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Problem 1

Prove that $L^\infty[a,b]$ is not divisible.

Solution *1.a* The metric ρ in $L^\infty[a,b]$ is defined to be

$$\rho(f,g) := \max_{x\in[a,b]} |f(x) - g(x)|. \tag{1}$$

If one considers

$$S := \left\{ \mathbb{1}_{[a,\epsilon]} : \epsilon \in [a,b] \right\} \tag{2}$$

one can check that $S \subset L^\infty[a,b]$. And the we have discrepancy property $\rho(\mathbb{1}_\epsilon, \mathbb{1}_{\epsilon'}) = 1$, for $\forall \epsilon \neq \epsilon'$. Suppose a subset D is dense in $L^\infty[a,b]$, then for any $\epsilon \in [a,b]$, there must be a sequence in D converges under ρ to $\mathbb{1}_\epsilon$, who does not converges to any other $\mathbb{1}_{\epsilon'}$ in S because of this discrepancy. Each point y in $[a,b]$ must be a limit point for some sequence $\{y_n\}$ in D. Since $[a,b]$ is not countable, D must also contains a subset, which is the union of those $\{x_n\}$. This subset must be uncountable since it is uncountable many union of sequence. So $L^\infty[a,b]$ is not divisible. $\square$

Problem 2

M is a compact metric space. Let $C(M) := \{f : M \to \mathbb{R}$ which is continuous.$\}$. Define $\rho(f,g) = \max_{x\in M} |f(x) - g(x)|$. prove that

1. *$(C(M), \rho)$ is a metric space.*
2. *$(C(M), \rho)$ is complete.*

Solution ***2.a*** To prove $(C(M),\rho)$ is a metric space, one needs to verify the three property of metric for ρ. $\rho(f,g)=\rho(g,f)$ and the property, $\rho(f,g)\geq 0$ where equality holds only when $f=g$, is trivial. It suffices to prove the triangular inequality.

$$\begin{aligned}\rho(f,g)+\rho(g,h) &= \max_{x\in M}|f(x)-g(x)|+\max_{x\in M}|g(x)-h(x)| \\ &\geq \max_{x\in M}(|f(x)-g(x)|+|g(x)-h(x)|) \\ &\geq \max_{x\in M}|f(x)-h(x)| \\ &= \rho(f,h)\end{aligned}$$

□

Solution ***2.b*** To prove $C(M)$ is complete, it suffices to prove for any Cauchy sequence $\{f_n\}\subset C(M)$, it converge under ρ in $C(M)$.

From the property of ρ, we have $\forall x\in M$ the sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, which has a limit denoted as $f(x)$. This would define a function f whose value is pointwise limit of the sequence. And it's automatically a limit of $\{f_n\}$ under ρ.

Now it suffices to prove $f\in C(M)$. $\forall\epsilon>0$, $\exists N$ s.t. $\forall n\geq N, \rho(f,f_n)\leq\frac{\epsilon}{3}$. The function f_n is continuous on a compact set for any n, hence it's also uniformly continuous for any n. So, there $\exists\delta>0$ s.t. $|f_n(x_1)-f_n(x_2)|\leq\frac{\epsilon}{3}$ uniformly for $d(x_1,x_2)\leq\delta$ by the compact

, where $x_1,x_2\in M$. We look at the difference

$$\begin{aligned}|f(x_1)-f(x_2)| &\leq |f(x_1)-f_n(x_1)|+|f_n(x_1)-f_n(x_2)|+|f_n(x_2)-f(x_2)| \\ &\leq 2\rho(f_n,f)+|f_n(x_1)-f_n(x_2)| \\ &\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon\end{aligned}$$

which complete the proof. Here it would be worth noticing we didn't use the property of compactness of (X,ρ). □

Problem 3

Define $W_0^{1\times 2}(0,1) := \left\{ \text{The completion of } C_0^1(0,1), \rho(f,g) := \left(\int_0^1 |f-g|^2 + |f'-g'|^2 \mathrm{d}x\right)^{1/2} \right\}$, *let* $M := \left\{ u \in W_0^{1\times 2}(0,1) \cap C_0^1(0,1), \int_0^1 (|u|^2 + |u'|^2)\mathrm{d}x \le c < +\infty \right\}$.

1. *prove* ρ *is a distance on* $C_0^1(0,1)$ *and it's not complete.*
2. M *is a sequentially compact on* $C[0,1]$.
3. M *is a sequentially compact on* $L^2(0,1)$.

Solution ***3.a*** Here we define a function

$$f_n(x) = \mathbb{1}_{\{|x-\frac{1}{2}|<\frac{1}{n}\}} \frac{1}{n\pi} \sin\left(n\pi\left[x - \left(\frac{1}{2} - \frac{1}{n}\right)\right]\right) + \mathbb{1}_{\{|x-\frac{1}{2}|<\frac{1}{n}\}^C} x \tag{3}$$

It's easy to verify any of f_n is in $C_0^1(0,1)$ since it's continuous at $x = \frac{1}{2} \pm \frac{1}{n}$, and derivitive at this point is 1. But this function converges to a spike function $f(x) = x\mathbb{1}_{x\in(0,1/2]} + (1-x)\mathbb{1}_{x\in(1/2,1)}$, because the indicater function converges to the spike function.

Spike function is not differentiable at $\frac{1}{2}$, but it's continuous on $[0,1]$. □

Solution ***3.b*** It's equivalent to prove that M is uniformly bounded and equicontinuous.

By Jensen's inequality, $\forall u \in M$, $|u(x)| = |\int_0^x u'(t)\mathrm{d}t| \le \int_0^x |u'(t)|\mathrm{d}t \le \int_0^1 |u'(t)|\mathrm{d}t \le \left(\int_0^1 |u'(t)|^2\mathrm{d}t\right)^{1/2} \le \sqrt{c}$. So it's uniformly bounded in $C[0,1]$.

And consider $x, y \in (0,1)$, we have

$$|u(x) - u(y)| = \left|\int_x^y u'(t)\mathrm{d}t\right| = \int_x^y |u'(t)|\mathrm{d}t \le \left(\int_x^y |u'(t)|^2\mathrm{d}t\right)^{1/2}\left(\int_x^y 1\mathrm{d}t\right)^{1/2} \le \sqrt{c}|y-x|^{\frac{1}{2}}$$

which converges to 0 as $|x-y|$ converges to 0 uniformly for $f \in M$. This would imply the equicontinuity.

□

Solution ***3.c*** $L^p[0,1]$ is a complete metric space, so sequential compactness is equivalent to totally bounded.

To prove it's totally bounded, we need to construct an finite ϵ-net in $L^2[0,1]$ for M. This is a more complicated problem. In short, we approximate it by step function. This is a result call Rellich-Kondrachev theorem imvolves imbeddings of Sobolev spaces.

Consider the piecewise-constant functions

$$F(x) = \begin{cases} c_1 & 0 \leq < \frac{1}{n} \\ c_2 & \frac{1}{n} \leq x < \frac{2}{n} \\ \cdots \\ c_n & \frac{n-1}{n} \leq x \leq 1 \end{cases} \tag{4}$$

We consider c_i in the range of $|c_i| \leq 2c$. Let $y = \arg\min_x |f(x)|$, then

$$|f(x)| \leq |f(y)| + |f(x) - f(y)| \leq ||f||_{L_1} + ||f'||_{L_2} \leq ||f||_{L_2} + ||f'||_{L_2} \leq 2c$$

For any $f \in M$, we choose c_k such that $|f(k/n) - c_k| < \epsilon$. Then

$$|f(x) - c_k| \leq \left| f\left(\frac{k}{n}\right) - c_k \right| + \left| f(x) - f\left(\frac{k}{n}\right) \right| \leq \epsilon + \sqrt{c} \left| x - \frac{k}{n} \right|^{1/2} \leq \epsilon + \frac{\sqrt{c}}{\sqrt{n}} \tag{5}$$

where $\frac{k}{n} \leq x \leq \frac{k+1}{n}$.

Then

$$\int_0^1 |f - F|^2 \leq \sum_{k=1}^{n} \int_{k/n}^{k+1/n} \left(\epsilon + \frac{\sqrt{c}}{\sqrt{n}} \right)^2 \leq n \frac{1}{n} \left(\epsilon + \frac{\sqrt{c}}{\sqrt{n}} \right)^2 \leq \left(\epsilon + \frac{\sqrt{c}}{\sqrt{n}} \right)^2 \tag{6}$$

For a ϵ small and a large but finite n, this term can be arbitrary small. □

Problem 4

Let F be all the real sequence that has only finite terms being non-zero, introduce distance $\rho(x, y) = \sup_{k \geq 1} |\xi_k - \eta_k|$, where $x = \{\xi_k\} \in F, y = \{\eta_k\} \in F$. Prove that (F, ρ) is not complete and find its completion.

Solution ***4.a*** This is trivial by considering the sequence $\{\xi_i\}_k = \left\{ \{\frac{1}{i}\}_{i \leq k} \right\}_{k \geq 0}$ which is inside F for any particular k. And it's a Cauchy sequence, since let $m \geq n$ the supremum of the distance between $\{\xi_i\}_m, \{\xi_i\}_n$ is $1/n$, which goes to zero by taking $n \to \infty$. But the limit of this sequence has infinite many non-zero terms, which is not inside F. The claim follows. □

Solution ***4.b*** The completion of F is the space of all real sequence that converges to 0 at infinity denoted by G. And ρ is defined identically. Isometry is identity mapping. Now we prove G is indeed a completion of F.

F itself is a dense subset of G. Subset property is trivial. For any point $x_k \in S$, any given error ϵ, one can always cut off x_k at some point to obtain x'_k, which is inside F, while retaining the error under ϵ by the converging nature of $\{x_k\}$.

Formally speaking, $\forall \{\xi_k\} \in C_0$, $\forall \epsilon > 0$ such that there exists $y \in F$ such that, $\rho(x, y) < \epsilon$. Notice that $\xi_k \to 0$, there exists $N \in \mathbb{N}$ such that whenever $k > N$, one has $|\xi_k| < \epsilon$. Let $y = (\xi_1, \dots, \xi_n, 0, \dots) \in F$. and $\rho(x, y) \leq \epsilon$.

Now we prove that (G, ρ) is a complete space which will complete the proof.

Let x_k^n be a Cauchy sequence inside G, where n is the index for elements and index k is the coordinate of this elements. Let $x = \lim_{n\to\infty} x^n$, which each coordinate x_k is defined to be $\lim_{n\to\infty} x_k^n$. And there is a limit because it's just a real Cauchy sequence and real numbers as a metric space is complete.

We have $|x_k| \leq |x_k^n| + |x_k - x_k^n|$. Let $k \to \infty$, $|x_k^n| \to 0$ since it's in G, and $|x_k - x_k^n|$ can be arbitrary small by taking n large. This would imply that $x \in G$ and the completeness follows. $\square$

Problem 5

Prove that all the polynomials on $[0, 1]$ *is not complete under metric*

$$\rho(p, q) = \int_0^1 |p(x) - q(x)| \mathrm{d}x \tag{7}$$

where p, q *are polynomials. And find its completion.*

Solution ***5.a*** Consider a counter example

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \forall x \tag{8}$$

Let $p_n = \sum_{k=0}^{n} \frac{x^k}{k}$. Then $\forall x, p_n \to e^x, n \to \infty$. p_n is a non-negative function sequence, hence $\sum_{k=0}^{\infty} \frac{x^k}{k}$ is non-decreasing. By monotone convergence theorem, point-wise convergence implies L^1

convergence. But e^x is not a polynomial.

The completion is $L^1[0,1]$. The isometry is identity mapping.

Polynomials are a dense subset under ρ for the set of continuous function by Weirestrass theorem, and the set of continuous function are dense under ρ in the set of $L^1[0,1]$. Since the topology in $C[0,1]$ is stronger that the one in $L^1[a,b]$, which means that the set of all polynomials on $[0,1]$ is a dense subset in $L^1[0,1]$. □

Problem 6

Prove that in a complete metric space, any subset A being sequentially compact is equivalent to prove $\forall \epsilon > 0$, there is a sequentially compact ϵ-net of A.

Solution ***6.a*** $\Rightarrow$: First, suppose there is such a ϵ that there is no sequential compact ϵ-net for A, which means that any ϵ-net of A is not sequential compact. This implies that there will be a sequence in on the net any of whose subsequence doesn't converge. An ϵ-net is a subset of A. Take this sequence, we get a contradiction.

$\Leftarrow$: Next, consider a $\epsilon/2$-net N for A, by assumption it's sequential compact. Then N must have a finite $\epsilon/2$-net N'. Then it suffices to prove N' is a finite ϵ-net for A, and A become totally bounded and it's in a complete metric space, then it's compact. $\forall x \in A$, we have a point $x' \in N'$ such that $\rho(x, x') \leq \epsilon/2$. And we have another point $x'' \in N'$ such that $\rho(x', x'') \leq \epsilon/2$. Then by triangular inequality, we proved the claim. □

Problem 7

Prove that in a metric space, continuous function on a compact set must be bounded and reaches its supremum and infinemum.

Solution ***7.a*** Suppose it's not bounded, then $\forall n \in \mathbb{N}$, $\exists x_n \in A$ such that $|f(x_n)| \geq A$. It follows that we constructed a sequence $\{x_n\} \subset A$ such that $f(x_n) \to \infty, n \to \infty$, which means that any of its subsequence $\{x_{n_m}\}$ will also bring f to infinity. However since A is compact, this sequence has a subsequence that converges to a point, on which the function f takes only finite value. Then exploiting continuity we can claim this subsequence does not bring f to infinity. This is a contradiction.

It's enough to prove it reaches it's supremum. Suppose the supremum of f on a compact set A is M. This mean that $\forall \epsilon > 0$, we have a point $x \in A$ such that $|f(x)| > M - \epsilon$. Then let $\epsilon = \frac{1}{n}$, we have a sequence $\{x_n\}$ such that $f(x_n) \uparrow M, n \to \infty$. Then for any subsequence $\{x_{n_m}\}$ of this sequence, one has $f(x_{n_m}) \to M, m \to \infty$. And by compactness we can find always such a subsequence and it converges to $x \in A$. Exploit continuity we find the supremum is taken at this point $x \in A$. $\square$

Problem 8

Prove that in a metric space, any totally bounded sets are bounded, and consider to use the subset $E = \{e_k\}_{k=1}^{\infty}$, *where*

$$e_k = \{0, 0, \ldots, 1, 0, \ldots\}$$

to prove that a set can be bounded but not totally bounded.

Solution ***8.a*** Totally bounded sets can be bounded by finite union of ball with radius 1. Finite union of bounded sets are still bounded.

Elements in E are away from each other by 1 under maximum distance of coordinates ρ,

$$\rho(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}) = \sup_{k\geq 1} |x_k - y_k|. \tag{9}$$

This implies that for $\forall \epsilon \leq 1$, for any ϵ-net X and any $e_i \in E$, there must be a point $x_i \in N$ such that $\rho(x_i, e_i) \leq \epsilon$ to make X a ϵ-net. Then X has cardinality of infinity. However they are all bounded under ρ by 1. $\square$

Problem 9

Let (X, ρ) be a metric space, F_1, F_2 are its two compact subset, prove that $\exists x_i \in F_i (i = 1, 2)$, let $\rho(F_1, F_2) = \rho(x_1, x_2)$, where

$$\rho(F_1, F_2) := \inf\{\rho(x, y) | x \in F_1, y \in F_2\} \tag{10}$$

Solution ***9.a*** Fixing one variable, the metric function is a continuous function itself for the alternative variable. And from what we have proved, the infinemum can be taken at some point. There $\exists x_1 \in F_1$ such that $\rho(x_1, x_2) = \rho(F_1, x_2)$. Then fix this particular x_1, repeat the argument again for x_2 would give the claim. □

Problem 10

Let M be a bounded set in $C[a, b]$, prove that the set

$$\left\{ F(x) = \int_a^x f(t) \mathrm{d}t \middle| f \in M \right\} \tag{11}$$

is sequential compact.

Solution ***10.a*** By Arzela-Ascoli theorem, it suffices to prove that this set is totally bounded and equicontinuous. Denote this set by I.

Since $f \in M$, $|f| \leq M_0$ for some $M_0 \geq 0$. And this would implies that $\forall F \in I$ one has $|F(x)| = \left|\int_a^x f(t)\mathrm{d}t\right| \leq \int_a^x |f(t)|\mathrm{d}t \leq M_0(b - a)$, which is independent of f.

And for equicontinuity, $\forall F \in I, \forall x_1 \leq x_2 \in [a, b]$, we have $|F(x_1) - F(x_2)| = |\int_{x_1}^{x_2} f(x)\mathrm{d}x| \leq \int_{x_1}^{x_2} |f(x)|\mathrm{d}x \leq M(x_2 - x_1)$. It converges to zero as $|x_1 - x_2|$ converges to 0 uniformly in f. □

Problem 11

Let $E = \{\sin nt\}_{n=1}^{\infty}$, prove that E is not compact in $C[0, \pi]$.

Solution ***11.a*** By Arzela-Ascoli theorem, it suffices to prove that this set is not uniformly bounded or not equicontinuous. The inherent distance here is supremum distance.

We will prove it's not equicontinuous. First notice that $\forall e \in E, \forall x_1, x_2 \in [0, \pi]$, we have

$$\begin{aligned}|e(x_1) - e(x_2)| &= |\sin nx_1 - \sin nx_2| \\ &= |2\cos\frac{n}{2}(x_1 + x_2)\sin\frac{n}{2}(x_1 - x_2)|\end{aligned}$$

If we set $x_2 = \frac{2}{n}$ and $x_1 = 0$, we have $|x_2 - x_1| \to 0, n \to \infty$. However, the above expression equals to $|2\cos 1 \sin 1|$ which is a non-zero constant. Hence it's not equicontinuous and therefore not compact in $C[0, \pi]$. □

Problem 12

Prove that the necessary and sufficient condition for a subset $A \subset S$ to be compact is $\forall n \in \mathbb{N}, \exists C_n > 0$, s.t. $\forall x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) \in A$, one has $|\xi_n| \le C_n (n = 1, 2, \ldots)$. And (S, ρ) is defined to be all the set of all the real sequence,

$$\rho(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k||\eta_k|}{1 + |\xi_k - \eta_k|}, \tag{12}$$

where $x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots)$, $y = (\eta_1, \eta_2, \ldots, \eta_k, \ldots)$.

Solution ***12.a*** We first prove the necessary condition. Since we have proved (S, ρ) is a complete metric space, A is compact is equivalent to A is uniformly bounded. Uniform boundedness implies boundedness aotumatically.

Then we prove it's sufficient. We will prove that we can find an sequential compact ϵ-net for A for any given ϵ. Consider $(\xi_1, \xi_2, \ldots, \xi_n)$ that satisfy the $|\xi_k| \le C_k (k = 1, 2, \ldots, n)$, which is a closed subset A' of $\mathbb{R}^n$. Then there exist a finite $\epsilon/2$-net N for it. We claim that N will also be a ϵ-net for A by taking n very large.

This comes from that A''s embedding in A, denoted by $\bar{A}$, is indeed an $\epsilon/2$-net for A by taking n large. $\forall x \in A$, let $\bar{x} \in \bar{A}$ be taking the first n coordinate leaving the rest to be 0. Then

$$\rho(x, \bar{x}) = \sum_{k=n+1}^{\infty} \frac{1}{2^k} \frac{|x_k||\bar{x}_k|}{1 + |x_k - \bar{x}_k|} \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} \tag{13}$$

The series is residues for a convergent series. Hence by taking n large, $\rho(x, \bar{x})$ can be arbitrarily small.

By a similar argument in problem 6, it's safe to claim N is a finite ϵ-net for A. Therefore, A is totally bounded. And it's a subset in a complete metric space, which means it's compact. $\square$