Shenduo Zhang

October 11, 2020 (GMT+8)

zhangshenduo@gmail.com

Problem 1

Let $f \in L^2[a,b]$, $K(\cdot,\cdot) \in L^2([a,b]\times[a,b])$, prove that

$$x(t) = f(t) + \lambda \int_a^b K(t,s)x(s)\mathrm{d}s \tag{1}$$

has a unique solution for λ is small enough.

Solution ***1.a*** Equation in this form is know to be second type Fredholm equation.

Define $Tx := f(t) + \lambda \int_a^b K(t,s)x(s)\mathrm{d}s$ on $x \in L^2[a,b]$. We prove that T is a contraction for λ small enough.

$$\begin{aligned}
\rho(Tx,Ty) &= \left(\int_a^b \lambda^2 \left(\int_a^b K(t,s)(x(s)-y(s)\mathrm{d}s)\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq \lambda\left(\int_a^b \int_a^b K(t,s)^2(x(s)-y(s))^2 \mathrm{d}s\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq \lambda\left(\int_{[a,b]^2} K(t,s)^2\right)^{\frac{1}{2}} \rho(x,y)
\end{aligned}$$

Since $K \in L^2$, for λ small enough, we have $\rho(Tx,Ty) \leq \alpha\rho(x,y), \alpha \in (0,1)$, which means T is a contraction. And by Banach's theorem, there exist a unique $x \in L^2[a,b]$ that is the solution to this second type Fredholm equation. $\square$

Problem 2

Let (X, ρ) be a complete metric space, and $X \neq \emptyset$. Let $T : X \mapsto X$ satisfy $\rho(Tx, Ty) \leq \alpha\rho(x, y), \alpha \in (0, 1)$. Let $R \geq 0$, define

$$A_R := \{x \in X, \rho(x, Tx) \leq R\} \tag{2}$$

Prove that,

1. *When $R > 0$, A_R is a non-empty closed set in X.*
2. *$\forall x, y \in A_R$, $\rho(x, y) \leq 2R + \rho(Tx, Ty) \Rightarrow \mathrm{diam} A_R \leq \frac{2R}{1-\alpha}$.*
3. *A_0 is not empty.*
4. *T has a unique fixed point.*

Solution ***2.a*** To prove the it's close, consider a convergent sequence $x_n \in A_R$ that is convergent, denote $\lim_{m\to\infty} x_n = x$. It suffices to prove $x \in A_R$. We give ourselves an ϵ of room. $\forall \epsilon > 0$,

$$\rho(x, Tx) \leq \rho(x, x_n) + \rho(x_n, Tx_n) + \rho(Tx_n, Tx) \tag{3}$$
$$\leq \frac{\epsilon}{2} + R + \frac{\epsilon}{2} \tag{4}$$
$$\leq R + \epsilon \tag{5}$$

where n is large enough. This claim hold $\forall\epsilon$, then it's safe to claim $\rho(x, Tx) \leq R$ which means $x \in A_R$.

We prove this is a non-empty set. Suppose it's not, then there exists $R > 0$ such that $\forall x \in X, \rho(x, Tx) > R$. Consider a fixed x, then

$$\rho(x, Tx) \geq \frac{1}{\alpha^m}\rho(T^m x, T^{m+1}x) > \frac{1}{\alpha^m}R \quad \forall m \in \mathbb{N} \tag{6}$$

Let $m \to \infty$, we have $\rho(x, Tx) = \infty, \forall x \in X$. We get a contradiction.

□

Solution ***2.b*** Observe that,

$$\rho(x, y) \leq \rho(x, Tx) + \rho(y, Ty) + \rho(Tx, Ty) \leq 2R + \alpha\rho(x, y) \quad \forall x, y \in A_R \tag{7}$$

Then take supremum on $x, y \in A_R$ to claim $\mathrm{A_R} \leq \frac{2R}{1-\alpha}$ $\square$

Solution ***2.c*** First observe that $A_R \subset A'_R, \forall 0 \leq R \leq R'$. Take $A_n := A_{\frac{1}{n}}$, we have a monotone decreasing closed non-empty set sequence A_n satisfy

$$\lim_{n\to\infty} \mathrm{diam} A_n = 0 \tag{8}$$

Then $\cap_{n\geq 1} A_n$ is a point by the completeness of (X, ρ). This point has the property $\rho(x, Tx) = 0$. And any fixed point shall be included in $A_R, \forall R$, which will give the uniqueness of fixed point. $\square$

Problem 3

Let $f \in C[a,b]$, $K(\cdot,\cdot) \in C([a,b]\times[a,b])$, such that

$$\sup_{a\leq x\leq b} \int_a^b |K(t,s)|\mathrm{d}s < 1. \tag{9}$$

prove that

$$x(t) = f(t) + \lambda \int_a^b K(t,s)x(s)\mathrm{d}s \quad \lambda \in \mathbb{R} \tag{10}$$

has a unique solution for $x_0 \in C[a,b]$.

Solution ***3.a*** We adopt a similar but somehow different approach to problem 1 by defining

$$Tx := f(t) + \lambda \int_a^b K(t,s)x(s)\mathrm{d}s$$

where $x \in C[a,b]$. and claim T^n is contraction for $n \geq N \in \mathbb{N}$ and $\forall \lambda \in \mathbb{R}$. Obeserve that,

$$\begin{aligned}
\rho(T^n x, T^n y) &= \sup_{t\in[a,b]} |T^n x(t) - T^n y(t)| \\
&\leq |\lambda| \sup_{t\in[a,b]} \left| \int_a^b K(t,s)\big(T^{n-1}x(s) - T^{n-1}y(s)\big)\mathrm{d}s \right| \\
&\leq |\lambda| \sup_{t\in[a,b]} \int_a^b \left| K(t,s)\big(T^{n-1}x(s) - T^{n-1}y(s)\big) \right| \mathrm{d}s \\
&\leq |\lambda| \sup_{t\in[a,b]} \int_a^b |K(t,s)\rho(T^{n-1}x, T^{n-1}y)|\mathrm{d}s \\
&\leq |\lambda| \rho(T^{n-1}x, T^{n-1}y) \sup_{t\in[a,b]} \int_a^b |K(t,s)|\mathrm{d}s \\
&\leq |\lambda| \left(\sup_{t\in[a,b]} \int_a^b |K(t,s)|\mathrm{d}s \right)^n \rho(x,y)
\end{aligned}$$

Since $\sup_{t\in[a,b]}\int_a^b |K(t,s)|\mathrm{d}s \in (0,1)$, for any given $\lambda \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, we have $|\lambda|\left(\sup_{t\in[a,b]}\int_a^b |K(t,s)|\mathrm{d}s\right)^n \in (0,1)$. Here we have proved that T^n is a contraction in a complete metric space $(C[a,b],\rho)$ for n large enough. Then by a variation of Banach's theorem, the claim follows. □

Problem 4

Let f be a twice continuously differentiable function on $[a,b]$ and $\hat{x} \in (a,b)$ such that $f(\hat{x}) = 0, f'(\hat{x}) \neq 0$. Prove that there exist a neighbor $U(\hat{x})$ of $\hat{x}$, such that $\forall x_0 \in U(\hat{x})$, the iteration sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (n = 0,1,2,\dots) \tag{11}$$

is convergent and

$$\lim_{n\to\infty} x_n = \hat{x}. \tag{12}$$

Solution ***4.a*** Define a mapping $Tx := x - \frac{f(x)}{f'(x)}$. Since f is smooth enough, on a small neighborhood of $\hat{x}$, we have

$$\begin{aligned}
Tx &= \hat{x} + x - \hat{x} + o(x-\hat{x}) - \frac{(x-\hat{x})f'(\hat{x}) + o(x-\hat{x})}{f'(\hat{x}) + o(x-\hat{x})} \\
&= \hat{x} + (x-\hat{x}) - \frac{x-\hat{x}+o(x-\hat{x})}{1+o(x-\hat{x})} + o(x-\hat{x}) \\
&= \hat{x} + (x-\hat{x}) - (x-\hat{x})(1+o(1)) + o(x-\hat{x}) \\
&= \hat{x} + o(x-\hat{x}) \\
&= \hat{x} + o(1) \text{ this hold only on a small neighborhood}
\end{aligned}$$

Then we have $|Tx - Ty| = o(1)$ on a neighborhood of $\hat{x}$, which is apparantly smaller than $|x-y|$. This prove T is a contraction. And by Banach's theorem, the limit exist. And iteration started in this neighborhood converges to $\hat{x}$ by the last equality above. □

Problem 5

Let (X, ρ) be a metric space, a mapping $T : X \mapsto X$ satisfy

$$\rho(Tx, Ty) < \rho(x, y) \quad \forall x \neq y, \tag{13}$$

Assume we know T has a fixed point, prove that such fixed point is unique.

Solution ***5.a*** If there's another two fixed points x, y such that $\rho(x, Tx) = 0, \rho(y, Ty) = 0$, then

$$\begin{aligned}\rho(x, y) &\leq \rho(x, Tx) + \rho(y, Ty) + \rho(Tx, Ty) \\ &\leq \rho(Tx, Ty) \\ &< \rho(x, y)\end{aligned}$$

which is a contradiction. □

Problem 6

Let M be a bounded closed set in $(\mathbb{R}^n, \rho)$, $T : M \mapsto M$ satisfy $\rho(Tx, Ty) < \rho(x, y), \forall x, y \in M, x \neq y$. Prove that there exists an unique fixed point for T in M.

Solution ***6.a*** This is an easy corallary of problem 7, since bounded closed sets are equivalent to compact sets in $\mathbb{R}^n$, and sequential compact is equivalent to compact in $\mathbb{R}^n$ too. We will prove it in problem 7. □

Problem 7

Let (X, ρ) be a metric space, and M be a sequential compact set in X. Let the mapping $f : X \mapsto M$ satisfy

$$\rho(f(x_1), f(x_2)) < \rho(x_1, x_2), \quad \forall x_1, x_2 \in X, x_1 \neq x_2. \tag{14}$$

Prove that there exists an unique fixed point of f in X.

Solution ***7.a*** ρ, f are continuous functions, so $\rho(x, f(x))$ is continuous in x, as well as its restriction on M. Consider this restriction, it's a continuous function on a compact set which has a infinimum of 0. Then this function must be 0 at some point $x \in M \subset X$. This would give the existence of fixed point $x \in X$. Now if there's another y such that $\rho(y, Ty) = 0$, then

$$\begin{aligned}
\rho(x, y) &\le \rho(x, Tx) + \rho(y, Ty) + \rho(Tx, Ty) \\
&\le \rho(Tx, Ty) \\
&< \rho(x, y)
\end{aligned}$$

which is a contradiction. $\square$