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Problem 1

1、试证明椭圆：

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

所包围的区域为凸集。

***Solution* 1.a**

For any $\lambda \in [0,1]$ and any pair of points $(x_1, y_1), (x_2, y_2) \in D$, where

$$D := \left\{ (x,y) \middle| \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} \tag{1}$$

the convex combination $(x, y) = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in D$, because

$$\begin{aligned}
&\frac{(\lambda x_1 + (1-\lambda)x_2)^2}{a^2} + \frac{(\lambda y_1 + (1-\lambda)y_2)^2}{b^2} \\
=&\lambda^2 \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \right) + (1-\lambda)^2 \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} \right) + 2\lambda(1-\lambda) \left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} \right) \\
\leq&\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} \right) \\
&\text{Since } x_1x_2 \leq \max\{x_1^2, x_2^2\} \\
\leq&(\lambda + (1-\lambda))^2 = 1.
\end{aligned}$$

□

Problem 2

2、判断下列函数的凸凹性，并说明理由

1）$f(x) = |x|^p$

2）$f(x) = x_1^2 + 2x_1^2 + x_3^2 + x_1x_2 - 7x_1 - 2x_3 + 12$

***Solution* 2.a**

The fuction is convex when $p \geq 1$ or $p = 0$, and not convex nor concave if not. First we observe that for any $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) = |\lambda x + (1-\lambda)y|^p \tag{2}$$

When $p \geq 1$, With the Newton's binary theorem

$$\begin{aligned}&\leq \lambda^p|x|^p + (1-\lambda)^p|y|^p\\&\leq \lambda f(x) + (1-\lambda)f(y)\end{aligned}$$

When $p = 0$, the function is a constant function which is indeed convex.

When $p < 0$, the domain is $(-\infty, 0) \cup (0, +\infty)$ which is not even convex set, hence it's not convex nor concave.

When $p \in (0, 1)$. It's not convex because one have $f(-1) = f(1) = 1$ and $f(0) = 0, f(1/2) > 1/2$. Hence by $\dfrac{f(1)+f(-1)}{2} > f(0)$, it's not concave. Nevertheless by $\dfrac{f(0)+f(1)}{2} = \dfrac{1}{2} \leq f(1/2)$, it's not convex.

□

Solution **2.b** The function is convex. Observe that

$$f(x) = x^T G x + (0, 5, 0)x \tag{3}$$

where

$$G = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

G is a positive definite matrix, f is a continuous differentiable function in $\mathbb{R}^3$ which is a convex set. Hence f is convex.

□

Problem 3

3、试求函数：

$$\mathrm{f(x) = x_1^2 - 4x_1x_2 + 6x_1x_3 + 5x_2 - 10x_2x_3 + 8x_3^2}$$

的驻点，并判定它们是极大点，还是极小点，还是鞍点。

Solution **3.a**

$$\nabla f = (2x_1 - 4x_2 + 6x_3, -4x_1 + 5 - 10x_3, 6x_1 - 10x_2 + 16x_3) \tag{5}$$

Let $nabla f = 0$ we obtain $x = \left(-\dfrac{5}{6}, \dfrac{5}{6}, \dfrac{5}{6}\right)^T$. This is a stationary point for f, because the Hesse matrix

$$H_f(x) = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 4 \end{pmatrix} \tag{6}$$

is indefinite. □

Problem 4

4、考虑约束优化问题：

$$\begin{array}{ll} \min & -x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \le 2 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

分别确定点(0,0),(0,1)处的可行方向与下降方向。

***Solution* 4.a** For the point (0,0),
the feasible direction is

$$S = \left\{ (\cos\theta, \sin\theta) \middle| \theta \in \left[0, \frac{\pi}{2}\right] \right\} \tag{7}$$

Consider the function $f(x_1, x_2) = -x_1 + x_2$, $\nabla f = (-1, 1)^T$. Denote the descending direction as D, i.e.

$$D := \left\{ s \in R^2 \text{ is a direction} \middle| \nabla f^T s < 0 \right\} \tag{8}$$

it follows that,

$$D = \left\{ (\cos\theta, \sin\theta) \middle| \theta \in \left[0, \frac{\pi}{4}\right) \right\} \tag{9}$$

□

***Solution* 4.b** For the point (0,1),
the feasible direction is

$$S = \left\{ (\cos\theta, \sin\theta) \middle| \theta \in \left[-\frac{\pi}{2}, -\arctan\left(\frac{1}{2}\right)\right] \right\} \tag{10}$$

Consider the function $f(x_1, x_2) = -x_1 + x_2$, $\nabla f = (-1, 1)^T$. Denote the descending direction as D, i.e.

$$D := \left\{ s \in S \subset R^2 \text{ is a direction} \middle| \nabla f^T s < 0 \right\} \tag{11}$$

it follows that,

$$D = \left\{ (\cos\theta, \sin\theta) \middle| \theta \in \left[-\frac{\pi}{2}, -\arctan\left(\frac{1}{2}\right)\right] \right\} \tag{12}$$

□

Problem 5

5、考虑以下两个数列：

$$u_k = \frac{1}{c^{2-k}} (c > 0), \ \ v_k = \frac{1}{k^k}$$

分别具有线性、超线性收敛速度。

Solution **5.a**
For the sequence u_k to converge when $k \to \infty$, one needs $c \leq 1$. Then

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{c^{k-1}}{c^{k-2}} = c \leq 1 \tag{13}$$

Hence the sequence has linear convergence rate if $c < 1$ or semi-linear convergence rate if $c = 1$. □

Solution **5.b** The sequence v_k has super-linear convergence rate. Check that

$$\begin{aligned}
\lim_{k\to\infty} \frac{v_{k+1}}{v_k} &= \lim_{k\to\infty} \frac{k^k}{(k+1)^{k+1}} \\
&= \lim_{k\to\infty} \exp\{k \log k - (k+1)\log(k+1)\} \\
&= \lim_{k\to\infty} \exp\{k \log k - k\log(k+1) - \log(k+1)\} \\
&= \lim_{k\to\infty} \exp\left\{k \log\left(1 - \frac{1}{k+1}\right) - \log(1+k)\right\} \\
&= \lim_{k\to\infty} \exp\left\{-\frac{k}{k+1} + o(1) - \log(1+k)\right\} \\
&= \lim_{k\to\infty} \frac{1}{1+k} \\
&=0
\end{aligned}$$

Hence the sequence has super-linear convergence rate.

□

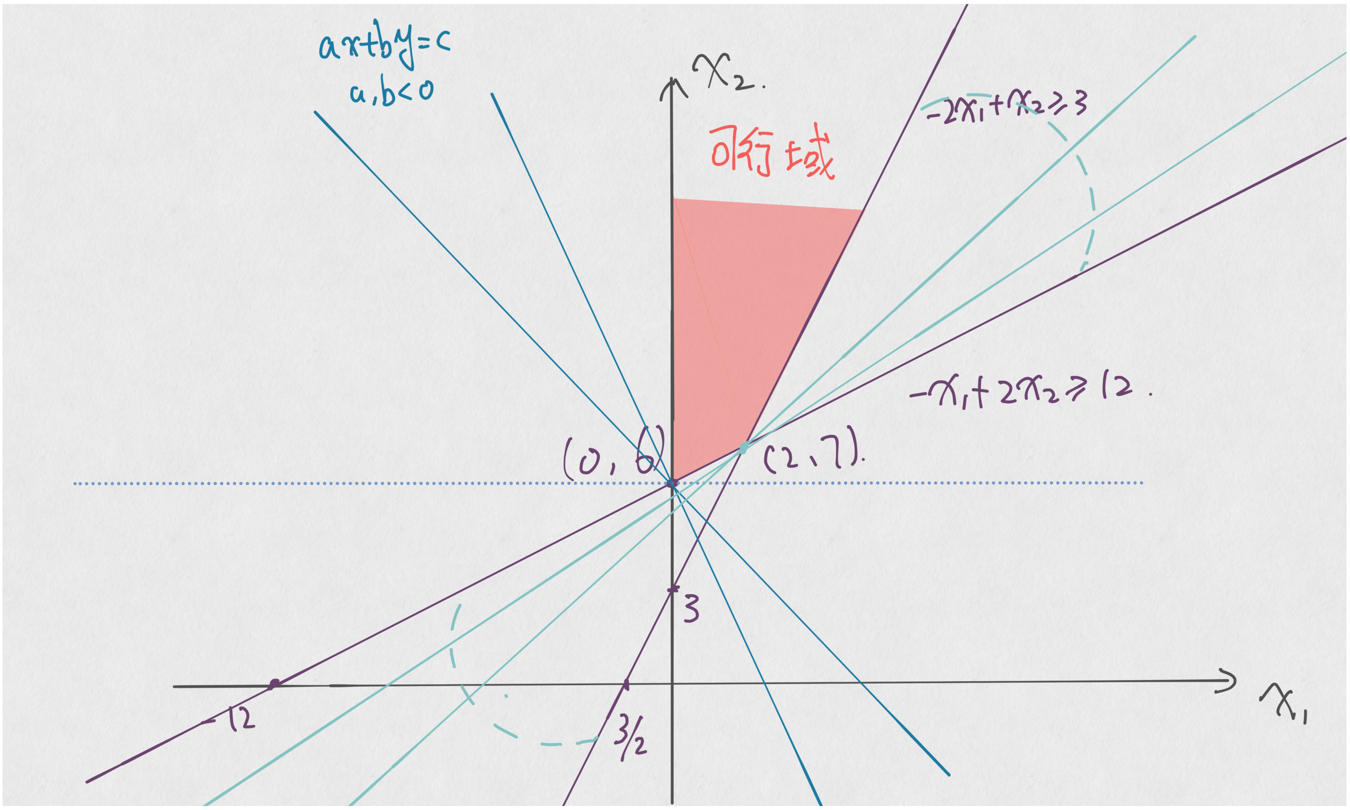
Problem 6

1、利用图解法分析以下线性规划问题最优解：

$$\begin{cases} max & 2x_1 + ax_3 \\ \text{s.t.} & -x_1 + 2x_2 \geq 12 \\ & -2x_1 + x_2 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

其中 a 为某参数。

Solution **6.a**



1. When $b \geq 0$

   There's no optimal solution since the maximum is infinity.

2. When $a \geq 0, b < 0$

   There's an optimal solution.

   (a) When $\dfrac{1}{2} < -\dfrac{a}{b} < 2$

   The optimal solution is $(2, 7)$, and the target function reaches 2a+7b.

   (b) When $-\dfrac{a}{b} \geq 2$

   There is no optimal solution.

   (c) When $0 \leq -\dfrac{a}{b} \leq \dfrac{1}{2}$

   The optimal solution is $(0, 6)$, and the target function reaches 6b.

3. When $a < 0, b < 0$

   The optimal solution is $(0, 6)$, the target function reaches 6b.

$\square$

Problem 7

2、确定以下线性规划问题：

$$\begin{cases} max & 3x_1 + 2x_2 \\ s.t. & x_1 + x_2 \leq 3 \\ & x_1 - 2x_2 \leq 4 \\ & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 1 \end{cases}$$

的所有基本解，并判断哪些是基本可行解，以及最优基本可行解。

***Solution* 7.a**
First, obtain the canonical form of this LP problem,

$$\begin{cases} \min & -3x_1 - 2x_2 & \\ \text{s.t} & x_1 + x_2 + x_3 = & 3 \\ & x_1 - 2x_2 + x_4 = & 4 \\ & x_1 + x_5 = & 4 \\ & x_2 + x_6 = & 1 \\ & x_1, x_2, ..., x_6 \geq & 0 \end{cases} \tag{14}$$

Find the basic solution for this problem,

$$\begin{cases} x_1 & = (4, 1, -2, 2, 0, 0) \\ x_2 & = (6, 1, -4, 0, -2, 0) \\ x_3 & = (4, 0, -1, 0, 0, 1) \\ x_4 & = (2, 1, 0, 4, 2, 0) \end{cases} \begin{cases} x_5 & = (4, -1, 0, -2, 0, 2) \\ x_6 & = (0, 0, 3, 4, 4, 1) \\ x_7 & = (\frac{10}{3}, -\frac{1}{3}, 0, 0, \frac{2}{3}, \frac{4}{3}) \\ x_8 & = (3, 0, 0, 1, 1, 1) \end{cases} \begin{cases} x_9 & = (0, 1, 2, 6, 4, 0) \\ x_{10} & = (0, -2, 5, 0, 4, 3) \\ x_{11} & = (0, 3, 0, 10, 4, -2) \end{cases} \tag{15}$$

The basic feasible solution are

$$\begin{aligned} x_4 & = (2, 1, 0, 4, 2, 0) \\ x_6 & = (0, 0, 3, 4, 4, 1) \\ x_8 & = (3, 0, 0, 1, 1, 1) \\ x_9 & = (0, 1, 2, 6, 4, 0) \end{aligned} \tag{16}$$

The basic optimal feasible solution is $x_8 = (3, 0, 0, 1, 1, 1)$ □