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# Video 2.1

CJ Taylor

# Linear Transformations

- Let $T$ be a function that takes elements in one vector space $\mathcal{V}_1$ and maps them onto elements in another vector space $\mathcal{V}_2$
- We will say that the function $T$ is a Linear Transformation iff it satisfies the following property:

$$T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$$

  Where $\alpha, \beta \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in \mathcal{V}_1$ and $T(\mathbf{v}), T(\mathbf{w}) \in \mathcal{V}_2$

- More formally, $T$ is a linear transformation iff:

$$T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$$

$$\forall \alpha, \beta \in \mathbb{R}, \ \ \forall \mathbf{v}, \mathbf{w} \in \mathcal{V}_1$$

# Some examples of linear transformations

- Scaling: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \end{pmatrix}$

- Reflection: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

- Projection: $T\begin{pmatrix} x \\ y \end{pmatrix} = (x + y)$

Video 2.2

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# Matrices and Linear Transforms

We can characterize the action of a Linear Transformation on vectors in $\mathbb{R}^n$ by reporting its action on the canonical basis vectors.

For example if $\mathbf{v} \in \mathbb{R}^2$

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$$

$$T(\mathbf{v}) = xT(\mathbf{e}_1) + yT(\mathbf{e}_2)$$

# Matrices and Linear Transforms

This observation motivates us to represent the action of a Linear Transformation on finite dimensional vector spaces with a matrix where the columns of the matrix represent the action of the function on the basis vectors.

For example consider the following operation on vectors in $\mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x + 6y \\ 2x - 3y \end{pmatrix}$$

This function, $f$, is a linear transformation so:

$$f\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = x\,f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\,f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} 5 \\ 2 \end{pmatrix} + y\begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 5 & 6 \\ 2 & -3 \end{pmatrix}$$

We then say that:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 5 \\ 2 \end{pmatrix} + y\begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

# Matrix Vector Multiplication

In general if $A \in \mathbb{R}^{m \times n}$ is a matrix with $m$ rows and $n$ columns and $\mathbf{v} \in \mathbb{R}^n$ we say that the matrix vector product:

$$\mathbf{w} = A\mathbf{v}$$

Produces a vector $\mathbf{w} \in \mathbb{R}^m$ whose elements are defined as follows:

$$w_i = \sum_{j=1}^{n} A_{ij} v_j$$

We can think of this product as a weighted sum of the columns of $A$ or as the column vector formed by taking the inner product of $\mathbf{v}$ with each row of the matrix $A$. Both interpretations are equivalent and we will use both at different times.

# Linear Transformations

- For any linear transformation that maps one finite dimensional vector space onto another you can construct a matrix that represents the action of that function

# Examples of Matrix Vector Multiplication

- $$\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} \tag{1}$$
- $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 21 \end{pmatrix} \tag{2}$$

# Composing Linear Transforms

A useful fact about Linear Transformations is that their compositions must also be linear. That is if $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are all vector spaces, $f$ is a linear transformation from $\mathcal{V}_2$ to $\mathcal{V}_3$ and $g$ is a transformation from $\mathcal{V}_1$ to $\mathcal{V}_2$, then their composition $(f \circ g)$ will also be a linear transformation:

$$\begin{aligned}(f \circ g)(\alpha\mathbf{v} + \beta\mathbf{w}) &= f(g(\alpha\mathbf{v} + \beta\mathbf{w})) \\ &= f(\alpha g(\mathbf{v}) + \beta g(\mathbf{w})) \\ &= \alpha f(g(\mathbf{v})) + \beta f(g(\mathbf{w}))\end{aligned}$$

So the composition $(f \circ g)$ is a linear transformation from $\mathcal{V}_1$ to $\mathcal{V}_3$.

# Composing Linear Transformation

- Consider a situation where we have two linear transformations, $T_1$ and $T_2$, that we want to apply one after the other to a vector in $\mathbb{R}^2$, $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.

$$
\begin{aligned}
T_2(T_1 \begin{pmatrix} v_x \\ v_y \end{pmatrix}) &= T_2(v_x T_1(\mathbf{e}_1) + v_y T_1(\mathbf{e}_2)) && (1) \\
&= v_x T_2(T_1(\mathbf{e}_1)) + v_y T_2(T_1(\mathbf{e}_2)) && (2) \\
&= [T_2(T_1(\mathbf{e}_1)), T_2(T_1(\mathbf{e}_2))] \begin{pmatrix} v_x \\ v_y \end{pmatrix} && (3)
\end{aligned}
$$

- If we represent the transformations $T_1$ and $T_2$ with matrices $M_1$ and $M_2$ we see that the net result of applying the two transformations back to back can be captured by a third matrix $M_3$ which is formed by applying $M_2$ to the columns of $M_1$.

$$
M_3 = [M_2 M_1(:,1), M_2 M_1(:,2)] \tag{4}
$$

# Composing Linear Transforms

Example: Consider the following transformations:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = (x+2y) = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+3y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Consider the action of their composition $(f \circ g)$:

$$\begin{aligned} (f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} &= x\,(f \circ g)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\,(f \circ g)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= x\left(\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}\right) + y\left(\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) \\ &= 9x + 1y \end{aligned}$$

So the matrix associated with $(f \circ g)$ is :

$$\begin{pmatrix} 9 & 1 \end{pmatrix}$$

# Matrix Multiplication

- In general, we can compute the product of two matrices: $A \in \mathbb{R}^{l \times m}$ and $B \in \mathbb{R}^{m \times n}$ as follows:

$$C = AB, \;\; C \in \mathbb{R}^{l \times n}$$

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

- The resulting matrix, C, represents the composition of the two linear transformations, $A$ and $B$.

- One way to think of C is as the matrix formed by applying the matrix $A$ to the columns of $B$ in turn.

# Examples of Matrix multiplication

- $$\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 0\end{pmatrix} = \left[\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} \quad \begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}3\\ 0\end{pmatrix}\right] = \begin{pmatrix}-5 & 3\\ 8 & 0\end{pmatrix} \tag{1}$$

- $$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ -1 & 4 & -1\\ 3 & -2 & 4\end{pmatrix} = \begin{pmatrix}9 & 3 & 10\\ 21 & 12 & 19\end{pmatrix} \tag{2}$$

# Matrix Operations

- We can multiply a matrix by a scalar in the usual manner by multiplying all of the elements.
- We can add two matrices of the same size element by element.
- These operations correspond to scaling or adding the corresponding linear transformations.

# Matrix Operations

If $A$, $B$ and $C$ are matrices and $\alpha, \beta \in \mathbb{R}$ are scalars then:

- Matrix multiplication is associative - follows from the associativity of function composition:

$$A(BC) = (AB)C$$

- Matrix Addition and scalar multiplication are distributive and commutative:

$$A + B = B + A$$

$$\alpha(A + B) = \alpha A + \alpha B$$

- Matrix Multiplication is Distributive

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$A(\alpha B + \beta C) = \alpha AB + \beta AC$$

# Matrix Multiplication IS NOT COMMUTATIVE

Note that Matrix Multiplication is NOT commutative. In general

$$AB \neq BA$$

In fact usually the reverse product isn't even possible because the matrix dimensions are wrong

# Matrix Multiplication in Matlab

- Matrix multiplication is built in to MATLAB.
- If you have an mxn matrix A and and nx1 vector x then typing y = A*x will compute the matrix vector product.
- Similarly if B is an nxk matrix then C = A*B will compute the matrix product
- Note the distinction between '*' and '.*', the former gives you matrix multiplication the latter gives you pointwise multiplication. They are very different operations.

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# Video 2.3

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# Matrix Multiplication Revisited

- When we consider the product of 2 matrices, $C = AB$ , we can compute the result by applying the first matrix, $A$, to each column of $B$ in turn.
- Equivalently we can think of the entries in the product, $C_{ij}$, as the inner product of the ith row of $A$ and the jth column of $B$

# Examples of Matrix multiplication

- $$\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 0\end{pmatrix} = \left[\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} \quad \begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}3\\ 0\end{pmatrix}\right] = \begin{pmatrix}-5 & 3\\ 8 & 0\end{pmatrix} \quad (1)$$
- $$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ -1 & 4 & -1\\ 3 & -2 & 4\end{pmatrix} = \begin{pmatrix}9 & 3 & 10\\ 21 & 12 & 19\end{pmatrix} \quad (2)$$

# Transposition

- Transposition is an operation that can be applied to a matrix to swap the roles of the rows and the columns.
- The transpose of a matrix $A$ is denoted $A^T$
- In Matlab you can produce the transpose of a matrix $A$ by typing $A'$
- Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 7 \end{pmatrix}$$

# Transposition and Matrix Multiplication

- We can show that
$$C = AB \;\Rightarrow\; C^T = B^T A^T$$

- To see this consider that $C_{ij} = C^T_{ji}$ can be seen as the inner product of the ith row of A and the jth column of B. After transposition we flip the rows and the columns and the order of multiplication but we get the same thing.

Video 2.4

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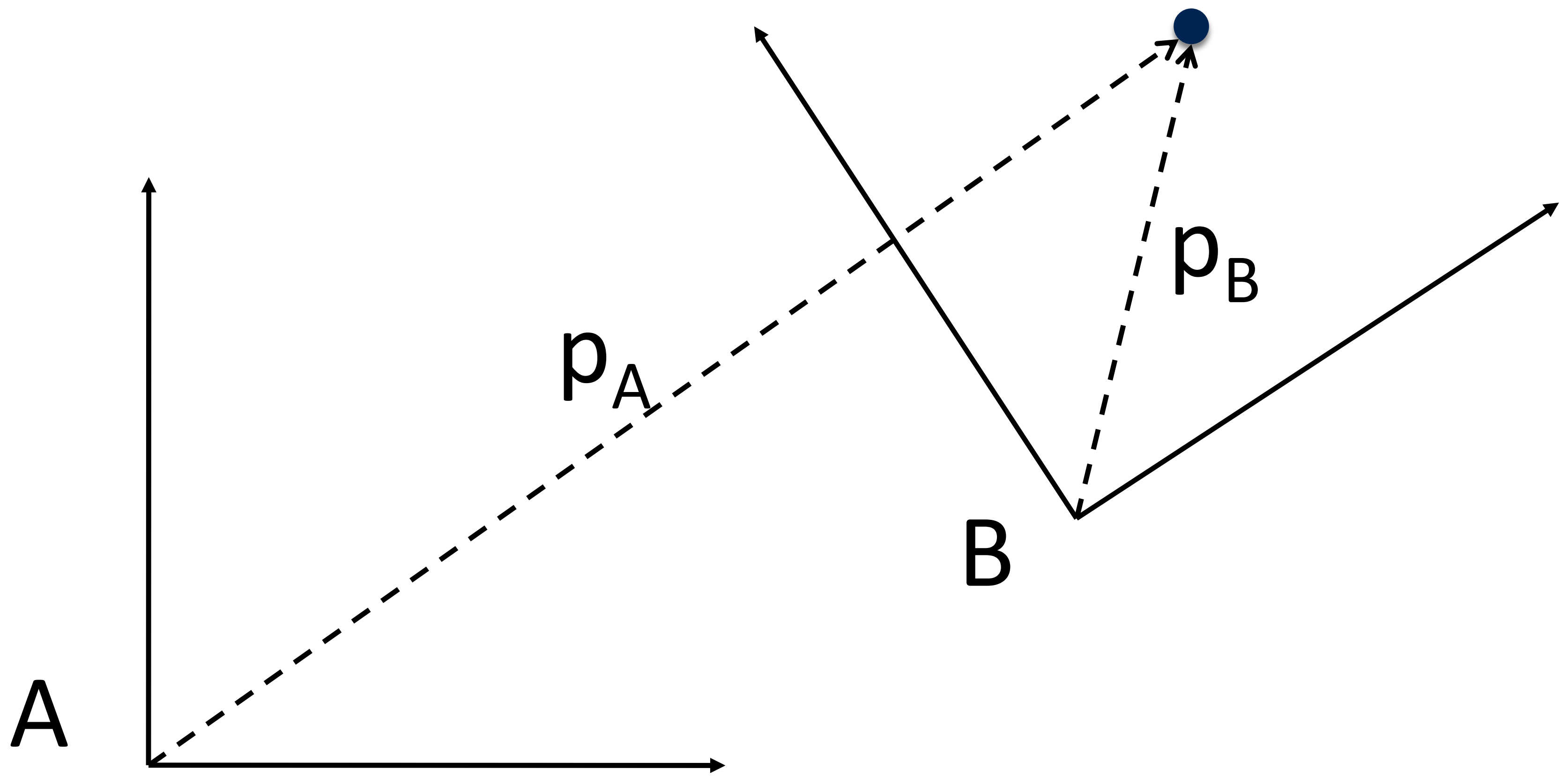
# Coordinate Transformation

- Let's begin by imagining 2 distinct coordinate frames in the plane, A and B.
- Let $p_A$ denote the coordinates of a point wrt frame A and $p_B$ the coordinates of the same point wrt frame B.



# Applications of Coordinate Transforms

- Robotic manipulators typically consist of a chain of links and we use coordinate transformations to determine where the links are wrt each other and a global frame

# Computer Graphics

- In virtual worlds and CAD systems we typically model scenes as a collection of parts and then use coordinate transformations to specify where things are with respect to each other and to the camera.

# Homogenous Coordinates

- In order to model the coordinate transformations that we will be interested in with matrix operations we will adopt homogenous coordinates.
- We will associate points in the plane with tuples consisting of 3 numbers where the last number is a 1.

$$P_A = \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix},\ P_B = \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$

- We will associate vectors in the plane with tuples consisting of 3 numbers where the last number is a 0.

$$V_A = \begin{pmatrix} v_A^x \\ v_A^y \\ 0 \end{pmatrix},\ V_B = \begin{pmatrix} v_B^x \\ v_B^y \\ 0 \end{pmatrix}$$

# Coordinate Transformation

- By adopting homogenous coordinates we will be able to associate the coordinate transformations that we are interested with matrix operations as follows:

$$P_A = g_{AB} P_B \tag{1}$$

Where $P_A = \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}$, $P_B = \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$ and $g_{AB}$ is a 3 by 3 matrix.

- Similarly for vectors

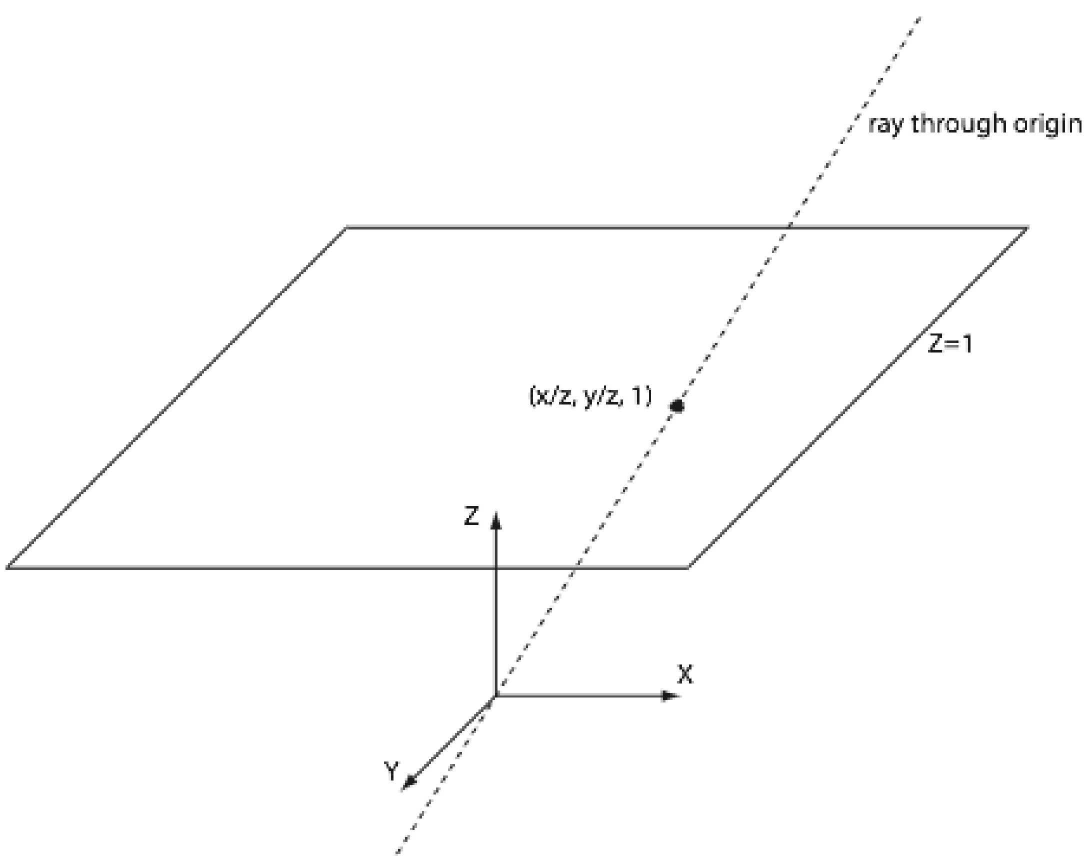
$$V_A = g_{AB} V_B \tag{2}$$

Where $V_A = \begin{pmatrix} v_A^x \\ v_A^y \\ 0 \end{pmatrix}$, $V_B = \begin{pmatrix} v_B^x \\ v_B^y \\ 0 \end{pmatrix}$

Video 2.5

CJ Taylor

# The Projective Plane




- Let's begin by considering the set of all rays through the origin in $\mathbb{R}^3$. (Note that we have gone up a dimension from $\mathbb{R}^2$ to $\mathbb{R}^3$ ). This set is referred to as the $RP(2)$, the real projective plane.

- Consider the intersection of the ray defined by the equation $\mathbf{P} = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with the plane $z = 1$.

- In general, the ray will intersect the plane at $\begin{pmatrix} (x/z) \\ (y/z) \\ 1 \end{pmatrix}$

- The exceptions are rays of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ which do not intersect the plane $z = 1$.

- We say there are two kinds of rays: rays that intersect the plane $z = 1$ with coordinates of the form $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ these rays can be associated with points in the plane $\mathbb{R}^2$, and rays that do not intersect the plane and have coordinates of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ these can be thought of as corresponding to directions in the plane $\mathbb{R}^2$ or <u>points at infinity</u>.

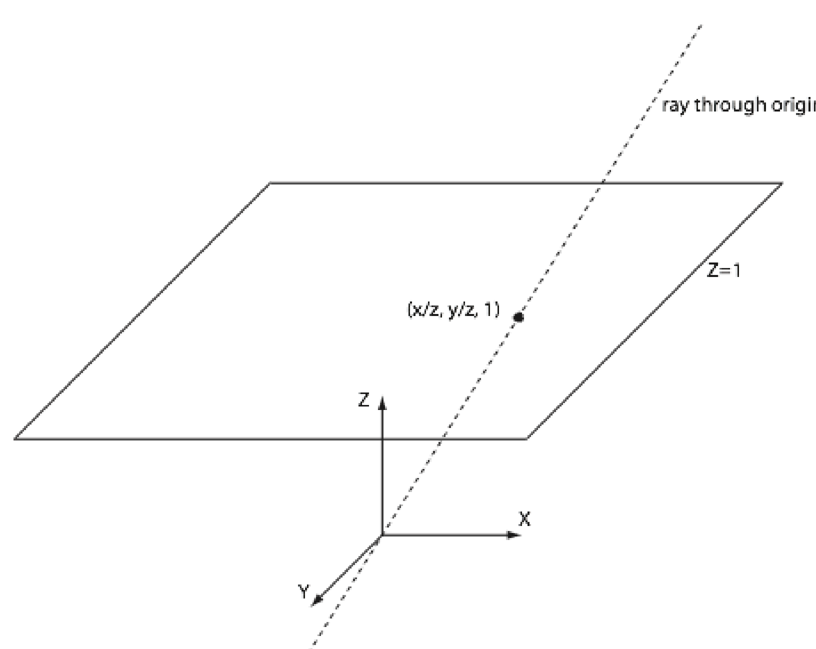
 

# Homogenous Coordinates

- Importantly when we are talking about projective coordinates we are referring to rays in space so absolute scale doesn't matter. As an example the following vectors all denote the same ray in space or projective point in $RP(2)$.

$$\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \propto \begin{pmatrix} 8 \\ 4 \\ -6 \end{pmatrix} \propto \begin{pmatrix} -12 \\ -6 \\ 9 \end{pmatrix}$$

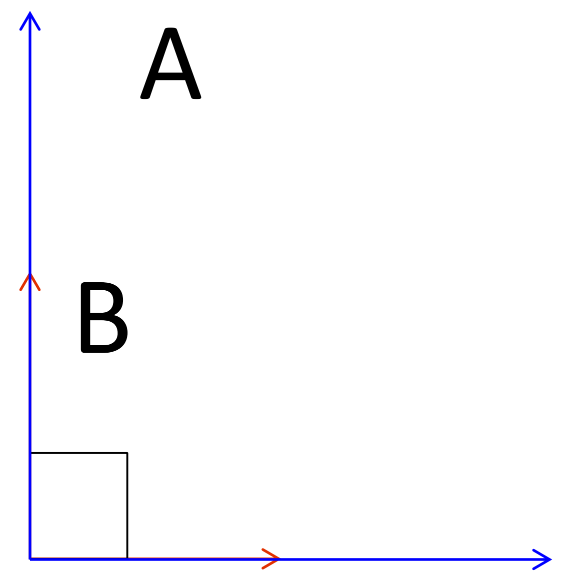
- This interpretation of rays in space is very useful for modeling **cameras** in computer vision and graphics since cameras effectively measure the intersection of light rays with an image plane.



# Projective Transformations

- From the perspective of projective geometry, the transformations that we are applying to the homogenous coordinates are simply examples of projective transformations that map the projective plane onto itself.

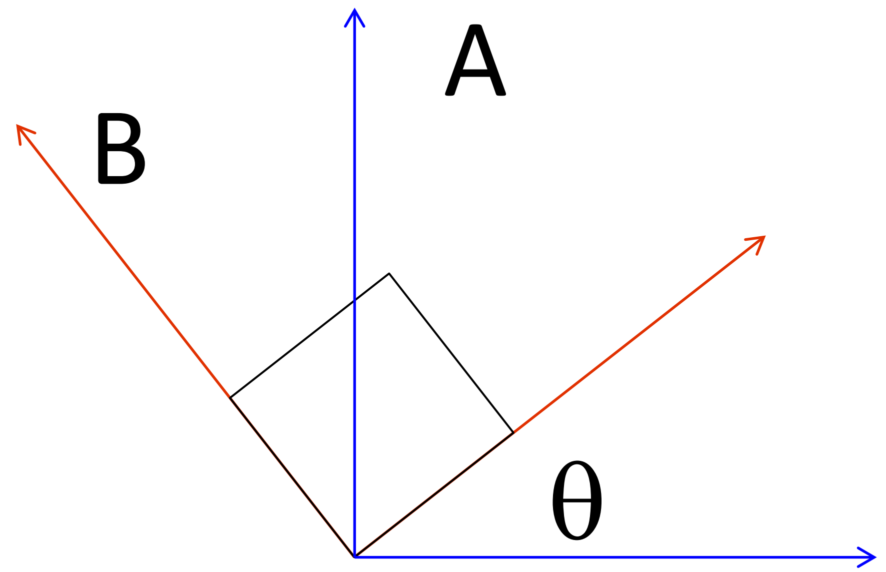
# Computing Elementary Transformations using Matrix Multiplication : Scaling




Scaling

$$\begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$

# Computing Elementary Transformations using Matrix Multiplication: Rotation



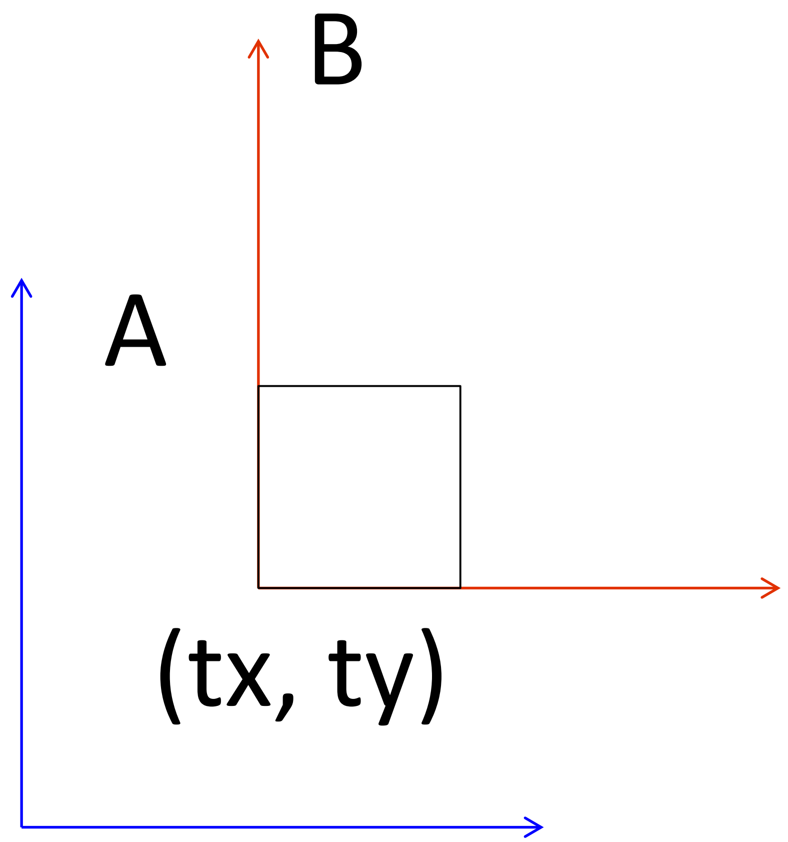

Rotation

$$\begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$

# Constructing Transformation Matrices

- A good way to construct a transformation matrix is by considering how it maps the coordinate axes and the origin from one frame of reference to another. These lead directly to the columns of the transformation matrix.

# Computing Elementary Transformations using Matrix Multiplication: Translation




Translation

$$\begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$