HOMEWORK NAME

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1. Problem 1

   (A) Let the random variables $Y_1, Y_2, \ldots$ be independent Binomial random variables $\mathrm{Bin}(n, p)$, i.e., each with n trials and with success probability $p \in (0,1)$. Show that $X_t := \min\{Y_1, \ldots, Y_t\}$ with $X_0 = n$ forms a Markov-Chain. Is the same true for $Z_t := \max\{Y_1, \ldots, Y_t\}$ with $Z_0 = 0$?

   (B) Prove or disprove: If $(X_t)_{t \geq 0}$ forms a Markov Chain, so does $(Y_t)_{t \geq 0}$ with $Y_t := |X_t|$.

   (C) Let S be a finite set. The sequence $(X_t)_{t \geq 0}$ of random variables with $X_t \in S$ satisfies for all $t \in \mathbb{N}$, $I \subseteq \{0, \ldots, t-1\}$ and $i, j, s_k \in S$ the condition

$$\mathbb{P}(X_{t+2} = j \mid X_t = i \text{ and } \forall k \in I : X_k = s_k) = \mathbb{P}(X_{t+2} = j \mid X_t = i).$$

   Is $(X_t)_{t \geq 0}$ a Markov Chain?

Solution. (A) First with a little obervation we can conclude that X_n is a monotone random variables.

$$\begin{aligned} X_t &= \min\{Y_1, Y_2, \ldots Y_t\} \\ &= \min\{\min\{Y_1, Y_2, \ldots, Y_{t-1}\}, Y_t\} \\ &= \min\{X_{t-1,} Y_t\} \end{aligned}$$

So X_{t-1} and Y_t actually completely determine X_t and Y_t is independent from X_{t-1}. Which implies that the X_t is a Markov Chain.

For a feasible integer finite sequence $0 \leq j \leq i \leq i_{t-1}, \cdots \leq i_1 \leq i_0 \leq n$ in the state space.

$$\begin{aligned} &\mathbb{P}(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0) \\ =&\mathbb{P}(\min\{X_t, Y_{t+1}\} = j | X_t = i, \ldots, X_0 = i_0) \\ &\text{Since } Y_i \text{ are independent random variables, the value of it does not depend on } X_t. \\ =&\mathbb{P}(\min\{X_t, Y_{t+1}\} = j | X_t = i) \\ =&\mathbb{P}(X_{t+1} = j | X_t = i) \end{aligned}$$

Which implies $\{X_t\}$ is a Markov Chain.

□

Solution. (B)

I will argue that it is a Markov Chain. X_t is a Markov chain,

$$\begin{aligned} &\mathbb{P}(Y_{t+1} = j | Y_t = i, Y_{t-1} = i_{t-1}, \ldots, Y_0 = i_0) \\ =&\mathbb{P}(X_{t+1} = \pm j | X_t = \pm i, X_{t-1} = \pm i_{t-1}, \ldots, X_0 = \pm i_0) \\ =&\mathbb{P}(X_{t+1} = \pm j | X_t = \pm i) \\ =&\mathbb{P}(Y_{t+1} = j | Y_t = i) \end{aligned}$$

□

Solution. (C) No, X_t is not a Markov Chain. But $X_{2k}, k = 0, 1, \dots, [n/2]$ and $X_{2k+1}, k = 0, 1, \dots, [n/2]$ are both Markov Chain. □

2. Problem 2

   Let k be a fixed positive integer. A stochastic process $(X_n)_{n\in\mathbb{N}}$ taking values in a discrete state space $\mathcal{X}$ is called a *k-th order Markov chain* if for all $n \in \mathbb{N}$ and all feasible sequences $x_0, x_1, \dots, x_{n+k} \in \mathcal{X}$ we have

$$\begin{aligned}&\mathbb{P}(X_{n+k} = x_{n+k} \mid X_0 = x_0, X_1 = x_1, \dots, X_{n+k-1} = x_{n+k-1})\\ &\quad = \mathbb{P}(X_k = x_{n+k} \mid X_n = x_n, X_{n+1} = x_{n+1}, \dots, X_{n+k-1} = x_{n+k-1}).\end{aligned}$$

   In other words, the transition to the next state depends only on the previous k states. Show that we can can 'embed' $(X_n)_{n\in\mathbb{N}}$ into a 'normal' (i.e., of order $k = 1$) Markov chain $(Z_n)_{n\in\mathbb{N}}$ with a larger state space, in the sense that X_n can be recovered from Z_n.

Solution. Consider the space $\mathcal{X}^k$ and $Z_n \in \mathcal{X}^k$ where $Z_0 = (X_0, X_1, \dots, X_{k-1})^T$, $Z_n = (X_n, X_{n+1}, \dots, X_{n+k-1})^T$. we first oberve that $Z_n = (X_n, X_{n+1}, \dots X_{n+k-1})^T$ and $Z_{n+1} = \mathbf{N}Z_n + (X_{n+k}, 0, \dots, 0)^T$ where Z_{n+1} is completely determined by Z_n, $\mathbf{N}$ and X_{n+k}

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & 0\dots & 0\\ 0 & 0 & 1\dots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & 0\dots & 1\\ 0 & 0 & 0\dots & 0 \end{pmatrix}$$

And X_{n+k} is independent from any entreis in Z_n, which implies it is independent from Z_n. Hence Z_{n+1} is a Markov Chain.

To recover X_n from Z_n, just take the first entry of Z_n

□

3. Problem 3

   (A) A Markov chain $(X_n)_{n\geq 0}$ with states $0, 1, 2$, has the transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6}\\ 0 & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

   If $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = 1/4$, find $\mathbb{E}(X_3)$.

   (B) Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

Solution. (A)

Obseve that $\mathbb{P}(X_0 = 2) = 1/2$, first compute P^3 we obtain

$$P^{(3)} = \begin{pmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{pmatrix}.$$

If $X_0 = 0$, $\mathbb{E}(X_3) = 11/54 + 47 * 2/108 = 29/27$. If $X_0 = 1$, $\mathbb{E}(X_3) = 4/27 + 11 * 2/27 = 26/27$. If$X_0 = 2$, $\mathbb{E}(X_3) = 2/9 + 13 * 2/36 = 17/18$. Then with the linearity and Law of total probability, we can conclude that $\mathbb{E}(X_3) = 1/4 * (29/27) + 1/4 * (26/27) + 1/2 * (17/18) = 53/54$ □

Solution. (B) Think it as a Markov Chain with 2 state, coin 1 or coin 2. The transition matrix will be

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}, \quad P^{(2)} = \begin{pmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{pmatrix}$$

Which implies $\mathbb{P}(\{\text{coin flipper on the third day is } 1\}) = (0.67 + 0.66)/2 = 0.665$

And to compute the probability if the coid flipped on Friday is 1, which is the same as the coin flipped on Saturday is coin 1. we neet to compute P^4 and take the first entry of the matrix.

$$P^4 = \begin{pmatrix} 0.6667 & 0.3333 \\ 0.6666 & 0.3334 \end{pmatrix}$$

Which implies

$$\mathbb{P}(\text{Friday flip end up with head}|\text{Monday flip is } 1) = 0.6667$$

□

4. Problem 4

(A) Let X_n be the weather on day n in Atlanta, GA, which we assume is either: 1 = rainy, or 2 = sunny. Even though the weather is not exactly a Markov chain, we can propose a Markov chain model for the weather by writing down a transition probability matrix

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

Determine whether the this Markov chain is irreducible and/or aperiodic (justify your answer). Furthermore, find the stationary distribution.

(B) Suppose there are three types of laundry detergent, 1, 2, and 3, and let X_n be the brand chosen on the n-th purchase. Customers who try these brands are satisfied and choose the same thing again with probabilities 0.8, 0.6, and 0.4, respectively. When they change they pick one of the other two brands at random. Model this as a Markov chain: write down the transition probability matrix P, and find the stationary distribution.

Solution. (A)

It is irreducible and aperiodic. It is irreducible because for each day the wheather can always go from state 1 to state 2 with probability 0.4 and go from state 2 to state 1 with probability 0.8. And this implies there's a loop between each state, which suffices to show this irreducible chain is aperiodic.

To find the stationary distribution. We need to solve the system $\Pi = \Pi\mathbf{P}$, together with $\Pi_1 + \Pi_2 = 1$, where Π is a distribution of the state. And through easy computation $\Pi = (1/3, 2/3)$ $\square$

Solution. (B)

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

Solve the system $\Pi = \Pi\mathbf{P}$, where Π is a probability distribution on the state space. We obtain $\Pi = (6/11, 3/11, 2/11)$ $\square$