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Problem 1

Let $S[a, b]$ be the set of all the measurable function taking finite value almost everywhere. Denote

$$\rho(f, g) := \int_a^b \frac{|f-g|}{1+|f-g|} d\mu \tag{1}$$

Prove that

1. convergence in ρ is equivalence in convergence in measure.
2. prove this metric space is complete.

***Solution* 1.a** $f_n \to f$ under $\rho \Leftrightarrow \forall \epsilon > 0, \int_{M_{\epsilon,n} \cup M_{\epsilon,n}^c} \frac{|f_n-f|}{1+|f_n-f|} d\mu \to 0$ where $M_{\epsilon,n} := \{x \in [a, b] : |f_n(x) - f(x)| > \epsilon\}$ and the complementary is defined with respect to $[a, b]$. Whereas $\frac{|f_n-f|}{1+|f_n-f|}$ is non-negative on $[a, b]$, one has $I_n := \int_{M_{\epsilon,n}} \frac{|f_n-f|}{1+|f_n-f|} d\mu \to 0$ as well as $\int_{M_{\epsilon,n}^c} \frac{|f_n-f|}{1+|f_n-f|} d\mu \to 0$. Also note that $\frac{|f_n-f|}{1+|f_n-f|} \geq \frac{\epsilon}{1+\epsilon} > 0$ on $M_{\epsilon,n}$ for the monotoncity of $x \mapsto 1 - \frac{1}{1+x}$. Combined we get

$$0 = \lim_{n\to\infty} I_n \geq \frac{\epsilon}{1+\epsilon} \lim_{n\to\infty} \mu(M_{\epsilon,n}) \geq 0 \tag{2}$$

We have $\lim_{n\to\infty} \mu(M_{\epsilon,n}) = 0$ which would give the claim. □

***Solution* 1.b**

Let f_n be a Cauchy sequence in $S[a, b]$, and from the proposition 1, we define f to be the pointwise limit of f_n under measure, which means that $|f_n(x) - f(x)| \to 0$ as $n \to \infty$ almost everywhere in $[a, b]$. So we only need to prove that this f is inside $S[a, b]$.

$$|f(x)| = |f(x) - f_n(x) + f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$$

$\forall x \in [a, b]$ let $\epsilon = 0.01$, we can take an n_0 such that $\forall n \geq n_0$, we have $|f(x) - f_n(x)| \leq \epsilon$. And use $f_n \in S[a, b]$ to conclude $|f_n|$ is bounded almost everywhere by some $M < \infty$. Hence $|f|$ is bounded by $M + 0.01$ almost everywhere. □

Problem 2

$L^p(E)$, $1 \leq p < \infty$, define

$$\rho(f, g) = (\int_E |f-g|^p d\mu)^{1/p} \tag{3}$$

prove ρ is a metric.

***Solution* 2.a** First we restrict our definition of the metric to classes of function, where a class if defined to be functions that equal to each other almost everywhere on E. This would satisty $\rho(f, g) \geq 0$ particularly for the equality hold implying $f = g$. $\rho(f, g) = \rho(g, f)$ is obvious. So it only suffices to verify the triangular inequality, which also follows from Minkovskii inequality

$$\rho(f,h)+\rho(h,g) = (\int_E |f-h|^p d\mu)^{1/p} + (\int_E |h-g|^p d\mu)^{1/p} \geq (\int_E (|f-h|+|h-g|)^p d\mu)^{1/p} \tag{4}$$

$$\geq (\int_E |f-g|^p d\mu)^{1/p} \tag{5}$$

where the second inequality follows from $|a|+|b| \geq |a-b|$.
Now we prove the Minkovskii inequality. Let $1/p+1/q=1$

$$\begin{aligned}
\int_a^b |f+g|^p \mathrm{d}t &= \int_a^b |f+g|^{p-1}|f+g|\mathrm{d}t \\
&\leq \int_a^b |f+g|^{p-1}(|f|+|g|)\mathrm{d}t \\
&\leq (\int_a^b |f|^p \mathrm{d}t)^{1/p}(\int_a^b |f+g|^{q(p-1)}\mathrm{d}t)^{1/p} + (\int_a^b |g|^p \mathrm{d}t)^{1/p}(\int_a^b |f+g|^{q(p-1)}\mathrm{d}t)^{1/p} \\
&= (\int_a^b |f+g|^p \mathrm{d}t)^{1/q}((\int_a^b |f|^p \mathrm{d}t)^{1/p} + (\int_a^b |f|^p \mathrm{d}t)^{1/p})
\end{aligned}$$

□

Problem 3

Suppose (X,ρ) is a metric space, $A \in X$ is a closed set $\iff$ A contains the limit points for all convergent sequence in A.

***Solution* 3.a** $\Rightarrow$: Let $A \in X$ be close, then A^c is open. For all convergent sequence $\{x_n\} \in A$, suppose it converge to a limit point $x \in A^c$. Due to the openess assumption, there is a ball $B(x,r)$ such that $B(x,r) \cap A = \emptyset$, hence $B(x,r) \cap \{x_n\} = \emptyset$, which would implies $\{x_n\} \nrightarrow x$, a contradiction.

$\Leftarrow$: Suppose $A \in X$ is not close, then A^c is not open. Which means there is some $y \in A^c$ such that $\forall r > 0, B(y,r) \cap A \neq \emptyset$. Take $r = \frac{1}{n}$, we obtain a sequence $\{y_n\}$ such that $y_n \to y$ and $\{y_n\} \in A, y \in A^c$ which contradicts A contains all limit points for convergent sequence. □

Problem 4

On $C[a,b]$, define

$$\rho(x,y) := \sup_{t \in [a,b]} |x(t)-y(t)| \tag{6}$$

prove the metric space is complete.

***Solution* 4.a** Let f_n be a Cauchy sequence in $C[a,b]$. Let $x \in [a,b], \epsilon > 0$, we have

$$|f_n(x)-f_m(x)| \leq d(f_n,f_m) \leq \epsilon, \quad \forall n,m \geq n_0 \in \mathbb{N} \tag{7}$$

So $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$ which is convergent. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Since $\forall x \in [a,b]$,

$$|f_n(x)-f_m(x)| \leq \epsilon \quad \forall n,m \geq n_0 \in \mathbb{N} \tag{8}$$

Let $m \to \infty$ one has

$$|f_n(x)-f(x)| \leq \epsilon \quad \forall n,m \geq n_0 \in \mathbb{N} \tag{9}$$

which indicates that

$$d(f_n,f) \leq \epsilon \quad \forall n \geq n_0 \tag{10}$$

Let $n \to \infty$ to obtain $f_n \to f$ under ρ.

Next it suffices to show f is continuous. Which means $\forall \epsilon > 0$, we need to find δ such that

$$|f(x) - f(x_0) < \epsilon, |x - x_0| < \delta \tag{11}$$

Use $3 - \epsilon$ technique,

$$\begin{aligned}|f(x) - f(x_0)| &= |f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0)| \\ &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon\end{aligned}$$

for the last inequality, we need to take the largest n_0 for both $|f_n(x) - f(x)| \leq \epsilon$ and $|f_n(x_0) - f(x_0)| \leq \epsilon$ as well as a δ such that $|f_m(x) - f_m(x_0) \leq \epsilon/3$. And we can do this for the claim we have proved and the fact $f_n \in C[a, b]$. □

Problem 5

(X, ρ), let $A \in X$ prove that $\text{diam}A = \text{diam}\bar{A}$.

***Solution* 5.a** $A \subset \bar{A} \Rightarrow \text{diam}A \leq \text{diam}\bar{A}$

For the equality to hold, it only suffices to prove the inequality from the other direction. $\forall x, y \in \bar{A}, \exists \{x_n\}, \{y_n\} \in A$ such that $x_n \to x, y_n \to y$. One has the following inequality from triangular inequality

$$\rho(x, y) \leq \rho(x, x_n) + \rho(y, y_n) + \rho(x_n, y_n) \tag{12}$$

Taking the supremum over $x, y \in \bar{A}$ and take the limit we have

$$\text{diam}\bar{A} \leq \lim_{n\to\infty} \rho(x_n, y_n) \leq \text{diam}A \tag{13}$$

for which the claim follows. □

Problem 6

Let (X, ρ) be a metric space, prove that,

$E \in X$ is no-where dense $\iff \forall \overline{B(x, r)}$, there is an open ball $B(x', r') \in B(x, r)$ such that $\overline{B(x', r')} \cap \bar{E} = \emptyset$.

***Solution* 6.a** $\Rightarrow$: Suppose it's not true, then we have a close ball $\overline{B(x, r)}$ such that $\forall B(x', r') \in B(x, r), \overline{B(x', r')} \cap \bar{E} \neq \emptyset$. Then let $r_n = \frac{1}{n}$, we can construct a sequence $\{x_n\} \in B(x', r') \subset X$ where $\lim_{n\to\infty} x_n = x \in \overline{B(x', r')}$ and $x \in E$ as well. This is contradiction to the fact that E shall be no-where dense.

$\Leftarrow$: This is trivial because if E is not nowhere dense, its closure $\bar{E}$ has an interior point x, which means there's an open ball $B(x, r) \subset \bar{E}$. Any closed ball contained in this open ball in also a subset of $\bar{E}$ which contradicts $\overline{B(x', r')} \cap \bar{E} = \emptyset$.

□