McGill University
Department of Mathematics and Statistics
MATH 254 Honours Analysis 1, Fall 2024
Assignment 3

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1 and 2.**

This assignment is due **Thursday, September 26, at 8:00pm** and is to be submitted on Crowdmark. **Late assignments will not be accepted!**

1. Let $S := \{\sqrt{k} - \sqrt{n} : k, n \in \mathbb{N}\}$. Prove that S is dense in $\mathbb{R}$, i.e. prove that for all $a, b \in \mathbb{R}$, with $a < b$, there exist $k, n \in \mathbb{N}$ such that $a < \sqrt{k} - \sqrt{n} < b$.

   Hint: Try to imitate the proof (not the details, only the main idea!) of the density of $\mathbb{Q}$ in $\mathbb{R}$.

2. Prove that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ i.e. prove that any interval $]a, b[$, $a, b \in \mathbb{R}$, $a < b$, contains at least one irrational number.

   Hint: Use the following two facts:

   - $\mathbb{Q}$ is dense in $\mathbb{R}$,
   - $\sqrt{2}$ is irrational.

3. Let x be a real number. Prove that for every $\epsilon > 0$ there exist two rational numbers q and q' such that $q < x < q'$ and $|q - q'| < \epsilon$.

4. Let A and B be finite, nonempty sets.

   (a) Prove that $|A| \leq |B|$ iff there exists an injective function $f : A \to B$.

   (b) Prove that $|A| \geq |B|$ iff there exists a surjective function $f : A \to B$.

5. (a) Let B be a subset of $\mathbb{N}$. Prove that B is countable.

   (b) Let A be a countably infinite set and let $B \subseteq A$. Prove that B is countable.

6. Prove that the set $\mathbb{N} \times \mathbb{N}$ is countably infinite.

7. (a) Let A and B be two countably infinite sets. Prove that $A \cup B$ is countably infinite.

   (b) Prove that the set of all irrational numbers is uncountable.

   (c) Let $A_1, A_2, A_3, \ldots$ be countably infinite sets. Prove that $\displaystyle\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

8. Let A be a finite set with $|A| = n$. Prove that the power set $\mathcal{P}(A)$ of A has cardinality 2^n.

   Remark: The *power set* $P(A)$ of a set A is defined to be the set of all subsets of A. For example, if $A = \{1, 2, 3\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

9. Prove that the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countably infinite.
   Hint: Consider for all $n \in \mathbb{N}$ the set S_n of all finite subsets of $\mathbb{N}$ whose maximal element is n. You are free to quote other questions from this assignment.