Robotics: Fundamentals

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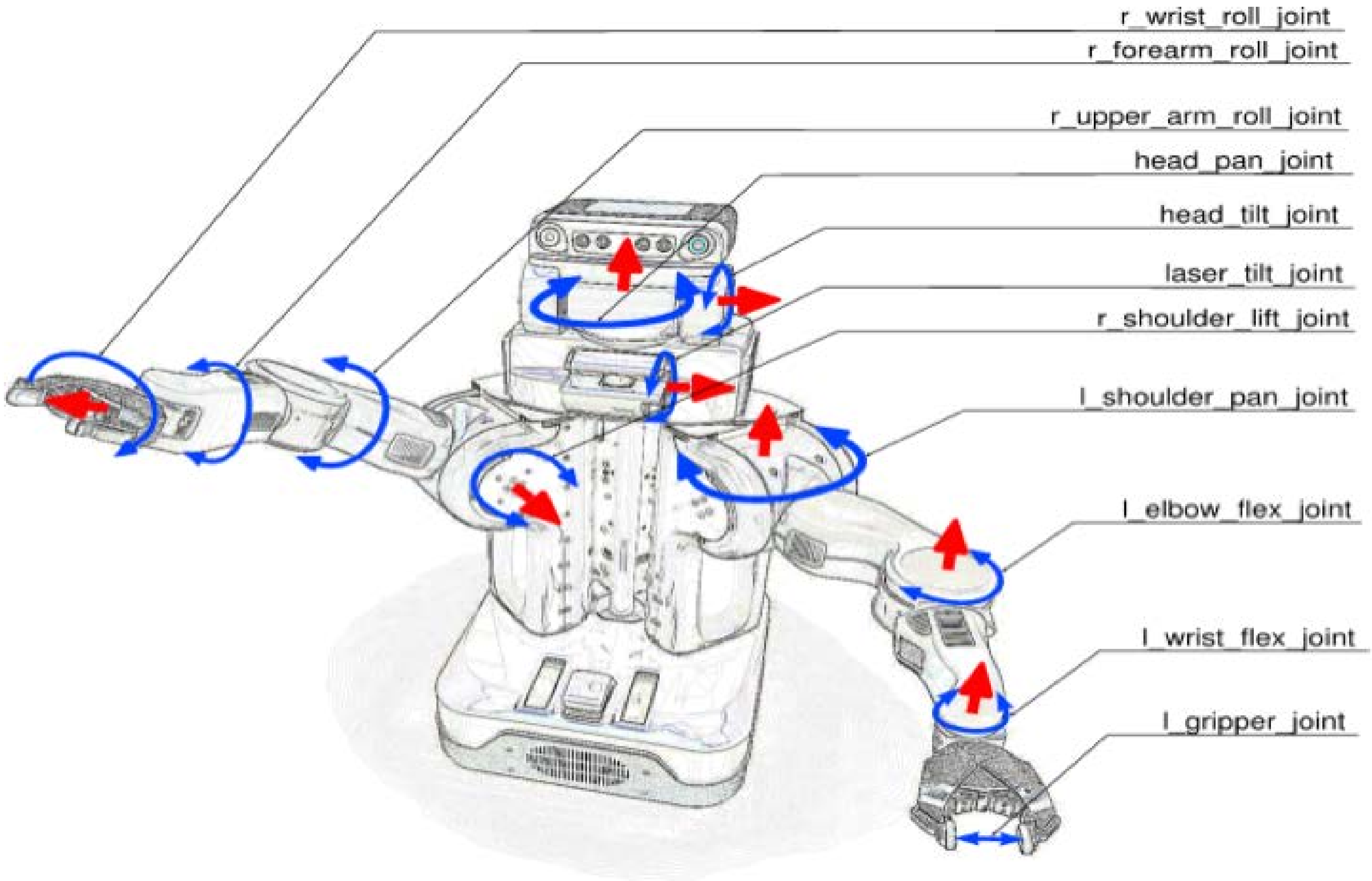
The Goal

- Develop the mathematical tools that we need to talk about where things are in space.
- Where things are depend on where you are looking at them from. How do we make this more precise?

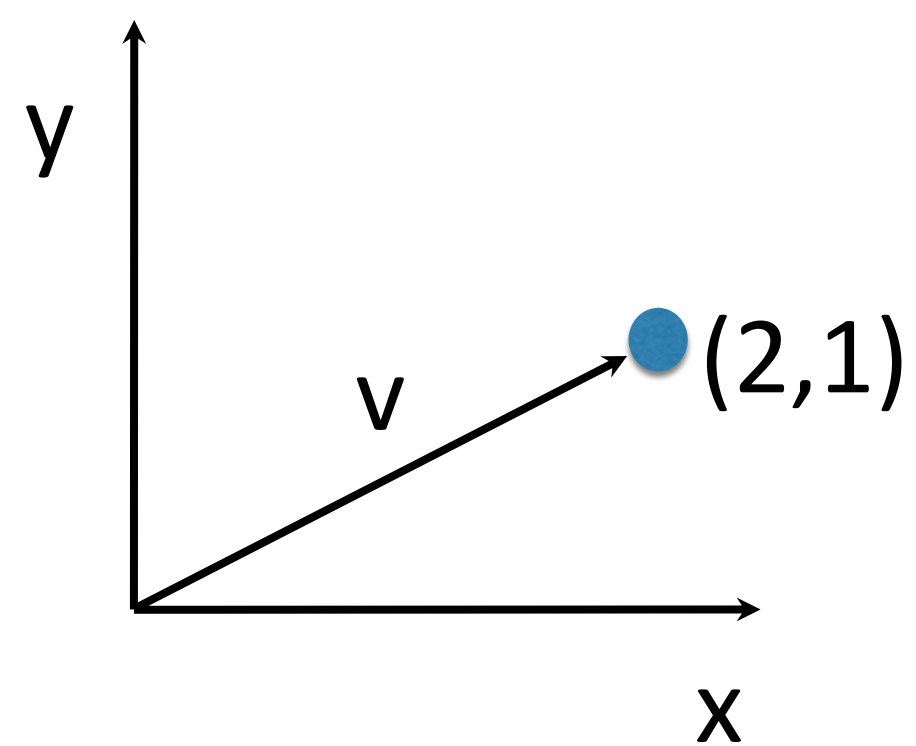
Coordinate Transforms in Robotics



PR2 Joints

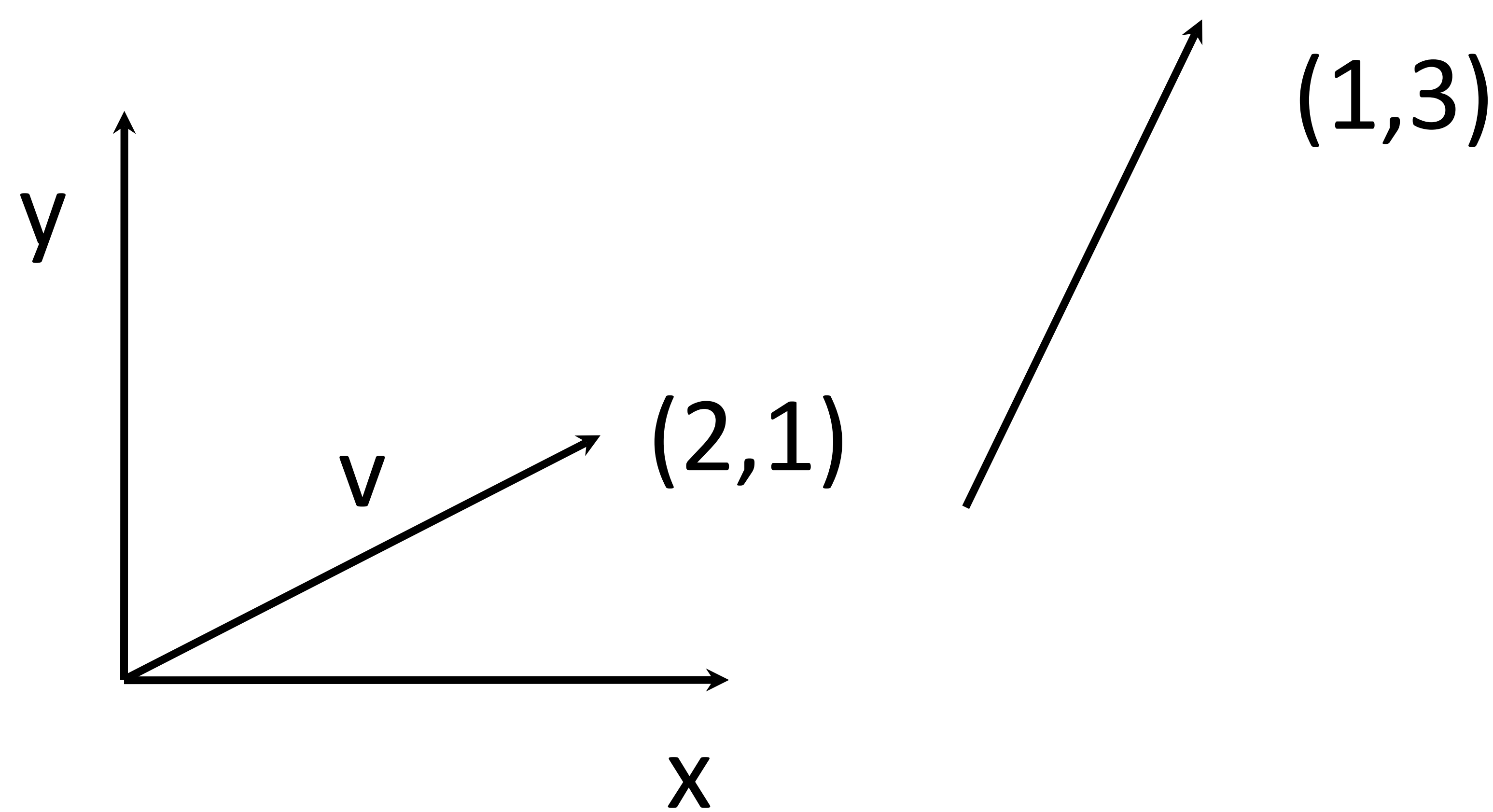
Descartes

- Consider the Cartesian coordinate plane
- Points in the plane can be denoted with tuples denoting their coordinates with respect to the frame of reference



Descartes

- Note that we can also talk about vectors. Corresponding to directions in the plane or to displacements between points. These can also be represented by tuples



Vectors

- Let's call these tuples of numbers, stacked up vertically **vectors**

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- We will say $\mathbf{v} \in \mathbb{R}^2$ to denote the fact that $\mathbf{v}$ is a vector composed of 2 real numbers, as opposed to two complex numbers

Operations on Vectors

- It is natural to imagine simple algebraic operations like multiplication by a scalar

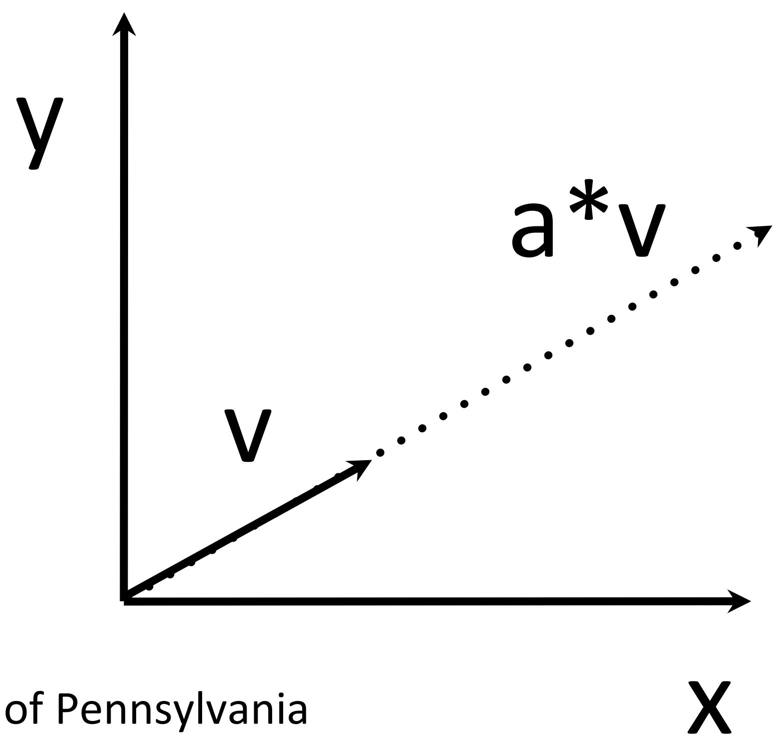
$$5 \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

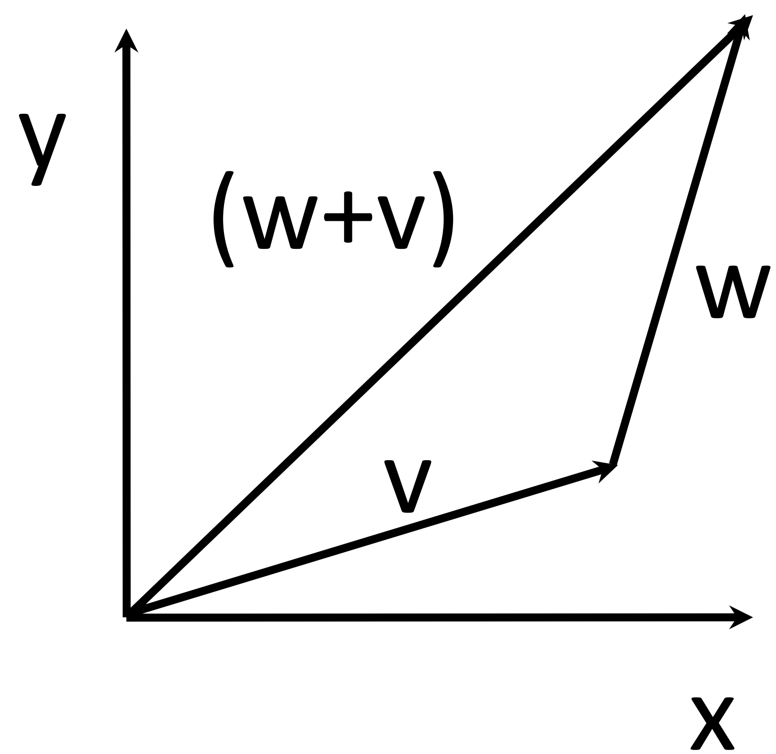
- or addition

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$$

Geometric Interpretation

- It is not hard to see that multiplication by a scalar value corresponds to scaling a vector by that amount.
- Similarly adding two tuples corresponds to adding the corresponding vectors head to tail





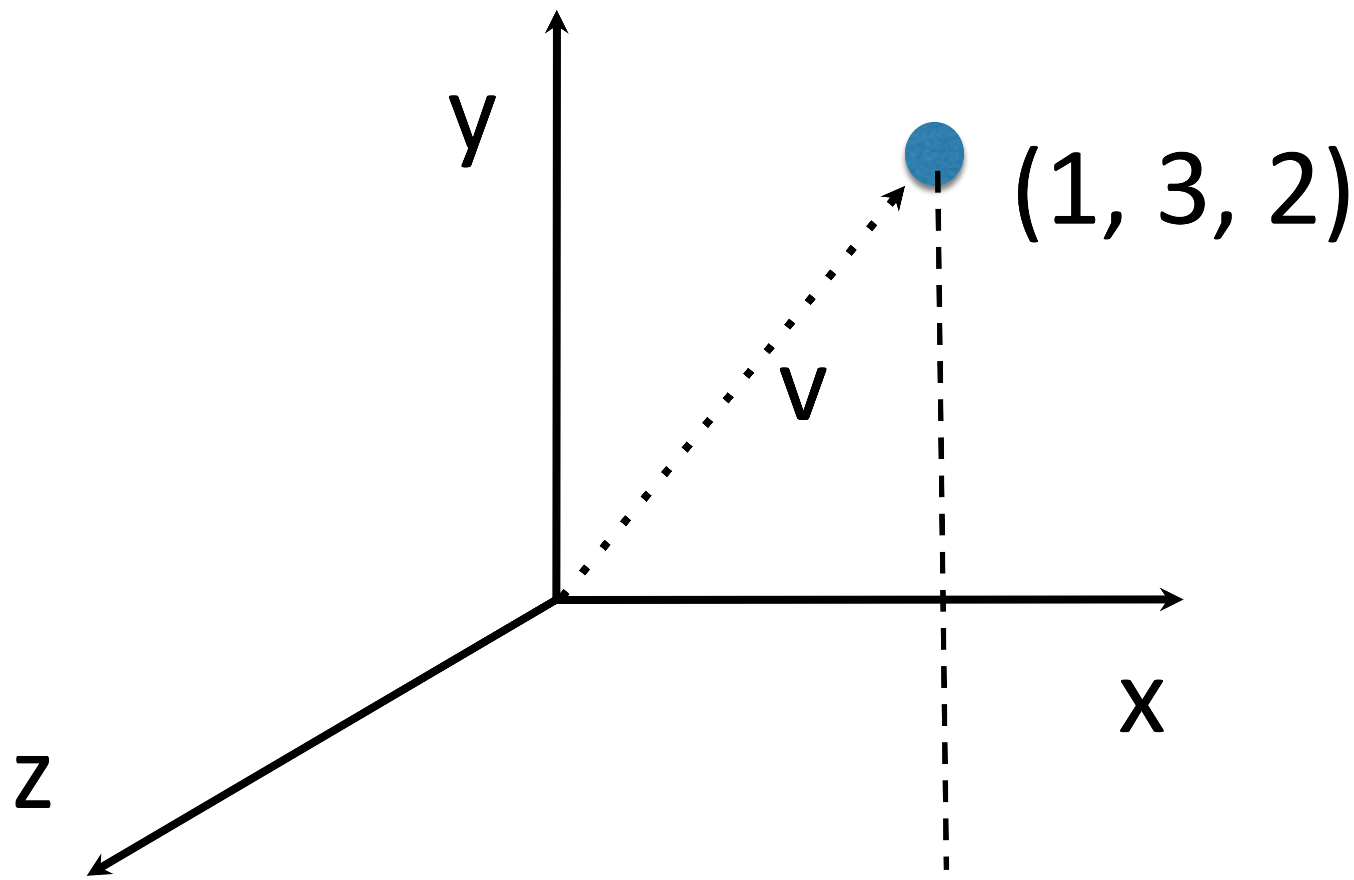
Adding Dimensions

- We refer to vectors of the form $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as elements of $\mathbb{R}^2$
- here the $\mathbb{R}$ reminds us that the entries in the vectors are real numbers, as opposed to complex numbers, while the 2 indicates that there are two entries in the vector.
- In general $\mathbb{R}^n$ denotes the set of real valued vectors with n entries, n is refered to as the *dimension* of the vector.
- examples:

$$\begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} \in \mathbb{R}^3, \begin{pmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 7 \\ -2 \end{pmatrix} \in \mathbb{R}^6$$

Descartes in 3D

- We will also be interested in talking about the location of objects in space. Here R^3 will be helpful.



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Inner Product

- Given two vectors in $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ we can define their inner product $\mathbf{v} \cdot \mathbf{w}$ as follows:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

  where:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

- Note that the inner product returns a single scalar value
- examples:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \times 3 + 2 \times 4 = 11$$

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3 \times 2 + (-5) \times 1 = 1$$

More Inner Products

- More generally, given two vectors in $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we can define their inner product $\mathbf{v} \cdot \mathbf{w}$ as follows:

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

where:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

- Note that the inner product returns a single scalar value
- examples:

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = 1 \times 3 + 2 \times 4 + 5 \times (-1) = 6$$

The Euclidean Norm

- If we take the inner product of a vector with itself we end up with a positive number, a sum of squares.
- For example if we take a vector in $\mathbb{R}^2$ we get:

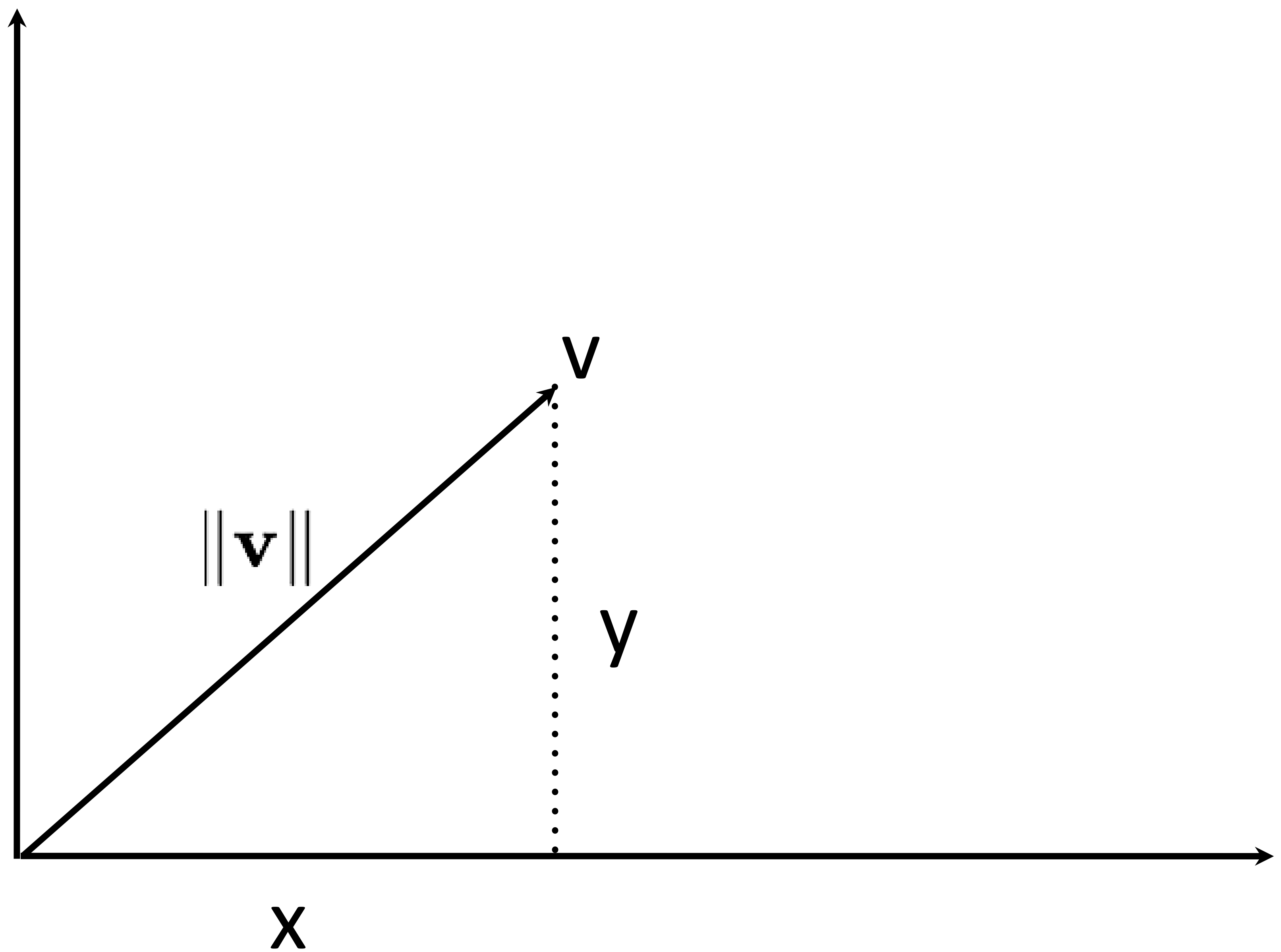
$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow (\mathbf{v} \cdot \mathbf{v}) = x^2 + y^2$$

- We can interpret this quantity as the square of the length of the vector as shown below. This quantity is referred to as the <u>Euclidean norm</u> of the vector and is denoted by $\|\mathbf{v}\|$
- In other words:

$$\|\mathbf{v}\|^2 = (\mathbf{v} \cdot \mathbf{v}) = x^2 + y^2$$

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v} \cdot \mathbf{v})} = \sqrt{(x^2 + y^2)}$$

Euclidean Norm



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Inner Product Properties

- The inner product is commutative
- If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ we have:

$$(\mathbf{v} \cdot \mathbf{w}) = \sum_{i=1}^{n} v_i w_i = \sum_{i=1}^{n} w_i v_i = (\mathbf{w} \cdot \mathbf{v})$$

- In conclusion:

$$(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{w} \cdot \mathbf{v})$$

Inner Product Properties

- The inner product is distributive with respect to scalar multiplication
- If $\alpha \in \mathbb{R}$ is a scalar and $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ we have:

$$(\mathbf{v} \cdot (\alpha \mathbf{w})) = \sum_{i=1}^{n} v_i(\alpha w_i) = \alpha \left(\sum_{i=1}^{n} v_i w_i \right) = \alpha(\mathbf{v} \cdot \mathbf{w})$$

- In conclusion:

$$(\mathbf{v} \cdot (\alpha \mathbf{w})) = \alpha(\mathbf{v} \cdot \mathbf{w})$$

Inner Product Properties

- The inner product is distributive with respect to vector addition
- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ we have:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \sum_{i=1}^{n} u_i(v_i + w_i) = \left(\sum_{i=1}^{n} u_i v_i\right) + \left(\sum_{i=1}^{n} u_i w_i\right) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$$

- In conclusion:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$$

Inner Product Properties

- As a consequence of the properties we just proved it is easy to prove the following identities:

- If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$:

$$
\begin{aligned}
(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c}+\mathbf{d}) &= \mathbf{a} \cdot (\mathbf{c}+\mathbf{d}) + \mathbf{b} \cdot (\mathbf{c}+\mathbf{d}) \\
&= (\mathbf{a} \cdot \mathbf{c}) + (\mathbf{a} \cdot \mathbf{d}) + (\mathbf{b} \cdot \mathbf{c}) + (\mathbf{b} \cdot \mathbf{d})
\end{aligned}
$$

- If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

$$
\begin{aligned}
\|(\mathbf{a}+\mathbf{b})\|^2 &= (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) \\
&= (\mathbf{a} \cdot \mathbf{a}) + (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{a}) + (\mathbf{b} \cdot \mathbf{b}) \\
&= \|\mathbf{a}\|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\|^2
\end{aligned}
$$

Cosine Rule

- Consider the figure below that shows two vectors in the plane, $\mathbf{v}$ and $\mathbf{w}$, separated by an angle θ.
- From our discussion of vector norms we know that:

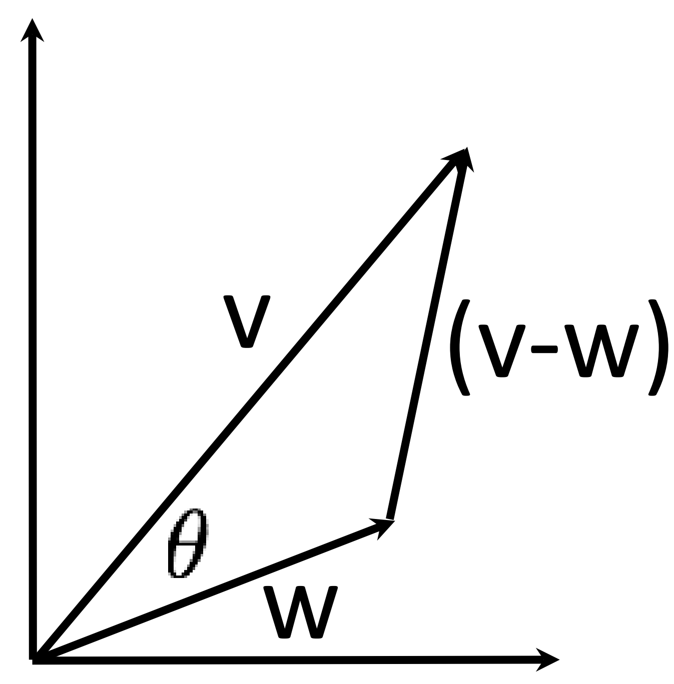
$$\|(\mathbf{v}-\mathbf{w})\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$$

- But from the Cosine Rule we also know that that the three sides of the triangle shown are related as follows:

$$\|(\mathbf{v}-\mathbf{w})\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2(\|\mathbf{v}\|\|\mathbf{w}\|)\cos\theta$$

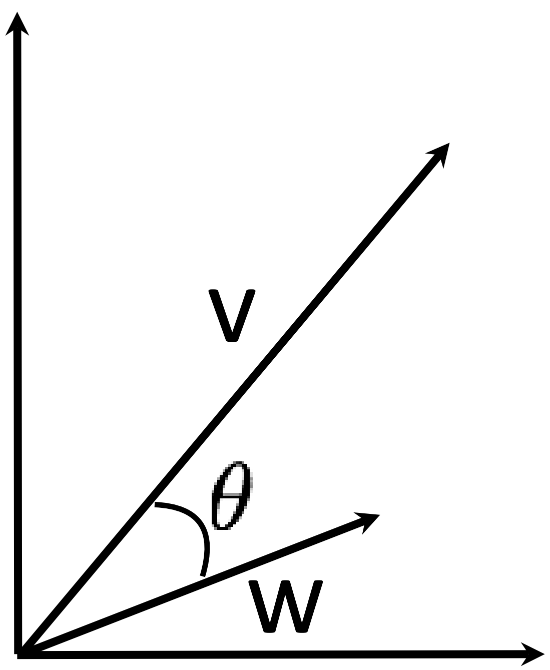
- So we can conclude that:

$$(\mathbf{v}\cdot\mathbf{w}) = (\|\mathbf{v}\|\|\mathbf{w}\|)\cos\theta$$



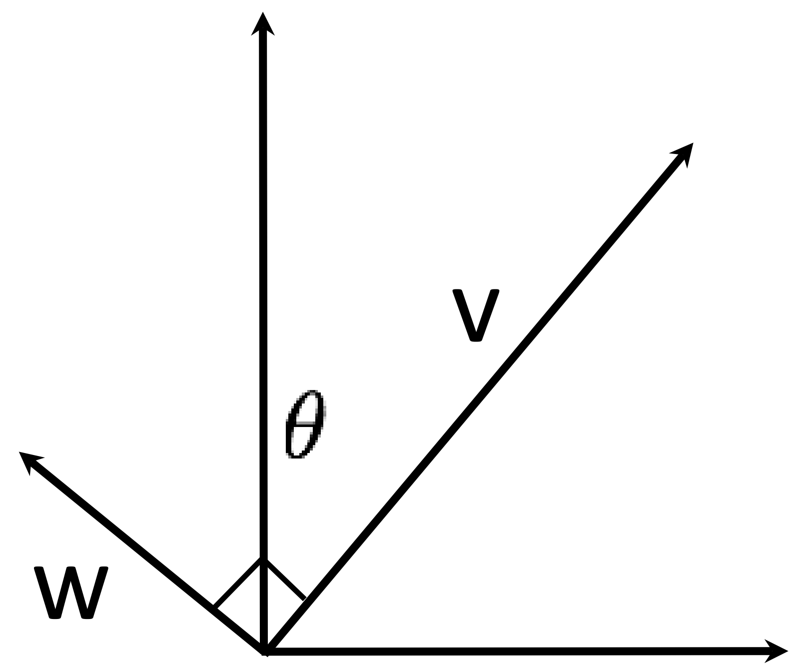
Geometric Interpretation

$\theta < \frac{\pi}{2}$



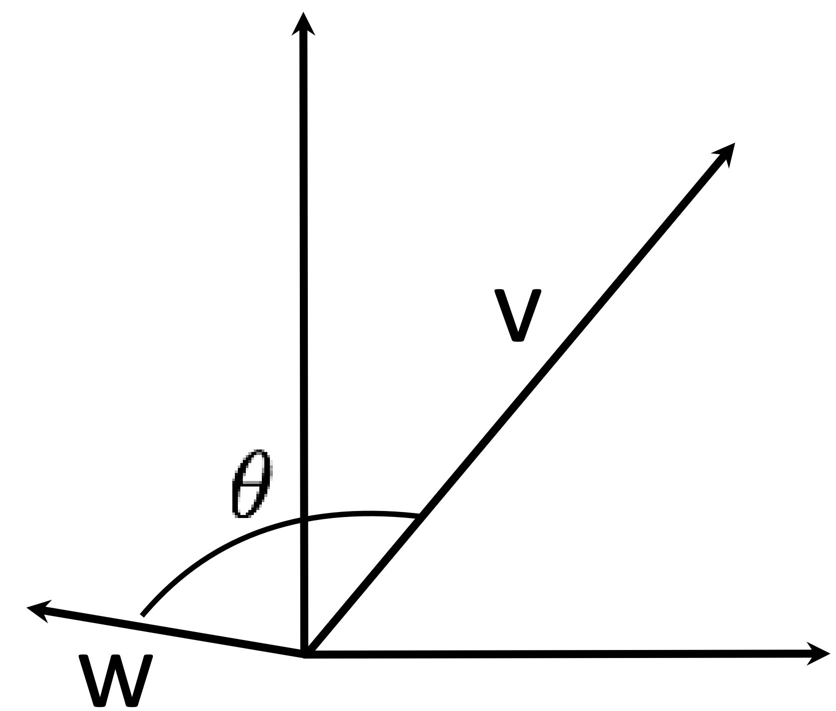
$(\mathbf{v} \cdot \mathbf{w}) > 0$

$\theta = \frac{\pi}{2}$



$(\mathbf{v} \cdot \mathbf{w}) = 0$

$\theta > \frac{\pi}{2}$

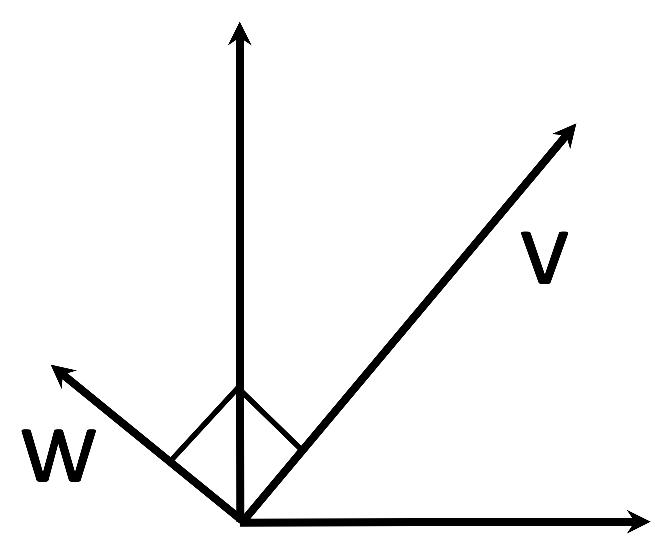


$(\mathbf{v} \cdot \mathbf{w}) < 0$

Orthogonality

- Whenever the inner product of two vectors is zero we say that they are orthogonal.
- The idea of two vectors at right angles generalizes to higher dimensions

$$\theta = \frac{\pi}{2}$$



$$(\mathbf{v} \cdot \mathbf{w}) = 0$$

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Vector Spaces

Definition adapted from "Matrix Analysis and Applied Linear Algebra" by Carl D. Meyer (Chapter 4 Pg 160)

The set $\mathcal{V}$ is called a **vector space** (over the reals) when we define two operations on the set, scalar multiplication, and vector addition that satisfy the following properties.

1. $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$
3. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$
4. There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$
5. For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
6. $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{R}$ and all $\mathbf{x} \in \mathcal{V}$. This is the closure property for scalar multiplication.
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{x} \in \mathcal{V}$.
8. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{x} \in \mathcal{V}$.
9. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for all $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.
10. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$

Vector Spaces

- Note that the set of tuples in $\mathbb{R}^n$ that we have been using with scalar multiplication and vector addition defined in the usual way is a canonical vector space
- Another example of a vector space is the set of continuous functions on the real numbers with scalar multiplication and function addition defined in the usual way

Basis

- Note that every element in $\mathbb{R}^n$ can be written as a linear combination of a set of n vectors
- For example:

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- In this example the vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are referred to as basis vectors. The two vectors taken together are referred to as a basis for $\mathbb{R}^2$
- Note that this choice of basis vectors is convenient but not unique, we could have done the same thing with many other pairs of vectors. We will have more to say about bases later.