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Problem 1

P87 2.2.1

Denote $f_1, f_2, \dots, f_n$ is a set of linear bounded functional on a Hilbert Space H.,

$$M \triangleq \cap_{k=1}^{n} N(f_k), \quad N(f_k) \triangleq \{x \in H | f_k(x) = 0\} \tag{1}$$

$k = 1, 2, \dots, n$. Denote y_0 as the orthogonal projection of x_0 on M. Prove that $\exists y_1, y_2, \dots, y_n \in H$ and $a_1, a_2, \dots, a_n \in \mathbb{K}$ such that

$$y_0 = x_0 - \sum_{k=1}^{n} a_k y_k. \tag{2}$$

Solution *1.a* It only suffices to prove $M = span\{y_1, y_2, \dots, y_k\}^{\perp}$ and M is a closed subspace. Denote the later space as $S^{\perp}$.

Both two spaces M, S are close because M is finite intersection of closed sets and S is union of finite closed sets.

Now we need to prove M, S together make up the whole space. Let y_k be chosen to as the Riesz's representation of linear functional f_k. If $x \in M$, then $f_k(x) = 0$. By Riesz's representation theorm $\langle x, y_k \rangle = 0, \forall k > 0$, i.e. $x \perp y_k, \forall k$. Then $x \in S^{\perp}$. For the other side of the equality, if $x \in S^{\perp}$, then $x \perp y_k, \forall k$, which means $\langle x, y_k \rangle = 0, \forall y_k \in S$. By Riesz's representation theorem, $f_k(x) = 0, \forall k$. In another word, $x \in N_k, \forall k$. Hence $x \in M$. $\square$

Problem 2

P103 2.3.1

Let $\mathcal{X}$ be a Banach Space, $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$. The map $\phi : \mathcal{X} \mapsto \mathcal{X}/\mathcal{X}_0$ is defined to be

$$\phi : x \mapsto [x] (\forall x \in \mathcal{X}) \tag{3}$$

where $[x]$ is the quatient class containing x. Prove that ϕ is an open mapping.

Solution ***2.a*** Since $\mathcal{X}$ is a Banach space and $\mathcal{X}_0$ is a closed subspace, the quotient space is also a Banach space. Hence, the map ϕ maps from a Banach space to another Banach space. To prove it's an open mapping, it only suffices to prove $R(\phi) = \mathcal{X}/\mathcal{X}_0$. This is automatic because once you choose a specific quotient class, it must have a representative element inside, which after affected by ϕ will be the quotient class. □

Problem 3

P103 2.3.2

Let $\mathcal{X}, \mathcal{Y}$ be Banach Space. Let the equation $Ux = y$ has a solution $x \in \mathcal{X}$ for all $y \in \mathbf{Y}$, where $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Suppose $\exists m > 0$ such that

$$\|Ux\| \geq m\|x\| (\forall x \in \mathcal{X}). \tag{4}$$

Prove that U has a continuous inverse U^{-1}, and $\|U^{-1}\| \leq 1/m$.

Solution ***3.a*** Since $U : \mathcal{X} \to \mathcal{Y}$, it suffices to prove U is a bijection to claim the existence of inverse. The surjection automatically follows since the equation has a solution for any $y \in \mathcal{Y}$. And this solution is unique. Suppose $Ux_1 = Ux_2$, then we have the following,

$$0 = \|U(x_1 - x_2)\| \geq m\|x_1 - x_2\| \geq 0 \tag{5}$$

Hence $x_1 = x_2$. By a corollary of open mapping theorem, U^{-1} exists b For all y such that $\|y\| = 1$, we have

$$\left\|U^{-1}y\right\| \leq \|y\|/m = \frac{1}{m} \tag{6}$$

Hence $\|U^{-1}\| \leq \frac{1}{m}$ b $\square$

Problem 4

P103 2.3.3

Let H be a Hilbert space, and $A \in \mathcal{L}(H)$. Suppose $\exists m > 0$ such that

$$|\langle Ax, x\rangle| \geq m\|x\|^2 \quad (\forall x \in H) \tag{7}$$

Prove that $\exists A^{-1} \in \mathcal{L}(H)$.

Solution ***4.a*** If H is a Hilbert space, then H is also a banach space. It only suffices to prove A is a bijection. The proof for injection is identical to above. The proof of surjection is also trivial since $R(A)$ is either H or a first catagory set. If it's first catagory, then let $\|x\| \to \infty$ to obtain a contradiction. $\square$

Problem 5

P103 2.3.5

Using equavalent norm to prove that $(C[0,1], \|\cdot\|_1)$ is not a B space, where

$$\|\cdot\|_1 = \int_0^1 |f(t)|\mathrm{d}t \quad (\forall f \in C[0,1]) \tag{8}$$

Solution ***5.a*** First we have $(C[0,1], \|\cdot\|_2)$ where $\|\cdot\|_2 := \sup_{t\in[0,1]}|f(t)|$. Then $\|\cdot\|_1$ is equavalent to $\|\cdot_2\|$, because

$$\|f\|_1 = \int_0^1 |f(t)|\mathrm{d}t \leq \int_0^1 \sup_{t\in[0,1]} |f(t)|\mathrm{d}x = \|f\|_2. \tag{9}$$

And there exists a Cauchy sequence under $\|\cdot\|_2$, such that it does not converge in $C[0,1]$. And by equavalence of norm, this Cauchy sequence is still a Cauchy sequence under $\|\cdot\|_1$. Hence we obtain a Cauchy sequence that does not converge. $\square$

Problem 6

P103 2.3.7

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach space, $A_n \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\forall x \in \mathcal{X}$, $\{A_n x\}$ is convergent in $\mathcal{Y}$. Prove that, $\exists A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that

$$A_n x \to Ax \quad (\forall x \in \mathcal{X}), \tag{10}$$

and $\|A\| \leq \liminf_{n\to\infty} \|A_n\|$.

Solution ***6.a*** From uniform boundedness principle (Banach-Steinhaus theorem), $\|A_n\|$ is uniformly bounded. Let $Ax = \lim_{n\to\infty} A_n x$. We need to prove that $\|A\|$ is bounded. For any $x \in \mathcal{X}$, we have

$$\|Ax\| = \lim_{n\to\infty} \|A_n x\| \leq \liminf_{n\to\infty} \|A_n\| \|x\| \leq M\|x\| \tag{11}$$

Hence $\|A\|$ is bounded. The inequality is also proved. $\square$

Problem 7

P103 2.3.8

Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. If $\{a_k\}$ is a sequence such that $\forall \{\xi_k\} \in l^p$, $\sum_{k=1}^{\infty} a_k \xi_k$ is convergent. Prove that $\{a_k\} \in l^p$.

Then if $f : x \mapsto \sum_{k=1}^{\infty} a_k x_k$, prove that f as a linear functional on l^p, we have

$$\|f\| = \left(\sum_{k=1}^{\infty} |a_k|^q \right)^{\frac{1}{q}.} \tag{12}$$

Solution ***7.a*** $f(x) = \langle a, x \rangle$. We need to prove $a \in l^q$, where a denotes the vector $(a_1, a_2, \dots)$. Here it suffice to prove $(l^p)^* = l^q$. Let

$$x_k^{(m)} = \begin{cases} |a_k|^{q-1} \mathrm{sign} a_k & 1 \leq k \leq m \\ 0 & k > m \end{cases} \tag{13}$$

Then

$$f(x^{(m)}) = \sum_{k=1}^{m} x_k^{(m)} a_k = \sum_{k=1}^{m} |a_k|^q \tag{14}$$

Then

$$\|f\| = \sup_{x\in l^p}\left|\frac{f(x)}{\|x\|_p}\right| \geq \frac{f(x^m)}{\|x^m\|_p} = \frac{\sum_{k=1}^m |a_k|^q}{\left(\sum_{k=1}^m |a_k|^{(q-1)p}\right)^{1/p}} = \frac{\sum_{k=1}^m |a_k|^q}{(\sum_{k=1}^m |a_k|^q)^{1-1/q}} = \|a\|_q \tag{15}$$

□

Solution ***7.b*** We have proved one direction of the equality, for the other direction of the equality,

$$\|f\| = \sup_{\|x\|=1} |\langle a, x\rangle| \leq \sup_{\|x\|=1} \|a\|\|x\| = \|a\| \tag{16}$$

□

Problem 8

P103 2.3.9

If there is a sequence $\{a_k\}$ such that $\forall x = \{\xi_k\} \in l^1$, $\sum_{k=1}^{\infty} a_k\xi_k$ is convergent. Prove that $\{a_k\} \in l^\infty$.

And if $f :\mapsto \sum_{k=1}^{\infty} a_k\xi_k$ is a linear functional on l^1. Prove that

$$\|f\| = \sup_{k\geq 1}|a_k| \tag{17}$$

Solution ***8.a*** First, we will prove $\|a\|_\infty \leq \|f\|_1$. Denote $e_k = (0, \ldots, 1, 0, \ldots)$, where the i element is 1.

$$\max_k |a_k| = \max_k |f(e_k)| \leq \|f\| \tag{18}$$

□

Solution ***8.b*** The other side follows from Holder inequality. □

Problem 9

P103 2.3.10

Prove the uniform bounded principle using Gelfrand lemma.

Solution ***9.a*** The Gelfrand lemma is as follow,

Lemma 1. Let $\mathcal{X}$ be a Banach space, $p : \mathcal{X} \to \mathbb{R}^1$ is a semi-linear functional satisfying,

1. $p(x) \geq 0 \quad \forall x \in \mathcal{X}$.
2. $p(\lambda x) = \lambda p(x) \quad (\forall \lambda > 0, \forall x \in \mathcal{X})$.
3. $p(x_1 + x_2) \leq p(x_1) + p(x_2) \quad (\forall x_1, x_2 \in \mathcal{X})$.
4. $x_n \to x \Rightarrow \liminf_{n\to\infty} p(x_n) \geq p(x)$.

Then $\exists M > 0$ such that $p(x) \leq M\|x\| \forall x \in \mathcal{X}$.

Let $W \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{Y}$ is a normed linear space. Define $p(\cdot) = \sup_{A\in W} \|A\cdot\|$. Then it's easy to verity p satisfy the first three property. It only suffice to prove the last property.

$$\|Ax\| \leq \|A(x - x_n)\| + \|Ax_n\| \leq \|A(x - x_n)\| + \sup_{A\in W} \|Ax_n\| \tag{19}$$

Then take the lower limit, since the limit of the second term in the last inequality might not exists. We prove the property 4. Then Gelfrand lemma implies that $\|A\| = \sup_{\|x\|=1} \|Ax\| \leq \sup_{\|x\|=1} \sup_{A\in W} \|Ax\| \leq M$, which is the statement of uniform boundedness principle. $\square$

Problem 10

P103 2.3.11

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces, $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjection. Prove that, if $y_n \to y_0$ in $\mathcal{Y}$. Then $\exists C > 0$ and $x_n \to x_0$, such that $Ax_n = y_n$, and $\|x_n\| \leq C\|y_n\|$.

Solution ***10.a*** The result of this problem would imply that once

Consider its quotient space $\mathcal{X}/\ker A$, which is a Banach space. Define $A' : \mathcal{X}/\ker A \to \mathcal{Y}; [x] \mapsto Ax$. We want to prove first A' is invertible and we can choose a convergent representitives in $\mathcal{X}$ such that the final inequality holds.

For the part of invertibility, it suffices to prove A' is a bijection. Since $\mathcal{X}/\ker A, \mathcal{Y}$ are all Banach spaces, the existence of $A'^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}/\ker A)$ is guaranteed. Since A is a surjection, then A' is automatically a surjection. Since we have ruled out the kernel of A in the quotient space, A' is also an injection. Then the existence of $[x_n] = A'^{-1}y_n$ is guaranteed and we also prove the following inequality on the fly,

$$\|[x_n]\| \leq C\|y_n\| \tag{20}$$

where $C = \|A'^{-1}\|$. $\|[x_n] - [x_0]\| \leq \|A'^{-1}\|\|y_n - y_0\|$ would implies the convergence of $\{[x_n]\}$.

Now the remaining question is if we can choose the right representitives x_n in $\mathcal{X}$ such that it converges and 20 holds without the bracket. The answer is yes because we can choose x_n in $\mathcal{X}$ such that

$$\|x_n\| \leq 2\|[x_n]\|. \tag{21}$$

□

Problem 11

P103 2.3.12

Let $\mathcal{X}, \mathcal{Y}$ be Banach Space, T be a closed linear operator, $D(T) \subset \mathcal{X}$, $R(T) \subset \mathcal{Y}$, $N(T) \triangleq \{x \in \mathcal{X} | Tx = \theta\}$.

1. *Prove that $N(T)$ is a closed linear subspace of $\mathcal{X}$.*
2. *If $N(T) = \{\theta\}$, prove that $R(T)$ being closed in $\mathcal{Y}$ is equivalent to $\exists a > 0$ such that*

$$\|x\| \leq a\|Tx\| \quad (\forall x \in D(T)); \tag{22}$$

3. *Denote the distance from $x \in \mathcal{X}$ to the set $N(T)$ as $d(x, N(T)) = \inf_{z \in N(T)} \|z - x\|$. Prove that $R(T)$ being close in $\mathcal{Y}$ is equivalent to $\exists a > 0$ such that*

$$d(x, N(T)) \leq a\|Tx\| \quad (\forall x \in D(T)). \tag{23}$$

Solution ***11.a*** The linear subspace is trivial. To prove it's closed, let $N(T) \ni x_n \to x_0$, it suffices to prove $Tx_0 = 0$.

$$\|Tx_0\| = \left\|\lim_{n\to\infty} Tx_n\right\| = \lim_{n\to\infty} \|Tx_n\| = 0 \tag{24}$$

$\square$

Solution ***11.b*** $\Rightarrow$: $R(T)$ is closed and $N(T) = \{\theta\}$ implies that $T : D(T) \to R(T)$ is a bijection from a Banach space to another Banach space. Hence T has an inverse T^{-1}. Then

$$\left\|T^{-1}y\right\| \leq \left\|T^{-1}\right\|\|y\| \quad (\forall y \in R(T)) \tag{25}$$

Hence let $a = \frac{1}{\|T^{-1}\|}$, the claim follows because T is a bijection.

$\Leftarrow$: To prove $R(T)$ is closed is equivalent to prove for any convergent series $\{y_k\} \in R(T)$, its limit y is still in $R(T)$. If we denote $Tx_n = y_n$, then the condition implies,

$$\|x_n - x_m\| \leq a\|Tx_n - Tx_m\|. \tag{26}$$

Hence $\{x_n\}$ is also a Cauchy sequence in $D(T)$. And by compleness of $\mathcal{X}$, we can find $x \in \mathcal{X}$ which is the limit of $\{x_n\}$. Then by the closeness of operator T, y is in $\mathcal{Y}$. $\square$

Solution ***11.c*** Notice that $d(x, N(T)) = \|[x]\|$, where $[x]$ is an element in the quotient space $X/\ker T$. Define $T' : \mathcal{X}/\ker T \to \mathcal{Y}$. It immediately follows that $R(T) = R(T')$. So it only suffices to prove $R(T')$ is closed. T' is a bijection, therefore to use the result from the previous question, it only suffices to prove $X/\ker T$ is a Banach space and T' is also a closed operator.

The first part is automatic because X is a Banach space.

For the second part, first we notice that T is a closed operator, so if $x_n \to x$ and $Tx_n \to y$ then $Tx = y$. The difference between T and T' is marginal in the sense that $T'[x_n] = Tx_n$. So if $T[x_n] \to y$, then $Tx_n \to y$. The convergence of $[x_n]$ would imply convergence of a sequence of representitives in X, because there exists representative $x_n - x$ such that $\|x_n - x\| \leq 2\|[x_n - x]\|$. Hence if $[x_n] \to [x]$ and $T'[x_n] \to y$ holds, then by choosing representitives, $x_n \to x$ and $Tx_n \to y$ holds. Then by the closeness of T, we conclude $Tx = y$. Adding any element in the kernel of T will not change its output, therefore $T[x] = y$. $\square$

Problem 12

P103 2.3.13

Suppose $a(x, y)$ is a adjoint bilinear functional on a Hilbert Space H, and it satisify

1. $\exists M > 0$ *such that* $|a(x, y)| \leq M\|x\|\|y\| \quad (\forall x, y \in H)$*;*
2. $\exists \delta > 0$ *such that* $|a(x, x)| \geq \delta\|x\|^2 \quad (\forall x \in H)$*.*

Prove that, $\forall f \in H^*, \exists! y_f \in H$ *such that*

$$a(x, y_f) = f(x) \quad (\forall x \in H), \tag{27}$$

and y_f depends on f continuously.

Solution ***12.a*** By Lax-Milgram theorem, $\exists! T \in \mathcal{L}(\mathcal{X})$ such that

$$a(x, y) = \langle x, Ty \rangle. \tag{28}$$

And the operator T has an inverse $T^{-1} \in \mathcal{L}(\mathcal{X})$. By Riesz's representation theorem, we know $\exists! \tilde{f} \in H$ such that the any linear functional $f \in \mathcal{L}(X)$ has a unique representation $\left\langle x, \tilde{f} \right\rangle$. Then $\left\langle x, \tilde{f} \right\rangle = \left\langle x, TT^{-1}\tilde{f} \right\rangle = a\left(x, T^{-1}\tilde{f}\right)$. This $y_f = T^{-1}\tilde{f}$ is unique since the Riesz representation is unique and T is a bijection. The continuous dependency follows from the continuity of $\langle x, \cdot \rangle$ and T^{-1}. □

Problem 13

Let $\mathcal{X}$, be a Hilbert Space, T be a linear operator satisfying $(Tx, y) = (x, Ty)$. Prove that $T \in \mathcal{L}(\mathcal{X})$.

Solution ***13.a*** If $x_n \to x$, it suffices to prove $Tx_n \to Tx$.

$$\begin{aligned} \|Tx_n - Tx\| &= \langle Tx_n - Tx, Tx_n - Tx \rangle \\ &= \langle Tx - Tx_n, Tx \rangle + \langle Tx_n, Tx_n - Tx \rangle \\ &= \langle x - x_n, TTx \rangle + \langle TTx_n, x_n - x \rangle \\ &\to 0 \end{aligned}$$

□