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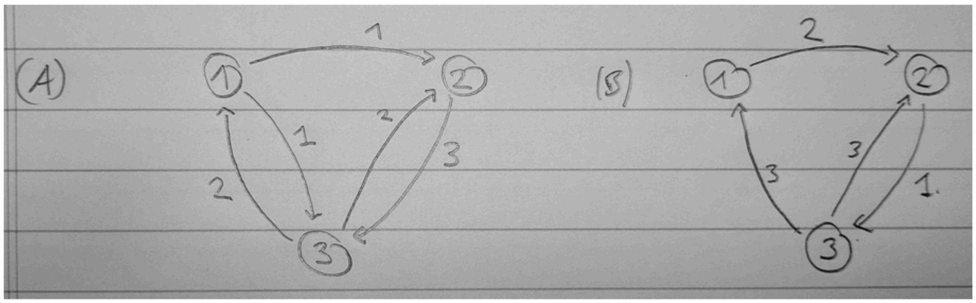
Problem 1

Calculate the stationary distribution of the following two continuous-time Markov chains:



***Solution* 1.a**

Denote stationary distribution for state 1,2,3 as $\pi = (\pi_1, \pi_2, \pi_3)$ respectively. Use the flow in and flow out, we have the following equation.

$$\begin{aligned} 0 &= 2\pi_3 - 2\pi_1 \\ 0 &= \pi_1 + 2\pi_3 - 3\pi_2 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{aligned}$$

By soving linear equation we have $\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ □

***Solution* 1.b**

Denote stationary distribution for state 1,2,3 as $\pi = (\pi_1, \pi_2, \pi_3)$ respectively. Use the flow in and flow out, we have the following equation.

$$\begin{aligned} 0 &= 3\pi_3 - 2\pi_1 \\ 0 &= 2\pi_1 + 3\pi_3 - \pi_2 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{aligned}$$

By soving linear equation we have $\pi = \left(\frac{3}{17}, \frac{12}{17}, \frac{2}{17}\right)$

□

Problem 2

This problem considers a continuous time Markov chain model for the changing pattern of relationships among members in a group. The group has four members: a, b, c, and d. Each pair of the group may or may not have a certain relationship with each other. If they have the relationship, we say that they are linked (for example, being linked may mean that the two members are communicating with each other). Suppose that, in a small time interval of length $\delta > 0$ the following holds: any pair of unlinked individuals will become linked with probability $\alpha\delta + o(\delta)$, and any pair of linked individuals will lose their link with probability $\beta\delta + o(\delta)$. Let X_t denote the number of linked pairs of individuals in the group at time t.

(A) Model X_t as a birth-and-death process, by specifying the birth/death parameters μ_k/λ_k.

(B) Determine the stationary distribution for the process.

Solution **2.a**

It clear that there're total $\binom{4}{2} = 6$ different relationship can appear in the group. And there appearance are independent.

For birth rate μ_n,

$$\begin{aligned}\mathbb{P}(X_{t+\delta} = n+1 | X_t = n) &= \binom{6-n}{1}\mathbb{P}(\text{a pair is linked}) \\ &= (6-n)\alpha\delta + o(\delta)\end{aligned}$$

Hence $\mu_n = (6-n)\alpha$.

$$\begin{aligned}\mathbb{P}(X_{t+\delta} = n-1 | X_t = n) &= \binom{n}{1}\mathbb{P}(\text{a linked pair is unlinked}) \\ &= n\alpha\delta + o(\delta)\end{aligned}$$

Hence $\lambda_n = n\beta$.

□

Solution **2.b**

Denote the stationary distribution as $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_6)$. And draw the diagram using the flow in and flow out, we will have a linear system.

$$\begin{pmatrix} 6\alpha & -\beta & & & & & \\ 6\alpha & -(\beta+5\alpha) & 2\beta & & & & \\ & 5\alpha & -(2\beta+4\alpha) & 3\beta & & & \\ & & 4\alpha & -(3\beta+3\alpha) & 4\beta & & \\ & & & 3\alpha & -(4\beta+2\alpha) & 5\beta & \\ & & & & 2\alpha & -(5\beta+1\alpha) & 6\beta \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6\alpha & -\beta & & & & & \\ & -5\alpha & 2\beta & & & & \\ & & -4\alpha & 3\beta & & & \\ & & & -3\alpha & 4\beta & & \\ & & & & -2\alpha & 5\beta & \\ & & & & & -\alpha & 6\beta \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(1+\frac{6\alpha}{\beta}+\frac{6\alpha 5\alpha}{\beta 2\beta}+\cdots+\left(\frac{\alpha}{\beta}\right)^6\right)\pi_0=1 \tag{1}$$

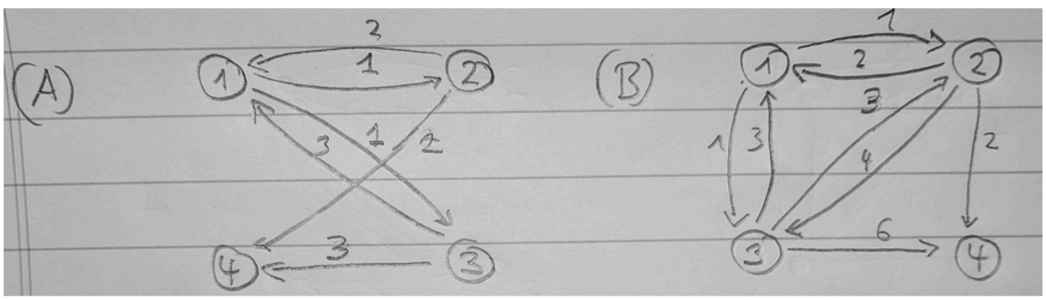
Then we can solve pi_j by the recurrency,

$$\pi_j=\frac{\binom{i}{6}\left(\frac{\alpha}{\beta}\right)^j}{\sum_{i=0}^{6}\binom{i}{6}\left(\frac{\alpha}{\beta}\right)^j} \tag{2}$$

□

Problem 3

Calculate in the following two continuous-time Markov chains (i) the probability of hitting state 3 starting from state 1, and (ii) the expected time to hit state 4 starting from state 1:



***Solution* 3.a** Let $p_k=\mathbb{P}(\text{hit 3 start from i})$ and $t_i=\mathbb{E}(\text{time it take to hit 4 from i})$, then

$$\begin{aligned}
p_4 &= 0, \quad p_3=1\\
p_2 &= \frac{2}{2+3}p_4+\frac{3}{2+3}p_1\\
p_1 &= \frac{1}{1+1}p_2+\frac{1}{1+1}
\end{aligned}$$

By solving the equation we have $\pi_1=\frac{10}{7}$.

$$\begin{aligned}
t_4 &= 0\\
t_3 &= \frac{1}{6}+\frac{1}{2}t_1+\frac{1}{2}t_4\\
t_2 &= \frac{1}{5}+\frac{3}{5}t_1+\frac{2}{5}t_4\\
t_1 &= \frac{1}{2}+\frac{1}{2}t_2+\frac{1}{2}t_3
\end{aligned}$$

By solving the equation we have $t_1=\frac{41}{27}$ □

***Solution* 3.b**

We use the same notation as above, here we have the equation

$$
\begin{aligned}
p_4 &= 0, \quad p_3 = 1 \\
p_1 &= \frac{1}{1+1} p_2 + \frac{1}{1+1} p_3 \\
p_2 &= \frac{2}{2+4+2} p_1 + \frac{4}{2+4+2} p_3 + \frac{2}{2+4+2} p_4
\end{aligned}
$$

By solving the equation we have $p_1 = \dfrac{6}{7}$

Together with the equation

$$
\begin{aligned}
t_4 &= 0 \\
t_3 &= \frac{1}{12} + \frac{1}{2} t_4 + \frac{1}{4} t_2 + \frac{1}{4} t_1 \\
t_2 &= \frac{1}{8} + \frac{1}{4} t_4 + \frac{1}{2} t_3 + \frac{1}{4} t_1 \\
t_1 &= \frac{1}{2} + \frac{1}{2} t_2 + \frac{1}{2} t_3
\end{aligned}
$$

By solving the equation we have $t_1 = \dfrac{37}{34}$ □

Problem 4

Consider a population consisting of a fixed number m of individuals. Suppose that at time $t = 0$ there are exactly one 'infected' and $m - 1$ susceptible individuals in the population. Once infected, an individual remains in the state forever. We suppose that, in any short time-interval of length δ, any given infected person will transmit the disease to any given susceptible individual with probability $\alpha\delta + o(\delta)$, where the 'infection rate' $\alpha > 0$ is fixed. Let X_t denote the number of infected individuals in the population at time t.

(A) Model X_t as a pure birth process, by specifying the birth parameters λ_k.

(B) Determine the expected time until the total population is infected.

Solution **4.a**

$$
\begin{aligned}
\mathbb{P}(X_{t+\delta} = n+1 | X_t = n) &= \mathbb{P}(\text{one is infected}) \binom{m-n}{1} \\
&= \left(\binom{n}{1} (\alpha\delta + o(\delta)) \right) (m-n) \\
&= (m-n) n \alpha \delta + o(\delta)
\end{aligned}
$$

Hence the birth rate at state n, $\lambda_n = (m-n)n\alpha$ for $1 \leq n \leq m$.

□

Solution **4.b**

Denote the expected time for the total population is infected start with i people infected as t_i.

And we will find the recurrent system

$$
\begin{aligned}
t_1 &= \frac{1}{\lambda_1} + \frac{1}{\lambda_1} t_2, \\
t_2 &= \frac{1}{\lambda_2} + \frac{1}{\lambda_2} t_3, \\
&\cdots \\
t_{m-1} &= \frac{1}{\lambda_{m-1}} + \frac{1}{\lambda_{m-1}} t_m
\end{aligned}
$$

Solving this recurrent system we will have

$$
\begin{aligned}
t_1 &= \left(\frac{1}{\lambda_1} \cdots \left(\frac{1}{\lambda_{m-2}} \left(\frac{1}{\lambda_{m-1}} + 1 \right) + 1 \right) + \cdots + 1 \right) \\
&= \prod_{i=1}^{m-1} \frac{1}{\lambda_i} + \prod_{i=1}^{m-2} \frac{1}{\lambda_i} + \cdots + \frac{1}{\lambda_1} \\
&= \sum_{j=1}^{m-1} \prod_{i=1}^{j} \frac{1}{(m-i)i\alpha}
\end{aligned}
$$

□

Problem 5

A new product is being introduced: the 'super home trainer'. The sales are expected to be determined by both media advertising and word-of-mouth advertising. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mouth advertising, assume that each purchaser of a super home trainer will generate new customers at a rate $\beta = 2$ customers per month. Let X_t denote the total number of super home trainer customers up to time t.

(A) Model X_t as a pure birth process, by specifying the birth parameters λ_k.

(B) Set up differential equations for $p_k(t) := \Pr(X_t = k)$ with $k \geq 0$, and verify that

$$p_0(t) = e^{-t} \quad \text{and} \quad p_1(t) = \tfrac{1}{2}e^{-t} - \tfrac{1}{2}e^{-3t}.$$

(C) What is the probability that exactly one super home trainer is sold during the first month?

(D) Optional Bonus Question: What is the probability that exactly two super home trainers are sold during the first month?

***Solution* 5.a** Let $Y_t = \#$ of customers created by media and $Z_t = \#$ of customers generated by word-of-mouth advertisement, it follows that $X_t = Y_t + Z_t$. And Y_t, Z_t is a Pure Birth process with constant rate $\alpha = 1$ and rate $n\beta = 2n$ respectively. Hence X_t is a Pure Birth process with rate $\lambda_n = 2n + 1$. □

***Solution* 5.b** Using flow in flow out we set up the differential equation

$$p_k'(t) = \lambda_{k-1} p_{k-1}(t) - \lambda_k p_k(t)$$

Plug in the given term to verify, $\lambda_0 = 1, \lambda_1 = 3$

$$-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} = e^{-t} - 3(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t})$$

Which is obviously true.

$\square$

***Solution* 5.c** Use the fact we derived previously. $p_1(1) = \frac{1}{2}e^{-1} - \frac{1}{2}e^{-3t}$. $\square$

***Solution* 5.d**

Basically we need to solve a differential equation

$$p_k'(t) = \lambda_{k-1}p_{k-1}(t) - \lambda_k p_k(t) \tag{3}$$

By the result we derived earlier, it simplifies to

$$p'(t) + 5p_2(t) = \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \tag{4}$$

Multiple each side with e^{5t},

$$\begin{aligned} p_2'(t)e^{5t} + 5e^{5t}p_2(t) &= \frac{3}{2}e^{4t} - \frac{3}{2}e^{2t} \\ (p_2(t)e^{5t})' &= \frac{3}{2}e^{4t} - \frac{3}{2}e^{2t} \\ p_2(t) &= \frac{3}{8}e^{-t} - \frac{3}{4}e^{-3t} \end{aligned}$$

Plug in $t = 1$

$p_2(1) = \frac{3}{8}e^{-1} - \frac{3}{4}e^{-3}$ $\square$

Problem 6

Given a copier, assume that a copy-job takes on average 2 minutes to process. In 10% of the cases the copier is left behind with a paper-jam, and then it takes on average 30 minutes until someone fixes the paper-jam and then immediately starts his copy-job. In all other cases, the average waiting time for the next job is 10 minutes.

(A) Approximate this scenario by a continuous time Markov chain (draw the rate-diagram).

(B) Determine the long-run probability that the copier is available (for accepting a new job).

(Hint: Three states are enough.)

***Solution* 6.a**

I think the figure is self explain. And the rate is taken with respect to per minute.

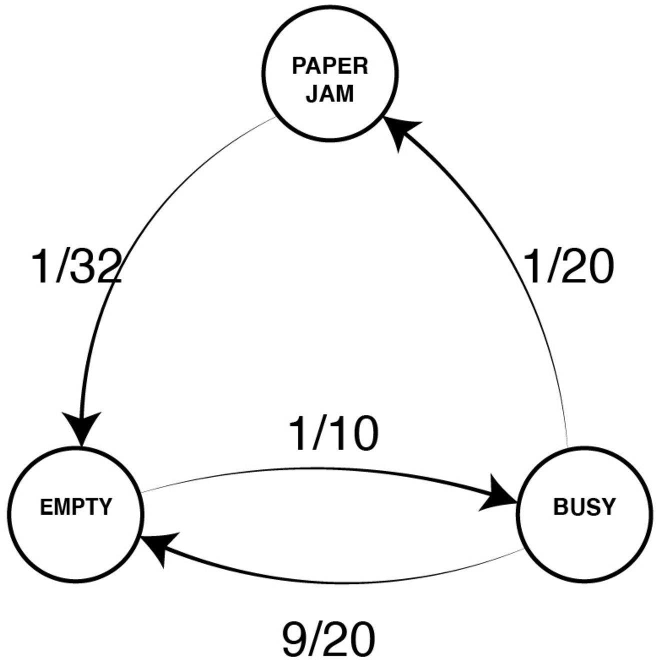
$\square$

***Solution* 6.b**

Denote the stationary distribution for empty busy and paper-jam as $\pi = (\pi_1, \pi_2, \pi_3)$ By flow in and flow out techniques, we obtain the following equations

$$\begin{aligned} 0 &= \frac{1}{32}\pi_3 + \frac{9}{20}\pi_2 - \frac{1}{10}\pi_1 \\ 0 &= \frac{1}{10}\pi_1 - \left(\frac{9}{20} + \frac{1}{20}\right)\pi_2 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{aligned}$$

Figure 1: Markov Chain



By solving those equations we derive $\pi_1 = \frac{25}{38}$, which means the probability that the copier is avaliable in the long run is $\frac{25}{38}$ $\square$