

1 Reproducing Kernel Hilbert Space

Definition 1.1 (Reproducing Kernel). *Let E be a nonempty set. A function K defined by*

$$K : E \times E \ni (x, y) \mapsto K(x, y) \in \mathbb{F}$$

is called a reproducing kernel of a Hilbert space H of functions on E if it satisfies the following conditions:

- (a) $K(\cdot, x) \in H$ for every $x \in E$
- (b) $\langle f, K(\cdot, x) \rangle_H = f(x)$ for every $x \in E$ and every $f \in H$.

Such Hilbert space is called a reproducing kernel hilbert space (RKHS, for short), and is denoted by $(H(E), K)$ or (H, K) .

Theorem 1.2. *A Hilbert space H of functions on a nonempty set E admits a reproducing kernel K if and only if all evaluation functionals $\{\text{ev}_x\}_{x \in E}$ are continuous on H .*

Proof. Suppose (H, K) is a RKHS. For $x \in E$ and for $f \in H$ we have

$$|\text{ev}_x(f)| = |f(x)| = |\langle f, K(\cdot, x) \rangle| \leq \|f\| \|K(\cdot, x)\| \leq \|f\| K(x, x)^{1/2} \rightarrow 0$$

as $\|f\| \rightarrow 0$. Thus, ev_x is continuous linear functional (with norm $K(x, x)^{1/2}$).

Conversely, if $\text{ev}_x : H \ni f \mapsto f(x) \in \mathbb{F}$ is continuous, then, by Riesz's representation theorem, there exists $r_x \in H$ such that

$$\langle f, r_x \rangle = f(x)$$

for every $f \in H$. If this happens for every $x \in E$, then $K(x, y) := r_x(y)$ is a reproducing kernel of H . \square

Corollary 1.3. *Every convergent sequence in RKHS converges pointwise to the same limit.*

Proof. $|f_n(x) - f(x)| = |\text{ev}_x(f_n - f)| \rightarrow 0$ when $f_n \rightarrow f$ in norm by continuity of evaluation functional. \square

Definition 1.4. (Positive definite function) *Let E be a nonempty set. A function $K : E \times E \rightarrow \mathbb{C}$ is called positive definite if for any $n \in \mathbb{N}$ and for any $a \in \mathbb{C}^n$ and $x \in E^n$ there holds*

$$\sum_{i=1}^n \sum_{j=1}^n a_i \bar{a}_j K(x_i, x_j) \geq 0,$$

where \bar{c} is the complex conjugate of c .

Proposition 1.5. Suppose φ is a mapping of a set E into a Hilbert space H . Then the mapping $K : E \times E \ni (x, y) \mapsto \langle \varphi(x), \varphi(y) \rangle \in \mathbb{C}$ is positive definite.

Proof. For a and x taken as in the definition, we have

$$\sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} K(x_i, x_j) = \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \langle \varphi(x_i), \varphi(x_j) \rangle = \left\| \sum_{i=1}^n a_i \varphi(x_i) \right\|^2 \geq 0.$$

□

Proposition 1.6. Every reproducing kernel is positive definite.

Proof. $\sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} K(x_i, x_j) = \left\| \sum_{i=1}^n K(\cdot, x_i) \right\|^2$.

□

Proposition 1.7. Every positive definite function $K : E \times E \rightarrow \mathbb{C}$ satisfies

- (a) $K(x, x) \geq 0$ for every $x \in E$
- (b) $K(x, y) = \overline{K(y, x)}$ for every $x, y \in E$
- (c) \overline{K} is also positive definite
- (d) $|K(x, y)| \leq K(x, x)K(y, y)$ for every $x, y \in E$.

Proof. (a) and (c) clearly hold. For $\alpha, \beta \in \mathbb{C}$ and $x, y \in E$, we have

$$g(\alpha, \beta) := |\alpha|^2 K(x, x) + \alpha \overline{\beta} K(x, y) + \overline{\alpha} \beta K(y, x) + |\beta|^2 K(y, y) \geq 0.$$

Choose $\alpha = \beta = 1$ and $\alpha = i, \beta = 1$ to get

$$\begin{aligned} K(x, y) + K(y, x) &= g(1, 1) - K(x, x) - K(y, y) =: A \in \mathbb{R} \\ iK(x, y) - iK(y, x) &= g(i, 1) - K(x, x) - K(y, y) =: B \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2K(y, x) &= A + iB \\ 2K(x, y) &= A - iB, \end{aligned}$$

which proves (b). Finally, for $x, y \in E$ with $K(x, y) \neq 0$ and for $r \in \mathbb{R}$, (b) gives

$$0 \geq g(r, K(x, y)) = r^2 K(x, x) + 2r |K(x, y)|^2 + |K(x, y)|^2 K(y, y).$$

As RHS is quadratic in r , it has to satisfy

$$|K(x, x)|^4 - |K(x, y)|^2 K(x, x) K(y, y) \leq 0,$$

from which (c) follows.

□