第六章

1. 考虑等式约束二次规划问题：

$$\min \quad q(x) = x_1^2 + x_2^2 + x_3^2$$
$$\text{s.t.} \quad x_1 + 2x_2 - x_3 - 4 = 0$$
$$x_1 - x_2 + x_3 + 2 = 0.$$

变量消去法

$$G = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}. \quad A^T = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_3 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 + x_3 \\ -2 - x_3 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 4 + x_3 \\ -2 - x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} x_3 \\ 2 + \frac{2}{3} x_3 \end{pmatrix}$$

$$\hat{q}(x) = \frac{1}{9} x_3^2 + \left(2 + \frac{2}{3} x_3\right)^2 + x_3^2 \qquad x_3^* = -\frac{6}{7}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 10/7 \\ -6/7 \end{pmatrix}$$

Lagrange 方法. G正定.

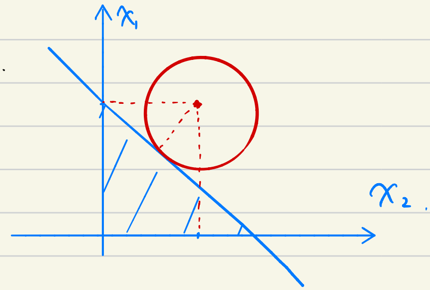
$$\begin{bmatrix} G & -A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = -\begin{bmatrix} g \\ b \end{bmatrix}$$

$$\begin{pmatrix} 2 & & & -1 & -1 \\ & 2 & & -2 & 1 \\ & & 2 & 1 & -1 \\ -1 & -2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 2/7 \\ 10/7 \\ -6/7 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 8/7 \\ -4/7 \end{pmatrix}$$

2. 考虑如下二次规划问题

$$\min \quad q(x) = x_1^2 + x_2^2 - 6x_1 - 4x_2 + 13$$

$$\text{s.t.} \begin{cases} x_1 + x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$



解: 图解法

$$q(x) = (x_1-3)^2 + (x_2-2)^2$$

设圆与可行域边界相切

$\Leftrightarrow$ 则 $x_1 = 3 - x_2$，有方程

$(3 - x_2 - 3)^2 + (x_2 - 2)^2 + c = 0$. 方程有唯一解

$\Leftrightarrow$ $2x_2^2 - 4x_2 + 4 + c = 0$. 有唯一解

$\Leftrightarrow$ $16 - 4\times 2\times(4+c) = 0$. $c = -2$ 此时 $\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$ $q(x) = 2$.

有效集法: $q(x) = \frac{1}{2} x^T \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} x + (-6, -4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 13$

$$\begin{cases} (-1,-1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ge -3 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases}$$

① 取顶点 $(0,3)$ 为初始点. $W_0 = \{1, 2\}$ $g_0 = G\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$.

$G = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}$ $A^T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} G & -A \\ -A^T & 0 \end{pmatrix}\begin{pmatrix} d^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -g_0 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 2 & & 1 & -1 \\ & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} d^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \lambda^* = \begin{pmatrix} -2 \\ -8 \end{pmatrix}.$$

故 $x_1 = (0, 3)^T$ $W_1 = \{1\}$.

② $A^T = (-1, -1)$. $\begin{pmatrix} 2 & & 1 \\ & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} d^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} \Rightarrow d_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ $\lambda^* = 2$.

此时 $b = -3$. $\Rightarrow (0, 1)\begin{pmatrix} 2 \\ -2 \end{pmatrix} = -2 < 0$ 故 $j = 3$.

因 $-3 - \frac{3}{-2} = -\frac{3}{2}$. 故 $\alpha_k = 1$ $W_2 = \{1\}$ $x_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$g_2 = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ $A^T = (-1, -1)$

$\begin{pmatrix} 2 & & 1 \\ & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} d^* \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \Rightarrow d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\lambda = 2$. 故终止.

极小值点为 $(2, 1)$. 极小值为 2.