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HOMEWORK Stochastic Process I MATH4221

Shenduo Zhang

1.

Let X and Y be discrete random variables taking values in $\mathcal{X}$ and $\mathcal{Y}$, respectively. Recalling that $\mathbb{E}X = \sum_{x\in\mathcal{X}} x\mathbb{P}(X = x)$ and $\mathbb{E}(X|Y = y) = \sum_{x\in\mathcal{X}} x\mathbb{P}(X = x|Y = y)$, verify by direct calculation the following identities:

(A) $\mathbb{E}X = \sum_{x\in\mathcal{X},y\in\mathcal{Y}} x\mathbb{P}(X = x \text{ and } Y = y)$

(B) $\mathbb{E}X = \sum_{y\in\mathcal{Y}} \mathbb{E}(X|Y = y)\mathbb{P}(Y = y)$

✓ *Solution.* (A)

$$\sum_{x\in\mathcal{X},y\in\mathcal{Y}} x\mathbb{P}(\mathbf{X} = x \textit{ and } \mathbf{Y} = y) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} x\mathbb{P}(\mathbf{Y} = y|\mathbf{X} = x)\mathbb{P}(\mathbf{X} = x)$$
$$= \sum_{x\in\mathcal{X}} x\mathbb{P}(\mathbf{X} = x)\sum_{y\in\mathcal{Y}} \mathbb{P}(\mathbf{Y} = y|\mathbf{X} = x)$$

Since the conditional probability is still a probability measure

$$= \sum_{x\in\mathcal{X}} x\mathbb{P}(\mathbf{X} = x)$$
$$= \mathbb{E}\mathbf{X}$$

□

✓ *Solution.* (B)

$$\sum_{y\in\mathcal{Y}} \mathbb{E}(\mathbf{X}|\mathbf{Y} = y)\mathbb{P}(\mathbf{Y} = y) = \sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}} x\mathbb{P}(\mathbf{X} = x|\mathbf{Y} = y)\right)\mathbb{P}(\mathbf{Y} = y)$$
$$= \sum_{x\in\mathcal{X}} x\sum_{y\in\mathcal{Y}} \mathbb{P}(\mathbf{X} = x|\mathbf{Y} = y)\mathbb{P}(\mathbf{Y} = y)$$
$$= \sum_{x\in\mathcal{X}} x\sum_{y\in\mathcal{Y}} \mathbb{P}(\mathbf{Y} = y|\mathbf{X} = x)\mathbb{P}(\mathbf{X} = x)$$
$$= \sum_{x\in\mathcal{X}} x\mathbb{P}(\mathbf{X} = x)\sum_{y\in\mathcal{Y}} \mathbb{P}(\mathbf{Y} = y|\mathbf{X} = x)$$
$$= \mathbb{E}\mathbf{X}$$

□

2. (A) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads.

(A.1) What is the probability that it is the fair coin?

(A.2) Suppose that he flips the same coin a second time and again it shows heads. Now what is the probability that it is the fair coin?

(A.3) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

(B) There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

Solution. (A.1)

To be more brief, we introduce the denotion **H** and $\mathbf{H_2}$ to refer to the event in which the coin shows one or two heads, and **F** means the coin is fair while **C** means it's a cheated two headed coin. And **B** means it is the biased random coin.

$$\mathbb{P}\left(\mathbf{F}|\mathbf{H}\right) = \frac{\mathbb{P}(\mathbf{H}|\mathbf{F})\mathbb{P}(\mathbf{F})}{\mathbb{P}(\mathbf{H}|\mathbf{F})\mathbb{P}(\mathbf{F}) + \mathbb{P}(\mathbf{H}|\mathbf{C})\mathbb{P}(\mathbf{C})} = \frac{1}{3}$$

□

Solution. (A.2)

$$\mathbb{P}(\mathbf{F}|\mathbf{H_2}) = \frac{\mathbb{P}(\mathbf{H_2}|\mathbf{F})\mathbb{P}(\mathbf{F})}{\mathbb{P}(\mathbf{H_2}|\mathbf{F})\mathbb{P}(\mathbf{F}) + \mathbb{P}(\mathbf{H_2}|\mathbf{C})\mathbb{P}(\mathbf{C})} = \frac{1}{5}$$

□

Solution. (A.3)

The two-headed coin do not have a tail. So it shows tail means it must be a fair coin. □

Solution. (B)

$$\mathbb{P}(C|H) = \frac{\mathbb{P}(\mathbf{H}|\mathbf{C})\mathbb{P}(\mathbf{C})}{\mathbb{P}(\mathbf{H}|\mathbf{C})\mathbb{P}(\mathbf{C}) + \mathbb{P}(\mathbf{H}|\mathbf{F})\mathbb{P}(\mathbf{F}) + \mathbb{P}(\mathbf{H}|\mathbf{B})\mathbb{P}(\mathbf{B})} = \frac{4}{9}$$

□

3. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

   (A) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?

   (B) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)

Solution. (A)

When the prison step into the door one and two, he will reach the state completely the the same with previous one, hence we will have,

$$\mathbb{E}\mathbf{X} = 0.5 * (\mathbb{E}\mathbf{X} + 2) + 0.3(\mathbb{E}\mathbf{X} + 3) + 0.2 * (0)$$

Solve for $\mathbb{E}\mathbf{X}$, we have $\mathbb{E}\mathbf{X} = \frac{19}{2}$ □

Solution. (B)

Consider the expected time it take for prison to go out

$$\begin{aligned}
\mathbb{EX} &= \frac{1}{3}\cdot 0 + \frac{2}{3}\left(\frac{1}{2}(\mathbb{EX}_1 + 2) + \frac{1}{2}(\mathbb{EX}_2 + 3)\right) \\
\mathbb{EX}_1 &= \frac{1}{2} * 3 + \frac{1}{2} * 0 \\
\mathbb{EX}_2 &= \frac{1}{2} * 2 + \frac{1}{2} * 0
\end{aligned}$$

where $\mathbf{X}_1$ stands for the time it takes after he took his first attempt into the first door. And $\mathbf{X}_2$ is analogous but with attempt into the second door.

Hence after easy computation, we have $\mathbb{EX} = \frac{5}{2}$. □

4. A fair coin (independently heads/tails with probability 1/2) is flipped repeatedly.
   (A) Find the expected number of flips needed to get three heads in a row.
   (B) Find the expected number of flips needed to get k heads in a row.

Solution. for both (A) and (B)

Instead of solving the probablem (A), I will prove problem (B) directly, since problem (A) is merely a straight forward conclusion of (B) letting $k = 3$.

Let $\mathcal{N}_i$, $i = 1, 2, 3, \ldots$ be the numbers of steps to get a k heads row while we already have a i heads row. Then $\mathcal{N}$ is a random variable from the event to $\mathbb{R}$.

Then we have

$$\begin{aligned}
\mathbb{E}\mathcal{N}_0 &= \frac{1}{2}(\mathbb{E}\mathcal{N}_1 + 1) + \frac{1}{2}(\mathbb{E}\mathcal{N}_0 + 1) \\
\mathbb{E}\mathcal{N}_1 &= \frac{1}{2}(\mathbb{E}\mathcal{N}_2 + 1) + \frac{1}{2}(\mathbb{E}\mathcal{N}_0 + 1) \\
&\vdots \\
\mathbb{E}\mathcal{N}_{k-1} &= \frac{1}{2}(\mathbb{E}\mathcal{N}_k + 1) + \frac{1}{2}(\mathbb{E}\mathcal{N}_0 + 1) \\
\mathbb{E}\mathcal{N}_k &= 0
\end{aligned} \tag{1}$$

Rewrite it in the form

$$\begin{cases}
\mathbb{E}\mathcal{N}_i - \mathbb{E}\mathcal{N}_0 - 2 = \frac{1}{2}(\mathbb{E}\mathcal{N}_{i+1} - \mathbb{E}\mathcal{N}_0 - 2), & i = 0, 1, 2, \ldots, k-1 \\
\mathbb{E}\mathcal{N}_k = 0
\end{cases} \tag{2}$$

We can easily conclude that

$$(\mathbb{E}\mathcal{N}_0 - \mathbb{E}\mathcal{N}_0 - 2) = (\frac{1}{2})^k(\mathbb{E}\mathcal{N}_k - \mathbb{E}\mathcal{N}_0 - 2) \tag{3}$$

Hence we have $\mathbb{E}\mathcal{N}_0 = 2^{k+1} - 2$ which is the expected step to have a k head in a row. Let $k = 3$, the answer to (A) is 14.

□