第四章

1. 用 DFP 拟牛顿法解如下优化问题.

$$\min\{x_1^2+4x_2^2\}$$

取 $x^{(0)}=(1,1)^T$ $\quad H^{(0)}=I_{\{2\times 2\}}$ $\quad G=\begin{pmatrix}2&\\&8\end{pmatrix}$.

解: 需计算 $f(x)=x_1^2+4x_2^2$ $\quad g_k=(2x_1,8x_2)^T$.

$s_k=x_{k+1}-x_k$ $\quad y_k=g_{k+1}-g_k$ $\quad \alpha_k=\dfrac{g_k^T H_k g_k}{d_k^T G d_k}=\dfrac{g_k^T H_k g_k}{g_k^T H_k G H_k g_k}$.

$$x_{k+1}=x_k-\alpha_k H_k g_k$$

$$H_{k+1}=H_k+\frac{s_k s_k^T}{s_k^T y_k}-\frac{H_k y_k y_k^T H_k}{y_k^T H_k y_k}$$

$g_0=(2,8)^T$ $\quad \alpha_0=\dfrac{68}{(2,8)\begin{pmatrix}2&\\&8\end{pmatrix}\begin{pmatrix}2\\8\end{pmatrix}}=\dfrac{68}{520}=\dfrac{17}{130}$.

$$x_1=\begin{pmatrix}1\\1\end{pmatrix}-\frac{17}{130}\begin{pmatrix}1&\\&1\end{pmatrix}\begin{pmatrix}2\\8\end{pmatrix}=\begin{pmatrix}\frac{48}{65}\\-\frac{3}{65}\end{pmatrix}.$$

$s_0=\begin{pmatrix}-17/65\\-68/65\end{pmatrix}$ $\quad g_1=\begin{pmatrix}96/65\\-24/65\end{pmatrix}$ $\quad y_0=\begin{pmatrix}-34/65\\-544/65\end{pmatrix}$

$$H_1=\begin{pmatrix}\frac{33537}{33410}, & -\frac{526}{16705}\\ -\frac{526}{16705}, & \frac{2121}{16705}\end{pmatrix}\qquad \alpha_1=\frac{257}{520}.$$

$$x_2=\begin{pmatrix}48/65\\-3/65\end{pmatrix}-\frac{257}{520}\begin{pmatrix}33537/33410, & -526/16705\\ -526/16705, & 2121/16705\end{pmatrix}\begin{pmatrix}96/65\\-24/65\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

而 $(0,0)$ 为 $f(x)$ 最小值 (二次终止性).

故 $(0,0)$ 是原问题的最优解

第五章

1. 利用KKT条件求解下列优化问题

$$\min \ (x_1-3)^2+(x_2-3)^2$$

$$\text{s.t.} \quad 4-x_1-x_2 \geq 0.$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$L(x,\lambda)=(x_1-3)^2+(x_2-3)^2-\lambda_1(4-x_1-x_2)-\lambda_2 x_1-\lambda_3 x_2.$$

KKT条件为

$$\begin{cases} \frac{\partial L}{\partial x_1}=2(x_1-3)+\lambda_1-\lambda_2=0 \\ \frac{\partial L}{\partial x_2}=2(x_2-3)+\lambda_1-\lambda_3=0 \\ 4-x_1-x_2 \geq 0. \\ x_1 \geq 0. \\ x_2 \geq 0. \\ \lambda_1(4-x_1-x_2)=\lambda_2 x_1=\lambda_3 x_2=0. \\ \lambda_i \geq 0. \quad i=1,2,3 \end{cases} \Rightarrow \begin{cases} x=(2,2) \\ \lambda=(2,0,0) \end{cases}$$

$$\nabla_{xx}^2 L(x,\lambda)=\begin{pmatrix} 2 & \\ & 2 \end{pmatrix} \quad \text{正定.}$$

故. $x=(2,2)$ 为最优解.

2. 考虑以下等式约束优化问题：

$$\min \quad x_2^2 - 3x_1$$
$$\text{s.t.} \quad x_1 + x_2 = 1$$
$$x_1 - x_2 = 0$$

应用外点二次罚函数法，当 $\mu \to 0$ 时，求出问题的最优解与相应的拉格朗日乘子。

解：$Q(x;\mu) = x_2^2 - 3x_1 + \frac{1}{2\mu}\left((x_1+x_2-1)^2 + (x_1-x_2)^2\right)$

$$\frac{\partial Q}{\partial x_1} = -3 + \frac{1}{2\mu}\left(2(x_1+x_2-1) + 2(x_1-x_2)\right)$$
$$= -3 + \frac{1}{\mu}(x_1 - 1 + x_1) = -3 + \frac{1}{\mu}(2x_1 - 1)$$

$$\frac{\partial Q}{\partial x_2} = 2x_2 + \frac{1}{\mu}(x_1 + x_2 - 1 + x_2 - x_1)$$
$$= 2x_2 + \frac{1}{\mu}(2x_2 - 1) = \left(2 + \frac{2}{\mu}\right)x_2 - \frac{1}{\mu}$$

$$\begin{cases} \frac{\partial Q}{\partial x_1} = 0 \\ \frac{\partial Q}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{3\mu+1}{2} \\ x_2 = \frac{1}{2\mu+2} \end{cases}$$

$\mu \downarrow 0$. 则 $(x_1, x_2)^* = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$L(x,\lambda) = x_2^2 - 3x_1 - \lambda_1(x_1 + x_2 - 1) - \lambda_2(x_1 - x_2)$$

$$\begin{cases} \frac{\nabla L}{\partial x_1} = -3 - \lambda_1 - \lambda_2 = 0 \\ \frac{\nabla L}{\partial x_2} = 2x_2 - \lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases}$$

其余KKT条件将 $(1/2, 1/2)$ 代入直接满足。

故其最优解为 $(1/2, 1/2)$，$f(x^*) = -\frac{5}{4}$，拉格朗日乘子为 $(-1,-2)$

3. 考虑以下等式约束优化问题.

$$\min \quad 2x_1 + 3x_2$$
$$\text{s.t.} \quad 1-2x_1^2-x_2^2 \geq 0.$$

应用对数障碍法。当$\mu \to 0$时，求出最优解与相应的拉格朗日乘子.

解: $P(x,\mu) = 2x_1+3x_2 - \mu \log(1-2x_1^2-x_2^2)$.

$$\frac{\partial P}{\partial x_1} = 2 + \mu \frac{4x_1}{1-2x_1^2-x_2^2}$$

$$\frac{\partial P}{\partial x_2} = 3 + \mu \frac{2x_2}{1-2x_1^2-x_2^2}$$

$$\begin{cases} \dfrac{\partial P}{\partial x_1} = 0 \\ \dfrac{\partial P}{\partial x_2} = 0 \end{cases} \xrightarrow{\text{舍去一根}} \begin{cases} x_1 = -\dfrac{\sqrt{11}}{11} \\ x_2 = -\dfrac{3\sqrt{11}}{11} \end{cases}$$

(通过KKT也可舍去一根，这里舍去因我们求的是P的极小值).

$$L(x,\lambda) = 2x_1+3x_2 - \lambda(1-2x_1^2-x_2^2)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2+4\lambda x_1 = 0 \\ \dfrac{\partial L}{\partial x_2} = 3+2\lambda x_2 = 0 \\ \lambda > 0. \end{cases} \Rightarrow \quad \lambda = \frac{\sqrt{11}}{2}$$

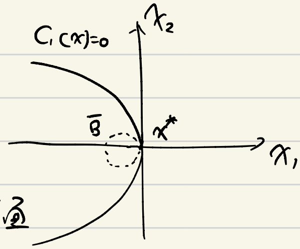
其余KKT条件代入$x=(-\frac{\sqrt{11}}{11}, -\frac{3}{11}\sqrt{11})$直接满足.

故 $x^* = (-\frac{\sqrt{11}}{11}, -\frac{3}{11}\sqrt{11})$ $\quad f(x^*) = -\sqrt{11}$, $\quad \lambda = \frac{\sqrt{11}}{2}$

附加题

考虑问题：

$$\begin{cases} \min & f(x) = x_2 \\ s.t. & c_1(x) = -x_1 - x_2^2 \geq 0 \end{cases}$$



$x^* = (0,0)^T$

① 可行方向 为 $\{d \mid d \in H\}$ H为不含y轴的左半平面

因为在 x^* 处的切线为y轴，x^*在D边界上，D边界光滑，D为凸集

故作x^*处的内切圆$\bar{B}$知存在 $x \in \bar{B}$ s.t. $x^* + (x - x^*) \in D$.

而D为凸集，故在x与x^*连线上任何一点都在D中。

故 $\mathcal{F}D = \{d \mid d \in H\}$ H为不含y轴的左半平面

② 线性化可行方向 $\nabla c(x) = (-1, 0)$ $\nabla c(x^*) = -1$

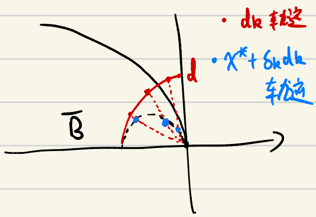
由 $d^T \nabla c(x^*) \geq 0$ 知 $d \in H \cup Y$.

故 $L\mathcal{F}D = \{d \mid d \in H \cup Y\}$, Y为y轴.

③ 序列可行方向. 首先 $\mathcal{F}D \subseteq S\mathcal{F}D \subseteq L\mathcal{F}D$.

故只需验证 $\{d \mid d \in Y\}$ 是否为序列可行的.

断言其是序列可行的. 在$\bar{B}$上取点列 $x_k \to x^*$，且x_k处于上半面



此时 $d_k = (x_k - x^*) / \|x_k - x^*\|$

$\delta_k = \|x_k - x^*\|$

首先 $d_k \to d$，d位于y轴正半轴 $d = (0,1)^T$

其次 $\delta_k \to 0$ 因 $x_k \to x^*$.

对y轴负半轴同理.

故 $S\mathcal{F}D = L\mathcal{F}D = \{d \mid d \in H \cup Y\}$, Y为y轴.