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Problem 1

P123 2.4.1

Let p be a semi linear functional on a real linear space $\mathcal{X}$, prove that:

1. $p(\theta) = 0$;
2. $p(-x) \geq -p(x)$;
3. *For arbitrary given $x_0 \in \mathcal{X}$, there must exists a real linear functinal f, satisfy $f(x_0) = p(x_0)$, and $f(x) \leq p(x)$ $(\forall x \in \mathcal{X})$.*

Solution ***1.a***

$$p(\theta) = 2p(\theta) \Rightarrow p(theta) = 0 \tag{1}$$

□

Solution ***1.b***

$$p(\theta) = p(x - x) \leq p(x) + p(-x) \tag{2}$$

□

Solution ***1.c*** Define $\mathcal{X}_0 = span\{x_0\}$, then if we can define such a functional on $\mathcal{X}_0$, by Hahn-Banach's theorem, the claim follows. We define $f(ax_0) = ap(x_0)$. Then we claim f is a linear functional that is dominated by p over $\mathcal{X}_0$. When $a \geq 0$, $f(ax) \leq p(ax)$. When $a \leq 0$, $f(-|a|x) = -p(|x|) \leq p(-|a|x) \leq p(ax)$. □

Problem 2

P124 2.4.4

Let $\mathcal{X}$ be a linear normed space, $\{x_n\}(n = 1, 2, \dots)$ be a sequence in $\mathcal{X}$. If $\forall f \in \mathcal{X}^*$, the sequence $\{f(x_n)\}$ is bounded, prove that: $\{x_n\}$ is bounded in $\mathcal{X}$.

Solution ***2.a*** Consider $span\{x_n\}$, then by Hahn-Banach's theorem, there $\exists f \in \mathcal{X}^*$ such that $\|x_n\| = f(x_n)$. Hence $\|x_n\| \leq \|f\| \leq M, \forall n$. □

Problem 3

P124 2.4.5

Let $\mathcal{X}_0$ be a closed subspace of a linear normed space $\mathcal{X}$, prove that:

$$\rho(x, \mathcal{X}_0) = \sup\{|f(x)|; f \in \mathcal{X}^*, \|f\| = 1, f(\mathcal{X}_0) = 0\} \quad (\forall x \in \mathcal{X}), \tag{3}$$

where $\rho(x, \mathcal{X}_0) = \inf_{y \in \mathcal{X}_0} \|x - y\|$.

Solution ***3.a*** From Hahn-Banach's theorem, we know there exists $f \in \mathcal{X}^*$ such that $f(x_0) = \rho(x_0, \mathcal{X})$. We have the following holds for any $x \notin \mathcal{X}_0$ because $\mathcal{X}_0$ is the kernel of the bounded linear functional f,

$$|f(x)| \leq \|f\| \rho(x, \mathcal{X}_0) \tag{4}$$

Hence the equality follows. □

Problem 4

P124 2.4.6

Let $\mathcal{X}$ be a linear normed space. Given n linear independent elements $x_1, x_2, \dots, x_n$ in $\mathcal{X}$ and n number $C_1, C_2, \dots, C_n$ in a number field $\mathbb{K}$, prove that: for the existence of $f \in \mathcal{X}^*$ such that $f(x_k) = C_k (k = 1, 2, \dots, n)$ and $\|f\| \leq M$, one must and only need to have for arbitrary

$a_1, a_2, \ldots, a_n \in \mathbb{K}$,

$$\left|\sum_{k=1}^{n} a_k C_k\right| \leq M \left\|\sum_{k=1}^{n} a_k x_k\right\|. \tag{5}$$

Solution ***4.a*** $\Rightarrow$:

$$\left|\sum_{k=1}^{n} a_k C_k\right| = \left|\sum_{k=1}^{n} a_k f(x_k)\right| \leq \|f\| \left\|\sum_{k=1}^{n} a_k x_k\right\| \leq M \left\|\sum_{k=1}^{n} a_k x_k\right\| \tag{6}$$

$\Leftarrow$: Let $M = span\{x_n, n \geq 1\}$. Let x_k be the basis for M, define the dual basis as $f(x_k) = c_k, \forall 1 \leq k \leq n$. The functional on the closed subspace M can be expressed as a linear combination of a_k and c_k. Hence for any $x \in M, x = \sum_{k=1}^{n} a_k x_k$,

$$|f_0(x)| = \left|\sum_{k=1}^{n} a_k C_k\right| \leq M\|x\|. \tag{7}$$

Then by Hahn-Banach's theorem, we can extend f to be a linear functional reserving its norm. □

Problem 5

P124 2.4.7

Given n linear independent elements $x_1, x_2, \ldots, x_n$ in a linear normed space $\mathcal{X}$. Prove that: $\exists f_1, f_2, \ldots, f_n \in \mathcal{X}^*$, such that

$$f_i(x_j) = \delta_{ij}(i, j = 1, 2, \ldots, n). \tag{8}$$

Solution ***5.a*** Consider the linear subspace M_i spaned by all those elements without x_i. Then, by Hahn-Banach's theorem, we know $\exists f_i \in \mathcal{X}^*$ such that $\|f_i\| = 1$ and $f_i(x_i) = d$, $f(M_j) = 0$. Divided the functional f_i by $\rho(x_i, M_i)$, we prove the result. □

Problem 6

P124 2.4.12

Let C be a convex set in a linear normed space. Let $x_0 \in \dot{C}, x_1 \in \partial C$, $x_2 = m(x_1 - x_0) + x_0 (m > 1)$. Prove that: $x_2 \notin C$.

Solution ***6.a*** Suppose it's inside C, it only suffices to prove then x_1 can not be on the boundary. x_0 is an interior point, hence exists $B(x_0, \delta) \subset C$. For all $y \in B(x_0, \delta)$, the convex combination of y and x_2 shall be inside C. Denote $z = \lambda x_2 + (1-\lambda)y$. Then $z - x_1 = (1-\lambda)(y - x_0)$. Let $d = (1-\lambda)\delta$, then for any such z inside $B(z_1, d)$, it can be expressed as a convex combination of y and x_2, which means it's inside C. This would imply a contradiction of x_1 being on ∂C. Indeed,

$$z = x_1 + (1-\lambda)(y - x_0) = \lambda x_2 + (1-\lambda)x_0 + (1-\lambda)(y - x_0) = \lambda x_2 + (1-\lambda)y \in C. \tag{9}$$

□

Problem 7

P124 2.4.13

Denote M to be a closed convex set in a linear normed space $\mathcal{X}$, prove that: $\forall x \in \mathcal{X} \backslash M$, there must $\exists f_1 \in \mathcal{X}^*$ satisfying $\|f_1\| = 1$, and

$$\sup_{y \in M} f_1(y) \leq f_1(x) - d(x), \tag{10}$$

where $d(x) = \inf_{z \in M} \|x - z\|$.

Solution ***7.a*** From the geometry form of Hahn-Banach's theorem, there exists $f \in \mathcal{X}^*$ such that H_f^a split M and $B(x, \delta)$, where $\delta = \rho(x, M)$ since x is an interior point.

$$\sup_{t \in M} f(t) \leq \inf_{\|e\|=1} f(x - \delta e) = f(x) - \delta \sup_{\|e\|=1} f(e) = f(x) - \delta \|f\| \tag{11}$$

Let $f_1 = \frac{f}{\|f\|}.$, we conclude the claim. □

Problem 8

P124 2.4.14

Let M be a closed convex set in a linear normed space $\mathcal{X}$, prove that

$$\inf_{z \in M} \|x - z\| = \sup_{f \in \mathcal{X}^*, \|f\|=1} \left\{ f(x) - \sup_{z \in M} f(z) \right\} \quad (\forall x \in \mathcal{X}). \tag{12}$$

Solution ***8.a***

From previous problem, we have proved the existence of $f_1 \in \mathcal{X}^*$ such that,

$$d(x) \leq f_1(x) - \sup_{x \in M} f_1(x). \tag{13}$$

And $\forall x \in \mathcal{X} \backslash M$, then $\forall \epsilon > 0, \exists z_\epsilon \in M$ such that

$$\|x - z\| < \rho(x, M) + \epsilon. \tag{14}$$

$$\sup_{f \in X^*, \|f\|=1} \left\{ f(x) - \sup_{x \in M} f(x) \right\} \leq \sup_{f \in X^*, \|f\|=1} \{ f(x) - f(z_\epsilon) \} \leq \|x - z_\epsilon\| \tag{15}$$

Hence

$$\sup_{f \in \mathcal{X}^*, \|f\|=1} \left\{ f(x) - \sup_{z \in M} f(z) \right\} \leq d(x) \tag{16}$$

Therefore, the equality is proved. □