24Fall MATH350 Honours Discrete Mathmetics

Assignment #3: Spanning trees and bipartite graph

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Problem 1.

Proof: First, let's prove the hint. $\forall v \in V(G)$, there are two cases:

1. There are even number of edges in $F_1 \cap F_2$ that is incident to v. Since F_1, F_2 are both even degree, we have even numbers of edges in both $F_1 - F_2$ and $F_2 - F_1$ that are incident to v. Thus, there are even number of edges in $F_1 \triangle F_2$ that is incident to v, i.e. $F_1 \triangle F_2$ is even-degree.

2. There are odd number of edges in $F_1 \cap F_2$ that is incident to v. Similarly, we have odd numbers of edges in both $F_1 - F_2$ and $F_2 - F_1$ that are incident to v. Thus, there are even number of edges in $F_1 \triangle F_2$ that is incident to v, i.e. $F_1 \triangle F_2$ is even-degree.

Let $F' := E(G) - E(T) = \{e_1, e_2, \ldots, e_k\}$. Let $F(e_i)$ be the edge set of the fundamental cycle $FC(T, e_i)$. Obviously, $F(e_i)$ is *even-degree*. By hint, $F := F(e_1) \triangle F(e_2) \triangle \ldots \triangle F(e_k)$ is *ever-degree*. And e_i for all $i \in \{1, 2, \ldots, k\}$ is in F, as $e_i \in F(e_i)$ and $e_i \notin F(e_j)$, where $j \neq i$. Thus, $F \cup E(T) = E(G)$. Therefore, there is an even-degree set $F = F(e_1) \triangle F(e_2) \triangle \ldots \triangle F(e_k)$ with $F \cup E(T) = E(G)$. $\square$

Problem 2.

(a) Let N be the number of spanning tree in K_n containing e. Now, let's try to present N', the number of all spanning tree in a complete graph of n vertices with N.

$$N' = \frac{N \times \frac{n(n-1)}{2}}{n-1}$$

For numerator, the number of spanning tree containing e is N and there is a total number of $\frac{n(n-1)}{2}$ in a complete graph of n vertices. For denominator, each spanning tree containing e is counted $n-1$ times, as each spanning tree contains $n-1$ edges.

By Cayley's theorem, $N' = n^{n-2}$. Therefore, $N = 2n^{n-3}$. $\square$

(b) $2n^{n-2}$

Problem 3.

Assume without losing generality, G is connected (If it is not connected, we can discuss in each components respectively and combine them together).

Obviously, for every graph G there exist at least one bipartite subgraph H such that $V(H) = V(G)$, as the spanning tree of G is bipartite. Let (A, B) be the bipartition of G such that the number of edges with one end in A and another in B is maximized. Let F be the edge set of edges described above. Let H' be the subgraph of G with $V(H') = V(G)$ and $E(H') = F$.

For H', we have $deg_{H'}(v) \geq deg_G(v)/2$ for every $v \in V(H')$. Suppose not, there exist a v such that $deg_{H'}(v) < deg_G(v)/2$. Assume without losing generality, $v \in A$. Then in original graph G the number of vertices adjacent to v in A is more than B . Consider a new bipartition $(A - \{v\}, B + \{v\})$, the number of edges with one end in $A - \{v\}$ and another in $B + \{v\}$ is larger than $|F|$, which contradicts to the maximum. Therefore, we have H is a bipartite subgraph of G, such that $V(H') = V(G)$ and $deg_{H'}(v) \geq deg_G(v)$ for every $v \in V(H')$. $\square$