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Problem 1

Prove $L^\infty[a,b]$ is a banach space.

Solution *1.a* A banach space is a complete normed metric space. Since the distance on L^∞ is defined as

$$\rho(f,g) := \sup_{x\in\{[a,b]\}} |f(x) - g(x)| \tag{1}$$

It only suffices to prove that ρ defines a norm on almost surely bounded measurable functions, and the completeness of space. The first requirement follows automatically by the invariance under translation. So it only suffices to prove the completeness.

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty[a,b]$, we have for any $\epsilon > 0$, for any point x almost everywhere, the following inequality holds for i,j large enough

$$|f_i(x) - f_j(x)| \leq \epsilon. \tag{2}$$

Then we can define a function f as the pointwise limit of $\{f_n\}$ a.e., since it's just Cauchy sequence in $\mathbb{R}$. Then we have $\forall\epsilon > 0$, there $\exists n \in \mathbb{N}$, such that $\forall n \geq N$, one has

$$|f(x) - f_n(x)| \leq \epsilon \quad a.e. \tag{3}$$

which implies that f is bounded a.e. by a essentially bounded function plus an arbitrary ϵ. Hence $f \in L^\infty[a,b]$.

□

Problem 2

$C_0^1(0,1)$, define two norm as

$$\|\cdot\|_1 := \left(\int_0^1 |f|^2 + |f'|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{4}$$

$$\|\cdot\|_2 := \left(\int_0^1 |f'|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{5}$$

prove that,

1. *$||\cdot||_1, ||\cdot||_2$ are equivalent.*
2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open region, define*

$$\|\cdot\|_1 := \left(\int_\Omega |f|^2 + |\nabla f|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{6}$$

$$\|\cdot\|_2 := \left(\int_0^1 |\nabla f|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{7}$$

then $\|f\|_1, \|f\|_2$ are equivalent.

Solution ***2.a*** Suppose $f \in C_0^1(0,1)$, we have $f(0) = f(1) = 0$ and,

$$\begin{aligned}
\|f\|_1 &= \left\| \int_0^x f'(t)\mathrm{d}t \right\|_1 \\
&= \left(\int_0^1 \left(\int_0^x f'(t)\mathrm{d}t\right)^2 + (|f'(x)|)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\|f\|_2^2 + \|f\|_2^2\right)^{\frac{1}{2}} \le \sqrt{2}\|f\|_2.
\end{aligned}$$

The reverse is trivial since $|f|^2$ is non-negative which would imply the equivalence between two norm. $\square$

Solution ***2.b*** Here the idea is similar but we need to recall some multivariate calculus.

I'm assuming the following to hold

$$|\nabla f|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2. \tag{8}$$

Now the proof is basically identical, $\|\cdot\|_1$ is obviously stronger than $\|\cdot\|_2$.

f is 0 on the boundary the region ω, which means Ω has finite Lebesgue measure. f''s second type intergral is consistent under different curves once the end points are fixed. For any interior point ω, we can construct a simple curve L connecting ω to a boundary point which would implies that,

$$\begin{aligned}
\|f\|_1 &= \left\|\int_L \nabla f \cdot \mathrm{d}\vec{l}\right\|_1 \\
&= \left(\int_\Omega \left|\int_L \nabla f \mathrm{d}\vec{l}\right|^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega \left|\int_L \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right|^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega \int_L \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right)^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega \int_L 2\left(\left(\frac{\partial f}{\partial x}\mathrm{d}x\right)^2 + \left(\frac{\partial f}{\partial y}\mathrm{d}y\right)^2\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega 2\left(\int_L \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega 2\left(\int_L \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega 2\|f\|_2^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\le \left((2\mathrm{Leb}(\Omega) + 1)\|f\|_2^2\right)^{\frac{1}{2}} \\
&= \sqrt{2\mathrm{Leb}(\Omega) + 1}\|f\|_2 \\
&= C_{\Omega,2}\|f\|_2
\end{aligned}$$

The proof is complete but it's worthy noticing that the Poincare inequalities of different dimension don't share a same constant. $\square$

Problem 3

Consider $\mathbb{R}^2$, *for any point* $z = (x, y)$, *define*

$$\|z\|_1 = |x| + |y| \tag{9}$$

$$\|z\|_2 = \sqrt{x^2 + y^2} \tag{10}$$

$$\|z\|_3 = \max(|x|, |y|) \tag{11}$$

$$\|z\|_4 = \left(x^4 + y^4\right)^{\frac{1}{4}} \tag{12}$$

prove that

1. $\|\cdot\|_i, (i = 1, 2, 3, 4)$ *are norm in* $\mathbb{R}^2$.
2. *Plot the unit sphere in* $(\mathbb{R}^2, \|\cdot\|)(i = 1, 2, 3, 4)$.
3. *Take three point* $O = (0, 0), A = (1, 0), B = (0, 1)$, *try to solve the length of each angle of* $\triangle OAB$ *under different norms.*

Solution ***3.a*** $\|\cdot\|_i, (i = 1, 2, 3, 4)$ are trivial since they are all positive definite functions.

$$\|kz\|_1 = |kx| + |ky| = k(|x| + |y|) = k\|z\|_1$$

$$\|kz\|_2 = \sqrt{(kx)^2 + (ky)^2} = k\sqrt{x^2 + y^2} = k\|z\|_2$$

$$\|kz\|_3 = \max(|kx|, |ky|) = k\max(|x|, |y|) = k\|z\|_3$$

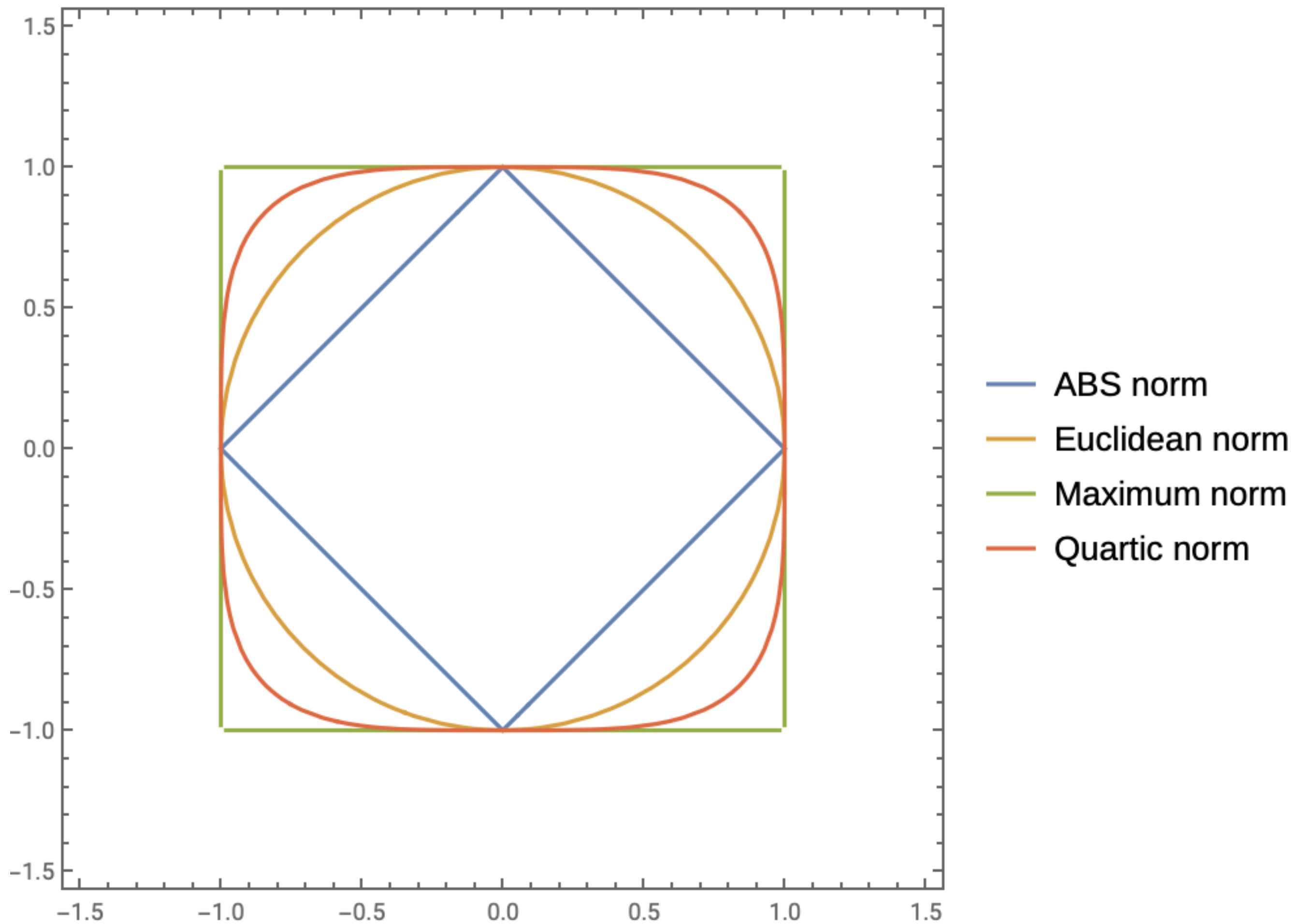
$$\|kz\|_4 = \left((kx)^4 + (ky)^4\right)^{\frac{1}{4}} = k\left((x)^4 + (y)^4\right)^{\frac{1}{4}} = \|z\|_4$$

What left is triangular inequalities. Let x, y be the points and subscript denote the coordinate. The computation here is quite tedious.

$$\begin{aligned}
\|x+y\|_1 &= |x_1+y_1|+|x_2+y_2| \le |x_1|+|x_2|+|y_1|+|y_2| \le \|x\|_1+\|y\|_1 \\
\|x+y\|_2 &= \sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \le \sqrt{\|x\|_2^2+2(x_1y_1+x_2y_2)+\|y\|_2^2} \\
&\le \sqrt{\|x\|_2^2+2\sqrt{x_1^2y_1^2+x_2^2y_2^2+2x_1x_2y_1y_2}+\|y\|_2^2} \\
&\le \sqrt{\|x\|_2^2+2\sqrt{x_1^2y_1^2+x_2^2y_2^2+x_1^2y_2^2+x_1^2y_2^2}+\|y\|_2^2} \\
&\le \sqrt{\|x\|_2^2+2\|x\|_2\|y\|_2+\|y\|_2^2} \le \|x\|_2+\|y\|_2 \\
\|x+y\|_3 &= \max(|x|,|y|) \le \max(|x_1+x_2|,|y_1+y_2|) \le \max(|x_1|+|x_2|,|y_1|+|y_2|) \\
&\le \max(|x_1|,|y_1|)+\max(|x_2|,|y_2|) = \|x\|_3+\|y\|_3 \\
\|x+y\|_4 &= \left((x_1+y_1)^4+(x_2+y_2)^4\right)^{\frac{1}{4}} \le \|x\|_4+\|y\|_4
\end{aligned}$$

Sorry, but I don't want to expand this quartic term. Maybe there's a better way to prove it. But since we haven't studied Cauchy-Swartz inequalities, i.e. Hilbert space, I will leave this proof to be obvious. $\square$

Solution ***3.b*** The plot is produced by mathematica.



□

Solution ***3.c***

$$\|OA\|_1 = 1, \quad \|AB\|_1 = 2, \|BO\|_1 = 1$$
$$\|OA\|_2 = 1, \quad \|AB\|_2 = \sqrt{2}, \|BO\|_2 = 1$$
$$\|OA\|_3 = 1, \quad \|AB\|_3 = 1, \|BO\|_3 = 1$$
$$\|OA\|_4 = 1, \quad \|AB\|_4 = 2^{\frac{1}{4}}, \|BO\|_4 = 1$$

□

Problem 4

Consider $C[0,1]$, prove that for $f \in C[0,1]$, let

$$\|f\|_1 = \left(\int_0^1 |f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}; \|f\|_2 = \left(\int_0^1 (1+x)|f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{13}$$

prove that, $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equavalent norm in $C[0,1]$.

Solution *4.a* First, $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ because its integrad is non-negative hence less than the one in $\|\cdot\|_1$.

And use the trivial estimation $1 + x \leq 2$ on $x \in [0,1]$ to obtain the other direction

$$\|f\|_2 \leq \left(\int_0^1 2|f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \sqrt{2}\|f\|_1 \tag{14}$$

which would give the claim. □

Problem 5

Let $BC[0,\infty]$ represent the union of function that are bounded and continuous on $[0,\infty)$. Then $\forall f \in BC[0,\infty)$ and $a > 0$, define

$$\|f\|_a = \left(\int_0^\infty e^{-ax}|f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{15}$$

prove that,

1. *$\|\cdot\|_a$ is a norm on $BC[0,\infty)$.*
2. *If $a, b > 0, a \neq b$, prove that $\|\cdot\|_a, \|\cdot\|_b$ are not equivalent as norm.*

Solution *5.a* First, the proof for non-negativity and homogenity are trivial. If $\|f\|_a = 0$, then $e^{-ax}|f(x)|^2$ is zero almost everywhere. Consider the set of irrational numbers bigger than zero smaller than K, where this function vanishes. One can exploit the continuity to conclude $f = 0$ on $[0, K)$. Then let K goes infinity and the claim follows. □

Solution *5.b* Suppose and they are equivalent. WLOG suppose $a > b$. Then for any Cauchy sequence under $\|\cdot\|_a$, it shall be a Cauchy sequence under $\|\cdot\|_b$. Now we construct a sequence by

stopping the exponential function

$$f_n(x) := \mathbb{1}_{x<n} e^{\frac{cx}{2}} + \mathbb{1}_{x\geq n} e^{\frac{cn}{2}}. \quad c \in (b, a) \tag{16}$$

First we claim this is a Cauchy sequence. By non-negativity of integrad, several elementary inequalities and monotoncity of x^2, we have the following estimate

$$\begin{aligned}
\|f_n - f_m\|_a &= \int_0^\infty e^{-ax}\big(\mathbb{1}_{n\leq x<m} e^{\frac{cx}{2}} + \mathbb{1}_{x\geq n} e^{\frac{cn}{2}} - \mathbb{1}_{x\geq m} e^{\frac{cm}{2}}\big)^2 \mathrm{d}x \\
&\leq 2\int_0^\infty \mathbb{1}_{n\leq x<m} e^{-ax} e^{cx} + e^{-ax}\big(\mathbb{1}_{x\geq n} e^{\frac{cn}{2}} - \mathbb{1}_{x\geq m} e^{\frac{cm}{2}}\big)^2 \mathrm{d}x \\
&\leq 2\int_0^\infty \mathbb{1}_{n\leq x<m} e^{-ax} e^{cx} + \mathbb{1}_{x\geq n} e^{-ax} e^{cn} \mathrm{d}x \\
&\leq 2\int_n^m e^{(c-a)x}\mathrm{d}x + \int_n^\infty e^{(c-a)x}\mathrm{d}x
\end{aligned}$$

Both two terms converges to zero as $n, m \to \infty$. Because the integral of $\int_0^\infty e^{-}x$ is convergent, the residue converges to zero.

However this is not a Cauchy sequence under $\|\cdot\|_b$. Note that

$$\|f_n - f_m\|_b = \int_0^\infty e^{-bx}\big(\mathbb{1}_{n\leq x<m} e^{\frac{cx}{2}} + \mathbb{1}_{x\geq n} e^{\frac{cn}{2}} - \mathbb{1}_{x\geq m} e^{\frac{cm}{2}}\big)^2 \mathrm{d}x \tag{17}$$

The integral $\int_0^\infty e^{(c-b)x}\mathrm{d}x$ is divergent, hence by the following estimate

$$\begin{aligned}
\|f_n - f_m\|_b &= \int_0^\infty e^{-bx}\big(\mathbb{1}_{n\leq x<m} e^{\frac{cx}{2}} + \mathbb{1}_{x\geq n} e^{\frac{cn}{2}} - \mathbb{1}_{x\geq m} e^{\frac{cm}{2}}\big)^2 \mathrm{d}x \\
&\geq \int_0^\infty e^{-bx}\big(\mathbb{1}_{n\leq x<m} e^{\frac{cx}{2}} - \mathbb{1}_{k\geq m} e^{\frac{cm}{2}}\big)^2 \mathrm{d}x \\
&= \int_0^\infty e^{-bx}(\mathbb{1}_{n\leq x<m} e^{cx} - \mathbb{1}_{k\geq m} e^{cm})\mathrm{d}x
\end{aligned}$$

the first term goes to infinity and the second term goes to zero as $n, m \to \infty$. Here we get a contracdiction, because $\{f_n\}$ is proved to be not a Cauchy sequence. $\square$

Problem 6

Let X_1, X_2 be two B^* space, the ordered pair (x_1, x_2) where $x_1 \in X_1$ and $x_2 \in X_2$ make up a space $X = X_1 \times X_2$. Define norm

$$\|x\| = \max(\|x_1\|_1, \|x_2\|_2), \tag{18}$$

where $x = (x_1, x_2), x_1 \in X_1, x_2 \in X_2$, $\|\cdot\|_1$ and $\|\cdot\|_2$ are also norms in X_1, X_2. Prove that, If X_1, X_2 are B space, then X must be a B space.

Solution ***6.a*** If we take any Cauchy sequence $\{x^n\} \subset X$, we have

$$\|x^n - x^m\| \to 0, \tag{19}$$

which would implies $\|x_1^n - x_1^m\|_1 \to 0$ and $\|x_2^n - x_2^m\|_2 \to 0$ as $n, m \to \infty$. Since X_1, X_2 are B space, there is a limit for coordinates in X_1, X_2. The pair of those coordinates is a point in the product space, which is also the limit or original sequence. Since if those finite many (specifically 2) coordinates converge, their maximum must converges, and that's convergence in $\|\cdot\|$. X is already a B^* space, we have proved its completeness, then it's a B space. □

Problem 7

Consider $\mathbb{R}^2$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ define norm

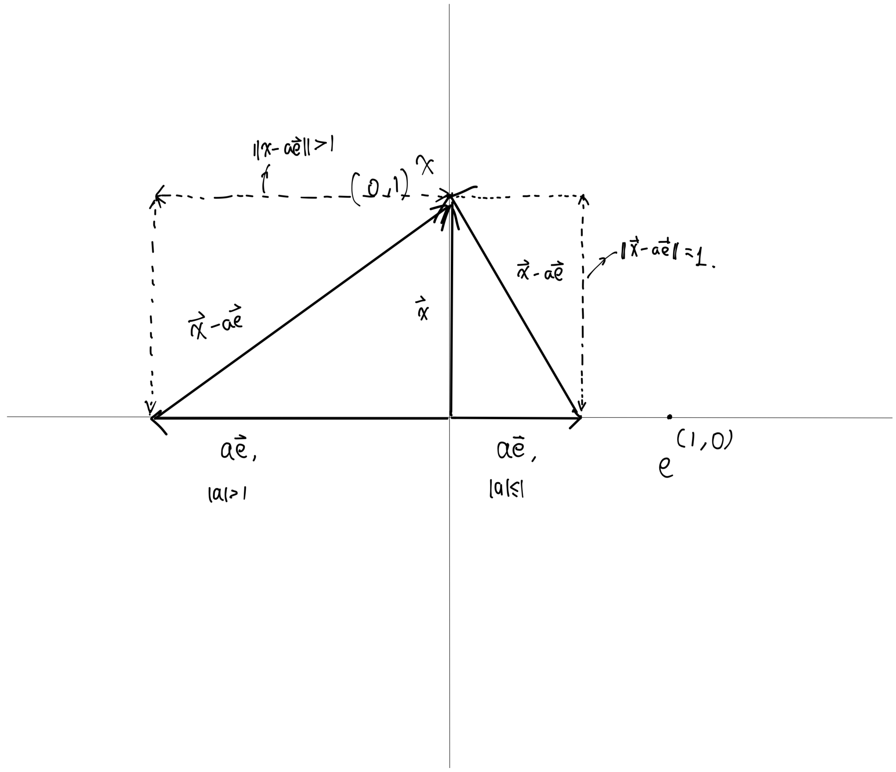
$$\|x\| := \max(|x_1|, |x_2|), \tag{20}$$

let $e_1 = (1, 0), x_0 = (0, 1)$. Find such an $a \in \mathbb{R}^1$ such that

$$\|x_0 - ae_1\| = \min_{\lambda \in \mathbb{R}^1} \|x_0 - \lambda e_1\|, \tag{21}$$

answer if such a is unique? Explain results in geometry.

Solution ***7.a*** For any $|a| \leq 1$, the condition is satisfied. See the following figure,



All the vectors on the boundary of a vertical aligned cube have the same norm. □

Problem 8

Prove that the strict convexity of norm is equivalent to the condition:

$$\|x + y\| = \|x\| + \|y\|, \quad (\forall x \neq 0, y \neq 0) \Rightarrow x = cy \quad (c > 0). \tag{22}$$

Solution ***8.a*** We use the mid point convexity. i.e.

$$\left\| \frac{1}{2}x + \frac{1}{2}y \right\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| \tag{23}$$

If a norm is mid-point strict convex, which means that the

$$\|x + y\| < \|x\| + \|y\| \quad \forall x \neq y. \tag{24}$$

If the equality holds, we can conclude $x = y$.

The converse is also true,

First we have $\|x+y\| = \|x\| + \|y\|$ only when $x = cy$. Let $x = \frac{1}{c+1}x'$, $y = \frac{c}{c+1}y'$, then the strict convexity follows because we ruled out the situation when the equality holds. $\square$

Problem 9

Let X be a B^* space, X_0 be a linear subspace of X, suppose that $\exists c \in (0,1)$ such that

$$\inf_{x \in X_0} \|y - x\| \leq c\|y\| \quad (\forall y \in X). \tag{25}$$

Prove that: X_0 is dense in X.

Solution ***9.a*** To prove X_0 is dense in X, it only suffices to prove that the $\bar{X}_0 = X$. If $X_0 = X$, the claim is trivial. First we have $\bar{X}_0 \subset X$ because X is a space, and the whole space is both closed and open, hence contains the closure of its subset. So it only suffices to prove $X \subset \bar{X}_0$. If $\forall y \in X$, we can construct a sequence $\{y_n\} \subset \bar{X}_0$ such that $y_n \to y$, then the claim shall follow.

Suppose we have a sequence $\{x_{n_1}\} \in x$ such that $\|y - x_{n_1}\| \to \rho(x, X_0)$, denote the limit point as y_1, and $y_1 \in \bar{X}_0$. We have the inequality

$$\|y - y_1\| \leq c\|y\|. \tag{26}$$

Now consider another point $y - y_1 \in X$, we repeat the above procedure to approximate this point with another sequence $\{x_{n_2}\} \subset X_0$, i.e. $\|y - y_1 - x_{n_2}\| \to \rho(y - y_1, X_0)$, such that $x_{n_2} \to y_2 \in \bar{X}_0$. And $\|y - y_1 - y_2\| \leq c\|y - y_1\| \leq c^2\|y\|$ by the uniformity of the constant c. Keep repeat this process, one will end up with a sequence $y_n \in \bar{X}_0$ and $\|\sum y_n - y\| \to 0$, the claim follows.

$\square$

Problem 10

Let X be a $B*$ space, and M be a finite dimensional proper subspace of X, prove that $\exists y \in X, \|y\| =$ 1

$$\|y - x\| \geq 1 \quad (\forall x \in M) \tag{27}$$

Solution ***10.a*** Consider $\forall y \in X \backslash M$, let $d_y = \rho(y, M)$. Hence $\forall \epsilon \in (0, 1)$, there exists $x_\epsilon \in M$ such that

$$d \leq \|y - x_\epsilon\| \leq (1 + \epsilon)d \tag{28}$$

Now let $\epsilon_n = \frac{1}{n}$, we have a sequence $\{x_{\epsilon_n}\}$ which is bounded and in a finite dimensional banach space, who must have a convergent sequence. Denote this limit by x_0, then we have

$$\|y - x_0\| = d_y \tag{29}$$

Let $z := \frac{y - x_0}{\|y - x_0\|}$, then $\|z\| = 1$. Then $\forall m \in M$ we have

$$\|z - m\| = \left\| \frac{y - x_0}{\|y - x_0\|} - m \right\| = \left\| \frac{y - x_0 + d_y m}{d_y} \right\| \geq \frac{d_y}{d_y} = 1 \tag{30}$$

$\square$