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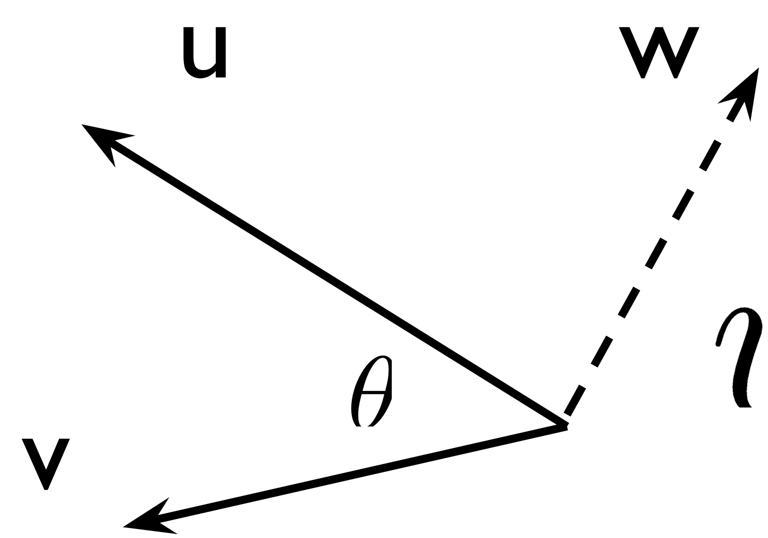
Video 3.1

CJ Taylor

Cross Product

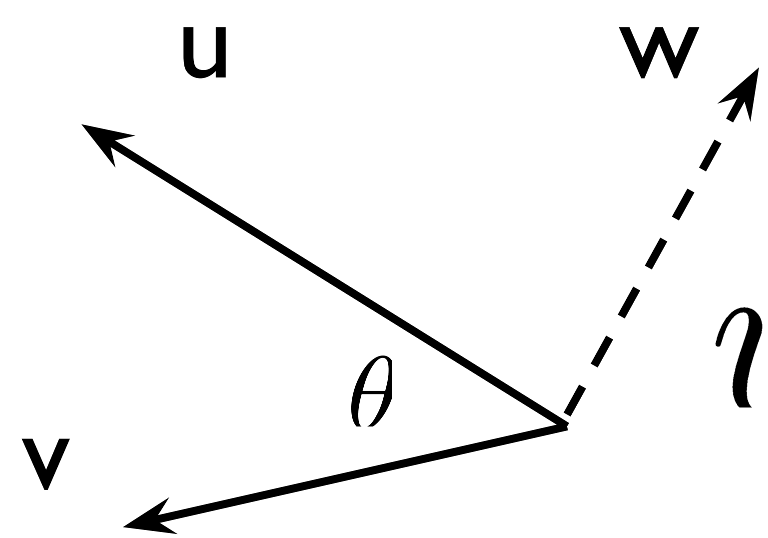


$$u \times v = w$$

$$\|u \times v\| = \|u\|\|v\| \sin\theta$$

$$u \times v = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Cross Product



$$u \times v = w$$

$$w = u \times v = J(u)v$$

$$J(u) = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

Rigid Transformations

Let g denote a function that maps $\mathbb{R}^3$ onto $\mathbb{R}^3$. This function is termed a rigid transformation iff it satisfies the following properties for all $u, v, w \in \mathbb{R}^3$

- $\|g(u) - g(v)\| = \|u - v\|$
- $g((u - w) \times (v - w)) = (g(u) - g(w)) \times (g(v) - g(w))$

Rigid Transformations

It can be shown that all rigid transformations can be expressed as follows.

$$g(v) = R * v + t, \quad R \in \mathbb{R}^{3\times 3}, t \in \mathbb{R}^3 \tag{1}$$

In this equation the matrix, R, is referred to as a rotation matrix and has the following special properties:

- $R = \begin{pmatrix} a & b & c \end{pmatrix}, \quad a, b, c \in \mathbb{R}^3$
- $\|a\| = \|b\| = \|c\| = 1$ - All columns are unit length
- $a \cdot b = b \cdot c = c \cdot a = 0$ - The columns are mutually orthogonal
- $(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b = 1$

Rotation Matrices

The set of 3×3 matrices with the following properties, coupled with the operation of matrix multiplication forms a group called $SO(3)$

$R = \begin{pmatrix} a & b & c \end{pmatrix}, \ \ a, b, c \in \mathbb{R}^3$ is a rotation matrix for $\mathbb{R}^3$ iff

- $R^T R = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\det(R) = (a \times b) \cdot c = 1$

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Rotation Matrices

The set of 3×3 matrices with the following properties, coupled with the operation of matrix multiplication forms a group called $SO(3)$

$R = \begin{pmatrix} a & b & c \end{pmatrix}, \;\; a, b, c \in \mathbb{R}^3$ is a rotation matrix for $\mathbb{R}^3$ iff

- $R^T R = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\det(R) = (a \times b) \cdot c = 1$

Groups

A *group* is an algebraic structure composed of a set, G, and a group operation, $\cdot$, that satisfies the following axioms:

- **Closure** : $a \cdot b \in G \;\; \forall a, b \in G$
- **Associativity** : $a \cdot (b \cdot c) = (a \cdot b) \cdot c \;\; \forall a, b, c \in G$
- **Identity and Inverse** : $\exists e \in G$ such that:

$$a \cdot e = a \;\; \forall a \in G$$

and

$$\forall a \in G \;\; \exists a^{-1} \in G \;\; \text{such that} \;\; a \cdot a^{-1} = e$$

The element e is referred to as an *identity* element.

Groups

Some examples of groups include:

- The set of all integers with the operation of addition
- The set of all real numbers except for zero with the operation of multiplication
- The set of all binary n-bit numbers with the binary operation xor.

SO(3)

The set $SO(3)$ with the operation of matrix multiplication forms a group

- **Identity** : $RI = IR = R \ \forall R \in SO(3)$ where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

SO(3)

The set $SO(3)$ with the operation of matrix multiplication forms a group

- **Closure** : $R_1, R_2 \in SO(3) \Rightarrow R_1 R_2 \in SO(3)$

$$\begin{aligned}(R_1 R_2)^T (R_1 R_2) &= R_2^T R_1^T R_1 R_2 && (1)\\ &= R_2^T I R_2 && (2)\\ &= R_2^T R_2 && (3)\\ &= I && (4)\end{aligned}$$

$$\det(R_1 R_2) = \det(R_1)\det(R_2) = 1 \times 1 = 1$$

SO(3)

The set $SO(3)$ with the operation of matrix multiplication forms a group

- **Associativity** : $R_1(R_2R_3) = (R_1R_2)R_3 \quad \forall\ R_1, R_2, R_3 \in SO(3)$: Since matrix multiplication is associative this holds true.

SO(3)

The set $SO(3)$ with the operation of matrix multiplication forms a group

- **Inverse**

$$\forall\, R \in SO(3) \;\; \exists\, R^T \;\in SO(3) \;\text{ such that }\; R^T R = R R^T = I$$

Group Properties

Observing that the set of rotation matrices forms a groups allows us to instantly take advantage of some useful results from group theory. For example:

- **Right Inverse is a Left Inverse**

$$a \cdot b = e \Rightarrow b \cdot a = e$$

$$R^T R = I \Rightarrow R R^T = I$$

- **Uniqueness of Identity Element**
- **Uniqueness of Inverse**

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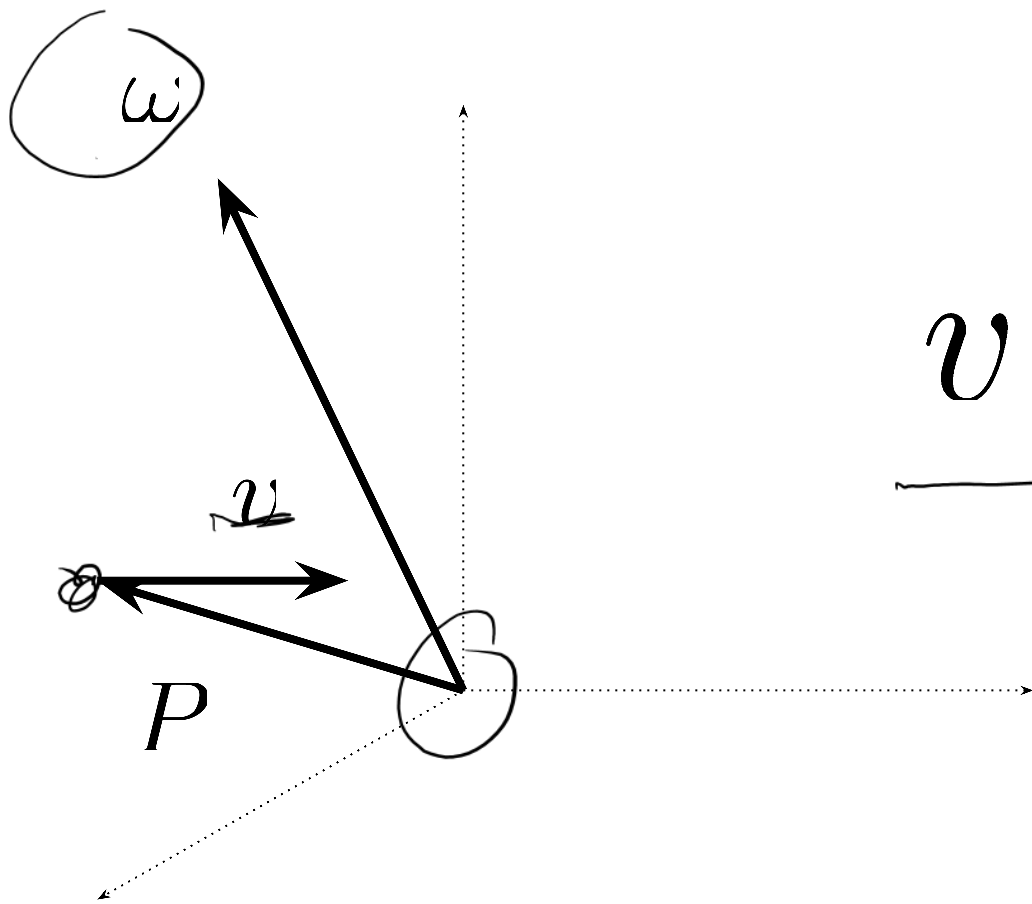
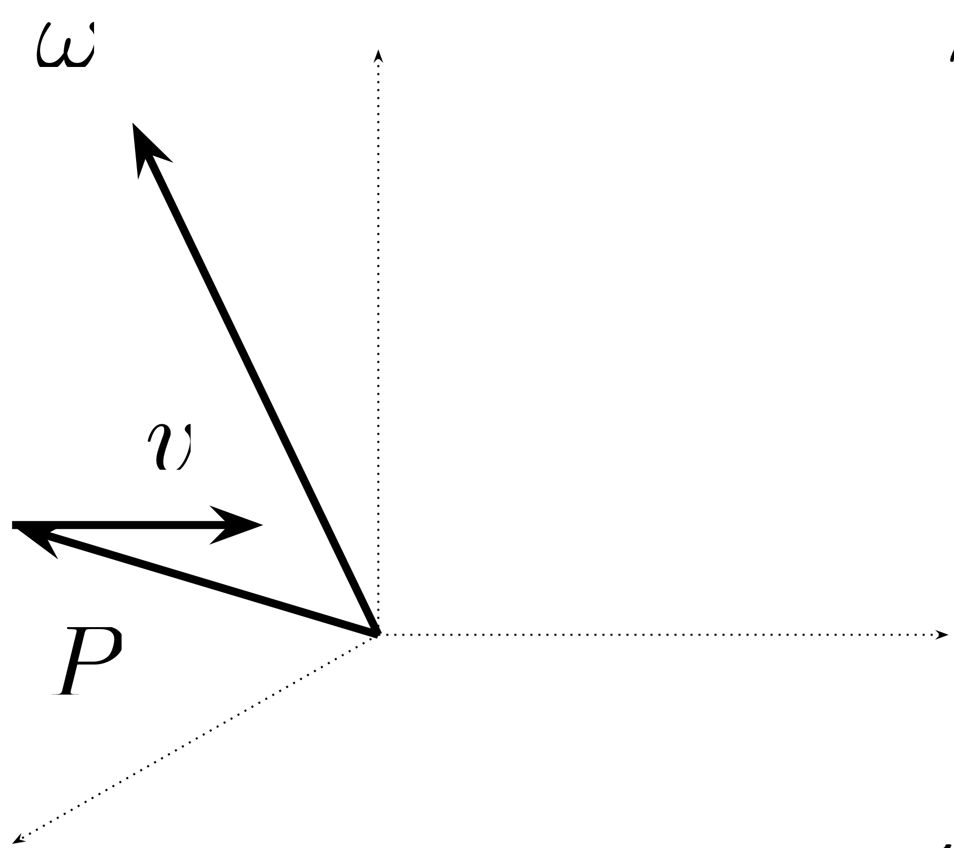
Angular Velocity



$$v = \omega \times P$$

$\omega \in \mathbb{R}^3$ denotes angular velocity. It is a vector whose direction indicates the axis of rotation and whose magnitude, $\|\omega\|$, indicates the rate of rotation in radians per second. $v \in \mathbb{R}^3$ denotes the instantaneous translational velocity of the point at $P \in \mathbb{R}^3$.

Angular Velocity



$$v = \frac{dP}{dt} = \omega \times P$$

$$\frac{dP}{dt} = J(\omega)P$$

$$J(\omega) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Linear Differential Equations

$$\frac{d}{dt}y(t) = ay(t)$$

$$\Rightarrow y(t) = \exp(at)y(0) = (1 + (at) + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \frac{(at)^4}{4!} + \cdots)y(0)$$

Linear Differential Equations

$$\frac{d}{dt}P(t) = J(\omega)P(t)$$

$$\Rightarrow P(t) = \exp(J(\omega)t)P(0) = (I + (J(\omega)t) + \frac{(J(\omega)t)^2}{2!} + \frac{(J(\omega)t)^3}{3!} + \frac{(J(\omega)t)^4}{4!} + \cdots)P(0)$$

$$\exp(J(\omega)t) = (I + (J(\omega)t) + \frac{(J(\omega)t)^2}{2!} + \frac{(J(\omega)t)^3}{3!} + \frac{(J(\omega)t)^4}{4!} + \cdots)$$

Rotations as Matrix Exponentials

All rotation matrices can be thought of as the action of some angular velocity, $\omega \in \mathbb{R}^3$, over a unit of time.

$$R(t) = \exp(J(\omega)t) = (I + (J(\omega)t) + \frac{(J(\omega)t)^2}{2!} + \frac{(J(\omega)t)^3}{3!} + \frac{(J(\omega)t)^4}{4!} + \cdots)$$

$$R = R(1) = \exp(J(\omega)1) = (I + (J(\omega)) + \frac{(J(\omega))^2}{2!} + \frac{(J(\omega))^3}{3!} + \frac{(J(\omega))^4}{4!} + \cdots)$$

$$R \in SO(3)$$

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Rotations as Matrix Exponentials

All rotation matrices can be thought of as the action of some angular velocity, $\omega \in \mathbb{R}^3$, over a unit of time.

$$R(t) = \exp(J(\omega)t) = \left(I + (J(\omega)t) + \frac{(J(\omega)t)^2}{2!} + \frac{(J(\omega)t)^3}{3!} + \frac{(J(\omega)t)^4}{4!} + \cdots\right)$$

$$R = R(1) = \exp(J(\omega)1) = \left(I + (J(\omega)) + \frac{(J(\omega))^2}{2!} + \frac{(J(\omega))^3}{3!} + \frac{(J(\omega))^4}{4!} + \cdots\right)$$

$$R \in SO(3)$$

Rotations as Matrix Exponentials

We can split the angular velocity, $\omega \in \mathbb{R}^3$, into two components, a unit vector indicating direction, $\hat{\omega}$, and a magnitude, $\theta = \|\omega\|$, where $\omega = \theta\hat{\omega}$

$$R = \exp(J(\theta\hat{\omega})) = \exp(\theta J(\hat{\omega})) = I + (\theta J(\hat{\omega})) + \frac{(\theta J(\hat{\omega}))^2}{2!} + \frac{(\theta J(\hat{\omega}))^3}{3!} + \frac{(\theta J(\hat{\omega}))^4}{4!} + \cdots$$

$$R \in SO(3)$$

Skew-Symmetric Products

$$J(u)J(v) = vu^T - (u^T v)I$$

$$J(u)u = u \times u = 0 \;\; \forall u \in \mathbb{R}^3$$

$$\Rightarrow J(\hat{\omega})^2 = J(\hat{\omega})J(\hat{\omega}) = \hat{\omega}\hat{\omega}^T - (\hat{\omega}^T \hat{\omega})I = \hat{\omega}\hat{\omega}^T - I$$

$$\Rightarrow J(\hat{\omega})^3 = J(\hat{\omega})J(\hat{\omega})^2 = J(\hat{\omega})(\hat{\omega}\hat{\omega}^T - I) = 0 - J(\hat{\omega}) = -J(\hat{\omega})$$

$$\Rightarrow J(\hat{\omega})^4 = J(\hat{\omega})J(\hat{\omega})^3 = -J(\hat{\omega})^2$$

$$\Rightarrow J(\hat{\omega})^{2n} = (-1)^{n-1}J(\hat{\omega})^2, \quad J(\hat{\omega})^{2n+1} = (-1)^n J(\hat{\omega})$$

The Rodrigues Formula

Applying this observation to our matrix exponential we note that the terms in the series can be grouped into even and odd powers as follows:

$$R = \exp(J(\theta\hat{\omega})) = \exp(\theta J(\hat{\omega})) = I + (\theta J(\hat{\omega})) + \frac{(\theta J(\hat{\omega}))^2}{2!} + \frac{(\theta J(\hat{\omega}))^3}{3!} + \frac{(\theta J(\hat{\omega}))^4}{4!} + \cdots$$

$$\exp(\theta J(\hat{\omega})) = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots) J(\hat{\omega}) + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} \cdots) J(\hat{\omega})^2$$

$$\exp(\theta J(\hat{\omega})) = I + \sin\theta J(\hat{\omega}) + (1 - \cos\theta) J(\hat{\omega})^2$$

The Rodrigues Formula

The Rodrigues Formula can be used to take a rotation expressed in terms of an axis, $\hat{\omega}$, and an angle, θ, and construct the corresponding rotation matrix, $R \in SO(3)$.

$$R = \exp(\theta J(\hat{\omega})) = I + \sin\theta J(\hat{\omega}) + (1 - \cos\theta) J(\hat{\omega})^2$$

It can also be used to extract the angle and axis from a given rotation matrix.

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Quaternions

- A quaternion can be thought of as a tuple consisting of a scalar part, $u_0 \in \mathbb{R}$, and a vector part, $u \in \mathbb{R}^3$, That is $q = (u_0, u)$.
- The product of two quaternions, (u_0, u), and (v_0, v), is defined as follows:

$$(u_0, u)(v_0, v) = (u_0 v_0 - (u^T v), u_0 v + v_0 u + u \times v)$$

Unit Quaternions

- A unit quaternion is a quaternion that satisfies the following constraint : $u_0^2 + u^T u = 1$. In other words the sum of the squares of the entries is 1.
- The set of unit quaternions along with the operation of quaternion multiplication forms a group.

Unit Quaternions and Rotations

- Given a rotation expressed in terms of an angle, θ, and an axis ,$\hat{\omega}$, we can construct a corresponding unit quaternion as follows.

$$(u_0, u) = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\omega})$$

- It is possible to construct the corresponding rotation matrix $R \in SO(3)$ from a unit quaternion as follows.

$$R = (u_0^2 - u^T u)I + 2u_0 J(u) + 2(uu^T)$$

  You can verify that this corresponds exactly to the Rodrigues Formula.

- Note that the unit quaternions (u_0, u) and $(-u_0, -u)$ produce exactly the same rotation matrix.

Quaternion Conjugate

- Given a quaternion $q = (u_0, u)$ we can compute it's conjugate q^* as follows $q^* = (u_0, -u)$.
- If the quaternion q corresponds to the rotation matrix R then it's conjugate q^* corresponds to the rotation matrix R^T, the inverse of R.

Unit Quaternions

- If the unit quaternions q_1 and q_2 correspond to the rotation matrices R_1 and R_2 respectively then you can show that the quaternion product $q_1 \cdot q_2$ produces another unit quaternion that corresponds to the product of the two rotations, $R_1 R_2$. In other words we can compute the product of two rotations via quaternion multiplication as an alternative to matrix multiplication.

$$q_1 \approx R_1,\ q_2 \approx R_2 \Rightarrow q_1 \cdot q_2 \approx R_1 R_2$$

Advantages of Quaternions

- One advantage of quaternions over Rotation matrices is that they are more compact since they allow us to represent rotations with 4 numbers instead of the 9 entries in a 3×3 matrix.

- More importantly when one multiplies rotation matrices together the resulting product will have imperfections due to numerical precision issues and it is difficult to renormalize the result. Renormalizing unit quaternions after quaternion multiplication is straightforward, simply scale all of the entries to achieve a unit norm.

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Summary of Rotation Representations

- A rotation can be expressed as a 3×3 matrix $R \in SO(3)$ where $R^T R = RR^T = I$ and $\det R = 1$

- A rotation can be expressed in terms of an angle θ and an axis $\hat{\omega} \in \mathbb{R}^3$ where $\|\hat{\omega}\| = 1$. We can relate this to the matrix form via the Rodriques formula.

$$R = \exp(\theta J(\hat{\omega})) = I + \sin\theta J(\hat{\omega}) + (1 - \cos\theta) J(\hat{\omega})^2$$

- A rotation matrix can be expressed as a unit quaternion:

$$(u_0, u) = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\omega})$$

- It is possible to construct the corresponding rotation matrix $R \in SO(3)$ from a unit quaternion as follows.

$$R = (u_0^2 - u^T u)I + 2u_0 J(u) + 2(uu^T)$$

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Rigid Transformations

Let g denote a function that maps $\mathbb{R}^3$ onto $\mathbb{R}^3$. This function is termed a rigid transformation iff it satisfies the following properties for all $u, v, w \in \mathbb{R}^3$

- $\|g(u) - g(v)\| = \|u - v\|$
- $g((u - w) \times (v - w)) = (g(u) - g(w)) \times (g(v) - g(w))$

Rigid Transformations

It can be shown that all rigid transformations can be expressed as follows.

$$g(v) = R * v + t, \quad R \in SO(3), t \in \mathbb{R}^3 \tag{1}$$

In this equation the matrix, R, is referred to as a rotation matrix and has the following special properties:

- $R^T R = I$
- $\det R = 1$

Rigid Transformations

If we employ homogenous coordinates as we did earlier we note that we can represent rigid transformations using 4×4 matrices as follows:

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P \\ 1 \end{pmatrix} = \begin{pmatrix} RP + t \\ 1 \end{pmatrix}$$

In other words rigid transformations are simply a class of coordinate transformations that preserve distances and cross products in the appropriate manner.

Composing Rigid Transformations

We can compose rigid transformations using matrix multiplication in the usual manner.

$$\begin{pmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_{BC} & t_{BC} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{AB}R_{BC} & R_{AB}t_{BC} + t_{AB} \\ 0 & 1 \end{pmatrix}$$

Note that the product of two rigid transformations is also a rigid transformation

Inverting Rigid Transformations

Given a rigid transformation between two frames, g_{AB}, we can readily compute the inverse transformation, g_{BA}, as follows

$$g_{AB} = \begin{pmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{pmatrix} \;\Rightarrow\; g_{BA} = \begin{pmatrix} R_{AB}^T & -R_{AB}^T t_{AB} \\ 0 & 1 \end{pmatrix}$$

Note that the inverse of a rigid transformation is also a rigid transformation

SE(3)

- The set of 4×4 matrices of the form $g_{AB} = \begin{pmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{pmatrix}$ where $R \in SO(3)$ and $t \in \mathbb{R}^3$ form a group with respect to the operation of matrix multiplication. This group is referred to as the Special Euclidean Group on $\mathbb{R}^3$ or $SE(3)$ and represents the set of rigid transformations on 3D space.

- Note that the product of two rigid transformations is a rigid transformation.

- Note that the inverse of a rigid transformation is a rigid transformation.

SE(3) and Matrix Exponentials

- Every element of $SE(3)$ can be expressed as the exponential of an appropriate matrix. More specifically every element of $SE(3)$ can be written as follows:

$$g_{AB} = \begin{pmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} J(w) & v \\ 0 & 0 \end{pmatrix}$$

Where $w, v \in \mathbb{R}^3$ and $J(w) = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$