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Problem 1

Prove $L^{\infty}[a,b]$ is a banach space.

Solution *1.a* A banach space is a complete normed metric space. Since the distance on L^{∞} is defined as

$$\rho(f,g) := \sup_{x \in \{[a,b]\}} |f(x) - g(x)| \tag{1}$$

It only suffices to prove that ρ defines a norm on almost surely bounded measurable functions, and the completeness of space. The first requirement follows automatically by the invariance under translation. So it only suffices to prove the completeness.

Let $\{f_n\}$ be a Cauchy sequence in $L^{\infty}[a,b]$, we have for any $\epsilon > 0$, for any point x almost everywhere, the following inequality holds for i, j large enough

$$|f_i(x) - f_j(x)| \leq \epsilon. \tag{2}$$

Then we can define a function f as the pointwise limit of $\{f_n\}$ a.e., since it's just Cauchy sequence in $\mathbb{R}$. Then we have $\forall \epsilon > 0$, there $\exists n \in \mathbb{N}$, such that $\forall n \geq N$, one has

$$|f(x) - f_n(x)| \leq \epsilon \quad a.e. \tag{3}$$

which implies that f is bounded a.e. by a essentially bounded function plus an arbitrary ϵ. Hence $f \in L^{\infty}[a,b]$.

□

Problem 2

$C_0^1(0,1)$, define two norm as

$$\|\cdot\|_1 := \left(\int_0^1 |f|^2 + |f'|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{4}$$

$$\|\cdot\|_2 := \left(\int_0^1 |f'|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{5}$$

prove that,

1. *$||\cdot||_1, ||\cdot||_2$ are equivalent.*
2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open region, define*

$$\|\cdot\|_1 := \left(\int_\Omega |f|^2 + |\nabla f|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{6}$$

$$\|\cdot\|_2 := \left(\int_0^1 |\nabla f|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{7}$$

then $\|f\|_1, \|f\|_2$ are equivalent.

Solution ***2.a*** Suppose $f \in C_0^1(0,1)$, we have $f(0) = f(1) = 0$ and,

$$\begin{aligned} \|f\|_1 &= \left\| \int_0^x f'(t)\mathrm{d}t \right\|_1 \\ &= \left(\int_0^1 \left(\int_0^x f'(t)\mathrm{d}t \right)^2 + (|f'(x)|)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \left(\|f\|_2^2 + \|f\|_2^2 \right)^{\frac{1}{2}} \leq \sqrt{2}\|f\|_2. \end{aligned}$$

The reverse is trivial since $|f|^2$ is non-negative which would imply the equivalence between two norm. $\square$

Solution ***2.b*** Here the idea is similar but we need to recall some multivariate calculus.

I'm assuming the following to hold

$$|\nabla f|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2. \tag{8}$$

Now the proof is basically identical, $\|\cdot\|_1$ is obviously stronger than $\|\cdot\|_2$.

f is 0 on the boundary the region ω, which means Ω has finite Lebesgue measure. f''s second type intergral is consistent under different curves once the end points are fixed. For any interior point ω, we can construct a simple curve L connecting ω to a boundary point which would implies that,

$$
\begin{aligned}
\|f\|_1 &= \left\|\int_L \nabla f \cdot \mathrm{d}\vec{l}\right\|_1 \\
&= \left(\int_\Omega \left|\int_L \nabla f \mathrm{d}\vec{l}\right|^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega \left|\int_L \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right|^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega \int_L \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right)^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega \int_L 2\left(\left(\frac{\partial f}{\partial x}\mathrm{d}x\right)^2 + \left(\frac{\partial f}{\partial y}\mathrm{d}y\right)^2\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega 2\left(\int_L \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega 2\left(\int_L \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right) + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega 2\|f\|_2^2 + |\nabla f|^2 \mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\leq \left((2\mathrm{Leb}(\Omega) + 1)\|f\|_2^2\right)^{\frac{1}{2}} \\
&= \sqrt{2\mathrm{Leb}(\Omega) + 1}\|f\|_2 \\
&= C_{\Omega,2}\|f\|_2
\end{aligned}
$$

The proof is complete but it's worthy noticing that the Poincare inequalities of different dimension don't share a same constant. □

Problem 3

Consider $\mathbb{R}^2$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ *define norm*

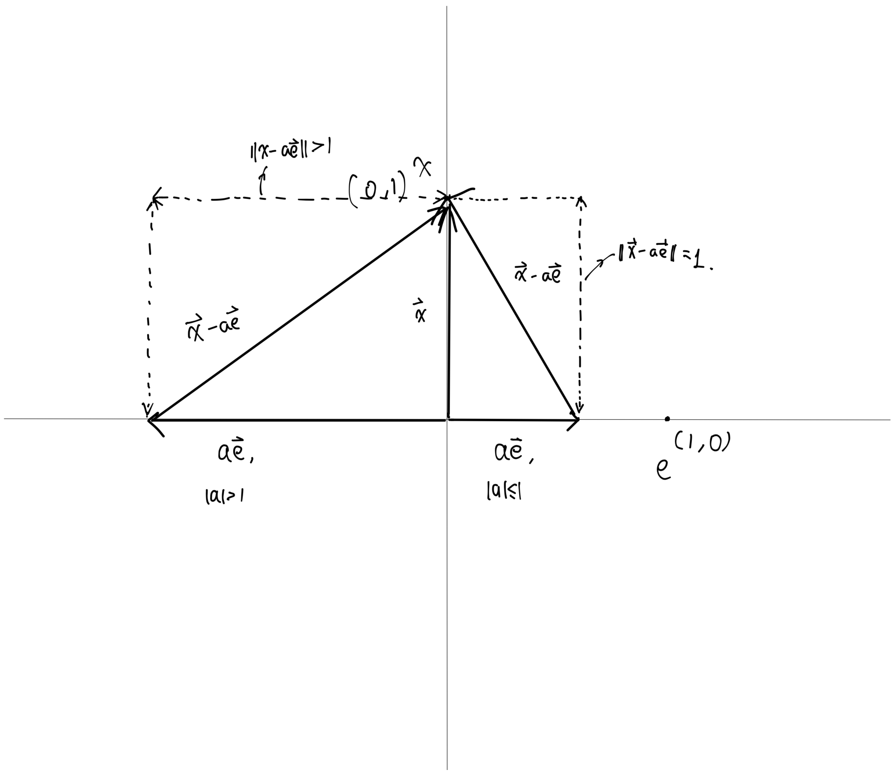
$$\|x\| := \max(|x_1|, |x_2|), \tag{9}$$

let $e_1 = (1, 0), x_0 = (0, 1)$. *Find such an* $a \in \mathbb{R}^1$ *such that*

$$\|x_0 - ae_1\| = \min_{\lambda \in \mathbb{R}^1} \|x_0 - \lambda e_1\|, \tag{10}$$

answer if such a *is unique? Explain results in geometry.*

Solution ***3.a*** For any $|a| \leq 1$, the condition is satisfied. See the following figure,



All the vectors on the boundary of a vertical aligned cube have the same norm. □

Problem 4

Let X *be a* $B*$ *space, and* M *be a finite dimensional proper subspace of* X, *prove that* $\exists y \in X, \|y\| =$ 1

$$\|y - x\| \geq 1 \quad (\forall x \in M) \tag{11}$$

Solution ***4.a*** Consider $\forall y \in X \backslash M$, let $d_y = \rho(y, M)$. Hence $\forall \epsilon \in (0, 1)$, there exists $x_\epsilon \in M$ such that

$$d \leq \|y - x_\epsilon\| \leq (1 + \epsilon)d \tag{12}$$

Now let $\epsilon_n = \frac{1}{n}$, we have a sequence $\{x_{\epsilon_n}\}$ which is bounded and in a finite dimensional banach space, who must have a convergent sequence. Denote this limit by x_0, then we have

$$\|y - x_0\| = d_y \tag{13}$$

Let $z := \frac{y-x_0}{\|y-x_0\|}$, then $\|z\| = 1$. Then $\forall m \in M$ we have

$$\|z - m\| = \left\| \frac{y - x_0}{\|y - x_0\|} - m \right\| = \left\| \frac{y - x_0 + d_y m}{d_y} \right\| \geq \frac{d_y}{d_y} = 1 \tag{14}$$

$\square$