Shenduo Zhang

1. Problem 1

(A) In unprofitable times corporations sometimes suspend dividend payments. Suppose that after a dividend has been paid the next one will be paid with probability 0.9, while after a dividend is suspended the next one will be suspended with probability 0.6. In the long run what is the fraction of dividends that will be paid?

(B) Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of k pairs of running shoes, what proportion of the time does he run barefooted?

(C) A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy?

Solution. (A)

Let state 1 be the dividend is paid and state 2 be the dividend is suspended. The corresponding transition matrix will be

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix}$$

In the long run we are interested in the stable distribution π of the process. Solve linear system $\pi = \pi P$ with condition $||\pi||_1 = 1$, we obtain $\pi = (0.8, 0.2)$. So $\frac{4}{5}$ of dividends will be paid.

□

Solution. (B)

This is equivalent with the model of random walk on one dimension with finite states. Let the number of shoes at the front door be X_t, then X_t travels on the integer points between 0 and k. And he has equal chance to move to each side when he's not in 0 or k. On 0 or k state, it has half probability to stay and to move. And the proportion of time that he runs barefooted is equal to $\mathbb{P}(X_t = 0) + \mathbb{P}(X_t = k)$. And the transition matrix of random process X_t is

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & \dots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 & 0 \\ 0 & 1/2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \dots & 0 & 1/2 & 1/2 \end{bmatrix}_{(k\times k)}$$

And we solve for the stationary distribution $\pi P = \pi$ with $||\pi||_1 = 1$, we obtain $\pi = (1/k, 1/k, \dots, 1/k)_k$. Hence the proportion that he runs with barefoot is $2/k$.

$\square$

Solution. (C)

This is a random process with 3 states. The transition matrix looks like

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

And let π be its stationary distribution, where $||\pi||_1 = 1$. Solve for $\pi P = \pi$, one obtain $\pi = (1/2, 1/4, 1/4)$.

$\square$

2. Problem 2

   (A) Jobs arrive at a central server. We distinguish complex and simple jobs. In each step, independently of all other jobs, with probability p a complex task arrives. Because the server is not powerful enough, it crashes if two complex tasks arrive in a row. How many steps does it take, on average, for the server to crash?

   (B) You throw five dice and set aside those dice that are sixes. Throw the remaining dice and again set aside the sixes. Continue until you get all sixes. How many turns does it take, on average, to get all sixes?

Solution. (A)

Let X_k be the numbers of steps it takes to crash when there were k complex tasks in a row. We are interested in $\mathbb{E}(X_0)$

$$\begin{aligned} &\mathbb{E}(X_2) = 0 \\ &\mathbb{E}(X_1) = p(\mathbb{E}(X_2) + 1) + (1-p)(\mathbb{E}(X_0) + 1) \\ &\mathbb{E}(X_0) = p(\mathbb{E}(X_1) + 1) + (1-p)(\mathbb{E}(X_0) + 1) \end{aligned}$$

We abtain $\mathbb{E}(X_0) = \dfrac{p+1}{p^2}$

$\square$

Solution. (B)
Let the number of dices we throw be the state space, we can have a Markov Chain. Denote the times we through to get all six when we had k dices to through as X_k.

$$
\begin{aligned}
\mathbb{E}X_1 &= 5 \\
\mathbb{E}X_2 &= \binom{2}{0}\left(\frac{1}{6}\right)^{0}\left(1-\frac{1}{6}\right)^{2-0}(\mathbb{E}X_2+1) \\
&\quad + \binom{2}{1}\left(\frac{1}{6}\right)^{1}\left(1-\frac{1}{6}\right)^{2-1}(\mathbb{E}X_1+1) + \binom{2}{2}\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{2-2}(0+1) \\
\mathbb{E}X_3 &= \binom{3}{0}\left(\frac{1}{6}\right)^{0}\left(1-\frac{1}{6}\right)^{3-0}(\mathbb{E}X_3+1) + \binom{3}{1}\left(\frac{1}{6}\right)^{1}\left(1-\frac{1}{6}\right)^{3-1}(\mathbb{E}X_2+1) \\
&\quad + \binom{3}{2}\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{3-2}(\mathbb{E}X_1+1) + \binom{3}{3}\left(\frac{1}{6}\right)^{3}\left(1-\frac{1}{6}\right)^{3-3}(0+1) \\
\mathbb{E}X_4 &= \binom{4}{0}\left(\frac{1}{6}\right)^{0}\left(1-\frac{1}{6}\right)^{4-0}(\mathbb{E}X_4+1) + \binom{4}{1}\left(\frac{1}{6}\right)^{1}\left(1-\frac{1}{6}\right)^{4-1}(\mathbb{E}X_3+1) \\
&\quad + \binom{4}{3}\left(\frac{1}{6}\right)^{3}\left(1-\frac{1}{6}\right)^{4-3}(\mathbb{E}X_2+1) + \binom{4}{2}\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{4-2}(\mathbb{E}X_1+1) \\
&\quad + \binom{4}{4}\left(\frac{1}{6}\right)^{4}\left(1-\frac{1}{6}\right)^{4-4}(0+1) \\
\mathbb{E}X_5 &= \binom{5}{0}\left(\frac{1}{6}\right)^{0}\left(1-\frac{1}{6}\right)^{5-0}(\mathbb{E}X_5+1) + \binom{5}{1}\left(\frac{1}{6}\right)^{1}\left(1-\frac{1}{6}\right)^{5-1}(\mathbb{E}X_4) \\
&\quad + \binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{5-2}(\mathbb{E}X_3) + \binom{5}{3}\left(\frac{1}{6}\right)^{3}\left(1-\frac{1}{6}\right)^{5-3}(\mathbb{E}X_2) \\
&\quad + \binom{5}{4}\left(\frac{1}{6}\right)^{4}\left(1-\frac{1}{6}\right)^{5-4}(\mathbb{E}X_1) + \binom{5}{5}\left(\frac{1}{6}\right)^{5}\left(1-\frac{1}{6}\right)^{5-5}(0+1)
\end{aligned}
$$

And by Mathematica solving the system above, we can abtain $\mathbb{E}X_5 = \dfrac{3698650986}{283994711} \approx 13$. (IT LOOKS RIDICULERS, ISN'T IT.)

$\square$

3. Problem 3

Two players, A and B, play the game of matching pennies: at each time n, each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player A wins the penny. If the pennies do not match (one heads and one tails), Player B wins the penny. Suppose the players have between them a total of 5 pennies. If at any time one player has all of the pennies, to keep the game going, he gives one back to the other player and the game will continue.

(A) Formulate this game as a Markov chain. Is the chain irreducible and aperiodic?

(B) If Player A starts with 3 pennies and Player B with 2, what is the probability that A will lose his pennies first?

Solution. (A)

Let the number of coins player A has be $\{X_t\}_{t\geq 0}$, then $\{X_t\}_{t\geq 0}$ is a Markov chain whose state space is $\{1, 2, 3, 4\}$. Because if one lost his last coin, one will receive a penny from the other player which is equivalent to stay in the current state. And two player have equal chances to win or to lose, hence the transition matrix has the form

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

And this Markov Chain is irreducible in the first place because it's always possible to go to any state from any state in the state space. Second there's two loops in such a irreducible chain, hence it's also aperiodic.

□

Solution. (B)

Suppose we have a state space $\{0, 1, 2, 3, 4, 5\}$, where state 0 and 5 is absorbing state. And for each non-absorbing state in space, it has equal probability to plus 1 or minus 1. So we are interested what is the probability that it reaches absorbing state 0 before he reaches absorbing state 5. Denote the probability that it reaches 0 before it reaches 6 on the condition that it starts at state i as $p_i \in \{0, 1, 2, 3, 4, 5\}$. And we observe the forllowing property,

$$p_i = \frac{p_{i-1} + p_{i+1}}{2}, \quad \forall i \in \{1, 2, 3, 4\}$$

$$p_0 = 1, \quad p_6 = 0$$

With easy observation that p_i forms an arithmetic progression. Hence $p_3 = \frac{2}{5}$.

□

4. Problem 4

Anna (A) and Bob (B) play the following game (starting with A's turn). When it's A's turn, she sequentially throws dices until either an odd number shows up (then it's B's turn) or three consecutive even numbers shows up (in which case the game ends and A wins). When it's B's turn, she throws a dice once. If the outcome is a six then then game ends and B wins; otherwise it's A turn. Model this as a Markov chain, and

(A) Calculate the probability that A wins.

(B) Determine how many turns the game lasts on average.

Solution. Model:

We construct our state space as $\Omega = \{SIX, B, A, E, EE, EEE\}$. SIX stands for B get a six in his turn and he win which is an absorbing state. B means that it is the player B's turn to throw a dice. A means that it is player A's turn to throw a dice. E stands for it's still player A's turn but he has already obtained one even number in a row. EE and EEE stands for it's still A's turn but he has outcome of 2 and 3 even numbers in a row respectively. And EEE is an another absorbing states. Every time a dice is thrown, we make one move in the state space Ω. Denote this Markov Chain as $\{\omega_t\}_{t\geq 0}$, $\omega_0 = A$. The transition matrix P has the form

$$P = \begin{array}{c} \\ SIX \\ B \\ A \\ E \\ EE \\ EEE \end{array} \begin{array}{c} \begin{array}{cccccc} SIX & B & A & E & EE & EEE \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{5}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

□

Solution. (A)

It will be easier in notation if we denote the state space as $\{0, 1, 2, 3, 4, 5\}$ respectively, with out changing the model. Let $p_i = \mathbb{P}(\text{The Markov Chain reaches state 5 before it reaches state } 0|i$ In our game the chain starts at state 3, so we are actually interested in p_3. From our model and the Markov property, the p_i satisfy the following condition,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & -1 & 5/6 & 0 & 0 & 0 \\ 0 & 1/2 & -1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & -1 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solve this system we can obtain a solution $(0, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{9}{13}, 1)$. So $p_3 = \frac{7}{13}$.

□

Solution. (B) We use the model in (A). Let h_i denote the expected value of the numbers of turns we played before somebody win when we start at state i. Hence $h_0 = h_5 = 0$. With Markov property and the model we have, h_i satisfy the following system,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & -1 & 5/6 & 0 & 0 & 0 \\ 0 & 1/2 & -1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & -1 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

We claim the solve is $\left(0, \frac{118}{13}, \frac{126}{13}, \frac{108}{13}, \frac{72}{13}, 0\right)$. Hence, $h_3 = \frac{108}{13}$

□

5. Problem 5

I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

(A) If the probability of rain is each time independently p, what is the probability that I get wet?

(B) Suppose that current estimates show that $p = 0.6$ in Atlanta. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

Solution. (A)

To solve this problem, first we observe that you go wet when and only when you left all the umbrellas at home and got wet on your way home. Or you left all your umbrellas at office and get wet on your way to the office. Denote the numbers of umbrellas that you left at home every time you go out before departure as X_t, then $\{X_t\}_{t\geq 0} \in \{0, 1, 2, 3, 4\}$ is a Markov Chain. We are first interested in probability you are in state 0 and state 4. Because that's where you have some probability to get wet. To obtain that, we first write the transition matrix for this Markov Chain,

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & 0 \\ p(1-p) & (1-p)^2+p^2 & p(1-p) & 0 & 0 \\ 0 & p(1-p) & (1-p)^2+p^2 & p(1-p) & 0 \\ 0 & 0 & p(1-p) & p^2+(1-p)^2 & p(1-p) \\ 0 & 0 & 0 & p(1-p) & p^2+(1-p)^2+(1-p)p \end{pmatrix}.$$

□

And solve for stationary distribution for this Markov Chain $\pi P = \pi, ||\pi||_1 = 1$. We obtain $\pi = (\frac{1-p}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p})$.

So in the long term, every time before our departure from home, we have probability $\frac{1-p}{5-p}$ that we don't have any umbrellas to take. Hence the the probability that we get on the way to work is $\frac{(1-p)}{5-p} * p$. And the probability that we get wet on the way home is the same as we left all the umbrellas at home and we didn't took one to work, just then on the way home it rained. So the probability will be $\frac{1}{5-p} * (1-p) * p$.

Hence the probability that we get wet is $\frac{2p(1-p)}{5-p}$

Solution. (B)

We use the model from (A) again, but now we have $k \geq 0$(the result is true even for $k = 0$) umbrellas, hence $\{X_t\}_{t\geq 0} \in \{0, 1, 2, \ldots, k\}$. So we will have a P with dimension k.

And denote the distribution π as $(\pi_1, \pi_2, \ldots, \pi_k)$. And we are interested in the null-space of $P^T - I$.

$$P^T - I = \begin{pmatrix} -p & p-p^2 & & & & & \\ p & -2p+2p^2 & p-p^2 & & & & \\ & p-p^2 & -2p+2p^2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & -2p+2p^2 & p-p^2 & \\ & & & & p-p^2 & -2p+2p^2 & p-p^2 \\ & & & & & p-p^2 & -p+p^2 \end{pmatrix}_{(k+1)\times(k+1)}$$

It's a nice recurrence system that we can solve by first add row 1 to row 2, and then add row 2 to row 3 and so on.

Eventually we will end up with solving the null space of the matrix down here,

$$\begin{pmatrix} -p & p-p^2 & & & & & \\ & -p+p^2 & p-p^2 & & & & \\ & & -p+p^2 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & -p+p^2 & p-p^2 & \\ & & & & & -p+p^2 & p-p^2 \\ & & & & & 0 & 0 \end{pmatrix}_{(k+1)\times(k+1)}$$

And the null space is

$$\pi^T = c \begin{pmatrix} 1-p \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}_{(k+1)\times 1}$$

And $||\pi^T||_1 = 1$ implies $\pi = (\frac{1-p}{k-p}, \frac{1}{1-p}, \ldots, \frac{1}{1-p}, \frac{1}{1-p})_{1\times(k+1)}$.

Hence with similar analysis as (A), the probability that we get wet in the long term while we have k umbrellas will be $\dfrac{2p(1-p)}{k-p+1}$.

Let $p = 0.6$ solve k from inequality $\dfrac{2p(1-p)}{k-p+1} < 0.1$, we obtain $k > 4.4$. Hence we need 5 umbrella at least.

$\square$