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Problem 1

On $L^2[a,b]$, consider the function set $\left\{e^{2\pi \mathbf{i} nx}\right\}_{n=-\infty}^{\infty}$,

1. *If $|b-a| \le 1$, prove that $S^{\perp} = \{\theta\}$;*
2. *If $|b-a| > 1$, prove that $S^{\perp} \ne \{\theta\}$.*

Solution ***1.a*** First, we know S forms a orthogonal system in $L^2[a,b]$. The elements in set S has periodic of length 1. And by Fourier analysis, S forms a complete basis for all integrable and quadratic integrable periodic function on $\mathbb{R}$ with periodic length 1, i.e. $L^2[T]$, where T is a length 1 interval. Let T be such an interval and $[a,b] \subset T$. Then by extending $L^2[T] \ni f\big|_{T\setminus[a,b]} := 0$ and completeness of S, the claim follows. $\square$

Solution ***1.b*** Take $f \in S^{\perp}$, we have the following equation

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x + \int_{a+1}^{a+1+\epsilon} f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x = 0, \forall n \in \mathbb{Z}. \tag{1}$$

We construct a function $f \ne 0$ such that,

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x = -\int_{a+1}^{a+1+\epsilon} f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x \forall n \in \mathbb{Z}. \tag{2}$$

Using the periodic nature of $e^{2\pi \mathbf{i} nx}$,

$$\int_a^{a+1} f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x = \int_a^{a+\epsilon} -f(x)\overline{e^{2\pi \mathbf{i} nx}}\mathrm{d}x \forall n \in \mathbb{Z}. \tag{3}$$

An obvious choice of f can be taken as,

$$f(x) = \begin{cases} 1, & x \in [a, a+\epsilon] \\ 0, & x \in (a+\epsilon, a+1] \\ -1, & x \in (a+1, a+1+\epsilon] \end{cases} \tag{4}$$

$\square$

Problem 2

Let $\{e_n\}_1^\infty$, $\{f_n\}_1^\infty$ be two orthonormal set, such that

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1. \tag{5}$$

Prove that completeness of one implies that of the other one.

Solution ***2.a*** Suppose $\{e_n\}$ is complete and $\{f_n\}$ is not. Completeness implies totalness. Then exist a f such that $f \neq 0$ and $\langle f, f_n \rangle = 0, \forall n \in \mathbb{N}$. Hence by completeness of $\{e_n\}$ we have

$$\begin{aligned} \|f\|^2 &= \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle f, e_n \rangle - \langle f, f_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle f, e_n - f_n \rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \|f\|^2 \|e_n - f_n\|^2 \\ &< \|f\|^2 \end{aligned}$$

$\square$

Problem 3

Let $\mathfrak{X}$ be a Hilbert space. Let $\mathfrak{X}_0$ be a closed linear subspace of $\mathfrak{X}$. Let $\{e_n\}$ and $\{f_n\}$ be orthonormal basis of $\mathfrak{X}_0$ and $\mathfrak{X}_0^\perp$. Prove that: $\{e_n\} \cup \{f_n\}$ is a set of orthonormal basis of $\mathfrak{X}$.

Solution ***3.a*** First, orthogonality and normality is trivial, because taking union does not change the norm of each element and the two basis are subset of orthogonal sets. It only suffices to prove $\{e_n\} \cup \{f_n\}$ is a set of basis for $\mathfrak{X}$. This can be done by orthogonal decomposition. For any given $x \in \mathfrak{X}$, $\mathfrak{X}_0$ is a closed subset of $\mathfrak{X}$, hence there exists a unique $x_0 \in \mathfrak{X}_0, x_1 \in \mathfrak{X}_0^\perp$ such that $x = x_0 + x_1$. And $\{e_n\}, \{f_n\}$ are basis of $\mathfrak{X}_0, \mathfrak{X}_0^\perp$, hence we have the follows identity,

$$x = x_0 + x_1 = \sum_{n=0}^{\infty} \langle e_n, x_0 \rangle e_n + \sum_{m}^{\infty} \langle f_m, x_1 \rangle f_m \tag{6}$$

which give the claim. □

Problem 4

Let $\mathfrak{X}$ be an inner product space, $\{e_n\}$ be an orthonormal set in $\mathfrak{X}$. Prove that,

$$\left| \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \right| \leq \|x\| \|y\|, \quad \forall x, y \in \mathfrak{X} \tag{7}$$

Proof.

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \right|^2 &\leq \left(\sum_{n=1}^{\infty} \langle x, e_n \rangle^2 \right) \left(\sum_{n=1}^{\infty} \langle y, e_n \rangle^2 \right) \\ &\leq \|x\| \|y\| \end{aligned}$$

□

Problem 5

Find $(a_0, a_1, a_2) \in \mathbb{R}^3$ minimizing $\int_0^1 |e^t - a_0 - a_1 t - a_2 t^2|^2 \mathrm{d}t$.

Solution ***5.a*** Expanding e^t to the polynomial orthogonal basis,

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots \tag{8}$$

each term is non-negative on $[0, 1]$. By the monotonicity of $|x|^2$, the choice of $\left(1, 1, \frac{1}{2}\right)$ minimizes the integral. □

Problem 6

Assume all $\mathcal{X}, \mathcal{Y}$ are Banach spaces.

Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, prove that

1. $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$,
2. $\|A\| = \sup_{\|x\| < 1} \|Ax\|$.

Solution ***6.a***

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \leq \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\| \leq 1} \left\| A \frac{x}{\|x\|} \right\| \|x\| \leq \sup_{\|x\| \leq 1} \left\| A \frac{x}{\|x\|} \right\| = \|A\| \tag{9}$$

□

Solution ***6.b*** It follows from that the unit ball is dense in its closure and norm of bounded operator is a uniformly continuous function. □

Problem 7

Let $f \in \mathcal{L}(\mathcal{X}, \mathbb{R}^1)$, prove that

1. $\|f\| = \sup_{\|x\|=1} f(x)$,
2. $\sup_{\|x\|<\delta} f(x) = \delta \|f\| (\forall \delta > 0)$.

Solution ***7.a***

$$\|f\| = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\|=1} \operatorname{sign}(f(x)) f(x) \leq \sup_{\|x\|=1} f(x) \tag{10}$$

The other direction is trivial because absolute value of f always dominates f. □

Solution ***7.b*** Assume $\delta > 0$, from what we have proved,

$$\delta \|f\| = \delta \sup_{\|x\|=1} f(x) = \sup_{\|x\|=1} \delta f(x) = \sup_{\|x\|=1} f(\delta x) = \sup_{\|x/\delta\|=1} f(x) = \sup_{\|x\|=\delta} f(x) \tag{11}$$

Then from what we have proved in problem 6,

$$\sup_{\|x\|=\delta} f(x) = \sup_{\|x\|<\delta} f(x) \tag{12}$$

□

Problem 8

Let $y(t) \in C[0,1]$, define a functional on $C[0,1]$

$$f(x) = \int_0^1 x(t)y(t)\mathrm{d}t \quad (\forall x \in C[0,1]), \tag{13}$$

find $\|f\|$.

Solution ***8.a*** The norm on $C[0,1]$ is defined to be

$$\|x\|_C = \sup_{t\in[0,1]} |x(t)| \tag{14}$$

Hence we have the following upper bound for f on unit sphere,

$$\|f\| = \sup_{\|x\|=1} \int_0^1 x(t)y(t)\mathrm{d}t \le \int_0^1 \|x\|_C |y(t)|\mathrm{d}t = \int_0^1 |y(t)|\mathrm{d}t \tag{15}$$

The lower bound can be implemented by Fourier approximation. Denote $\bar{y} = \mathrm{sign} y$ Observe that

$$\int_0^1 \bar{y}(t)y(t)\mathrm{d}t \tag{16}$$

However, $\bar{y}$ is not a continuous function. It's a step function. Consider the Fourier series of $\bar{y}$, we can construct the following uniformly convergence,

$$\sum_{n=0}^{\infty} \langle \bar{y}, e_i \rangle e_i \to \bar{y} \tag{17}$$

almost everywhere, and e_i is the triangular Fourier basis aligned for interval $[0,1]$. Then $\forall \epsilon > 0$, there exists such an N that

$$\sup_{t\in T} \left| |y(t)| - \sum_{n=0}^{N} \langle \bar{y}, e_i \rangle e_i(t) y(t) \right| \le \epsilon \tag{18}$$

where T is interval $[0,1]$ minus a set of 0 Lebesgue measure. Integral the above inequality and the claim follows.

□

Problem 9

Let f be a non-zero bounded linear functional on $\mathcal{X}$, let $d = \inf\left\{\|x\|\middle|f(x) = 1\right\}$, prove that: $\|f\| = 1/\|d\|$.

Solution ***9.a***

$$\|f\| = \inf\left\{\frac{|f(x)|}{\|x\|}\right\} \tag{19}$$

$$\frac{1}{\|d\|} = \inf\left\{\frac{1}{\|x\|}\middle|f(x) = 1\right\} = \inf\left\{\frac{f(x)}{\|x\|}\middle|f(x) = 1\right\} = \|f\| \tag{20}$$

The last equality holds because the linearity of f and $\|\cdot\|$ allow us to change the contour without changing the ratio. $\square$

Problem 10

Let $f \in \mathcal{X}^*$, prove that $\forall\epsilon > 0, \exists x_0 \in \mathcal{X}$, such that $f(x_0) = \|f\|$ and $\|x_0\| \leq 1 + \epsilon$.

Solution ***10.a***

$$\|f\| = \sup_{\|x\|=1} |f(x)| \tag{21}$$

$\forall\epsilon > 0$, we have $x_1 \in \mathcal{X}$ such that $f(x_1) > \|f\| - \epsilon$. Let $x_0 = x_1\frac{\|f\|}{f(x_1)}$, then $f(x_0) = \|f\|$ and $\|x_0\| \leq \frac{\|f\|}{\|f\|-\epsilon}$. Then let $\epsilon = \frac{\delta}{1+\delta}\|f\|$, where $\delta > 0$, would give the claim. $\square$