Robotics: Fundamentals

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Week 7: Manipulator Jacobian

Kinematics

Joint space

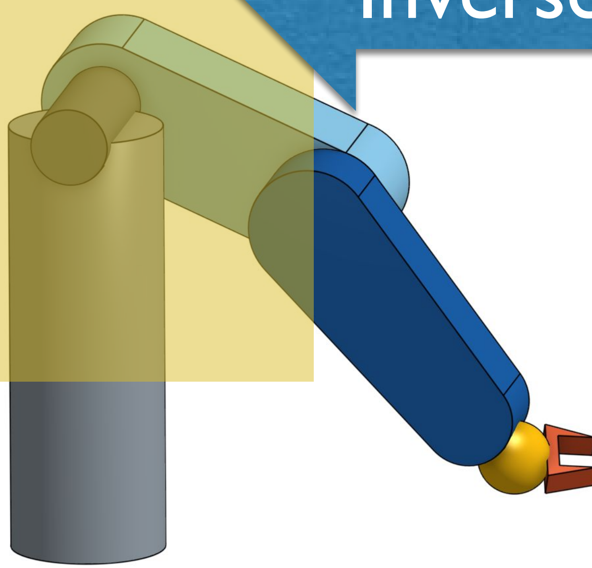
$$\boldsymbol{q} = [q_1, q_2, \cdots q_n]^T$$

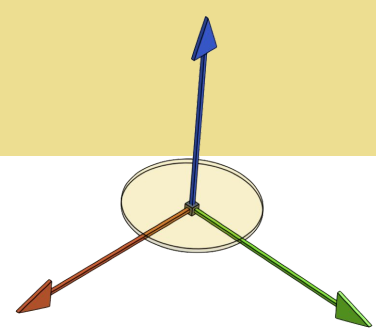
Forward

Inverse

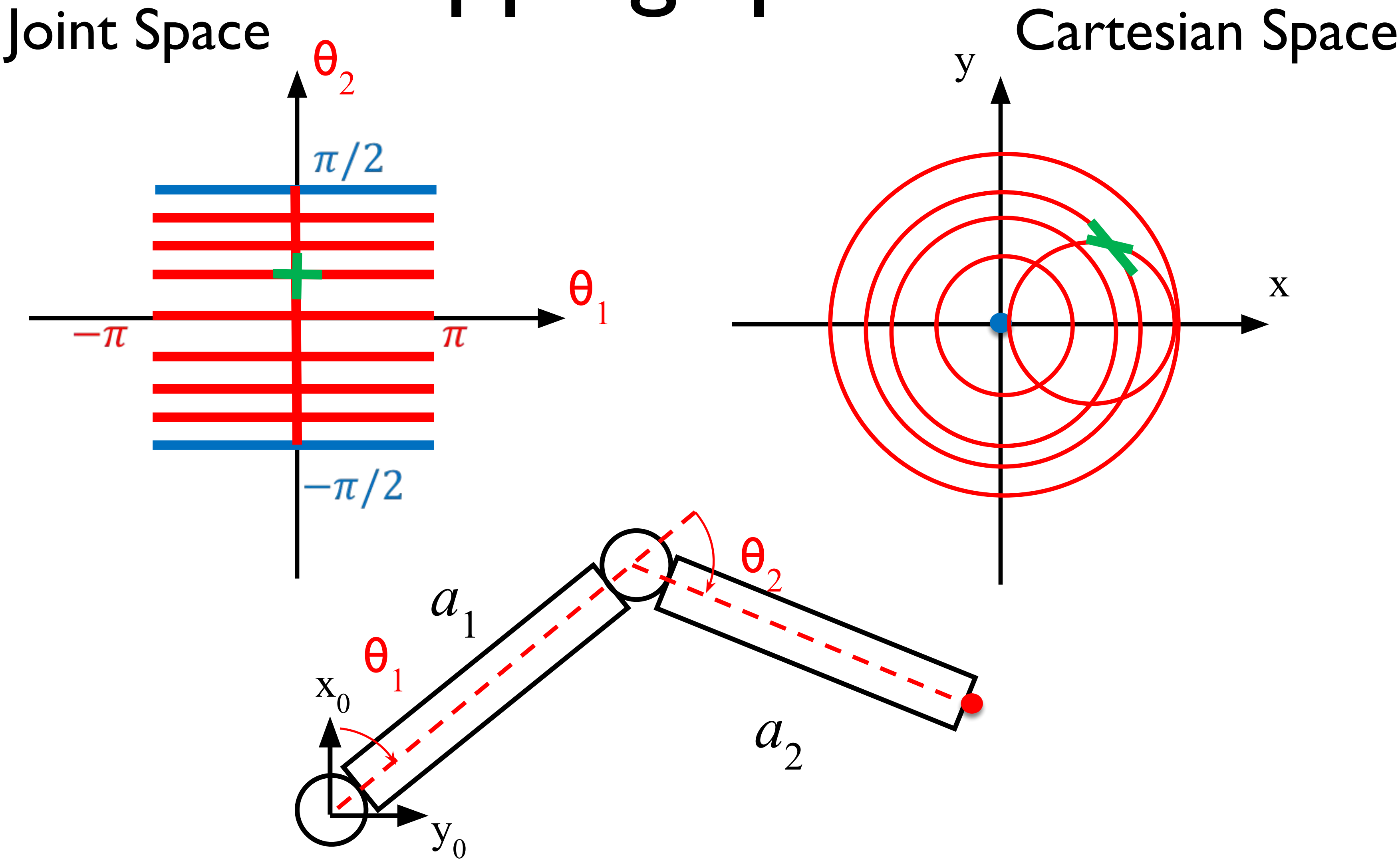
Cartesian space

$$\boldsymbol{H} = \begin{bmatrix} [\boldsymbol{R}] & [\boldsymbol{P}] \\ [\mathbf{0}] & 1 \end{bmatrix}$$





Mapping spaces



Manipulator Jacobian

$$\dot{x} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\dot{y} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\dot{z} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\dot{\phi} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\dot{\theta} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\dot{\psi} = f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n$$

$$\xi = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{J}(\mathbf{q})\ \dot{\boldsymbol{q}}$$

Jacobian Matrix

$$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{q} \in \mathbb{R}^n \quad \mathbf{f}(\mathbf{q}) \in \mathbb{R}^m$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial q_1} & \cdots & \dfrac{\delta \mathbf{f}}{\partial q_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \cdots & \dfrac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial q_1} & \cdots & \dfrac{\partial f_m}{\partial q_n} \end{bmatrix}$$

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial q_j}$$

Manipulator Jacobian

$$\left.\begin{aligned}\dot{x} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n \\ \dot{y} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n \\ \dot{z} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n\end{aligned}\right\} \boldsymbol{v} = \mathbf{J}_v\, \dot{\boldsymbol{q}}$$

$$\left.\begin{aligned}\dot{\phi} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n \\ \dot{\theta} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n \\ \dot{\psi} &= f_1(\boldsymbol{q})\dot{q}_1 + \cdots + f_n(\boldsymbol{q})\dot{q}_n\end{aligned}\right\} \boldsymbol{\omega} = \mathbf{J}_\omega\, \dot{\boldsymbol{q}}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \mathbf{J}(\mathbf{q})\, \dot{\boldsymbol{q}}$$

Position Jacobian

$$x(t) = f\left(q(t)_1, q(t)_2 \ldots q(t)_n\right)$$

$$\frac{\partial x}{\partial t} = \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} \frac{\partial q_i}{\partial t}$$

$$\mathbf{J}_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial z}{\partial q_1} & \cdots & \frac{\partial z}{\partial q_n} \end{bmatrix}$$

$$\boldsymbol{v} = \mathbf{J}_v \, \dot{\boldsymbol{q}}$$

Position Jacobian

$$\boldsymbol{T_{0n}} = \boldsymbol{A}_0 \ldots \boldsymbol{A}_n = \begin{bmatrix} [\boldsymbol{R}] & [\boldsymbol{P}] \\ [\boldsymbol{0}] & 1 \end{bmatrix}$$

$$\mathbf{J}_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial z}{\partial q_1} & \cdots & \frac{\partial z}{\partial q_n} \end{bmatrix}$$

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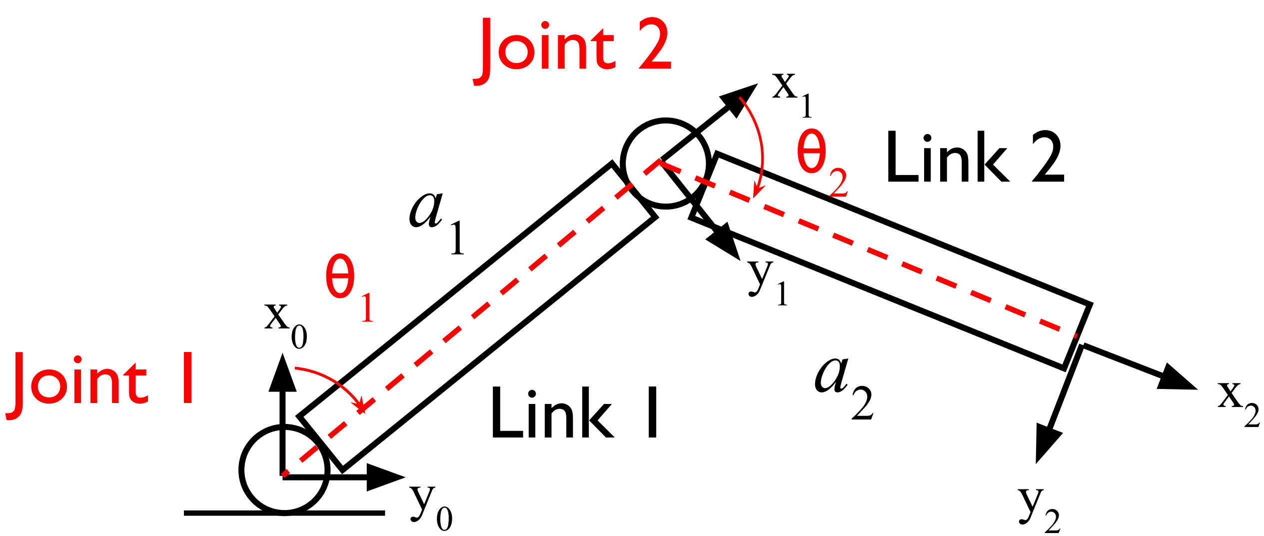
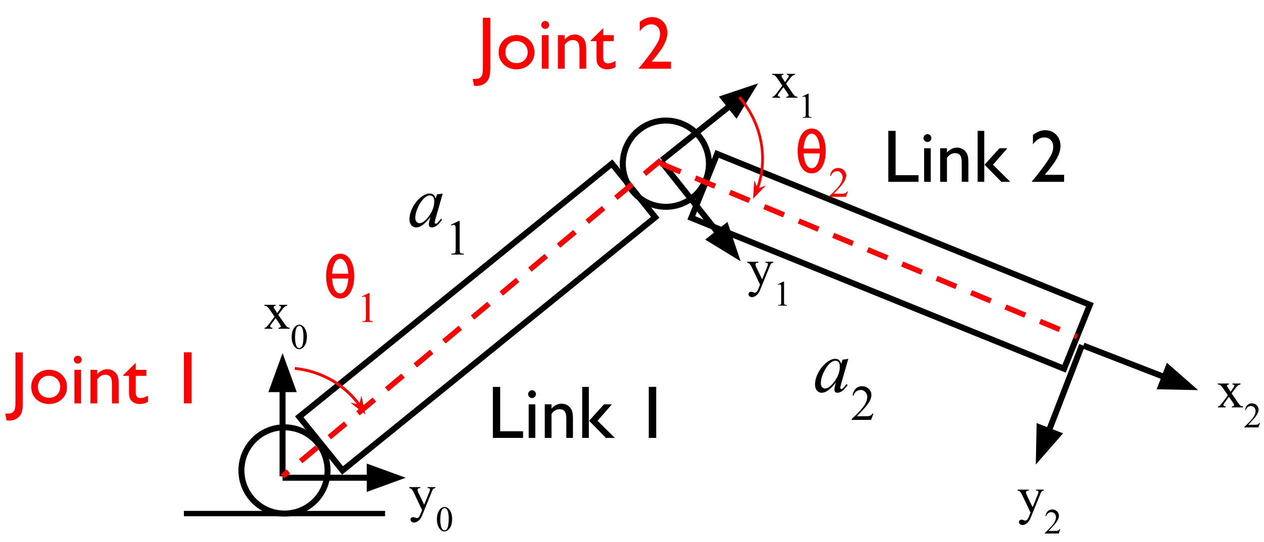
Position Jacobian

$$P = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \qquad T_{0n} = \begin{bmatrix} [R] & [P] \\ [0] & 1 \end{bmatrix}$$



Position Jacobian

$$\boldsymbol{P} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \quad \frac{\partial x}{\partial q_1} = -a_1 s_1 - a_2 s_{12} \quad \frac{\partial x}{\partial q_2} = -a_2 s_{12}$$



Position Jacobian

$$\boldsymbol{P} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$\frac{\partial x}{\partial q_1} = -a_1 s_1 - a_2 s_{12} \qquad \frac{\partial x}{\partial q_2} = -a_2 s_{12}$$

$$\frac{\partial y}{\partial q_1} = a_1 c_1 + a_2 c_{12} \qquad \frac{\partial y}{\partial q_2} = a_2 c_{12}$$

$$\frac{\partial z}{\partial q_1} = 0 \qquad \frac{\partial z}{\partial q_2} = 0$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_1 c_{12} & a_2 c_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\boldsymbol{v} = \mathbf{J}_v \, \dot{\boldsymbol{q}}$$

$\mathbf{J}_v$ in a serial chain

$$\boldsymbol{v} = \dot{\boldsymbol{P}} = \mathbf{J}_v \, \dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{P}}_{0n} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{P}_{0n}}{\partial q_i} \dot{q}_i$$

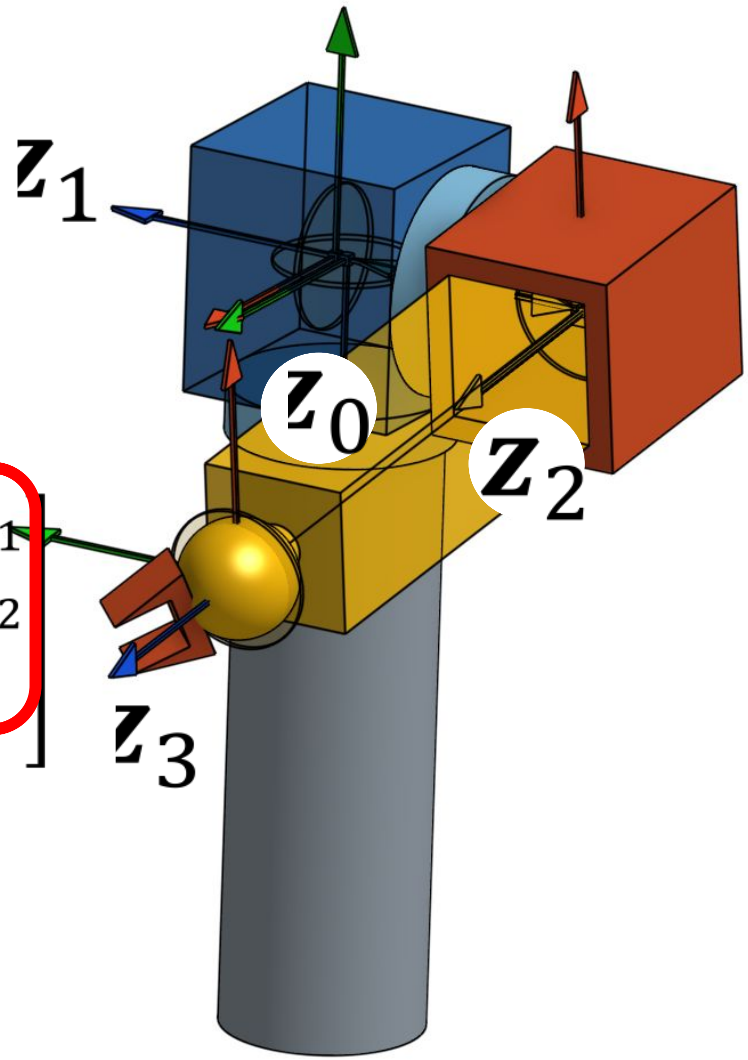
$$\mathbf{J}_{v_i} = \frac{\partial \boldsymbol{P}_{0n}}{\partial q_i}$$

$$\mathbf{J}_v = \left[[\mathbf{J}_{v_1}] \quad \cdots \quad [\mathbf{J}_{v_n}] \right]$$

Prismatic Joints

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	$\underline{\boldsymbol{\theta}_1}$
2	0	90	d_2	$\underline{\boldsymbol{\theta}_2}$
3	0	0	$\underline{\boldsymbol{d}_3}$	0

$$\mathbf{J}_{v_i} = \frac{\partial \boldsymbol{P}_{0n}}{\partial q_i}$$



$$\dot{\boldsymbol{P}}_{03} = \frac{\partial \boldsymbol{P}_{01}}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{P}_{02}}{\partial q_2}\dot{q}_2 + \frac{\partial \boldsymbol{P}_{03}}{\partial q_3}\dot{q}_3$$

$$\boldsymbol{T}_{03} = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 - d_2s_1 \\ c_2s_1 & c_1 & s_1s_2 & d_3s_1s_2 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

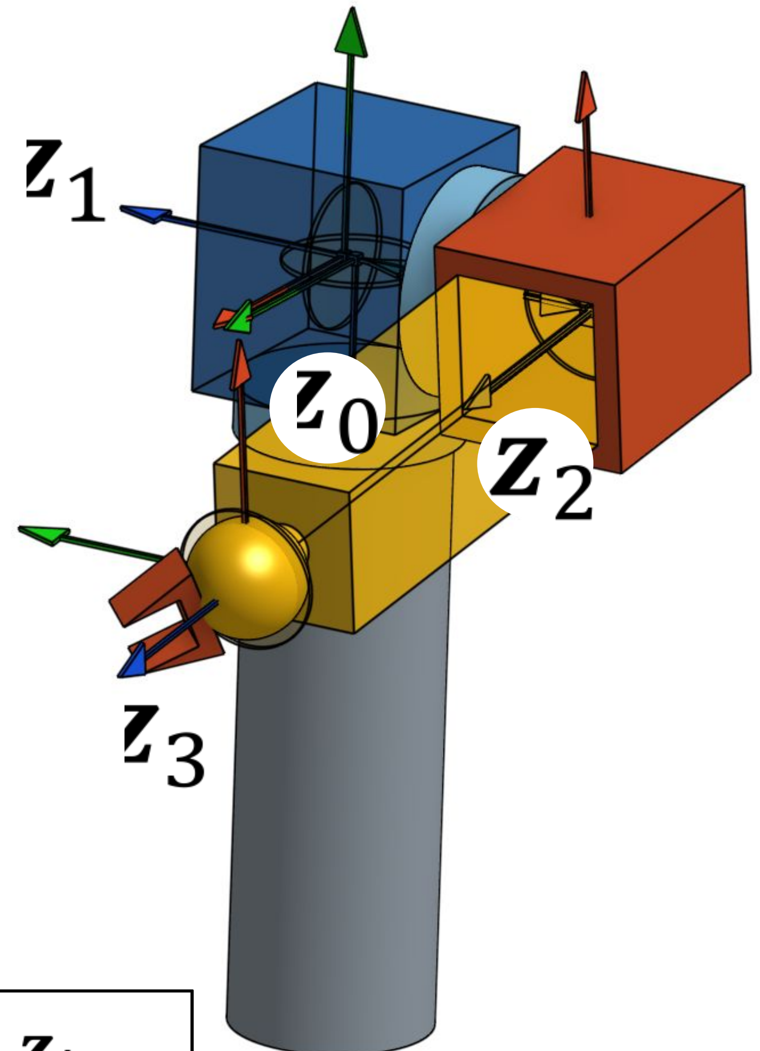
$$\dot{\boldsymbol{P}}_{03} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix} \dot{d}_3$$

Prismatic Joints

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	$\underline{\boldsymbol{\theta_1}}$
2	0	90	d_2	$\underline{\boldsymbol{\theta_2}}$
3	0	0	$\underline{\boldsymbol{d_3}}$	0

$$\dot{\boldsymbol{P}}_{03} = \frac{\partial \boldsymbol{P}_{01}}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{P}_{02}}{\partial q_2}\dot{q}_2 + \frac{\partial \boldsymbol{P}_{03}}{\partial q_3}\dot{q}_3$$

$$\dot{\boldsymbol{P}}_{03} = \boldsymbol{z}_2\dot{q}_3 \qquad \mathbf{J}_{v_3} = \boldsymbol{z}_2$$



$$\dot{\boldsymbol{P}}_{03} = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \dot{d}_3$$

$$\boxed{\mathbf{J}_{v_i} = \boldsymbol{z}_{i-1}}$$

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Prismatic Joints

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	$\underline{\boldsymbol{\theta}_1}$
2	0	90	d_2	$\underline{\boldsymbol{\theta}_2}$
3	0	0	$\underline{\boldsymbol{d}_3}$	0

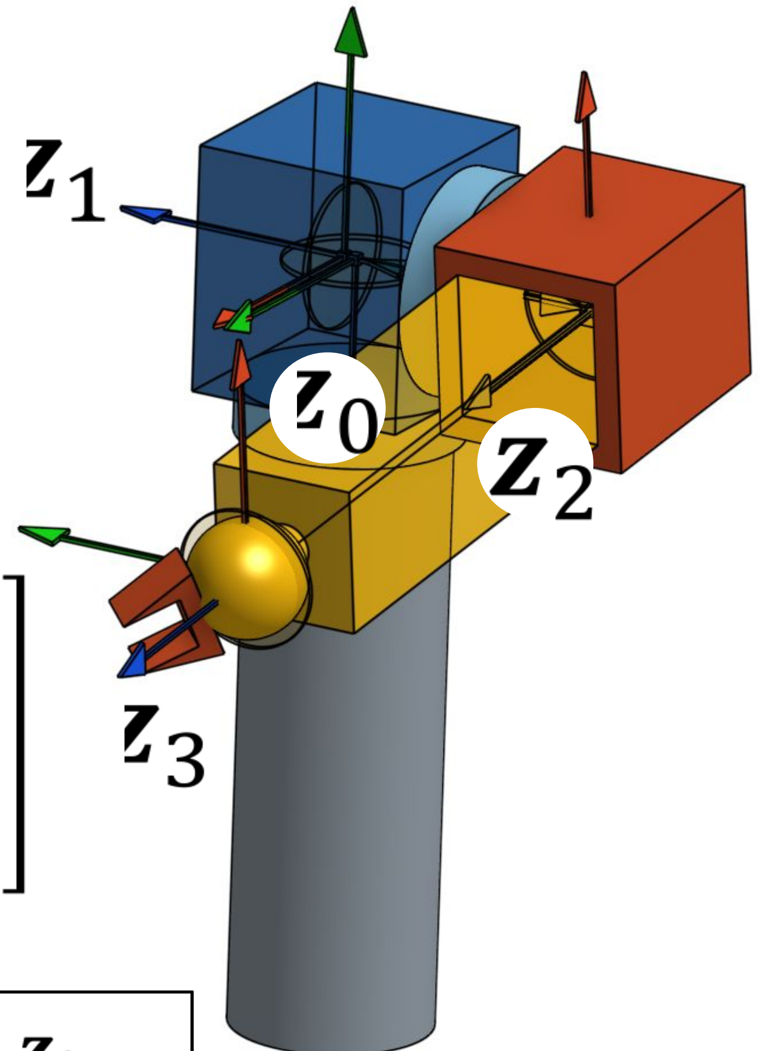
$$\dot{\boldsymbol{P}}_{03} = \frac{\partial \boldsymbol{P}_{01}}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{P}_{02}}{\partial q_2}\dot{q}_2 + \frac{\partial \boldsymbol{P}_{03}}{\partial q_3}\dot{q}_3$$

$$\dot{\boldsymbol{P}}_{03} = \boldsymbol{z}_2\dot{q}_3 \qquad \mathbf{J}_{v_3} = \boldsymbol{z}_2$$

$$T_{02} = \begin{bmatrix} c_1 s_2 & -s_1 & \boxed{\begin{matrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{matrix}} & -d_2 s_1 \\ c_2 s_1 & c_1 & & d_2 c_1 \\ -s_2 & 0 & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\boldsymbol{P}}_{03} = \boxed{\begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}} \dot{d}_3$$

$$\boxed{\mathbf{J}_{v_i} = \boldsymbol{z}_{i-1}}$$



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Revolute Joints

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	$\underline{\boldsymbol{\theta_1}}$
2	0	90	d_2	$\underline{\boldsymbol{\theta_2}}$
3	0	0	$\underline{\boldsymbol{d_3}}$	0

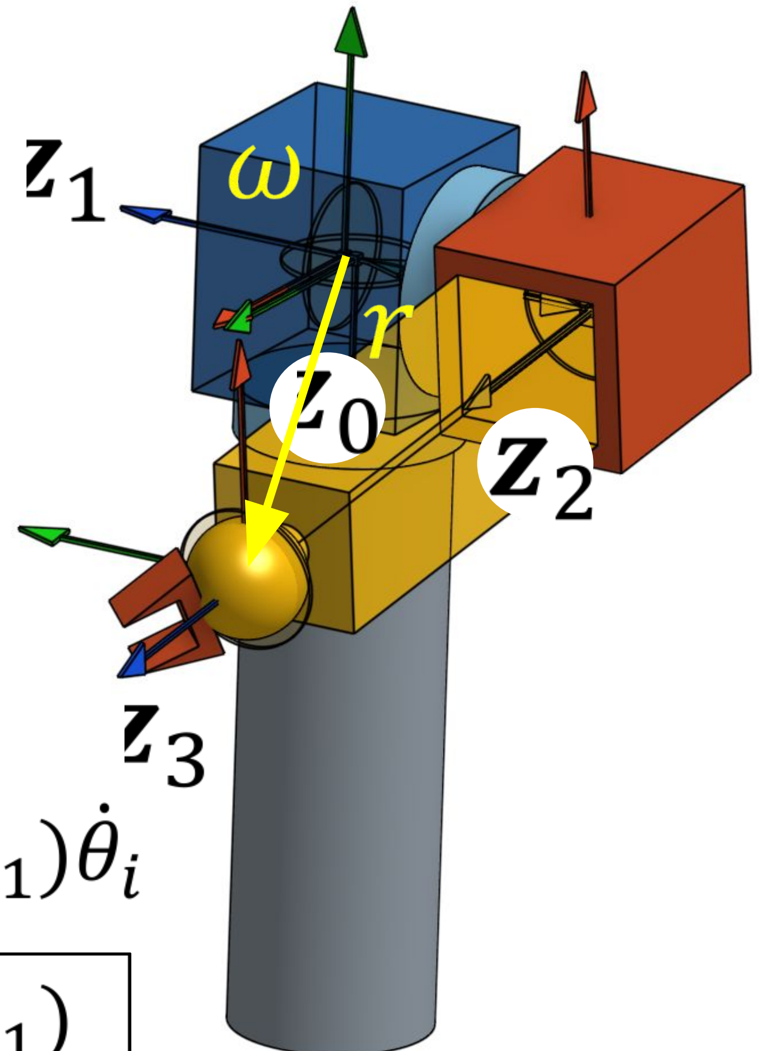
$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{\omega} = \dot{\theta}_i \mathbf{z}_{i-1}$$

$$\boldsymbol{r} = \left(\boldsymbol{P}_n - \boldsymbol{P}_{i-1}\right)$$

$$\boldsymbol{v} = \mathbf{z}_{i-1} \times \left(\boldsymbol{P}_n - \boldsymbol{P}_{i-1}\right)\dot{\theta}_i$$

$$\boxed{\mathbf{J}_{\boldsymbol{v}_i} = \mathbf{z}_{i-1} \times \left(\boldsymbol{P}_n - \boldsymbol{P}_{i-1}\right)}$$



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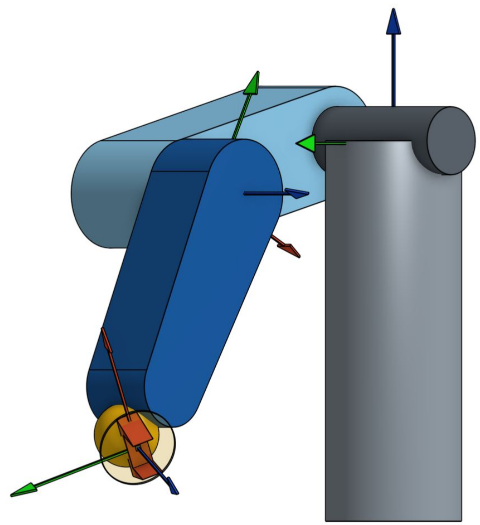
Orientation Jacobian

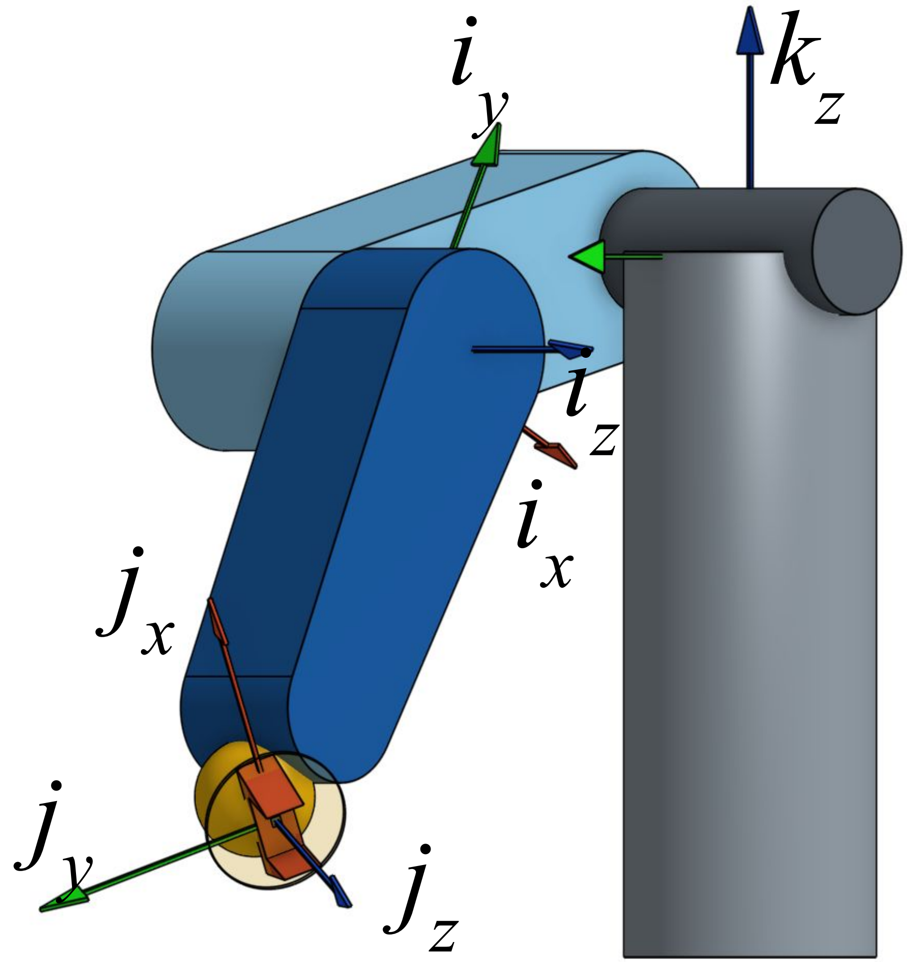
$$\frac{\partial x}{\partial t} = \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} \frac{\partial q_i}{\partial t}$$



Angular Velocity

$$\omega = \mathbf{J}_\omega(q)\dot{\boldsymbol{q}}$$

$$\omega_{ij}^{k}$$



ω is the angular velocity of *j* with respect to *i* expressed in frame *k*

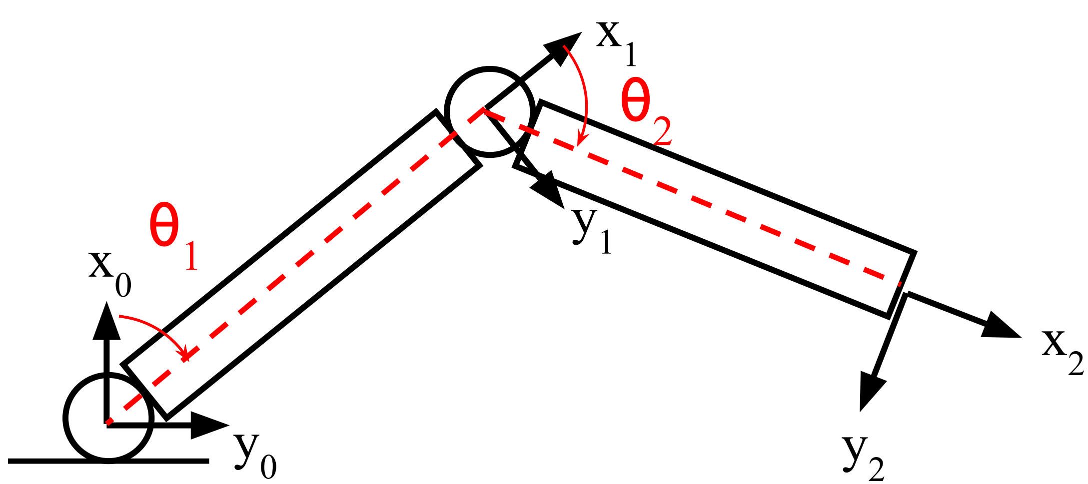
Orientation Jacobian

$$\omega_{01}^{0} = 0\hat{x}_0 + 0\hat{y}_0 + \dot{\theta}_1\hat{z}_0$$

$$\omega_{12}^{1} = 0\hat{x}_1 + 0\hat{y}_1 + \dot{\theta}_2\hat{z}_1$$

$$\omega_{12}^{0} = \boldsymbol{R}_{01}\omega_{12}^{1}$$

$$\omega_{02}^{0} = \omega_{01}^{0} + \boldsymbol{R}_{01}\omega_{12}^{1} = (\dot{\theta}_1 + \dot{\theta}_1)\hat{z}_0$$



Orientation Jacobian

$$\omega_{01}^{0} = 0\hat{x}_0 + 0\hat{y}_0 + \dot{\theta}_1\hat{z}_0$$

$$\omega_{12}^{1} = 0\hat{x}_1 + 0\hat{y}_1 + \dot{\theta}_2\hat{z}_1$$

$$\omega_{12}^{0} = \boldsymbol{R}_{01}\omega_{12}^{1}$$

$$\omega_{02}^{0} = \omega_{01}^{0} + \boldsymbol{R}_{01}\omega_{12}^{1} = (\dot{\theta}_1 + \dot{\theta}_1)\hat{z}_0$$

$$\omega_{0n}^{0} = \sum_{i=1}^{n} \boldsymbol{R}_{0(i-1)}\omega_{i-1\,i}^{i-1}$$

for revolute:

$$\omega_{0n}^{0} = \sum_{i=1}^{n} \hat{z}_{i-1}\dot{\theta}_i$$

Orientation Jacobian

for prescription: $\mathbf{J}_\omega = 0$

$$\rho_i = \begin{array}{l} 0 \text{ if } i \text{ is prismatic} \\ 1 \text{ if } i \text{ is revolute} \end{array}$$

$$\boxed{\mathbf{J}_\omega = [[\rho_1 \hat{\mathbf{z}}_0] \quad \cdots \quad [\rho_n \hat{\mathbf{z}}_{n-1}]]}$$

$$\omega_{0n}^0 = \sum_{i=1}^{n} \hat{z}_{i-1} \dot{\theta}_i$$

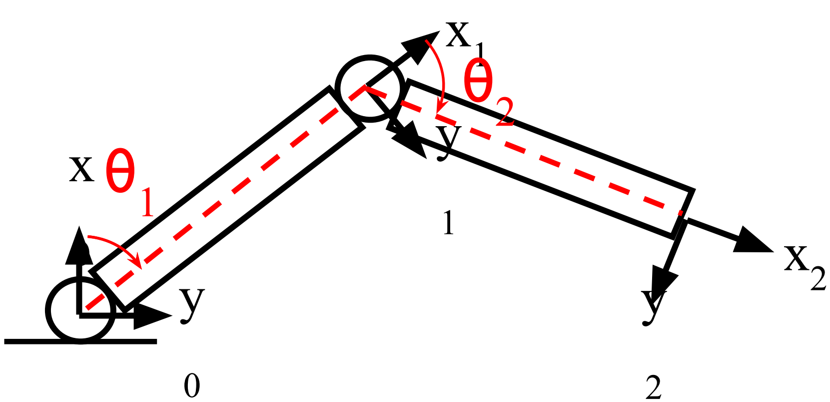
Combining Linear and Angular Velocity

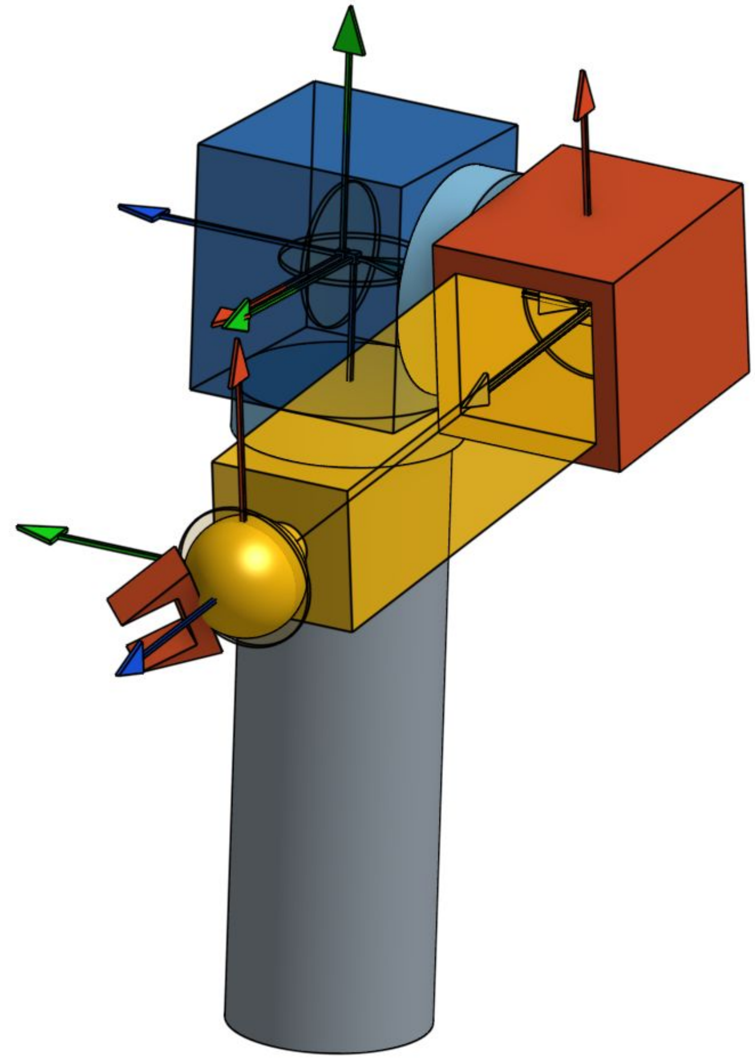
$$\mathbf{J} = [[\mathbf{J}_1][\mathbf{J}_2] \quad \cdots \quad [\mathbf{J}_n]]$$

If Revolute: $$\mathbf{J}_i = \begin{bmatrix} \hat{\mathbf{z}}_{i-1} \times (\boldsymbol{P}_n - \boldsymbol{P}_{i-1}) \\ \hat{\mathbf{z}}_{i-1} \end{bmatrix}$$

If Prismatic: $$\mathbf{J}_i = \begin{bmatrix} \hat{\mathbf{z}}_{i-1} \\ 0 \end{bmatrix}$$

Jacobian Examples

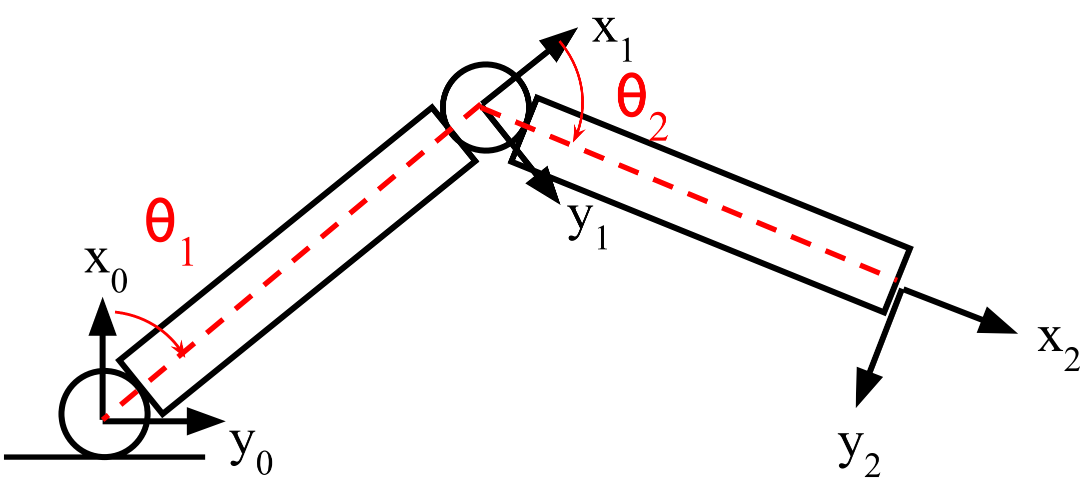




2 Link Arm Jacobian

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} \hat{\mathbf{z}}_0 \times (\boldsymbol{P}_2 - \boldsymbol{P}_0) & \hat{\mathbf{z}}_1 \times (\boldsymbol{P}_2 - \boldsymbol{P}_1) \\ \hat{\mathbf{z}}_0 & \hat{\mathbf{z}}_1 \end{bmatrix}$$

If Revolute: $$\mathbf{J}_i = \begin{bmatrix} \hat{\mathbf{z}}_{i-1} \times (\boldsymbol{P}_n - \boldsymbol{P}_{i-1}) \\ \hat{\mathbf{z}}_{i-1} \end{bmatrix}$$

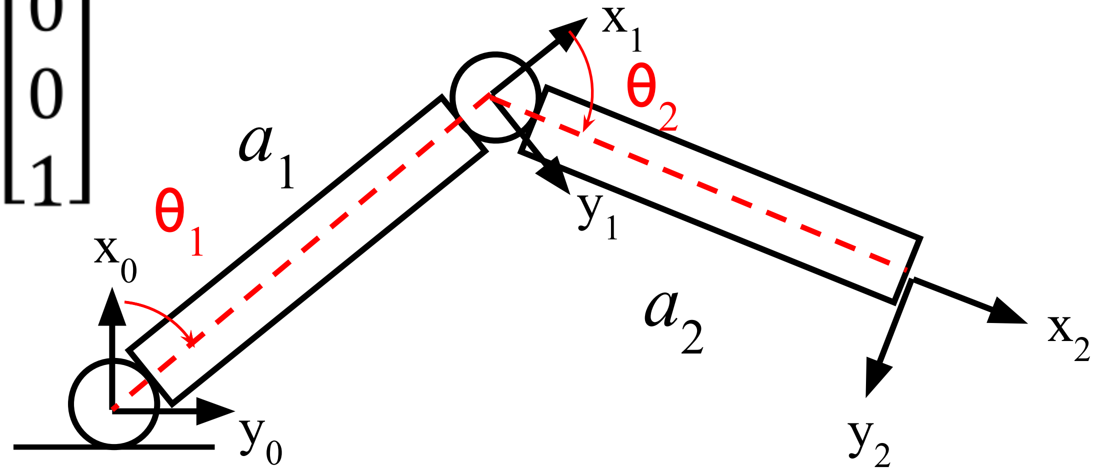


2 Link Arm Jacobian

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} \hat{\mathbf{z}}_0 \times (\boldsymbol{P}_2 - \boldsymbol{P}_0) & \hat{\mathbf{z}}_1 \times (\boldsymbol{P}_2 - \boldsymbol{P}_1) \\ \hat{\mathbf{z}}_0 & \hat{\mathbf{z}}_1 \end{bmatrix}$$

$$\boldsymbol{P}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{P}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{P}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{z}}_0 = \hat{\mathbf{z}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



2 Link Arm Jacobian

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c_{12} \\ a_2 s_{12} \\ 0 \end{bmatrix} \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

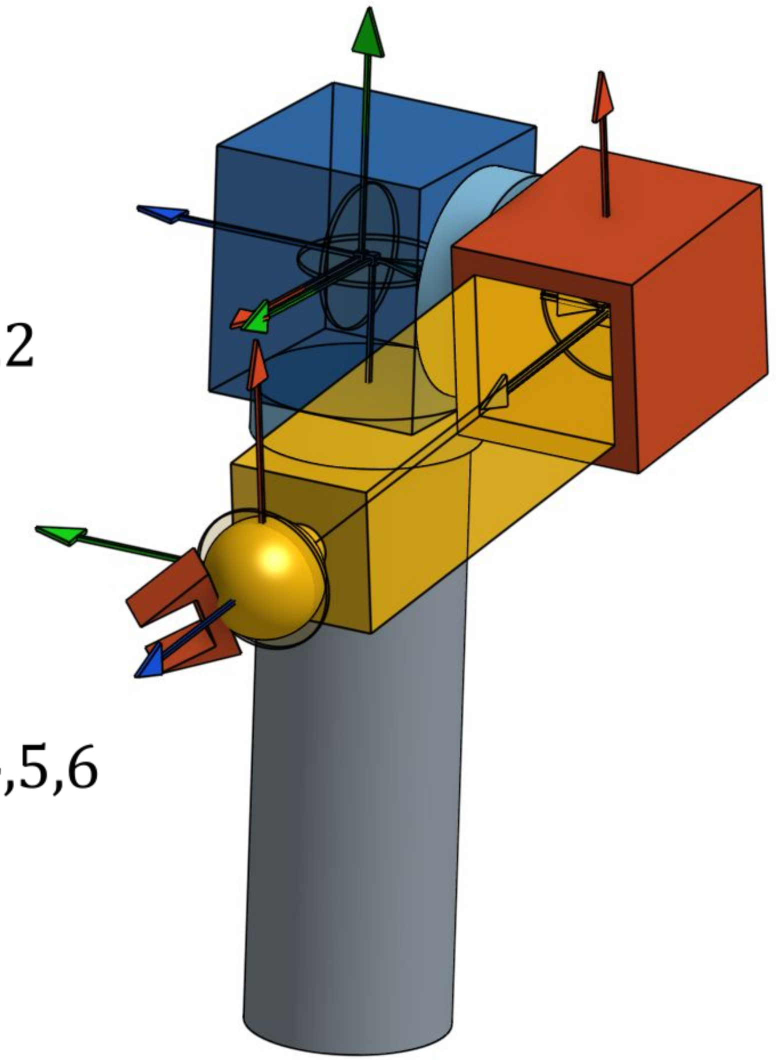
Stanford Arm Jacobian

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	$\underline{\boldsymbol{\theta}_1}$
2	0	90	d_2	$\underline{\boldsymbol{\theta}_2}$
3	0	0	$\underline{\boldsymbol{d_3}}$	0

$$\mathbf{J}_i = \begin{bmatrix} \hat{\mathbf{z}}_{i-1} \times (\boldsymbol{P}_6 - \boldsymbol{P}_{i-1}) \\ \hat{\mathbf{z}}_{i-1} \end{bmatrix}, \qquad i = 1,2$$

$$\mathbf{J}_3 = \begin{bmatrix} \hat{\mathbf{z}}_2 \\ 0 \end{bmatrix}$$

$$\mathbf{J}_i = \begin{bmatrix} \hat{\mathbf{z}}_{i-1} \times (\boldsymbol{P}_6 - \boldsymbol{P}_{i-1}) \\ \hat{\mathbf{z}}_{i-1} \end{bmatrix}, \qquad i = 4,5,6$$



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Stanford Arm

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stanford Arm MATLAB

```
T01 = a1
T02 = T01*a2
T03 = T02*a3
T04 = T03*a4
T05 = T04*a5
T06 = T05*a6
```

```
Z0 = [0;0;1]; P0 = [0;0;0]
Z1 = T01(1:3, 3)
Z2 = T02(1:3, 3)
Z3 = T03(1:3, 3)
Z4 = T04(1:3, 3)
Z5 = T05(1:3, 3)
Z6 = T06(1:3, 3)
P1 = T01(1:3, 4)
P2 = T02(1:3, 4)
P3 = T03(1:3, 4)
P4 = T04(1:3, 4)
P5 = T05(1:3, 4)
P6 = T06(1:3, 4)
```

```
Jp1 = cross(Z0,P6-P0)
Jo1 = Z0
Jp2 = cross(Z1,P6-P1)
Jo2 = Z1
Jp3 = Z2
Jo3 = [0;0;0]
Jp4 = cross(Z3,P6-P3)
Jo4 = Z3
Jp5 = cross(Z4,P6-P4)
Jo5 = Z4
Jp6 = cross(Z5,P6-P5)
Jo6 = Z5
```

```
J = [Jp1 Jp2 Jp3 Jp4 Jp5 Jp6 ; Jo1 Jo2 Jo3 Jo4 Jo5 Jo6]
```

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Results

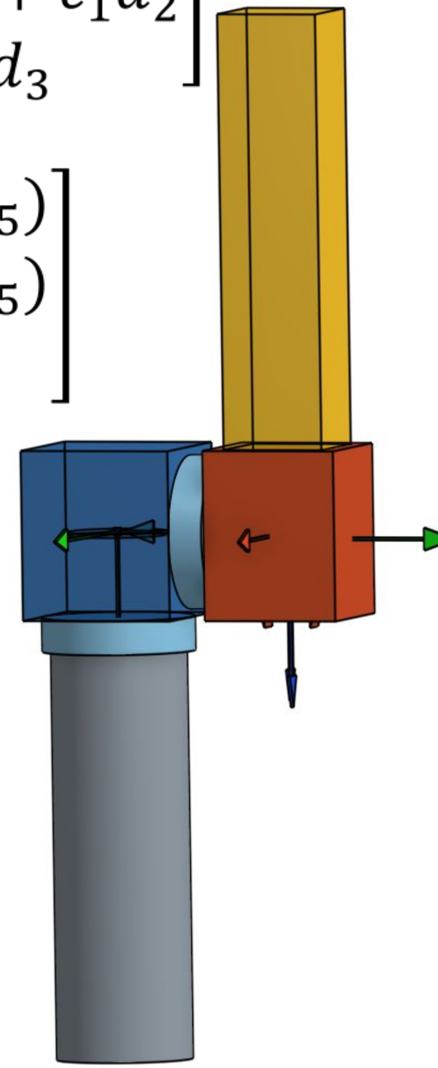
$$p_0 = p_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} -s_1 d_2 \\ c_1 d_2 \\ 0 \end{bmatrix} \quad p_3 = p_4 = p_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$p_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6(s_1 c_2 c_4 s_5 + s_1 c_5 s_2 + c_1 s_4 s_5) \\ c_1 d_3 + d_6(c_2 c_5 - s_2 c_4 s_5) \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

$$z_2 = z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \quad z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$



Results

```
J11 - c1*d2 - d6*(s5*(c1*s4 + c2*c4*s1) + c5*s1*s2) - d3*s1*s2
J12 c1*(c2*d3 + d6*(c2*c5 - c4*s2*s5))
J13 c1*s2
J14 d6*s1*s2*(c2*c5 - c4*s2*s5) - c2*d6*(s5*(c1*s4 + c2*c4*s1) + c5*s1*s2)
J15 d6*(c1*c4 - c2*s1*s4)*(c2*c5 - c4*s2*s5) - d6*s2*s4*(s5*(c1*s4 + c2*c4*s1) + c5*s1*s2)
J16 0

J21 c1*d3*s2 - d6*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2) - d2*s1
J22 s1*(c2*d3 + d6*(c2*c5 - c4*s2*s5))
J23 s1*s2
J24 - c2*d6*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2) - c1*d6*s2*(c2*c5 - c4*s2*s5)
J25 d6*(c4*s1 + c1*c2*s4)*(c2*c5 - c4*s2*s5) - d6*s2*s4*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2)
J26 0

J31 0
J32 c1*(d2*s1 + d6*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2) - c1*d3*s2) - s1*(c1*d2 + d6*(s5*(c1*s4
    + c2*c4*s1) + c5*s1*s2) + d3*s1*s2)
J33 c2
J34 c1*d6*s2*(s5*(c1*s4 + c2*c4*s1) + c5*s1*s2) + d6*s1*s2*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2)
J35 d6*(c1*c4 - c2*s1*s4)*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2) - d6*(s5*(c1*s4 + c2*c4*s1) +
    + c5*s1*s2)*(c4*s1 + c1*c2*s4)
J36 0

J41 0
J42 -s1
J43 0
J44 c1*s2
J45 - c4*s1 - c1*c2*s4
J46 c1*c5*s2 - s5*(s1*s4 - c1*c2*c4)

J51 0
J52 c1
J53 0
J54 s1*s2
J55 c1*c4 - c2*s1*s4
J56 s5*(c1*s4 + c2*c4*s1) + c5*s1*s2

J61 1
J62 0
J63 0
J64 c2
J65 s2*s4
J66 c2*c5 - c4*s2*s5
```

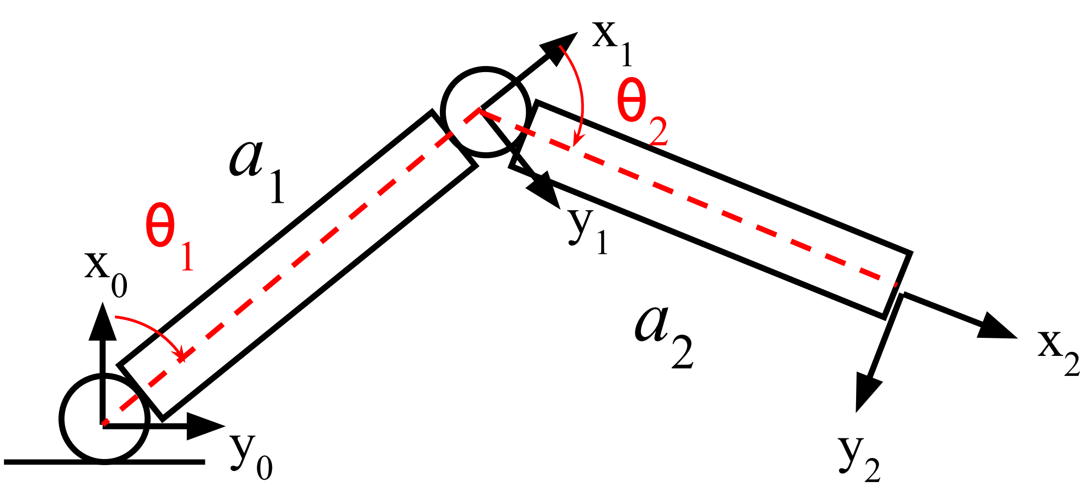
Robotics: Fundamentals

Prof. Mark Yim
University of Pennsylvania

Week 7: Singularities, Manipulability, Forces, Torques

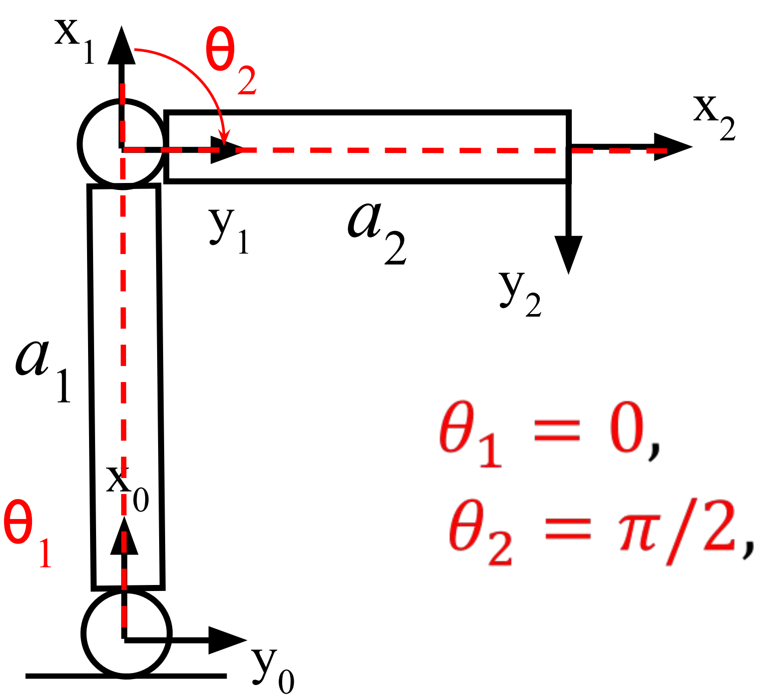
Singularities

Configurations for which the rank $\mathbf{J}(q)$ is less than its maximum value are called **singularities** or **singular configurations**.



Penn Engineering

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_2 \\ a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

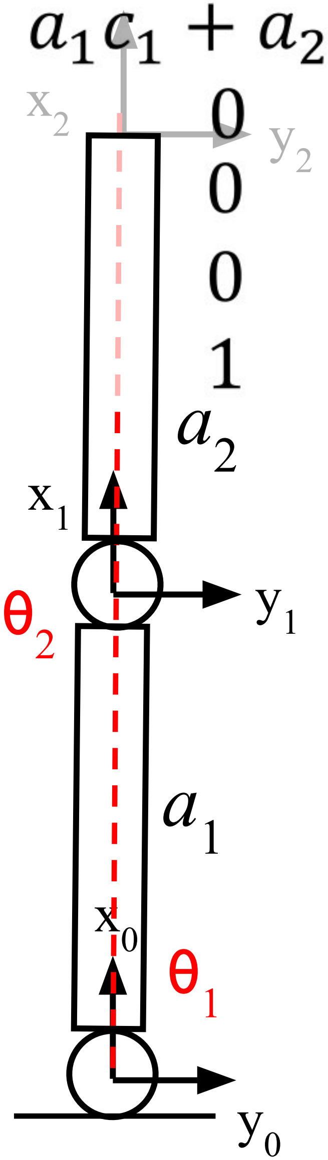


$\theta_1 = 0,\quad s_1 = 0, c_1 = 1$

$\theta_2 = \pi/2,\quad s_2 = 1, c_2 = 0$

$s_{12} = 1, c_{12} = 0$

$$
\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix} \end{bmatrix}
$$



$$\boxed{\det(\mathbf{J}) = 0}$$

$$\theta_1 = 0,\ s_1 = 0, c_1 = 1$$
$$\theta_2 = 0,\quad s_2 = 0, c_2 = 1$$
$$s_{12} = 0, c_{12} = 1$$

Characteristics at Singular Configurations

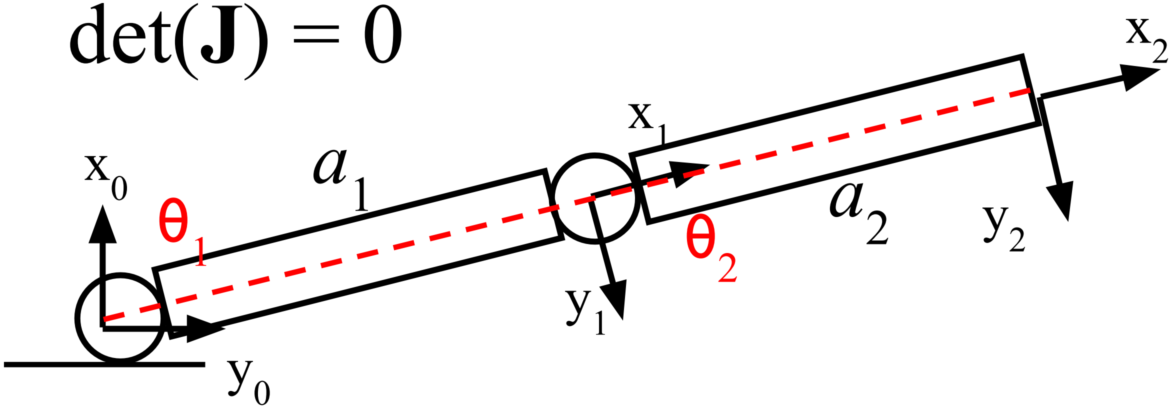
- Directions of motion may be lost
- Infinite joint velocities may be required for finite end-effector velocities
- Theoretically infinite end-effector forces may result from finite joint forces
- Often correspond to points on the boundary of the manipulator workspace.
- There may be no IK solution or there may be infinitely many IK solutions

Using determinant

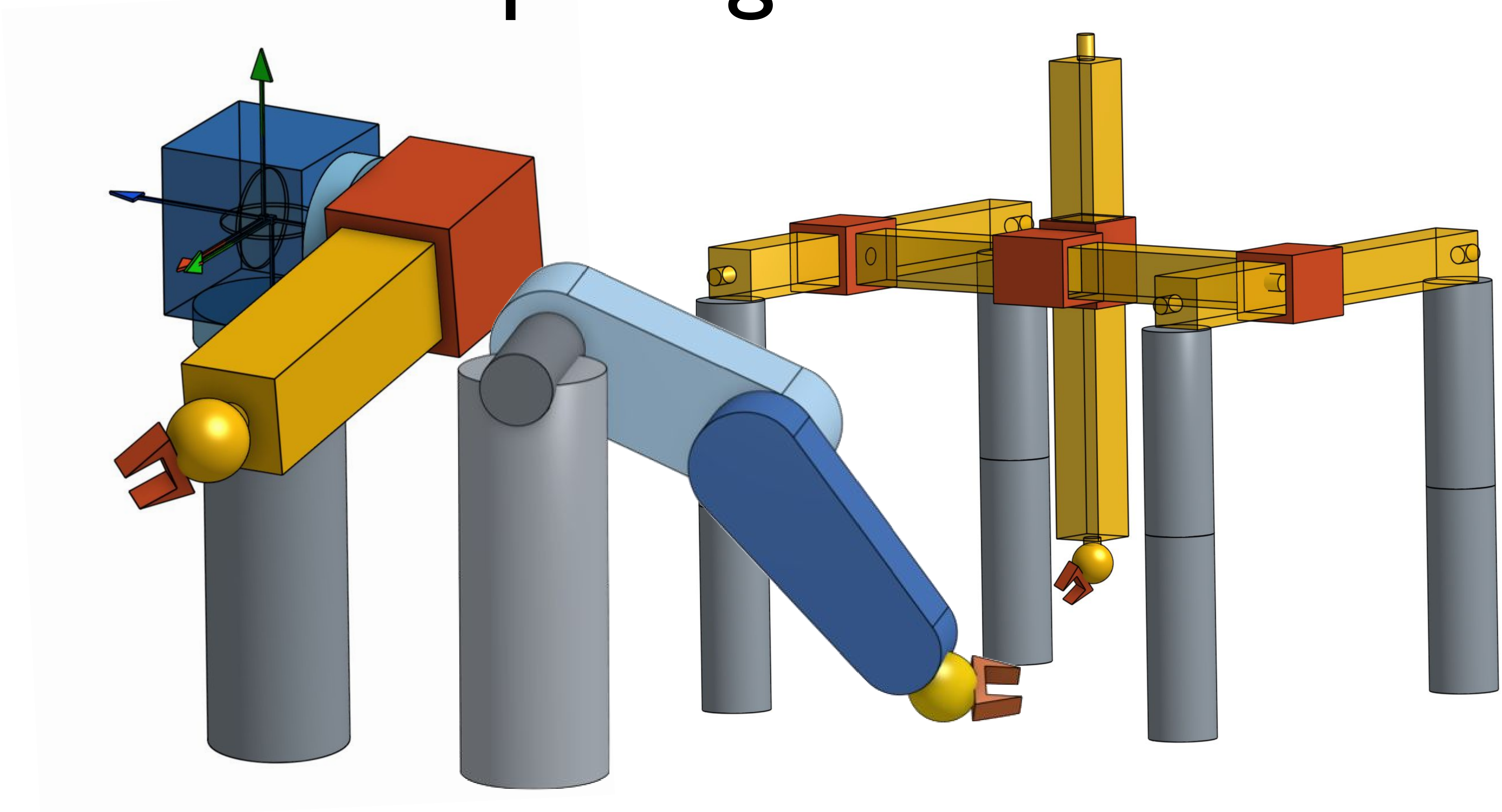
$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_1 & -a_2 s_1 \\ a_1 c_1 + a_2 c_1 & a_2 c_1 \end{bmatrix} \qquad \theta_2 = 0, \pi$$

$$\det(\mathbf{J}) = 0$$



Decomposing 6DOF arms



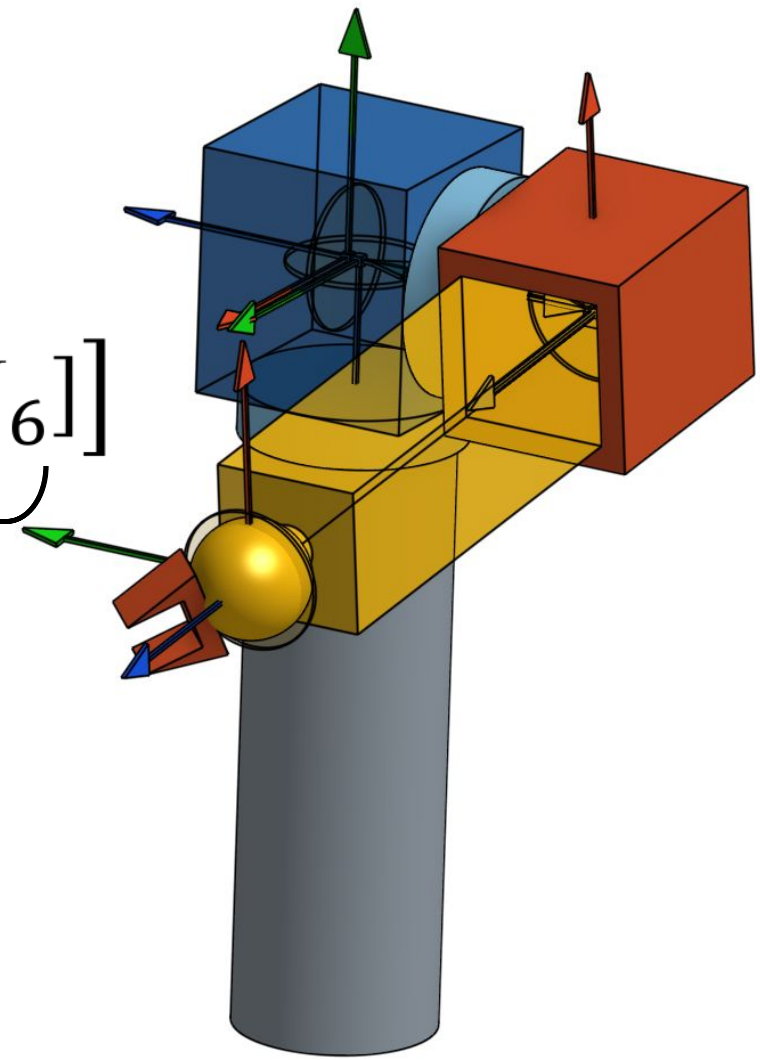
Arm singularities   Wrist singularities

Decomposing 6DOF arms

$\mathbf{J}(q)$ is 6x6 and is singular
if and only if
det $(\mathbf{J}) = 0$

$$\mathbf{J} = \Big[\underbrace{[\mathbf{J}_1][\mathbf{J}_2][\mathbf{J}_3]}\underbrace{[\mathbf{J}_4][\mathbf{J}_5][\mathbf{J}_6]}\Big]$$

$$\mathbf{J} = [\mathbf{J}_P|\mathbf{J}_O]$$



Penn Engineering

Decomposing 6DOF arms

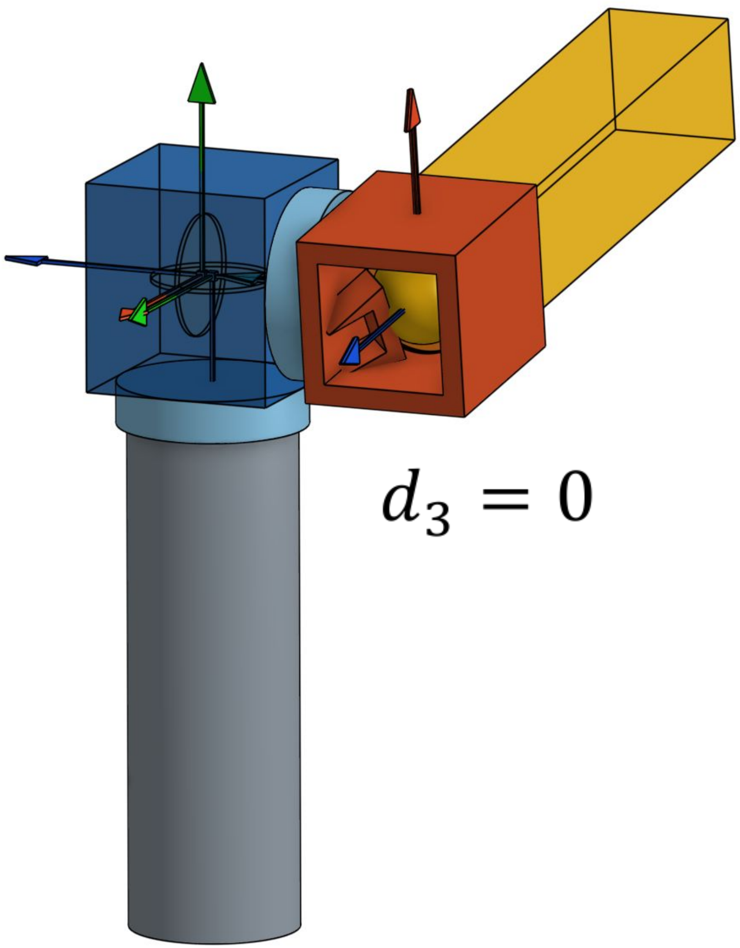
$$\mathbf{J}_O = \begin{bmatrix} \hat{\mathbf{z}}_3 \times (\boldsymbol{P}_6 - \boldsymbol{P}_3) & \hat{\mathbf{z}}_4 \times (\boldsymbol{P}_6 - \boldsymbol{P}_4) & \hat{\mathbf{z}}_5 \times (\boldsymbol{P}_6 - \boldsymbol{P}_5) \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

$$P_3 = P_4 = P_5 = P_6$$

Choose $P_6 = P_c$

$$\mathbf{J}_O = \begin{bmatrix} [0] & [0] & [0] \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

$$\mathbf{J} = [\mathbf{J}_P | \boxed{\mathbf{J}_O}]$$

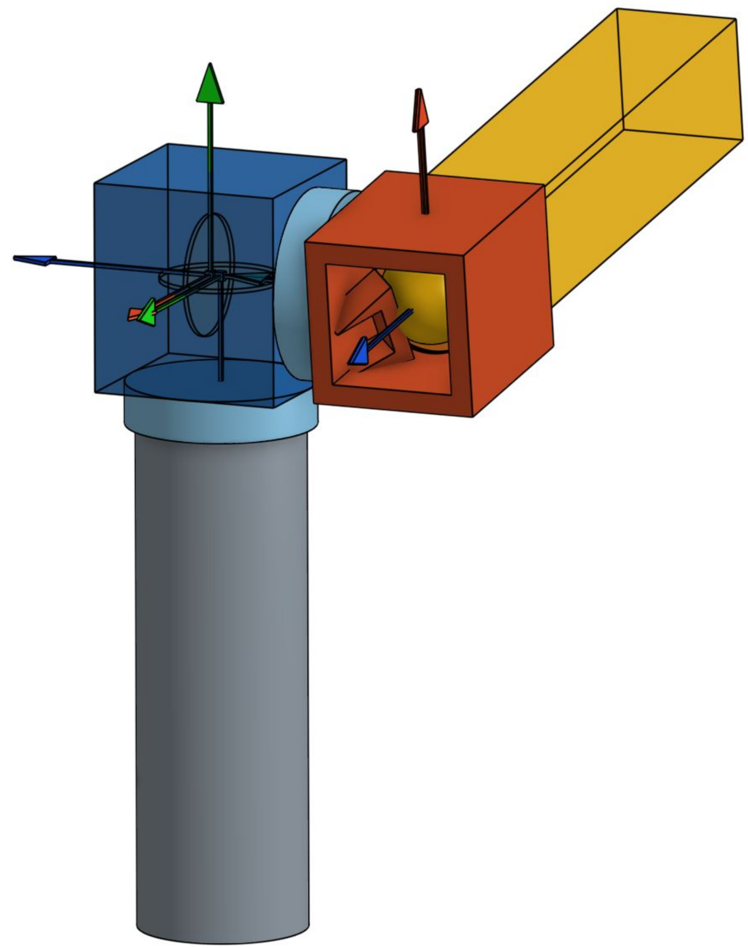


$d_3 = 0$

Decomposing 6DOF arms

$$\mathbf{J}_O = \begin{bmatrix} \hat{\mathbf{z}}_3 \times (\boldsymbol{P}_6 - \boldsymbol{P}_3) & \hat{\mathbf{z}}_4 \times (\boldsymbol{P}_6 - \boldsymbol{P}_4) & \hat{\mathbf{z}}_5 \times (\boldsymbol{P}_6 - \boldsymbol{P}_5) \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

$$\mathbf{J}_O = \begin{bmatrix} [0] & [0] & [0] \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$



$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}$$

$$\det(\mathbf{J}) = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{0} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}$$
$$= \det(\mathbf{J}_{11}) \det(\mathbf{J}_{22})$$

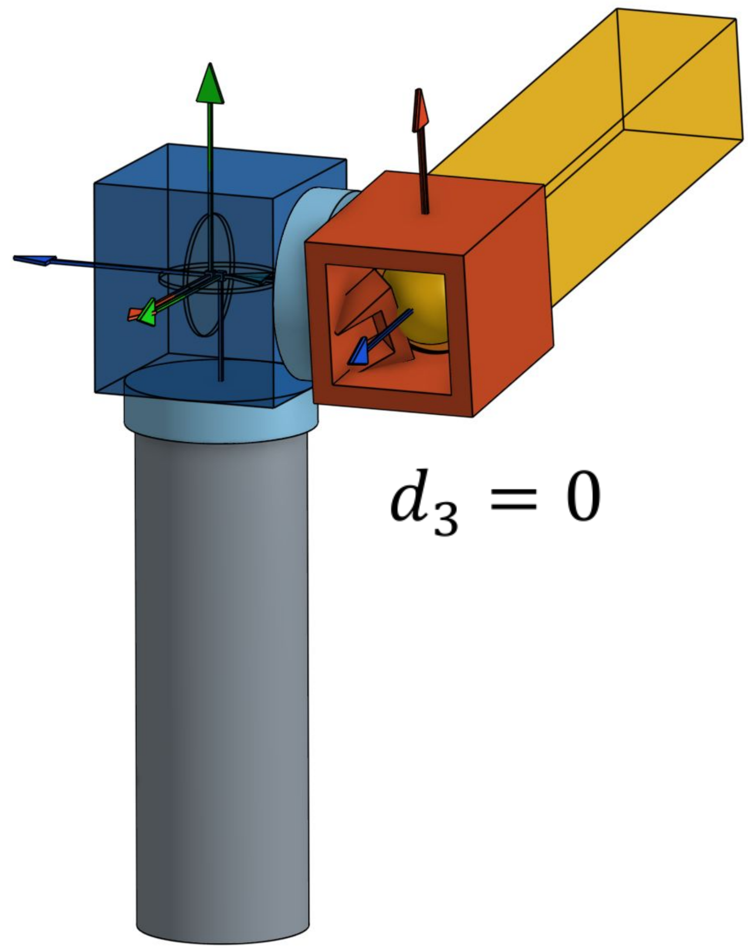
Penn Engineering

Decomposing 6DOF arms

$\mathbf{J}_{22}$ is orientation part of wrist joints

$$\mathbf{J}_{22} = [\hat{\mathbf{z}}_3 \quad \hat{\mathbf{z}}_4 \quad \hat{\mathbf{z}}_5]$$

$$= \det(\mathbf{J}_{11}) \det(\mathbf{J}_{22})$$



$d_3 = 0$

Wrist Singularities

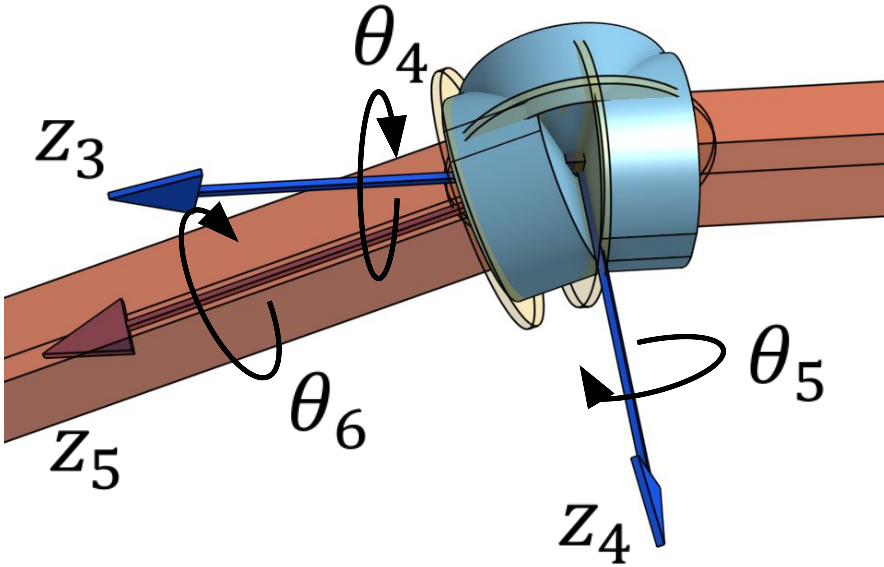
Singularities occur in the wrist:

- if and only if joint axis are collinear (0 or π radians)
- unavoidable if moving through that point.

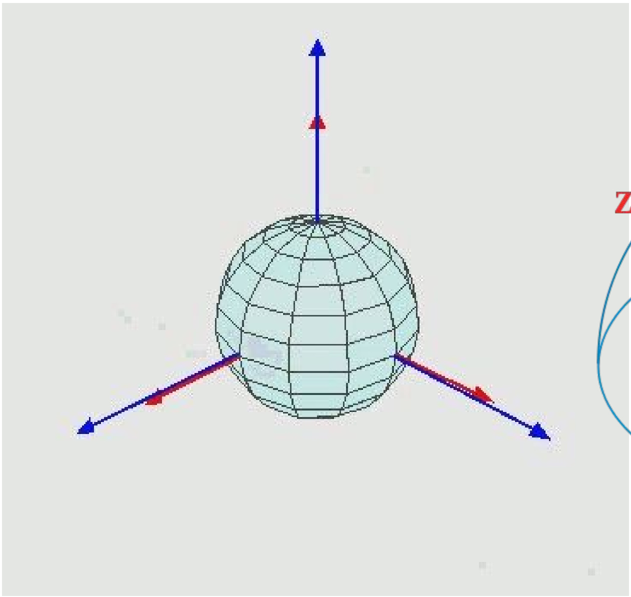
$\mathbf{J}_{22}$ is orientation part of wrist joints

$$\mathbf{J}_{22} = [\hat{\mathbf{z}}_3 \quad \hat{\mathbf{z}}_4 \quad \hat{\mathbf{z}}_5]$$

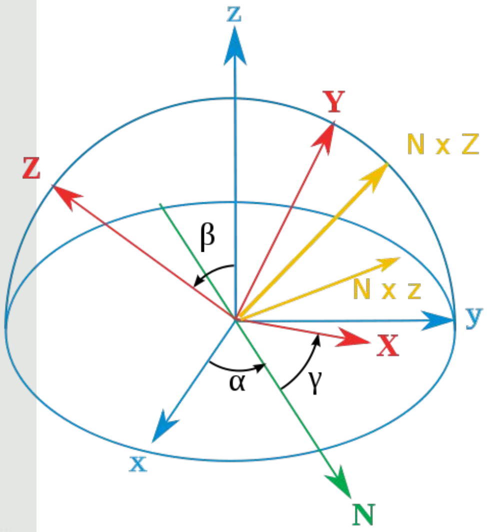
$$\theta_5 = 0 \; or \; \pi$$



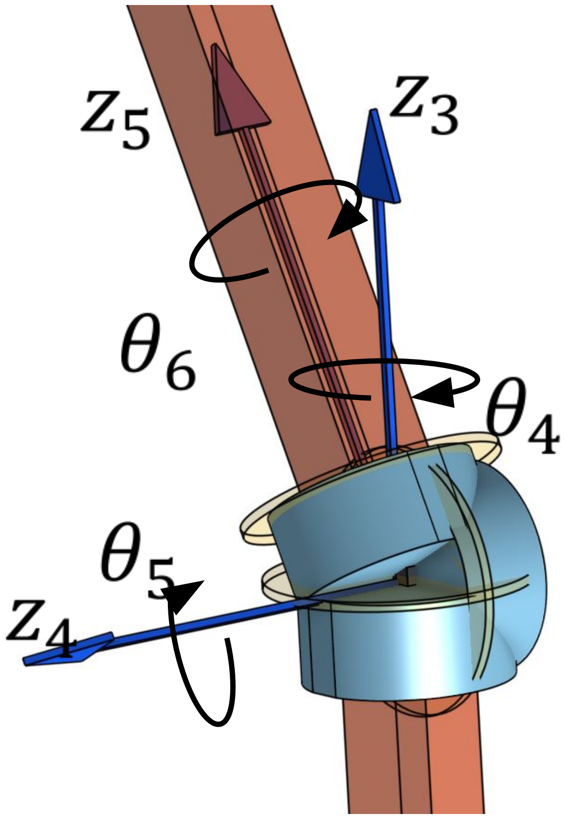
Representational Singularities



By Euler2.gif: Juansemperederivative work: Xavax - This file was derived from Euler2.gif:, CC BY-SA 3.0,



By Juansempere - Own work, GFDL,

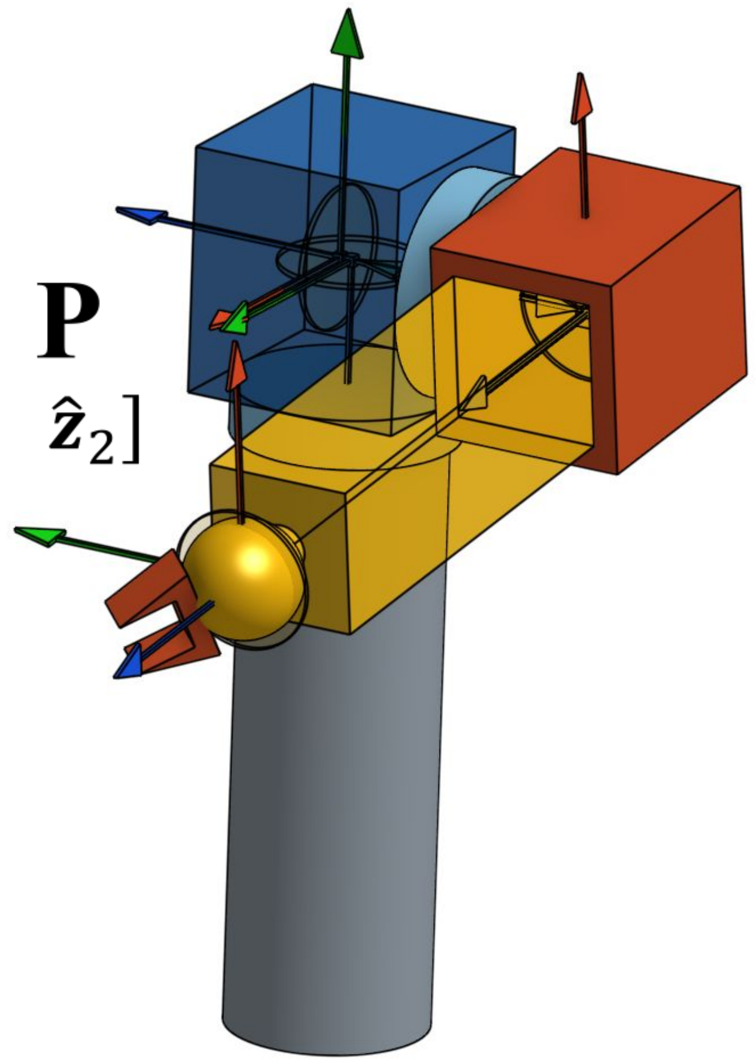


Arm Singularities

$\mathbf{J}_{11}$ is the position jacobian of links 1-3

$$\begin{array}{cccc} & \mathbf{R} & \mathbf{R} & \mathbf{P} \\ \mathbf{J}_{11} = [& \hat{\boldsymbol{z}}_0 \times (\boldsymbol{P}_6 - \boldsymbol{P}_0) & \hat{\boldsymbol{z}}_1 \times (\boldsymbol{P}_6 - \boldsymbol{P}_1) & \hat{\boldsymbol{z}}_2] \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 d_3 s_2 - d_2 s_1 \\ c_1 d_2 + d_3 s_1 s_2 \\ c_2 d_3 \end{bmatrix}$$



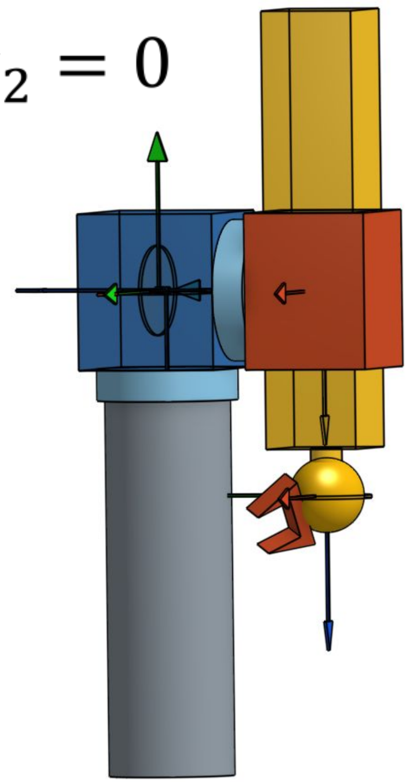
Penn Engineering

Arm Singularities

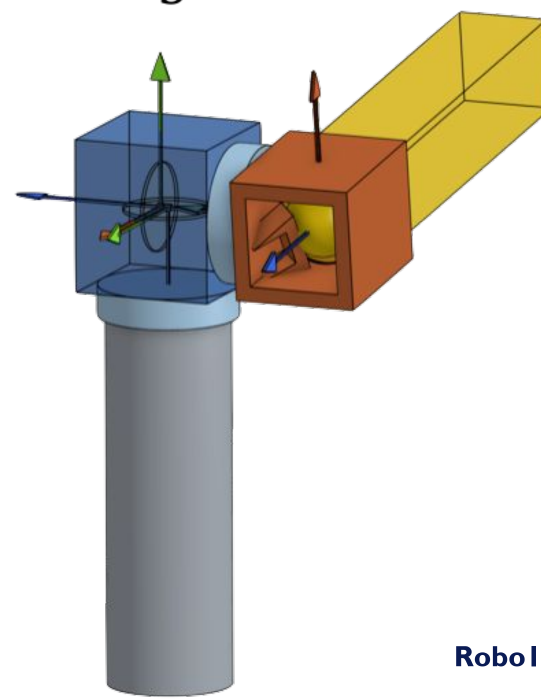
$$\mathbf{J}_{11} = \begin{bmatrix} -s_1 s_2 d_3 - c_1 d_2 & c_1 c_2 d_3 & c_1 s_2 \\ c_1 s_2 d_3 - s_1 d_2 & s_1 c_2 d_3 & s_1 s_2 \\ 0 & -s_2 d_3 & c_2 \end{bmatrix}$$

$$\det(\mathbf{J}) = -s_2 d_3^2$$

$-s_2 = 0$

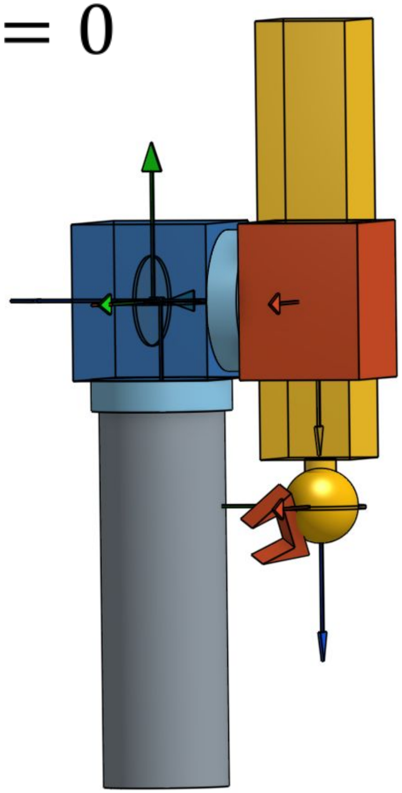


$d_3^2 = 0$

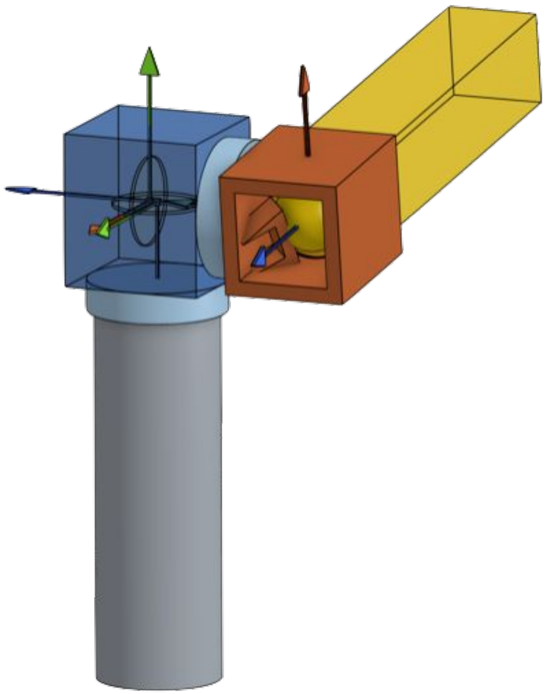


Stanford arm Singularities

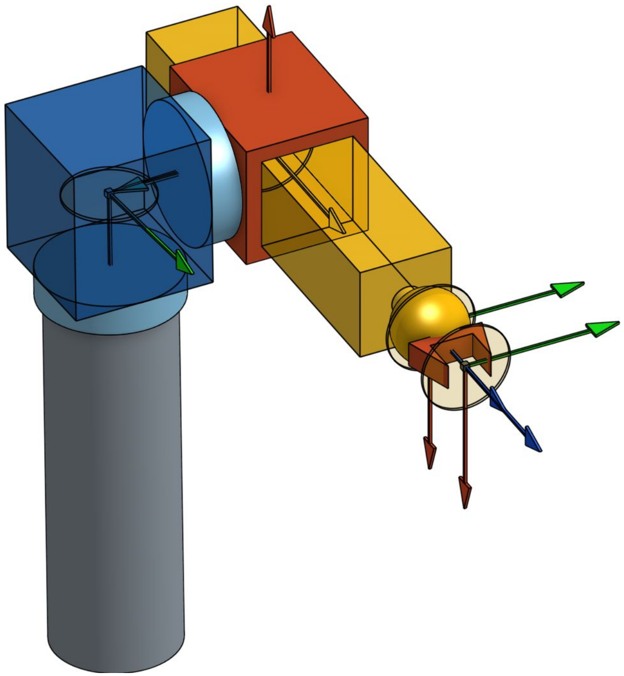
$-s_2 = 0$



$d_3^2 = 0$

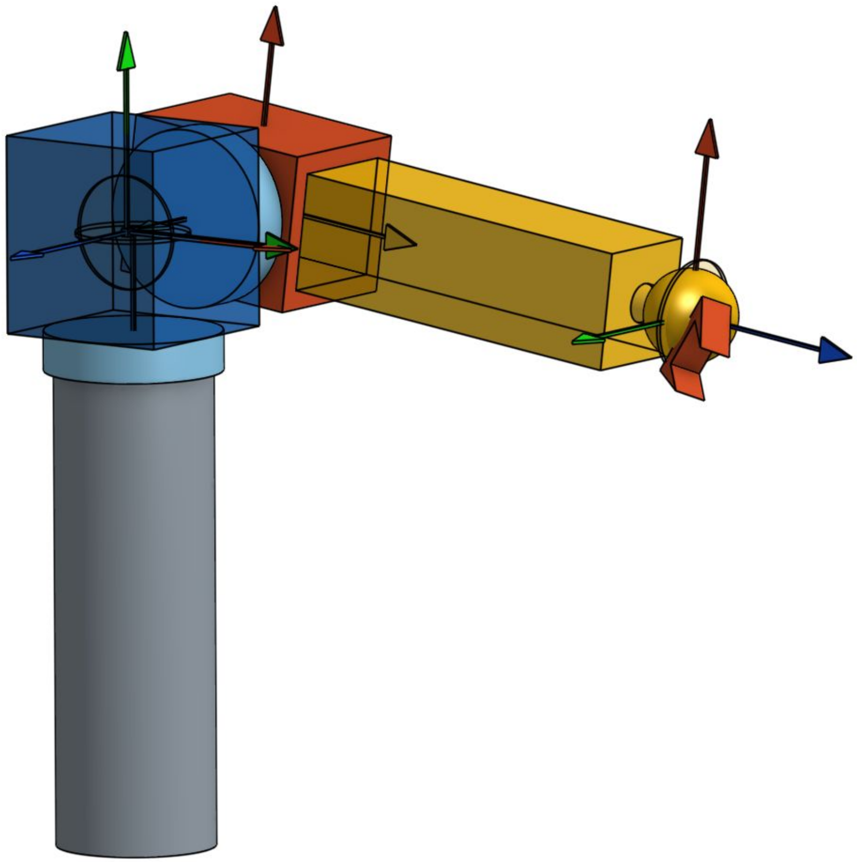


$\theta_5 = 0$



Manipulability

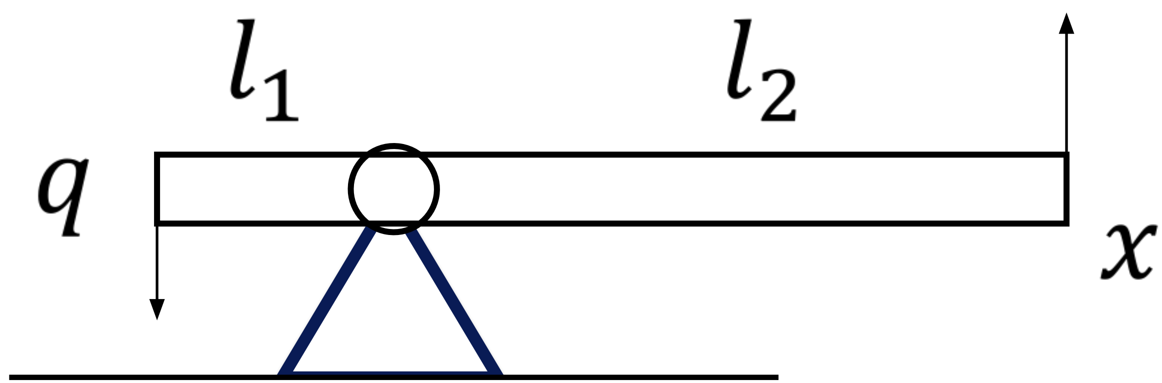
$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$



Manipulability

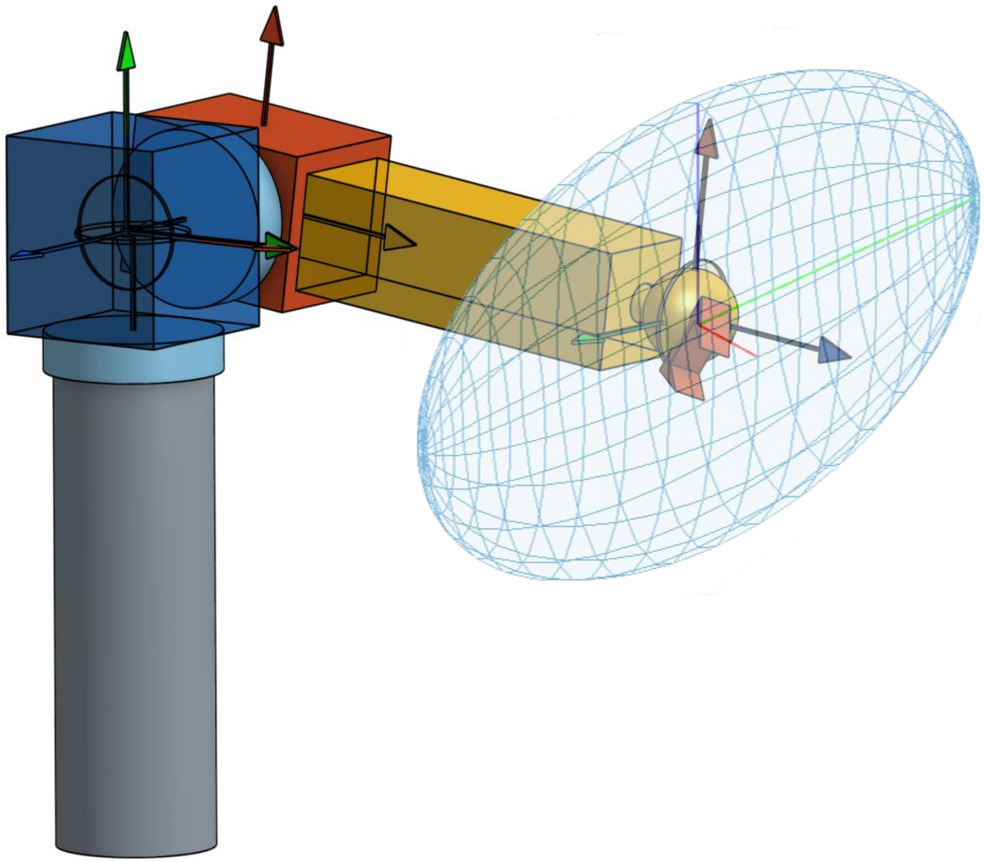
$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$

$$\dot{x} = \frac{l_2}{l_1}\dot{q}$$



Manipulability

$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$



Manipulability

For unit input $\|\dot{\boldsymbol{q}}\|$=1 and minimum norm solution

$$\|\dot{q}\| = \dot{q}^T \dot{q}$$

$$\xi^T \left(\mathbf{J}\mathbf{J}^T\right)^{-1} \xi \leq 1$$

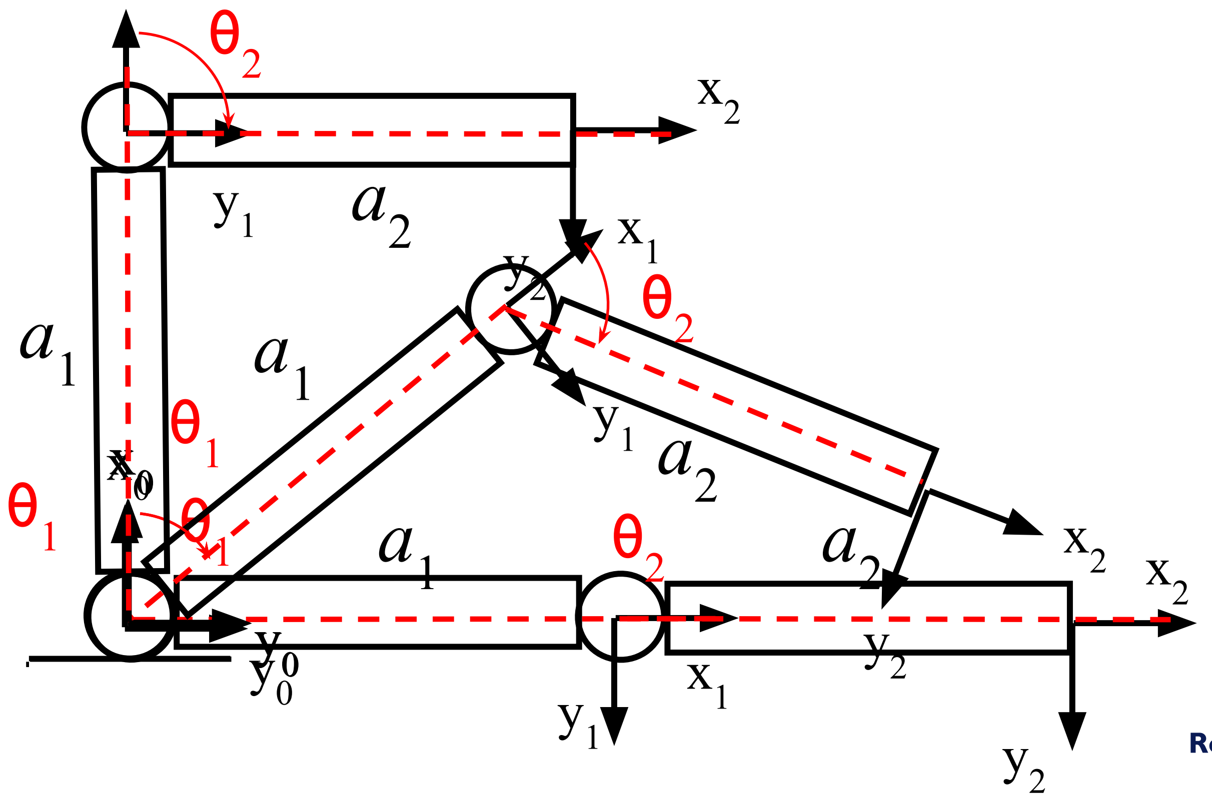
$$\det\left(\mathbf{J}\mathbf{J}^T\right) = \lambda_1^2 \lambda_2^2 \dots \lambda_n^2$$

- Axis of ellipsoid are eigenvalues
- Volume $= |K \lambda_1 \lambda_2 \dots \lambda_n| = |K \det(\mathbf{J})|$
- Manipulability := $\boxed{\mu = |\det(\mathbf{J})|}$

Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$

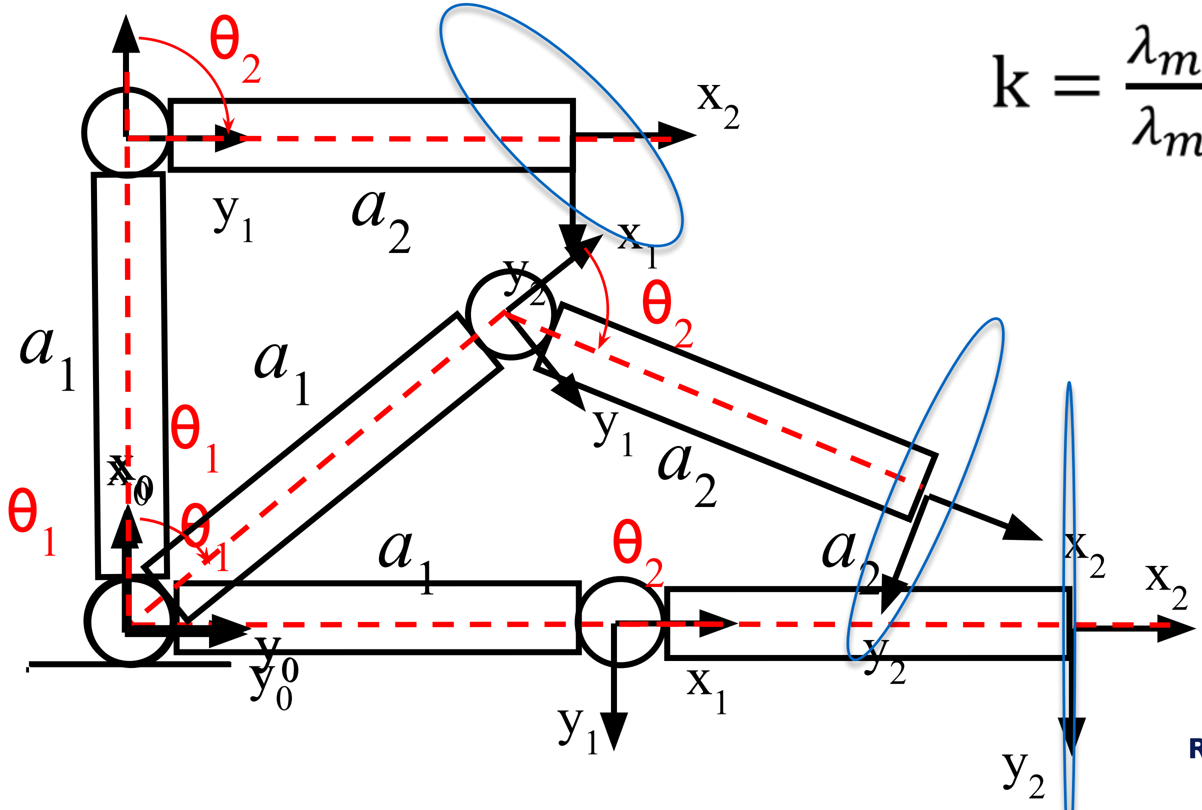


Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

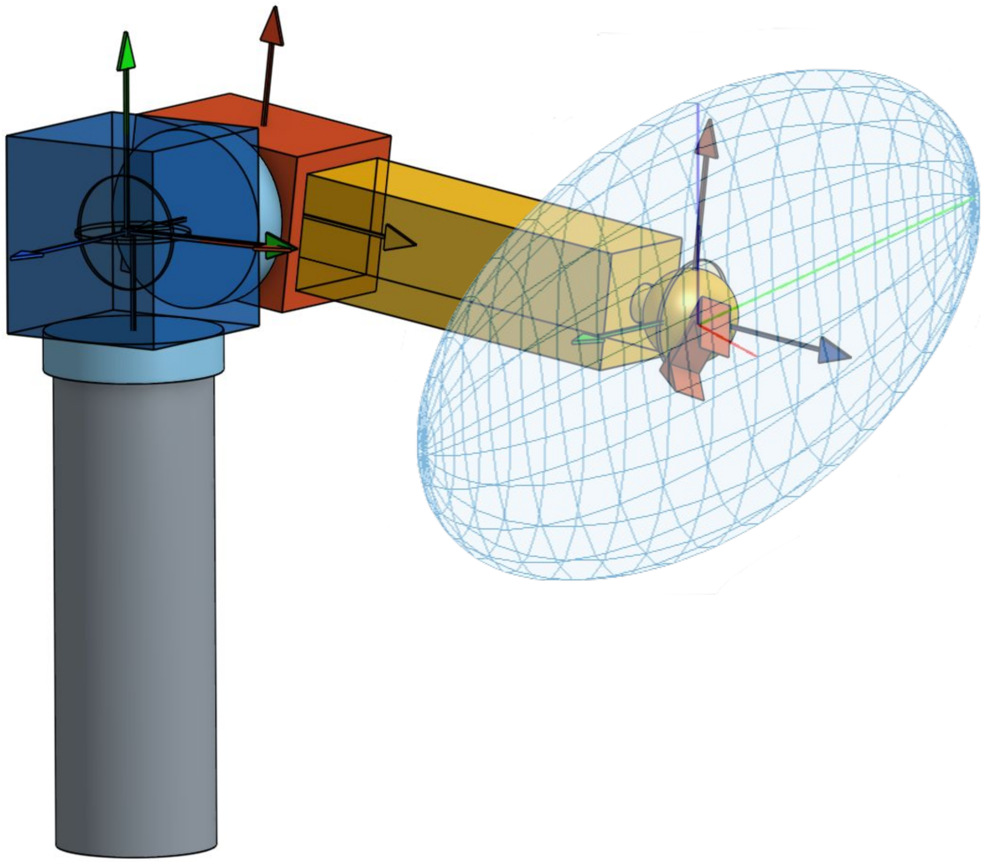
$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$

$$\mathrm{k} = \frac{\lambda_{max}}{\lambda_{min}}$$

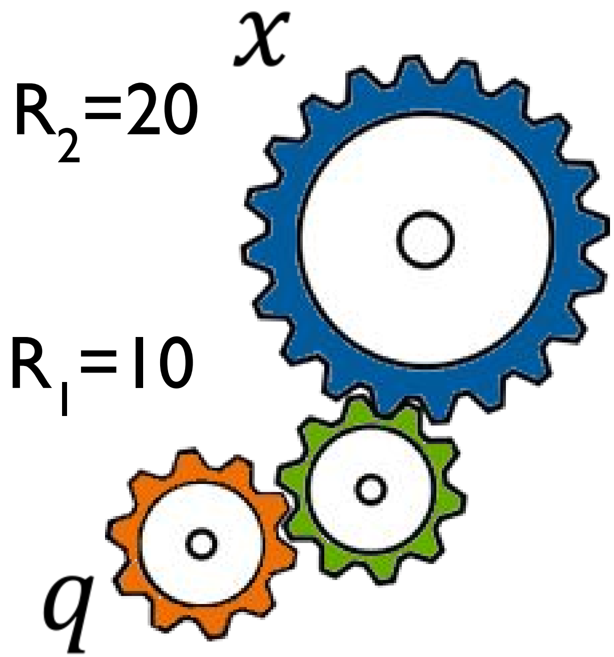


Manipulability

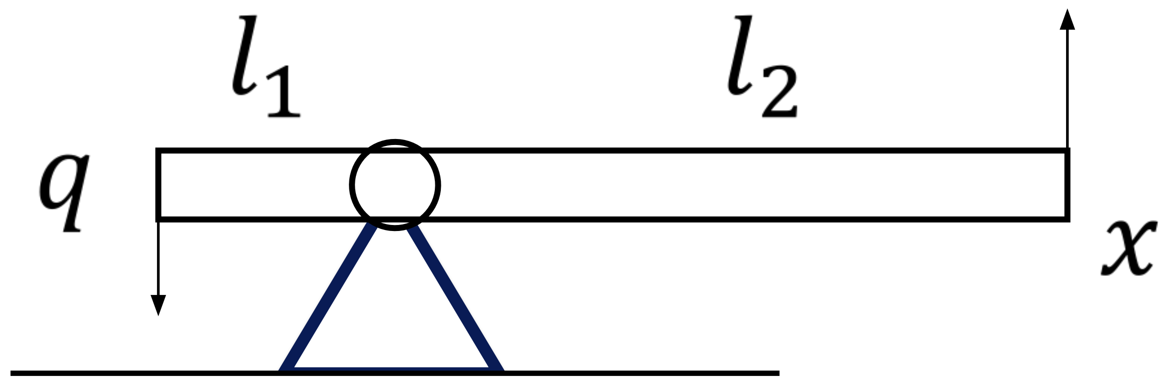
For unit norm input $||\dot{\boldsymbol{q}}||=1$ $\quad \dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$



Jacobian Transpose



$$\frac{F_{in}}{F_{out}} = \frac{V_{out}}{V_{in}}$$



Principle of Virtual Work

$$\boldsymbol{F} \cdot \delta \boldsymbol{x} = \boldsymbol{\tau} \cdot \delta \boldsymbol{q}$$

$$\boldsymbol{F}^T \delta \boldsymbol{x} = \boldsymbol{\tau}^T \delta \boldsymbol{q}$$

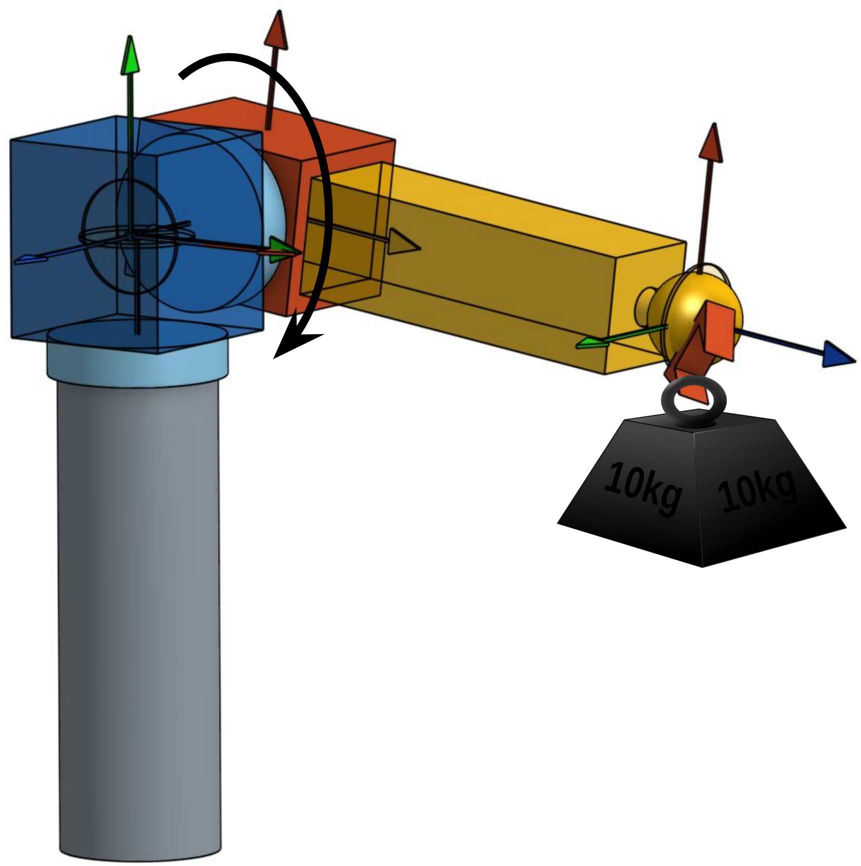
$$\boldsymbol{F}^T \mathbf{J} \delta \boldsymbol{q} = \boldsymbol{\tau}^T \delta \boldsymbol{q}$$

$$\boldsymbol{F}^T \mathbf{J} = \boldsymbol{\tau}^T$$

$$\boxed{\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}}$$

Static Forces

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ F_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix}$$



$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

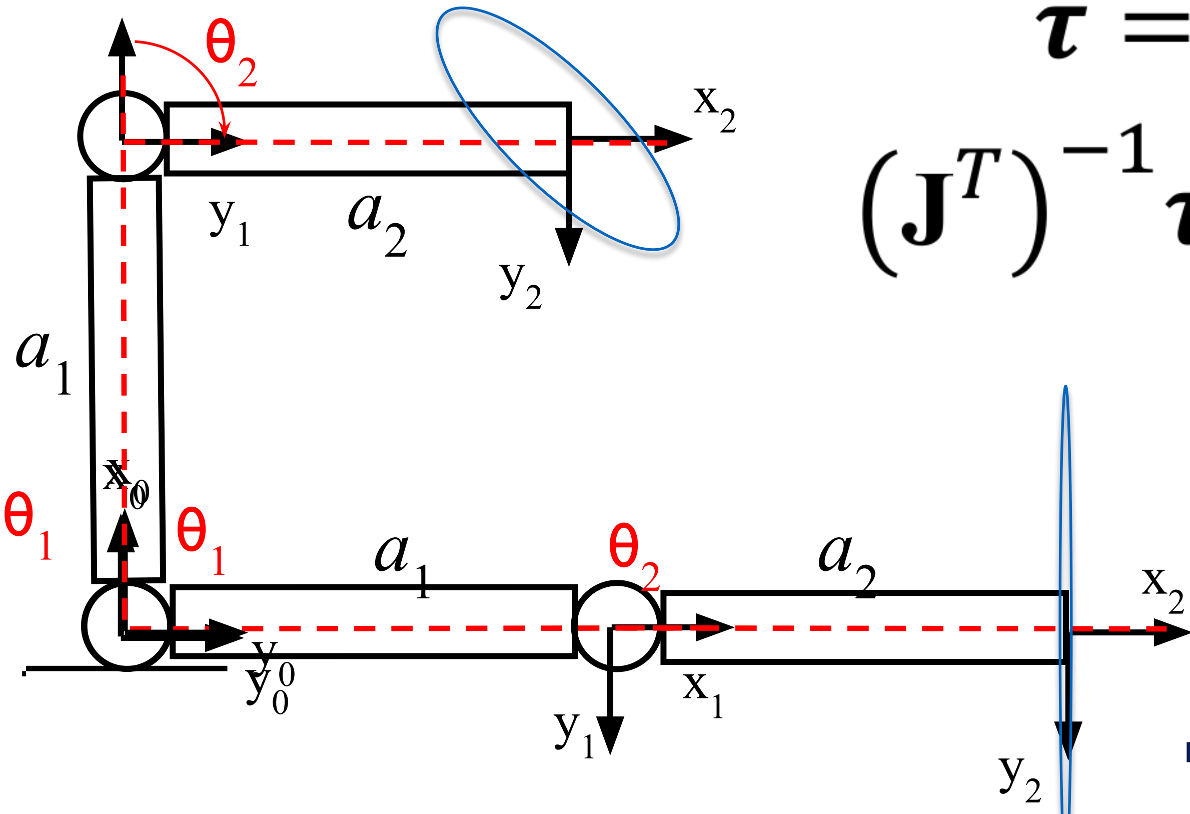
Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

$$\left(\mathbf{J}^T\right)^{-1} \boldsymbol{\tau} = \mathbf{F}$$

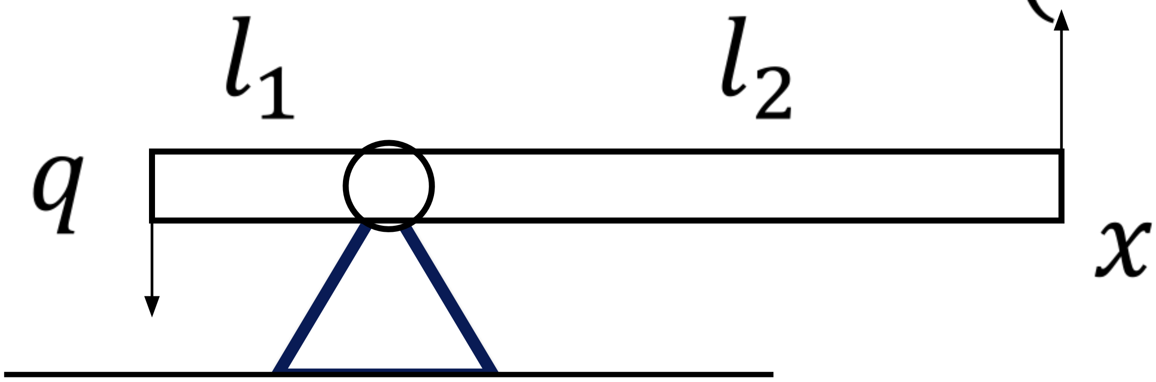


Manipulability

$$\dot{x} = \frac{l_2}{l_1}\dot{q} \qquad \mathbf{F} = \frac{l_1}{l_2}\boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \mathbf{J}^T\mathbf{F}$$

$$\left(\mathbf{J}^T\right)^{-1}\boldsymbol{\tau} = \mathbf{F}$$

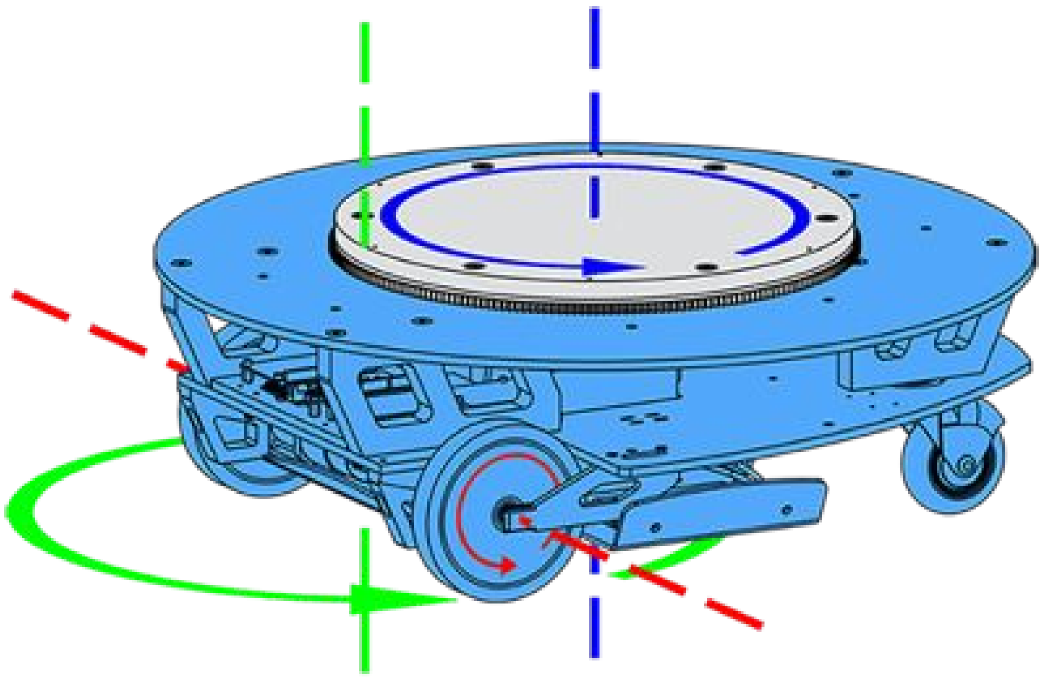


Robotics: Fundamentals

Prof. Mark Yim
University of Pennsylvania

Week 7: Mobile Robot Jacobian

Jacobian for Mobile Robots

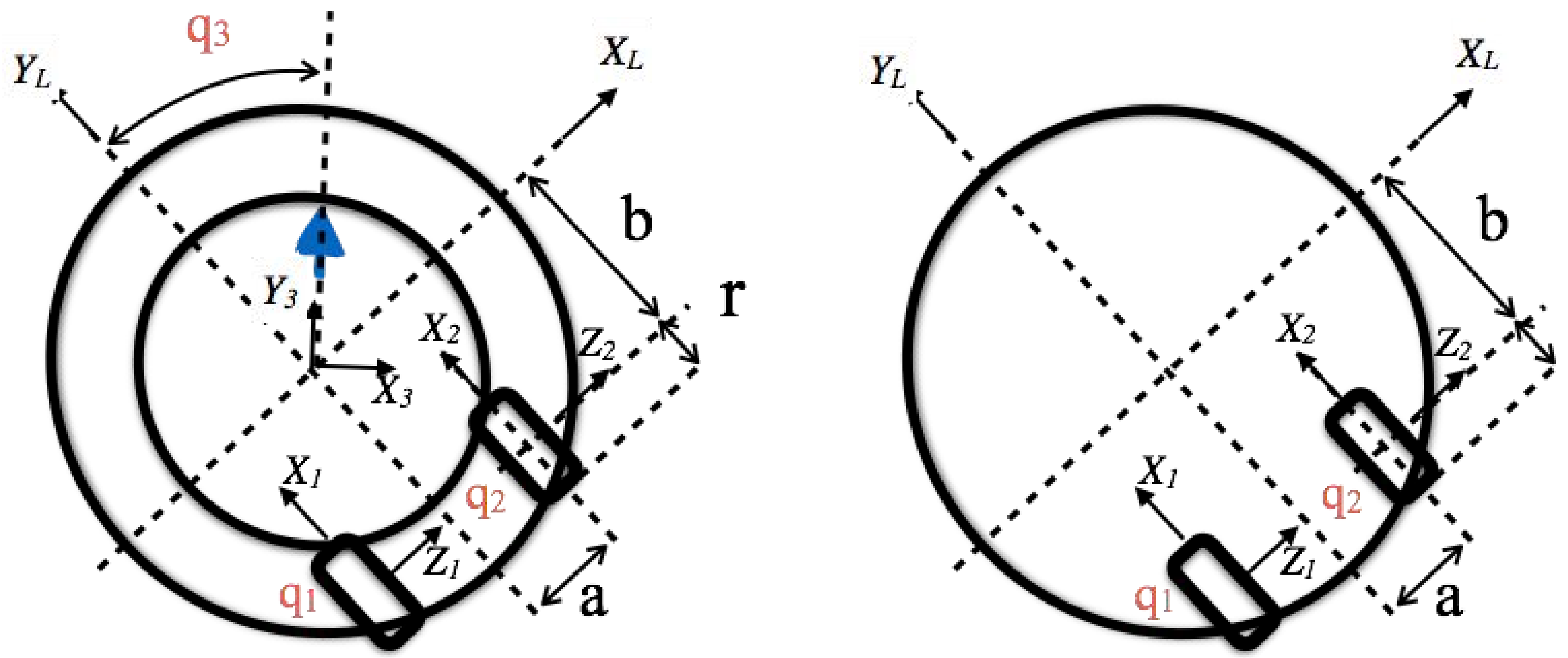


$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

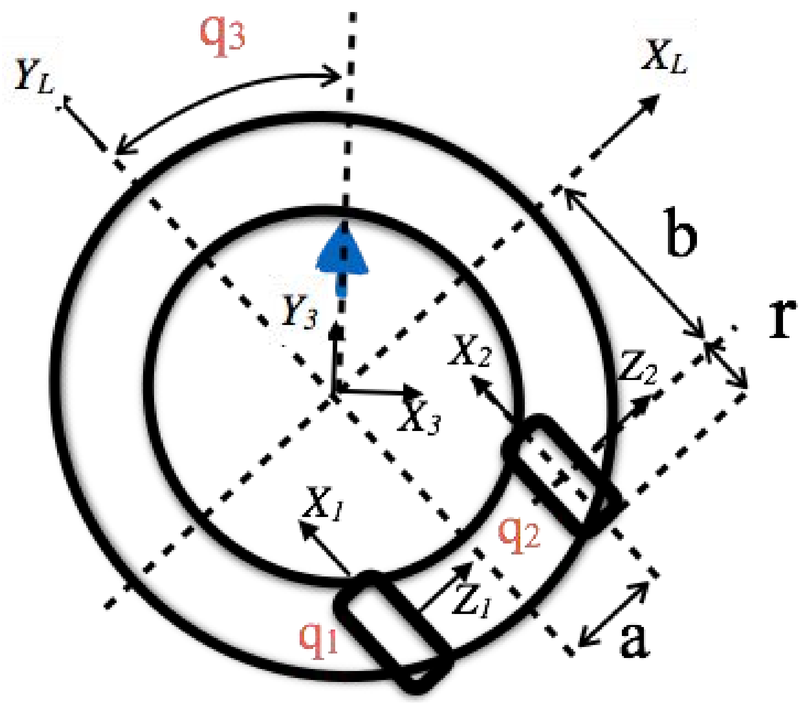
Jacobian for Mobile Robots

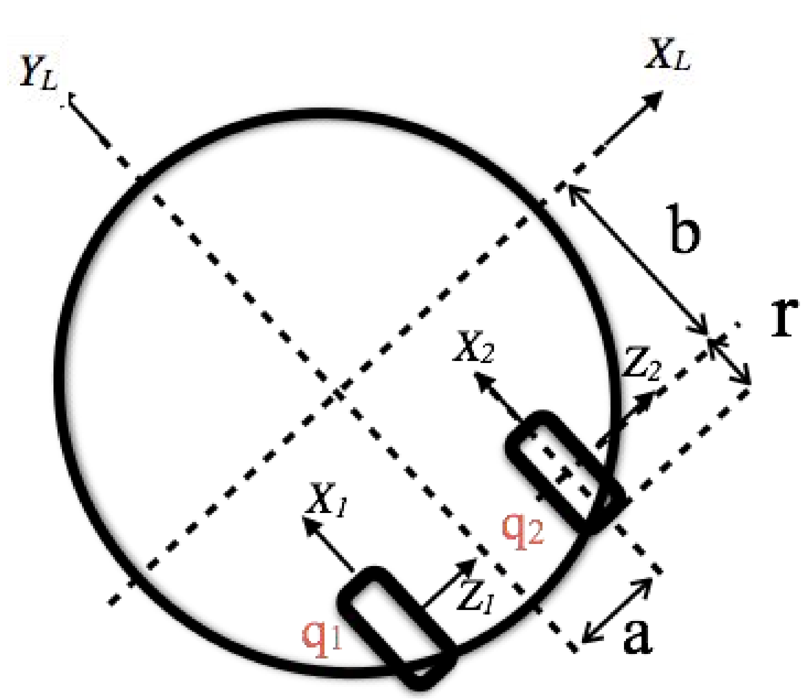
$$\mathbf{J} = [\mathbf{J}_D | \mathbf{J}_3]$$



Jacobian for Mobile Robots

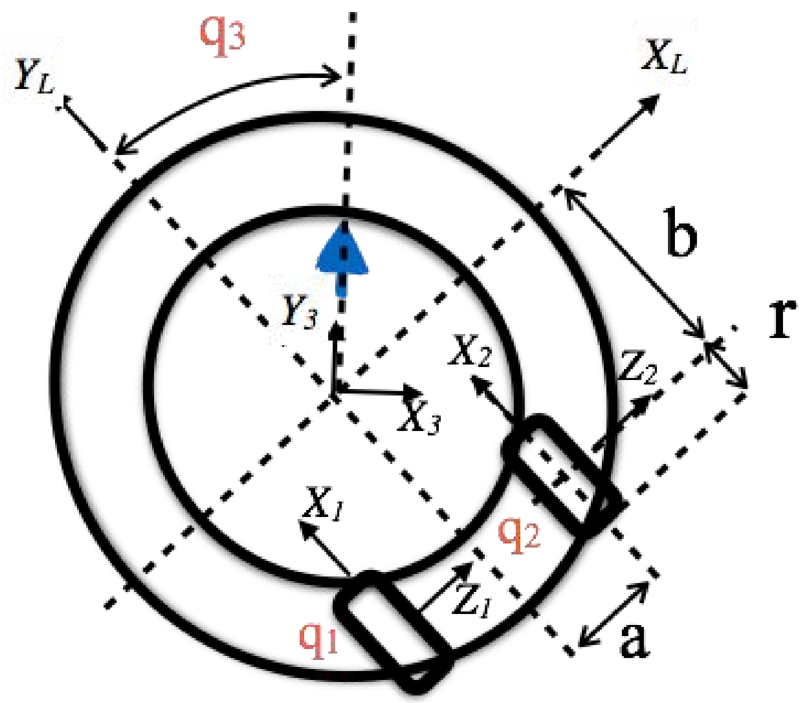
$$\mathbf{J}_D = R_{3L}\mathbf{J}_D^L \qquad \mathbf{J} = [\mathbf{J}_D | \mathbf{J}_3]$$

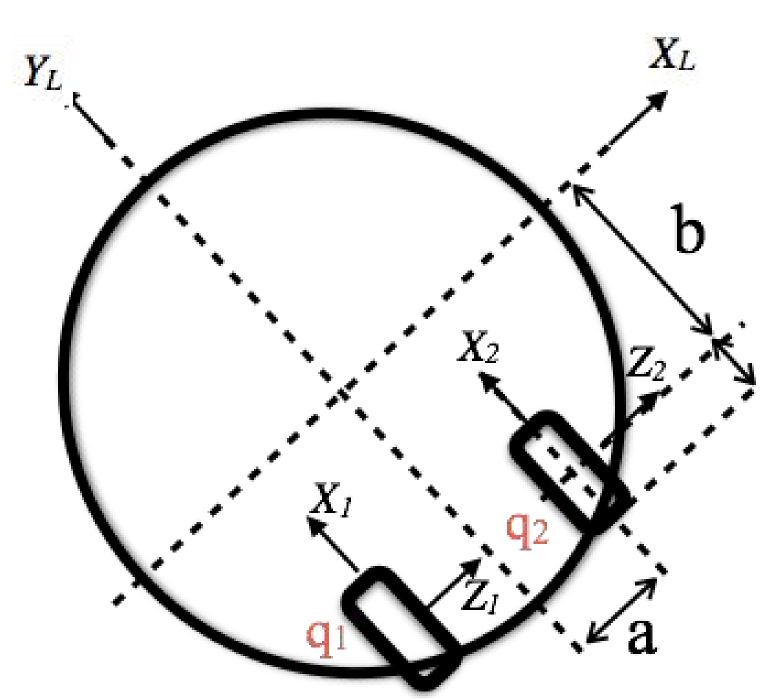




Jacobian for Mobile Robots

$$\mathbf{J}_D = R_{3L}\mathbf{J}_D^L \qquad \mathbf{J}_i = R_{3L}\mathbf{J}_i^L$$





Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

Penn Engineering

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \qquad \mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix}$$

$$R_{3L} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix}$$

$$\mathbf{J}_1 = \begin{bmatrix} -\frac{rbc_3}{2a} - \frac{rs_3}{2} \\ -\frac{rbs_3}{2a} + \frac{rc_3}{2} \\ -\frac{r}{2a} \end{bmatrix}$$

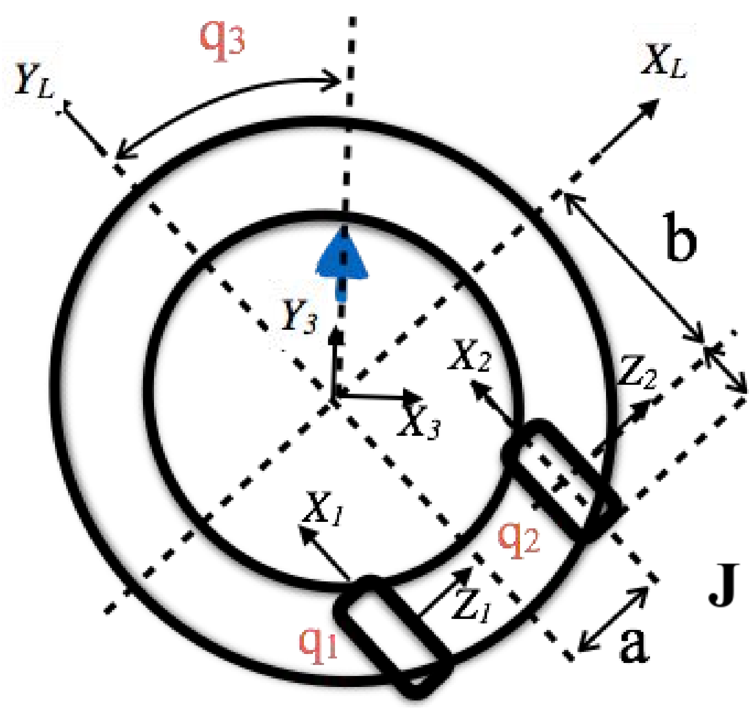
Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \qquad \mathbf{J}_2 = \begin{bmatrix} \frac{rbc_3}{2a} - \frac{rs_3}{2} \\ \frac{rbs_3}{2a} + \frac{rc_3}{2} \\ \frac{r}{2a} \end{bmatrix}$$

Penn Engineering

Jacobian for Mobile Robots



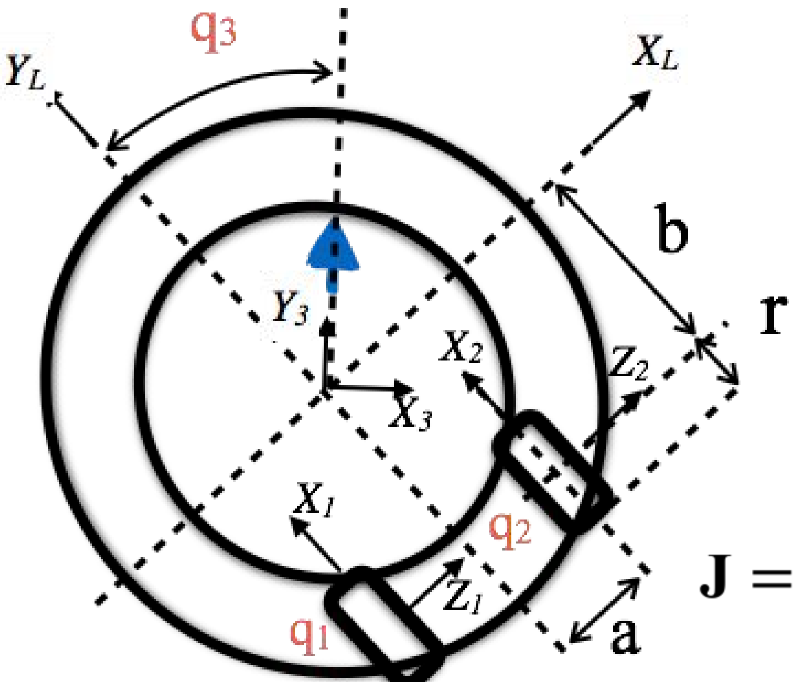
$$\mathbf{J} = [\mathbf{J_1} \quad \mathbf{J_2} \quad \mathbf{J_3}]$$

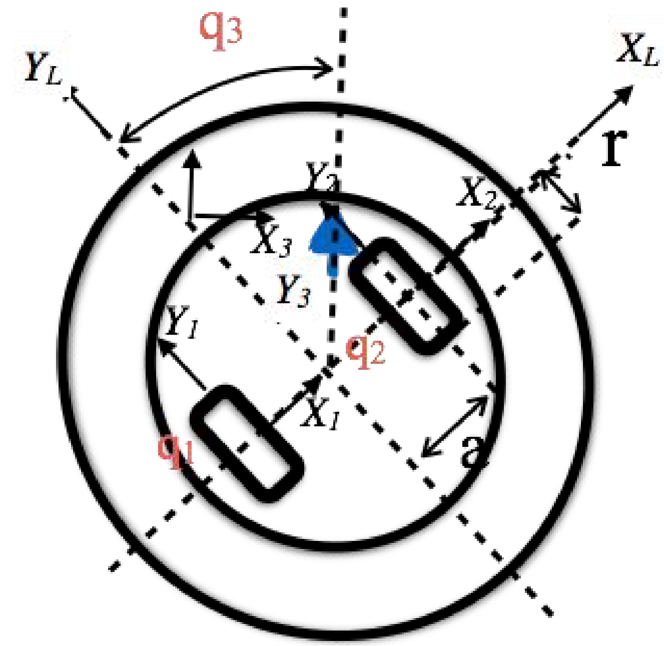
$$\mathbf{J} = \begin{bmatrix} -\frac{rbc_3}{2a} - \frac{rs_3}{2} & \frac{rbc_3}{2a} - \frac{rs_3}{2} & 0 \\ -\frac{rbs_3}{2a} + \frac{rc_3}{2} & \frac{rbs_3}{2a} + \frac{rc_3}{2} & 0 \\ -\frac{r}{2a} & \frac{r}{2a} & 1 \end{bmatrix}$$

$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -rbc_3 - ras_3 & rbc_3 - ras_3 & 0 \\ -rbs_3 + rac_3 & rbs_3 + rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$b = 0?$$



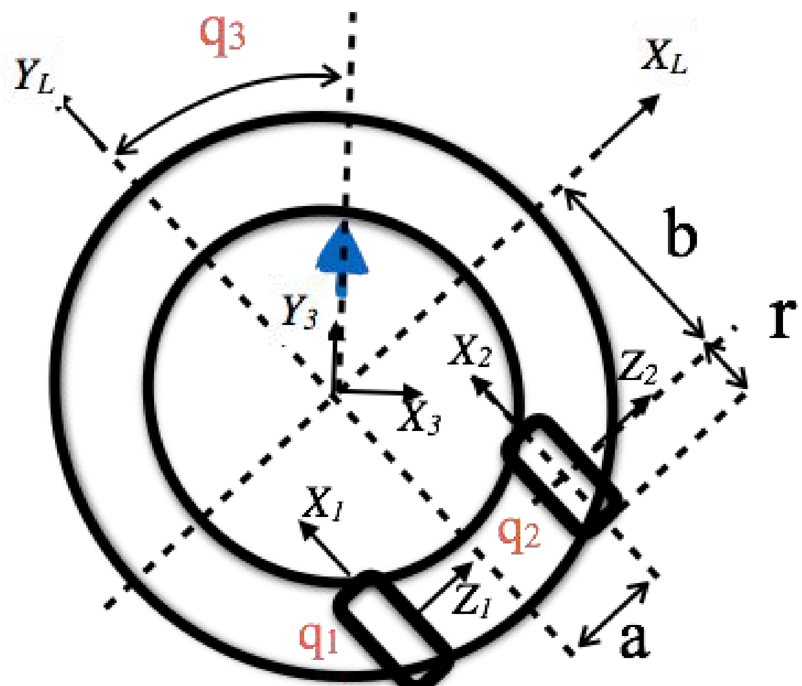


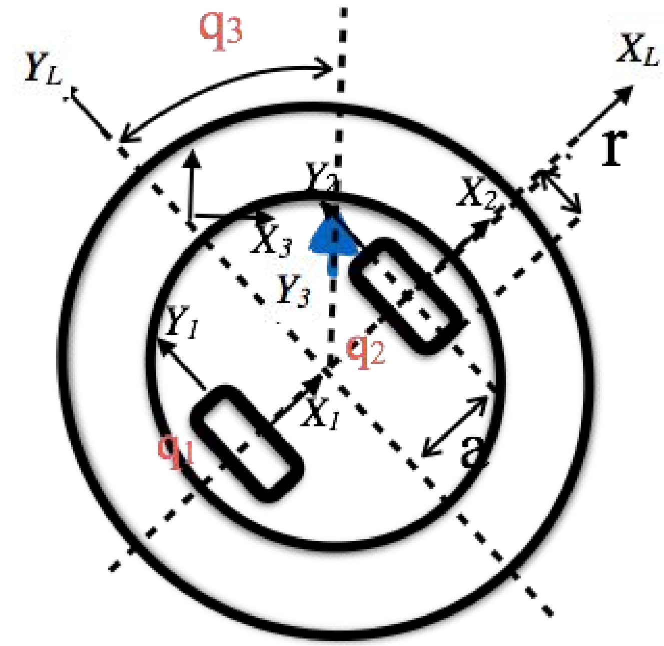
$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -rbc_3 - ras_3 & rbc_3 - ras_3 & 0 \\ -rbs_3 + rac_3 & rbs_3 + rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$b = 0?$$

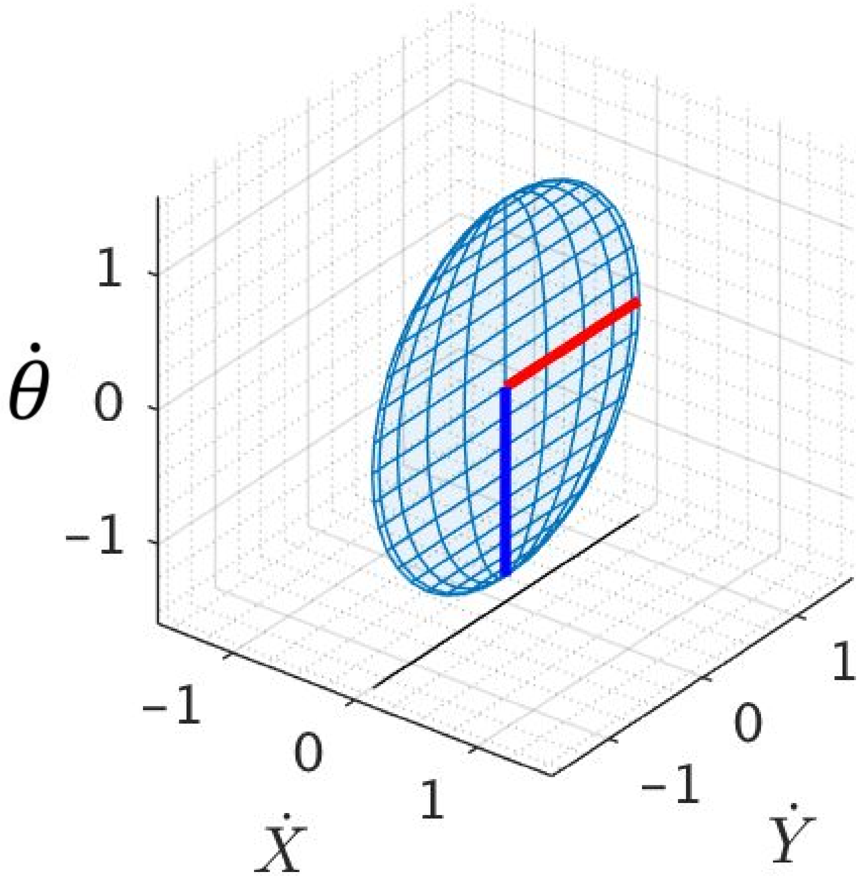
Singularity!

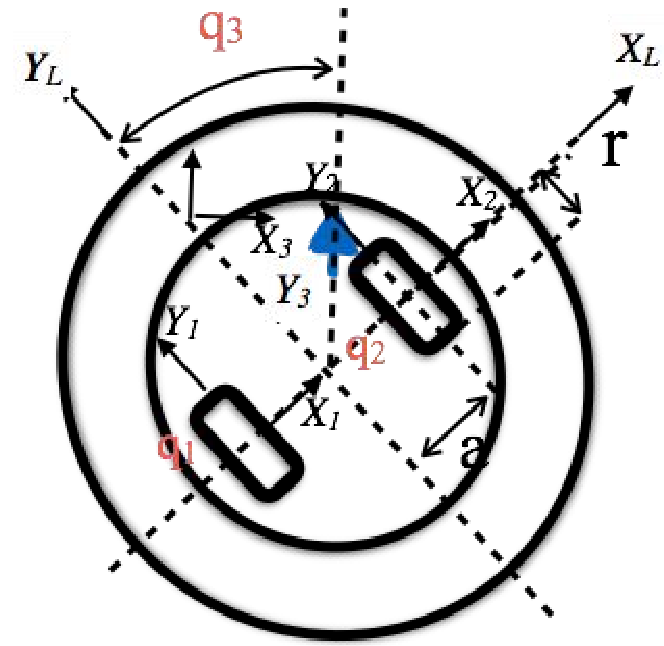




$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -ras_3 & -ras_3 & 0 \\ rac_3 & rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

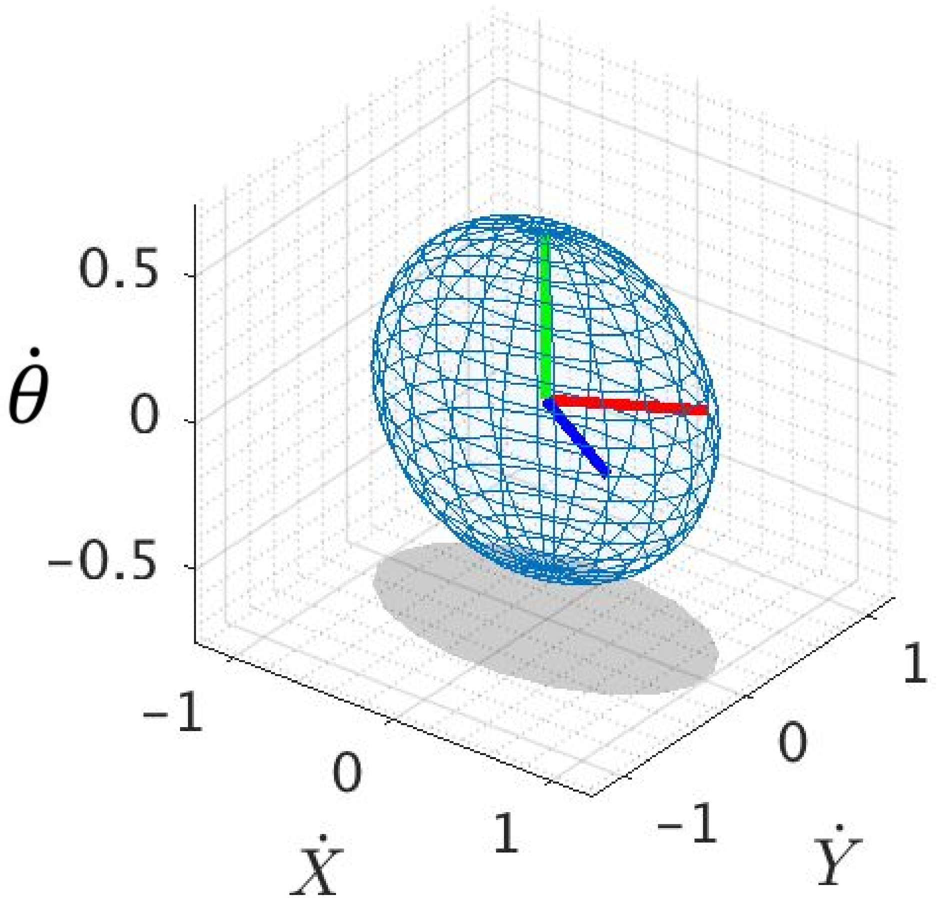
Mobility Ellipsoid

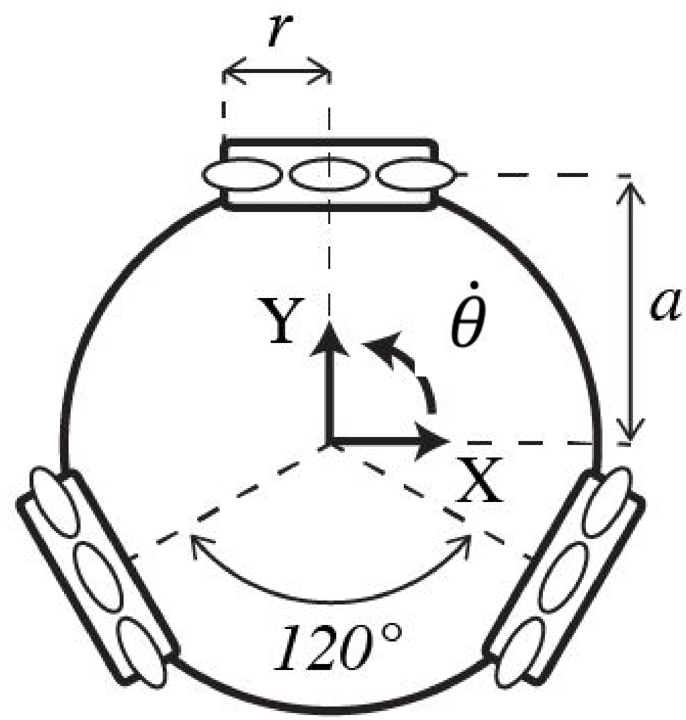




$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -ras_3 & -ras_3 & 0 \\ rac_3 & rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Mobility Ellipsoid





Mobility Ellipsoid