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Problem 1

Let X be a B^* space and its norm $\|\cdot\|$ satisfy the parallelogram law. Prove that when the inherent number field $\mathbb{K} = \mathbb{C}$, one can define an inner product $\langle \cdot, \cdot \rangle$ in X such that $\|x\| = \sqrt{\langle x, x \rangle}, \forall x \in X$.

Solution ***1.a*** The inner product is defined as

$$\langle x, y \rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + \mathbf{i}\|x+\mathbf{i}y\|^2 - \mathbf{i}\|x-\mathbf{i}y\|^2\big) \tag{1}$$

If we write $\langle x, y \rangle = \langle \Re x + \mathbf{i}\Im x, \Re y + \mathbf{i}\Im y \rangle = \langle \Re x, \Re y \rangle + \langle \Im x, \Im y \rangle - i\langle \Re x, \Im y \rangle + i\langle \Re x, \Im y \rangle$

First, we want to verify the conjugate symmetry.

$$\begin{aligned}
\langle y, x \rangle &= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + \mathbf{i}\|y+\mathbf{i}x\|^2 - \mathbf{i}\|y-\mathbf{i}x\|^2\big) \\
&= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - \mathbf{i}\|x-\mathbf{i}x\|^2 + \mathbf{i}\|\mathbf{i}y-\mathbf{i}x\|^2\big) \\
&= \overline{\langle x, y \rangle}
\end{aligned}$$

The positive definite is obvious since it's induced by a norm. So it only suffices to prove the linearity of first variable.

Note that we have

$$\langle x, y \rangle = \langle x, y \rangle' + \mathbf{i}\langle x, \mathbf{i}y \rangle' \tag{2}$$

where $\langle x, y \rangle' = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$ and satisfy linearity in first variable. First, notice that, Hence to prove linearity in first variable, first prove the additivity

$$\begin{aligned}
\langle x+y,z\rangle &= \langle x+y,z\rangle' + \mathbf{i}\langle x+y,\mathbf{i}z\rangle' \\
&= \langle x,z\rangle' + \langle y,z\rangle' + \mathbf{i}\langle x,\mathbf{i}z\rangle' + \mathbf{i}\langle y,\mathbf{i}z\rangle' \\
&= \langle x,z\rangle + \langle y,z\rangle.
\end{aligned}$$

And for scalar multiplication, we start with the case when c is $\mathbf{i}$,

$$\begin{aligned}
\langle \mathbf{i}x,y\rangle &= \frac{1}{4}\left(\|\mathbf{i}x+y\|^2 - \|\mathbf{i}x-y\|^2 + \mathbf{i}\|\mathbf{i}x+\mathbf{i}y\|^2 - \mathbf{i}\|\mathbf{i}x-\mathbf{i}y\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \mathbf{i}\|x+y\|^2 - \mathbf{i}\|x-y\|^2\right) \\
&= \mathbf{i}\langle x,y\rangle.
\end{aligned}$$

If c is real, it immediately follows from the linearity of $\langle\cdot,\cdot\rangle'$ and 2. Hence if c is purely imaginary, it follows by considering multiply $\mathbf{i}$ and $\Im c$ sequentially, which we have both all already proved. Then if $c \in \mathbb{C}$, the claim will follow with additivity and above.

$\square$

Problem 2

Prove that the norm $\|\cdot\|_p$ in l_p space $(1 \le p < \infty, p \ne 2)$ can not be determined by any inner product.

Solution ***2.a*** This is trivial by consider $x = (1,0,0,\dots), y = (0,1,0,\dots)$. We have

$$\begin{aligned}
\|x+y\|_p^2 + \|x-y\|_p^2 &= 2^{1+\frac{2}{p}} \\
2\left(\|x\|_p^2 + \|y\|_p^2\right) &= 4
\end{aligned}$$

When $p \ne 2$, the norm does not satisfy parallelogram law. Hence, it can't be induced by any inner product, since all norms induced by inner product follow this law. $\square$

Problem 3

Prove that Hilbert space is a strict convex space.

Solution ***3.a*** $\forall 0 < \lambda < 1$, when $x \neq cy, x \neq 0, y \neq 0$ one has

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &= \sqrt{\lambda^2\|x\|^2 + 2\lambda(1-\lambda)\Re\langle x, y\rangle + (1-\lambda)^2\|y\|^2} \\ &< \sqrt{\lambda^2\|x\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| + (1-\lambda)^2\|y\|^2} \\ &= \lambda\|x\| + (1-\lambda)\|y\|\end{aligned}$$

$\square$

Problem 4

Let a be a conjugate bilinear function on linear space X and q be the quadratic form determined by a. Prove the polarization inequality. In another word, $\forall x, y \in X$, one has

$$a(x,y) = \frac{1}{4}\{q(x+y) - q(x-y) + \mathbf{i}q(x+\mathbf{i}y) - \mathbf{i}q(x-\mathbf{i}y)\}. \tag{3}$$

$$\begin{aligned}RHS &= \frac{1}{4}\Big(2\Big(a(x,y) + \overline{a(x,y)}\Big) + 2\mathbf{i}\Big(-\mathbf{i}a(x,y) + \mathbf{i}\overline{a(x,y)}\Big)\Big) \\ &= a(x,y)\end{aligned}$$

Problem 5

Prove that it's impossible to induce an inner product $\langle\cdot,\cdot\rangle$ on $C[a,b]$ such that,

$$\langle f, f\rangle^{\frac{1}{2}} = \max_{a\le x\le b}|f(x)| \quad (\forall f \in C[a,b]). \tag{4}$$

Solution ***5.a*** It only suffice to find a function that does not satisfy parallelogram law, since once an inner product is induced from a norm, the norm induced by the induced inner product is the original one. And it shall satisfy parallelogram law. WOLG, assume $[a,b] = [0,1]$. Consider the following two function,

$$\begin{aligned}x(t) &= \frac{1}{3}t & t \in [0,1] \\ y(t) &= -\frac{2}{3}t + 1 & t \in [0,1]\end{aligned}$$

Then the $\|x+y\|^2 < 1$ and $\|x-y\|^2 = \max_{0\le t\le 1}|x(t)-y(t)|^2 \le 1$. Hence LHS of parallelogram is smaller than 2. However, its RHS is strictly larger than 2. Indeed, $\|x\|^2 = 1, \|y\|^2 = \frac{1}{9}$. Therefore, it's impossible to induce an inner product on $C[a,b]$ by its norm. $\square$

Problem 6

Let M, N be two subsets on an inner product space. Prove that

$$M \subset N \Rightarrow N^\perp \subset M^\perp. \tag{5}$$

Solution ***6.a*** $\forall x \in N^\perp \Rightarrow \forall y \in N, y \perp x$. $M \subset N \Rightarrow \forall y' \in M, y' \perp x$. Hence $N^\perp \subset M^\perp$. $\square$

Problem 7

Let M be a subset of a Hilber space X. Prove that,

$$\left(M^\perp\right)^\perp = \overline{\mathrm{span} M} \tag{6}$$

Solution ***7.a*** Using the remark made in class, we have $M^\perp = \left(\overline{\mathrm{span} M}\right)^\perp = (\mathrm{span} M)^\perp$. So it only suffice to prove that $\left(\left(\overline{\mathrm{span} M}\right)^\perp\right)^\perp = \overline{\mathrm{span} M}$. Notice that X is a Hilbert space, and $\overline{\mathrm{span} M}$ is a closed subspace, hence the closure of $\overline{\mathrm{span} M}$ is itself. By another remark on orthogonal decomposition made in class, the claim follows. $\square$