$

Lemma 1.10. *(A) Let $f \in H$ and let $\{f_n\} \subset H_0$ be a Cauchy sequence converging pointwise to f. Then $f_n \to f$ also in H-norm.*

(B) H_0 is dense in H.

Proof. (A): Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that

$$\|f_n - f_m\|_{H_0} < \epsilon$$

for all $n, m > N$. For fixed n, $\{f_n - f_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence converging pointwise to $f_n - f$. Therefore, by definition of $\langle\cdot,\cdot\rangle_H$,

$$\|f - f_n\|_H = \lim_{n\to\infty} \|f_n - f_m\|_{H_0} \le \epsilon.$$

(B) is obvious from (A). $\square$