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Problem 1

(A) If $(X_i)_{1\le i\le 3}$ are independent exponential random variables with rates $(\lambda_i)_{1\le i\le 3}$, find $\mathbb{P}(X_1 < X_2 < X_3)$ and $\mathbb{P}(X_1 < X_2 \mid \max\{X_1, X_2, X_3\} = X_3)$
(B) Machine 1 is currently working. Machine 2 will be put in use at a time t from now. If the lifetime of machine i is exponential with rate λ_i (both independent), what is the probability that machine 1 is the first machine to fail?

***Solution* 1.a** $X_i \sim Exp(\lambda_i), i = 1, 2, 3.$

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_0^\infty \int_0^{x_3} \int_0^{x_2} \lambda_1 e^{-\lambda_1 x_1} \mathrm{d}x_1 \lambda_2 e^{-\lambda_2 x_2} \mathrm{d}x_2 \lambda_3 e^{-\lambda_3 x_3} \mathrm{d}x_3 \\
&= \int_0^\infty \int_0^{x_3} \lambda_2\lambda_3 e^{-(\lambda_1+\lambda_2)x_2 - \lambda_3 x_3} (e^{\lambda_1 x_2} - 1) \mathrm{d}x_2 \mathrm{d}x_3 \\
&= \int_0^\infty \lambda_3 e^{-(\lambda_2+\lambda_3)x_3} (e^{\lambda_2 x_3} - 1) - \frac{\lambda_2\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda 2+\lambda_3)x_3} (e^{(\lambda_1+\lambda_2)x_3} - 1) \mathrm{d}x_3 \\
&= \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}
\end{aligned}$$

This is the answer to the first question.

$$\mathbb{P}(X_1 < X_2 | \max\{X_1, X_2, X_3\} = X_3) = \frac{\mathbb{P}(X_3 > X_2 > X_1)}{\mathbb{P}(X_3 > X_1, X_3 > X_2)} \tag{1}$$

And the events on the denominator are not independent which gives us a lot of troubles.

$$\begin{aligned}
\mathbb{P}(X_3 > X_1, X_3 > X_2) &= \int_0^\infty \int_0^{x_3} \int_0^{x_3} \lambda_1 e^{-\lambda_1 x_1} \mathrm{d}x_1 \lambda_2 e^{-\lambda_2 x_2} \mathrm{d}x_2 \lambda_3 e^{-\lambda_3 x_3} \mathrm{d}x_3 \\
&= \int_0^\infty \lambda_3 (1 - e^{-\lambda_1 x_3})(1 - e^{-\lambda_2 x_3}) e^{-\lambda_3 x_3} \mathrm{d}x_3 \qquad (2) \\
&= \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}
\end{aligned}$$

Hence from (2) and (3) we derive

$$\mathbb{P}(X_1 < X_2 | \max\{X_1, X_2, X_3\} = X_3) = \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3} \tag{3}$$

Denote the time it take for machine i to crash as T_i, then $T_i \sim \mathrm{Exp}(\lambda_i)$. The machine 1 is the first one to fail is equivalence to $T_1 - t < T_2$.

$$
\begin{aligned}
\mathbb{P}(T_1 - t < T_2) &= \int_0^\infty \int_0^{t_2+t} \lambda_1 e^{-\lambda_1 t_1} \mathrm{d}t_1 \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \\
&= \int_0^\infty (1 - e^{-\lambda_1(t_2+t)}) \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \\
&= 1 - e^{-\lambda_1 t} \frac{\lambda_2}{\lambda_1 + \lambda_2}
\end{aligned}
$$

□

Problem 2

In a certain system, a customer must first be served by server 1 and then by server 2. The service times at server i are exponential with rate λ_i (both independent). An arrival finding server 1 busy waits in line for that server. Upon completion ofservice at server 1, a customer either enters service with server 2 if that server is free or else remains with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2.

(A) Suppose that when you arrive there is one customer in the system and that customer is being served by server 1. What is the expected total time you spend in the system?

(B) Suppose you arrive to find two others in the system, one being served by server 1 and one by server 2. What is the expected time you spend in the system? Recall that if server 1 finishes before server 2, then server 1s customer will remain with him (thus blocking your entrance) until server 2 becomes free.

Solution **2.a** Denote the time customer arrived ealier as A and the time he spend in server i as T_i^A. And denote myself as B. Denote the time I spend in the sever as T then, it's equal to the time it take for me to enter service 1 and enter service 2 and then leave service 2.

$$
T = T_1^A + \max\{T_2^A, T_1^B\} + T_2^B \tag{4}
$$

where $T_1^A, T_1^B \sim \mathrm{Exp}(\lambda_1)$ and $T_2^A, T_2^B \sim \mathrm{Exp}(\lambda_2)$.

$$
\begin{aligned}
\mathbb{E}T &= \mathbb{E}T_1^A + \mathbb{E}\max\{T_2^A, T_1^B\} + \mathbb{E}T_2^B \\
&= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \mathbb{E}\max\{T_2^A, T_1^B\} \\
&= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \mathbb{E}[T_2^A + T_1^B] - \mathbb{E}[\min\{T_2^A, T_1^B\}] \\
&= \frac{2}{\lambda_1} + \frac{2}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}
\end{aligned}
$$

□

Solution **2.b** Denote the time as P, and here we use the similar notation but denote ourselves as A and the one at server 1,2 as B,C respectively. Still the time we spend in the system is equal to the time it take for us to enter server 1 and enter sever 2 and leave.

$$
T = \max\{T_1^B, T_2^C\} + \max\{T_2^B, T_1^A\} + T_2^A \tag{5}
$$

where all of the components above follows $\mathrm{Exp}(\lambda_i)$ where i is the subsript. And using the same techniques we derive

$$\begin{aligned}\mathbb{E}T &= (\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2})+(\frac{1}{\lambda_2}+\frac{1}{\lambda_1}-\frac{1}{\lambda_2+\lambda_1})+\frac{1}{\lambda_2}\\ &= \frac{2}{\lambda_1}+\frac{3}{\lambda_2}-\frac{2}{\lambda_1+\lambda_2}\end{aligned}$$

Here we complete the proof.

□

Problem 3

Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with $\lambda = 4$.

(A) What is the probability that fewer than two calls come in the first hour?

(B) Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?

(C) The person answering the phones waits until fifteen calls have arrived before going on a break. What is the expected amount of time that the person will wait?

(D) Suppose that it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly five of them arrived in the first hour?

(E) Suppose that it is known that exactly k calls arrived in the first four hours. What is the probability that exactly j of them arrived in the first hour?

(F) Finally, let $(X_t)_{t\in[0,\infty)}$ be a Poisson Process with arbitrary rate $\lambda \in (0,\infty)$. Given $s < t$, show that for integers $0 \le k \le n$ we have

$$\mathbb{P}(X_s = k \mid X_t = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}.$$

***Solution* 3.a**
Denote the number of calls by i hour as X_i.

$$\mathbb{P}(X_1 < 2) = \mathbb{P}(X_1 = 0) + \mathbb{P}(X_1 = 1) = 5e^{-4}$$

□

***Solution* 3.b** By the restart property of poisson process,

$$\mathbb{P}(X_2 - X_1 \ge 2|X_1 = 6) = \mathbb{P}(X_1 \ge 2) = 1 - \mathbb{P}(X_1 < 2) = 1 - 5e^{-4}$$

□

***Solution* 3.c** Denote the k-th arrival time as $Y_k, Y_0 = 0$. Then,

$$Y_k = \sum_{j=1}^{k} T_j,$$

where T_j denote the interarrival time, and $T_j \sim$Exp$(\frac{1}{\lambda})$ independently. Hence

$$\mathbb{E}Y_{15} = \sum_{i=1}^{15} \mathbb{E}T_k = \frac{15}{4}$$

$\square$

***Solution* 3.d**

Using the independence between events happens in disjoint interval and the restart property, we can derive

$$\begin{aligned}\mathbb{P}(X_1 = 5|X_2 = 8) &= \frac{\mathbb{P}(X_2 = 8|X_1 = 5)\mathbb{P}(X_1 = 5)}{\mathbb{P}(X_2 = 8)} \\ &= \frac{\mathbb{P}(X_2 - X_1 = 3)\mathbb{P}(X_1 = 5)}{\mathbb{P}(X_2 = 8)} \\ &= \frac{1}{2^8}\binom{8}{3}\end{aligned}$$

$\square$

***Solution* 3.e**

$$\begin{aligned}\mathbb{P}(X_1 = j|X_4 = k) &= \frac{\mathbb{P}(X_4 - X_1 = k - j)\mathbb{P}(X_1 = j)}{\mathbb{P}(X_4 = k)} \\ &= \frac{\mathbb{P}(X_3 = k - j)\mathbb{P}(X_1 = j)}{\mathbb{P}(X_4 = k)} \\ &= \frac{3^{k-j}}{4^k}\binom{k}{j}\end{aligned}$$

$\square$

***Solution* 3.f**

$$\begin{aligned}\mathbb{P}(X_s = k|X_t = n) &= \frac{\mathbb{P}(X_t = n|X_s = k)\mathbb{P}(X_s = k)}{\mathbb{P}(X_t = n)} \\ &= \frac{\mathbb{P}(X_{t-s} = n - k)\mathbb{P}(X_s = k)}{\mathbb{P}(X_t = n)} \\ &= \frac{e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}e^{-\lambda s}\frac{(\lambda s)^k}{k!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}} \\ &= \binom{n}{k}\frac{(t-s)^{n-k}s^k}{t^n} \\ &= \binom{n}{k}\left(1 - \frac{s}{t}\right)^{n-k}\left(\frac{s}{t}\right)^k\end{aligned}$$

$\square$

Problem 4

(A) Let $(X_t)_{t\in[0,\infty)}$ and $(Y_t)_{t\in[0,\infty)}$ be two independent Poisson processes with rates λ and μ, respectively. Show that $(Z_t)_{t\in[0,\infty)}$ with $Z_t := X_t+Y_t$ forms a Poisson process with rate $\lambda+\mu$.

(B) Let airplanes arrive at Atlanta airport according to Poisson process $(X_t)_{t\in[0,\infty)}$ with rate λ, and each such airplane is a jumbo-jet with probability p, independently of the type of all other arriving airplanes. Show that the number Z_t of arriving jumbo-jets (arriving by time t) forms a Poisson process with rate λp.

(Hint for (A)+(B): calculate suitable transition-probabilities $\mathbb{P}(Z_{t+\delta} = n+m \mid Z_t = n) = \cdots + o(\delta)$.)

***Solution* 4.a**

1. Proving the independence of numbers of arrivals in disjoint interval.
   Let $T(a,b) = Z_b - Z_a$, $T_X(a,b) = X_b - X_a$, $T_Y(a,b) = Y_b - Y_a$, where $0 \le a < b < c < d$. It follows from that $T(a,b) = T_X(a,b) + T_Y(a,b)$. For $n \ge 0$, $m \ge 0$,

$$
\begin{aligned}
&\mathbb{P}\left(T(a,b) = n, T(c,d) = m\right)\\
=&\mathbb{P}(T_X(a,b) + T_Y(a,b) = n, T_X(c,d) + T_Y(c,d) = m)\\
=&\sum_{n_1+n_2=n}\sum_{m_1+m_2=m}\mathbb{P}(T_X(a,b) = n_1, T_Y(a,b) = n_2, T_X(c,d) = m_1, T_Y(c,d) = m_2)\\
&\text{by the independence of two process and their poisson property}\\
=&\sum_{n_1+n_2=n}\sum_{m_1+m_2=m}\mathbb{P}(T_X(a,b) = n_1)\mathbb{P}(T_Y(a,b) = n_2)\mathbb{P}(T_X(c,d) = m_1)\mathbb{P}(T_Y(c,d) = m_2)\\
=&\sum_{n_1+n_2=n}\mathbb{P}(T_X(a,b) = n_1, T_Y(a,b) = n_2)\sum_{m_1+m_2=m}\mathbb{P}(T_X(a,b) = m_1, T_Y(c,d) = m_2)\\
=&\mathbb{P}(T(a,b) = n)\mathbb{P}(T(c,d) = m)
\end{aligned}
$$

   Hence, the numbers of events happens in disjoint interval are independent.

2. Proving the Time-Homogeneuty.
   By the "independence" property we proved in 1. ,

$$
\begin{aligned}
\mathbb{P}(Z_{t+d} = n+m|Z_t = n) &= \mathbb{P}(Z_d = m)\\
&= \mathbb{P}(X_d + Y_d = m)\\
&= \sum_{k=0}^{m}\mathbb{P}(X_d = k)\mathbb{P}(Y_d = m-k)
\end{aligned}
$$

   (a) m=0

$$
\begin{aligned}
\mathbb{P}(Z_{t+d} = Z_t) &= \mathbb{P}(X_d = 0)\mathbb{P}(Y_d = 0)\\
&= (1 - \lambda d + o(d))(1 - \mu d + o(d))\\
&= 1 - \lambda d - \mu d + o(d)\\
&= 1 - (\lambda+\mu)d + o(d)
\end{aligned}
$$

(b) m=1

$$\begin{aligned}\mathbb{P}(Z_{t+d} = Z_t + 1) &= \mathbb{P}(X_d = 0)\mathbb{P}(Y_d = 1) + \mathbb{P}(X_d = 1)\mathbb{P}(Y_d = 0)\\ &= (1 - \lambda d + o(d))\mu d + (1 - \mu d + o(d))\lambda d\\ &= (\mu + \lambda)d - 2\lambda\mu d^2 + o(d^2)\\ &= (\mu + \lambda)d + o(d)\end{aligned}$$

Here we have completed the proof, because the third property can be easily derived by subtracting. $\mathbb{P}(Z_{t+d} \geq X_t + 2) = 1 - \mathbb{P}(Z_{t+d} = Z_t + 1) - \mathbb{P}(Z_{t+d} = Z_t) = o(d)$.

Here we have prove that $\{Z_t\}_{[0,=\infty)}$ is a Poisson process with rate $\mu + \lambda$.

□

***Solution* 4.b**
The key observation here is that $Z_T = \sum_{i-1}^{X_t} Y_i$, where $Y_i \sim Ber(p)$ and they are i.i.d..
The proof will be two parts using the same techniques as 4.a. For $n, m \geq 0$, $0 \leq a < b < c < d$,

1. Proving "independence"

$$\begin{aligned}&\mathbb{P}(Z_b - Z_a = n, Z_d - Z_c = m)\\ =&\mathbb{P}(\sum_{i=X_a}^{X_b} Y_i = n, \sum_{i=X_c}^{X_d} Y_i = m)\\ =&\mathbb{P}(\sum_{i=1}^{X_b-X_a} Y_i = n, \sum_{j=1}^{X_d-X_c} \tilde{Y}_j = m)\end{aligned}$$

Above follows from the "independence" property of X_t and Y_i i.i.d.

where, $Y_i, \tilde{Y}_i$ are i.i.d., hence by the independence of Y_i,

$$\begin{aligned}=&\mathbb{P}(\sum_{i=1}^{X_b-X_a} Y_i = n)\mathbb{P}(\sum_{j=1}^{X_d-X_c} \tilde{Y}_j = m)\\ =&\mathbb{P}(Z_b - Z_a = n)\mathbb{P}(Z_d - Z_c = m)\end{aligned}$$

2. Proving the time homogeneoty.

$$\begin{aligned}\mathbb{P}(Z_{t+d} = n + m | Z_t = n) &= \mathbb{P}(\sum_{i=1}^{X_{t+d}} Y_i = \sum_{i=1}^{X_t} Y_i + m)\\ &= \mathbb{P}(\sum_{i=1}^{X_d} Y_i = m)\end{aligned}$$

(a) m=0

$$\begin{aligned}\mathbb{P}(\sum_{i=1}^{X_d} Y_i = 0) &= \mathbb{P}(X_d = 0) + \mathbb{P}(X_d = 1)(1 - p) + \mathbb{P}(X_d = 2)(1 - p)^2 + \cdots\\ &= (1 - \lambda d + o(d)) + (\lambda d + o(d))(1 - p) + o(d)[(1 - p)^2 + \cdots]\\ &= 1 - p\lambda d + o(d)\end{aligned}$$

(b) m=1

$$
\begin{aligned}
\mathbb{P}(\sum_{i=1}^{X_d} Y_i = 1) &= \mathbb{P}(X_d = 0) \times 0 + \mathbb{P}(X_d = 1)p + \mathbb{P}(X_d = 3)p(1-p) + \cdots \\
&= \lambda dp + o(d) \left[p(1-p) + \binom{2}{1} p(1-p)^2 + \dots \right] \\
&= \lambda dp + o(d)
\end{aligned}
$$

And for the situation when $m \geq 2$, just use the same subtraction will show it is nothing but $o(d)$ again.

Here we have completed the proof that $\{Z_t\}_{[0,=\infty)}$ is a Poisson process with rate $p\lambda$.

$\square$