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Problem 1

On $L^2[a,b]$, consider the function set $\left\{e^{2\pi \mathbf{i} n x}\right\}_{n=-\infty}^{\infty}$,

1. *If $|b-a| \leq 1$, prove that $S^{\perp} = \{\theta\}$;*
2. *If $|b-a| > 1$, prove that $S^{\perp} \neq \{\theta\}$.*

Solution ***1.a*** First, we know S forms a orthogonal system in $L^2[a,b]$. The elements in set S has periodic of length 1. And by Fourier analysis, S forms a complete basis for all integrable and quadratic integrable periodic function on $\mathbb{R}$ with periodic length 1, i.e. $L^2[T]$, where T is a length 1 interval. Let T be such an interval and $[a,b] \subset T$. Then by extending $L^2[T] \ni f\big|_{T\setminus[a,b]} := 0$ and completeness of S, the claim follows. □

Solution ***1.b*** Take $f \in S^{\perp}$, we have the following equation

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x + \int_{a+1}^{a+1+\epsilon} f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x = 0, \forall n \in \mathbb{Z}. \tag{1}$$

We construct a function $f \neq 0$ such that,

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x = -\int_{a+1}^{a+1+\epsilon} f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x \forall n \in \mathbb{Z}. \tag{2}$$

Using the periodic nature of $e^{2\pi \mathbf{i} n x}$,

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x = \int_a^{a+\epsilon} -f(x)\overline{e^{2\pi \mathbf{i} n x}}\mathrm{d}x \forall n \in \mathbb{Z}. \tag{3}$$

An obvious choice of f can be taken as,

$$f(x) = \begin{cases} 1, & x \in [a, a+\epsilon] \\ 0, & x \in (a+\epsilon, a+1] \\ -1, & x \in (a+1, a+1+\epsilon] \end{cases} \tag{4}$$

□

Problem 2

Let $\{e_n\}_1^\infty$, $\{f_n\}_1^\infty$ be two orthonormal set, such that

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1. \tag{5}$$

Prove that completeness of one implies that of the other one.

Solution ***2.a*** Suppose $\{e_n\}$ is complete and $\{f_n\}$ is not. Completeness implies totalness. Then exist a f such that $f \neq 0$ and $\langle f, f_n \rangle = 0, \forall n \in \mathbb{N}$. Hence by completeness of $\{e_n\}$ we have

$$\begin{aligned} \|f\|^2 &= \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle f, e_n \rangle - \langle f, f_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle f, e_n - f_n \rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \|f\|^2 \|e_n - f_n\|^2 \\ &< \|f\|^2 \end{aligned}$$

□

Problem 3

Let $\mathfrak{X}$ be a Hilbert space. Let $\mathfrak{X}_0$ be a closed linear subspace of $\mathfrak{X}$. Let $\{e_n\}$ and $\{f_n\}$ be orthonormal basis of $\mathfrak{X}_0$ and $\mathfrak{X}_0^\perp$. Prove that: $\{e_n\} \cup \{f_n\}$ is a set of orthonormal basis of $\mathfrak{X}$.

Solution ***3.a*** First, orthogonality and normality is trivial, because unioning does not change the norm of each element and the two basises are subset of orthogonal sets. It only suffices to prove $\{e_n\} \cup \{f_n\}$ is a set of basis for $\mathfrak{X}$. This can be done by othogonoal decomposition. For any given $x \in \mathfrak{X}$, $\mathfrak{X}_0$ is a closed subset of $\mathfrak{X}$, hence there exists a unique $x_0 \in \mathfrak{X}_0, x_1 \in \mathfrak{X}_0^\perp$ such that $x = x_0 + x_1$. And $\{e_n\}, \{f_n\}$ are basis of $\mathfrak{X}_0, \mathfrak{X}_0^\perp$, hence we have the follows identity,

$$x = x_0 + x_1 = \sum_{n=0}^{\infty} \langle e_n, x_0 \rangle e_n + \sum_{m}^{\infty} \langle f_m, x_1 \rangle f_m \tag{6}$$

which give the claim. □

Problem 4

Let $\mathfrak{X}$ be an inner product space, $\{e_n\}$ be an orthonormal set in $\mathfrak{X}$. Prove that,

$$\left| \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \right| \leq \|x\| \|y\|, \quad \forall x, y \in \mathfrak{X} \tag{7}$$

Proof.

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \right|^2 &\leq \left(\sum_{n=1}^{\infty} \langle x, e_n \rangle^2 \right) \left(\sum_{n=1}^{\infty} \langle y, e_n \rangle^2 \right) \\ &\leq \|x\| \|y\| \end{aligned}$$

□

Problem 5

Find $(a_0, a_1, a_2) \in \mathbb{R}^3$ minimizing $\int_0^1 |e^t - a_0 - a_1 t - a_2 t^2|^2 \mathrm{d}t$.

Solution ***5.a*** Expanding e^t to the polynomial orthogonal basis,

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots \tag{8}$$

each term is non-negative on $[0, 1]$. By the monotoncity of $|x|^2$, the choice of $\left(1, 1, \frac{1}{2}\right)$ minimizes the integral. □