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Problem 1

Let $X_1, \dots, X_n$ and be i.i.d. real valued random variables such that $\mathbf{E}(X_1^2) < \infty$. Denote $\mu := \mathbf{E}(X_1)$, $\sigma^2 := \mathrm{Var}(X_1) > 0$. Denote $\overline{X}_n := n^{-1} \sum_{i=1}^n X_i$.

(a) (2 points) Write Chebyshev's inequality for $\mathbf{P}\left(|\overline{X}_n - \mu| \geq t\right)$ for an arbitrary $t > 0$.

(b) (4 points) Let $\varepsilon > 0$ and $\alpha \in (0,1)$ be fixed numbers. Find the smallest number $N = N(\varepsilon, \alpha) \in \mathbb{N}$ such that $\forall n \geq N$

$$\mathbf{P}\left(|\overline{X}_n - \mu| \geq \varepsilon\right) \leq \alpha.$$

Solution **1.a**

$$\mathbb{E}\bar{X}_n = \frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i = \mathbb{E}X_i = \mu$$

$$\mathbb{P}(|\bar{X}_n - \mu| \geq t) \leq \frac{\mathbf{Var}(\bar{X}_n)}{t^2} \quad t \geq 0 \tag{1}$$

$$\begin{aligned}
\mathbf{Var}(\bar{X}_n) &= \mathbb{E}(\bar{X}_n{}^2) - (\mathbb{E}\bar{X}_n)^2 \\
&= \mathbb{E}(\frac{1}{n}\sum_{i=1}^n X_i)^2 - \mu^2 \\
&= \frac{1}{n^2}\mathbb{E}(\sum_{i=1, j\neq i}^n X_i X_j + \sum_{j=1}^n X_j^2) - \mu^2 \\
&\text{because of the independence of } X_i \\
&= \frac{1}{n^2}(n(n-1)\mu^2 + n(\mu^2 + \sigma^2)) - \mu^2 \\
&= \frac{n-1}{n}\mu^2 + \frac{1}{n}\mu^2 + \frac{\sigma^2}{n} - \mu^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

$$\mathbb{P}(|\bar{X}_n - \mu| \geq t) \leq \frac{\sigma^2}{nt^2} \tag{2}$$

□

Solution **1.b** Use the result from the last problem one can easily derive one special N such that, $\forall n > N$, $\frac{\sigma^2}{n\epsilon} \leq \alpha$, we conclude $N = \lceil \frac{\sigma^2}{\epsilon^2 \alpha} \rceil$. But i failed to prove any optimality result when one can show for $n = N - 1$ there exist a counter example. □

Problem 2

(a) (2 points) Let X_1, X_2 be two i.i.d. random variables with $\mathcal{N}(0,1)$ distribution for some $\sigma^2 > 0$. Let $Y := a_1 X_1^2 + a_2 X_2^2$ for some fixed (not necessarily positive) $a_1, a_2 \in \mathbf{R}$. Find moment generating function of Y, specify a domain (a subset of $\mathbf{R}$) where this function is defined.

(b) (4 points) Let $Y_1, Y_2, \ldots, Y_n$ be mutually independent r.v. such that $Y_i \sim \chi^2(k_i)$ for $i = 1, \ldots, n$. Let $T := \sum_{i=1}^n b_i Y_i$ for fixed numbers $b_1, \ldots, b_n > 0$. Find moment generating function of T.

(c) (4 points) Establish conditions on $b_1, \ldots, b_n > 0$ sufficient for T to have Gamma distribution, specify parameters of this distribution

***Solution* 2.a**

Since X_1, X_2 are to i.i.d. random variables, their joint distribution is equal to the product of the marginal distribution.

$$
\begin{aligned}
M_{\mathbf{Y}}(t) &= \mathbb{E}\exp[t\mathbf{Y}] = \mathbb{E}\exp[t(a_1X_1^2 + a_2X_2^2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[t(a_1x_1^2 + a_2x_2^2)]p_1(x_1)p_2(x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_{-\infty}^{\infty} \exp[ta_1x_1^2]p_1(x_1)\mathrm{d}x_1 \int_{-\infty}^{\infty} \exp[ta_2x_2^2]p_2(x_2)\mathrm{d}x_2 \\
&= \frac{1}{2\pi\sigma^2}\frac{1}{\sqrt{\frac{1}{2\sigma^2} - ta_1}}\frac{1}{\sqrt{\frac{1}{2\sigma^2} - ta_2}}\left(\int_{-\infty}^{\infty} \exp[-\mu^2]\mathrm{d}\mu\right)^2 \\
&= \frac{1}{\sqrt{1 - 2\sigma^2 ta_1}}\frac{1}{\sqrt{1 - 2\sigma^2 ta_2}} \qquad \forall t \le \min\{\frac{1}{2\sigma^2 a_1}, \frac{1}{2\sigma^2 a_2}\}
\end{aligned}
$$

□

***Solution* 2.b**

$$\mathbb{E}e^{tT} = \mathbb{E}e^{t(b_1Y_1 + b_2Y_2 + \cdots + b_nY_n)} \tag{3}$$

Since $Y_1, Y_2, \ldots, Y_n$ are mutually independent, let $\tilde{Y} = (Y_1, Y_2, \ldots, Y_n)^\intercal$ then

$$
\begin{aligned}
\mathbb{E}e^{t(\frac{t_1}{t}Y_1 + \frac{t_2}{t}Y_2 + \cdots + \frac{t_n}{t}Y_n)} &= M_Y(t_1, t_2, \ldots, t_n) \\
&= M_{Y_1}(t_1)M_{Y_2}(t_2)\ldots M_{Y_n}(t_n)
\end{aligned} \tag{4}
$$

Let $t_i = b_i t, \quad \forall 1 \le i \le N$, and because $Y_i \sim \chi^2(k_i)$, $M_{Y_i}(t) = (1-2t)^{\frac{k_i}{2}}$, then we have

$$M_T(t) = \mathbb{E}e^{tT} = M_{\tilde{Y}}(b_1t, b_2t, \ldots, b_nt) = \prod_{i=1}^n (1 - 2b_it)^{-\frac{k_i}{2}} \qquad \forall t < \min\{\frac{1}{2b_1}, \frac{1}{2b_2}, \ldots, \frac{1}{2b_n}\} \tag{5}$$

□

***Solution* 2.c**

$$T \sim \Gamma(\alpha, \beta) \Leftrightarrow M_T(t) = (1 - \beta t)^{-\alpha} \quad \forall t < \frac{1}{\beta} \tag{6}$$

Hence let $b_1 = b_2 = \cdots = b_n = \dfrac{\beta}{2}$,

$$M_T(t) = \prod_{i=1}^{n}(1-\beta t)^{-\frac{k_i}{2}} = (1-\beta t)^{-\frac{1}{2}\sum_{i=1}^{n} k_i} \tag{7}$$

Therefore, $T \sim \Gamma(\dfrac{1}{2}\sum_{i=1}^{n} k_i, 2b_1)$.

□

Problem 3

The joint density of two random variables X_1 and X_2 is

$$f(x_1, x_2) = \begin{cases} 2\mathrm{e}^{-x_1}\mathrm{e}^{-x_2}, & \text{for } 0 < x_1 < x_2 < +\infty; \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the transformation $Y_1 = 2X_1$, $Y_2 = X_2 - X_1$. Find the joint density of Y_1 and Y_2, and show whether Y_1 and Y_2 are independent or not.

Solution **3.a**

$$\begin{cases} X_1 = \dfrac{1}{2}Y_1 \\ X_2 = Y_2 + \dfrac{1}{2}Y_1 \end{cases} \qquad J = \begin{vmatrix} 1/2 & 0 \\ 1/2 & 1 \end{vmatrix} = \frac{1}{2} \qquad Y_1 > 0, \quad Y_2 > 0 \tag{8}$$

$$f_{Y_1,Y_2}(y_1, y_2) = 2e^{-\frac{1}{2}y_1}e^{-\frac{1}{2}y_1 - y_2}\frac{1}{2} = e^{-y_1}e^{-y_2} \qquad \forall y_1, y_2 > 0 \tag{9}$$

And $f_{Y_1}(t) = f_{Y_2}(t) = e^{-t}$. Hence the joint density is equal to the product of marginal density, which implies the independency of Y_1 and Y_2.

□

Problem 4

(a) (3 points) Let $Z \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_d)$ be a random vector for some fixed $\sigma^2 > 0$. Let $a, b \in \mathbf{R}^d$ be some fixed d-dimensional vectors. Show necessary and sufficient conditions on vectors a, b for independence of $a^\top Z$ and $b^\top Z$ from each other.
(b) (4 points) Consider the problem in part (a) for $Z \sim \mathcal{N}(0, \Sigma)$, where Σ is an arbitrary positive-definite covariance matrix in $\mathbf{R}^{d\times d}$.

Solution **4.a**
Let $A = (a^\intercal, b^\intercal)^\intercal$, then $AZ = (a^\intercal Z, b^\intercal Z)^\intercal$ is normal. Hence its coordinates are jointly normal, which implies *independent* ⇔ *uncorrelated.*

$$
\begin{aligned}
&\mathbf{COV}(a^\intercal, b^\intercal) = 0 \\
\Leftrightarrow & a^\intercal \mathbf{COV}(Z, Z) b = 0 \\
\Leftrightarrow & \sigma^2 a^\intercal \mathbf{I}_d b = 0 \\
\Leftrightarrow & \text{a and b are orthogonal in } \mathbb{R}^d
\end{aligned}
$$

□

Solution **4.b**

independent $\Leftrightarrow a^\intercal \Sigma b = 0$

Since $\Sigma \succ 0$ and $\Sigma^\intercal = \Sigma$, Σ has d positive eigenvalues and there exists an orthogonal matrix $\mathbf{U}$ such that, $\Sigma = \mathbf{U}^\intercal \mathbf{\Lambda} \mathbf{U}$ where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_i)_d$, λ_i is the eigenvalue of Σ.

$$
\begin{aligned}
& a^\intercal \Sigma b = 0 \\
\Leftrightarrow & a^\intercal \mathbf{U}^\intercal \Lambda \mathbf{U} b = 0 \\
\Leftrightarrow & a^\intercal \mathbf{U}^\intercal \mathbf{U} b = 0 \\
\Leftrightarrow & a^\intercal b = 0.
\end{aligned}
$$

Hence the necessary and sufficient condition for $a^\intercal Z, b^\intercal Z$ to be independent is still a and b are orthogonal in $\mathbb{R}^d$ □