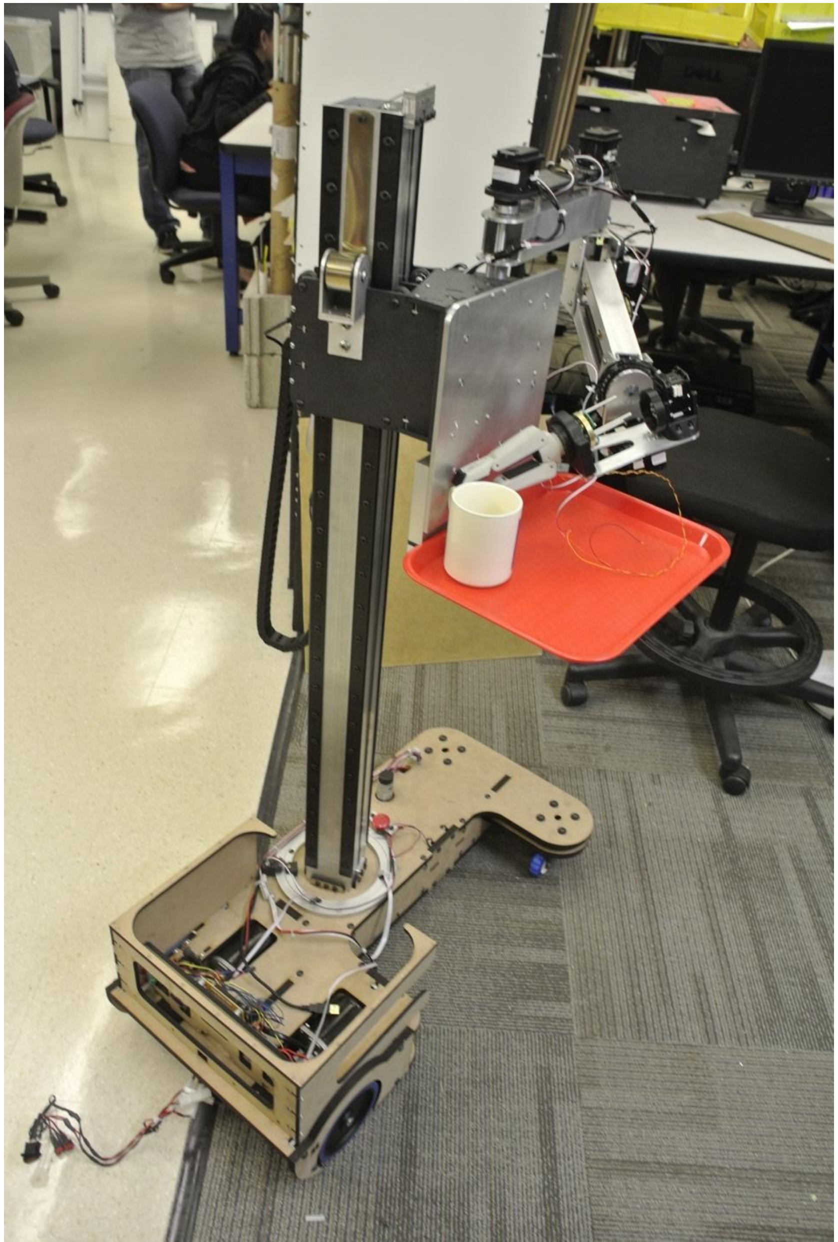
Robotics: Fundamentals

Prof. Mark Yim
University of Pennsylvania

Week 6: Inverse Kinematics

Inverse Kinematics



Inverse Kinematics Definition

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{R} & \mathbf{P} \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}_{0n}(q_1 \cdots q_n) = \boldsymbol{H}$$

$$\boldsymbol{T}_{0n}(q_1 \cdots q_n) = \boldsymbol{A}_1 \cdots \boldsymbol{A}_n$$

$$T_{ij}(q_1 \cdots q_n) = H_{ij}$$

Where, T_{ij}, H_{ij} refer to the 12 nontrivial entries of $\boldsymbol{T}_{0n}$, and $\boldsymbol{H}$

Stanford Arm Transform

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{11} = c_1[c_2(c_4c_5c_6 - s_4c_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6)$$
$$r_{21} = s_1[c_2(c_4c_5c_6 - s_4c_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6)$$
$$r_{31} = -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6$$
$$r_{12} = c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5c_6 + c_4s_6)$$
$$r_{22} = s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5c_6 + c_4s_6)$$
$$r_{32} = s_2(c_4c_5c_6 + s_4s_6) + c_2s_5c_6$$
$$r_{13} = c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5$$
$$r_{23} = s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5$$
$$r_{33} = -s_2c_4s_5 + c_2c_5$$
$$P_x = c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1s_2c_5 - s_1s_4s_5)$$
$$P_y = s_1s_2d_3 + c_1d_2 + d_6(s_1c_2c_4s_5 + s_1s_2c_5 + c_1s_4s_5)$$
$$P_z = c_2d_3 + d_6(c_2c_5 - s_2c_4s_5)$$

12 non-linear eqns, 7 unknowns

Approaches

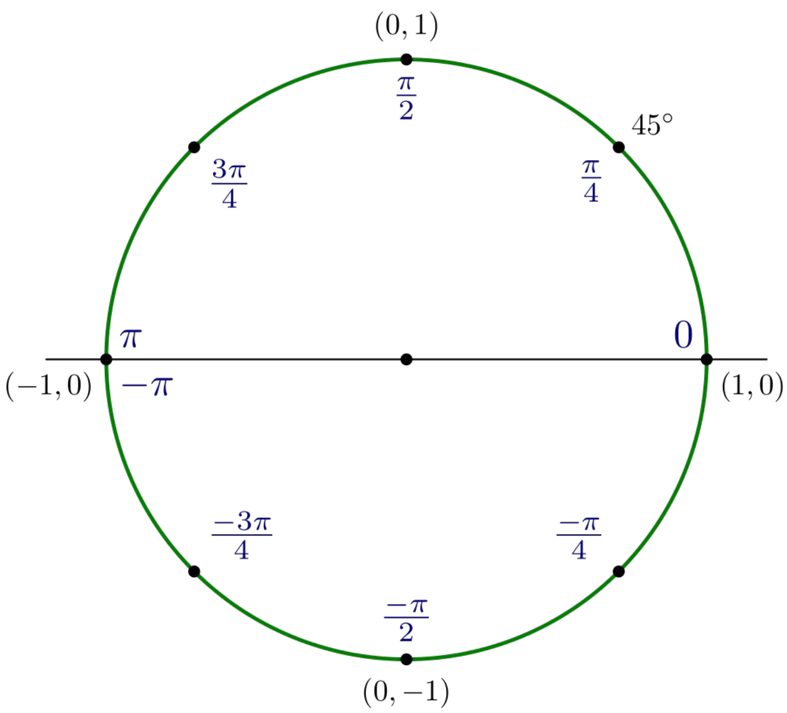
Closed form

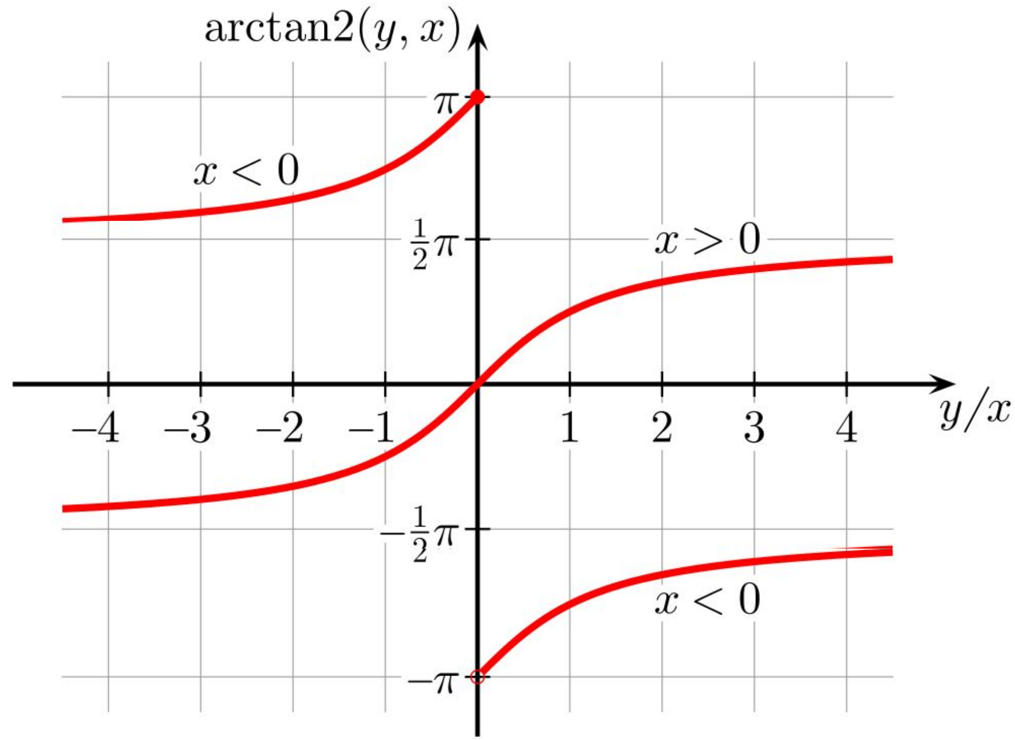
- $q_k = f_k(h_{11,} \cdots, h_{34}), k = 1, \cdots, n$
- Geometric approach
- *IKFast* (open source C++ code)

Approximate Solutions

- Iterative optimization
- Jacobian Inverse

`atan(y/x)` **vs** `atan2(y,x)`





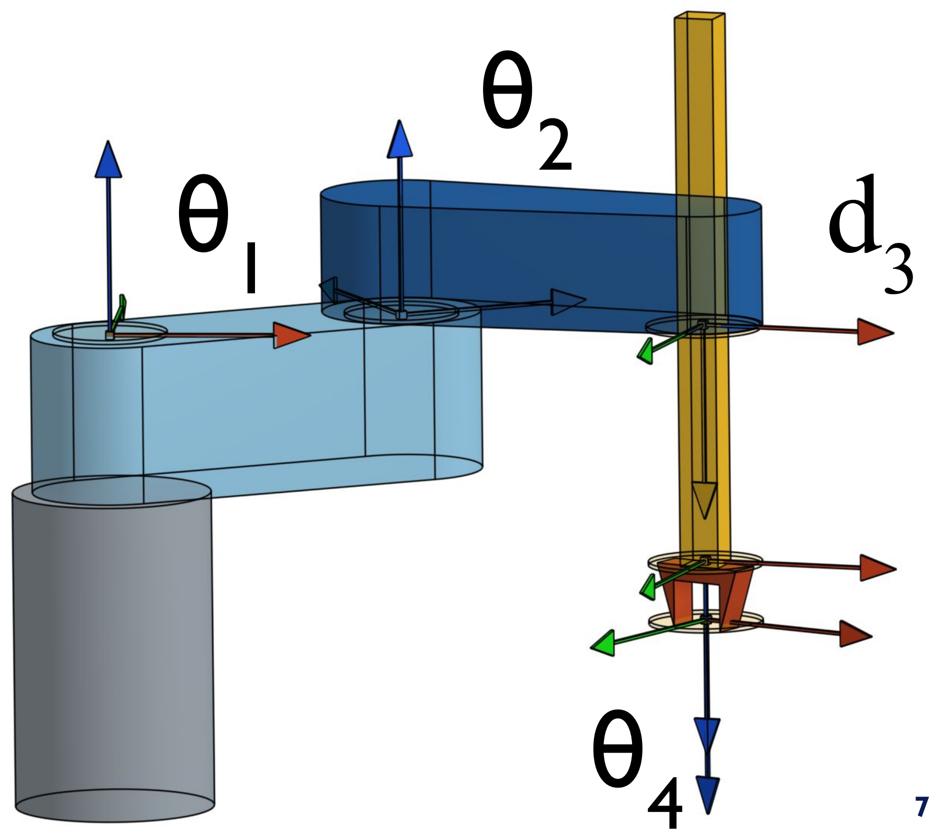
By Qef (self-made using design of the bitmap version) [CC0]

By Geek3 (Own work) [GFDL]

SCARA IK

$$\begin{bmatrix} \boldsymbol{c_{12}c_4} + \boldsymbol{s_{12}s_4} & -\boldsymbol{c_{12}s_4} + \boldsymbol{s_{12}c_4} & 0 & a_1\boldsymbol{c_1} + a_2\boldsymbol{c_{12}} \\ \boldsymbol{s_{12}c_4} - \boldsymbol{c_{12}s_4} & -\boldsymbol{s_{12}s_4} - \boldsymbol{c_{12}c_4} & 0 & a_1\boldsymbol{s_1} + a_2\boldsymbol{s_{12}} \\ 0 & 0 & -1 & -\boldsymbol{d_3} - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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SCARA IK

$$\begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

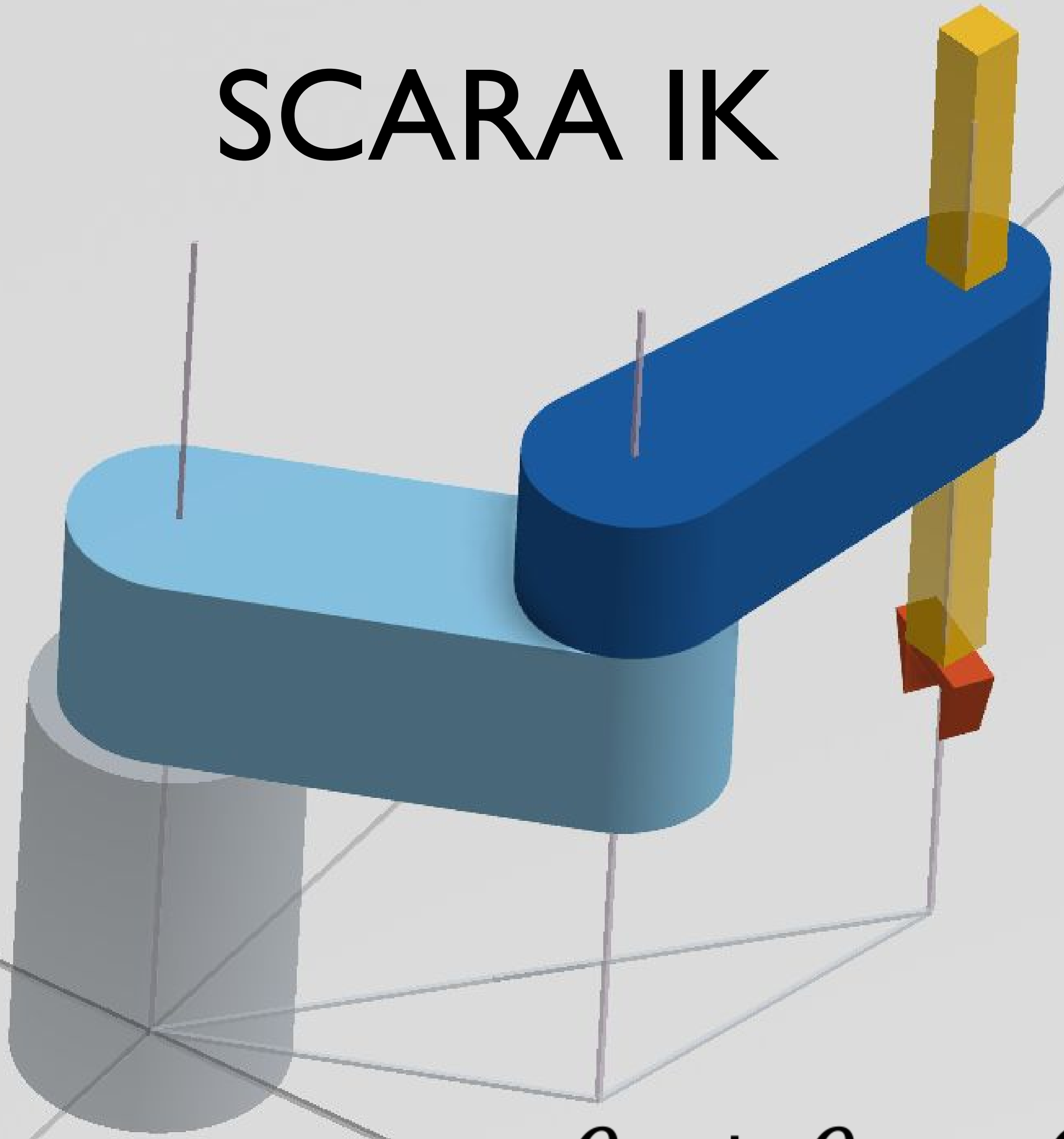
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\psi & s_\psi & 0 \\ s_\psi & -s_\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$angle = \operatorname{atan2}(y, x)$$

$$\psi = \operatorname{atan2}(r_{12}, r_{11})$$

Penn Engineering

SCARA IK



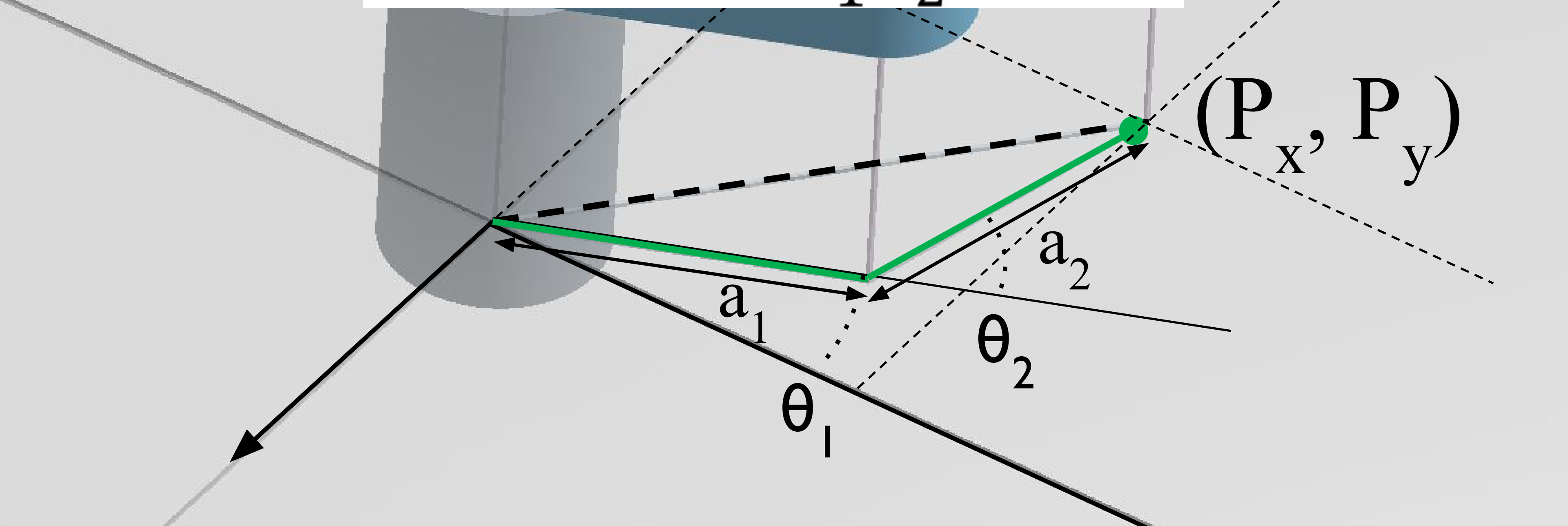
$$\theta_1 + \theta_2 - \theta_4 = \psi$$

SCARA IK

$$r^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\theta)$$

$$P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2\, c_2$$

$$c_2 = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

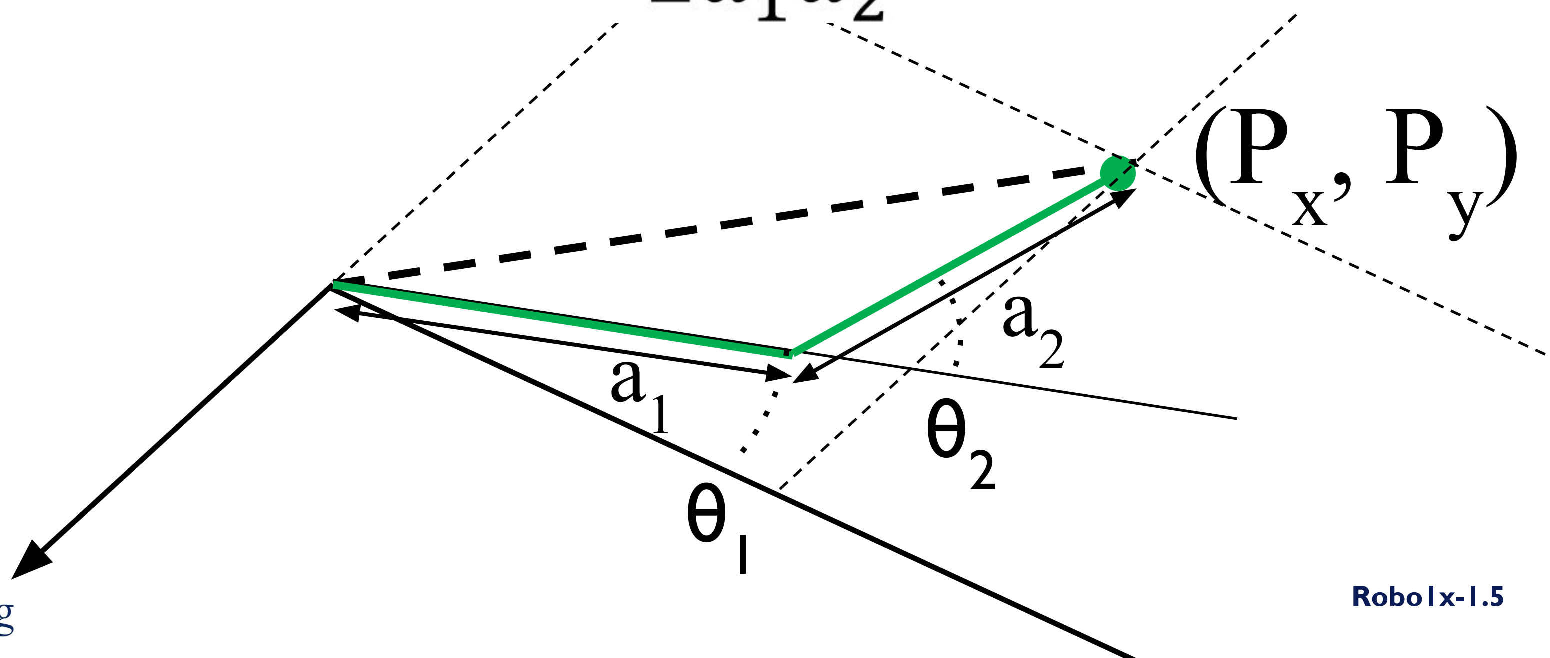


SCARA IK θ_2

$$r^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos(\theta_2)$$

$$P_x^2 + P_y^2 = a_1^2 + a_2^2 - 2a_1 a_2\, c_2$$

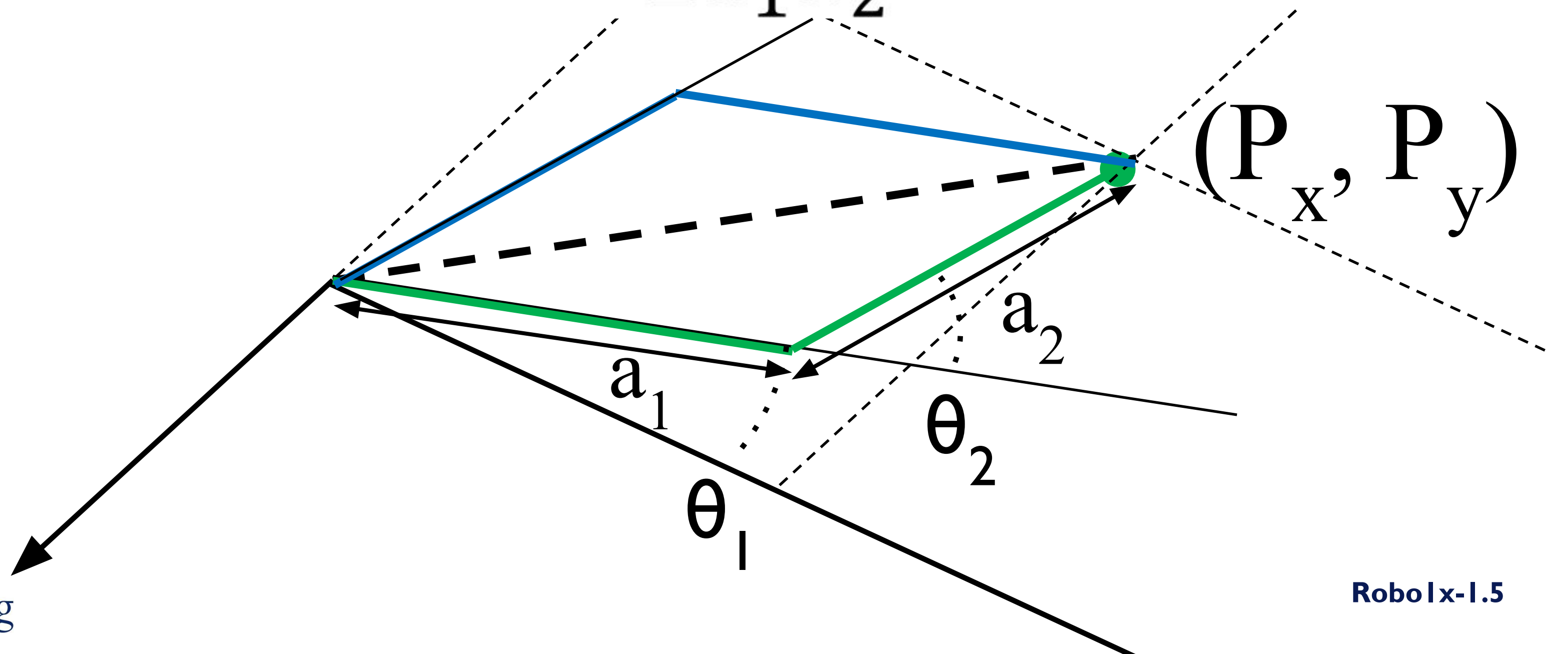
$$c_2 = \frac{P_x^2 + P_y^2 - (a_1^2 + a_2^2)}{2a_1 a_2}$$



SCARA IK θ_2

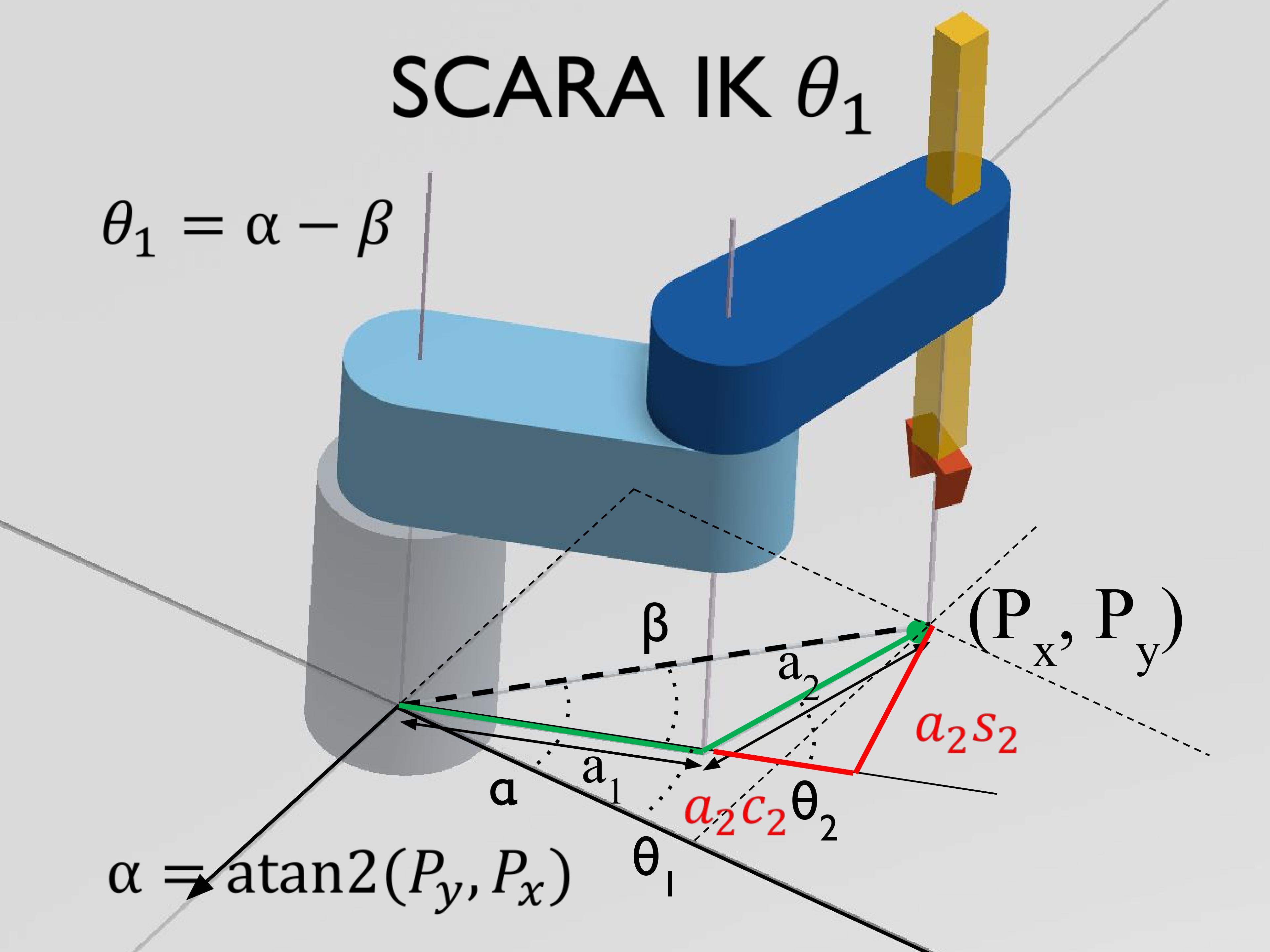
$$\boxed{\theta_2 = \pm\cos^{-1}\left(\frac{P_x^2 + P_y^2 - (a_1^2 + a_2^2)}{2a_1a_2}\right)}$$

$$c_2 = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$



SCARA IK θ_1

$$\theta_1 = \alpha - \beta$$

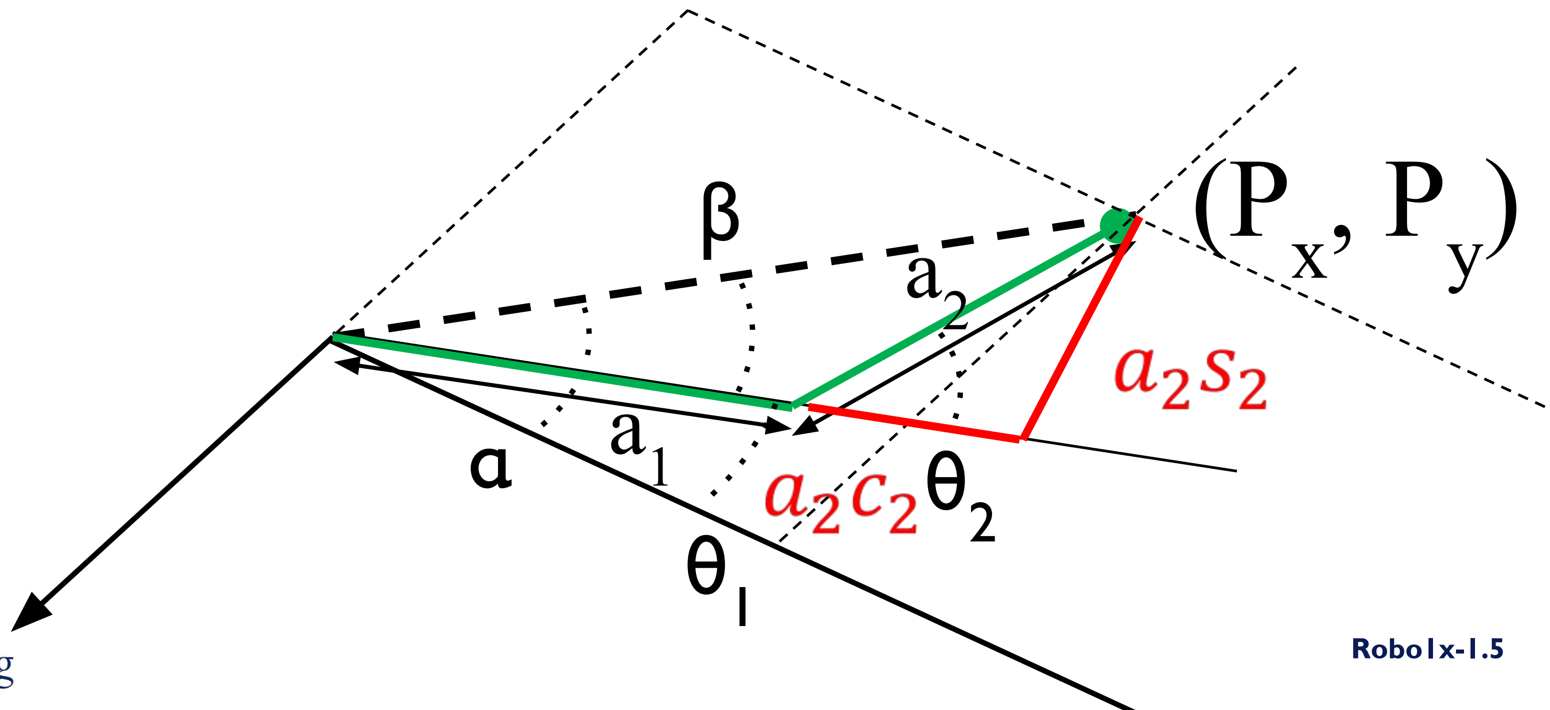


$$\alpha = \text{atan2}(P_y, P_x)$$

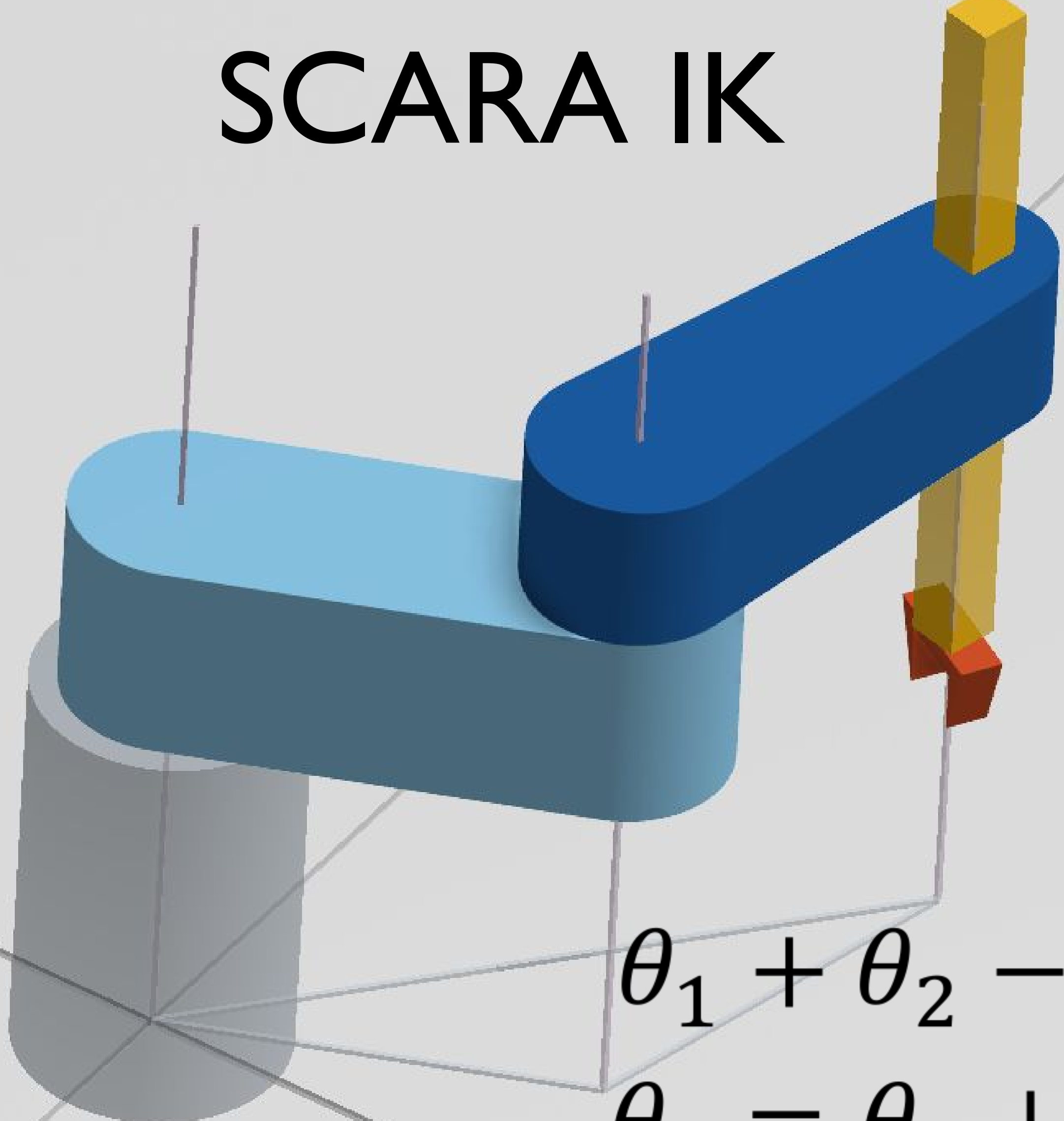
SCARA IK θ_1

$$\theta_1 = \alpha - \beta$$

$$\boxed{\theta_1 = \text{atan2}(P_y, P_x) - \text{atan2}(a_2 s_2,\ a_1 + a_2 c_2)}$$



SCARA IK

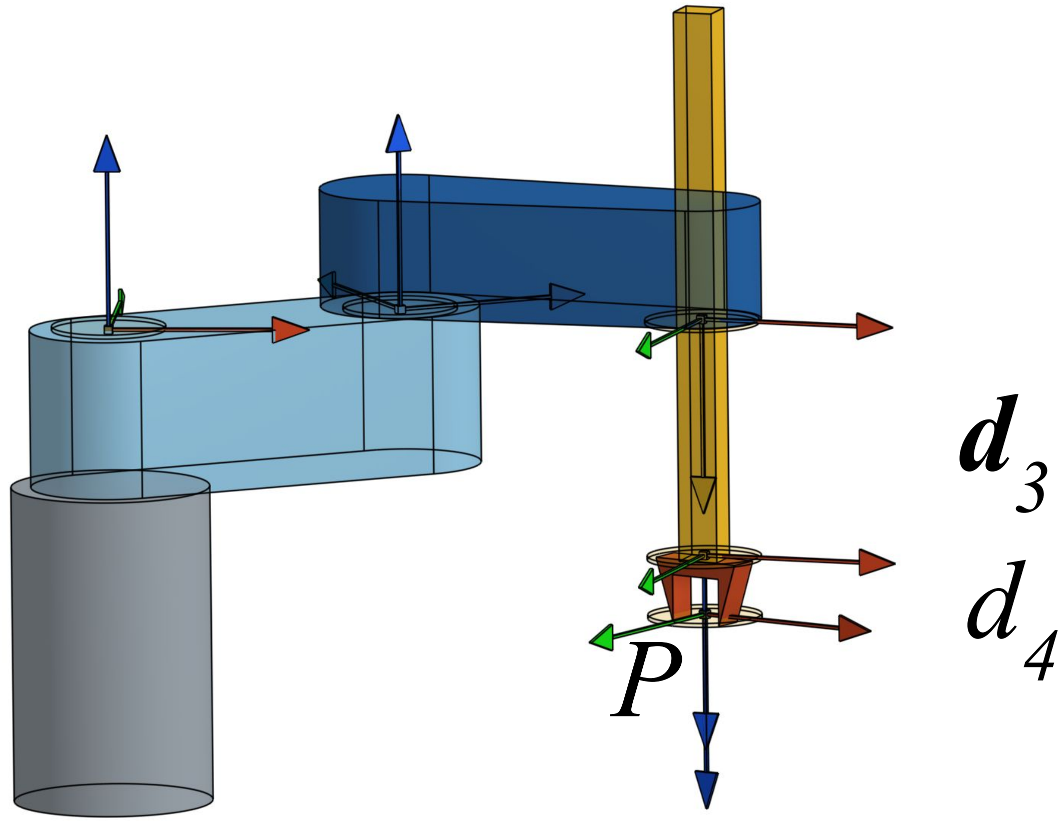


$$\theta_1 + \theta_2 - \theta_4 = \psi$$

$$\theta_4 = \theta_1 + \theta_2 - \psi$$

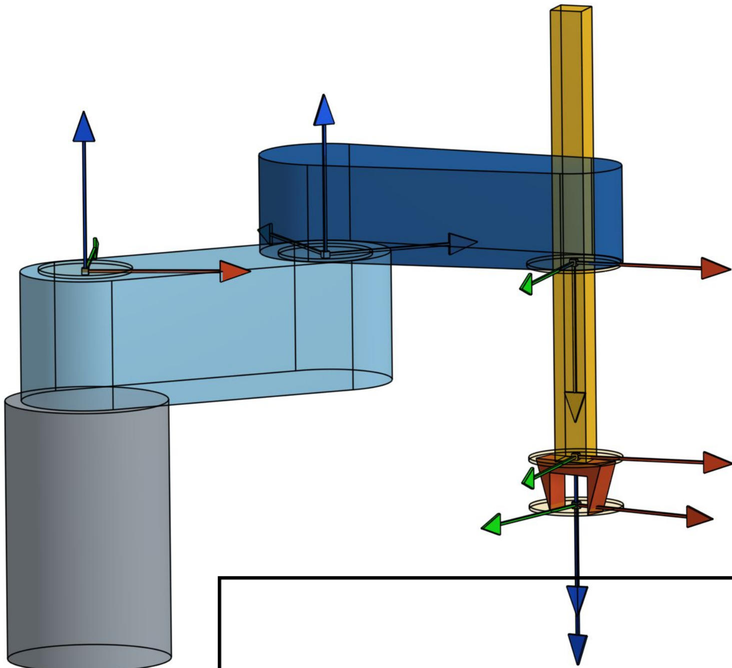
$$\boxed{\theta_4 = \theta_1 + \theta_2 - \operatorname{atan}(r_{12}, r_{11})}$$

SCARA IK d_3



$$\boxed{\boldsymbol{d}_3 = P_z + d_4}$$

SCARA IK



$$\theta_2 = \pm\cos^{-1}\left(\frac{P_x^2 + P_y^2 - (a_1^2 + a_2^2)}{2a_1a_2}\right)$$

$$\theta_1 = \text{atan2}(P_y, P_x) - \text{atan2}(a_2 s_2,\ a_1 + a_2 c_2)$$

$$\theta_4 = \theta_1 + \theta_2 - \text{atan}(r_{12}, r_{11})$$

$$\boldsymbol{d}_3 = P_z + d_4$$

SCARA IK

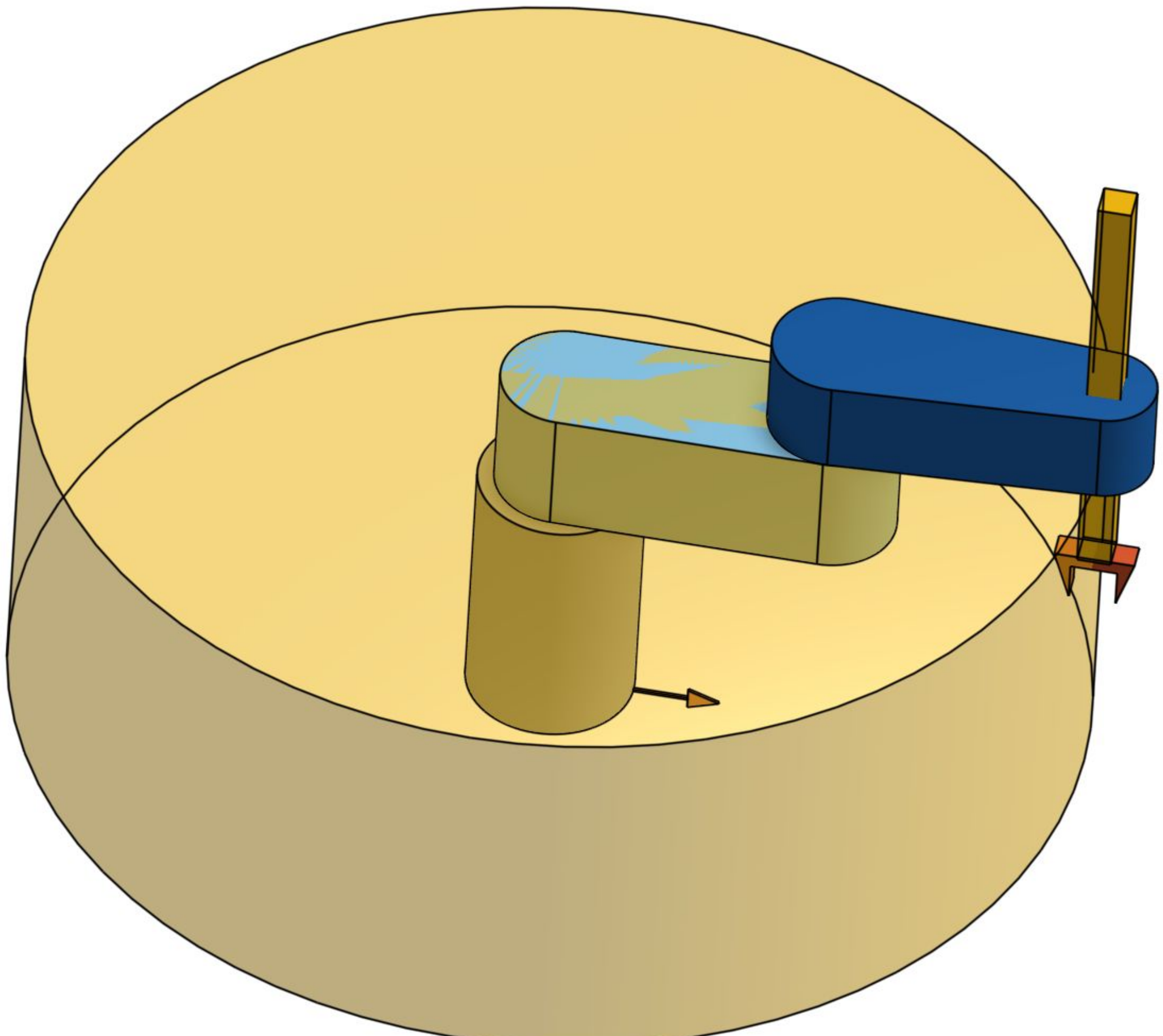
$$\begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boxed{\theta_2 = \pm\cos^{-1}\left(\frac{P_x^2 + P_y^2 - (a_1^2+a_2^2)}{2a_1a_2}\right)}$$

$$\sqrt{P_x^2 + P_y^2} = \sqrt{(a_1 + a_2)^2}$$

$$P_x^2 + P_y^2 = a_1^2 + 2a_1a_2 + a_2^2$$

Workspace



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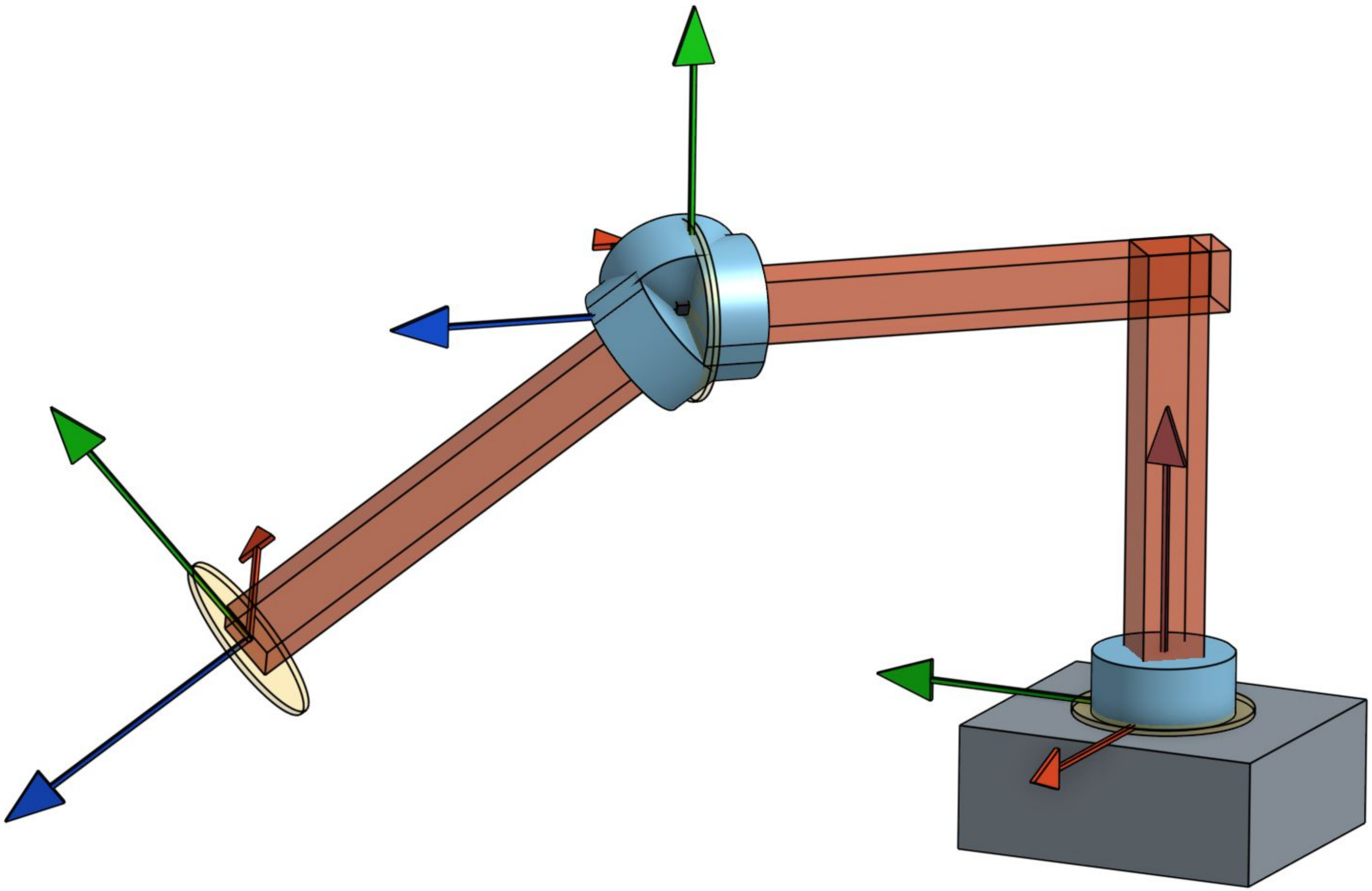
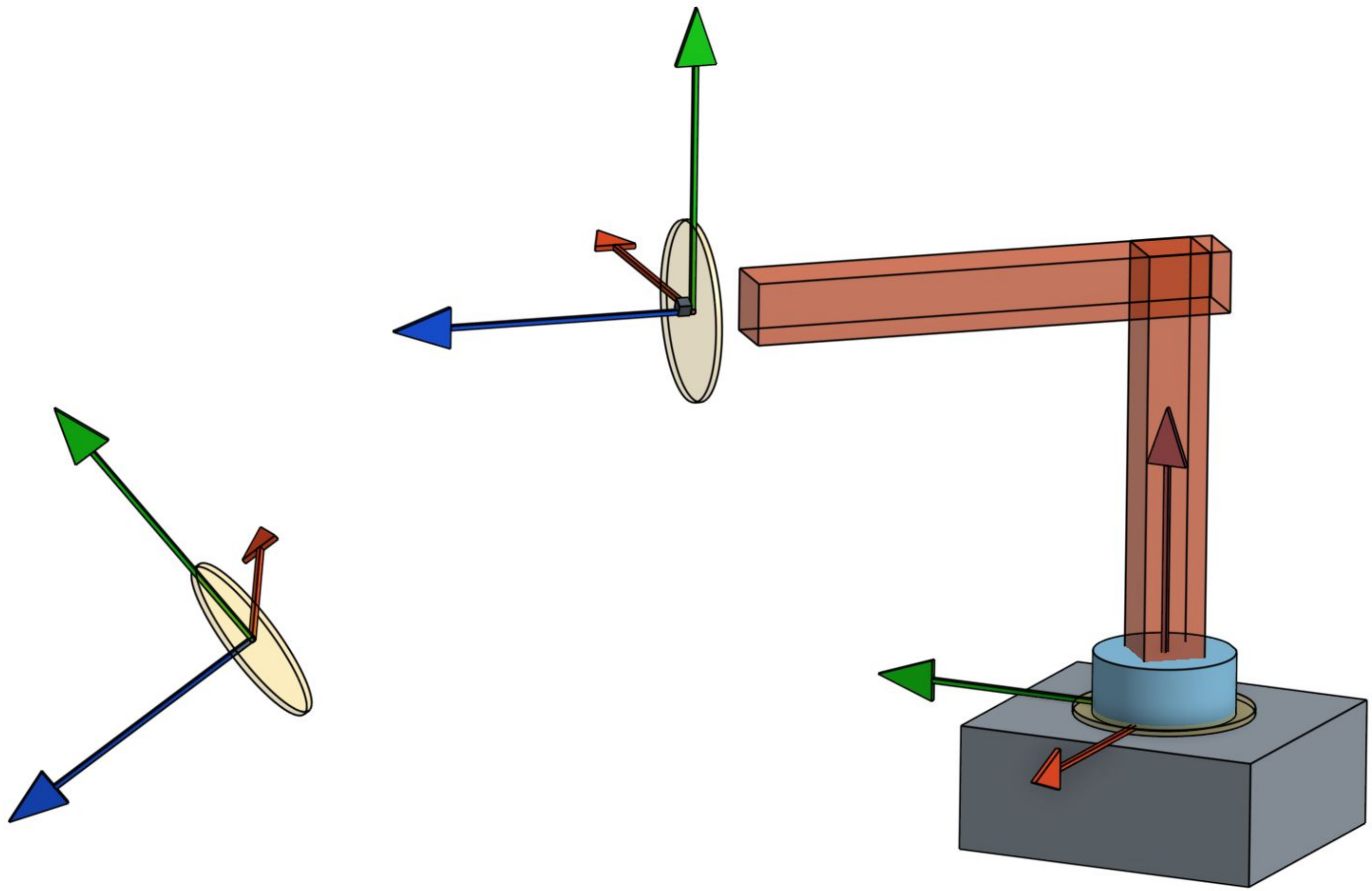
Week 6: Kinematic Decoupling and Inverse Position

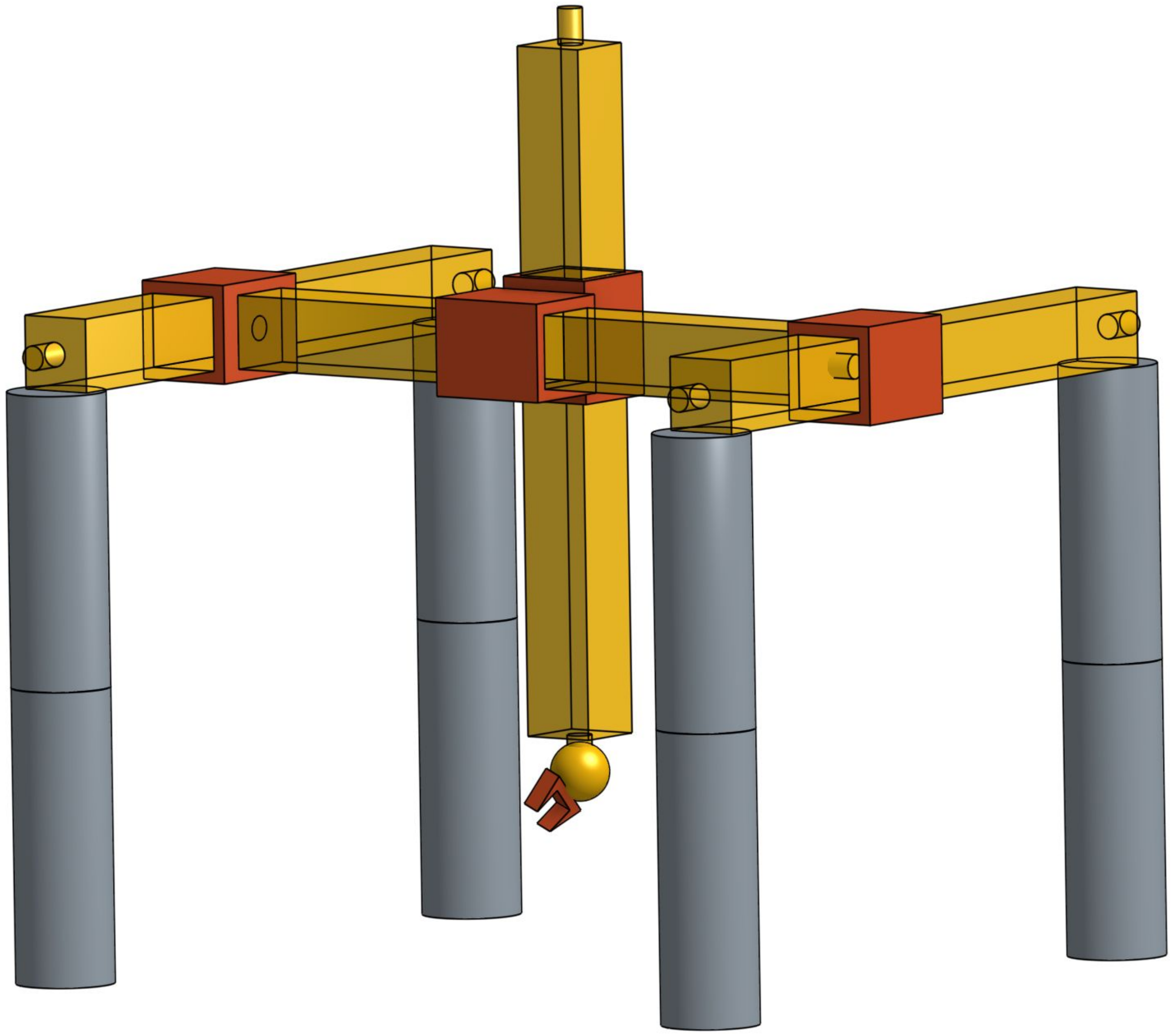
Kinematic Decoupling



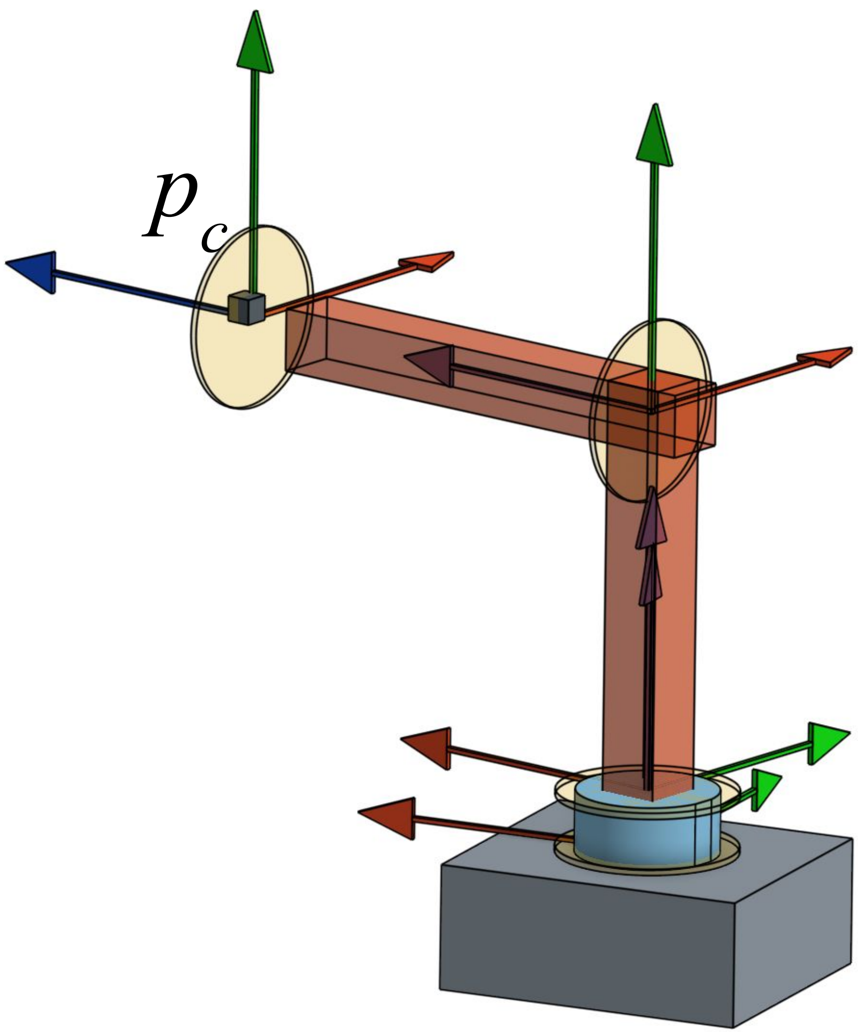
Kinematic Decoupling



Kinematic Decoupling



Inverse Position:
Cylindrical robot case

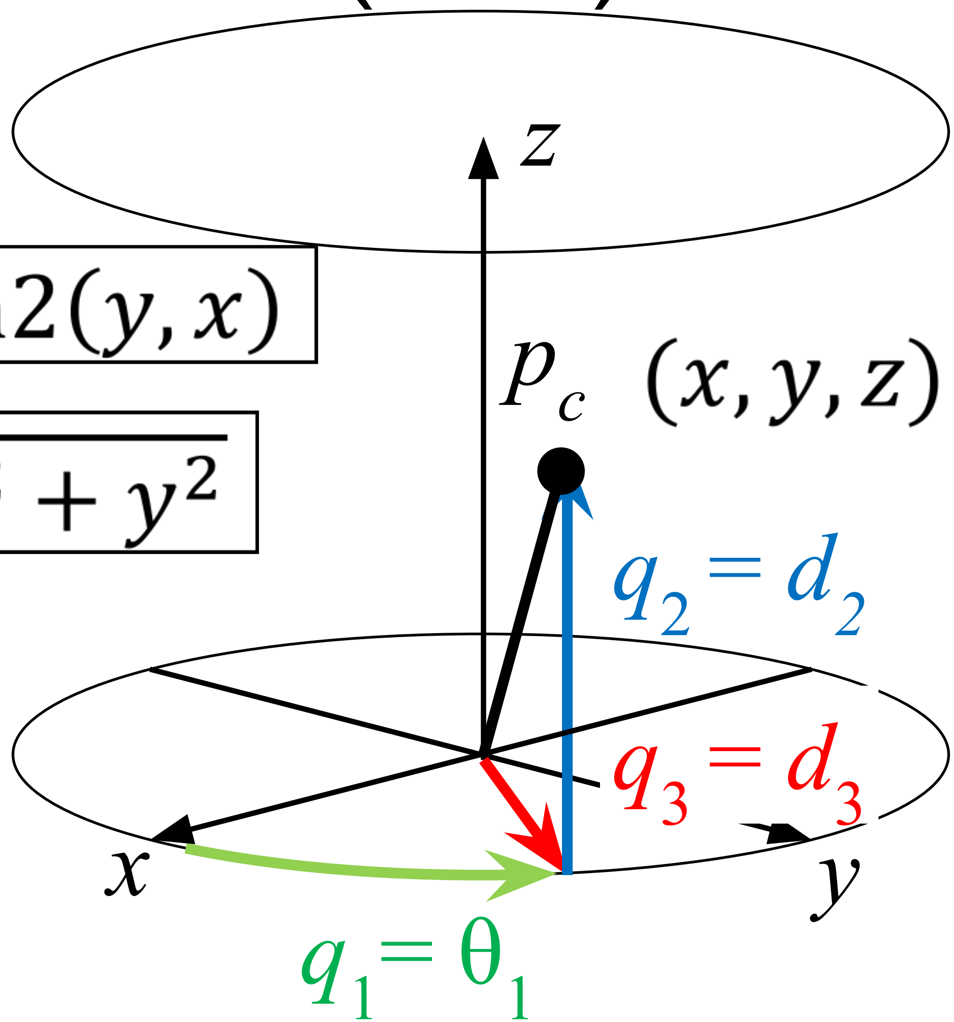


Cylindrical coordinates (RPP)

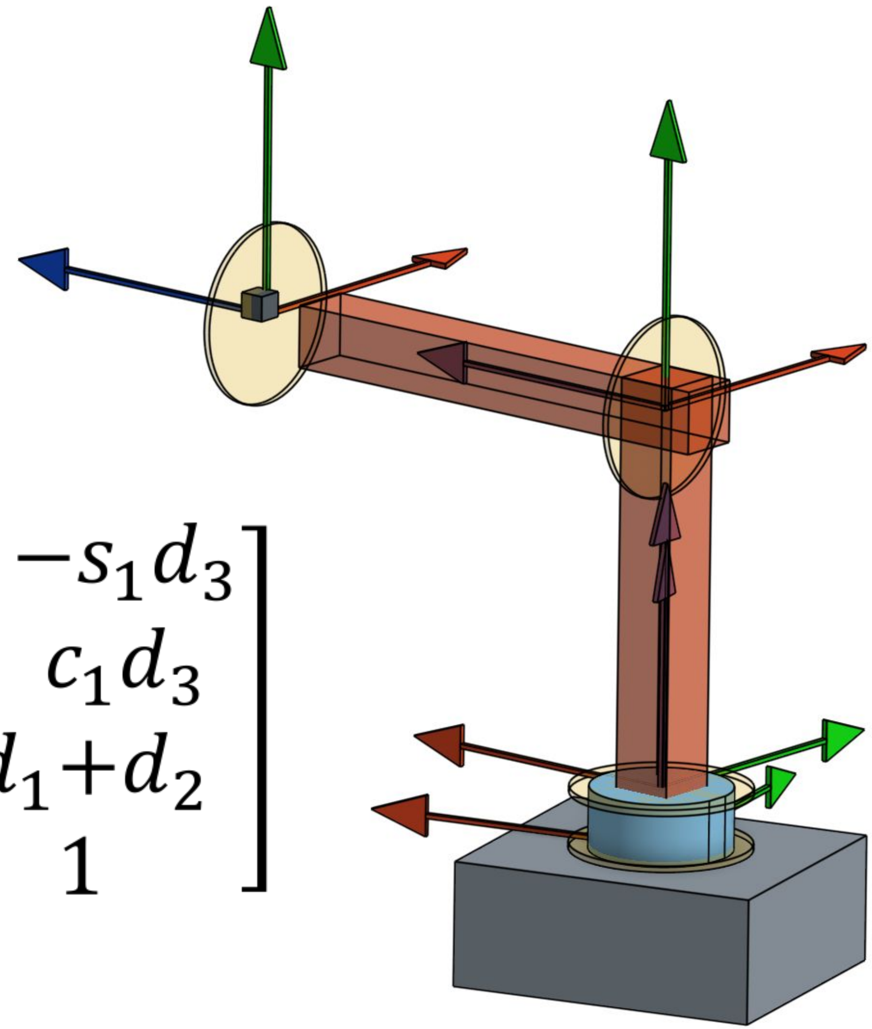
$$\boxed{\theta_1 = \mathrm{atan2}(y, x)}$$

$$\boxed{d_2 = \sqrt{x^2 + y^2}}$$

$$\boxed{d_3 = z}$$

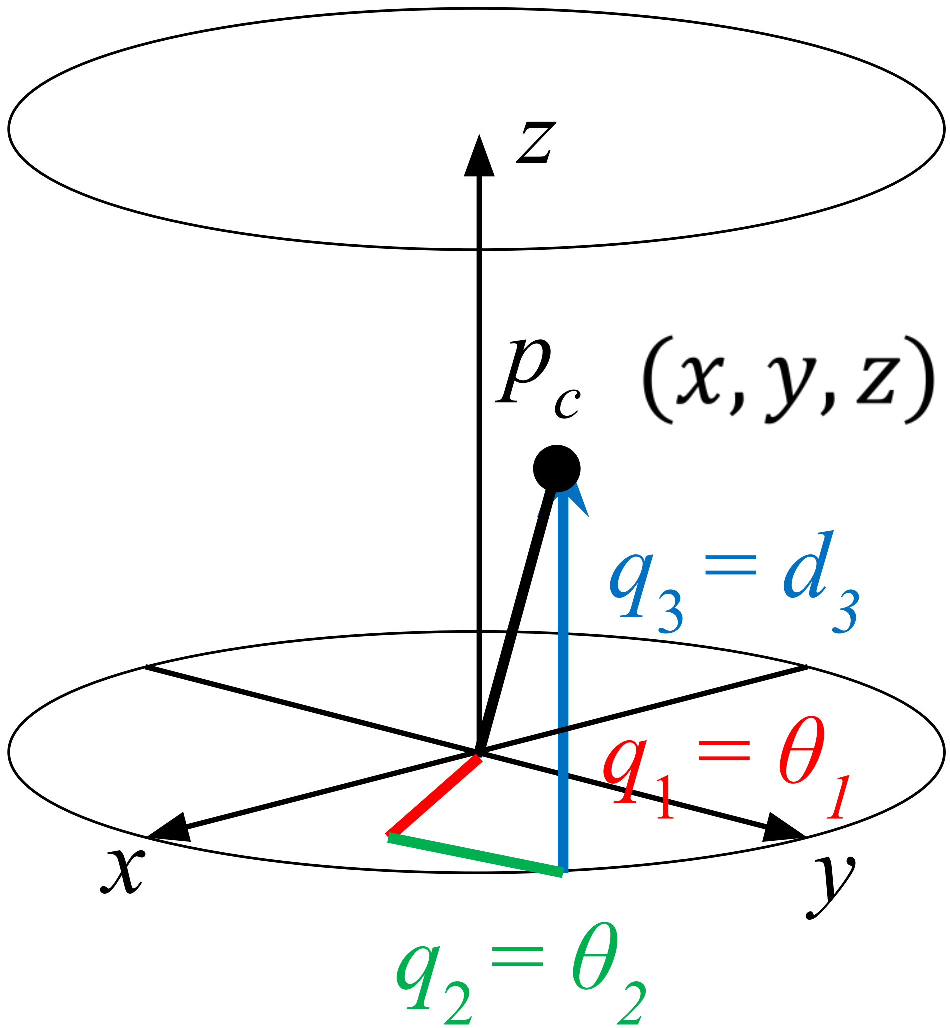


Wrist Center Rotation



$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1st part of SCARA (RRP)



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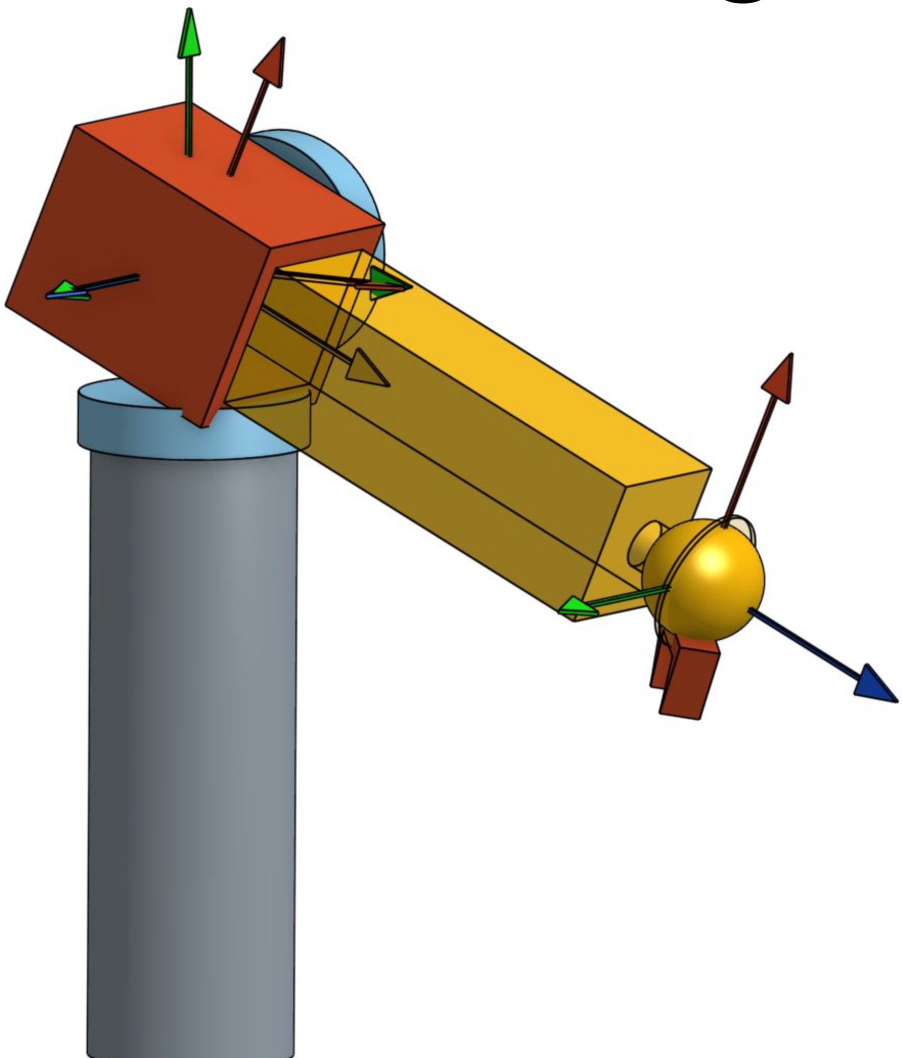
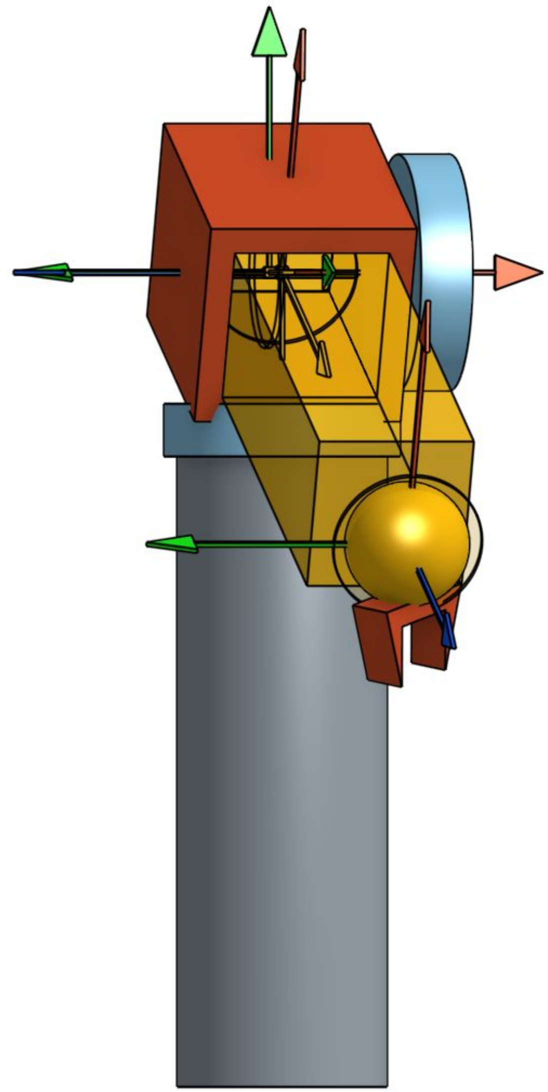
Inverse Position
Spherical Configuration (RRP)





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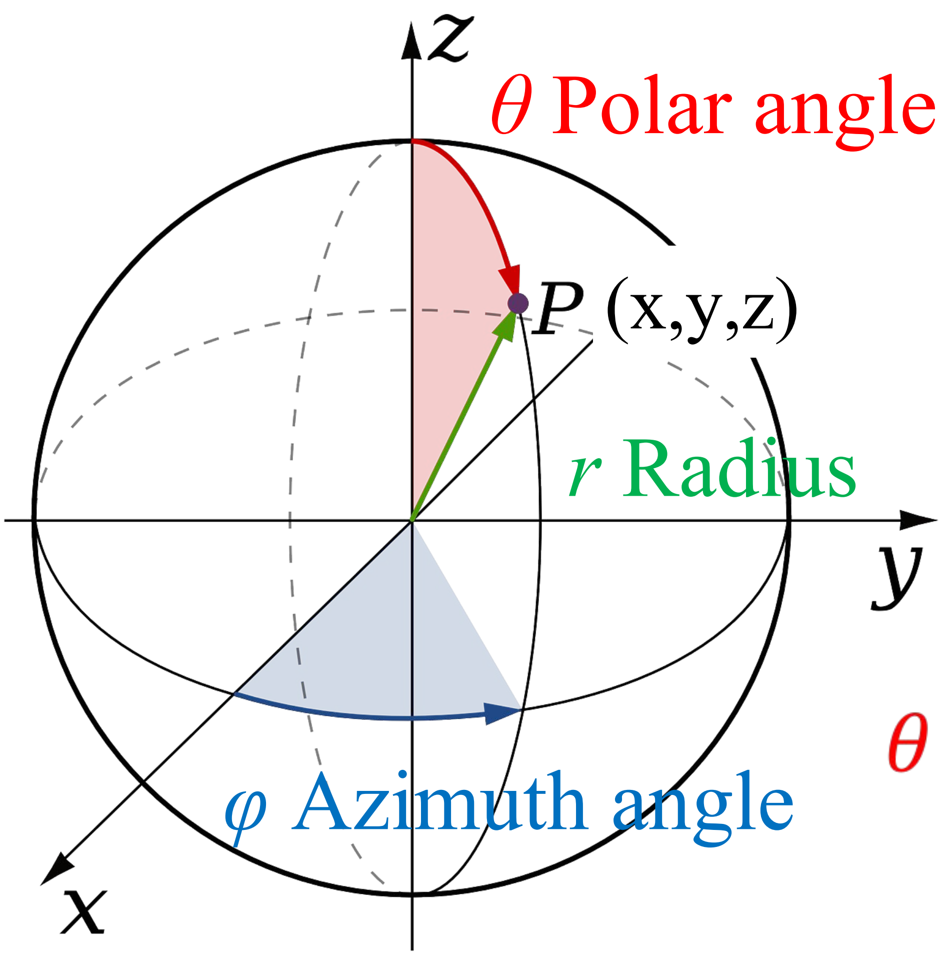
Spherical Coordinates



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = atan2(y, x)$$

$$\theta = atan2\left(\sqrt{x^2 + y^2}, z\right)$$

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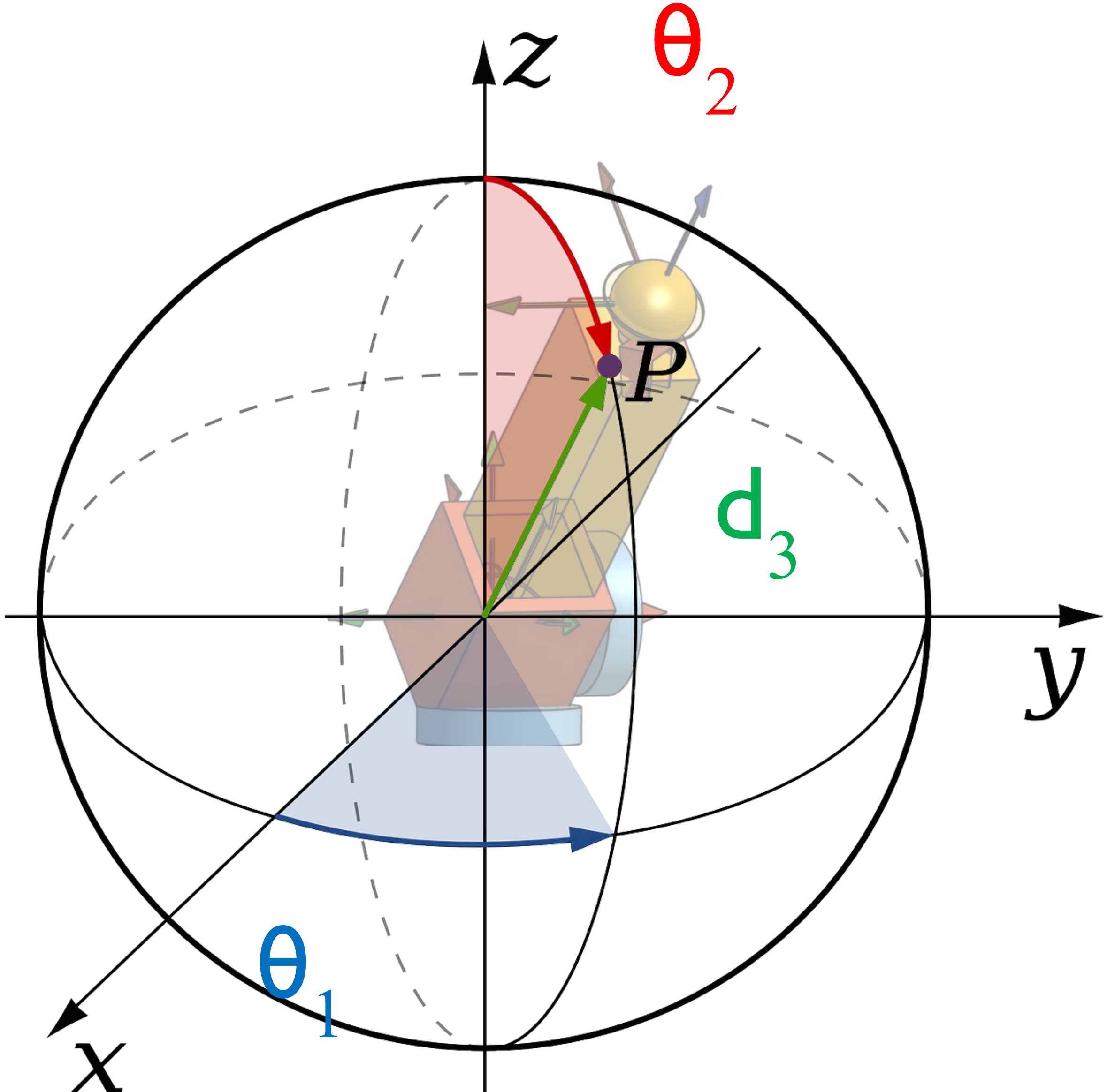
Inv. Pos. Spherical (RRP)

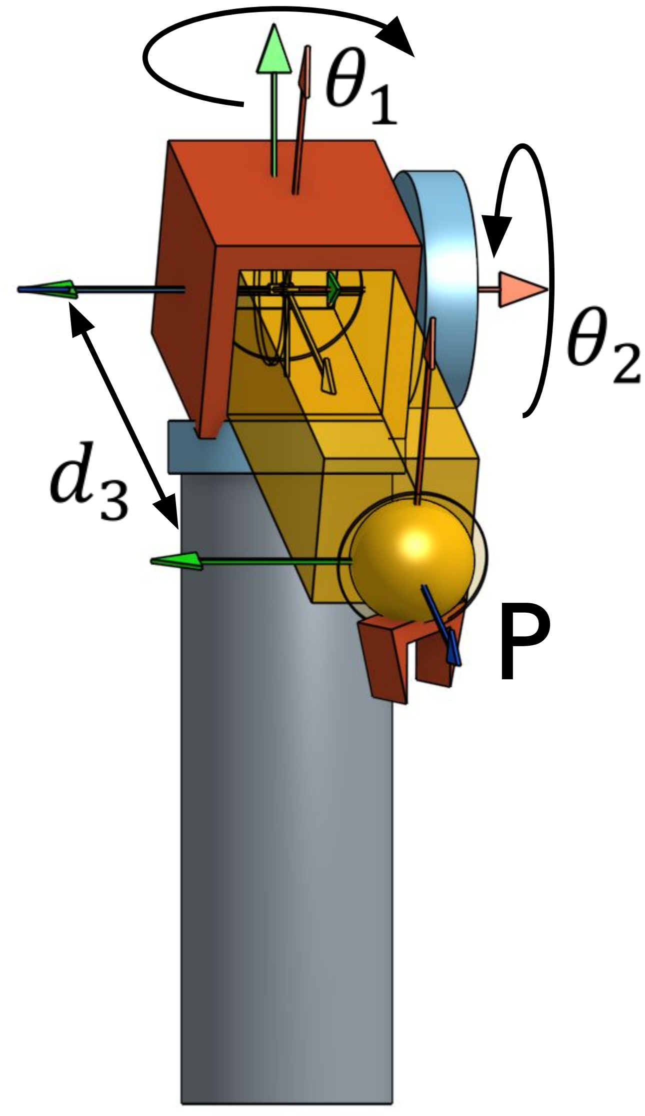


Inv. Pos. Spherical (RRP)

$$\boxed{\theta_1 = atan2\left(P_y, P_x\right)}$$

$$\boxed{\theta_2 = atan2\left(\sqrt{P_x^2 + P_y^2}, P_z\right)}$$

$$\boxed{d_3 = \sqrt{P_x^2 + P_y^2 + P_z^2}}$$



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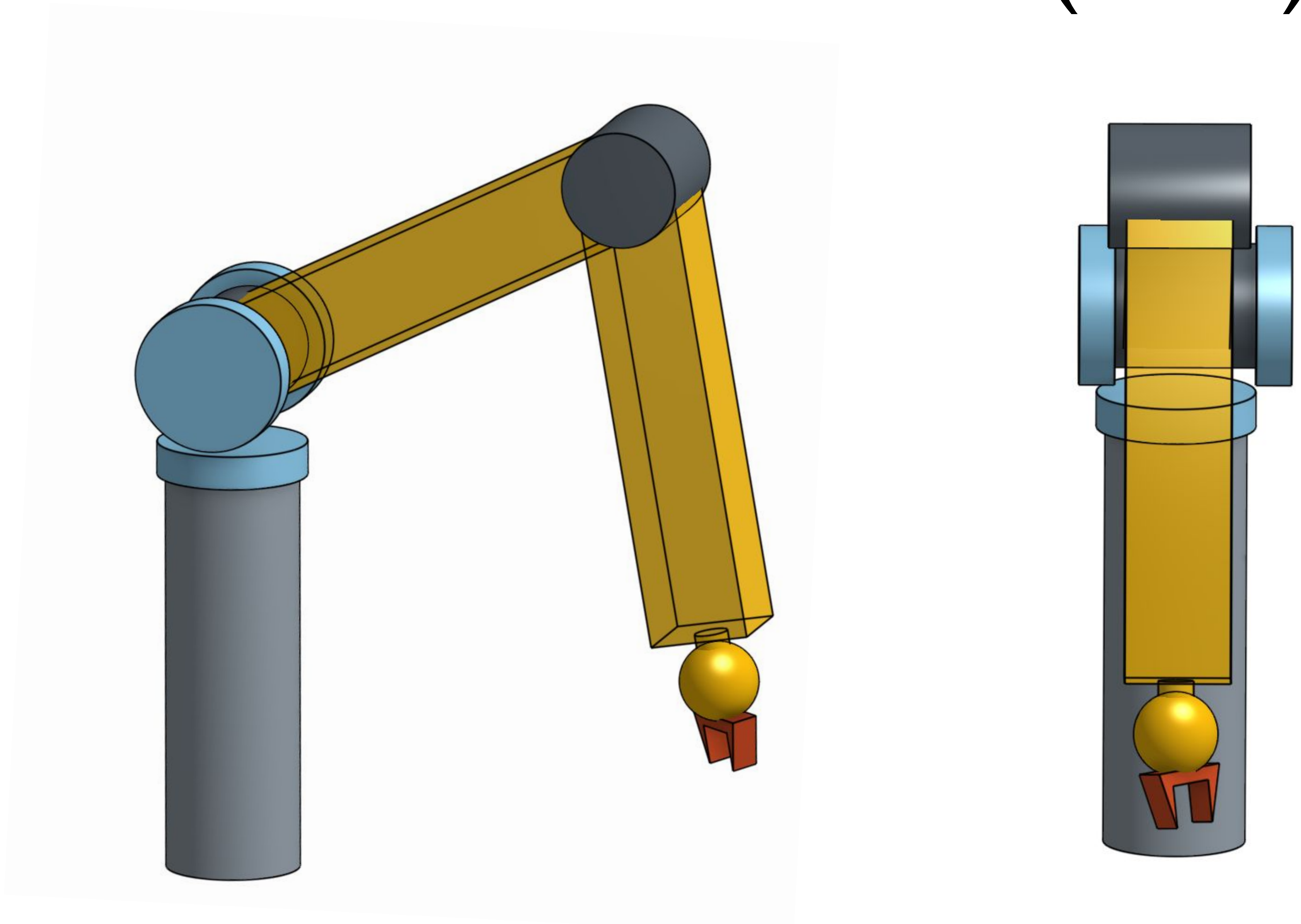
Inv. Pos. Articulated (RRR)



Inv. Pos. Articulated (RRR)

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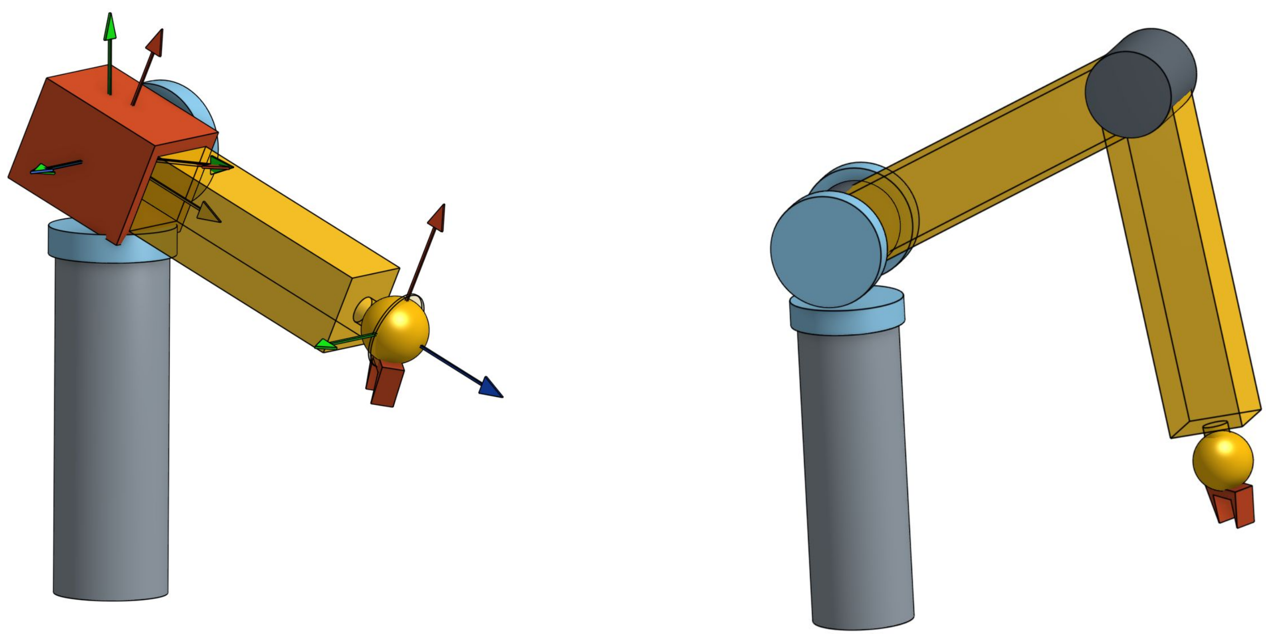
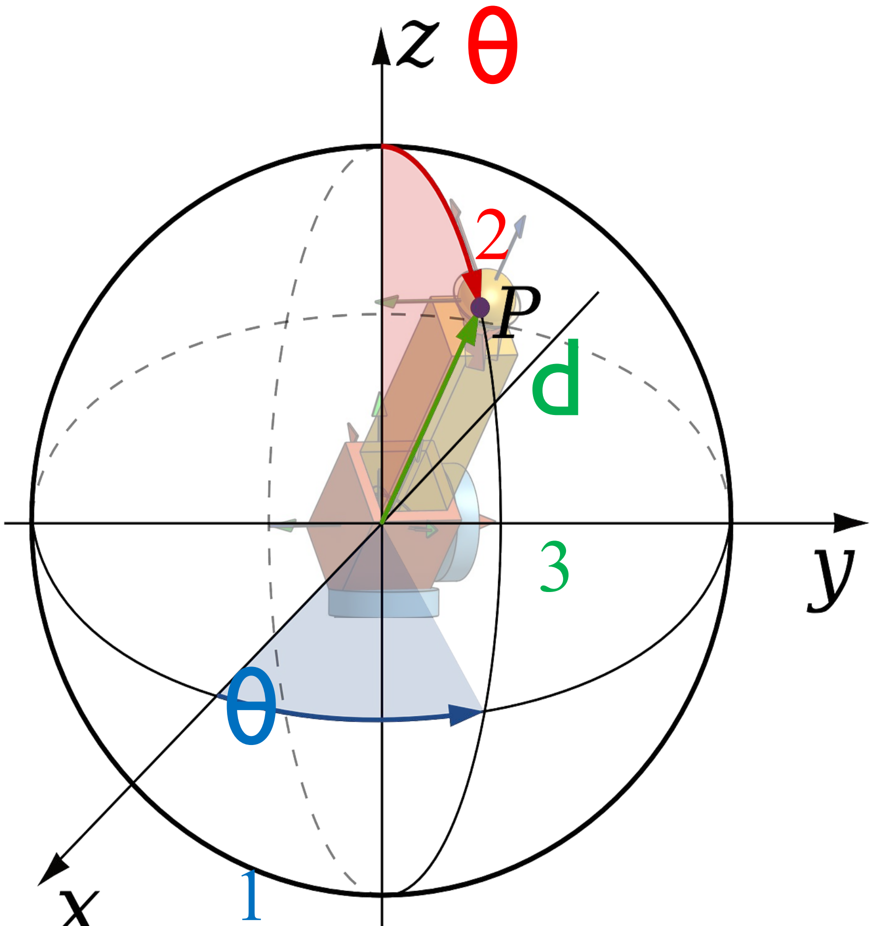
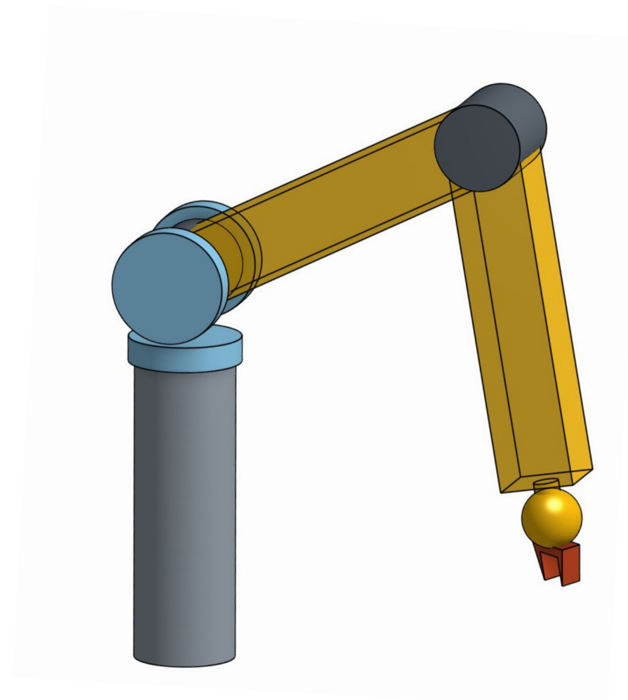
To each other simultaneously]



Inv. Pos. Articulated (RRR)



$\phi = atan2(y, x)$



$$\boxed{\theta_1 = atan2(y, x)}$$

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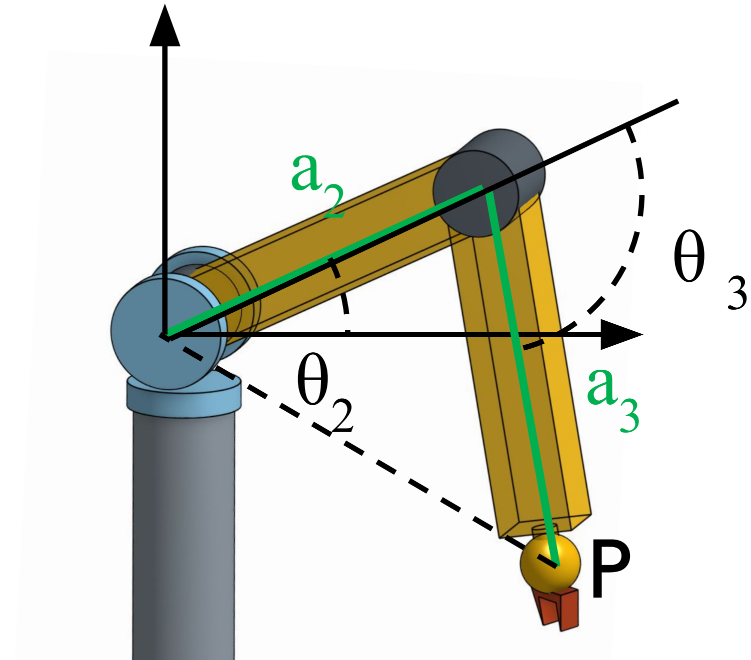
Inv. Pos. Articulated (RRR)



$$\theta_1 = atan2(P_y, P_x)$$

$$\theta_2 = \mathrm{atan2}(P_y, P_x) - \mathrm{atan2}(a_3 s_3,\, a_2 + a_3 c_3)$$

$$\theta_3 = \pm\cos^{-1}\left(\frac{P_x^2 + P_y^2 - (a_2^2 + a_3^2)}{2a_2 a_3}\right)$$

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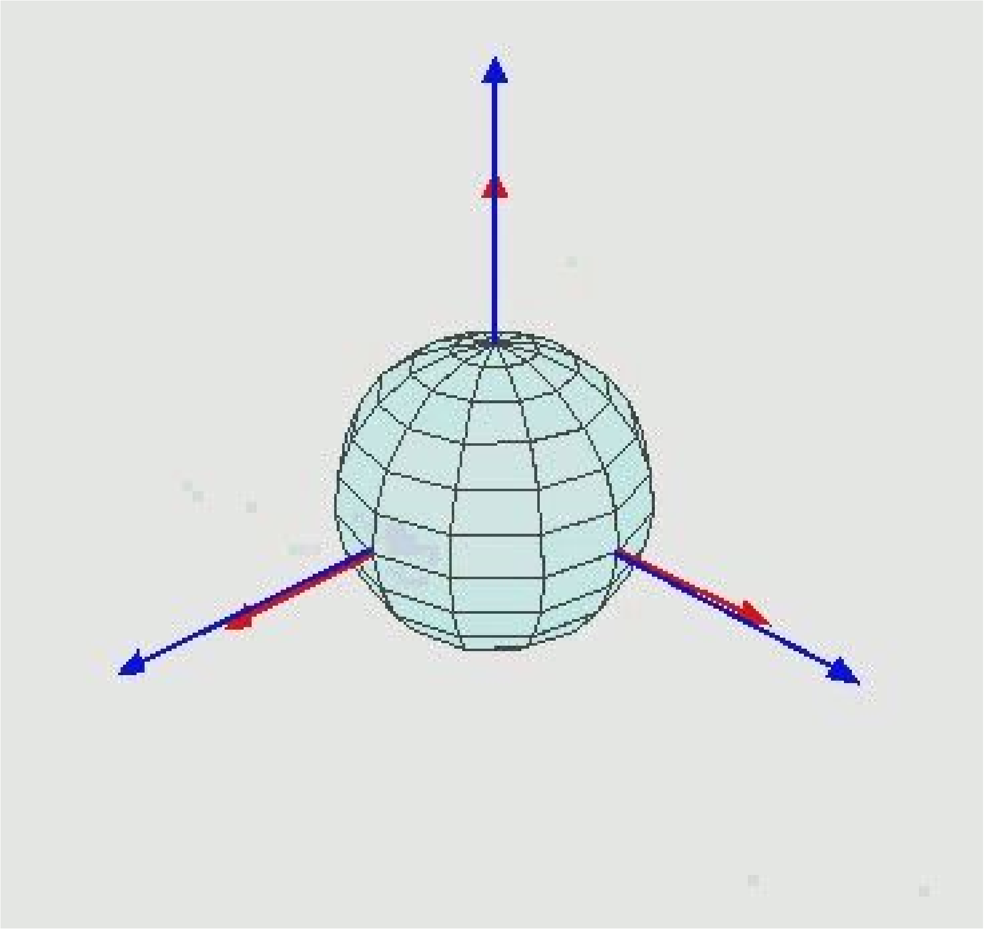
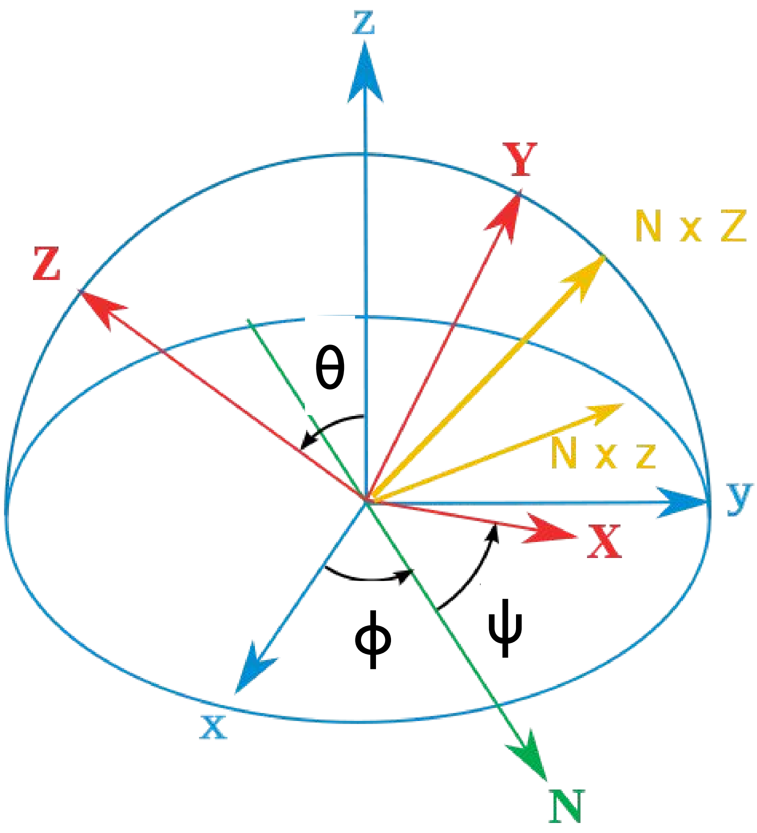
Week 6: Spherical Wrist and Inverse Orientation

Sequential Rotations



By Euler2.gif: Juansemperederivative work: Xavax - This file was derived from Euler2.gif:, CC BY-SA 3.0,

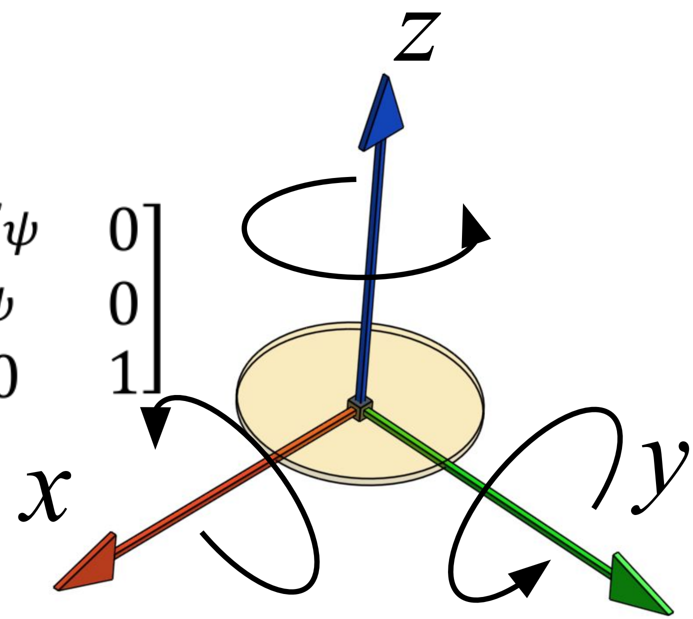


By Juansempere - Own work, GFDL,

Euler Angle Rotation

$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

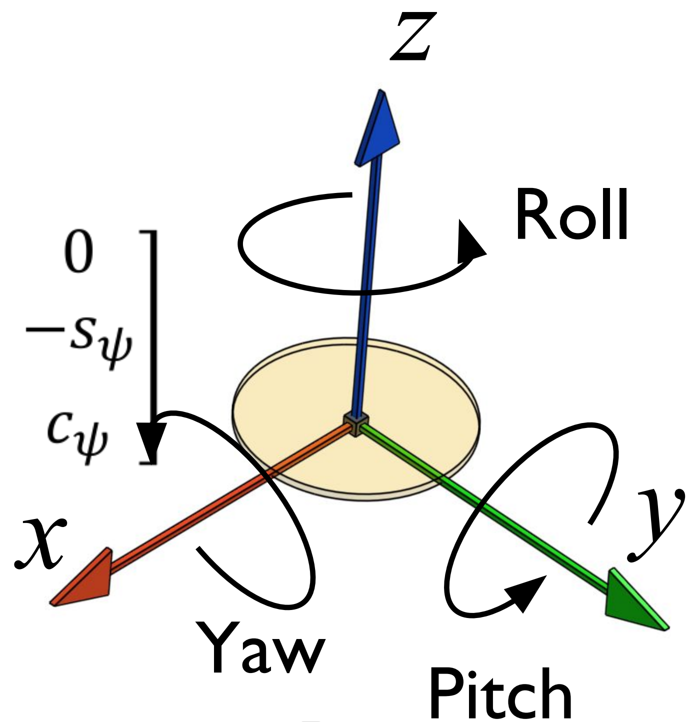


$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Roll Pitch Yaw Rotation

$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{X,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$



$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi\, s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\psi s_\theta & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

12 possible orderings

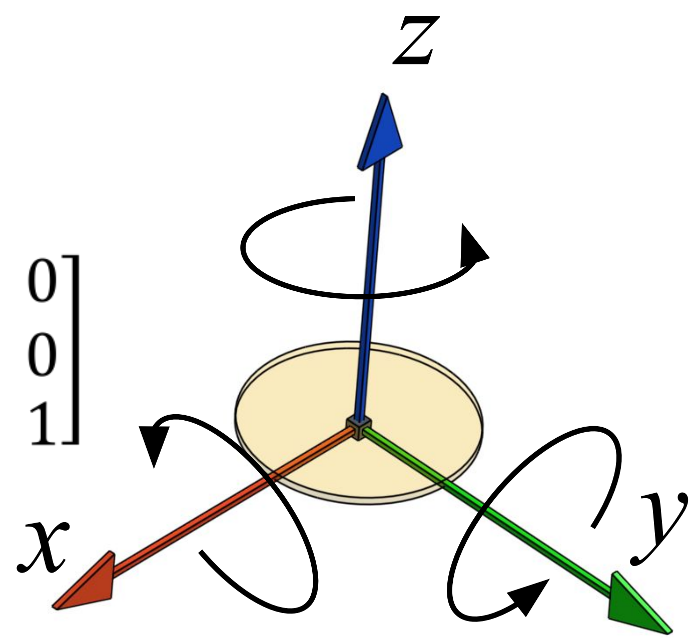
Proper Euler angles	Tait-Bryan angles
$X_1 Z_2 X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$X_1 Z_2 Y_3 = \begin{bmatrix} c_2 c_3 & -s_2 & c_2 s_3 \\ s_1 s_3 + c_1 c_3 s_2 & c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 \\ c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 \end{bmatrix}$
$X_1 Y_2 X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$X_1 Y_2 Z_3 = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 \\ s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$Y_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 \\ c_2 s_3 & c_2 c_3 & -s_2 \\ c_1 s_2 s_3 - c_3 s_1 & c_1 c_3 s_2 + s_1 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 Z_2 Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$Y_1 Z_2 X_3 = \begin{bmatrix} c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 \\ s_2 & c_2 c_3 & -c_2 s_3 \\ -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 \end{bmatrix}$
$Z_1 Y_2 Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$	$Z_1 Y_2 X_3 = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix}$
$Z_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$	$Z_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 \\ c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 \\ -c_2 s_3 & s_2 & c_2 c_3 \end{bmatrix}$

CCA https://en.wikipedia.org/wiki/Euler_angles#Tait.E2.80.93Bryan_angles

Euler Angle Rotation

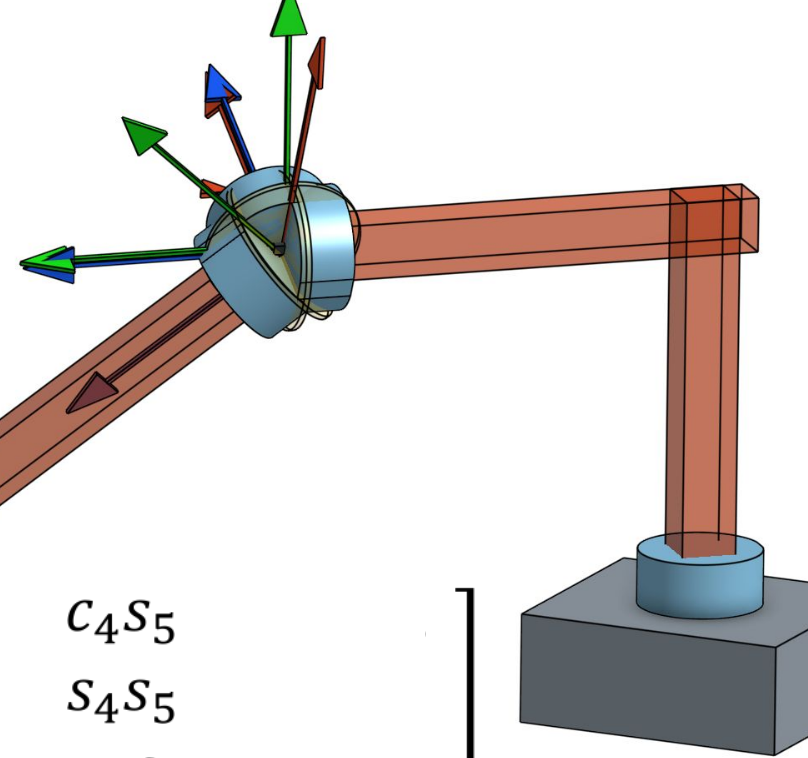
$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Euler Angle vs Spherical Wrist



$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\theta_4 = \phi, \quad \theta_5 = \theta, \quad \theta_6 = \psi$$

Inverse Orientation (ZYZ)

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 = ?$$
$$\theta_5 = ?$$
$$\theta_6 = ?$$

$$c_5 = r_{33}$$

$$\theta_5 = \pm cos^{-1}(r_{33})$$

Euler Angle Rotation

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 = ?$$

$$\boxed{\theta_5 = \pm cos^{-1}(r_{33})}$$

$$\theta_6 = ?$$

$$atan2(y, x) = atan2(Cy, Cx), C > 0$$

$$\theta_4 = atan2(r_{23}, r_{13}), \quad s_5 > 0$$

$$\theta_4 = atan2(-r_{23}, -r_{13}), s_5 < 0$$

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Euler Angle Rotation

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\theta_4 = \ atan2(\pm r_{23}, \pm r_{13})}$$

$$\boxed{\theta_5 = \ \pm \cos^{-1}(r_{33})}$$

$$\theta_6 = ?$$

$$atan2(y, x) = atan2(Cy, Cx), C > 0$$

$$\theta_6 = \ atan2(-r_{31}, r_{32}), \quad s_5 > 0$$

$$\theta_6 = \ atan2(r_{31}, -r_{32}), \quad s_5 < 0$$

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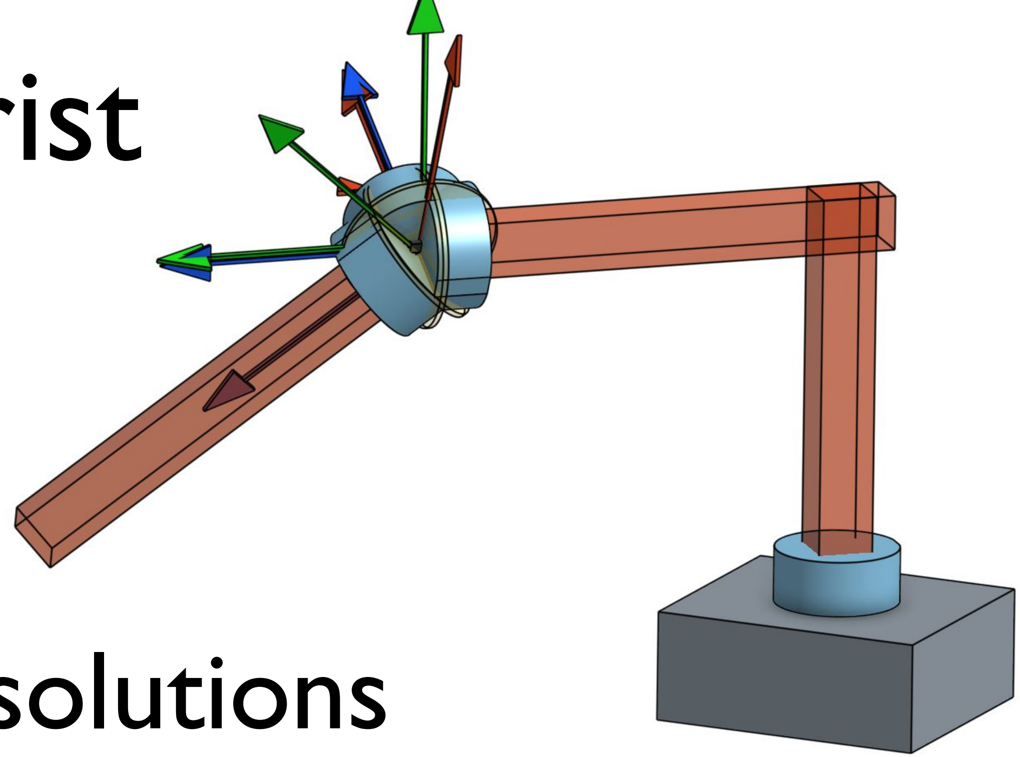
Spherical Wrist (ZYZ)



Two solutions

$$\boxed{\theta_4 = atan2(r_{23}, r_{13})}$$

$$\boxed{\theta_5 = cos^{-1}(r_{33})}$$

$$\boxed{\theta_6 = atan2(r_{32}, -r_{31})}$$

$$\boxed{\theta_4 = atan2(-r_{23}, -r_{13})}$$

$$\boxed{\theta_5 = -cos^{-1}(r_{33})}$$

$$\boxed{\theta_6 = atan2(-r_{32}, r_{31})}$$

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Inverse Pos. + Ori. For P_c

- From Inverse Position we have the joint variables $q_{1\ldots3}$ for $T_{03} = \begin{bmatrix} R_{03} & P_c \\ 0 & 1 \end{bmatrix}$
- From Inverse Orientation we have the joint variables $q_{4\ldots6}$ for a given arbitary R
- Can we put the two together to solve our initial problem for 6DOF?

$$\boldsymbol{T}_{06}(q_1 \cdots q_6) = \boldsymbol{H}$$

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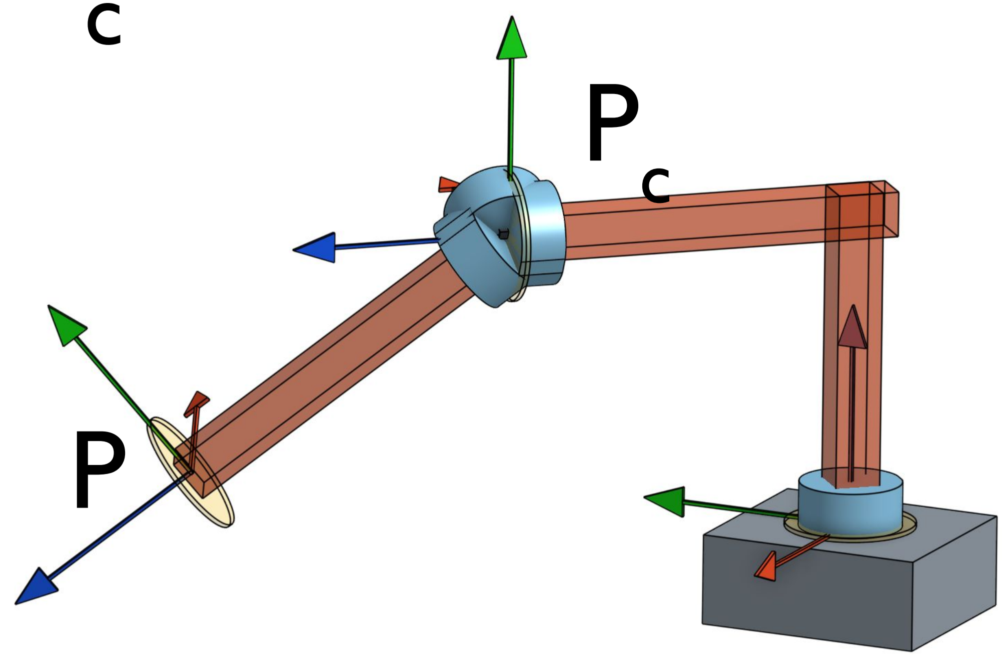
Offset from P_c wrist center

$$H = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\boldsymbol{P_c} = \boldsymbol{P} - d_6\hat{\boldsymbol{k}}$$

$$\begin{bmatrix} P_{cx} \\ P_{cy} \\ P_{cz} \end{bmatrix} = \begin{bmatrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{bmatrix}$$

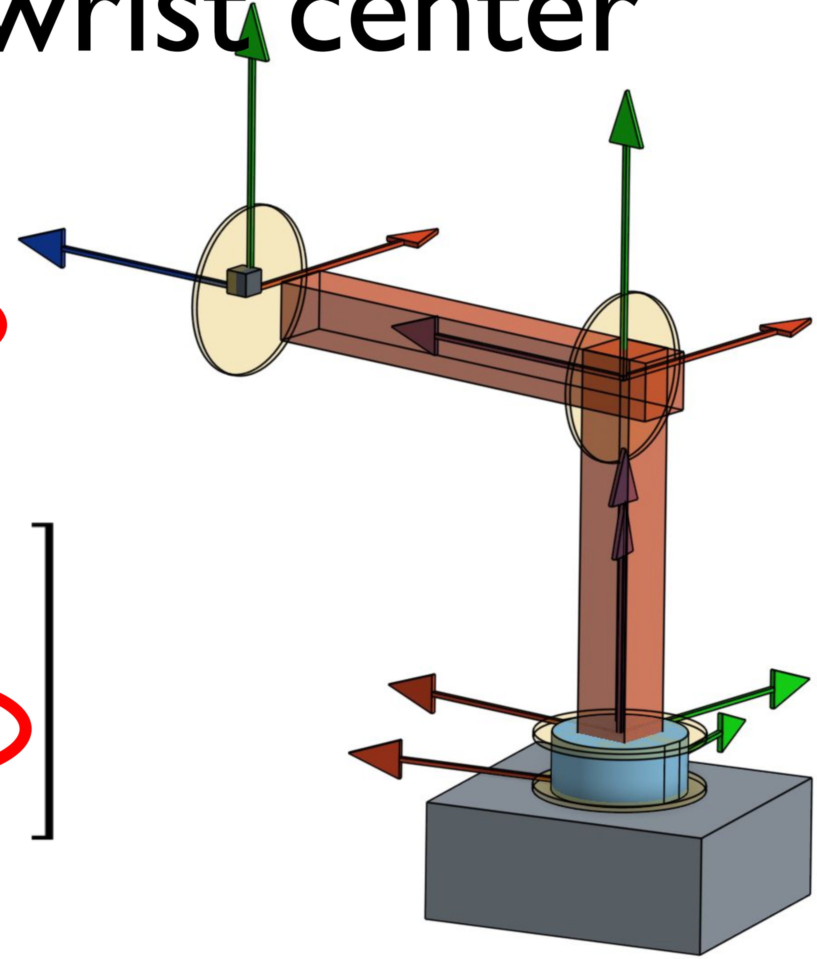
$$\boldsymbol{H_c} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} - d_6\hat{\boldsymbol{k}} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} & P_x - d_6 r_{13} \\ [\boldsymbol{R}] & P_y - d_6 r_{23} \\ & P_z - d_6 r_{33} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix}$$



Offset from P_c wrist center

$$H_c = \begin{bmatrix} & [\boldsymbol{R}] & & P_x - d_6 r_{13} \\ & & & P_y - d_6 r_{23} \\ & & & P_z - d_6 r_{33} \\ [0 & 0 & 0] & & & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\boxed{d_2 = P_z - d_6 r_{33} - d_1}$$

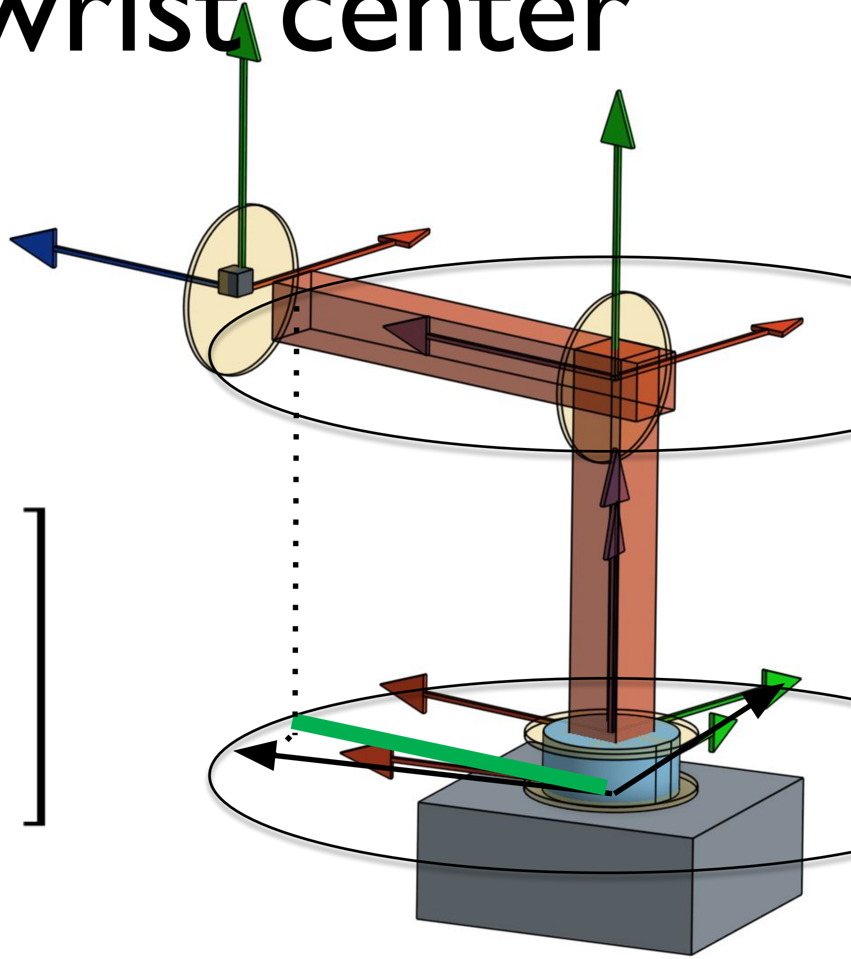
Offset from P_c wrist center

$$H_c = \begin{bmatrix} & [\boldsymbol{R}] & & \begin{matrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{matrix} \\ [0 & 0 & 0] & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_3 = \sqrt{x^2 + y^2}$$

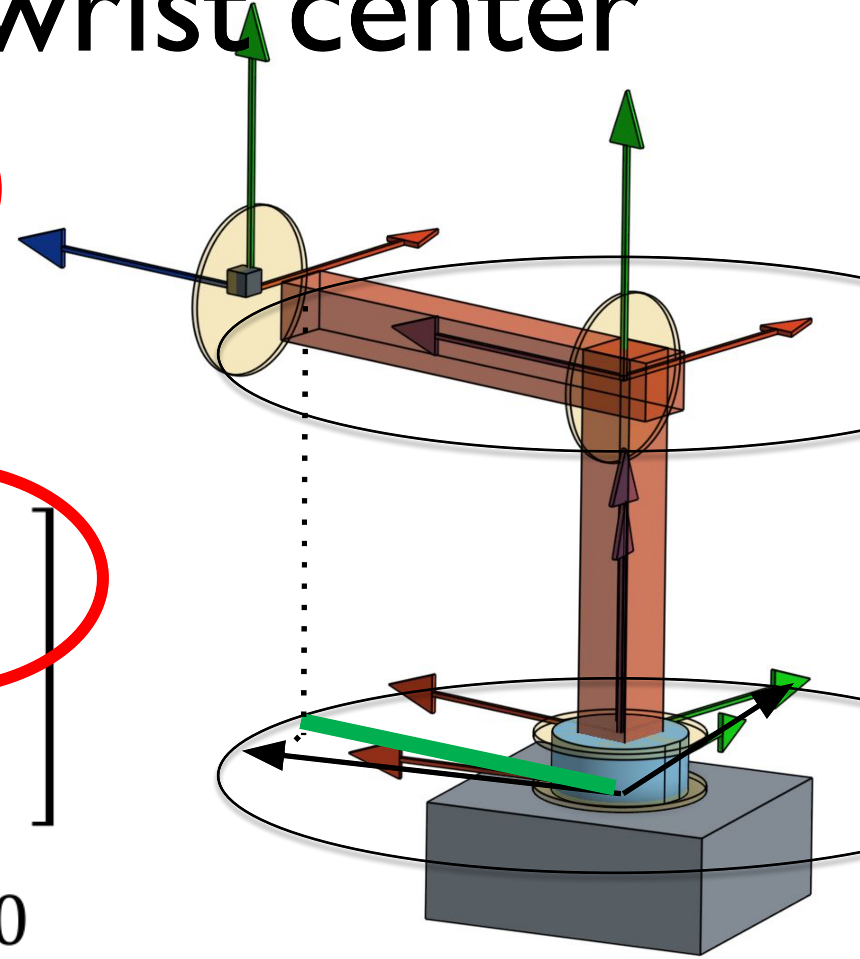
$$\boxed{d_3 = \sqrt{\left(P_y - d_6 r_{23}\right)^2 + \left(P_x - d_6 r_{13}\right)^2}}$$



Offset from P_c wrist center

$$\boldsymbol{H_c} = \begin{bmatrix} & [\boldsymbol{R}] & & P_x - d_6 r_{13} \\ & & & P_y - d_6 r_{23} \\ & & & P_z - d_6 r_{33} \\ [0 & 0 & 0] & & & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\theta_1 = atan2(Csin, \mathrm{Ccos}), C > 0$$

$$\boxed{\theta_1 = atan2(-(P_x - d_6 r_{13}), P_y - d_6 r_{23}), d_3 > 0}$$

$$\boxed{\theta_1 = atan2(P_x - d_6 r_{13}, -P_y + d_6 r_{23}), d_3 < 0}$$

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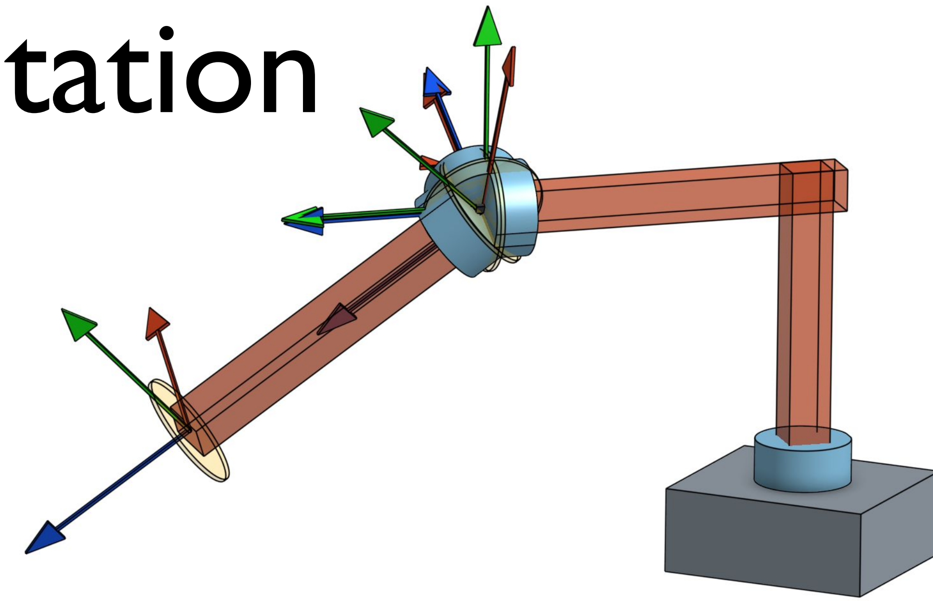
Inverse Orientation

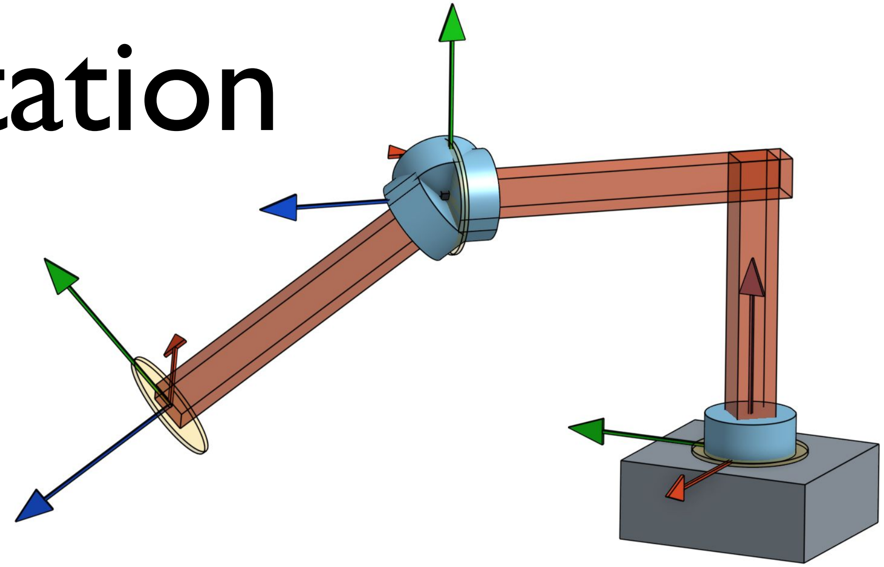


$$H_c = \begin{bmatrix} [R] & [P - d_6 \widehat{k}] \\ [0] & 1 \end{bmatrix}$$

Spherical Wrist solution for *R?*

$$\theta_4 = \ atan2(\pm r_{23}, \pm r_{13})$$
$$\theta_5 = \ \pm\, cos^{-1}\,(r_{33})$$
$$\theta_6 = \ atan2(\mp r_{31}, \pm r_{32})$$

Inverse Orientation



$$H_c = \begin{bmatrix} [R] & [P - d_6\hat{k}] \\ [0] & 1 \end{bmatrix}$$

$$T_{06} = \begin{bmatrix} [R_{03}][R_{36}] & P_c \\ [0] & 1 \end{bmatrix}$$

$$R_{06} = R_{03}R_{36} = R$$

$$R_{36} = (R_{03})^{-1}R$$

$$R_{36} = (R_{03})^{T}R$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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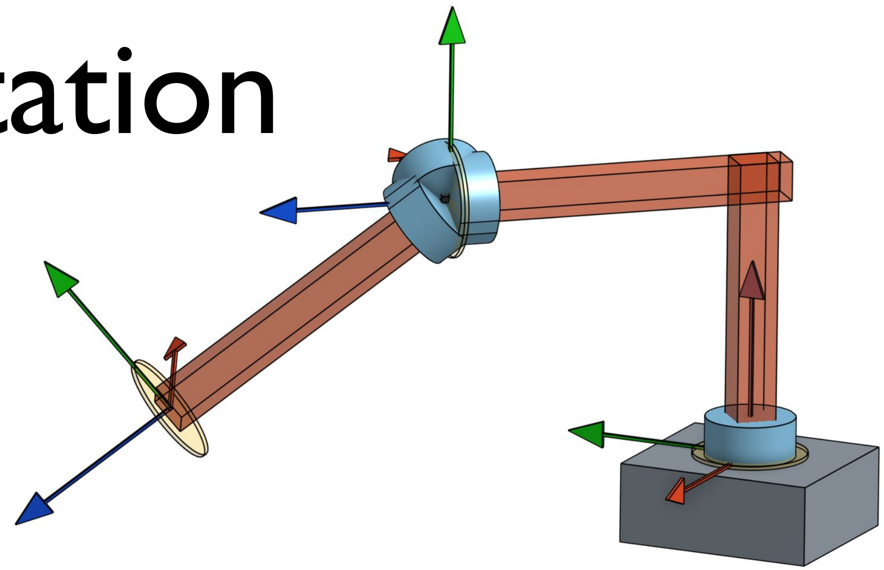
Inverse Orientation



$$\boldsymbol{R_{36}} = \begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} \\ -r_{31} & -r_{32} & -r_{33} \\ c_1 r_{21} - s_1 r_{11} & c_1 r_{22} - s_1 r_{12} & c_1 r_{23} - s_1 r_{13} \end{bmatrix}$$

$$\boldsymbol{R_{36}} = (\boldsymbol{R_{03}})^T \boldsymbol{R}$$

$$T_{03} = \begin{bmatrix} \color{red}{c_1} & 0 & \color{red}{-s_1} & \color{red}{-s_1 d_3} \\ \color{red}{s_1} & 0 & \color{red}{c_1} & \color{red}{c_1 d_3} \\ 0 & -1 & 0 & d_1 + \color{red}{d_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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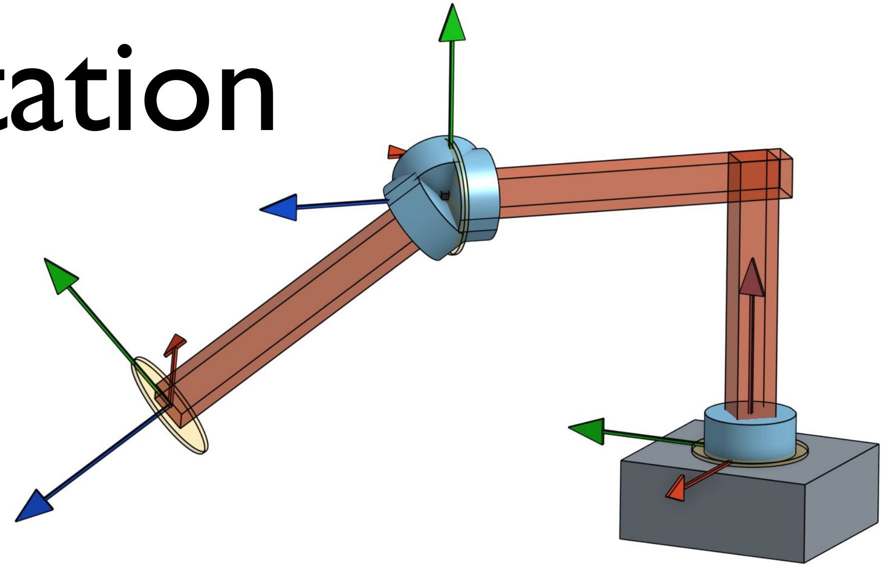
Inverse Orientation

$$\theta_4 = \ atan2(r_{23}, r_{13})$$

$$\theta_5 = \ \pm\, cos^{-1}\,(r_{33})$$

$$\theta_6 = \ atan2(-r_{31}, r_{32})$$



$$\boldsymbol{R_{36}} = \begin{bmatrix} c_1r_{11} + s_1r_{21} & c_1r_{12} + s_1r_{22} & c_1r_{13} + s_1r_{23} \\ -r_{31} & -r_{32} & -r_{33} \\ c_1r_{21} - s_1r_{11} & c_1r_{22} - s_1r_{12} & c_1r_{23} - s_1r_{13} \end{bmatrix}$$

$$\boxed{\theta_4 = \ atan2(-r_{33}, c_1r_{13} + s_1r_{23})}$$

$$\boxed{\theta_5 = \pm\, cos^{-1}(c_1r_{23} - s_1r_{13})}$$

$$\boxed{\theta_6 = atan2(\pm(s_1r_{11} - c_1\, r_{21}), \pm(c_1r_{22} - s_1r_{12}))}$$

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IK for Cylindrical RPP

$$\text{2 soln}\ \left\{ \begin{array}{l} \boxed{\theta_1 = atan2\left(\mp(P_x - d_6 r_{13}), \pm(P_y - d_6 r_{13})\right)} \\[1em] \boxed{d_2 = P_z - d_6 r_{33}} \\[1em] \boxed{d_3 = \sqrt{\left(P_y - d_6 r_{23}\right)^2 + (P_x - d_6 r_{13})^2}} \end{array} \right.$$

$$\text{2 soln}\ \left\{ \begin{array}{l} \boxed{\theta_4 = atan2(\mp r_{33}, \pm c_1 r_{13} \pm s_1 r_{23})} \\[1em] \boxed{\theta_5 = \pm\, cos^{-1}(c_1 r_{23} - s_1 r_{13})} \\[1em] \boxed{\theta_6 = atan2(\pm(s_1 r_{11} - c_1 r_{21}), \pm(c_1 r_{22} - s_1 r_{12}))} \end{array} \right.$$

Robotics: Fundamentals

Prof. Mark Yim
University of Pennsylvania

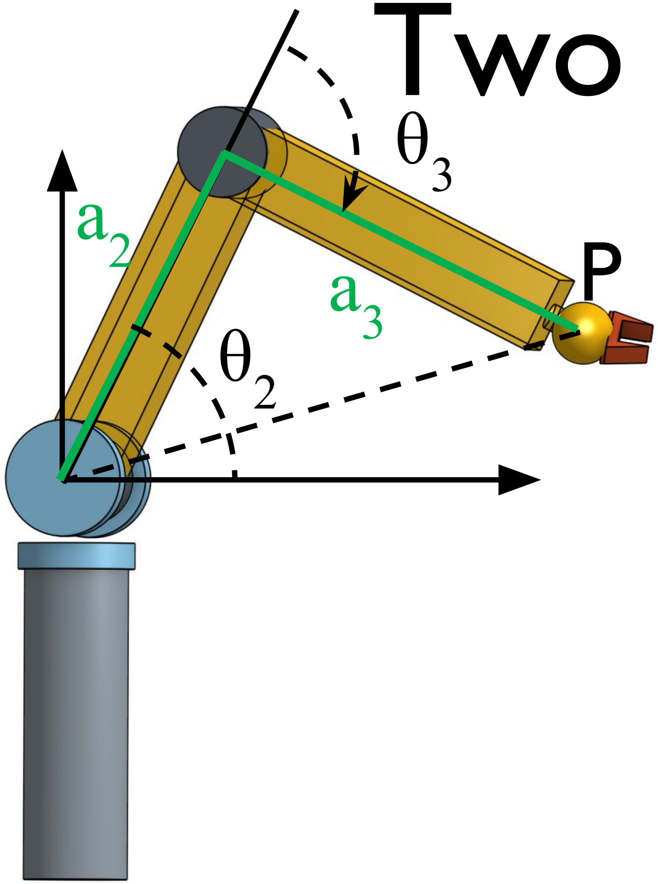
Week 6: Multiple Inverse Kinematic Solutions

Penn Engineering

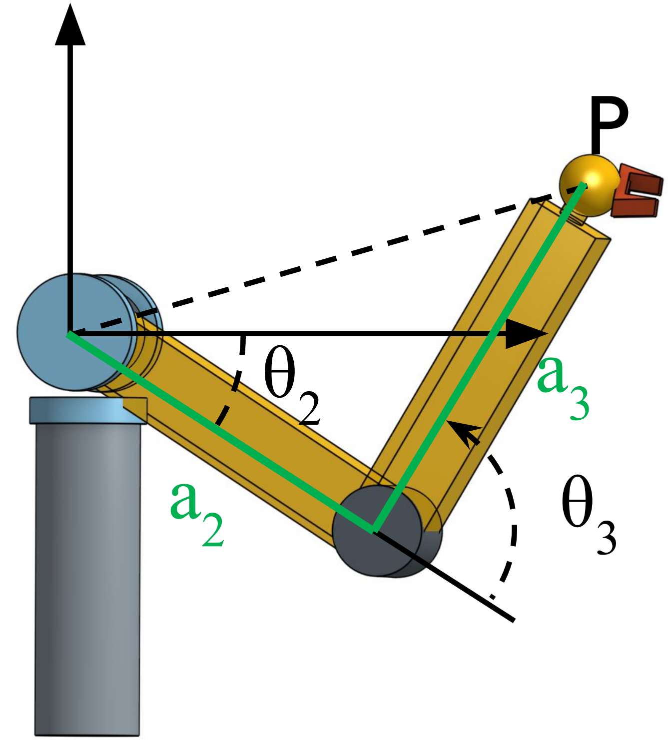
Special cases

- Multiple solutions
- Gimbal lock
- Representation singularity
- Joint singularity
- Dextrous workspace

Two solutions

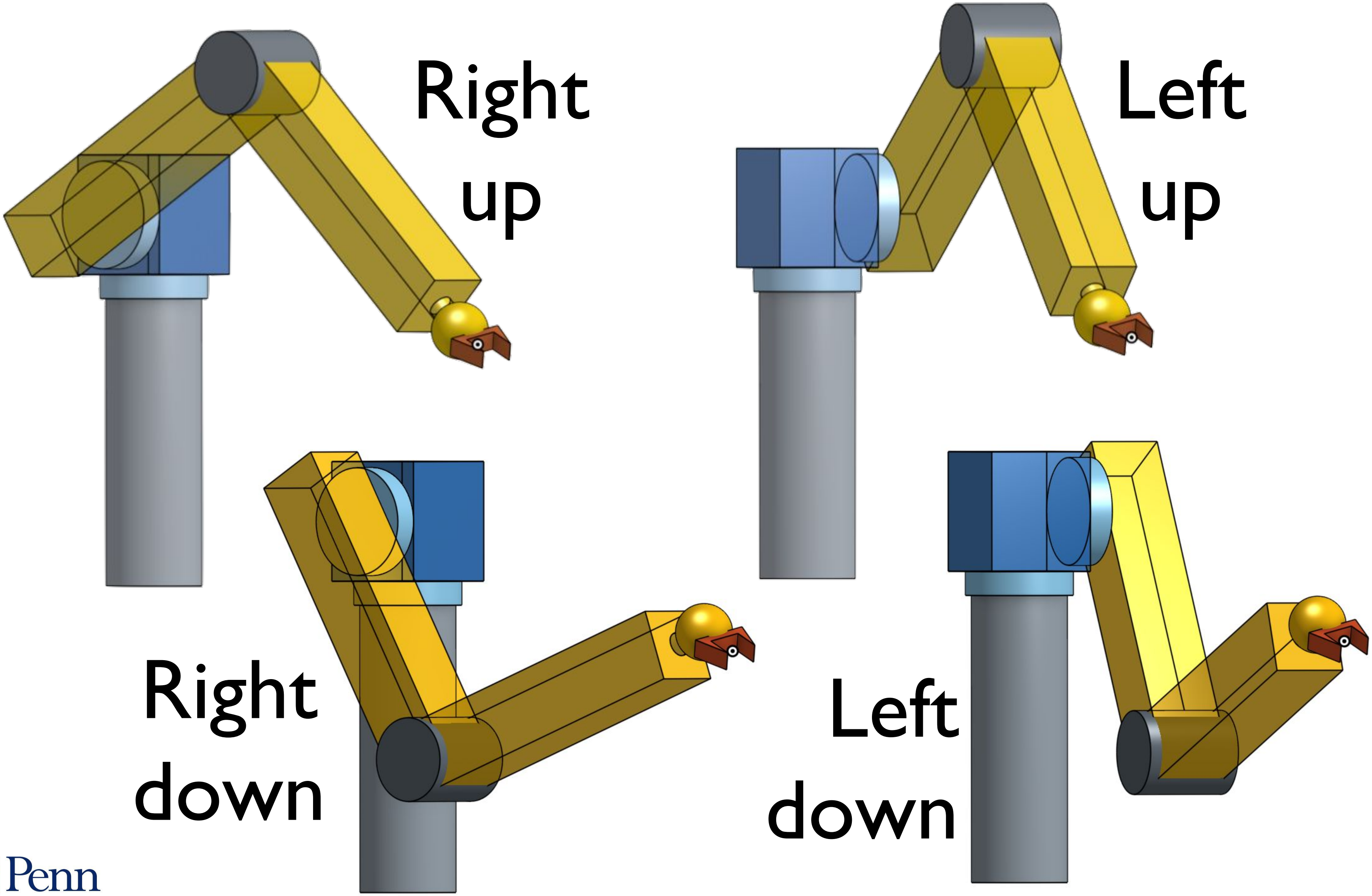


Positive θ_3



Negative θ_3

Four Solutions to 6R robot



Wrist multiple solution

$$\boxed{\theta_4 = \ atan2(\mp r_{33}, \pm c_1 r_{13} \pm s_1 r_{23})}$$

$$\boxed{\theta_5 = \pm \, cos^{-1}(c_1 r_{23} - s_1 r_{13})}$$

$$\boxed{\theta_6 = atan2(\pm(s_1 r_{11} - c_1 \, r_{21}), \pm(c_1 r_{22} - s_1 r_{12}))}$$

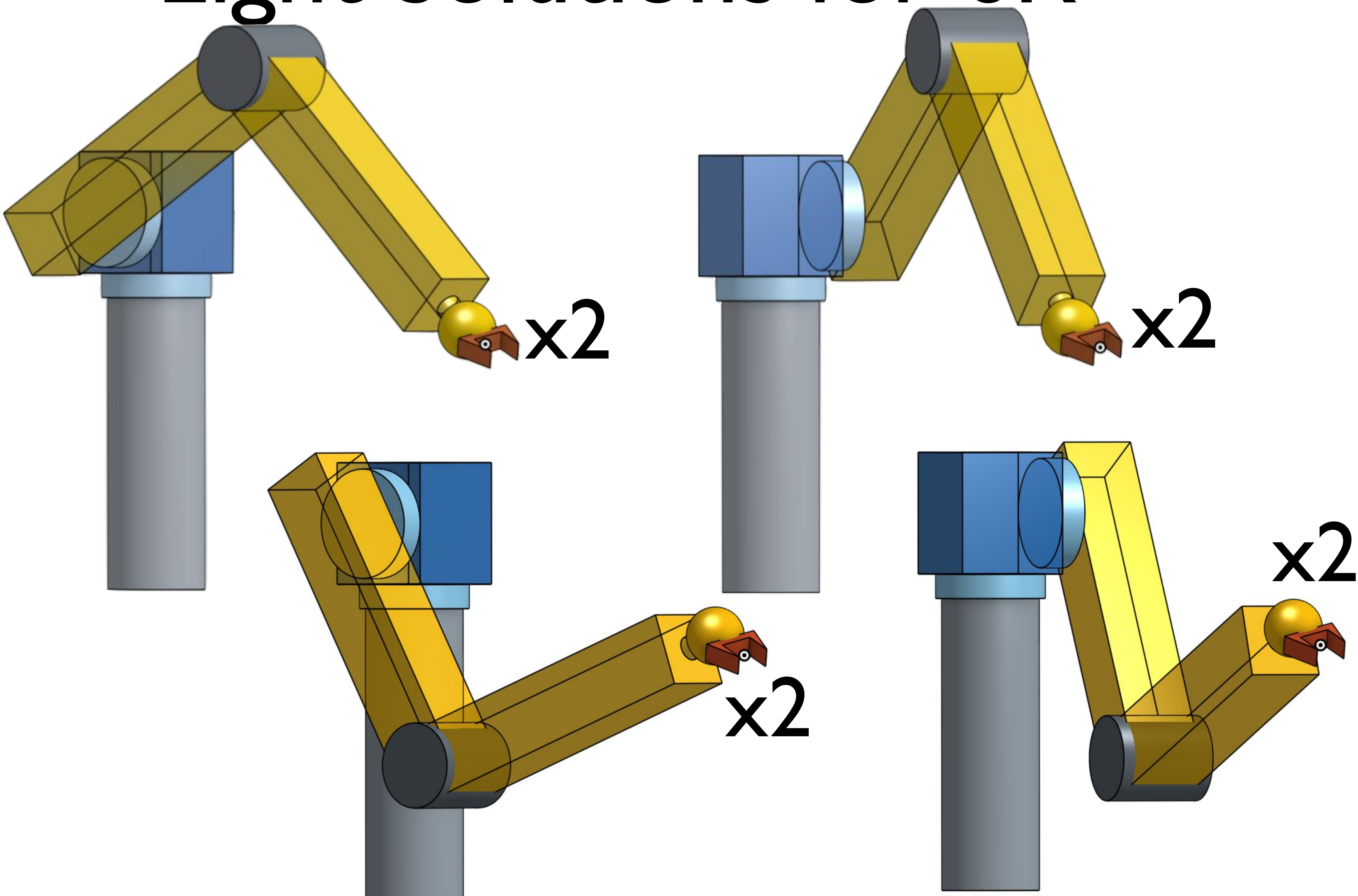
$$\boxed{\theta_4 = \ atan2(-r_{33}, \, c_1 r_{13} + s_1 r_{23})} - \pi$$

$$\boxed{\theta_5 = \quad cos^{-1}(c_1 r_{23} - s_1 r_{13})} * (-1)$$

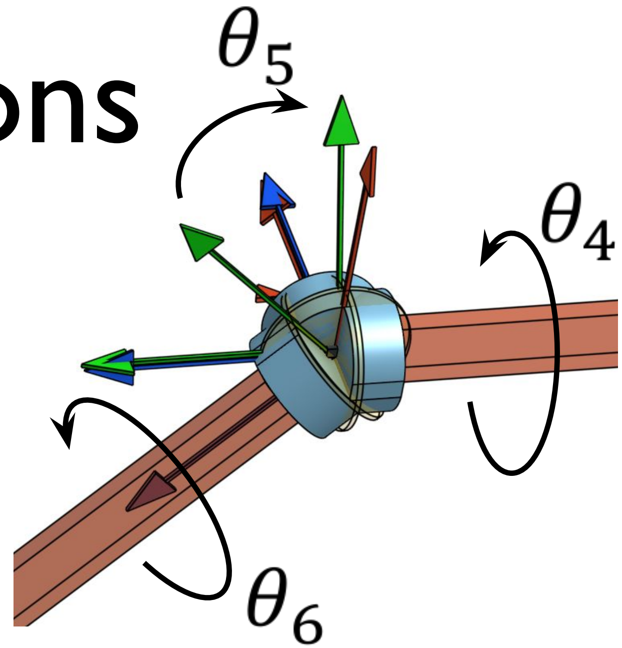
$$\boxed{\theta_6 = atan2(s_1 r_{11} - c_1 \, r_{21}, \ c_1 r_{22} - s_1 r_{12})} - \pi$$

1 st solution   2nd solution

Eight Solutions for 6R



Infinite solutions

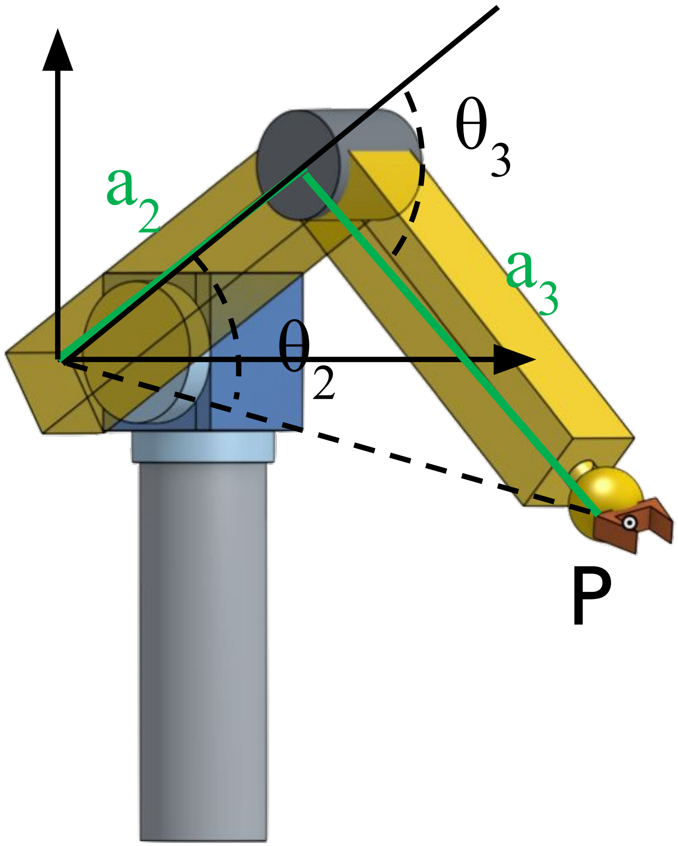


$$\theta_5 = 0,$$
$$\psi = \theta_4 + \theta_6$$

Two Solutions

$s_5 > 0$	$s_5 < 0$
$\boxed{\theta_4 = atan2(r_{23}, r_{13})}$	$\boxed{\theta_4 = atan2(-r_{23}, -r_{13})}$
$\boxed{\theta_5 = cos^{-1}(r_{33})}$	$\boxed{\theta_5 = -cos^{-1}(r_{33})}$
$\boxed{\theta_6 = atan2(-r_{31}, r_{32})}$	$\boxed{\theta_6 = atan2(r_{31}, -r_{32})}$

Elbow Solutions



Positive θ_3