The null graph is the graph with no vertices.

maximum number of edges in a graph with n vertices?

the first vertex of a edge

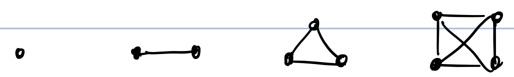
$n=0$	0
$n=1$	0
$n=2$	1

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

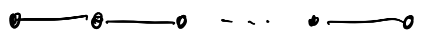
the second vertex

every edge is counted twice

The complete graph on n vertices K_n is the graph in which every two distinct vertices are adjacent.



K_1 K_2 K_3 K_4



v_1 v_2 v_3 v_{n-1} v_n

A path P_n on n vertices is a graph with

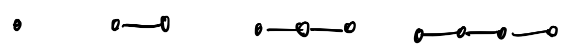
$$V(P_n) = \{v_1, v_2, \cdots v_n\}$$

$$E(P_n) = \{v_1v_2,\ v_2v_3, \cdots\ v_{n-1}v_n\}$$

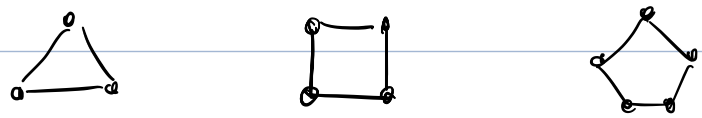
v_1, v_n is the ends of P_n

$$= \{v_iv_{i+1},\ 1 \le i \le n-1\}$$

P_1 P_2 P_3 P_4



For $n \geq 3$ the cycle C_n on n vertices is a graph with $V(C_n) = \{v_1, v_2, \dots v_n\}$ $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$



C_3 C_4 C_5

Adjacency matrix has rows and columns indexed by vertices and an entry in row v & column u is: 1 if $uv \in E(G)$
0, otherwise.

	1	2	3	4	5
1					
2					
3					
4					
5					

An edge is <u>incident</u> to a vertex v if v is an <u>end</u> of e, i.e. $v \in e$

Incidence Matrix of G has rows indexed by $V(G)$,
columns indexed by $E(G)$.
the entry in row v & column e is 1 if v is incident to e
0 otherwise

The <u>degree</u> of a vertex v in a graph G is the number of vertices is adjacent to (the number of edges v is incident to)

Notation. $\deg_G(v)$ or $\deg(v)$

Theorem 1.1.: In every G $\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|$
(Handshake Lemma)

Proof: For $e \in E(G)$ and $v \in V(G)$ let $i(e,v) = \begin{cases} 1, & v \in e \\ 0, & v \notin e \end{cases}$

$$\sum_{v \in V(G)} \deg_G(v) = \sum_{v \in V(G)} \Big(\sum_{e \in E(G)} i(e,v) \Big) = \sum_{e \in E(G)} \Big(\sum_{v \in V(G)} i(e,v) \Big) = 2|E(G)|$$

A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$
$E(H) \subseteq E(H)$

A path (or a cycle) in a graph G is a path or a cycle that is a subgraph in G

The union $H \cup G$ of graphs H & G is a graph with
$V(H \cup G) = V(H) \cup V(G)$
$E(H \cup G) = E(H) \cup E(G)$

The intersection ...

How many graphs are there on $\{1, 2, \cdots n\}$?
$2^{\binom{n}{2}}$

An isomorphism between graphs H & G is a bijection

$\varphi: V(H) \to V(G)$ such that $uv \in E(H)$ iff $\varphi(u)\varphi(v) \in E(G)$

2. Connectivity

$u \in V(G)$ is connected to $v \in V(G)$ if we can travel from u to v along edges

A walk in G with ends u_0 and u_k is a sequence $(u_0, u_1, u_2, \cdots, u_k)$ of vertices of G s.t. $u_i u_{i+1} \in E(G)$ for all $0 \leq i \leq k-1$.

Length of such walk is k

Lemma 2.1: Let u & v be vertices of a graph G
There exists a walk in G with ends u & v iff there exists a path in G with ends u & v

Proof:

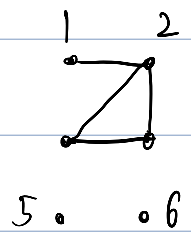
"If" Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, \dots, v_k = v$ so that $v_i v_{i+1} \in E(G)$ for all $0 \leq i \leq k-1$ i.e. $(v_0, v_1, \cdots v_k)$ is a walk

"Only If" Let $(u = v_0, v_1, \cdots, v_k = v)$ be a walk with ends u & v of minimum length. If $v_0, v_1, \cdots, v_k$ are pairwise distinct then it corresponds to path Otherwise, $v_i = v_j$ for some $i < j$

then $(v_0, v_1, \cdots, v_i, v_{j+1}, \cdots, v_k)$ is a walk with ends v & u of length $k+i-j < k$ contradiction to minimum assumption

A graph G is connected if for all $u, v \in V(G)$ there exists a walk with ends u & v in G

$\Leftrightarrow$ i.e. path



an example of not connected

Lemma 2.2 A graph is not connected iff there exists a partition (X, Y) of $V(G)$ s.t. $X, Y \neq \emptyset$, no edge of G has one end in X and another in Y.

Definition of partition. $(X_1, X_2, \cdots, X_k)$ is a partition of S if $\bigcup_{i=1}^{k} X_i = S$ and $X_i \cap X_j = \emptyset$ for all i, j.

Proof:

"If" Choose $u \in X$ and $v \in Y$. We will show that there is no walk in G from X to Y. Suppose not.

Let $(u = u_0, u_1, \cdots, u_k = v)$ be such a walk

Let i be minimum s.t. $u_i \in Y$ then $u_{i-1} \in X$ and $u_{i-1}u_i \in E(G)$ contradicts the choice of (X, Y)

"Only if" Let $u, v \in V(G)$ be such that there is no walk in G from u to v.

Let $X \subseteq V(G)$ be the set of all $w \in V(G)$ s.t. there is a walk with ends u & w in G and let $Y = V(G) - X$

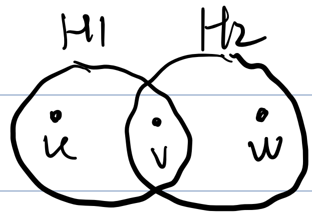
Suppose for a contradiction that there exists $w \in X$ $y \in Y$ s.t. $wy \in E(G)$

Appending y to a walk from u to w we get a walk from u to y, and so $y \in X$ contradiction

Let G be a graph $H \subseteq G$ is a connected component of G if H is a maximal connected subgraph of G

(i.e. H is connected if $H \subseteq H' \subseteq G$ and H' is connected then $H' = H$)

Lemma 2.3 If H_1, H_2 are connected $V(H_1) \cap V(H_2) \neq \emptyset$ then $H_1 \cup H_2$ is connected



Proof Let $v \in V(H_1) \cap V(H_2)$

Let $u, w \in V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$

$H_1 \cup H_2$ contains a walk from u to v

(if $u \in V(H_1)$ then H_1 contains such walk otherwise H_2 contains it)

Similarly $H_1 \cup H_2$ contains a walk from v to w

concatenating them we get a walk from u to w,

As this holds all $u, w \in V(H_1 \cup H_2)$, $H_1 \cup H_2$ is connected

Corollary 2.4: Let G be a graph. Then every $v \in V(G)$ belongs to a unique connected component of G

Proof: Every $v \in V(G)$ belongs to some connected subgraph of G (i.e. a one ~~vertex~~ subgraph)

so to some maximal such subgraph. If H_1, H_2 are two connected components of G s.t. $v \in V(H_1) \cap V(H_2)$ and $H_1 \neq H_2$ then $H_1 \cup H_2$ is connected by 2.3. and $H_1, H_2 \subseteq H_1 \cup H_2$.

so by maximality we must have $H_1 = H_1 \cup H_2 = H_2$

a contradiction