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Problem 1

Let X, Y be a Banach space and $T \in L(X, Y)$, then T^{**} is a extension of T.

Solution ***1.a*** $T^{**} \in L(X^{**}, Y^{**}), T \in L(X, Y)$. Let J be the natural mapping. The natural mapping embeds X to X^{**} and Y to Y^{**}. On this subspace, i.e. the union of all elements J_x and J_y.

$T^{**}J_x(y^*) = J_x(T^*y^*) = T^*y^*(x) = y^*(Tx) = J_{Tx}y$. Hence $T^{**}J = JT$.

$\square$

Problem 2

Let X be a Banach Space. Prove that $T \in L(X)$ has a bounded inverse is equivalent to T^* has a bounded inverse. Moreover,

$$\left(T^{-1}\right)^* = (T^*)^{-1} \tag{1}$$

Solution ***2.a*** $\Rightarrow$: If T has a bounded inverse, then for arbitrary functional f, we have

$$f(Tx) = T^*f(x) = T^*f(T^{-1}Tx) = T^*\left(T^{-1}\right)^*f(Tx). \tag{2}$$

Since T is a bijection, then we proved $Id = T^*(T^{-1})^*$ for any functional f. Therefore the T^* also has an inverse, which is $(T^{-1})^*$.

$\Leftarrow$: Now it only suffice to prove the existence of T^{-1}. Since X is a Banach Space, then $T^{-1} \in$

$L(X)$ when T is a bijection. The injection follows from the lower bound of

$$\|Tx\| \geq \|T\|\|x\| = \|T^*\|\|x\|. \tag{3}$$

Then it only suffices to prove T is a surjection. If not, there will exist a non-zero functional which vanish on the range of T and not vanish on its kernel (By corollary of Hahn-Banach). This will result in for a non-zero functional f, $0 = f(Tx) = T^* f(x), \forall x$. This contradicts with the invertibility of T^*. $\square$

Problem 3

Prove that when X is a complex Hilbert space, $T^* = T \Leftrightarrow \langle Tx, x\rangle \in \mathbb{R}$.

Solution ***3.a***

$\Rightarrow$ is easy. It only suffices to prove $\overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$. This is immediate, since $\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$.

$\Leftarrow$: This is more complicated, it's easy to see that

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle T^*x, x\rangle. \tag{4}$$

which would imply $\langle (T^* - T)x, x\rangle = 0, \forall x \in X$. The question remaining is whether this condition would implies $T = T^*$.

Sadly, in the case when the space is real, this will not be true. If A is a rotation operator of 90 degree in $\mathbb{R}^2$, then we have $\langle Ax, x\rangle = 0$ for any $x \in \mathbb{R}^2$ but apparently A is not a zero operator.

But this is true in complex case due the the extra two terms in the polarization equality.

$$\begin{aligned}
0 &= \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + \mathbf{i}\langle A(x+\mathbf{i}y), x+y\rangle - \mathbf{i}\langle A(x-\mathbf{i}y), x-\mathbf{i}y\rangle \\
&= 2\langle Ax, y\rangle + 2\mathbf{i}\langle Ax, \mathbf{i}y\rangle + a\langle Ay, x\rangle + 2\mathbf{i}\langle A\mathbf{i}y, x\rangle \\
&= 2(\langle Ax, y\rangle + (-\mathbf{i})\mathbf{i}\langle Ax, y\rangle) + 2(\langle Ay, x\rangle + \mathbf{i}\cdot\mathbf{i}\langle Ay, x\rangle) \\
&= 4\langle Ax, y\rangle
\end{aligned}$$

for any x, y. Hence A is a zero operator. $\square$

Problem 4

Let X be a Hilbert space, $T \in L(X)$, $T = T^*$. Then

$$\ker T = R(T)^{\perp} \tag{5}$$

Solution ***4.a*** For any $x \in \ker T$, we have $Tx = 0$. And for any $y \in R(T)$, there's such a z in X such that $Tz = y$. $\langle x, y\rangle = \langle x, Tz\rangle = \langle Tx, z\rangle = 0, \forall y \in X$. Hence $\ker T \subset R(T)^{\perp}$.

$\forall x \in R(T)^{\perp}$, $x \perp y, \forall y \in R(T)$. This would imply $\forall z \in X, \langle x, Tz\rangle = 0$. Since T is adjoint, $\langle Tx, z\rangle = 0, \forall z \in X$. Therefore the functional $\langle Tx, \cdot\rangle$ is a zero functional. Hence $Tx = 0$, which would imply $x \in \ker T$. $\square$

Problem 5

Let X, Y be Banach spaces and $T \in L(X, Y)$, then

$$^{\perp}R(T) = \ker(T^*)$$

$$\ker T = R(T^*)^{\perp}$$

.

Solution ***5.a***

$$\begin{aligned}
^{\perp}R(T) &= \{f \in X^* : f(x) = 0, \forall x \in R(T)\} \\
&= \{f \in X^* : f(Tx) = 0, \forall x \in X\} \\
&= \{f \in X^* : T^* f(x) = 0, x \in X\} \\
&= \ker T
\end{aligned}$$

$\square$

Solution ***5.b***

$$\begin{aligned} R(T^*)^{\perp} &= \{x \in X : f(x) = 0, \forall f \in R(T^*)\} \\ &= \{x \in X : T^* g(x), \forall g \in Y^*\} \\ &= \{x \in X : g(Tx) = 0, \forall g \in Y^*\} \\ &= \{x \in X : Tx = 0\} \\ &= \ker T \end{aligned}$$

□

Problem 6

Let $X := \{u \in C^2(B_{R_1}(\theta)), \Delta u = 0$ *on* $B_{R_1}(\theta)\}$ *and* $\tilde{X}$ *be the completion of* X *under* L^2 *norm. Let*

$$\Gamma_R = \big\{x \in \mathbb{R}^2 : |x| = R > R_1\big\}. \tag{6}$$

Define $T : L^2(\Gamma_R) \mapsto \tilde{X}$*,*

$$(T\phi)(x) := \int_{\Gamma_R} \frac{1}{4\pi|x-y|}\phi(y)\mathrm{d}s(y), \quad x \in B_{R_1}(\theta), \tag{7}$$

prove that $\overline{R(T)} = \tilde{X}$*.*

Solution ***6.a***

□

Problem 7

Prove that the weak limit is unique.

Solution ***7.a*** This is a result of Hahn-Banach. If $\{T_n\}$ has two limit T, G, then $\forall f \in Y^*, \forall x \in X$, we have $f(Tx) = f(Gx)$. This would imply $f((T-G)x) = 0$ for any functional in $f \in Y^*$. Then by Hahn-Banach, $(T-G)x = 0$ and this holds for any $x \in X$. Therefore $T = G$. □

Problem 8

Let X be a normed linear space, and $\dim X < \infty$. Prove that

$$\|x_n - x\| \to 0 \Leftrightarrow x_n \rightharpoonup x. \tag{8}$$

Solution ***8.a*** $\Rightarrow$: This is trivial since $|f(x_n) - f(x)| \leq \|f\|\|x_n - x\|$.

$\Leftarrow$: By considering the dual basis of X^*, we would obtain that for any coordinate of x_n converge to the corresponding one of x. This would imply the convergence under norm, since all norm in finite dimensional linear space is equivalent to Euclidean norm, which will converge if coordinate converges. $\square$