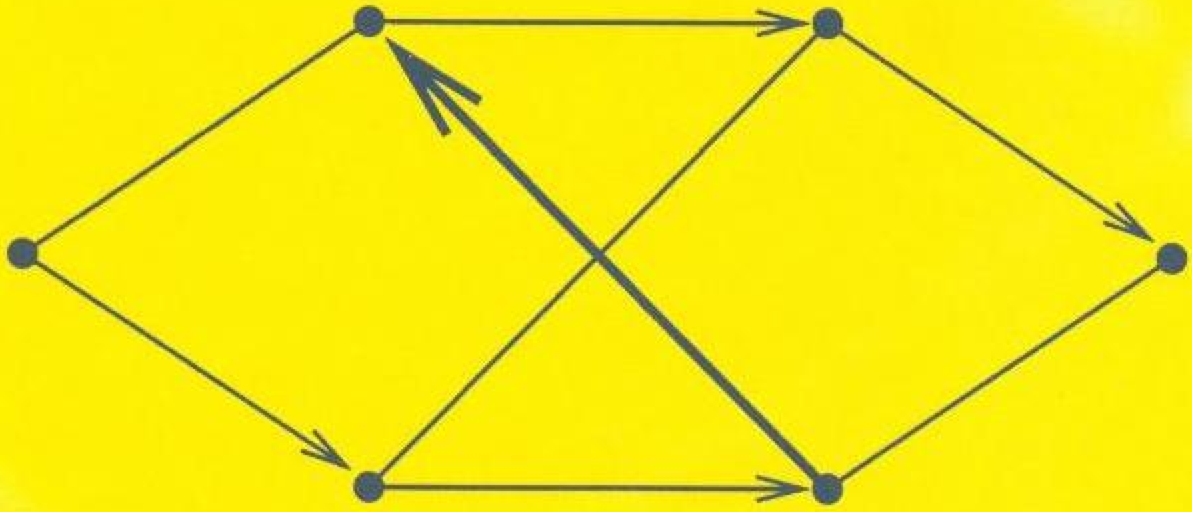
24 Algorithms and Combinatorics Volume A

Combinatorial Optimization

Polyhedra and Efficiency

Alexander Schrijver



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Preface

The book by Gene Lawler from 1976 was the first of a series of books all entitled 'Combinatorial Optimization', some embellished with a subtitle: 'Networks and Matroids', 'Algorithms and Complexity', 'Theory and Algorithms'. Why adding another book to this illustrious series? The justification is contained in the subtitle of the present book, 'Polyhedra and Efficiency'. This is shorthand for Polyhedral Combinatorics and Efficient Algorithms.

Pioneered by the work of Jack Edmonds, polyhedral combinatorics has proved to be a most powerful, coherent, and unifying tool throughout combinatorial optimization. Not only it has led to efficient (that is, polynomial-time) algorithms, but also, conversely, efficient algorithms often imply polyhedral characterizations and related min-max relations. It makes the two sides closely intertwined.

We aim at offering both an introduction to and an in-depth survey of polyhedral combinatorics and efficient algorithms. Within the span of polyhedral methods, we try to present a broad picture of polynomial-time solvable combinatorial optimization problems — more precisely, of those problems that have been proved to be polynomial-time solvable. Next to that, we go into a few prominent NP-complete problems where polyhedral methods were successful in obtaining good bounds and approximations, like the stable set and the traveling salesman problem. Nevertheless, while we obviously hope that the question 'NP=P?' will be settled soon one way or the other, we realize that in the astonishing event that NP=P will be proved, this book will be highly incomplete.

By definition, being in P means being solvable by a 'deterministic sequential polynomial-time' algorithm, and in our discussions of algorithms and complexity we restrict ourselves mainly to this characteristic. As a consequence, we do not cover (but yet occasionally touch or outline) the important work on approximative, randomized, and parallel algorithms and complexity, areas that are recently in exciting motion. We also neglect applications, modelling, and computational methods for NP-complete problems. Advanced data structures are treated only moderately. Other underexposed areas include semidefinite programming and graph decomposition. 'This all just to keep size under control.'

Although most problems that come up in practice are NP-complete or worse, recognizing those problems that are polynomial-time solvable can be very helpful: polynomial-time (and polyhedral) methods may be used in preprocessing, in obtaining approximative solutions, or as a subroutine, for instance to calculate bounds in a branch-and-bound method. A good understanding of what is in the polynomial-time tool box is essential also for the NP-hard problem solver.

* * *

This book is divided into eight main parts, each discussing an area where polyhedral methods apply:

I. Paths and Flows
II. Bipartite Matching and Covering
III. Nonbipartite Matching and Covering
IV. Matroids and Submodular Functions
V. Trees, Branchings, and Connectors
VI. Cliques, Stable Sets, and Colouring
VII. Multiflows and Disjoint Paths
VIII. Hypergraphs

Volume A contains Parts I–III, Volume B Parts IV–VI, and Volume C Parts VII and VIII, the list of References, and the Name and Subject Indices.

Each of the eight parts starts with an elementary exposition of the basic results in the area, and gradually evolves to the more elevated regions. Subsections in smaller print go into more specialized topics. We also offer several references for further exploration of the area.

Although we give elementary introductions to the various areas, this book might be less satisfactory as an introduction to combinatorial optimization. Some mathematical maturity is required, and the general level is that of graduate students and researchers. Yet, parts of the book may serve for undergraduate teaching.

The book does not offer exercises, but, to stimulate research, we collect open problems, questions, and conjectures that are mentioned throughout this book, in a separate section entitled ‘Survey of Problems, Questions, and Conjectures’ (in Volume C). It is not meant as a complete list of all open problems that may live in the field, but only of those mentioned in the text.

We assume elementary knowledge of and familiarity with graph theory, with polyhedra and linear and integer programming, and with algorithms and complexity. To support the reader, we survey the knowledge assumed in the introductory chapters, where we also give additional background references. These chapters are meant mainly just for consultation, and might be less attractive to read from front to back. Some less standard notation and terminology are given on the inside back cover of this book.

For background on polyhedra and linear and integer programming, we also refer to our earlier book *Theory of Linear and Integer Programming*

(Wiley, Chichester, 1986). This might seem a biased recommendation, but this 1986 book was partly written as a preliminary to the present book, and it covers anyway the author's knowledge on polyhedra and linear and integer programming.

Incidentally, the reader of this book will encounter a number of concepts and techniques that regularly crop up: total unimodularity, total dual integrality, duality, blocking and antiblocking polyhedra, matroids, submodularity, hypergraphs, uncrossing. It makes that the meaning of 'elementary' is not unambiguous. Especially for the basic results, several methods apply, and it is not in all cases obvious which method and level of generality should be chosen to give a proof. In some cases we therefore will give several proofs of one and the same theorem, just to open the perspective.

* * *

While I have pursued great carefulness and precision in composing this book, I am quite sure that much room for corrections and additions has remained. To inform the reader about them, I have opened a website at the address

`www.cwi.nl/~lex/co`

Any corrections (including typos) and other comments and suggestions from the side of the reader are most welcome at

`lex@cwi.nl`

I plan to provide those who have contributed most to this, with a complimentary copy of a potential revised edition.

* * *

In preparing this book I have profited greatly from the support and help of many friends and colleagues, to whom I would like to express my gratitude.

I am particularly much obliged to Sasha Karzanov in Moscow, who has helped me enormously by tracking down ancient publications in the (former) Lenin Library in Moscow and by giving explanations and interpretations of old and recent Russian papers. I also thank Sasha's sister Irina for translating Tolstoĭ's 1930 article for me.

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Sincere thanks are due as well to Truus W. Koopmans for sharing with me her 'memories and stories' and sections from her late husband's war diary, and to Herb Scarf for his kind mediation in this. I am indebted to Steve Brady (RAND) and Dick Cottle for their successful efforts in obtaining classic RAND Reports for me, and to Richard Bancroft and Gustave Shubert of RAND Corporation for their help in downgrading the secret Harris-Ross report.

The assistance of my institute, CWI in Amsterdam, has been indispensable in writing this book. My special thanks go to Karin van Gemert of the CWI Library for her indefatigable efforts in obtaining rare publications from every corner of the world, always in sympathizing understanding for my often extravagant requests. Also Thea de Hoog and Rick Ooteman were very helpful in this. Other members of CWI's staff whose assistance is highly appreciated include Miente Bakker, Susanne van Dam, Lieke van den Eersten, Jacqueline de Klerk, Wouter Mettrop, Ay Ong, and Jos van der Werf.

In the technical realization of this book, I thankfully enjoyed the first-rate workmanship of the staff of Springer-Verlag in Heidelberg. I thank in particular Frank Holzwarth, Claudia Kehl, Leonie Kunz, Ute McCrory, and Martin Peters for their skilful and enthusiastic commitment in finalizing this out-size project.

As it has turned out, it was only by gravely neglecting my family that I was able to complete this project. I am extremely grateful to Monique, Nella, and Juliette for their perpetual understanding and devoted support. Now comes the time for the pleasant fulfilment of all promises I made for 'when my book will be finished'.

Amsterdam
November 2002

Alexander Schrijver

Table of Contents

Volume A

Volume B

Volume C

Chapter 1

Introduction

1.1. Introduction

Combinatorial optimization searches for an optimum object in a finite collection of objects. Typically, the collection has a concise representation (like a graph), while the number of objects is huge — more precisely, grows exponentially in the size of the representation (like all matchings or all Hamiltonian circuits). So scanning all objects one by one and selecting the best one is not an option. More efficient methods should be found.

In the 1960s, Edmonds advocated the idea to call a method efficient if its running time is bounded by a polynomial in the size of the representation. Since then, this criterion has won broad acceptance, also because Edmonds found polynomial-time algorithms for several important combinatorial optimization problems (like the matching problem). The class of polynomial-time solvable problems is denoted by P.

Further relief in the landscape of combinatorial optimization was discovered around 1970 when Cook and Karp found out that several other prominent combinatorial optimization problems (including the traveling salesman problem) are the hardest in a large natural class of problems, the class NP. The class NP includes most combinatorial optimization problems. Any problem in NP can be reduced to such 'NP-complete' problems. All NP-complete problems are equivalent in the sense that the polynomial-time solvability of one of them implies the same for all of them.

Almost every combinatorial optimization problem has since been either proved to be polynomial-time solvable or NP-complete — and none of the problems have been proved to be both. This spotlights the big mystery: are the two properties disjoint (equivalently, P$\neq$NP), or do they coincide (P=NP)?

This book focuses on those combinatorial optimization problems that have been proved to be solvable in polynomial time, that is, those that have been proved to belong to P. Next to polynomial-time solvability, we focus on the related polyhedra and min-max relations.

These three aspects have turned out to be closely related, as was shown also by Edmonds. Often a polynomial-time algorithm yields, as a by-product, a description (in terms of inequalities) of an associated polyhedron. Conversely, an appropriate description of the polyhedron often implies the polynomial-time solvability of the associated optimization problem, by applying linear programming techniques. With the duality theorem of linear programming, polyhedral characterizations yield min-max relations, and vice versa.

So the span of this book can be portrayed alternatively by those combinatorial optimization problems that yield well-described polyhedra and min-max relations. This field of discrete mathematics is called *polyhedral combinatorics*. In the following sections we give some basic, illustrative examples.[1]

1.2. Matchings

Let $G = (V, E)$ be an undirected graph and let $w : E \to \mathbb{R}_+$. For any subset F of E, denote

(1.1) $$w(F) := \sum_{e \in F} w(e).$$

We will call $w(F)$ the *weight* of F.

Suppose that we want to find a *matching* (= set of disjoint edges) M in G with weight $w(M)$ as large as possible. In notation, we want to 'solve'

(1.2) $$\max\{w(M) \mid M \text{ matching in } G\}.$$

We can formulate this problem equivalently as follows. For any matching M, denote the incidence vector of M in $\mathbb{R}^E$ by χ^M; that is,

(1.3) $$\chi^M(e) := \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{if } e \notin M, \end{cases}$$

for $e \in E$. Considering w as a *vector* in $\mathbb{R}^E$, we have $w(M) = w^\mathsf{T}\chi^M$. Hence problem (1.2) can be rewritten as

(1.4) $$\max\{w^\mathsf{T}\chi^M \mid M \text{ matching in } G\}.$$

This amounts to maximizing the linear function $w^\mathsf{T}x$ over a finite set of vectors. Therefore, the optimum value does not change if we maximize over the *convex hull* of these vectors:

(1.5) $$\max\{w^\mathsf{T}x \mid x \in \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}\}.$$

The set

(1.6) $$\text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}$$

is a polytope in $\mathbb{R}^E$, called the *matching polytope* of G. As it is a polytope, there exist a matrix A and a vector b such that

[1] Terms used but not introduced yet can be found later in this book — consult the Subject Index.

(1.7) $\quad \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\} = \{x \in \mathbb{R}^E \mid x \geq \mathbf{0}, Ax \leq b\}.$

Then problem (1.5) is equivalent to

(1.8) $\quad \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Ax \leq b\}.$

In this way we have formulated the original combinatorial problem (1.2) as a *linear programming* problem. This enables us to apply linear programming methods to study the original problem.

The question at this point is, however, how to find the matrix A and the vector b. We know that A and b do exist, but we must know them in order to apply linear programming methods.

If G is bipartite, it turns out that the matching polytope of G is equal to the set of all vectors $x \in \mathbb{R}^E$ satisfying

(1.9)
$$\begin{aligned} x(e) &\geq 0 && \text{for } e \in E, \\ \sum_{e \ni v} x(e) &\leq 1 && \text{for } v \in V. \end{aligned}$$

(The sum ranges over all edges e containing v.) That is, for A we can take the $V \times E$ incidence matrix of G and for b the all-one vector $\mathbf{1}$ in $\mathbb{R}^V$.

It is not difficult to show that the matching polytope for bipartite graphs is indeed completely determined by (1.9). First note that the matching polytope is contained in the polytope determined by (1.9), since χ^M satisfies (1.9) for each matching M. To see the reverse inclusion, we note that, if G is bipartite, then the matrix A is *totally unimodular*, i.e., each square submatrix has determinant belonging to $\{0, +1, -1\}$. (This easy fact will be proved in Section 18.2.) The total unimodularity of A implies that the vertices of the polytope determined by (1.9) are *integer* vectors, i.e., belong to $\mathbb{Z}^E$. Now each integer vector satisfying (1.9) must trivially be equal to χ^M for some matching M. Hence, if G is bipartite, the matching polytope is determined by (1.9).

We therefore can apply linear programming techniques to handle problem (1.2). Thus we can find a maximum-weight matching in a bipartite graph in polynomial time, with any polynomial-time linear programming algorithm. Moreover, the duality theorem of linear programming gives

(1.10)
$$\begin{aligned} \max\{w(M) \mid M \text{ matching in } G\} &= \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} \\ &= \min\{y^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, y^\mathsf{T} A \geq w^\mathsf{T}\}. \end{aligned}$$

If we take for w the all-one vector $\mathbf{1}$ in $\mathbb{R}^E$, we can derive from this Kőnig's matching theorem (Kőnig [1931]):

(1.11) the maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover,

where a *vertex cover* is a set of vertices intersecting each edge. Indeed, the left-most expression in (1.10) is equal to the maximum size of a matching. The minimum can be seen to be attained by an integer vector y, again by the total unimodularity of A. This vector y is a $0,1$ vector in $\mathbb{R}^V$, and hence is the incidence vector χ^U of some subset U of V. Then $y^{\mathsf{T}}A \geq \mathbf{1}^{\mathsf{T}}$ implies that U is a vertex cover. Therefore, the right-most expression is equal to the minimum size of a vertex cover.

Kőnig's matching theorem (1.11) is an example of a *min-max formula* that can be derived from a polyhedral characterization. Conversely, min-max formulas (in particular in a weighted form) often give polyhedral characterizations.

The polyhedral description together with linear programming duality also gives a *certificate* of optimality of a matching M: to convince your 'boss' that a certain matching M has maximum size, it is possible and sufficient to display a vertex cover of size $|M|$. In other words, it yields a *good characterization* for the maximum-size matching problem in bipartite graphs.

1.3. But what about nonbipartite graphs?

If G is *nonbipartite*, the matching polytope is not determined by (1.9): if C is an odd circuit in G, then the vector $x \in \mathbb{R}^E$ defined by $x(e) := \frac{1}{2}$ if $e \in EC$ and $x(e) := 0$ if $e \notin EC$, satisfies (1.9) but does not belong to the matching polytope of G.

A pioneering and central theorem in polyhedral combinatorics of Edmonds [1965b] gives a complete description of the inequalities needed to describe the matching polytope for arbitrary graphs: one should add to (1.9) the inequalities

(1.12) $$\sum_{e \subseteq U} x(e) \leq \lfloor \tfrac{1}{2}|U| \rfloor \text{ for each odd-size subset } U \text{ of } V.$$

Trivially, the incidence vector χ^M of any matching M satisfies (1.12). So the matching polytope of G is contained in the polytope determined by (1.9) and (1.12). The content of Edmonds' theorem is the converse inclusion. This will be proved in Chapter 25.

In fact, Edmonds designed a polynomial-time algorithm to find a maximum-weight matching in a graph, which gave this polyhedral characterization as a by-product. Conversely, from the characterization one may derive the polynomial-time solvability of the weighted matching problem, with the ellipsoid method. In applying linear programming methods for this, one will be faced with the fact that the system $Ax \leq b$ consists of exponentially many inequalities, since there exist exponentially many odd-size subsets U of V. So in order to solve the problem with linear programming methods, we cannot just list all inequalities. However, the ellipsoid method does not require that all inequalities are listed a priori. It suffices to have a polynomial-time algorithm answering the question:

(1.13) given $x \in \mathbb{R}^E$, does x belong to the matching polytope of G?

Such an algorithm indeed exists, as it has been shown that the inequalities (1.9) and (1.12) can be checked in time bounded by a polynomial in $|V|$, $|E|$, and the size of x. This method obviously should avoid testing all inequalities (1.12) one by one.

Combining the description of the matching polytope with the duality theorem of linear programming gives a min-max formula for the maximum weight of a matching. It again yields a certificate of optimality: if we have a matching M, we can convince our 'boss' that M has maximum weight, by supplying a dual solution y of objective value $w(M)$. So the maximum-weight matching problem has a good characterization — i.e., belongs to NP∩co-NP.

This gives one motivation for studying polyhedral methods. The ellipsoid method proves polynomial-time solvability, it however does not yield a practical method, but rather an incentive to search for a practically efficient algorithm. The polyhedral method can be helpful also in this, e.g., by imitating the simplex method with a constraint generation technique, or by a primal-dual approach.

1.4. Hamiltonian circuits and the traveling salesman problem

As we discussed above, matching is an area where the search for an inequality system determining the corresponding polytope has been successful. This is in contrast with, for instance, Hamiltonian circuits. No full description in terms of inequalities of the convex hull of the incidence vectors of Hamiltonian circuits — the *traveling salesman polytope* — is known. The corresponding optimization problem is the traveling salesman problem: 'find a Hamiltonian circuit of minimum weight', which problem is NP-complete. This implies that, unless NP=co-NP, there exist facet-inducing inequalities for the traveling salesman polytope that have no polynomial-time certificate of validity. Otherwise, linear programming duality would yield a good characterization. So unless NP=co-NP there is no hope for an appropriate characterization of the traveling salesman polytope.

Moreover, unless NP=P, there is no polynomial-time algorithm answering the question

(1.14) given $x \in \mathbb{R}^E$, does x belong to the traveling salesman polytope?

Otherwise, the ellipsoid method would give the polynomial-time solvability of the traveling salesman problem.

Nevertheless, polyhedral combinatorics can be applied to the traveling salesman problem in a positive way. If we include the traveling salesman polytope in a larger polytope (a *relaxation*) over which we *can* optimize in polynomial time, we obtain a polynomial-time computable bound for the traveling salesman problem. The closer the relaxation is to the traveling salesman polytope, the better the bound is. This can be very useful in a branch-and-bound algorithm. This idea originates from Dantzig, Fulkerson, and Johnson [1954b].

1.5. Historical and further notes

1.5a. Historical sketch on polyhedral combinatorics

The first min-max relations in combinatorial optimization were proved by Dénes Kőnig [1916,1931], on edge-colouring and matchings in bipartite graphs, and by Karl Menger [1927], on disjoint paths in graphs. The matching theorem of Kőnig was extended to the weighted case by Egerváry [1931]. The proofs by Kőnig and Egerváry were in principal algorithmic, and also for Menger's theorem an algorithmic proof was given in the 1930s. The theorem of Egerváry may be seen as polyhedral.

Applying linear programming techniques to combinatorial optimization problems came along with the introduction of linear programming in the 1940s and 1950s. In fact, linear programming forms the hinge in the history of combinatorial optimization. Its initial conception by Kantorovich and Koopmans was motivated by combinatorial applications, in particular in transportation and transshipment.

After the formulation of linear programming as generic problem, and the development in 1947 by Dantzig of the simplex method as a tool, one has tried to attack about all combinatorial optimization problems with linear programming techniques, quite often very successfully. In the 1950s, Dantzig, Ford, Fulkerson, Hoffman, Kuhn, and others studied problems like the transportation, maximum flow, and assignment problems. These problems can be reduced to linear programming by the total unimodularity of the underlying matrix, thus yielding extensions and polyhedral and algorithmic interpretations of the earlier results of Kőnig, Egerváry, and Menger. Kuhn realized that the polyhedral methods of Egerváry for weighted bipartite matching are in fact algorithmic, and yield the efficient 'Hungarian' method for the assignment problem. Dantzig, Fulkerson, and Johnson gave a solution method for the traveling salesman problem, based on linear programming with a rudimentary, combinatorial version of a cutting plane technique.

A considerable extension and deepening, and a major justification, of the field of polyhedral combinatorics was obtained in the 1960s and 1970s by the work and pioneering vision of Jack Edmonds. He characterized basic polytopes like the matching polytope, the arborescence polytope, and the matroid intersection polytope; he introduced (with Giles) the important concept of total dual integrality; and he advocated the interconnections between polyhedra, min-max relations, good characterizations, and efficient algorithms. We give a few quotes in which Edmonds enters into these issues.

In his paper presenting a maximum-size matching algorithm, Edmonds [1965d] gave a polyhedral argument why an algorithm can lead to a min-max theorem:

> It is reasonable to hope for a theorem of this kind because any problem which involves maximizing a linear form by one of a discrete set of non-negative vectors has associated with it a dual problem in the following sense. The discrete set of vectors has a convex hull which is the intersection of a discrete set of half-spaces. The value of the linear form is as large for some vector of the discrete set

> as it is for any other vector in the convex hull. Therefore, the discrete problem is equivalent to an ordinary linear programme whose constraints, together with non-negativity, are given by the half-spaces. The dual (more precisely, a dual) of the discrete problem is the dual of this ordinary linear programme.
> For a class of discrete problems, formulated in a natural way, one may hope then that equivalent linear constraints are pleasant enough though they are not explicit in the discrete formulation.

In another paper (characterizing the matching polytope), Edmonds [1965b] stressed that the number of inequalities is not relevant:

> The results of this paper suggest that, in applying linear programming to a combinatorial problem, the number of relevant inequalities is not important but their combinatorial structure is.

Also in a discussion at the IBM Scientific Computing Symposium on Combinatorial Problems (March 1964 in Yorktown Heights, New York), Edmonds emphasized that the number of facets of a polyhedron is not a measure of the complexity of the associated optimization problem (see Gomory [1966]):

> I do not believe there is any reason for taking as a measure of the algorithmic difficulty of a class of combinatorial extremum problems the number of faces in the associated polyhedra. For example, consider the generalization of the assignment problem from bipartite graphs to arbitrary graphs. Unlike the case of bipartite graphs, the number of faces in the associated polyhedron increases exponentially with the size of the graph. On the other hand, there is an algorithm for this generalized assignment problem which has an upper bound on the work involved just as good as the upper bound for the bipartite assignment problem.

After having received support from H.W. Kuhn and referring to Kuhn's maximum-weight bipartite matching algorithm, Edmonds continued:

> This algorithm depends crucially on what amounts to knowing all the bounding inequalities of the associated convex polyhedron—and, as I said, there are many of them. The point is that the inequalities are known by an easily verifiable characterization rather than by an exhaustive listing—so their number is not important.
> This sort of thing should be expected for a class of extremum problems with a combinatorially special structure. For the traveling salesman problem, the vertices of the associated polyhedron have a simple characterization despite their number—so might the bounding inequalities have a simple characterization despite their number. At least we should hope they have, because finding a really good traveling salesman algorithm is undoubtedly equivalent to finding such a characterization.

So Edmonds was aware of the correlation of good algorithms and polyhedral characterizations, which later got further support by the ellipsoid method.

Also during the 1960s and 1970s, Fulkerson designed the clarifying framework of blocking and antiblocking polyhedra, throwing new light by the classical polarity of vertices and facets of polyhedra on combinatorial min-max relations and enabling, with a theorem of Lehman, the deduction of one polyhedral characterization from another. It stood at the basis of the solution of Berge's perfect graph conjecture in 1972 by Lovász, and it also inspired Seymour to obtain several other basic results in polyhedral combinatorics.

1.5b. Further notes

Raghavan and Thompson [1987] showed that randomized rounding of an optimum fractional solution to a combinatorial optimization problem yields, with high probability, an integer solution with objective value close to the value of the fractional solution (hence at least as close to the optimum value of the combinatorial problem). Related results were presented by Raghavan [1988], Plotkin, Shmoys, and Tardos [1991,1995], and Srinivasan [1995,1999].

Introductions to combinatorial optimization (and more than that) can be found in the books by Lawler [1976b], Papadimitriou and Steiglitz [1982], Sysło, Deo, and Kowalik [1983], Nemhauser and Wolsey [1988], Parker and Rardin [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Mehlhorn and Näher [1999], and Korte and Vygen [2000]. Focusing on applying geometric algorithms in combinatorial optimization are Lovász [1986] and Grötschel, Lovász, and Schrijver [1988]. Bibliographies on combinatorial optimization were given by Kastning [1976], Golden and Magnanti [1977], Hausmann [1978b], von Randow [1982,1985,1990], and O'hEigeartaigh, Lenstra, and Rinnooy Kan [1985].

Survey papers on polyhedral combinatorics and min-max relations were presented by Hoffman [1979], Pulleyblank [1983,1989], Schrijver [1983a,1986a,1987, 1995], and Grötschel [1985], on geometric methods in combinatorial optimization by Grötschel, Lovász, and Schrijver [1984b], and on polytopes and complexity by Papadimitriou [1984].

Chapter 2

General preliminaries

We give general preliminaries on sets, numbers, orders, vectors, matrices, and functions, we discuss how to interpret maxima, minima, and infinity, and we formulate and prove Fekete's lemma.

2.1. Sets

A large part of the sets considered in this book are finite. We often neglect mentioning this when introducing a set. For instance, graphs in this book are finite graphs, except if we explicitly mention otherwise. Similarly for other structures like hypergraphs, matroids, families of sets, etc. Obvious exceptions are the sets of reals, integers, etc.

We call a subset Y of a set X *proper* if $Y \neq X$. Similarly, any other substructure like subgraph, minor, etc. is called *proper* if it is not equal to the structure of which it is a substructure.

A *family* is a set in which elements may occur more than once. More precisely, each element has a *multiplicity* associated. Sometimes, we indicate a family by $(A_1, \ldots, A_n)$ or $(A_i \mid i \in I)$.

A *collection* is synonymous with *set*, but is usually used for a set whose elements are sets. Also *class* and *system* are synonyms of set, and are usually used for sets of structures, like a set of graphs, inequalities, or curves.

A set is called *odd* (*even*) if its size is odd (even). We denote for any set X:

(2.1) $$\begin{aligned} \mathcal{P}(X) &:= \text{collection of all subsets of } X, \\ \mathcal{P}_{\text{odd}}(X) &:= \text{collection of all odd subsets } Y \text{ of } X, \\ \mathcal{P}_{\text{even}}(X) &:= \text{collection of all even subsets } Y \text{ of } X. \end{aligned}$$

Odd and even are called *parities*.

We sometimes say that if $s \in U$, then U *covers* s and s *covers* U. A set U is said to *separate* s and t if $s \neq t$ and $|U \cap \{s,t\}| = 1$. Similarly, a set U is said to *separate* sets S and T if $S \cap T = \emptyset$ and $U \cap (S \cup T) \in \{S, T\}$.

We denote the *symmetric difference* of two sets S and T by $S \triangle T$:

(2.2) $$S \triangle T = (S \setminus T) \cup (T \setminus S).$$

We sometimes use the following shorthand notation, where X is a set and y an 'element':

(2.3) $\quad X + y := X \cup \{y\}$ and $X - y := X \setminus \{y\}$.

We say that sets $S_1, S_2, \ldots, S_k$ are *disjoint* if they are *pairwise* disjoint:

(2.4) $\quad S_i \cap S_j = \emptyset$ for distinct $i, j \in \{1, \ldots, k\}$.

A *partition* of a set X is a collection of disjoint subsets of X with union X. The elements of the partition are called its *classes*.

As usual:

(2.5) $\quad X \subseteq Y$ means that X is a subset of Y,
$X \subset Y$ means that X is a *proper* subset of Y, that is: $X \subseteq Y$ and $X \neq Y$.

Two sets X, Y are *comparable* if $X \subseteq Y$ or $Y \subseteq X$. A collection of pairwise comparable sets is called a *chain*.

Occasionally, we need the following inequality:

Theorem 2.1. *If T and U are subsets of a set S with $T \not\subseteq U$ and $U \not\subseteq T$, then*

(2.6) $$|T||\overline{T}| + |U||\overline{U}| > |T \cap U||\overline{T \cap U}| + |T \cup U||\overline{T \cup U}|,$$

where $\overline{X} := S \setminus X$ for any $X \subseteq S$.

Proof. Define $\alpha := |T \cap U|$, $\beta := |T \setminus U|$, $\gamma := |U \setminus T|$, and $\delta := |\overline{T \cup U}|$. Then:

(2.7) $$\begin{aligned} |T||\overline{T}| + |U||\overline{U}| &= (\alpha + \beta)(\gamma + \delta) + (\alpha + \gamma)(\beta + \delta) \\ &= 2\alpha\delta + 2\beta\gamma + \alpha\gamma + \beta\delta + \alpha\beta + \gamma\delta \end{aligned}$$

and

(2.8) $$\begin{aligned} |T \cap U||\overline{T \cap U}| + |T \cup U||\overline{T \cup U}| &= \alpha(\beta + \gamma + \delta) + (\alpha + \beta + \gamma)\delta \\ &= 2\alpha\delta + \alpha\gamma + \beta\delta + \alpha\beta + \gamma\delta. \end{aligned}$$

Since $\beta\gamma > 0$, (2.6) follows. ∎

A set U is called an *inclusionwise minimal* set in a collection $\mathcal{C}$ of sets if $U \in \mathcal{C}$ and there is no $T \in \mathcal{C}$ with $T \subset U$. Similarly, U is called an *inclusionwise maximal* set in $\mathcal{C}$ if $U \in \mathcal{C}$ and there is no $T \in \mathcal{C}$ with $T \supset U$.

We sometimes use the term *minimal* for *inclusionwise* minimal, and *minimum* for minimum-*size*. Similarly, we sometimes use *maximal* for *inclusionwise* maximal, and *maximum* for maximum-*size* (or maximum-*value* for flows).

A *metric* on a set V is a function $\mu : V \times V \to \mathbb{R}_+$ such that $\mu(v, v) = 0$, $\mu(u, v) = \mu(v, u)$, and $\mu(u, w) \leq \mu(u, v) + \mu(v, w)$ for all $u, v, w \in V$.

2.2. Orders

A relation $\leq$ on a set X is called a *pre-order* if it is reflexive ($x \leq x$ for all $x \in X$) and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). It is a *partial order* if it is moreover anti-symmetric ($x \leq y$ and $y \leq x$ implies $x = y$). The pair $(X, \leq)$ is called a *partially ordered set* if $\leq$ is a partial order.

A partial order $\leq$ is a *linear order* or *total order* if $x \leq y$ or $y \leq x$ for all $x, y \in X$. If $X = \{x_1, \ldots, x_n\}$ and $x_1 < x_2 < \cdots < x_n$, we occasionally refer to the linear order $\leq$ by $x_1, \ldots, x_n$ or $x_1 < \cdots < x_n$. A linear order $\preceq$ is called a *linear extension* of a partial order $\leq$ if $x \leq y$ implies $x \preceq y$.

In a partially ordered set $(X, \leq)$, a *lower ideal* is a subset Y of X such that if $y \in Y$ and $z \leq y$, then $z \in Y$. Similarly, an *upper ideal* is a subset Y of X such that if $y \in Y$ and $z \geq y$, then $z \in Y$. Alternatively, Y is called *down-monotone* if Y is a lower ideal, and *up-monotone* if Y is an upper ideal.

If $(X, \leq)$ is a linearly ordered set, then the *lexicographic order* $\preceq$ on $\bigcup_{k \geq 0} X^k$ is defined by:

(2.9) $\quad (v_1, \ldots, v_t) \prec (u_1, \ldots, u_s) \iff$ the smallest i with $v_i \neq u_i$ satisfies $v_i < u_i$,

where we set $v_i :=$void if $i > t$, $u_i :=$void if $i > s$, and void$< x$ for all $x \in X$.

2.3. Numbers

$\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of integers, rational numbers, and real numbers, respectively. The subscript $+$ restricts the sets to the nonnegative numbers:

(2.10) $\quad \mathbb{Z}_+ := \{x \in \mathbb{Z} \mid x \geq 0\}$, $\mathbb{Q}_+ := \{x \in \mathbb{Q} \mid x \geq 0\}$,
$\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$.

Further we denote for any $x \in \mathbb{R}$:

(2.11) $\quad \lfloor x \rfloor :=$ largest integer y satisfying $y \leq x$,
$\lceil x \rceil :=$ smallest integer y satisfying $y \geq x$.

2.4. Vectors, matrices, and functions

All vectors are assumed to be *column* vectors. The *components* or *entries* of a vector $x = (x_1, \ldots, x_n)^{\mathsf{T}}$ are $x_1, \ldots, x_n$. The *support* of x is the set of indices i with $x_i \neq 0$. The *size* of a vector x is the sum of its components.

A $0, 1$ *vector*, or a $\{0, 1\}$-*valued vector*, or a *simple vector*, is a vector with all entries in $\{0, 1\}$. An *integer vector* is a vector with all entries integer.

We identify the concept of a *function* $x : V \to \mathbb{R}$ with that of a *vector* x in $\mathbb{R}^V$. Its components are denoted equivalently by $x(v)$ or x_v. An *integer function* is an integer-valued function.

For any $U \subseteq V$, the *incidence vector* of U (in $\mathbb{R}^V$) is the vector χ^U defined by:

$$\chi^U(s) := \begin{cases} 1 & \text{if } s \in U, \\ 0 & \text{if } s \notin U. \end{cases} \tag{2.12}$$

For any $u \in V$ we set

$$\chi^u := \chi^{\{u\}}. \tag{2.13}$$

This is the uth *unit base vector*. Given a vector space $\mathbb{R}^V$ for some set V, the all-one vector is denoted by $\mathbf{1}_V$ or just by $\mathbf{1}$, and the all-zero vector by $\mathbf{0}_V$ or just by $\mathbf{0}$. Similarly, $\mathbf{2}_V$ or $\mathbf{2}$ is the all-two vector. We use $\boldsymbol{\infty}$ for the all-∞ vector.

If $a = (a_1, \ldots, a_n)^\mathsf{T}$ and $b = (b_1, \ldots, b_n)^\mathsf{T}$ are vectors, we write $a \leq b$ if $a_i \leq b_i$ for $i = 1, \ldots, n$, and $a < b$ if $a_i < b_i$ for $i = 1, \ldots, n$.

If A is a matrix and x, b, y, and c are vectors, then when using notation like

$$Ax = b,\ Ax \leq b,\ y^\mathsf{T} A = c^\mathsf{T},\ c^\mathsf{T} x, \tag{2.14}$$

we often implicitly assume compatibility of dimensions.

For any vector $x = (x_1, \ldots, x_n)^\mathsf{T}$:

$$\|x\|_1 := |x_1| + \cdots + |x_n| \text{ and } \|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}. \tag{2.15}$$

A *hyperplane* in $\mathbb{R}^n$ is a set H with $H = \{x \in \mathbb{R}^n \mid c^\mathsf{T} x = \delta\}$ for some $c \in \mathbb{R}^n$ with $c \neq \mathbf{0}$ and some $\delta \in \mathbb{R}$.

If U and V are sets, then a $U \times V$ *matrix* is a matrix whose rows are indexed by the elements of U and whose columns are indexed by the elements of V. Generally, when using this terminology, the order of the rows or columns is irrelevant. For a $U \times V$ matrix M and $u \in U$, $v \in V$, the entry in position u, v is denoted by $M_{u,v}$. The all-one $U \times V$ matrix is denoted by $J_{U \times V}$, or just by J.

The *tensor product* of vectors $x \in \mathbb{R}^U$ and $y \in \mathbb{R}^V$ is the vector $x \circ y$ in $\mathbb{R}^{U \times V}$ defined by:

$$(x \circ y)_{(u,v)} := x_u y_v \tag{2.16}$$

for $u \in U$ and $v \in V$.

The *tensor product* of a $W \times X$ matrix M and a $Y \times Z$ matrix N (where W, X, Y, Z are sets), is the $(W \times Y) \times (X \times Z)$ matrix $M \circ N$ defined by

$$(M \circ N)_{(w,y),(x,z)} := M_{w,x} N_{y,z} \tag{2.17}$$

for $w \in W$, $x \in X$, $y \in Y$, $z \in Z$.

The $\mathcal{C} \times V$ *incidence matrix* of a collection or family $\mathcal{C}$ of subsets of a set V is the $\mathcal{C} \times V$ matrix M with $M_{C,v} := 1$ if $v \in C$ and $M_{C,v} := 0$ if $v \notin C$ (for $C \in \mathcal{C}$, $v \in V$). Similarly, the $V \times \mathcal{C}$ incidence matrix is the transpose of this matrix.

For any function $w : V \to \mathbb{R}$ and any $U \subseteq V$, we denote

$$w(U) := \sum_{v \in U} w(v). \tag{2.18}$$

If U is a family, we take multiplicities into account (so if v occurs k times in U, $w(v)$ occurs k times in sum (2.18)).

If w is introduced as a 'weight function', then $w(v)$ is called the *weight* of v, and for any $U \subseteq V$, $w(U)$ is called the *weight* of U. Moreover, for any $x : V \to \mathbb{R}$, we call $w^{\mathsf{T}}x$ the *weight* of x. If confusion may arise, we call $w(U)$ and $w^{\mathsf{T}}x$ the w-*weight* of U and x, respectively.

The adjective 'weight' to 'function' has no mathematical meaning, and implies no restriction, but is just introduced to enable referring to $w(v)$ or $w(U)$ as the weight of v or U. Similarly, for 'length function', 'cost function', 'profit function', 'capacity function', 'demand function', etc., leading to the *length*, *cost*, *profit*, *capacity*, *demand*, etc. of elements or of subsets. Obviously, *shortest* and *longest* are synonyms for 'minimum-length' and 'maximum-length'.

A *permutation matrix* is a square $\{0, 1\}$ matrix, with exactly one 1 in each row and in each column.

Vectors $x_1, \dots, x_k$ are called *affinely independent* if there do not exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 x_1 + \cdots + \lambda_k x_k = \mathbf{0}$ and $\lambda_1 + \cdots + \lambda_k = 0$ and such that the λ_i are not all equal to 0.

Vectors $x_1, \dots, x_k$ are called *linearly independent* if there do not exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 x_1 + \cdots + \lambda_k x_k = \mathbf{0}$ and such that the λ_i are not all equal to 0. The linear hull of a set X is denoted by lin.hull$X$ or lin.hull(X).

If X and Y are subsets of a linear space L over a field $\mathbb{F}$, $z \in L$, and $\lambda \in \mathbb{F}$, then

(2.19) $$z + X := \{z + x \mid x \in X\},\ X + Y := \{x + y \mid x \in X, y \in Y\}, \text{ and } \lambda X = \{\lambda x \mid x \in X\}.$$

If X and Y are subspaces of L, then

(2.20) $$X/Y := \{x + Y \mid x \in X\}$$

is a *quotient space*, which is again a linear space, with addition and scalar multiplication given by (2.19). The dimension of X/Y is equal to $\dim(X) - \dim(X \cap Y)$.

A function $f : X \to Y$ is called an *injection* or an *injective function* if it is one-to-one: if $x, x' \in X$ and $x \neq x'$, then $f(x) \neq f(x')$. The function f is a *surjection* if it is onto: for each $y \in Y$ there is an $x \in X$ with $f(x) = y$. It is a *bijection* if it is both an injection and a surjection.

For a vector $x = (x_1, \dots, x_n)^{\mathsf{T}} \in \mathbb{R}^n$, we denote

(2.21) $$\lfloor x \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)^{\mathsf{T}} \text{ and } \lceil x \rceil := (\lceil x_1 \rceil, \dots, \lceil x_n \rceil)^{\mathsf{T}}.$$

If $f, g : X \to \mathbb{R}$ are functions, we say that $f(x)$ is $O(g(x))$, in notation

(2.22) $$f(x) = O(g(x)) \text{ or } O(f(x)) = O(g(x)),$$

if there exists a constant $c \geq 0$ with $f(x) \leq cg(x) + c$ for all $x \in X$. Hence the relation $=$ given in (2.22) is transitive, but not symmetric. We put

(2.23) $$g(x) = \Omega(f(x))$$

if $f(x) = O(g(x))$.

2.5. Maxima, minima, and infinity

In this book, when speaking of a maximum or minimum, we often implicitly assume that the optimum is finite. If the optimum is not finite, consistency in min-max relations usually can be obtained by setting a minimum over the empty set to $+\infty$, a maximum over a set without upper bound to $+\infty$, a maximum over the empty set to 0 or $-\infty$ (depending on what is the universe), and a minimum over a set without lower bound to $-\infty$. This usually leads to trivial, or earlier proved, statements.

When we speak of making a value infinite, usually large enough will suffice.

If we consider maximizing a function $f(x)$ over $x \in X$, we call any $x \in X$ a *feasible solution*, and any $x \in X$ maximizing $f(x)$ an *optimum solution*. Similarly for minimizing.

2.6. Fekete's lemma

We will need the following result called Fekete's lemma, due to Pólya and Szegő [1925] (motivated by a special case proved by Fekete [1923]):

Theorem 2.2 (Fekete's lemma)**.** *Let $a_1, a_2, \ldots$ be a sequence of reals such that $a_{n+m} \geq a_n + a_m$ for all positive $n, m \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \sup_n \frac{a_n}{n}. \tag{2.24}$$

Proof. For all $i, j, k \geq 1$ we have $a_{jk+i} \geq ja_k + a_i$, by the inequality prescribed in the theorem. Hence for all fixed $i, k \geq 1$ we have

$$\begin{aligned}\liminf_{j\to\infty} \frac{a_{jk+i}}{jk+i} \geq \liminf_{j\to\infty} \frac{ja_k + a_i}{jk+i} &= \liminf_{j\to\infty}\left(\frac{a_k}{k}\frac{jk}{jk+i} + \frac{a_i}{jk+i}\right)\\ &= \frac{a_k}{k}.\end{aligned} \tag{2.25}$$

As this is true for each i, we have for each fixed $k \geq 1$:

$$\liminf_{n\to\infty} \frac{a_n}{n} = \inf_{i=1,\ldots,k} \liminf_{j\to\infty} \frac{a_{jk+i}}{jk+i} \geq \frac{a_k}{k}. \tag{2.26}$$

So

$$\liminf_{n\to\infty} \frac{a_n}{n} \geq \sup_k \frac{a_k}{k}, \tag{2.27}$$

implying (2.24). ∎

We sometimes use the multiplicative version of Fekete's lemma:

Corollary 2.2a. *Let $a_1, a_2, \ldots$ be a sequence of positive reals such that $a_{n+m} \geq a_n a_m$ for all positive $n, m \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \sup_n \sqrt[n]{a_n}. \tag{2.28}$$

Proof. Directly from Theorem 2.2 applied to the sequence $\log a_1, \log a_2, \ldots$. ∎

Chapter 3

Preliminaries on graphs

This chapter is not meant as a rush course in graph theory, but rather as a reference guide and to settle notation and terminology.
To promote readability of the book, nonstandard notation and terminology will be, besides below in this chapter, also explained on the spot in later chapters.

3.1. Undirected graphs

A *graph* or *undirected graph* is a pair $G = (V, E)$, where V is a finite set and E is a family of *unordered* pairs from V. The elements of V are called the *vertices*, sometimes the *nodes* or the *points*. The elements of E are called the *edges*, sometimes the *lines*. We use the following shorthand notation for edges:

(3.1) $\qquad uv := \{u, v\}.$

We denote

(3.2) $\qquad VG :=$ set of vertices of G,
$\qquad EG :=$ family of edges of G.

In running time estimates of algorithms, we denote:

(3.3) $\qquad n := |VG|$ and $m := |EG|$.

In the definition of graph we use the term 'family' rather than 'set', to indicate that the same pair of vertices may occur several times in E. A pair occurring more than once in E is called a *multiple* edge, and the number of times it occurs is called its *multiplicity*. Two edges are called *parallel* if they are represented by the same pair of vertices. A *parallel class* is a maximal set of pairwise parallel edges.

So distinct edges may be represented in E by the same pair of vertices. Nevertheless, we will often speak of 'an edge uv' or even of 'the edge uv', where 'an edge of type uv' would be more correct.

Also *loops* are allowed: edges that are families of the form $\{v, v\}$. Graphs without loops and multiple edges are called *simple*, and graphs without loops are called *loopless*. A vertex v is called a *loopless vertex* if $\{v, v\}$ is not a loop.

An edge uv is said to *connect* u and v. The vertices u and v are called the *ends* of the edge uv. If there exists an edge connecting vertices u and v, then u and v are called *adjacent* or *connected*, and v is called a *neighbour* of u. The edge uv is said to be *incident* with, or to *meet*, or to *cover*, the vertices u and v, and conversely. The edges e and f are said to be *incident*, or to *meet*, or to *intersect*, if they have a vertex in common. Otherwise, they are called *disjoint*.

If $U \subseteq V$ and both ends of an edge e belong to U, then we say that U *spans* e. If at least one end of e belongs to U, then U is said to be *incident with* e. An edge connecting a vertex in a set S and a vertex in a set T is said to *connect* S and T. A set F of edges is said to *cover* a vertex v if v is covered by at least one edge in F, and to *miss* v otherwise.

For a vertex v, we denote:

(3.4) $$\begin{aligned}\delta_G(v) &:= \delta_E(v) := \delta(v) := \text{family of edges incident with } v,\\ N_G(v) &:= N_E(v) := N(v) := \text{set of neighbours of } v.\end{aligned}$$

Here and below, notation with the subscript deleted is used if the graph is clear from the context. We speak in the definition of $\delta(v)$ of the *family* of edges incident with v, since any loop at v occurs twice in $\delta(v)$.

The *degree* $\deg_G(v)$ of a vertex v is the number of edges incident with v. In notation,

(3.5) $$\deg_G(v) := \deg_E(v) := \deg(v) := |\delta_G(v)|.$$

A vertex of degree 0 is called *isolated*, and a vertex of degree 1 an *end vertex*. A vertex of degree k is called *k-valent*. So isolated vertices are loopless.

We denote

(3.6) $$\begin{aligned}\varDelta(G) &:= \text{maximum degree of the vertices of } G,\\ \delta(G) &:= \text{minimum degree of the vertices of } G.\end{aligned}$$

$\varDelta(G)$ and $\delta(G)$ are called the *maximum degree* and *minimum degree* of G, respectively.

If $\varDelta(G) = \delta(G)$, that is, if all degrees are equal, G is called *regular*. If all degrees are equal to k, the graph is called *k-regular*. A 3-regular graph is also called a *cubic graph*.

If $G = (V, E)$ and $G' = (V', E')$ are graphs, we denote by $G + G'$ the graph

(3.7) $$G + G' := (V \cup V', E \cup E')$$

where $E \cup E'$ is the union of E and E' as families (taking multiplicities into account).

Complementary, complete, and line graph

The *complementary graph* or *complement* of a graph $G = (V, E)$ is the simple graph with vertex set V and edges all pairs of distinct vertices that are nonadjacent in G. In notation,

(3.8) $\overline{G}$:= the complementary graph of G.

So if G is simple, then $\overline{\overline{G}} = G$.

A graph G is called *complete* if G is simple and any two distinct vertices are adjacent. In notation,

(3.9) K_n := complete graph with n vertices.

As K_n is unique up to isomorphism, we often speak of *the* complete graph on n vertices.

The *line graph* of a graph $G = (V, E)$ is the simple graph with vertex set E, where two elements of E are adjacent if and only if they meet. In notation,

(3.10) $L(G)$:= the line graph of G.

Subgraphs

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If H is a subgraph of G, we say that G *contains* H. If $G' \neq G$, then G' is called a *proper subgraph* of G. If $V' = V$, then G' is called a *spanning subgraph* of G. If E' consists of all edges of G spanned by V', G' is called an *induced subgraph*, or the *subgraph induced by* V'. In notation,

(3.11) $G[V']$:= subgraph of G induced by V',
$E[V']$:= family of edges spanned by V'.

So $G[V'] = (V', E[V'])$. We further denote for any graph $G = (V, E)$ and for any vertex v, any subset U of V, any edge e, and any subset F of E,

(3.12) $G - v := G[V \setminus \{v\}]$, $G - U := G[V \setminus U]$, $G - e := (V, E \setminus \{e\})$,
$G - F := (V, E \setminus F)$.

We say that these graphs arise from G by *deleting* v, U, e, or F. (We realize that, since an edge e is a set of two vertices, the notation $G - e$ might be ambiguous (if we would consider $U := e$). We expect, however, that the appropriate interpretation will be clear from the context.)

Two subgraphs of G are called *edge-disjoint* if they have no edge in common, and *vertex-disjoint* or *disjoint*, if they have no vertex in common.

In many cases we deal with graphs *up to isomorphism*. For instance, if G and H are graphs, we say that a subgraph G' of G is *an H subgraph* if G' is isomorphic to H.

Paths and circuits

A *walk* in an undirected graph $G = (V, E)$ is a sequence

(3.13) $\qquad P = (v_0, e_1, v_1, \ldots, e_k, v_k),$

where $k \geq 0$, $v_0, v_1, \ldots, v_k$ are vertices, and e_i is an edge connecting v_{i-1} and v_i (for $i = 1, \ldots, k$). If $v_0, v_1, \ldots, v_k$ are all distinct, the walk is called a *path*. (Hence $e_1, \ldots, e_k$ are distinct.)

The vertex v_0 is called the *starting vertex* or *first vertex* of P and the vertex v_k the *end vertex* or *last vertex* of P. Sometimes, both v_0 and v_k are called the *end vertices*, or just the *ends* of P. Similarly, edge e_1 is called the *starting edge* or *first edge* of P, and edge e_k the *end edge* or *last edge* of P. Sometimes, both e_1 and e_k are called the *end edges*.

The walk P is said to *connect* v_0 and v_k, to *run from* v_0 *to* v_k (or *between* v_0 *and* v_k), and to *traverse* $v_0, e_1, v_1, \ldots, e_k, v_k$. The vertices $v_1, \ldots, v_{k-1}$ are called the *internal vertices* of P. For $s, t \in V$, the walk P is called an $s - t$ *walk* if it runs from s to t, and for $S, T \subseteq V$, it is called an $S - T$ *walk* if it runs from a vertex in S to a vertex in T. Similarly, $s - T$ *walks* and $S - t$ *walks* run form s to a vertex in T and from a vertex in S to t, respectively.

The number k is called the *length* of P. (We deviate from this in case a function $l : E \to \mathbb{R}$ has been introduced as a length function. Then the *length* of P is equal to $l(e_1) + \cdots + l(e_k)$.) A walk is called *odd* (*even*, respectively) if its length is odd (even, respectively).

The minimum length of a path connecting u and v is called the *distance* of u and v. The maximum distance over all vertices u, v of G is called the *diameter* of G.

The *reverse walk* P^{-1} of P is the walk obtained from (3.13) by reversing the order of the elements:

(3.14) $\qquad P^{-1} := (v_k, e_k, v_{k-1}, \ldots, e_1, v_0).$

If $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ and $Q = (u_0, f_1, u_1, \ldots, f_l, u_l)$ are walks satisfying $u_0 = v_k$, the *concatenation* PQ of P and Q is the walk

(3.15) $\qquad PQ := (v_0, e_1, v_1, \ldots, e_k, v_k, f_1, u_1, \ldots, f_l, u_l).$

For any walk P, we denote by VP and EP the families of vertices and edges, respectively, occurring in P:

(3.16) $\qquad VP := \{v_0, v_1, \ldots, v_k\}$ and $EP := \{e_1, \ldots, e_k\}$.

A *chord* of P is an edge of G that is not in EP and that connects two vertices of P. The path P is called *chordless* if P has no chords.

If no confusion may arise, we sometimes identify the walk P with the subgraph (VP, EP) of G, or with the set VP of vertices in P, or with the family EP of edges in P. If the graph is simple or if the edges traversed are irrelevant, we indicate the walk just by the sequence of vertices traversed:

(3.17) $\qquad P = (v_0, v_1, \ldots, v_k)$ or $P = v_0, v_1, \ldots, v_k$.

A simple path may be identified by the sequence of edges:

(3.18) $\qquad P = (e_1, \ldots, e_k)$ or $P = e_1, \ldots, e_k$.

We denote

(3.19) $\qquad P_n :=$ a path with n vertices,

usually considered as the *graph* (VP_n, EP_n). This graph is unique up to isomorphism.

Two walks P and Q are called *vertex-disjoint* or *disjoint* if VP and VQ are disjoint, *internally vertex-disjoint* or *internally disjoint* if the set of internal vertices of P is disjoint from the set of internal vertices of Q, and *edge-disjoint* if EP and EQ are disjoint.

The walk P in (3.13) is called *closed* if $v_k = v_0$. It is called a *circuit* if $v_k = v_0$, $k \geq 1$, $v_1, \ldots, v_k$ are all distinct, and $e_1, \ldots, e_k$ are all distinct.

The circuit is also called a k-*circuit*. If $k = 1$, then e_1 must be a loop, and if $k = 2$, e_1 and e_2 are (distinct) parallel edges. If $k = 3$, the circuit is sometimes called a *triangle*.

The above definition of chord of a walk implies that an edge e of G is a *chord* of a circuit C if e connects two vertices in VC but does not belong to EC. A *chordless circuit* is a circuit without chords.

We denote

(3.20) $\qquad C_n :=$ a circuit with n edges,

usually considered as the *graph* (VC_n, EC_n). Again, this graph is unique up to isomorphism.

For any graph $G = (V, E)$, a subset F of E is called a *cycle* if each degree of the subgraph (V, F) is even. One may check that for any $F \subseteq E$:

(3.21) $\qquad F$ is a cycle $\iff$ F is the symmetric difference of the edge sets of a number of circuits.

Connectivity and components

A graph $G = (V, E)$ is *connected* if for any two vertices u and v there is a path connecting u and v. A maximal connected nonempty subgraph of G is called a *connected component*, or just a *component*, of G. Here ‘maximal’ is taken with respect to taking subgraphs. Each component is an induced subgraph, and each vertex and each edge of G belong to exactly one component.

We often identify a component K with the set VK of its vertices. Then the components are precisely the equivalence classes of the equivalence relation $\sim$ on V defined by: $u \sim v \iff$ there exists a path connecting u and v.

A component is called *odd* (*even*) if it has an odd (even) number of vertices.

Cuts

Let $G = (V, E)$ be a graph. For any $U \subseteq V$, we denote

(3.22) $\delta_G(U) := \delta_E(U) := \delta(U) :=$ set of edges of G connecting U and $V \setminus U$.

A subset F of E is called a *cut*, if $F = \delta(U)$ for some $U \subseteq V$. In particular, $\emptyset$ is a cut. If $\emptyset \neq U \neq V$, then $\delta(U)$ is called a *nontrivial cut*. (So $\emptyset$ is a nontrivial cut if and only if G is disconnected.) It is important to observe that for any two sets $T, U \subseteq V$:

(3.23) $\delta(T) \triangle \delta(U) = \delta(T \triangle U)$.

Hence the collection of cuts is closed under taking symmetric differences.

If $s \in U$ and $t \notin U$, then $\delta(U)$ is called an $s - t$ *cut*. If $S \subseteq U$ and $T \subseteq V \setminus U$, $\delta(U)$ is called an $S - T$ *cut*. An edge-cut of size k is called a k*-cut*.

A subset F of E is called a *disconnecting edge set* if $G - F$ is disconnected. For $s, t \in V$, if F intersects each $s - t$ path, then F is said to *disconnect* or to *separate* s and t, or to be $s - t$ *disconnecting* or $s - t$ *separating*. For $S, T \subseteq V$, if F intersects each $S - T$ path, then F is said to *disconnect* or to *separate* S and T, or to be $S - T$ *disconnecting* or $S - T$ *separating*.

One may easily check that for all $s, t \in V$:

(3.24) each $s - t$ cut is $s - t$ disconnecting; each inclusionwise minimal $s - t$ disconnecting edge set is an $s - t$ cut.

An edge e of G is called a *bridge* if $\{e\}$ is a cut. A graph having no bridges is called *bridgeless*.

For any subset U of V we denote

(3.25) $d_G(U) := d_E(U) := d(U) := |\delta(U)|$.

Moreover, for subsets U, W of V:

(3.26) $E[U, W] := \{e \in E \mid \exists u \in U, w \in W : e = uw\}$.

The following is straightforward and very useful:

Theorem 3.1. *For all $U, W \subseteq V$:*

(3.27) $d(U) + d(W) = d(U \cap W) + d(U \cup W) + 2|E[U \setminus W, W \setminus U]|$.

Proof. Directly by counting edges. ∎

This in particular gives:

Corollary 3.1a. *For all $U, W \subseteq V$:*

(3.28) $d(U) + d(W) \geq d(U \cap W) + d(U \cup W)$.

Proof. Directly from Theorem 3.1. ∎

A cut of the form $\delta(v)$ for some vertex v is called a *star*.

Neighbours and vertex-cuts

Let $G = (V, E)$ be a graph. For any $U \subseteq V$, we call a vertex v a *neighbour* of U if $v \notin U$ and v has a neighbour in U. We denote

(3.29) $\qquad N_G(U) := N_E(U) := N(U) :=$ set of neighbours of U.

We further denote

(3.30) $\qquad N^2(v) := N(N(v)) \setminus \{v\}$.

A subset U of V is called a *disconnecting vertex set*, or a *vertex-cut*, if $G - U$ is disconnected. A vertex-cut of size k is called a *k-vertex-cut*. A *cut vertex* is a vertex v of G for which $G - v$ has more components than G has.

For $s, t \in V$, if U intersects each $s - t$ path, then U is said to *disconnect* s and t, or called *$s - t$ disconnecting*. If moreover $s, t \notin U$, then U is said to *separate* s and t, or called *$s - t$ separating*, or an *$s - t$ vertex-cut*. It can be shown that if U is an inclusionwise minimal $s-t$ vertex-cut, then $U = N(K)$ for the component K of $G - U$ that contains s.

For $S, T \subseteq V$, if U intersects each $S-T$ path, then U is said to *disconnect* S and T, or called *$S - T$ disconnecting*. If moreover U is disjoint from $S \cup T$, then U is said to *separate* S and T, or called *$S - T$ separating* or an *$S - T$ vertex-cut*.

A pair of subgraphs (V_1, E_1), (V_2, E_2) of a graph $G = (V, E)$ is called a *separation* if $V_1 \cup V_2 = V$ and $E_1 \cup E_2 = E$. So G has no edge connecting $V_1 \setminus V_2$ and $V_2 \setminus V_1$. Therefore, if these sets are nonempty, $V_1 \cap V_2$ is a vertex-cut of G.

Trees and forests

A graph is called a *forest* if it has no circuits. For any forest (V, E),

(3.31) $\qquad |E| = |V| - \kappa$,

where κ is the number of components of (V, F). A *tree* is a connected forest. So for any tree (V, E),

(3.32) $\qquad |E| = |V| - 1$.

Any forest with at least one edge has an end vertex. A connected subgraph of a tree T is called a *subtree* of T.

The notions of forest and tree extend to subsets of edges of a graph $G = (V, E)$ as follows. A subset F of E is called a *forest* if (V, F) is a forest, and a *spanning tree* if (V, F) is a tree. Then for any graph $G = (V, E)$:

(3.33) $\qquad G$ has a spanning tree $\iff$ G is connected.

For any connected graph $G = (V, E)$ and any $F \subseteq E$:

(3.34) $\qquad F$ is a spanning tree $\iff$ F is an inclusionwise maximal forest $\iff$ F is an inclusionwise minimal edge set with (V, F) connected.

Cliques, stable sets, matchings, vertex covers, edge covers

Let $G = (V, E)$ be a graph. A subset C of V is called a *clique* if any two vertices in V are adjacent, a *stable set* if any two vertices in C are nonadjacent, and a *vertex cover* if C intersects each edge of G.

A subset M of E is called a *matching* if any two edges in M are disjoint, an *edge cover* if each vertex of G is covered by at least one edge in M, and a *perfect matching* if it is both a matching and an edge cover. So a perfect matching M satisfies $|M| = \frac{1}{2}|V|$.

We denote and define:

(3.35) $\omega(G) :=$ *clique number* of $G :=$ maximum size of a clique in G,
$\alpha(G) :=$ *stable set number* of $G :=$ maximum size of a stable set in G,
$\tau(G) :=$ *vertex cover number* of $G :=$ minimum size of a vertex cover in G,
$\nu(G) :=$ *matching number* of $G :=$ maximum size of a matching in G,
$\rho(G) :=$ *edge cover number* of $G :=$ minimum size of an edge cover in G.

(We will recall this notation where used.)

Given a matching M in a graph $G = (V, E)$, we will say that a vertex u is *matched to* a vertex v, or u is the *mate* of v, if $uv \in M$. A subset U of V is called *matchable* if the subgraph $G[U]$ of G induced by U has a perfect matching.

Colouring

A *vertex-colouring*, or just a *colouring*, is a partition of V into stable sets. We sometimes consider a colouring as a function $\phi : V \to \{1, \ldots, k\}$ such that $\phi^{-1}(i)$ is a stable set for each $i = 1, \ldots, k$.

Each of the stable sets in a colouring is called a *colour* of the colouring. The *vertex-colouring number*, or just the *colouring number*, is the minimum number of colours in a vertex-colouring. In notation,

(3.36) $\chi(G) :=$ vertex-colouring number of G.

A graph G is called *k-colourable*, or *k-vertex-colourable*, if $\chi(G) \leq k$, and *k-chromatic* if $\chi(G) = k$. A vertex-colouring is called a *minimum vertex-colouring*, or a *minimum colouring*, if it uses the minimum number of colours.

Similar terminology holds for edge-colouring. An *edge-colouring* is a partition of E into matchings. Each of these matchings is called a *colour* of the edge-colouring. An edge-colouring can also be described by a function $\phi : E \to \{1, \ldots, k\}$ such that $\phi^{-1}(i)$ is a matching for each $i = 1, \ldots, k$.

The *edge-colouring number* is the minimum number of colours in an edge-colouring. In notation, (3.37) $\chi'(G) :=$ edge-colouring number of G.

So $\chi'(G) = \chi(L(G))$.

A graph G is called *k-edge-colourable* if $\chi'(G) \le k$, and *k-edge-chromatic* if $\chi'(G) = k$. An edge-colouring is called a *minimum edge-colouring* if it uses the minimum number of colours.

Bipartite graphs

A graph $G = (V, E)$ is called *bipartite* if $\chi(G) \le 2$. Equivalently, G is bipartite if V can be partitioned into two sets U and W such that each edge of G connects U and W. We call the sets U and W the *colour classes* of G (although they generally need not be unique).

Bipartite graphs are characterized by:

(3.38) G is bipartite $\iff$ each circuit of G has even length.

A graph $G = (V, E)$ is called a *complete bipartite graph* if G is simple and V can be partitioned into sets U and W such that E consists of all pairs $\{u, w\}$ with $u \in U$ and $w \in W$. If $|U| = m$ and $|W| = n$, the graph is denoted by $K_{m,n}$:

(3.39) $K_{m,n} :=$ the complete bipartite graph with colour classes of size m and n.

The graphs $K_{1,n}$ are called *stars* or (when $n \ge 3$) *claws*.

Hamiltonian and Eulerian graphs

A *Hamiltonian circuit* in a graph G is a circuit C satisfying $VC = VG$. A graph is *Hamiltonian* if it has a Hamiltonian circuit. A *Hamiltonian path* is a path P with $VP = VG$.

A walk P is called *Eulerian* if each edge of G is traversed exactly once by P. A graph G is called *Eulerian* if it has a closed Eulerian walk. The following is usually attributed to Euler [1736] (although he only proved the 'only if' part):

(3.40) a graph $G = (V, E)$ without isolated vertices is Eulerian if and only if G is connected and all degrees of G are even.

Sometimes, we call a graph Eulerian if all degrees are even, ignoring connectivity. This will be clear from the context.

Contraction and minors

Let $G = (V, E)$ be a graph and let $e = uv \in E$. *Contracting* e means deleting e and identifying u and v. We denote (for $F \subseteq E$):

(3.41) $G/e :=$ graph obtained from G by contracting e,
$G/F :=$ graph obtained from G by contracting all edges in F.

The *image* of a vertex v of G in G/F is the vertex of G/F to which v is contracted.

A graph H is called a *minor* of a graph G if H arises from G by a series of deletions and contractions of edges and deletions of vertices. A minor H of G is called a *proper minor* if $H \neq G$. If G and H are graphs, we say that a minor G' of G is *an H minor* of G if G' is isomorphic to H.

Related is the following contraction. Let $G = (V, E)$ be a graph and let $S \subseteq V$. The graph G/S (obtained by *contracting* S) is obtained by identifying all vertices in S to one new vertex, called S, deleting all edges contained in S, and redefining any edge uv with $u \in S$ and $v \notin S$ to Sv.

Homeomorphic graphs

A graph G is called a *subdivision* of a graph H if G arises from H by replacing edges by paths of length at least 1. So it arises from H by iteratively choosing an edge uv, introducing a new vertex w, deleting edge uv, and adding edges uw and wv. If G is a subdivision of H, we call G an *H-subdivision*.

Two graphs G and G' are called *homeomorphic* if there exists a graph H such that both G and G' are subdivisions of H. The graph G is called a *homeomorph* of G' if G and G' are homeomorphic.

Homeomorphism can be described topologically. For any graph $G = (V, E)$, the *topological graph* $|G|$ associated with G is the topological space consisting of V and for each edge e of G a curve $|e|$ connecting the ends of e, such that for any two edges e, f one has $|e| \cap |f| = e \cap f$. Then

(3.42) G and H are homeomorphic graphs $\iff$ $|G|$ and $|H|$ are homeomorphic topological spaces.

Planarity

An *embedding* of a graph G in a topological space S is an embedding (continuous injection) of the topological graph $|G|$ in S. A graph G is called *planar* if it has an embedding in the plane $\mathbb{R}^2$.

Often, when dealing with a planar graph G, we assume that it *is* embedded in the plane $\mathbb{R}^2$. The topological components of $\mathbb{R}^2 \setminus |G|$ are called the *faces* of G. A vertex or edge is said to be *incident* with a face F if it is contained in the boundary of F. Two faces are called *adjacent* if they are incident with some common edge.

There is a unique *unbounded face*, all other faces are *bounded*. The boundary of the unbounded face is part of $|G|$, and is called the *outer boundary* of G.

Euler's formula states that any connected planar graph $G = (V, E)$, with face collection $\mathcal{F}$, satisfies:

$$(3.43) \qquad |V| + |\mathcal{F}| = |E| + 2.$$

Kuratowski [1930] found the following characterization of planar graphs:

Theorem 3.2 (Kuratowski's theorem)**.** *A graph G is planar $\Longleftrightarrow$ no subgraph of G is homeomorphic to K_5 or to $K_{3,3}$.*

(See Thomassen [1981b] for three short proofs, and for history and references to other proofs.)

As Wagner [1937a] noticed, the following is an immediate consequence of Kuratowski's theorem (since planarity is maintained under taking minors, and since any graph without K_5 minor has no subgraph homeomorphic to K_5):

(3.44) A graph G is planar $\Longleftrightarrow$ G has no K_5 or $K_{3,3}$ minor.

(In turn, with a little more work, this equivalence can be shown to imply Kuratowski's theorem.)

The *four-colour theorem* of Appel and Haken [1977] and Appel, Haken, and Koch [1977] states that each loopless planar graph is 4-colourable. (Robertson, Sanders, Seymour, and Thomas [1997] gave a shorter proof.)

Tait [1878b] showed that the four-colour theorem is equivalent to: each cubic bridgeless planar graph is 3-edge-colourable. Petersen [1898] gave the example of the now-called *Petersen graph* (Figure 3.1), to show that not every bridgeless cubic graph is 3-edge-colourable. (This graph was also given by Kempe [1886], for a different purpose.)

Wagner's theorem

We will use occasionally an extension of Kuratowski's theorem, proved by Wagner [1937a]. For this we need the notion of a k-sum of graphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and let $k := |V_1 \cap V_2|$. Suppose that $(V_1 \cap V_2, E_1 \cap E_2)$ is a (simple) complete graph. Then the graph

$$(3.45) \qquad (V_1 \cup V_2, E_1 \triangle E_2)$$

is called a *k-sum* of G_1 and G_2. We allow multiple edges, so the k-sum might keep edges spanned by $V_1 \cap V_2$.

To formulate Wagner's theorem, we also need the graph denoted by V_8, given in Figure 3.2.

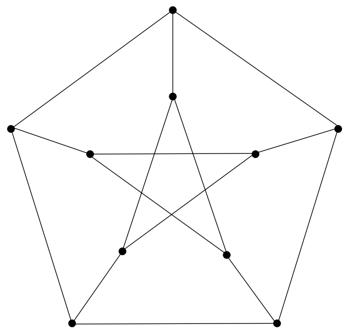

Figure 3.1
The Petersen graph

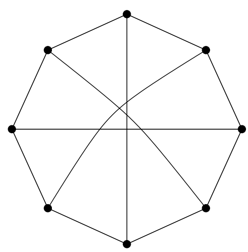

Figure 3.2
V_8

Theorem 3.3 (Wagner's theorem). *A graph G has no K_5 minor $\iff$ G can be obtained from planar graphs and from copies of V_8 by taking* 1-, 2-, *and* 3-*sums.*

As Wagner [1937a] pointed out, this theorem implies that the four-colour theorem is equivalent to: each graph without K_5 minor is 4-colourable. This follows from the fact that k-colourability is maintained under taking k'-sums for all $k' \leq k$.

The dual graph

The *dual* G^* of an embedded planar graph $G = (V, E)$ is the graph having as vertex set the set of faces of G and having, for each $e \in E$, an edge e^* connecting the two faces incident with e. Then G^* again is planar, and $(G^*)^*$ is isomorphic to G, if G is connected. For any $C \subseteq E$, C is a circuit in G if and only if $C^* := \{e^* \mid e \in C\}$ is an inclusionwise minimal nonempty cut in

G^*. Moreover, C is a spanning tree in G if and only if $\{e^* \mid e \in E \setminus C\}$ is a spanning tree in G^*.

Series-parallel and outerplanar graphs

A graph is called a *series-parallel graph* if it arises from a forest by repeated replacing edges by parallel edges or by edges in series. It was proved by Duffin [1965] that a graph is series-parallel if and only if it has no K_4 minor.

A graph is called *outerplanar* if it can be embedded in the plane such that each vertex is on the outer boundary. It can be easily derived from Kuratowski's theorem that a graph is outerplanar if and only if it has no K_4 or $K_{2,3}$ minor.

Adjacency and incidence matrix

The *adjacency matrix* of a graph $G = (V, E)$ is the $V \times V$ matrix A with

(3.46) $$A_{u,v} := \text{number of edges connecting } u \text{ and } v$$

for $u, v \in V$.

The *incidence matrix*, or $V \times E$ *incidence matrix*, of G is the $V \times E$ matrix B with

(3.47) $$B_{v,e} := \begin{cases} 1 & \text{if } v \in e \text{ and } e \text{ is not a loop,} \\ 2 & \text{if } v \in e \text{ and } e \text{ is a loop,} \\ 0 & \text{if } v \notin e, \end{cases}$$

for $v \in V$ and $e \in E$. The transpose of B is called the $E \times V$ incidence matrix of G, or just the incidence matrix, if no confusion is expected.

The concepts from graph theory invite to a less formal, and more expressive language, which appeals to the intuition, and whose formalization will be often tedious rather than problematic. Thus we say 'replace the edge uv by two edges in series', which means deleting uv and introducing a new vertex, w say, and new edges uw and wv. Similarly, 'replacing the edge uv by a path' means deleting uv, and introducing new vertices $w_1, \ldots, w_k$ say, and new edges uw_1, $w_1w_2, \ldots, w_{k-1}w_k$, w_kv.

3.2. Directed graphs

A *directed graph* or *digraph* is a pair $D = (V, A)$ where V is a finite set and A is a family of *ordered* pairs from V. The elements of V are called the *vertices*, sometimes the *nodes* or the *points*. The elements of A are called the *arcs* (sometimes *directed edges*). We denote:

(3.48) $VD :=$ set of vertices of D and $AD :=$ family of arcs of D.

In running time estimates of algorithms we denote:

(3.49) $n := |VD|$ and $m := |AD|$.

Again, the term 'family' is used to indicate that the same pair of vertices may occur several times in A. A pair occurring more than once in A is called a *multiple* arc, and the number of times it occurs is called its *multiplicity*. Two arcs are called *parallel* if they are represented by the same ordered pair of vertices.

Also *loops* are allowed, that is, arcs of the form (v, v). In our discussions, loops in directed graphs will be almost always irrelevant, and it will be clear from the context if they may occur. Directed graphs without loops and multiple arcs are called *simple*, and directed graphs without loops are called *loopless*.

Each directed graph $D = (V, A)$ gives rise to an *underlying undirected graph*, which is the graph $G = (V, E)$ obtained by ignoring the orientation of the arcs:

(3.50) $E := \{\{u, v\} \mid (u, v) \in A\}$.

We often will transfer undirected terminology to the directed case. Where appropriate, the adjective 'undirected' is added to a term if it refers to the underlying undirected graph.

If G is the underlying undirected graph of a directed graph D, we call D an *orientation* of G.

An arc (u, v) is said to *connect* u and v, and to *run from* u *to* v. For an arc $a = (u, v)$, u and v are called the *ends* of a, and u is called the *tail* of a, and v the *head* of a. We say that $a = (u, v)$ *leaves* u and *enters* v. For $U \subseteq V$, an arc $a = (u, v)$ is said to *leave* U if $u \in U$ and $v \notin U$. It is said to *enter* U if $u \notin U$ and $v \in U$.

If there exists an arc connecting vertices u and v, then u and v are called *adjacent* or *connected*. If there exists an arc (u, v), then v is called an *outneighbour* of u, and u is called an *inneighbour* of v.

The arc (u, v) is said to be *incident* with, or to *meet*, or to *cover*, the vertices u and v, and conversely. The arcs a and b are said to be *incident*, or to *meet*, or to *intersect*, if they have a vertex in common. Otherwise, they are called *disjoint*. If $U \subseteq V$ and both ends of an arc a belong to U, then we say that U *spans* a.

For any vertex v, we denote:

(3.51)
$$
\begin{aligned}
&\delta^{\text{in}}_D(v) := \delta^{\text{in}}_A(v) := \delta^{\text{in}}(v) := \text{set of arcs entering } v,\\
&\delta^{\text{out}}_D(v) := \delta^{\text{out}}_A(v) := \delta^{\text{out}}(v) := \text{set of arcs leaving } v,\\
&N^{\text{in}}_D(v) := N^{\text{in}}_A(v) := N^{\text{in}}(v) := \text{set of inneighbours of } v,\\
&N^{\text{out}}_D(v) := N^{\text{out}}_A(v) := N^{\text{out}}(v) := \text{set of outneighbours of } v.
\end{aligned}
$$

The *indegree* $\deg^{\text{in}}_D(v)$ of a vertex v is the number of arcs entering v. The *outdegree* $\deg^{\text{out}}_D(v)$ of a vertex v is the number of arcs leaving v. In notation,

(3.52)
$$\begin{aligned}&\deg_D^{\mathrm{in}}(v) := \deg_A^{\mathrm{in}}(v) := \deg^{\mathrm{in}}(v) := |\delta_D^{\mathrm{in}}(v)|,\\ &\deg_D^{\mathrm{out}}(v) := \deg_A^{\mathrm{out}}(v) := \deg^{\mathrm{out}}(v) := |\delta_D^{\mathrm{out}}(v)|.\end{aligned}$$

A vertex of indegree 0 is called a *source* and a vertex of outdegree 0 a *sink*. For any arc $a = (u, v)$ we denote

(3.53) $$a^{-1} := (v, u).$$

For any digraph $D = (V, A)$ the *reverse digraph* D^{-1} is defined by

(3.54) $$D^{-1} = (V, A^{-1}), \text{ where } A^{-1} := \{a^{-1} \mid a \in A\}.$$

A *mixed graph* is a triple (V, E, A) where (V, E) is an undirected graph and (V, A) is a directed graph.

The complete directed graph and the line digraph

The *complete directed graph* on a set V is the simple directed graph with vertex set V and arcs all pairs (u, v) with $u, v \in V$ and $u \neq v$. A *tournament* is any simple directed graph (V, A) such that for all distinct $u, v \in V$ precisely one of (u, v) and (v, u) belongs to A.

The *line digraph* of a directed graph $D = (V, A)$ is the digraph with vertex set A and arc set

(3.55) $$\{((u, v), (x, y)) \mid (u, v), (x, y) \in A, v = x\}.$$

Subgraphs of directed graphs

A digraph $D' = (V', A')$ is called a *subgraph* of a digraph $D = (V, A)$ if $V' \subseteq V$ and $A' \subseteq A$. If $D' \neq D$, then D' is called a *proper subgraph* of D. If $V' = V$, then D' is called a *spanning subgraph* of D. If A' consists of all arcs of D spanned by V', D' is called an *induced subgraph*, or the *subgraph induced by* V'. In notation,

(3.56)
$$\begin{aligned}&D[V'] := \text{subgraph of } D \text{ induced by } V',\\ &A[V'] := \text{family of arcs spanned by } V'.\end{aligned}$$

So $D[V'] = (V', A[V'])$. We further denote for any vertex v, any subset U of V, any arc a, and any subset B of A,

(3.57)
$$\begin{aligned}&D - v := D[V \setminus \{v\}],\ D - U := D[V \setminus U],\ D - a := (V, A \setminus \{a\}),\\ &D - B := (V, A \setminus B).\end{aligned}$$

We say that these graphs arise from D by *deleting* v, U, a, or B.

Two subgraphs of D are called *arc-disjoint* if they have no arc in common, and *vertex-disjoint* or *disjoint*, if they have no vertex in common.

Directed paths and circuits

A *directed walk*, or just a *walk*, in a directed graph $D = (V, A)$ is a sequence

(3.58) $\quad P = (v_0, a_1, v_1, \ldots, a_k, v_k),$

where $k \geq 0$, $v_0, v_1, \ldots, v_k \in V$, $a_1, \ldots, a_k \in A$, and $a_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, k$. The path is called a *directed path*, or just a *path*, if $v_0, \ldots, v_k$ are distinct. (Hence $a_1, \ldots, a_k$ are all distinct.)

The vertex v_0 is called the *starting vertex* or the *first vertex* of P, and the vertex v_k the *end vertex* or the *last vertex* of P. Sometimes, both v_0 and v_k are called the *end vertices*, or just the *ends* of P. Similarly, arc a_1 is called the *starting arc* or *first arc* of P and arc a_k the *end arc* or *last arc* of P. Sometimes, both a_1 and a_k are called the *end arcs*.

The walk P is said to *connect* the vertices v_0 and v_k, to *run from v_0 to v_k* (or *between v_0 and v_k*), and to *traverse* $v_0, a_1, v_1, \ldots, a_k, v_k$. The vertices $v_1, \ldots, v_{k-1}$ are called the *internal vertices* of P. For $s, t \in V$, a walk P is called an *$s - t$ walk* if it runs from s to t, and for $S, T \subseteq V$, P is called an *$S - T$ walk* if it runs from a vertex in S to a vertex in T. If P is a path, we obviously speak of an *$s - t$ path* and an *$S - T$ path*, respectively.

A vertex t is called *reachable from* a vertex s (or from a set S) if there exists a directed $s - t$ path (or directed $S - t$ path). Similarly, a vertex s is said to *reach*, or to be *reachable to*, a vertex t (or to a set T) if there exists a directed $s - t$ path (or directed $s - T$ path).

The number k in (3.58) is called the *length* of P. (We deviate from this in case a function $l : A \to \mathbb{R}$ has been introduced as a length function. Then the *length* of P is equal to $l(a_1) + \cdots + l(a_k)$.)

The minimum length of a path from u to v is called the *distance* from u to v.

An *undirected walk* in a directed graph $D = (V, A)$ is a walk in the underlying undirected graph; more precisely, it is a sequence

(3.59) $\quad P = (v_0, a_1, v_1, \ldots, a_k, v_k)$

where $k \geq 0$, $v_0, v_1, \ldots, v_k \in V$, $a_1, \ldots, a_k \in A$, and $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$ for $i = 1, \ldots, k$. The arcs a_i with $a_i = (v_{i-1}, v_i)$ are called the *forward arcs* of P, and the arcs a_i with $a_i = (v_i, v_{i-1})$ the *backward arcs* of P.

If $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$ and $Q = (u_0, b_1, u_1, \ldots, b_l, u_l)$ are walks satisfying $u_0 = v_k$, the *concatenation* PQ of P and Q is the walk

(3.60) $\quad PQ := (v_0, a_1, v_1, \ldots, a_k, v_k, b_1, u_1, \ldots, b_l, u_l).$

For any walk P, we denote by VP and AP the families of vertices and arcs, respectively, occurring in P:

(3.61) $\quad VP := \{v_0, v_1, \ldots, v_k\}$ and $AP := \{a_1, \ldots, a_k\}.$

If no confusion may arise, we sometimes identify the walk P with the subgraph (VP, AP) of D, or with the set VP of vertices in P, or with the family AP of arcs in P.

If the digraph is simple or (more generally) if the arcs traversed are irrelevant, we indicate the walk just by the sequence of vertices traversed:

(3.62) $P = (v_0, v_1, \ldots, v_k)$ or $P = v_0, v_1, \ldots, v_k$.

A path may be identified by the sequence of arcs:

(3.63) $P = (a_1, \ldots, a_k)$ or $P = a_1, \ldots, a_k$.

Two walks P and Q are called *vertex-disjoint* or *disjoint* if VP and VQ are disjoint, *internally vertex-disjoint* or *internally disjoint* if the set of internal vertices of P is disjoint from the set of internal vertices of Q, and *arc-disjoint* if AP and AQ are disjoint.

The directed walk P in (3.13) is called a *closed directed walk* or *directed cycle* if $v_k = v_0$. It is called a *directed circuit*, or just a *circuit*, if $v_k = v_0$, $k \geq 1$, $v_1, \ldots, v_k$ are all distinct, and $a_1, \ldots, a_k$ are all distinct. An *undirected circuit* is a circuit in the underlying undirected graph.

Connectivity and components of digraphs

A digraph $D = (V, A)$ is called *strongly connected* if for each two vertices u and v there is a directed path from u to v. The digraph D is called *weakly connected* if the underlying undirected graph is connected; that is, for each two vertices u and v there is an undirected path connecting u and v.

A maximal strongly connected nonempty subgraph of a digraph $D = (V, A)$ is called a *strongly connected component*, or a *strong component*, of D. Again, 'maximal' is taken with respect to taking subgraphs. A *weakly connected component*, or a *weak component*, of D is a component of the underlying undirected graph.

Each strong component is an induced subgraph. Each vertex belongs to exactly one strong component, but there may be arcs that belong to no strong component. One has:

(3.64) arc (u, v) belongs to a strong component $\Longleftrightarrow$ there exists a directed path in D from v to u.

We sometimes identify a strong component K with the set VK of its vertices. Then the strong components are precisely the equivalence classes of the equivalence relation $\sim$ defined on V by: $u \sim v \Longleftrightarrow$ there exist a directed path from u to v and a directed path from v to u.

Cuts

Let $D = (V, A)$ be a directed graph. For any $U \subseteq V$, we denote:

(3.65) $\delta_D^{\text{in}}(U) := \delta_A^{\text{in}}(U) := \delta^{\text{in}}(U) :=$ set of arcs of D entering U,
$\delta_D^{\text{out}}(U) := \delta_A^{\text{out}}(U) := \delta^{\text{out}}(U) :=$ set of arcs of D leaving U.

A subset B of A is called a *cut* if $B = \delta^{\text{out}}(U)$ for some $U \subseteq V$. In particular, $\emptyset$ is a cut. If $\emptyset \neq U \neq V$, then $\delta^{\text{out}}(U)$ is called a *nontrivial cut.*

If $s \in U$ and $t \notin U$, then $\delta^{\text{out}}(U)$ is called an $s-t$ *cut.* If $S \subseteq U$ and $T \subseteq V \setminus U$, $\delta^{\text{out}}(U)$ is called an $S-T$ *cut.* A cut of size k is called a k*-cut.*

A subset B of A is called a *disconnecting arc set* if $D - B$ is not strongly connected. For $s,t \in V$, it is said to be $s-t$ *disconnecting*, if B intersects each directed $s-t$ path. For $S,T \subseteq V$, B is said to be $S-T$ *disconnecting*, if B intersects each directed $S-T$ path.

One may easily check that for all $s,t \in V$:

(3.66) each $s-t$ cut is $s-t$ disconnecting; each inclusionwise minimal $s-t$ disconnecting arc set is an $s-t$ cut.

For any subset U of V we denote

(3.67) $d_D^{\text{in}}(U) := d_A^{\text{in}}(U) := d^{\text{in}}(U) := |\delta^{\text{in}}(U)|$,
$d_D^{\text{out}}(U) := d_A^{\text{out}}(U) := d^{\text{out}}(U) := |\delta^{\text{out}}(U)|$.

The following inequalities will be often used:

Theorem 3.4. *For any digraph $D = (V,A)$ and $X,Y \subseteq V$:*

(3.68) $d^{\text{in}}(X) + d^{\text{in}}(Y) \geq d^{\text{in}}(X \cap Y) + d^{\text{in}}(X \cup Y)$ *and*
$d^{\text{out}}(X) + d^{\text{out}}(Y) \geq d^{\text{out}}(X \cap Y) + d^{\text{out}}(X \cup Y)$,

Proof. The first inequality follows directly from the equation

(3.69) $d^{\text{in}}(X) + d^{\text{in}}(Y) =$
$d^{\text{in}}(X \cap Y) + d^{\text{in}}(X \cup Y) + |A[X \setminus Y, Y \setminus X]| + |A[Y \setminus X, X \setminus Y]|$,

where $A[S,T]$ denotes the set of arcs with tail in S and head in T. The second inequality follows similarly. ∎

A cut C is called a *directed cut* if $C = \delta^{\text{in}}(U)$ for some $U \subseteq V$ with $\delta^{\text{out}}(U) = \emptyset$ and $\emptyset \neq U \neq V$. An arc is called a *cut arc* if $\{a\}$ is a directed cut; equivalently, if a is a bridge in the underlying undirected graph.

Vertex-cuts

Let $D = (V,A)$ be a digraph. A subset U of V is called a *disconnecting vertex set*, or a *vertex-cut*, if $D - U$ is disconnected. A vertex-cut of size k is called a k*-vertex-cut.*

For $s,t \in V$, if U intersects each directed $s-t$ path in D, then U is said to *disconnect* s and t, or called $s-t$ *disconnecting*. If moreover $s,t \notin U$, then U is said to *separate* s and t, or called $s-t$ *separating*, or an $s-t$ *vertex-cut.*

For $S, T \subseteq V$, if U intersects each directed $S - T$ path, then U is said to *disconnect* S and T, or called $S - T$ *disconnecting*. If moreover U is disjoint from $S \cup T$, then U is said to *separate* S and T, or called $S - T$ *separating* or an $S - T$ *vertex-cut*.

Acyclic digraphs and directed trees

A directed graph $D = (V, A)$ is called *acyclic* if it has no directed circuits. It is easy to show that

(3.70) an acyclic digraph has at least one source and at least one sink,

provided that it has at least one vertex.

A directed graph is called a *directed tree* if the underlying undirected graph is a tree; that is, if D is weakly connected and has no undirected circuits. It is called a *rooted tree* if moreover D has precisely one source, called the *root*. If r is the root, we say that the rooted tree is *rooted at* r. If a rooted tree $D = (V, A)$ has root r, then each vertex $v \neq r$ has indegree 1, and for each vertex v there is a unique directed $r - v$ path. An *arborescence* in a digraph $D = (V, A)$ is a set B of arcs such that (V, B) is a rooted tree. If the rooted tree has root r, it is called an r*-arborescence*.

A directed graph is called a *directed forest* if the underlying undirected graph is a forest; that is, if D has no undirected circuits. It is called a *rooted forest* if moreover each weak component is a rooted tree. The roots of the weak components are called the *roots* of the rooted forest. A *branching* in a digraph $D = (V, A)$ is a set B of arcs such that (V, B) is a rooted forest.

Hamiltonian and Eulerian digraphs

A *Hamiltonian circuit* in a directed graph $D = (V, A)$ is a directed circuit C with $VC = VD$. A digraph is *Hamiltonian* if it has a Hamiltonian circuit. A *Hamiltonian path* is a directed path P with $VP = VD$.

A directed walk P is called *Eulerian* if each arc of D is traversed exactly once by P. A digraph D is called *Eulerian* if it has a closed Eulerian directed walk. Then a digraph $D = (V, A)$ is Eulerian if and only if D is weakly connected and $\deg^{\text{in}}(v) = \deg^{\text{out}}(v)$ for each vertex v. Sometimes, we call a digraph Eulerian if each weak component is Eulerian. This will be clear from the context.

An *Eulerian orientation* of an undirected graph $G = (V, E)$ is an orientation (V, A) of G with $\deg_A^{\text{in}}(v) = \deg_A^{\text{out}}(v)$ for each $v \in V$. A classical theorem in graph theory states that an undirected graph G has an Eulerian orientation if and only if all degrees of G are even.

Contraction

Contraction of directed graphs is similar to contraction of undirected graphs. Let $D = (V, A)$ be a digraph and let $a = (u, v) \in A$. *Contracting* a means deleting a and identifying u and v. We denote:

(3.71) $\quad D/a :=$ digraph obtained from D by contracting a.

Related is the following contraction. Let $D = (V, A)$ be a digraph and let $S \subseteq V$. The digraph D/S (obtained by *contracting* S) is obtained by identifying all vertices in S to one new vertex, called S, deleting all arcs contained in S, and redefining any arc (u, v) to (S, v) if $u \in S$ and to (u, S) if $v \in S$.

Planar digraphs and their duals

A digraph D is called *planar* if its underlying undirected graph G is planar. There is a natural way of making the dual graph G^* of G into a directed graph D^*, the *dual*: if arc $a = (u, v)$ of D separates faces F and F', such that, when following a from u to v, F is at the left and F' is at the right of a, then the dual edge is oriented from F to F', giving the arc a^* of D^*. Then D^{**} is isomorphic to D^{-1}, if D is weakly connected. One may check that a subset C of D is a directed circuit in D if and only if the set $\{a^* \mid a \in C\}$ is an inclusionwise minimal directed cut in D^*.

Adjacency and incidence matrix

The *adjacency matrix* of a digraph $D = (V, A)$ is the $V \times V$ matrix M with

(3.72) $\quad M_{u,v} :=$ number of arcs from u to v

for $u, v \in V$.

The *incidence matrix*, or $V \times A$ *incidence matrix*, of D is the $V \times A$ matrix B with

(3.73)
$$B_{v,a} := \begin{cases} -1 & \text{if } v \text{ is tail of } a, \\ +1 & \text{if } v \text{ is head of } a, \\ 0 & \text{otherwise,} \end{cases}$$

for any $v \in V$ and any nonloop $a \in A$. If a is a loop, we set $B_{v,a} := 0$ for each vertex v.

The transpose of B is called the $A \times V$ incidence matrix of D, or just the incidence matrix, if no confusion is expected.

3.3. Hypergraphs

Part VIII is devoted to hypergraphs, but we occasionally need the terminology of hypergraphs in earlier parts. A *hypergraph* is a pair $H = (V, \mathcal{E})$ where V is a finite set and $\mathcal{E}$ is a family of subsets of V. The elements of V and $\mathcal{E}$ are called the *vertices* and the *edges* respectively. If $|F| = k$ for each $F \in \mathcal{E}$, the hypergraph is called *k-uniform*.

A hypergraph $H = (V, \mathcal{E})$ is called *connected* if there is no $U \subseteq V$ such that $\emptyset \neq U \neq V$ and such that $F \subseteq U$ or $F \subseteq V \setminus U$ for each edge F. A *(connected) component* of H is a hypergraph $K = (V', \mathcal{E}')$ with $V' \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$, such that V' and $\mathcal{E}'$ are inclusionwise maximal with the property that K is connected. A component is uniquely identified by its set of vertices.

Packing and covering

A family $\mathcal{F}$ of sets is called a *packing* if the sets in $\mathcal{F}$ are pairwise disjoint. For $k \in \mathbb{Z}_+$, $\mathcal{F}$ is called a *k-packing* if each element of $\bigcup \mathcal{F}$ is in at most k sets in $\mathcal{F}$ (counting multiplicities). In other words, any $k+1$ sets from $\mathcal{F}$ have an empty intersection. If each set in $\mathcal{F}$ is a subset of some set S, and $c : S \to \mathbb{R}$, then $\mathcal{F}$ is called a *c-packing* if each element $s \in S$ is in at most $c(s)$ sets in $\mathcal{F}$ (counting multiplicities).

A *fractional packing* is a function $\lambda : \mathcal{F} \to \mathbb{R}_+$ such that, for each $s \in S$,

$$\sum_{\substack{U \in \mathcal{F} \\ s \in U}} \lambda_U \leq 1. \tag{3.74}$$

For $c : S \to \mathbb{R}$, the function $\lambda : \mathcal{F} \to \mathbb{R}_+$ is called a *fractional c-packing* if

$$\sum_{U \in \mathcal{F}} \lambda_U \chi^U \leq c. \tag{3.75}$$

The *size* of $\lambda : \mathcal{F} \to \mathbb{R}$ is, by definition,

$$\sum_{U \in \mathcal{F}} \lambda_U. \tag{3.76}$$

Similarly, a family $\mathcal{F}$ of sets is called a *covering* of a set S if S is contained in the union of the sets in $\mathcal{F}$. For $k \in \mathbb{Z}_+$, $\mathcal{F}$ is called a *k-covering* of S if each element of S is in at least k sets in $\mathcal{F}$ (counting multiplicities). For $c : S \to \mathbb{R}$, $\mathcal{F}$ is called a *c-covering* if each element $s \in S$ is in at least $c(s)$ sets in $\mathcal{F}$ (counting multiplicities).

A *fractional covering* of S is a function $\lambda : \mathcal{F} \to \mathbb{R}_+$ such that, for each $s \in S$,

$$\sum_{\substack{U \in \mathcal{F} \\ s \in U}} \lambda_U \geq 1. \tag{3.77}$$

For $c : S \to \mathbb{R}$, the function $\lambda : \mathcal{F} \to \mathbb{R}_+$ is called a *fractional c-covering* if

$$(3.78) \qquad \sum_{U \in \mathcal{F}} \lambda_U \chi^U \geq c.$$

Again, the *size* of $\lambda : \mathcal{F} \to \mathbb{R}$ is, by definition,

$$(3.79) \qquad \sum_{U \in \mathcal{F}} \lambda_U.$$

Cross-free and laminar families

A collection $\mathcal{C}$ of subsets of a set V is called *cross-free* if for all $T, U \in \mathcal{C}$:

$$(3.80) \qquad T \subseteq U \text{ or } U \subseteq T \text{ or } T \cap U = \emptyset \text{ or } T \cup U = V.$$

$\mathcal{C}$ is called *laminar* if for all $T, U \in \mathcal{C}$:

$$(3.81) \qquad T \subseteq U \text{ or } U \subseteq T \text{ or } T \cap U = \emptyset.$$

There is the following upper bound on the size of a laminar family:

Theorem 3.5. *If $\mathcal{C}$ is laminar and $V \neq \emptyset$, then $|\mathcal{C}| \leq 2|V|$.*

Proof. By induction on $|V|$. We can assume that $|V| \geq 2$ and that $V \in \mathcal{C}$. Let U be an inclusionwise minimal set in $\mathcal{C}$ with $|U| \geq 2$. Resetting $\mathcal{C}$ to $\mathcal{C} \setminus \{\{v\} \mid v \in U\}$, and identifying all elements in U, $|\mathcal{C}|$ decreases by at most $|U|$, and $|V|$ by $|U| - 1$. Since $|U| \leq 2(|U| - 1)$ (as $|U| \geq 2$), induction gives the required inequality. ∎

3.3a. Background references on graph theory

For background on graph theory we mention the books by Kőnig [1936] (historical), Harary [1969] (classical reference book), Wilson [1972b] (introductory), Bondy and Murty [1976], and Diestel [1997].

Chapter 4

Preliminaries on algorithms and complexity

This chapter gives an introduction to algorithms and complexity, in particular to polynomial-time solvability and NP-completeness. We restrict ourselves to a largely informal outline and keep formalisms at a low level. Most of the formalisms described in this chapter are not needed in the remaining of this book. A rough understanding of algorithms and complexity suffices.

4.1. Introduction

An informal, intuitive idea of what is an algorithm will suffice to understand the greater part of this book. An algorithm can be seen as a finite set of instructions that perform operations on certain data. The input of the algorithm will give the initial data. When the algorithm stops, the output will be found in prescribed locations of the data set. The instructions need not be performed in a linear order: an instruction determines which of the instructions should be followed next. Also, it can prescribe to stop the algorithm.

While the set of instructions constituting the algorithm is finite and fixed, the size of the data set may vary, and will depend on the input. Usually, the data are stored in arrays, that is, finite sequences. The lengths of these arrays may depend on the input, but the number of arrays is fixed and depends only on the algorithm. (A more-dimensional array like a matrix is stored in a linear fashion, in accordance with the linear order in which computer memory is organized.)

The data may consist of numbers, letters, or other symbols. In a computer model they are usually stored as finite strings of 0's and 1's (*bits*). The *size* of the data is the total length of these strings. In this context, the *size* of a rational number p/q with $p, q \in \mathbb{Z}$, $q \geq 1$, and g.c.d.$(p.q) = 1$, is equal to $1 + \lceil \log(|p| + 1) \rceil + \lceil \log q \rceil$.

4.2. The random access machine

We use the algorithmic model of the *random access machine*, sometimes abbreviated to *RAM*. It operates on entries that are $0, 1$ strings, representing abstract objects (like vertices of a graph) or rational numbers. An instruction can read several (but a fixed number of) entries simultaneously, perform arithmetic operations on them, and store the answers in array positions prescribed by the instruction[2]. The array positions that should be read and written, are given in locations prescribed by the instruction.

We give a more precise description. The random access machine has a finite set of variables $z_0, \ldots, z_k$ and one array, f say, of length depending on the input. Each array entry is a $0, 1$ string. They can be interpreted as rationals, in some binary encoding, but can also have a different meaning. Initially, $z_0, \ldots, z_k$ are set to 0, and f contains the input.

Each instruction is a finite sequence of resettings of one the following types, for $i, j, h \in \{1, \ldots, k\}$:

(4.1) $$z_i := f(z_j);\ f(z_j) := z_i;\ z_i := z_j + z_h;\ z_i := z_j - z_h;\ z_i := z_j z_h;$$
$$z_i := z_j / z_h;\ z_i := z_i + 1;\ z_i := 1 \text{ if } z_j > 0 \text{ and } z_i := 0 \text{ otherwise.}$$

These include the *elementary arithmetic operations*: addition, subtraction, multiplication, division, comparison. (One may derive other arithmetic operations from this like rounding and taking logarithm or square root, by performing $O(\sigma + |\log \varepsilon|)$ elementary arithmetic operations, where σ is the size of the rational number and ε is the required precision.)

The instructions are numbered $0, 1, \ldots, t$, and z_1 is the number of the instruction to be executed. If $z_1 > t$ we stop and return the contents of the array f as output.

4.3. Polynomial-time solvability

A *polynomial-time algorithm* is an algorithm that terminates after a number of steps bounded by a polynomial in the input size. Here a *step* consists of performing one instruction. Such an algorithm is also called a *good algorithm* or an *efficient algorithm*.

In this definition, the *input size* is the size of the input, that is, the number of bits that describe the input. We say that a problem is *polynomial-time solvable*, or is *solvable in polynomial time*, if it can be solved by a polynomial-time algorithm.

This definition may depend on the chosen algorithmic model, but it has turned out that for most models the set of problems solvable by a polynomial-time algorithm is the same. However, in giving order estimates of running

[2] This property has caused the term 'random' in random access machine: the machine has access, in constant time, to the data in *any* (however, well-determined) position. This is in contrast with the Turing machine, which can only move to adjacent positions.

times and in considering the concept of 'strongly polynomial-time' algorithm (cf. Section 4.12), we fix the above algorithmic model of the random access machine.

4.4. P

P, NP, and co-NP are collections of *decision problems*: problems that can be answered by 'yes' or 'no', like whether a given graph has a perfect matching or a Hamiltonian circuit. An optimization problem is no decision problem, but often can be reduced to it in a certain sense — see Section 4.7 below.

A decision problem is completely described by the inputs for which the answer is 'yes'. To formalize this, fix some finite set Σ, called the *alphabet*, of size at least 2 — for instance $\{0,1\}$ or the ASCII-set of symbols. Let Σ^* denote the set of all finite strings (*words*) of letters from Σ. The *size* of a word is the number of letters (counting multiplicities) in the word. We denote the size of a word w by $\text{size}(w)$.

As an example, an undirected graph can be represented by the word

$$(\{a,b,c,d\},\{\{a,b\},\{b,c\},\{a,d\},\{b,d\},\{a,c\}\}) \tag{4.2}$$

(assuming that Σ contains each of these symbols). Its size is 43.

A *problem* is any subset Π of Σ^*. The corresponding 'informal' problem is:

$$\text{given a word } x \in \Sigma^*\text{, does } x \text{ belong to } \Pi? \tag{4.3}$$

As an example, the problem if a given graph is Hamiltonian is formalized by the collection of all strings representing a Hamiltonian graph.

The string x is called the *input* of the problem. One speaks of an *instance* of a problem Π if one asks for one concrete input x whether x belongs to Π.

A problem Π is called *polynomial-time solvable* if there exists a polynomial-time algorithm that decides whether or not a given word $x \in \Sigma^*$ belongs to Π. The collection of all polynomial-time solvable problems $\Pi \subseteq \Sigma^*$ is denoted by P.

4.5. NP

An easy way to characterize the class NP is: NP is the collection of decision problems that can be reduced in polynomial time to the satisfiability problem — that is, to checking if a Boolean expression can be satisfied. For instance, it is not difficult to describe the conditions for a perfect matching in a graph by a Boolean expression, and hence reduce the existence of a perfect matching to the satisfiability of this expression. Also the problem of finding a Hamiltonian circuit, or a clique of given size, can be treated this way.

However, this is not the definition of NP, but a theorem of Cook. Roughly speaking, NP is defined as the collection of all decision problems for which each input with positive answer, has a polynomial-time checkable 'certificate' of correctness of the answer. Consider, for instance, the question:

(4.4) Is a given graph Hamiltonian?

A positive answer can be 'certified' by giving a Hamiltonian circuit in the graph. The correctness of it can be checked in polynomial time. No such certificate is known for the opposite question:

(4.5) Is a given graph non-Hamiltonian?

Checking the certificate in polynomial time means: checking it in time bounded by a polynomial in the original input size. In particular, it implies that the certificate itself has size bounded by a polynomial in the original input size.

This can be formalized as follows. NP is the collection of problems $\Pi \subseteq \Sigma^*$ for which there is a problem $\Pi' \in$P and a polynomial p such that for each $w \in \Sigma^*$ one has:

(4.6) $w \in \Pi \iff$ there exists a word x of size at most $p(\text{size}(w))$ with $wx \in \Pi'$.

The word x is called a *certificate* for w. (NP stands for *nondeterministically polynomial-time*, since the string x could be chosen by the algorithm by guessing. So guessing well leads to a polynomial-time algorithm.)

For instance, the collection of Hamiltonian graphs belongs to NP since the collection Π' of strings GC, consisting of a graph G and a Hamiltonian circuit C in G, belongs to P. (Here we take graphs and circuits as strings like (4.2).)

Trivially, we have P$\subseteq$NP, since if $\Pi \in$P, we can take $\Pi' = \Pi$ and $p \equiv 0$ in (4.6).

About all problems that ask for the existence of a structure of a prescribed type (like a Hamiltonian circuit) belong to NP. The class NP is apparently much larger than the class P, and there might be not much reason to believe that the two classes are the same. But, as yet, nobody has been able to prove that they really are different. This is an intriguing mathematical question, but besides, answering the question might also have practical significance. If P=NP can be shown, the proof might contain a revolutionary new algorithm, or alternatively, it might imply that the concept of 'polynomial-time' is completely meaningless. If P$\neq$NP can be shown, the proof might give us more insight in the reasons why certain problems are more difficult than other, and might guide us to detect and attack the kernel of the difficulties.

4.6. co-NP and good characterizations

The collection co-NP consists of all problems Π for which the complementary problem $\Sigma^* \setminus \Pi$ belongs to NP. Since for any problem $\Pi \in$P, also $\Sigma^* \setminus \Pi$ belongs to P, we have

(4.7) P⊆NP∩co-NP.

The problems in NP∩co-NP are those for which both a positive answer and a negative answer have a polynomial-time checkable certificate. In other words, any problem Π in NP∩co-NP has a *good characterization*: there exist $\Pi', \Pi'' \in$P and a polynomial p such that for each $w \in \Sigma^*$:

(4.8) there is an $x \in \Sigma^*$ with $wx \in \Pi'$ and $\text{size}(x) \leq p(\text{size}(w)) \iff$ there is no $y \in \Sigma^*$ with $wy \in \Pi''$ and $\text{size}(y) \leq p(\text{size}(w))$.

Therefore, the problems in NP∩co-NP are called *well-characterized*.

A typical example is Tutte's 1-factor theorem:

(4.9) a graph $G = (V, E)$ has a perfect matching if and only if there is no $U \subseteq V$ such that $G - U$ has more than $|U|$ odd components.

So in this case Π consists of all graphs having a perfect matching, Π' of all strings GM where G is a graph and M a perfect matching in G, and Π'' of all strings GU where G is a graph and U is a subset of the vertex set of G such that $G - U$ has more than $|U|$ odd components. (To be more precise, since Σ^* is the universe, we must add all strings $w\{\}$ to Π'' where w is a word in Σ^* that does not represent a graph.) This is why Tutte's theorem is said to be a good characterization.

In fact, there are very few problems known that have been proved to belong to NP∩co-NP, but that are not known to belong to P. Most problems having a good characterization, have been proved to be solvable in polynomial time. So one may ask: is P=NP∩co-NP?

4.7. Optimization problems

Optimization problems can be transformed to decision problems as follows. Consider a *minimization* problem: minimize $f(x)$ over $x \in X$, where X is a collection of elements derived from the input of the problem, and where f is a rational-valued function on X. (For instance, minimize the length of a Hamiltonian circuit in a given graph, for a given length function on the edges.) This can be transformed to the following decision problem:

(4.10) given a rational number r, is there an $x \in X$ with $f(x) \leq r$?

If we have an upper bound β on the size of the minimum value (being proportional to the sum of the logarithms of the numerator and the denominator), then by asking question (4.10) for $O(\beta)$ choices of r, we can find the optimum value (by binary search). In this way we usually can derive a polynomial-time algorithm for the minimization problem from a polynomial-time algorithm for the decision problem. Similarly, for maximization problems.

About all combinatorial optimization problems, when framed as a decision problem like (4.10), belong to NP, since a positive answer to question (4.10) can often be certified by just specifying an $x \in X$ satisfying $f(x) \leq r$.

If a combinatorial optimization problem is characterized by a min-max relation like

$$(4.11) \qquad \min_{x \in X} f(x) = \max_{y \in Y} g(y),$$

this often leads to a good characterization of the corresponding decision problem. Indeed, if $\min_{x \in X} f(x) \leq r$ holds, it can be certified by an $x \in X$ satisfying $f(x) \leq r$. On the other hand, if $\min_{x \in X} f(x) > r$ holds, it can be certified by a $y \in Y$ satisfying $g(y) > r$. If these certificates can be checked in polynomial time, we say that the min-max relation is a *good characterization*, and that the optimization problem is *well-characterized*.

4.8. NP-complete problems

The NP-complete problems are the problems that are the hardest in NP: every problem in NP can be reduced to them. We make this more precise.

Problem $\Pi \subseteq \Sigma^*$ is said to be *reducible* to problem $\Lambda \subseteq \Sigma^*$ if there exists a polynomial-time algorithm that returns, for any input $w \in \Sigma^*$, an output $x \in \Sigma^*$ with the property:

$$(4.12) \qquad w \in \Pi \iff x \in \Lambda.$$

This implies that if Π is reducible to Λ and Λ belongs to P, then also Π belongs to P. Similarly, one may show that if Π is reducible to Λ and Λ belongs to NP, then also Π belongs to NP.

A problem Π is said to be NP-*complete* if each problem in NP is reducible to Π. Hence

(4.13) if some NP-complete problem belongs to P, then P=NP.

Surprisingly, there exist NP-complete problems (Cook [1971]). Even more surprisingly, several prominent combinatorial optimization problems, like the traveling salesman problem, the maximum clique problem, and the maximum cut problem, are NP-complete (Karp [1972b]).

Since then one generally distinguishes between the polynomial-time solvable problems and the NP-complete problems, although there is no proof that these two concepts really are distinct. For almost every combinatorial optimization problem (and many other problems) one has been able to prove either that it is solvable in polynomial time, or that it is NP-complete — and no problem has been proved to be both. But it still has not been excluded that these two concepts are just the same!

The usual approach to prove NP-completeness of problems is to derive it from the NP-completeness of one basic problem, often the satisfiability problem. To this end, we prove NP-completeness of the satisfiability problem in the coming sections.

4.9. The satisfiability problem

To formulate the satisfiability problem, we need the notion of a *Boolean expression*. Examples are:

$$((x_2 \wedge x_3) \vee \neg(x_3 \vee x_5) \wedge x_2),\ ((\neg x_{47} \wedge x_2) \wedge x_{47}),\ \text{and}\ \neg(x_7 \wedge \neg x_7). \tag{4.14}$$

Boolean expressions can be defined inductively. We work with an alphabet Σ containing the 'special' symbols '(', ')', '$\wedge$', '$\vee$', '$\neg$', and ',', and not containing the symbols 0 and 1. Then any word not containing any special symbol is a Boolean expression, called a *variable*. Next, if v and w are Boolean expressions, then also $(v \wedge w)$, $(v \vee w)$, and $\neg v$ are Boolean expressions. These rules give us all Boolean expressions. We denote a Boolean expression f by $f(x_1, \ldots, x_k)$ if $x_1, \ldots, x_k$ are the variables occurring in f.

A Boolean expression $f(x_1, \ldots, x_k)$ is called *satisfiable* if there exist $\alpha_1, \ldots, \alpha_k \in \{0, 1\}$ such that $f(\alpha_1, \ldots, \alpha_k) = 1$, using the well-known identities

$$\begin{aligned} &0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0, 1 \wedge 1 = 1,\\ &0 \vee 0 = 0, 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1,\\ &\neg 0 = 1, \neg 1 = 0, (0) = 0, (1) = 1. \end{aligned} \tag{4.15}$$

Now let SAT$\subseteq \Sigma^*$ be the collection of satisfiable Boolean expressions. SAT is called the *satisfiability problem*.

The satisfiability problem SAT trivially belongs to NP: to certify that $f(x_1, \ldots, x_k)$ belongs to SAT, we can take the equations $x_i = \alpha_i$ that give f the value 1.

4.10. NP-completeness of the satisfiability problem

Let an algorithm be represented by the random access machine (we use notation as in Section 4.2). Consider the performance of the algorithm for some input w of size s (in the alphabet $\{0, 1\}$). We may assume that all entries in the random access machine are stored with the same number of bits, α say, only depending on s. Let q be the length of the array f. We may assume that q is invariant throughout the algorithm, and that q only depends on s. (So the initial input w is extended to an array f of length q.) Let r be the number of iterations performed by the algorithm. We may assume that r only depends on s.

Let m_i be the following word in $\{0, 1\}^*$:

(4.16) $z_0 z_1 \ldots z_k f(0) f(1) \ldots f(q)$

after performing i iterations (where each z_j and each $f(j)$ is a word in $\{0,1\}^*$ of size α). So it is the content of the machine memory after i iterations. We call the word

(4.17) $h = m_0 m_1 \ldots m_r$

the *history.* The size of h is equal to

(4.18) $T := (r+1)(k+q+2)\alpha.$

We call a word h *correct* if there is an input w of size s that leads to history h.

The following observation is basic:

(4.19) given the list of instructions describing the random access machine and given s, we can construct, in time bounded by a polynomial in T, a Boolean expression $g(x_1, \ldots, x_T)$ such that any 0,1 word $h = \alpha_1 \ldots \alpha_T$ is correct if and only if $g(\alpha_1, \ldots, \alpha_T) = 1$.

To see this, we must observe that each of the instructions (4.1) can be described by Boolean expressions in the 0,1 variables describing the corresponding entries.

We can permute the positions in g such that the first s variables correspond to the s input bits, and that the last variable gives the output bit (0 or 1). Let it give the Boolean expression $\tilde{g}(y_1, \ldots, y_T)$. Then input $\beta_1 \ldots \beta_s$ leads to output 1 if and only if

(4.20) $\tilde{g}(\beta_1, \ldots, \beta_s, y_{s+1}, \ldots, y_{T-1}, 1) = 1$

has a solution in the variables $y_{s+1}, \ldots, y_{T-1}$.

Consider now a problem Π in NP. Let Π' be a problem in P and p a polynomial satisfying (4.6). We can assume that x has size precisely $p(\text{size}(w))$. So if input w of Π has size u, then wx has size $s := u + p(u)$. Let A be a polynomial-time algorithm as described above solving Π' and let $\tilde{g}$ be the corresponding Boolean expression as above. Let $w = \beta_1 \ldots \beta_u$. Then w belongs to Π if and only if

(4.21) $\tilde{g}(\beta_1, \ldots, \beta_u, y_{u+1}, \ldots, y_s, y_{s+1}, \ldots, y_{T-1}, 1) = 1$

is solvable. This reduces Π to the satisfiability problem. Hence we have the main result of Cook [1971] (also Levin [1973]):

Theorem 4.1. *The satisfiability problem is* NP*-complete.*

Proof. See above. ∎

4.11. NP-completeness of some other problems

For later reference, we derive from Cook's theorem the NP-completeness of some other problems. First we show that the 3-*satisfiability problem* 3-SAT is NP-complete (Cook [1971], cf. Karp [1972b]). Let B_1 be the set of all words $x_1, \neg x_1, x_2, \neg x_2, \ldots$, where the x_i are words not containing the symbols '$\neg$', '$\wedge$', '$\vee$', '(', ')'. Let B_2 be the set of all words $(w_1 \vee \cdots \vee w_k)$, where $w_1, \cdots, w_k$ are words in B_1 and $1 \leq k \leq 3$. Let B_3 be the set of all words $w_1 \wedge \ldots \wedge w_k$, where $w_1, \ldots, w_k$ are words in B_2. Again, we say that a word $f(x_1, x_2, \ldots) \in B_3$ is *satisfiable* if there exists an assignment $x_i := \alpha_i \in \{0, 1\}$ $(i = 1, 2, \ldots)$ such that $f(\alpha_1, \alpha_2, \ldots) = 1$ (using the identities (4.15)).

Now the 3-satisfiability problem 3-SAT is: given a word $f \in B_3$, decide if it is satisfiable. More formally, 3-SAT is the set of all satisfiable words in B_3.

Corollary 4.1a. *The* 3-*satisfiability problem* 3-SAT *is* NP-*complete.*

Proof. We give a polynomial-time reduction of SAT to 3-SAT. Let $f(x_1, x_2, \ldots)$ be a Boolean expression. Introduce a variable y_g for each subword g of f that is a Boolean expression (not splitting variables).

Now f is satisfiable if and only if the following system is satisfiable:

$$\text{(4.22)} \qquad \begin{array}{ll} y_g = y_{g'} \vee y_{g''} & (\text{if } g = (g' \vee g'')), \\ y_g = y_{g'} \wedge y_{g''} & (\text{if } g = (g' \wedge g'')), \\ y_g = \neg y_{g'} & (\text{if } g = \neg g'), \\ y_f = 1. & \end{array}$$

Now $y_g = y_{g'} \vee y_{g''}$ can be equivalently expressed by: $y_g \vee \neg y_{g'} = 1, y_g \vee \neg y_{g''} = 1, \neg y_g \vee y_{g'} \vee y_{g''} = 1$. Similarly, $y_g = y_{g'} \wedge y_{g''}$ can be equivalently expressed by: $\neg y_g \vee y_{g'} = 1, \neg y_g \vee y_{g''} = 1, y_g \vee \neg y_{g'} \vee \neg y_{g''} = 1$. The expression $y_g = \neg y_{g'}$ is equivalent to: $y_g \vee y_{g'} = 1, \neg y_g \vee \neg y_{g'} = 1$.

By renaming variables, we thus obtain words $w_1, \ldots, w_k$ in B_2, such that f is satisfiable if and only if the word $w_1 \wedge \ldots \wedge w_k$ is satisfiable. ∎

(As Cook [1971] mentioned, a method of Davis and Putnam [1960] solves the 2-satisfiability problem in polynomial time.)

We next derive that the *partition problem* is NP-complete (Karp [1972b]). This is the problem:

(4.23) Given a collection of subsets of a finite set X, does it contain a subcollection that is a partition of X?

Corollary 4.1b. *The partition problem is* NP-*complete.*

Proof. We give a polynomial-time reduction of 3-SAT to the partition problem. Let $f = w_1 \wedge \ldots \wedge w_k$ be a word in B_3, where $w_1, \ldots, w_k$ are words in B_2. Let $x_1, \ldots, x_m$ be the variables occurring in f. Make a bipartite graph G with colour classes $\{w_1, \ldots, w_k\}$ and $\{x_1, \ldots, x_m\}$, by joining w_i and x_j by an edge if and only if x_j or $\neg x_j$ occurs in w_i. Let X be the set of all vertices and edges of G.

Let $\mathcal{C}'$ be the collection of all sets $\{w_i\} \cup E'$, where E' is a nonempty subset of the edge set incident with w_i. Let $\mathcal{C}''$ be the collection of all sets $\{x_j\} \cup E'_j$ and $\{x_j\} \cup E''_j$, where E'_j is the set of all edges $\{w_i, x_j\}$ such that x_j occurs in w_i and where E''_j is the set of all edges $\{w_i, x_j\}$ such that $\neg x_j$ occurs in w_i.

Now f is satisfiable if and only if the collection $\mathcal{C}' \cup \mathcal{C}''$ contains a subcollection that partitions X. Thus we have a reduction of 3-SAT to the partition problem. ∎

In later chapters we derive from these results the NP-completeness of several other combinatorial optimization problems.

4.12. Strongly polynomial-time

Roughly speaking, an algorithm is strongly polynomial-time if the number of elementary arithmetic and other operations is bounded by a polynomial in the size of the input, where any number in the input is counted only for 1. Strong polynomial-timeness of an algorithm is of relevance only for problems that have numbers among its input data. (Otherwise, strongly polynomial-time coincides with polynomial-time.)

Consider a problem that has a number k of input parts, like a vertex set, an edge set, a length function. Let $f : \mathbb{Z}_+^{2k} \to \mathbb{R}$. We say that an algorithm takes $O(f)$ time if the algorithm terminates after

(4.24) $$O(f(n_1, s_1, \ldots, n_k, s_k))$$

operations (including elementary arithmetic operations), where the ith input part consists of n_i numbers of maximum size s_i ($i = 1, \ldots, k$), and if the numbers occurring during the execution of the algorithm have size

(4.25) $$O(\max\{s_1, \ldots, s_k\}).$$

The algorithm is called a *strongly polynomial-time algorithm* if the algorithm takes $O(f)$ time for some polynomial f in the array lengths $n_1, \ldots, n_k$, where f is independent of $s_1, \ldots, s_k$. If a problem can be solved by a strongly polynomial-time algorithm, we say that it is *solvable in strongly polynomial time* or *strongly polynomial-time solvable.*

An algorithm is called *linear-time* if f can be taken linear in $n_1, \ldots, n_k$, and independent of $s_1, \ldots, s_k$. If a problem can be solved by a linear-time algorithm, we say that it is *solvable in linear time* or *linear-time solvable.*

Rounding a rational x to $\lfloor x \rfloor$ can be done in polynomial-time, by $O(\text{size}(x))$ elementary arithmetic operations. It however cannot be done in strongly polynomial time. In fact, even checking if an integer k is odd or even cannot be done in strongly polynomial time: for any strongly polynomial-time algorithm with one integer k as input, there is a number L and a rational function $q : \mathbb{Z} \to \mathbb{Q}$ such that if $k > L$, then the output equals $q(k)$. (This can be proved by induction on the number of steps of the algorithm.) However, there do not exist a rational function q and number L such that for $k > L$, $q(k) = 0$ if k is even, and $q(k) = 1$ if k is odd.

We say that an algorithm is *semi-strongly polynomial-time* if we count rounding a rational as one step (one time-unit). We sometimes say *weakly polynomial-time* for polynomial-time, to distinguish from strongly polynomial-time.

4.13. Lists and pointers

Algorithmically, sets (of vertices, edges, etc.) are often introduced and handled as *ordered* sets, called *lists*. Their elements can be indicated just by their positions (*addresses*) in the order: $1, 2, \ldots$. Then attributes (like the capacity, or the ends, of an edge) can be specified in arrays.

Arrays represent functions, and such functions are also called *pointers* if their value is taken as an address. Such functions also allow the value *void*, where the function is undefined. Pointers can be helpful to shorten the running time of an algorithm.

One way to store a list is just in an array. But then updating may take (relatively) much time, for instance, if we would like to perform operations on lists, such as removing or inserting elements or concatenating two lists.

A better way to store a list $S = \{s_1, \ldots, s_k\}$ is as a *linked list*. This is given by a pointer $f : S \setminus \{s_k\} \to S$ where $f(s_i) = s_{i+1}$ for $i = 1, \ldots, k-1$, together with the first element s_1 given by the variable b say (a fixed array of length 1). It makes that S can be scanned in time $O(|S|)$.

If we need to update the list after removing an element from S, it is convenient to store S as a *doubly linked list*. Then we keep, next to f and b, a pointer $g : S \setminus \{s_1\} \to S$ where $g(s_i) = s_{i-1}$ for $i = 2, \ldots, k$, and a variable l say, with $l := s_k$. The virtue of this data structure is that it can be restored in constant time if we remove some element s_j from S. Also concatenating two doubly linked lists can be done in constant time. It is usually easy to build up the doubly linked list along with reading the input, taking time $O(|S|)$.

A convenient (but usually too abundant) way to store a directed graph $D = (V, A)$ using these data structures is as follows. For each $v \in V$, order the sets $\delta^{\text{in}}(v)$ and $\delta^{\text{out}}(v)$. Store V as a doubly linked list. Give pointers $t, h : A \to V$, where $t(a)$ and $h(a)$ are the tail and head of a. Give four pointers $V \to A$, indicating the first and last (respectively) arc in the lists $\delta^{\text{in}}(v)$ and $\delta^{\text{out}}(v)$ (respectively). Give four pointers $A \to A$, indicating for each $a \in A$, the previous and next (respectively) arc in the lists $\delta^{\text{in}}(h(a))$ and $\delta^{\text{out}}(t(a))$ (respectively). (Values may be 'void'. One can avoid the value 'void' by merging the latter eight pointers described into four pointers $V \cup A \to V \cup A$.)

If, in the input of a problem, a directed graph is given as a string (or file), like

(4.26) $$(\{a, b, c, d\}, \{(a, c), (a, d), (b, d), (c, d)\}),$$

we can build up the above data structure in time linear in the length of the string. Often, when implementing a graph algorithm, a subset of this structure will be sufficient. Undirected graphs can be handled similarly by choosing an arbitrary orientation of the edges. (So each edge becomes a list.)

4.14. Further notes

4.14a. Background literature on algorithms and complexity

Background literature on algorithms and complexity includes Knuth [1968] (data structures), Garey and Johnson [1979] (complexity, NP-completeness), Papadimitriou and Steiglitz [1982] (combinatorial optimization and complexity), Aho, Hopcroft, and Ullman [1983] (data structures and complexity), Tarjan [1983] (data structures), Cormen, Leiserson, and Rivest [1990] (algorithms), Papadimitriou [1994] (complexity), Sipser [1997] (algorithms, complexity), and Mehlhorn and Näher [1999] (data structures, algorithms and algorithms).

In this book we restrict algorithms and complexity to deterministic, sequential, and exact. For other types of algorithms and complexity we refer to the books by Motwani and Raghavan [1995] (randomized algorithms and complexity), Leighton [1992,2001] (parallel algorithms and complexity), and Vazirani [2001] (approximation algorithms and complexity). A survey on practical problem solving with cutting planes was given by Jünger, Reinelt, and Thienel [1995].

4.14b. Efficiency and complexity historically

In the history of complexity, more precisely, in the conception of the notions 'polynomial-time' and 'NP-complete', two lines loom up: one motivated by questions in logic, recursion, computability, and theorem proving, the other more down-to-earth focusing on the complexity of some concrete problems, with background in discrete mathematics and operations research.

Until the mid-1960s, the notions of efficiency and complexity were not formalized. The notion of algorithm was often used for a method that was better than brute-force enumerating. We focus on how the ideas of polynomial-time and NP-complete got shape. We will not go into the history of data structures, abstract computational complexity, or the subtleties inside and beyond NP (for which we refer to Papadimitriou [1994]).

We quote references in chronological order. This order is quite arbitrary, since the papers mostly seem to be written isolated from each other and they react very seldom to each other.

Maybe the first paper that was concerned with the complexity of computation is an article by Lamé [1844], who showed that the number of iterations in the Euclidean g.c.d. algorithm is linear in the logarithm of the smallest of the two (natural) numbers:

> Dans les traités d'Arithmétique, on se contente de dire que le nombre des divisions à effectuer, dans la recherche du plus grand commun diviseur entre deux entiers, *ne pourra pas surpasser la moitié du plus petit.* Cette limite, qui peut être dépassée si les nombres sont petits, s'éloigne outre mesure quand ils ont plusieurs chiffres. L'exagération est alors semblable à celle qui assignerait la moitié d'un nombre comme la limite de son logarithme; l'analogie devient évidente quand on connaît le théorème suivant:
>
> Théorème. *Le nombre des divisions à effectuer, pour trouver le plus grand commun diviseur entre deux entiers* A, *et* B<A, *est toujours moindre que cinq fois le nombre des chiffres de* B.[3]

The first major combinatorial optimization problem for which a polynomial-time algorithm was given is the shortest spanning tree problem, by Borůvka [1926a, 1926b] and Jarník [1930], but these papers do not discuss the complexity issue — the efficiency of the method might have been too obvious. Choquet [1938] mentioned explicitly an estimate for the number of iterations in finding a shortest spanning tree:

> Le réseau cherché sera tracé après $2n$ opérations élémentaires au plus, en appelant opération élémentaire la recherche du continu le plus voisin d'un continu donné.[4]

The traveling salesman and the assignment problem

The traveling salesman problem and the assignment problem have been long-term bench-marks that gave shape to the ideas on efficiency and complexity.

Menger might have been the first to ask attention for the complexity of the traveling salesman problem. In the session of 5 February 1930 of his *mathematische Kolloquium* in Vienna (as reported in Menger [1932a]), he introduced *das Botenproblem*, later called the traveling salesman problem and raised the question for a better-than-finite algorithm:

> Dieses Problem ist natürlich stets durch endlichviele Versuche lösbar. Regeln, welche die Anzahl der Versuche unter die Anzahl der Permutationen der gegebenen Punkte herunterdrücken würden, sind nicht bekannt.[5]

[3] In the handbooks of Arithmetics, one contents oneself with saying that, in the search for the greatest common divisor of two integers, the number of divisions to execute *could not surpass half of the smallest [integer].* This bound, that can be exceeded if the numbers are small, goes away beyond measure when they have several digits. The exaggeration then is similar to that which would assign half of a number as bound of its logarithm; the analogy becomes clear when one knows the following theorem:

Theorem. *The number of divisions to execute, to find the greatest common divisor of two integers* A, *and* B<A, *is always smaller than five times the number of digits of* B.

[4] The network looked for will be traced after at most $2n$ elementary operations, calling the search for the continuum closest to a given continuum an elementary operation.

[5] Of course, this problem is solvable by finitely many trials. Rules which would push the number of trials below the number of permutations of the given points, are not known.

Ghosh [1949] observed that the problem of finding a shortest tour along n random points in the plane (which is the traveling salesman problem) is hard:

> After locating the n random points in a map of the region, it is very difficult to find out *actually* the shortest path connecting the points, unless the number n is very small, which is seldom the case for a large-scale survey.

We should realize however that at that time also the (now known to be polynomial-time solvable) assignment problem was considered to be hard. In an Address delivered on 9 September 1949 at a meeting of the American Psychological Association at Denver, Colorado, Thorndike [1950] studied the problem of the 'classification' of personnel:

> There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

But, in a RAND Report dated 5 December 1949, Robinson [1949] reported that an 'unsuccessful attempt' to solve the traveling salesman problem, led her to a 'cycle-cancelling' method for the optimum assignment problem, which in fact stands at the basis of efficient algorithms for network problems. She gave an optimality criterion for the assignment problem (absence of negative-length cycles in the residual graph). As for the traveling salesman problem she mentions:

> Since there are only a finite number of paths to consider, the problem consists in finding a method for picking out the optimal path when n is moderately large, say $n = 50$. In this case, there are more than 10^{62} possible paths, so we can not simply try them all. Even for as few as 10 points, some short cuts are desirable.

She also observed that the number of feasible solutions is not a measure for the complexity (where 'it' refers to the assignment problem):

> However at first glance, it looks more difficult than the traveling salesman probl[e]m, for there are obviously many more systems of circuits than circuits.

The development of the simplex method for linear programming, and its, in practice successful, application to combinatorial optimization problems like assignment and transportation, led to much speculation on the theoretical efficiency of the simplex method. In his paper describing the application of the simplex method to the transportation problem, Dantzig [1951a] mentioned (after giving a variable selection criterion that he speculates to lead to favourable computational experience for large-scale practical problems):

> This does not mean that theoretical problems could not be "cooked up" where this criterion is weak, but that in practical problems the number of steps has not been far from $m + n - 1$.

(Here n and m are the numbers of vertices and arcs, respectively.)

At the Symposium on Linear Inequalities and Programming in Washington, D.C. in 1951, Votaw and Orden [1952] reported on early computational results with the simplex method (on the SEAC), and claimed (without proof) that the simplex method is polynomial-time for the transportation problem (a statement refuted by Zadeh [1973a]):

> As to computation time, it should be noted that for moderate size problems, say $m \times n$ up to 500, the time of computation is of the same order of magnitude as the time required to type the initial data. The computation time on a sample computation in which m and n were both 10 was 3 minutes. The time of computation can be shown by study of the computing method and the code to be proportional to $(m+n)^3$.

Another early mention of polynomial-time as efficiency criterion is by von Neumann, who considered the complexity of the assignment problem. In a talk in the Princeton University Game Seminar on 26 October 1951, he described a method which is equivalent to finding a best strategy in a certain zero-sum two-person game. According to a transcript of the talk (cf. von Neumann [1951,1953]), von Neumann noted the following on the number of steps:

> It turns out that this number is a moderate power of n, i.e., considerably smaller than the "obvious" estimate $n!$ mentioned earlier.

However, no further argumentation is given.

In a Cowles Commission Discussion Paper of 2 April 1953, also Beckmann and Koopmans [1953] asked for better-than-finite methods for the assignment problem, but no explicit complexity measure was proposed, except that the work should be reduced to 'manageable proportions':

> It should be added that in all the assignment problems discussed, there is, of course, the obvious brute force method of enumerating all assignments, evaluating the maximand at each of these, and selecting the assignment giving the highest value. This is too costly in most cases of practical importance, and by a method of solution we have meant a procedure that reduces the computational work to manageable proportions in a wider class of cases.

During the further 1950s, better-than-finite methods were developed for the assignment and several other problems like shortest path and maximum flow. These methods turned out to give polynomial-time algorithms (possibly after modification), and several speedups were found — but polynomial-time was, as yet, seldom marked as efficiency criterion. The term 'algorithm' was often used just to distinguish from complete enumeration, but no mathematical characterization was given.

Kuhn [1955b,1956] introduced the 'Hungarian method' for the assignment problem (inspired by the proof method of Egerváry [1931]). Kuhn contented himself with showing finiteness of the method, but Munkres [1957] showed that it is strongly polynomial-time:

> The final maximum on the number of operations needed is
>
> $$(11n^3 + 12n^2 + 31n)/6.$$
>
> This maximum is of theoretical interest, since it is much smaller than the $n!$ operations necessary in the most straightforward attack on the problem.

As for the maximum flow problem, Ford and Fulkerson [1955,1957b] showed that their augmenting path method is finite, but only Dinits [1970] and Edmonds and Karp [1970,1972] showed that it can be adapted to be (strongly) polynomial-time.

Several algorithms were given for finding shortest paths (Shimbel [1955], Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957], Bellman [1958], Dantzig [1958,1960], Dijkstra [1959], Moore [1959]), and most of them are obviously strongly polynomial-time. (Ford [1956] gave a liberal shortest path algorithm that may require exponential time (Johnson [1973a,1973b,1977a]).)

Similarly, the interest in the shortest spanning tree problem revived, leading to old and new strongly polynomial-time algorithms (Kruskal [1956], Loberman and Weinberger [1957], Prim [1957], and Dijkstra [1959]).

The traveling salesman problem resisted these efforts. In the words of Dantzig, Fulkerson, and Johnson [1954a,1954b]:

> Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much;

The papers [3,7,8] referred to, are the papers Dantzig [1951a], Votaw and Orden [1952], and von Neumann [1953], quoted above.

The use of the word 'Although' in the above quote makes it unclear what Dantzig, Fulkerson, and Johnson considered to be an algorithm. Their algorithm uses polyhedral methods to solve the traveling salesman problem, while Dantzig [1951a] and Votaw and Orden [1952] apply the simplex method to solve the assignment and transportation problems. In a follow-up paper, Dantzig, Fulkerson, and Johnson [1959] seem to have come to the conclusion that both methods are of a comparable level:

> Neither does the example, as we have solved it, indicate how one could make the combinatorial analysis a routine procedure. This can certainly be done (by enumeration, if nothing else)—but the fundamental question is: does the use of a few linear inequalities in general reduce the combinatorial magnitude of such problems significantly?
>
> We do not know the answer to this question in any theoretical sense, but it is our feeling, based on our experience in using the method, that it does afford a practical means of computing optimal tours in problems that are not too huge. It should be noted that a similar question, for example, arises when one uses the simplex method to find optimal solutions to linear programs, since no one has yet proved that the simplex method cuts down the computational task significantly from the crude method of examining all basic solutions, say. Nonetheless, people do use the simplex method because of successful experience with many hundreds of practical problems.

The feeling that the traveling salesman problem is more complex than the assignment problem was stated by Tompkins [1956]:

> A traveling-salesman problem is in some respects similar to the assignment problem. It seems definitely more difficult, however.

Tompkins described a branch-and-bound scheme to the permutation problem (including assignment and traveling salesman), but said:

> It must be noted, however, that this is not a completely satisfactory scheme for solution of such problems. In a few important cases (such as the assignment problem) more efficient machine methods have been devised.

The available algorithms for the traveling salesman problem were also not acceptable to Flood [1956]:

> There are as yet no acceptable computational methods, and surprisingly few mathematical results relative to the problem.

He mentioned that the problem might be 'fundamentally complex':

> Very recent mathematical work on the traveling-salesman problem by I. Heller, H.W. Kuhn, and others indicates that the problem is fundamentally complex. It seems very likely that quite a different approach from any yet used may be required for succesful treatment of the problem. In fact, there may well be no general method for treating the problem and impossibility results would also be valuable.

Logic and computability

Parallel to those motivated by concrete combinatorial problems, interest in complexity arose in the circles of logicians and recursion theorists.

A first quote is from a letter of K. Gödel to J. von Neumann of 20 March 1956. (The letter was reviewed by Hartmanis [1989], to whose attention it was brought by G. Heise. A reproduction and full translation was given by Sipser [1992].)

Turing [1937] proved that there is no algorithm that decides if a given statement in full first-order predicate logic has a proof (the unsolvability of Hilbert's *Entscheidungsproblem* of the *engere Funktionskalkül* (which is the term originally used by Hilbert for full first-order predicate calculus; Turing [1937] translated it into restricted functional calculus)). It implies the result of Gödel [1931] that there exist propositions A such that neither A nor $\neg A$ is provable (in the formalism of the Principia Mathematica).

But a *given* proof can algorithmically be checked, hence there is a finite algorithm to check if there exists a proof of any prescribed length n (simply by enumeration). Nowadays it is known that this is in fact an NP-complete problem (the satisfiability problem is a special case). Gödel asked for the opinion of von Neumann on whether a proof could be found algorithmically in time linear (or else quadratic) in the length of the proof — quite a bold statement, which Gödel yet seemed to consider plausible:

> Man kann offenbar leicht eine Turingmaschine konstruieren, welche von jeder Formel F des engeren Funktionenkalküls u. jeder natürl. Zahl n zu entscheiden gestattet ob F einen Beweis der Länge n hat [Länge = Anzahl der Symbole]. Sei $\psi(F, n)$ die Anzahl der Schritte die die Maschine dazu benötigt u. sei $\varphi(n) = \max_F \psi(F, n)$. Die Frage ist, wie rasch $\varphi(n)$ für eine optimale Maschine wächst. Man kann zeigen $\varphi(n) \geq Kn$. Wenn es wirklich eine Maschine mit $\varphi(n) \sim K.n$ (oder auch nur $\sim Kn^2$) gäbe, hätte das Folgerungen von der grössten Tragweite. Es würde nämlich offenbar bedeuten, dass man trotz der Unlösbarkeit des Entscheidungsproblems die Denkarbeit des Mathematikers bei ja-oder-nein Fragen vollständig* durch Maschinen ersetzen könnte. Man müsste ja bloss das n so gross wählen, dass, wenn die Maschine kein Resultat liefert es auch keinen Sinn hat über das Problem nachzudenken. Nun scheint es mir aber durchaus im Bereich der Möglichkeit zu liegen, dass $\varphi(n)$ so langsam wächst. Denn 1.) scheint $\varphi(n) \geq Kn$ die einzige Abschätzung zu sein, die man durch eine Verallgemeinerung des Beweises für die Unlösbarkeit des Entscheidungsproblems erhalten kann; 2. bedeutet ja $\varphi(n) \sim K.n$ (oder $\sim Kn^2$) bloss, dass die Anzahl der Schritte gegenüber <u>dem blossen Probieren</u> von N auf $\log N$ (oder $(\log N)^2$) verringert werden kann. So starke Verringerungen kommen aber bei andern finiten Problemen durchaus vor, z.B. bei der Berechnung eines quadratischen Restsymbols durch wiederholte Anwendung des Reziprozitätsgesetzes. Es wäre interessant zu wissen, wie es damit z.B. bei der Feststellung, ob eine Zahl Primzahl ist, steht u. wie stark <u>im allgemeinen</u> bei finiten kombinatorischen Problemen die Anzahl der Schritte <u>gegenüber dem</u> blossen Probieren verringert werden kann.
>
> * abgesehen von der Aufstellung der Axiome[6]

[6] Clearly, one can easily construct a Turing machine, which makes it possible to decide, for each formula F of the restricted functional calculus and each natural number n, whether F has a proof of length n [length = number of symbols]. Let $\psi(F, n)$ be the number of steps that the machine needs for that and let $\varphi(n) = \max_F \psi(F, n)$. The question is, how fast $\varphi(n)$ grows for an optimal machine. One can show $\varphi(n) \geq Kn$.

(For integers a, p with p prime, the Legendre symbol $\left(\frac{a}{p}\right)$ indicates if a is a quadratic residue mod p (that is, if $x^2 = a \pmod{p}$ has an integer solution x), and can be calculated by $\log a + \log p$ arithmetic operations (using the Jacobi symbol and the reciprocity law) — so Gödel took the logarithms of the numbers as size.)

The unavoidability of brute-force search for finding the smallest Boolean representation for a function was claimed by Yablonskiĭ [1959] (cf. Trakhtenbrot [1984]).

Davis and Putnam [1960] gave a method for the satisfiability problem (in reaction to earlier, exponential-time methods of Gilmore [1960] and Wang [1960] based on elimination of variables), which they claimed to have some (not exactly formulated) efficiency:

> In the present paper, a uniform proof procedure for quantification theory is given which is feasible for use with some rather complicated formulas and which does not ordinarily lead to exponentiation.

(It was noticed later by Cook [1971] that Davis and Putnam's method gives a polynomial-time method for the 2-satisfiability problem.)

A mathematical framework for computational complexity of algorithms was set up by Hartmanis and Stearns [1965]. They counted the number of steps made by a multitape Turing machine to solve a decision problem. They showed that for all 'real-time countable' functions f, g (which include all functions n^k, k^n, $n!$, and sums, products, and compositions of them) the following holds: if each problem solvable in time $O(f)$ is also solvable in time $O(g)$, then $f = O(g^2)$. This implies, for instance, that there exist problems solvable in time $O(n^5)$ but not in time $O(n^2)$, and problems solvable in time $O(2^n)$ but not in time $O(2^{n/3})$ (hence not in polynomial time).

Polynomial-time

In the summer of 1963, at a Workshop at the RAND Corporation, Edmonds discovered that shrinking leads to a polynomial-time algorithm to find a maximum-size matching in any graph — a basic result in graph algorithmics. It was described in the paper Edmonds [1965d] (received November 22, 1963), in which he also gave his views on algorithms and complexity:

When really there were a machine with $\varphi(n) \sim K.n$ (or even just $\sim Kn^2$), that would have consequences of the largest impact. In particular, it would obviously mean that, despite the unsolvability of the Entscheidungsproblem, one could replace the brainwork of the mathematician in case of yes-or-no questions fully* by machines. One should indeed only choose n so large that if the machine yields no result, there is also no sense in thinking about the problem. Now it seems to me, however, to lie completely within the range of possibility that $\varphi(n)$ grows that slowly. Because 1.) $\varphi(n) \geq Kn$ seems to be the only estimate that one can obtain by a generalization of the proof for the unsolvability of the Entscheidungsproblem; 2. $\varphi(n) \sim K.n$ (or $\sim Kn^2$) means indeed only that the number of steps can be reduced compared to mere trying from N to $\log N$ (or $(\log N)^2$). Such strong reductions occur however definitely at other finite problems, e.g. at the calculation of a quadratic residue symbol by repeated application of the reciprocity law. It would be interesting to know how this is e.g. for the decision if a number is prime, and how strong in general, for finite combinatorial problems, the number of steps can be reduced compared to mere trying.

* apart from the set-up of the axioms

> For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance." This is roughly the meaning I want—in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good."
> I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum size matching in a graph.
> There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

Moreover:

> For practical purposes the difference between algebraic and exponential order is often more crucial than the difference between finite and non-finite.

In another paper, Edmonds [1965c] introduced the term *good characterization*:

> We seek a good characterization of the minimum number of independent sets into which the columns of a matrix of M_F can be partitioned. As the criterion of "good" for the characterization we apply the "principle of the absolute supervisor." The good characterization will describe certain information about the matrix which the supervisor can require his assistant to search out along with a minimum partition and which the supervisor can then use with ease to verify with mathematical certainty that the partition is indeed minimum. Having a good characterization does not mean necessarily that there is a good algorithm. The assistant might have to kill himself with work to find the information and the partition.

Further motivation for polynomial-time solvability was given by Edmonds [1967b]:

> An algorithm which is good in the sense used here is not necessarily very good from a practical viewpoint. However, the good-versus-not-good dichotomy is useful. It is easily formalized (say, relative to a Turing machine, or relative to a typical digital computer with an unlimited supply of tape), and usually it is easily recognized informally. Within limitations it does have practical value, and it does admit refinements to "how good" and "how bad". The classes of problems which are respectively known and not known to have good algorithms are very interesting theoretically.

Edmonds [1967a] conjectured that there is no polynomial-time algorithm for the traveling salesman problem — in language developed later, this is equivalent to NP$\neq$P:

> I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know.

Also Cobham [1965] singled out polynomial-time as a complexity criterion, in a paper on Turing machines and computability, presented at the 1964 International Congress on Logic, Methodology and Philosophy of Science in Jerusalem (denoting the size of n by $l(n)$):

> To obtain some idea as to how we might go about the further classification of relatively simple functions, we might take a look at how we ordinarily set about computing some of the more common of them. Suppose, for example, that m and n are two numbers given in decimal notation with one written above the other and their right ends aligned. Then to add m and n we start at the right and

> proceed digit-by-digit to the left writing down the sum. No matter how large m and n, this process terminates with the answer after a number of steps equal at most to one greater than the larger of $l(m)$ and $l(n)$. Thus the process of adding m and n can be carried out in a number of steps which is bounded by a linear polynomial in $l(m)$ and $l(n)$. Similarly, we can multiply m and n in a number of steps bounded by a quadratic polynomial in $l(m)$ and $l(n)$. So, too, the number of steps involved in the extraction of square roots, calculation of quotients, etc., can be bounded by polynomials in the lengths of the numbers involved, and this seems to be a property of simple functions in general. This suggests that we consider the class, which I will call $\mathcal{L}$, of all functions having this property.

At a symposium in New York in 1966, also Rabin [1967] noted the importance of polynomial-time solvability:

> In the following we shall consider an algorithm to be practical if, for automata with n states, it requires at most cn^k (k is a fixed integer and c a fixed constant) computational steps. This stipulation is, admittedly, both vague and arbitrary. We do not, in fact cannot, define what is meant by a computational step, thus have no precise and general measure for the complexity of algorithms. Furthermore, there is no compelling reason to classify algorithms requiring cn^k steps as practical.
> Several points may be raised in defense of the above stipulation. In every given algorithm the notion of a computational step is quite obvious. Hence there is not much vagueness about the measure of complexity of existing algorithms. Another significant pragmatic fact is that all existing algorithms either require up to about n^4 steps or else require 2^n or worse steps. Thus drawing the line of practicality between algorithms requiring n^k steps and algorithms for which no such bound exists seems to be reasonable.

NP-completeness

Cook [1971] proved the NP-completeness of the satisfiability problem ('Theorem 1') and of the 3-satisfiability problem and the subgraph problem ('Theorem 2') and mentioned (the class of polynomial-time solvable problems is denoted by $\mathcal{L}_*$; { tautologies } is the satisfiability problem):

> Theorem 1 and its corollary give strong evidence that it is not easy to determine whether a given proposition formula is a tautology, even if the formula is in normal disjunctive form. Theorems 1 and 2 together suggest that it is fruitless to search for a polynomial decision procedure for the subgraph problem, since success would bring polynomial decision procedures to many other apparently intractable problems. Of course, the same remark applies to any combinatorial problem to which { tautologies } is P-reducible.
> Furthermore, the theorems suggest that { tautologies } is a good candidate for an interesting set not in $\mathcal{L}_*$, and I feel it is worth spending considerable effort trying to prove this conjecture. Such a proof would be a major breakthrough in complexity theory.

So Cook conjectured that NP$\neq$P.

Also Levin [1973] considered the distinction between NP and P:

> After the concept of the algorithm had been fully refined, the algorithmic unsolvability of a number of classical large-scale problems was proved (including the problems of the identity of elements of groups, the homeomorphism of varieties, the solvability of the Diophantine equations, etc.). These findings dispensed with the question of finding a practical technique for solving the indicated problems. However, the existence of algorithms for the solution of other problems does not

> eliminate the analogous question, because the volume of work mandated by those algorithms is fantastically large. This is the situation with so-called sequential (or exhaustive) search problems, including: the minimization of Boolean functions, the search for proofs of finite length, the determination of the isomorphism of graphs, etc. All of these problems are solved by trivial algorithms entailing the sequential scanning of all possibilities. The operating time of the algorithms, however, is exponential, and mathematicians nurture the conviction that it is impossible to find simpler algorithms.

Levin next announced that any problem in NP (in his terminology, any 'sequential search problem') can be reduced to the satisfiability problem, and to a few other problems.

The wide extent of NP-completeness was disclosed by Karp [1972b], by showing that a host of prominent combinatorial problems is NP-complete, therewith revealing the fissure in the combinatorial optimization landscape. According to Karp, his theorems

> strongly suggest, but do not imply, that these problems, as well as many others, will remain intractable perpetually.

Karp also introduced the notation P and NP, and in a subsequent paper, Karp [1975] introduced the term NP-complete.

Sipser [1992] gave an extensive account on the history of the P=NP question. Hartmanis [1989] reviewed the historic setting of 'Gödel, von Neumann and the P=?NP Problem'. Other papers on the history of complexity are Hartmanis [1981], Trakhtenbrot [1984] (Russian approaches), Karp [1986], and Iri [1987] (the Japanese view).

Chapter 5

Preliminaries on polyhedra and linear and integer programming

This chapter surveys what we need on polyhedra and linear and integer programming. Most background can be found in Chapters 7–10, 14, 16, 19, 22, and 23 of Schrijver [1986b]. We give proofs of a few easy further results that we need in later parts of the present book.

The results of this chapter are mostly formulated for real space, but are maintained when restricted to rational space. So the symbol $\mathbb{R}$ can be replaced by the symbol $\mathbb{Q}$. In applying these results, we add the adjective *rational* when we restrict ourselves to rational numbers.

5.1. Convexity and halfspaces

A subset C of $\mathbb{R}^n$ is *convex* if $\lambda x + (1-\lambda)y$ belongs to C for all $x, y \in C$ and each λ with $0 \leq \lambda \leq 1$. A *convex body* is a compact convex set.

The *convex hull* of a set $X \subseteq \mathbb{R}^n$, denoted by conv.hull$X$, is the smallest convex set containing X. Then:

(5.1) $$\text{conv.hull}X = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \geq 1, x_1, \ldots, x_k \in X, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+, \lambda_1 + \cdots + \lambda_k = 1\}.$$

A useful fundamental result was proved by Carathéodory [1911]:

Theorem 5.1 (Carathéodory's theorem)**.** *For any $X \subseteq \mathbb{R}^n$ and $x \in$ conv.hull$X$, there exist affinely independent vectors $x_1, \ldots, x_k$ in X with $x \in \text{conv.hull}\{x_1, \ldots, x_k\}$.*

(Corollary 7.1f in Schrijver [1986b].)

A subset H of $\mathbb{R}^n$ is called an *affine halfspace* if $H = \{x \mid c^\mathsf{T} x \leq \delta\}$, for some $c \in \mathbb{R}^n$ with $c \neq \mathbf{0}$ and some $\delta \in \mathbb{R}$. If $\delta = 0$, then H is called a *linear halfspace*.

Let $X \subseteq \mathbb{R}^n$. The set conv.hull$X + \mathbb{R}^n_+$ is called the *up hull* of X, and the set conv.hull$X - \mathbb{R}^n_+$ the *down hull* of X.

5.2. Cones

A subset C of $\mathbb{R}^n$ is called a *(convex) cone* if $C \neq \emptyset$ and $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \in \mathbb{R}_+$. The cone *generated by* a set X of vectors is the smallest cone containing X:

(5.2) $$\text{cone}X = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \geq 0, \lambda_1, \ldots, \lambda_k \geq 0, x_1, \ldots, x_k \in X\}.$$

There is a variant of Carathéodory's theorem:

Theorem 5.2. *For any $X \subseteq \mathbb{R}^n$ and $x \in \text{cone}X$, there exist linearly independent vectors $x_1, \ldots, x_k$ in X with $x \in \text{cone}\{x_1, \ldots, x_k\}$.*

A cone C is *polyhedral* if there is a matrix A such that

(5.3) $$C = \{x \mid Ax \leq \mathbf{0}\}.$$

Equivalently, C is polyhedral if it is the intersection of finitely many linear halfspaces.

Results of Farkas [1898,1902], Minkowski [1896], and Weyl [1935] imply that

(5.4) a convex cone is polyhedral if and only if it is finitely generated,

where a cone C is *finitely generated* if $C = \text{cone}X$ for some finite set X. (Corollary 7.1a in Schrijver [1986b].)

5.3. Polyhedra and polytopes

A subset P of $\mathbb{R}^n$ is called a *polyhedron* if there exists an $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$ (for some $m \geq 0$) such that

(5.5) $$P = \{x \mid Ax \leq b\}.$$

So P is a polyhedron of and only if it is the intersection of finitely many affine halfspaces. If (5.5) holds, we say that $Ax \leq b$ *determines* P. Any inequality $c^{\mathsf{T}}x \leq \delta$ is called *valid* for P if $c^{\mathsf{T}}x \leq \delta$ holds for each $x \in P$.

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. Motzkin [1936] showed:

(5.6) a set P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some cone C.

(Corollary 7.1b in Schrijver [1986b].) If $P \neq \emptyset$, then C is unique and is called the *characteristic cone* char.cone(P) of P. Then:

(5.7) $$\text{char.cone}(P) = \{y \in \mathbb{R}^n \mid \forall x \in P \forall \lambda \geq 0 : x + \lambda y \in P\}.$$

If $P = \emptyset$, then by definition its characteristic cone is char.cone$(P) := \{\mathbf{0}\}$.

(5.6) implies the following fundamental result (Minkowski [1896], Steinitz [1916], Weyl [1935]):

(5.8) a set P is a polytope if and only if P is a bounded polyhedron.

(Corollary 7.1c in Schrijver [1986b].)

A polyhedron P is called *rational* if it is determined by a rational system of linear inequalities. Then a rational polytope is the convex hull of a finite number of rational vectors.

5.4. Farkas' lemma

A system $Ax \leq b$ is called *feasible* (or *solvable*) if it has a solution x. Feasibility of a system $Ax \leq b$ of linear inequalities is characterized by *Farkas' lemma* (Farkas [1894,1898], Minkowski [1896]):

Theorem 5.3 (Farkas' lemma). *$Ax \leq b$ is feasible $\iff$ $y^{\mathsf{T}} b \geq 0$ for each $y \geq \mathbf{0}$ with $y^{\mathsf{T}} A = \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1e in Schrijver [1986b].) Theorem 5.3 is equivalent to:

Corollary 5.3a (Farkas' lemma — variant). *$Ax = b$ has a solution $x \geq \mathbf{0}$ $\iff$ $y^{\mathsf{T}} b \geq 0$ for each y with $y^{\mathsf{T}} A \geq \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1d in Schrijver [1986b].) A second equivalent variant is:

Corollary 5.3b (Farkas' lemma — variant). *$Ax \leq b$ has a solution $x \geq \mathbf{0}$ $\iff$ $y^{\mathsf{T}} b \geq 0$ for each $y \geq \mathbf{0}$ with $y^{\mathsf{T}} A \geq \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1f in Schrijver [1986b].) A third equivalent, affine variant of Farkas' lemma is:

Corollary 5.3c (Farkas' lemma — affine variant). *Let $Ax \leq b$ be a feasible system of inequalities and let $c^{\mathsf{T}} x \leq \delta$ be an inequality satisfied by each x with $Ax \leq b$. Then for some $\delta' \leq \delta$, the inequality $c^{\mathsf{T}} x \leq \delta'$ is a nonnegative linear combination of the inequalities in $Ax \leq b$.*

(Corollary 7.1h in Schrijver [1986b].)

5.5. Linear programming

Linear programming, abbreviated to *LP*, concerns the problem of maximizing or minimizing a linear function over a polyhedron. Examples are

(5.9) $\max\{c^{\mathsf{T}} x \mid Ax \leq b\}$ and $\min\{c^{\mathsf{T}} x \mid x \geq \mathbf{0}, Ax \geq b\}$.

If a supremum of a linear function over a polyhedron is finite, then it is attained as a maximum. So a maximum is finite if the value set is nonempty and has an upper bound. Similarly for infimum and minimum.

The *duality theorem of linear programming* says (von Neumann [1947], Gale, Kuhn, and Tucker [1951]):

Theorem 5.4 (duality theorem of linear programming)**.** *Let A be a matrix and b and c be vectors. Then*

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}, \tag{5.10}$$

if at least one of these two optima is finite.

(Corollary 7.1g in Schrijver [1986b].) So, in particular, if at least one of the optima is finite, then both are finite.

Note that the inequality $\leq$ in (5.10) is easy, since $c^{\mathsf{T}}x = y^{\mathsf{T}}Ax \leq y^{\mathsf{T}}b$. This is called *weak duality*.

There are several equivalent forms of the duality theorem of linear programming, like

$$\begin{aligned}
&\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},\\
&\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax = b\} = \min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},\\
&\min\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \geq b\} = \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \leq c^{\mathsf{T}}\},\\
&\min\{c^{\mathsf{T}}x \mid Ax \geq b\} = \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}.
\end{aligned} \tag{5.11}$$

Any of these equalities holds if at least one of the two optima is finite (implying that both are finite).

A most general formulation is: let $A, B, C, D, E, F, G, H, K$ be matrices and let a, b, c, d, e, f be vectors; then

$$\begin{aligned}
&\max\{d^{\mathsf{T}}x + e^{\mathsf{T}}y + f^{\mathsf{T}}z \mid x \geq \mathbf{0}, z \leq \mathbf{0},\\
&Ax + By + Cz \leq a,\\
&Dx + Ey + Fz = b,\\
&Gx + Hy + Kz \geq c\}\\
&= \min\{u^{\mathsf{T}}a + v^{\mathsf{T}}b + w^{\mathsf{T}}c \mid u \geq \mathbf{0}, w \leq \mathbf{0},\\
&u^{\mathsf{T}}A + v^{\mathsf{T}}D + w^{\mathsf{T}}G \geq d^{\mathsf{T}},\\
&u^{\mathsf{T}}B + v^{\mathsf{T}}E + w^{\mathsf{T}}H = e^{\mathsf{T}},\\
&u^{\mathsf{T}}C + v^{\mathsf{T}}F + w^{\mathsf{T}}K \leq f^{\mathsf{T}}\},
\end{aligned} \tag{5.12}$$

provided that at least one of the two optima is finite (cf. Section 7.4 in Schrijver [1986b]).

So there is a one-to-one relation between constraints in a problem and variables in its dual problem. The objective function in one problem becomes the right-hand side in the dual problem. We survey these relations in the following table:

maximize	minimize
$\leq$ constraint	variable ≥ 0
$\geq$ constraint	variable ≤ 0
$=$ constraint	unconstrained variable
variable ≥ 0	$\geq$ constraint
variable ≤ 0	$\leq$ constraint
unconstrained variable	$=$ constraint
right-hand side	objective function
objective function	right-hand side

Some LP terminology. Linear programming concerns maximizing or minimizing a linear function $c^{\mathsf{T}}x$ over a polyhedron P. The polyhedron P is called the *feasible region*, and any vector in P a *feasible solution*. If the feasible region is nonempty, the problem is called *feasible*, and *infeasible* otherwise. The function $x \to c^{\mathsf{T}}x$ is called the *objective function* or the *cost function*. Any feasible solution attaining the optimum value is called an *optimum solution*. An inequality $c^{\mathsf{T}}x \leq \delta$ is called *tight* or *active* for some x^* if $c^{\mathsf{T}}x^* = \delta$.

Equations like (5.10), (5.11), and (5.12) are called *linear programming duality equations*. The minimization problem is called the *dual problem* of the maximization problem (which problem then is called the *primal problem*), and conversely. A feasible solution of the dual problem is called a *dual solution*.

Complementary slackness. The following *complementary slackness conditions* characterize optimality of a pair of feasible solutions x, y of the linear programs (5.10):

(5.13) x and y are optimum solutions if and only if $(Ax)_i = b_i$ for each i with $y_i > 0$.

Similar conditions can be formulated for other pairs of dual linear programs (cf. Section 7.9 in Schrijver [1986b]).

Carathéodory's theorem. A consequence of Carathéodory's theorem (Theorem 5.1 above) is:

Theorem 5.5. *If the optimum value in the LP problems* (5.10) *is finite, then the minimum is attained by a vector* $y \geq \mathbf{0}$ *such that the rows of* A *corresponding to positive components of* y *are linearly independent.*

(Corollary 7.1l in Schrijver [1986b].)

5.6. Faces, facets, and vertices

Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. If c is a nonzero vector and $\delta = \max\{c^{\mathsf{T}}x \mid Ax \leq b\}$, the affine hyperplane $\{x \mid c^{\mathsf{T}}x = \delta\}$ is called a *supporting hyperplane* of P. A subset F of P is called a *face* if $F = P$ or if $F = P \cap H$ for some supporting hyperplane H of P. So

(5.14) F is a face of P $\iff$ F is the set of optimum solutions of $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ for some $c \in \mathbb{R}^n$.

An inequality $c^{\mathsf{T}}x \leq \delta$ is said to *determine* or to *induce* face F of P if

(5.15) $F = \{x \in P \mid c^{\mathsf{T}}x = \delta\}$.

Alternatively, F is a face of P if and only if

(5.16) $F = \{x \in P \mid A'x = b'\}$

for some subsystem $A'x \leq b'$ of $Ax \leq b$ (cf. Section 8.3 in Schrijver [1986b]). So any face of a nonempty polyhedron is a nonempty polyhedron. We say that a constraint $a^{\mathsf{T}}x \leq \beta$ from $Ax \leq b$ is *tight* or *active* in a face F if $a^{\mathsf{T}}x = \beta$ holds for each $x \in F$.

An inequality $a^{\mathsf{T}}x \leq \beta$ from $Ax \leq b$ is called an *implicit equality* if $Ax \leq b$ implies $a^{\mathsf{T}}x = \beta$. Then:

Theorem 5.6. *Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. Let $A'x \leq b'$ be the subsystem of implicit inequalities in $Ax \leq b$. Then* $\dim P = n - \text{rank}A'$.

(Cf. Section 8.2 in Schrijver [1986b].)

A *facet* of P is an inclusionwise maximal face F of P with $F \neq P$. An inequality determining a facet is called *facet-determining* or *facet-inducing*. Any facet has dimension one less than the dimension of P.

A system $Ax \leq b$ is called *minimal* or *irredundant* if each proper subsystem $A'x \leq b'$ has a solution x not satisfying $Ax \leq b$. If $Ax \leq b$ is irredundant and P is full-dimensional, then $Ax \leq b$ is the unique minimal system determining P, up to multiplying inequalities by positive scalars.

If $Ax \leq b$ is irredundant, then there is a one-to-one relation between the facets F of P and those inequalities $a^{\mathsf{T}}x \leq \beta$ in $Ax \leq b$ that are not implicit equalities, given by:

(5.17) $F = \{x \in P \mid a^{\mathsf{T}}x = \beta\}$

(cf. Theorem 8.1 in Schrijver [1986b]). This implies that each face $F \neq P$ is the intersection of facets.

A face of $P = \{x \mid Ax \leq b\}$ is called a *minimal face* if it is an inclusionwise minimal face. Any minimal face is an affine subspace of $\mathbb{R}^n$, and all minimal faces of P are translates of each other. They all have dimension $n - \text{rank}A$.

If each minimal face has dimension 0, P is called *pointed*. A *vertex* of P is an element z such that $\{z\}$ is a minimal face. A polytope is the convex hull of its vertices.

For any element z of $P = \{x \mid Ax \leq b\}$, let $A_z x \leq b_z$ be the system consisting of those inequalities from $Ax \leq b$ that are satisfied by z with equality. Then:

Theorem 5.7. *Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$ and let $z \in P$. Then z is a vertex of P if and only if* $\text{rank}(A_z) = n$.

An *edge* of P is a bounded face of dimension 1. It necessarily connects two vertices of P. Two vertices connected by an edge are called *adjacent*. An *extremal ray* is a face of dimension 1 that forms a halfline.

The 1-*skeleton* of a pointed polyhedron P is the union of the vertices, edges, and extremal rays of P. If P is a polytope, the 1-skeleton is a topological graph. The *diameter* of P is the diameter of the associated (combinatorial) graph.

The *Hirsch conjecture* states that a d-dimensional polytope with m facets has diameter at most $m - d$. Naddef [1989] proved this for polytopes with $0, 1$ vertices. We refer to Kalai [1997] for a survey of bounds on the diameter and on the number of pivot steps in linear programming.

5.7. Polarity

(For the results of this section, see Section 9.1 in Schrijver [1986b].) For any subset C of $\mathbb{R}^n$, the *polar* of C is

(5.18) $$C^* := \{z \in \mathbb{R}^n \mid x^\mathsf{T} z \leq 1 \text{ for all } x \in C\}.$$

If C is a cone, then C^* is again a cone, the *polar cone* of C, and satisfies

(5.19) $$C^* := \{z \in \mathbb{R}^n \mid x^\mathsf{T} z \leq 0 \text{ for all } x \in C\}.$$

Let C be a polyhedral cone; so $C = \{x \mid Ax \leq \mathbf{0}\}$ for some matrix A. Trivially, if C is generated by the vectors $x_1, \ldots, x_k$, then C^* is equal to the cone determined by the inequalities $x_i^\mathsf{T} z \leq 0$ for $i = 1, \ldots, k$. It is less trivial, and can be derived from Farkas' lemma, that:

(5.20) the polar cone C^* is equal to the cone generated by the transposes of the rows of A.

This implies

(5.21) $$C^{**} = C \text{ for each polyhedral cone } C.$$

So there is a symmetric duality relation between finite sets of vectors generating a cone and finite sets of vectors generating its polar cone.

5.8. Blocking polyhedra

(For the results of this section, see Section 9.2 in Schrijver [1986b].) A duality relation similar to polarity holds between convex sets 'of blocking type', and also between convex sets 'of antiblocking type'. This was shown by Fulkerson [1970b,1971a,1972a], who found several applications in combinatorial optimization.

We say that a subset P of $\mathbb{R}^n$ is *up-monotone* if $x \in P$ and $y \geq x$ imply $y \in P$. Similarly, P is *down-monotone* if $x \in P$ and $y \leq x$ imply $y \in P$.

Moreover, P is *down-monotone in* $\mathbb{R}^n_+$ if $x \in P$ and $\mathbf{0} \leq y \leq x$ imply $y \in P$. For any $P \subseteq \mathbb{R}^n$ we define

(5.22) $$P^{\uparrow} := \{y \in \mathbb{R}^n \mid \exists x \in P : y \geq x\} = P + \mathbb{R}^n_+ \text{ and}$$
$$P^{\downarrow} := \{y \in \mathbb{R}^n \mid \exists x \in P : y \leq x\} = P - \mathbb{R}^n_+.$$

$P^{\uparrow}$ is called the *dominant* of P. So P is up-monotone if and only if $P = P^{\uparrow}$, and P is down-monotone if and only if $P = P^{\downarrow}$.

We say that a convex set $P \subseteq \mathbb{R}^n$ is of *blocking type* if P is a closed convex up-monotone subset of $\mathbb{R}^n_+$. Each polyhedron P of blocking type is pointed. Moreover, P is a polyhedron of blocking type if and only if there exist vectors $x_1, \ldots, x_k \in \mathbb{R}^n_+$ such that

(5.23) $$P = \text{conv.hull}\{x_1, \ldots, x_k\}^{\uparrow};$$

and also, if and only if

(5.24) $$P = \{x \in \mathbb{R}^n_+ \mid Ax \geq \mathbf{1}\}$$

for some nonnegative matrix A.

For any polyhedron P in $\mathbb{R}^n$, the *blocking polyhedron* $B(P)$ of P is defined by

(5.25) $$B(P) := \{z \in \mathbb{R}^n_+ \mid x^{\mathsf{T}} z \geq 1 \text{ for each } x \in P\}.$$

Fulkerson [1970b,1971a] showed:

Theorem 5.8. *Let $P \subseteq \mathbb{R}^n_+$ be a polyhedron of blocking type. Then $B(P)$ is again a polyhedron of blocking type and $B(B(P)) = P$. Moreover, for any $x_1, \ldots, x_k \in \mathbb{R}^n_+$:*

(5.26) (5.23) *holds if and only if* $B(P) = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}} z \geq 1$ *for* $i = 1, \ldots, k\}$.

Here the only if part is trivial, while the if part requires Farkas' lemma.

Theorem 5.8 implies that for vectors $x_1, \ldots, x_k \in \mathbb{R}^n_+$ and $z_1, \ldots, z_d \in \mathbb{R}^n_+$ one has:

(5.27) $$\text{conv.hull}\{x_1, \ldots, x_k\} + \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid z_j^{\mathsf{T}} x \geq 1 \text{ for } j = 1, \ldots, d\}$$

if and only if

(5.28) $$\text{conv.hull}\{z_1, \ldots, z_d\} + \mathbb{R}^n_+ = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}} z \geq 1 \text{ for } i = 1, \ldots, k\}.$$

Two polyhedra P, R are called a *blocking pair (of polyhedra)* if they are of blocking type and satisfy $R = B(P)$. So if P, R is a blocking pair, then so is R, P.

5.9. Antiblocking polyhedra

(For the results of this section, see Section 9.3 in Schrijver [1986b].) The theory of antiblocking polyhedra is almost fully analogous to the blocking case and arises mostly by reversing inequality signs.

We say that a set $P \subseteq \mathbb{R}^n$ is of *antiblocking type* if P is a nonempty closed convex subset of $\mathbb{R}^n_+$ that is down-monotone in $\mathbb{R}^n_+$. Then P is a polyhedron of antiblocking type if and only if

(5.29) $\qquad P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$

for some nonnegative matrix A and nonnegative vector b.

For any subset P of $\mathbb{R}^n$, the *antiblocking set* $A(P)$ of P is defined by

(5.30) $\qquad A(P) := \{z \in \mathbb{R}^n_+ \mid x^{\mathsf{T}}z \leq 1 \text{ for each } x \in P\}.$

If $A(P)$ is a polyhedron we speak of the *antiblocking polyhedron*, and if $A(P)$ is a convex body, of the *antiblocking body.*

Fulkerson [1971a,1972a] showed:

Theorem 5.9. *Let $P \subseteq \mathbb{R}^n_+$ be of antiblocking type. Then $A(P)$ is again of antiblocking type and $A(A(P)) = P$.*

The antiblocking analogue of (5.26) is a little more complicated to formulate, but we need it only for full-dimensional polytopes. For any full-dimensional polytope $P \subseteq \mathbb{R}^n$ of antiblocking type and $x_1, \ldots, x_k \in \mathbb{R}^n_+$ we have:

(5.31) $\qquad P = \text{conv.hull}\{x_1, \ldots, x_k\}^{\downarrow} \cap \mathbb{R}^n_+$ if and only if $A(P) = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}}z \leq 1 \text{ for } i = 1, \ldots, k\}$.

Two convex sets P, R are called an *antiblocking pair (of polyhedra)* if they are of antiblocking type and satisfy $R = A(P)$. So if P, R is an antiblocking pair, then so is R, P.

5.10. Methods for linear programming

The simplex method was designed by Dantzig [1951b] to solve linear programming problems. It is in practice and on average quite efficient, but no polynomial-time worst-case running time bound has been proved (most of the pivot selection rules that have been proposed have been proved to take exponential time in the worst case).

The simplex method consists of finding a path in the 1-skeleton of the feasible region, ending at an optimum vertex (in preprocessing, the problem first is transformed to one with a pointed feasible region). An important issue when implementing this is that the LP problem is not given by vertices and edges, but by linear inequalities, and that vertices are determined by a, not necessarily unique, 'basis' among the inequalities.

The first polynomial-time method for linear programming was given by Khachiyan [1979,1980], by adapting the 'ellipsoid method' for nonlinear programming of Shor [1970a,1970b,1977] and Yudin and Nemirovskiĭ [1976]. The method consists of finding a sequence of shrinking ellipsoids each containing at least one optimum solution, until we have an ellipsoid that is small enough so as to derive an optimum solution. The method however is practically quite infeasible.

Karmarkar [1984a,1984b] showed that 'interior point' methods can solve linear programming in polynomial time, and moreover that they have efficient implementations, competing with the simplex method. Interior point methods make a tour not along vertices and edges, but across the feasible region.

5.11. The ellipsoid method

While the ellipsoid method is practically infeasible, it turned out to have features that are useful for deriving complexity results in combinatorial optimization. Specifically, the ellipsoid method does not require listing all constraints of an LP problem a priori, but allows that they are generated when needed. In this way, one can derive the polynomial-time solvability of a number of combinatorial optimization problems. This should be considered as existence proofs of polynomial-time algorithms — the algorithms are not practical.

This application of the ellipsoid method was described by Karp and Papadimitriou [1980,1982], Padberg and Rao [1980], and Grötschel, Lovász, and Schrijver [1981]. The book by Grötschel, Lovász, and Schrijver [1988] is devoted to it. We refer to Chapter 6 of this book or to Chapter 14 of Schrijver [1986b] for proofs of the results that we survey below.

The ellipsoid method applies to classes of polyhedra (and more generally, classes of convex sets) which are described as follows.

Let Σ be a finite alphabet and let Π be a subset of the set Σ^* of words over Σ. In applications, we take for Π very simple sets like the set of strings representing a graph or the set of strings representing a digraph.

For each $\sigma \in \Pi$, let E_σ be a finite set and let P_σ be a rational polyhedron in $\mathbb{Q}^{E_\sigma}$. (When we apply this, E_σ is often the vertex set or the edge or arc set of the (di)graph represented by σ.) We make the following assumptions:

(5.32)
- (i) there is a polynomial-time algorithm that, given $\sigma \in \Sigma^*$, tests if σ belongs to Π and, if so, returns the set E_σ;
- (ii) there is a polynomial p such that, for each $\sigma \in \Pi$, P_σ is determined by linear inequalities each of size at most $p(\text{size}(\sigma))$.

Here the *size* of a rational linear inequality is proportional to the sum of the sizes of its components, where the *size* of a rational number p/q (for integers

p, q) is proportional to $\log(|p|+1)+\log q$. Condition (5.32)(ii) is equivalent to (cf. Theorem 10.2 in Schrijver [1986b]):

(5.33) there is a polynomial q such that, for each $\sigma \in \Pi$, we can write $P_\sigma = Q + C$, where Q is a polytope with vertices each of input size at most $q(\text{size}(\sigma))$ and where C is a cone generated by vectors each of input size at most $q(\text{size}(\sigma))$.

(The *input size*[7] of a vector is the sum of the sizes of its components.) In most applications, the existence of the polynomial p in (5.32)(ii) or of the polynomial q in (5.33) is obvious.

We did not specify how the polyhedra P_σ are given algorithmically. In applications, they might have an exponential number of vertices or facets, so listing them would not be an algorithmic option. To handle this, we formulate two, in a sense dual, problems. An algorithm for either of them would determine the polyhedra P_σ.

First, the *optimization problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.34) given: $\sigma \in \Pi$ and $c \in \mathbb{Q}^{E_\sigma}$,
find: $x \in P_\sigma$ maximizing $c^{\mathsf{T}}x$ over P_σ or $y \in \text{char.cone}(P_\sigma)$ with $c^{\mathsf{T}}y > 0$, if either of them exists.

Second, the *separation problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.35) given: $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$,
find: $c \in \mathbb{Q}^{E_\sigma}$ such that $c^{\mathsf{T}}x < c^{\mathsf{T}}z$ for all $x \in P_\sigma$ (if such a c exists).

So c gives a separating hyperplane if $z \notin P_\sigma$.

Then the ellipsoid method implies that these two problems are 'polynomial-time equivalent':

Theorem 5.10. *Let $\Pi \subseteq \Sigma^*$ and let $(P_\sigma \mid \sigma \in \Pi)$ satisfy* (5.32). *Then the optimization problem for $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable if and only if the separation problem for $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable.*

(Cf. Theorem (6.4.9) in Grötschel, Lovász, and Schrijver [1988] or Corollary 14.1c in Schrijver [1986b].)

The equivalence in Theorem 5.10 makes that we call $(P_\sigma \mid \sigma \in \Pi)$ *polynomial-time solvable* if it satisfies (5.32) and the optimization problem (equivalently, the separation problem) for it is polynomial-time solvable.

Using simultaneous diophantine approximation based on the basis reduction method given by Lenstra, Lenstra, and Lovász [1982], Frank and Tardos [1985,1987] extended these results to strong polynomial-time solvability:

[7] We will use the term *size* of a vector for the sum of its components.

Theorem 5.11. *The optimization problem and the separation problem for any polynomial-time solvable system of polyhedra are solvable in strongly polynomial time.*

(Theorem (6.6.5) in Grötschel, Lovász, and Schrijver [1988].)

For polynomial-time solvable classes of polyhedra, the separation problem can be strengthened so as to obtain a facet as separating hyperplane:

Theorem 5.12. *Let $(P_\sigma \mid \sigma \in \Pi)$ be a polynomial-time solvable system of polyhedra. Then the following problem is strongly polynomial-time solvable:*

(5.36) given: $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$,
find: $c \in \mathbb{Q}^{E_\sigma}$ and $\delta \in \mathbb{Q}$ such that $c^\mathsf{T} z > \delta$ and such that $c^\mathsf{T} x \leq \delta$ is facet-inducing for P_σ (if it exists).

(Cf. Theorem (6.5.16) in Grötschel, Lovász, and Schrijver [1988].) Also a weakening of the separation problem turns out to be equivalent, under certain conditions. The *membership problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.37) given $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$, does z belong to P_σ?

Theorem 5.13. *Let $(P_\sigma \mid \sigma \in \Pi)$ be a system of full-dimensional polytopes satisfying* (5.32)*, such that there is a polynomial-time algorithm that gives for each $\sigma \in \Pi$ a vector in the interior of P_σ. Then $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable if and only if the membership problem for $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable.*

(This follows from Corollary (4.3.12) and Theorem (6.3.2) in Grötschel, Lovász, and Schrijver [1988].)

The theorems above imply:

Theorem 5.14. *Let $(P_\sigma \mid \sigma \in \Pi)$ and $(Q_\sigma \mid \sigma \in \Pi)$ be polynomial-time solvable classes of polyhedra, such that for each $\sigma \in \Pi$, the polyhedra P_σ and Q_σ are in the same space $\mathbb{R}^{E_\sigma}$. Then also $(P_\sigma \cap Q_\sigma \mid \sigma \in \Pi)$ and $(\text{conv.hull}(P_\sigma \cup Q_\sigma) \mid \sigma \in \Pi)$ are polynomial-time solvable.*

(Corollary 14.1d in Schrijver [1986b].)

Corollary 5.14a. *Let $(P_\sigma \mid \sigma \in \Pi)$ be a polynomial-time solvable system of polyhedra, all of blocking type. Then also the system of blocking polyhedra $(B(P_\sigma) \mid \sigma \in \Pi)$ is polynomial-time solvable.*

(Corollary 14.1e in Schrijver [1986b].) Similarly:

Corollary 5.14b. *Let $(P_\sigma \mid \sigma \in \Pi)$ be a polynomial-time solvable system of polyhedra, all of antiblocking type. Then also the system of antiblocking polyhedra $(A(P_\sigma) \mid \sigma \in \Pi)$ is polynomial-time solvable.*

(Corollary 14.1e in Schrijver [1986b].)

Also the following holds:

Theorem 5.15. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a polynomial-time solvable system of polyhedra, where each* P_σ *is a polytope. Then the following problems are strongly polynomial-time solvable:*

(5.38)
- (i) *given* $\sigma \in \Pi$, *find an internal vector, a vertex, and a facet-inducing inequality of* P_σ*;*
- (ii) *given* $\sigma \in \Pi$ *and* $x \in P_\sigma$, *find affinely independent vertices* $x_1, \ldots, x_k$ *of* P_σ *and write* x *as a convex combination of* $x_1, \ldots, x_k$*;*
- (iii) *given* $\sigma \in \Pi$ *and* $c \in \mathbb{R}^{E_\sigma}$, *find facet-inducing inequalities* $c_1^{\mathsf{T}}x \le \delta_1, \ldots, c_k^{\mathsf{T}}x \le \delta_k$ *of* P_σ *with* $c_1, \ldots, c_k$ *linearly independent, and find* $\lambda_1, \ldots, \lambda_k \ge 0$ *such that* $\lambda_1 c_1 + \cdots + \lambda_k c_k = c$ *and* $\lambda_1\delta_1 + \cdots + \lambda_k\delta_k = \max\{c^{\mathsf{T}}x \mid x \in P_\sigma\}$ *(i.e., find an optimum dual solution).*

(Corollary 14.1f in Schrijver [1986b].)

The ellipsoid method can be applied also to nonpolyhedral convex sets, in which case only approximative versions of the optimization and separation problems can be shown to be equivalent. We only need this in Chapter 67 on the convex body TH(G), where we refer to the appropriate theorem in Grötschel, Lovász, and Schrijver [1988].

5.12. Polyhedra and NP and co-NP

An appropriate polyhedral description of a combinatorial optimization problem relates to the question NP≠co-NP. More precisely, unless NP=co-NP, the polyhedra associated with an NP-complete problem cannot be described by 'certifiable' inequalities. (These insights go back to observations in the work of Edmonds of the 1960s.)

Again, let $(P_\sigma \mid \sigma \in \Pi)$ be a system of polyhedra satisfying (5.32). Consider the decision version of the optimization problem:

(5.39) given $\sigma \in \Pi$, $c \in \mathbb{Q}^{E_\sigma}$, and $k \in \mathbb{Q}$, is there an $x \in P_\sigma$ with $c^{\mathsf{T}}x > k$?

Then:

Theorem 5.16. *Problem* (5.39) *belongs to* co-NP *if and only if for each* $\sigma \in \Pi$, *there exists a collection* $\mathcal{I}_\sigma$ *of inequalities determining* P_σ *such that the problem:*

(5.40) *given* $\sigma \in \Pi$, $c \in \mathbb{Q}^{E_\sigma}$, *and* $\delta \in \mathbb{Q}$, *does* $c^{\mathsf{T}}x \le \delta$ *belong to* $\mathcal{I}_\sigma$, *belongs to* NP.

Proof. To see necessity, we can take for $\mathcal{I}_\sigma$ the collection of *all* valid inequalities for P_σ. Then co-NP-membership of (5.39) is equivalent of NP-membership of (5.40).

To see sufficiency, a negative answer to question (5.39) can be certified by giving inequalities $c_i^\mathsf{T} x \leq \delta_i$ from $\mathcal{I}_\sigma$ and $\lambda_i \in \mathbb{Q}_+$ $(i = 1, \ldots, k)$ such that $c = \lambda_1 c_1 + \cdots + \lambda_k c_k$ and $\delta \geq \lambda_1 \delta_1 + \cdots + \lambda_k \delta_k$. As we can take $k \leq |E_\sigma|$, and as each inequality in $\mathcal{I}_\sigma$ has a polynomial-time checkable certificate (as (5.40) belongs to NP), this gives a polynomial-time checkable certificate for the negative answer. Hence (5.39) belongs to co-NP. ∎

This implies for NP-complete problems:

Corollary 5.16a. *Let* (5.39) *be* NP*-complete and suppose* NP≠co-NP. *For each $\sigma \in \Pi$, let $\mathcal{I}_\sigma$ be a collection of inequalities determining P_σ. Then problem* (5.40) *does not belong to* NP.

Proof. If problem (5.40) would belong to NP, then by Theorem 5.16, problem (5.39) belongs to co-NP. If (5.39) is NP-complete, this implies NP=co-NP. ∎

Roughly speaking, this implies that if (5.39) is NP-complete and NP≠co-NP, then P_σ has 'difficult' facets, that is, facets which have no polynomial-time checkable certificate of validity for P_σ.

(Related work on the complexity of facets was reported in Karp and Papadimitriou [1980,1982] and Papadimitriou and Yannakakis [1982,1984].)

5.13. Primal-dual methods

As a generalization of similar methods for network flow and transportation problems, Dantzig, Ford, and Fulkerson [1956] designed the 'primal-dual method' for linear programming. The general idea is as follows. Starting with a dual feasible solution y, the method searches for a primal feasible solution x satisfying the complementary slackness condition with respect to y. If such a primal feasible solution x is found, x and y form a pair of optimum solutions (by (5.13)). If no such primal solution is found, the method prescribes a modification of y, after which the method iterates.

The problem now is how to find a primal feasible solution x satisfying the complementary slackness condition, and how to modify the dual solution y if no such primal solution is found. For general linear programs this problem can be seen to amount to another linear program, generally simpler than the original linear program. To solve the simpler linear program we could use any LP method. In many combinatorial applications, however, this simpler linear program is a simpler combinatorial optimization problem, for which direct methods are available. Thus, if we can describe a combinatorial optimization problem as a linear program, the primal-dual method gives us a scheme for reducing one combinatorial problem to an easier combinatorial problem. The efficiency of the method depends on the complexity of the easier problem and on the number of primal-dual iterations.

We describe the primal-dual method more precisely. Suppose that we wish to solve the LP problem

(5.41) $$\min\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax = b\},$$

where A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and where $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The dual problem is

(5.42) $$\max\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A \leq c^{\mathsf{T}}\}.$$

The primal-dual method consists of repeating the following *primal-dual iteration*. Suppose that we have a feasible solution y_0 for problem (5.42). Let A' be the submatrix of A consisting of those columns a_j of A for which $y_0^{\mathsf{T}}a_j = c_j$ holds. To find a feasible primal solution satisfying the complementary slackness, solve the *restricted linear program*

(5.43) $$x' \geq \mathbf{0}, A'x' = b.$$

If such an x' exists, by adding components 0, we obtain a vector $x \geq \mathbf{0}$ such that $Ax = b$ and such that $x_j = 0$ if $y_0^{\mathsf{T}}a_j < c_j$. By complementary slackness ((5.13)), it follows that x and y_0 are optimum solutions for problems (5.41) and (5.42).

On the other hand, if no x' satisfying (5.43) exists, by Farkas' lemma (Corollary 5.3a), there exists a y' such that $y'^{\mathsf{T}}A' \leq 0$ and $y'^{\mathsf{T}}b > 0$. Let α be the largest real number satisfying

(5.44) $$(y_0 + \alpha y')^{\mathsf{T}}A \leq c^{\mathsf{T}}.$$

(Note that $\alpha > 0$.) Reset $y_0 := y_0 + \alpha y'$, and start the iteration anew. (If $\alpha = \infty$, (5.42) is unbounded, hence (5.41) is infeasible.)

This describes the primal-dual method. It reduces problem (5.41) to (5.43), which often is an easier problem.

The primal-dual method can equally well be considered as a *gradient method*. Suppose that we wish to solve problem (5.42), and we have a feasible solution y_0. This y_0 is not optimum if and only if there exists a vector y' such that $y'^{\mathsf{T}}b > 0$ and y' is a *feasible direction* at y_0 (that is, $(y_0 + \alpha y')^{\mathsf{T}}A \leq c^{\mathsf{T}}$ for some $\alpha > 0$). If we let A' consist of those columns of A in which $y_0^{\mathsf{T}}A \leq c^{\mathsf{T}}$ has equality, then y' is a feasible direction if and only if $y'^{\mathsf{T}}A' \leq 0$. So y' can be found by solving (5.43).

5.14. Integer linear programming

A vector $x \in \mathbb{R}^n$ is called *integer* if each component is an integer, i.e., if x belongs to $\mathbb{Z}^n$. Many combinatorial optimization problems can be described as maximizing a linear function $c^{\mathsf{T}}x$ over the *integer* vectors in some polyhedron $P = \{x \mid Ax \leq b\}$.

So this type of problems can be described as:

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \in \mathbb{Z}^n\}. \tag{5.45}$$

Such problems are called *integer linear programming*, or *ILP*, problems. They consist of maximizing a linear function over the intersection $P \cap \mathbb{Z}^n$ of a polyhedron P with the set $\mathbb{Z}^n$ of integer vectors.

Clearly, always the following inequality holds:

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \text{ integer}\} \leq \max\{c^{\mathsf{T}}x \mid Ax \leq b\}. \tag{5.46}$$

It is easy to make an example where strict inequality holds. This implies, that generally one will have strict inequality in the following duality relation:

$$\begin{aligned}&\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \text{ integer}\}\\ &\leq \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}; y^{\mathsf{T}}A = c^{\mathsf{T}}; y \text{ integer}\}.\end{aligned} \tag{5.47}$$

No polynomial-time algorithm is known to exist for solving an integer linear programming problem in general. In fact, the general integer linear programming problem is NP-complete (since the satisfiability problem is easily transformed to an integer linear programming problem). However, for special classes of integer linear programming problems, polynomial-time algorithms have been found. These classes often come from combinatorial problems.

5.15. Integer polyhedra

A polyhedron P is called an *integer polyhedron* if it is the convex hull of the integer vectors contained in P. This is equivalent to: P is rational and each face of P contains an integer vector. So a polytope P is integer if and only if each vertex of P is integer. If a polyhedron $P = \{x \mid Ax \leq b\}$ is integer, then the linear programming problem

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} \tag{5.48}$$

has an integer optimum solution if it is finite. Hence, in that case,

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b;\ x \text{ integer}\} = \max\{c^{\mathsf{T}}x \mid Ax \leq b\}. \tag{5.49}$$

This in fact characterizes integer polyhedra, since:

Theorem 5.17. *Let P be a rational polyhedron in $\mathbb{Q}^n$. Then P is integer if and only if for each $c \in \mathbb{Q}^n$, the linear programming problem* $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ *has an integer optimum solution if it is finite.*

A stronger characterization is (Edmonds and Giles [1977]):

Theorem 5.18. *A rational polyhedron P in $\mathbb{Q}^n$ is integer if and only if for each $c \in \mathbb{Z}^n$ the value of* $\max\{c^{\mathsf{T}}x \mid x \in P\}$ *is an integer if it is finite.*

(Corollary 22.1a in Schrijver [1986b].) We also will use the following observation:

Theorem 5.19. *Let P be an integer polyhedron in $\mathbb{R}^n_+$ with $P+\mathbb{R}^n_+ = P$ and let $c \in \mathbb{Z}^n_+$ be such that $x \leq c$ for each vertex x of P. Then $P \cap \{x \mid x \leq c\}$ is an integer polyhedron again.*

Proof. Let $Q := P \cap \{x \mid x \leq c\}$ and let R be the convex hull of the integer vectors in Q. We must show that $Q \subseteq R$.

Let $x \in Q$. As $P = R + \mathbb{R}^n_+$ there exists a $y \in R$ with $y \leq x$. Choose such a y with $y_1 + \cdots + y_n$ maximal. Suppose that $y_i < x_i$ for some component i. Since $y \in R$, y is a convex combination of integer vectors in Q. Since $y_i < x_i \leq c_i$, at least one of these integer vectors, z say, has $z_i < c_i$. But then the vector $z' := z+\chi^i$ belongs to R. Hence we could increase y_i, contradicting the maximality of y. ∎

We call a polyhedron P *box-integer* if $P \cap \{x \mid d \leq x \leq c\}$ is an integer polyhedron for each choice of integer vectors d, c. The set $\{x \mid d \leq x \leq c\}$ is called a *box*.

A $0,1$ *polytope* is a polytope with all vertices being 0,1 vectors.

5.16. Totally unimodular matrices

Total unimodularity of matrices is an important tool in integer programming. A matrix A is called *totally unimodular* if each square submatrix of A has determinant equal to 0, +1, or −1. In particular, each entry of a totally unimodular matrix is 0, +1, or −1.

An alternative way of characterizing total unimodularity is by requiring that the matrix is integer and that each nonsingular submatrix has an integer inverse matrix. This implies the following easy, but fundamental result:

Theorem 5.20. *Let A be a totally unimodular $m \times n$ matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron*

(5.50) $$P := \{x \mid Ax \leq b\}$$

is integer.

(Cf. Theorem 19.1 in Schrijver [1986b].) It follows that each linear programming problem with integer data and totally unimodular constraint matrix has integer optimum primal and dual solutions:

Corollary 5.20a. *Let A be a totally unimodular $m \times n$ matrix, let $b \in \mathbb{Z}^m$, and let $c \in \mathbb{Z}^n$. Then both optima in the LP duality equation*

(5.51) $$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$$

have integer optimum solutions (if the optima are finite).

(Corollary 19.1a in Schrijver [1986b].) Hoffman and Kruskal [1956] showed that this property is close to a characterization of total unimodularity.

Corollary 5.20a implies:

Corollary 5.20b. *Let A be an $m \times n$ matrix, let $b \in \mathbb{Z}^m$, and let $c \in \mathbb{R}^n$. Suppose that*

(5.52) $$\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\}$$

has an optimum solution x^* such that the columns of A corresponding to positive components of x^* form a totally unimodular matrix. Then (5.52) *has an integer optimum solution.*

Proof. Since x^* is an optimum solution, we have

(5.53) $$\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\} = \max\{c'^{\mathsf{T}}x' \mid x' \geq \mathbf{0}, A'x' \leq b\},$$

where A' and c' are the parts of A and c corresponding to the support of x^*. As A' is totally unimodular, the right-hand side maximum in (5.53) has an integer optimum solution x'^*. Extending x'^* by components 0, we obtain an integer optimum solution of the left-hand side maximum in (5.53). ∎

We will use the following characterization of Ghouila-Houri [1962b] (cf. Theorem 19.3 in Schrijver [1986b]):

Theorem 5.21. *A matrix M is totally unimodular if and only if each collection R of rows of M can be partitioned into classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries* $0, \pm 1$ *only.*

5.17. Total dual integrality

Edmonds and Giles [1977] introduced the powerful notion of total dual integrality. It is not only useful as a tool to derive combinatorial min-max relation, but also it gives an efficient way of expressing a whole bunch of min-max relations simultaneously.

A system $Ax \leq b$ in n dimensions is called *totally dual integral*, or just *TDI*, if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$:

(5.54) $$\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$$

has an integer optimum solution y, if it is finite.

By extension, a system $A'x \leq b', A''x = b''$ is defined to be TDI if the system $A'x \leq b', A''x \leq b'', -A''x \leq -b''$ is TDI. This is equivalent to requiring that A', A'', b', b'' are rational and for each $c \in \mathbb{Z}^n$ the dual of maximizing $c^{\mathsf{T}}x$ over $A'x \leq b'$, $A''x = b''$ has an integer optimum solution, if finite.

Problem (5.54) is the problem dual to $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$, and Edmonds and Giles showed that total dual integrality implies that also this primal problem has an integer optimum solution, if b is integer. In fact, they showed Theorem 5.18, which implies (since if (5.54) has an integer optimum solution, the optimum value is an integer):

Theorem 5.22. *If $Ax \leq b$ is TDI and b is integer, then $Ax \leq b$ determines an integer polyhedron.*

So total dual integrality implies 'primal integrality'. For combinatorial applications, the following observation is useful:

Theorem 5.23. *Let A be a nonnegative integer $m \times n$ matrix such that the system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ is TDI. Then also the system $\mathbf{0} \leq x \leq \mathbf{1}, Ax \geq \mathbf{1}$ is TDI.*

Proof. Choose $c \in \mathbb{Z}^n$. Let c_+ arise from c by setting negative components to 0. By the total dual integrality of $x \geq \mathbf{0}$, $Ax \geq \mathbf{1}$, there exist integer optimum solutions x, y of

(5.55) $$\min\{c_+^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\} = \max\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \leq c_+^{\mathsf{T}}\}.$$

As A is nonnegative and integer and as $c_+ \geq \mathbf{0}$, we may assume that $x \leq \mathbf{1}$. Moreover, we can assume that $x_i = 1$ if $(c_+)_i = 0$, that is, if $c_i \leq 0$.

Let $z := c - c_+$. So $z \leq \mathbf{0}$. We show that x, y, z are optimum solutions of

(5.56) $$\begin{aligned}&\min\{c^{\mathsf{T}}x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \geq \mathbf{1}\}\\ &= \max\{y^{\mathsf{T}}\mathbf{1} + z^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, z \leq \mathbf{0}, y^{\mathsf{T}}A + z^{\mathsf{T}} \leq c^{\mathsf{T}}\}.\end{aligned}$$

Indeed, x is feasible, as $\mathbf{0} \leq x \leq \mathbf{1}$ and $Ax \geq \mathbf{1}$. Moreover, y, z is feasible, as $y^{\mathsf{T}}A + z^{\mathsf{T}} \leq c_+^{\mathsf{T}} + z^{\mathsf{T}} = c^{\mathsf{T}}$. Optimality of x, y, z follows from

(5.57) $$c^{\mathsf{T}}x = c_+^{\mathsf{T}}x + z^{\mathsf{T}}x = y^{\mathsf{T}}\mathbf{1} + z^{\mathsf{T}}x = y^{\mathsf{T}}\mathbf{1} + z^{\mathsf{T}}\mathbf{1}.$$ ∎

In certain cases, to obtain total dual integrality one can restrict oneself to nonnegative objective functions:

Theorem 5.24. *Let A be a nonnegative $m \times n$ matrix and let $b \in \mathbb{R}_+^m$. Then $x \geq \mathbf{0}, Ax \leq b$ is TDI if and only if $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\}$ is attained by an integer optimum solution (if finite), for each $c \in \mathbb{Z}_+^n$.*

Proof. Necessity is trivial. To see sufficiency, let $c \in \mathbb{Z}^n$ with $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\}$ finite. Let it be attained by y. Let c_+ arise from c by setting negative components to 0. Then

(5.58) $$\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c_+^{\mathsf{T}}\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},$$

since $y^{\mathsf{T}}A \geq \mathbf{0}$ if $y \geq \mathbf{0}$. As the first minimum has an integer optimum solution, also the second minimum has an integer optimum solution. ∎

Total dual integrality is maintained under setting an inequality to an equality (Theorem 22.2 in Schrijver [1986b]):

Theorem 5.25. *Let $Ax \leq b$ be TDI and let $A'x \leq b'$ arise from $Ax \leq b$ by adding $-a^{\mathsf{T}}x \leq -\beta$ for some inequality $a^{\mathsf{T}}x \leq \beta$ in $Ax \leq b$. Then also $A'x \leq b'$ is TDI.*

Total dual integrality is also maintained under translation of the solution set, as follows directly from the definition of total dual integrality:

Theorem 5.26. *If $Ax \leq b$ is TDI and $w \in \mathbb{R}^n$, then $Ax \leq b - Aw$ is TDI.*

For future reference, we prove:

Theorem 5.27. *Let $A_{11}, A_{12}, A_{21}, A_{22}$ be matrices and let b_1, b_2 be column vectors, such that the system*

(5.59) $$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 &= b_1, \\ A_{2,1}x_1 + A_{2,2}x_2 &\leq b_2 \end{aligned}$$

is TDI and such that $A_{1,1}$ is nonsingular. Then also the system

(5.60) $$(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2})x_2 \leq b_2 - A_{2,1}A_{1,1}^{-1}b_1$$

is TDI.

Proof. We may assume that $b_1 = \mathbf{0}$, since by Theorem 5.26 total dual integrality is invariant under replacing (5.59) by

(5.61) $$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 &= b_1 - A_{1,1}A_{1,1}^{-1}b_1 = \mathbf{0}, \\ A_{2,1}x_1 + A_{2,2}x_2 &\leq b_2 - A_{2,1}A_{1,1}^{-1}b_1. \end{aligned}$$

Let x_2 minimize $c^{\mathsf{T}}x_2$ over (5.60), for some integer vector c of appropriate dimension. Define $x_1 := -A_{1,1}^{-1}A_{1,2}x_2$. Then x_1, x_2 minimizes $c^{\mathsf{T}}x_2$ over (5.59), since any solution x_1', x_2' of (5.59) satisfies $x_1' = -A_{1,1}^{-1}A_{1,2}x_2'$, and therefore x_2' satisfies (5.60); hence $c^{\mathsf{T}}x_2' \geq c^{\mathsf{T}}x_2$.

Let y_1, y_2 be an integer optimum solution of the problem dual to maximizing $c^{\mathsf{T}}x_2$ over (5.59). So y_1, y_2 satisfy

(5.62) $$y_1^{\mathsf{T}}A_{1,1} + y_2^{\mathsf{T}}A_{2,1} = \mathbf{0},\ y_1^{\mathsf{T}}A_{1,2} + y_2^{\mathsf{T}}A_{2,2} = c^{\mathsf{T}},\ y_2^{\mathsf{T}}b_2 = c^{\mathsf{T}}x_2.$$

Hence

(5.63) $$y_2^{\mathsf{T}}(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}) = y_2^{\mathsf{T}}A_{2,2} + y_1^{\mathsf{T}}A_{1,2} = c^{\mathsf{T}}$$

and

(5.64) $\qquad y_2^{\mathsf{T}} b_2 = c^{\mathsf{T}} x_2.$

So y_2 is an integer optimum solution of the problem dual to maximizing $c^{\mathsf{T}} x_2$ over (5.60). ∎

This has as consequence (where a_0 is a column vector):

Corollary 5.27a. *If $x_0 = \beta$, $a_0 x_0 + Ax \leq b$ is TDI, then $Ax \leq b - \beta a_0$ is TDI.*

Proof. This is a special case of Theorem 5.27. ∎

We also have:

Theorem 5.28. *Let $A = [a_1 \quad a_2 \quad A'']$ be an integer $m \times n$ matrix and let $b \in \mathbb{R}^m$. Let A' be the $m \times (n-1)$ matrix $[a_1 + a_2 \quad A'']$. Then $A'x' \leq b$ is TDI if and only if $Ax \leq b, x_1 - x_2 = 0$ is TDI.*

Proof. To see necessity, choose $c \in \mathbb{Z}^n$. Let $c' := (c_1 + c_2, c_3, \ldots, c_n)^{\mathsf{T}}$. Then

(5.65) $\qquad \mu := \max\{c^{\mathsf{T}} x \mid Ax \leq b, x_1 - x_2 = 0\} = \max\{c'^{\mathsf{T}} x' \mid A'x' \leq b\}.$

Let $y \in \mathbb{Z}_+^m$ be an integer optimum dual solution of the second maximum. So $y^{\mathsf{T}} A' = c'$ and $y^{\mathsf{T}} b = \mu$. Then $y^{\mathsf{T}} a_1 + y^{\mathsf{T}} a_2 = c_1 + c_2$. Hence $y^{\mathsf{T}} A = c^{\mathsf{T}} + \lambda(1, -1, 0, \ldots, 0)$ for some $\lambda \in \mathbb{Z}$. So y, λ form an integer optimum dual solution of the first maximum.

To see sufficiency, choose $c' = (c_2, \ldots, c_n)^{\mathsf{T}} \in \mathbb{Z}^{n-1}$. Define $c := (0, c_2, \ldots, c_n)^{\mathsf{T}}$. Again we have (5.65). Let $y \in \mathbb{Z}_+^m$, $\lambda \in \mathbb{Z}$ constitute an integer optimum dual solution of the first maximum, where λ corresponds to the constraint $x_1 - x_2 = 0$. So $y^{\mathsf{T}} A + \lambda(1, -1, 0, \ldots, 0) = c$ and $y^{\mathsf{T}} b = \mu$. Hence $y^{\mathsf{T}} A' = c^{\mathsf{T}}$, and therefore, y is an integer optimum dual solution of the second maximum. ∎

Let A be a rational $m \times n$ matrix and let $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$. Consider the following series of inequalities (where a vector z is *half-integer* if $2z$ is integer):

(5.66)
$$\begin{aligned} &\max\{c^{\mathsf{T}} x \mid Ax \leq b, x \text{ integer}\} \leq \max\{c^{\mathsf{T}} x \mid Ax \leq b\} \\ &= \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}\} \\ &\leq \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \text{ half-integer}\} \\ &\leq \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \text{ integer}\}. \end{aligned}$$

Under certain circumstances, equality in the last inequality implies equality throughout:

Theorem 5.29. *Let $Ax \leq b$ be a system with A and b rational. Then $Ax \leq b$ is TDI if and only if*

(5.67) $\qquad \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \textit{ half-integer}\}$

is finite and is attained by an integer optimum solution y, for each integer vector c with $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ *finite.*

Proof. Necessity follows directly from (5.66). To see sufficiency, choose $c \in \mathbb{Z}^n$ with $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ finite. We must show that $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$ is attained by an integer optimum solution.

For each $k \geq 1$, define

(5.68) $$\alpha_k = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = kc^{\mathsf{T}}, y \text{ integer}\}.$$

This is well-defined, as $\max\{kc^{\mathsf{T}}x \mid Ax \leq b\}$ is finite.

The condition in the theorem gives that, for each $t \geq 0$,

(5.69) $$\frac{\alpha_{2^t}}{2^t} = \alpha_1.$$

This can be shown by induction on t, the case $t = 0$ being trivial. If $t \geq 1$, then

(5.70) $$\begin{aligned}\alpha_{2^t} &= \min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^t c^{\mathsf{T}}, y \in \mathbb{Z}^m_+\}\\ &= 2\min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^{t-1}c^{\mathsf{T}}, y \in \tfrac{1}{2}\mathbb{Z}^m_+\}\\ &= 2\min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^{t-1}c^{\mathsf{T}}, y \in \mathbb{Z}^m_+\} = 2\alpha_{2^{t-1}},\end{aligned}$$

implying (5.69) by induction.

Now $\alpha_{k+l} \leq \alpha_k + \alpha_l$ for all k, l. Hence we can apply Fekete's lemma, and get:

(5.71) $$\begin{aligned}&\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\} = \min_k \frac{\alpha_k}{k} = \lim_{k\to\infty}\frac{\alpha_k}{k} = \lim_{t\to\infty}\frac{\alpha_{2^t}}{2^t}\\ &= \alpha_1.\end{aligned}$$ ∎

The following analogue of Carathéodory's theorem holds (Cook, Fonlupt, and Schrijver [1986]):

Theorem 5.30. *Let $Ax \leq b$ be a totally dual integral system in n dimensions and let $c \in \mathbb{Z}^n$. Then $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\}$ has an integer optimum solution y with at most $2n-1$ nonzero components.*

(Theorem 22.12 in Schrijver [1986b].)

We also will need the following substitution property:

Theorem 5.31. *Let $A_1x \leq b_1, A_2x \leq b_2$ be a TDI system with A_1 integer, and let $A_1' \leq b_1'$ be a TDI system with*

(5.72) $$\{x \mid A_1x \leq b_1\} = \{x \mid A_1'x \leq b_1'\}.$$

Then the system $A_1'x \leq b_1', A_2x \leq b_2$ is TDI.

Proof. Let $c \in \mathbb{Z}^n$ with

(5.73) $$\begin{aligned}&\max\{c^{\mathsf{T}}x \mid A_1'x \leq b_1', A_2x \leq b_2\}\\ &= \min\{y^{\mathsf{T}}b_1' + z^{\mathsf{T}}b_2 \mid y, z \geq \mathbf{0}, y^{\mathsf{T}}A_1' + z^{\mathsf{T}}A_2 = c^{\mathsf{T}}\}\end{aligned}$$

finite. By (5.72), also

(5.74) $$\max\{c^{\mathsf{T}}x \mid A_1x \leq b_1, A_2x \leq b_2\}$$
$$= \min\{y^{\mathsf{T}}b_1 + z^{\mathsf{T}}b_2 \mid y, z \geq \mathbf{0}, y^{\mathsf{T}}A_1 + z^{\mathsf{T}}A_2 = c^{\mathsf{T}}\}$$

is finite. Hence, since $A_1x \leq b_1, A_2x \leq b_2$ is TDI, the minimum in (5.74) has an integer optimum solution y, z. Set $d := y^{\mathsf{T}}A_1$. Then, as d is an integer vector,

(5.75) $$y^{\mathsf{T}}b_1 = \min\{u^{\mathsf{T}}b_1 \mid u \geq \mathbf{0}, u^{\mathsf{T}}A_1 = d^{\mathsf{T}}\}$$
$$= \max\{d^{\mathsf{T}}x \mid A_1x \leq b_1\} = \max\{d^{\mathsf{T}}x \mid A_1'x \leq b_1'\}$$
$$= \min\{v^{\mathsf{T}}b_1' \mid v \geq \mathbf{0}, v^{\mathsf{T}}A_1' = d^{\mathsf{T}}\}$$

is finite. Hence, since $A_1'x \leq b_1'$ is TDI, the last minimum in (5.75) has an integer optimum solution v. Then v, z is an integer optimum solution of the minimum in (5.73). ∎

A system $Ax \leq b$ is called *totally dual half-integral* if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$ has a half-integer optimum solution, if it is finite. Similarly, $Ax \leq b$ is called *totally dual quarter-integral* if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$ has a quarter-integer optimum solution y, if it is finite.

5.18. Hilbert bases and minimal TDI systems

For any $X \subseteq \mathbb{R}^n$ we denote

(5.76) $$\text{lattice}X := \{\lambda_1x_1 + \cdots + \lambda_kx_k \mid k \geq 0, \lambda_1, \ldots, \lambda_k \in \mathbb{Z}, x_1, \ldots, x_k \in X\}.$$

A subset L of $\mathbb{R}^n$ is called a *lattice* if $L = \text{lattice}X$ for some base X of $\mathbb{R}^n$. So for general X, lattice$X$ need not be a lattice.

The *dual lattice* of X is, by definition:

(5.77) $$\{x \in \mathbb{R}^n \mid y^{\mathsf{T}}x \in \mathbb{Z} \text{ for each } y \in X\}.$$

Again, this need not be a lattice in the proper sense.

A set X of vectors is called a *Hilbert base* if each vector in lattice$X \cap$cone$X$ is a nonnegative integer combination of vectors in X. The Hilbert base is called *integer* if it consists of integer vectors only.

One may show:

(5.78) Each rational polyhedral cone C is generated by an integer Hilbert base. If C is pointed, there exists a unique inclusionwise minimal integer Hilbert base generating C.

(Theorem 16.4 in Schrijver [1986b].)

There is a close relation between Hilbert bases and total dual integrality:

Theorem 5.32. *A rational system $Ax \leq b$ is TDI if and only if for each face F of $P := \{x \mid Ax \leq b\}$, the rows of A which are active in F form a Hilbert base.*

(Theorem 22.5 in Schrijver [1986b].)

(5.78) and Theorem 5.32 imply (Giles and Pulleyblank [1979], Schrijver [1981b]):

Theorem 5.33. *Each rational polyhedron P is determined by a TDI system $Ax \leq b$ with A integer. If moreover P is full-dimensional, there exists a unique minimal such system.*

(Theorem 22.6 in Schrijver [1986b].)

5.19. The integer rounding and decomposition properties

A system $Ax \leq b$ is said to have the *integer rounding property* if $Ax \leq b$ is rational and

(5.79)
$$\begin{aligned}&\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}, y \text{ integer}\}\\ &= \lceil \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}\rceil\end{aligned}$$

for each integer vector c for which $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$ is finite. So any TDI system has the integer rounding property.

A polyhedron P is said to have the *integer decomposition property* if for each natural number k, each integer vector in $k \cdot P$ is the sum of k integer vectors in P.

Baum and Trotter [1978] showed that an integer matrix A is totally unimodular if and only if the polyhedron $\{x \mid x \geq \mathbf{0}, Ax \leq b\}$ has the integer decomposition property for each integer vector b. In another paper, Baum and Trotter [1981] observed the following relation between the integer rounding and the integer decomposition property:

(5.80) Let A be a nonnegative integer matrix. Then the system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ has the integer rounding property if and only if the blocking polyhedron $B(P)$ of $P := \{x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\}$ has the integer decomposition property and all minimal integer vectors in $B(P)$ are transposes of rows of A (minimal with respect to $\leq$).

Similarly,

(5.81) Let A be a nonnegative integer matrix. Then the system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ has the integer rounding property if and only if the antiblocking polyhedron $A(P)$ of $P := \{x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$ has the integer decomposition property and all maximal integer vectors in $A(P)$ are transposes of rows of A (maximal with respect to $\leq$).

(Theorem 22.19 in Schrijver [1986b].)

5.20. Box-total dual integrality

A system $Ax \leq b$ is called *box-totally dual integral*, or just *box-TDI*, if the system $d \leq x \leq c, Ax \leq b$ is totally dual integral for each choice of vectors $d, c \in \mathbb{R}^n$. By Theorem 5.22,

(5.82) if $Ax \leq b$ is box-totally dual integral, then the polyhedron $\{x \mid Ax \leq b\}$ is box-integer.

We will need the following two results.

Theorem 5.34. *If $Ax \leq b$ is box-TDI in n dimensions and $w \in \mathbb{R}^n$, then $Ax \leq b - Aw$ is box-TDI.*

Proof. Directly from the definition of box-total dual integrality. ∎

Theorem 5.35. *Let $Ax \leq b$ be a system of linear inequalities, with A an $m \times n$ matrix. Suppose that for each $c \in \mathbb{R}^n$, $\max\{c^\mathsf{T}x \mid Ax \leq b\}$ has (if finite) an optimum dual solution $y \in \mathbb{R}^m_+$ such that the rows of A corresponding to positive components of y form a totally unimodular submatrix of A. Then $Ax \leq b$ is box-TDI.*

Proof. Choose $d, c \in \mathbb{R}^n$, with $d \leq c$, and choose $c \in \mathbb{Z}^n$. Consider the dual of maximizing $c^\mathsf{T}x$ over $Ax \leq b$, $d \leq x \leq c$:

(5.83) $$\min\{y^\mathsf{T}b + z_1^\mathsf{T}c - z_2^\mathsf{T}d \mid y \in \mathbb{R}^m_+, z_1, z_2 \in \mathbb{R}^n_+, y^\mathsf{T}A + z_1^\mathsf{T} - z_2^\mathsf{T} = c^\mathsf{T}\}.$$

Let y, z_1, z_2 attain this optimum. Define $c' := c - z_1 + z_2$. By assumption, $\min\{y'^\mathsf{T}b \mid y' \in \mathbb{R}^m_+, y'^\mathsf{T}A = c'^\mathsf{T}\}$ has an optimum solution such that the rows of A corresponding to positive components of y' form a totally unimodular matrix. Now y', z_1, z_2 is an optimum solution of (5.83). Also, the rows in $Ax \leq b$, $d \leq x \leq c$ corresponding to positive components of y', z_1, z_2 form a totally unimodular matrix. Hence by Corollary 5.20b, (5.83) has an integer optimum solution. ∎

5.21. The integer hull and cutting planes

Let P be a rational polyhedron. The *integer hull* P_I of P is the convex hull of the integer vectors in P:

(5.84) $P_{\mathrm{I}} = \text{conv.hull}(P \cap \mathbb{Z}^n)$.

It can be shown that P_{I} is a rational polyhedron again.

Consider any rational affine halfspace $H = \{x \mid c^{\mathsf{T}}x \leq \delta\}$, where c is a nonzero integer vector such that the g.c.d. of its components is equal to 1 and where $\delta \in \mathbb{Q}$. Then it is easy to show that

(5.85) $H_{\mathrm{I}} = \{x \mid c^{\mathsf{T}}x \leq \lfloor\delta\rfloor\}$.

The inequality $c^{\mathsf{T}}x \leq \lfloor\delta\rfloor$ (or, more correctly, the hyperplane $\{x \mid c^{\mathsf{T}}x = \lfloor\delta\rfloor\}$) is called a *cutting plane*.

Define for any rational polyhedron P:

(5.86) $P' := \bigcap_{H \supseteq P} H_{\mathrm{I}},$

where H ranges over all rational affine halfspaces H containing P. Then P' is a rational polyhedron contained in P. Since $P \subseteq H$ implies $P_{\mathrm{I}} \subseteq H_{\mathrm{I}}$, we know

(5.87) $P_{\mathrm{I}} \subseteq P' \subseteq P$.

For $k \in \mathbb{Z}_+$, define $P^{(k)}$ inductively by:

(5.88) $P^{(0)} := P$ and $P^{(k+1)} := (P^{(k)})'$.

Then (Gomory [1958,1960], Chvátal [1973a], Schrijver [1980b]):

Theorem 5.36. *For each rational polyhedron there exists a $k \in \mathbb{Z}_+$ with $P_{\mathrm{I}} = P^{(k)}$.*

(For a proof, see Theorem 23.2 in Schrijver [1986b].)

5.21a. Background literature

Most background on polyhedra and linear and integer programming needed for this book can be found in Schrijver [1986b].

More background can be found in Dantzig [1963] (linear programming), Grünbaum [1967] (polytopes), Hu [1969] (integer programming), Garfinkel and Nemhauser [1972a] (integer programming), Brøndsted [1983] (polytopes), Chvátal [1983] (linear programming), Lovász [1986] (ellipsoid method), Grötschel, Lovász, and Schrijver [1988] (ellipsoid method), Nemhauser and Wolsey [1988] (integer programming), Padberg [1995] (linear programming), Ziegler [1995] (polytopes), and Wolsey [1998] (integer programming).

Part I

Paths and Flows

Paths belong to the most basic and important objects in combinatorial optimization. First of all, paths are of direct practical use, to make connections and to search. One can imagine that even in very primitive societies, finding short paths and searching (for instance, for food) is essential. 'Short' need not be just in terms of geometric distance, but might observe factors like differences in height, crosscurrent, and wind. In modern societies, searching is also an important issue in several kinds of networks, like in communication webs and data structures. Several other nonspatial problems (like the knapsack problem, dynamic programming) can be modelled as a shortest path problem. Also planning tools like PERT and CPM are based on shortest paths.

Next to that, paths form an important tool in solving other combinatorial optimization problems. In a large part of combinatorial algorithms, finding an appropriate path is the main issue in a subroutine or in the iterations. Several combinatorial optimization problems can be solved by iteratively finding a shortest path.

Disjoint paths were first investigated in a topological setting by Menger starting in the 1920s, leading to Menger's theorem, a min-max relation equating the maximum number of disjoint $s-t$ paths and the minimum size of an $s-t$ cut. The theorem is fundamental to graph theory, and provides an important tool to handle the connectivity of graphs.

In a different environment, the notion of *flow* in a graph came up, namely at RAND in the 1950s, motivated by a study of the capacity of the Soviet and East European railway system. It inspired Ford and Fulkerson to develop a maximum flow algorithm based on augmenting paths and to prove the max-flow min-cut theorem. As flows can be considered as linear combinations of incidence vectors of paths, there is a close connection between disjoint paths and flow problems, and it turned out that the max-flow min-cut theorem and Menger's theorem can be derived from each other.

Minimum-cost flows can be considered as the common generalization of shortest paths and disjoint paths/flows. Related are minimum-cost circulations and transshipments. This connects the topic to the origins of linear programming in the 1940s, when Koopmans designed pivot-like procedures for minimum-cost transshipment in order to plan protected ship convoys during World War II.

Actually, linear programming and polyhedral methods apply very favourably to path and flow problems, by the total unimodularity of the underlying constraint matrices. They lead to fast, strongly polynomial-time algorithms for such problems, while also fast direct, combinatorial algorithms have been found.

Chapters:

Chapter 6

Shortest paths: unit lengths

The first three chapters of this part are devoted to the shortest path problem. A number of simple but fundamental methods have been developed for it.
The division into the three chapters is by increasing generality of the length function. In the present chapter we take unit lengths; that is, each edge or arc has length 1. Equivalently, we search for paths with a minimum number of edges or arcs. We also consider 'zero length', equivalently, searching for *any* path.
Next, in Chapter 7, we consider nonnegative lengths, where Dijkstra's method applies. Finally, in Chapter 8, we go over to arbitrary lengths. If we put no further constraints, the shortest path problem is NP-complete (in fact, even if all lengths are -1). But if there are no negative-length directed circuits, the problem is polynomial-time solvable, by the Bellman-Ford method.
The methods and results in this chapter generally apply to directed and undirected graphs alike; however, in case of an undirected graph with length function such that each circuit has nonnegative length, the problem is polynomial-time, but the method is much more involved. It can be solved in polynomial time with nonbipartite matching methods, and for this we refer to Section 29.2.
In this chapter, graphs can be assumed to be simple.

6.1. Shortest paths with unit lengths

Let $D = (V, A)$ be a digraph. In this chapter, the length of any path in D is the number of its arcs. For $s, t \in V$, the *distance* from s to t is the minimum length of any $s-t$ path. If no $s-t$ path exists, we set the distance from s to t equal to ∞.

There is an easy min-max relation, due to Robacker [1956b], characterizing the minimum length of an $s-t$ path. Recall that a subset C of A is an $s-t$ *cut* if $C = \delta^{\text{out}}(U)$ for some subset U of V satisfying $s \in U$ and $t \notin U$.[1]

[1] $\delta^{\text{out}}(U)$ and $\delta^{\text{in}}(U)$ denote the sets of arcs leaving and entering U, respectively.

Theorem 6.1. *Let $D = (V, A)$ be a digraph and let $s, t \in V$. Then the minimum length of an $s-t$ path is equal to the maximum number of disjoint $s-t$ cuts.*

Proof. Trivially, the minimum is at least the maximum, since each $s-t$ path intersects each $s-t$ cut in at least one arc. That the minimum is equal to the maximum follows by considering the $s-t$ cuts $\delta^{\text{out}}(U_i)$ for $i = 1, \ldots, d$, where d is the distance from s to t and where U_i is the set of vertices of distance less than i from s. ∎

(This is the proof of Robacker [1956b].)

Dantzig [1957] observed the following. Let $D = (V, A)$ be a digraph and let $s \in V$. A rooted tree $T = (V', A')$ rooted at s is called a *shortest paths tree* (*rooted at* s) if V' is the set of vertices in D reachable from s and $A' \subseteq A$, such that for each $t \in V'$, the $s-t$ path in T is a shortest $s-t$ path in D.

Theorem 6.2. *Let $D = (V, A)$ be a digraph and let $s \in V$. Then there exists a shortest paths tree rooted at s.*

Proof. Let V' be the set of vertices reachable in D from s. Choose, for each $t \in V' \setminus \{s\}$, an arc a_t that is the last arc of some shortest $s-t$ path in D. Then $A' := \{a_t \mid t \in V' \setminus \{s\}\}$ gives the required rooted tree. ∎

The above trivially applies also to *undirected* graphs.

6.2. Shortest paths with unit lengths algorithmically: breadth-first search

The following algorithm of Berge [1958b] and Moore [1959], essentially *breadth-first search*, determines the distance from s to t. Let V_i denote the set of vertices of D at distance i from s. Then $V_0 = \{s\}$, and for each i:

(6.1) V_{i+1} is equal to the set of vertices $v \in V \setminus (V_0 \cup V_1 \cup \cdots \cup V_i)$ for which $(u, v) \in A$ for some $u \in V_i$.

This gives us directly an algorithm for determining the sets V_i: we set $V_0 := \{s\}$ and next we determine with rule (6.1) the sets $V_1, V_2, \ldots$ successively, until $V_{i+1} = \emptyset$.

In fact, it gives a linear-time algorithm, and so:

Theorem 6.3. *Given a digraph $D = (V, A)$ and $s, t \in V$, a unit-length shortest $s-t$ path can be found in linear time.*

Proof. Directly from the description. ∎

In fact, the algorithm finds the distance from s to all vertices reachable from s. Moreover, it gives the shortest paths, even the shortest paths tree:

Theorem 6.4. *Given a digraph $D = (V, A)$ and $s \in V$, a shortest path tree rooted at s can be found in linear time.*

Proof. Use the algorithm described above. ∎

6.3. Depth-first search

In certain cases it is more useful to scan a graph not by breadth-first search as in Section 6.2, but by *depth-first search* (a variant of which goes back to Tarry [1895]).

Let $D = (V, A)$ be a digraph. Define the operation of *scanning* a vertex v recursively by:

(6.2) For each arc $a = (v, w) \in \delta^{\text{out}}(v)$: delete all arcs entering w and scan w.

Then depth-first search from a vertex s amounts to scanning s. If each vertex of D is reachable from s, then all arcs a chosen in (6.2) form a rooted tree with root s. This tree is called a *depth-first search tree*.

This can be applied to find a path and to sort and order the vertices of a digraph $D = (V, A)$. We say that vertices $v_1, \ldots, v_n$ are in *topological order* if $i < j$ for all i, j with $(v_i, v_j) \in A$. So a subset of V can be topologically ordered only if it induces no directed circuit.

To grasp the case where directed circuits occur, we say that vertices $v_1, \ldots, v_n$ are in *pre-topological order* if for all i, j, if v_j is reachable from v_i and $j < i$, then v_i is reachable from v_j. So if D is acyclic, any pre-topological order is topological.

We can interpret a pre-topological order as a linear extension of the partial order $\prec$ defined on V by:

(6.3) $v \prec w \iff w$ is reachable from v, but v is not reachable from w.

Thus a pre-topological ordering is one satisfying: $v_i \prec v_j \Rightarrow i < j$.

The following was shown by Knuth [1968] and Tarjan [1974d] (cf. Kahn [1962]):

Theorem 6.5. *Given a digraph $D = (V, A)$ and $s \in V$, the vertices reachable from s can be ordered pre-topologically in time $O(m')$, where m' is the number of arcs reachable from s.*

Proof. Scan s. Then recursively all vertices reachable from s will be scanned, and the order in which we *finish* scanning them is the opposite of a pre-topological order: for vertices v, w reachable from s, if D has a $v - w$ path but no $w - v$ path, then scanning w is finished before scanning v. ∎

This implies:

Corollary 6.5a. *Given a digraph $D = (V, A)$, the vertices of D can be ordered pre-topologically in linear time.*

Proof. Add a new vertex s and arcs (s, v) for each $v \in V$. Applying Theorem 6.5 gives the required pre-topological ordering. ∎

For acyclic digraphs, it gives a topological order:

Corollary 6.5b. *Given an acyclic digraph $D = (V, A)$, the vertices of D can be ordered topologically in linear time.*

Proof. Directly from Corollary 6.5a, as a pre-topological order of the vertices of an acyclic graph is topological. ∎

(The existence of a topological order for an acyclic digraph is implicit in the work of Szpilrajn [1930].)

One can also use a pre-topological order to identify the strong components of a digraph in linear time (Karzanov [1970], Tarjan [1972]). We give a method essentially due to S.R. Kosaraju (cf. Aho, Hopcroft, and Ullman [1983]) and Sharir [1981].

Theorem 6.6. *Given a digraph $D = (V, A)$, the strong components of D can be found in linear time.*

Proof. First order the vertices of D pre-topologically as $v_1, \ldots, v_n$. Next let V_1 be the set of vertices reachable to v_1. Then V_1 is the strong component containing v_1: each v_j in V_1 is reachable from v_1, by the definition of pre-topological order.

By Theorem 6.5, the set V_1 can be found in time $O(|A_1|)$, where A_1 is the set of arcs with head in V_1. Delete from D and from $v_1, \ldots, v_n$ all vertices in V_1 and all arcs in A_1, yielding the subgraph D' and the ordered vertices $v'_1, \ldots, v'_{n'}$. This is a pre-topological order for D', for suppose that $i < j$ and that v'_i is reachable from v'_j in D' while v'_j is not reachable in D' from v'_i. Then v'_j is also not reachable in D from v'_i, since otherwise V_1 would be reachable in D from v'_i, and hence $v'_i \in V_1$, a contradiction.

So recursion gives all strong components, in linear time. ∎

As a consequence one has for undirected graphs (Shirey [1969]):

Corollary 6.6a. *Given a graph $G = (V, E)$, the components of G can be found in linear time.*

Proof. Directly from Theorem 6.6. ∎

If we apply depth-first search to a connected undirected graph $G = (V, E)$, starting from a vertex s, then the depth-first search tree T has the property that

(6.4) for each edge $e = uv$ of G, u is on the $s - v$ path in T or v is on the $s - u$ path in T.

So the ends of each edge e of G are connected by a directed path in T.

6.4. Finding an Eulerian orientation

An orientation $D = (V, A)$ of an undirected graph $G = (V, E)$ is called an *Eulerian orientation* if

(6.5) $\deg_A^{\text{in}}(v) = \deg_A^{\text{out}}(v)$

for each $v \in V$. As is well-known, an undirected graph $G = (V, E)$ has an Eulerian orientation if and only if each $v \in V$ has even degree in G. (We do not require connectivity.)

An Eulerian orientation can be found in linear time:

Theorem 6.7. *Given an undirected graph $G = (V, E)$ with all degrees even, an Eulerian orientation of G can be found in $O(m)$ time.*

Proof. We assume that we have a list of vertices, and, with each vertex v, a list of edges incident with v.

Consider the first nonisolated vertex in the list, v say. Starting at v, we make a walk such that no edge is traversed more than once. We make this walk as long as possible. Since all degrees are even, we terminate at v.

We orient all edges traversed in the direction as traversed, delete them from G, find the next nonisolated vertex, and iterate the algorithm. We stop if all vertices are isolated. Then all edges are oriented as required. ∎

6.5. Further results and notes

6.5a. All-pairs shortest paths in undirected graphs

Theorem 6.3 directly gives that determining the distances in a digraph $D = (V, A)$ between all pairs of vertices can be done in time $O(nm)$; similarly for undirected graphs.

Seidel [1992,1995] gave the following faster method for dense undirected graphs $G = (V, E)$, assuming without loss of generality that G is connected (by Corollary 6.6a).

For any undirected graph $G = (V, E)$, let $\mathrm{dist}_G(v, w)$ denote the distance between v and w in G (with unit-length edges). Moreover, let G^2 denote the graph with vertex set V, where two vertices v, w are adjacent in G^2 if and only if $\mathrm{dist}_G(v, w) \leq 2$.

Let $M(n)$ denote the time needed to determine the product $A \cdot B$ of two $n \times n$ matrices A and B, each with entries in $\{0, \ldots, n\}$.

It is not difficult to see that the following problem can be solved in time $O(M(n))$:

(6.6) given: an undirected graph $G = (V, E)$;
find: the undirected graph G^2.

Moreover, also the following problem can be solved in time $O(M(n))$:

(6.7) given: an undirected graph $G = (V, E)$ and the function dist_{G^2};
find: the function dist_G.

Lemma 6.8α. *Problem* (6.7) *can be solved in time* $O(M(n))$.

Proof. Let A be the adjacency matrix of G and let T be the $V \times V$ matrix with $T_{v,w} = \mathrm{dist}_{G^2}(v, w)$ for all $v, w \in V$. Note that $\mathrm{dist}_{G^2} = \lceil \mathrm{dist}_G/2 \rceil$. Let $B := T \cdot A$. So for all $u, w \in V$:

(6.8) $$B_{u,w} = \sum_{v \in N(w)} \mathrm{dist}_{G^2}(u, v).$$

Now if $\mathrm{dist}_G(u, w) = 2\lceil \mathrm{dist}_G(u, w)/2 \rceil$, then $\lceil \mathrm{dist}_G(u, v)/2 \rceil \geq \lceil \mathrm{dist}_G(u, w)/2 \rceil$ for each neighbour v of w, and hence $B_{u,w} \geq \deg(v)\lceil \mathrm{dist}_G(u, w)/2 \rceil$. On the other hand, if $\mathrm{dist}_G(u, w) = 2\lceil \mathrm{dist}_G(u, w)/2 \rceil - 1$, then $\lceil \mathrm{dist}_G(u, v)/2 \rceil \leq \lceil \mathrm{dist}_G(u, w)/2 \rceil$ for each neighbour v of w, with strict inequality for at least one neighbour v of w (namely any neighbour of w on a shortest $u - w$ path). So we have $B_{u,w} < \deg(v)\lceil \mathrm{dist}_G(u, w)/2 \rceil$. We conclude that having G, $\mathrm{dist}_{G^2} = \lceil \mathrm{dist}_G/2 \rceil$, and B, we can derive dist_G. As we can calculate B in time $O(M(n))$, we have the lemma. ∎

The all-pairs shortest paths algorithm now is described recursively as follows:

(6.9) If G is a complete graph, the distance between any two distinct vertices is 1. If G is not complete, determine G^2 and from this (recursively) dist_{G^2}. Next determine dist_G.

Theorem 6.8. *Given an undirected graph G on n vertices, the function* dist_G *can be determined in time $O(M(n) \log n)$. Here $M(n)$ denotes the time needed to multiply two $n \times n$ matrices each with entries in* $\{0, \ldots, n\}$.

Proof. Determining G^2 from G and determining dist_G from G and dist_{G^2} can be done in time $O(M(n))$. Since the depth of the recursion is $O(\log n)$, the algorithm has running time $O(M(n) \log n)$. ∎

The results on fast matrix multiplication of Coppersmith and Winograd [1987, 1990] give $M(n) = o(n^{2.376})$ (extending earlier work of Strassen [1969]).

Seidel [1992,1995] showed in fact that also shortest paths can be found in this way. More precisely, for all $u, w \in V$ with $u \neq w$, a neighbour v of w can be found such that $\mathrm{dist}_G(u, v) = \mathrm{dist}_G(u, w) - 1$, in time $O(M(n) \log n + n^2 \log^2 n)$. Having this, one can find, for any $u, w \in V$, a shortest $u - w$ path in time $O(\mathrm{dist}_G(u, w))$.

6.5b. Complexity survey

Complexity survey for all-pairs shortest paths with unit lengths (∗ indicates an asymptotically best bound in the table):

	$O(nm)$	Berge [1958b], Moore [1959]
	$O(n^{\frac{3+\omega}{2}} \log^3 n)$	Alon, Galil, and Margalit [1991,1997]
∗	$O(nm \log_n (n^2/m))$	Feder and Motwani [1991,1995]
∗	$O(n^\omega \log n)$	*undirected* Seidel [1992,1995]

Here ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^\omega)$ arithmetic operations (e.g. $\omega = 2.376$).

Alon, Galil, and Margalit [1991,1997] extended their method to digraphs with arc lengths in $\{-1, 0, 1\}$.

Related work was done by Fredman [1976], Yuval [1976], Romani [1980], Aingworth, Chekuri, and Motwani [1996], Zwick [1998,1999a,2002], Aingworth, Chekuri, Indyk, and Motwani [1999], and Shoshan and Zwick [1999].

6.5c. Ear-decomposition of strongly connected digraphs

Let $D = (V, A)$ be a directed graph. An *ear* of D is a directed path or circuit P in D such that all internal vertices of P have indegree and outdegree equal to 1 in D. The path may consist of a single arc — so any arc of D is an ear. If I is the set of internal vertices of an ear P, we say that D arises from $D - I$ by *adding ear* P. An *ear-decomposition* of D is a series of digraphs $D_0, D_1, \ldots, D_k$, where $D_0 = K_1$, $D_k = D$, and D_i arises from D_{i-1} by adding an ear ($i = 1, \ldots, k$).

Digraphs having an ear-decomposition are be characterized by:

Theorem 6.9. *A digraph $D = (V, A)$ is strongly connected if and only if D has an ear-decomposition.*

Proof. Sufficiency of the condition is easy, since adding an ear to a strongly connected graph maintains strong connectivity.

To see necessity, let $D = (V, A)$ be strongly connected. Let $D' = (V', A')$ be a subgraph of D which has an ear-decomposition and with $|V'| + |A'|$ as large as possible. (Such a subgraph exists, as any single vertex has an ear-decomposition.)

Then $D' = D$, for otherwise there exists an arc $a \in A \setminus A'$ with tail in V'. Then a is contained in a directed circuit C (as D is strongly connected). This circuit C contains a subpath (or circuit) P such that P can be added as an ear to D'. This contradicts the maximality of $|V'| + |A'|$. ∎

A related decomposition of strongly connected digraphs was described by Knuth [1974]. Related work was done by Grötschel [1979].

6.5d. Transitive closure

Complexity survey for finding the transitive closure of a directed graph ($*$ indicates an asymptotically best bound in the table):

	$O(n^3)$	Warshall [1962]
$*$	$O(nm)$	Purdom [1970], cf. Coffy [1973] (also Ebert [1981])
	$O(n^3/\log n)$	Arlazarov, Dinits, Kronrod, and Faradzhev [1970]
	$\widetilde{O}(n^\omega)$	Furman [1970], Munro [1971]
	$O(n^\omega \log n \log\log n \log\log\log n)$	Aho, Hopcroft, and Ullman [1974]
$*$	$O(n^\omega \log n \log\log\log n \log\log\log\log n)$	Adleman, Booth, Preparata, and Ruzzo [1978]

Again, ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^\omega)$ arithmetic operations (e.g. $\omega = 2.376$). Moreover, $f = \widetilde{O}(g)$ if $f = O(g \log^k g)$ for some k.

For more on finding the transitive closure we refer to Fischer and Meyer [1971], Munro [1971], O'Neil and O'Neil [1973], Dzikiewicz [1975], Sysło and Dzikiewicz [1975], Warren [1975], Eve and Kurki-Suonio [1977], Adleman, Booth, Preparata, and Ruzzo [1978], Schnorr [1978a], Schmitz [1983], Ioannidis and Ramakrishnan [1988], Jakobsson [1991], Ullman and Yannakakis [1991], and Cohen [1994a,1997].

Aho, Garey, and Ullman [1972] showed that finding a minimal directed graph having the same transitive closure as a given directed graph, has the same time complexity as finding the transitive closure.

6.5e. Further notes

For the decomposition of graphs into 3-connected graphs, see Cunningham and Edmonds [1980]. Karzanov [1970] and Tarjan [1974b] gave linear-time algorithms (based on a search method) to find the bridges of an undirected graph.

Theorem 6.1 implies the result of Moore and Shannon [1956] that if $D = (V, A)$ is a digraph, $s, t \in V$, and l is the minimum length of an $s-t$ path and w is the minimum size of an $s-t$ cut, then $|A| \geq lw$ (the *length-width inequality*).

Finding a shortest (directed) circuit in a (directed) graph can be reduced to finding a shortest path. More efficient algorithms were given by Itai and Rodeh [1978].

Barnes and Ruzzo [1991,1997] gave a polynomial-time algorithm to test if there exists an $s-t$ path in an undirected graph, using sublinear space only. This was extended to directed graphs by Barnes, Buss, Ruzzo, and Schieber [1992,1998]. Related work was done by Savitch [1970], Cook and Rackoff [1980], Beame, Borodin, Raghavan, Ruzzo, and Tompa [1990,1996], Nisan [1992,1994], Nisan, Szemeredi, and Wigderson [1992], Broder, Karlin, Raghavan, and Upfal [1994], and Armoni, Ta-Shma, Wigderson, and Zhou [1997,2000].

Karp and Tarjan [1980a,1980b] gave algorithms for finding the connected components of an undirected graph, and the strong components of a directed graph, in $O(n)$ expected time. More on finding strong components can be found in Gabow [2000a].

Books discussing algorithmic problems on paths with unit lengths (reachability, closure, etc.) include Even [1973], Aho, Hopcroft, and Ullman [1974,1983], Christofides [1975], Cormen, Leiserson, and Rivest [1990], Lengauer [1990], Jungnickel [1999], and Mehlhorn and Näher [1999]. Berge [1958b] gave an early survey on shortest paths.

Chapter 7

Shortest paths: nonnegative lengths

In this chapter we consider the shortest path problem in graphs where each arc has a nonnegative length, and describe Dijkstra's algorithm, together with a number of speedups based on heaps.

In this chapter, graphs can be assumed to be simple. If not mentioned explicitly, *length* is taken with respect to a given function l.

7.1. Shortest paths with nonnegative lengths

The methods and results discussed in Chapter 6 for unit-length arcs can be generalized to the case where arcs have a not necessarily unit length. For any 'length' function $l : A \to \mathbb{R}$ and any path $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$, the *length* $l(P)$ of P is defined by:

$$l(P) := \sum_{i=1}^{m} l(a). \tag{7.1}$$

The *distance* from s to t (with respect to l), denoted by $\mathrm{dist}_l(s,t)$, is equal to the minimum length of any $s-t$ path. If no $s-t$ path exists, $\mathrm{dist}_l(s,t)$ is set to $+\infty$.

A weighted version of Theorem 6.1 is as follows, again due to Robacker [1956b] (sometimes called the 'max-potential min-work theorem' (Duffin [1962])):

Theorem 7.1. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $l : A \to \mathbb{Z}_+$. Then the minimum length of an $s-t$ path is equal to the maximum size k of a family of $s-t$ cuts $C_1, \ldots, C_k$ such that each arc a is in at most $l(a)$ of the cuts C_i.*

Proof. Again, the minimum is not smaller than the maximum, since if P is any $s-t$ path and $C_1, \ldots, C_k$ is any collection as described in the theorem, then

$$(7.2) \qquad l(P) = \sum_{a \in AP} l(a) \geq \sum_{a \in AP} (\text{number of } i \text{ with } a \in C_i)$$
$$= \sum_{i=1}^{k} |C_i \cap AP| \geq \sum_{i=1}^{k} 1 = k.$$

To see equality, let d be the distance from s to t and let U_i be the set of vertices at distance less than i from s, for $i = 1, \ldots, d$. Taking $C_i := \delta^{\text{out}}(U_i)$, we obtain a collection $C_1, \ldots, C_d$ as required. ∎

A rooted tree $T = (V', A')$, with root s, is called a *shortest paths tree* for a length function $l : A \to \mathbb{R}_+$, if V' is the set of vertices reachable from s and $A' \subseteq A$ such that for each $t \in V'$, the $s - t$ path in T is a shortest $s - t$ path in D. Again, Dantzig [1957] showed:

Theorem 7.2. *Let $D = (V, A)$ be a digraph, let $s \in V$, and let $l : A \to \mathbb{R}_+$. Then there exists a shortest paths tree for l, with root s.*

Proof. Let V' be the set of vertices reachable from s. Let A' be an inclusionwise minimal set containing for each $t \in V'$ a shortest $s - t$ path of D. Suppose that some vertex v is entered by two arcs in A'. Then at least one of these arcs can be deleted, contradicting the minimality of A'. One similarly sees that no arc in A' enters s. ∎

7.2. Dijkstra's method

Dijkstra [1959] gave an $O(n^2)$ algorithm to find a shortest $s - t$ path for nonnegative length functions — in fact, the output is a shortest paths tree with root s. We describe Dijkstra's method (the idea of this method was also described by Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957]).

We keep a subset U of V and a function $d : V \to \mathbb{R}_+$ (the *tentative distance*). Start with $U := V$ and set $d(s) := 0$ and $d(v) = \infty$ if $v \neq s$. Next apply the following iteratively:

(7.3) Find $u \in U$ minimizing $d(u)$ over $u \in U$. For each $a = (u, v) \in A$ for which $d(v) > d(u) + l(a)$, reset $d(v) := d(u) + l(a)$. Reset $U := U \setminus \{u\}$.

We stop if $d(u) = \infty$ for all $u \in U$. The final function d gives the distance from s. Moreover, if we store for each $v \neq s$ the last arc $a = (u, v)$ for which we have reset $d(v) := d(u) + l(a)$, we obtain a shortest path tree with root s.

Clearly, the number of iterations is at most $|V|$, while each iteration takes $O(n)$ time. So the algorithm has running time $O(n^2)$. Thus:

Theorem 7.3. *Given a digraph $D = (V, A)$, $s \in V$, and a length function $l : A \to \mathbb{Q}_+$, a shortest paths tree with root s can be found in time $O(n^2)$.*

Proof. We show correctness of the algorithm. Let $\text{dist}(v)$ denote the distance from s to v, for any vertex v. Trivially, $d(v) \geq \text{dist}(v)$ for all v, throughout the iterations. We prove that throughout the iterations, $d(v) = \text{dist}(v)$ for each $v \in V \setminus U$. At the start of the algorithm this is trivial (as $U = V$).

Consider any iteration (7.3). It suffices to show that $d(u) = \text{dist}(u)$ for the chosen $u \in U$. Suppose $d(u) > \text{dist}(u)$. Let $s = v_0, v_1, \ldots, v_k = u$ be a shortest $s - u$ path. Let i be the smallest index with $v_i \in U$.

Then $d(v_i) = \text{dist}(v_i)$. Indeed, if $i = 0$, then $d(v_i) = d(s) = 0 = \text{dist}(s) = \text{dist}(v_i)$. If $i > 0$, then (as $v_{i-1} \in V \setminus U$):

(7.4) $$d(v_i) \leq d(v_{i-1}) + l(v_{i-1}, v_i) = \text{dist}(v_{i-1}) + l(v_{i-1}, v_i) = \text{dist}(v_i).$$

This implies $d(v_i) \leq \text{dist}(v_i) \leq \text{dist}(u) < d(u)$, contradicting the choice of u. ∎

7.3. Speeding up Dijkstra's algorithm with *k*-heaps

If $|A|$ is asymptotically smaller than $|V|^2$, one may expect faster methods than $O(n^2)$. Such a method based on 'heaps' (introduced by Williams [1964] and Floyd [1964]), was given by Murchland [1967b] and sharpened by Johnson [1972], Johnson [1973b,1977a] and Tarjan [1983] (see Section 8.6g).

In Dijkstra's algorithm, we spend (in total) $O(m)$ time on updating the values $d(u)$, and $O(n^2)$ time on finding a $u \in U$ minimizing $d(u)$. As $m \leq n^2$, a decrease in the running time bound requires a speedup in finding a u minimizing $d(u)$.

A way of doing this is based on storing U in some order such that a $u \in U$ minimizing $d(u)$ can be found quickly and such that it does not take too much time to restore the order if we delete a u minimizing $d(u)$ or if we decrease some $d(u)$.

This can be done by using 'heaps', two forms of which we consider: k-heaps (in this section) and Fibonacci heaps (in the next section).

A *k-heap* is an ordering $u_0, \ldots, u_n$ of the elements of U such that for all i, j, if $ki < j \leq k(i+1)$, then $d(u_i) \leq d(u_j)$.

This is a convenient way of defining (and displaying) the heap, but it is helpful to imagine the heap as a rooted tree on U: its arcs are the pairs (u_i, u_j) with $ki < j \leq k(i+1)$. So u_i has outdegree k if $k(i+1) \leq n$. The root of this rooted tree is u_0.

If one has a k-heap, one easily finds a u minimizing $d(u)$: it is the root u_0. The following two theorems are basic for estimating the time needed for updating the k-heap if we change U or values of $d(u)$. To *swap* u_i and u_j means exchanging the positions of u_i and u_j in the order (that is, resetting $u_j := u_i$ and $u_i :=$ the old u_j).

Theorem 7.4. *If u_0 is deleted, the k-heap can be restored in time $O(k \log_k n)$.*

Proof. Reset $u_0 := u_n$ and $n := n-1$. Let $i = 0$. While there is a j with $ki < j \leq ki + k$, $j \leq n-1$, and $d(u_j) < d(u_i)$, choose such a j with smallest $d(u_j)$, swap u_i and u_j, and reset $i := j$.

The final k-heap is as required. ∎

The operation described is called *sift-down*. The following theorem describes the operation *sift-up*.

Theorem 7.5. *If $d(u_i)$ is decreased, the k-heap can be restored in time $O(\log_k n)$.*

Proof. While $i > 0$ and $d(u_j) > d(u_i)$ for $j := \lfloor \frac{i-1}{k} \rfloor$, swap u_i and u_j, and reset $i := j$. The final k-heap is as required. ∎

In Dijkstra's algorithm, we delete at most $|V|$ times a u minimizing $d(u)$ and we decrease at most $|A|$ times any $d(u)$. So using a k-heap, the algorithm can be done in time $O(nk \log_k n + m \log_k n)$. This implies:

Theorem 7.6. *Given a digraph $D = (V, A)$, $s \in V$, and a length function $l : A \to \mathbb{Q}_+$, a shortest paths tree with root s can be found in time $O(m \log_k n)$, where $k := \max\{2, \frac{m}{n}\}$.*

Proof. See above. ∎

This implies that if for some class of digraphs $D = (V, A)$ one has $|A| \geq |V|^{1+\varepsilon}$ for some fixed $\varepsilon > 0$, then there is a linear-time shortest path algorithm for these graphs.

7.4. Speeding up Dijkstra's algorithm with Fibonacci heaps

Using a more sophisticated heap, the 'Fibonacci heap', Dijkstra's algorithm can be speeded up to $O(m + n \log n)$, as was shown by Fredman and Tarjan [1984,1987].

A *Fibonacci forest* is a rooted forest (U, A), such that for each $v \in U$ the children of v can be ordered in such a way that the ith child has at least $i - 2$ children. (If $(u, v) \in A$, v is called a *child* of u, and u the *parent* of v.)

Lemma 7.7α. *In a Fibonacci forest (U, A), each vertex has outdegree at most $2 \log_2 |U|$.*

Proof. We show:

(7.5) if u has outdegree at least k, then at least $\sqrt{2}^k$ vertices are reachable from u.

This implies the lemma, since $\sqrt{2}^k \le |U|$ is equivalent to $k \le 2\log_2|U|$.

In proving (7.5), we may assume that u is a root. We prove (7.5) by induction on k, the case $k = 0$ being trivial. If $k \ge 1$, let v be the highest ordered child of u. So v has outdegree at least $k - 2$. Then by induction, at least $\sqrt{2}^{k-2}$ vertices are reachable from v. Next delete arc (u, v). We keep a Fibonacci forest, in which u has outdegree at least $k - 1$. By induction, at least $\sqrt{2}^{k-1}$ vertices are reachable from u in the new forest. Hence at least

(7.6) $$\sqrt{2}^{k-2} + \sqrt{2}^{k-1} \ge \sqrt{2}^k$$

vertices are reachable from u in the original forest. ∎

(The recursion (7.6) shows that the Fibonacci numbers give the best bound, justifying the name Fibonacci forest. The weaker bound given, however, is sufficient for our purposes.)

A *Fibonacci heap* consists of a rooted forest $F = (U, A)$ and functions $d : U \to \mathbb{R}$ and $\phi : U \to \{0, 1\}$, such that:

(7.7)
(i) if $(u, v) \in A$, then $d(u) \le d(v)$;
(ii) for each $u \in U$, the children of u can be ordered such that the ith child v satisfies $\deg^{\text{out}}(v) + \phi(v) \ge i - 1$;
(iii) if u and v are distinct roots of F, then $\deg^{\text{out}}(u) \ne \deg^{\text{out}}(v)$.

Condition (7.7)(ii) implies that F is a Fibonacci forest. So, by Lemma 7.7α, condition (7.7)(iii) implies that F has at most $1 + 2\log_2|U|$ roots.

The Fibonacci heap will be specified by the following data structure, where $t := \lfloor 2\log_2|U|\rfloor$:

(7.8)
(i) for each $u \in U$, a doubly linked list of the children of u (in any order);
(ii) the function parent : $U \to U$, where parent(u) is the parent of u if it has one, and parent$(u) = u$ otherwise;
(iii) the functions $\deg^{\text{out}} : U \to \mathbb{Z}_+$, $\phi : U \to \{0, 1\}$, and $d : U \to \mathbb{R}$;
(iv) a function $b : \{0, \ldots, t\} \to U$ with $b(\deg^{\text{out}}(u)) = u$ for each root u.

Theorem 7.7. *When inserting p times a new vertex, finding and deleting n times a root u minimizing $d(u)$, and decreasing m times the value of $d(u)$, the structure can be restored in time $O(m + p + n \log p)$.*

Proof. Inserting a new vertex v, with value $d(v)$, can be done by setting $\phi(v) := 0$ and by applying:

(7.9)
plant(v):
Let $r := b(\deg^{\text{out}}(v))$.
If r is a root with $r \ne v$, then:

$$\begin{cases} \text{if } d(r) \le d(v), \text{ add arc } (r,v) \text{ to } A \text{ and plant}(r); \\ \text{if } d(r) > d(v), \text{ add arc } (v,r) \text{ to } A \text{ and plant}(v); \end{cases}$$
else define $b(\deg^{\text{out}}(v)) := v$.

Throughout we update the lists of children and the functions parent, $\deg^{\text{out}}$, ϕ, and b.

A root u minimizing $d(u)$ can be found in time $O(\log p)$, by scanning $d(b(i))$ for $i = 0, \ldots, t$ where $b(i)$ is a root.

The root u can be deleted as follows. Let $v_1, \ldots, v_k$ be the children of u. First delete u and all arcs leaving u from the forest. This maintains conditions (7.7)(i) and (ii). Next, condition (7.7)(iii) can be restored by applying plant(v) for each $v = v_1, \ldots, v_k$.

If we decrease the value of $d(u)$ for some $u \in U$ we do the following:

(7.10) Determine the longest directed path P in F ending at u such that each internal vertex v of P satisfies $\phi(v) = 1$. Reset $\phi(v) := 1-\phi(v)$ for each $v \in VP \setminus \{u\}$. Delete all arcs of P from A. Apply plant(v) to each $v \in VP$ that is a root of the new forest.

The fact that this maintains (7.7) uses that if the starting vertex q of P is not a root of the original forest, then $q \neq u$ and $\phi(q)$ is reset from 0 to 1 — hence $\deg^{\text{out}}(q) + \phi(q)$ is not changed, and we maintain (7.7)(ii).

We estimate the running time. Throughout all iterations, ϕ increases at most m times (at most once in each application of (7.10)). Hence ϕ decreases at most m times. So the sum of the lengths of the paths P in (7.10) is at most $2m$. So A decreases at most $2m + 2n \log_2 p$ times (since each time we delete a root we delete at most $2 \log_2 p$ arcs). Therefore, A increases at most $2m + 2n \log_2 p + p$ times (since the final $|A|$ is less than p). This gives the running time bound. ∎

This implies for the shortest path problem:

Corollary 7.7a. *Given a digraph $D = (V, A)$, $s \in V$, and a length function $l : A \to \mathbb{Q}_+$, a shortest paths tree with root s can be found in time $O(m+n \log n)$.*

Proof. Directly from Dijkstra's method and Theorem 7.7. ∎

7.5. Further results and notes

7.5a. Weakly polynomial-time algorithms

The above methods all give a strongly polynomial-time algorithm for the shortest path problem, with best running time bound $O(m + n \log n)$. If we allow also the size of the numbers to occur in the running time bound, some other methods are of interest that are in some cases (when the lengths are small integers) faster than the above methods.

In Dijkstra's algorithm, we must select a $u \in U$ with $d(u)$ minimum. It was observed by Dial [1969] that partitioning U into 'buckets' according to the values of d gives a competitive running time bound. The method also gives the following result of Wagner [1976]:

Theorem 7.8. *Given a digraph $D = (V, A)$, $s \in V$, $l : A \to \mathbb{Z}_+$, and an upper bound Δ on $\max\{\mathrm{dist}_l(s, v) \mid v$ reachable from $s\}$, a shortest path tree rooted at s can be found in time $O(m + \Delta)$.*

Proof. Apply Dijkstra's algorithm as follows. Next to the function $d : U \to \mathbb{Z}_+ \cup \{\infty\}$, we keep doubly linked lists $L_0, \ldots, L_\Delta$ such that if $d(u) \le \Delta$, then u is in $L_{d(u)}$. We keep also, for each $i = 0, \ldots, \Delta$, the first element u_i of L_i. If L_i is empty, then u_i is void. Moreover, we keep a 'current minimum' $\mu \in \{0, \ldots, \Delta\}$.

The initialization follows directly from the initialization of d: we set $L_0 := \{s\}$, $u_0 := s$, while L_i is empty and u_i void for $i = 1, \ldots, \Delta$. Initially, $\mu := 0$.

The iteration is as follows. If $L_\mu = \emptyset$ and $\mu \le \Delta$, reset $\mu := \mu + 1$. If $L_\mu \ne \emptyset$, apply Dijkstra's iteration to u_μ: We remove u_μ from L_μ. When decreasing some $d(u)$ from d to d', we delete u from L_d (if $d \le \Delta$) and insert it into $L_{d'}$ (if $d' \le \Delta$).

We stop if $\mu = \Delta + 1$. With each removal or insertion, we can update the lists and the u_i in constant time. Hence we have the required running time. ∎

A consequence is the bound of Dial [1969]:

Corollary 7.8a. *Given a digraph $D = (V, A)$, $s \in V$, and $l : A \to \mathbb{Z}_+$, a shortest path tree rooted at s can be found in time $O(m+nL)$ where $L := \max\{l(a) \mid a \in A\}$.*

Proof. We can take $\Delta = nL$ in Theorem 7.8. ∎

One can derive from Theorem 7.8 also the following result of Gabow [1985b]:

Theorem 7.9. *Given a digraph $D = (V, A)$, $s \in V$, and a length function $l : A \to \mathbb{Z}_+$, a shortest paths tree rooted at s can be found in time $O(m \log_d L)$, where $d = \max\{2, m/n\}$, and $L := \max\{l(a) \mid a \in A\}$.*

Proof. For each $a \in A$, let $l'(a) := \lfloor l(a)/d \rfloor$. Recursively we find $\mathrm{dist}_{l'}(s, v)$ for all $v \in V$, in time $O(m \log_d L')$ where $L' := \lfloor L/d \rfloor$. Note that $\log_d L' \le (\log_d L) - 1$. Now set

$$\tilde{l}(a) := l(a) + d \cdot \mathrm{dist}_{l'}(s, u) - d \cdot \mathrm{dist}_{l'}(s, v) \tag{7.11}$$

for each $a = (u, v) \in A$. Then $\tilde{l}(a) \ge 0$, since

$$l(a) \ge d \cdot l'(a) \ge d(\mathrm{dist}_{l'}(s, v) - \mathrm{dist}_{l'}(s, u)). \tag{7.12}$$

Moreover, $\mathrm{dist}_{\tilde{l}}(s, v) \le nd$ for each v reachable from s, since if P is an $s - v$ path with $l'(P) = \mathrm{dist}_{l'}(s, v)$, then $\tilde{l}(P) = l(P) - dl'(P) \le nd$. So by Theorem 7.8 we can find $\mathrm{dist}_{\tilde{l}}(s, v)$ for all $v \in V$ in time $O(m)$, since $nd \le 2m$. As $\mathrm{dist}_l(s, v) = \mathrm{dist}_{\tilde{l}}(s, v) - d \cdot \mathrm{dist}_{l'}(s, v)$, we find the required data. ∎

(This improves a result of Hansen [1980a].)

7.5b. Complexity survey for shortest paths with nonnegative lengths

The following gives a survey of the development of the running time bound for the shortest path problem for a digraph $D = (V, A)$, $s, t \in V$, and nonnegative length-function l, where $n := |V|$, $m := |A|$, and $L := \max\{l(a) \mid a \in A\}$ (assuming l integer). As before, $*$ indicates an asymptotically best bound in the table.

	$O(n^4)$	Shimbel [1955]
	$O(n^2 mL)$	Ford [1956]
	$O(nm)$	Bellman [1958], Moore [1959]
	$O(n^2 \log n)$	Dantzig [1958,1960], Minty (cf. Pollack and Wiebenson [1960]), Whiting and Hillier [1960]
	$O(n^2)$	Dijkstra [1959]
	$O(m + nL)$	Dial [1969] (cf. Wagner [1976], Filler [1976](=E.A. Dinits))
	$O(m \log(2 + (n^2/m)))$	Johnson [1972]
	$O(dn \log_d n + m + m \log_d(n^2/m))$	(for any $d \geq 2$) Johnson [1973b]
	$O(m \log_{m/n} n)$	Johnson [1973b,1977a], Tarjan [1983]
	$O(L + m \log \log L)$	van Emde Boas, Kaas, and Zijlstra [1977]
	$O(m \log \log L + n \log L \log \log L)$	Johnson [1977b]
	$O(\min_{k \geq 2}(nkL^{1/k} + m \log k))$	Denardo and Fox [1979]
	$O(m \log L)$	Hansen [1980a]
$*$	$O(m \log \log L)$	Johnson [1982], Karlsson and Poblete [1983]
$*$	$O(m + n \log n)$	Fredman and Tarjan [1984,1987][2]
	$O(m \log_{m/n} L)$	Gabow [1983b,1985b]
$*$	$O(m + n\sqrt{\log L})$	Ahuja, Mehlhorn, Orlin, and Tarjan [1990]

Fredman and Willard [1990,1994] gave an $O(m + n\frac{\log n}{\log \log n})$-time algorithm for shortest paths with nonnegative lengths, utilizing nonstandard capabilities of a RAM like addressing. This was extended to $O(m + n\sqrt{\log n \log \log n})$ by Raman [1996], to $O(m \log \log n)$ and $O(m \log \log L)$ by Thorup [1996,2000b] (cf. Hagerup [2000]), and to $O(m+n(\log L \log \log L)^{1/3})$ by Raman [1997]. For undirected graphs,

[2] Aho, Hopcroft, and Ullman [1983] (p. 208) claimed to give an $O(m + n \log n)$-time shortest path algorithm based on 2-heaps, but they assume that, after resetting a value, the heap can be restored in constant time.

a bound of $O(m)$ was achieved by Thorup [1997,1999,2000a]. Related results were given by Pettie and Ramachandran [2002b].

The expected complexity of Dijkstra's algorithm is investigated by Noshita, Masuda, and Machida [1978], Noshita [1985], Cherkassky, Goldberg, and Silverstein [1997,1999], Goldberg [2001a,2001b], and Meyer [2001].

In the special case of *planar* directed graphs:

	$O(n\sqrt{\log n})$	Frederickson [1983b,1987b]
*	$O(n)$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]

For the all-pairs shortest paths problem with nonnegative lengths one has:

	$O(n^4)$	Shimbel [1955]
	$O(n^3 \log n)$	Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957]
	$O(n^2 m)$	Bellman [1958], Moore [1959]
	$O(n^3 \log n)$	Dantzig [1958,1960], Minty (cf. Pollack and Wiebenson [1960]), Whiting and Hillier [1960]
	$O(n^3)$	Dijkstra [1959]
	$O(nm + n^2 L)$	Dial [1969] (cf. Wagner [1976])
	$O(nm \log n)$	Johnson [1972]
	$O(n^3 (\log\log n / \log n)^{1/3})$	Fredman [1976]
	$O(nm \log_{m/n} n)$	Johnson [1973b,1977a], Tarjan [1983]
	$O(nL + nm \log\log L)$	van Emde Boas, Kaas, and Zijlstra [1977]
	$O(nm \log\log L + n^2 \log L \log\log L)$	Johnson [1977b]
	$O(nm \log L)$	Hansen [1980a]
*	$O(nm \log\log L)$	Johnson [1982], Karlsson and Poblete [1983]
*	$O(n(m + n \log n))$	Fredman and Tarjan [1984,1987]
	$O(nm \log_{m/n} L)$	Gabow [1983b,1985b]
*	$O(nm + n^2 \sqrt{\log L})$	Ahuja, Mehlhorn, Orlin, and Tarjan [1990]
*	$O(n^3 (\log\log n / \log n)^{1/2})$	Takaoka [1992a,1992b]
*	$\widetilde{O}(n^{\frac{5\omega-3}{\omega+1}} L + n^{\frac{3+\omega}{2}} L^{\frac{\omega-1}{2}})$	Galil and Margalit [1997a,1997b]
*	$\widetilde{O}(n^{\omega} L^{\frac{\omega+1}{2}})$	*undirected* Galil and Margalit [1997a,1997b]

Here ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^\omega)$ arithmetic operations (e.g. $\omega = 2.376$). Moreover, $f = \widetilde{O}(g)$ if $f = O(g \log^k g)$ for some k. (At the negative side, Kerr [1970] showed that matrix multiplication of $n \times n$ matrices with addition and multiplication replaced by minimization and addition, requires time $\Omega(n^3)$.)

Spira [1973] gave an $O(n^2 \log^2 n)$ expected time algorithm for all-pairs shortest paths with nonnegative lengths. This was improved to $O(n^2 \log n \log\log n)$ by Takaoka and Moffat [1980], to $O(n^2 \log n \log^* n)$ by Bloniarz [1980,1983] (defining $\log^* n := \min\{i \mid \log^{(i)} n \leq 1\}$, where $\log^{(0)} n := n$ and $\log^{(i+1)} n := \log(\log^{(i)} n))$, and to $O(n^2 \log n)$ by Moffat and Takaoka [1985,1987]. Related work includes Carson and Law [1977], Frieze and Grimmett [1985], Hassin and Zemel [1985], Walley and Tan [1995], and Mehlhorn and Priebe [1997].

Yen [1972] (cf. Williams and White [1973]) described an all-pairs shortest paths method (based on Dijkstra's method) using $\frac{1}{2}n^3$ additions and n^3 comparisons. Nakamori [1972] gave a lower bound on the number of operations. Yao, Avis, and Rivest [1977] gave a lower bound of $\Omega(n^2 \log n)$ for the time needed for the all-pairs shortest paths problem.

Karger, Koller, and Phillips [1991,1993] and McGeoch [1995] gave an $O(n(m^* + n \log n))$ algorithm for all-pairs shortest paths, where m^* is the number of arcs that belong to at least one shortest path. See also Yuval [1976] and Romani [1980] for relations between all-pairs shortest paths and matrix multiplication.

Frederickson [1983b,1987b] showed that in a *planar* directed graph, with nonnegative lengths, the all-pairs shortest paths problem can be solved in $O(n^2)$ time.

7.5c. Further notes

Spira and Pan [1973,1975], Shier and Witzgall [1980], and Tarjan [1982] studied the sensitivity of shortest paths trees under modifying arc lengths. Fulkerson and Harding [1977] studied the problem of lengthening the arc lengths within a given budget (where each arc has a given cost for lengthening the arc length) so as to maximize the distance from a given source to a given sink. They reduced this problem to a parametric minimum-cost flow problem. Land and Stairs [1967] and Hu [1968] studied decomposition methods for finding all-pairs shortest paths (cf. Farbey, Land, and Murchland [1967], Hu and Torres [1969], Yen [1971b], Shier [1973], and Blewett and Hu [1977]).

Frederickson [1989,1995] gave a strongly polynomial-time algorithm to find an $O(n)$ encoding of shortest paths between all pairs in a directed graph with nonnegative length function. (It extends earlier work of Frederickson [1991] for planar graphs.)

Algorithms for finding the k shortest paths between pairs of vertices in a directed graph were given by Clarke, Krikorian, and Rausen [1963], Yen [1971a], Minieka [1974], Weigand [1976], Lawler [1977], Shier [1979], Katoh, Ibaraki, and Mine [1982], Byers and Waterman [1984], Perko [1986], Chen [1994], and Eppstein [1994b,1999].

Mondou, Crainic, and Nguyen [1991] gave a survey of shortest paths methods, with computational results, and Raman [1997] on 'recent' results on shortest paths with nonnegative lengths.

Books covering shortest path methods for nonnegative lengths include Berge [1973b], Aho, Hopcroft, and Ullman [1974,1983], Christofides [1975], Lawler [1976b], Minieka [1978], Even [1979], Hu [1982], Papadimitriou and Steiglitz [1982], Smith [1982], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Rockafellar [1984], Nemhauser and Wolsey [1988], Bazaraa, Jarvis, and Sherali [1990], Chen [1990], Cormen, Leiserson, and Rivest [1990], Lengauer [1990], Ahuja, Magnanti, and Orlin [1993], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].

Chapter 8

Shortest paths: arbitrary lengths

We now go over to the shortest path problem for the case where negative lengths are allowed, but where each directed circuit has nonnegative length (with no restriction, the problem is NP-complete). The basic algorithm here is the Bellman-Ford method.

In this chapter, graphs can be assumed to be simple. If not mentioned explicitly, *length* is taken with respect to a given function l.

8.1. Shortest paths with arbitrary lengths but no negative circuits

If lengths of arcs may take negative values, finding a shortest $s-t$ path is NP-complete — see Theorem 8.11 below. Negative-length directed circuits seem to be the source of the trouble: if no negative-length directed circuits exist, there is a polynomial-time algorithm — mainly due to the fact that running into loops cannot give shortcuts. So a shortest *walk* (nonsimple path) exists and yields a shortest *path*.

We first observe that if no negative-length directed circuits exists, then the existence of a shortest paths tree is easy:

Theorem 8.1. *Let $D = (V, A)$ be a digraph, let $s \in V$, and let $l : A \to \mathbb{R}$ be such that each directed circuit reachable from s has nonnegative length. Then there exists a shortest paths tree with root s.*

Proof. As the proof of Theorem 7.2. ∎

8.2. Potentials

The following observation of Gallai [1958b] is very useful. Let $D = (V, A)$ be a digraph and let $l : A \to \mathbb{R}$. A function $p : V \to \mathbb{R}$ is called a *potential* if for each arc $a = (u, v)$:

(8.1) $\qquad l(a) \geq p(v) - p(u).$

Theorem 8.2. *Let $D = (V, A)$ be a digraph and let $l : A \to \mathbb{R}$ be a length function. Then there exists a potential if and only if each directed circuit has nonnegative length. If moreover l is integer, the potential can be taken integer.*

Proof. *Suffiency.* Suppose that a function p as described exists. Let $C = (v_0, a_1, v_1, \ldots, a_m, v_m)$ be a directed circuit ($v_m = v_0$). Then

$$l(C) = \sum_{i=1}^{m} l(a_i) \geq \sum_{i=1}^{m} (p(v_i) - p(v_{i-1})) = 0. \tag{8.2}$$

Necessity. Suppose that each directed circuit has nonnegative length. For each $t \in V$, let $p(t)$ be the minimum length of any path ending at t (starting wherever). This function satisfies the required condition. ∎

Theorem 8.2 gives a good characterization for the problem of deciding if there exists a negative-length directed circuit.

A potential is useful in transforming a length function to a nonnegative length function: if we define $\tilde{l}(a) := l(a) - p(v) + p(u)$ for each arc $a = (u, v)$, then we obtain a nonnegative length function $\tilde{l}$ such that each $s - t$ path is shortest with respect to l if and only if it is shortest with respect to $\tilde{l}$. So once we have a potential p, we can find shortest paths with Dijkstra's algorithm. This can be used for instance in finding shortest paths between all pairs of vertices — see Section 8.4.

One can also formulate a min-max relation in terms of functions that are potentials on an appropriate subgraph. This result is sometimes called the 'max-potential min-work theorem' (Duffin [1962]).

Theorem 8.3. *Let $D = (V, A)$ be a digraph, let $s, t \in V$ and let $l : A \to \mathbb{R}$. Then $\mathrm{dist}_l(s, t)$ is equal to the maximum value of $p(t) - p(s)$, where $p : V \to \mathbb{R}$ is such that $l(a) \geq p(v) - p(u)$ for each arc $a = (u, v)$ traversed by at least one $s - t$ walk. If l is integer, we can restrict p to be integer.*

Proof. Let $p(v) := \mathrm{dist}_l(s, v)$ if v belongs to at least one $s - t$ walk, and $p(v) := 0$ otherwise. This p is as required. ∎

The following observation can also be of use, for instance when calculating shortest paths by linear programming:

Theorem 8.4. *Let $D = (V, A)$ be a digraph, let $s \in V$ be such that each vertex of D is reachable from s, and let $l : A \to \mathbb{R}$ be such that each directed circuit has nonnegative length. Let p be a potential with $p(s) = 0$ and $\sum_{v \in V} p(v)$ maximal. Then $p(t) = \mathrm{dist}_l(s, t)$ for each $t \in V$.*

Proof. One easily shows that for any potential p with $p(s) = 0$ one has $p(t) \leq \text{dist}_l(s,t)$ for each $t \in V$. As $\text{dist}_l(s,\cdot)$ is a potential, the theorem follows. ∎

8.3. The Bellman-Ford method

Also in the case of a length function without negative-length directed circuit, there is a polynomial-time shortest path algorithm, the *Bellman-Ford method* (Shimbel [1955], Ford [1956], Bellman [1958], Moore [1959]). Again, it finds a shortest paths tree for any root s.

To describe the method, define for $t \in V$ and $k \geq 0$:

(8.3) $d_k(t) :=$ minimum length of any $s-t$ walk traversing at most k arcs,

setting $d_k(t) := \infty$ if no such walk exists.

Clearly, if there is no negative-length directed circuit reachable from s, the distance from s to t is equal to $d_n(v)$, where $n := |V|$.

Algorithmically, the function d_0 is easy to set: $d_0(s) = 0$ and $d_0(t) = \infty$ if $t \neq s$. Next $d_1, d_2, \ldots$ can be successively computed by the following rule:

$$d_{k+1}(t) = \min\{d_k(t), \min_{(u,t)\in A}(d_k(u) + l(u,t))\} \tag{8.4}$$

for all $t \in V$.

This method gives us the distance from s to t. It is not difficult to derive a method finding a shortest paths tree with root s. Thus:

Theorem 8.5. *Given a digraph $D = (V,A)$, $s \in V$, and a length function $l : A \to \mathbb{Q}$ such that each directed circuit reachable from s has nonnegative length, a shortest paths tree rooted at s can be found in time $O(nm)$.*

Proof. There are at most n iterations, each of which can be performed in time $O(m)$. ∎

A negative-length directed circuit can be detected similarly:

Theorem 8.6. *Given a digraph $D = (V,A)$, $s \in V$, and a length function $l : A \to \mathbb{Q}$, a directed circuit of negative length reachable from s (if any exists) can be found in time $O(nm)$.*

Proof. If $d_n \neq d_{n-1}$, then $d_n(t) < d_{n-1}(t)$ for some $t \in V$. So the algorithm finds an $s-t$ walk P of length $d_n(t)$, traversing n arcs. As P traverses n arcs, it contains a directed circuit C. Removing C gives an $s-t$ walk P' with less than n arcs. So $l(P') \geq d_{n-1}(t) > d_n(t) = l(P)$ and hence $l(C) < 0$.

If $d_n = d_{n-1}$, then there is no negative-length directed circuit reachable from s. ∎

Also a potential can be found with the Bellman-Ford method:

Theorem 8.7. *Given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{Q}$ such that each directed circuit has nonnegative length, a potential can be found in time $O(nm)$.*

Proof. Extend D by a new vertex s and arcs (s, v) for $v \in V$, each of length 0. Then setting $p(v)$ equal to the distance from s to v (which can be determined with the Bellman-Ford method) gives a potential. ▮

We remark that the shortest path problem for *undirected* graphs, for length functions without negative-length circuits, also can be solved in polynomial time. However, the obvious reduction — replacing every undirected edge uv by two arcs (u, v) and (v, u) each of length $l(uv)$ — may yield a negative-length directed circuit. So in this case, the undirected case does not reduce to the directed case, and we cannot apply the Bellman-Ford method. The undirected problem can yet be solved in polynomial time, with the methods developed for the matching problem — see Section 29.2.

8.4. All-pairs shortest paths

Let $D = (V, A)$ be a digraph and $l : A \to \mathbb{Q}$ be a length function such that each directed circuit has nonnegative length. By applying $|V|$ times the Bellman-Ford method one can find shortest $s-t$ paths for all $s, t \in V$. As the Bellman-Ford method takes time $O(nm)$, this makes an $O(n^2m)$ algorithm.

A more efficient algorithm, the *Floyd-Warshall method* was described by Floyd [1962b], based on an idea of Warshall [1962], earlier found by Kleene [1956], Roy [1959], and McNaughton and Yamada [1960]: Order the vertices of D (arbitrarily) as $v_1, \ldots, v_n$. Define for $s, t \in V$ and $k \in \{0, \ldots, n\}$:

(8.5) $d_k(s, t) :=$ minimum length of an $s - t$ walk using only vertices in $\{s, t, v_1, \ldots, v_k\}$.

Clearly, $d_0(s, t) = l(s, t)$ if $(s, t) \in A$, while $d_0(s, t) = \infty$ otherwise. Moreover:

(8.6) $$d_{k+1}(s, t) = \min\{d_k(s, t), d_k(s, v_{k+1}) + d_k(v_{k+1}, t)\}$$

for all $s, t \in V$ and $k < n$.

This gives:

Theorem 8.8. *Given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{Q}$ with no negative-length directed circuit, all distances $\mathrm{dist}_l(s, t)$ can be determined in time $O(n^3)$.*

Proof. Note that $\mathrm{dist}_l = d_n$ and that d_n can be determined in n iterations, each taking $O(n^2)$ time. ▮

The Floyd-Warshall method can be adapted so as to find for all $s \in V$, a shortest $s-t$ paths tree rooted at s.

A faster method was observed by Johnson [1973b,1977a] and Bazaraa and Langley [1974]. Combined with the Fibonacci heap implementation of Dijkstra's algorithm, it gives all-pairs shortest paths in time $O(n(m + n\log n))$, which is, if $n\log n = O(m)$, of the same order as the Bellman-Ford for *single*-source shortest path. The idea is to preprocess the data by a potential function, so as to make the length function nonnegative, and next to apply Dijkstra's method:

Theorem 8.9. *Given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{Q}$ with no negative-length directed circuit, a family $(T_s \mid s \in V)$ of shortest paths trees T_s rooted at s can be found in time $O(n(m + n\log n))$.*

Proof. With the Bellman-Ford method one finds a potential p in time $O(nm)$ (Theorem 8.7). Set $\tilde{l}(a) := l(a) - p(v) + p(u)$ for each arc $a = (u, v)$. So $\tilde{l}(a) \geq 0$ for each arc a. Next with Dijkstra's method, using Fibonacci heaps, one can determine for each $s \in V$ a shortest paths tree T_s for $\tilde{l}$, in time $O(m + n\log n)$ (Corollary 7.7a). As these are shortest paths trees also for l, we have the current theorem. ∎

8.5. Finding a minimum-mean length directed circuit

Let $D = (V, A)$ be a directed graph (with n vertices) and let $l : A \to \mathbb{R}$. The *mean length* of a directed cycle (directed closed walk) $C = (v_0, a_1, v_1, \ldots, a_t, v_t)$ with $v_t = v_0$ and $t > 0$ is $l(C)/t$. Karp [1978] gave the following polynomial-time method for finding a directed cycle of minimum mean length. For each $v \in V$ and each $k = 0, 1, 2, \ldots$, let $d_k(v)$ be the minimum length of a walk with exactly k arcs, ending at v. So for each v one has

(8.7) $$d_0(v) = 0 \text{ and } d_{k+1}(v) = \min\{d_k(u) + l(a) \mid a = (u, v) \in \delta^{\text{in}}(v)\}.$$

Now Karp [1978] showed:

Theorem 8.10. *The minimum mean length of a directed cycle in D is equal to*

(8.8) $$\min_{v \in V} \max_{0 \leq k \leq n-1} \frac{d_n(v) - d_k(v)}{n-k}.$$

Proof. We may assume that the minimum mean length is 0, since adding ε to the length of each arc increases both minima in the theorem by ε. So we must show that (8.8) equals 0.

First, let minimum (8.8) be attained by v. Let P_n be a walk with n arcs ending at v, of length $d_n(v)$. So P_n can be decomposed into a path P_k, say, with k arcs ending at v, and a directed cycle C with $n-k$ arcs (for some $k < n$). Hence $d_n(v) = l(P_n) = l(P_k) + l(C) \geq l(P_k) \geq d_k(v)$ and so $d_n(v) - d_k(v) \geq 0$. Therefore, (8.8) is nonnegative.

To see that it is 0, let $C = (v_0, a_1, v_1, \ldots, a_t, v_t)$ be a directed cycle of length 0. Then $\min_r d_r(v_0)$ is attained by some r with $n - t \leq r < n$ (as it is attained by some $r < n$ (since each circuit has nonnegative length), and as we can add C to the shortest walk ending at v_0). Fix this r.

Let $v := v_{n-r}$, and split C into walks

(8.9) $$P := (v_0, a_1, v_1, \ldots, a_{n-r}, v_{n-r}) \text{ and}$$
$$Q := (v_{n-r}, a_{n-r+1}, v_{n-r+1}, \ldots, a_t, v_t).$$

Then $d_n(v) \leq d_r(v_0) + l(P)$, and therefore for each k:

(8.10) $$d_k(v) + l(Q) \geq d_{k+(t-(n-r))}(v_0) \geq d_r(v_0) \geq d_n(v) - l(P).$$

This implies $d_n(v) - d_k(v) \leq l(C) = 0$. So the minimum (8.8) is at most 0. ∎

Algorithmically, it gives:

Corollary 8.10a. *A minimum-mean length directed circuit can be found in time $O(nm)$.*

Proof. See the method above. ∎

Notes. Karp and Orlin [1981] and Karzanov [1985c] gave generalizations. Orlin and Ahuja [1992] gave an $O(\sqrt{n}\, m \log(nL))$ algorithm for the minimum-mean length directed circuit problem (cf. McCormick [1993]). Early work on this problem includes Lawler [1967], Shapiro [1968], and Fox [1969].

8.6. Further results and notes

8.6a. Complexity survey for shortest path without negative-length circuits

The following gives a survey of the development of the running time bound for the shortest path problem for a digraph $D = (V, A)$, $s, t \in V$, and $l : A \to \mathbb{Z}$ (without negative-length directed circuits), where $n := |V|$, $m := |A|$, $L := \max\{|l(a)| \mid a \in A\}$, and $L' := \max\{-l(a) \mid a \in A\}$ (assuming $L' \geq 2$). As before, $*$ indicates an asymptotically best bound in the table.

	$O(n^4)$	Shimbel [1955]
	$O(n^2 mL)$	Ford [1956]
$*$	$O(nm)$	Bellman [1958], Moore [1959]
	$O(n^{3/4} m \log L')$	Gabow [1983b,1985b]

≫

continued

	$O(\sqrt{n}\, m \log(nL'))$	Gabow and Tarjan [1988b,1989]
*	$O(\sqrt{n}\, m \log L')$	Goldberg [1993b,1995]

(Gabow [1983b,1985b] and Gabow and Tarjan [1988b,1989] give bounds with L instead of L', but Goldberg [1995] mentioned that an anonymous referee of his paper observed that L can be replaced by L'.)

Kolliopoulos and Stein [1996,1998b] proved a bound of $o(n^3)$ for the average-case complexity.

For the special case of *planar* directed graphs:

	$O(n^{3/2})$	Lipton, Rose, and Tarjan [1979]
	$O(n^{4/3} \log(nL'))$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]
*	$O(n \log^3 n)$	Fakcharoenphol and Rao [2001]

For the all-pairs shortest paths problem, with no negative-length directed circuits, one has:

	$O(n^4)$	Shimbel [1955]
	$O(n^3 \log n)$	Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957]
	$O(n^2 m)$	Bellman [1958], Moore [1959]
	$O(n^3)$	Floyd [1962b]
*	$O(n \cdot \mathrm{SP}_+(n, m, L))$	Johnson [1973b,1977a], Bazaraa and Langley [1974]
	$O(nm + n^3 (\log \log n / \log n)^{1/3})$	Fredman [1976]
	$O(nm \log \log L + n^2 \log L \log \log L)$	Johnson [1977b]
	$O(nL + nm \log \log L)$	van Emde Boas, Kaas, and Zijlstra [1977]
*	$O(nm \log \log L)$	Johnson [1982]
	$O(nm \log_{m/n} L)$	Gabow [1985b]
*	$O(nm + n^2 \sqrt{\log L})$	Ahuja, Mehlhorn, Orlin, and Tarjan [1990]
*	$O((nL)^{\frac{3+\omega}{2}} \log^3 n)$	Alon, Galil, and Margalit [1991, 1997], Galil and Margalit [1997a, 1997b]

Here $\mathrm{SP}_+(n, m, L)$ denotes the time needed to find a shortest path in a digraph with n vertices and m arcs, with *nonnegative* integer lengths on the arcs, each at most L. ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^\omega)$ arithmetic operations (e.g. $\omega = 2.376$).

Frederickson [1983b,1987b] showed that for *planar* directed graphs, the all-pairs shortest paths problem, with no negative-length directed circuits, can be solved in $O(n^2)$ time.

8.6b. NP-completeness of the shortest path problem

In full generality — that is, not requiring that each directed circuit has nonnegative length — the shortest path problem is NP-complete, even if each arc has length -1. Equivalently, finding a longest path in a graph (with unit length arcs) is NP-complete. This is a result of E.L. Lawler and R.E. Tarjan (cf. Karp [1972b]).

This directly follows from the NP-completeness of finding a Hamiltonian path in a graph. Let $D = (V, A)$ be a digraph. (A directed path P is called *Hamiltonian* if each vertex of D is traversed exactly once.)

We show the NP-completeness of the *directed Hamiltonian path problem*: Given a digraph $D = (V, A)$ and $s, t \in V$, is there a Hamiltonian $s - t$ path?

Theorem 8.11. *The directed Hamiltonian path problem is* NP*-complete.*

Proof. We give a polynomial-time reduction of the partition problem (Section 4.11) to the directed Hamiltonian path problem. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a collection of subsets of the set $X = \{1, \ldots, k\}$. Introduce vertices $r_0, r_1, \ldots, r_m, 0, 1, \ldots, k$.

For each $i = 1, \ldots, m$, we do the following. Let $C_i = \{j_1, \ldots, j_t\}$. We construct a digraph on the vertices r_{i-1}, r_i, $j_h - 1, j_h$ (for $h = 1, \ldots, t$) and $3t$ new vertices, as in Figure 8.1. Moreover, we make an arc from r_m to 0.

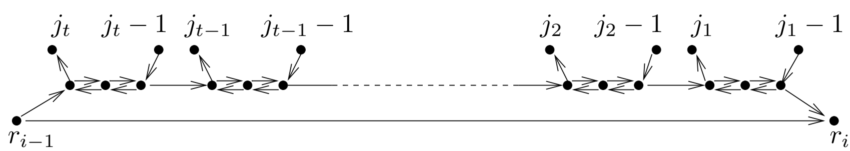

Figure 8.1

Let D be the digraph arising in this way. Then it is not difficult to check that there exists a subcollection $\mathcal{C}'$ of $\mathcal{C}$ that partitions X if and only if D has a directed Hamiltonian $r_0 - k$ path P. (Take: $(r_{i-1}, r_i) \in P \iff C_i \in \mathcal{C}'$.) ∎

Hence:

Corollary 8.11a. *Given a digraph $D = (V, A)$ and $s, t \in V$, finding a longest $s - t$ path is* NP*-complete.*

Proof. This follows from the fact that there exists an $s - t$ path of length $|V| - 1$ if and only if there is a directed Hamiltonian $s - t$ path. ∎

From this we derive the NP-completeness of the *undirected Hamiltonian path problem*: Given a graph $G = (V, E)$ and $s, t \in V$, does G have a Hamiltonian $s - t$ path? (R.E. Tarjan (cf. Karp [1972b])).

Corollary 8.11b. *The undirected Hamiltonian path problem is* NP*-complete.*

Proof. We give a polynomial-time reduction of the directed Hamiltonian path problem to the undirected Hamiltonian path problem. Let D be a digraph. Replace each vertex v by three vertices v', v'', v''', and make edges $\{v', v''\}$ and $\{v'', v'''\}$. Moreover, for each arc (v_1, v_2) of D, make an edge $\{v_1''', v_2'\}$. Delete the vertices s', s'', t'', t'''. This makes the undirected graph G. One easily checks that D has a directed Hamiltonian $s-t$ path if and only if G has an (undirected) Hamiltonian $s''' - t'$ path. ∎

Again it implies:

Corollary 8.11c. *Given an undirected graph $G = (V, E)$ and $s, t \in V$, finding a longest $s-t$ path is* NP*-complete.*

Proof. This follows from the fact that there exists an $s-t$ path of length $|V|-1$ if and only if there is a Hamiltonian $s-t$ path. ∎

Notes. Corollary 8.11b implies that finding a Hamiltonian circuit in an undirected graph is NP-complete: just add a new vertex r and edges rs and rt. This reduces finding a Hamiltonian $s-t$ path in the original graph to finding a Hamiltonian circuit in the extended graph.

Also the directed Hamiltonian circuit problem is NP-complete, as the undirected version can be reduced to it by replacing each edge uv by two oppositely oriented arcs (u, v) and (v, u).

8.6c. Nonpolynomiality of Ford's method

The method originally described by Ford [1956] consists of the following. Given a digraph $D = (V, A)$, $s, t \in V$, and a length function $l : A \to \mathbb{Q}$, define $d(s) := 0$ and $d(v) := \infty$ for all $v \neq s$; next perform the following iteratively:

(8.11) choose an arc (u, v) with $d(v) > d(u) + l(u, v)$, and reset $d(v) := d(u) + l(u, v)$.

Stop if no such arc exists.

If there are no negative-length directed circuits, this is a finite method, since at each iteration $\sum_v d(v)$ decreases, while it is at least $\sum_v \text{dist}_l(s, v)$ and it is an integer multiple of the g.c.d. of the $l(a)$.

In fact, it can be shown that the number of iterations is at most $2^{|V|}$, if l is nonnegative. Moreover, if l is arbitrary (without negative directed circuit), there are at most $2|V|^2 L$ iterations, where $L := \max\{|l(a)| \bigm| a \in A\}$.

However, Johnson [1973a,1973b] showed that the number of iterations is $\Omega(n2^n)$, even if we prescribe to choose (u, v) in (8.11) with $d(u)$ minimal. For nonnegative l, Johnson [1973b,1977a] showed that the number of iterations is $\Omega(2^n)$ if we prescribe no selection rule of u.

8.6d. Shortest and longest paths in acyclic graphs

Let $D = (V, A)$ be a digraph. A subset C of A is called a *directed cut* if there is a subset U of V with $\emptyset \neq U \neq V$ such that $\delta^{\text{out}}(U) = C$ and $\delta^{\text{in}}(U) = \emptyset$. So each directed cut is a cut.

It is easy to see that, if D is acyclic, then a set B of arcs is contained in a directed cut if and only if no two arcs in B are contained in a directed path. Similarly, if D is acyclic, a set B of arcs is contained in a directed path if and only if no two arcs in B are contained in a directed cut.

Theorem 8.12. *Let $D = (V, A)$ be an acyclic digraph and let $s, t \in V$. Then the maximum length of an $s - t$ path is equal to the minimum number of directed $s - t$ cuts covering all arcs that are on at least one $s - t$ path.*

Proof. Any $s-t$ path of length k needs at least k directed $s-t$ cuts to be covered, so the maximum cannot exceed the minimum.

To see equality, let for each $v \in V$, $d(v)$ be equal to the length of a longest $s - v$ path. Let $k := d(t)$. For $i = 1, \ldots, k$, let $U_i := \{v \in V \mid d(v) < i\}$. Then the $\delta^{\text{out}}(U_i)$ form k directed $s - t$ cuts covering all arcs that are on at least one $s - t$ path. ∎

One similarly shows for paths not fixing its ends (Vidyasankar and Younger [1975]):

Theorem 8.13. *Let $D = (V, A)$ be an acyclic digraph. Then the maximum length of any path is equal to the minimum number of directed cuts covering A.*

Proof. Similar to the proof above. ∎

Also weighted versions hold, and may be derived similarly. A weighted version of Theorem 8.12 is:

Theorem 8.14. *Let $D = (V, A)$ be an acyclic digraph, let $s, t \in V$ and let $l : A \to \mathbb{Z}_+$ be a length function. Then the maximum length of an $s - t$ path is equal to the minimum number of directed $s - t$ cuts covering each arc a that is on at least one $s - t$ path, at least $l(a)$ times.*

Proof. Any $s-t$ path of length k needs at least k directed $s-t$ cuts to be covered appropriately, so the maximum cannot exceed the minimum.

To see equality, let for each $v \in V$, $d(v)$ be equal to the length of a longest $s - v$ path. Let $k := d(t)$. For $i = 1, \ldots, k$, let $U_i := \{v \in V \mid d(v) < i\}$. Then the $\delta^{\text{out}}(U_i)$ form k directed $s - t$ cuts covering each arc a on any $s - t$ path at least $l(a)$ times. ∎

Similarly, a weighted version of Theorem 8.13 is:

Theorem 8.15. *Let $D = (V, A)$ be an acyclic digraph and $l : A \to \mathbb{Z}_+$ be a length function. Then the maximum length of any path is equal to the minimum number of directed cuts such that any arc a is in at least $l(a)$ of these directed cuts.*

Proof. Similar to the proof above. ∎

In acyclic graphs one can find shortest paths in *linear* time (Morávek [1970]):

Theorem 8.16. *Given an acyclic digraph $D = (V, A)$, $s, t \in V$, and a length function $l : A \to \mathbb{Q}$, a shortest $s-t$ path can be found in time $O(m)$.*

Proof. First order the vertices reachable from s topologically as $v_1, \ldots, v_n$ (cf. Corollary 6.5b). So $v_1 = s$. Set $d(v_1) := 0$ and determine

(8.12) $$d(v) := \min\{d(u) + l(u, v) \mid (u, v) \in \delta^{\mathrm{in}}(v)\}$$

for $v = v_2, \ldots, v_n$ (in this order). Then for each v reachable from s, $d(v)$ is the distance from s to v. ∎

Note that this implies that also a *longest* path in an acyclic digraph can be found in linear time.

Johnson [1973b] showed that, in a not necessarily acyclic digraph, an $O(m)$-time algorithm for the single-source shortest path problem exists if the number of directed circuits in any strongly connected component is bounded by a constant. Related work was reported by Wagner [2000].

More on longest paths and path covering in acyclic graphs can be found in Chapter 14.

8.6e. Bottleneck shortest path

Pollack [1960] observed that several of the shortest path algorithms can be modified to the following maximum-capacity path problem. For any digraph $D = (V, A)$ and 'capacity' function $c : A \to \mathbb{Q}$, the *capacity* of a path P is the minimum of the capacities of the arcs in P. (This is also called sometimes the *reliability* of P — cf. Section 50.6c.)

Then the *maximum-capacity path problem* (also called the *maximum reliability problem*), is:

(8.13) given: a digraph $D = (V, A)$, $s, t \in V$, and a 'capacity' function $c : A \to \mathbb{Q}$;
find: an $s - t$ path of maximum capacity.

To this end, one should appropriately replace min by max and + by min in these algorithms. Applying this to Dijkstra's algorithm gives, with Fibonacci heaps, a running time of $O(m + n \log n)$.

In fact, the following 'bottleneck' min-max relation holds (Fulkerson [1966]):

Theorem 8.17. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $c : A \to \mathbb{R}$ be a capacity function. Then:*

(8.14) $$\max_{P} \min_{a \in AP} c(a) = \min_{C} \max_{a \in C} c(a),$$

where P ranges over all $s - t$ paths and C over all $s - t$ cuts.

Proof. To see $\leq$ in (8.14), let P be an $s-t$ path and let C be an $s-t$ cut. Since P and C have at least one arc in common, say a_0, we have $\min_{a \in AP} c(a) \leq c(a_0) \leq \max_{a \in C} c(a)$.

To see $\geq$ in (8.14), let $\gamma := \max_P \min_{a \in AP} c(a)$. Let $A' := \{a \in A \mid c(a) \leq \gamma\}$. Then A' intersects each $s-t$ path. So A' contains an $s-t$ cut C. Therefore, $c(a) \leq \gamma$ for all $a \in C$; that is, $\max_{a \in C} c(a) \leq \gamma$. Hence $\min_C \max_{a \in C} c(a) \leq \gamma$. ∎

It is easy to solve the bottleneck shortest path problem by binary search in time $O(m \log L)$, where $L := \|l\|_\infty$ assuming l integer. This was improved by Gabow [1985b] to $O(m \log_n L)$.

8.6f. Further notes

Further analyses of shortest path methods were given by Pollack and Wiebenson [1960], Hoffman and Winograd [1972], Tabourier [1973], Pape [1974], Kershenbaum [1981], Glover, Glover, and Klingman [1984], Pallottino [1984], Glover, Klingman, and Phillips [1985], Glover, Klingman, Phillips, and Schneider [1985], Desrochers [1987], Bertsekas [1991], Goldfarb, Hao, and Kai [1991], Sherali [1991], Cherkassky, Goldberg, and Radzik [1994,1996], and Cherkassky and Goldberg [1999] (negative circuit detection).

Spirakis and Tsakalidis [1986] gave an average-case analysis of an $O(nm)$-time negative circuit detecting algorithm, and Tarjan [1982] a sensitivity analysis of shortest paths trees.

Fast approximation algorithms for shortest paths were given by Klein and Sairam [1992], Cohen [1994b,2000], Aingworth, Chekuri, and Motwani [1996], Dor, Halperin, and Zwick [1996,2000], Cohen and Zwick [1997,2001], and Aingworth, Chekuri, Indyk, and Motwani [1999].

Dantzig [1957] observed that the shortest path problem can be formulated as a linear programming problem, and hence can be solved with the simplex method. Edmonds [1970a] showed that this may take exponentially many pivot steps, even for nonnegative arc lengths. On the other hand, Dial, Glover, Karney, and Klingman [1979] and Zadeh [1979] gave pivot rules that solve the shortest path problem with nonnegative arc lengths in $O(n)$ pivots. For arbitrary length Akgül [1985] and Goldfarb, Hao, and Kai [1990b] gave strongly polynomial simplex algorithms. Akgül [1993] gave an algorithm using $O(n^2)$ pivots, yielding an $O(n(m + n \log n))$-time algorithm. An improvement to $O(nm)$ (with $(n-1)(n-2)/2$ pivots) was given by Goldfarb and Jin [1999b]. Related work was done by Dantzig [1963], Orlin [1985], and Ahuja and Orlin [1988,1992].

Computational results were presented by Pape [1974,1980], Golden [1976], Carson and Law [1977], Kelton and Law [1978], van Vliet [1978], Denardo and Fox [1979], Dial, Glover, Karney, and Klingman [1979], Glover, Glover, and Klingman [1984], Imai and Iri [1984], Glover, Klingman, Phillips, and Schneider [1985], Gallo and Pallottino [1988], Mondou, Crainic, and Nguyen [1991], Goldberg and Radzik [1993] (Bellman-Ford method), Cherkassky, Goldberg, and Radzik [1996], Goldberg and Silverstein [1997], and Zhan and Noon [1998].

Frederickson [1987a,1991] gave an algorithm that gives a succint encoding of all pairs shortest path information in a directed planar graph (with arbitrary lengths, but no negative directed circuits).

Surveys and bibliographies on shortest paths were presented by Pollack and Wiebenson [1960], Murchland [1967a], Dreyfus [1969], Gilsinn and Witzgall [1973], Pierce [1975], Yen [1975], Lawler [1976b], Golden and Magnanti [1977], Deo and Pang [1984], Gallo and Pallottino [1986], and Mondou, Crainic, and Nguyen [1991].

Books covering shortest path methods include Christofides [1975], Lawler [1976b], Even [1979], Hu [1982], Papadimitriou and Steiglitz [1982], Smith [1982], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Nemhauser and Wolsey [1988], Bazaraa, Jarvis, and Sherali [1990], Chen [1990], Cormen, Leiserson, and Rivest [1990], Lengauer [1990], Ahuja, Magnanti, and Orlin [1993], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].

8.6g. Historical notes on shortest paths

Compared with other combinatorial optimization problems like the minimum spanning tree, assignment, and transportation problems, research on the shortest path problem started relatively late. This might be due to the fact that the problem is elementary and relatively easy, which is also illustrated by the fact that at the moment that the problem came into the focus of interest, several researchers independently developed similar methods. Yet, the problem has offered some substantial difficulties, as is illustrated by the fact that heuristic, nonoptimal approaches have been investigated (cf. for instance Rosenfeld [1956], who gave a heuristic approach for determining an optimal trucking route through a given traffic congestion pattern).

Search methods

Depth-first search methods were described in the 19th century in order to traverse all lanes in a maze without knowing its plan. Wiener [1873] described the following method:

> Man markire sich daher den Weg, den man zurücklegt nebst dem Sinne, in welchem es geschieht. Sobald man auf einen schon markirten Weg stösst, kehre man um und durchschreite den schon beschriebenen Weg in umgekehrtem Sinne. Da man, wenn man nicht ablenkte, denselben hierbei in seiner ganzen Ausdehnung nochmals zurücklegen würde, so muss mann nothwendig hierbei auf einen noch nicht markirten einmündenden Weg treffen, den man dann verfolge, bis man wieder auf einen markirten trifft. Hier kehre man wieder um und verfahre wie vorher. Es werden dadurch stets neue Wegtheile zu den beschriebenen zugefügt, so dass man nach einer endlichen Zeit das ganze Labyrinth durchwandern würde und so jedenfalls den Ausgang fände, wenn er nicht schon vorher erreicht worden wäre.[3]

[3] One therefore marks the road that one traverses together with the direction in which it happens. As soon as one hits a road already marked, one turns and traverses the road already followed in opposite direction. As one, if one would not deviate, would traverse it to its whole extent another time, by this one should necessarily meet a road running to a not yet marked one, which one next follows, until one again hits a marked one. Here one turns again and proceeds as before. In that road always new road parts are added to those already followed, so that after a finite time one would walk through the whole labyrinth and in this road in any case would find the exit, if it would not have been reached already before.

In his book *Recréations mathématiques*, Lucas [1882] described a method due to C.P. Trémaux to traverse all lanes of a maze exactly twice, starting at a vertex A: First traverse an arbitrary lane starting at A. Next apply the following rule iteratively when arriving through a lane L at a vertex N:

(8.15) if you did not visit N before, traverse next an arbitrary other lane at N, except if L is the only lane at N, in which case you return through L;
if you have visited N before, return through L, except if you have traversed L already twice; in that case traverse another lane at N not traversed before; if such a lane does not exist, traverse a lane at L that you have traversed before once.

The method stops if no lane can be chosen by this rule at N. It is not hard to show that then one is at the starting vertex A and that all lanes of the maze have been traversed exactly twice (if the maze is connected).

A simpler rule was given by Tarry [1895]:

> Tout labyrinthe peut être parcouru en une seule course, en passant deux fois en sens contraire par chacune des allées, sans qu'il soit nécessaire d'en connaître le plan.
> Pour résoudre ce problème, il suffit d'observer cette règle unique:
> *Ne reprendre l'allée initiale qui a conduit à un carrefour pour la première fois que lorsqu'on ne peut pas faire autrement.*[4]

This is equivalent to depth-first search.

Alternate routing

Path problems were also studied at the beginning of the 1950s in the context of 'alternate routing', that is, finding a second shortest route if the shortest route is blocked. This applies to freeway usage (cf. Trueblood [1952]), but also to telephone call routing. At that time making long-distance calls in the U.S.A. was automatized, and alternate routes for telephone calls over the U.S. telephone network nation-wide should be found automatically:

> When a telephone customer makes a long-distance call, the major problem facing the operator is how to get the call to its destination. In some cases, each toll operator has two main routes by which the call can be started towards this destination. The first-choice route, of course, is the most direct route. If this is busy, the second choice is made, followed by other available choices at the operator's discretion. When telephone operators are concerned with such a call, they can exercise choice between alternate routes. But when operator or customer toll dialing is considered, the choice of routes has to be left to a machine. Since the "intelligence" of a machine is limited to previously "programmed" operations, the choice of routes has to be decided upon, and incorporated in, an automatic alternate routing arrangement.

(Jacobitti [1955], cf. Myers [1953], Clos [1954], and Truitt [1954]).

[4] Each maze can be traversed in one single run, by passing each of the corridors twice in opposite direction, without that it is necessary to know its plan.
To solve this problem, it suffices to observe this only rule:
Retake the initial corridor that has led to a crossing for the first time only when one cannot do otherwise.

Matrix methods

Matrix methods were developed to study relations in networks, like finding the transitive closure of a relation; that is, identifying in a digraph the pairs of vertices s, t such that t is reachable from s. Such methods were studied because of their application to communication nets (including neural nets) and to animal sociology (e.g. peck rights).

The matrix methods consist of representing the relation by a matrix, and then taking iterative matrix products to calculate the transitive closure. This was studied by Landahl and Runge [1946], Landahl [1947], Luce and Perry [1949], Luce [1950], Lunts [1950,1952], and by A. Shimbel.

Shimbel's interest in matrix methods was motivated by their applications to neural networks. He analyzed with matrices which sites in a network can communicate to each other, and how much time it takes. To this end, let S be the $0,1$ matrix indicating that if $S_{i,j} = 1$, then there is direct communication from i to j (including $i = j$). Shimbel [1951] observed that the positive entries in S^t correspond to the pairs between which there exists communication in t steps. An *adequate* communication system is one for which S^t is positive for some t. One of the other observations of Shimbel [1951] is that in an adequate communication system, the time it takes that all sites have all information, is equal to the minimum value of t for which S^t is positive. (A related phenomenon was observed by Luce [1950].)

Shimbel [1953] mentioned that the distance from i to j is equal to the number of zeros in the i, j position in the matrices $S^0, S^1, S^2, \ldots, S^t$. So essentially he gave an $O(n^4)$ algorithm to find all distances in a (unit-length) digraph.

Shortest paths

The basic methods for the shortest path problem are the Bellman-Ford method and Dijkstra's method. The latter one is faster but is restricted to nonnegative length functions. The former method only requires that there is no directed circuit of negative length.

The general framework for both methods is the following scheme, described in this general form by Ford [1956]. Keep a provisional distance function d. Initially, set $d(s) := 0$ and $d(v) := \infty$ for each $v \neq s$. Next, iteratively,

(8.16) choose an arc (u, v) with $d(v) > d(u) + l(u, v)$ and reset $d(v) := d(u) + l(u, v)$.

If no such arc exists, d is the distance function.

The difference in the methods is the rule by which the arc (u, v) with $d(v) > d(u) + l(u, v)$ is chosen. The Bellman-Ford method consists of considering all arcs consecutively and applying (8.16) where possible, and repeating this (at most $|V|$ rounds suffice). This is the method described by Shimbel [1955], Bellman [1958], and Moore [1959].

Dijkstra's method prescribes to choose an arc (u, v) with $d(u)$ smallest (then each arc is chosen at most once, if the lengths are nonnegative). This was described by Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957] and Dijkstra [1959]. A related method, but slightly slower than Dijkstra's method when implemented, was given by Dantzig [1958], G.J. Minty, and Whiting and Hillier [1960], and chooses an arc (u, v) with $d(u) + l(u, v)$ smallest.

Parallel to this, a number of further results were obtained on the shortest path problem, including a linear programming approach and 'good characterizations'.

We now describe the developments in greater detail.

The Bellman-Ford method: Shimbel

In April 1954, Shimbel [1955] presented at the Symposium on Information Networks in New York some observations on calculating distances, which amount to describing a 'min-addition' algebra and a method which later became known as the Bellman-Ford method. He introduced:

> Arithmetic
> For any arbitrary real or infinite numbers x and y
>
> $$x + y \equiv \min(x, y) \text{ and}$$
> $$xy \equiv \text{the algebraic sum of } x \text{ and } y.$$

He extended this arithmetic to the matrix product. Calling the distance matrix associated with a given length matrix S the 'dispersion', he stated:

> It follows trivially that S^k $k \geq 1$ is a matrix giving the shortest paths from site to site in S given that $k-1$ other sites may be traversed in the process. It also follows that for any S there exists an integer k such that $S^k = S^{k+1}$. Clearly, the dispersion of S (let us label it $D(S)$) will be the matrix S^k such that $S^k = S^{k+1}$.

Although Shimbel did not mention it, one trivially can take $k \leq |V|$, and hence the method yields an $O(n^4)$ algorithm to find the distances between all pairs of vertices.

Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957] noted that Shimbel's method can be speeded up by calculating S^k by iteratively raising the current matrix to the square (in the min-addition matrix algebra). This solves the all-pairs shortest paths problem in time $O(n^3 \log n)$.

The Bellman-Ford method: Ford

In a RAND paper dated 14 August 1956, Ford [1956] described a method to find a shortest path from P_0 to P_N, in a network with vertices $P_0, \ldots, P_N$, where l_{ij} denotes the length of an arc from i to j:

> Assign initially $x_0 = 0$ and $x_i = \infty$ for $i \neq 0$. Scan the network for a pair P_i and P_j with the property that $x_i - x_j > l_{ji}$. For this pair replace x_i by $x_j + l_{ji}$. Continue this process. Eventually no such pairs can be found, and x_N is now minimal and represents the minimal distance from P_0 to P_N.

Ford next argues that the method terminates. It was shown by Johnson [1973a, 1973b,1977a] that Ford's liberal selection rule can require exponential time.

In their book *Studies in the Economics of Transportation*, Beckmann, McGuire, and Winsten [1956] showed that the distance matrix $D = (d_{i,j})$ is the unique matrix satisfying

(8.17) $$d_{i,i} = 0 \text{ for all } i;$$
$$d_{i,k} = \min_j (l_{i,j} + d_{j,k}) \text{ for all } i, k \text{ with } i \neq k.$$

The Bellman-Ford method: Bellman

We next describe the work of Bellman on shortest paths. After publishing several papers on dynamic programming (in a certain sense a generalization of shortest path methods), Bellman [1958] eventually focused on the shortest path problem by itself. He described the following 'functional equation approach' (originating from dynamic programming) for the shortest path problem, which is the same as that of Shimbel [1955].

Bellman considered N cities, numbered $1, \ldots, N$, every two of which are linked by a direct road, together with an $N \times N$ matrix $T = (t_{i,j})$, where $t_{i,j}$ is the time required to travel from i to j (not necessarily symmetric). Find a path between 1 and N which consumes minimum time. First, Bellman remarked that the problem is finite:

> Since there are only a finite number of paths available, the problem reduces to choosing the smallest from a finite set of numbers. This direct, or enumerative, approach is impossible to execute, however, for values of N of the order of magnitude of 20.

Next he gave a 'functional equation approach':

> The basic method is that of successive approximations. We choose an initial sequence $\{f_i^{(0)}\}$, and then proceed iteratively, setting
>
> $$f_i^{(k+1)} = \underset{j \neq i}{\text{Min}} \; (t_{ij} + f_j^{(k)}], \qquad i = 1, 2, \cdots, N-1,$$
>
> $$f_N^{(k+1)} = 0,$$
>
> for $k = 0, 1, 2 \cdots,$.

For the initial function $f_i^{(0)}$, Bellman proposed (upon a suggestion of F. Haight) to take $f_i^{(0)} = t_{i,N}$ for all i. Bellman observed that, for each fixed i, starting with this choice of $f_i^{(0)}$ gives that $f_i^{(k)}$ is monotonically nonincreasing in k, and states:

> It is clear from the physical interpretation of this iterative scheme that at most $(N-1)$ iterations are required for the sequence to converge to the solution.

Since each iteration can be done in time $O(N^2)$, the algorithm takes time $O(N^3)$. Bellman also remarks:

> It is easily seen that the iterative scheme discussed above is a feasible method for either hand or machine computation for values of N of the order of magnitude of 50 or 100.

In a footnote, Bellman says:

> *Added in proof* (*December* 1957): After this paper was written, the author was informed by Max Woodbury and George Dantzig that the particular iterative scheme discussed in Sec. 5 had been obtained by them from first principles.

Bellman [1958] mentioned that one could also start with $f_i^{(0)} = \min_{j \neq i} t_{i,j}$ (if $i \neq N$) and $f_N^{(0)} = 0$. In that case for each fixed i, the value of $f_i^{(k)}$ is monotonically *nondecreasing* in k, and converges to the distance from i to N. (Indeed, $f_i^{(k)}$ is equal to the shortest length of all those paths starting at i that have either exactly $k+1$ arcs, or have at most k arcs and end at N.)

The Bellman-Ford method: Moore

At the International Symposium on the Theory of Switching at Harvard University in April 1957, Moore [1959] of Bell Laboratories presented a paper 'The shortest path through a maze':

> The methods given in this paper require no foresight or ingenuity, and hence deserve to be called algorithms. They would be especially suited for use in a machine, either a special-purpose or a general-purpose digital computer.

The motivation of Moore was the routing of toll telephone traffic. He gave algorithms A, B, and C, and D.

First, Moore considered the case of an undirected graph $G = (V, E)$ with no length function, where a path from vertex A to vertex B should be found with a minimum number of edges. Algorithm A is: first give A label 0. Next do the following for $k = 0, 1, \ldots$: give label $k+1$ to all unlabeled vertices that are adjacent to some vertex labeled k. Stop as soon as vertex B is labeled.

> If it were done as a program on a digital computer, the steps given as single steps above would be done serially, with a few operations of the computer for each city of the maze; but, in the case of complicated mazes, the algorithm would still be quite fast compared with trial-and-error methods.

In fact, a direct implementation of the method would yield an algorithm with running time $O(m)$. It is essentially breadth-first search. Algorithms B and C differ from A in a more economical labeling (by fewer bits).

Moore's algorithm D finds a shortest route for the case where each edge of the graph has a nonnegative length. This method gives a refinement of Bellman's method described above: (i) it extends to the case that not all pairs of vertices have a direct connection; that is, if there is an underlying graph $G = (V, E)$ with length function; (ii) at each iteration only those $d_{i,j}$ are considered for which u_i has been decreased in the previous iteration.

The method has running time $O(nm)$. Moore observed that the algorithm is suitable for parallel implementation, yielding a decrease in the running time bound to $O(n\Delta(G))$, where $\Delta(G)$ is the maximum degree of G. He concluded:

> The origin of the present methods provides an interesting illustration of the value of basic research on puzzles and games. Although such research is often frowned upon as being frivolous, it seems plausible that these algorithms might eventually lead to savings of very large sums of money by permitting more efficient use of congested transportation or communication systems. The actual problems in communication and transportation are so much complicated by timetables, safety requirements, signal-to-noise ratios, and economic requirements that in the past those seeking to solve them have not seen the basic simplicity of the problem, and have continued to use trial-and-error procedures which do not always give the true shortest path. However, in the case of a simple geometric maze, the absence of these confusing factors permitted algorithms A, B, and C to be obtained, and from them a large number of extensions, elaborations, and modifications are obvious.
> The problem was first solved in connection with Claude Shannon's maze-solving machine. When this machine was used with a maze which had more than one solution, a visitor asked why it had not been built to always find the shortest path. Shannon and I each attempted to find economical methods of doing this by machine. He found several methods suitable for analog computation, and I obtained these algorithms. Months later the applicability of these ideas to practical problems in communication and transportation systems was suggested.

Among the further applications of his method, Moore described the example of finding the fastest connections from one station to another in a given railroad timetable (cf. also Levin and Hedetniemi [1963]). A similar method was given by Minty [1958].

Berge [1958b] described a breadth-first search method similar to Moore's algorithm A, to find the shortest paths from a given vertex a, for unit lengths, but he described it more generally for directed graphs: let $A(0) := \{a\}$; if $A(k)$ has been found, let $A(k+1)$ be the set of vertices x for which there is a $y \in A(k)$ with (y,x) an arc and with $x \notin A(i)$ for all $i \leq k$. One directly finds a shortest $a-b$ path from the $A(k)$. This gives an $O(m)$ algorithm.

D.A. D'Esopo (cf. the survey of Pollack and Wiebenson [1960]) proposed the following sharpening of Moore's version of the Bellman-Ford method, by indexing the vertices during the algorithm. First define index$(s) := 1$, and let $i := 1$. Then apply the following iteratively:

(8.18) Let v be the vertex with index i. For each arc (v,w) leaving v, reset $d(w) := d(v)+l(v,w)$ if it decreases $d(w)$; if w is not indexed give it the smallest unused index; if some $d(w)$ has been reset with index$(w) < i$, choose the w minimizing index(w), and let $i :=$index(w); otherwise, let $i := i+1$.

In May 1958, Hoffman and Pavley [1959b] reported, at the Western Joint Computer Conference in Los Angeles, the following computing time for finding the distances between all pairs of vertices by Moore's algorithm (with nonnegative lengths):

> It took approximately three hours to obtain the minimum paths for a network of 265 vertices on an IBM *704*.

Linear programming and transportation

Orden [1955] observed that the shortest path problem is a special case of the transshipment problem: let be given an $n \times n$ matrix $(c_{i,j})$ and a vector $g \in \mathbb{R}^n$

(8.19)
$$\begin{aligned} &\text{minimize } \textstyle\sum_{i,j} c_{i,j}x_{i,j} \\ &\text{subject to } \sum_{j=1}^{n}(x_{i,j} - x_{j,i}) = g_i \text{ for } i = 1,\ldots,n \\ &\text{and } x_{i,j} \geq 0 \text{ for } i,j = 1,\ldots n, \end{aligned}$$

and showed that it can be reduced to a 'transportation problem', and hence to a linear programming problem. If one wants to find a shortest $1-n$ path, set $g_1 = 1$, $g_n = -1$, and $g_i = 0$ for all other i.

In a paper presented at the Summer 1955 meeting of ORSA at Los Angeles, Dantzig [1957] formulated the shortest path problem as an integer linear programming problem 'very similar to the system for the assignment problem', and similar to Orden's formulation. Dantzig observed that replacing the condition $x_{i,j} \geq 0$ by $x_{i,j} \in \{0,1\}$ does not change the minimum value. (Dantzig assumed $d_{i,j} = d_{j,i}$ for all i,j.)

He described a graphical procedure for the simplex method applied to this problem. Let T be a rooted tree on $\{1,\ldots,n\}$, with root 1. For each $i = 1,\ldots,n$, let u_i be equal to the length of the path from 1 to i in T. Now if $u_j \leq u_i + d_{i,j}$ for all i, j, then for each i, the $1 - i$ path in T is a shortest path. If $u_j > u_i + d_{i,j}$, replace the arc of T entering j by the arc (i, j), and iterate with the new tree.

Trivially, this process terminates (as $\sum_{j=1}^{n} u_j$ decreases at each iteration, and as there are only finitely many rooted trees). (Edmonds [1970a] showed that the method may take exponential time.) Dantzig illustrated his method by an example of sending a package from Los Angeles to Boston.

Good characterizations

Robacker [1956b] observed that the minimum length of a $P_0 - P_n$ path in a graph N is equal to the maximum number of disjoint $P_0 - P_n$ cuts:

> the maximum number of mutually disjunct cuts of N is equal to the length of the shortest chain of N from P_0 to P_n.

Gallai [1958b] noticed that if the length function $l : A \to \mathbb{Z}$ on the arcs of a digraph (V, A) gives no negative-length directed circuits, then there is a 'potential' $p : V \to \mathbb{Z}$ with $l(u, v) \geq p(v) - p(u)$ for each arc (u, v).

Case Institute of Technology 1957

In the *First Annual Report* of the project *Investigation of Model Techniques*, carried out by the Case Institute of Technology in Cleveland, Ohio for the Combat Development Department of the Army Electronic Proving Ground, Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957] describe (rudimentarily) a shortest path algorithm similar to Dijkstra's algorithm:

> (1) All the links joined to the origin, a, may be given an outward orientation. ...
> (2) Pick out the link or links radiating from a, $a_{a\alpha}$, with the smallest delay. ... Then it is impossible to pass from the origin to any other node in the network by any "shorter" path than $a_{a\alpha}$. Consequently, the minimal path to the general node α is $a_{a\alpha}$.
> (3) All of the other links joining α may now be directed outward. Since $a_{a\alpha}$ must necessarily be the minimal path to α, there is no advantage to be gained by directing any other links toward α. ...
> (4) Once α has been evaluated, it is possible to evaluate immediately all other nodes in the network whose minimal values do not exceed the value of the second-smallest link radiating from the origin. Since the minimal values of these nodes are less than the values of the second-smallest, third-smallest, and all other links radiating directly from the origin, only the smallest link, $a_{a\alpha}$, can form a part of the minimal path to these nodes. Once a minimal value has been assigned to these nodes, it is possible to orient all other links except the incoming link in an outward direction.
> (5) Suppose that all those nodes whose minimal values do not exceed the value of the second-smallest link radiating from the origin have been evaluated. Now it is possible to evaluate the node on which the second-smallest link terminates. At this point, it can be observed that if conflicting directions are assigned to a link, in accordance with the rules which have been given for direction assignment, that link may be ignored. It will not be a part of the minimal path to either of the two nodes it joins. ...
> Following these rules, it is now possible to expand from the second-smallest link as well as the smallest link so long as the value of the third-smallest link radiating from the origin is not exceeded. It is possible to proceed in this way until the entire network has been solved.

(In this quotation we have deleted sentences referring to figures.)

Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, and Seitz [1957] also described a speedup of solving the all-pairs shortest paths problem by matrix-multiplication:

> This process of multiplication may be simplified somewhat by squaring the original structure matrix to obtain a dispersion matrix which is the second power of the structure matrix; squaring the second-power matrix to obtain the fourth power of the structure matrix; and so forth.

This gives an $O(n^3 \log n)$-time all-pairs shortest paths algorithm.

Analog computing

In a reaction to the linear programming approach of Dantzig [1957] discussed above, Minty [1957] proposed an 'analog computer' for the shortest path problem:

> Build a string model of the travel network, where knots represent cities and string lengths represent distances (or costs). Seize the knot 'Los Angeles' in your left hand and the knot 'Boston' in your right and pull them apart. If the model becomes entangled, have an assistant untie and re-tie knots until the entanglement is resolved. Eventually one or more paths will stretch tight — they then are alternative shortest routes.
> Dantzig's 'shortest-route tree' can be found in this model by weighting the knots and picking up the model by the knot 'Los Angeles'.
> It is well to label the knots since after one or two uses of the model their identities are easily confused.

A similar method was proposed by Bock and Cameron [1958] (cf. Peart, Randolph, and Bartlett [1960]). The method was extended to the directed case by Klee [1964].

Rapaport and Abramson [1959] described an electric analog computer for solving the shortest path problem.

Dantzig's $O(n^2 \log n)$ algorithm

Dantzig [1958,1960] gave an $O(n^2 \log n)$ algorithm for the shortest path problem with nonnegative length function. A set X is updated throughout, together with a function $d : X \to \mathbb{Q}_+$. Initially $X = \{s\}$ and $d(s) = 0$. Then do the following iteratively:

(8.20) for each $v \in X$, let w_v be a vertex not in X with $d(w_v)$ minimal. Choose a $v \in X$ minimizing $d(v) + l(v, w_v)$. Add w_v to X and set $d(w_v) := d(v) + l(v, w_v)$.

Stop if $X = V$.

Note that throughout the iterations, the function d is only extended, and not updated. Dantzig assumed

> (a) that one can write down without effort for each node the arcs leading to other nodes in increasing order of length and (b) that it is no effort to ignore an arc of the list if it leads to a node that has been reached earlier.

Indeed, in a preprocessing the arcs can be ordered in time $O(n^2 \log n)$, and, for instance by using doubly linked lists, an arc can be deleted from the appropriate list in time $O(1)$. As each iteration can be done in time $O(n)$ (identifying a v

minimizing $d(v) + l(v, w_v)$ and deleting all arcs entering w_v from each list of arcs leaving x for $x \in X$), Dantzig's method can be performed in time $O(n^2 \log n)$.

Dantzig [1958,1960] mentioned that, beside Bellman, Moore, Ford, and himself, also D. Gale and Fulkerson proposed shortest path methods, 'in informal conversations'.

The same method as that of Dantzig (however without the observations concerning storing the outgoing arcs from any vertex in a list) was given by G.J. Minty (cf. Pollack and Wiebenson [1960]) and by Whiting and Hillier [1960].

Dijkstra's $O(n^2)$ algorithm

Dijkstra [1959] gave an $O(n^2)$ method which is slightly different from that of Dantzig [1958,1960]. Let $D = (V, A)$ be a graph and let a length function $l : A \to \mathbb{R}_+$ be given. Dijkstra's method consists of repeatedly updating a set X and a function $d : V \to \mathbb{R}_+$ as follows.

Initially, set $X = \emptyset$, $d(s) = 0$, $d(v) = \infty$ if $v \neq s$. Next move s into X. Then do the following iteratively: Let v be the vertex just moved into X;

(8.21) for each arc (v, w) with $w \notin X$, reset $d(w) := d(v) + l(v, w)$ if this would decrease $d(w)$. Choose $v' \notin X$ with minimum $d(v')$, and move v' into X.

Stop if no such v' exists.

Since each iteration can be done in time $O(n)$ and since there are at most $|V|$ iterations, the algorithm runs in time $O(n^2)$. Dijkstra states:

> The solution given above is to be preferred to the solution by L.R. Ford *[3]* as described by C. Berge *[4]*, for, irrespective of the number of branches, we need not store the data for all branches simultaneously but only those for the branches in sets I and II, and this number is always less than n. Furthermore, the amount of work to be done seems to be considerably less.

(Dijkstra's references [3] and [4] are Ford [1956] and Berge [1958b].)

Dijkstra's method is easier to implement (as an $O(n^2)$ algorithm) than Dantzig's, since we need not store the information in lists: in order to find $v' \notin X$ minimizing $d(v')$ we can just scan all vertices that are not in X.

Whiting and Hillier [1960] described the same method as Dijkstra.

Heaps

The 2-heap was introduced by Williams [1964] (describing it as an array, with subroutines to insert and extract elements of the heap), as a major improvement to the sorting algorithm 'treesort' of Floyd [1962a]. The 2-heap of Williams was next extended by Floyd [1964] to the sorting algorithm 'treesort3'.

In an erratum of 24 October 1968 to a report of the London Business School, Murchland [1967b] seems to be the first to use heaps for finding shortest paths, although he concludes to a time bound of $O(n^2 \log n)$ only — worse than Dijkstra's bound $O(n^2)$. E.L. Johnson [1972] improved Murchland's method to $O(m \log(n^2/m))$. He also considers the k-heap for arbitrary k ('k-tree').

In his Ph.D. thesis, D.B. Johnson [1973b], using a sharper analysis and k-heaps, obtains an $O((nd + m) \log_d n)$-time algorithm, implying algorithms with running time $O(m \log n)$ and $O(n^{1+\varepsilon} + m)$ (for each $\varepsilon > 0$) (published in D.B. Johnson [1977a]). Tarjan [1983] observed that taking $d := m/n$ gives $O(m \log_{m/n} n)$. Next, Fredman and Tarjan [1984,1987] showed that Fibonacci heaps give $O(m + n \log n)$.

All pairs: Roy, Warshall, Floyd, Dantzig

Based on a study of Kleene [1951,1956], McNaughton and Yamada [1960] gave a formula to calculate a 'regular expression' associated with a 'state graph' (essentially describing all paths from a given source) that is quite similar to the fast method of Roy [1959] and Warshall [1962] to compute the transitive closure $\overline{A}$ of a digraph $D = (V, A)$: Assume that the vertices are ordered $1, \ldots, n$. First set $\widetilde{A} := A$. Next, for $k = 1, \ldots, n$, add to $\widetilde{A}$ all pairs (i, j) for which both (i, k) and (k, j) belong to $\widetilde{A}$. The final $\widetilde{A}$ equals $\overline{A}$. This gives an $O(n^3)$ algorithm, which is faster than iterative matrix multiplication.

Floyd [1962b] extended this method to an algorithm to find all distances $d(i, j)$ given a length function $l : V \times V \to \mathbb{Q}_+$: First set $d(i, j) := l(i, j)$ for all i, j. Next, for $k = 1, \ldots, n$, reset, for all i, j, $d(i, j) := d(i, k) + d(k, j)$ if it decreases $d(i, j)$. This gives an $O(n^3)$ transitive closure algorithm for finding the distances between all pairs of vertices.

Dantzig [1967] proposed a variant of this method. For i, j, k with $i \leq k$ and $j \leq k$, let $d^k_{i,j}$ be the length of the shortest $i - j$ path in the graph induced by $\{1, \ldots, k\}$. Then there is an easy iterative scheme to determine the $d^k_{i,j}$ from the $d^{k-1}_{i,j}$: first $d^k_{i,k} = \min_{j<k}(d^{k-1}_{i,j} + l_{j,k})$ and $d^k_{k,i} = \min_{j<k}(l_{k,j} + d^{k-1}_{j,i})$. Next, for all $i, j < k$, $d^k_{i,j} = \min(d^{k-1}_{i,j}, d^k_{i,k} + d^k_{k,j})$.

Negative lengths

Ford and Fulkerson [1962] seem to be the first to observe that the Bellman-Ford method also works for arbitrary lengths as long as each directed circuit has non-negative length. It is also implicit in the paper of Iri [1960].

PERT and CPM

The application of shortest path (and other) methods in the form of PERT (*Program Evaluation and Review Technique*, originally called *Program Evaluation Research Task*) started in 1958, and was reported by Malcolm, Roseboom, Clark, and Fazar [1959]. The use of the *Critical Path Method* (CPM) was described by Kelley [1957, 1961], and Kelley and Walker [1959].

The kth shortest path

Bock, Kantner, and Haynes [1957,1958] described a method to find the kth shortest path in a graph, based essentially on enumerating. Hoffman and Pavley [1959a] described an adaptation of Dantzig's tree method to obtain the kth shortest path. Bellman and Kalaba [1960] gave a method to find the kth shortest paths from a given vertex simultaneously to all other vertices. Also Pollack [1961b] described a method for the kth shortest path problem, especially suitable if k is small. A survey was given by Pollack [1961a].

Bottleneck path problems

Pollack [1960] modified the shortest path algorithm so as to obtain a path with maximum capacity (the capacity of a path is equal to the minimum of the capacities of the arcs in the path). Related work was done by Amara, Lindgren, and Pollack [1961].

Fanning out from both ends

Berge and Ghouila-Houri [1962] and Dantzig [1963] proposed to speed up Dijkstra's method by fanning out at both ends simultaneously. Berge and Ghouila-Houri [1962] proposed to stop as soon as a vertex is permanently labeled from both ends; however, one may see that this need not yield a shortest path.

Dantzig [1963] proposed to add an arc (s, v) with length $d(s, v)$ as soon as v is permanently labeled when fanning out from s, and similarly add an arc (w, t) with length $d(w, t)$ if a vertex w is labeled permanently when fanning out from t:

> The algorithm terminates whenever the fan of one of the problems reaches its terminal in the other.

Chapter 9

Disjoint paths

Having done with *shortest* paths, we now arrive at *disjoint* paths. We consider disjoint $s-t$ paths, where s and t are the same for all paths. The more general problem where we prescribe for each path a (possibly different) pair of ends, will be discussed in Part VII.

Menger's theorem equates the maximum number of disjoint $s-t$ paths to the minimum size of a cut separating s and t. There are several variants of Menger's theorem, all about equivalent: undirected, directed, vertex-disjoint, arc- or edge-disjoint. The meaning of 'cut' varies accordingly.

Next to Menger's min-max relation, we consider the algorithmic side of disjoint $s-t$ paths. This will be an extract from the related maximum flow algorithms to be discussed in the next chapter. (Maximum integer flow can be viewed as the capacitated version of disjoint $s-t$ paths.)

9.1. Menger's theorem

Menger [1927] gave a min-max theorem for the maximum number of disjoint $S-T$ paths in an undirected graph. It was observed by Grünwald [1938] (= T. Gallai) that the theorem also holds for directed graphs. We follow the proof given by Göring [2000].

Recall that a path is an $S-T$ *path* if it runs from a vertex in S to a vertex in T. A set C of vertices is called $S-T$ *disconnecting* if C intersects each $S-T$ path (C may intersect $S \cup T$).

Theorem 9.1 (Menger's theorem (directed vertex-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint $S-T$ paths is equal to the minimum size of an $S-T$ disconnecting vertex set.*

Proof. Obviously, the maximum does not exceed the minimum. Equality is shown by induction on $|A|$, the case $A = \emptyset$ being trivial.

Let k be the minimum size of an $S-T$ disconnecting vertex set. Choose $a = (u, v) \in A$. If each $S-T$ disconnecting vertex set in $D-a$ has size at least k, then inductively there exist k vertex-disjoint $S-T$ paths in $D-a$, hence in D.

So we can assume that $D - a$ has an $S - T$ disconnecting vertex set C of size $\leq k - 1$. Then $C \cup \{u\}$ and $C \cup \{v\}$ are $S - T$ disconnecting vertex sets of D of size k.

Now each $S - (C \cup \{u\})$ disconnecting vertex set B of $D - a$ has size at least k, as it is $S - T$ disconnecting in D. Indeed, each $S - T$ path P in D intersects $C \cup \{u\}$, and hence P contains an $S - (C \cup \{u\})$ path in $D - a$. So P intersects B.

So by induction, $D - a$ contains k disjoint $S - C \cup \{u\}$ paths. Similarly, $D - a$ contains k disjoint $C \cup \{v\} - T$ paths. Any path in the first collection intersects any path in the second collection only in C, since otherwise $D - a$ contains an $S - T$ path avoiding C.

Hence, as $|C| = k - 1$, we can pairwise concatenate these paths to obtain disjoint $S - T$ paths, inserting arc a between the path ending at u and starting at v. ∎

A consequence of this theorem is a variant on *internally vertex-disjoint* $s - t$ paths, that is, $s - t$ paths having no vertex in common except for s and t. Recall that a set U of vertices is called an $s - t$ *vertex-cut* if $s, t \notin U$ and each $s - t$ path intersects U.

Corollary 9.1a (Menger's theorem (directed internally vertex-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let s and t be two nonadjacent vertices of D. Then the maximum number of internally vertex-disjoint $s - t$ paths is equal to the minimum size of an $s - t$ vertex-cut.*

Proof. Let $D' := D - s - t$ and let S and T be the sets of outneighbours of s and of inneighbours of t, respectively. Then Theorem 9.1 applied to D', S, T gives the corollary. ∎

In turn, Theorem 9.1 follows from Corollary 9.1a by adding two new vertices s and t and arcs (s, v) for all $v \in S$ and (v, t) for all $v \in T$.

Also an arc-disjoint version can be derived (where paths are *arc-disjoint* is they have no arc in common). This version was first formulated by Dantzig and Fulkerson [1955,1956] for directed graphs and by Kotzig [1956] for undirected graphs.

Recall that a set C of arcs is an $s - t$ *cut* if $C = \delta^{\text{out}}(U)$ for some subset U of V with $s \in U$ and $t \notin U$.

Corollary 9.1b (Menger's theorem (directed arc-disjoint version))**.** *Let $D = (V, A)$ be a digraph and $s, t \in V$. Then the maximum number of arc-disjoint $s - t$ paths is equal to the minimum size of an $s - t$ cut.*

Proof. Let $L(D)$ be the line digraph of D and let $S := \delta_A^{\text{out}}(s)$ and $T := \delta_A^{\text{in}}(t)$. Then Theorem 9.1 for $L(D), S, T$ implies the corollary. Note that a minimum-size set of arcs intersecting each $s - t$ path necessarily is an $s - t$ cut. ∎

The internally vertex-disjoint version of Menger's theorem can be derived in turn from the arc-disjoint version: make digraph D' as follows from D: replace any vertex v by two vertices v', v'' and make an arc (v', v''); moreover, replace each arc (u, v) by (u'', v'). Then Corollary 9.1b for D', s'', t' gives Corollary 9.1a for D, s, t.

Similar theorems hold for *undirected* graphs. The undirected vertex-disjoint version follows immediately from Theorem 9.1 by replacing each undirected edge by two oppositely oriented arcs. Next, the undirected edge-disjoint version follows from the undirected vertex-disjoint version applied to the line graph (like the proof of Corollary 9.1b).

9.1a. Other proofs of Menger's theorem

The proof above of Theorem 9.1b was given by Göring [2000], which curtails the proof of Pym [1969a], which by itself is a simplification of a proof of Dirac [1966]. The basic idea (decomposition into two subproblems determined by a minimum-size cut) is due to Menger [1927], for the undirected vertex-disjoint version. (Menger's original proof contains a hole, closed by Kőnig [1931] — see Section 9.6e.)

Hajós [1934] gave a different proof for the undirected vertex-disjoint case, based on intersections and unions of sets determining a cut. Also the proofs given in Nash-Williams and Tutte [1977] are based on this. We give the first of their proofs.

Let $G = (V, E)$ be an undirected graph and let $s, t \in V$. Suppose that the minimum size of an $s-t$ vertex-cut is k. We show by induction on $|E|$ that there exist k vertex-disjoint $s-t$ paths.

The statement is trivial if each edge is incident with at least one of s and t. So we can consider an edge $e = xy$ incident with neither s nor t.

We can assume that $G - e$ has an $s-t$ vertex-cut C of size $k-1$ — otherwise the statement follows by induction. Similarly, we can assume that G/e has an $s-t$ vertex-cut of size $k-1$ — otherwise the statement follows by induction again. Necessarily, this cut contains the new vertex obtained by contracting e. Hence G has an $s-t$ vertex-cut C' of size k containing both x and y.

Now let C_s be the set of vertices in $C \cup C'$ that are reachable in G from s by a path with no internal vertex in $C \cup C'$. Similarly, let C_t be the set of vertices in $C \cup C'$ that are reachable in G from t by a path with no internal vertex in $C \cup C'$.

Trivially, C_s and C_t are $s-t$ vertex-cuts, and $C_s \cup C_t \subseteq C \cup C'$. Moreover, $C_s \cap C_t \subseteq C \cap C'$, since for any $v \in C_s \cap C_t$ there is an $s-t$ path P intersecting $C \cup C'$ only in v. As $x, y \in C'$, P does not traverse edge e. Hence $v \in C \cap C'$.

Therefore,

$$(9.1) \qquad |C_s| + |C_t| \leq |C| + |C'| \leq 2k - 1,$$

contradicting the fact that C_s and C_t each have size at least k.

An augmenting path proof for the directed vertex-disjoint version was given by Grünwald [1938] (= T. Gallai), and for the directed arc-disjoint version by Ford and Fulkerson [1955,1957b] — see Section 9.2. (O'Neil [1978] gave a proof similar to that of Grünwald [1938].)

More proof ideas were given by Halin [1964,1968,1989], Hajós [1967], McCuaig [1984], and Böhme, Göring, and Harant [2001].

9.2. Path packing algorithmically

A specialization of the maximum flow algorithm of Ford and Fulkerson [1955, 1957b] (to be discussed in the next chapter) yields a polynomial-time algorithm to find a maximum number of disjoint $s-t$ paths and a minimum-size $s-t$ cut.

Define for any digraph D and any path P in D:

(9.2) $D \leftarrow P :=$ the digraph arising from D by reversing the orientation of each arc occurring in P.

Note that if P is an $s-t$ path in $D = (V, A)$, then for each $U \subseteq V$ with $s \in U$, $t \notin U$, we have

(9.3) $$\delta^{\text{out}}_{A'}(U) = \delta^{\text{out}}_{A}(U) - 1,$$

where A' is the arc set of $D \leftarrow P$.

Determine $D_0, D_1, \ldots$ as follows.

(9.4) Set $D_0 := D$. If D_k has been found and contains an $s-t$ path P, set $D_{k+1} := D_k \leftarrow P$. If D_k contains no $s-t$ path we stop.

The path P is called an *augmenting path*.

Now finding a minimum-size $s-t$ cut is easy: let U be the set of vertices reachable in the final D_k from s. Then $\delta^{\text{out}}_{A}(U)$ is a minimum-size $s-t$ cut, by (9.3).

Also a maximum packing of $s-t$ paths can be derived. Indeed, the set B of arcs of D that are reversed in the final D_k contains k arc-disjoint $s-t$ paths in D. This can be seen as follows.

Let B_i be the set of arcs of D that are reversed in D_i, added with i parallel arcs from t to s. We show by induction on i that (V, B_i) is Eulerian. For $i = 0$, this is trivial. Suppose that it has been proved for i. Let P be the $s-t$ path in D_i with $D_{i+1} = D_i \leftarrow P$. Then $(V, B_i \cup AP \cup \{(t,s)\})$ is Eulerian. Since B_{i+1} arises from $B_i \cup AP \cup \{(t,s)\}$ by deleting pairs a, a^{-1} with $a \in B_i$ and $a^{-1} \in AP$, also (V, B_{i+1}) is Eulerian.

A consequence is that k arc-disjoint $s-t$ paths in B can be found in linear time.

Since an $s-t$ path in D_k can be found in time $O(m)$, and since there are at most $|A|$ arc-disjoint $s-t$ paths, one has:

Theorem 9.2. *A maximum collection of arc-disjoint $s-t$ paths and a minimum-size $s-t$ cut can be found in time $O(m^2)$.*

Proof. See above. ∎

Similarly one has for the vertex-disjoint variant:

Theorem 9.3. *A maximum collection of internally vertex-disjoint $s-t$ paths and a minimum-size $s-t$ vertex-cut can be found in time $O(nm)$.*

Proof. Apply the reduction described after Corollary 9.1b. In this case the number of iterations is at most $|V|$. ∎

One similarly derives for a fixed number k of arc-disjoint paths:

Corollary 9.3a. *Given a digraph $D = (V, A)$, $s, t \in V$, and a natural number k, we can find k arc-disjoint $s - t$ paths (if they exist) in time $O(km)$.*

Proof. Directly from the fact that the path P can be found in time $O(m)$. ∎

9.3. Speeding up by blocking path packings

The algorithm might be speeded up by selecting, at each iteration, not just *one* path P, but several arc-disjoint paths $P_1, \ldots, P_l$ in D_i at one go, and setting

(9.5) $$D_{i+1} := D_i \leftarrow P_1 \leftarrow \cdots \leftarrow P_l.$$

This might reduce the number of iterations — but of course this should be weighed against the increase in complexity of each iteration.

Such a speedup is obtained by a method of Dinits [1970] as follows. For any digraph $D = (V, A)$ and $s, t \in V$, let $\mu(D)$ denote the minimum length of an $s - t$ path in D. If no such path exists, set $\mu(D) = \infty$. If we choose the paths $P_1, \ldots, P_l$ in such a way that $\mu(D_{i+1}) > \mu(D_i)$, then the number of iterations clearly is not larger than $|V|$ (as $\mu(D_i) < |V|$ if finite).

We show that a collection $P_1, \ldots, P_l$ with the property that $\mu(D \leftarrow P_1 \leftarrow \cdots \leftarrow P_l) > \mu(D)$ indeed can be found quickly, namely in linear time.

To that end, call a collection of arc-disjoint $s-t$ paths $P_1, \ldots, P_l$ *blocking* if D contains no $s - t$ path arc-disjoint from $P_1, \ldots, P_l$. This is weaker than a maximum number of arc-disjoint paths, but a blocking collection can be found in linear time (Dinits [1970]):

Theorem 9.4. *Given an acyclic digraph $D = (V, A)$ and $s, t \in V$, a blocking collection of arc-disjoint $s - t$ paths can be found in time $O(m)$.*

Proof. With depth-first search we can find in time $O(|A'|)$ a subset A' of A and an $s - t$ path P_1 in A' such that no arc in $A' \setminus AP_1$ is contained in any $s - t$ path: scan s (cf. (6.2)) and stop as soon as t is reached; let A' be the set of arcs considered so far (as D is acyclic).

Next we find (recursively) a blocking collection $P_2, \ldots, P_k$ of arc-disjoint $s - t$ paths in the graph $D' := (V, A \setminus A')$. Then $P_1, \ldots, P_k$ is blocking in D. For suppose that D contains an $s - t$ path Q that is arc-disjoint from $P_1, \ldots, P_k$. Then $AQ \cap A' \neq \emptyset$, since $P_2, \ldots, P_k$ is blocking in D'. So AQ intersects AP_1, a contradiction. ∎

We also need the following. Let $\alpha(D)$ denote the set of arcs contained in at least one shortest $s-t$ path. Then:

Theorem 9.5. *Let $D = (V, A)$ be a digraph and let $s, t \in V$. Define $D' := (V, A \cup \alpha(D)^{-1})$. Then $\mu(D') = \mu(D)$ and $\alpha(D') = \alpha(D)$.*

Proof. It suffices to show that $\mu(D)$ and $\alpha(D)$ are invariant if we add a^{-1} to D for one arc $a \in \alpha(D)$. Suppose not. Then there is a directed $s-t$ path P in $A \cup \{a^{-1}\}$ traversing a^{-1}, of length at most $\mu(D)$. As $a \in \alpha(D)$, there is an $s-t$ path Q traversing a, of length $\mu(D)$. Hence $AP \cup AQ \setminus \{a, a^{-1}\}$ contains an $s-t$ path of length less than $\mu(D)$, a contradiction. ∎

The previous two theorems imply:

Corollary 9.5a. *Given a digraph $D = (V, A)$ and $s, t \in V$, a collection of arc-disjoint $s-t$ paths $P_1, \ldots, P_l$ with $\mu(D \leftarrow P_1 \leftarrow \cdots \leftarrow P_l) > \mu(D)$ can be found in time $O(m)$.*

Proof. Let $\widetilde{D} = (V, \alpha(D))$. (Note that $\alpha(D)$ can be identified in time $O(m)$ and that $(V, \alpha(D_f))$ is acyclic.) By Theorem 9.4, we can find in time $O(m)$ a blocking collection $P_1, \ldots, P_l$ in $\widetilde{D}$. Define:

$$(9.6) \qquad D' := (V, A \cup \alpha(D)^{-1}) \text{ and } D'' := D \leftarrow P_1 \leftarrow \cdots \leftarrow P_l.$$

We show $\mu(D'') > \mu(D)$. As D'' is a subgraph of D', we have $\mu(D'') \geq \mu(D') = \mu(D)$, by Theorem 9.5. Suppose that $\mu(D'') = \mu(D')$. Then $\alpha(D'') \subseteq \alpha(D') = \alpha(D)$ (again by Theorem 9.5). Hence, as $\alpha(D'')$ contains an $s-t$ path (of length $\mu(D'')$), $\alpha(D)$ contains an $s-t$ path arc-disjoint from $P_1, \ldots, P_l$. This contradicts the fact that $P_1, \ldots, P_l$ is blocking in $\widetilde{D}$. ∎

This gives us the speedup in finding a maximum packing of $s-t$ paths:

Corollary 9.5b. *Given a digraph $D = (V, A)$ and $s, t \in V$, a maximum number of arc-disjoint $s-t$ paths and a minimum-size $s-t$ cut can be found in time $O(nm)$.*

Proof. Directly from Corollary 9.5a with the iterations (9.5). ∎

9.4. A sometimes better bound

Since $\mu(D_i)$ is at most $|V|$ (as long as it is finite), the number k of iterations is at most $|V|$. But Karzanov [1973a], Tarjan [1974e], and Even and Tarjan [1975] showed that an alternative, often tighter bound on k holds.

To see this, it is important to observe that, for each i, the set of arcs of D_i that are reversed in the final D_k (compared with D_i) forms a maximum number of arc-disjoint $s-t$ paths in D_i.

Theorem 9.6. *If $\mu(D_{i+1}) > \mu(D_i)$ for each $i < k$, then $k \leq 2|A|^{1/2}$. If moreover D is simple, then $k \leq 2|V|^{2/3}$.*

Proof. Let $p := \lfloor |A|^{1/2} \rfloor$. Then each $s-t$ path in D_p has length at least $p+1 \geq |A|^{1/2}$. Hence D_p contains at most $|A|/|A|^{1/2} = |A|^{1/2}$ arc-disjoint $s-t$ paths. Therefore $k - p \leq |A|^{1/2}$, and hence $k \leq 2|A|^{1/2}$.

If D is simple, let $p := \lfloor |V|^{2/3} \rfloor$. Then each $s-t$ path in D_p has length at least $p+1 \geq |V|^{2/3}$. Then D_p contains at most $|V|^{2/3}$ arc-disjoint $s-t$ paths. Indeed, let U_i denote the set of vertices at distance i from s in D_p. Then

$$\sum_{i=0}^{p}(|U_i| + |U_{i+1}|) \leq 2|V|. \tag{9.7}$$

Hence there is an $i \leq p$ with $|U_i|+|U_{i+1}| \leq 2|V|^{1/3}$. This implies $|U_i|\cdot|U_{i+1}| \leq \frac{1}{4}(|U_i| + |U_{i+1}|)^2 \leq |V|^{2/3}$. So D_p contains at most $|V|^{2/3}$ arc-disjoint $s-t$ paths. Therefore $k - p \leq |V|^{2/3}$, and hence $k \leq 2|V|^{2/3}$. ∎

This gives the following time bounds (Karzanov [1973a], Tarjan [1974e], Even and Tarjan [1975]):

Corollary 9.6a. *Given a digraph $D = (V, A)$ and $s, t \in V$, a maximum number of arc-disjoint $s-t$ paths and a minimum-size $s-t$ cut can be found in time $O(m^{3/2})$. If D is simple, the paths and the cut can be found also in time $O(n^{2/3}m)$.*

Proof. Directly from Corollary 9.5a and Theorem 9.6. ∎

(Related work was presented in Ahuja and Orlin [1991].)

9.5. Complexity of the vertex-disjoint case

If we are interested in *vertex*-disjoint paths, the results can be sharpened. Recall that if $D = (V, A)$ is a digraph and $s, t \in V$, then the problem of finding a maximum number of internally vertex-disjoint $s-t$ paths can be reduced to the arc-disjoint case by replacing each vertex $v \neq s, t$ by two vertices v', v'', while each arc with head v is redirected to v' and each arc with tail v is redirected from v''; moreover, an arc (v', v'') is added.

By Corollary 9.6a, this construction directly yields algorithms for vertex-disjoint paths with running time $O(m^{3/2})$ and $O(n^{2/3}m)$. But one can do better. Note that, with this construction, each of the digraphs D_i has the property that each vertex has indegree at most 1 or outdegree at most 1. Under this condition, the bound in Theorem 9.6 can be improved to $2|V|^{1/2}$:

Theorem 9.7. *If each vertex $v \neq s, t$ has indegree or outdegree equal to 1, and if $\mu(D_{i+1}) > \mu(D_i)$ for each $i \leq k$, then $k \leq 2|V|^{1/2}$.*

Proof. Let $p := \lceil |V|^{1/2} \rceil$. Then each $s-t$ path in D_p has length at least $p+1$. Let U_i be the set of vertices at distance i from s in D_i. Then $\sum_{i=1}^{p} |U_i| \leq |V|$. This implies that $|U_i| \leq |V|^{1/2}$ for some i. Hence D_i has at most $|V|^{1/2}$ arc-disjoint $s-t$ paths. So $k + 1 - p \leq |V|^{1/2}$. Hence $k \leq 2|V|^{1/2}$. ∎

This gives, similarly to Corollary 9.6a, another result of Karzanov [1973a], Tarjan [1974e], and Even and Tarjan [1975] (which can be derived also from Theorem 16.4 due to Hopcroft and Karp [1971,1973] and Karzanov [1973b], with the method of Hoffman [1960] given in Section 16.7c):

Corollary 9.7a. *Given a digraph $D = (V, A)$ and $s, t \in V$, a maximum number of internally vertex-disjoint $s-t$ paths and a minimum-size $s-t$ vertex-cut can be found in time $O(n^{1/2}m)$.*

Proof. Directly from Corollary 9.5a and Theorem 9.7. ∎

In fact one can reduce n in this bound to the minimum number $\tau(D)$ of vertices intersecting each arc of D (this bound will be used in deriving bounds for bipartite matching (Theorem 16.5)):

Theorem 9.8. *Given a digraph $D = (V, A)$ and $s, t \in V$, a maximum number of internally vertex-disjoint $s-t$ paths and a minimum-size $s-t$ vertex-cut can be found in time $O(\tau(D)^{1/2}m)$.*

Proof. Similar to Corollary 9.7a, by taking $p := \lfloor \tau(D)^{1/2} \rfloor$ in Theorem 9.7: Finding D_p takes $O(pm)$ time. Let W be a set of vertices intersecting each arc of D, of size $\tau(D)$. In D_p there are at most $2\tau(D)^{1/2}$ internally vertex-disjoint $s-t$ paths, since each $s-t$ path contains at least $p/2$ vertices in W. ∎

9.6. Further results and notes

9.6a. Complexity survey for the disjoint $s-t$ paths problem

For finding arc-disjoint $s-t$ paths we have the following survey of running time bounds ($*$ indicates an asymptotically best bound in the table):

	$O(m^2)$	Ford and Fulkerson [1955,1957b]
$*$	$O(nm)$	Dinits [1970]
$*$	$O(m^{3/2})$	Karzanov [1973a], Tarjan [1974e], Even and Tarjan [1975]
$*$	$O(n^{2/3}m)$	Karzanov [1973a], Tarjan [1974e], Even and Tarjan [1975] *D simple*

≫

continued

*	$O(k^2 n)$	Nagamochi and Ibaraki [1992a] finding k arc-disjoint paths
*	$O(kn^{5/3})$	Nagamochi and Ibaraki [1992a] finding k arc-disjoint paths
*	$O(kn^2)$	Nagamochi and Ibaraki [1992a] finding k arc-disjoint paths; D *simple*
*	$O(k^{3/2}n^{3/2})$	Nagamochi and Ibaraki [1992a] finding k arc-disjoint paths; D *simple*

For *undirected* simple graphs, Goldberg and Rao [1997b,1999] gave an $O(n^{3/2}m^{1/2})$ bound, and Karger and Levine [1998] gave $(m + nk^{3/2})$ and $O(nm^{2/3}k^{1/6})$ bounds, where k is the number of paths.

For vertex-disjoint paths:

	$O(nm)$	Grünwald [1938], Ford and Fulkerson [1955, 1957b]
	$O(\sqrt{n}\,m)$	Karzanov [1973a], Tarjan [1974e], Even and Tarjan [1975]
*	$O(\sqrt{n}\,m \log_n(n^2/m))$	Feder and Motwani [1991,1995]
*	$O(k^2 n)$	Nagamochi and Ibaraki [1992a] finding k vertex-disjoint paths
*	$O(kn^{3/2})$	Nagamochi and Ibaraki [1992a] finding k vertex-disjoint paths

For edge-disjoint $s-t$ paths in simple undirected *planar* graphs:

	$O(n^2 \log n)$	Itai and Shiloach [1979]
	$O(n^2)$	Cheston, Probert, and Saxton [1977]
	$O(n^{3/2} \log n)$	Johnson and Venkatesan [1982]
	$O(n \log^2 n)$	Reif [1983] (minimum-size $s-t$ cut), Hassin and Johnson [1985] (edge-disjoint $s-t$ paths)
	$O(n \log n \log^* n)$	Frederickson [1983b]
	$O(n \log n)$	Frederickson [1987b]
*	$O(n)$	Weihe [1994a,1997a]

For arc-disjoint $s-t$ paths in simple *directed* planar graphs:

$O(n^{3/2} \log n)$	Johnson and Venkatesan [1982]
$O(n^{4/3} \log^2 n)$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]

≫

continued

	$O(n \log n)$	Weihe [1994b,1997b]
*	$O(n)$	Brandes and Wagner [1997]

For vertex-disjoint $s-t$ paths in *undirected* planar graphs:

	$O(n \log n)$	Suzuki, Akama, and Nishizeki [1990]
*	$O(n)$	Ripphausen-Lipa, Wagner, and Weihe [1993b,1997]

Orlova and Dorfman [1972] and Hadlock [1975] showed, with matching theory, that in planar undirected graphs also a *maximum*-size cut can be found in polynomial-time (Barahona [1990] gave an $O(n^{3/2} \log n)$ time bound) — see Section 29.1. Karp [1972b] showed that in general finding a maximum-size cut is NP-complete — see Section 75.1a.

9.6b. Partially disjoint paths

For any digraph $D = (V, A)$ and any $B \subseteq A$, call two paths *disjoint on* B if they have no common arc in B. One may derive from Menger's theorem a more general min-max relation for such partially disjoint paths:

Theorem 9.9. *Let $D = (V, A)$ be a digraph, $s, t \in V$, and $B \subseteq A$. Then the maximum number of $s-t$ paths such that any two are disjoint on B is equal to the minimum size of an $s-t$ cut contained in B.*

Proof. If there is no $s-t$ cut contained in B, then clearly the maximum is infinite. If B contains any $s-t$ cut, let k be its minimum size. Replace any arc $a \in A \setminus B$ by $|A|$ parallel arcs. Then by Menger's theorem there exist k arc-disjoint $s-t$ paths in the extended graph. This gives k $s-t$ paths in the original graph that are disjoint on B. ∎

The construction given in this proof can also be used algorithmically, but making $|A|$ parallel arcs takes $\Omega(m^2)$ time. However, one can prove:

Theorem 9.10. *Given a digraph $D = (V, A)$, $s, t \in V$, and $B \subseteq A$, a maximum number of $s-t$ paths such that any two are disjoint on B can be found in time $O(nm)$.*

Proof. The theorem follows from Corollary 10.11a below. ∎

9.6c. Exchange properties of disjoint paths

Disjoint paths have a number of exchange properties that imply a certain matroidal structure (cf. Section 39.4).

Let $D = (V, A)$ be a directed graph and let $X, Y \subseteq V$. Call X *linked to* Y if $|X| = |Y|$ and D has $|X|$ vertex-disjoint $X - Y$ paths. Note that the following theorem follows directly from the algorithm for finding a maximum set of disjoint paths (since any vertex in $S \cup T$, once covered by the disjoint paths, remains covered during the further iterations):

Theorem 9.11. *Let $D = (V, A)$ be a digraph, let $S, T \subseteq V$, and suppose that $X \subseteq S$ and $Y \subseteq T$ are linked. Then there exists a maximum number of vertex-disjoint $S-T$ paths covering $X \cup Y$.*

Proof. Directly from the algorithm. ∎

The following result, due to Perfect [1968], says that for two distinct maximum packings $\mathcal{P}, \mathcal{Q}$ of $S-T$ paths there exists a maximum packing of $S-T$ paths whose starting vertices are equal to those of $\mathcal{P}$ and whose end vertices are equal to those of $\mathcal{Q}$.

Theorem 9.12. *Let $D = (V, A)$ be a digraph and $S, T \subseteq V$. Let k be the maximum number of disjoint $S - T$ paths. Let $X \subseteq S$ be linked to $Y \subseteq T$, and let $X' \subseteq S$ be linked to $Y' \subseteq T$, with $|X| = |X'| = k$. Then X is linked to Y'.*

Proof. Let C be a minimum-size vertex set intersecting each $S - T$ path. So by Menger's theorem, $|C| = |X| = |Y| = |X'| = |Y'| = k$. Let $P'_1, \ldots, P'_k$ be vertex-disjoint $X - Y$ paths. Similarly, let $P''_1, \ldots, P''_k$ be vertex-disjoint $X' - Y'$ paths. We may assume that, for each i, P'_i and P''_i have a vertex in C in common. Let P_i be the path obtained by traversing P'_i until it reaches C, after which it traverses P''_i. Then $P_1, \ldots, P_k$ are vertex-disjoint $X - Y'$ paths. ∎

The previous two theorems imply:

Corollary 9.12a. *Let $D = (V, A)$ be a digraph, let X' be linked to Y', and let X'' be linked to Y''. Then there exist X and Y with $X' \subseteq X \subseteq X' \cup X''$ and $Y'' \subseteq Y \subseteq Y' \cup Y''$ such that X is linked to Y.*

Proof. Directly from Theorems 9.11 and 9.12. ∎

Other proofs of this corollary were given by Pym [1969b,1969c], Brualdi and Pym [1971], and McDiarmid [1975b].

9.6d. Further notes

Lovász, Neumann-Lara, and Plummer [1978] proved the following on the maximum number of disjoint paths of bounded length. Let $G = (V, E)$ be an undirected graph, let s and t be two distinct and nonadjacent vertices, and let $k \geq 2$. Then the minimum number of vertices ($\neq s, t$) intersecting all $s - t$ paths of length at most k is at most $\lfloor \frac{1}{2}k \rfloor$ times the maximum number of internally vertex-disjoint $s - t$ paths each of length at most k. (A counterexample to a conjecture on this raised by Lovász, Neumann-Lara, and Plummer [1978] was given by Boyles and Exoo [1982]. Related results are given by Galil and Yu [1995].)

On the other hand, when taking lower bounds on the path lengths, Montejano and Neumann-Lara [1984] showed that, in a directed graph, the minimum number of vertices ($\neq s, t$) intersecting all $s-t$ paths of length at least k is at most $3k-5$ times the maximum number of internally vertex-disjoint such paths. For $k=3$, the factor was improved to 3 by Hager [1986] and to 2 by Mader [1989].

Egawa, Kaneko, and Matsumoto [1991] gave a version of Menger's theorem in which vertex-disjoint and edge-disjoint are mixed: Let $G=(V,E)$ be an undirected graph, let $s,t \in V$, and $k,l \in \mathbb{Z}_+$. Then G contains l disjoint edge sets, each containing k vertex-disjoint $s-t$ paths if and only if for each $U \subseteq V \setminus \{s,t\}$ there exist $l(k-|U|)$ edge-disjoint $s-t$ paths in $G-U$. Similarly, for directed graphs. The proof is by reduction to Menger's theorem using integer flow theory.

Bienstock and Diaz [1993] showed that the problem of finding a minimum-weight subset of edges intersecting all $s-t$ cuts of size at most k is polynomial-time solvable if k is fixed, while it is NP-complete if k is not fixed.

Motwani [1989] investigated the expected running time of Dinits' disjoint paths algorithm.

Extensions of Menger's theorem to the infinite case were given by P. Erdős (cf. Kőnig [1932]), Grünwald [1938] (= T. Gallai), Dirac [1960,1963,1973], Halin [1964], McDiarmid [1975b], Podewski and Steffens [1977], Aharoni [1983a,1987], and Polat [1991].

Halin [1964], Lovász [1970b], Escalante [1972], and Polat [1976] made further studies of (the lattice of) $s-t$ cuts.

9.6e. Historical notes on Menger's theorem

The topologist Karl Menger published his theorem in an article called *Zur allgemeinen Kurventheorie* (On the general theory of curves) (Menger [1927]) in the following form:

> Satz β. *Ist K ein kompakter regulär-eindimensionaler Raum, welcher zwischen den beiden endlichen Mengen P und Q n-punktig zusammenhängend ist, dann enthält K n paarweise fremde Bögen, von denen jeder einen Punkt von P und einen Punkt von Q verbindet.*[5]

It can be formulated equivalently in terms of graphs as: Let $G=(V,E)$ be an undirected graph and let $P,Q \subseteq V$. Then the maximum number of disjoint $P-Q$ paths is equal to the minimum size of a set W of vertices such that each $P-Q$ path intersects W.

The result became known as the *n-chain theorem*. Menger's interest in this question arose from his research on what he called 'curves': a *curve* is a connected compact topological space X with the property that for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x with $\mathrm{bd}(N')$ totally disconnected. (Here bd stands for boundary; a space is *totally disconnected* if each point forms an open set.)

The curve is called *regular* if for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x with $|\mathrm{bd}(N')|$ finite. The *order* of a point $x \in X$ is equal to the minimum natural number n such that for each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x satisfying $|\mathrm{bd}(N')| \leq n$.

According to Menger:

[5] Theorem β. *If K is a compact regularly one-dimensional space which is n-point connected between the two finite sets P and Q, then K contains n pairwise disjoint curves, each of which connects a point in P and a point in Q.*

> Eines der wichtigsten Probleme der Kurventheorie ist die Frage nach den Beziehungen zwischen der Ordnungszahl eines Punktes der regulären Kurve K und der Anzahl der im betreffenden Punkt zusammenstossenden und sonst fremden Teilbögen von K.[6]

In fact, Menger used 'Satz β' to show that if a point in a regular curve K has order n, then there exists a topological *n-leg* with p as top; that is, K contains n arcs $P_1, \ldots, P_n$ such that $P_i \cap P_j = \{p\}$ for all i, j with $i \neq j$.

The proof idea is as follows. There exists a series $N_1 \supset N_2 \supset \cdots$ of open neighbourhoods of p such that $N_1 \cap N_2 \cap \cdots = \{p\}$ and $|\text{bd}(N_i)| = n$ for all $i = 1, 2, \ldots$ and such that

$$(9.8) \qquad |\text{bd}(N)| \geq n \text{ for each neighbourhood } N \subseteq N_1.$$

This follows quite directly from the definition of order.

Now Menger showed that we may assume that the space $G_i := \overline{N_i} \setminus N_{i+1}$ is a (topological) graph. For each i, let $Q_i := \text{bd}(N_i)$. Then (9.8) gives with Menger's theorem that there exist n disjoint paths $P_{i,1}, \ldots, P_{i,n}$ in G such that each $P_{i,j}$ runs from Q_i to Q_{i+1}. Properly connecting these paths for $i = 1, 2, \ldots$ we obtain n arcs forming the required n-leg.

It was however noticed by Kőnig [1932] that Menger's proof of 'Satz β' is incomplete. Menger applied induction on $|E|$, where E is the edge set of the graph G. Menger first claimed that one easily shows that $|E| \geq n$, and that if $|E| = n$, then G consists of n disjoint edges connecting P and Q. He stated that if $|E| > n$, then there exists a vertex $s \notin P \cup Q$, or in his words (where the 'Grad' denotes $|E|$):

> Wir nehmen also an, der irreduzibel n-punktig zusammenhängende Raum K' besitze den Grad $g(> n)$. Offenbar enthält dann K' ein punktförmiges Stück s, welches in der Menge $P + Q$ nicht enthalten ist.[7]

Indeed, as Menger showed, if such a vertex s exists one is done: If s is contained in no set W intersecting each $P - Q$ path with $|W| = n$, then we can delete s and the edges incident with s without decreasing the minimum in the theorem. If s is contained in some set W intersecting each $P - Q$ path such that $|W| = n$, then we can split G into two subgraphs G_1 and G_2 that intersect in W in such a way that $P \subseteq G_1$ and $Q \subseteq G_2$. By the induction hypothesis, there exist n disjoint $P - W$ paths in G_1 and n disjoint $W - Q$ paths in G_2. By pairwise sticking these paths together at W we obtain paths as required.

However, such a vertex s need not exist. It might be that V is the disjoint union of P and Q in such a way that each edge connects P and Q, and that there are more than n edges. In that case, G is a bipartite graph, with colour classes P and Q, and what should be shown is that G contains a matching of size n. This is a nontrivial basis of the proof.

At the meeting of 26 March 1931 of the Eötvös Loránd Matematikai és Fizikai Társulat (Loránd Eötvös Mathematical and Physical Society) in Budapest, Kőnig [1931] presented a new result that formed the missing basis for Menger's theorem:

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[6] One of the most important problems of the theory of curves is the question of the relations between the order of a point of a regular curve K and the number of subarcs of K meeting in that point and disjoint elsewhere.

[7] Thus we assume that the irreducibly n-point-connected space K' has degree $g(> n)$. Obviously, in that case K' contains a point-shaped piece s, that is not contained in the set $P + Q$.

> Páros körüljárású graphban az éleket kimerítő szögpontok minimális száma megegyezik a páronként közös végpontot nem tartalmazó élek maximális számával.[8]

In other words, in a bipartite graph $G = (V, E)$, the maximum size of a matching is equal to the minimum number of vertices needed to cover all edges, which is Kőnig's matching theorem — see Theorem 16.2.

Kőnig did not mention in his 1931 paper that this result provided the missing element in Menger's proof, although he finishes with:

> Megemlítjük végül, hogy eredményeink szorosan összefüggnek FROBENIUSnak determinánsokra és MENGERnek graphokra vonatkozó némely vizsgálatával. E kapcsolatokra másutt fogunk kiterjeszkedni.[9]

'Elsewhere' is Kőnig [1932], in which paper he gave a full proof of Menger's theorem. The hole in Menger's original proof is discussed in a footnote:

> Der Beweis von MENGER enthält eine Lücke, da es vorausgesetzt wird (S. 102, Zeile 3–4) daß „K' ein punktförmiges Stück s enthält, welches in der Menge $P+Q$ nicht enthalten ist“, während es recht wohl möglich ist, daß —mit der hier gewählten Bezeichnungsweise ausgedrückt—jeder Knotenpunkt von G zu $H_1 + H_2$ gehört. Dieser—keineswegs einfacher—Fall wurde in unserer Darstellung durch den Beweis des Satzes 13 erledigt. Die weiteren—hier folgenden—Überlegungen, die uns zum Mengerschen Satz führen werden, stimmen im Wesentlichen mit dem—sehr kurz gefaßten—Beweis von MENGER überein. In Anbetracht der Allgemeinheit und Wichtigkeit des Mengerschen Satzes wird im Folgenden auch dieser Teil ganz ausführlich und den Forderungen der *rein-kombinatorischen* Graphentheorie entsprechend dargestellt.
> [Zusatz bei der Korrektur, 10.V.1933] Herr MENGER hat die Freundlichkeit gehabt—nachdem ich ihm die Korrektur meiner vorliegenden Arbeit zugeschickt habe—mir mitzuteilen, daß ihm die oben beanstandete Lücke seines Beweises schon bekannt war, daß jedoch sein vor Kurzem erschienenes Buch *Kurventheorie* (Leipzig, 1932) einen vollkommen lückenlosen und rein kombinatorischen Beweis des Mengerschen Satzes (des "n-Kettensatzes") enthält. Mir blieb dieser Beweis bis jetzt unbekannt.[10]

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[8] In an even circuit graph, the minimal number of vertices that exhaust the edges agrees with the maximal number of edges that pairwise do not contain any common end point.

[9] We finally mention that our results are closely connected to some investigations of FROBENIUS on determinants and of MENGER on graphs. We will enlarge on these connections elsewhere.

[10] The proof of MENGER contains a hole, as it is assumed (page 102, line 3–4) that 'K' contains a point-shaped piece s that is not contained in the set $P + Q$', while it is quite well possible that—expressed in the notation chosen here—every node of G belongs to $H_1 + H_2$. This—by no means simple—case is settled in our presentation by the proof of Theorem 13. The further arguments following here that will lead us to Menger's theorem, agree essentially with the—very briefly couched—proof of MENGER. In view of the generality and the importance of Menger's theorem, also this part is exhibited in the following very extensively and conforming to the progress of the *purely combinatorial* graph theory.

[Added in proof, 10 May 1933] Mr. MENGER has had the kindness—after I have sent him the galley proofs of my present work—to inform me that the hole in his proof objected above, was known to him already, but that his, recently appeared, book *Kurventheorie* (Leipzig, 1932) contains a completely holeless and purely combinatorial proof of the Menger theorem (the 'n-chain theorem'). As yet, this proof remained unknown to me.

The book *Kurventheorie* (Curve Theory) mentioned is Menger [1932b], which contains a complete proof of Menger's theorem. Menger did not refer to any hole in his original proof, but remarked:

> Über den n-Kettensatz für Graphen und die im vorangehenden zum Beweise verwendete Methode vgl. Menger (Fund. Math. *10*, 1927, S. 101 f.). Die obige detaillierte Ausarbeitung und Darstellung stammt von Nöbeling.[11]

In his book *Theorie der endlichen und unendlichen Graphen* (Theory of finite and infinite graphs), Kőnig [1936] called his theorem *ein wichtiger Satz* (an important theorem), and he emphasized the chronological order of the proofs of Menger's theorem and of Kőnig's theorem (which is implied by Menger's theorem):

> Ich habe diesen Satz 1931 ausgesprochen und bewiesen, s. Kőnig [9 und 11]. 1932 erschien dann der erste lückenlose Beweis des Mengerschen Graphensatzes, von dem in §4 die Rede sein wird und welcher als eine Verallgemeinerung dieses Satzes 13 (falls dieser *nur für endliche* Graphen formuliert wird) angesehen werden kann.[12]

([9 und 11] are Kőnig [1931] and Kőnig [1932].)

In his reminiscences on the origin of the n-arc theorem, Menger [1981] wrote:

> In the spring of 1930, I came through Budapest and met there a galaxy of Hungarian mathematicians. In particular, I enjoyed making the acquaintance of Dénes Kőnig, for I greatly admired the work on set theory of his father, the late Julius Kőnig—to this day one of the most significant contributions to the continuum problem—and I had read with interest some of Dénes' papers. Kőnig told me that he was about to finish a book that would include all that was known about graphs. I assured him that such a book would fill a great need; and I brought up my n-Arc Theorem which, having been published as a lemma in a curve-theoretical paper, had not yet come to his attention. Kőnig was greatly interested, but did not believe that the theorem was correct. "This evening," he said to me in parting, "I won't go to sleep before having constructed a counterexample." When we met again the next day he greeted me with the words, "A sleepless night!" and asked me to sketch my proof for him. He then said that he would add to his book a final section devoted to my theorem. This he did; and it is largely thanks to Kőnig's valuable book that the n-Arc Theorem has become widely known among graph theorists.

Related work

In a paper presented 7 May 1927 to the American Mathematical Society, Rutt [1927, 1929] gave the following variant of Menger's theorem, suggested by J.R. Kline. Let $G = (V, E)$ be a planar graph and let $s, t \in V$. Then the maximum number of internally vertex-disjoint $s - t$ paths is equal to the minimum number of vertices in $V \setminus \{s, t\}$ intersecting each $s - t$ path.

---

[11] On the n-chain theorem for graphs and the method used in the foregoing for the proof, compare Menger (Fund. Math. *10*, 1927, p. 101 ff.). The detailed elaboration and explanation above originates from Nöbeling.

[12] I have enunciated and proved this theorem in 1931, see Kőnig [9 and 11]. Next, in 1932, the first holeless proof of the Menger theorem appeared, of which will be spoken in §4 and which can be considered as a generalization of this Theorem 13 (in case this is formulated *only for finite* graphs).

In fact, the theorem follows quite easily from Menger's version of his theorem by deleting s and t and taking for P and Q the sets of neighbours of s and t respectively. (Rutt referred to Menger and gave an independent proof of the theorem.)

This construction was also observed by Knaster [1930] who showed that Menger's theorem would follow from Rutt's theorem for general (not necessarily planar) graphs. A similar theorem was published by Nöbeling [1932], using Menger's result.

A result implied by Menger's theorem was presented by Whitney [1932a] on 28 February 1931 to the American Mathematical Society: a graph is n-connected if and only if any two vertices are connected by n internally disjoint paths. While referring to the papers of Menger and Rutt, Whitney gave a direct proof. Kőnig [1932] remarked on Whitney's theorem:

> Das interessante Hauptresultat einer Abhandlung von Whitney [10], nämlich sein Theorem 7, folgt unmittelbar aus diesem Mengerschen Satz, jedoch, wie es scheint, nicht umgekehrt.[13]

In the 1930s, other proofs of Menger's theorem were given by Hajós [1934] and Grünwald [1938] (= T. Gallai) — the latter paper gives an essentially algorithmic proof based on augmenting paths, and it observes, in a footnote, that the theorem also holds for directed graphs:

> Die ganze Betrachtung lässt sich auch bei orientierten Graphen durchführen und liefert dann eine Verallgemeinerung des Mengerschen Satzes.[14]

The arc-disjoint version of Menger's theorem seems to be first shown by Ford and Fulkerson [1954,1956b] and Kotzig [1956] for undirected graphs and by Dantzig and Fulkerson [1955,1956] for directed graphs.

In his dissertation for the degree of Academical Doctor, Kotzig [1956] defined, for any undirected graph G and vertices u, v of G, $\sigma_G(u, v)$ to be the minimum size of a $u - v$ cut. Then he states:

> Veta 35. Nech G je l'ubovol'ný graf obsahujúci uzly $u \neq v$, o ktorých platí $\sigma_G(u, v) = k > 0$, potom existuje systém ciest $\{C_1, C_2, \ldots, C_k\}$ taký že každa cesta spojuje uzly u, v a žiadne dve rôzne cesty systému nemajú spoločnej hrany. Takýto systém ciest v G existuje len vtedy, keď je $\sigma_G(u, v) \geq k$.[15]

In Theorems 33 and 34 of the dissertation, methods are developed for the proof of Theorem 35. The method is to consider a minimal graph satisfying the cut condition, and next to orient it so as to make a directed graph in which each vertex w (except u and v) has indegree = outdegree, while u has outdegree k and indegree 0. This then yields the required paths.

Although the dissertation has several references to Kőnig's book, which contains the undirected vertex-disjoint version of Menger's theorem, Kotzig did not link his

[13] The interesting main result of an article of Whitney [10], namely his Theorem 7, follows immediately from this theorem of Menger, however, as it seems, not conversely.

[14] The whole argument lets itself carry out also for oriented graphs and then yields a generalization of Menger's theorem.

[15] Theorem 35. Let G be an arbitrary graph containing vertices $u \neq v$ for which $\sigma_G(u, v) = k > 0$, then there exists a system of paths $\{C_1, C_2, \ldots, C_k\}$ such that each path connects vertices u, v and no two distinct paths have an edge in common. Such a system of paths in G exists only if $\sigma_G(u, v) \geq k$.

result to that of Menger. (Kotzig [1961a] gave a proof of the directed arc-disjoint version of Menger's theorem, without reference to Menger.)

We refer to the historical notes on maximum flows in Section 10.8e for further notes on the work of Dantzig, Ford, and Fulkerson on Menger's theorem.

Chapter 10

Maximum flow

An $s-t$ flow is defined as a nonnegative real-valued function on the arcs of a digraph satisfying the 'flow conservation law' at each vertex $\neq s, t$. In this chapter we consider the problem of finding a maximum-value flow subject to a given capacity function. Basic results are Ford and Fulkerson's max-flow min-cut theorem and their augmenting path algorithm to find a maximum flow.

Each $s-t$ flow is a nonnegative linear combination of incidence vectors of $s-t$ paths and of directed circuits. Moreover, an integer flow is an integer such combination. This makes flows tightly connected to disjoint paths. Thus, maximum integer flow corresponds to a capacitated version of a maximum packing of disjoint paths, and the max-flow min-cut theorem is equivalent to Menger's theorem on disjoint paths.

Distinguishing characteristic of flow is however that it is not described by a combination of paths but by a function on the arcs. This promotes the algorithmic tractability.

In this chapter, graphs can be assumed to be simple.

10.1. Flows: concepts

Let $D = (V, A)$ be a digraph and let $s, t \in V$. A function $f : A \to \mathbb{R}$ is called a *flow from s to t*, or an *$s-t$ flow*, if:

$$(10.1)\qquad \begin{array}{rll} \text{(i)} & f(a) \geq 0 & \text{for each } a \in A, \\ \text{(ii)} & f(\delta^{\text{out}}(v)) = f(\delta^{\text{in}}(v)) & \text{for each } v \in V \setminus \{s, t\}. \end{array}$$

Condition (10.1)(ii) is called the *flow conservation law*: the amount of flow entering a vertex $v \neq s, t$ should be equal to the amount of flow leaving v.

The *value* of an $s-t$ flow f is, by definition:

$$(10.2)\qquad \text{value}(f) := f(\delta^{\text{out}}(s)) - f(\delta^{\text{in}}(s)).$$

So the value is the net amount of flow leaving s. This is equal to the net amount of flow entering t (this follows from (10.5) below).

Let $c : A \to \mathbb{R}_+$ be a *capacity* function. We say that a flow f is *under c* (or *subject to c*) if

$$(10.3)\qquad f(a) \leq c(a) \text{ for each } a \in A.$$

A *maximum $s-t$ flow*, or just a *maximum flow*, is an $s-t$ flow under c, of maximum value. The *maximum flow problem* is to find a maximum flow.

By compactness and continuity, a maximum flow exists. It will follow from the results in this chapter (in particular, Theorem 10.4), that if the capacities are rational, then there exists a rational-valued maximum flow.

It will be convenient to make an observation on general functions $f : A \to \mathbb{R}$. For any $f : A \to \mathbb{R}$, the *excess function* is the function $\text{excess}_f : \mathcal{P}(V) \to \mathbb{R}$ defined by

(10.4) $$\text{excess}_f(U) := f(\delta^{\text{in}}(U)) - f(\delta^{\text{out}}(U))$$

for $U \subseteq V$. Set $\text{excess}_f(v) := \text{excess}_f(\{v\})$ for $v \in V$. Then:

Theorem 10.1. *Let $D = (V, A)$ be a digraph, let $f : A \to \mathbb{R}$, and let $U \subseteq V$. Then:*

(10.5) $$\text{excess}_f(U) = \sum_{v \in U} \text{excess}_f(v).$$

Proof. This follows directly by counting, for each $a \in A$, the multiplicity of $f(a)$ at both sides of (10.5). ∎

To formulate a min-max relation, define the *capacity* of a cut $\delta^{\text{out}}(U)$ by $c(\delta^{\text{out}}(U))$. Then:

Theorem 10.2. *Let $D = (V, A)$ be a digraph, $s, t \in V$, and $c : A \to \mathbb{R}_+$. Then*

(10.6) $$\text{value}(f) \leq c(\delta^{\text{out}}(U)),$$

for each $s-t$ flow $f \leq c$ and each $s-t$ cut $\delta^{\text{out}}(U)$. Equality holds in (10.6) *if and only if $f(a) = c(a)$ for each $a \in \delta^{\text{out}}(U)$ and $f(a) = 0$ for each $a \in \delta^{\text{in}}(U)$.*

Proof. Using (10.5) we have

(10.7) $$\begin{aligned}\text{value}(f) &= -\text{excess}_f(s) = -\text{excess}_f(U) = f(\delta^{\text{out}}(U)) - f(\delta^{\text{in}}(U)) \\ &\leq c(\delta^{\text{out}}(U)),\end{aligned}$$

with equality if and only if $f(\delta^{\text{out}}(U)) = c(\delta^{\text{out}}(U))$ and $f(\delta^{\text{in}}(U)) = 0$. ∎

Finally, we consider a concept that turns out to be important in studying flows. Let $D = (V, A)$ be a digraph. For each $a = (u, v) \in A$, let $a^{-1} := (v, u)$. Define

(10.8) $$A^{-1} := \{a^{-1} \mid a \in A\}.$$

Fix a lower bound function $d : A \to \mathbb{R}$ and an upper bound function $c : A \to \mathbb{R}$. Then for any $f : A \to \mathbb{R}$ satisfying $d \leq f \leq c$ we define

(10.9) $$A_f := \{a \mid a \in A, f(a) < c(a)\} \cup \{a^{-1} \mid a \in A, f(a) > d(a)\}.$$

Clearly, A_f depends not only on f, but also on D, d, and c, but in the applications below D, d, and c are fixed, while f is variable. The digraph

$$D_f = (V, A_f) \tag{10.10}$$

is called the *residual graph* of f. So D_f is a subgraph of the directed graph $(V, A \cup A^{-1})$. As we shall see, the residual graph is very useful in studying flows and circulations, both theoretically and algorithmically.

In the context of flows we take $d = \mathbf{0}$. We observe:

Corollary 10.2a. *Let f be an $s - t$ flow in D with $f \leq c$. Suppose that D_f has no $s - t$ path. Define U as the set of vertices reachable in D_f from s. Then* $\text{value}(f) = c(\delta^{\text{out}}_A(U))$. *In particular, f has maximum value.*

Proof. We apply Theorem 10.2. For each $a \in \delta^{\text{out}}_A(U)$, one has $a \notin A_f$, and hence $f(a) = c(a)$. Similarly, for each $a \in \delta^{\text{in}}_A(U)$ one has $a^{-1} \notin A_f$, and hence $f(a) = 0$. So $\text{value}(f) = c(\delta^{\text{out}}_A(U))$ and f has maximum value by Theorem 10.2. ∎

Any directed path P in D_f gives an undirected path in $D = (V, A)$. Define $\chi^P \in \mathbb{R}^A$ by:

$$\chi^P(a) := \begin{cases} 1 & \text{if } P \text{ traverses } a, \\ -1 & \text{if } P \text{ traverses } a^{-1}, \\ 0 & \text{if } P \text{ traverses neither } a \text{ nor } a^{-1}, \end{cases} \tag{10.11}$$

for $a \in A$.

10.2. The max-flow min-cut theorem

The following theorem was proved by Ford and Fulkerson [1954,1956b] for the undirected case and by Dantzig and Fulkerson [1955,1956] for the directed case. (According to Robacker [1955a], the max-flow min-cut theorem was conjectured first by D.R. Fulkerson.)

Theorem 10.3 (max-flow min-cut theorem). *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $c : A \to \mathbb{R}_+$. Then the maximum value of an $s - t$ flow subject to c is equal to the minimum capacity of an $s - t$ cut.*

Proof. Let f be an $s - t$ flow subject to c, of maximum value. By Theorem 10.2, it suffices to show that there is an $s - t$ cut $\delta^{\text{out}}(U)$ with capacity equal to $\text{value}(f)$.

Consider the residual graph D_f (for lower bound $d := \mathbf{0}$). Suppose that it contains an $s - t$ path P. Then $f' := f + \varepsilon\chi^P$ is again an $s - t$ flow subject to c, for $\varepsilon > 0$ small enough, with $\text{value}(f') = \text{value}(f) + \varepsilon$. This contradicts the maximality of $\text{value}(f)$.

So D_f contains no $s-t$ path. Let U be the set of vertices reachable in D_f from s. Then value$(f) = c(\delta^{\text{out}}(U))$ by Corollary 10.2a. ∎

This 'constructive' proof method is implied by the algorithm of Ford and Fulkerson [1955,1957b], to be discussed below.

Moreover, one has (Dantzig and Fulkerson [1955,1956])[16]:

Corollary 10.3a (integrity theorem)**.** *If c is integer, there exists an integer maximum flow.*

Proof. Directly from the proof of the max-flow min-cut theorem, where we can take $\varepsilon = 1$. ∎

10.3. Paths and flows

The following observation gives an important link between flows at one side and paths at the other side.

Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $f : A \to \mathbb{R}_+$ be an $s-t$ flow. Then f is a nonnegative linear combination of at most $|A|$ vectors χ^P, where P is a directed $s-t$ path or a directed circuit. If f is integer, we can take the linear combination integer-scalared.

Conversely, if $P_1, \ldots, P_k$ are $s-t$ paths in D, then $f := \chi^{AP_1} + \cdots + \chi^{AP_k}$ is an integer $s-t$ flow of value k.

With this observation, Corollary 10.3a implies the arc-disjoint version of Menger's theorem (Corollary 9.1b). Conversely, Corollary 10.3a (the integrity theorem) can be derived from the arc-disjoint version of Menger's theorem by replacing each arc a by $c(a)$ parallel arcs.

10.4. Finding a maximum flow

The proof idea of the max-flow min-cut theorem can also be used algorithmically to find a maximum $s-t$ flow, as was shown by Ford and Fulkerson [1955,1957b]. Let $D = (V, A)$ be a digraph and $s, t \in V$ and let $c : A \to \mathbb{Q}_+$ be a 'capacity' function.

Initially set $f := \mathbf{0}$. Next apply the following *flow-augmenting algorithm* iteratively:

(10.12) let P be a directed $s-t$ path in D_f and reset $f := f + \varepsilon\chi^P$, where ε is as large as possible so as to maintain $\mathbf{0} \leq f \leq c$.

If no such path exists, the flow f is maximum, by Corollary 10.2a.

The path P is called a *flow-augmenting path* or an *f-augmenting path*, or just an *augmenting path*.

[16] The name 'integrity theorem' was used by Ford and Fulkerson [1962].

As for termination, we have:

Theorem 10.4. *If all capacities $c(a)$ are rational, the algorithm terminates.*

Proof. If all capacities are rational, there exists a natural number K such that $Kc(a)$ is an integer for each $a \in A$. (We can take for K the l.c.m. of the denominators of the $c(a)$.)

Then in the flow-augmenting iterations, every $f_i(a)$ and every ε is a multiple of $1/K$. So at each iteration, the flow value increases by at least $1/K$. Since the flow value cannot exceed $c(\delta^{\text{out}}(\{s\}))$, there are only finitely many iterations. ∎

If we delete the rationality condition, this theorem is not maintained — see Section 10.4a. On the other hand, in Section 10.5 we shall see that if we always choose a *shortest possible* flow-augmenting path, then the algorithm terminates in a polynomially bounded number of iterations, regardless whether the capacities are rational or not.

10.4a. Nontermination for irrational capacities

Ford and Fulkerson [1962] showed that Theorem 10.4 is not maintained if we allow arbitrary real-valued capacities. The example is as follows.

Let $D = (V, A)$ be the complete directed graph on 8 vertices, with $s, t \in V$. Let $A_0 = \{a_1, a_2, a_3\}$ consist of three disjoint arcs of D, each disjoint from s and t. Let r be the positive root of $r^2 + r - 1 = 0$; that is, $r = (-1 + \sqrt{5})/2 < 1$. Define a capacity function c on A by

(10.13) $$c(a_1) := 1,\ c(a_2) := 1,\ c(a_3) := r,$$

and $c(a)$ at least

(10.14) $$q := \frac{1}{1-r} = 1 + r + r^2 + \cdots$$

for each $a \in A \setminus A_0$. Apply the flow-augmenting algorithm iteratively as follows.

In step 0, choose, as flow-augmenting path, the $s - t$ path of length 3 traversing a_1. After this step, the flow f satisfies, for $k = 1$:

(10.15)
(i) f has value $1 + r + r^2 + \cdots + r^{k-1}$,
(ii) $\{c(a) - f(a) \mid a \in A_0\} = \{0, r^{k-1}, r^k\}$,
(iii) $f(a) \leq 1 + r + r^2 + \cdots + r^{k-1}$ for each $a \in A$.

We describe the further steps. In each step k, for $k \geq 1$, the input flow f satisfies (10.15). Choose a flow-augmenting path P in D_f that contains the arc $a \in A_0$ satisfying $c(a) - f(a) = 0$ in backward direction, and the other two arcs in A_0 in forward direction; all other arcs of P are arcs of D in forward direction. Since $r^k < r^{k-1}$, and since $(1 + r + \cdots + r^{k-1}) + r^k < q$, the flow augmentation increases the flow value by r^k. Since $r^{k-1} - r^k = r^{k+1}$, the new flow satisfies (10.15) with k replaced by $k + 1$.

We can keep iterating this, making the flow value converge to $1+r+r^2+r^3+\cdots = q$. So the algorithm does not terminate, and the flow value does not converge to the optimum value, since, trivially, the maximum flow value is more than q.

(Zwick [1995] gave the smallest directed graph (with 6 vertices and 8 arcs) for which the algorithm (with irrational capacities) need not terminate.)

10.5. A strongly polynomial bound on the number of iterations

We saw in Theorem 10.4 that the number of iterations in the maximum flow algorithm is finite, if all capacities are rational. But if we choose as our flow-augmenting path P in the auxiliary graph D_f an *arbitrary* $s-t$ path, the number of iterations yet can get quite large. For instance, in the graph in Figure 10.1 the number of iterations, at an unfavourable choice of paths, can become $2 \cdot 10^k$, so exponential in the size of the input data (which is $O(k)$).

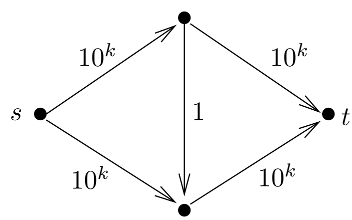

Figure 10.1

However, if we choose always a *shortest* $s-t$ path in D_f as our flow-augmenting path P (that is, with a minimum number of arcs), then the number of iterations is at most $|V| \cdot |A|$ (also if capacities are irrational). This was shown by Dinits [1970] and Edmonds and Karp [1972]. (The latter remark that this refinement 'is so simple that it is likely to be incorporated innocently into a computer implementation.')

To see this bound on the number of iterations, let again, for any digraph $D = (V, A)$ and $s, t \in V$, $\mu(D)$ denote the minimum length of an $s-t$ path. Moreover, let $\alpha(D)$ denote the set of arcs contained in at least one shortest $s-t$ path. Recall that by Theorem 9.5:

(10.16) for $D' := (V, A \cup \alpha(D)^{-1})$, one has $\mu(D') = \mu(D)$ and $\alpha(D') = \alpha(D)$.

This implies the result of Dinits [1970] and Edmonds and Karp [1972]:

Theorem 10.5. *If we choose in each iteration a shortest $s-t$ path in D_f as flow-augmenting path, the number of iterations is at most $|V| \cdot |A|$.*

Proof. If we augment flow f along a shortest $s-t$ path P in D_f, obtaining flow f', then $D_{f'}$ is a subgraph of $D' := (V, A_f \cup \alpha(D_f)^{-1})$. Hence $\mu(D_{f'}) \geq \mu(D') = \mu(D_f)$ (by (10.16)). Moreover, if $\mu(D_{f'}) = \mu(D_f)$, then $\alpha(D_{f'}) \subseteq \alpha(D') = \alpha(D_f)$ (again by (10.16)). As at least one arc in P belongs to D_f but not to $D_{f'}$, we have a strict inclusion. Since $\mu(D_f)$ increases at most $|V|$ times and, as long as $\mu(D_f)$ does not change, $\alpha(D_f)$ decreases at most $|A|$ times, we have the theorem. ∎

Since a shortest path can be found in time $O(m)$ (Theorem 6.3), this gives:

Corollary 10.5a. *A maximum flow can be found in time $O(nm^2)$.*

Proof. Directly from Theorem 10.5. ∎

10.6. Dinits' $O(n^2m)$ algorithm

Dinits [1970] observed that one can speed up the maximum flow algorithm, by not augmenting simply along *paths* in D_f, but along *flows* in D_f. The approach is similar to that of Section 9.3 for path packing.

To describe this, define a capacity function c_f on A_f by, for each $a \in A$:

(10.17) $$\begin{array}{ll} c_f(a) := c(a) - f(a) & \text{if } a \in A_f \text{ and} \\ c_f(a^{-1}) := f(a) & \text{if } a^{-1} \in A_f. \end{array}$$

Then for any flow g in D_f subject to c_f,

(10.18) $$f'(a) := f(a) + g(a) - g(a^{-1})$$

gives a flow f' in D subject to c. (We define $g(a)$ or $g(a^{-1})$ to be 0 if a or a^{-1} does not belong to A_f.)

Now we shall see that, given a flow f in D, one can find in time $O(m)$ a flow g in D_f such that the flow f' arising by (10.18) satisfies $\mu(D_{f'}) > \mu(D_f)$. It implies that there are at most n iterations.

The basis of the method is the concept of 'blocking flow'. An $s-t$ flow f is called *blocking* if for each $s-t$ flow f' with $f \leq f' \leq c$ one has $f' = f$.

Theorem 10.6. *Given an acyclic graph $D = (V, A)$, $s, t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$, a blocking $s-t$ flow can be found in time $O(nm)$.*

Proof. By depth-first search we can find, in time $O(|A'|)$, a subset A' of A and an $s-t$ path P in A' such that no arc in $A' \setminus AP$ is contained in any $s-t$ path: just scan s (cf. (6.2)) until t is reached; then A' is the set of arcs considered so far.

Let f be the maximum flow that can be sent along P, and reset $c := c - f$. Delete all arcs in $A' \setminus AP$ and all arcs a with $c(a) = 0$, and recursively find a blocking $s-t$ flow f' in the new network. Then $f'+f$ is a blocking $s-t$ flow for the original data, as is easily checked.

The running time of the iteration is $O(n+t)$, where t is the number of arcs deleted. Since there are at most $|A|$ iterations and since at most $|A|$ arcs can be deleted, we have the required running time bound. ∎

Hence we have an improvement on the running time for finding a maximum flow:

Corollary 10.6a. *A maximum flow can be found in time $O(n^2m)$.*

Proof. It suffices to describe an $O(nm)$ method to find, for given flow f, a flow f' with $\mu(D_{f'}) > \mu(D_f)$.

Find a blocking flow g in $(V, \alpha(D_f))$. (Note that $\alpha(D_f)$ can be determined in $O(m)$ time.) Let $f'(a) := f(a)+g(a)-g(a^{-1})$, taking values 0 if undefined. Then $D_{f'}$ is a subgraph of $D' := (V, A_f \cup \alpha(D_f)^{-1})$, and hence by (10.16), $\mu(D_{f'}) \geq \mu(D') = \mu(D_f)$. If $\mu(D_{f'}) = \mu(D_f)$, $D_{f'}$ has a path P of length $\mu(D_f)$, which (again (10.16)) should also be a path in $\alpha(D_f)$. But then g could have been increased along this path, contradicting the fact that g is blocking in D_f. ∎

10.6a. Karzanov's $O(n^3)$ algorithm

Karzanov [1974] gave a faster algorithm to find a blocking flow, thus speeding up the maximum flow algorithm. We give the short proof of Malhotra, Kumar, and Maheshwari [1978] (see also Cherkasskiĭ [1979] and Tarjan [1984]).

Theorem 10.7. *Given an acyclic digraph $D = (V, A)$, $s, t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$, a blocking $s-t$ flow can be found in time $O(n^2)$.*

Proof. First order the vertices reachable from s as $s = v_1, v_2, \ldots, v_{n-1}, v_n$ *topologically*; that is, if $(v_i, v_j) \in A$, then $i < j$. This can be done in time $O(m)$ (see Corollary 6.5b).

We describe the algorithm recursively. Consider the minimum of the values $c(\delta^{\text{in}}(v))$ for all $v \in V \setminus \{s\}$ and $c(\delta^{\text{out}}(v))$ for all $v \in V \setminus \{t\}$. Let the minimum be attained by v_i and $c(\delta^{\text{out}}(v_i))$ (without loss of generality). Define $f(a) := c(a)$ for each $a \in \delta^{\text{out}}(v_i)$ and $f(a) := 0$ for all other a.

Next for $j = i+1, \ldots, n-1$, redefine $f(a)$ for each $a \in \delta^{\text{out}}(v_j)$ such that $f(a) \leq c(a)$ and such that $f(\delta^{\text{out}}(v_j)) = f(\delta^{\text{in}}(v_j))$. By the minimality of $c(\delta^{\text{out}}(v_i))$, we can always do this, as initially $f(\delta^{\text{in}}(v_j)) \leq c(\delta^{\text{out}}(v_i)) \leq c(\delta^{\text{out}}(v_j))$. We do this in such a way that finally $f(a) \in \{0, c(a)\}$ for all but at most one a in $\delta^{\text{out}}(v_j)$.

After that, for $j = i, i-1, \ldots, 2$, redefine similarly $f(a)$ for $a \in \delta^{\text{in}}(v_j)$ such that $f(a) \leq c(a)$, $f(\delta^{\text{in}}(v_j)) = f(\delta^{\text{out}}(v_j))$, and $f(a) \in \{0, c(a)\}$ for all but at most one a in $\delta^{\text{in}}(v_j)$.

If $v_i \in \{s, t\}$ we stop, and f is a blocking flow. If $v_i \notin \{s, t\}$, set $c'(a) := c(a) - f(a)$ for each $a \in A$, and delete all arcs a with $c'(a) = 0$ and delete v_i and all arcs incident with v_i, thus obtaining the directed graph $D' = (V', A')$. Obtain (recursively) a blocking flow f' in D' subject to the capacity function c'. Define $f''(a) := f(a) + f'(a)$ for $a \in A'$ and $f''(a) = f(a)$ for $a \in A \setminus A'$. Then f'' is a blocking flow in D.

This describes the algorithm. The correctness can be seen as follows. If $v_i \in \{s,t\}$ the correctness is immediate. If $v_i \notin \{s,t\}$, suppose that f'' is not a blocking flow in D, and let P be an $s-t$ path in D with $f''(a) < c(a)$ for each arc a in P. Then each arc of P belongs to A', since $f''(a) = f(a) = c(a)$ for each $a \in A \setminus (A' \cup \delta^{\text{in}}(v_i))$. So for each arc a of P one has $c'(a) = c(a) - f(a) > f''(a) - f(a) = f'(a)$. This contradicts the fact that f' is a blocking flow in D'.

The running time of the algorithm is $O(n^2)$, since the running time of the iteration is $O(n + |A \setminus A'|)$, and since there are at most $|V|$ iterations. ∎

Theorem 10.7 improves the running time for finding a maximum flow as follows:

Corollary 10.7a. *A maximum flow can be found in time $O(n^3)$.*

Proof. Similar to the proof of Corollary 10.6a. ∎

Sharper blocking flow algorithms were found by Cherkasskiĭ [1977a] ($O(n\sqrt{m})$), Galil [1978,1980a] ($O((nm)^{2/3})$), Shiloach [1978] and Galil and Naamad [1979,1980] ($O(m \log^2 n)$), Sleator [1980] and Sleator and Tarjan [1981,1983a] ($O(m \log n)$), and Goldberg and Tarjan [1990] ($O(m \log(n^2/m))$), each yielding a maximum flow algorithm with running time bound a factor of n higher.

An alternative approach finding a maximum flow in time $O(nm \log(n^2/m))$, based on the 'push-relabel' method, was developed by Goldberg [1985,1987] and Goldberg and Tarjan [1986,1988a], and is described in the following section.

10.7. Goldberg's push-relabel method

The algorithms for the maximum flow problem described above are all based on flow augmentation. The basis is updating a flow f until D_f has no $s-t$ path. Goldberg [1985,1987] and Goldberg and Tarjan [1986,1988a] proposed a different, in a sense dual, method, the 'push-relabel' method: update a 'preflow' f, maintaining the property that D_f has no $s-t$ path, until f is a flow. (Augmenting flow methods are 'primal' as they maintain feasibility of the primal linear program, while the push-relabel method maintains feasibility of the dual linear program.)

Let $D = (V, A)$ be a digraph, $s, t \in V$, and $c : A \to \mathbb{Q}_+$. A function $f : A \to \mathbb{Q}$ is called an $s-t$ *preflow*, or just a *preflow*, if

(10.19) (i) $0 \leq f(a) \leq c(a)$ for each $a \in A$,
(ii) $\text{excess}_f(v) \geq 0$ for each vertex $v \neq s$.

(Preflows were introduced by Karzanov [1974]. excess_f was defined in Section 10.1.)

Condition (ii) says that at each vertex $v \neq s$, the outgoing preflow does not exceed the ingoing preflow. For any preflow f, call a vertex v *active* if

$v \neq t$ and $\text{excess}_f(v) > 0$. So f is an $s-t$ flow if and only if there are no active vertices.

The *push-relabel method* consists of keeping a pair f, p, where f is a preflow and $p : V \to \mathbb{Z}_+$ such that

(10.20) (i) if $(u, v) \in A_f$, then $p(v) \geq p(u) - 1$,
(ii) $p(s) = n$ and $p(t) = 0$.

Note that for any given f, such a function p exists if and only if D_f has no $s-t$ path. Hence, if a function p satisfying (10.20) exists and f is an $s-t$ flow, then f is an $s-t$ flow of maximum value (Corollary 10.2a).

Initially, f and p are set by:

(10.21) $f(a) := c(a)$ if $a \in \delta^{\text{out}}(s)$ and $f(a) := 0$ otherwise;
$p(v) := n$ if $v = s$ and $p(v) := 0$ otherwise.

Next, while there exist active vertices, choose an active vertex u maximizing $p(u)$, and apply the following iteratively, until u is inactive:

(10.22) choose an arc $(u, v) \in A_f$ with $p(v) = p(u) - 1$ and *push* over (u, v); if no such arc exists, *relabel* u.

Here to *push* over $(u, v) \in A_f$ means:

(10.23) if $(u, v) \in A$, reset $f(u, v) := f(u, v) + \varepsilon$, where $\varepsilon := \min\{c(u, v) - f(u, v), \text{excess}_f(u)\}$;
if $(v, u) \in A$, reset $f(v, u) := f(v, u) - \varepsilon$, where $\varepsilon := \min\{f(v, u), \text{excess}_f(u)\}$.

To *relabel* u means:

(10.24) reset $p(u) := p(u) + 1$.

Note that if A_f has no arc (u, v) with $p(v) = p(u) - 1$, then we can relabel u without violating (10.20).

This method terminates, since:

Theorem 10.8. *The number of pushes is $O(n^3)$ and the number of relabels is $O(n^2)$.*

Proof. First we show:

(10.25) throughout the process, $p(v) < 2n$ for each $v \in V$.

Indeed, if v is active, then D_f contains a $v-s$ path (since f can be decomposed as a sum of incidence vectors of $s-v$ paths, for $v \in V$, and of directed circuits). So by (10.20)(i), $p(v) - p(s) \leq \text{dist}_{D_f}(v, s) < n$. As $p(s) = n$, we have $p(v) < 2n$. This gives (10.25), which directly implies:

(10.26) the number of relabels is at most $2n^2$.

To estimate the number of pushes, call a push (10.23) *saturating* if after the push one has $f(u,v) = c(u,v)$ (if $(u,v) \in A$) or $f(v,u) = 0$ (if $(v,u) \in A$). Then:

(10.27) the number of saturating pushes is $O(nm)$.

For consider any arc $a = (u,v) \in A$. If we increase $f(a)$, then $p(v) = p(u) - 1$, while if we decrease $f(a)$, then $p(u) = p(v) - 1$. So meantime $p(v)$ should have been relabeled at least twice. As p is nondecreasing (in time), by (10.25) we have (10.27).

Finally:

(10.28) the number of nonsaturating pushes is $O(n^3)$.

Between any two relabels the function p does not change. Hence there are $O(n)$ nonsaturating pushes, as each of them makes an active vertex v maximizing $p(v)$ inactive (while possibly a vertex v' with $p(v') < p(v)$ is activated). With (10.26) this gives (10.28). ∎

There is an efficient implementation of the method:

Theorem 10.9. *The push-relabel method finds a maximum flow in time* $O(n^3)$.

Proof. We order the vertex set V as a doubly linked list, in order of increasing value $p(v)$. Moreover, for each $u \in V$ we keep the set L_u of arcs (u,v) in A_f with $p(v) = p(u) - 1$, ordered as a doubly linked list. We also keep with each vertex v the value $\text{excess}_f(v)$, and we keep linked lists of arcs of D incident with v.

Throughout the iterations, we choose an active vertex u maximizing $p(u)$, and we process u, until u becomes inactive. Between any two relabelings, this searching takes $O(n)$ time, since as long as we do not relabel, we can continue searching the list V in order. As we relabel $O(n^2)$ times, we can do the searching in $O(n^3)$ time.

Suppose that we have found an active vertex u maximizing $p(u)$. We next push over each of the arcs in L_u. So finding an arc $a = (u,v)$ for pushing takes time $O(1)$. If it is a saturating push, we can delete (u,v) from L_u in time $O(1)$. Moreover, we can update $\text{excess}_f(u)$ and $\text{excess}_f(v)$ in time $O(1)$. Therefore, as there are $O(n^3)$ pushes, they can be done in $O(n^3)$ time.

We decide to relabel u if $L_u = \emptyset$. When relabeling, updating the lists takes $O(n)$ time: When we reset $p(u)$ from i to $i+1$, then for each arc (u,v) or (v,u) of D, we add (u,v) to L_u if $p(v) = i$ and $(u,v) \in A_f$, and we remove (v,u) from L_v if $p(v) = i+1$ and $(v,u) \in A_f$; moreover, we move u to its new rank in the list V. This all takes $O(n)$ time. Therefore, as there are $O(n^2)$ relabels, they can be done in $O(n^3)$ time. ∎

Further notes on the push-relabel method. If we allow any active vertex u to be chosen for (10.22) (not requiring maximality of $p(u)$), then the bounds of

$O(n^2)$ on the number of relabels and $O(nm)$ on the number of saturating pushes are maintained, while the number of nonsaturating pushes is $O(n^2m)$.

A first-in first-out selection rule was studied by Goldberg [1985], also yielding an $O(n^3)$ algorithm. Theorem 10.9 (using the largest-label selection) is due to Goldberg and Tarjan [1986,1988a], who also showed an implementation of the push-relabel method with dynamic trees, taking $O(nm \log(n^2/m))$ time. Cheriyan and Maheshwari [1989] and Tunçel [1994] showed that the bound on the number of pushes in Theorem 10.8 can be improved to $O(n^2\sqrt{m})$, yielding an $O(n^2\sqrt{m})$ running time bound. Further improvements are given in Ahuja and Orlin [1989] and Ahuja, Orlin, and Tarjan [1989]. The worst-case behaviour of the push-relabel method was studied by Cheriyan and Maheshwari [1989].

10.8. Further results and notes

10.8a. A weakly polynomial bound

Edmonds and Karp [1972] considered the following *fattest augmenting path* rule: choose a flow-augmenting path for which the flow value increase is maximal. They showed that, if all capacities are integer, it terminates in at most $1 + m' \log \phi$ iterations, where ϕ is the maximum flow value and where m' is the maximum number of arcs in any $s - t$ cut. This gives a maximum flow algorithm of running time $O(n^2 m \log nC)$, where C is the maximum capacity (assuming all capacities are integer). (For irrational capacities, Queyranne [1980] showed that the method need not terminate.)

Edmonds and Karp [1970,1972] and Dinits [1973a] introduced the idea of *capacity-scaling*, which gives the following stronger running time bound:

Theorem 10.10. *For integer capacities, a maximum flow can be found in time $O(m^2 \log C)$.*

Proof. Let $L := \lceil \log_2 C \rceil + 1$. For $i = L, L-1, \ldots, 0$, we can obtain a maximum flow f' for capacity function $c' := \lfloor c/2^i \rfloor$, from a maximum flow f'' for capacity function $c'' := \lfloor c/2^{i+1} \rfloor$ as follows. Observe that the maximum flow value for c' differs by at most m from that of the maximum flow value ϕ for $2c''$. For let $\delta^{\text{out}}(U)$ be a cut with $2c''(\delta^{\text{out}}(U)) = \phi$. Then $c'(\delta^{\text{out}}(U)) - \phi \leq |\delta^{\text{out}}(U)| \leq m$. So a maximum flow with respect to c' can be obtained from $2f''$ by at most m augmenting path iterations. As each augmenting path iteration can be done in $O(m)$ time, and as $\lfloor c/2^L \rfloor = 0$, we have the running time bound given. ∎

With methods similar to those used in Corollary 10.6a, the bound in Theorem 10.10 can be improved to $O(nm \log C)$, a result of Dinits [1973a] and Gabow [1985b]. To see this, observe that the proof of Theorem 10.6 also yields:

Theorem 10.11. *Given an acyclic graph $D = (V, A)$, $s, t \in V$, and a capacity function $c : A \to \mathbb{Z}_+$, an integer blocking flow f can be found in time $O(n\phi + m)$, where ϕ is the value of f.*

Proof. Consider the proof of Theorem 10.6. We do at most ϕ iterations, while each iteration takes $O(n+t)$ time, where t is the number of arcs deleted. ∎

Hence, similarly to Corollary 10.6a one has:

Corollary 10.11a. *For integer capacities, a maximum flow can be found in time $O(n(\phi+m))$, where ϕ is the maximum flow value.*

Proof. Similar to the proof of Corollary 10.6a. ∎

Therefore,

Corollary 10.11b. *For integer capacities, a maximum flow can be found in time $O(nm\log C)$.*

Proof. In the proof of Theorem 10.10, a maximum flow with respect to c' can be obtained from $2f''$ in time $O(nm)$ (by Corollary 10.11a), since the maximum flow value in the residual graph $D_{f''}$ is at most m. ∎

10.8b. Complexity survey for the maximum flow problem

Complexity survey ($*$ indicates an asymptotically best bound in the table):

$O(n^2mC)$	Dantzig [1951a] simplex method
$O(nmC)$	Ford and Fulkerson [1955,1957b] augmenting path
$O(nm^2)$	Dinits [1970], Edmonds and Karp [1972] shortest augmenting path
$O(n^2m\log nC)$	Edmonds and Karp [1972] fattest augmenting path
$O(n^2m)$	Dinits [1970] shortest augmenting path, layered network
$O(m^2\log C)$	Edmonds and Karp [1970,1972] capacity-scaling
$O(nm\log C)$	Dinits [1973a], Gabow [1983b,1985b] capacity-scaling
$O(n^3)$	Karzanov [1974] (preflow push); cf. Malhotra, Kumar, and Maheshwari [1978], Tarjan [1984]
$O(n^2\sqrt{m})$	Cherkasskiĭ [1977a] blocking preflow with long pushes
$O(nm\log^2 n)$	Shiloach [1978], Galil and Naamad [1979,1980]
$O(n^{5/3}m^{2/3})$	Galil [1978,1980a]

≫

continued

	$O(nm \log n)$	Sleator [1980], Sleator and Tarjan [1981,1983a] dynamic trees
*	$O(nm \log(n^2/m))$	Goldberg and Tarjan [1986,1988a] push-relabel+dynamic trees
	$O(nm + n^2 \log C)$	Ahuja and Orlin [1989] push-relabel + excess scaling
	$O(nm + n^2\sqrt{\log C})$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved
*	$O(nm \log((n/m)\sqrt{\log C} + 2))$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved + dynamic trees
*	$O(n^3/\log n)$	Cheriyan, Hagerup, and Mehlhorn [1990,1996]
	$O(n(m + n^{5/3} \log n))$	Alon [1990] (derandomization of Cheriyan and Hagerup [1989,1995])
	$O(nm + n^{2+\varepsilon})$	(for each $\varepsilon > 0$) King, Rao, and Tarjan [1992]
*	$O(nm \log_{m/n} n + n^2 \log^{2+\varepsilon} n)$	(for each $\varepsilon > 0$) Phillips and Westbrook [1993,1998]
*	$O(nm \log_{\frac{m}{n \log n}} n)$	King, Rao, and Tarjan [1994]
*	$O(m^{3/2} \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]
*	$O(n^{2/3} m \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]

Here $C := \|c\|_\infty$ for integer capacity function c. For a complexity survey for unit capacities, see Section 9.6a.

Research problem: Is there an $O(nm)$-time maximum flow algorithm?

For the special case of *planar* undirected graphs:

	$O(n^2 \log n)$	Itai and Shiloach [1979]
	$O(n \log^2 n)$	Reif [1983] (minimum cut), Hassin and Johnson [1985] (maximum flow)
	$O(n \log n \log^* n)$	Frederickson [1983b]
*	$O(n \log n)$	Frederickson [1987b]

For *directed* planar graphs:

	$O(n^{3/2} \log n)$	Johnson and Venkatesan [1982]
	$O(n^{4/3} \log^2 n \log C)$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]
*	$O(n \log n)$	Weihe [1994b,1997b]

Itai and Shiloach [1979] and Hassin [1981b] showed that if s and t both are on the outer boundary, then a shortest path algorithm applied to the dual gives an $O(n \log n)$ algorithm for finding a minimum-capacity $s-t$ cut and a maximum-value $s-t$ flow, also for the directed case. This extends earlier work of Hu [1969].

Khuller, Naor, and Klein [1993] studied the lattice structure of the integer $s-t$ flows in a planar directed graph. More on planar maximum flow can be found in Khuller and Naor [1990,1994].

10.8c. An exchange property

Dinits [1973b] and Minieka [1973] observed the following analogue of Theorem 9.12:

Theorem 10.12. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, let $c : A \to \mathbb{R}_+$, and let f_1 and f_2 be maximum $s-t$ flows subject to c. Then there exists a maximum $s-t$ flow f subject to c such that $f(a) = f_1(a)$ for each arc a incident with s and $f(a) = f_2(a)$ for each arc a incident with t.*

Proof. Let $\delta^{\text{out}}(U)$ be an $s-t$ cut of minimum capacity, with $U \subseteq V$ and $s \in U$,$t \notin U$. So f_1 and f_2 coincide on $\delta^{\text{out}}(U)$ and on $\delta^{\text{in}}(U)$. Define $f(a) := f_1(a)$ if a is incident with U and $f(a) := f_2(a)$ if a is incident with $V \setminus U$. This defines a maximum $s-t$ flow as required. ∎

This was also shown by Megiddo [1974], who used it to prove the following. Let $D = (V, A)$ be a directed graph, let $c : A \to \mathbb{R}_+$ be a capacity function, and let $s, t \in V$, where s is a source, and t is a sink. An $s-t$ flow $f \leq c$ is called *source-optimal* if the vector $(f(a) \mid a \in \delta^{\text{out}}(s))$ is lexicographically maximal among all $s-t$ flows subject to c (ordering the $a \in \delta^{\text{out}}(s)$ by nonincreasing value of $f(a)$). The maximum flow algorithm implies that a source-optimal $s-t$ flow is a maximum-value $s-t$ flow.

One similarly defines *sink-optimal*, and Theorem 10.12 implies that there exists an $s-t$ flow that is both source- and sink-optimal. The proof shows that this flow can be found by combining a source-optimal and a sink-optimal flow appropriately.

As Megiddo showed, a source-optimal flow can be found iteratively, by updating a flow f (starting with $f = \mathbf{0}$), by determining an arc $a \in \delta^{\text{out}}(s)$ with $f(a) = 0$, on which $f(a)$ can be increased most. Making this increase, gives the next f. Stop if no increase is possible anymore.

10.8d. Further notes

Simplex method. The maximum flow problem is a linear programming problem, and hence it can be solved with the simplex method of Dantzig [1951b] (this paper includes an anti-cycling rule based on perturbation). This was elaborated by Fulkerson and Dantzig [1955a,1955b]. A direct, combinatorial anti-cycling rule for flow problems was given by Cunningham [1976]. Goldfarb and Hao [1990] gave a pivot rule that leads to at most nm pivots, yielding an algorithm of running time $O(n^2m)$. Goldberg, Grigoriadis, and Tarjan [1991] showed that with the help of dynamic trees there is an $O(nm \log n)$ implementation. See also Gallo, Grigoriadis, and Tarjan [1989], Plotkin and Tardos [1990], Goldfarb and Hao [1991], Orlin, Plotkin, and Tardos [1993], Ahuja and Orlin [1997], Armstrong and Jin [1997], Goldfarb and Chen [1997], Tarjan [1997], Armstrong, Chen, Goldfarb, and Jin [1998], and Hochbaum [1998].

Worst-case analyses of maximum flow algorithms were given by Zadeh [1972, 1973b] (shortest augmenting path rule), Dinits [1973b] (shortest augmenting path rule), Tarjan [1974e], Even and Tarjan [1975] (Dinits $O(n^2m)$ algorithm), Baratz [1977] (Karzanov's $O(n^3)$ algorithm), Galil [1981], Cheriyan [1988] (push-relabel method), Cheriyan and Maheshwari [1989] (push-relabel method), and Martel [1989] (push-relabel method). For further analysis of maximum flow algorithms, see Tucker [1977a] and Ahuja and Orlin [1991].

Computational studies were reported by Cherkasskiĭ [1979], Glover, Klingman, Mote, and Whitman [1979,1980,1984], Hamacher [1979] (Karzanov's method), Cheung [1980], Imai [1983b], Goldfarb and Grigoriadis [1988] (Dinits' method and the simplex method), Derigs and Meier [1989] (push-relabel method), Alizadeh and Goldberg [1993] (push-relabel in parallel), Anderson and Setubal [1993] (push-relabel), Gallo and Scutellà [1993], Nguyen and Venkateswaran [1993] (push-relabel), and Cherkassky and Goldberg [1995,1997] (push-relabel method). Consult also Johnson and McGeoch [1993].

A probabilistic analysis was presented by Karp, Motwani, and Nisan [1993]. A randomized approximation algorithm for minimum $s-t$ cut was given by Benczúr and Karger [1996].

Fulkerson [1959b] gave a labeling algorithm for finding the minimum cost of capacities to be added to make an $s-t$ flow of given value possible. Wollmer [1964] studied which k arcs to remove from a capacitated digraph so as to reduce the maximum $s-t$ flow value as much as possible. McCormick [1997] studied the problem of computing 'least infeasible' flows. Akers [1960] described the effect of ΔY operations on max-flow computations.

Ponstein [1972] gave another rule guaranteeing termination of the augmenting path iterations in Ford and Fulkerson's algorithm. Karp [1972b] showed that the *maximum*-cut problem is NP-complete — see Section 75.1a. Work on flows with small capacities was reported by Fernández-Baca and Martel [1989] and Ahuja and Orlin [1991]. Decomposition algorithms for locating minimal cuts were studied by Jarvis and Tufekci [1982]. The kth best cut algorithm was given by Hamacher [1982].

The problem of determining a flow along odd paths in an undirected graph was considered by Schrijver and Seymour [1994] — see Section 29.11e.

For an in-depth survey of network flows, see Ahuja, Magnanti, and Orlin [1993]. Other surveys were given by Ford and Fulkerson [1962], Dantzig [1963], Busacker and Saaty [1965], Fulkerson [1966], Hu [1969,1982], Iri [1969], Frank and Frisch [1971], Berge [1973b], Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975] (for a review, see Goldberg and Gusfield [1991]), Christofides [1975], Lawler [1976b], Bazaraa and Jarvis [1977], Minieka [1978], Even [1979], Jensen and Barnes [1980], Papadimitriou and Steiglitz [1982], Smith [1982], Chvátal [1983], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Rockafellar [1984], Tarjan [1986], Nemhauser and Wolsey [1988], Ahuja, Magnanti, and Orlin [1989,1991], Chen [1990], Cormen, Leiserson, and Rivest [1990], Goldberg, Tardos, and Tarjan [1990], Frank [1995], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].

Golden and Magnanti [1977] gave a bibliography and Slepian [1968] discussed the algebraic theory of flows.

10.8e. Historical notes on maximum flow

The problem of sending flow through a network was considered by Kantorovich [1939]. In fact, he considered multicommodity flows — see the historical notes in Section 70.13g.

The foundations for one-commodity maximum flow were laid during the period November 1954-December 1955 at RAND Corporation in Santa Monica, California. We will review the developments in a chronological order, by the date of the RAND Reports.

In their basic report *Maximal Flow through a Network* dated 19 November 1954, Ford and Fulkerson [1954,1956b] showed the max-flow min-cut theorem for undirected graphs:

> Theorem 1. (Minimal cut theorem). The maximal flow value obtainable in a network N is the minimum of $v(D)$ taken over all disconnecting sets D.

(Robacker [1955a] wrote that the max-flow min-cut theorem was conjectured first by Fulkerson.)

Ford and Fulkerson were motivated by flows and cuts in railway networks — see below. In the same report, also a simple algorithm was described for the maximum flow problem in case the graph, added with an extra edge connecting s and t, is planar.

The authors moreover observed that the maximum flow problem is a special case of a linear programming problem and that hence it can be solved by Dantzig's simplex method.

In a report of 1 January 1955 (revised 15 April 1955), Dantzig and Fulkerson [1955,1956] showed that the max-flow min-cut theorem can also be deduced from the duality theorem of linear programming (they mention that also A.J. Hoffman did this), they generalized it to the directed case, and they observed, using results of Dantzig [1951a], that if the capacities are integer, there is an integer maximum flow (the 'integrity theorem'). Hence (as they mention) Menger's theorem follows as a consequence. A simple computational method for the maximum flow problem based on the simplex method was described in a report of 1 April 1955 by Fulkerson and Dantzig [1955a,1955b].

Conversely, in a report of 26 May 1955, Robacker [1955a] derived the undirected max-flow min-cut theorem from the undirected vertex-disjoint version of Menger's theorem.

Boldyreff's heuristic

While the maximum flow algorithms found so far were derived from the simplex method, the quest for combinatorial methods remained vivid. A heuristic for the maximum flow problem, the 'flooding technique', was presented by Boldyreff [1955c, 1955b] on 3 June 1955 at the New York meeting of the Operations Research Society of America (published as a RAND Report of 5 August 1955 (Boldyreff [1955a])). The method is intuitive and the author did not claim generality (we quote from Boldyreff [1955b]):

> It has been previously assumed that a highly complex railway transportation system, too complicated to be amenable to analysis, can be represented by a much simpler model. This was accomplished by representing each complete railway operating division by a point, and by joining pairs of such points by arcs (lines) with traffic carrying capacities equal to the maximum possible volume of traffic (expressed in some convenient unit, such as trains per day) between the corresponding operating divisions.
> In this fashion, a network is obtained consisting of three sets of points — points of origin, intermediate or junction points, and the terminal points (or points of destination) — and a set of arcs of specified traffic carrying capacities, joining these points to each other.

Boldyreff's arguments for designing a heuristic procedure are formulated as follows:

> In the process of searching for the methods of solving this problem the following objectives were used as a guide:
> **1.** That the solution could be obtained quickly, even for complex networks.
> **2.** That the method could be explained easily to personnel without specialized technical training and used by them effectively.
> **3.** That the validity of the solution be subject to easy, direct verification.
> **4.** That the method would not depend on the use of high-speed computing or other specialized equipment.

Boldyreff's 'flooding technique' pushes as much flow as possible greedily through the network. If at some vertex a 'bottleneck' arises (i.e., more trains arrive than can be pushed further through the network), it is eliminated by returning the excess trains to the origin.

The method is empirical, not using backtracking, and not leading to an optimum solution in all cases:

> Whenever arbitrary decisions have to be made, ordinary common sense is used as a guide. At each step the guiding principle is to move forward the maximum possible number of trains, and to maintain the greatest flexibility for the remaining network.

Boldyreff speculates:

> In dealing with the usual railway networks a single flooding, followed by removal of bottlenecks, should lead to a maximal flow.

In the abstract of his lecture, Boldyreff [1955c] mentions:

> The mechanics of the solutions is formulated as a simple game which can be taught to a ten-year-old boy in a few minutes.

In his article, Boldyreff [1955b] gave as example the model of a real, comprehensive, railway transportation system with 41 vertices and 85 arcs:

> The total time of solving the problem is less than thirty minutes.

His closing remarks are:

> Finally there is the question of a systematic formal foundation, the comprehensive mathematical basis for empiricism and intuition, and the relation of the present techniques to other processes, such as, for instance, the multistage decision process (a suggestion of Bellman's).
> All this is reserved for the future.

Ford and Fulkerson's motivation: The Harris-Ross report

In their first report on maximum flow, *Maximal Flow through a Network*, Ford and Fulkerson [1954,1956b] mentioned that the maximum flow problem was formulated by T.E. Harris as follows:

> Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.

Later, in their book *Flows in Networks*, Ford and Fulkerson [1962] gave a more precise reference[17]:

> It was posed to the authors in the spring of 1955 by T.E. Harris, who, in conjunction with General F.S. Ross (Ret.), had formulated a simplified model of railway traffic flow, and pinpointed this particular problem as the central one suggested by the model [11].

Ford and Fulkerson's reference [11] here is the secret report by Harris and Ross [1955] entitled *Fundamentals of a Method for Evaluating Rail Net Capacities*, dated 24 October 1955[18], and written for the Air Force. The report was downgraded to 'unclassified' on 21 May 1999.

Unlike what Ford and Fulkerson write, the interest of Harris and Ross was not to find a maximum flow, but rather a minimum cut ('interdiction') of the Soviet railway system. We quote:

> Air power is an effective means of interdicting an enemy's rail system, and such usage is a logical and important mission for this Arm.
> As in many military operations, however, the success of interdiction depends largely on how complete, accurate, and timely is the commander's information, particularly concerning the effect of his interdiction-program efforts on the enemy's capability to move men and supplies. This information should be available at the time the results are being achieved.
> The present paper describes the fundamentals of a method intended to help the specialist who is engaged in estimating railway capabilities, so that he might more readily accomplish this purpose and thus assist the commander and his staff with greater efficiency than is possible at present.

In the Harris-Ross report, first much attention is given to modelling a railway network: taking each railway junction as a vertex would give a too refined network (for their purposes). Therefore, Harris and Ross proposed to take 'railway divisions' (organizational units based on geographical areas) as vertices, and to estimate the capacity of the connections between any two adjacent railway divisions. In an interview with Alexander [1996], Harris remembered:

> We were studying rail transportation in consultation with a retired army general, Frank Ross, who had been chief of the Army's Transportation Corps in Europe. We thought of modeling a rail system as a network. At first it didn't make sense, because there's no reason why the crossing point of two lines should be a special

[17] There seems to be some discrepancy between the date of the RAND Report of Ford and Fulkerson (19 November 1954) and the date mentioned in the quotation (spring of 1955).

[18] In their book, Ford and Fulkerson incorrectly date the Harris-Ross report 24 October 1956.

> sort of node. But Ross realized that, in the region we were studying, the “divisions” (little administrative districts) should be the nodes. The link between two adjacent nodes represents the total transportation capacity between them. This made a reasonable and manageable model for our rail system.

The Harris-Ross report stresses that specialists remain needed to make up the model (which seems always a good strategy to get new methods accepted):

> It is not the purpose that the highly specialized individual who estimates track and network capacities should be replaced by a novice with a calculating machine. Rather, it is accepted that the evaluation of track capacities remains a task for the specialist.
>
> [...]
>
> The ability to estimate with relative accuracy the capacity of single railway lines is largely an art. Specialists in this field have no authoritative text (insofar as the authors are informed) to guide their efforts, and very few individuals have either the experience or talent for this type of work. The authors assume that this job will continue to be done by the specialist.

The authors next disputed the naive belief that a railway network is just a set of disjoint through lines, and that cutting these lines would imply cutting the network:

> It is even more difficult and time-consuming to evaluate the capacity of a railway network comprising a multitude of rail lines which have widely varying characteristics. Practices among individuals engaged in this field vary considerably, but all consume a great deal of time. Most, if not all, specialists attack the problem by viewing the railway network as an aggregate of through lines.
> The authors contend that the foregoing practice does not portray the full flexibility of a large network. In particular it tends to gloss over the fact that even if every one of a set of independent through lines is made inoperative, there may exist alternative routings which can still move the traffic.
> This paper proposes a method that departs from present practices in that it views the network as an aggregate of railway operating divisions. All trackage capacities within the divisions are appraised, and these appraisals form the basis for estimating the capability of railway operating divisions to receive trains from and concurrently pass trains to each neighboring division in 24-hour periods.

Whereas experts are needed to set up the model, to solve it is routine (when having the ‘work sheets’):

> The foregoing appraisal (accomplished by the expert) is then used in the preparation of comparatively simple work sheets that will enable relatively inexperienced assistants to compute the results and thus help the expert to provide specific answers to the problems, based on many assumptions, which may be propounded to him.

While Ford and Fulkerson flow-augmenting path algorithm for the maximum flow problem was not found yet, the Harris-Ross report suggests applying Boldyreff’s flooding technique described above. The authors preferred this above the simplex method for maximum flow:

> The calculation would be cumbersome; and, even if it could be performed, sufficiently accurate data could not be obtained to justify such detail.

However, later in the report their assessment of the simplex method is more favourable:

> These methods do not require elaborate computations and can be performed by a relatively untrained person.

The Harris-Ross report applies Boldyreff's flooding technique to a network model of the Soviet and East European railways. For the data it refers to several secret reports of the Central Intelligence Agency (C.I.A.) on sections of the Soviet and East European railway networks. After the aggregation of railway divisions to vertices, the network has 44 vertices and 105 (undirected) edges.

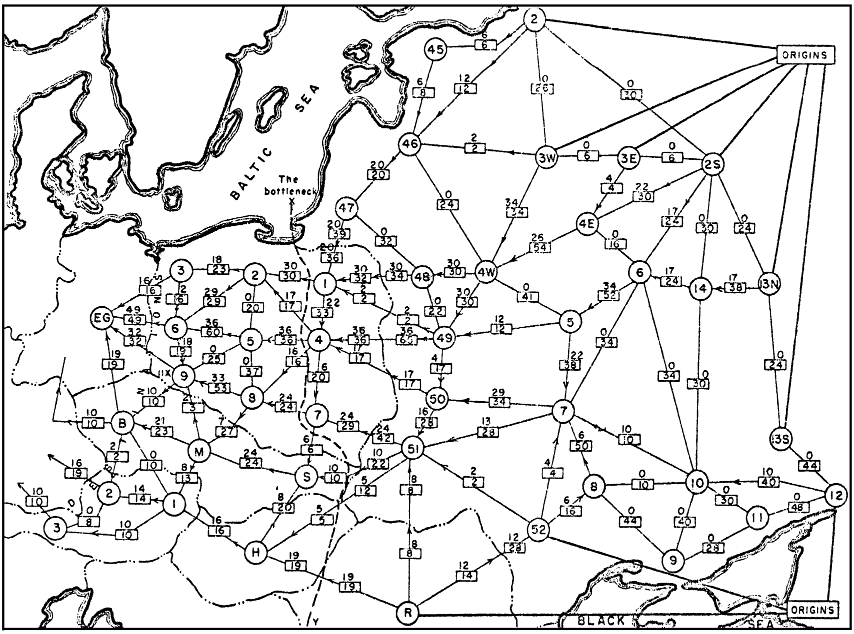

Figure 10.2

From Harris and Ross [1955]: Schematic diagram of the railway network of the Western Soviet Union and East European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as 'The bottleneck'.

The application of the flooding technique to the problem is displayed step by step in an appendix of the report, supported by several diagrams of the railway network. (Also work sheets are provided, to allow for future changes in capacities.) It yields a flow of value 163,000 tons from sources in the Soviet Union to destinations in East European 'satellite' countries, together with a cut with a capacity of, again, 163,000 tons. So the flow value and the cut capacity are equal, hence optimum. In the report, the minimum cut is indicated as 'the bottleneck' (Figure 10.2).

Further developments

Soon after the Harris-Ross report, Ford and Fulkerson [1955,1957b] presented in a RAND Report of 29 December 1955 their 'very simple algorithm' for the maximum flow problem, based on finding 'augmenting paths' as described in Section 10.4 above. The algorithm finds in a finite number of steps a maximum flow, if all capacities have integer values. We quote:

> This problem is of course a linear programming problem, and hence may be solved by Dantzig's simplex algorithm. In fact, the simplex computation for a problem of this kind is particularly efficient, since it can be shown that the sets of equations one solves in the process are always triangular [2]. However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm; it is, moreover, readily learned by a person with no special training, and may easily be mechanized for handling large networks. We believe that problems involving more than 500 nodes and 4,000 arcs are within reach of present computing machines.

(Reference [2] is Dantzig and Fulkerson [1955].)

In the RAND Report, Ford and Fulkerson [1955] mention that Boldyreff's flooding technique might give a good starting flow, but in the final paper (Ford and Fulkerson [1957b]) this suggestion has been omitted.

An alternative proof of the max-flow min-cut theorem was given by Elias, Feinstein, and Shannon [1956] ('manuscript received by the PGIT, July 11,1956'), who claimed that the result was known by workers in communication theory:

> This theorem may appear almost obvious on physical grounds and appears to have been accepted without proof for some time by workers in communication theory. However, while the fact that this flow cannot be exceeded is indeed almost trivial, the fact that it can actually be achieved is by no means obvious. We understand that proofs of the theorem have been given by Ford and Fulkerson and Fulkerson and Dantzig. The following proof is relatively simple, and we believe different in principle.

The proof of Elias, Feinstein, and Shannon is based on a reduction technique similar to that used by Menger [1927] in proving his theorem.

Chapter 11

Circulations and transshipments

Circulations and transshipments are variants of flows. Circulations have no source or sink — so flow conservation holds in each vertex — while transshipments have several sources and sinks — so *any* nonnegative function is a transshipment (however, the problem is to find a transshipment with prescribed excess function).
Problems on circulations and transshipments can be reduced to flow problems, or can be treated with similar methods.

11.1. A useful fact on arc functions

Recall that for any digraph $D = (V, A)$ and any $f : A \to \mathbb{R}$, the *excess* function $\text{excess}_f : V \to \mathbb{R}$ is defined by

$$\text{(11.1)} \qquad \text{excess}_f(v) := f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v))$$

for $v \in V$.

The following theorem of Gallai [1958a,1958b] will turn out to be useful in this chapter:

Theorem 11.1. *Let $D = (V, A)$ be a digraph and let $f : A \to \mathbb{R}_+$. Then f is a nonnegative linear combination of at most $|A|$ vectors χ^P, where P is a directed path or circuit. If P is a path, it starts at a vertex v with $\text{excess}_f(v) < 0$ and ends at a vertex with $\text{excess}_f(v) > 0$. If f is integer, we can take the linear combination integer-scalared. The combination can be found in $O(nm)$ time.*

Proof. We may assume that $\text{excess}_f = \mathbf{0}$, since we can add a new vertex u, for each $v \in V$ with $\text{excess}_f(v) > 0$, an arc (v, u), and for each $v \in V$ with $\text{excess}_f(v) < 0$, an arc (u, v). Define $f(v, u) := \text{excess}_f(v)$ and $f(u, v) := -\text{excess}_f(v)$ for any new arc (u, v) or (v, u). Then the new f satisfies $\text{excess}_f = \mathbf{0}$, and a decomposition for the new f gives a decomposition for the original f.

Define $A' := \{a \mid f(a) > 0\}$. We apply induction on $|A'|$. We may assume that $A' \neq \emptyset$. Then A' contains a directed circuit, C say. Let τ be the minimum of the $f(a)$ for $a \in AC$ and let $f' := f - \tau\chi^C$. Then the theorem follows by induction.

Since we can find C in $O(n)$ time, we can find the decomposition in $O(nm)$ time. ∎

11.2. Circulations

Let $D = (V, A)$ be a digraph. A function $f : A \to \mathbb{R}$ is called a *circulation* if

(11.2) $$f(\delta^{\text{in}}(v)) = f(\delta^{\text{out}}(v))$$

for each vertex $v \in V$. So now the flow conservation law holds in *each* vertex v. By Theorem 11.1,

(11.3) each nonnegative circulation is a nonnegative linear combination of incidence vectors of directed circuits; each nonnegative integer circulation is the sum of incidence vectors of directed circuits.

Hoffman [1960] mentioned that he proved the following characterization of the existence of circulations in 1956[19]:

Theorem 11.2 (Hoffman's circulation theorem)**.** *Let $D = (V, A)$ be a digraph and let $d, c : A \to \mathbb{R}$ with $d \leq c$. Then there exists a circulation f satisfying $d \leq f \leq c$ if and only if*

(11.4) $$d(\delta^{\text{in}}(U)) \leq c(\delta^{\text{out}}(U))$$

for each subset U of V. If moreover d and c are integer, f can be taken integer.

Proof. To see necessity of (11.4), suppose that a circulation f satisfying $d \leq f \leq c$ exists. Then for each $U \subseteq V$,

(11.5) $$d(\delta^{\text{in}}(U)) \leq f(\delta^{\text{in}}(U)) = f(\delta^{\text{out}}(U)) \leq c(\delta^{\text{out}}(U)).$$

To see sufficiency, choose a function f satisfying $d \leq f \leq c$ and minimizing $\|\text{excess}_f\|_1$. Let $S := \{v \in V \mid \text{excess}_f(v) > 0\}$ and $T := \{v \in V \mid \text{excess}_f(v) < 0\}$. Suppose that $S \neq \emptyset$. Let $D_f = (V, A_f)$ be the residual graph (defined in (10.9)). If D_f contains an $S-T$ path P, we can modify f along P so as to reduce $\|\text{excess}_f\|_1$. So D_f contains no $S-T$ path. Let U be the set of vertices reachable in D_f from S. Then for each $a \in \delta_A^{\text{out}}(U)$ we have $a \notin A_f$ and hence $f(a) = c(a)$. Similarly, for each $a \in \delta_A^{\text{in}}(U)$ we have $a^{-1} \notin A_f$ and hence $f(a) = d(a)$. Therefore,

(11.6) $$d(\delta^{\text{in}}(U)) - c(\delta^{\text{out}}(U)) = f(\delta^{\text{in}}(U)) - f(\delta^{\text{out}}(U)) = \text{excess}_f(U) = \text{excess}_f(S) > 0,$$

[19] In fact, A.J. Hoffman [1960] attributes it to A.H. Hoffman, but this is a misprint (A.J. Hoffman, personal communication 1995).

contradicting (11.4). ∎

(One can derive this theorem also from the max-flow min-cut theorem, with the methods described in Section 11.6 below.)

Theorem 11.2 implies that any circulation can be rounded:

Corollary 11.2a. *Let $D = (V, A)$ be a digraph and let $f : A \to \mathbb{R}$ be a circulation. Then there exists an integer circulation f' with $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$ for each arc a.*

Proof. Take $d := \lfloor f \rfloor$ and $c := \lceil f \rceil$ in Theorem 11.2. ∎

Another consequence is:

Corollary 11.2b. *Let $D = (V, A)$ be a digraph, let $k \in \mathbb{Z}_+$ (with $k \geq 1$), and let $f : A \to \mathbb{Z}$ be a circulation. Then $f = f_1 + \cdots + f_k$ where each f_i is an integer circulation satisfying*

(11.7) $\lfloor \frac{1}{k} f \rfloor \leq f_i \leq \lceil \frac{1}{k} f \rceil.$

Proof. By induction on k. Define $d := \lfloor \frac{1}{k} f \rfloor$ and $c := \lceil \frac{1}{k} f \rceil$. It suffices to show that there exists an integer circulation f_k such that

(11.8) $d \leq f_k \leq c$ and $(k-1)d \leq f - f_k \leq (k-1)c,$

equivalently,

(11.9) $$\max\{d(a), f(a) - (k-1)c(a)\} \leq f_k(a) \\ \leq \min\{c(a), f(a) - (k-1)d(a)\}$$

for each $a \in A$. Since these bounds are integer, by Corollary 11.2a it suffices to show that there is any circulation obeying these bounds. For that we can take $\frac{1}{k} f$. ∎

This corollary implies that the set of circulations f satisfying $d \leq f \leq c$ for some integer bounds d, c, has the integer decomposition property.

11.3. Flows with upper and lower bounds

We can derive from Corollary 11.2a that flows can be rounded similarly:

Corollary 11.2c. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let f be an $s-t$ flow of value $k \in \mathbb{Z}$. Then there exists an integer $s-t$ flow f' of value k with $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$ for each arc a.*

Proof. Add an arc (t, s) and define $f(t, s) := k$. We obtain a circulation to which we can apply Corollary 11.2a. Deleting the new arc, we obtain an $s-t$ flow as required. ∎

According to Berge [1958b], A.J. Hoffman showed the following on the existence of a flow obeying both an upper bound (capacity) and a lower bound (demand):

Corollary 11.2d. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $d, c : A \to \mathbb{R}_+$ with $d \leq c$. Then there exists an $s - t$ flow f with $d \leq f \leq c$ if and only if*

(11.10) $\quad c(\delta^{\text{out}}(U)) \geq d(\delta^{\text{in}}(U))$

for each $U \subseteq V$ not separating s and t. If moreover d and c are integer, f can be taken integer.

Proof. Necessity being direct, we show sufficiency. Identify s and t. By Theorem 11.2, there exists a circulation in the shrunk network. This gives a flow as required in the original network. ∎

(Berge [1958b] wrote that this result was shown by Hoffman with linear programming techniques, and was reduced to network theory by L.R. Ford, Jr.)

Moreover, a min-max relation for the maximum value of a flow obeying upper and lower bounds can be derived:

Corollary 11.2e. *Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $d, c : A \to \mathbb{R}_+$ with $d \leq c$, such that there exists an $s - t$ flow f with $d \leq f \leq c$. Then the maximum value of an $s - t$ flow f with $d \leq f \leq c$ is equal to the minimum value of*

(11.11) $\quad c(\delta^{\text{out}}(U)) - d(\delta^{\text{in}}(U))$

taken over $U \subseteq V$ with $s \in U$ and $t \notin U$. If d and c are integer, the maximum is attained by an integer flow f.

Proof. Let μ be the minimum value of (11.11). Add to D an arc (t, s), with $d(t, s) = c(t, s) = \mu$. Then the extended network has a circulation by Theorem 11.2. Indeed, condition (11.4) for U not separating s and t follows from (11.10). If $s \in U$, $t \notin U$, (11.4) follows from the definition of μ. If $s \notin U$, $t \in U$, then $\mu \geq \text{value}(f)$. Hence

(11.12)
$$\begin{aligned} \mu + c(\delta^{\text{out}}(U)) - d(\delta^{\text{in}}(U)) &\geq \mu + f(\delta^{\text{out}}(U)) - f(\delta^{\text{in}}(U)) \\ &= \mu - \text{value}(f) \geq 0. \end{aligned}$$

Therefore, the original network has a flow as required. ∎

11.4. b-transshipments

Let $D = (V, A)$ be a digraph and let $b \in \mathbb{R}^V$. A function $f : A \to \mathbb{R}$ is called a *b-transshipment* if $\text{excess}_f = b$. (So each function $f : A \to \mathbb{R}$ is a

b-transshipment for some b. excess$_f$ is defined in Section 10.1.) By reduction to Hoffman's circulation theorem, one may characterize the existence of a b-transshipment obeying given upper and lower bounds on the arcs:

Corollary 11.2f. *Let $D = (V, A)$ be a digraph, let $d, c : A \to \mathbb{R}$ with $d \leq c$, and let $b : V \to \mathbb{R}$ with $b(V) = 0$. Then there exists a b-transshipment f with $d \leq f \leq c$ if and only if*

$$c(\delta^{\text{in}}(U)) - d(\delta^{\text{out}}(U)) \geq b(U) \tag{11.13}$$

for each $U \subseteq V$. If moreover b, c, and d are integer, f can be taken integer.

Proof. The corollary can be reduced to Hoffman's circulation theorem (Theorem 11.2). Add a new vertex u, and for each $v \in V$ an arc (v, u) with $d(v, u) := c(v, u) := b(v)$. Then a function f as required exists if and only if the extended graph has a circulation f' satisfying $d \leq f' \leq c$. The condition in Hoffman's circulation theorem is equivalent to (11.13). ∎

Conversely, Hoffman's circulation theorem is the special case $b = \mathbf{0}$. The special case $d = \mathbf{0}$ is the following result of Gale [1956,1957]:

Corollary 11.2g (Gale's theorem)**.** *Let $D = (V, A)$ be a digraph and let $c : A \to \mathbb{R}$ and $b : V \to \mathbb{R}$ with $b(V) = 0$. Then there exists a b-transshipment f satisfying $\mathbf{0} \leq f \leq c$ if and only if*

$$c(\delta^{\text{in}}(U)) \geq b(U) \tag{11.14}$$

for each $U \subseteq V$. If moreover b and c are integer, f can be taken integer.

Proof. Take $d = \mathbf{0}$ in Corollary 11.2f. ∎

The proof of Gale is by reduction to the max-flow min-cut theorem. Conversely, the max-flow min-cut theorem follows easily: if $s, t \in V$ and ϕ is the minimum capacity of an $s - t$ cut, then set $b(s) := -\phi$, $b(t) := \phi$, and $b(v) := 0$ for each $v \neq s, t$. As (11.14) is satisfied, by Gale's theorem there exists a b-transshipment f with $\mathbf{0} \leq f \leq c$. This is an $s - t$ flow of value ϕ.

Taking $c = \infty$ in Gale's theorem gives the following result of Rado [1943]:

Corollary 11.2h. *Let $D = (V, A)$ be a digraph and let $b : V \to \mathbb{R}$ with $b(V) = 0$. Then there exists a b-transshipment $f \geq \mathbf{0}$ if and only if $b(U) \leq 0$ for each $U \subseteq V$ with $\delta^{\text{in}}(U) = \emptyset$.*

Proof. This is Gale's theorem (Corollary 11.2g) for $c = \infty$. ∎

11.5. Upper and lower bounds on excess$_f$

Instead of equality constraints on excess$_f$ one may put upper and lower bounds b and a. This has the following characterization:

Corollary 11.2i. *Let $D = (V, A)$ be a digraph, let $d, c : A \to \mathbb{R}$ with $d \leq c$, and let $a, b : V \to \mathbb{R}$ with $a \leq b$. Then there exists a z-transshipment f with $d \leq f \leq c$ for some z with $a \leq z \leq b$ if and only if*

(11.15) $$c(\delta^{\text{in}}(U)) - d(\delta^{\text{out}}(U)) \geq \max\{a(U), -b(V \setminus U)\}$$

for each $U \subseteq V$. If moreover a, b, c, and d are integer, f can be taken integer.

Proof. The corollary can be reduced to Hoffman's circulation theorem: Add a new vertex u, and for each $v \in V$ an arc (v, u) with $d(v, u) := a(v)$ and $c(v, u) := b(v)$. Then a function f as required exists if and only if the extended graph has a circulation f' satisfying $d \leq f' \leq c$. ∎

This characterization can be formulated equivalently as:

Corollary 11.2j. *Let $D = (V, A)$ be a digraph, let $d, c : A \to \mathbb{R}$ with $d \leq c$, and let $a, b : V \to \mathbb{R}$ with $a \leq b$. Then there exists a z-transshipment f with $d \leq f \leq c$ for some z with $a \leq z \leq b$ if and only if there exists a z-transshipment f' with $d \leq f' \leq c$ for some $z \geq a$ and there exists a z-transshipment f'' with $d \leq f'' \leq c$ for some $z \leq b$.*

Proof. Directly from Corollary 11.2i, since (11.15) can be split into a condition on a and one on b. ∎

For $d = \mathbf{0}$, Corollary 11.2i gives a result of Fulkerson [1959a]:

Corollary 11.2k. *Let $D = (V, A)$ be a digraph, let $c : A \to \mathbb{R}_+$, and let $a, b : V \to \mathbb{R}$ with $a \leq b$. Then there exists a z-transshipment f satisfying $\mathbf{0} \leq f \leq c$, for some z with $a \leq z \leq b$ if and only if*

(11.16) $$c(\delta^{\text{in}}(U)) \geq \max\{a(U), -b(V \setminus U)\}$$

for each $U \subseteq V$. If moreover a, b, and c are integer, f can be taken integer.

Proof. Directly from Corollary 11.2i, taking $d = \mathbf{0}$. ∎

11.6. Finding circulations and transshipments algorithmically

Algorithmic and complexity results for circulations and transshipments follow directly from those for the maximum flow problem, by the following construction.

Let $D = (V, A)$ be a digraph and let $d, c : A \to \mathbb{Q}$ with $d \leq c$. Then a circulation f satisfying $d \leq f \leq c$ can be found as follows. Give each arc a a new capacity

(11.17) $$c'(a) := c(a) - d(a).$$

Add two new vertices s and t. For each $v \in V$ with $\text{excess}_d(v) > 0$, add an arc (s, v) with capacity $c'(s, v) := \text{excess}_d(v)$. For each $v \in V$ with $\text{excess}_d(v) < 0$, add an arc (v, t) with capacity $c'(v, t) := -\text{excess}_d(v)$. This makes the extended graph $D' = (V', A')$.

Then D has a circulation f satisfying $d \leq f \leq c$ if and only if D' has an $s - t$ flow $f' \leq c'$ of value

$$\text{(11.18)} \qquad \sum_{\substack{v \in V \\ \text{excess}_d(v) > 0}} \text{excess}_d(v)$$

(by taking $f(a) = f'(a) + d(a)$ for each $a \in A$).

This yields:

Theorem 11.3. *If a maximum flow can be found in time* $\text{MF}(n, m)$, *then a circulation can be found in time* $O(\text{MF}(n, m))$.

Proof. Apply the above construction. ∎

If, in addition, functions $a, b : V \to \mathbb{Q}$ are given, we can reduce the problem of finding a transshipment f satisfying $d \leq f \leq c$ and $a \leq \text{excess}_f \leq b$ to finding a circulation in a slightly larger graph — see the proof of Corollary 11.2i. This gives:

Corollary 11.3a. *If a maximum flow can be found in time* $\text{MF}(n, m)$, *then (given* a, b, d, c*) a* z*-transshipment* f *satisfying* $d \leq f \leq c$ *and* $a \leq z \leq b$ *can be found in time* $O(\text{MF}(n, m))$.

Proof. Reduce the problem with the construction of Corollary 11.2i to the circulation problem, and use Theorem 11.3. ∎

11.6a. Further notes

The results on flows, circulations, and transshipments extend directly to the case where also each vertex has an upper and/or lower bound on the amount of flow traversing that vertex. We can reduce this to the cases considered above by splitting any vertex v into two vertices v' and v'', adding an arc from v' to v'' with bounds equal to the vertex bounds, and replacing any arc (u, v) by (u'', v').

The results of this chapter also apply to characterizing the existence of a subgraph $D' = (V, A')$ of a given graph $D = (V, A)$, where D' has prescribed bounds on the indegrees and outdegrees (cf. Hakimi [1965]).

Chapter 12

Minimum-cost flows and circulations

Minimum-cost flows can be seen to generalize both shortest path and maximum flow. A shortest $s-t$ path can be deduced from a minimum-cost $s-t$ flow of value 1, while a maximum $s-t$ flow is a minimum-cost $s-t$ flow if we take cost -1 on arcs leaving s and 0 on all other arcs (assuming no arc enters s).
Minimum-cost flows, circulations, and transshipments are closely related, and when describing algorithms, we will choose the most suitable variant.
In this chapter, graphs can be assumed to be simple.

12.1. Minimum-cost flows and circulations

Let $D = (V, A)$ be a digraph and let $k : A \to \mathbb{R}$, called the *cost* function. For any function $f : A \to \mathbb{R}$, the *cost* of f is, by definition,

(12.1) $$\operatorname{cost}(f) := \sum_{a \in A} k(a) f(a).$$

The *minimum-cost $s-t$ flow problem* is: given a digraph $D = (V, A)$, $s, t \in V$, a 'capacity' function $c : A \to \mathbb{Q}_+$, a 'cost' function $k : A \to \mathbb{Q}$, and a value $\phi \in \mathbb{Q}_+$, find an $s-t$ flow $f \leq c$ of value ϕ that minimizes $\operatorname{cost}(f)$. This problem includes the problem of finding a maximum-value $s-t$ flow that has minimum cost among all maximum-value $s-t$ flows.

Related is the *minimum-cost circulation problem:* given a digraph $D = (V, A)$, a 'demand' function $d : A \to \mathbb{Q}$, a 'capacity' function $c : A \to \mathbb{Q}$, and a 'cost' function $k : A \to \mathbb{Q}$, find a circulation f subject to $d \leq f \leq c$, minimizing $\operatorname{cost}(f)$.

One can easily reduce the minimum-cost flow problem to the minimum-cost circulation problem: just add an arc $a_0 := (t, s)$ with $d(a_0) := c(a_0) = \phi$ and $k(a_0) := 0$. Also, let $d(a) := 0$ for each arc $a \neq a_0$. Then a minimum-cost circulation in the extended digraph gives a minimum-cost flow of value ϕ in the original digraph.

Also the problem of finding a maximum-value $s-t$ flow can be reduced easily to a minimum-cost circulation problem in the extended digraph: now define $d(a_0) := 0$, $c(a_0) := \infty$, and $k(a_0) := -1$. Moreover, set $k(a) := 0$ for each $a \neq a_0$. Then a minimum-cost circulation gives a maximum-value $s-t$ flow.

Edmonds and Karp [1970,1972] showed that the minimum-cost circulation problem is solvable in polynomial time. Their algorithm is based on a technique called *capacity-scaling* and can be implemented to run in $O(m(m + n \log n) \log C)$ time, where $C := \|c\|_\infty$ (assuming c integer). So it is weakly polynomial-time. They raised the question of the existence of a *strongly* polynomial-time algorithm.

Tardos [1985a] answered this question positively. Her algorithm has resulted in a stream of further research on strongly polynomial-time algorithms for the minimum-cost circulation problem. It stood at the basis of the strongly polynomial-time algorithms discussed in this chapter.

12.2. Minimum-cost circulations and the residual graph D_f

It will be useful again to consider the residual graph $D_f = (V, A_f)$ of $f : A \to \mathbb{R}$ (with respect to d and c), where

(12.2) $$A_f := \{a \mid a \in A, f(a) < c(a)\} \cup \{a^{-1} \mid a \in A, f(a) > d(a)\}.$$

Here $a^{-1} := (v, u)$ if $a = (u, v)$.

We extend any cost function k to A^{-1} by defining the *cost* $k(a^{-1})$ of a^{-1} by:

(12.3) $$k(a^{-1}) := -k(a)$$

for each $a \in A$.

We also use the following notation. Any directed circuit C in D_f gives an undirected circuit in $D = (V, A)$. We define $\chi^C \in \mathbb{R}^A$ by:

(12.4) $$\chi^C(a) := \begin{cases} 1 & \text{if } C \text{ traverses } a, \\ -1 & \text{if } C \text{ traverses } a^{-1}, \\ 0 & \text{if } C \text{ traverses neither } a \text{ nor } a^{-1}, \end{cases}$$

for $a \in A$.

Given $D = (V, A)$, $d, c : A \to \mathbb{R}$, a circulation f in D is called *feasible* if $d \leq f \leq c$. The following observation is fundamental[20]:

Theorem 12.1. *Let $D = (V, A)$ be a digraph and let $d, c, k : A \to \mathbb{R}$. Let $f : A \to \mathbb{R}$ be a feasible circulation. Then f has minimum cost among all feasible circulations if and only if each directed circuit of D_f has nonnegative cost.*

[20] The idea goes back to Tolstoǐ [1930,1939] (for the transportation problem), and was observed also by Robinson [1949,1950] (for the transportation problem), Gallai [1957, 1958b], Busacker and Gowen [1960], Fulkerson [1961], and Klein [1967].

Proof. *Necessity.* Let C be a directed circuit in D_f of negative cost. Then for small enough $\varepsilon > 0$, $f' := f + \varepsilon\chi^C$ is again a circulation satisfying $d \leq f' \leq c$. Since $\text{cost}(f') < \text{cost}(f)$, f is not minimum-cost.

Sufficiency. Suppose that each directed circuit in D_f has nonnegative cost. Let f' be any feasible circulation. Then $f' - f$ is a circulation, and hence

(12.5) $$f' - f = \sum_{j=1}^{m} \lambda_j \chi^{C_j}$$

for some directed circuits $C_1, \ldots, C_m$ in D_f and $\lambda_1, \ldots, \lambda_m > 0$. Hence

(12.6) $$\text{cost}(f') - \text{cost}(f) = \text{cost}(f' - f) = \sum_{j=1}^{m} \lambda_j k(C_j) \geq 0.$$

So $\text{cost}(f') \geq \text{cost}(f)$. ∎

This directly implies a strong result: optimality of a given feasible circulation f can be checked in polynomial time, namely in time $O(nm)$ (with the Bellman-Ford method). It also implies the following good characterization (Gallai [1957,1958b], Ford and Fulkerson [1962]; for the simpler case of a symmetric cost function satisfying the triangle inequality it was shown by Kantorovich [1942], Kantorovich and Gavurin [1949], and Koopmans and Reiter [1951]):

Corollary 12.1a. *Let $D = (V, A)$ be a digraph, let $d, c, k : A \to \mathbb{R}$, and let f be a feasible circulation. Then f is minimum-cost if and only if there exists a function $p : V \to \mathbb{R}$ such that*

(12.7) $$\begin{aligned} k(a) &\geq p(v) - p(u) \text{ if } f(a) < c(a),\\ k(a) &\leq p(v) - p(u) \text{ if } f(a) > d(a), \end{aligned}$$

for each arc $a = (u, v) \in A$.

Proof. Directly from Theorem 12.1 with Theorem 8.2. ∎

From this characterization, a min-max relation for minimum-cost circulations can be derived — see Section 12.5b. It also follows directly from the duality theorem of linear programming — see Chapter 13.

12.3. Strongly polynomial-time algorithm

Theorem 12.1 gives us a method to improve a given circulation f:

(12.8) Choose a negative-cost directed circuit C in the residual graph D_f, and reset $f := f + \tau\chi^C$ where τ is maximal subject to $d \leq f \leq c$. If no such directed circuit exists, f is a minimum-cost circulation.

It is not difficult to see that for rational data this leads to a finite algorithm.

However, if we just select circuits in an arbitrary fashion, the algorithm may take exponential time, as follows by application to the maximum flow problem given in Figure 10.1 (by adding an arc from t to s of cost -1). Zadeh [1973a,1973b] showed that several strategies of selecting a circuit do not lead to a strongly polynomial-time algorithm.

Goldberg and Tarjan [1988b,1989] were able to prove that one obtains a strongly polynomial-time algorithm if one chooses in (12.8) a directed circuit C of minimum mean cost, that is, one minimizing

(12.9) $$\frac{k(C)}{|C|}.$$

(We identify C with its set AC of arcs.) In Corollary 8.10a we saw that such a circuit can be found in time $O(nm)$.

Note that if we formulate a maximum $s-t$ flow problem as a minimum-cost circulation problem, by adding an arc (t,s) of cost -1 and capacity $+\infty$, then the minimum-mean cost cycle-cancelling algorithm reduces to the shortest augmenting path method of Dinits [1970] and Edmonds and Karp [1972] (Section 10.5).

We now prove the result of Goldberg and Tarjan (as usual, $n := |V|$, $m := |A|$):

Theorem 12.2. *Choosing a minimum-mean cost cycle C in* (12.8)*, the number of iterations is at most* $4nm^2\lceil \ln n\rceil$*.*

Proof. Let $f_0, f_1, \ldots$ be the circulations found. For each $i \geq 0$, define $A_i := A_{f_i}$, let ε_i be *minus* the minimum of (12.9) in (V, A_i), and let C_i be the directed circuit in A_i chosen to obtain f_{i+1} (taking circuits as arc sets). So

(12.10) $$k(C_i) = -\varepsilon_i|C_i|.$$

ε_i is the smallest value such that if we would add ε_i to the cost of each arc of A_i, then each directed circuit has nonnegative cost. So ε_i is the smallest value for which there exists a function $p_i : V \to \mathbb{Q}$ such that

(12.11) $$k(a) + \varepsilon_i \geq p_i(v) - p_i(u) \text{ for each } a = (u,v) \in A_i.$$

The proof of the theorem is based on the following two facts on the decrease of the ε_i:

(12.12) $$\text{(i) } \varepsilon_{i+1} \leq \varepsilon_i \text{ and (ii) } \varepsilon_{i+m} \leq (1 - \tfrac{1}{n})\varepsilon_i$$

(assuming in (ii) that we reach iteration $i+m$). To prove (12.12), we may assume that $i = 0$ and $p_0 = \mathbf{0}$. Then $k(a) \geq -\varepsilon_0$ for each $a \in A_0$, with equality if $a \in C_0$.

Since $A_1 \subseteq A_0 \cup C_0^{-1}$ and since each arc in C_0^{-1} has cost $\varepsilon_0 \geq 0$, we know that $k(a) \geq -\varepsilon_0$ for each $a \in A_1$. Hence $\varepsilon_1 \leq \varepsilon_0$. This proves (12.12)(i).

To prove (ii), we may assume $\varepsilon_m > 0$. We first show that at least one of the directed circuits $C_0, \ldots, C_{m-1}$ contains an arc a with $k(a) \geq 0$. Otherwise each A_h arises from A_{h-1} by deleting at least one negative-cost arc and adding only positive-cost arcs. This implies that A_m contains no negative-cost arc, and hence f_m has minimum cost; so $\varepsilon_m = 0$, contradicting our assumption.

Let h be the smallest index such that C_h contains an arc a with $k(a) \geq 0$. So all negative-cost arcs in C_h also belong to A_0, and hence have cost at least $-\varepsilon_0$. So $k(C_h) \geq -(|C_h|-1)\varepsilon_0$ and therefore $\varepsilon_h = -k(C_h)/|C_h| \leq (1-\frac{1}{n})\varepsilon_0$. This proves (12.12).

Now define

(12.13) $$t := 2nm\lceil \ln n \rceil.$$

Then by (12.12)(ii):

(12.14) $$\varepsilon_t \leq (1-\frac{1}{n})^{2n\lceil \ln n\rceil}\varepsilon_0 < \varepsilon_0/2n,$$

since $(1-\frac{1}{n})^n < e^{-1}$ and $e^{-2\ln n} = n^{-2} \leq \frac{1}{2n}$.

We finally show that for each i there exists an arc a in C_i such that $a \notin C_h$ for each $h \geq i+t$. Since $|A \cup A^{-1}| = 2m$, this implies that the number of iterations is at most $4mt$, as required.

To prove this, we may assume that $i = 0$ and that $p_t = \mathbf{0}$. As $k(C_0) = -\varepsilon_0|C_0|$, C_0 contains an arc a_0 with $k(a_0) \leq -\varepsilon_0 < -2n\varepsilon_t$. Without loss of generality, $a_0 \in A$.

Suppose that $f_h(a_0) \neq f_t(a_0)$ for some $h > t$. Since $k(a_0) < -2n\varepsilon_t \leq -\varepsilon_t$, we have that $a_0 \notin A_t$ (by (12.11)). So $f_t(a_0) = c(a_0)$, and hence $f_h(a_0) < f_t(a_0)$. Then, by (11.3) applied to $f_t - f_h$, A_h has a directed circuit C containing a_0 such that A_t contains C^{-1}. By (12.11), $-k(a) = k(a^{-1}) \geq -\varepsilon_t$ for each $a \in C$. This gives (using (12.12)(i)):

(12.15) $$\begin{aligned} k(C) &= k(a_0) + k(C \setminus \{a_0\}) < -2n\varepsilon_t + (|C|-1)\varepsilon_t \leq -n\varepsilon_t \\ &\leq -n\varepsilon_h \leq -|C|\varepsilon_h, \end{aligned}$$

contradicting the definition of ε_h. ∎

This gives for finding a minimum-cost circulation:

Corollary 12.2a. *A minimum-cost circulation can be found in $O(n^2m^3\log n)$ time. If d and c are integer, an integer minimum-cost circulation is found.*

Proof. Directly from Theorem 12.2 and Corollary 8.10a. Note that if d and c are integer and we start with an integer circulation f_0, all further circulations obtained by (12.8) are integer. ∎

So we have the theorem of Tardos [1985a]:

Corollary 12.2b. *A minimum-cost circulation can be found in strongly polynomial time. If d and c are integer, an integer circulation is found.*

Proof. Directly from Corollary 12.2a. ∎

Notes. Goldberg and Tarjan [1988b,1989] showed that, with the help of dynamic trees, the running time of the minimum-mean cost cycle-cancelling method can be improved to $O(nm \log n \min\{\log(nK), m \log n\})$, where $K := \|k\|_\infty$, assuming k to be integer.

Weintraub [1974] showed that if we take always a directed circuit C in D_f such that, by resetting f to $f + \tau\chi^C$ as in (12.8), the cost decreases most, then the number of iterations (12.8) is polynomially bounded. However, finding such a circuit is NP-complete (finding a Hamiltonian circuit in a directed graph is a special case). Weintraub [1974] also proposed a heuristic of finding a short (negative) circuit by finding a minimum-cost set of vertex-disjoint circuits in D_f (by solving an assignment problem), and choosing the shortest among them. Barahona and Tardos [1989] showed that this also leads to a (weakly) polynomial-time algorithm.

12.4. Related problems

Corollary 12.2b concerns solving the minimization problem in the following LP-duality equation, where M denotes the $V \times A$ incidence matrix of D:

(12.16)
$$\begin{aligned}&\min\{k^\mathsf{T} x \mid d \leq x \leq c, Mx = \mathbf{0}\}\\ &= \max\{z_1^\mathsf{T} d - z_2^\mathsf{T} c \mid z_1, z_2 \geq \mathbf{0}, \exists y : z_1^\mathsf{T} - z_2^\mathsf{T} + y^\mathsf{T} M = k^\mathsf{T}\}.\end{aligned}$$

It implies that also the maximization problem can be solved in strongly polynomial time:

Corollary 12.2c. *The maximization problem in* (12.16) *can be solved in strongly polynomial time. If k is integer, an integer optimum solution is found.*

Proof. Let x be an optimum solution of the minimization problem, that is, a minimum-cost circulation. Since x is extreme, the digraph D_x has no negative-cost directed circuits. Hence we can find a function ('potential') $y : V \to \mathbb{Q}$ such that $k(u, v) \geq y(v) - y(u)$ if $x(u, v) < c(u, v)$ and $k(u, v) \leq y(v) - y(u)$ if $x(u, v) > d(u, v)$, in strongly polynomial time (Theorem 8.7). If k is integer, we find an integer y.

Let z_1 and z_2 be the unique vectors with $z_1, z_2 \geq \mathbf{0}$, $z_1^\mathsf{T} - z_2^\mathsf{T} = k^\mathsf{T} - y^\mathsf{T} M$ and $z_1(a)z_2(a) = 0$ for each $a \in A$. So $z_1(a) = 0$ if $x(a) > d(a)$ and $z_2(a) = 0$ if $x(a) < c(a)$. Hence

(12.17)
$$z_1^\mathsf{T} d - z_2^\mathsf{T} c = z_1^\mathsf{T} x - z_2^\mathsf{T} x = (k^\mathsf{T} - y^\mathsf{T} M)x = k^\mathsf{T} x.$$

So z_1, z_2 form an optimum solution for the maximization problem. ∎

By an easy construction, Corollary 12.2b implies that a more general problem is solvable in strongly polynomial time:

(12.18) input: a digraph $D = (V, A)$ and functions $a, b : V \to \mathbb{Q}$ and $d, c, k : A \to \mathbb{Q}$, find: a z-transshipment x with $a \leq z \leq b$ and $d \leq x \leq c$, minimizing $k^{\mathsf{T}}x$.

Corollary 12.2d. *Problem* (12.18) *is solvable in strongly polynomial time. If a, b, d, and c are integer, an integer optimum solution is found.*

Proof. Extend D by a new vertex u and arcs (v, u) for each $v \in V$. Extend d, c, and k by defining $d(v, u) := a(v)$, $c(v, u) := b(v)$ and $k(v, u) := 0$ for each $v \in V$. By Corollary 12.2b, we can find a minimum-cost circulation x with $d \leq x \leq c$ in the extended digraph in strongly polynomial time. It gives a b-transshipment in the original graph as required. ∎

For later reference, we derive that also the dual problem (in the LP sense) can be solved in strongly polynomial time. If M denotes $V \times A$ incidence matrix of D, problem (12.18) corresponds to the minimum in the LP-duality equation:

(12.19)
$$\begin{aligned}&\min\{k^{\mathsf{T}}x \mid d \leq x \leq c, a \leq Mx \leq b\}\\ &= \max\{y_1^{\mathsf{T}}b - y_2^{\mathsf{T}}a + z_1^{\mathsf{T}}d - z_2^{\mathsf{T}}c \mid y_1, y_2, z_1, z_2 \geq \mathbf{0},\\ &\qquad (y_1 - y_2)^{\mathsf{T}}M + (z_1 - z_2)^{\mathsf{T}} = k^{\mathsf{T}}\}.\end{aligned}$$

Corollary 12.2e. *An optimum solution for the maximum in* (12.19) *can be found in strongly polynomial time. If k is integer, we find an integer optimum solution.*

Proof. By reduction to Corollary 12.2c, using a reduction similar to that given in the proof of Corollary 12.2d. ∎

12.4a. A dual approach

The approach above consists of keeping a feasible circulation, and throughout improving its cost. A dual approach can best be described in terms of b-transshipments: we keep a b'-transshipment f such that D_f has no negative-cost directed circuits, and improve b' until $b' = b$. This can be studied with the concept of 'extreme function'.

Let $D = (V, A)$ be a digraph and let $d, c, k : A \to \mathbb{R}$ be given, the lower bound function, the capacity function, and the cost function, respectively. Let $f : A \to \mathbb{R}$ be such that $d \leq f \leq c$. We call f *extreme* if $\text{cost}(f') \geq \text{cost}(f)$ for each function f' satisfying $d \leq f' \leq c$ and $\text{excess}_{f'} = \text{excess}_f$; in other words, setting $b := \text{excess}_f$, f is a minimum-cost b-transshipment subject to $d \leq f \leq c$. (excess_f is defined in Section 10.1.)

Note that the concept of extreme depends on k, d, and c. So it might be better to define a function to be extreme *with respect to* k, d, and c. However, when considering extreme functions f, the functions k, d, and c are generally fixed, or follow from the context.

Again it will be useful to consider the residual graph $D_f = (V, A_f)$ of f (with respect to d and c), where

(12.20) $$A_f := \{a \mid a \in A, f(a) < c(a)\} \cup \{a^{-1} \mid a \in A, f(a) > d(a)\}.$$

Here $a^{-1} := (v, u)$ if $a = (u, v)$.

We extend k to $A^{-1} := \{a^{-1} \mid a \in A\}$ by defining

(12.21) $$k(a^{-1}) := -k(a)$$

for each $a \in A$. We call $k(a^{-1})$ the *cost* of a^{-1}.

We also use the following notation. Any directed path P in D_f gives an undirected path in $D = (V, A)$. We define $\chi^P \in \mathbb{R}^A$ by:

(12.22) $$\chi^P(a) := \begin{cases} 1 & \text{if } P \text{ traverses } a, \\ -1 & \text{if } P \text{ traverses } a^{-1}, \\ 0 & \text{if } P \text{ traverses neither } a \text{ nor } a^{-1}, \end{cases}$$

for $a \in A$.

Theorem 12.1 for minimum-cost circulations can be directly extended to extreme functions:

Theorem 12.3. *Let $D = (V, A)$ be a digraph and let $d, c, k, f : A \to \mathbb{R}$ with $d \le f \le c$. Then f is extreme if and only if each directed circuit of D_f has nonnegative cost.*

Proof. Like the proof of Theorem 12.1. ∎

This implies that the optimality of a given feasible solution f of a b-transshipment problem can be checked in polynomial time, namely in time $O(nm)$ (with the Bellman-Ford method). It also implies the following good characterization (Kantorovich [1942], Gallai [1957,1958b], Ford and Fulkerson [1962]):

Corollary 12.3a. *Let $D = (V, A)$ be a digraph and let $d, c, k, f : A \to \mathbb{R}$ with $d \le f \le c$. Then f is extreme if and only if there exists a function $p : V \to \mathbb{R}$ such that*

(12.23) $$\begin{aligned} k(a) &\ge p(v) - p(u) \text{ if } f(a) < c(a), \\ k(a) &\le p(v) - p(u) \text{ if } f(a) > d(a), \end{aligned}$$

for each arc $a = (u, v) \in A$.

Proof. Directly from Theorem 12.3 with Theorem 8.2. ∎

As for the algorithmic side, the following observation (Jewell [1958], Busacker and Gowen [1960], Iri [1960]) is very useful in analyzing algorithms ('This theorem may properly be regarded as the central one concerning minimal cost flows' — Ford and Fulkerson [1962]):

Theorem 12.4. *Let $D = (V, A)$ be a digraph and let $d, c, k, f : A \to \mathbb{R}$ with $d \le f \le c$ and with f extreme. Let P be a minimum-cost $s-t$ path in D_f, for some $s, t \in V$, and let $\varepsilon > 0$ be such that $f' := f + \varepsilon\chi^P$ satisfies $d \le f' \le c$. Then f' is extreme again.*

Proof. Let f'' satisfy $d \le f'' \le c$ and $\text{excess}_{f''} = \text{excess}_{f'}$. Then by Theorem 11.1,

$$f'' - f = \sum_{i=1}^{n} \mu_i \chi^{P_i} + \sum_{j=1}^{m} \lambda_j \chi^{C_j}, \tag{12.24}$$

where $P_1, \ldots, P_n$ are $s-t$ paths in D_f, $C_1, \ldots, C_m$ are directed circuits in D_f, $\mu_1, \ldots, \mu_n > 0$, and $\lambda_1, \ldots, \lambda_m > 0$, with $\sum_i \mu_i = \varepsilon$. Then

$$\text{cost}(f'' - f) = \sum_i \mu_i \cdot \text{cost}(P_i) + \sum_j \lambda_j \cdot \text{cost}(C_j) \geq \sum_i \mu_i \tau = \varepsilon\tau, \tag{12.25}$$

where $\tau := \text{cost}(P)$. As $\text{cost}(f' - f) = \varepsilon\tau$, we have $\text{cost}(f'' - f) \geq \text{cost}(f' - f)$, and therefore $\text{cost}(f'') \geq \text{cost}(f')$. ∎

We will refer to updating f to $f + \varepsilon\chi^P$ as in Theorem 12.4 as to *sending a flow of value ε over P.*

Also the following observation is useful in algorithms (Edmonds and Karp [1970], Tomizawa [1971]):

Theorem 12.5. *In Theorem 12.4, if p is a potential for D_f such that $p(t) - p(s) = \text{dist}_k(s,t)$, then p is also a potential for $D_{f'}$.*

Proof. Choose $a = (u, v) \in A_{f'}$. If $a \in A_f$, then $p(v) \leq p(u) + k(a)$. If $a \notin A_f$, then a^{-1} is traversed by P, and hence $p(u) = p(v) + k(a^{-1}) = p(v) - k(a)$. Therefore $p(v) \leq p(u) + k(a)$. ∎

These theorems lead to the following minimum-cost $s-t$ flow algorithm due to Ford and Fulkerson [1958b], Jewell [1958], Busacker and Gowen [1960], and Iri [1960] (an equivalent 'primal-dual' algorithm was given by Fujisawa [1959]).

Let be given $D = (V, A)$, $s, t \in V$, and $c, k : A \to \mathbb{Q}_+$, the capacity and cost function, respectively.

Algorithm for minimum-cost $s-t$ flow
Starting with $f = \mathbf{0}$ apply the following iteratively:
Iteration: Let P be an $s-t$ path in D_f minimizing $k(P)$. Reset $f := f + \varepsilon\chi^P$, where ε is maximal subject to $\mathbf{0} \leq f + \varepsilon \cdot \chi^P \leq c$.

Termination of this algorithm for rational capacities follows similarly as for the maximum flow algorithm (Theorem 10.4).

One may use the Bellman-Ford method to obtain the path P, since D_f has no negative-cost directed circuits (by Theorems 12.3 and 12.4). However, using a trick of Edmonds and Karp [1970] and Tomizawa [1971], one can use Dijkstra's algorithm, since by Theorem 12.5 we can maintain a potential that makes all lengths (= costs) nonnegative. This leads to the following theorem (where $\text{SP}_+(n, m, K)$ denotes the time needed to find a shortest path in a digraph with n vertices, m arcs, and *nonnegative* integer lengths, each at most K):

Theorem 12.6. *For $c, k : A \to \mathbb{Z}_+$ and $\phi \in \mathbb{Z}_+$, a minimum-cost $s-t$ flow $f \leq c$ of value ϕ can be found in time $O(\phi \cdot \text{SP}_+(n, m, K))$, where $K := \|k\|_\infty$.*

Proof. Note that each iteration consists of finding a shortest path in D_f. Simultaneously we can find a potential for D_f satisfying $p(t) - p(s) = \text{dist}_k(s,t)$. Since by Theorem 12.5, p is a potential also for $D_{f'}$, we can perform each iteration in time $\text{SP}_+(n, m, K)$. ∎

12.4b. A strongly polynomial-time algorithm using capacity-scaling

The algorithm given in Theorem 12.6 is not polynomial-time, but several improvements leading to a polynomial-time algorithm have been found. Orlin [1988,1993] gave the currently fastest strongly polynomial-time algorithm for minimum-cost circulation, which is based on this dual approach: while keeping an extreme b'-transshipment, it throughout improves b', until $b' = b$. This implies an algorithm for minimum-cost circulation.

Let $D = (V, A)$ be a digraph, let $k : A \to \mathbb{Q}_+$ be a cost function, and let $b : V \to \mathbb{Q}$ be such that there exists a nonnegative b-transshipment. For any $f : A \to \mathbb{Q}$ define $\mathrm{def}_f : V \to \mathbb{Q}$ by

(12.26) $\qquad \mathrm{def}_f := b - \mathrm{excess}_f.$

So $\mathrm{def}_f(v)$ is the 'deficiency' of f at v. Then $\mathrm{def}_f(V) = b(V) - \mathrm{excess}_f(V) = 0$.

The algorithm determines a sequence of functions $f_i : A \to \mathbb{Q}_+$ and rationals β_i $(i = 0, 1, 2, \ldots)$. Initially, set $f_0 := \mathbf{0}$ and $\beta_0 := \|b\|_\infty$. If f_i and β_i have been found, we find f_{i+1} and β_{i+1} by the following iteration (later referred to as *iteration* i).

Let A_i be the set of arcs a with $f_i(a) > 12n\beta_i$ and let $\mathcal{K}_i$ be the collection of weak components of the digraph (V, A_i). We are going to update a function g_i starting with $g_i := f_i$.

(12.27) If there exists a component $K \in \mathcal{K}_i$ and distinct $u, v \in K$ with $|\mathrm{def}_{g_i}(u)| \geq |\mathrm{def}_{g_i}(v)| > 0$, then update g_i by sending a flow of value $|\mathrm{def}_{g_i}(v)|$ from v to u or conversely along a path in A_i, so as to make $\mathrm{def}_{g_i}(v)$ equal to 0.

(In (12.32) it is shown that this is possible, and that it does not modify D_{g_i}, hence g_i remains extreme.) We iterate (12.27), so that finally each $K \in \mathcal{K}_i$ contains at most one vertex u with $\mathrm{def}_{g_i}(u) \neq 0$.

Next do the following repeatedly, as long as there exists a $u \in V$ with $|\mathrm{def}_{g_i}(u)| > \frac{n-1}{n}\beta_i$:

(12.28) If $\mathrm{def}_{g_i}(u) > \frac{n-1}{n}\beta_i$, then there exists a $v \in V$ such that $\mathrm{def}_{g_i}(v) < -\frac{1}{n}\beta_i$ and such that u reachable from v in the residual graph D_{g_i}. Update g_i by sending a flow of value β_i along a minimum-cost $v - u$ path in D_{g_i}.
If $\mathrm{def}_{g_i}(u) < -\frac{n-1}{n}\beta_i$, proceed symmetrically.

(The existence of v in (12.28) follows from the assumption that there exists a nonnegative b-transshipment, f say, by applying Theorem 11.1 to $f - g_i$. The fact that we can send a flow of value β_i in the residual graph follows from (12.36).)

When we cannot apply (12.27) anymore, we define $f_{i+1} := g_i$. Let $T := \|\mathrm{def}_{f_{i+1}}\|_\infty$. If $T = 0$ we stop. Otherwise, define:

(12.29) $$\beta_{i+1} := \begin{cases} \frac{1}{2}\beta_i & \text{if } T \geq \frac{1}{12n}\beta_i, \\ T & \text{if } 0 < T < \frac{1}{12n}\beta_i, \end{cases}$$

and iterate.

Theorem 12.7. *The algorithm stops after at most n iterations of* (12.27) *and at most $O(n \log n)$ iterations of* (12.28).

Proof. Throughout the proof we assume $n \geq 2$. We first observe that for each i:

(12.30) $\|\text{def}_{f_{i+1}}\|_\infty \le \frac{n-1}{n}\beta_i,$

since otherwise we could have applied (12.28) to the final $g_i(= f_{i+1})$. This implies that for each i:

(12.31) $\|\text{def}_{f_i}\|_\infty \le 2\beta_i.$

This is direct for $i = 0$. If $\beta_{i+1} = \frac{1}{2}\beta_i$, then, by (12.30), $\|\text{def}_{f_{i+1}}\|_\infty \le \frac{n-1}{n}\beta_i \le \beta_i = 2\beta_{i+1}$. If $\beta_{i+1} < \frac{1}{2}\beta_i$, then $\|\text{def}_{f_{i+1}}\|_\infty = T = \beta_{i+1} \le 2\beta_{i+1}$. This proves (12.31).

We next show, that, for any i:

(12.32) in the iterations (12.27) and (12.28), for any $a \in A_i$ the value of $g_i(a)$ remains more than $6n\beta_i$.

In each iteration (12.27), for any arc $a \in A_i$, the value of $g_i(a)$ changes by at most $\|\text{def}_{f_i}\|_\infty$, which is at most $2\beta_i$ (by (12.31)). For any fixed i, we apply (12.27) at most n times. So the value of $g_i(a)$ on any arc $a \in A_i$ changes by at most $2n\beta_i$.

In the iterations (12.28), the value of $g_i(a)$ changes by at most $4n\beta_i$. To see this, consider the sum

(12.33) $$\sum_{\substack{v \in V \\ |\text{def}_{g_i}(v)| > \frac{n-1}{n}\beta_i}} |\text{def}_{g_i}(v)|.$$

In each iteration (12.28), this sum decreases by at least $\frac{n-1}{n}\beta_i$, which is at least $\frac{1}{2}\beta_i$. On the other hand, $g_i(a)$ changes by at most β_i. Since (12.33) initially is at most $\|\text{def}_{g_i}\|_1 \le \|\text{def}_{f_i}\|_1 \le 2n\beta_i$, we conclude that in the iterations (12.28), $g_i(a)$ changes by at most $4n\beta_i$.

Concluding, in the iterations (12.27) and (12.28), any $g_i(a)$ changes by at most $6n\beta_i$. Since at the beginning of these iterations we have $g_i(a) > 12n\beta_i$ for $a \in A_i$, this proves (12.32).

(12.32) implies that in iteration (12.27) we can make $\text{def}_{g_i}(v)$ equal to 0. (After that it will remain 0.) Hence, iteration (12.27) can be applied at most n times in total (over all i), since each time the number of vertices v with $\text{def}_{f_i}(v) \neq 0$ drops.

(12.32) also implies:

(12.34) each f_i is extreme.

This is clearly true if $i = 0$ (since the cost function k is nonnegative). Suppose that f_i is extreme. Then also g_i is extreme initially, and remains extreme during the iterations (12.27) (since by (12.32) the residual graph D_{g_i} does not change during the iterations (12.27)). Moreover, also during the iterations (12.28) the function g_i remains extreme, since we send flow over a minimum-cost path in D_{g_i} (Theorem 12.4). This proves (12.34).

Directly from (12.32) we have, for each i:

(12.35) $A_i \subseteq A_{i+1},$

since $\beta_{i+1} \le \frac{1}{2}\beta_i$. This implies that each set in $\mathcal{K}_i$ is contained in some set in $\mathcal{K}_{i+1}$.

Next, throughout iteration i,

(12.36) If $a \in A \setminus A_i$, then $\beta_i | g_i(a)$.

The proof is by induction on i, the case $i = 0$ being trivial (since for $i = 0$ we do not apply (12.27), as $A_0 = \emptyset$). Suppose that we know (12.36). Choose $a \in A \setminus A_{i+1}$. Then $a \in A \setminus A_i$ by (12.35). Hence $\beta_i | g_i(a)$, and so $\beta_i | f_{i+1}(a)$. If $f_{i+1}(a) > 0$ and $\beta_{i+1} < \frac{1}{2}\beta_i$, then $\beta_i > 12nT = 12n\|\text{def}_{f_{i+1}}\|_\infty$, and hence

(12.37) $\quad f_{i+1}(a) \geq \beta_i > 12n\|\mathrm{def}_{f_{i+1}}\|_\infty = 12n\beta_{i+1},$

contradicting the fact that a does not belong to A_{i+1}.

So $f_{i+1}(a) = 0$ or $\beta_{i+1} = \frac{1}{2}\beta_i$. This implies that $\beta_{i+1}|f_{i+1}(a)$. In iteration $i+1$, only flow packages of size β_{i+1} are sent over arc a (since $a \notin A_{i+1}$). Therefore, throughout iteration $i+1$ we have $\beta_{i+1}|g_{i+1}(a)$, which proves (12.36).

So in iteration (12.28) we indeed can send a flow of value β_i in the residual graph D_{g_i}. Since f_{i+1} is equal to the final g_i, (12.36) also implies:

(12.38) $\quad$ if $a \in A \setminus A_i$, then $\beta_i|f_{i+1}(a)$.

Next we come to the kernel in the proof, which gives two bounds on $b(K)$ for $K \in \mathcal{K}_i$:

Claim 1. For each i and each $K \in \mathcal{K}_i$:

(12.39) $\quad$ (i) $|b(K)| \leq 13n^3\beta_i$;
(ii) *suppose $i > 0$, $K \in \mathcal{K}_{i-1}$, and* (12.28) *is applied to a vertex u in K; then* $|b(K)| \geq \frac{1}{n}\beta_i$.

Proof of Claim 1. I. We first show (12.39)(i). If $i = 0$, then $|b(K)| \leq n\|b\|_\infty = n\beta_0 \leq 13n^3\beta_0$. If $i > 0$, then, by (12.31),

(12.40) $\quad |\mathrm{def}_{f_i}(K)| \leq n\|\mathrm{def}_{f_i}\|_\infty \leq 2n\beta_i.$

Moreover, since $f_i(a) \leq 12n\beta_i$ for each $a \in \delta^{\mathrm{in}}(K)$,

(12.41) $\quad |\mathrm{excess}_{f_i}(K)| \leq 12n\beta_i \cdot |\delta^{\mathrm{in}}(K)| \leq 12n^3\beta_i.$

Hence

(12.42) $\quad |b(K)| \leq |\mathrm{def}_{f_i}(K)| + |\mathrm{excess}_{f_i}(K)| \leq 2n\beta_i + 12n^3\beta_i \leq 13n^3\beta_i.$

This proves (12.39)(i).

II. Next we show (12.39)(ii). Since $K \in \mathcal{K}_{i-1} \cap \mathcal{K}_i$, u is the only vertex in K with $\mathrm{def}_{f_i}(u) \neq 0$. So, in iteration i, we do not apply (12.27) to a vertex in K. Moreover, by applying (12.28), $|\mathrm{def}_{g_i}(K)|$ does not increase. This gives

(12.43) $$\frac{n-1}{n}\beta_i < |\mathrm{def}_{g_i}(K)| \leq |\mathrm{def}_{f_i}(K)| \leq \frac{n-1}{n}\beta_{i-1}.$$

The first inequality holds as we apply (12.28) to u, and the last inequality follows from (12.30).

To prove (12.39)(ii), first assume $\beta_i = \frac{1}{2}\beta_{i-1}$. Since $\mathrm{def}_{f_i}(K) = \mathrm{def}_{f_i}(u)$, we have, by (12.43),

(12.44) $$\frac{n-1}{2n}\beta_{i-1} = \frac{n-1}{n}\beta_i \leq |\mathrm{def}_{f_i}(K)| \leq \frac{n-1}{n}\beta_{i-1}.$$

So $|\mathrm{def}_{f_i}(K)|/\beta_{i-1}$ has distance at least $1/2n$ to $\mathbb{Z}$. Since $f_i(a) \equiv 0 \pmod{\beta_{i-1}}$ by (12.38), we have

(12.45) $\quad |b(K)|/\beta_{i-1} \equiv |\mathrm{def}_{f_i}(K)|/\beta_{i-1} \pmod 1.$

Hence also $|b(K)|/\beta_{i-1}$ has distance at least $1/2n$ to $\mathbb{Z}$. So $|b(K)| \geq \frac{1}{2n}\beta_{i-1} \geq \frac{1}{n}\beta_i$, as required.

Second assume $\beta_i < \frac{1}{2}\beta_{i-1}$. Then $\beta_i = \|\mathrm{def}_{f_i}\|_\infty < \frac{1}{12n}\beta_{i-1}$. Now as $K \in \mathcal{K}_i$, we have for each $a \in \delta(K)$: $0 \leq f_i(a) < 12n\beta_i < \beta_{i-1}$, while $f_i(a) \equiv 0 \pmod{\beta_{i-1}}$ by (12.38). So $f_i(a) = 0$ for each $a \in \delta(K)$. Hence by (12.43), (12.46) $\quad |b(K)| = |\mathrm{def}_{f_i}(K)| \geq \frac{n-1}{n}\beta_i \geq \frac{1}{n}\beta_i,$

which proves (12.39)(ii). *End of Proof of Claim 1*

Define $\mathcal{K}^* := \bigcup_i \mathcal{K}_i$, and consider any $K \in \mathcal{K}^*$. Let I be the set of i with $K \in \mathcal{K}_i$. Let t be the smallest element of I. Let λ_K be the number of components in $\mathcal{K}_{t-1}$ contained in K. (Set $\lambda_K := 1$ if $t = 0$.) Then in iteration t, (12.28) is applied at most $4\lambda_K$ times to a vertex u in K, since (at the start of applying (12.28))

(12.47) $\quad |\mathrm{def}_{g_t}(K)| = |\mathrm{def}_{f_t}(K)| \leq \lambda_K \|\mathrm{def}_{f_t}\|_\infty \leq 2\lambda_K\beta_t$

(by (12.31)). In any further iteration $i > t$ with $i \in I$, (12.28) is applied at most twice to u (again by (12.31)).

We estimate now the number of iterations $i \in I$ in which (12.28) is applied to $u \in K$. Consider the smallest such i with $i > t$. Then:

(12.48) $\quad$ if $j > i + \log_2(13n^4)$, then $j \notin I$.

For suppose to the contrary that $K \in \mathcal{K}_j$. Since $\beta_i \geq 2^{j-i}\beta_j > 13n^4\beta_j$, Claim 1 gives the contradiction $b(K) \leq 13n^3\beta_j < \frac{1}{n}\beta_i \leq b(K)$. This proves (12.48).

So (12.28) is applied at most $4\lambda_K + 2\log_2(13n^4)$ times to a vertex $u \in K$, in iterations $i \in I$. Since

(12.49) $\quad \sum_{K \in \mathcal{K}^*} \lambda_K \leq |\mathcal{K}^*| < 2n$

(as $\mathcal{K}^*$ is laminar — cf. Theorem 3.5), (12.28) is applied at most $8n + 4n\log_2(13n^4)$ times in total. ∎

This bound on the number of iterations gives:

Corollary 12.7a. *A minimum-cost nonnegative b-transshipment can be found in time $O(n\log n(m + n\log n))$.*

Proof. Directly from Theorem 12.7, since any iteration (12.27) or (12.28) takes $O(m + n\log n)$ time, using Fibonacci heaps (Corollary 7.7a) and maintaining a potential as in Theorem 12.5. ∎

We can derive a bound for finding a minimum-cost circulation:

Corollary 12.7b. *A minimum-cost circulation can be found in time $O(m\log n(m + n\log n))$.*

Proof. The minimum-cost circulation problem can be reduced to the minimum-cost transshipment problem as follows. Let $D = (V, A)$, $d, c, k : A \to \mathbb{Q}$ be the input for the minimum-cost circulation problem. Define

(12.50) $\quad b(v) := d(\delta^{\mathrm{out}}(v)) - d(\delta^{\mathrm{in}}(v))$

for each $v \in V$. Then any minimum-cost b-transshipment x satisfying $\mathbf{0} \leq x \leq c - d$ gives a minimum-cost circulation $x' := x + d$ satisfying $d \leq x' \leq c$. So we can assume $d = \mathbf{0}$.

Now replace each arc $a = (u, v)$ by three arcs (u, u_a), (v_a, u_a), and (v_a, v), where u_a and v_a are new vertices. This makes the digraph D' say.

Define $b(u_a) := c(a)$ and $b(v_a) := -c(a)$. Moreover, define a cost function k' on the arcs of D' by $k'(u, u_a) := k(a)$, $k'(v_a, u_a) := 0$, $k'(v_a, v) := 0$ if $k(a) \geq 0$, and $k'(u, u_a) := 0$, $k'(v_a, u_a) := -k(a)$, $k'(v_a, v) := 0$ if $k(a) < 0$. Then a minimum-cost b-transshipment $x \geq \mathbf{0}$ in D' gives a minimum-cost b-transshipment x satisfying $\mathbf{0} \leq x \leq c$ in the original digraph D.

By Theorem 12.7, a minimum-cost b-transshipment $x \geq \mathbf{0}$ in D' can be found by finding $O(n \log n)$ times a shortest path in a residual graph D'_x. While this digraph has $2m + n$ vertices, it can be reduced in $O(m)$ time to finding a shortest path in an auxiliary digraph with $O(n)$ vertices only. Hence again it takes $O(m + n \log n)$ time by using Fibonacci heaps (Corollary 7.7a) and maintaining a potential as in Theorem 12.5. ∎

12.5. Further results and notes

12.5a. Complexity survey for minimum-cost circulation

Complexity survey for minimum-cost circulation ($*$ indicates an asymptotically best bound in the table):

	$O(n^4 CK)$	Ford and Fulkerson [1958b] labeling
	$O(m^3 C)$	Yakovleva [1959], Minty [1960], Fulkerson [1961] out-of-kilter method
	$O(nm^2 C)$	Busacker and Gowen [1960], Iri [1960] successive shortest paths
$*$	$O(nC \cdot \text{SP}_+(n, m, K))$	Edmonds and Karp [1970], Tomizawa [1971] successive shortest paths with nonnegative lengths using vertex potentials
	$O(nK \cdot \text{MF}(n, m, C))$	Edmonds and Karp [1972]
$*$	$O(m \log C \cdot \text{SP}_+(n, m, K))$	Edmonds and Karp [1972] capacity-scaling
	$O(nm \log(nC))$	Dinits [1973a] capacity-scaling
	$O(n \log K \cdot \text{MF}(n, m, C))$	Röck [1980] (cf. Bland and Jensen [1992]) cost-scaling
	$O(m^2 \log n \cdot \text{MF}(n, m, C))$	Tardos [1985a]
	$O(m^2 \log n \cdot \text{SP}_+(n, m, K))$	Orlin [1984a], Fujishige [1986]
	$O(n^2 \log n \cdot \text{SP}_+(n, m, K))$	Galil and Tardos [1986,1988]
	$O(n^3 \log(nK))$	Goldberg and Tarjan [1987], Bertsekas and Eckstein [1988]
$*$	$O(n^{5/3} m^{2/3} \log(nK))$	Goldberg and Tarjan [1987] generalized cost-scaling

≫

continued

	$O(nm \log n \log(nK))$	Goldberg and Tarjan [1987] generalized cost-scaling; Goldberg and Tarjan [1988b,1989] minimum-mean cost cycle-cancelling
*	$O(m \log n \cdot \mathrm{SP}_+(n, m, K))$	Orlin [1988,1993]
*	$O(nm \log(n^2/m) \log(nK))$	Goldberg and Tarjan [1990] generalized cost-scaling
*	$O(nm \log \log C \log(nK))$	Ahuja, Goldberg, Orlin, and Tarjan [1992] double scaling
*	$O(n \log C(m + n \log n))$	*circulations with lower bounds only* Gabow and Tarjan [1989]
*	$O((m^{3/2}C^{1/2} + \gamma \log \gamma) \log(nK))$	Gabow and Tarjan [1989]
*	$O((nm + \gamma \log \gamma) \log(nK))$	Gabow and Tarjan [1989]

Here $K := \|k\|_\infty$, $C := \|c\|_\infty$, and $\gamma := \|c\|_1$, for integer cost function k and integer capacity function c. Moreover, $\mathrm{SP}_+(n, m, K)$ denotes the running time of any algorithm finding a shortest path in a digraph with n vertices, m arcs, and nonnegative integer length function l with $K = \|l\|_\infty$. Similarly, $\mathrm{MF}(n, m, C)$ denotes the running time of any algorithm finding a maximum flow in a digraph with n vertices, m arcs, and nonnegative integer capacity function c with $C = \|c\|_\infty$.

Complexity survey for minimum-cost nonnegative transshipment:

*	$O(n \log B \cdot \mathrm{SP}_+(n, m, K))$	Edmonds and Karp [1970,1972]
	$O(n^2 \log n \cdot \mathrm{SP}_+(n, m, K))$	Galil and Tardos [1986,1988]
*	$O(n \log n \cdot \mathrm{SP}_+(n, m, K))$	Orlin [1988,1993]

Here $B := \|b\|_\infty$ for integer b.

12.5b. Min-max relations for minimum-cost flows and circulations

From Corollary 12.1a, the following min-max equality for minimum-cost circulation can be derived. The equality also follows directly from linear programming duality and total unimodularity. Both approaches were considered by Gallai [1957,1958a, 1958b].

Theorem 12.8. *Let $D = (V, A)$ be a digraph and let $c, d, k : A \to \mathbb{R}$. Then the minimum of $\sum_{a \in A} k(a)f(a)$ taken over all circulations f in D satisfying $d \leq f \leq c$ is equal to the maximum value of*

(12.51) $$\sum_{a \in A} (y(a)d(a) - z(a)c(a)),$$

where $y, z : A \to \mathbb{R}_+$ are such that there exists a function $p : V \to \mathbb{R}$ with the property that

(12.52) $\quad y(a) - z(a) = k(a) - p(v) + p(u)$

for each arc $a = (u, v)$ of D.

If d and c are integer, we can take f integer. If k is integer, we can take y and z integer.

Proof. The minimum is not less than the maximum, since if f is any circulation in D with $d \leq f \leq c$ and y, z, p satisfy (12.52), then

(12.53)
$$\sum_{a \in A} k(a)f(a) = \sum_{a=(u,v) \in A} (k(a)+p(u)-p(v))f(a) = \sum_{a \in A} (y(a)-z(a))f(a) \geq \sum_{a \in A} (y(a)d(a) - z(a)c(a)).$$

To see equality, let f be a minimum-cost circulation. By Corollary 12.1a, there is a function $p : V \to \mathbb{R}$ satisfying (12.7). Define for each arc $a = (u, v)$:

(12.54)
$$y(a) := \max\{0, k(a) - p(v) + p(u)\},$$
$$z(a) := \max\{0, -k(a) + p(v) - p(u)\}.$$

So y and z satisfy (12.52). Moreover, we have by (12.7) that $y(a)(f(a) - d(a)) = 0$ and $z(a)(c(a) - f(a)) = 0$ for each arc a. Hence we have equality throughout in (12.53). ∎

We consider a special case (Gallai [1957,1958a,1958b]). Let $D = (V, A)$ be a strongly connected digraph, let $d : A \to \mathbb{Z}_+$, and let $k : A \to \mathbb{Z}_+$ be a cost function, with $k(C) \geq 0$ for each directed circuit C. Then:

(12.55) the minimum cost $\sum_{a \in A} k(a)f(a)$ of an integer circulation f in D with $f \geq d$ is equal to the maximum value of $\sum_{a \in A} d(a)y(a)$ where $y : A \to \mathbb{Z}_+$ with $y(C) = k(C)$ for each directed circuit C in D.

A consequence of this applies to the 'directed Chinese postman problem': given a strongly connected directed graph, find a shortest directed closed path traversing each arc at least once. If we take unit length, we obtain the following. The minimum number of arcs in any closed directed path traversing each arc at least once is equal to the maximum value of

(12.56)
$$|A| + \sum_{U \in \mathcal{U}} (|\delta^{\text{in}}(U)| - |\delta^{\text{out}}(U)|),$$

where $\mathcal{U}$ is a collection of subsets U of V such that the $\delta^{\text{out}}(U)$ are disjoint.

This equality follows from (12.55), by taking $d = \mathbf{1}$ and $k = \mathbf{1}$: then there is a $p : V \to \mathbb{Z}$ with $y(a) = 1 - p(v) + p(u)$ for each arc $a = (u, v)$. So $p(v) \leq p(u) + 1$ for each arc $a = (u, v)$. Taking $U_i := \{v \in V \mid p(v) \leq i\}$ for each $i \in \mathbb{Z}$ gives the required cuts $\delta^{\text{out}}(U_i)$.

12.5c. Dynamic flows

A minimum-cost flow algorithm (in disguised form) was given by Ford and Fulkerson [1958b]. They considered the following 'dynamic flow' problem. Let $D = (V, A)$ be a digraph and let $r, s \in V$ (for convenience we assume that r is a source and s is a sink of D). Let $c : A \to \mathbb{Z}_+$ be a capacity function. Moreover, let a 'traversal time' function $\tau : A \to \mathbb{Z}_+$ be given, and a 'time limit' T.

The problem now is to send a maximum amount of flow from r to s, such that, for each arc a, at each time unit at most $c(a)$ flow is sent over a; it takes $\tau(a)$ time to traverse a. All flow is sent from r at one of the times $1, 2, \ldots, T$, while it reaches s at time at most T.

More formally, for any arc $a = (u, v)$ and any $t \in \{1, 2, \ldots, T\}$, let $x(a, t)$ denote the amount of flow sent from u over a at time t, reaching v at time $t + \tau(a)$. A first constraint is:

$$(12.57) \qquad 0 \leq x(a, t) \leq c(a)$$

for each $a \in A$ and each $t \in \{1, \ldots, T\}$. Next a flow conservation law can be formulated. Flow may 'wait' at any vertex until there is capacity enough to be transmitted further. This can be described as follows:

$$(12.58) \qquad \sum_{a \in \delta^{\text{in}}(v)} \sum_{t=1}^{t'-\tau(a)} x(a, t) \geq \sum_{a \in \delta^{\text{out}}(v)} \sum_{t=1}^{t'} x(a, t)$$

for each $v \in V \setminus \{r, s\}$ and each $t' \in \{1, \ldots, T\}$. We maximize the amount of flow reaching s not later than time T; that is, we

$$(12.59) \qquad \text{maximize} \sum_{a \in \delta^{\text{in}}(s)} \sum_{t=1}^{T-\tau(a)} x(a, t).$$

Since we may assume that we do not send flow from r that will not reach s, we may assume that we have equality in (12.58) if $t' = T$.

As Ford and Fulkerson [1958b] observed, this 'dynamic flow' problem can be transformed to a 'static' flow problem as follows. Let D' be the digraph with vertices all pairs (v, t) with $v \in V$ and $t \in \{1, \ldots, T\}$, and arcs

(12.60) (i) $((u, t), (v, t+\tau(a)))$ for each $a = (u, v) \in A$ and $t \in \{1, \ldots, T-\tau(a)\}$,
(ii) $((v, t), (v, t + 1))$ for each $v \in V$ and $t \in \{1, \ldots, T - 1\}$.

Let any arc of type (i) have capacity $c(a)$ and let any arc of type (ii) have capacity $+\infty$. Then the maximum dynamic flow problem is equivalent to finding a maximum flow in the new network from $(r, 1)$ to (s, T).

By this construction, a maximum dynamic flow can be found by solving a maximum flow problem in the large graph D'. Ford and Fulkerson [1958b] however described an alternative way of finding a dynamic flow that has a number of advantages. First of all, no 'large' graph D' has to be constructed (and the final algorithm can be modified with the scaling method of Edmonds and Karp [1972] to a method that is polynomial also in $\log T$). Second, the solution can be represented as a relatively small number of paths over which flow is transmitted repeatedly. Finally, the method shows that at intermediate vertices hold-over of flows is not necessary (that is, all arcs of type (12.60)(ii) with $v \neq r, s$ can be deleted).

Ford and Fulkerson [1958b] showed that a solution of the dynamic flow problem can be found by solving the following problem:

$$(12.61) \qquad \text{maximize} \sum_{a \in \delta^{\text{in}}(s)} T x(a) - \sum_{a \in A} \tau(a) x(a),$$
$$\text{where } x \text{ is an } r - s \text{ flow satisfying } \mathbf{0} \leq x \leq c.$$

This is equivalent to a minimum-cost flow problem, with cost $k(a) := \tau(a) - T$ for $a \in \delta^{\text{in}}(s)$, and $k(a) := \tau(a)$ for all other a. Note that there are arcs of negative cost (generally), and that the value of the flow is not prescribed. So by adding an arc (s, r) we obtain a minimum-cost circulation problem.

How is problem (12.61) related to the dynamic flow problem? Given an optimum solution $x : A \to \mathbb{Z}_+$ of (12.61), there exist $r - s$ paths $P_1, \ldots, P_m$ in D such that

(12.62) $$x \geq \sum_{i=1}^{m} \chi^{P_i}$$

where m is the value of x. (We identify a path P and its set of arcs.) For any path P, let $\tau(P)$ be the traversal time of P (= the sum of the traversal times of the arcs in P). Then $\tau(P_i) \leq T$ for each i, since otherwise we could replace x by $x - \chi^{P_i}$, while increasing the objective value in (12.61).

Now send, for each $i = 1, \ldots, m$, a flow of value 1 along P_i at times $1, \ldots, T - \tau(P_i)$. It is not difficult to describe this in terms of the $x(a, t)$, yielding a feasible solution for the dynamic flow problem, of value

(12.63) $$\sum_{i=1}^{m} (T - \tau(P_i)) \geq mT - \sum_{a \in A} \tau(a) x(a),$$

which is the optimum value of (12.61).

In fact, this dynamic flow is optimum. Indeed, by Theorem 12.8 (alternatively, by LP-duality and total unimodularity), the optimum value of (12.61) is equal to that of:

(12.64) $$\text{minimize} \sum_{a \in A} c(a) y(a)$$

where $y : A \to \mathbb{Z}_+$ such that there exists $p : V \to \mathbb{Z}$ satisfying:

(12.65) $$p(u) - p(v) + y(a) \geq -\tau(a) \text{ for each } a = (u, v) \in A,$$

where $p(r) = 0$ and $p(s) = T$.

Now if $x(a, t)$ is a feasible solution of the dynamic flow problem, then by (12.58), (12.57) and (12.63),

(12.66) $$\begin{aligned}
\sum_{a \in \delta^{\text{in}}(s)} \sum_{t=1}^{T-\tau(a)} x(a, t) &\leq \sum_{a \in \delta^{\text{in}}(s)} \sum_{t=1}^{T-\tau(a)} x(a, t) \\
&+ \sum_{v \neq s} \Big(\sum_{a \in \delta^{\text{in}}(v)} \sum_{t=1}^{p(v)-\tau(a)} x(a, t) - \sum_{a \in \delta^{\text{out}}(v)} \sum_{t=1}^{p(v)} x(a, t) \Big) \\
&= \sum_{a=(u,v) \in A} \Big(\sum_{t=1}^{p(v)-\tau(a)} x(a, t) - \sum_{t=1}^{p(u)} x(a, t) \Big) \\
&\leq \sum_{a=(u,v) \in A} \sum_{t=p(u)+1}^{p(v)-\tau(a)} x(a, t) \leq \sum_{a=(u,v) \in A} c(a)(p(v) - \tau(a) - p(u)) \\
&\leq \sum_{a \in A} c(a) y(a).
\end{aligned}$$

Therefore, the dynamic flow constructed is optimum.

Ford and Fulkerson [1958b] described a method for solving (12.61) which essentially consists of repeatedly finding a shortest $r-s$ path in the residual graph, making costs nonnegative by translating the cost with the help of the current potential p (this is 'Routine I' of Ford and Fulkerson [1958b]). In this formulation, it is a primal-dual method.

The method of Ford and Fulkerson [1958b] improves the algorithm of Ford [1956] for the dynamic flow problem. More on this and related problems can be found in Wilkinson [1971], Minieka [1973], Orlin [1983,1984b], Aronson [1989], Burkard, Dlaska, and Klinz [1993], Hoppe and Tardos [1994,1995,2000], Klinz and Woeginger [1995], Fleischer and Tardos [1998], and Fleischer [1998b,1999c,2001b,2001a].

12.5d. Further notes

The minimum-cost flow problem is a linear programming problem, and hence it can be solved with the primal simplex method or the dual simplex method. Strongly polynomial *dual* simplex algorithms for minimum-cost flow have been given by Orlin [1985] ($O(m^3)$ pivots) and Plotkin and Tardos [1990] ($O(m^2/\log n)$ pivots) (cf. Orlin, Plotkin, and Tardos [1993]). No pivot rule is known however that finds a minimum-cost flow with the *primal* simplex method in polynomial time. Partial results were found by Goldfarb and Hao [1990] and Tarjan [1991].

Further work on the primal simplex method applied to the minimum-cost flow problem is discussed by Dantzig [1963], Gassner [1964], Johnson [1966b], Grigoriadis and Walker [1968], Srinivasan and Thompson [1973], Glover, Karney, and Klingman [1974], Glover, Karney, Klingman, and Napier [1974], Glover, Klingman, and Stutz [1974], Ross, Klingman, and Napier [1975], Cunningham [1976,1979], Bradley, Brown, and Graves [1977], Gavish, Schweitzer, and Shlifer [1977], Barr, Glover, and Klingman [1978,1979], Mulvey [1978a], Kennington and Helgason [1980], Chvátal [1983], Cunningham and Klincewicz [1983], Gibby, Glover, Klingman, and Mead [1983], Grigoriadis [1986], Ahuja and Orlin [1988,1992], Goldfarb, Hao, and Kai [1990a], Tarjan [1991,1997], Eppstein [1994a,2000], Orlin [1997], and Sokkalingam, Sharma, and Ahuja [1997].

Further results on the dual simplex method applied to minimum-cost flows are given by Dantzig [1963], Helgason and Kennington [1977b], Armstrong, Klingman, and Whitman [1979], Orlin [1984a], Ikura and Nemhauser [1986], Adler and Cosares [1990], Plotkin and Tardos [1990], Orlin, Plotkin, and Tardos [1993], Eppstein [1994a,2000], and Armstrong and Jin [1997].

Further algorithmic work is presented by Briggs [1962], Pla [1971] (dual out-of-kilter), Barr, Glover, and Klingman [1974] (out-of-kilter), Hassin [1983,1992], Bertsekas [1985], Kapoor and Vaidya [1986] (interior-point method), Bertsekas and Tseng [1988] ('relaxation method'), Masuzawa, Mizuno, and Mori [1990] (interior-point method), Cohen and Megiddo [1991], Bertsekas [1992] ('auction algorithm'), Norton, Plotkin, and Tardos [1992], Wallacher and Zimmermann [1992] (combinatorial interior-point method), Ervolina and McCormick [1993a,1993b], Fujishige, Iwano, Nakano, and Tezuka [1993], McCormick and Ervolina [1994], Hadjiat and Maurras [1997], Goldfarb and Jin [1999a], McCormick and Shioura [2000a,2000b] (cycle canceling), Shigeno, Iwata, and McCormick [2000] (cycle- and cut-canceling), and Vygen [2000].

Worst-case studies are made by Zadeh [1973a,1973b,1979] (cf. Niedringhaus and Steiglitz [1978]), Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975], Dinits and Karzanov [1974], Radzik and Goldberg [1991,1994] (minimum-mean cost cycle-cancelling), and Hadjiat [1998] (dual out-of-kilter).

Computational studies were presented by Glover, Karney, and Klingman [1974] (simplex method), Glover, Karney, Klingman, and Napier [1974], Harris [1976], Karney and Klingman [1976], Bradley, Brown, and Graves [1977], Helgason and Kennington [1977b] (dual simplex method), Ali, Helgason, Kennington, and Lall [1978], Mulvey [1978b] (simplex method), Armstrong, Klingman, and Whitman [1979], Monma and Segal [1982] (simplex method), Gibby, Glover, Klingman, and Mead [1983], Grigoriadis [1986] (simplex method), Ikura and Nemhauser [1986] (dual simplex method), Bertsekas and Tseng [1988], Bland and Jensen [1992], Bland, Cheriyan, Jensen, and Ladányi [1993], Fujishige, Iwano, Nakano, and Tezuka [1993], Goldberg [1993a,1997] (push-relabel and successive approximation), Goldberg and Kharitonov [1993] (push-relabel), and Resende and Veiga [1993] (interior-point). Consult also Johnson and McGeoch [1993].

Bein, Brucker, and Tamir [1985] and Hoffman [1988] considered minimum-cost circulation for series-parallel digraphs. Wagner and Wan [1993] gave a polynomial-time simplex method for the maximum k-flow problem (with a profit for every unit of flow sent, and a cost for every unit capacity added to any arc a), which can be reduced to a minimum-cost circulation problem.

Maximum flows where the cost may not exceed a given 'budget' were considered by Fulkerson [1959b] and Ahuja and Orlin [1995].

'Unsplittable' flows (with one source and several sinks, where all flow from the source to any sink follows the same path) were investigated by Kleinberg [1996, 1998], Kolliopoulos and Stein [1997,1998a,1999,2002], Srinivasan [1997], Dinitz, Garg, and Goemans [1998,1999], Skutella [2000,2002], Azar and Regev [2001], Erlebach and Hall [2002], and Kolman and Scheideler [2002].

For generalized flows (with 'gains' on arcs), see Jewell [1962], Fujisawa [1963], Eisemann [1964], Mayeda and van Valkenburg [1965], Charnes and Raike [1966], Onaga [1966,1967], Glover, Klingman, and Napier [1972], Maurras [1972], Glover and Klingman [1973], Grinold [1973], Truemper [1977], Minieka [1978], Elam, Glover, and Klingman [1979], Jensen and Barnes [1980], Gondran and Minoux [1984], Kapoor and Vaidya [1986], Bertsekas and Tseng [1988], Ruhe [1988], Vaidya [1989c], Goldberg, Tardos, and Tarjan [1990], Goldberg, Plotkin, and Tardos [1991], Cohen and Megiddo [1994], Goldfarb and Jin [1996], Tseng and Bertsekas [1996,2000], Goldfarb, Jin, and Orlin [1997], Radzik [1998], Tardos and Wayne [1998], Oldham [1999,2001], Wayne [1999,2002], Wayne and Fleischer [1999], Fleischer and Wayne [2002], and Goldfarb, Jin, and Lin [2002].

For convex costs,see Charnes and Cooper [1958], Beale [1959], Shetty [1959], Berge [1960b], Minty [1960,1961,1962], Tụy [1963,1964], Menon [1965], Hu [1966], Weintraub [1974], Jensen and Barnes [1980], Kennington and Helgason [1980], Dembo and Klincewicz [1981], Hassin [1981a], Klincewicz [1983], Ahuja, Batra, and Gupta [1984], Minoux [1984,1986], Rockafellar [1984], Florian [1986], Bertsekas, Hosein, and Tseng [1987], Katsura, Fukushima, and Ibaraki [1989], Karzanov and McCormick [1995,1997], Tseng and Bertsekas [1996,2000], and Ahuja, Hochbaum, and Orlin [1999].

Concave costs were studied by Zangwill [1968], Rothfarb and Frisch [1970], Daeninck and Smeers [1977], Jensen and Barnes [1980], Graves and Orlin [1985], and Erickson, Monma, and Veinott [1987].

The basic references on network flows are the books of Ford and Fulkerson [1962] (historical) and Ahuja, Magnanti, and Orlin [1993]. Minimum-cost flow problems are discussed also in the books of Hu [1969,1982], Iri [1969], Frank and Frisch [1971], Potts and Oliver [1972], Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975] (for a review, see Goldberg and Gusfield [1991]), Christofides [1975], Lawler [1976b], Murty [1976], Bazaraa and Jarvis [1977], Minieka [1978], Jensen and Barnes [1980], Kennington and Helgason [1980], Phillips and Garcia-Diaz [1981], Swamy and Thulasiraman [1981], Papadimitriou and Steiglitz [1982], Smith [1982], Chvátal [1983], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Rockafellar [1984], Derigs [1988a], Nemhauser and Wolsey [1988], Bazaraa, Jarvis, and Sherali [1990], Chen [1990], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], and Korte and Vygen [2000].

Survey papers include Glover and Klingman [1977], Ahuja, Magnanti, and Orlin [1989,1991], Goldberg, Tardos, and Tarjan [1990], and Frank [1995]. A bibliography was given by Golden and Magnanti [1977].

The history of the minimum-cost flow, circulation, and transshipment problems is closely intertwined with that of the transportation problem — see Section 21.13e.

Chapter 13

Path and flow polyhedra and total unimodularity

A large part of the theory of paths and flows can be represented geometrically by polytopes and polyhedra, and can be studied with methods from geometry and linear programming. Theorems like Menger's theorem, the max-flow min-cut theorem, and Hoffman's circulation theorem can be derived and interpreted with elementary polyhedral tools.
This can be done with the help of the total unimodularity of the incidence matrices of directed graphs, and of the more general network matrices. It again yields proofs of basic flow theorems and implies the polynomial-time solvability of flow problems.

13.1. Path polyhedra

Let $D = (V, A)$ be a digraph and let $s, t \in V$. The *$s-t$ path polytope* $P_{s-t \text{ path}}(D)$ is the convex hull of the incidence vectors in $\mathbb{R}^A$ of $s-t$ paths in D. (We recall that paths are simple, by definition.) So $P_{s-t \text{ path}}(D)$ is a polytope in the space $\mathbb{R}^A$. Since finding a maximum-length $s-t$ path in D is NP-complete, we may not expect to have a decent description of the inequalities determining $P_{s-t \text{ path}}(D)$ (cf. Corollary 5.16a). That is, the separation problem for $P_{s-t \text{ path}}(D)$ is NP-hard.

However, if we extend the $s-t$ path polytope to its dominant, it becomes more tractable. This leads to an illuminating geometric framework, in which the (easy) max-potential min-work theorem (Theorem 7.1) and the (more difficult) max-flow min-cut theorem (Theorem 10.3) show up as polars of each other, and can be derived from each other. This duality forms a prototype for many other dual theorems and problems.

The dominant $P^{\uparrow}_{s-t \text{ path}}(D)$ of $P_{s-t \text{ path}}(D)$ is the set of vectors $x \in \mathbb{R}^A$ with $x \geq y$ for some $y \in P_{s-t \text{ path}}(D)$. So

$$(13.1) \qquad P^{\uparrow}_{s-t \text{ path}}(D) = P_{s-t \text{ path}}(D) + \mathbb{R}^A_+.$$

An alternative way of describing this polyhedron is as the set of all capacity functions $c : A \to \mathbb{R}_+$ for which there exists a flow $x \leq c$ of value 1.

It is not difficult to derive from the (easy) max-potential min-work theorem that the following inequalities determine $P^{\uparrow}_{s-t \text{ path}}(D)$:

(13.2) (i) $x(a) \geq 0$ for each $a \in A$,
(ii) $x(C) \geq 1$ for each $s-t$ cut C.

Theorem 13.1. *$P^{\uparrow}_{s-t \text{ path}}(D)$ is determined by* (13.2).

Proof. Clearly, each vector in $P^{\uparrow}_{s-t \text{ path}}(D)$ satisfies (13.2). So $P^{\uparrow}_{s-t \text{ path}}(D)$ is contained in the polyhedron Q determined by (13.2). Suppose that the reverse inclusion does not hold. Then there is an $l \in \mathbb{Z}^A$ such that the minimum of $l^{\mathsf{T}}x$ over $x \in Q$ is smaller than over $x \in P^{\uparrow}_{s-t \text{ path}}(D)$. If $l \notin \mathbb{Z}^A_+$, the minimum in both cases is $-\infty$; so $l \in \mathbb{Z}^A_+$. Then the minimum over $P^{\uparrow}_{s-t \text{ path}}(D)$ is equal to the minimum length k of an $s-t$ path, taking l as length function. By Theorem 7.1, there exist $s-t$ cuts $C_1, \ldots, C_k$ such that each arc a is in at most $l(a)$ of the C_i. Hence for any $x \in Q$ one has

$$(13.3) \qquad l^{\mathsf{T}}x \geq \Big(\sum_{i=1}^{k} \chi^{C_i}\Big)^{\mathsf{T}} x = \sum_{i=1}^{k} x(C_i) \geq k,$$

by (13.2)(ii). So the minimum over Q is at least k, contradicting our assumption. ∎

So the characterization of the dominant $P^{\uparrow}_{s-t \text{ path}}(D)$ of the $s-t$ path polytope follows directly from the easy Theorem 7.1 (the max-potential min-work theorem).

Note that Theorem 13.1 is equivalent to:

Corollary 13.1a. *The polyhedron determined by* (13.2) *is integer.*

Proof. The vertices are integer, as they are incidence vectors of paths. ∎

Next, the theory of blocking polyhedra implies a similar result when we interchange 'paths' and 'cuts', thus deriving the max-flow min-cut theorem.

The *$s-t$ cut polytope* $P_{s-t \text{ cut}}(D)$ is the convex hull of the incidence vectors of $s-t$ cuts in D. Again, $P_{s-t \text{ cut}}(D)$ is a polytope in the space $\mathbb{R}^A$. Since finding a maximum-size $s-t$ cut in D is NP-complete (Theorem 75.1), we may not expect to have a decent description of inequalities determining $P_{s-t \text{ cut}}(D)$. That is, the separation problem for $P_{s-t \text{ cut}}(D)$ is NP-hard.

Again, a polyhedron that behaves more satisfactorily is the dominant $P^{\uparrow}_{s-t \text{ cut}}(D)$ of the $s-t$ cut polytope, which is the set of vectors $x \in \mathbb{R}^A$ with $x \geq y$ for some $y \in P_{s-t \text{ cut}}(D)$. That is,

$$(13.4) \qquad P^{\uparrow}_{s-t \text{ cut}}(D) = P_{s-t \text{ cut}}(D) + \mathbb{R}^A_+.$$

Now the following inequalities determine $P^{\uparrow}_{s-t \text{ cut}}(D)$:

(13.5) (i) $x(a) \geq 0$ for $a \in A$,
(ii) $x(AQ) \geq 1$ for each $s-t$ path Q.

Corollary 13.1b. $P^{\uparrow}_{s-t \text{ cut}}(D)$ *is determined by* (13.5).

Proof. Directly with the theory of blocking polyhedra (Theorem 5.8) from Theorem 13.1. ∎

Equivalently:

Corollary 13.1c. *The polyhedron determined by* (13.5) *is integer.*

Proof. The vertices are integer, as they are incidence vectors of $s-t$ cuts. ∎

The two polyhedra are connected by the blocking relation:

Corollary 13.1d. *The polyhedra $P^{\uparrow}_{s-t \text{ path}}(D)$ and $P^{\uparrow}_{s-t \text{ cut}}(D)$ form a blocking pair of polyhedra.*

Proof. Directly from the above. ∎

With linear programming duality, this theorem implies the max-flow min-cut theorem:

Corollary 13.1e (max-flow min-cut theorem)**.** *Let $D=(V,A)$, let $s,t \in V$ and let $c : A \to \mathbb{R}_+$ be a capacity function. Then the maximum value of an $s-t$ flow $f \leq c$ is equal to the minimum capacity of an $s-t$ cut.*

Proof. The minimum capacity of an $s-t$ cut is equal to the minimum of $c^{\mathsf{T}}x$ over $x \in P^{\uparrow}_{s-t \text{ cut}}(D)$ (by definition (13.4)). By Corollary 13.1b, this is equal to the minimum value μ of $c^{\mathsf{T}}x$ where x satisfies (13.5). By linear programming duality, μ is equal to the maximum value of $\sum_Q \lambda_Q$, where $\lambda_Q \geq 0$ for each $s-t$ path Q, such that

$$\sum_Q \lambda_Q \chi^{AQ} \leq c. \tag{13.6}$$

Then $f := \sum_Q \lambda_Q \chi^{AQ}$ is an $s-t$ flow of value μ. ∎

Thus the theory of blocking polyhedra links minimum-length paths and minimum-capacity cuts.

Algorithmic duality. The duality of paths and cuts can be extended to the polynomial-time solvability of the corresponding optimization problems. Indeed, Corollary 5.14a implies that the following can be derived from the fact that $P^{\uparrow}_{s-t \text{ path}}(D)$ and $P^{\uparrow}_{s-t \text{ cut}}(D)$ form a blocking pair of polyhedra (Corollary 13.1d):

(13.7) the minimum-length path problem is polynomial-time solvable
$\Longleftrightarrow$
the minimum-capacity cut problem is polynomial-time solvable.

(Here the length and capacity functions are restricted to be nonnegative.)

By Theorem 5.15, one can also find a dual solution, which implies:

(13.8) the minimum-capacity cut problem is polynomial-time solvable
$\Longleftrightarrow$
the maximum flow problem is polynomial-time solvable.

The statements in (13.7) by themselves are not surprising, since the polynomial-time solvability of neither of the problems has turned out to be hard, although finding a shortest path in polynomial time is easier than finding a maximum flow in polynomial time.

However, it is good to realize that the equivalence has been derived purely from the theoretical fact that the two polyhedra form a blocking pair. In further chapters we will see more sophisticated applications of this principle.

Dual integrality. The fact that $P^{\uparrow}_{s-t \text{ path}}(D)$ and $P^{\uparrow}_{s-t \text{ cut}}(D)$ form a blocking pair of polyhedra, is equivalent to the fact that the polyhedra determined by (13.2) and (13.5) each are integer (that is, have integer vertices only). More precisely, blocking polyhedra theory tells us:

(13.9) the polyhedron determined by (13.2) is integer $\Longleftrightarrow$ the polyhedron determined by (13.5) is integer.

In other words, minimizing any linear function over (13.2) gives an integer optimum solution if and only if minimizing any linear function over (13.5) gives an integer optimum solution. Thus there is an equivalence of the existence of integer optimum solutions between two classes of linear programming problems. What can be said about the *dual* linear programs?

There is no general theorem known that links the existence of integer optimum dual solutions of blocking pairs of polyhedra. In fact, it is not the case that if two systems $Ax \leq b$ and $A'x \leq b'$ of linear inequalities represent a blocking pair of polyhedra, then the existence of integer optimum dual solutions for one system implies the existence of integer optimum dual solutions for the other. That is, the total dual integrality of $Ax \leq b$ is not equivalent to the total dual integrality of $A'x \leq b'$ (even not if one puts strong conditions on the two systems, like A, b, A', and b' being $0, 1$).

Yet, in the special case of paths and cuts the systems *are* totally dual integral, as follows directly from theorems proved in previous chapters. (In particular, total dual integrality of (13.5) amounts to the integrity theorem for flows.)

Theorem 13.2. *The systems* (13.2) *and* (13.5) *are totally dual integral.*

Proof. Total dual integrality of (13.2) is equivalent to Theorem 7.1, and total dual integrality of (13.5) is equivalent to Corollary 10.3a. ∎

By Theorem 5.22, total dual integrality of a system $Ax \leq b$ (with b integer) implies total *primal* integrality; that is, integrality of the polyhedron determined by $Ax \leq b$. So general polyhedral theory gives the following implications:

$$\begin{array}{ccc}
\boxed{\text{(13.2) determines an integer polyhedron}} & \Longleftrightarrow & \boxed{\text{(13.5) determines an integer polyhedron}} \\
\Uparrow & & \Uparrow \\
\boxed{\text{(13.2) is totally dual integral}} & & \boxed{\text{(13.5) is totally dual integral}}
\end{array} \tag{13.10}$$

13.1a. Vertices, adjacency, and facets

Vertices of the dominant of the $s-t$ path polytope have a simple characterization:

Theorem 13.3. *A vector x is a vertex of $P^{\uparrow}_{s-t\text{ path}}$ if and only if $x = \chi^{\pi}$ for some $s-t$ path π.*

Proof. If $x = \chi^{\pi}$ for some $s-t$ path π, then x is a vertex of $P^{\uparrow}_{s-t\text{ path}}$, as for the length function l defined by $l(a) := 0$ if $a \in A\pi$ and $l(a) := 1$ otherwise, the path π is the unique shortest $s-t$ path.

Conversely, let x be a vertex. As x is integer, $x \geq \chi^{\pi}$ for some $s-t$ path π. Then χ^{π} and $2x - \chi^{\pi} = x + (x - \chi^{\pi})$ belong to $P^{\uparrow}_{s-t\text{ path}}$. As $x = (\chi^{\pi} + (2x - \chi^{\pi}))/2$, we have $x = \chi^{\pi}$. ∎

As for adjacency, one has:

Theorem 13.4. *Let π and π' be two distinct $s-t$ paths in D. Then χ^{π} and $\chi^{\pi'}$ are adjacent vertices of $P^{\uparrow}_{s-t\text{ path}}$ if and only if $A\pi \triangle A\pi'$ is an undirected circuit consisting of two internally vertex-disjoint directed paths.*

Proof. If $A\pi \triangle A\pi'$ is an undirected circuit consisting of two internally vertex-disjoint paths, define the length function l by $l(a) := 0$ if $a \in A\pi \cup A\pi'$ and $l(a) := 1$ otherwise. Then π and π' are the only two shortest $s-t$ paths.

Conversely, let χ^{π} and $\chi^{\pi'}$ be adjacent. Suppose that $A\pi \cup A\pi'$ contains an $s-t$ path π'' different from π and π'. Then $\chi^{\pi} + \chi^{\pi'} - \chi^{\pi''} = \chi^{\pi'''}$ for some $s-t$ path π''', contradicting the adjacency of χ^{π} and $\chi^{\pi'}$. This implies that $A\pi \triangle A\pi'$ is an undirected circuit consisting of two internally vertex-disjoint directed paths. ∎

Finally, for the facets we have:

Theorem 13.5. *Let C be an $s-t$ cut. Then the inequality $x(C) \geq 1$ determines a facet of $P^{\uparrow}_{s-t\text{ path}}$ if and only if C is an inclusionwise minimal $s-t$ cut.*

Proof. *Necessity.* Suppose that there is an $s-t$ cut $C' \subset C$. Then the inequalities $x \geq \mathbf{0}$ and $x(C') \geq 1$ imply $x(C) \geq 1$, and hence $x(C) \geq 1$ is not facet-inducing.

Sufficiency. If the inequality $x(C) \geq 1$ is not facet-inducing, it is a nonnegative linear combination of other inequalities in system (13.2). At least one of them is of the form $x(C') \geq 1$ for some $s-t$ cut C'. Then necessarily $C' \subset C$. ∎

Garg and Vazirani [1993,1995] characterized the vertices of and adjacency on a variant of the $s-t$ cut polytope.

13.1b. The $s-t$ connector polytope

There are a number of related polyhedra for which similar results hold. Call a subset A' of A an *$s-t$ connector* if A' contains the arc set of an $s-t$ path as a subset. The *$s-t$ connector polytope* $P_{s-t \text{ connector}}(D)$ is the convex hull of the incidence vectors of the $s-t$ connectors.

This polytope turns out to be determined by the following system of linear inequalities:

(13.11) (i) $0 \leq x(a) \leq 1$ for each $a \in A$,
(ii) $x(C) \geq 1$ for each $s-t$ cut C.

Again, the fact that the $s-t$ connector polytope is contained in the polytope determined by (13.11) follows from the fact that χ^P satisfies (13.11) for each $s-t$ connector P.

Also in this case one has:

Theorem 13.6. *System* (13.11) *is totally dual integral.*

Proof. Directly from Theorem 13.2, using Theorem 5.23. ∎

It implies primal integrality:

Corollary 13.6a. *The $s-t$ connector polytope is equal to the solution set of* (13.11).

Proof. Directly from Theorem 13.6. ∎

The dimension of $P_{s-t \text{ connector}}(D)$ is easily determined:

Theorem 13.7. *Let A' be the set of arcs a for which there exists an $s-t$ path not traversing a. Then* $\dim P_{s-t \text{ connector}}(D) = |A'|$.

Proof. We use Theorem 5.6. Clearly, no inequality $x_a \geq 0$ is an implicit equality. Moreover, the inequality $x_a \leq 1$ is an implicit equality if and only if $a \in A \setminus A'$. For distinct arcs $a \in A \setminus A'$, these equalities are independent.

Suppose that there is a $U \subseteq V$ with $s \in U$, $t \notin U$, such that $x(\delta^{\text{out}}(U)) \geq 1$ is an implicit equality. Then $|\delta^{\text{out}}(U)| = 1$, since the all-one vector belongs to the polytope. So the arc in $\delta^{\text{out}}(U)$ belongs to $A \setminus A'$.

We conclude that the maximum number of independent implicit equalities is equal to $|A \setminus A'|$. Hence $\dim P = |A'|$. ∎

Let $D = (V, A)$ be a digraph, let $s, t \in V$, and let $k \in \mathbb{Z}_+$. By the results above, the convex hull P of the incidence vectors χ^B of those subsets B of A that contain k arc-disjoint $s-t$ paths, is determined by

$$\text{(13.12)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x(a) \leq 1 & \text{for each } a \in A, \\ \text{(ii)} & x(C) \geq k & \text{for each } s-t \text{ cut } C. \end{array}$$

L.E. Trotter, Jr observed that this polytope P has the *integer decomposition property*; that is, for each $l \in \mathbb{Z}_+$, any integer vector $x \in l \cdot P$ is the sum of l integer vectors in P.

Theorem 13.8. *The polytope P determined by* (13.12) *has the integer decomposition property.*

Proof. Let $l \in \mathbb{Z}_+$ and let $x \in l \cdot P$. Then there exists an integer $s-t$ flow $f \leq x$ of value $l \cdot k$ (by the max-flow min-cut theorem). We can assume that $x = f$. As $\frac{1}{l}f$ is an $s-t$ flow of value k, by Corollary 11.2c there exists an integer $s-t$ flow f' of value k with

$$\text{(13.13)} \qquad \lfloor \frac{1}{l} f(a) \rfloor \leq f'(a) \leq \lceil \frac{1}{l} f(a) \rceil$$

for each arc a. Then f' is an integer vector in P, since $f'(a) \leq 1$ for each arc a, as $f(a) \leq l$ for each arc a. Moreover, $f - f'$ is an integer vector belonging to $(l-1) \cdot P$, as $f - f'$ is an $s-t$ flow of value $(l-1) \cdot k$ and as $(f - f')(a) \leq l - 1$ for each arc a, since if $f(a) = l$, then $f'(a) = 1$ by (13.13). ∎

13.2. Total unimodularity

Let $D = (V, A)$ be a digraph. Recall that the $V \times A$ *incidence matrix* M of D is defined by $M_{v,a} := -1$ if a leaves v, $M_{v,a} := +1$ if a enters v, and $M_{v,a} := 0$ otherwise. So each column of M contains exactly one $+1$ and exactly one -1, while all other entries are 0. The following basic statement follows from a theorem of Poincaré [1900][21] (we follow the proofs of Chuard [1922] and Veblen and Franklin [1921]):

Theorem 13.9. *The incidence matrix M of any digraph D is totally unimodular.*

Proof. Let B be a square submatrix of M, of order k say. We prove that $\det B \in \{0, \pm 1\}$ by induction on k, the case $k = 1$ being trivial. Let $k > 1$. We distinguish three cases.

[21] Poincaré [1900] showed the total unimodularity of any $\{0, \pm 1\}$ matrix $M = (M_{i,j})$ with the property that for each k and all distinct row indices $i_1, \ldots, i_k$ and all distinct column indices $j_1, \ldots, j_k$, the product

$$M_{i_1,j_1} M_{i_1,j_2} M_{i_2,j_2} M_{i_2,j_3} \cdots M_{i_{k-1},j_{k-1}} M_{i_{k-1},j_k} M_{i_k,j_k} M_{i_k,j_1}$$

belongs to $\{0, 1\}$ if k is even and to $\{0, -1\}$ if k is odd. Incidence matrices of digraphs have this property.

Case 1: B has a column with only zeros. Then $\det B = 0$.

Case 2: B has a column with exactly one nonzero. Then we can write (up to permuting rows and columns):

$$(13.14) \qquad B = \begin{pmatrix} \pm 1 & b^{\mathsf{T}} \\ \mathbf{0} & B' \end{pmatrix},$$

for some vector b and matrix B'. Then by the induction hypothesis, $\det B' \in \{0, \pm 1\}$, and hence $\det B \in \{0, \pm 1\}$.

Case 3 : Each column of B contains two nonzeros. Then each column of B contains one $+1$ and one -1, while all other entries are 0. So each row of B adds up to 0, and hence $\det B = 0$. ∎

One can derive several results on circulations, flows, and transshipments from the total unimodularity of the incidence matrix of a digraph, like the max-flow min-cut theorem (see Section 13.2a below) and theorems characterizing the existence of a circulation or a b-transshipment (Theorem 11.2 and Corollary 11.2f). Moreover, min-max equalities for minimum-cost flow, circulation (cf. Theorem 12.8), and b-transshipment follow. We discuss some of the previous and some new results in the following sections.

13.2a. Consequences for flows

We show that the max-flow min-cut theorem can be derived from the total unimodularity of the incidence matrix of a digraph:

Corollary 13.9a (max-flow min-cut theorem)**.** *Let $D = (V, A)$, let $s, t \in V$, and let $c : A \to \mathbb{R}_+$ be a capacity function. Then the maximum value of an $s-t$ flow $f \leq c$ is equal to the minimum capacity of an $s-t$ cut.*

Proof. Since the maximum clearly cannot exceed the minimum, it suffices to show that there exists an $s-t$ flow $x \leq c$ and an $s-t$ cut, whose capacity is not more than the value of x.

Let M be the incidence matrix of D and let M' arise from M by deleting the rows corresponding to s and t. So the condition $M'x = \mathbf{0}$ means that the flow conservation law should hold at any vertex $v \neq s, t$.

Let w be the row of M corresponding to vertex t. So for any arc a, $w_a = +1$ if a enters t, $w_a = -1$ if a leaves t, and $w_a = 0$ otherwise.

Now the maximum value of an $s-t$ flow subject to c is equal to

$$(13.15) \qquad \max\{w^{\mathsf{T}} x \mid \mathbf{0} \leq x \leq c; M'x = \mathbf{0}\}.$$

By LP-duality, this is equal to

$$(13.16) \qquad \min\{y^{\mathsf{T}} c \mid y \geq \mathbf{0}; \exists z : y^{\mathsf{T}} + z^{\mathsf{T}} M' \geq w^{\mathsf{T}}\}.$$

Since M' is totally unimodular by Theorem 13.9 and since w is an integer vector, minimum (13.16) is attained by *integer* vectors y and z. Extend z by defining $z_t := -1$ and $z_s := 0$. Then $y^{\mathsf{T}} + z^{\mathsf{T}} M \geq \mathbf{0}$.

Now define

(13.17) $U := \{v \in V \mid z_v \geq 0\}$.

Then U is a subset of V containing s and not containing t.

It suffices to show that

(13.18) $c(\delta^{\text{out}}(U)) \leq y^{\mathsf{T}}c$,

since $y^{\mathsf{T}}c$ is equal to the maximum flow value (13.15).

To prove (13.18), it suffices to show that

(13.19) if $a = (u, v) \in \delta^{\text{out}}(U)$, then $y_a \geq 1$.

To see this, note that $z_u \geq 0$ and $z_v \leq -1$. Moreover, $y^{\mathsf{T}} + z^{\mathsf{T}}M \geq \mathbf{0}$ implies $y_a + z_v - z_u \geq 0$. So $y_a \geq z_u - z_v \geq 1$. This proves (13.19). ∎

It follows similarly that if all capacities are integers, then there exists a maximum *integer* flow; that is, we have the integrity theorem (Corollary 10.3a).

Let $D = (V, A)$ be a digraph and $s, t \in V$. The set of all $s - t$ flows of value 1 is a polyhedron $P_{s-t \text{ flow}}(D)$, determined by:

(13.20)
(i) $x(a) \geq 0$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(v)) = x(\delta^{\text{out}}(v))$ for each $v \in V \setminus \{s, t\}$,
(iii) $x(\delta^{\text{out}}(s)) - x(\delta^{\text{in}}(s)) = 1$.

The set of all $s - t$ flows of value ϕ trivially equals $\phi \cdot P_{s-t \text{ flow}}(D)$.

The total unimodularity of M gives that, for integer ϕ, the intersection of $\phi \cdot P_{s-t \text{ flow}}(D)$ with an integer box $\{x \mid \mathbf{0} \leq x \leq c\}$ is an integer polytope. In other words:

Theorem 13.10. *Let $D = (V, A)$ be a digraph, $s, t \in V$, $c : A \to \mathbb{Z}$, and $\phi \in \mathbb{Z}_+$. Then the set of $s - t$ flows $x \leq c$ of value ϕ forms an integer polytope.*

Proof. Directly from the total unimodularity of the incidence matrix of a digraph (using Theorem 5.20). ∎

In particular this gives:

Corollary 13.10a. *Let $D = (V, A)$ be a digraph, $s, t \in V$, $c : A \to \mathbb{Z}$ and $\phi \in \mathbb{Z}$. If there exists an $s - t$ flow $x \leq c$ of value ϕ, then there exists an integer such flow.*

Proof. Directly from Theorem 13.10. ∎

Notes. A relation of $P_{s-t \text{ flow}}(D)$ with the polytope $P_{s-t \text{ path}}(D)$ is that

(13.21) $P_{s-t \text{ path}}(D) \subseteq P_{s-t \text{ flow}}(D) \subseteq P^{\uparrow}_{s-t \text{ path}}(D)$.

Hence

(13.22) $P^{\uparrow}_{s-t \text{ path}}(D) = P^{\uparrow}_{s-t \text{ flow}}(D)$.

Dantzig [1963] (pp. 352–366) showed that each vertex x of $P_{s-t \text{ flow}}(D)$ is the incidence vector χ^P of some $s - t$ path P. It can be shown that, if each arc of D is in some $s - t$ path, then $P_{s-t \text{ flow}}(D)$ is the topological closure of the convex hull of the vectors $\chi^P \in \mathbb{R}^A$ where P is an $s - t$ walk and where

(13.23) $\chi^P(a) :=$ number of times P traverses a,

for $a \in A$.

For two distinct $s-t$ paths, the vertices χ^P and $\chi^{P'}$ are adjacent if and only if the symmetric difference $AP \triangle AP'$ forms an undirected circuit consisting of two internally vertex disjoint directed paths.

Saigal [1969] proved that any two vertices of $P_{s-t \text{ flow}}(D)$ are connected by a path on the 1-skeleton of $P_{s-t \text{ flow}}(D)$ with at most $|A|-1$ edges — this implies the Hirsch conjecture for this class of polyhedra. (The Hirsch conjecture (cf. Dantzig [1963,1964]) says that the 1-skeleton of a polytope in $\mathbb{R}^n$ determined by m inequalities has diameter at most $m-n$.) In fact, Saigal showed more strongly that for any two feasible bases B, B' of (13.20), there is a series of at most $|A|-1$ pivots bringing B to B'. It amounts to the following. Call a spanning tree T of D *feasible* if it contains a directed $s-t$ path. Call two feasible spanning trees T, T' *adjacent* if $|AT \setminus AT'| = 1$. Then for any two feasible spanning trees T, T' there exists a sequence $T_0, \ldots, T_k$ of feasible spanning trees such that $T_0 = T$, $T_k = T'$, $k \leq |A|-1$, and T_{i-1} and T_i adjacent for $i = 1, \ldots, k$.

Rispoli [1992] showed that if D is the complete directed graph, then for each length function l and each vertex x_0 of (13.20), there is a path $x_0, x_1, \ldots, x_d$ on the 1-skeleton of (13.20), where $l^\mathsf{T} x_i \leq l^\mathsf{T} x_{i-1}$ for $i = 1, \ldots, d$, where x_d minimizes $l^\mathsf{T} x$ over (13.20), and where $d \leq \frac{2}{3}(|V|-1)$.

13.2b. Consequences for circulations

Another consequence is:

Corollary 13.10b. *Let $D = (V, A)$ be a digraph and let $c, d : A \to \mathbb{Z}$. Then the set of circulations x satisfying $d \leq x \leq c$ forms an integer polytope.*

Proof. Directly from the total unimodularity of the incidence matrix of a digraph. ∎

In particular this implies:

Corollary 13.10c. *Let $D = (V, A)$ be a digraph and let $c, d : A \to \mathbb{Z}$. If there exists a circulation x satisfying $d \leq x \leq c$, then there exists an integer such circulation.*

Proof. Directly from Corollary 13.10b. ∎

Another consequence is the integer decomposition property as in Corollary 11.2b.

13.2c. Consequences for transshipments

Let $D = (V, A)$ be a digraph, $d, c \in \mathbb{R}_+^A$, and $b \in \mathbb{R}^V$. The *b-transshipment polytope* is the set of all b-transshipments x with $d \leq x \leq c$. So it is equal to

(13.24) $P := \{x \in \mathbb{R}^A \mid d \leq x \leq c, Mx = b\}$, where M is the $V \times A$ incidence matrix of D.

Again, the total unimodularity of M (Theorem 13.9) implies:

Theorem 13.11. *Let $D = (V, A)$ be a digraph, $b : V \to \mathbb{Z}$, and $c, d : A \to \mathbb{Z}$. Then the b-transshipment polytope is an integer polytope.*

Proof. Directly from the total unimodularity of the incidence matrix of a digraph. ∎

In particular this gives:

Corollary 13.11a. *Let $D = (V, A)$ be a digraph, $b : V \to \mathbb{Z}$, and $c, d : A \to \mathbb{Z}$. If there exists a b-transshipment x with $d \leq x \leq c$, then there exists an integer such b-transshipment.*

Proof. Directly from Theorem 13.11. ∎

Also Corollary 11.2f can be derived:

Theorem 13.12. *There exists a b-transshipment x satisfying $d \leq x \leq c$ if and only if $b(V) = 0$, $d \leq c$ and $c(\delta^{\mathrm{in}}(U)) - d(\delta^{\mathrm{out}}(U)) \geq b(U)$ for each $U \subseteq V$.*

Proof. Necessity being easy, we show sufficiency. If no b-transshipment as required exists, then by Farkas' lemma, there exist vectors $y \in \mathbb{R}^V$ and $z', z'' \in \mathbb{R}^A_+$ such that $y^{\mathsf{T}} M + z'^{\mathsf{T}} - z''^{\mathsf{T}} = \mathbf{0}$ and $y^{\mathsf{T}} b + z'^{\mathsf{T}} c - z''^{\mathsf{T}} d < 0$. By adding a multiple of $\mathbf{1}$ to y we can assume that $y \geq \mathbf{0}$ (since $\mathbf{1}^{\mathsf{T}} M = \mathbf{0}$ and $\mathbf{1}^{\mathsf{T}} b = 0$). Next, by scaling we can assume that $\mathbf{0} \leq y \leq \mathbf{1}$. As M is totally unimodular, we can assume moreover that y is integer. So $y = \chi^U$ for some $U \subseteq V$. Since $d \leq c$, we can assume that $z'(a) = 0$ or $z''(a) = 0$ for each $a \in A$. Hence $z' = \chi^{\delta^{\mathrm{out}}(U)}$ and $z'' = \chi^{\delta^{\mathrm{in}}(U)}$. Then $y^{\mathsf{T}} b + z'^{\mathsf{T}} c - z''^{\mathsf{T}} d < 0$ contradicts the condition for $V \setminus U$. ∎

For any digraph $D = (V, A)$ and $b \in \mathbb{R}^V$, let P_b denote the set of b-transshipments. So

(13.25) $\quad P_b = \{x \mid Mx = b\}$

where M is the $V \times A$ incidence matrix of D. Koopmans and Reiter [1951] characterized the dimension of the transshipment space:

(13.26) if P_b is nonempty, then it has dimension $|A| - |V| + k$, where k is the number of weak components of D.

(A *weak component* of a digraph is a component of the underlying undirected graph.)

To see (13.26), let $F \subseteq A$ form a spanning forest in the underlying undirected graph. So (V, F) has k weak components and contains no undirected circuit. Then $|F| = |V| - k$. Now each $x \in \mathbb{R}^{A \setminus F}$ can be extended uniquely to a b-transshipment $x \in \mathbb{R}^A$. Hence P_b has dimension $|A \setminus F| = |A| - |V| + k$.

Consider next the polyhedron

(13.27) $\quad Q_b := \{x \in \mathbb{R}^A \mid$ there exists a nonnegative b-transshipment $f \leq x\}$

So

(13.28) $$Q_b = (P_b \cap \mathbb{R}_+^A) + \mathbb{R}_+^A.$$

By Gale's theorem (Corollary 11.2g), Q_b is determined by:

(13.29)
$$\begin{array}{llll} \text{(i)} & x(a) \geq 0 & \text{for each } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(U)) \geq b(U) & \text{for each } U \subseteq V. \end{array}$$

Fulkerson and Weinberger [1975] showed that this system is TDI:

Theorem 13.13. *System* (13.29) *is TDI.*

Proof. Choose $w \in \mathbb{Z}_+^A$. We must show that the dual of minimizing $w^\mathsf{T} x$ over (13.29) has an integer optimum solution.

Let μ be the minimum value of $w^\mathsf{T} x$ over (13.29). As (13.29) determines Q_b, μ is equal to the minimum value of $w^\mathsf{T} x$ over $x \geq \mathbf{0}$, $Mx = b$, where M is the $V \times A$ incidence matrix of D. Since M is totally unimodular, this LP-problem has an integer optimum dual solution. That is, there exists a $y \in \mathbb{Z}^V$ such that $y^\mathsf{T} M \leq w^\mathsf{T}$ and $y^\mathsf{T} b = \mu$. We can assume that $y \geq \mathbf{0}$, since $\mathbf{1}^\mathsf{T} M = \mathbf{0}$ and $\mathbf{1}^\mathsf{T} b = 0$ (we can add a multiple of $\mathbf{1}$ to y). For each $i \in \mathbb{Z}_+$, let $U_i := \{v \mid y_v \geq i\}$. (So $U_i = \emptyset$ for i large enough.) Then

(13.30) $$\sum_{i=1}^{\infty} \chi^{\delta^{\text{out}}(U_i)} \leq w,$$

since for each arc $a = (u, v)$ we have $y_v - y_u \leq w(a)$, implying that the number of i such that a enters U_i is at most $\max\{0, y_v - y_u\}$, which is at most $w(a)$. So this gives a feasible integer dual solution to the problem of minimizing $w^\mathsf{T} x$ over (13.29). It is in fact optimum, since

(13.31) $$\sum_{i=1}^{\infty} b(U_i) = y^\mathsf{T} b = \mu.$$

This proves the theorem. ∎

This implies for primal integrality:

Corollary 13.13a. *If b is integer, then Q_b is integer.*

Proof. Directly from Theorem 13.13. ∎

Fulkerson and Weinberger [1975] also showed an integer decomposition theorem for nonnegative b-transshipments (it also follows directly from the total unimodularity of the incidence matrix M of D):

Theorem 13.14. *Let $D = (V, A)$, $b \in \mathbb{Z}^V$, and $k \in \mathbb{Z}_+$, with $k \geq 1$, and let $f : A \to \mathbb{Z}_+$ be a $k \cdot b$-transshipment. Then there exist b-transshipments $f_1, \ldots, f_k : A \to \mathbb{Z}_+$ such that $f = f_1 + \cdots + f_k$.*

Proof. It suffices to show that there exists a b-transshipment $g : A \to \mathbb{Z}_+$ such that $g \leq f$ — the theorem then follows by induction on k.

The existence of g follows from Gale's theorem (Corollary 11.2g), since $b(U) \leq f(\delta^{\text{in}}(U))$ for each $U \subseteq V$, as either $b(U) < 0$, or $b(U) \leq kb(U) \leq f(\delta^{\text{in}}(U))$. ∎

Theorem 13.14 implies the integer decomposition property for the polyhedron Q_b:

Corollary 13.14a. *If b is integer, the polyhedron Q_b has the integer decomposition property.*

Proof. Let $k \in \mathbb{Z}_+$ and let c be an integer vector in kQ_b. So $c \in Q_{k \cdot b}$, implying that there exists an integer $k \cdot b$-transshipment $f \leq c$. By Theorem 13.14, there exist integer b-transshipments $f_1, \ldots, f_k \geq \mathbf{0}$ with $f = f_1 + \cdots + f_k$. Define $f_1' := f_1 + (c - f)$. Then $f_1', f_2, \ldots, f_k \in Q_b$ and $c = f_1' + f_2 + \cdots + f_k$. ∎

If b is integer, we know:

(13.32) $\quad Q_b = \text{conv.hull}\{f \mid f \text{ nonnegative integer } b\text{-transshipment}\} + \mathbb{R}_+^A.$

Hence the blocking polyhedron $B(Q_b)$ of Q_b is determined by:

(13.33) (i) $x(a) \geq 0$ for each $a \in A$,
(ii) $f^{\mathsf{T}} x \geq 1$ for each nonnegative integer b-transshipment f.

Fulkerson and Weinberger [1975] derived from Corollary 13.14a that this system has the integer rounding property if b is integer:

Corollary 13.14b. *If b is integer, system* (13.33) *has the integer rounding property.*

Proof. Choose $c \in \mathbb{Z}^A$. Let

(13.34) $\quad \mu := \max\{\sum_f z_f \mid z_f \geq 0, \sum_f z_f f \leq c\}$

and let μ' be the maximum in which the z_f are restricted to nonnegative integers (here f ranges over minimal nonnegative integer b-transshipments). Let $k := \lfloor \mu \rfloor$. We must show that $\mu' = k$.

As $\mu = \min\{c^{\mathsf{T}} x \mid x \in B(Q_b)\}$, we know that $c \in \mu \cdot Q_b$. Hence, as $k \leq \mu$, $c \in kQ_b$. By Corollary 13.14a, there exist nonnegative integer b-transshipments $f_1, \ldots, f_k$ with $f_1 + \cdots + f_k \leq c$. Hence $\mu' \geq k$. Since $\mu' \leq \mu$, we have $\mu' = k$. ∎

Generally, we cannot restrict (13.33) to those f that form a vertex of Q_b while maintaining the integer rounding property, as was shown by Fulkerson and Weinberger [1975]. Trotter and Weinberger [1978] extended these results to b-transshipments with upper and lower bounds on the arcs. For related results, see Bixby, Marcotte, and Trotter [1987].

13.2d. Unions of disjoint paths and cuts

The total unimodularity of the incidence matrix of a digraph can also be used to derive min-max relations for the minimum number of arcs covered by l arc-disjoint $s - t$ paths:

Theorem 13.15. *Let $D = (V, A)$ be a digraph, $s, t \in V$, and $l \in \mathbb{Z}_+$. Then the minimum value of $|AP_1| + \cdots + |AP_l|$ where $P_1, \ldots, P_l$ are arc-disjoint $s - t$ paths is equal to the maximum value of*

(13.35) $\qquad |\bigcup \mathcal{C}| - \sum_{C \in \mathcal{C}} (|C| - l),$

where $\mathcal{C}$ ranges over all collections of $s-t$ cuts.

Proof. The minimum in the theorem is equal to the minimum value of $\sum_{a \in A} f(a)$ subject to

(13.36) $$\begin{array}{ll} 0 \leq f(a) \leq 1 & \text{for each } a \in A, \\ f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v)) = 0 & \text{for each } v \in V \setminus \{s,t\}, \\ f(\delta^{\text{in}}(t)) - f(\delta^{\text{out}}(t)) = l. & \end{array}$$

By LP-duality and total unimodularity of the constraint matrix, this minimum value μ is equal to the maximum value of $l \cdot p(t) - \sum_{a \in A} y(a)$, where $y \in \mathbb{Z}_+^A$ and $p \in \mathbb{Z}^V$ satisfy:

(13.37) $$\begin{array}{l} p(s) = 0; \\ p(v) - p(u) - y(a) \leq 1 \text{ for each } a = (u,v) \in A. \end{array}$$

As $\mu \geq 0$, we know $p(t) \geq 0$. Let $r := p(t)$, and for $j = 1, \ldots, r$, let $U_j := \{v \in V \mid p(v) < j\}$ and $C_j := \delta^{\text{out}}(U_j)$. Then

(13.38) $$\begin{aligned} \sum_{j=1}^{r} |C_j| &\leq \sum_{\substack{a = (u,v) \in A \\ p(v) > p(u) \\ p(v) \geq 0 \\ p(u) < r}} (p(v) - p(u)) \leq \sum_{\substack{a = (u,v) \in A \\ p(v) > p(u) \\ p(v) \geq 0 \\ p(u) < r}} (1 + y(a)) \\ &\leq \left| \bigcup_{j=1}^{r} C_j \right| + \sum_{a \in A} y(a) = \left| \bigcup_{j=1}^{r} C_j \right| + l \cdot r - \mu. \end{aligned}$$

So

(13.39) $$\mu \leq \left| \bigcup_{j=1}^{r} C_j \right| - \sum_{j=1}^{r} (|C_j| - l),$$

and we have the required min-max equality. ∎

A similar formula holds for unions of arc-disjoint $s-t$ cuts:

Theorem 13.16. *Let $D = (V,A)$ be a digraph, $s,t \in V$, and $l \in \mathbb{Z}_+$. Then the minimum value of $|C_1| + \cdots + |C_l|$ where $C_1, \ldots, C_l$ are disjoint $s-t$ cuts is equal to the maximum value of*

(13.40) $\qquad |\bigcup \mathcal{P}| - \sum_{P \in \mathcal{P}} (|AP| - l),$

where $\mathcal{P}$ ranges over all collections of $s-t$ paths.

Proof. By total unimodularity, the minimum size of the union of l disjoint $s-t$ cuts is equal to the minimum value of $\sum_{a \in A} x(a)$ where $x \in \mathbb{R}^A$ and $p \in \mathbb{R}^V$ such that

(13.41) $$\begin{array}{ll} 0 \leq x(a) \leq 1 & \text{for each } a \in A, \\ p(v) - p(u) - x(a) \leq 0 & \text{for each } a = (u,v) \in A; \\ p(t) - p(s) = l. & \end{array}$$

By LP-duality and total unimodularity, this is equal to the maximum value of $l \cdot r - \sum_{a \in A} y(a)$, where $r \in \mathbb{Z}$, $y \in \mathbb{Z}_+^A$, $f \in \mathbb{Z}_+^A$ such that

(13.42)
$$\begin{aligned} &f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v)) = 0 && \text{for each } v \in V \setminus \{s,t\}, \\ &f(\delta^{\text{in}}(t)) - f(\delta^{\text{out}}(t)) = r, && \\ &f(a) - y(a) \leq 1 && \text{for each } a \in A. \end{aligned}$$

As f is an $s-t$ flow of value r, it is the sum of the incidence vectors of $s-t$ paths $P_1, \ldots, P_r$, say. Then

(13.43)
$$\sum_{a \in A} y(a) \geq \sum_{j=1}^{r} |AP_j| - \Big| \bigcup_{j=1}^{r} AP_j \Big|.$$

Hence we have the required equality. ∎

One may derive similarly min-max formulas for the minimum number of vertices in l internally vertex-disjoint $s-t$ paths and for the minimum number of vertices in l disjoint $s-t$ vertex-cuts.

Minimum-cost flow methods also provide fast algorithms to find optimum unions of disjoint paths:

Theorem 13.17. *Given a digraph $D = (V, A)$, $s, t \in V$, and $l \in \mathbb{Z}_+$, a collection of arc-disjoint $s-t$ paths $P_1, \ldots, P_l$ minimizing $|AP_1| + \cdots + |AP_l|$ can be found in time $O(lm)$.*

Proof. Directly from Theorems 12.6 and 11.1 and Corollary 7.8a. ∎

Similarly for disjoint cuts:

Theorem 13.18. *Given a digraph $D = (V, A)$, $s, t \in V$, $l \in \mathbb{Z}_+$, and a length function $k : A \to \mathbb{Q}_+$, a collection of arc-disjoint $s-t$ paths $P_1, \ldots, P_l$ minimizing $k(P_1) + \cdots + k(P_l)$ can be found in time $O(l(m + n \log n))$.*

Complexity survey for finding k arc-disjoint $s-t$ paths of minimum total length ($*$ indicates an asymptotically best bound in the table):

$*$	$O(k \cdot \text{SP}(n, m, L))$	Ford and Fulkerson [1958b], Jewell [1958], Busacker and Gowen [1960], Iri [1960]
	$O(nL \cdot \text{DP}_k(n, m))$	Edmonds and Karp [1972]
$*$	$O(n \log L \cdot \text{DP}_k(n, m))$	Röck [1980] (cf. Bland and Jensen [1992])

Here $\text{DP}_k(n, m)$ denotes the time needed to find k arc-disjoint disjoint $s-t$ paths in a digraph with n vertices and m edges.

Suurballe and Tarjan [1984] described an $O(m \log_{m/n} n)$ algorithm for finding, in a digraph with nonnegative length function and fixed vertex s, for all v a pair of edge-disjoint $s-v$ paths P_v, Q_v with $\text{length}(P_v) + \text{length}(Q_v)$ minimum.

Gabow [1983b,1985b] described minimum-cost flow algorithms for networks with unit capacities. The running times are $O(m^{7/4} \log L)$ and, if D is simple, $O(n^{1/3} m^{3/2} \log L)$. For the vertex-disjoint case, he gave algorithms with running time $O(n^{3/4}m \log L)$ and $O(nm \log_{2+\frac{m}{n}} L)$. Goldberg and Tarjan [1990] gave an $O(nm \log(nL))$ algorithm for minimum-cost flow with unit capacities. More complexity results follow from the table in Section 12.5a. Disjoint $s-t$ cuts were considered by Wagner [1990] and Talluri and Wagner [1994].

13.3. Network matrices

Let $D = (V, A)$ be a digraph and let $T = (V, A')$ be a directed tree. Let C be the $A' \times A$ matrix defined as follows. Take $a' \in A'$ and $a = (u, v) \in A$ and let P be the undirected $u - v$ path in T. Define

$$(13.44) \qquad C_{a',a} := \begin{cases} +1 & \text{if } a' \text{ occurs in forward direction in } P, \\ -1 & \text{if } a' \text{ occurs in backward direction in } P, \\ 0 & \text{if } a' \text{ does not occur in } P. \end{cases}$$

Matrix C is called a *network matrix, generated by* $T = (V, A')$ and $D = (V, A)$.

Theorem 13.19. *Any submatrix of a network matrix is again a network matrix.*

Proof. Deleting column indexed by $a \in A$ corresponds to deleting a from $D = (V, A)$. Deleting the row indexed by $a' = (u, v) \in A'$ corresponds to contracting a' in the tree $T = (V, A')$ and identifying u and v in D. ∎

The following theorem is implicit in Tutte [1965a]:

Theorem 13.20. *A network matrix is totally unimodular.*

Proof. By Theorem 13.19, it suffices to show that any square network matrix C has determinant 0, 1, or -1. We prove this by induction on the size of C, the case of 1×1 matrices being trivial. We use notation as above.

Assume that $\det C \neq 0$. Let u be an end vertex of T and let a' be the arc in T incident with u. By reversing orientations, we can assume that each arc in A and A' incident with u, has u as tail. Then, by definition of C, the row indexed by a' contains only 0's and 1's.

Consider two 1's in row a'. That is, consider two columns indexed by arcs $a_1 = (u, v_1)$ and $a_2 = (u, v_2)$ in A. Subtracting column a_1 from column a_2, has the effect of resetting a_2 to (v_1, v_2). So after that, column a_2 has a 0 in position a'. Since this subtraction does not change the determinant, we can assume that there is exactly one arc in A incident with u; that is, row a' has exactly one nonzero. Then by expanding the determinant by row a', we obtain inductively that $\det C = \pm 1$. ∎

The incidence matrix of a digraph $D = (V, A)$ is a network matrix: add a new vertex u to D giving digraph $D' = (V \cup \{u\}, A)$. Let T be the directed tree on $V \cup \{u\}$ with arcs (u, v) for $v \in V$. Then the network matrix generated by T and D' is equal to the incidence matrix of D.

Notes. Recognizing whether a given matrix is a network matrix has been studied by Gould [1958], Auslander and Trent [1959,1961], Tutte [1960,1965a,1967], Tomizawa [1976b], and Fujishige [1980a] (cf. Bixby and Wagner [1988] and Section 20.1 in Schrijver [1986b]).

Seymour [1980a] showed that all totally unimodular matrices can be obtained by glueing network matrices and copies of a certain 5×5 matrix together (cf. Schrijver [1986b] and Truemper [1992]).

13.4. Cross-free and laminar families

We now show how cross-free and laminar families of sets give rise to network matrices. The results in this section will be used mainly in Part V.

A family $\mathcal{C}$ of subsets of a finite set S is called *cross-free* if for all $X, Y \in \mathcal{C}$ one has

(13.45) $\quad X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$ or $X \cup Y = S$.

$\mathcal{C}$ is called *laminar* if for all $X, Y \in \mathcal{C}$ one has

(13.46) $\quad X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$.

So each laminar family is cross-free.

Cross-free families could be characterized geometrically as having a 'Venn-diagram' representation on the sphere without crossing lines. If the family is laminar we have such a representation in the plane.

A laminar collection $\mathcal{C}$ can be partitioned into 'levels': the ith level consists of all sets $X \in \mathcal{C}$ such that there are $i-1$ sets $Y \in \mathcal{C}$ satisfying $Y \supset X$. Then each level consists of disjoint sets, and for each set X of level $i+1$ there is a unique set of level i containing X.

Note that if $\mathcal{C}$ is a cross-free family, then adding, for each set $X \in \mathcal{C}$, the complement $S \setminus X$ to $\mathcal{C}$ maintains cross-freeness. Moreover, for any fixed $s \in S$, the family $\{X \in \mathcal{C} \mid s \notin X\}$ is laminar.

In order to relate cross-free families and directed trees, suppose that we have a directed tree $T = (V, A)$ and a function $\pi : S \to V$, for some set S. Then the pair T, π defines a family $\mathcal{C}$ of subsets of S as follows. Define for each arc $a = (u, v)$ of T the subset X_a of S by:

(13.47) $\quad X_a :=$ the set of vertices in the weak component of $T - a$ containing v.

So X_a is the set of $s \in S$ for which arc a 'points' in the direction of $\pi(s)$ in T.

Let $\mathcal{C}_{T,\pi}$ be the family of sets X_a; that is,

(13.48) $\quad \mathcal{C}_{T,\pi} := \{X_a \mid a \in A\}$.

If $\mathcal{C} = \mathcal{C}_{T,\pi}$, the pair T, π is called a *tree-representation* for $\mathcal{C}$. If moreover T is a rooted tree, then T, π is called a *rooted tree-representation* for $\mathcal{C}$.

It is easy to see that $\mathcal{C}_{T,\pi}$ is cross-free. Moreover, if T is a rooted tree, then $\mathcal{C}_{T,\pi}$ is laminar. In fact, each cross-free family has a tree-representation, and each laminar family has a rooted tree-representation, as is shown by the following theorem of Edmonds and Giles [1977]:

Theorem 13.21. *A family $\mathcal{C}$ of subsets of S is cross-free if and only if $\mathcal{C}$ has a tree-representation. Moreover, $\mathcal{C}$ is laminar if and only if $\mathcal{C}$ has a rooted tree-representation.*

Proof. As sufficiency of the conditions is easy, we show necessity. We first show that each laminar family $\mathcal{C}$ of subsets of a set S has a rooted tree-representation. The proof is by induction on $|\mathcal{C}|$, the case $\mathcal{C} = \emptyset$ being trivial. If $\mathcal{C} \neq \emptyset$, choose an inclusionwise minimal $X \in \mathcal{C}$. By induction, the family $\mathcal{C}' := \mathcal{C} \setminus \{X\}$ has a rooted tree-representation $T = (V, A)$, $\pi : S \to V$.

If $X = \emptyset$, then we can add to T a new arc from any vertex to a new vertex, to obtain a rooted tree-representation T', π of $\mathcal{C}$. So we can assume that $X \neq \emptyset$.

Now $|\pi(X)| = 1$, since if $\pi(x) \neq \pi(y)$ for some $x, y \in X$, then there is an arc a of T separating $\pi(x)$ and $\pi(y)$. Hence the set $X_a \in \mathcal{C}'$ contains one of $\pi(x)$ and $\pi(y)$, say $\pi(y)$. As $\mathcal{C}$ is laminar, this implies that X_a is properly contained in X, contradicting the minimality of X.

This proves that $|\pi(X)| = 1$. Let v be the vertex of T with $\pi(X) = \{v\}$. Augment T by a new vertex w and a new arc $b = (v, w)$. Reset $\pi(z) := w$ for each $z \in X$. Then the new tree and π form a rooted tree-representation for $\mathcal{C}$. This shows that each laminar family has a rooted tree-representation.

To see that each cross-free family $\mathcal{C}$ has a tree-representation, choose $s \in S$, and let $\mathcal{G}$ be obtained from $\mathcal{C}$ by replacing any set containing s by its complement. Then $\mathcal{G}$ is laminar, and hence it has a rooted tree-representation by the foregoing. Reversing arcs in the tree if necessary, it gives a tree-representation for $\mathcal{C}$. ∎

From Theorems 13.20 and 13.21 we derive the total unimodularity of certain matrices. Let $D = (V, A)$ be a directed graph and let $\mathcal{C}$ be a family of subsets of V. Let N be the $\mathcal{C} \times A$ matrix defined by:

$$(13.49) \qquad N_{X,a} := \begin{cases} 1 & \text{if } a \text{ enters } X, \\ -1 & \text{if } a \text{ leaves } X, \\ 0 & \text{otherwise,} \end{cases}$$

for $X \in \mathcal{C}$ and $a \in A$.

Corollary 13.21a. *If $\mathcal{C}$ is cross-free, then N is a network matrix, and hence N is totally unimodular.*

Proof. Let $T = (W, B)$, $\pi : V \to W$ be a tree-representation for $\mathcal{C}$. Let $D' = (W, A')$ be the directed graph with

$$A' := \{(\pi(u), \pi(v)) \mid (u, v) \in A\}. \tag{13.50}$$

Then N is equal to the network matrix generated by T and D' (up to identifying any arc b of T with the set X_b in $\mathcal{C}$ determined by b, and any arc (u, v) of D with the arc $(\pi(u), \pi(v))$ of D'). Hence by Theorem 13.20, N is totally unimodular. ∎

Chapter 14

Partially ordered sets and path coverings

Partially ordered sets can be considered as a special type of networks, and several optimization problems on partially ordered sets can be handled with flow techniques. Basic theorem is Dilworth's min-max relation for the maximum size of an antichain.

14.1. Partially ordered sets

A *partially ordered set* is a pair $(S, \leq)$ where S is a set and where $\leq$ is a relation on S satisfying:

(14.1) (i) $s \leq s$,
(ii) if $s \leq t$ and $t \leq s$, then $s = t$,
(iii) if $s \leq t$ and $t \leq u$, then $s \leq u$,

for all $s, t, u \in S$. We put $s < t$ if $s \leq t$ and $s \neq t$. We restrict ourselves to *finite* partially ordered sets; that is, with S finite.

A subset C of S is called a *chain* if $s \leq t$ or $t \leq s$ for all $s, t \in C$. A subset A of S is called an *antichain* if $s \not< t$ and $t \not< s$ for all $s, t \in A$. Hence if C is a chain and A is an antichain, then

(14.2) $|C \cap A| \leq 1$.

First we notice the following easy min-max relation:

Theorem 14.1. *Let $(S, \leq)$ be a partially ordered set. Then the minimum number of antichains covering S is equal to the maximum size of a chain.*

Proof. That the maximum cannot be larger than the minimum follows easily from (14.2). To see that the two numbers are equal, define for any element $s \in S$ the *height* of s as the maximum size of any chain in S with maximum s. For any $i \in \mathbb{Z}_+$, let A_i denote the set of elements of height i. Let k be the maximum height of the elements of S. Then $A_1, \ldots, A_k$ are antichains covering S, and moreover there exists a chain of size k, since there exists an element of height k. ∎

This result can also be formulated in terms of graphs. Let $D = (V, A)$ be a digraph. A subset C of A is called a *directed cut* if there exists a subset U of V such that $\emptyset \neq U \neq V$, $\delta^{\text{out}}(U) = C$, and $\delta^{\text{in}}(U) = \emptyset$. Then Vidyasankar and Younger [1975] observed:

Corollary 14.1a. *Let $D = (V, A)$ be an acyclic digraph. Then the minimum number of directed cuts covering A is equal to the maximum length of a directed path.*

Proof. Define a partial order $\leq$ on A by: $a < a'$ if there exists a directed path traversing a and a', in this order. Applying Theorem 14.1 gives the Corollary. ∎

14.2. Dilworth's decomposition theorem

Dilworth [1950] proved that Theorem 14.1 remains true after interchanging the terms 'chain' and 'antichain', which is less simple to prove:

Theorem 14.2 (Dilworth's decomposition theorem)**.** *Let $(S, \leq)$ be a partially ordered set. Then the minimum number of chains covering S is equal to the maximum size of an antichain.*

Proof. That the maximum cannot be larger than the minimum follows easily from (14.2). To see that the two numbers are equal, we apply induction on $|S|$. Let α be the maximum size of an antichain and let A be an antichain of size α. Define

(14.3) $$A^{\downarrow} := \{s \in S \mid \exists t \in A : s \leq t\},$$
$$A^{\uparrow} := \{s \in S \mid \exists t \in A : s \geq t\}.$$

Then $A^{\downarrow} \cap A^{\uparrow} = A$ and $A^{\downarrow} \cup A^{\uparrow} = S$ (otherwise we can augment A).

First assume that $A^{\downarrow} \neq S$ and $A^{\uparrow} \neq S$. Then, by induction, $A^{\downarrow}$ can be covered by α chains. Since $A \subseteq A^{\downarrow}$, each of these chains contains exactly one element in A. For each $s \in A$, let C_s denote the chain containing s. Similarly, there exist α chains C'_s (for $s \in A$) covering $A^{\uparrow}$, where C'_s contains s. Then for each $s \in A$, $C_s \cup C'_s$ forms a chain in S, and moreover these chains cover S.

So we may assume that $A^{\downarrow} = S$ or $A^{\uparrow} = S$ for each antichain A of size α. It means that each antichain A of size α is either the set of minimal elements of S or the set of maximal elements of S. Now choose a minimal element s and a maximal element t of S with $s \leq t$. Then the maximum size of an antichain in $S \setminus \{s, t\}$ is equal to $\alpha - 1$ (since each antichain in S of size α contains s or t). By induction, $S \setminus \{s, t\}$ can be covered by $\alpha - 1$ chains. Adding the chain $\{s, t\}$ yields a covering of S by α chains. ∎

Notes. This proof is due to Perles [1963]. Dilworth original proof is based on a different induction. For a proof using linear programming duality, see Dantzig and Hoffman [1956]. For a deduction of Dilworth's decomposition theorem from Kőnig's matching theorem (Theorem 16.2), see Fulkerson [1956], Ford and Fulkerson [1962] (pp. 61–64), and Mirsky and Perfect [1966]. Further proofs were given by Dilworth [1960], Tverberg [1967], and Pretzel [1979].

14.3. Path coverings

Dilworth's decomposition theorem can be formulated equivalently in terms of covering vertices of a digraph[22]:

Corollary 14.2a. *Let $D = (V, A)$ be an acyclic digraph. Then the minimum number of paths covering all vertices is equal to the maximum number of vertices no two of which belong to a directed path.*

Proof. Apply Dilworth's decomposition theorem to the partially ordered set $(V, \leq)$ where $u \leq v$ if and only if v is reachable in D from u. ∎

As for covering the arcs, we have:

Corollary 14.2b. *Let $D = (V, A)$ be an acyclic digraph. Then the minimum number of paths covering all arcs is equal to the maximum size of a directed cut.*

Proof. Apply Dilworth's decomposition theorem to the partially ordered set $(A, \leq)$ where $a < a'$ if and only if there exists a directed path traversing a and a', in this order. ∎

Similarly, for $s - t$ paths:

Corollary 14.2c. *Let $D = (V, A)$ be an acyclic digraph with exactly one source, s, and exactly one sink, t. Then the minimum number of $s - t$ paths covering A is equal to the maximum size of a directed $s - t$ cut.*

Proof. Apply Dilworth's decomposition theorem to the partially ordered set $(A, \leq)$ defined by: $a \leq a'$ if and only if there exists an $s - t$ path traversing a and a', in this order. ∎

If only a subset of the arcs has to be covered, one has more generally:

Corollary 14.2d. *Given an acyclic digraph $D = (V, A)$ and $B \subseteq A$, the minimum number of paths covering B is equal to the maximum of $|C \cap B|$ where C is a directed cut.*

[22] Gallai and Milgram [1960] claim to have found this result in 1947.

Proof. Consider the partially ordered set $(B, \leq)$ with $a < a'$ if there exists a directed path traversing a and a', in this order. Then for each chain K in $(B, \leq)$ there is a path in D covering K, and for each antichain L in $(B, \leq)$ there is a directed cut C in D with $L \subseteq C \cap B$. Hence the theorem follows from Dilworth's decomposition theorem. ∎

14.4. The weighted case

Dilworth's decomposition theorem has a self-refining nature, and implies a weighted version. Let $(S, \leq)$ be a partially ordered set. Let $\mathcal{C}$ and $\mathcal{A}$ denote the collections of chains and antichains in $(S, \leq)$, respectively. Let $w : S \to \mathbb{Z}_+$ be a 'weight' function. Then:

Theorem 14.3. *The maximum weight $w(A)$ of an antichain A is equal to the minimum size of a family of chains covering each element s exactly $w(s)$ times.*

Proof. Replace each element s of S by $w(s)$ copies, making the set S'. For any copy s' of s and t' of t, define $s' <' t'$ if and only if $s < t$. This gives the partially ordered set $(S', \leq')$. Note that the copies of one element of S form an antichain in S'.

Then the maximum weight $w(A)$ of an antichain A in S is equal to the maximum size $|A'|$ of an antichain A' in S'. By Dilworth's decomposition theorem, S' can be covered by a collection Λ of $|A'|$ chains. Replacing the elements of each chain by their originals in S, gives the required equality. ∎

In terms of digraphs this gives the following result of Gallai [1958a,1958b]:

Corollary 14.3a. *Let $D = (V, A)$ be an acyclic digraph and let S and T be subsets of V such that each vertex is on at least one $S - T$ path. Let $c \in \mathbb{Z}_+^V$. Then the minimum number k of $S - T$ paths $P_1, \ldots, P_k$ such that each vertex v is covered at least $c(v)$ times by the P_i is equal to the maximum of $c(U)$ where U is a set of vertices intersecting each $S - T$ path at most once.*

Proof. Directly from Theorem 14.3 by defining the partially ordered set $(V, \leq)$ by: $u \leq v$ if and only if there exists a $u - v$ path. ∎

Similarly, there is the following 'min-flow max-cut theorem':

Corollary 14.3b. *Let $D = (V, A)$ be an acyclic digraph with exactly one source, s, and exactly one sink, t. Let $d : A \to \mathbb{R}_+$. Then the minimum value of any $s - t$ flow f satisfying $f \geq d$ is equal to the maximum value of $d(C)$ where C is a directed $s - t$ cut. If d is integer, we can take f integer.*

Proof. Define a partial order $\leq$ on A by: $a < a'$ if there is an $s-t$ path traversing a and a' in this order. Then any chain in A is contained in some $s-t$ path and any antichain is contained in some directed $s-t$ cut. Hence this Corollary can be derived from Theorem 14.3 (using continuity, compactness, and scaling). ∎

A similar weighted variant of the easier Theorem 14.1 holds: Again, let $(S, \leq)$ be a partially ordered set. Let $w : S \to \mathbb{Z}_+$ be a 'weight' function.

Theorem 14.4. *The maximum weight $w(C)$ of any chain is equal to the minimum size of a family of antichains covering each element s exactly $w(s)$ times.*

Proof. Similar to the proof of Theorem 14.3. ∎

The following 'length-width inequality' follows similarly:

Theorem 14.5. *Let $(S, \leq)$ be a partially ordered set and let $l, w : S \to \mathbb{R}_+$. Then*

$$\max_{C \text{ chain}} l(C) \cdot \max_{A \text{ antichain}} w(A) \geq \sum_{s \in S} l(s)w(s). \tag{14.4}$$

Proof. We can assume that l and w are rational (by continuity), and hence integer. Let t be the maximum of $l(C)$ taken over all chains C.

For each $s \in S$, let $h(s)$ be the maximum of $l(C)$ taken over all chains C with maximum element s. For each $k \in \mathbb{Z}$, let A_k be the set of those elements $s \in S$ with $h(s) - l(s) < k \leq h(s)$.

Then each A_k is an antichain. Moreover, $A_k = \emptyset$ if $k > t$, and

$$\sum_{k=1}^{t} \chi^{A_k} = l. \tag{14.5}$$

Therefore,

$$\begin{aligned} \max_{C} l(C) \cdot \max_{A} w(A) &= t \cdot \max_{A} w(A) \geq \sum_{k=1}^{t} w(A_k) = \sum_{k=1}^{t} w^{\mathsf{T}} \chi^{A_k} \\ &= w^{\mathsf{T}} l, \end{aligned} \tag{14.6}$$

where C and A range over chains and antichains, respectively. ∎

14.5. The chain and antichain polytopes

Let $(S, \leq)$ be a partially ordered set. The *chain polytope* $P_{\text{chain}}(S)$ of S is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of chains in S. Similarly, the *antichain polytope* $P_{\text{antichain}}(S)$ of S is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of antichains in S.

These two polytopes turn out to form an antiblocking pair of polyhedra. To see this, we first show:

Corollary 14.5a. *The chain polytope of S is determined by*

(14.7)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for each } s \in S, \\ \text{(ii)} & x(A) \leq 1 & \text{for each antichain } A, \end{array}$$

and this system is TDI.

Proof. Directly from Theorem 14.4, by LP-duality: for any chain C, χ^C is a feasible solution of (14.7). An antichain family covering each element s precisely $w(s)$ times gives a dual feasible solution. As the minimum of $w(C)$ is equal to the maximum value of these dual feasible solutions (by Theorem 14.4), the linear program of minimizing $w^{\mathsf{T}}x$ over (14.7) has integer optimum primal and dual solutions. ∎

Similarly for the antichain polytope:

Theorem 14.6. *The antichain polytope of S is determined by the inequalities*

(14.8)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for each } s \in S, \\ \text{(ii)} & x(C) \leq 1 & \text{for each chain } C, \end{array}$$

and this system is TDI.

Proof. Similar to the previous proof, now using Theorem 14.3. ∎

This implies that the chain and antichain polytope are related by the antiblocking relation:

Corollary 14.6a. *$P^{\uparrow}_{\text{chain}}(S)$ and $P^{\uparrow}_{\text{antichain}}(S)$ form an antiblocking pair of polyhedra.*

Proof. Directly from the previous results. ∎

14.5a. Path coverings algorithmically

When studying partially ordered sets $(S, \leq)$ algorithmically, we should know how these are represented. Generally, giving all pairs (s, t) with $s \leq t$ yields a large, redundant input. It suffices to give an acyclic digraph $D = (S, A)$ such that $s \leq t$ if and only if t is reachable from s. So it is best to formulate the algorithmic results in terms of acyclic digraphs.

The strong polynomial-time solvability of the problems discussed below follow from the strong polynomial-time solvability of minimum-cost circulation. We give some better running time bounds.

Theorem 14.7. *Given an acyclic digraph $D = (V, A)$ and $B \subseteq A$, a minimum number of paths covering B can be found in time $O(nm)$.*

Proof. Add two vertices s and t to D, and, for each vertex v, make $\deg_B^{\text{in}}(v)$ parallel arcs from s to v and $\deg_B^{\text{out}}(v)$ parallel arcs from v to t. Let $D' = (V', A')$ be the extended graph. By Theorem 9.10, we can find in time $O(nm)$ a maximum collection $P_1, \ldots, P_k$ of $s-t$ paths that are disjoint on the set $A' \setminus A$ of new arcs.

We make another auxiliary graph $\widetilde{D} = (V, \widetilde{A})$ as follows. Each arc in B belongs to $\widetilde{A}$. Moreover, for each $a = (u, v) \in A$, we make r parallel arcs from u to v, where r is the number of times a is traversed by the P_i. (So if $a \in B$, there are $r+1$ parallel arcs from u to v.) This gives the acyclic graph $\widetilde{D}$. Now choose, repeatedly as long as possible, in $\widetilde{D}$ a path from a (current) source to a (current) sink and remove its arcs. This gives us a collection of paths in $\widetilde{D}$ and hence also in D, covering all arcs in B. We claim that it has minimum size.

For each i, let P_i' be the path obtained from P_i by deleting the first and last arc. For each $v \in V$, let $\sigma(v)$ be the number of P_i that start with (s, v) and let $\tau(v)$ be the number of P_i that and with (v, t).

Let $U \subseteq V$ give a minimum $s-t$ cut $\delta_{A'}^{\text{out}}(U \cup \{s\})$ in D' with $\delta_A^{\text{out}}(U) = \emptyset$. As $P_1, \ldots, P_k$ form a maximum $s-t$ path packing in D', we have for each $v \in V$:

(14.9) $\quad \sigma(v) \leq \delta_B^{\text{in}}(v)$, with equality if $v \in V \setminus U$,
$\quad\quad\quad \tau(v) \leq \delta_B^{\text{out}}(v)$, with equality if $v \in U$.

So

$$\deg_{\widetilde{A}}^{\text{in}}(v) = \deg_B^{\text{in}}(v) + \sum_{i=1}^{k} \deg_{AP_i'}^{\text{in}}(v) \geq \sigma(v) + \sum_{i=1}^{k} \deg_{AP_i'}^{\text{in}}(v) = \sum_{i=1}^{k} \deg_{AP_i}^{\text{in}}(v), \tag{14.10}$$

with equality if $v \in V \setminus U$. Similarly,

$$\deg_{\widetilde{A}}^{\text{out}}(v) = \deg_B^{\text{out}}(v) + \sum_{i=1}^{k} \deg_{AP_i'}^{\text{out}}(v) \geq \tau(v) + \sum_{i=1}^{k} \deg_{AP_i'}^{\text{out}}(v) = \sum_{i=1}^{k} \deg_{AP_i}^{\text{out}}(v), \tag{14.11}$$

with equality if $v \in U$. Hence, since $\deg_{AP_i}^{\text{in}}(v) = \deg_{AP_i}^{\text{out}}(v)$ for each $v \in V$ and each $i = 1, \ldots, k$:

(14.12) $\quad \deg_{\widetilde{D}}^{\text{in}}(v) \geq \deg_{\widetilde{D}}^{\text{out}}(v)$ for each $v \in U$ and
$\quad\quad\quad \deg_{\widetilde{D}}^{\text{in}}(v) \leq \deg_{\widetilde{D}}^{\text{out}}(v)$ for each $v \in V \setminus U$.

Now deleting any source-sink path P in $\widetilde{D}$ does not invalidate (14.12). Moreover, P runs from $V \setminus U$ to U. Since none of the arcs in $\delta_A^{\text{in}}(U)$ are traversed by any P_i, we know that P should use an arc of $B \cap \delta^{\text{in}}(U)$. So the number of paths found is at most $|B \cap \delta_A^{\text{in}}(U)|$. Therefore, by Corollary 14.2d, the paths form a minimum-size collection of paths covering B. ∎

The special case where *all* arcs must be covered, is:

Corollary 14.7a. *Given an acyclic digraph $D = (V, A)$, a minimum collection of paths covering all arcs can be found in time $O(nm)$.*

Proof. Directly from the foregoing, by taking $B := A$. ∎

The theorem also applies to vertex coverings:

Corollary 14.7b. *Given an acyclic digraph $D = (V, A)$, a minimum number of paths covering all vertices can be found in time $O(nm)$.*

Proof. Introduce for each vertex v of D vertices v' and v''. Define $V' := \{v' \mid v \in V\}$, $V'' := \{v'' \mid v \in V\}$, $A' := \{(v', v'') \mid v \in V\}$, and $A'' := \{(u'', v') \mid (u, v) \in A\}$.

Then a minimum cover of A' by paths in the new graph $(V' \cup V'', A' \cup A'')$ gives a minimum cover of V by paths in the original graph. ∎

It is trivial to extend the results to $s - t$ paths:

Corollary 14.7c. *Given an acyclic digraph $D = (V, A)$ and $s, t \in V$, a minimum collection of $s - t$ paths covering all arcs can be found in time $O(nm)$.*

Proof. We may assume that each arc of D is contained in at least one $s-t$ path. But then each path can be extended to an $s - t$ path, and hence a minimum collection of paths gives a minimum collection of $s - t$ paths. ∎

One similarly has for covering the vertices:

Corollary 14.7d. *Given an acyclic digraph $D = (V, A)$ and $s, t \in V$, a minimum collection of $s - t$ paths covering all vertices can be found in time $O(nm)$.*

Proof. Similar to the proof of Corollary 14.7b. ∎

These bounds are best possible, as the size of the output is $\Omega(nm)$. As for paths covering the arcs, this can be seen by taking vertices $v_1, \ldots, v_n$, with r parallel arcs from v_1 to v_2, r parallel arcs from v_{n-1} to v_n, and one arc from v_{i-1} to v_i for each $i = 2, \ldots, n - 1$. Then the number of arcs is $2r + n - 2$, while any minimum path covering of the arcs consists of r paths of length $n - 1$ each.

14.6. Unions of directed cuts and antichains

The following theorem is (in the terminology of partially ordered sets — see Corollary 14.8a) due to Greene and Kleitman [1976]. We follow the proof method of Fomin [1978] and Frank [1980a] based on minimum-cost circulations.

Theorem 14.8. *Let $D = (V, A)$ be an acyclic digraph, let $B \subseteq A$, and let $k \in \mathbb{Z}_+$. Then the maximum of $|B \cap \bigcup \mathcal{C}|$, where $\mathcal{C}$ is a collection of at most k directed cuts is equal to the minimum value of*

(14.13) $\qquad |B \setminus \bigcup \mathcal{P}| + k \cdot |\mathcal{P}|,$

where $\mathcal{P}$ is a collection of directed paths.

Proof. To see max $\leq$ min, let $\mathcal{C}$ be a collection of at most k directed cuts and let $\mathcal{P}$ be a collection of directed paths. Then, setting $\Gamma := \bigcup \mathcal{C}$ and $\Pi := \bigcup \mathcal{P}$,

(14.14) $\qquad |B \cap \Gamma| \leq |B \setminus \Pi| + |\Gamma \cap \Pi| \leq |B \setminus \Pi| + k \cdot |\mathcal{P}|.$

(Note that any directed path P intersects any directed cut in at most one edge; hence $|\Gamma \cap AP| \leq k$ for each $P \in \mathcal{P}$.)

In proving equality, we may assume that D has exactly one source, s say, and exactly one sink, t say. (Adding s to V, and all arcs (s, v) for $v \in V$ does not change the theorem. Similarly for adding a sink.)

Define for each $a \in A$, a capacity $c(a) := \infty$ and a cost $l(a) := 0$. For each arc $a = (u, v) \in B$, introduce a new arc $a' = (u, v)$ parallel to a, with $c(a) := 1$ and $l(a) := -1$. Finally, add an arc (t, s), with $c(t, s) := \infty$ and $l(t, s) := k$. This makes the digraph $\widetilde{D} = (V, \widetilde{A})$.

Let $f : \widetilde{A} \to \mathbb{Z}$ be a minimum-cost nonnegative circulation in $\widetilde{D}$ subject to c. As D_f has no negative-cost directed circuits (Theorem 12.1), there exists a function $p : V \to \mathbb{Z}$ such that for each $a = (u, v) \in A$:

(14.15) $\qquad p(v) \leq p(u)$, with equality if $f(a) \geq 1$.

Moreover, for each $a = (u, v) \in B$:

(14.16) $\qquad p(v) \leq p(u) - 1$ if $f(a') = 0$,
$\qquad\qquad\quad p(v) \geq p(u) - 1$ if $f(a') = 1$.

Finally,

(14.17) $\qquad p(s) \leq p(t) + k$, with equality if $f(t, s) \geq 1$.

We may assume that $p(t) = 0$. So by (14.15), $p(s) \geq 0$ and by (14.17), $p(s) \leq k$. For each $i = 1, \ldots, p(s)$, let $U_i := \{v \in V \mid p(v) \geq i\}$. Then for each i, $\delta^{\text{out}}_A(U_i)$ is a directed $s - t$ cut, since $s \in U_i$, $t \notin U_i$, and no arc in A enters U_i: if $(u, v) \in A$ with $v \in U_i$, then $p(v) \geq i$, and hence by (14.15), $p(u) \geq i$, that is $u \in U_i$.

Let $\mathcal{C}$ be the collection of these directed cuts and let $\Gamma := \bigcup \mathcal{C}$. We can decompose f as a sum of incidence vectors of directed circuits in $\widetilde{D}$. Each of these circuits contains exactly one arc (t, s). Deleting it, and identifying any a' with a (for $a \in B$) gives a path collection $\mathcal{P}$ in D. Let $\Pi := \bigcup \mathcal{P}$.

Then $B \setminus \Pi = B \cap \Gamma \setminus \Pi$. For let $a = (u, v) \in B \setminus \Pi$. Then $f(a') = 0$, and hence by (14.16), $p(v) \leq p(u) - 1$. Hence $a \in \delta^{\text{out}}(U_i)$ for $i := p(u)$. So $a \in \Gamma$.

Moreover,

(14.18)
$$\begin{aligned} k \cdot |\mathcal{P}| &= (p(s) - p(t)) f(t, s) \\ &= \sum_{a=(u,v) \in A} (p(u) - p(v)) f(a) + \sum_{a=(u,v) \in B} (p(u) - p(v)) f(a') \\ &= \sum_{a=(u,v) \in B} (p(u) - p(v)) f(a') = |B \cap \Gamma \cap \Pi|. \end{aligned}$$

Thus $|B\setminus\Pi|+k\cdot|\mathcal{P}| = |B\setminus\Pi|+|B\cap\Gamma\cap\Pi| = |B\cap\Gamma\setminus\Pi|+|B\cap\Gamma\cap\Pi| = |B\cap\Gamma|$. ∎

This implies for partially ordered sets:

Corollary 14.8a. *Let $(S,\leq)$ be a partially ordered set and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k antichains is equal to the minimum value of*

$$(14.19)\qquad \sum_{C\in\mathcal{C}} \min\{k, |C|\},$$

where $\mathcal{C}$ ranges over partitions of S into chains.

Proof. This can be reduced to Theorem 14.8, by making a digraph $D = (V, A)$ as follows. Let for each $s \in S$, s' be a copy of s, and let $V := S \cup \{s' \mid s \in S\}$. Let A consist of all pairs (s, s') with $s \in S$ and all pairs (s', t) with $s < t$. Taking $B := \{(s, s') \mid s \in S\}$ reduces Corollary 14.8a to Theorem 14.8. (For each arcs (s, s') in $B \setminus \bigcup\mathcal{P}$, we take a singleton $C = \{s\}$.) ∎

Corollary 14.8a can be stated in a slightly different form. For any partially ordered set $(S,\leq)$, any $Y \subseteq S$, and any $k \in \mathbb{Z}_+$, let

$$(14.20)\qquad a_k(Y) := \max\{|Z| \bigm| Z \subseteq Y \text{ is the union of } k \text{ antichains}\}.$$

Then:

Corollary 14.8b. $a_k(S) = \min_{Y\subseteq S}(|S \setminus Y| + k \cdot a_1(Y))$.

Proof. The inequality $\leq$ follows from the fact that if Z is the union of k antichains and $Y \subseteq S$, then $|Z| \leq |Z \setminus Y| + a_k(Z \cap Y) \leq |S \setminus Y| + k \cdot a_1(Y)$.

To obtain equality, let $\mathcal{C}$ be a partition of S into chains attaining the minimum in Corollary 14.8a. Let $\mathcal{C}'$ be the collection of those chains $C \in \mathcal{C}$ with $|C| \geq k$. Let Y be the union of the chains in $\mathcal{C}'$. Then (14.19) is equal to $|S \setminus Y| + k|\mathcal{C}'|$. This is at least $|S \setminus Y| + k \cdot a_1(Y)$ (as $|\mathcal{C}'| \geq a_1(Y)$). Thus we have equality. ∎

Note that the proof of Theorem 14.8 gives a polynomial-time algorithm to find a maximum union of k directed cuts or antichains. For a proof of the results in this section based on LP-duality, see Hoffman and Schwartz [1977]. For other proofs, see Saks [1979]. For extensions, see Linial [1981] and Cameron [1986].

14.6a. Common saturating collections of chains

Greene and Kleitman [1976] also showed that for each h there is a chain partition $\mathcal{C}$ of a partially ordered set $(S,\leq)$ attaining the minimum of (14.19) both for $k = h$ and for $k = h + 1$.

More generally, in terms of acyclic digraphs, there is the following result of Greene and Kleitman [1976] on the minimum in Theorem 14.8:

Theorem 14.9. *Let $D = (V, A)$ be an acyclic digraph, let $B \subseteq A$, and let $h \in \mathbb{Z}_+$. Then there is a collection $\mathcal{P}$ of directed paths attaining*

(14.21) $$\min_{\mathcal{P}}(|B \setminus \bigcup\mathcal{P}| + k \cdot |\mathcal{P}|)$$

both for $k = h$ and for $k = h + 1$.

Proof. It suffices to show that in the proof of Theorem 14.8 the minimum-cost circulation f can be chosen such that it has minimum cost simultaneously with respect to the given cost function l, and with respect to the cost function l' which is the same as l except that $l'(t, s) = k + 1$.

For choose f such that it has minimum cost with respect to l, and with $l'^{\mathsf{T}} f$ as small as possible. Suppose that f does *not* have minimum cost with respect to l'. Then D_f has a directed circuit C with $l'(C) < 0$. As $l(C) \geq 0$, C traverses (s, t). So $l(C) - l'(C) = l(s, t) - l'(s, t) = 1$, and therefore $l(C) = 0$. So $f' := f + \chi^C$ is a feasible circulation with $l^{\mathsf{T}} f' = l^{\mathsf{T}} f$ and $l'^{\mathsf{T}} f' < l'^{\mathsf{T}} f$. This contradicts our assumption. ∎

For partially ordered sets it gives (using definition (14.20)):

Corollary 14.9a. *Let $(S, \leq)$ be a partially ordered set and let $k \in \mathbb{Z}_+$. Then there exists a chain partition $\mathcal{C}$ of S such that*

(14.22) $$a_k(S) = \sum_{C \in \mathcal{C}} \min\{k, |C|\} \text{ and } a_{k+1}(S) = \sum_{C \in \mathcal{C}} \min\{k + 1, |C|\}.$$

Proof. Directly from Theorem 14.9. ∎

(For a linear programming proof and an extension, see Hoffman and Schwartz [1977]. For another proof, see Perfect [1984]. Denig [1981] showed that the common saturating chain collections determine a matroid.)

14.7. Unions of directed paths and chains

Results dual to those of the previous sections were obtained by Greene [1976] and Edmonds and Giles [1977]. They can be formulated by interchanging the terms 'chain' and 'antichain'. Again we follow the proof method of Fomin [1978] and Frank [1980a] based on minimum-cost flows.

Theorem 14.10. *Let $D = (V, A)$ be an acyclic digraph, let $B \subseteq A$, and let $k \in \mathbb{Z}_+$. Then the maximum of $|B \cap \bigcup\mathcal{P}|$, where $\mathcal{P}$ is a collection of the arc sets of at most k directed paths, is equal to the minimum value of*

(14.23) $$|B \setminus \bigcup\mathcal{C}| + k \cdot |\mathcal{C}|,$$

where $\mathcal{C}$ is a collection of directed cuts.

Proof. The inequality max $\leq$ min is shown similarly as in Theorem 14.8. In proving the theorem, we may again assume that D has only one source, s say, and only one sink, t say.

To obtain equality, we again consider the extended graph $\widetilde{D}$ as in the proof of Theorem 14.8, with capacity c and cost l, except that we delete arc (t,s).

Let $f : \widetilde{A} \to \mathbb{Z}$ be a minimum-cost $s-t$ flow in $\widetilde{D}$ of value k subject to c. As D_f has no negative-cost directed circuits, there exists a function $p : V \to \mathbb{Z}$ such that for each $a = (u,v) \in A$:

(14.24) $\qquad p(v) \leq p(u)$, with equality if $f(a) \geq 1$.

Moreover, for each $a = (u,v) \in B$:

(14.25) $$\begin{aligned} &p(v) \leq p(u) - 1 \text{ if } f(a') = 0,\\ &p(v) \geq p(u) - 1 \text{ if } f(a') = 1. \end{aligned}$$

We may assume that $p(t) = 0$. By (14.24), $p(s) \geq 0$. For each $i = 1, \ldots, p(s)$, let $U_i := \{v \in V \mid p(v) \geq i\}$. Then for each i, $\delta^{\text{out}}(U_i)$ is a directed cut, since $s \in U_i$, $t \notin U_i$, and no arc in A enters U_i: if $(u,v) \in A$ with $v \in U_i$, then $p(v) \geq i$, and hence by (14.24), $p(u) \geq i$, that is $u \in U_i$.

Let $\mathcal{C}$ be the collection of these directed cuts and let $\varGamma := \bigcup \mathcal{C}$. We can decompose f as a sum of incidence vectors of k directed paths in $\widetilde{D}$. Identifying any a' with a (for $a \in B$) this gives a collection $\mathcal{P}$ of $s-t$ paths. Let $\varPi := \bigcup_{P \in \mathcal{P}} AP$.

Then $B \setminus \varPi \subseteq \varGamma$. For let $a = (u,v) \in B \setminus \varPi$. Then $f(a') = 0$, and hence by (14.25), $p(v) \leq p(u) - 1$. Hence $a \in \delta^{\text{out}}(U_i)$ for $i = p(u)$. So $a \in \varGamma$.

Moreover,

(14.26) $$\begin{aligned} k \cdot |\mathcal{C}| &= k(p(s) - p(t))\\ &= \sum_{a=(u,v)\in A} (p(u)-p(v))f(a) + \sum_{a=(u,v)\in B} (p(u)-p(v))f(a')\\ &= \sum_{a\in B} (p(u)-p(v))f(a') = |B \cap \varGamma \cap \varPi|. \end{aligned}$$

So $|B\setminus\varGamma| + k|\mathcal{C}| = |B\setminus\varGamma| + |B\cap\varGamma\cap\varPi| = |B\cap\varPi\setminus\varGamma| + |B\cap\varGamma\cap\varPi| = |B\cap\varPi|$. ∎

This implies for partially ordered sets:

Corollary 14.10a. *Let $(S,\leq)$ be a partially ordered set and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k chains is equal to the minimum value of*

(14.27) $$\sum_{A\in\mathcal{A}} \min\{k, |A|\},$$

where $\mathcal{A}$ ranges over partitions of S into antichains.

Proof. Similar to the proof of Corollary 14.8a. ∎

Like the result in the previous section, also this theorem can be stated in a different form. For any partially ordered set $(S, \leq)$, $Y \subseteq S$ and $k \in \mathbb{Z}_+$, let

$$c_k(Y) := \max\{|Z| \mid Z \subseteq Y \text{ is the union of } k \text{ chains}\}. \tag{14.28}$$

Then:

Corollary 14.10b. $c_k(S) = \min\limits_{Y \subseteq S}(|S \setminus Y| + k \cdot c_1(Y))$.

Proof. Similar to the proof of Corollary 14.8b. ∎

Note that the proof method gives a polynomial-time algorithm to find a maximum union of k paths or chains. A weighted version was given by Edmonds and Giles [1977]. For an extension, see Hoffman [1983].

14.7a. Common saturating collections of antichains

Similar results to those in Section 14.6a were obtained for antichain partitions by Greene [1976]. Consider the proof of Theorem 14.10. By Theorem 12.5, there exist minimum-cost flows f and f' of values k and $k+1$ respectively and a function $p : V \to \mathbb{Z}$ that is both a potential for f and for f'.

This implies the following result of Greene [1976] on the minimum in Theorem 14.10:

Theorem 14.11. *Let $D = (V, A)$ be an acyclic digraph, let $B \subseteq A$, and let $h \in \mathbb{Z}_+$. Then there is a collection $\mathcal{C}$ of directed cuts attaining*

$$\min_{\mathcal{C}}(|B \setminus \bigcup \mathcal{C}| + k \cdot |\mathcal{C}|) \tag{14.29}$$

both for $k = h$ and for $k = h + 1$.

Proof. Directly from the foregoing observation. ∎

For partially ordered sets it gives (using definition (14.28)):

Corollary 14.11a. *Let $(S, \leq)$ be a partially ordered set and let $k \in \mathbb{Z}_+$. Then there exists an antichain partition $\mathcal{A}$ of S such that*

$$c_k(S) = \sum_{A \in \mathcal{A}} \min\{k, |A|\} \text{ and } c_{k+1}(S) = \sum_{A \in \mathcal{A}} \min\{k+1, |A|\}. \tag{14.30}$$

Proof. Directly from Theorem 14.11. ∎

More on this can be found in Perfect [1984]. For more on chain and antichain partitions, see Frank [1980a].

14.7b. Conjugacy of partitions

Greene [1976] showed that the numbers studied above give so-called 'conjugate' partitions, implying that in fact the results of Section 14.6 and those of 14.7 can be derived from each other.

Fix a partially ordered set $(S, \leq)$. For each $k = 0, 1, 2, \ldots$, let a_k be the maximum size of the union of k antichains in S and let c_k be the maximum size of the union of k chains in S.

Then Corollary 14.8a is equivalent to:

(14.31) $$a_k = |S| + \min_{p \geq 0}(kp - c_p).$$

Similarly, Corollary 14.10a is equivalent to:

(14.32) $$c_k = |S| + \min_{p \geq 0}(kp - a_p).$$

Define for each $k = 1, 2, \ldots$:

(14.33) $$\alpha_k := a_k - a_{k-1} \text{ and } \gamma_k := c_k - c_{k-1}.$$

Trivially, each α_k and γ_k is nonnegative, and both $\alpha_1, \alpha_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$ are partitions of the number $|S|$. In fact:

Theorem 14.12. $\alpha_1 \geq \alpha_2 \geq \cdots$ *and* $\gamma_1 \geq \gamma_2 \geq \cdots$.

Proof. For each $k \geq 1$, one has $\alpha_k \geq \alpha_{k+1}$; equivalently

(14.34) $$a_{k+1} + a_{k-1} \leq 2a_k.$$

Indeed, by Corollary 14.8a there is a collection $\mathcal{C}$ of chains satisfying

(14.35) $$a_k = \sum_{C \in \mathcal{C}} \min\{k, |C|\}.$$

Then

(14.36) $$2a_k = \sum_{C \in \mathcal{C}} 2\min\{k, |C|\} \geq \sum_{C \in \mathcal{C}} (\min\{k-1, |C|\} + \min\{k+1, |C|\}) \geq a_{k-1} + a_{k+1}.$$

The second part of the theorem is shown similarly (with Corollary 14.10a). ∎

In fact, the partitions $(\alpha_1, \alpha_2, \ldots)$ and $(\gamma_1, \gamma_2, \ldots)$ of $|S|$ are *conjugate*. To show this, we mention some of the theory of partitions of numbers.

Let $\nu_1 \geq \nu_2 \geq \cdots$ be integers forming a partition of the number n; that is, $\nu_1 + \nu_2 + \cdots = n$. (So $\nu_k = 0$ for almost all k.) The *conjugate* partition (ν_p^*) of (ν_k) is defined by:

(14.37) $$\nu_p^* := \max\{k \mid \nu_k \geq p\}$$

for $p = 1, 2, \ldots$. Then it is easy to see that $\nu_1^* \geq \nu_2^* \geq \cdots$, and that

(14.38) $$\text{for all } p, k \geq 1\text{: } p \leq \nu_k \iff k \leq \nu_p^*.$$

The conjugate partition can be interpreted in terms of the 'Young diagram'. The *Young diagram* F of (ν_k) is the collection of pairs (x, y) of natural numbers $x, y \geq 1$ satisfying $y \leq \nu_x$. So the Young diagram uniquely determines the sequence (ν_k). The number of pairs in the Young diagram is equal to n. Now the Young diagram F^* of the conjugate partition (ν_p^*) satisfies

(14.39) $$F^* = \{(y,x) \mid (x,y) \in F\}.$$

This follows directly from (14.38). It implies that the conjugate partition (ν_p^*) is again a partition of n, and that the conjugate of (ν_p^*) is (ν_k).

The following interprets conjugacy of partitions in terms of their partial sums. Let $\nu_1 \geq \nu_2 \geq \cdots$ and $\nu'_1 \geq \nu'_2 \geq \cdots$ be partitions of n. For each $k = 0, 1, \ldots$, let

(14.40) $$n_k := \nu_1 + \cdots + \nu_k \text{ and } n'_k := \nu'_1 + \cdots + \nu'_k.$$

Then:

Lemma 14.13α. *(ν_k) and (ν'_k) are conjugate partitions if and only if*

(14.41) $$n'_p = n + \min_{k\geq 0}(pk - n_k)$$

for each $p = 0, 1, 2, \ldots$.

Proof. First note that, for each $p = 1, 2, \ldots$,

(14.42) $$\min_{k\geq 0}(pk - n_k) \text{ is attained by } k = \nu_p^*.$$

Indeed, choose $k \geq 0$ attaining $\min_{k\geq 0}(pk - n_k)$, with k as large as possible. Then $(k+1)p - n_{k+1} > pk - n_k$, and hence $\nu_{k+1} < p$. Moreover, if $k \geq 1$, then $(k-1)p - n_{k-1} \geq pk - n_k$, and hence $\nu_k \geq p$, implying $\nu_p^* = k$. If $k = 0$, then $\nu_1 < p$, again implying $\nu_p^* = 0 = k$. This shows (14.42).

Moreover,

(14.43) $$\sum_{q=1}^{p} \nu_q^* = n + \min_{k\geq 0}(pk - n_k),$$

since

(14.44) $$\begin{aligned}\sum_{q=1}^{p} \nu_q^* &= \sum_{q=1}^{p} \max\{k \mid \nu_k \geq q\} = \sum_{q=1}^{p} \sum_{\substack{k=1 \\ \nu_k \geq q}}^{\infty} 1 = \sum_{k=1}^{\infty} \min\{\nu_k, p\} \\ &= \sum_{k=1}^{\infty} \nu_k - \sum_{k=1}^{\nu_p^*} (\nu_k - p) = n + p\nu_p^* - \sum_{k=1}^{\nu_p^*} \nu_k = n + p\nu_p^* - n_{\nu_p^*} \\ &= n + \min_{k\geq 0}(pk - n_k),\end{aligned}$$

by (14.42).

By (14.43), condition (14.41) is equivalent to

(14.45) $$n'_p = \sum_{q=1}^{p} \nu_q^* \text{ for } p = 0, 1, \ldots.$$

Hence it is equivalent to: $\nu'_q = \nu_q^*$ for each $q = 1, 2, \ldots$; that is to: (ν_k) and (ν'_k) are conjugate. ∎

This yields the conjugacy of the α_k and γ_p:

Theorem 14.13. *(α_k) and (γ_p) are conjugate partitions of $|S|$.*

Proof. Directly from Lemma 14.13α and Corollary 14.10b. ∎

Lemma 14.13α gives the equivalence of the Corollaries 14.10b and 14.8b. For other proofs of the conjugacy of (α_k) and (γ_p), see Fomin [1978] and Frank [1980a].

14.8. Further results and notes

14.8a. The Gallai-Milgram theorem

Gallai and Milgram [1960] showed the following generalization of Dilworth's decomposition theorem. It applies to any directed graph, but generally is not a min-max relation.

Theorem 14.14 (Gallai-Milgram theorem)**.** *Let $D = (V, A)$ be a digraph and let $\alpha(D)$ be the maximum number of vertices that are pairwise nonadjacent in the underlying undirected graph. Then V can be partitioned into $\alpha(D)$ directed paths.*

Proof. For any partition Π of V into directed paths, let C_Π be the set of end vertices of the paths in Π. For any subset U of V, let $\alpha(U)$ be the maximum number of pairwise nonadjacent vertices in U. We show by induction on $|V|$ that

(14.46) for each partition Π of V into directed paths there is a partition Π' into directed paths with $C_{\Pi'} \subseteq C_\Pi$ and $|C_{\Pi'}| \leq \alpha(V)$.

This implies the theorem.

Let Π be a partition of V into directed paths. To prove (14.46), we may assume that C_Π is inclusionwise minimal among all such partitions.

If C_Π is a stable set, then (14.46) is trivial. If C_Π is not a stable set, take $u, v \in C_\Pi$ with $(v, u) \in A$. By the minimality of C_Π, the path P_u in Π ending at u consists of more than u alone, since otherwise we could extend the path ending at v by u. So P_u has a one but last vertex, w say.

Let $\widetilde{\Pi}$ be obtained from Π by deleting u from P_u. So $\widetilde{\Pi}$ is a partition of $V \setminus \{u\}$ into directed paths. By induction, there is a partition $\widetilde{\Pi}'$ of $V \setminus \{u\}$ into directed paths such that $C_{\widetilde{\Pi}'} \subseteq C_{\widetilde{\Pi}}$ and such that $|C_{\widetilde{\Pi}'}| \leq \alpha(V \setminus \{u\})$.

If one of the paths in $\widetilde{\Pi}'$ ends at w, we can extend it with u, and obtain a partition Π' of V as required. If none of the paths in $\widetilde{\Pi}'$ end at w, but one of the paths ends at v, we can extend it with u, again obtaining a partition Π' as required. If none of the paths in $\widetilde{\Pi}'$ end at v or w, then augmenting $\widetilde{\Pi}'$ by a path consisting of u alone, gives a partition Π' of V with $C_{\Pi'} \subset C_\Pi$, contradicting the minimality of C_Π. ∎

Theorem 14.14 gives no min-max relation, as is shown by a directed circuit of length $2k$: the vertices can be covered by one directed path, while there exist k pairwise nonadjacent vertices.

A consequence of Theorem 14.14 is Dilworth's decomposition theorem: for any partially ordered set $(S, \leq)$ take $V := S$ and $A := \{(x, y) \mid x < y\}$. Another consequence is the graph-theoretical result of Rédei [1934] that each tournament has a Hamiltonian path:

Corollary 14.14a (Rédei's theorem)**.** *Each tournament has a Hamiltonian path.*

Proof. This is the special case where $\alpha(D) = 1$ in the Gallai-Milgram theorem. ∎

Berge [1982b] posed the following conjecture generalizing the Gallai-Milgram theorem. Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then for each path collection $\mathcal{P}$ partitioning V and minimizing

(14.47) $$\sum_{P \in \mathcal{P}} \min\{|VP|, k\},$$

there exist disjoint stable sets $C_1, \ldots, C_k$ in D such that each $P \in \mathcal{P}$ intersects $\min\{|VP|, k\}$ of them. This was proved by Saks [1986] for acyclic graphs. For extensions and related results, see Linial [1978,1981], Saks [1986], and Thomassé [2001].

14.8b. Partially ordered sets and distributive lattices

There is a strong relation between the class of partially ordered sets and the class of partially ordered sets of a special type, the *distributive lattices*. It formed the original motivation for Dilworth to study minimum chain partitions of partially ordered sets.

Let $(S, \leq)$ be a partially ordered set and let $a, b \in S$. Then an element $c \in S$ is called the *meet* of a and b if for each $s \in S$:

(14.48) $s \leq c$ if and only if $s \leq a$ and $s \leq b$.

Note that if the meet of a and b exists, it is unique. Similarly, c is called the *join* of a and b if for each $s \in S$:

(14.49) $s \geq c$ if and only if $s \geq a$ and $s \geq b$.

Again, if the join of a and b exists, it is unique.

S is called a *lattice* if each pair s, t of elements of S has a meet and a join; they are denoted by $s \wedge t$ and $s \vee t$, respectively. The lattice is called *distributive* if

(14.50) $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$ and $s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u)$

for all $s, t, u \in S$. (In fact it suffices to require only one of the two equalities.)

Each partially ordered set $(S, \leq)$ gives a distributive lattice in the following way. Call a subset $I \subseteq S$ a *lower ideal*, or just an *ideal*, if $t \in I$ and $s \leq t$ implies that $s \in I$. Let $\mathcal{I}_S$ be the collection of ideals in S. Then $(\mathcal{I}_S, \subseteq)$ is a distributive lattice. This follows directly from the fact that for $I, J \in \mathcal{I}_S$ one has $I \wedge J = I \cap J$ and $I \vee J = I \cup J$, and hence (14.50) is elementary set theory. Thus

(14.51) $(S, \leq) \to (\mathcal{I}_S, \subseteq)$

associates a distributive lattice with any partially ordered set.

In fact, each finite distributive lattice can be obtained in this way; that is, we can reverse (14.51). For any lattice $(L, \leq)$, call an element $u \in L$ *join-irreducible* if there exist no $s, t \in L$ with $s \neq u$, $t \neq u$, and $u = s \vee t$. Let J_L be the set of join-irreducible elements in L. Trivially,

(14.52) $(L, \leq) \to (J_L, \leq)$

associates a partially ordered set with any distributive lattice.

Theorem 14.15. *Functions* (14.51) *and* (14.52) *are inverse to each other.*

Proof. First, let $(S \leq)$ be a partially ordered set. For each $s \in S$, let $I_s := \{t \in S \mid t \leq s\}$. Then an element I of $\mathcal{I}_S$ is join-irreducible if and only if there exists an $s \in S$ with $I = I_s$. Moreover, $s \leq t$ if and only if $I_s \subseteq I_t$. Thus we have an isomorphism of $(S, \leq)$ and $(J_{\mathcal{I}_S}, \subseteq)$.

Conversely, let $(L, \leq)$ be a distributive lattice. For each $s \in L$, let $J_s := \{t \in J_L \mid t \leq s\}$. This gives a one-to-one relation between elements of L and ideals in J_L. Indeed, if I is an ideal in J_L, let $s := \bigvee I$. Then $I = J_s$. Clearly, $I \subseteq J_s$, since $t \leq s$ for each $t \in I$. Conversely, let $u \in J_s$. Then, as L is distributive,

$$u = u \wedge s = u \wedge \bigvee_{t \in I} t = \bigvee_{t \in I} (u \wedge t). \tag{14.53}$$

Since u is join-irreducible, $u = u \wedge t$ for some $t \in I$. Hence $u \leq t$, and therefore $u \in I$.

Moreover, for any $s, t \in L$ one has: $s \leq t$ if and only if $J_s \subseteq J_t$. So we have an isomorphism of $(L, \leq)$ and $(\mathcal{I}_{J_L}, \subseteq)$. ∎

There is moreover a one-to-one relation between ideals I in a partially ordered set $(S, \leq)$ and antichains A in S, given by:

$$A = I^{\max} \text{ and } I = A^{\downarrow}. \tag{14.54}$$

Here, for any $Y \subseteq S$, $Y^{\max}$ denotes the set of maximal elements of Y and

$$Y^{\downarrow} := \{s \in S \mid \exists t \in Y : s \leq t\}. \tag{14.55}$$

For each d, the set $\mathbb{Z}^d$ is a distributive lattice, under the usual order: $x \leq y$ if and only if $x_i \leq y_i$ for each $i = 1, \ldots, d$. Any finite distributive lattice $(L, \leq)$ is a sublattice of $\mathbb{Z}^d$ for some d (as will follow from the next theorem). That is, there is an injection $\phi : L \to \mathbb{Z}^d$ such that $\phi(s \wedge t) = \phi(s) \wedge \phi(t)$ and $\phi(s \vee t) = \phi(s) \vee \phi(t)$ for all $s, t \in L$.

Let $d(L)$ be the minimum number d for which L is (isomorphic to) a sublattice $\mathbb{Z}^d$. As Dilworth [1950] showed, the number $d(L)$ can be characterized with Dilworth's decomposition theorem.

To this end, an element s of a partially ordered set $(S, \leq)$ is said to *cover* $t \in S$ if $s > t$ and there is no $u \in S$ with $s > u > t$. For any $s \in S$, let cover(s) be the number of elements covered by s.

Then the following result of Dilworth [1950] can be derived from Dilworth's decomposition theorem:

Theorem 14.16. *Let L be a finite distributive lattice. Then*

$$d(L) = \max_{s \in L} \operatorname{cover}(s). \tag{14.56}$$

Proof. We first show that $d(L) \geq \operatorname{cover}(s)$ for each $s \in L$. Let $d := d(L)$, let $L \subset \mathbb{Z}^d$, and choose $s \in L$. Let Y be the set of elements covered by s. For each $t \in Y$, let $U_t := \{i \mid t_i < s_i\}$. Now for all $t, u \in Y$ with $t \neq u$ one has $t \vee u = s$; hence $U_t \cap U_u = \emptyset$. As $U_t \neq \emptyset$ for all $t \in Y$, we have $|Y| \leq d$.

So $d(L) \geq \max_{s \in L} \operatorname{cover}(s)$. To see equality, by Theorem 14.15 we may assume that $L = \mathcal{I}_S$ (the set of ideals in S) for some partially ordered set $(S, \leq)$, ordered by inclusion. For any ideal I in S, cover(I) is the number of inclusionwise maximal ideals $J \subset I$. Each such ideal J is equal to $I \setminus \{t\}$ for some $t \in I^{\max}$. So cover$(I) = |I^{\max}|$. Hence $\max_{s \in L} \operatorname{cover}(s)$ is equal to the maximum antichain size in S. Let this be d, say.

By Dilworth's decomposition theorem, S can be covered by d chains, $C_1, \ldots, C_d$ say. For each j, the collection $\mathcal{I}_{C_j}$ of ideals in C_j is again a chain (ordered by inclusion). Now $I \to (I \cap C_1, \ldots, I \cap C_d)$ embeds $\mathcal{I}_S$ into the product $\mathcal{I}_{C_1} \times \cdots \times \mathcal{I}_{C_d}$. Therefore $d(\mathcal{I}_S) \leq d$. ∎

The following was noted by Dilworth [1960]. The relations (14.54) give the following partial order on the collection $\mathcal{A}_S$ of antichains in a partially ordered set $(S, \leq)$:

$$A \preceq B \text{ if and only if } A^{\downarrow} \subseteq B^{\downarrow} \tag{14.57}$$

for $A, B \in \mathcal{A}_S$. As $(\mathcal{I}_S, \subseteq)$ is a lattice, also $(\mathcal{A}_S, \preceq)$ is a lattice.

Theorem 14.17. *Let $(S, \leq)$ be a partially ordered set and let A and B be maximum-size antichains. Then also $A \wedge B$ and $A \vee B$ are maximum-size antichains.*

Proof. One has $|A \wedge B| + |A \vee B| \geq |A| + |B|$. Indeed, $A \cup B \subseteq (A \wedge B) \cup (A \vee B)$ and $A \cap B \subseteq (A \wedge B) \cap (A \vee B)$. So

$$\begin{aligned} |A \wedge B| + |A \vee B| &= |(A \wedge B) \cup (A \vee B)| + |(A \wedge B) \cap (A \vee B)| \\ &\geq |A \cup B| + |A \cap B| = |A| + |B|. \end{aligned} \tag{14.58}$$

As $|A \wedge B| \leq |A| = |B|$ and $|A \vee B| \leq |A| = |B|$, we have $|A \wedge B| = |A \vee B| = |A| = |B|$. ∎

In terms of distributive lattices this gives:

Corollary 14.17a. *Let L be a finite distributive lattice. Then the elements s maximizing* cover(s) *form a sublattice of L.*

Proof. We can represent L as the set $\mathcal{I}_S$ of ideals in a partially ordered set $(S, \leq)$. As one has $\text{cover}(I) = |I^{\max}|$ for any $I \in \mathcal{I}_S$, the result follows from Theorem 14.17. ∎

Theorem 14.17 led Dilworth [1960] to derive an alternative proof of Dilworth's decomposition theorem. For another proof and application of Theorem 14.17, see Freese [1974]. Theorem 14.17 was extended to maximum-size unions of k antichains by Greene and Kleitman [1976].

14.8c. Maximal chains

Let $(S, \leq)$ be a partially ordered set. Call a chain *maximal* if it is contained in no other chain. As 'complementary' to Dilworth's decomposition theorem, Greene and Kleitman [1976] observed:

Theorem 14.18. *The maximum number of disjoint maximal chains is equal to the minimum size of a set intersecting all maximal chains.*

Proof. Define a digraph $D = (S, A)$ where A consists of all pairs (s, t) where t covers s. Let U and W be the sets of minimal and maximal elements of S, respectively. So maximal chains correspond to $U - W$ paths, and the theorem follows from Menger's theorem. ∎

14.8d. Further notes

Related results were given by Bogart [1970], Saks [1986], and Behrendt [1988], extensions and algorithms by Cameron and Edmonds [1979], Frank [1980a], Linial [1981], Cameron [1982,1985,1986], and Hoffman [1983], surveys by Greene [1974b], Hoffman [1982], West [1982], and Bogart, Greene, and Kung [1990], and an introduction to and historical account of Dilworth's decomposition theorem by Dilworth [1990].

Fleiner [1997] proved the following conjecture of A. Frank: Let $(S, \leq)$ be a partially ordered set and let M be a perfect matching on S such that for any two $uu', vv' \in M$ with $u \leq v$, one has $v' \leq u'$. (This is called a *symmetric partially ordered set.*) Call a subset C of M a *symmetric chain* if S has a chain intersecting each edge in C. Then the minimum number of symmetric chains covering M is equal to the maximum value of

$$\sum_{i=1}^{k} \lceil \tfrac{1}{2}|X_i| \rceil, \tag{14.59}$$

where $X_1, \ldots, X_k$ are disjoint subsets of M (for some k) with the properties that (i) no X_i contains a symmetric chain of size 3, and (ii) if $i \neq j$, then there exist no $x \in e \in X_i$ and $y \in f \in X_j$ with $x \leq y$.

Chapter 15

Connectivity and Gomory-Hu trees

Since a minimum $s - t$ cut can be found in polynomial time, also the connectivity of a graph can be determined in polynomial time, just by checking all pairs s, t. However, there are more economical methods, which we discuss in this chapter.
A finer description of the edge-connectivity of a graph G is given by the function $r_G(s,t)$, defined as the minimum size of a cut separating s and t. A concise description of the corresponding minimum-size cuts is given by the Gomory-Hu tree — see Section 15.4.

15.1. Vertex-, edge-, and arc-connectivity

For any undirected graph $G = (V, E)$, the *vertex-connectivity*, or just *connectivity*, of G is the minimum size of a subset U of V for which $G - U$ is not connected. A subset U of V attaining the minimum is called a *minimum vertex-cut*. If no such U exists (namely, if G is complete), then the (vertex-)connectivity is ∞.

The connectivity of G is denoted by $\kappa(G)$. If $\kappa(G) \geq k$, G is called *k-vertex-connected*, or just *k-connected*.

The following direct consequence of Menger's theorem was formulated by Whitney [1932a]:

Theorem 15.1. *An undirected graph $G = (V, E)$ is k-connected if and only if there exist k internally vertex-disjoint paths between any two nonadjacent vertices s and t.*

Proof. Directly from the vertex-disjoint version of Menger's theorem (Corollary 9.1a). ∎

Similarly, the *edge-connectivity* of G is the minimum size of a subset C of E for which $G - C$ is not connected. So it is the minimum size of any cut $\delta(U)$ with $\emptyset \neq U \neq V$. A cut C attaining the minimum size is called a *minimum cut*. The edge-connectivity of G is denoted by $\lambda(G)$. If $\lambda(G) \geq k$, G is called *k-edge connected*. Then:

Theorem 15.2. *An undirected graph $G = (V, E)$ is k-edge-connected if and only if there exist k edge-disjoint paths between any two vertices s and t.*

Proof. Directly from the edge-disjoint version of Menger's theorem (Corollary 9.1b). ∎

Similar terminology and characterizations apply to digraphs. For any digraph $D = (V, A)$ the *vertex-connectivity*, or just *connectivity*, of D is the minimum size of a subset U of V for which $D - U$ is not strongly connected. A set U attaining the minimum is called a *minimum vertex-cut*. If no such U exists (that is, if each pair (u, v) of vertices is an arc), then the (vertex-)connectivity of D is ∞.

The connectivity of D is denoted by $\kappa(D)$. If $\kappa(D) \geq k$, D is called *k-vertex-connected*, or just *k-connected*. Now one has:

Theorem 15.3. *A digraph $D = (V, A)$ is k-connected if and only if there exist k internally vertex-disjoint $s - t$ paths for any $s, t \in V$ for which there is no arc from s to t.*

Proof. Directly from the vertex-disjoint version of Menger's theorem (Corollary 9.1a). ∎

Finally, the *arc-connectivity* of D is the minimum size of a subset C of A for which $D - C$ is not strongly connected. That is, it is the minimum size of any cut $\delta^{\text{out}}(U)$ with $\emptyset \neq U \neq V$. Any cut attaining the minimum size is called a *minimum cut*. The arc-connectivity of D is denoted by $\lambda(D)$. If $\lambda(D) \geq k$, D is called *k-arc-connected* or *strongly k-connected*. (So D is 1-arc-connected if and only if D is strongly connected.) Then:

Theorem 15.4. *A digraph $D = (V, A)$ is k-arc-connected if and only if there exist k arc-disjoint paths between any two vertices s and t.*

Proof. Directly from the arc-disjoint version of Menger's theorem (Corollary 9.1b). ∎

Shiloach [1979a] observed that Edmonds' disjoint arborescences theorem (to be discussed in Chapter 53) implies a stronger characterization of k-arc-connectivity: a digraph $D = (V, A)$ is k-arc-connected if and only if for all $s_1, t_1, \ldots, s_k, t_k \in V$ there exist arc-disjoint paths $P_1, \ldots, P_k$, where P_i runs from s_i to t_i $(i = 1, \ldots, k)$ — see Corollary 53.1d.

A similar characterization does not hold for the undirected case, as is shown by the 2-edge-connected graph in Figure 15.1.

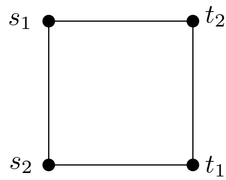

Figure 15.1

15.2. Vertex-connectivity algorithmically

It is clear that the vertex-connectivity of a directed or undirected graph can be determined in polynomial time, just by finding a minimum-size $s-t$ vertex-cut for each pair s,t of vertices. Since by Corollary 9.7a, a minimum-size $s-t$ vertex-cut can be found in $O(n^{1/2}m)$ time, this yields an $O(n^{5/2}m)$ algorithm. In fact, Podderyugin [1973] (for undirected graphs) and Even and Tarjan [1975] observed that one need not consider every pair of vertices:

Theorem 15.5. *A minimum-size vertex-cut in a digraph D can be found in $O(\kappa(D)n^{3/2}m)$ time.*

Proof. Let $D = (V, A)$ be a digraph. We may assume that D is simple. Order V arbitrarily as $v_1, \ldots, v_n$. For $i = 1, 2, \ldots$, determine, for each $v \in V$, a minimum $v_i - v$ vertex-cut $C_{v_i,v}$ and a minimum $v - v_i$ vertex-cut C_{v,v_i}. (This takes $O(n^{3/2}m)$ time by Corollary 9.7a.) At any moment, let c be the minimum size of the cuts found so far. We stop if $i > c+1$. Then c is the vertex-connectivity of D.

Indeed let C be a minimum-size vertex-cut. Then for $i := \kappa(D)+2$, there is a $j < i$ with $v_j \notin C$. Hence there is a vertex v such that C is a $v_j - v$ or a $v - v_j$ vertex-cut. Assume without loss of generality that C is a $v_j - v$ vertex-cut. Then $c \le |C_{v_j,v}| = |C| = \kappa(D)$. ∎

Since $\kappa(D) \le m/n$ if D is not complete, this implies:

Corollary 15.5a. *A minimum-size vertex-cut in a digraph can be found in $O(n^{1/2}m^2)$ time.*

Proof. Immediately from Theorem 15.5, since if D is not complete, $\kappa(D)$ is at most the minimum outdegree of D, and hence $\kappa(D) \le m/n$. ∎

If we want to test the k-connectivity for some fixed k, we may use the following result given by Even [1975]:

Theorem 15.6. *Given a digraph D and an integer k, one can decide in $O((k + \sqrt{n})k\sqrt{n}m)$ time if D is k-connected, and if not, find a minimum cut.*

Proof. Let $V = \{1, \ldots, n\}$. Determine

(15.1) (i) for all $i, j \in \{1, \ldots, k\}$ with $(i, j) \notin A$, a minimum-size $i - j$ vertex-cut if it has size less than k;
(ii) for each $i = k + 1, \ldots, n$ a minimum-size $\{1, \ldots, i - 1\} - i$ vertex-cut if it has size less than k, and a minimum-size $i - \{1, \ldots, i - 1\}$ vertex-cut if it has size less than k.

We claim that if we find any vertex-cut, the smallest among them is a minimum-size vertex-cut. If we find no vertex-cuts, then D is k-connected.

To see this, let U be a minimum-size vertex-cut with $|U| < k$. Suppose that each vertex-cut found has size $> |U|$. Then for all distinct $i, j \in \{1, \ldots, k\} \setminus U$ there is an $i - j$ path avoiding U. So $\{1, \ldots, k\} \setminus U$ is contained in some strong component K of $D - U$. As $D - U$ is not strongly connected, $D - U$ has a vertex not in K. Let i be the smallest index $i \notin K \cup U$. As there exist $|U| + 1$ disjoint $\{1, \ldots, i - 1\} - i$ paths, $D - U$ has a $j - i$ path for some $j < i$; then $j \in K$. Similarly, $D - U$ contains an $i - j'$ path for some $j' \in K$. This contradicts the fact that $i \notin K$ and K is a strong component.

This implies the theorem. Indeed, by Corollaries 9.3a and 9.7a one can find a vertex-cut as in (15.1)(i) or (ii) in time $O(\min\{k, \sqrt{n}\}m)$ time. So in total it takes $O((k^2 + n) \min\{k, \sqrt{n}\}m) = O((k + \sqrt{n})k\sqrt{n}\, m)$ time. ∎

This implies:

Corollary 15.6a. *A minimum-size vertex-cut in a digraph can be found in time* $O(\max\{\frac{m^3}{n\sqrt{n}}, m^2\})$.

Proof. From Theorem 15.6 by taking $k := \lfloor m/n \rfloor$. ∎

Matula [1987] showed that Theorem 15.6 implies the following result of Galil [1980b], where $\kappa(D)$ is the vertex-connectivity of D:

Corollary 15.6b. *A minimum-size vertex-cut in a digraph D can be found in* $O((\kappa(D) + \sqrt{n})\kappa(D)\sqrt{n}\, m)$ *time.*

Proof. For $k = 2, 2^2, 2^3, \ldots$ test with the algorithm of Theorem 15.6 if D is k-connected. Stop if D is not k-connected; then the algorithm gives a minimum-size vertex-cut. Let l be such that $k = 2^l$. As $2^l \leq 2\kappa(D)$, this takes time

(15.2)
$$O\Big(\sum_{i=1}^{l}((2^i + \sqrt{n})2^i\sqrt{n}\, m)\Big) = O((4^{l+1} + 2^{l+1}\sqrt{n})\sqrt{n}\, m)$$
$$= O((\kappa(D)^2 + \kappa(D)\sqrt{n})\sqrt{n}\, m).$$
∎

Since vertex-cuts in an undirected graph G are equal to vertex-cuts in the digraph obtained from G by replacing each edge by two oppositely oriented arcs, the results above immediately imply similar results for vertex-cuts and vertex-connectivity in undirected graphs.

15.2a. Complexity survey for vertex-connectivity

Complexity survey for vertex-connectivity (in directed graphs, unless stated otherwise; again, $*$ indicates an asymptotically best bound in the table):

	$O(n^2 \cdot \mathrm{VC}(n, m))$	(trivial)
	$O(kn \cdot \mathrm{VC}_k(n, m))$	Kleitman [1969]
	$O(\kappa n \cdot \mathrm{VC}(n, m))$	Podderyugin [1973], Even and Tarjan [1975]
	$O((k^2 + n) \cdot \mathrm{VC}_k(n, m))$	Even [1975] (cf. Esfahanian and Hakimi [1984])
	$O((\kappa^2 + n) \cdot \mathrm{VC}_\kappa(n, m))$	Galil [1980b] (cf. Matula [1987])
	$O((\kappa + \sqrt{n})\kappa^2 n^{3/2})$	*undirected* Nagamochi and Ibaraki [1992a], Cheriyan and Thurimella [1991]
	$O((\kappa + \sqrt{n})\kappa^2 n^{3/2} \log_n(n^2/m))$	*undirected* Feder and Motwani [1991, 1995]
	$O((\kappa^3 + n)m)$	Henzinger, Rao, and Gabow [1996, 2000]
	$O(\kappa nm)$	Henzinger, Rao, and Gabow [1996, 2000]
	$O((\kappa^3 + n)\kappa n)$	*undirected* Henzinger, Rao, and Gabow [1996,2000]
	$O(\kappa^2 n^2)$	*undirected* Henzinger, Rao, and Gabow [1996,2000]
$*$	$O((\kappa^{5/2} + n)m)$	Gabow [2000b]
$*$	$O((\kappa + n^{1/4})n^{3/4}m)$	Gabow [2000b]
$*$	$O((\kappa^{5/2} + n)\kappa n)$	*undirected* Gabow [2000b]
$*$	$O((n^{1/4} + \kappa)\kappa n^{7/4})$	*undirected* Gabow [2000b]

Here κ denotes the vertex-connectivity of the graph. Note that $\kappa \leq m/n$. If k is involved, the time bound is for determining $\min\{\kappa, k\}$. By $\mathrm{VC}(n, m)$ we denote the time needed to find the minimum size of an $s-t$ vertex-cut for fixed s, t. Moreover, $\mathrm{VC}_k(n, m)$ denotes the time needed to find the minimum size of an $s-t$ vertex-cut if this size is less than k. We refer to Sections 9.5 and 9.6a for bounds on $\mathrm{VC}(n, m)$ and $\mathrm{VC}_k(n, m)$. Note that $\mathrm{VC}_k(n, m) = O(\min\{k, \sqrt{n}\}m)$.

By the observation of Matula [1987] (cf. Corollary 15.6b above), if $\min\{k, \kappa\}$ can be determined in time $O(k^\alpha f(n, m))$ (for some $\alpha \geq 1$), then κ can be determined in time $O(\kappa^\alpha f(n, m))$.

15.2b. Finding the 2-connected components

In this section we show the result due to Paton [1971], Tarjan [1972], and Dinits, Zajtsev, and Karzanov [1974] (cf. Hopcroft and Tarjan [1973a]) that the 2-vertex-connected components of an undirected graph can be found in linear time. Hence, the 2-connectivity of an undirected graph can be tested in linear time.

Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. A *k-connected component* is an inclusionwise maximal subset U of V for which $G[U]$ is k-connected. A *block* is a 2-connected component U with $|U| \geq 2$.

We note

(15.3) if U and W are two different k-connected components, then $|U \cap W| < k$.

Indeed, as $G[U \cup W]$ is not k-connected, there is a subset C of $U \cup W$ with $G[(U \cup W) \setminus C]$ disconnected and $|C| < k$. As $(U \cup W) \setminus C = (U \setminus C) \cup (W \setminus C)$ and as $G[U \setminus C]$ and $G[W \setminus C]$ are connected, it follows that $(U \setminus C) \cap (W \setminus C) = \emptyset$. Hence $C \supseteq U \cap W$, and therefore $|U \cap W| < k$.

(15.3) implies that each edge of G is contained in a unique 2-connected component. So the 2-connected components partition the edge set. One may show:

(15.4) edges e and e' are contained in the same 2-connected component if and only if G has a circuit C containing both e and e'.

Indeed, if C exists, it forms a 2-connected subgraph of G, and hence e and e' are contained in some 2-connected component. Conversely, if $e = uv$ and $e' = u'v'$ are contained in a 2-connected component H, by Menger's theorem H has two vertex-disjoint $\{u, v\} - \{u', v'\}$ paths; with e and e' these paths form a circuit C as required.

Theorem 15.7. *The collection of blocks of a graph $G = (V, E)$ can be identified in linear time.*

Proof. By Corollary 6.6a, we may assume that G is connected. Choose $s \in V$ arbitrarily. Apply depth-first search starting at s. If we orient the final tree to become a rooted tree (V, T) with root s, all further edges of G connect two vertices u, v such that T has a directed $u - v$ path P. For each such edge, make an arc (v, u'), where u' is the second vertex of P. The arcs in T and these new arcs form a directed graph denoted by $D = (V, A)$.

By adapting the depth-first search, we can find A is linear time. Indeed, while scanning s, we keep a directed path Q in T formed by the vertices whose scanning has begun but is not yet finished. If we scan v and meet an edge uv with u on Q, we can find u' and construct the arc (v, u').

Since no arc of D enters s, $\{s\}$ is a strong component. For any strong component K of D one has:

(15.5) the subgraph of T induced by K is a subtree.

Indeed, for any arc $(u, v) \in A \setminus T$ spanned by K, the $v - u$ path in T is contained in K, since it forms a directed circuit with (u, v) and since K is a strong component. This proves (15.5).

(15.5) implies that for each strong component K of D with $K \neq \{s\}$, there is a unique arc of T entering K; let u_K be its tail and define $K' := K \cup \{u_K\}$. We finally show

(15.6) $\{K' \mid K$ strong component of D, $K \neq \{s\}\}$ is equal to the collection of blocks of G.

This proves the theorem (using Theorem 6.6).

Let (t, u, v) be a directed path in T. Then

(15.7) tu and uv are contained in the same block of G if and only if D has a directed $v - u$ path.

To see this, let W be the set of vertices reachable in (V, T) from v. Then:

(15.8) D has a directed $v - u$ path $\Longleftrightarrow$ D has an arc leaving W $\Longleftrightarrow$ G has an edge leaving W and not containing u $\Longleftrightarrow$ G has a $v - t$ path not traversing u $\Longleftrightarrow$ tu and uv are contained in a circuit of G $\Longleftrightarrow$ tu and uv are contained in the same block of G.

This proves (15.7).

(15.7) implies that for each strong component $K \subseteq V \setminus \{s\}$ of D, the set K' is contained in a block of G. Conversely, let B be a block of G. Then B induces a subtree of T. Otherwise, B contains two vertices u and v such that the undirected $u - v$ path P in T has length at least two and such that no internal vertex of P belongs to B. Let Q be a $u - v$ path in B. Then P and Q form a circuit, hence a 2-connected graph. So P is contained in B, a contradiction.

So B induces a subtree of T. Let u be its root. As $B \setminus \{u\}$ induces a connected subgraph of G, there is a unique arc in T entering $B \setminus \{u\}$. So also $B \setminus \{u\}$ induces a subtree of T. Then (15.7) implies that $B \setminus \{u\}$ is contained in a strong component of D. ∎

Corollary 15.7a. *The* 2*-connectivity of an undirected graph can be tested in linear time.*

Proof. A graph (V, E) is 2-connected if and only if V is a 2-connected component. So Theorem 15.7 implies the result. ∎

Hopcroft and Tarjan [1973b] gave a linear-time algorithm to test 3-connectivity of an undirected graph; more generally, to decompose an undirected graph into 3-connected components (cf. Miller and Ramachandran [1987,1992]). Kanevsky and Ramachandran [1987,1991] gave an $O(n^2)$ algorithm to test 4-connectivity of an undirected graph.

Finding all 2- and 3-vertex-cuts of an undirected graph has been investigated by Tarjan [1972], Hopcroft and Tarjan [1973b], Kanevsky and Ramachandran [1987, 1991], Miller and Ramachandran [1987,1992], and Kanevsky [1990a]. Kanevsky [1990b] showed that for each fixed k, the number of vertex-cuts of size k in a k-vertex-connected graph is $O(n^2)$ (cf. Kanevsky [1993]). Related results can be found in Gusfield and Naor [1990], Cohen, Di Battista, Kanevsky, and Tamassia [1993], Gabow [1993b,1995c], and Cheriyan and Thurimella [1996b,1999].

15.3. Arc- and edge-connectivity algorithmically

Denote by $\mathrm{EC}(n, m)$ the time needed to find a minimum-size $s - t$ cut for any given pair of vertices s, t. Even and Tarjan [1975] (and Podderyugin [1973] for undirected graphs) observed that one need not check all pairs of vertices to find a minimum cut:

Theorem 15.8. *A minimum-size cut in a digraph can be found in* $O(n \cdot \mathrm{EC}(n,m))$ *time.*

Proof. Choose $s \in V$. For each $t \neq s$, determine a minimum $s-t$ cut $C_{s,t}$ and a minimum $t-s$ cut $C_{t,s}$. The smallest among all these cuts is a minimum cut. ∎

Hence we have for the arc-connectivity:

Corollary 15.8a. *The arc-connectivity of a digraph can be determined in* $O(n \cdot \mathrm{EC}(n,m))$ *time.*

Proof. Directly from Theorem 15.8. ∎

As $\mathrm{EC}(n,m) = O(m^{3/2})$ by Corollary 9.6a, it follows that the arc-connectivity can be determined in time $O(nm^{3/2})$. Actually, also in time $O(m^2)$, since we need to apply the disjoint paths algorithm only until we have at most $k := \lfloor m/n \rfloor$ arc-disjoint paths, as the arc-connectivity is at most m/n (there is a $v \in V$ with $|\delta^{\mathrm{out}}(v)| \leq \lfloor m/n \rfloor$).

Moreover, again by Corollary 9.6a, $\mathrm{EC}(n,m) = O(n^{2/3}m)$ for simple digraphs, and hence the arc-connectivity of a simple directed graph can be determined in time $O(n^{5/3}m)$ (cf. Esfahanian and Hakimi [1984]).

Schnorr [1978b,1979] showed that in fact:

Theorem 15.9. *Given a digraph D and an integer k, one can decide in $O(knm)$ time if D is k-arc-connected, and if not, find a minimum cut.*

Proof. In Theorem 15.8 one needs to check only if there exist k arc-disjoint $s-t$ paths, and if not find a minimum-size $s-t$ cut. This can be done in time $O(km)$, as we saw in Corollary 9.3a. ∎

With a method of Matula [1987] this implies, where $\lambda(D)$ is the arc-connectivity of D:

Corollary 15.9a. *A minimum-size cut in a digraph D can be found in time $O(\lambda(D)nm)$.*

Proof. For $k = 2, 2^2, 2^3, \ldots$ test if D is k-arc-connected, until we find that D is not k-arc-connected, and have a minimum-size cut. With the method of Theorem 15.9 this takes time $O((2+2^2+2^3+\cdots+2^l)nm)$, with $2^l \leq 2\lambda(D)$. So $2+2^2+2^3+\cdots+2^l \leq 2^{l+1} \leq 4\lambda(D)$, and the result follows. ∎

For undirected graphs, Nagamochi and Ibaraki [1992b] showed that the edge-connectivity of an undirected graph can be determined in time $O(nm)$ (for simple graphs this bound is due to Podderyugin [1973]). We follow the shortened algorithm described by Frank [1994b] and Stoer and Wagner [1994, 1997].

Theorem 15.10. *Given an undirected graph G, a minimum cut in G can be found in time $O(nm)$.*

Proof. Let $G = (V, E)$ be a graph. For $U \subseteq V$ and $v \in V \setminus U$, let $d(U, v)$ denote the number of edges connecting U and v. Let $r(u, v)$ denote the minimum capacity of a $u - v$ cut.

Call an ordering $v_1, \ldots, v_n$ of the vertices of G a *legal order* for G if $d(\{v_1, \ldots, v_{i-1}\}, v_i) \geq d(\{v_1, \ldots, v_{i-1}\}, v_j)$ for all i, j with $1 \leq i < j \leq n$. Then:

(15.9) If $v_1, \ldots, v_n$ is a legal order for $G = (V, E)$, then $r(v_{n-1}, v_n) = d(v_n)$.

To see this, let C be any $v_{n-1} - v_n$ cut. Define $u_0 := v_1$. For $i = 1, \ldots, n-1$, define $u_i := v_j$, where j is the smallest index such that $j > i$ and C is a $v_i - v_j$ cut. Note that for each $i = 1, \ldots, n-1$ one has

(15.10) $$d(\{v_1, \ldots, v_{i-1}\}, u_i) \leq d(\{v_1, \ldots, v_{i-1}\}, u_{i-1}),$$

since if $u_{i-1} \neq u_i$, then $u_{i-1} = v_i$, in which case (15.10) follows from the legality of the order.

Then we have

(15.11) $$\begin{aligned} d(C) &\geq \sum_{i=1}^{n-1} d(v_i, u_i) \\ &= \sum_{i=1}^{n-1} (d(\{v_1, \ldots, v_i\}, u_i) - d(\{v_1, \ldots, v_{i-1}\}, u_i)) \\ &\geq \sum_{i=1}^{n-1} (d(\{v_1, \ldots, v_i\}, u_i) - d(\{v_1, \ldots, v_{i-1}\}, u_{i-1})) \\ &= d(\{v_1, \ldots, v_{n-1}\}, v_n) = d(v_n), \end{aligned}$$

showing (15.9).

Next one has that a legal order for a given graph G can be found in time $O(m)$. Indeed, one can find $v_1, v_2, v_3, \ldots$ successively: if $v_1, \ldots, v_{i-1}$ have been found, we need to find a $v \in V \setminus \{v_1, \ldots, v_{i-1}\}$ maximizing $d(\{v_1, \ldots, v_{i-1}\}, v)$. With a 'bucket' data structure this can be done in $O(m)$ time.[23]

[23] Suppose that we have a set V and a function $\phi : V \to \mathbb{Z}$. We can select and delete a v maximizing $\phi(v)$ in $O(1)$ time and to reset $\phi(v)$ from k to k' in $O(|k' - k|)$ time.

Concluding, in any given graph $G = (V, E)$ with $|V| \geq 2$, we can find two vertices s and t with $r(s,t) = d(s)$ in time $O(m)$. Identify s and t, and find recursively a minimum cut C in the new graph. Then $\delta(s)$ is a minimum cut separating s and t, and C is a minimum cut not separating s and t. Hence, the smallest of the two is a minimum cut. ∎

(Another correctness proof was given by Fujishige [1998].)

Theorem 15.10 extends to capacitated graphs (Nagamochi and Ibaraki [1992b]):

Theorem 15.11. *Given an undirected graph $G = (V, E)$ and a capacity function $c : E \to \mathbb{Q}_+$, a minimum-capacity cut can be found in time $O(n(m + n\log n))$.*

Proof. This can be shown in the same way as Theorem 15.10, using Fibonacci heaps for finding a legal order. ∎

15.3a. Complexity survey for arc- and edge-connectivity

Survey for arc-complexity (in uncapacitated directed graphs, unless stated otherwise):

	$O(n \cdot \mathrm{EC}(n,m))$	(trivial)
	$O(nm)$	*simple undirected* Podderyugin [1973]
	$O(n \cdot \mathrm{EC}_l(n,m))$	Schnorr [1978b,1979]
	$O(\lambda^3 n^2 + \lambda m)$	Timofeev [1982]
	$O(\lambda n^2)$	*simple undirected* Karzanov and Timofeev [1986], Matula [1987]
	$O(n \cdot \mathrm{EC}_\lambda(n,m))$	Matula [1987]
	$O(nm)$	*simple* Mansour and Schieber [1989]
	$O(\lambda^2 n^2)$	*simple* Mansour and Schieber [1989]
*	$O(n\frac{\log\delta}{\delta} \cdot \mathrm{EC}(n,m))$	*simple* N. Alon, 1988 (cf. Mansour and Schieber [1989])
	$O(nm \log_n(n^2/m))$	*undirected* Feder and Motwani [1991,1995]

≫

To this end, partition V into classes V_j, where $V_j := \{v \in V \mid \phi(v) = j\}$. Each nonempty V_j is ordered as a doubly linked list, and the nonempty V_j among them are ordered as a doubly linked list L, in increasing order of j. Then in $O(1)$ time we can choose the largest j for which V_j is nonempty, choose $v \in V_j$, delete v from V_j, and possibly delete V_j from L (if V_j has become empty). Resetting $\phi(v)$ from k to k' can be done by finding or creating $V_{k'}$ in L, which takes $O(|k'-k|)$ time.

Having this data structure, throughout let $U := V \setminus \{v_1, \ldots, v_i\}$, and for each $v \in U$ let $\phi(v) := d(\{v_1, \ldots, v_i\}, v)$. If v_{i+1} has been found, we must delete v_{i+1} from U, and reset, for each neighbour v of v_{i+1} in U, $\phi(v)$ to $d(\{v_1, \ldots, v_{i+1}\}, v)$. This gives an $O(m)$-time algorithm.

continued

	$O(nm)$	*undirected* Nagamochi and Ibaraki [1992b]
	$O(m+\lambda^2 n^2)$	*undirected* Nagamochi and Ibaraki [1992b]
	$O(m+\tilde{m}n+n^2\log n)$	*undirected* Nagamochi and Ibaraki [1992b]
*	$O(n(m+n\log n))$	*capacitated undirected* Nagamochi and Ibaraki [1992b]
*	$O(nm\log(n^2/m))$	*capacitated* Hao and Orlin [1992,1994]
*	$O(\lambda m\log(n^2/m))$	Gabow [1991a,1995a]
*	$O(m+\lambda^2 n\log(n/\lambda))$	*undirected* Gabow [1991a,1995a]

Here λ denotes the arc- or edge-connectivity of the graph, $\tilde{m}$ the number of parallel classes of edges, and δ the minimum (out-)degree. Note that $\lambda \leq \delta \leq 2m/n$. If l is involved, the time bound is for determining $\min\{\lambda, l\}$.

$\mathrm{EC}(n,m)$ denotes the time needed to find the minimum size of an $s-t$ cut, for fixed s,t. Moreover, $\mathrm{EC}_l(n,m)$ denotes the time needed to find the minimum size of an $s-t$ cut (for fixed s,t) if this size is less than l. We refer to Sections 9.4 and 9.6a for bounds on $\mathrm{EC}(n,m)$ and $\mathrm{EC}_l(n,m)$.

By the observation of Matula [1987] (cf. Corollary 15.9a above), if $\min\{\lambda, l\}$ can be determined in time $O(l^\alpha f(n,m))$ (for some $\alpha \geq 1$), then λ can be determined in time $O(\lambda^\alpha f(n,m))$.

Matula [1993] gave a linear-time $2+\varepsilon$-approximative algorithm determining the edge-connectivity. (Related work was done by Henzinger [1997].)

Galil and Italiano [1991] described a linear-time method to make from a graph G a graph $\phi_k(G)$, with $m+(k-2)n$ vertices and $(2k-3)m$ edges such that: G is k-edge-connected $\iff$ $\phi_k(G)$ is k-vertex-connected. This implies, for instance, that 3-edge-connectivity can be tested in linear time (as Hopcroft and Tarjan [1973b] showed that 3-vertex-connectivity can be tested in linear time). Related work was reported by Esfahanian and Hakimi [1984] and Padberg and Rinaldi [1990a].

Karger and Stein [1993,1996] gave a randomized minimum cut algorithm for undirected graphs, with running time $O(n^2\log^3 n)$. Karger [1996,2000] gave an improvement to $O(m\log^3 n)$.

Nagamochi, Ono, and Ibaraki [1994] report on computational experiments with the Nagamochi-Ibaraki algorithm. An experimental study of several minimum cut algorithms was presented by Chekuri, Goldberg, Karger, Levine, and Stein [1997].

15.3b. Finding the 2-edge-connected components

Let $G=(V,E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. Consider the relation $\sim$ on V defined by:

(15.12) $\quad u \sim v \iff G$ has k edge-disjoint $u-v$ paths.

Then $\sim$ is an equivalence relation. This can be seen with Menger's theorem. If $u \sim v$ and $v \sim w$, then $u \sim w$; otherwise, there is a $u-w$ cut C of size less than k. Then C is also a $u-v$ cut or a $v-w$ cut, contradicting the fact that $u \sim v$ and $v \sim w$.

The equivalence classes are called the *k-edge-connected components* of G. So the 1-edge connected components of G coincide with the components of G, and can be found in linear time by Corollary 6.6a. Also for $k = 2$, the k-edge-connected components can be found in linear time (Karzanov [1970]; we follow the proof of Tarjan [1972]):

Theorem 15.12. *Given an undirected graph $G = (V, E)$, its 2-edge-connected components can be found in linear time.*

Proof. We may assume that G is connected, since by Corollary 6.6a, the components of G can be found in linear time.

Choose $s \in V$ arbitrarily, and consider a depth-first search tree T starting at s. Orient each edge in T away from s. For each remaining edge $e = uv$, there is a directed path in T that connects u and v. Let the path run from u to v. Then orient e from v to u. This gives the orientation D of G.

Then any edge not in T belongs to a directed circuit in D. Moreover, any edge in T that is not a cut edge, belongs to a directed circuit in D. Then the 2-edge-connected components of G coincide with the strong components of D. By Theorem 6.6, these components can be found in linear time. ∎

More on finding 2-edge-connected components can be found in Gabow [2000a].

15.4. Gomory-Hu trees

In previous sections of this chapter we have considered the problem of determining a minimum cut in a graph, where the minimum is taken over all pairs s, t. The *all-pairs minimum-size cut problem* asks for a minimum $s - t$ cut for all pairs of vertices s, t. Clearly, this can be solved in time $O(n^2\tau)$, where τ is the time needed for finding a minimum $s - t$ cut for any given s, t.

Gomory and Hu [1961] showed that for *undirected* graphs it can be done faster, and that there is a concise structure, the Gomory-Hu tree, to represent all minimum cuts. Similarly for the capacitated case.

Fix an undirected graph $G = (V, E)$ and a capacity function $c : E \to \mathbb{R}_+$. A *Gomory-Hu tree* (*for G and c*) is a tree $T = (V, F)$ such that for each edge $e = st$ of T, $\delta(U)$ is a minimum-capacity $s - t$ cut of G, where U is any of the two components of $T - e$. (Note that it is not required that T is a subgraph of G.)

Gomory and Hu [1961] showed that for each G, c there indeed exists a Gomory-Hu tree, and that it can be found by $n - 1$ minimum-cut computations.

For distinct $s, t \in V$, define $r(s, t)$ as the minimum capacity of an $s - t$ cut. The following triangle inequality holds:

$$(15.13) \qquad r(u, w) \geq \min\{r(u, v), r(v, w)\}$$

for all distinct $u, v, w \in G$. Now a Gomory-Hu tree indeed describes concisely minimum-capacity $s - t$ cuts for all s, t:

Theorem 15.13. *Let $T = (V, F)$ be a Gomory-Hu tree. Consider any $s, t \in V$, the $s-t$ path P in T, an edge $e = uv$ on P with $r(u, v)$ minimum, and any component K of $T-e$. Then $r(s, t) = r(u, v)$ and $\delta(K)$ is a minimum-capacity $s - t$ cut.*

Proof. Inductively, (15.13) gives $r(s, t) \geq r(u, v)$. Moreover, $\delta(K)$ is an $s - t$ cut, and hence $r(s, t) \leq c(\delta(K)) = r(u, v)$. ∎

To show that a Gomory-Hu tree does exist, we first prove:

Lemma 15.14α. *Let $s, t \in V$, let $\delta(U)$ be a minimum-capacity $s - t$ cut in G, and let $u, v \in U$ with $u \neq v$. Then there exists a minimum-capacity $u - v$ cut $\delta(W)$ with $W \subseteq U$.*

Proof. Consider a minimum-capacity $u - v$ cut $\delta(X)$. By symmetry we may assume that $s \in U$ (otherwise interchange s and t), $t \notin U$, $s \in X$ (otherwise replace X by $V \setminus X$), $u \in X$ (otherwise interchange u and v), and $v \notin X$. So one of the diagrams of Figure 15.2 applies.

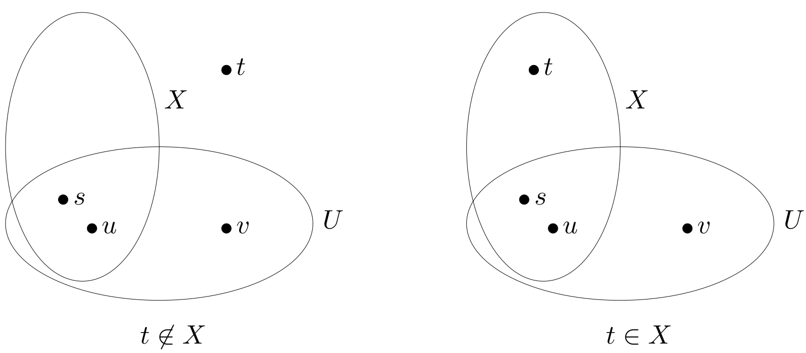

Figure 15.2

In particular, $\delta(U \cap X)$ and $\delta(U \setminus X)$ are $u-v$ cuts. If $t \notin X$, then $\delta(U \cup X)$ is an $s - t$ cut. As

(15.14) $$c(\delta(U \cap X)) + c(\delta(U \cup X)) \leq c(\delta(U)) + c(\delta(X))$$

and

(15.15) $$c(\delta(U \cup X)) \geq c(\delta(U)),$$

we have $c(\delta(U \cap X)) \leq c(\delta(X))$. So $\delta(U \cap X)$ is a minimum-capacity $u - v$ cut.

If $t \in X$, then $\delta(X \setminus U)$ is an $s - t$ cut. As

(15.16) $$c(\delta(U \setminus X)) + c(\delta(X \setminus U)) \leq c(\delta(U)) + c(\delta(X))$$

and

(15.17) $\quad c(\delta(X \setminus U)) \geq c(\delta(U)),$

we have $c(\delta(U \setminus X)) \leq c(\delta(X))$. So $\delta(U \setminus X)$ is a minimum-capacity $u-v$ cut. ∎

This lemma is used in proving the existence of Gomory-Hu trees:

Theorem 15.14. *For each graph $G=(V,E)$ and each capacity function $c: E \to \mathbb{R}_+$ there exists a Gomory-Hu tree.*

Proof. Define a *Gomory-Hu tree for* a set $R \subseteq V$ to be a pair of a tree (R,T) and a partition $(C_r \mid r \in R)$ of V such that:

(15.18) (i) $r \in C_r$ for each $r \in R$,
(ii) $\delta(U)$ is a minimum-capacity $s-t$ cut for each edge $e=st \in T$, where $U := \bigcup_{u \in K} C_u$ and K is a component of $T-e$.

We show by induction on $|R|$ that for each nonempty $R \subseteq V$ there exists a Gomory-Hu tree for R. Then for $R=V$ we have a Gomory-Hu tree.

If $|R|=1$, (15.18) is trivial, so assume $|R| \geq 2$. Let $\delta(W)$ be a minimum-capacity cut separating at least one pair of vertices in R. Contract $V \setminus W$ to one vertex, v' say, giving graph G'. Let $R' := R \cap W$. By induction, G' has a Gomory-Hu tree (R',T'), $(C'_r \mid r \in R')$ for R'.

Similarly, contract W to one vertex, v'' say, giving graph G''. Let $R'' := R \setminus W$. By induction, G'' has a Gomory-Hu tree (R'',T''), $(C''_r \mid r \in R'')$ for R''.

Now let $r' \in R'$ be such that $v' \in C'_{r'}$. Similarly, let $r'' \in R''$ be such that $v'' \in C''_{r''}$. Let $T := T' \cup T'' \cup \{r'r''\}$, Let $C_{r'} := C'_{r'} \setminus \{v'\}$ and let $C_r := C'_r$ for all other $r \in R'$. Similarly, let $C_{r''} := C''_{r''} \setminus \{v''\}$ and let $C_r := C''_r$ for all other $r \in R''$.

Now (R,T) and the C_r form a Gomory-Hu tree for R. Indeed, for any $e \in T$ with $e \neq r'r''$, (15.18) follows from Lemma 15.14α. If $e = r'r''$, then $U = W$ and $\delta(W)$ is a minimum-capacity $r'-r''$ cut (as it is minimum-capacity over all cuts separating at least one pair of vertices in R). ∎

The method can be sharpened to give the following algorithmic result:

Theorem 15.15. *A Gomory-Hu tree can be found by $n-1$ applications of a minimum-capacity cut algorithm.*

Proof. In the proof of Theorem 15.14, it suffices to take for $\delta(W)$ just a minimum-capacity $s-t$ cut for at least one pair $s,t \in R$. Then $\delta(W)$ is also a minimum-capacity $r'-r''$ cut. For suppose that there exists an $r'-r''$ cut $\delta(X)$ of smaller capacity. We may assume that $s \in W$ and $t \notin W$. As $\delta(W)$ is a minimum-capacity $s-t$ cut, $\delta(X)$ is not an $s-t$ cut. So it should separate

s and r' or t and r''. By symmetry, we may assume that it separates s and r'. Then it also as is a $u-v$ cut for some edge uv on the $s-r'$ path in T'. Let uv determine cut $\delta(U)$. This cut is an $s-t$ cut, and hence $c(\delta(U)) \geq c(\delta(W))$. On the other hand, $c(\delta(U)) \leq c(\delta(X))$, as $\delta(U)$ is a minimum-capacity $u-v$ cut. This contradicts our assumption that $c(\delta(X)) < c(\delta(W))$. ∎

This implies for the running time:

Corollary 15.15a. *A Gomory-Hu tree can be found in time $O(n\tau)$ time, if for any $s, t \in V$ a minimum-capacity $s-t$ cut can be found in time τ.*

Proof. Directly from Theorem 15.15. ∎

Notes. The method gives an $O(m^2)$ method to find a Gomory-Hu tree for the capacity function $c = \mathbf{1}$, since $O(m^2) = O(\sum_v d(v)m)$, and for each new vertex v a minimum cut can be found in time $O(d(v)m)$. Hao and Orlin [1992,1994] gave an $O(n^3)$-time method to find, for given graph $G = (V, E)$ and $s \in V$, all minimum-size $s-t$ cuts for all $t \neq s$ (with push-relabel). Shiloach [1979b] gave an $O(n^2m)$ algorithm to find a maximum number of edge-disjoint paths between all pairs of vertices in an undirected graph. Ahuja, Magnanti, and Orlin [1993] showed that the best directed all-pairs cut algorithm takes $\Omega(n^2)$ max-flow iterations.

For planar graphs, Hartvigsen and Mardon [1994] gave an $(n^2 \log n + m)$ algorithm to find a Gomory-Hu tree (they observed that this bound can be derived also from Frederickson [1987b]). This improves a result of Shiloach [1980a], who gave an $O(n^2(\log n)^2)$-time algorithm to find minimum-size cuts between all pairs of vertices in a planar graph.

Theorem 15.13 implies that a Gomory-Hu tree for a graph $G = (V, E)$ is a maximum-weight spanning tree in the complete graph on V, for weight function $r(u, v)$. However, not every maximum-weight spanning tree is a Gomory-Hu tree (for $G = K_{1,2}$, $c = \mathbf{1}$, only G itself is a Gomory-Hu tree, but all spanning trees on $VK_{1,2}$ have the same weight).

More on Gomory-Hu trees can be found in Elmaghraby [1964], Hu and Shing [1983], Agarwal, Mittal, and Sharma [1984], Granot and Hassin [1986], Hassin [1988], Chen [1990], Gusfield [1990], Hartvigsen and Margot [1995], Talluri [1996], Goldberg and Tsioutsiouliklis [1999,2001], and Hartvigsen [2001b]. Generalizations were given by Cheng and Hu [1990,1991,1992] and Hartvigsen [1995] (to matroids).

15.4a. Minimum-requirement spanning tree

Hu [1974] gave the following additional application of Gomory-Hu trees. Let $G = (V, E)$ be an undirected graph and let $r : E \to \mathbb{R}_+$ be a 'requirement' function (say, the number of telephone calls to be made between the end vertices of e).

We want to find a tree T on V minimizing

$$\sum_{e \in E} r(e)\mathrm{dist}_T(e), \tag{15.19}$$

where $\text{dist}_T(e)$ denotes the distance in T between the end vertices of e.

Now any Gomory-Hu tree T for G and capacity function r indeed minimizes (15.19). To see this, let for any edge f of T, $R_T(f)$ be equal to the requirement (= capacity) of the cut determined by the two components of $T - f$. Then (15.19) is equal to

$$(15.20) \qquad \sum_{f \in T} R_T(f).$$

Now T minimizes (15.20), as was shown by Adolphson and Hu [1973]. For let T' be any other spanning tree on V. Then for each $f = st \in T$ and each edge f' on the $s - t$ path in T' one has

$$(15.21) \qquad R_{T'}(f') \geq R_T(f),$$

since the components of $T - f$ determine a minimum-capacity $s - t$ cut, and since the components of $T' - f'$ determine an $s - t$ cut. Since T and T' are spanning trees, there exists a one-to-one function $\phi : T \to T'$ such that for each $f = st \in T$, $\phi(f)$ is an edge on the $s - t$ path in T'.

To see this, let u be an end vertex of T. Let $f = uv$ be the edge of T incident with u, and define $\phi(f)$ to be the first edge of the $u - v$ path in T'. Delete f and contract $\phi(f)$. Then induction gives the required function.

So (15.21) implies that (15.20) is not decreased by replacing T by T'. Hence T minimizes (15.20), and therefore also (15.19).

15.5. Further results and notes

15.5a. Ear-decomposition of undirected graphs

In Section 6.5c we characterized the strongly connected digraphs as those digraphs having an ear-decomposition. We now consider the undirected case, and we will see a correspondence between ear-decompositions and 2-(edge-)connected graphs.

Let $G = (V, E)$ be an undirected graph. An *ear* of G is a path or circuit P in G, of length ≥ 1, such that all internal vertices of P have degree 2 in G. The path may consist of a single edge — so any edge of G is an ear. A *proper ear* is an ear that is a path, that is, has two different ends.

If I is the set of internal vertices of an ear P, we say that G arises from $G - I$ by *adding ear*. An *ear-decomposition* of G is a series of graphs $G_0, G_1, \ldots, G_k$, where $G_0 = K_1$, $G_k = G$, and G_i arises from G_{i-1} by adding an ear ($i = 1, \ldots, k$). If $G_0 = K_2$ and G_i arises from G_{i-1} by adding a proper ear, it is a *proper ear-decomposition*.

Graphs with a *proper* ear-decomposition were characterized by Whitney [1932b]:

Theorem 15.16. *A graph $G = (V, E)$ with $|V| \geq 2$ has a proper ear-decomposition if and only if G is 2-vertex-connected.*

Proof. Necessity follows from the facts that K_2 is 2-vertex-connected and that 2-vertex-connectivity is maintained under adding proper ears. To see sufficiency, let G be 2-vertex-connected, and let $G' = (V', E')$ be a subgraph of G that has a proper ear-decomposition, with $|E'|$ as large as possible. Suppose that $E' \neq E$, and let $e = uv$ be an edge in $E \setminus E'$ incident with V'; say $u \in V'$. By the 2-connectivity of G, there is a path from v to V' avoiding u. Let P be a shortest such path. Then path e, P is a proper ear that can be added to G', contradicting the maximality of $|E'|$. ∎

Similarly, graphs having an ear-decomposition are characterized by being 2-edge-connected (this is implicit in Robbins [1939]):

Theorem 15.17. *A graph $G = (V, E)$ has an ear-decomposition if and only if G is 2-edge-connected.*

Proof. Necessity follows from the facts that K_1 is 2-edge-connected and that 2-edge-connectivity is maintained under adding ears. To see sufficiency, let G be 2-edge-connected, and let $G' = (V', E')$ be a subgraph of G that has an ear-decomposition, with $|E'|$ as large as possible. Suppose that $E' \neq E$, and let $e = uv$ be an edge in $E \setminus E'$ incident with V'; say $u \in V'$. Let C be a circuit in G traversing e. Let C start with $u, e, \ldots$. Let s be the first vertex in C, after e, that belongs to V'. Then subpath $P = u, e \ldots, w$ of C is an ear that can be added to G', contradicting the maximality of $|E'|$. ∎

15.5b. Further notes

Dinits, Karzanov, and Lomonosov [1976] showed that the set of all minimum-capacity cuts of an undirected graph (with positive capacities on the edges) has the "cactus structure", as follows. A *cactus* is a connected graph such that each edge belongs to at most one circuit. Let $G = (V, E)$ be a graph with a capacity function $c : E \to \mathbb{Z}_+$ such that the minimum cut capacity λ is positive. Then there exist a cactus K with $O(|V|)$ vertices and a function $\phi : V \to VK$ such that for each inclusionwise minimal cut $\delta_K(U)$ of K, the set $\delta_G(\phi^{-1}(U))$ is a cut of capacity λ, and such that each minimum-capacity cut in G can be obtained this way. Moreover, K is a tree when λ is odd. It follows that the number of minimum-capacity cuts is at most $\binom{n}{2}$ (and at most $n - 1$ when λ is odd), and that the vertices of G can be ordered as $v_1, \ldots, v_n$ so that each minimum-capacity cut is of the form $\delta(\{v_i, v_{i+1}, \ldots, v_j\})$ for some $i \leq j$. Related results can be found in Picard and Queyranne [1980], Karzanov and Timofeev [1986], Gabow [1991b,1993b,1995c], Gusfield and Naor [1993], Karger and Stein [1993,1996], Nagamochi, Nishimura, and Ibaraki [1994,1997], Benczúr [1995], Henzinger and Williamson [1996], Karger [1996,2000], Fleischer [1998a,1999b], Dinitz and Vainshtein [2000], and Nagamochi, Nakao, and Ibaraki [2000].

Gusfield and Naor [1990,1991] considered the analogue of the Gomory-Hu tree for *vertex*-cuts.

A theorem of Mader [1971] implies that each k-connected graph $G = (V, E)$ contains a k-connected spanning subgraph with $O(k|V|)$ edges — similarly for k-edge-connected. This was extended by Nagamochi and Ibaraki [1992a], showing that for each k, each graph $G = (V, E)$ has a subgraph $G_k = (V, E_k)$ such that $|E_k| = O(k|V|)$ and such that for all $s, t \in V$:

(15.22) (i) $\lambda_{G_k}(s, t) \geq \min\{\lambda_G(s, t), k\}$,

(ii) $\kappa_{G_k}(s,t) \geq \min\{\kappa_G(s,t), k\}$ if $st \notin E$.

Here $\lambda_H(s,t)$ ($\kappa_H(s,t)$, respectively) denotes the maximum number of edge-disjoint (internally vertex-disjoint, respectively) $s-t$ paths in H. They also gave a linear-time algorithm finding G_k. A shorter proof and a generalization was given by Frank, Ibaraki, and Nagamochi [1993].

Frank [1995] showed that the following is implied by the existence of a Gomory-Hu tree. Let $G = (V, E)$ be an undirected graph of minimum degree k. Then there exist two distinct vertices $s, t \in V$ connected by k edge-disjoint paths. This follows by taking for s a vertex of degree 1 in the Gomory-Hu tree, and for t its neighbour in this tree. Then $\delta_E(s)$ is a minimum-size cut separating s and t.

Tamir [1994] observed that the up hull P of the incidence vectors of the nontrivial cuts of an undirected graph $G = (V, E)$ can be described as follows. Let $D = (V, A)$ be the digraph with A being the set of all ordered pairs (u, v) for adjacent $u, v \in V$. Choose $r \in V$ arbitrarily. Then P is equal to the projection to x-space of the polyhedron in the variables $x \in \mathbb{R}^E$ and $y \in \mathbb{R}^A$ determined by:

(15.23)
(i) $y(a) \geq 0$ for each $a \in A$,
(ii) $y(B) \geq 1$ for each r-arborescence B,
(iii) $x(e) = y(u,v) + y(v,u)$ for each edge $e = uv$ of G.

(Here an *r-arborescence* is a subset B of A such that (V, B) is a rooted tree rooted at r.) This can be shown with the help of the results to be discussed in Chapter 53. To see this, consider any $c \in \mathbb{R}^E_+$. Then the minimum value of $c^{\mathsf{T}}x$ over all x, y satisfying (15.23), is equal to the minimum value of $d^{\mathsf{T}}y$ over all x, y satisfying (15.23), where $d(u,v) := c(uv)$ for each $(u,v) \in A$. This is equal to the minimum value of $d^{\mathsf{T}}y$ over all y satisfying (i) and (ii) of (15.23). By Corollary 53.1f, below this is equal to the minimum d-weight of an r-cut in D, which is equal to the minimum c-weight of a nontrivial cut in G.

No explicit description in terms of linear inequalities is known for the up hull of the incidence vectors of nontrivial cuts. Alevras [1999] gave descriptions for small instances (up to seven vertices for undirected graphs and up to five vertices for directed graphs).

The minimum k-cut problem: 'find a partition of the vertex set of a graph into k nonempty classes such that the number of edges connecting different classes is minimized', is NP-complete if k is part of the input (there is an easy reduction from the maximum clique problem, as the problem is equivalent to maximizing the number of edges spanned by the classes in the partition). For fixed k however, it was shown to be polynomial-time solvable by Goldschmidt and Hochbaum [1988,1994]. If we prescribe certain vertices to belong to the classes, the problem is NP-complete even for $k = 3$ (Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [1992, 1994]). More on this problem can be found in Hochbaum and Shmoys [1985], Lee, Park, and Kim [1989], Chopra and Rao [1991], Cunningham [1991], He [1991], Saran and Vazirani [1991,1995], Garg, Vazirani, and Yannakakis [1994], Kapoor [1996], Burlet and Goldschmidt [1997], Kamidoi, Wakabayashi, and Yoshida [1997], Călinescu, Karloff, and Rabani [1998,2000], Hartvigsen [1998b], Karzanov [1998c], Cunningham and Tang [1999], Karger, Klein, Stein, Thorup, and Young [1999], Nagamochi and Ibaraki [1999a,2000], Nagamochi, Katayama, and Ibaraki [1999, 2000], Goemans and Williamson [2001], Naor and Rabani [2001], Zhao, Nagamochi, and Ibaraki [2001], and Ravi and Sinha [2002].

Surveys on connectivity are given by Even [1979], Mader [1979], Frank [1995], and Subramanian [1995] (edge-connectivity). For the decomposition of 3-connected graphs into 4-connected graphs, see Coullard, Gardner, and Wagner [1993].

Part II

Bipartite Matching and Covering

Part II: Bipartite Matching and Covering

A second classical area of combinatorial optimization is formed by bipartite matching. The area gives rise to a number of basic problems and techniques, and has an abundance of applications in various forms of assignment and transportation.
Work of Frobenius in the 1910s on the decomposition of matrices formed the incentive to Kőnig to study matchings in bipartite graphs. An extension by Egerváry in the 1930s to weighted matchings inspired Kuhn in the 1950s to design the 'Hungarian method' for the assignment problem (which is equivalent to finding a minimum-weight perfect matching in a complete bipartite graph).
Parallel to this, Tolstoĭ, Kantorovich, Hitchcock, and Koopmans had investigated the transportation problem. It motivated Kantorovich and Dantzig to consider more general problems, culminating in the development of linear programming. It led in turn to solving the assignment problem by linear programming, and thus to a polyhedral approach.
Several variations and extensions of bipartite matching, like edge covers, factors, and transversals, can be handled similarly. Major explanation is the total unimodularity of the underlying matrices.
Bipartite matching and transportation can be considered as special cases of disjoint paths and of transshipment, studied in the previous part — just consider a bipartite graph as a directed graph, by orienting all edges from one colour class to the other. It was however observed by Hoffman and Orden that this can be turned around, and that disjoint paths and transshipment problems can be reduced to bipartite matching and transportation problems. So several results in this part on bipartite matching are matched by results in the previous part on paths and flows. Viewed this way, the present part forms a link between the previous part and the next part on *nonbipartite* matching, where the underlying matrices generally are not totally unimodular.

Chapters:

Chapter 16

Cardinality bipartite matching and vertex cover

'Cardinality matching' deals with maximum-size matchings. In this chapter we give the theorems of Frobenius on the existence of a perfect matching in a bipartite graph, and the extension by Kőnig on the maximum size of a matching in a bipartite graph. We also discuss finding a maximum-size matching in a bipartite graph algorithmically.

We start with an easy but fundamental theorem relating maximum-size matchings and M-alternating paths, that applies to any graph and that will also be important for nonbipartite matching.

In this chapter, graphs can be assumed to be simple.

16.1. M-augmenting paths

Let $G = (V, E)$ be an undirected graph. A *matching* in G is a set of disjoint edges. An important concept in finding a maximum-size matching, both in bipartite and in nonbipartite graphs, is that of an 'augmenting path' (introduced by Petersen [1891]).

Let M be a matching in a graph $G = (V, E)$. A path P in G is called M*-augmenting* if P has odd length, its ends are not covered by M, and its edges are alternatingly out of and in M.

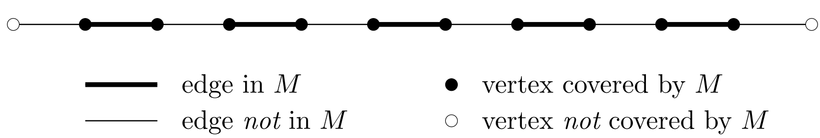

Figure 16.1
An M-augmenting path

Clearly, if P is an M-augmenting path, then

$$M' := M \triangle EP \tag{16.1}$$

is again a matching and satisfies $|M'| = |M| + 1$.[1] In fact, it is not difficult to show (Petersen [1891]):

Theorem 16.1. *Let $G = (V, E)$ be a graph and let M be a matching in G. Then either M is a matching of maximum size or there exists an M-augmenting path.*

Proof. If M is a maximum-size matching, there cannot exist an M-augmenting path P, since otherwise $M \triangle EP$ would be a larger matching.

Conversely, if M' is a matching larger than M, consider the components of the graph $G' := (V, M \cup M')$. Then G' has maximum degree two. Hence each component of G' is either a path (possibly of length 0) or a circuit. Since $|M'| > |M|$, at least one of these components should contain more edges in M' than in M. Such a component forms an M-augmenting path. ∎

So in any graph, if we have an algorithm finding an M-augmenting path for any matching M, then we can find a maximum-size matching: we iteratively find matchings $M_0, M_1, \ldots$, with $|M_i| = i$, until we have a matching M_k such that there exists no M_k-augmenting path. (Also this was observed by Petersen [1891].)

16.2. Frobenius' and Kőnig's theorems

A classical min-max relation due to Kőnig [1931] characterizes the maximum size of a matching in a bipartite graph. To this end, call a set C of vertices of a graph G a *vertex cover* if each edge of G intersects C. Define

(16.2) $$\begin{aligned} \nu(G) &:= \text{the maximum size of a matching in } G,\\ \tau(G) &:= \text{the minimum size of a vertex cover in } G. \end{aligned}$$

These numbers are called the *matching number* and the *vertex cover number* of G, respectively. It is easy to see that, for any graph G,

(16.3) $$\nu(G) \leq \tau(G),$$

since any two edges in any matching contain different vertices in any vertex cover. The graph K_3 has strict inequality in (16.3). However, if G is bipartite, equality holds, which is the content of Kőnig's matching theorem (Kőnig [1931]). It can be seen to be equivalent to a theorem of Frobenius [1917] (Corollary 16.2a below).

Theorem 16.2 (Kőnig's matching theorem)**.** *For any bipartite graph $G = (V, E)$ one has*

(16.4) $$\nu(G) = \tau(G).$$

[1] EP denotes the set of edges in P. $\triangle$ denotes symmetric difference.

That is, the maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover.

Proof. By (16.3) it suffices to show that $\nu(G) \geq \tau(G)$. We may assume that G has at least one edge. Then:

(16.5) $\quad G$ has a vertex u covered by each maximum-size matching.

To see this, let $e = uv$ be any edge of G, and suppose that there are maximum-size matchings M and N missing u and v respectively[2]. Let P be the component of $M \cup N$ containing u. So P is a path with end vertex u. Since P is not M-augmenting (as M has maximum size), P has even length, and hence does not traverse v (otherwise, P ends at v, contradicting the bipartiteness of G). So $P \cup e$ would form an N-augmenting path, a contradiction (as N has maximum size). This proves (16.5).

Now (16.5) implies that for the graph $G' := G - u$ one has $\nu(G') = \nu(G) - 1$. Moreover, by induction, G' has a vertex cover C of size $\nu(G')$. Then $C \cup \{u\}$ is a vertex cover of G of size $\nu(G') + 1 = \nu(G)$. ∎

(This proof is due to De Caen [1988]. For Kőnig's original, algorithmic proof, see the proof of Theorem 16.6. Note that also Menger's theorem implies Kőnig's matching theorem (using the construction given in the proof of Theorem 16.4 below). For a proof based on showing that any minimum bipartite graph with a given vertex cover number is a matching, see Lovász [1975d]. For another proof (of Rizzi [2000a]), see Section 16.2c. As we will see in Chapter 18, Kőnig's matching theorem also follows from the total unimodularity of the incidence matrix of a bipartite graph. (Flood [1960] and Entringer and Jackson [1969] gave proofs similar to Kőnig's proof.))

A consequence of Theorem 16.2 is a theorem of Frobenius [1917] that characterizes the existence of a perfect matching in a bipartite graph. (A matching is *perfect* if it covers all vertices.) Actually, this theorem motivated Kőnig to study matchings in graphs, and in turn it can be seen to imply Kőnig's matching theorem.

Corollary 16.2a (Frobenius' theorem)**.** *A bipartite graph $G = (V, E)$ has a perfect matching if and only if each vertex cover has size at least $\frac{1}{2}|V|$.*

Proof. Directly from Kőnig's matching theorem, since G has a perfect matching if and only if $\nu(G) \geq \frac{1}{2}|V|$. ∎

This implies an earlier theorem of Kőnig [1916] on *regular* bipartite graphs:

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[2] M *misses* a vertex u if $u \notin \bigcup M$. Here $\bigcup M$ denotes the union of the edges in M; that is, the set of vertices covered by the edges in M.

Corollary 16.2b. *Each regular bipartite graph (of positive degree) has a perfect matching.*

Proof. Let $G = (V, E)$ be a k-regular bipartite graph. So each vertex is incident with k edges. Since $|E| = \frac{1}{2}k|V|$, we need at least $\frac{1}{2}|V|$ vertices to cover all edges. Hence Corollary 16.2a implies the existence of a perfect matching. ∎

Let A be the $V \times E$ incidence matrix of the bipartite graph $G = (V, E)$. Kőnig's matching theorem (Theorem 16.2) states that the optima in the linear programming duality equation

(16.6) $$\max\{\mathbf{1}^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq \mathbf{1}^{\mathsf{T}}\}$$

are attained by integer vectors x and y. This can also be derived from the total unimodularity of A — see Section 18.3.

16.2a. Frobenius' proof of his theorem

The proof method given by Frobenius [1917] of Corollary 16.2a is in terms of matrices, but can be formulated in terms of graphs as follows. Necessity of the condition being easy, we prove sufficiency. Let U and W be the colour classes of G. As both U and W are vertex covers, and hence have size at least $\frac{1}{2}|V|$, we have $|U| = |W| = \frac{1}{2}|V|$.

Choose an edge $e = \{u, w\}$ with $u \in U$ and $w \in W$. We may assume that $G - u - w$ has no perfect matching. So, inductively, $G - u - w$ has a vertex cover C' with $|C'| < |U| - 1$. Then $C := C' \cup \{u, w\}$ is a vertex cover of G, with $|C| \leq |U|$, and hence $|C| = |U|$.

Now $U \triangle C$ and $W \triangle C$ partition V (where $\triangle$ denotes symmetric difference). If both $U \triangle C$ and $W \triangle C$ are matchable[3], then G has a perfect matching. So, by symmetry, we may assume that $U \triangle C$ is not matchable. Now $U \triangle C \neq V$ as $u \notin U \triangle C$. Hence we can apply induction, giving that $G[U \triangle C]$ has a vertex cover D with $|D| < \frac{1}{2}|U \triangle C|$. Then the set $D \cup (U \cap C)$ is a vertex cover of G (since each edge of G intersects both U and C, and hence it either intersects $U \cap C$, or is contained in $U \triangle C$ and hence intersects D). However, $|D| + |U \cap C| < \frac{1}{2}|U \triangle C| + |U \cap C| = \frac{1}{2}(|U| + |C|) = \frac{1}{2}|V|$, a contradiction.

(This is essentially also the proof method of Rado [1933] and Dulmage and Halperin [1955].)

16.2b. Linear-algebraic proof of Frobenius' theorem

Frobenius [1917] was motivated by a determinant problem, namely by the following direct consequence of his theorem. Let $A = (a_{i,j})$ be an $n \times n$ matrix in which each entry $a_{i,j}$ is either 0 or a variable $x_{i,j}$ (where the variables $x_{i,j}$ are independent). Then Frobenius' theorem is equivalent to: $\det A = 0$ if and only if A has a $k \times l$ all-zero submatrix with $k + l > n$. (Earlier, Frobenius [1912] showed that for such

[3] A set T of vertices is called *matchable* if there exists a matching M with $T = \bigcup M$.

a matrix A, $\det A$ is reducible (that is, there exist nonconstant polynomials p and q with $\det A = p \cdot q$) if and only if A has a $k \times l$ all-zero submatrix with $k + l = n$ and $k, l \geq 1$.)

Edmonds [1967b] showed that the argumentation can be applied also the other way around. This gives the following linear-algebraic proof of Frobenius' theorem (implying linear-algebraic proofs also of other bipartite matching theorems).

Let $G = (V, E)$ be a bipartite graph not having a perfect matching. Let U and W be the colour classes of G. We may assume that $|U| = |W|$ (otherwise the smaller colour class is a vertex cover of size less than $\frac{1}{2}|V|$).

Make a $U \times W$ matrix $A = (a_{u,w})$, where $a_{u,w} = 0$ if u and w are not adjacent, and $a_{u,w} = x_{u,w}$ otherwise, where the $x_{u,w}$ are independent variables.

As G has no perfect matching, we know that $\det A = 0$, and hence the columns of A are linearly dependent. Let $W' \subseteq W$ be the index set of a minimal set of linearly dependent columns of A. Then there is a subset U' of U with $|U'| = |W'| - 1$ such that the $U' \times W'$ submatrix A' of A has rank $|U'|$. Hence there is a vector y such that $A'y = \mathbf{0}$ and such that each entry in y is a nonzero polynomial in those variables $x_{u,w}$ that occur in A'. Let A'' be the $U \times W'$ submatrix of A. Then $A''y = \mathbf{0}$, and hence all entries in the $(U \setminus U') \times W'$ submatrix of A are 0. Hence the rows in U' and columns in $W \setminus W'$ cover all nonzeros. As $|U'| + |W \setminus W'| < |W|$, we have Frobenius' theorem.

16.2c. Rizzi's proof of Kőnig's matching theorem

Rizzi [2000a] gave the following short proof of Kőnig's matching theorem. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimal. Then G has a vertex u of degree at least 3. Let v be a neighbour of u. By the minimality of G, $G - v$ has a vertex cover U of size $\nu(G - v)$. Then $U \cup \{v\}$ is a vertex cover of G. As G is a counterexample, we have $|U \cup \{v\}| \geq \nu(G) + 1$, and so $\nu(G - v) = |U| \geq \nu(G)$. Therefore, G has a maximum-size matching M not covering v. Let $f \in E \setminus M$ be incident with u and not with v. Then $\nu(G - f) \geq |M| = \nu(G)$. Let W be a vertex cover of $G - f$ of size $\nu(G - f) = \nu(G)$. Then $v \notin W$, since v is not covered by M. Hence $u \in W$, as W covers edge uv of $G - f$. Therefore, W also covers f, and hence it is a vertex cover of G of size $\nu(G)$.

16.3. Maximum-size bipartite matching algorithm

We now focus on the problem of finding a maximum-size matching in a bipartite graph algorithmically. In view of Theorem 16.1, this amounts to finding an augmenting path. In the bipartite case, this can be done by finding a directed path in an auxiliary directed graph. This method is essentially due to van der Waerden [1927] and Kőnig [1931].

Matching augmenting algorithm for bipartite graphs

input: a bipartite graph $G = (V, E)$ and a matching M,
output: a matching M' satisfying $|M'| > |M|$ (if there is one).

description of the algorithm: Let G have colour classes U and W. Make a directed graph D_M by orienting each edge $e = \{u, w\}$ of G (with $u \in U, w \in W$) as follows:

(16.7) if $e \in M$, then orient e from w to u,
if $e \notin M$, then orient e from u to w.

Let U_M and W_M be the sets of vertices in U and W (respectively) missed by M.

Now an M-augmenting path (if any) can be found by finding a directed path in D_M from U_M to W_M. This gives a matching larger than M. ∎

The correctness of this algorithm is immediate. Since a directed path can be found in time $O(m)$, we can find an augmenting path in time $O(m)$. Hence we have the following result (implicit in Kuhn [1955b]):

Theorem 16.3. *A maximum-size matching in a bipartite graph can be found in time $O(nm)$.*

Proof. Note that we do at most n iterations, each of which can be done in time $O(m)$ by breadth-first search (Theorem 6.3). ∎

16.4. An $O(n^{1/2}m)$ algorithm

Hopcroft and Karp [1971,1973] and Karzanov [1973b] proved the following sharpening of Theorem 16.3, which we derive from the (equivalent) result of Karzanov [1973a], Tarjan [1974e], and Even and Tarjan [1975] on the complexity of finding a maximum number of vertex-disjoint paths (Corollary 9.7a).

Theorem 16.4. *A maximum-size matching in a bipartite graph can be found in $O(n^{1/2}m)$ time.*

Proof. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W. Make a directed graph $D = (V, A)$ as follows. Orient all edges from U to W. Moreover, add a new vertex s, with arcs (s, u) for all $u \in U$, and a new vertex t, with arcs (w, t) for all $w \in W$. Then the maximum number of internally vertex-disjoint $s - t$ paths in D is equal to the maximum size of a matching in G. The result now follows from Corollary 9.7a. ∎

In fact, the factor $n^{1/2}$ can be reduced to $\nu(G)^{1/2}$ (as before, $\nu(G)$ and $\tau(G)$ denote the maximum size of a matching and the minimum size of a vertex cover, respectively):

Theorem 16.5. *A maximum-size matching in a bipartite graph G can be found in $O(\nu(G)^{1/2}m)$ time.*

Proof. Similar to the proof of Theorem 16.4, using Theorem 9.8 and the fact that $\nu(G) = \tau(G)$. ∎

Gabow and Tarjan [1988a] observed that the method of Corollary 9.7a applied to the bipartite matching problem implies that for each k one can find in time $O(km)$ a matching of size at least $\nu(G) - \frac{n}{k}$.

16.5. Finding a minimum-size vertex cover

From a maximum-size matching in a bipartite graph, one can derive a minimum-size vertex cover. The method gives an alternative proof of Kőnig's matching theorem (in fact, this is the original proof of Kőnig [1931]):

Theorem 16.6. *Given a bipartite graph G and a maximum-size matching M in G, we can find a minimum-size vertex cover in G in time $O(m)$.*

Proof. Make D_M, U_M, and W_M as in the matching-augmenting algorithm, and let R_M be the set of vertices reachable in D_M from U_M. So $R_M \cap W_M = \emptyset$. Then each edge uw in M is either contained in R_M or disjoint from R_M (that is, $u \in R_M \iff w \in R_M$). Moreover, no edge of G connects $U \cap R_M$ and $W \setminus R_M$, as no arc of D_M leaves R_M. So $C := (U \setminus R_M) \cup (W \cap R_M)$ is a vertex cover of G. Since C is disjoint from $U_M \cup W_M$ and since no edge in M is contained in C, we have $|C| \leq |M|$. Therefore, C is a minimum-size vertex cover. ∎

Hence:

Corollary 16.6a. *A minimum-size vertex cover in a bipartite graph can be found in $O(n^{1/2}m)$ time.*

Proof. Directly from Theorems 16.4 and 16.6. ∎

16.6. Matchings covering given vertices

The following theorem characterizes when one of the colour classes of a bipartite graph can be covered by a matching, and is a direct consequence of Kőnig's matching theorem (where $N(S)$ denotes the set of vertices not in S that have a neighbour in S):

Theorem 16.7. *Let $G = (V, E)$ be a bipartite graph with colour classes U and W. Then G has a matching covering U if and only if $|N(S)| \geq |S|$ for each $S \subseteq U$.*

Proof. Necessity being trivial, we show sufficiency. By Kőnig's matching theorem (Theorem 16.2) it suffices to show that each vertex cover C has $|C| \geq |U|$. This indeed is the case, since $N(U \setminus C) \subseteq C \cap W$, and hence

$$(16.8) \qquad |C| = |C \cap U| + |C \cap W| \geq |C \cap U| + |N(U \setminus C)| \geq |C \cap U| + |U \setminus C| = |U|. \qquad \blacksquare$$

This can be extended to general subsets of V. First, Hoffman and Kuhn [1956b] and Mendelsohn and Dulmage [1958a] showed:

Theorem 16.8. *Let $G = (V, E)$ be a bipartite graph with colour classes U and W and let $R \subseteq V$. Then there exists a matching covering R if and only if there exist a matching M covering $R \cap U$ and a matching N covering $R \cap W$.*

Proof. Necessity being trivial, we show sufficiency. We may assume that G is connected, that $E = M \cup N$, and that neither M nor N covers R. This implies that there is a $u \in R \cap U$ missed by N and a $w \in R \cap W$ missed by M. So G is an even-length $u - w$ path, a contradiction, since $u \in U$ and $w \in W$. $\blacksquare$

(This theorem goes back to theorems of F. Bernstein (cf. Borel [1898] p. 103), Banach [1924], and Knaster [1927] on injective mappings between two sets.)

Theorem 16.8 implies a characterization of sets that are covered by some matching:

Corollary 16.8a. *Let $G = (V, E)$ be a bipartite graph with colour classes U and W and let $R \subseteq V$. Then there is a matching covering R if and only if $|N(S)| \geq |S|$ for each $S \subseteq R \cap U$ and for each $S \subseteq R \cap W$.*

Proof. Directly from Theorems 16.7 and 16.8. $\blacksquare$

It also gives the following exchange property:

Corollary 16.8b. *Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, let M and N be maximum-size matchings, let U' be the set of vertices in U covered by M, and let W' be the set of vertices in W covered by N. Then there exists a maximum-size matching covering $U' \cup W'$.*

Proof. Directly from Theorem 16.8: the matching found is maximum-size since $|U'| = |W'| = \nu(G)$. $\blacksquare$

Notes. These results also are special cases of the exchange results on paths discussed in Section 9.6c. Perfect [1966] gave the following linear-algebraic argument for Corollary 16.8b. Make a $U \times W$ matrix A with $a_{u,w} = x_{u,w}$ if $uw \in E$ and $a_{u,w} := 0$ otherwise, where the $x_{u,w}$ are independent variables. Let U' be any maximum-size subset of U covered by some matching and let W' be any maximum-size subset of W covered by some matching. Then U' gives a maximum-size set of

linearly independent rows of A and W' gives a maximum-size set of linearly independent columns of A. Then the $U' \times W'$ submatrix of A is nonsingular, hence of nonzero determinant. It implies (by the definition of determinant) that G has a matching covering $U' \cup W'$.
(Related work includes Perfect and Pym [1966], Pym [1967], Brualdi [1969b,1971b], and Mirsky [1969].)

16.7. Further results and notes

16.7a. Complexity survey for cardinality bipartite matching

Complexity survey for cardinality bipartite matching ($*$ indicates an asymptotically best bound in the table):

	$O(nm)$	Kőnig [1931], Kuhn [1955b]
	$O(\sqrt{n}\,m)$	Hopcroft and Karp [1971,1973], Karzanov [1973a]
$*$	$\widetilde{O}(n^{\omega})$	Ibarra and Moran [1981]
	$O(n^{3/2}\sqrt{\frac{m}{\log n}})$	Alt, Blum, Mehlhorn, and Paul [1991]
$*$	$O(\sqrt{n}\,m \log_n(n^2/m))$	Feder and Motwani [1991,1995]

Here ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^{\omega})$ arithmetic operations (e.g. $\omega = 2.376$).

Goldberg and Kennedy [1997] described a bipartite matching algorithm based on the push-relabel method, of complexity $O(\sqrt{n}\,m \log_n(n^2/m))$. Balinski and Gonzalez [1991] gave an alternative $O(nm)$ bipartite matching algorithm (not using augmenting paths).

16.7b. Finding perfect matchings in regular bipartite graphs

By Kőnig's matching theorem, each k-regular bipartite graph has a perfect matching (if $k \geq 1$). One can use the regularity also to find quickly a perfect matching. This will be used in Chapter 20 on bipartite edge-colouring.

First we show the following result of Cole and Hopcroft [1982] (which will not be used any further in this book):

Theorem 16.9. *A perfect matching in a regular bipartite graph can be found in $O(m \log n)$ time.*

Proof. We first describe an $O(m \log n)$-time algorithm for the following problem:

(16.9) given: a k-regular bipartite graph $G = (V, E)$ with $k \geq 2$,
find: a nonempty proper subset F of E with (V, F) regular.

Let G have colour classes U and W. First let k be even. Then find an Eulerian orientation of the edges of G (this can be done in $O(m)$ time (Theorem 6.7)). Let F be the set of edges oriented from U to W.

Next let k be odd. Call a subset F of E *almost regular* if $|\deg_F(v)-\deg_F(u)| \leq 1$ for all $u, v \in V$. (Here $\deg_F(v)$ is the degree of v in the graph (V, F).)

Moreover, let $\mathrm{odd}(F)$ and $\mathrm{even}(F)$ denote the sets of vertices v with $\deg_F(v)$ odd and even, respectively, and let $\Delta(F)$ denote the maximum degree of the graph (V, F). We give an $O(m)$ algorithm for the following problem:

(16.10) given: an almost regular subset F of E with $\Delta(F) \geq 2$,
find: an almost regular subset F' of E with $\Delta(F') \geq 2$ and $|\mathrm{odd}(F')| \leq \frac{1}{2}|\mathrm{odd}(F)|$.

In time $O(m)$ we can find a subset F'' of F such that

(16.11) $$\lfloor \tfrac{1}{2} \deg_F(v) \rfloor \leq \deg_{F''}(v) \leq \lceil \tfrac{1}{2} \deg_F(v) \rceil$$

for each vertex v: make an Eulerian orientation in the graph obtained from (V, F) by adding edges so as to make all degrees even, and choose for F'' the subset of all edges oriented from U to W. So F'' and $F \setminus F''$ are almost regular.

We choose F'' such that

(16.12) $$|\mathrm{odd}(F'') \cap \mathrm{odd}(F)| \leq \tfrac{1}{2}|\mathrm{odd}(F)|$$

(otherwise replace F'' by $F \setminus F''$). Let $2l$ be the degree of the even-degree vertices of (V, F). We consider two cases.

Case 1: l is even. Define $F' := F''$. By (16.11), F' is almost regular. Moreover, as l is even, $\mathrm{odd}(F') \subseteq \mathrm{odd}(F)$, implying (with (16.12)) that $|\mathrm{odd}(F')| \leq \frac{1}{2}|\mathrm{odd}(F)|$. Finally, $\Delta(F') \geq 2$, since otherwise $\Delta(F) \leq 3$ and hence $l = 0$, implying $\Delta(F) \leq 1$, a contradiction.

Case 2: l is odd. Define $F' := F'' \cup (E \setminus F)$. Then F' is almost regular, since each $\deg_{F'}(v)$ is either $\lfloor \frac{1}{2} \deg_F(v) \rfloor + k - \deg_F(v) = k - \lceil \frac{1}{2} \deg_F(v) \rceil$ or $\lceil \frac{1}{2} \deg_F(v) \rceil + k - \deg_F(v) = k - \lfloor \frac{1}{2} \deg_F(v) \rfloor$.

Since k is odd, one also has (by definition of F'): $\deg_{F'}(v)$ is odd $\iff$ $\deg_{F''}(v) + k - \deg_F(v)$ is odd $\iff$ $\deg_{F''}(v) \equiv \deg_F(v) \pmod 2$ $\iff$ $v \in \mathrm{odd}(F'') \cap \mathrm{odd}(F)$ (since $\mathrm{even}(F) \subseteq \mathrm{odd}(F'')$, as l is odd). So $|\mathrm{odd}(F')| = |\mathrm{odd}(F'') \cap \mathrm{odd}(F)| \leq \frac{1}{2}|\mathrm{odd}(F)|$, by (16.12).

Finally, suppose that $\Delta(F') \leq 1$. Choose $v \in \mathrm{odd}(F) \setminus \mathrm{odd}(F')$. So $v \in \mathrm{even}(F')$, hence $\deg_{F'}(v) = 0$, implying $\deg_{F''}(v) = 0$ and $\deg_F(v) = k$. But then $0 = \lfloor \frac{1}{2} k \rfloor$, and so $k \leq 1$, a contradiction.

This describes the $O(m)$-time algorithm for problem (16.10). It implies that one can find an almost regular subset F of E with $\Delta(F) \geq 2$ and $\mathrm{odd}(F) = \emptyset$ in $O(m \log n)$ time. So (V, F) is a regular subgraph of G, and we have solved (16.9).

This implies an $O(m \log n)$ algorithm for finding a perfect matching: First find a subset F of E as in (16.9). Without loss of generality, $|F| \leq \frac{1}{2}|E|$. Recursively, find a perfect matching in (V, F). The time is bounded by $O((m+\frac{1}{2}m+\frac{1}{4}m+\cdots)\log n) = O(m \log n)$. ∎

In fact, as was shown by Cole, Ost, and Schirra [2001], one can find a perfect matching in a regular bipartite graph in $O(m)$ time. To explain this algorithm, we first describe an algorithm that finds a perfect matching in a k-regular bipartite graph in $O(km)$ time (Schrijver [1999]). So for each fixed degree k one can find a perfect matching in a k-regular graph in linear time, which is also a consequence of an $O(n2^{2^{O(k)}})$-time algorithm of Cole [1982].

Theorem 16.10. *A perfect matching in a k-regular bipartite graph can be found in time $O(km)$.*

Proof. Let $G = (V, E)$ be a k-regular bipartite graph. For any function $w : E \to \mathbb{Z}_+$, define $E_w := \{e \in E \mid w_e > 0\}$.

Initially, set $w_e := 1$ for each $e \in E$. Next apply the following iteratively:

(16.13) Find a circuit C in E_w. Let $C = M \cup N$ for matchings M and N with $w(M) \geq w(N)$. Reset $w := w + \chi^M - \chi^N$.

Note that at any iteration, the equation $w(\delta(v)) = k$ is maintained for all v.

To see that the process terminates, note that at any iteration the sum

$$\sum_{e \in E} w_e^2 \tag{16.14}$$

increases by

$$\sum_{e \in M}((w_e+1)^2 - w_e^2) + \sum_{e \in N}((w_e-1)^2 - w_e^2) = 2w(M) + |M| - 2w(N) + |N|, \tag{16.15}$$

which is at least $|M| + |N| = |C|$. Since $w_e \leq k$ for each $e \in E$, (16.14) is bounded, and hence the process terminates. We now estimate the running time.

At termination, we have that the set E_w contains no circuit, and hence is a perfect matching (since $w(\delta(v)) = k$ for each vertex v). So at termination, the sum (16.14) is equal to $\frac{1}{2}nk^2 = km$.

Now we can find a circuit C in E_w in $O(|C|)$ time on average. Indeed, keep a path P in E_w such that $w_e < k$ for each e in P. Let v be the last vertex of P. Then there is an edge $e = vu$ not occurring in P, with $0 < w_e < k$. Reset $P := P \cup \{e\}$. If P is not a path, it contains a circuit C, and we can apply (16.13) to C, after which we reset $P := P \setminus C$. We continue with P.

Concluding, as each step increases the sum (16.14) by at least $|C|$, and takes $O(|C|)$ time on average, the algorithm terminates in $O(km)$ time. ∎

The bound given in this theorem was improved to linear time *independent of* the degree, by Cole, Ost, and Schirra [2001]. Their method forms a sharpening of the method described in the proof of Theorem 16.10, utilizing the fact that when breaking a circuit, the path segments left ('chains') can be used in the further path search to extend the path by chains, rather than just edge by edge. To this end, these chains need to be supplied with some extra data structure, the 'self-adjusting binary trees', in order to avoid that we have to run through the chain to find an end of the chain where it can be attached to the path. The basic operation is the 'splay'.

The main technique of Cole, Ost, and Schirra's theorem is contained in the proof of the following theorem. For any graph $G = (V, E)$ call a ('weight') function $w : E \to \mathbb{R}$ *k-regular* if $w(\delta(v)) = k$ for all $v \in V$.

Theorem 16.11. *Given a bipartite graph $G = (V, E)$ and a k-regular $w : E \to \mathbb{Z}_+$, for some $k \geq 2$, a perfect matching in G can be found in time $O(m \log^2 k)$.*

Proof. I. *Conceptual outline.* We first give a conceptual description, as extension of the algorithm described in the previous proof. First delete all edges e with $w_e = 0$.

We keep a set F of edges such that each component of (V, F) is a path (possibly a singleton) with at most k^2 vertices, and we keep a path

(16.16) $$Q = (P_0, e_1, P_1, \ldots, e_t, P_t),$$

where each P_j is a (path) component of (V, F). Let v be the last vertex of Q and let $e = vu$ be an edge in $E \setminus F$ incident with v with $w_e < k$. Let P be the component of (V, F) containing u.

If u is not on Q, let R be a longest segment of P starting from u. Delete the first edge of the other segment of P (if any) from F. If $|P_t| + |R| \leq k^2$, add e to F, and reset P_t to P_t, e, R. (Here and below, $|X|$ denotes the number of vertices of a path X.) Otherwise, extend Q by e, R.

If u is on Q, then:

(16.17) split Q into a part Q_1 from the beginning to u, and a part Q_2 from u to the end;
split the circuit Q_2, e into two matchings M and N, such that $w(M) \geq w(N)$;
let α be the minimum of the weights in N;
reset $w := w + \alpha(\chi^M - \chi^N)$;
delete the edges g with $w(g) = 0$ or $w(g) = k$ (in the latter case, also delete the two ends of g);
delete the first edge of Q_2 from F if it was in F;
reset $Q := Q_1$;
iterate.

If v is incident with no edge $e \in E \setminus F$ satisfying $w_e < k$, start Q in a new vertex that is incident with an edge e with $w_e < k$. If no such vertex exists, we are done: the edges left form a perfect matching.

II. *Data structure.* In order to make profit of storing paths, we need additional data structure (based on 'self-adjusting binary trees', analyzed by Sleator and Tarjan [1983b,1985], cf. Tarjan [1983]).

We keep a collection $\mathcal{P}$ of paths (possibly singletons), each being a subpath of a component of F, such that

(16.18)
(i) each component of F itself is a path in $\mathcal{P}$;
(ii) $\mathcal{P}$ is *laminar*, that is, any two paths in $\mathcal{P}$ are vertex disjoint, or one is a subpath of the other;
(iii) any nonsingleton path $P \in \mathcal{P}$ has an edge e_P such that the two components of $P - e_P$ again belong to $\mathcal{P}$.

With any path $P \in \mathcal{P}$ we keep the following information:

(16.19)
(i) the number $|P|$ of vertices in P;
(ii) a list ends(P) of the ends of P (so ends(P) contains one or two vertices);

(iii) if P is not a singleton, the edge e_P, and a list subpaths(P) of the two components of $P - e_P$;
(iv) the smallest path parent(P) in $\mathcal{P}$ that properly contains P (null if there is no such path).

Then for each edge $e \in F$ there is a unique path $P_e \in \mathcal{P}$ traversing e such that both components of $P_e - e$ again belong to $\mathcal{P}$ (that is, $e_{P_e} = e$). We keep with any $e \in F$ the path P_e.

Call a path $P \in \mathcal{P}$ a *root* if parent(P) = null. So the roots correspond to the components of the graph (V, F). Along a path $P \in \mathcal{P}$ we call edges alternatingly *odd* and *even in* P in such a way that e_P is odd.

We also store information on the current values of the w_e. Algorithmically, we only reset explicitly those w_e for which e is not in F. For $e \in F$, these values are stored implicitly, such that it takes only $O(1)$ time to update w_e for all e in a root when adding α to the odd edges and $-\alpha$ to the even edges in it. This can be done as follows.

If P is a root, we store $w(e_P)$ at P. If P has a parent Q, we store

(16.20) $$w(e_P) \pm w(e_Q)$$

at P, where $\pm$ is $-$ if e_P is odd in Q, and $+$ otherwise.

We also need the following values for any $P \in \mathcal{P}$ with $EP \neq \emptyset$:

(16.21) minodd$(P) := \min\{w_e \mid e$ odd in $P\}$, mineven$(P) := \min\{w_e \mid e$ even in $P\}$,
diffsum$(P) := \sum(w_e \mid e$ odd in $P) - \sum(w_e \mid e$ even in $P)$

(taking a minimum ∞ if the range is empty). When storing these data, we relate them to $w(e_P)$, again so as to make them invariant under updates. Thus we store

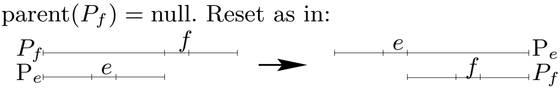
(16.22) $$\text{diffsum}(P) - |EP|w(e_P), \text{minodd}(P) - w(e_P), \text{mineven}(P) + w(e_P)$$

at P. So for any root P we have diffsum(P), minodd(P), and mineven(P) ready at hand, as we know $w(e_P)$.

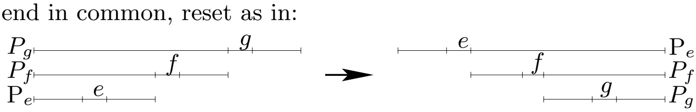
III. *The splay.* We now describe *splaying* an edge $e \in F$. It changes the data structure so that P_e becomes a root, keeping F invariant. It modifies the tree associated with the laminar family through three generations at a time, so as to attain efficiency on average. (The adjustments make future searches more efficient.)

The splay is as follows. While parent$(P_e) \neq$ null, do the following:

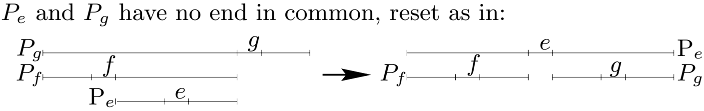
(16.23) Let $P_f :=$ parent(P_e).
Case 1: parent(P_f) = null. Reset as in:



Case 2: parent$(P_f) \neq$ null. Let $P_g :=$ parent(P_f). If P_e and P_g have an end in common, reset as in:



If P_e and P_g have no end in common, reset as in:



Note that Case 1 applies only in the last iteration of the while loop. It is straightforward to check that the data associated with the paths can be restored in $O(1)$ time at any iteration.

IV. *Running time of one splay.* To estimate the running time of a splay, define:

(16.24) $$\gamma := \sum_{P \in \mathcal{P}} \log |P|,$$

taking logarithms with base 2 (again, $|P|$ denotes the number of vertices of P).

For any splay of e one has (adding $'$ to parameters after the splay):

(16.25) the number of iterations of (16.23) is at most $\gamma - \gamma' + 3(\log |P'_e| - \log |P_e|) + 1$.

To show this, consider any iteration (16.23) (adding $'$ to parameters after the iteration).

If Case 1 applies, then

(16.26) $$\begin{aligned}&\gamma - \gamma' + 3(\log |P'_e| - \log |P_e|) + 1 \\ &= \log |P_e| + \log |P_f| - \log |P'_e| - \log |P'_f| + 3 \log |P'_e| - 3 \log |P_e| + 1 \\ &= 3 \log |P_f| - \log |P'_f| - 2 \log |P_e| + 1 \geq 1,\end{aligned}$$

since $P'_e = P_f$ and since P'_f and P_e are subpaths of P_f. If Case 2 applies, then

(16.27) $$\begin{aligned}&\gamma - \gamma' + 3(\log |P'_e| - \log |P_e|) = \log |P_e| + \log |P_f| + \log |P_g| \\ &- \log |P'_e| - \log |P'_f| - \log |P'_g| + 3 \log |P'_e| - 3 \log |P_e| \\ &= 3 \log |P_g| + \log |P_f| - \log |P'_f| - \log |P'_g| - 2 \log |P_e| \geq 1.\end{aligned}$$

The last equality follows from $P'_e = P_g$. The last inequality holds since P_e is a subpath of P_f, and P'_f, P'_g, and P_e are subpaths of P_g, and since, if the first alternative in Case 2 holds, then P_e and P'_g are vertex-disjoint (implying $2 \log |P_g| \geq \log |P_e| + \log |P'_g| + 1$), and, if the second alternative in Case 2 holds, then P'_f and P'_g are vertex-disjoint (implying $2 \log |P_g| \geq \log |P'_f| + \log |P'_g| + 1$).

(16.26) and (16.27) imply (16.25).

V. *The algorithm.* Now we use the splay to perform the conceptual operations described in the conceptual outline (proof section I above). Thus, let v be the last vertex of the current path Q (cf. (16.16)) and let $e = vu$ be an edge in $E \setminus F$ incident with u. Determine the root $P \in \mathcal{P}$ containing u (possibly by splaying an edge in F incident with u).

Case A: P is not on Q. (We keep a pointer to indicate if a root belongs to Q.) Find a root R as follows. If u is incident with no edge in F, then $R := \{u\}$. If u is incident with exactly one edge $f \in F$, splay f and let $R := P_f$. If u is incident with two edges in F, by splaying find $f \in F$ incident with u such that (after splaying f) $\text{subpaths}(P_f) = \{R, R'\}$ where $u \in \text{ends}(R)$ and $|R| > |R'|$; then delete P_f from $\mathcal{P}$, and f from F.

This determines R. If $|P_t| + |R| \leq k^2$, add e to F, let P_e be the join of P_t, e, and R, and reset P_t in Q to P_e. If $|P_t| + |R| > k^2$, extend Q by $e, P_{t+1} := R$.

Case B: P is on Q, say $P = P_j$. By (possibly) splaying, we can decide if u is at the end of P_j or not. In the former case, reset $Q := P_0, e_1, P_1, \ldots, e_j, P_j$ and let $C := e_{j+1}, P_{j+1}, \ldots, P_t, e$. In the latter case, split P_j to P'_j, f, P''_j in such a way that

$Q := P_0, e_1, P_1, \ldots, e_j, P'_j$ is the initial segment of the original Q ending at u, and let $C := f, P''_j, e_{j+1}, P_{j+1}, \ldots, P_t, e$.

Determine the difference of the sum of the w_e over the odd edges in C and that over the even edges in C. As we know diffsum(S) for any root S, this can be done in time $O(t - j + 1)$. Depending on whether this difference is positive or not, we know (implicitly) which splitting of the edges on C into matchings M and N gives $w(M) \geq w(N)$. From the values of minodd and mineven for the paths $P \in \mathcal{P}$ on C and from the values of w_e for the edges $e_{j+1}, \ldots, e_t$, e on C (and possibly f), we can find the maximum decrease α on the edges in N, and reset the parameters.

Next, for any $P \in \mathcal{P}$ on C with minodd$(P) = 0$ or mineven$(P) = 0$, determine the edges on P of weight 0, delete them after splaying, and decompose P accordingly. Delete any edge e_i on C with $w(e_i) = 0$ (similarly f).

This describes the iteration.

VI. *Running time of the algorithm.* We finally estimate the running time. In any iteration, let γ be the number of roots of $\mathcal{P}$ that are not on Q. Initially, $\gamma \leq n$. During the algorithm, γ only increases when we are in Case B and break a circuit C, in which case γ increases by at most

$$2\frac{L_C}{k^2} + m_C + 2, \tag{16.28}$$

where L_C is the length of C in G (that is, the number of edges e_i plus the sum of the lengths of the paths P_i in C), and where m_C is the number of edges of weight 0 deleted at the end of the iteration. Bound (16.28) uses the fact that the sizes of any two consecutive paths along C sum up to more than k^2, except possibly at the beginning and the end of the circuit, and that any edge of weight 0 can split a root into two new roots.

Now if we sum bound (16.28) over all circuits C throughout the iterations, we have

$$\sum_C (2\frac{L_C}{k^2} + m_C + 2) = O(m), \tag{16.29}$$

since $\sum_C L_C \leq nk^2$, like in the proof of the previous theorem (note that $m_C \geq 1$ for each C, so the term 2 is absorbed by m_C). So the number of roots created throughout the Case B iterations is $O(m)$. Now at each Case A iteration, we split off a part of a root of size less than half the size of the root; the split off part can be used again by Q some time in later iterations. Hence any root can be split at most $\log k^2$ times, and therefore, the number of Case A iterations is $O(m \log k)$. In particular, the number of times we join two paths in $\mathcal{P}$ and make a new path is $O(m \log k)$.

Next consider γ as defined in (16.24). Note that at any iteration except for joins and splays, γ does not increase. At any join, γ increases by at most $\log k^2$, and hence the total increase of γ during joins is $O(m \log^2 k)$.

Now the number of splays during any Case A iteration is $O(1)$, and during any Case B iteration $O(L_C/k^2 + m_C + 1)$. Hence by (16.29), the total number of splays is $O(m \log k)$. By (16.25), each splay takes time $O(\delta + \log k)$, where δ is the decrease of γ (possibly $\delta < 0$). The sum of δ over all splays is $O(m \log^2 k)$, as this is the total increase of γ during joins. So all splays take time $O(m \log^2 k)$. As the number of splits is proportional to the number of splays, and each takes $O(1)$ time, we have the overall time bound of $O(m \log^2 k)$. ∎

This implies a linear-time perfect matching algorithm for regular bipartite graphs:

Corollary 16.11a. *A perfect matching in a regular bipartite graph can be found in linear time.*

Proof. Let $G = (V, E)$ be a k-regular bipartite graph. We keep a weight function $w : E \to \mathbb{Z}_+$, with the property that $w(\delta(v)) = k$ for each $v \in V$. Throughout the algorithm, let G_i be the subgraph of G consisting of those edges e of G with $w_e = 2^i$ (for $i = 1, \ldots$).

Initially, define a weight $w_e := 1$ for each edge e. For $i = 0, 1, \ldots, \lfloor \log_2 k \rfloor$ do the following. Perform a depth-first search in G_i. If we meet a circuit C in G_i, then split C arbitrarily into matchings M and N, reset $w := w + 2^i(\chi^M - \chi^N)$, delete the edges in N, and update G_i (that is, delete the edges of C from G_i).

As G_i has at most $m/2^i$ edges (since $w(E) = \frac{1}{2}kn = m$), and as depth-first search can be done in time linear in the number of edges, this can be done in $O(m + \frac{1}{2}m + \frac{1}{4}m + \cdots) = O(m)$ time.

For the final G and w, all weights are a power of 2 and each graph G_i has no circuits, and hence has at most $|V| - 1$ edges. So G has at most $|V| \log_2 k$ edges. As w is k-regular, by Theorem 16.11 we can find a perfect matching in G in time $O(|V| \log^3 k)$, which is linear in the number of edges of the original graph G. ∎

This result will be used in obtaining a fast edge-colouring algorithm for bipartite graphs (Section 20.9a).

Notes. Alon [2000] gave the following easy $O(m \log m)$-time method for finding a perfect matching in a regular bipartite graph $G = (V, E)$. Let k be the degree, and choose t with $2^t \geq kn$. Let $\alpha := \lfloor 2^t/k \rfloor$ and $\beta := 2^t - k\alpha$. So $\beta < k$. Let H be the graph obtained from G by replacing each edge by α parallel edges, and by adding a β-regular set F of (new) edges, consisting of $\frac{1}{2}n$ disjoint classes, each consisting of β parallel edges. So H is 2^t-regular.

Iteratively, split H into two regular graphs of equal degree (by determining an Eulerian orientation), and reset H to the graph that has a least number of edges in F.

As $|F| = \frac{1}{2}\beta n < 2^t$, after $\log_2 |F| < t$ iterations, H contains no edge in F. Hence after t iterations we have a perfect matching in H not intersecting F; that is, we have a perfect matching in G.

This gives an $O(m \log m)$-time method, provided that we do not display the graph H fully, but handle the parallel edges implicitly (by the sizes as a function of the underlying edges).

Note that $O(m \log m) = O(nk(\log k + \log n))$. An $O(nk + n \log n \log k)$-time algorithm finding a perfect matching in a k-regular bipartite graph was given by Rizzi [2002].

(Csima and Lovász [1992] described a space-efficient $O(n^2 k \log k)$-time algorithm for finding a perfect matching in a k-regular bipartite graph.)

16.7c. The equivalence of Menger's theorem and Kőnig's theorem

We have seen that Kőnig's matching theorem can be derived from Menger's theorem (by the construction given in the proof of Theorem 16.4) — in fact it forms the induction basis in Menger's proof. The interrelation however is even stronger, as was noticed by Hoffman [1960] (cf. Orden [1955], Ford and Fulkerson [1958c], Hoffman and Markowitz [1963], Ingleton and Piff [1973]): in turn Menger's theorem (in the form of Theorem 9.1) can be derived from Kőnig's matching theorem by a direct (noninductive) construction.

Let $D = (V, A)$ be a directed graph and let $S, T \subseteq V$. We may assume that $S \cap T = \emptyset$. For each $v \in V \setminus S$ introduce a vertex v' and for each $v \in V \setminus T$ introduce a vertex v''. Let E be the set of pairs $\{u', v''\}$ with $u \in V \setminus S$ and $v \in V \setminus T$ with the property that $(u, v) \in A$ or $u = v$. This makes the bipartite graph G, containing the matching

(16.30) $$M := \{\{v', v''\} \mid v \in V \setminus (S \cup T)\}.$$

For any $X \subseteq V$, let $X' := \{v' \mid v \in X\}$ and $X'' := \{v'' \mid v \in X\}$.

Now let M' be a matching in G of size $\nu(G)$. For each component of $M \triangle M'$ having more than one vertex, we may assume that it is an M-augmenting path (since any other component K has an equal number of edges in M and in M', and hence we can replace M' by $M' \triangle K$). Each M-augmenting path is an $S'' - T'$ path. Hence there exist $|M'| - |M| = \nu(G) - |V \setminus (S \cup T)|$ vertex-disjoint $S - T$ paths.

Let $U \subseteq V \setminus T$ and $W \subseteq V \setminus S$ be such that $D := U'' \cup W'$ is a vertex cover of G, with $|U| + |W| = \tau(G)$. Then

(16.31) $$C := (U \cap S) \cup (U \cap W) \cup (W \cap T)$$

intersects each $S - T$ path in D. Indeed, suppose $P = (v_0, v_1, \ldots, v_k)$ is an $S - T$ path not intersecting C. We may assume that P intersects S and T only at v_0 and v_k, respectively. Now

(16.32) $$Q := (v_0'', v_1', v_1'', \ldots, v_{k-1}', v_{k-1}'', v_k')$$

is a path in G of odd length $2k - 1$. Hence D intersects Q in at least k vertices. Therefore, $v_0'' \in D$ (hence $v_0 \in U \cap S \subseteq C$), or $v_k' \in D$ (hence $v_k \in W \cap T \subseteq C$), or $v_i', v_i'' \in D$ for some $i \in \{1, \ldots, k-1\}$ (hence $v_i \in U \cap W \subseteq C$). So C intersects each $S - T$ path in D.

As

(16.33) $$\begin{aligned} |C| &= |U \cap S| + |U \cap W| + |W \cap T| = |U \cap S| + |U| + |W| - |U \cup W| + |W \cap T| \\ &= |U| + |W| - |V \setminus (S \cup T)| \end{aligned}$$

(since $(U \cup W) \setminus (S \cup T) = V \setminus (S \cup T)$), and as $|U| + |W| = \tau(G) = \nu(G)$, we have that the size of C is at most the number of disjoint $S - T$ paths found above.

The converse construction (described by Kuhn [1956]) also applies. Let be given a bipartite graph $G = (V, E)$, with colour classes U and W, and a matching M in G. Orient each edge from U to W, and next contract all edges in M. This gives a directed graph $D = (V', A)$. Let S and T be the sets of vertices in U and W missed by M. Then the maximum number of vertex-disjoint $S - T$ paths in D is equal to $\nu(G) - |M|$.

These constructions also imply:

Theorem 16.12. *For any function $\phi(n,m)$ one has: the bipartite matching problem with n vertices and m edges is solvable in time $O(\phi(n,m)) \iff$ the disjoint $s-t$ paths problem with n vertices and m arcs is solvable in time $O(\phi(n,m))$.*

Proof. See above. ∎

16.7d. Equivalent formulations in terms of matrices

Frobenius [1917] proved his theorem (Corollary 16.2a) in terms of matrices, in the following form:

(16.34) Each diagonal of an $n \times n$ matrix has product 0 if and only if M has a $k \times l$ all-zero submatrix with $k + l > n$.

Similarly, Kőnig's matching can be formulated in matrix terms as follows:

(16.35) In a matrix, the maximum number of nonzero entries with no two in the same line (=row or column) is equal to the minimum number of lines that include all nonzero entries.

An equivalent form of Kőnig's theorem on the existence of a perfect matching in a regular bipartite graph (Corollary 16.2b) is:

(16.36) If in a nonnegative matrix each row and each column has the same positive sum, then it has a diagonal with positive entries.

16.7e. Equivalent formulations in terms of partitions

Bipartite graphs can be studied also as unions of two partitions of a given set. Indeed, let $G = (V, E)$ be a bipartite graph. Then the family $(\delta(v) \mid v \in V)$ is a union of two partitions of E. Since each union of two partitions arises in this way, we can formulate theorems on bipartite graphs equivalently as theorems on unions of two partitions of a set.

The following equivalent form of Frobenius' theorem (Corollary 16.2a) was given by Maak [1936]:

(16.37) Let $\mathcal{A}$ and $\mathcal{B}$ be two partitions of the finite set X. Then there is a subset Y of X intersecting each set in $\mathcal{A} \cup \mathcal{B}$ in exactly one element if and only if for each natural number k, the union of any collection of k classes of $\mathcal{A}$ intersects at least k classes of $\mathcal{B}$.

This implies the following equivalent form of Corollary 16.2b, given by van der Waerden [1927] (with short proof by Sperner [1927] — see Section 22.7d):

(16.38) Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be two partitions of a finite set X with $|A_1| = \cdots = |A_n| = |B_1| = \cdots = |B_n|$. Then there is a subset Y of X intersecting each A_i and each B_i in exactly one element.

Some of the matching results can be formulated in terms of (common) transversals. We will discuss this more extensively in Chapters 22 and 23.

16.7f. On the complexity of bipartite matching and vertex cover

In a *bipartite* graph we can derive a minimum-size vertex cover from a maximum-size matching in linear time (for general graphs this would imply NP=P) — see Theorems 16.6.

So knowing a maximum-size matching in a bipartite graph gives us a minimum-size vertex cover in linear time. The reverse, however, is unlikely, unless there would exist an algorithm to find a perfect matching in a bipartite graph in linear time. To see this, suppose that there is an algorithm $\mathcal{A}$ to derive from a minimum-size vertex cover a maximum-size matching in linear time. Now let $G = (V, E)$ be a bipartite graph in which we want to find a perfect matching. Then we may *assume* that G has a perfect matching. So we may assume by Frobenius' theorem that the colour classes U and W are minimum-size vertex covers. Then apply $\mathcal{A}$ to G and U. Then either we obtain a perfect matching if U indeed is a minimum-size vertex cover, or else (if our assumption is wrong) the algorithm gets stuck, in which case we may conclude that G has no perfect matching.

16.7g. Further notes

Extensions of Frobenius' and Kőnig's theorems to the infinite case were considered by Kőnig and Valkó [1925], Shmushkovich [1939], de Bruijn [1943], Rado [1949b], Brualdi [1971f], Aharoni [1983b,1984b], and Aharoni, Magidor, and Shore [1992].

Itai, Rodeh, and Tanimoto [1978] showed that, given a bipartite graph $G = (V, E)$, $F \subseteq E$, and $k \in \mathbb{Z}_+$, one can find a perfect matching M with $|M \cap F| \leq k$ (or decide that no such perfect matching exists) in time $O(nm)$. (This amounts to a minimum-cost flow problem.)

Karp, Vazirani, and Vazirani [1990] gave an optimal on-line bipartite matching algorithm. Motwani [1989,1994] investigated the expected running time of matching algorithms.

The following question was posed by A. Frank: Given a bipartite graph $G = (V, E)$ whose edges are coloured red and blue, and given k and l; when does there exist a matching containing k red edges and l blue edges? This problem is NP-complete, but for complete bipartite graphs it was characterized by Karzanov [1987c].

An extension of Frobenius' theorem to more general matrices than described in Section 16.2b was given by Hartfiel and Loewy [1984].

Dulmage and Mendelsohn [1958] study minimum-size vertex covers in a bipartite graph as a lattice. For maintaining perfect matchings 'in the presence of failure', see Sha and Steiglitz [1993]. Lovász [1970a] gave a generalization of Kőnig's matching theorem — see Section 60.1a. Uniqueness of a maximum-size matching in a bipartite graph was investigated by Cechlárová [1991], and related work was reported by Costa [1994]. A variant of Kőnig's matching theorem was given by de Werra [1984].

For surveys on matching algorithms, see Galil [1983,1986a,1986b]. For surveys on bipartite matching, see Woodall [1978a,1978b]. Books discussing bipartite matching include Ford and Fulkerson [1962], Ore [1962], Dantzig [1963], Christofides [1975], Lawler [1976b], Even [1979], Papadimitriou and Steiglitz [1982], Tarjan [1983], Tutte [1984], Halin [1989], Cook, Cunningham, Pulleyblank, and Schrijver

[1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].

16.7h. Historical notes on bipartite matching

The fundaments of matching theory in bipartite graphs were laid by Frobenius (in terms of matrices and determinants) and Kőnig. In his article *Über Matrizen aus nicht negativen Elementen* (On matrices with nonnegative elements), Frobenius [1912] investigated the decomposition of matrices:

> In §11 dehne ich die Untersuchung auf zerlegbare Matrizen aus, und in §12 zeige ich, daß eine solche nur auf eine Art in unzerlegbare Teile zerfällt werden kann. Dabei ergibt sich der merkwürdige Determinantensatz:
> I. *Die Elemente einer Determinante $n$ten Grades seien n^2 unabhängige Veränderliche. Man setze einige derselben Null, doch so, daß die Determinante nicht identisch verschwindet. Dann bleibt sie eine irreduzible Funktion, außer wenn für einen Wert $m < n$ alle Elemente verschwinden, die m Zeilen mit $n - m$ Spalten gemeinsam haben.*[4]

Frobenius gave a combinatorial and an algebraic proof.

In a reaction to Frobenius' paper, Kőnig [1915] ('presented to Class III of the Hungarian Academy of Sciences on 16 November 1914') next gave a proof of Frobenius' result with the help of graph theory:

> A graphok alkalmazásával e tételnek egyszerű és szemléletes új bizonyitását adjuk a következőkben.[5]

He introduced a now quite standard construction of making a bipartite graph from a matrix $(a_{i,j})$: for each row index i there is a vertex A_i and for each column index j there is a vertex B_j; then vertices A_i and B_j are connected by an edge if and only if $a_{i,j} \neq 0$.

Kőnig was interested in graphs because of his interest in set theory, especially cardinal numbers (cf. footnotes in Kőnig [1916]). In proving Schröder-Bernstein type results on the equivalence of sets, graph-theoretic arguments (in particular: matchings) can be illustrative. This led Kőnig to studying graphs (in particular bipartite graphs) and its applications in other areas of mathematics.

Kőnig's work on matchings in regular bipartite graphs

Earlier, on 7 April 1914, Kőnig had presented the following theorem at the *Congrès de Philosophie mathématique* in Paris (cf. Kőnig [1923]):

> *A. Chaque graphe régulier à circuits pairs possède un facteur du premier degré.*[6]

[4] In §11, I extend the investigation to decomposable matrices, and in §12, I show that such a matrix can be decomposed in only one way into indecomposable parts. With that, the [following] curious determinant theorem comes up:
I. *Let the elements of a determinant of degree n be n^2 independent variables. One sets some of them equal to zero, but such that the determinant does not vanish identically. Then it remains an irreducible function, except when for some value $m < n$ all elements vanish that have m rows in common with $n - m$ columns.*

[5] In the following we will give a simple and clear new proof by applying graphs to this theorem.

[6] *A. Each regular graph with even circuits has a factor of the first degree.*

That is, every regular bipartite graph has a perfect matching (= factor of degree 1). As a corollary, Kőnig derived:

> *B. Chaque graphe régulier à circuits pairs est le produit de facteurs du premier degré; le nombre de ces facteurs est égal au degré du graphe.*[7]

That is, each k-regular bipartite graph is k-edge-colourable (cf. Chapter 20). Kőnig did not give a proof of the theorem in the Paris paper, but expressed the hope to give a complete proof 'at another occasion'.

This occasion came in Kőnig [1916] ('presented to Class III of the Hungarian Academy of Sciences on 15 November 1915') where next to the above mentioned Theorems A and B, Kőnig gave the following result:

> C) *Ha egy páros körüljárású graph bármelyik csúcsába legfeljebb k-számú él fut, akkor minden éléhez oly módon lehet k-számú index valamelyikét hozzárendelni, hogy ugyanabba a csúcsba futó két élhez mindenkor két különböző index legyen rendelve.*[8]

In other words, the edge-colouring number of a bipartite graph is equal to its maximum degree. Kőnig gave a proof of result C), and derived A and B. (See the proof of Theorem 20.1 below of Kőnig's proof.)

In §2 of Kőnig [1916], applications of his results to matrices and determinants are studied. First:

> D) *Ha egy nem negativ [egész számú] elemekből álló determináns minden sora és minden oszlopa ugyanazt a positiv összeget adja, akkor van a determinánsnak legalább egy el nem tünő tagja.*[9]

Next:

> E) *Ha egy determináns minden sorában és oszlopában pontosan k-számú el nem tünő elem van, akkor legalább k-számú determinánstag nem tünik el.*[10]

Third:

> F) *Ha egy n^2 mezejű quadratikus táblán kn-számú figura úgy van elhelyezve (ugyanazon a mezőn több figura is lehet), hogy minden sorban és oszlopban pontosan k-számú figura fordul elő, akkor e konfiguráczió mindig mint k-számú ugyancsak n^2 mezejű oly konfiguráczió egyesítése keletkeztethető, melyek mindegyikében egy-egy figura van minden sorban és minden oszlopban.*[11]

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[7] *B. Each regular graph with even circuits is the product of factors of the first degree; the number of these factors is equal to the degree of the graph.*

[8] C) *If in each vertex of an even circuit graph at most k edges meet, then one can assign to each of the edges of the graph one from k indices in such a way that two edges that meet in a point always obtain different indices.*

[9] D) *If in a determinant of nonnegative [integer] numbers each row and each column yield the same positive sum, then at least one member of the determinant is different from zero.*

[10] E) *If the number of nonvanishing elements in each row and column of a determinant is exactly equal to k, then there are at least k nonvanishing determinant members.*

[11] F) *If kn pieces are placed on a quadratic board with n^2 fields (where several pieces may stand in the same field), such that each row and each column contains exactly k pieces, then this configuration always arises by joining k such configurations with also n^2 fields, in which each row and each column contains exactly one piece.*

Frobenius' theorem

Chronologically next is a paper of Frobenius [1917]. In order to give an elementary proof of his result in Frobenius [1912] quoted above, he proved the following 'Hilfssatz':

> II. *Wenn in einer Determinante nten Grades alle Elemente verschwinden, welche p $(\leq n)$ Zeilen mit $n-p+1$ Spalten gemeinsam haben, so verschwinden alle Glieder der entwickelten Determinante.*
> *Wenn alle Glieder einer Determinante nten Grades verschwinden, so verschwinden alle Elemente, welche p Zeilen mit $n-p+1$ Spalten gemeinsam haben für $p = 1$ oder $2, \cdots$ oder n.*[12]

That is, if $A = (a_{i,j})$ is an $n \times n$ matrix, and if $\prod_{i=1}^{n} a_{i,j} = 0$ for each permutation π of $\{1, \ldots, n\}$, then for some p there exist p rows and $n-p+1$ columns of A such that each element that is both in one of these rows and in one of these columns, is equal to 0.

In other words, a bipartite graph $G = (V, E)$ with colour classes V_1 and V_2 satisfying $|V_1| = |V_2| = n$ has a perfect matching if and only if one cannot select p vertices in V_1 and $n-p+1$ vertices in V_2 such that no edge is connecting two of these vertices.

Frobenius noticed with respect to Kőnig's work:

> Aus dem Satze II ergibt sich auch leicht ein Ergebnis der Hrn. DÉNIS KŐNIG, *Uber Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. Bd. 77.*
> *Wenn in einer Determinante aus nicht negativen Elementen die Größen jeder Zeile und jeder Spalte dieselbe, von Null verschiedene Summe haben, so können ihre Glieder nicht sämtlich verschwinden.*[13]

Frobenius gave a short combinatorial proof of his theorem — see Section 16.2a. His proof is in terms of determinants, and he offered his opinion on graph-theoretic methods:

> Die Theorie der Graphen, mittels deren Hr. KŐNIG den obigen Satz abgeleitet hat, ist nach meiner Ansicht ein wenig geeignetes Hilfsmittel für die Entwicklung der Determinantentheorie. In diesem Falle führt sie zu einem ganz speziellen Satze von geringem Werte. Was von seinem Inhalt Wert hat, ist in dem Satze II ausgesprochen.[14]

(See Schneider [1977] for some comments.)

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[12] II. *If in a determinant of the nth degree all elements vanish that p $(\leq n)$ rows have in common with $n-p+1$ columns, then all members of the expanded determinant vanish.*
If all members of a determinant of degree n vanish, then all elements vanish that p rows have in common with $n-p+1$ columns for $p = 1$ or $2, \cdots$ or n.

[13] From Theorem II, a result of Mr DÉNIS KŐNIG, *Uber Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. Vol. 77* follows also easily.
If in a determinant of nonnegative elements the quantities of each row and of each column have the same nonzero sum, then its members cannot vanish altogether.

[14] The theory of graphs, by which Mr KŐNIG has derived the theorem above, is to my opinion of little appropriate help for the development of determinant theory. In this case it leads to a very special theorem of little value. What from its contents has value, is enunciated in Theorem II.

Equivalent formulations in terms of partitions

In October 1926, van der Waerden [1927] presented the following theorem at the *Mathematisches Seminar* in Hamburg:

> *Es seien zwei Klasseneinteilungen einer endlichen Menge $\mathcal{M}$ gegeben. Die eine soll die Menge in μ zueinander fremde Klassen $\mathcal{A}_1, \ldots, \mathcal{A}_\mu$ zu je n Elementen zerlegen, die andere ebenfalls in μ fremde Klassen $\mathcal{B}_1, \ldots, \mathcal{B}_\mu$ zu je n Elementen. Dann gibt es ein System von Elementen $x_1, \ldots, x_\mu$, derart, daß jede A-Klasse und ebenso jede B-Klasse under den x_i durch ein Element vertreten wird.*[15]

The proof of van der Waerden is based on an augmenting path argument. Moreover, van der Waerden remarked that E. Artin has communicated orally to him that the result can be sharpened to the existence of n disjoint such common transversals.

In the article, the following note is added in proof:

> **Zusatz bei der Korrektur.** Ich bemerke jetzt, daß der hier bewiesene Satz mit einem Satz von DÉNES KŐNIG über reguläre Graphen äquivalent ist.[16]

The article of van der Waerden is followed by an article of Sperner [1927] (presented at the *Mathematisches Seminar* in Januari 1927), which gives a 'simple proof' of van der Waerden's result — we quote the full paper in Section 22.7d.

Kőnig's matching theorem

At the meeting of 26 March 1931 of the Eötvös Loránd Matematikai és Fizikai Társulat (Loránd Eötvös Mathematical and Physical Society) in Budapest, Kőnig [1931] presented a new result that formed the basis for Menger's theorem:

> Páros körüljárású graphban az éleket kimerítő szögpontok minimális száma megegyezik a páronként közös végpontot nem tartalmazó élek maximális számával.[17]

In other words, the maximum size of a matching in a bipartite graph is equal to the minimum number of vertices needed to cover all edges. As we discussed in Section 9.6e, Kőnig's proof formed the missing basis for Menger's theorem. Kőnig also referred to the work of Frobenius (but did not notice that his theorem can be derived from Frobenius' theorem).

The proof of Kőnig [1931] is based on an augmenting path argument. A German version of it was published in Kőnig [1932] (stating that another proof was given by L. Kalmár), in which paper he described several other results as consequences of the theorem. First he derived his theorem on the existence of a perfect matching in a regular bipartite graph:

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[15] Let be given two partitions of a finite set $\mathcal{M}$. One of them should decompose the set into μ mutually disjoint classes $\mathcal{A}_1, \ldots, \mathcal{A}_\mu$ each of n elements, the other likewise in μ disjoint classes $\mathcal{B}_1, \ldots, \mathcal{B}_\mu$ each of n elements. Then there exists a system of elements $x_1, \ldots, x_\mu$ such that each $\mathcal{A}$-class and likewise each $\mathcal{B}$-class is represented by one element among the x_i.

[16] **Note added in proof.** I now notice that the theorem proved here is equivalent to a theorem of DÉNES KŐNIG on regular graphs.

[17] In an even circuit graph, the minimal number of vertices that exhaust the edges agrees with the maximal number of edges that pairwise do not contain any common end point.

> Um die Tragweite dieses Satzes zu beleuchten, wollen wir noch zeigen, daß ein von mir schon vor längerer Zeit bewiesener Satz über die Faktorenzerlegung von regulären endlichen paaren Graphen aus Satz 13 unmittelbar abgeleitet werden kann.
> Der betreffende Satz lautet:
> 14. *Jeder endliche paare reguläre Graph besitzt einen Faktor ersten Grades.*[18]

In a footnote, Kőnig mentioned:

> Später wurden für diesen Satz, bzw. für seine Interpretation in der Determinantentheorie und in der Kombinatorik verschiedene Beweise gegeben, so durch FROBENIUS, SAINTE-LAGUË, VAN DER WAERDEN, SPERNER, SKOLEM, EGERVÁRY.[19]

Another consequence is a graph-theoretic variant of the result of Frobenius [1912] on reducible determinants:

> 16. *Im (paaren) Graphen G soll jede Kante einen der Punkte von $\Pi_1 = (P_1, P_2, \ldots, P_n)$ mit einem der Punkte von $\Pi_2 = (Q_1, Q_2, \ldots, Q_n)$ verbinden $(P_i \neq Q_j)$ und diejenigen Kanten von G, die in einem Faktor ersten Grades von G enthalten sind, sollen einen nichtzusammenhängenden Graphen G^* bilden. Dann kann man $r(>0)$ Punkte aus Π_1 und $n-r(>0)$ Punkte aus Π_2 so auswählen, daß keine Kante von G zwei ausgewählte Punkte verbinde.*[20]

As consequences in matrix theory, Kőnig [1932] gave:

> 17. *Verschwinden sämtliche Entwicklungsglieder aller Unterdeterminanten n-ter Ordnung einer Matrix von p Zeilen und q Spalten (wo $n \leq p$, $n \leq q$ ist), so verschwinden alle Elemente, welche r Zeilen mit $(p+q-n+1)-r$ Spalten gemeinsam haben für $r = 1$, oder $2, \ldots,$ oder p.*[21]

and

> 18. *Die Minimalzahl der Reihen (Zeilen und Spalten), welche in ihr Gesamtheit jedes nicht-verschwindende Element einer Matrix enthalten, ist gleich der Maximalzahl von nicht-verschwindenden Elementen, welche paarweise verschiedenen Zeilen und verschiedenen Spalten angehören.*[22]

Again, a footnote is added:

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[18] To illustrate the bearing of this theorem, we want to show that a theorem, proved by me already long ago, on the factorization of regular finite bipartite graphs, can be derived immediately from Theorem 13.
The theorem referred to reads:
14. *Every finite bipartite regular graph possesses a factor of first degree.*

[19] Later, several proofs were given for this theorem, respectively for its interpretation in determinant theory and in combinatorics, so by FROBENIUS, SAINTE-LAGUË, VAN DER WAERDEN, SPERNER, SKOLEM, EGERVÁRY.

[20] 16. *Let every edge in the (bipartite) graph G connect a vertex of $\Pi_1 = (P_1, \ldots, P_n)$ with a vertex of $\Pi_2 = (Q_1, \ldots, Q_n)$ $(P_i \neq Q_j)$, and let those edges of G that are contained in a factor of first degree form a disconnected graph G^*. Then one can choose $r(>0)$ vertices in Π_1 and $n-r(>0)$ vertices in Π_2 such that no edge of G connects two of the chosen vertices.*

[21] 17. *If all expansion terms of all underdeterminants of the order n of a matrix with p rows and q columns vanish (where $n \leq p$, $n \leq q$), then all entries vanish that r rows have in common with $(p+q-n+1)-r$ columns, for $r = 1$, or $2, \ldots,$ or p.*

[22] 18. *The minimum number of lines (rows and columns) that together contain each nonvanishing entry of a matrix, is equal to the maximum number of nonvanishing entries that pairwise belong to different rows and different columns.*

> Die Sätze 17 und 18 hat der Verfasser, mit den hier gegebenen Beweisen, am 26. März 1931 in der Budapester Mathematischen und Physikalischen Gesellschaft vorgetragen, s. [6]. Hieran anschließend hat dann E. Egerváry [1] für den Satz 18 einen anderen Beweis und eine interessante Verallgemeinerung gegeben.[23]

(We note that references [6] and [1] in Kőnig's article correspond to our references Kőnig [1931] and Egerváry [1931].)

Kőnig also derived the theorems of Frobenius [1912,1917] mentioned above:

> 19. *Wenn alle Glieder einer Determinante n-ter Ordnung verschwinden, so verschwinden alle Elemente, welche r Zeilen mit $n-r+1$ Spalten gemeinsam haben, für $r = 1$ oder $2, \ldots,$ oder n.*[24]

> 20. *In einer Determinante n-ter Ordnung D seien die nichtverschwindenden Elemente unabhängige Veränderliche. Ist D eine reduzible Funktion ihrer (nichtverschwindenden) Elemente, so verschwinden alle Elemente von D, welche r Zeilen mit $n-r$ Spalten gemeinsam haben für $r = 1$ oder $2, \ldots,$ oder $n-1$.*[25]

With respect to Frobenius [1912], Kőnig noticed in a footnote:

> Dort wird dieser Satz "aus verborgenen Eigenschaften der Determinanten mit nichtnegativen Elementen" durch komplizierte Betrachtungen bewiesen. Ich gab dann in 1915 in meiner Arbeit [4] einen elementaren graphentheoretischen Beweis (welcher hier durch einen noch einfacheren ersetzt wird). In 1917 hat dann auch Frobenius [3] einen elementaren Beweis publiziert, und zwar nach dem ich ihm meinen Beweis (in deutscher Übersetzung) zugeschickt hatte. Frobenius hat es dort unterlassen, diese Tatsache, sowie überhaupt meine Arbeit [4] zu erwähnen. Jedoch zitiert er meine Arbeit [5] und zwar mit folgender Bemerkung: "Die Theorie der Graphen, mittels deren Hr. Kőnig den obigen Satz [dies ist die determinantentheoretische Interpretation von Satz 14] abgeleitet hat, ist nach meiner Ansicht ein wenig geeignetes Hilfsmittel für die Entwicklung der Determinantentheorie. In diesem Falle führt sie zu einem ganz speziellen Satz vom geringem Werte. Was von seinem Inhalt Wert hat, ist in dem Satze II [dies ist der Frobeniussche Satz 19] ausgesprochen."
> Es ist wohl natürlich, daß der Verfasser vorliegender Abhandlung diese Meinung nicht unterschreiben wird. Die Gründe, die man für oder gegen den Wert oder Unwert eines Satzes oder eine Methode anführen könnte, haben stets, mehr oder weniger, einen subjektiven Character, so daß es vom geringen wissenschaftlichen Wert wäre, wenn wir hier den Standpunkt von Frobenius zu bekämpfen versuchten. Wollte aber Frobenius seine verwerfende Kritik über die Anwendbarkeit der Graphen auf Determinantentheorie damit begründen, daß sein tatsächlich "wertvoller" Satz 19 nicht graphentheoretisch bewiesen werden kann, so ist seine Begründung—wie wir gesehen haben—sicherlich nicht stichhaltig. Der graphentheoretische Beweis, den wir für Satz 19 gegeben haben, scheint uns ein einfacher und anschaulicher Beweis zu sein, der dem *kombinatorischen* Character der Satzes in natürlicher Weise entspricht und auch zu einer bemerkenswerten Verallgemeinerung (Satz 17) führt.

[23] The author has presented Theorems 17 and 18, with the proofs given here, on 26 March 1931 to the Budapest Mathematical and Physical Society, see [6]. Following this, E. Egerváry [1] has next given another proof for Theorem 18 and an interesting generalization.

[24] 19. *When all members of a determinant of the order n vanish, then all elements vanish that have r rows in common with $n-r+1$ columns, for $r = 1$ or $2, \ldots,$ or n.*

[25] 20. *Let, in a determinant D of order n, the nonvanishing entries be independent variables. If D is a reducible function of its (nonvanishing) entries, then all entries of D vanish that have r rows in common with $n-r$ columns for $r = 1$ or $2, \ldots,$ or $n-1$.*

> Es sei noch erwähnt, daß wir oben, im §2, beim Beweis des Satzes 16 einen Gedanken von FROBENIUS benützt haben, den er bei seiner Zurückführung des Satzes 20 auf Satz 19 angewendet hat.[26]

(We note that Kőnig's quotation 'aus verborgenen Eigenschaften der Determinanten mit nichtnegativen Elementen' is from Frobenius [1917]. The references [3], [4], and [5] in Kőnig's article correspond to our references Frobenius [1917], Kőnig [1915], and Kőnig [1916], respectively.)

In terms of transversals, the theorems of Frobenius and Kőnig have been rediscovered by Hall [1935] — see the historical notes on transversals in Section 22.7d. Other developments are mentioned in Section 19.5a.

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[26] This theorem was proved there 'from hidden properties of determinants with nonnegative elements' by complicated arguments. Next, I gave in 1915, in my work [4], an elementary, graph-theoretic proof (which was replaced here by an even simpler one). Next, in 1917, also FROBENIUS [3] has published an elementary proof, and that after I had sent him my proof (in German translation). FROBENIUS has refrained from mentioning this fact there, as well as my work [4] at all. Yet, he quotes my work [5], and that with the following remark: 'The theory of graphs, by which Mr. KŐNIG has derived the theorem above [this is the determinant-theoretic interpretation of Theorem 14], is, to my opinion, of little appropriate help for the development of determinant theory. In this case it leads to a very special theorem of little value. What from its contents has value, is expressed in Theorem II [this is Theorem 19 of Frobenius]'.

Obviously, the author of the present treatise will not subscribe to this opinion. The arguments that one can produce for or against the value or valuelessness of a theorem or a method, have always, more or less, a subjective character, so that it would be of little scientific value when we here tried to fight the point of view of FROBENIUS. But if FROBENIUS wants to base his rejecting criticism about the applicability of graphs to determinant theory on the fact that his actually 'more valuable' Theorem 19 cannot be proved graph-theoretically, then his ground is—as we have seen—certainly not solid. The graph-theoretic proof that we have given for Theorem 19 seems to us to be a simple and illustrative proof, that corresponds naturally to the *combinatorial* character of the theorem and also leads to a remarkable generalization (Theorem 17).

Let it finally be mentioned that above, in §2, in the proof of Theorem 16, we have used an idea of FROBENIUS, which he has applied at his reduction of Theorem 20 to Theorem 19.

Chapter 17

Weighted bipartite matching and the assignment problem

The methods and results of the previous chapter can be extended to handle maximum-*weight* matchings. Egerváry's theorem is the weighted version of Kőnig's matching theorem. It led Kuhn to develop the 'Hungarian method' for the assignment problem. This problem is equivalent to finding a minimum-weight perfect matching in a complete bipartite graph.

17.1. Weighted bipartite matching

For bipartite graphs, Egerváry [1931] characterized the maximum weight of a matching by the following duality relation:

Theorem 17.1 (Egerváry's theorem)**.** *Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{R}_+$ be a weight function. Then the maximum weight of a matching in G is equal to the minimum value of $y(V)$, where $y : V \to \mathbb{R}_+$ is such that*

$$(17.1) \qquad y_u + y_v \geq w_e$$

for each edge $e = uv$. If w is integer, we can take y integer.

Proof. The maximum is not more than the minimum, since for any matching M and any $y \in \mathbb{R}^V_+$ satisfying (17.1) for each edge $e = uv$, one has

$$(17.2) \qquad w(M) \leq \sum_{e=uv \in M} (y_u + y_v) \leq \sum_{v \in V} y_v.$$

To see equality, choose a $y \in \mathbb{R}^V_+$ attaining the minimum value. Let F be the set of edges e having equality in (17.1) and let R be the set of vertices v with $y_v > 0$.

If F contains a matching M covering R, we have equality throughout in (17.2), showing that the maximum is equal to the minimum value.

So we may assume that no such matching exists. Then by Corollary 16.8a there exists a stable set $S \subseteq R$ containing no edge and such that $|N(S)| < |S|$.

Then there is an $\alpha > 0$ such that decreasing y_v by α for $v \in S$ and increasing y_v by α for $v \in N(S)$ gives a better y — a contradiction.

If w is integer we can keep y integer, by taking $\varepsilon = 1$ throughout. ∎

(This is essentially the proof method of Egerváry [1931].)

We can formulate Egerváry's theorem in combinatorial terms. Let $G = (V, E)$ be a graph and let $w \in \mathbb{Z}_+^E$. A *w-vertex cover* is a vector $y \in \mathbb{Z}_+^V$ such that

(17.3) $$y_u + y_v \geq w_e$$

for each edge $e = uv$ of G. The *size* of any vector $y \in \mathbb{R}^V$ is the sum of its components.

Corollary 17.1a. *Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{Z}_+$ be a weight function. Then the maximum weight of a matching in G is equal to the minimum size of a w-vertex cover.*

Proof. The corollary is a reformulation of the integer part of Egerváry's theorem (Theorem 17.1). ∎

Let A be the $V \times E$ incidence matrix of G. Egerváry's theorem states that for $w \in \mathbb{Z}_+^V$, the optima in the linear programming duality equation

(17.4) $$\max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^\mathsf{T} \mathbf{1} \mid y \geq \mathbf{0}, y^\mathsf{T} A \geq w^\mathsf{T}\}$$

are attained by integer vectors x and y. This also follows from the total unimodularity of A — see Section 18.3.

17.2. The Hungarian method

We describe the *Hungarian method* for the maximum-weight matching problem. In its basic form it is due to Kuhn [1955b], based on Egerváry's proof above. Sharpenings were given by Munkres [1957] (yielding a polynomial-time method), Iri [1960], Edmonds and Karp [1970], and Tomizawa [1971].

Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, and let $w : E \to \mathbb{Q}$ be a weight function.

We start with matching $M = \emptyset$. If we have found a matching M, let D_M be the directed graph obtained from G by orienting each edge e in M from W to U, with length $l_e := w_e$, and orienting each edge e not in M from U to W, with length $l_e := -w_e$. Let U_M and W_M be the set of vertices in U and W, respectively, missed by M. If there is a $U_M - W_M$ path, find a shortest such path, P say, and reset $M' := M \triangle EP$.

We iterate until no $U_M - W_M$ path exists in D_M (whence M is a maximum-*size* matching). The maximum-weight matching among the matchings found, has maximum weight among all matchings.

To see this, call a matching M *extreme* if it has maximum weight among all matchings of size $|M|$. Then, inductively:

Theorem 17.2. *Each matching M found is extreme.*

Proof. This is clearly true if $M = \emptyset$. Suppose next that M is extreme, and let P and M' be the path and matching found in the iteration. Consider any extreme matching N of size $|M|+1$. As $|N| > |M|$, $M \cup N$ has a component Q that is an M-augmenting path. As P is a shortest M-augmenting path, we know $l(Q) \geq l(P)$. As $N \triangle Q$ is a matching of size $|M|$, and as M is extreme, we have $w(N \triangle Q) \leq w(M)$. Hence $w(N) = w(N \triangle Q) - l(Q) \leq w(M) - l(P) = w(M')$. ∎

If M is extreme, then D_M has no negative-length circuit C (otherwise $M \triangle C$ is a matching of size $|M|$ and larger weight than M). So by the theorem, we can find with the Bellman-Ford method a shortest $U_M - W_M$ path in time $O(nm)$, yielding an $O(n^2m)$ method overall (Iri [1960]).

But in fact one may apply Dijkstra's method (Edmonds and Karp [1970], Tomizawa [1971]) and obtain a better time bound:

Theorem 17.3. *The method can be performed in time $O(n(m + n \log n))$.*

Proof. Let R_M denote the set of vertices reachable in D_M from U_M. We show that along with M we can keep a potential p for the subgraph $D_M[R_M]$ of D_M induced by R_M (with respect to the length function l defined above).[27]

When $M = \emptyset$ we take $p(v) := \max\{w_e \mid e \in E\}$ if $v \in U$ and $p(v) := 0$ if $v \in W$.

Suppose next that for given extreme M we have a potential p for $D_M[R_M]$. Then define $p'(v) := \mathrm{dist}_l(U_M, v)$ for each $v \in R_M$. Note that having p, one can determine p' in $O(m + n \log n)$ time (cf. Section 8.2).

Then p' is a potential for $D_{M'}[R_{M'}]$. To see this, let P be the path in D_M with $M' = M \triangle EP$. Trivially, $U_{M'} \subseteq U_M$. Moreover, $R_{M'} \subseteq R_M$. Indeed, otherwise some arc of $D_{M'}$ leaves R_M. As no arc of D_M leaves R_M, this implies that P has an arc entering R_M. So P has an arc leaving R_M, contradicting the definition of R_M. Concluding, $R_{M'} \subseteq R_M$.

Finally consider an arc (u, v) of $D_{M'}[R_{M'}]$. If (u, v) is also an arc of D_M, then $p'(v) \leq p'(u) + l(u, v)$. If (u, v) is not an arc of D_M, then (v, u) belongs to P, and hence (as P is shortest) $p'(u) = p'(v) + l(v, u)$. So $p'(v) - p'(u) = -l(v, u) = l(u, v)$. ∎

Observe that in the Hungarian method one can stop as soon as matching M' has no larger weight than M; that is, D_M has no $U_M - W_M$ path of negative length. For let N be a matching with $w(N) > w(M)$. So $|N| > |M|$

[27] A *potential* for a digraph $D = (V, A)$ with respect to a length function $l : A \to \mathbb{R}$ is a function $p : V \to \mathbb{R}$ satisfying $p(v) - p(u) \leq l(a)$ for each arc $a = (u, v)$.

(since all matchings of size $\leq |M|$ have weight $\leq w(M)$). Choose N with $|N \triangle M|$ minimal. By similar arguments as used in the proof of Theorem 17.2, we may assume that $N \triangle M$ has $|N| - |M|$ nontrivial components, each having one more edge in N than in M. So each component gives a $U_M - W_M$ path in D_M. As none of them have negative length, we have $w(N) \leq w(M)$, a contradiction.

Hence we can reduce the factor n in the time bound:

Theorem 17.4. *In a weighted bipartite graph, a maximum-weight matching can be found in time $O(n'(m + n \log n))$, where n' is the minimum size of a maximum-weight matching.*

Proof. See above. ∎

17.3. Perfect matching and assignment problems

The methods described above also find a maximum-weight perfect matching in a bipartite graph. This follows from the fact that a maximum-weight perfect matching is an extreme matching of size $\frac{1}{2}|V|$.

By multiplying all weights by -1, this problem can be seen to be equivalent to finding a *minimum*-weight perfect matching. Hence:

Corollary 17.4a. *A minimum-weight perfect matching can be found in time $O(n(m + n \log n))$.*

Proof. Directly from the above. ∎

This in turn gives an algorithm for the *assignment problem*:

(17.5) given: a rational $n \times n$ matrix $A = (a_{i,j})$;

find: a permutation π of $\{1, \ldots, n\}$ minimizing $\sum_{i=1}^{n} a_{i,\pi(i)}$.

Corollary 17.4b. *The assignment problem can be solved in time $O(n^3)$.*

Proof. Take $G = K_{n,n}$ in Corollary 17.4a. ∎

The following characterization of the minimum weight of a perfect matching can be derived from Egerváry's theorem — we however give a direct proof that might be illuminating:

Theorem 17.5. *Let $G = (V, E)$ be a bipartite graph having a perfect matching and let $w : E \to \mathbb{Q}$ be a weight function. The minimum weight of a perfect matching is equal to the maximum value of $y(V)$ taken over $y : V \to \mathbb{Q}$ with*

(17.6) $\quad y_u + y_v \leq w_e$ *for each edge* $e = uv$.

If w *is integer, we can take* y *integer.*

Proof. Clearly, the minimum is not less than the maximum, since for any perfect matching M and any $y \in \mathbb{Q}^V$ satisfying (17.6) one has

$$(17.7) \qquad w(M) = \sum_{e \in M} w_e \geq \sum_{v \in V} y_v = y(V).$$

To see the reverse inequality, let M be a minimum-weight perfect matching. Make a digraph $D = (V, A)$, with length function, as follows. Orient any edge e of G from one colour class, U say, to the other, W say, with length w_e. Moreover, add for each edge e in M an arc parallel to e oriented from W to U, with length $-w_e$. As M is minimum-weight, the digraph has no negative-weight directed circuits (otherwise we could make a perfect matching of smaller weight). Hence, by Theorem 8.2, there exists a function $p : V \to \mathbb{Q}$ such that $w(a) \geq p(v) - p(u)$ for each arc $a = (u, v)$ of D. Defining $y_v := -p(v)$ for $v \in U$ and $y_v := p(v)$ for $v \in W$, we obtain a function y satisfying (17.6). For each edge $e = uv$ in M, the arcs (u, v) and (v, u) form a zero-length directed circuit in D, and therefore $w_e = y_u + y_v$. This gives equality in (17.7).

If w is integer, we can take p and hence y integer. ∎

17.4. Finding a minimum-size w-vertex cover

Given a maximum-weight matching M in a bipartite graph $G = (V, E)$ with weight $w : E \to \mathbb{Z}_+$, we can find a minimum-size w-vertex cover as follows. Let U and W be the colour classes of G. As before, define $U_M := U \setminus \bigcup M$ and $W_M := W \setminus \bigcup M$.

For any edge $e = uv$, with $u \in U$, $v \in W$, make an arc $a = (u, v)$, of length $l(a) := -w_e$. If $e \in M$, make also an arc $a' = (v, u)$, of length $l(a') := w_e$. We obtain a directed graph $D = (V, A)$ without negative-length directed circuits and no negative-length directed path from $U_M \cup (W \setminus W_M)$ to $W_M \cup (U \setminus U_M)$ (otherwise we can improve M). Then we can find a potential $p : V \to \mathbb{Z}$ such that $l(a) \geq p(v) - p(u)$ for each arc $a = (u, v)$ of D and such that $p(v) = 0$ for each $v \in U_M \cup W_M$, $p(v) \geq 0$ for each $v \in U$, and $p(v) \leq 0$ for each $v \in W$. To see this, add an extra vertex r, and arcs (r, v) for each $v \in U_M \cup (W \setminus W_M)$ and (v, r) for each $v \in W_M \cup (U \setminus U_M)$. Let the new arcs have length 0. Then the extended digraph D' has no negative-length circuits. Let p be a potential for D'. By translating, we can assume $p(r) = 0$. Resetting $p(v)$ to 0 if $v \in U_M \cup W_M$ maintains that p is a potential. This gives a potential for D as described.

Now set $y_v := -p(v)$ if $v \in U$ and $y_v := p(v)$ if $v \in W$. Then y is a w-vertex cover of size $w(M)$, and hence it is a minimum-size w-vertex cover. Therefore (Iri [1960]):

Theorem 17.6. *A minimum-size w-vertex cover in a bipartite graph can be found in $O(n(m + n \log n))$ time.*

Proof. See above. ▮

17.5. Further results and notes

17.5a. Complexity survey for maximum-weight bipartite matching

Complexity survey for the maximum-weight bipartite matching ($*$ indicates an asymptotically best bound in the table):

	$O(nW \cdot \mathrm{VC}(n, m))$	Egerváry [1931] (implicitly)
	$O(2^n n^2)$	Easterfield [1946]
	$O(nW \cdot \mathrm{DC}(n, m, W))$	Robinson [1949]
	$O(n^4)$	Kuhn [1955b], Munkres [1957][28] Hungarian method
	$O(n^2 m)$	Iri [1960]
	$O(n^3)$	Dinits and Kronrod [1969]
$*$	$O(n \cdot \mathrm{SP}_+(n, m, W))$	Edmonds and Karp [1970], Tomizawa [1971]
	$O(n^{3/4} m \log W)$	Gabow [1983b,1985a,1985b]
$*$	$O(\sqrt{n}\, m \log(nW))$	Gabow and Tarjan [1988b,1989] (cf. Orlin and Ahuja [1992])
	$O(\sqrt{n}\, mW)$	Kao, Lam, Sung, and Ting [1999]
$*$	$O(\sqrt{n}\, mW \log_n(n^2/m))$	Kao, Lam, Sung, and Ting [2001]

Here $W := \|w\|_\infty$ (assuming w to be integer-valued). Moreover, $\mathrm{SP}_+(n, m, W)$ is the time needed to find a shortest path in a directed graph with n vertices and m arcs, with nonnegative integer lengths on the arcs, each at most W. Similarly, $\mathrm{DC}(n, m, W)$ is the time required to find a negative-length directed circuit in a directed graph with n vertices and m arcs, with integer lengths on the arcs, each at most W in absolute value. Moreover, $\mathrm{VC}(n, m)$ is the time required to find a minimum-size vertex cover in a bipartite graph with n vertices and m edges.

Dinits [1976] gave an algorithm for finding a minimum-weight matching in $K_{p,q}$ of size p, with time bound $O(|p|^3 + pq)$ (taking $p \leq q$).

17.5b. Further notes

Simplex method. Finding a maximum-weight matching in a bipartite graph is a special case of a linear programming problem (see Chapter 18), and hence linear programming methods like the simplex method apply.

[28] Munkres showed that Kuhn's 'Hungarian method' takes $O(n^4)$ time.

Gassner [1964] studied cycling of the simplex method when applied to the assignment problem. Using the 'strongly feasible' trees of Cunningham [1976], Roohy-Laleh [1980] showed that a version of the simplex method solves the assignment problem in less than n^3 pivots (cf. Hung [1983], Orlin [1985], and Akgül [1993]; the last paper gives a method with $O(n^2)$ pivots, yielding an $O(n(m + n \log n))$ algorithm).

Balinski [1985] (cf. Goldfarb [1985]) showed that a version of the dual simplex method (the *signature method*) solves the assignment problem in strongly polynomial time ($O(n^2)$ pivots, yielding an $O(n^3)$ algorithm). More can be found in Dantzig [1963], Barr, Glover, and Klingman [1977], Balinski [1986], Ahuja and Orlin [1988,1992], Akgül [1988], Paparrizos [1988], and Akgül and Ekin [1991].

For further algorithmic studies of the assignment problem, consult Flood [1960], Kurtzberg [1962], Hoffman and Markowitz [1963], Balinski and Gomory [1964], Tabourier [1972], Carpaneto and Toth [1980a,1983,1987], Hung and Rom [1980], Karp [1980], Bertsekas [1981,1987,1992] ('auction method'), Engquist [1982], Avis [1983], Avis and Devroye [1985], Derigs [1985b,1988a], Carraresi and Sodini [1986], Derigs and Metz [1986a], Glover, Glover, and Klingman [1986], Jonker and Volgenant [1986], Kleinschmidt, Lee, and Schannath [1987], Avis and Lai [1988], Bertsekas and Eckstein [1988], Motwani [1989,1994], Kalyanasundaram and Pruhs [1991, 1993], Khuller, Mitchell, and Vazirani [1991,1994], Goldberg and Kennedy [1997] (push-relabel), and Arora, Frieze, and Kaplan [1996,2002].

For computational studies, see Silver [1960], Florian and Klein [1970], Barr, Glover, and Klingman [1977] (simplex method), Gavish, Schweitzer, and Shlifer [1977] (simplex method), Bertsekas [1981], Engquist [1982], McGinnis [1983], Lindberg and Ólafsson [1984] (simplex method), Glover, Glover, and Klingman [1986], Jonker and Volgenant [1987], Bertsekas and Eckstein [1988], and Goldberg and Kennedy [1995] (push-relabel). Consult also Johnson and McGeoch [1993].

Linear-time algorithm for weighted bipartite matching problems satisfying a quadrangle or other inequality were given by Karp and Li [1975], Buss and Yianilos [1994,1998], and Aggarwal, Bar-Noy, Khuller, Kravets, and Schieber [1995].

For generating *all* minimum-weight perfect matchings, see Fukuda and Matsui [1992]. For studies of the 'most vital' edges in a weighted bipartite graph, see Hung, Hsu, and Sung [1993].

Aráoz and Edmonds [1985] gave an example showing that iterative dual improvements in the linear programming problem dual to the assignment problem, need not converge for irrational data.

For the 'bottleneck' assignment problem, see Gross [1959] and Garfinkel [1971]. An algebraic approach to assignment problems was described by Burkard, Hahn, and Zimmermann [1977].

For surveys on matching algorithms, see Galil [1983,1986a,1986b]. Books covering the weighted bipartite matching and assignment problems include Ford and Fulkerson [1962], Dantzig [1963], Christofides [1975], Lawler [1976b], Bazaraa and Jarvis [1977], Burkard and Derigs [1980], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Rockafellar [1984], Derigs [1988a], Bazaraa, Jarvis, and Sherali [1990], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].

17.5c. Historical notes on weighted bipartite matching and optimum assignment

Monge: optimum assignment

The assignment problem is one of the first studied combinatorial optimization problems. It was investigated by Monge [1784], albeit camouflaged as a continuous problem, and often called a transportation problem.

Monge was motivated by transporting earth, which he considered as the discontinuous, combinatorial problem of transporting molecules:

> Lorsqu'on doit transporter des terres d'un lieu dans un autre, on a coutime de donner le nom de *Déblai* au volume des terres que l'on doit transporter, & le nom de *Remblai* à l'espace qu'elles doivent occuper après le transport.
> Le prix du transport d'une molécule étant, toutes choses d'ailleurs égales, proportionnel à son poids & à l'espace qu'on lui fait parcourir, & par conséquent le prix du transport total devant être proportionnel à la somme des produits des molécules multipliées chacune par l'espace parcouru, il s'ensuit que le déblai & le remblai étant donnés de figure & de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, & le prix du transport total sera un *minimum*.[29]

Monge described an interesting geometric method to solve this problem. Consider a line that is tangent to both areas, and move the molecule m touched in the first area to the position x touched in the second area, and repeat, until all earth has been transported. Monge's argument that this would be optimum is simple: if molecule m would be moved to another position, then another molecule should be moved to position x, implying that the two routes traversed by these molecules cross, and that therefore a shorter assignment exists:

> Étant données sur un même plan deux aires égales $ABCD$, & $abcd$, terminées par des contours quelconques, continus ou discontinus, trouver la route que doit suivre chaque molécule M de la premiere, & le point m où elle doit arriver dans la seconde, pour que tous les points étant semblablement transportés, ils replissent exactement la seconde aire, & que la somme des produits de chaque molécule multipliée par l'espace parcouru soit un *minimum*.
> Si par un point M quelconque de la première aire, on mène une droite Bd, telle que le segment BAD soit égal au segment bad, je dis que pour satisfaire à la question, il faut que toutes les molécules du segment BAD, soient portées sur le segment bad, & que par conséquent les molécules du segment BCD soient portées

[29] When one must transport earth from one place to another, one usually gives the name of *Déblai* to the volume of earth that one must transport, & the name of *Remblai* to the space that they should occupy after the transport.

The price of the transport of one molecule being, if all the rest is equal, proportional to its weight & to the distance that one makes it covering, & hence the price of the total transport having to be proportional to the sum of the products of the molecules each multiplied by the distance covered, it follows that, the déblai & the remblai being given by figure and position, it makes difference if a certain molecule of the déblai is transported to one or to another place of the remblai, but that there is a certain distribution to make of the molecules from the first to the second, after which the sum of these products will be as little as possible, & the price of the total transport will be a *minimum*.

> sur le segment égal *bcd*; car si un point K quelconque du segment BAD, étoit porté sur un point k de *bcd*, il faudroit nécessairement qu'un point égal L, pris quelque part dans BCD, fût transporté dans un certain point l de *bad*, ce qui ne pourroit pas se faire sans que les routes Kk, Ll, ne se coupassent entre leurs extrémités, & la somme des produits des molécules par les espaces parcourus ne seroit pas un *minimum*. Pareillement, si par un point M' infiniment proche du point M, on mène la droite $B'd'$, telle qu'on ait encore le segment $B'A'D'$, égal au segment $b'a'd'$, il faut pour que la question soit satisfaite, que les molécules du segment $B'A'D'$ soient transportées sur $b'a'd'$. Donc toutes les molécules de l'élément $BB'D'D$ doivent être transportées sur l'élément égal $bb'd'd$. Ainsi en divisant le déblai & le remblai en une infinité d'élémens par des droites qui coupent dans l'un & dans l'autre des segmens égaux entr'eux, chaque élément du déblai doit être porté sur l'élément correspondant du remblai.
> Les droites Bd & $B'd'$ étant infiniment proches, il est indifférent dans quel ordre les molécules de l'élément $BB'D'D$ se distribuent sur l'élément $bb'd'd$; de quelque manière en effet que se fasse cette distribution, la somme des produits des molécules par les espaces parcourus, est toujours la même, mais si l'on remarque que dans la pratique il convient de débleyer premièrement les parties qui se trouvent sur le passage des autres, & de n'occuper que les dernières les parties du remblai qui sont dans le même cas; la molécule MM' ne devra se transporter que lorsque toute la partie $MM'D'D$ qui la précêde, aura été transportée en $mm'd'd$; donc dans cette hypothèse, si l'on fait $mm'd'd = MM'D'D$, le point m sera celui sur lequel le point M sera transporté.[30]

Although geometrically intuitive, the method is however not fully correct, as was noted by Appell [1928]:

---

[30] Being given, in the same plane, two equal areas $ABCD$ & *abcd*, bounded by arbitrary contours, continuous or discontinuous, find the route that every molecule M of the first should follow & the point m where it should arrive in the second, so that, all points being transported likewise, they fill precisely the second area & so that the sum of the products of each molecule multiplied by the distance covered, is *minimum*.

If one draws a straight line Bd through an arbitrary point M of the first area, such that the segment BAD is equal to the segment *bad*, I assert that, in order to satisfy the question, all molecules of the segment BAD should be carried on the segment *bad*, & hence the molecules of the segment BCD should be carried on the equal segment *bcd*; for, if an arbitrary point K of segment BAD, is carried to a point k of *bcd*, then necessarily some point L somewhere in BCD is transported to a certain point l in *bad*, which cannot be done without that the routes Kk, Ll cross each other between their end points, & the sum of the products of the molecules by the distances covered would not be a *minimum*. Likewise, if one draws a straight line $B'd'$ through a point M' infinitely close to point M, in such a way that one still has that segment $B'A'D'$ is equal to segment $b'a'd'$, then in order to satisfy the question, the molecules of segment $B'A'D'$ should be transported to $b'a'd'$. So all molecules of the element $BB'D'D$ must be transported to the equal element $bb'd'd$. Dividing the déblai & the remblai in this way into an infinity of elements by straight lines that cut in the one & in the other segments that are equal to each other, every element of the déblai must be carried to the corresponding element of the remblai.

The straight lines Bd & $B'd'$ being infinitely close, it does not matter in which order the molecules of element $BB'D'D$ are distributed on the element $bb'd'd$; indeed, in whatever manner this distribution is being made, the sum of the products of the molecules by the distances covered is always the same; but if one observes that in practice it is convenient first to dig off the parts that are in the way of others, & only at last to cover similar parts of the remblai; the molecule MM' must be transported only when the whole part $MM'D'D$ that precedes it will have been transported to $mm'd'd$; hence with this hypothesis, if one has $mm'd'd = MM'D'D$, point m will be the one to which point M will be transported.

> Il est bien facile de faire la figure de manière que les chemins suivis par les deux parcelles dont parle Monge ne se croisent pas.[31]

(cf. Taton [1951]).

Egerváry

Egerváry [1931] published a weighted version of Kőnig's theorem:

> *Ha az $\|a_{ij}\|$ n-edrendű matrix elemei adott nem negatív egész számok, úgy a*
>
> $$\lambda_i + \mu_j \geq a_{ij}, \qquad (i, j = 1, 2, \ldots n),$$
> *(λ_i, μ_j nem negatív egész számok)*
>
> *feltételek mellett*
>
> $$\min . \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max . (a_{1\nu_1} + a_{2\nu_2} + \cdots + a_{n\nu_n}).$$
>
> *hol $\nu_1, \nu_2, \ldots \nu_n$ az $1, 2, \ldots n$ számok összes permutációit befutják.*[32]

The proof method of Egerváry is essentially algorithmic. Assume that the $a_{i,j}$ are integer. Let λ_i^*, μ_j^* attain the minimum. If there is a permutation ν of $\{1, \ldots, n\}$ with $\lambda_i^* + \mu_{\nu_i}^* = a_{i,\nu_i}$ for all i, then this permutation attains the maximum, and we have the required equality. If no such permutation exists, by Frobenius' theorem there are subsets I, J of $\{1, \ldots, n\}$ such that

$$\lambda_i^* + \mu_j^* > a_{i,j} \text{ for all } i \in I, \, j \in J \tag{17.8}$$

and such that $|I| + |J| = n + 1$. Resetting $\lambda_i^* := \lambda_i^* - 1$ if $i \in I$ and $\mu_j^* := \mu_j^* + 1$ if $j \notin J$, would give feasible values for the λ_i and μ_j, however with their total sum being decreased. This is a contradiction.

Translated into an algorithm, it consists of applying $O(nW)$ times a cardinality bipartite matching algorithm, where W is the maximum weight. So its running time is $O(nW \cdot B(n))$, where $B(n)$ is a bound on the running time of any algorithm finding a maximum-size matching and a minimum-size vertex cover in a bipartite graph with n vertices.

This method forms the basis for the *Hungarian method* of Kuhn [1955b,1956] — see below.

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[31] It is very easy to make the figure in such a way that the routes followed by the two particles of which Monge speaks, do not cross each other.

[32] *If the elements of the matrix $\|a_{ij}\|$ of order n are given nonnegative integers, then under the assumption*

$$\lambda_i + \mu_j \geq a_{ij}, \qquad (i, j = 1, 2, \ldots n),$$
(λ_i, μ_j nonnegative integers)

we have

$$\min . \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max . (a_{1\nu_1} + a_{2\nu_2} + \cdots + a_{n\nu_n}).$$

where $\nu_1, \nu_2, \ldots \nu_n$ run over all possible permutations of the numbers $1, 2, \ldots n$.

The 1940s

The first algorithm for the assignment problem might have been published by Easterfield [1946], who described his motivation as follows:

> In the course of a piece of organisational research into the problems of demobilisation in the R.A.F., it seemed that it might be possible to arrange the posting of men from disbanded units into other units in such a way that they would not need to be posted again before they were demobilised; and that a study of the numbers of men in the various release groups in each unit might enable this process to be carried out with a minimum number of postings. Unfortunately the unexpected ending of the Japanese war prevented the implications of this approach from being worked out in time for effective use. The algorithm of this paper arose directly in the course of the investigation.

Easterfield seems to have worked without knowledge of the existing literature. He formulated and proved a theorem equivalent to Hall's marriage theorem (see Section 22.1a) and he described a primal-dual type method for the assignment problem from which Egerváry's result given above follows. The idea of the method can be described as follows.

Let $A = (a_{i,j})$ be an $n \times n$ matrix and let for each column index j, I_j be the set of row indices i for which $a_{i,j}$ is minimum among all entries in row i. If the collection $(I_1, \ldots, I_n)$ has a transversal, say $i_1, \ldots, i_n$ (with $i_j \in I_j$), then $i_j \to j$ is an optimum assignment.

If $(I_1, \ldots, I_n)$ has no transversal, let $\mathcal{J}$ be the collection of subsets J of $\{1, \ldots, n\}$ for which $(I_j \mid j \in J)$ has a transversal. Select an inclusionwise minimal set J that is not in $\mathcal{J}$. Then there exists an $\varepsilon > 0$ such that subtracting ε from each entry in each of the columns in J extends $\mathcal{J}$ by (at least) J. (This can be seen using Hall's condition.)

Easterfield described an implementation (including scanning all subsets in lexicographic order), that has running time $O(2^n n^2)$. (This is better than scanning all permutations, which takes time $\Omega(n!)$.) The algorithm was explained again by Easterfield [1960].

Birkhoff [1946] derived from Hall's marriage theorem that each doubly stochastic matrix is a convex combination of permutation matrices. Birkhoff's motivation was:

> Estas matrices son interesantes para la probabilidad, y los cuadrados mágicos son múltiplos escalares de estas matrices.[33]

A breakthrough in solving the assignment problem came when Dantzig [1951a] showed that the assignment problem can be formulated as a linear programming problem that automatically has an integer optimum solution. Indeed, by Birkhoff's theorem, minimizing a linear functional over the set of doubly stochastic matrices (which is a linear programming problem) gives a permutation matrix, being the optimum assignment. So the assignment problem can be solved with the simplex method.

In an address delivered on 9 September 1949 at a meeting of the American Psychological Association at Denver, Colorado, Thorndike [1950] studied the problem of the 'classification' of personnel:

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[33] These matrices are interesting because of the probability, and the magic squares are scalar multiples of these matrices.

> The past decade, and particularly the war years, have witnessed a great concern about the classification of personnel and a vast expenditure of effort presumably directed towards this end.

He exhibited little trust in mathematicians:

> There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

Thorndike next presented three heuristics for the assignment problem, the *Method of Divine Intuition*, the *Method of Daily Quotas*, and the *Method of Predicted Yield*.

In a RAND Report dated 5 December 1949, Robinson [1949] reported that an 'unsuccessful attempt' to solve the traveling salesman problem, led her to the following 'cycle-cancelling' method for the optimum assignment problem.

Let matrix $(a_{i,j})$ be given, and consider any permutation π. Define for all i, j a 'length' $l_{i,j}$ by: $l_{i,j} := a_{j,\pi(i)} - a_{i,\pi(i)}$ if $j \neq \pi(i)$ and $l_{i,\pi(i)} = \infty$. If there exists a negative-length directed circuit, there is a straightforward way to improve π. If there is no such circuit, then π is an optimal permutation.

This clearly is a finite method. Robinson remarked:

> I believe it would be feasible to apply it to as many as 50 points provided suitable calculating equipment is available.

The early 1950s

Von Neumann considered the complexity of the assignment problem. In a talk in the Princeton University Game Seminar on 26 October 1951, he showed that the assignment problem can be reduced to finding an optimum column strategy in a certain zero-sum two-person game, and that it can be found by a method given by Brown and von Neumann [1950]. We give first the mathematical background.

A zero-sum two-person game is given by a matrix A, the 'pay-off matrix'. The interpretation as a game is that a 'row player' chooses a row index i and a 'column player' chooses simultaneously a column index j. After that, the column player pays the row player $A_{i,j}$. The game is played repeatedly, and the question is what is the best strategy.

Let A have order $m \times n$. A *row strategy* is a vector $x \in \mathbb{R}^m_+$ satisfying $\mathbf{1}^\mathsf{T} x = 1$. Similarly, a *column strategy* is a vector $y \in \mathbb{R}^n_+$ satisfying $\mathbf{1}^\mathsf{T} y = 1$. Then

$$\max_x \min_j (x^\mathsf{T} A)_j = \min_y \max_i (Ay)_i, \tag{17.9}$$

where x ranges over row strategies, y over column strategies, i over row indices, and j over column indices. Equality (17.9) follows from LP-duality.

It implies that the best strategy for the row player is to choose rows with distribution an optimum x in (17.9). Similarly, the best strategy for the column player is to choose columns with distribution an optimum y in (17.9). The average pay-off then is the value of (17.9).

The method of Brown [1951] to determine the optimum strategies is that each player chooses in turn the line that is best with respect to the distribution of the lines chosen by the opponent so far. It was proved by Robinson [1951] that this converges to optimum strategies. The method of Brown and von Neumann [1950] is a continuous version of this, and amounts to solving a system of linear differential equations.

Now von Neumann noted that the following reduces the assignment problem to the problem of finding an optimum column strategy. Let $C = (c_{i,j})$ be an $n \times n$ cost matrix, as input for the assignment problem. We may assume that C is positive. Consider the following pay-off matrix A, of order $2n \times n^2$, with columns indexed by ordered pairs (i, j) with $i, j = 1, \ldots, n$. The entries of A are given by: $A_{i,(i,j)} := 1/c_{i,j}$ and $A_{n+j,(i,j)} := 1/c_{i,j}$ for $i, j = 1, \ldots, n$, and $A_{k,(i,j)} := 0$ for all i, j, k with $k \neq i$ and $k \neq n + j$. Then any minimum-cost assignment, of cost γ say, yields an optimum column strategy y by: $y_{(i,j)} := c_{i,j}/\gamma$ if i is assigned to j, and $y_{(i,j)} := 0$ otherwise. Any optimum column strategy is a convex combination of strategies obtained this way from optimum assignments. So an optimum assignment can in principle be found by finding an optimum column strategy.

According to a transcript of the talk (cf. von Neumann [1951,1953]), von Neumann noted the following on the number of steps:

> It turns out that this number is a moderate power of n, i.e., considerably smaller than the "obvious" estimate n! mentioned earlier.

However, no further argumentation is given. (Related observations were given by Dulmage and Halperin [1955] and Koopmans and Beckmann [1955,1957].)

Beckmann and Koopmans [1952] studied the quadratic assignment problem, and they noted that the traveling salesman problem is a special case. In a Cowles Commission Discussion Paper of 2 April 1953, Beckmann and Koopmans [1953] mentioned applying polyhedral methods to solve the assignment problem, and noted:

> It should be added that in all the assignment problems discussed, there is, of course, the obvious brute force method of enumerating all assignments, evaluating the maximand at each of these, and selecting the assignment giving the highest value. This is too costly in most cases of practical importance, and by a method of solution we have meant a procedure that reduces the computational work to manageable proportions in a wider class of cases.

Geometric methods were proposed by Lord [1952] and Dwyer [1954] (the 'method of optimal regions') and other heuristics by Votaw and Orden [1952] and Törnqvist [1953]. A survey of developments on the assignment problem until 1955 was given by Motzkin [1956].

Computational results of the early 1950s

In a paper presented at the Symposium on Linear Inequalities and Linear Programming (14–16 June 1951 in Washington, D.C.), Votaw and Orden [1952] mentioned that solving a 10×10 transportation problem took 3 minutes on the SEAC (National Bureau of Standards Eastern Automatic Computer). However, in a later paper (submitted 1 November 1951), Votaw [1952] said that solving a 10×10 assignment problem with the simplex method on the SEAC took 20 minutes.

Moreover, in his reminiscences, Kuhn [1991] mentioned:

> The story begins in the summer of 1953 when the National Bureau of Standards and other US government agencies had gathered an outstanding group of combinatorialists and algebraists at the Institute for Numerical Analysis (INA) located on the campus of the University of California at Los Angeles. Since space was tight, I shared an office with Ted Motzkin, whose pioneering work on linear inequalities and related systems predates linear programming by more than ten years. A rather unique feature of the INA was the presence of the Standards Western Automatic Computer (SWAC), the entire memory of which consisted of 256 Williamson cathode ray tubes. The SWAC was faster but smaller than its sibling machine, the Standards Eastern Automatic Computer (SEAC), which boasted a liquid mercury memory and which had been coded to solve linear programs.
>
> During the summer, C.B. Tompkins was attempting to solve 10 by 10 assignment problems by programming the SWAC to enumerate the $10! = 3,628,800$ permutations of 10 objects. He never succeeded in this project.
>
> Thus, the 10 by 10 assignment problem is a linear program with 100 nonnegative variables and 20 equation constraints (of which only 19 are needed). In 1953, there was no machine in the world that had been programmed to solve a linear program this large!

If 'the world' includes the Eastern Coast of the U.S.A., there seems to be some discrepancy with the remarks of Votaw [1952] mentioned above.

On 23 April 1954, Gleyzal [1955] wrote that a code of his algorithm for the transportation problem, for the special case of the assignment problem with an 8×8 matrix, had just been composed for the SWAC.

Tompkins [1956] mentioned the following 'branch-and-bound' approach to the assignment problem:

> Benjamin Handy, on the suggestion of D.H. Lehmer and with advice from T.S. Motzkin [1], coded this problem for SWAC; he used exhaustive search including rejection of blocks of permutations when the first few elements of the trace led to a hopelessly low contribution. The problem worked for a problem whose matrix had 12 rows and 12 columns and was composed of random three-digit numbers. The solution in this case took three hours. Some restrictions which had been imposed concerning the types of problems to which the code should be applicable led to some inefficiencies; however, the simplex method of G.B. Dantzig [7] and various other methods of solution of this problem seem greatly superior to this method of exhaustive search;

(References [1] and [7] in this quotation are Motzkin [1956] and Dantzig [1951b].)

Kuhn, Munkres: the Hungarian method

Kuhn [1955b,1956] developed a new combinatorial procedure for solving the assignment problem. The method is based on the work of Egerváry [1931], and therefore Kuhn introduced the name *Hungarian method* for it. (According to Kuhn [1955b], the algorithm is 'latent in work of D. Kőnig and J. Egerváry'.) The method was sharpened by Munkres [1957].

In an article *On the origin of the Hungarian method*, Kuhn [1991] presented the following reminiscences on the Hungarian method, from the time starting Summer 1953:

> During this period, I was reading Kőnig's classical book on the theory of graphs and realized that the matching problem for a bipartite graph on two sets of n vertices was exactly the same as an n by n assignment problem with all $a_{ij} = 0$ or 1. More significantly, Kőnig had given a combinatorial algorithm (based on augmenting paths) that produces optimal solutions to the matching problem and its combinatorial (or linear programming) dual. In one of the several formulations given by Kőnig (p. 240, Theorem D), given an n by n matrix $A = (a_{ij})$ with all $a_{ij} = 0$ or 1, the maximum number of 1's that can be chosen with no two in the same line (horizontal row or vertical column) is equal to the minimum number of lines that contain all of the 1's. Moreover, the algorithm seemed to be 'good' in a sense that will be made precise later. The problem then was: how could the general assignment problem be reduced to the 0-1 special case?
> Reading Kőnig's book more carefully, I was struck by the following footnote (p. 238, footnote 2): "... Eine Verallgemeinerung dieser Sätze gab Egerváry, Matrixok kombinatorius tulajdonságairól (Über kombinatorische Eigenschaften von Matrizen), Matematikai és Fizikai Lapok, *38*, 1931, S. 16-28 (ungarisch mit einem deutschen Auszug) ..." This indicated that the key to the problem might be in Egerváry's paper. When I returned to Bryn Mawr College in the fall, I obtained a copy of the paper together with a large Hungarian dictionary and grammar from the Haverford College library. I then spent two weeks learning Hungarian and translated the paper [1]. As I had suspected, the paper contained a method by which a general assignment problem could be reduced to a finite number of 0-1 assignment problems.
> Using Egerváry's reduction and Kőnig's maximum matching algorithm, in the fall of 1953 I solved several 12 by 12 assignment problems (with 3-digit integers as data) by hand. Each of these examples took under two hours to solve and I was convinced that the combined algorithm was 'good'. This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world.

(Reference [1] is the English translation of the paper of Egerváry [1931].)

The method described by Kuhn is a sharpening of the method of Egerváry sketched above, in two respects: (i) it gives an (augmenting path) method to find either a perfect matching or sets I and J as required, and (ii) it improves the λ_i and μ_j not by 1, but by the largest value possible.

Kuhn [1955b] described the method in terms of matrices — in terms of graphs it amounts to the following algorithm for the maximum weighted perfect matching problem in a complete bipartite graph $G = (V, E)$, with weight function $w : E \to \mathbb{Z}_+$. Let U and W be the colour classes of G. Throughout there is a function $p : V \to \mathbb{Z}$ satisfying

(17.10) $\qquad p(u) + p(v) \geq w(uv)$ for each edge uv

and a matching M in the subgraph $G' = (V, E')$ of G consisting of those edges having equality in (17.10).

If M is not a perfect matching, orient each edge in M from W to U, and every other edge of G' from U to W, giving graph D'_M. Let U_M and W_M be the sets of vertices in U and W missed by M.

Kuhn [1955b] described a depth-first search to find the set R_M of vertices that are reachable by a directed path in D'_M from U_M. (In a subsequent paper, Kuhn [1956] described a breadth-first search, starting at only one vertex in U_M.)

Case 1: $R_M \cap W_M \neq \emptyset$. We have an M-augmenting path in G', by which we increase M.

Case 2: $R_M \cap U_M = \emptyset$. Determine

(17.11) $\mu := \min\{p(u) + p(v) - w(u, v) \mid u \in U \cap R_M, v \in W \setminus R_M\}.$

This number is positive, since no edge of G' connects $U \cap R_M$ and $W \setminus R_M$. Decrease $p(u)$ by μ if $u \in U \cap R_M$ and increase $p(v)$ by μ if $v \in W \cap R_M$. Then (17.10) is maintained, while the sum $\sum_{v \in V} p(v)$ decreases (as $|U \cap R_M| > |W \cap R_M|$).

After this we iterate, until we have a perfect matching M in G', which is a maximum-weight perfect matching.

Kuhn [1955b] contented himself with stating that the number of iterations is finite (since the number of iterations where Case 2 applies is finite (as $\sum_v p(v)$ is nonnegative)).

It was observed by Munkres [1957] that the method runs in strongly polynomial time, since, between any two occurrences of Case 1, the number of iterations where Case 2 applies is at most n, as at each such iteration $R_M \cap W$ increases (namely by all vertices v that attain the minimum (17.11)).

So the number of iterations is at most n^2 (since M can increase at most n times). As the (depth- or breadth-first) search takes $O(n^2)$ this gives an $O(n^4)$ algorithm.

Munkres [1957] observed also that after an occurrence of Case 2 one can continue the search of the previous iteration, since edges of G' traversed in the search from U_M, remain edges of the new graph G'. Hence between any two occurrences of Case 1, the depth-first search takes time $O(n^2)$. This still gives an $O(n^4)$ algorithm, since calculating the minimum (17.11) takes $O(n^2)$ time. (Munkres claimed that his algorithm takes $O(n^3)$ operations, but he takes 'scanning a line' (that is, considering all edges incident with a given vertex) as one operation.)

(However, all Case 2-iterations can be combined to one iteration, by finding distances from U_M, with respect to the length function w in the oriented G'. It amounts to including a Dijkstra-like labeling, yielding an $O(n^3)$ time bound. This is the method we described in Section 17.2. This principle was noticed by Edmonds and Karp [1970] and Tomizawa [1971].)

Ford and Fulkerson [1955,1957b] (cf. Ford and Fulkerson [1956c,1956d]) extended the Hungarian method to general transportation problems. They state in Ford and Fulkerson [1956c,1956d]:

> Large systems involving hundreds of equations in thousands of unknowns have been successfully solved by hand using the simplex computation. The procedure of this paper has been compared with the simplex method on a number of randomly chosen problems and has been found to take roughly half the effort for small problems. We believe that as the size of the problem increases, the advantages of the present method become even more marked.

In a footnote, the authors add as to the assignment problem:

> The largest example tried was a 20×20 optimal assignment problem. For this example, the simplex method required well over an hour, the present method about thirty minutes of hand computation.

Chapter 18

Linear programming methods and the bipartite matching polytope

The weighted matching problem for bipartite graphs discussed in the previous chapter is related to the 'matching polytope' and the 'perfect matching polytope', and can be handled with linear programming methods by the total unimodularity of the incidence matrix of a bipartite graph.
In this chapter, graphs can be assumed to be simple.

18.1. The matching and the perfect matching polytope

Let $G = (V, E)$ be a graph. The *perfect matching polytope* $P_{\text{perfect matching}}(G)$ of G is defined as the convex hull of the incidence vectors of perfect matchings in G. So $P_{\text{perfect matching}}(G)$ is a polytope in $\mathbb{R}^E$.

The perfect matching polytope is a polyhedron, and hence can be described by linear inequalities. The following are clearly valid inequalities:

(18.1) (i) $x_e \geq 0$ for each edge e,
(ii) $x(\delta(v)) = 1$ for each vertex v.

These inequalities are generally not enough (for instance, not for K_3). However, as Birkhoff [1946] showed, for bipartite graphs they are enough:

Theorem 18.1. *If G is bipartite, the perfect matching polytope of G is determined by* (18.1)*.*

Proof. Let x be a vertex of the polytope determined by (18.1). Let F be the set of edges e with $x_e > 0$. Suppose that F contains a circuit C. As C has even length, $EC = M \cup N$ for two disjoint matchings M and N. Then for ε close enough to 0, both $x + \varepsilon(\chi^M - \chi^N)$ and $x - \varepsilon(\chi^M - \chi^N)$ satisfy (18.1), contradicting the fact that x is a vertex of the polytope. So (V, F) is a forest, and hence by (18.1), F is a perfect matching. ∎

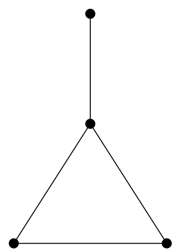


Figure 18.1

The implication cannot be reversed, as is shown by the graph in Figure 18.1.

Theorem 18.1 was shown by Birkhoff in the terminology of doubly stochastic matrices. A matrix A is called *doubly stochastic* if A is nonnegative and each row sum and each column sum equals 1. A *permutation matrix* is an integer doubly stochastic matrix (so it is $\{0,1\}$-valued, and has precisely one 1 in each row and in each column). Then:

Corollary 18.1a (Birkhoff's theorem)**.** *Each doubly stochastic matrix is a convex combination of permutation matrices.*

Proof. Directly from Theorem 18.1, by taking $G = K_{n,n}$. ∎

Theorem 18.1 also implies a characterization of the matching polytope for bipartite graphs. For any graph $G = (V, E)$, the *matching polytope* $P_{\text{matching}}(G)$ of G is the convex hull of the incidence vectors of matchings in G. So again it is a polytope in $\mathbb{R}^E$. The following are valid inequalities for the matching polytope:

$$
\text{(18.2)} \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each edge } e, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for each vertex } v. \end{array}
$$

Then:

Corollary 18.1b. *The matching polytope of G is determined by* (18.2) *if and only if G is bipartite.*

Proof. To see necessity, suppose that G is not bipartite, and let C be an odd circuit in G. Define $x_e := \frac{1}{2}$ if $e \in C$ and $x_e := 0$ otherwise. Then x satisfies (18.2) but does not belong to the matching polytope of G.

To see sufficiency, let G be bipartite and let x satisfy (18.2). Let G' and x' be a copy of G and x, and add edges vv', where v' is the copy of $v \in V$. Define $y(vv') := 1 - x(\delta(v))$. Then x, x', y satisfy (18.1) with respect to the new graph, and hence by Theorem 18.1, it is a convex combination of incidence vectors of perfect matchings in the new graph. Hence x is a convex combination of incidence vectors of matchings in G. ∎

Notes. Birkhoff derived Corollary 18.1a from Hall's marriage theorem (Theorem 22.1), which is equivalent to Kőnig's matching theorem. (Also Dulmage and Halperin [1955] derived Birkhoff's theorem from Kőnig's matching theorem.) Other proofs were given by von Neumann [1951,1953], Dantzig [1952], Hoffman and Wielandt [1953], Koopmans and Beckmann [1955,1957], Hammersley and Mauldon [1956] (a polyhedral proof based on total unimodularity), Tompkins [1956], Mirsky [1958], and Vogel [1961]. A survey was given by Mirsky [1962]. More can be found in Johnson, Dulmage, and Mendelsohn [1960], Nishi [1979], and Brualdi [1982].

18.2. Totally unimodular matrices from bipartite graphs

In this section we show that the results on matchings discussed above can also be derived from linear programming duality with total unimodularity (Hoffman [1956b]).

Let A be the $V \times E$ incidence matrix of a graph $G = (V, E)$. The matrix A generally is not totally unimodular. E.g., if G is the complete graph K_3 on three vertices, then the determinant of A is equal to $+2$ or -2.

However, the following can be proved (necessity can also be derived directly from the total unimodularity of the incidence matrix of a directed graph (Theorem 13.9) — we give a direct proof):

Theorem 18.2. *A graph $G = (V, E)$ is bipartite if and only if its incidence matrix A is totally unimodular.*

Proof. *Sufficiency.* Assume that A is totally unimodular and G is not bipartite. Then G has a circuit of odd length, t say. The submatrix of A induced by the vertices and edges in C is a $t \times t$ matrix with exactly two ones in each row and each column. As t is odd, the determinant of this matrix is ± 2, contradicting the total unimodularity of A.

Necessity. Let G be bipartite. We show that A is totally unimodular. Let B be a square submatrix of A, of order $t \times t$ say. We show that $\det B$ equals 0 or ± 1 by induction on t. If $t = 1$, the statement is trivial. So let $t > 1$. We distinguish three cases.

Case 1: B has a column with only 0*'s.* Then $\det B{=}0$.

Case 2: B has a column with exactly one 1. In that case we can write (possibly after permuting rows or columns):

$$(18.3) \qquad B = \begin{pmatrix} 1 & b^{\mathsf{T}} \\ \mathbf{0} & B' \end{pmatrix},$$

for some matrix B' and vector b, where $\mathbf{0}$ denotes the all-zero vector in $\mathbb{R}^{t-1}$. By the induction hypothesis, $\det B' \in \{0, \pm 1\}$. Hence, by (18.3), $\det B \in \{0, \pm 1\}$.

Case 3. Each column of B contains exactly two 1*'s.* Then, since G is bipartite, we can write (possibly after permuting rows):

(18.4) $$B = \begin{pmatrix} B' \\ B'' \end{pmatrix},$$

in such a way that each column of B' contains exactly one 1 and each column of B'' contains exactly one 1. So adding up all rows in B' gives the all-one vector, and also adding up all rows in B'' gives the all-one vector. The rows of B therefore are linearly dependent, and hence $\det B{=}0$. ∎

18.3. Consequences of total unimodularity

Let $G = (V, E)$ be a bipartite graph and let A be its $V \times E$ incidence matrix. Consider Kőnig's matching theorem (Theorem 16.2): the maximum size of a matching in G is equal to the minimum size of a vertex cover in G. This can be derived from the total unimodularity of A as follows. By Corollary 5.20a, both optima in the LP-duality equation

(18.5) $$\max\{\mathbf{1}^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq \mathbf{1}^{\mathsf{T}}\}$$

have integer optimum solutions x^* and y^*. Now x^* necessarily is the incidence vector of a matching and y^* is the incidence vector of a vertex cover. So we have Kőnig's matching theorem.

One can also derive the weighted version of Kőnig's matching theorem, Egerváry's theorem (Theorem 17.1): for any weight function $w : E \to \mathbb{Z}_+$, the maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in V} y_v$, where y ranges over all $y : V \to \mathbb{Z}_+$ with $y_u + y_v \geq w_e$ for each edge $e = uv$ of G. To derive this, consider the LP-duality equation

(18.6) $$\max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq w^{\mathsf{T}}\}.$$

By the total unimodularity of A, these optima are attained by integer x^* and y^*, and we have the theorem.

The min-max relation for minimum-weight *perfect* matching (Theorem 17.5) follows similarly.

One can also derive the characterizations of the matching polytope and perfect matching polytope of a bipartite graph (Theorem 18.1 and Corollary 18.1b) from the total unimodularity of the incidence matrix of a bipartite graph. This amounts to the fact that the polyhedra

(18.7) $$\{x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$$

and

(18.8) $$\{x \mid x \geq \mathbf{0}, Ax = \mathbf{1}\}$$

are integer polyhedra, by the total unimodularity of A.

18.4. The vertex cover polytope

One can similarly derive, from the total unimodularity, a description of the vertex cover polytope of a bipartite graph. The *vertex cover polytope* of a graph G is the convex hull of the incidence vectors of vertex covers. It is a polytope in $\mathbb{R}^V$.

For bipartite graphs, it is determined by:

$$\text{(18.9)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq y_v \leq 1 & \text{for each } v \in V, \\ \text{(ii)} & y_u + y_v \geq 1 & \text{for each } e = uv \in E. \end{array}$$

In fact, this characterizes bipartiteness:

Theorem 18.3. *A graph G is bipartite if and only if the vertex cover polytope of G is determined by* (18.9)*.*

Proof. Necessity follows from the total unimodularity of the incidence matrix of A (Theorem 18.2). Sufficiency can be seen as follows. Suppose that G contains an odd circuit C. Define $y_v := \frac{1}{2}$ for each $v \in V$. Then y satisfies (18.9) but does not belong to the vertex cover polytope, as each vertex cover contains more than $\frac{1}{2}|VC|$ vertices in C. ∎

The total unimodularity of A also yields descriptions of the edge cover and stable set polytopes of a bipartite graph — see Section 19.5.

18.5. Further results and notes

18.5a. Derivation of Kőnig's matching theorem from the matching polytope

We note here that Kőnig's matching theorem quite easily follows from description (18.2) of the matching polytope of a bipartite graph.

Since the matching polytope of a bipartite graph $G = (V, E)$ is determined by (18.2), the maximum size of a matching in G is equal to the minimum value of $\sum_{v \in V} y_v$ where $y_v \geq 0$ ($v \in V$) such that $y_u + y_v \leq 1$ for each edge $e = uv$.

Now consider any vertex u with $y_u > 0$. Then by complementary slackness, each maximum-size matching covers u. That is, we have (16.5), which (as we saw) directly implies Kőnig's matching theorem, by applying induction to $G - u$.

18.5b. Dual, primal-dual, primal?

The Hungarian method is considered as the first so-called 'primal-dual' method. It maintains a feasible dual solution, and tries to build up a feasible primal solution fulfilling the complementary slackness conditions. We will show that in a certain sense the method can also be considered as just dual or just primal.

We consider the problem of finding a minimum-weight perfect matching in a bipartite graph $G = (V, E)$, with weight function $w : E \to \mathbb{Q}_+$. Let U and W be the colour classes of G, with $|U| = |W|$. The corresponding LP-duality equation is

(18.10) $$\min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax = \mathbf{1}\} = \max\{y^{\mathsf{T}}\mathbf{1} \mid y^{\mathsf{T}}A \leq w^{\mathsf{T}}\},$$

where A is the $V \times E$ incidence matrix of G.

To describe the Hungarian method as a purely dual method one can start with $y = \mathbf{0}$. So y satisfies

(18.11) $$y_u + y_v \leq w_e$$

for each edge $e = uv$ of G. Consider the subset

(18.12) $$F := \{e = uv \in E \mid y_u + y_v = w_e\}$$

of E. If F contains a perfect matching M, then M is a minimum-weight perfect matching, by complementary slackness applied to (18.10). If F contains no perfect matching, by Frobenius' theorem (Corollary 16.2a) there exist $U' \subseteq U$ and $W' \subseteq W$ such that each edge in F intersecting U' also intersects W' and such that $|W'| < |U'|$. Now we can reset

(18.13) $$y_v := \begin{cases} y_v + \alpha & \text{if } v \in U', \\ y_v - \alpha & \text{if } v \in W', \end{cases}$$

choosing α as large as possible while maintaining (18.11). That is, α is equal to the minimum of $w_e - y_u - y_v$ over all edges $e = uv \in E$ with $u \in U'$ and $v \notin W'$. So $\alpha > 0$, and hence $y^{\mathsf{T}}\mathbf{1}$ increases. After that we iterate.

Described in this way it is a purely *dual method*, since only in the last iteration we see a primal solution. In each iteration we test the existence of a perfect matching from scratch. We could, however, remember our work of the previous iteration in our search for a perfect matching in F.

To this end, we keep at any iteration a maximum-size matching M in F. Let D_M be the directed graph obtained from (V, F) by orienting each edge in M from W to U and each edge in $F \setminus M$ from U to W. Let U_M and W_M be the set of vertices in U and W, respectively, missed by M. We also keep, throughout the iterations, the set R_M of vertices reachable in D_M from U_M.

Then we can take $U' := U \cap R_M$ and $W' := W \cap R_M$. Resetting (18.13) of y increases R_M, since at least one edge connecting U' and $W \setminus W'$ is added to F, while all edges in F that were contained in $U' \cup W'$ remain in F. So after at most n iterations, R_M contains a vertex in W_M, in which case we can augment M.

Described in this way it is a *primal-dual method*. Throughout the iterations we keep a feasible dual solution y and a partially feasible primal solution M.

We could however combine all updates of y, between any two augmentations of M, by taking $l_e := w_e - y_u - y_v$ as a length function, and by determining, for each vertex v, the distance $d(v)$ from v to W_M in D_M with respect to length function l. Resetting

(18.14) $$y_v := \begin{cases} y_v + d(v) & \text{if } v \in U, \\ y_v - d(v) & \text{if } v \in W, \end{cases}$$

maintains (18.10), while the new F contains an M-augmenting path (namely, any shortest $U_M - W_M$ path in D_M). Note that this updating of y is the same as the aggregated updating of y (in (18.13)) between any two matching augmentations.

This still is a primal-dual method, since we keep sequences of vectors y and matchings M. It enables us to apply Dijkstra's method to find the distances and the shortest path, since the length function l is nonnegative. We can however do without y, at the cost of an increase in the complexity, since we then must use the Bellman-Ford method (like in our description in Section 17.2). We can use this method since D_M has no negative-length directed circuit, because M is an extreme matching (that is, a matching of minimum weight among all matchings M' with $|M'| = |M|$).

Indeed, we can define the length function l by $l_e := w_e$ if $e \in E \setminus M$ and $l_e := -w_e$ if $e \in M$. Then D_M has no negative-length directed circuits. Any shortest $U_M - W_M$ path is an M-augmenting path yielding an extreme matching M' with $|M'| = |M| + 1$.

Described in this way we have a purely *primal method*, since we keep no vector $y \in \mathbb{Q}^V$ anymore.

18.5c. Adjacency and diameter of the matching polytope

Clearly, for each perfect matching M, the incidence vector χ^M is a vertex of the perfect matching polytope. Adjacency is also easily characterized (Balinski and Russakoff [1974]):

Theorem 18.4. *Let M and N be perfect matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the perfect matching polytope if and only if $M \triangle N$ is a circuit.*

Proof. To see necessity, let χ^M and χ^N be adjacent. Then $M \triangle N$ is the vertex-disjoint union of circuits $C_1, \ldots, C_k$. If $k = 1$ we are done so assume $k \geq 2$. Let $M' := M \triangle C_1$ and $N' := N \triangle C_1$. Then $\frac{1}{2}(\chi^M + \chi^N) = \frac{1}{2}(\chi^{M'} + \chi^{N'})$. This contradicts the adjacency of χ^M and χ^N.

To see sufficiency, define a weight function $w : E \to \mathbb{R}$ by $w_e := 0$ if $e \in M \cup N$ and $w_e := 1$ otherwise. Then M and N are the only two perfect matchings in G of minimum weight. Hence χ^M and χ^N are adjacent. ∎

This gives for the diameter:

Corollary 18.4a. *The perfect matching polytope of a graph $G = (V, E)$ has diameter at most $\frac{1}{2}|V|$. If G is simple, the diameter is at most $\frac{1}{4}|V|$.*

Proof. Let M and N be perfect matchings of G. Let $M \triangle N$ be the vertex-disjoint union of circuits $C_1, \ldots, C_k$. Define $M_i := M \triangle (C_1 \cup \cdots \cup C_i)$, for $i = 0, \ldots, k$. Then $M = M_0$, $N = M_k$, and M_i and M_{i+1} give adjacent vertices of the perfect matching polytope of G (by Theorem 18.4). As each C_i has at least two vertices, we have $k \leq \frac{1}{2}|V|$. If G is simple, each C_i has at least four vertices, and hence $k \leq \frac{1}{4}|V|$. ∎

For complete bipartite graphs, this bound can be strengthened. The *assignment polytope* is the perfect matching polytope of a complete bipartite graph $K_{n,n}$. So in matrix terms, it is the polytope of the $n \times n$ doubly stochastic matrices. Balinski and Russakoff [1974] showed:

Theorem 18.5. *The diameter of the assignment polytope is 2 (if $n \geq 4$).*

Proof. Let U and W be the two colour classes of $K_{n,n}$. Let M and N be two distinct perfect matchings in $K_{n,n}$. Assume that $M \neq N$ and that M and N are not adjacent. Let $M \triangle N$ be the vertex-disjoint union of the circuits $C_1, \ldots, C_k$. As M and N are not adjacent, $k \geq 2$. For each $i = 1, \ldots, k$, choose an edge $u_i w_i \in C_i \cap M$, with $u_i \in U$ and $w_i \in W$. Let C be the circuit

(18.15) $$C := \{u_1 w_1, u_2 w_1, u_2 w_2, u_3 w_2, \ldots, u_n w_n, u_1 w_n\}$$

and let $L := M \triangle C$. As $M \triangle L = C$, L is a perfect matching adjacent to M. Now L is adjacent also to N as well, since $N \triangle L = (C_1 \cup \cdots \cup C_k) \triangle C$, which is a circuit. ∎

Naddef [1982] characterized the dimension of the perfect matching polytope of a bipartite graph (cf. Lovász and Plummer [1986]):

Theorem 18.6. *Let $G = (V, E)$ be a bipartite graph with at least one perfect matching. Then the dimension of the perfect matching polytope of G is equal to $|E_0| - |V| + k$, where E_0 is the set of edges contained in at least one perfect matching and where k is the number of components of the graph (V, E_0).*

Proof. It is easy to see that we may assume that $E_0 = E$ and that G is connected and has at least four vertices. Let T be the edge set of a spanning tree in G. So $|E \backslash T| = |E| - |V| + 1$. Now for any $x \in P_{\text{perfect matching}}(G)$, the values x_e with $e \in T$ are determined by the values x_e with $e \in E \setminus T$. Hence $\dim(P_{\text{perfect matching}}(G)) \leq |E \setminus T| = |E| - |V| + 1$.

To see the reverse inequality, choose a vector x in the relative interior of $P_{\text{perfect matching}}(G)$. So $0 < x_e < 1$ for each $e \in E$ (as each edge is contained in some perfect matching and is missed by some perfect matching). Then any small enough change of x_e for any $e \in E \setminus T$ can be corrected by changing values of $x(e')$ with $e' \in T$. Therefore $\dim(P_{\text{perfect matching}}(G)) \geq |E \setminus T|$. ∎

Rispoli [1992] showed that the 'monotonic diameter' (that is, the maximum length of a shortest path on the polytope where a given objective function is monotonically increasing) of the assignment polytope is equal to $\lfloor \frac{n}{2} \rfloor$. More can be found in Balinski and Russakoff [1974], Padberg and Rao [1974], Brualdi and Gibson [1976,1977a,1977b,1977c], Roohy-Laleh [1980], Hung [1983], Balinski [1985], and Goldfarb [1985].

18.5d. The perfect matching space of a bipartite graph

The *perfect matching space* of a graph $G = (V, E)$ is the linear hull of the incidence vectors of perfect matchings:

(18.16) $$S_{\text{perfect matching}}(G) := \text{lin.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

(Here lin.hull denotes linear hull.)

Note that Theorem 18.6 directly implies the dimension of the perfect matching space of a bipartite graph:

Corollary 18.6a. *Let $G = (V, E)$ be a bipartite graph with at least one perfect matching. Then the dimension of the perfect matching space of G is equal to $|E_0| -$*

$|V|+k+1$, where E_0 is the set of edges contained in at least one of perfect matching, and where k is the number of components of the graph (V, E_0).

Proof. The dimension of the perfect matching space is 1 more than the dimension of the perfect matching polytope (as $\mathbf{0}$ does not belong to the affine hull of the incidence vectors of perfect matchings). So the Corollary follows from Theorem 18.6. ∎

With the help of the description of the perfect matching polytope we can similarly describe the perfect matching space in terms of equations:

Theorem 18.7. *The perfect matching space of a bipartite graph $G = (V, E)$ is equal to the set of vectors $x \in \mathbb{R}^E$ such that*

(18.17) (i) $x_e = 0$ if e is contained in no perfect matching,
(ii) $x(\delta(u)) = x(\delta(v))$ for all $u, v \in V$.

Proof. (18.17) clearly is a necessary condition for each vector x in the perfect matching space. To see sufficiency, let $x \in \mathbb{R}^E$ satisfy (18.17). We can assume that G has at least one perfect matching.

By adding sufficiently many incidence vectors of perfect matchings to x, we can achieve that $x_e \geq 0$ for all $e \in E$. By scaling we can achieve that $x(\delta(v)) = 1$ for each $v \in V$. Then x belongs to the perfect matching polytope of G, and hence to the perfect matching space. ∎

This theorem has as direct consequence a characterization of the linear space orthogonal to the perfect matching space:

Corollary 18.7a. *Let $G = (V, E)$ be a bipartite graph and let $w \in \mathbb{R}^E$. Then $w(M) = 0$ for each perfect matching M if and only if there exists a vector $b \in \mathbb{R}^V$ with $b(V) = 0$ such that $w_e = b_u + b_v$ for each edge $e = uv$ contained in at least one perfect matching.*

Proof. Directly by orthogonality from Theorem 18.7. ∎

18.5e. Up and down hull of the perfect matching polytope

Fulkerson [1970b] studied the up hull of the perfect matching polytope of a graph $G = (V, E)$, that is,

(18.18) $$P^{\uparrow}_{\text{perfect matching}}(G) = P_{\text{perfect matching}}(G) + \mathbb{R}^E_+.$$

Any x in this polyhedron satisfies:

(18.19) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(E[S]) \geq |S| - \frac{1}{2}|V|$ for each $S \subseteq V$.

Here $E[S]$ denotes the set of edges spanned by S. Inequality (18.19)(ii) follows from the fact that any perfect matching M has at most $|V \setminus S|$ edges not contained in S, and hence at least $\frac{1}{2}|V| - |V \setminus S| = |S| - \frac{1}{2}|V|$ edges contained in S.

Fulkerson [1970b] showed that for bipartite graphs these inequalities are enough to characterize polyhedron (18.18):

Theorem 18.8. *If G is bipartite, then $P^{\uparrow}_{\text{perfect matching}}(G)$ is determined by* (18.19).

Proof. Let U and W be the colour classes of G. Let $x \in \mathbb{R}^E$ satisfy (18.19). Note that this implies that $|U| = |W| = \frac{1}{2}|V|$, for if (say) $|U| > \frac{1}{2}|V|$, then (18.19) implies that $0 = x(E[U]) \geq |U| - \frac{1}{2}|V| > 0$, a contradiction.

We must show that there exists a vector y such that $\mathbf{0} \leq y \leq x$ and such that $y(\delta(v)) = 1$ for each $v \in V$. This can be shown quite directly with flow theory, for instance with Gale's theorem (Corollary 11.2g): Make a directed graph by orienting each edge from U to W. Then by Gale's theorem (taking $b(v) := -1$ if $v \in U$ and $b(v) := 1$ if $v \in W$), it suffices to show that $|W'| - |U'| \leq x(\delta^{\text{in}}(U' \cup W'))$ for each $U' \subseteq U$ and $W' \subseteq W$. Let $S := (U \setminus U') \cup W'$. Then $\delta^{\text{in}}(U' \cup W') = E[S]$ and $|W'| - |U'| = |S| - \frac{1}{2}|V|$, giving the required inequality. ∎

(Fulkerson [1970b] derived Theorem 18.8 from an earlier result in Fulkerson [1964b], which is Corollary 20.9a below. Related results were given by O'Neil [1971,1975], Cruse [1975], and Houck and Pittenger [1979].)

Note that the theorem gives also a characterization of the convex hull of the incidence vectors of edge sets containing a perfect matching in a bipartite graph:

Corollary 18.8a. *Let $G = (V, E)$ be a bipartite graph. Then the convex hull of the incidence vectors of edge sets containing a perfect matching is determined by* (18.19) *together with $x_e \leq 1$ for each $e \in E$.*

Proof. Directly from Theorem 18.8. ∎

One can similarly characterize the convex hull of the incidence vectors of subsets of perfect matchings in a bipartite graph. Consider:

(18.20) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(E[S]) \leq |S| - \frac{1}{2}|V|$ for each vertex cover S.

Theorem 18.9. *The convex hull of the incidence vectors of subsets of perfect matchings in a bipartite graph is determined by* (18.20).

Proof. Similar to the proof of Theorem 18.8. ∎

(Alternative proofs of Theorems 18.8 and 18.9 were given by Cunningham and Green-Krótki [1986].)

See Section 20.6a for more results on $P^{\uparrow}_{\text{perfect matching}}(G)$.

18.5f. Matchings of given size

Let $G = (V, E)$ be a graph and let $k, l \in \mathbb{Z}_+$ with $k \leq l$. It is easy to derive from the description of the matching polytope, a description of the convex hull of incidence vectors of matchings M satisfying $k \leq |M| \leq l$. To this end we show:

Theorem 18.10. *Let $G = (V, E)$ be an undirected graph and let $x \in P_{\text{matching}}(G)$. Then x is a convex combination of incidence vectors of matchings M satisfying*

(18.21) $\lfloor \mathbf{1}^{\mathsf{T}} x \rfloor \leq |M| \leq \lceil \mathbf{1}^{\mathsf{T}} x \rceil.$

Proof. Write $x = \sum_M \lambda_M \chi^M$, where M ranges over all matchings in G and where $\lambda_M \geq 0$ with $\sum_M \lambda_M = 1$. Assume that we have chosen the λ_M such that

(18.22) $$\sum_M \lambda_M |M|^2$$

is as small as possible. We show that if M and N are matchings with $\lambda_M > 0$ and $\lambda_N > 0$, then $\big||M| - |N|\big| \leq 1$. This implies the theorem.

Suppose that $|M| \geq |N| + 2$. Let P be a component of $M \cup N$ having more elements in M than in N. Let $M' := M \triangle EP$ and $N' := N \triangle EP$. Then $\chi^{M'} + \chi^{N'} = \chi^M + \chi^N$ and $|M'|^2 + |N'|^2 < |M|^2 + |N|^2$. So decreasing λ_M and λ_N by ε, and increasing $\lambda_{M'}$ and $\lambda_{N'}$ by ε, where $\varepsilon := \min\{\lambda_M, \lambda_N\}$, would decrease sum (18.22), contradicting our assumption. ∎

This implies that certain slices of the matching polytope are again integer polytopes:

Corollary 18.10a. *Let $G = (V, E)$ be an undirected graph and let $k, l \in \mathbb{Z}_+$ with $k \leq l$. Then the convex hull of the incidence vectors of matchings M satisfying $k \leq |M| \leq l$ is equal to the set of those vectors x in the matching polytope of G satisfying $k \leq \mathbf{1}^{\mathsf{T}} x \leq l$.*

Proof. Directly from Theorem 18.10. ∎

A special case is the following result of Mendelsohn and Dulmage [1958b]. Call a matrix a *subpermutation matrix* if it is a $\{0, 1\}$-valued matrix with at most one 1 in each row and in each column. Then:

Corollary 18.10b. *A matrix M belongs to the convex hull of the subpermutation matrices of rank r if and only if M is nonnegative, each row and column sum is at most 1, and the sum of the entries in M is equal to r.*

Proof. Directly from Theorem 18.10. ∎

18.5g. Stable matchings

Let $G = (V, E)$ be a graph and let for each $v \in V$, $\leq_v$ be a total order on $\delta(v)$. Put $e \preceq f$ if e and f have a vertex v in common with $e \leq_v f$. Call a set M of edges *stable* if for each $e \in E$ there exists an $f \in M$ with $e \preceq f$.

In general, stable matchings need not exist (e.g., generally not for K_3). However, Gale and Shapley [1962] showed that if G is bipartite, they do exist:

Theorem 18.11 (Gale-Shapley theorem)**.** *If G is bipartite, then there exists a stable matching.*

Proof. Let U and W be the colour classes of G. For each edge $e = uw$ with $u \in U$ and $w \in W$, let $\phi(e)$ be the height of e in $(\delta(w), \leq_w)$. (The *height* of e is the maximum size of a chain with maximum e.) Choose a matching M in G such that for each edge $e = uw$ of G, with $u \in U$ and $w \in W$,

(18.23) if $f \leq_u e$ for some $f \in M$, then $e \leq_w g$ for some $g \in M$,

and such that $\sum_{e\in M} \phi(e)$ is as large as possible. (Such a matching exists, since $M = \emptyset$ satisfies (18.23).) We show that M is stable.

Choose $e = uw \in E$ with $u \in U$ and $w \in W$ and suppose that there is no $e' \in M$ with $e \preceq e'$. Choose e largest in $\leq_u$ with this property. Then by (18.23) there is no $f \in M$ with $f \leq_u e$; and moreover, there is no $f \in M$ with $e \leq_u f$. Hence u is missed by M.

Since also there is no $g \in M$ with $e \leq_w g$, we can remove any edge in M incident with w and add e to M, so as to obtain a matching satisfying (18.23) with larger $\sum_{e\in M} \phi(e)$, a contradiction. ∎

This proof also gives a polynomial-time algorithm to find a stable matching[34]. The following fact was shown by McVitie and Wilson [1970]:

Theorem 18.12. *Each two stable matchings cover the same set of vertices.*

Proof. Let M and N be two stable matchings, and suppose that there exists a vertex v covered by M but not by N. Let P be the path component of $M \cup N$ starting at v. Denote $P = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$ with $v = v_0$. As v_0 is missed by N, $e_1 <_{v_1} e_2$. As M and N are stable, if $e_{i-1} <_{v_{i-1}} e_i$, then $e_i <_{v_i} e_{i+1}$ for each $i < k$. So $e_{k-1} <_{v_{k-1}} e_k$. However, as v_k is missed by M or N, $e_k <_{v_{k-1}} e_{k-1}$. So we have a contradiction. ∎

In particular:

Corollary 18.12a. *All stable matchings have the same size.*

Proof. Directly from Theorem 18.12. ∎

In order to find a maximum-*weight* stable matching, we consider the *stable matching polytope* $P_{\text{stable matching}}(G)$ of G, which is defined as the convex hull of the incidence vectors of the stable matchings. Vande Vate [1989] (also Rothblum [1992]) characterized the inequalities determining the stable matching polytope if G is bipartite. In that case it suffices to add the following inequalities to the system defining the matching polytope:

(18.24) $$\sum_{f \succeq e} x(f) \geq 1 \text{ for each } e \in E.$$

Theorem 18.13. *If G is bipartite, then $x \in P_{\text{stable matching}}(G)$ if and only if $x \in P_{\text{matching}}(G)$ and x satisfies* (18.24).

[34] It was noted by Roth [1984] that this algorithm is in fact in use in practice since 1951 in the U.S., to match hospitals and medical students (cf. Roth and Sotomayor [1990]).

Proof. Necessity is easy, since the incidence vector of any stable matching satisfies (18.24). To see sufficiency, let x be a vertex of the polytope of all vectors in $P_{\text{matching}}(G)$ satisfying (18.24). Define E^+ to be the set of edges e with $x_e > 0$, and V^+ the set of vertices covered by E^+. For each $v \in V^+$, let e_v be the maximum element of $(\delta(v) \cap E^+, \leq_v)$.

We first show that for each $v \in V^+$, with say $e_v = vv'$,

(18.25) e_v is the minimum element in $(\delta(v') \cap E^+, \leq_{v'})$ and that $x(\delta(v')) = 1$.

Indeed, (18.24) implies (writing $e := e_v$):

$$(18.26) \qquad 1 \leq \sum_{f \succeq e} x(f) = \sum_{f \geq_{v'} e} x(f) = x(\delta(v')) - \sum_{f <_{v'} e} x(f) \leq 1 - \sum_{f <_{v'} e} x(f).$$

Hence we have equality throughout in (18.26). This implies that $x(f) = 0$ for each $f <_{v'} e$ and that $x(\delta(v')) = 1$. This proves (18.25).

It follows that for each $v' \in V^+$ there is exactly one $v \in V^+$ with $e_v = vv'$. Now let U and W be the colour classes of G. The sets $M := \{e_v \mid v \in U \cap V^+\}$ and $N := \{e_v \mid v \in W \cap V^+\}$ are matchings covering V^+. Consider the vector $x' = x + \varepsilon\chi^M - \varepsilon\chi^N$, with ε close enough to 0 (positive or negative). It is easy to see that x' again belongs to the matching polytope. To see that x' satisfies (18.24) for ε close enough to 0, let e be an edge of G attaining equality in (18.24). We show that $e \preceq f$ for exactly one $f \in M$. If $e \in M$, this is trivial, so assume that $e \notin M$. Let $e = uw$ with $u \in U$ and $w \in W$. Then

$$(18.27) \qquad \text{there is an } f \in M \text{ with } e <_u f \iff \sum_{f >_u e} x(f) > 0 \iff \sum_{g \geq_w e} x(g) < 1$$
$$\iff \text{there is no } g \in M \text{ with } e <_w g.$$

Similarly, $e \preceq f$ for exactly one $f \in N$. Concluding,

$$(18.28) \qquad \sum_{f \succeq e} x'(f) = \sum_{f \succeq e} x(f) = 1$$

if ε is close enough to 0. So x' again satisfies (18.24). Since x is a vertex, we have $\chi^M = \chi^N$, that is, $M = N$. So $E^+ = M$, and hence $x = \chi^M$, and therefore x is $\{0,1\}$-valued. ∎

As for algorithms, this theorem directly implies:

Corollary 18.13a. *A maximum-weight stable matching can be found in polynomial time.*

Proof. This follows from the fact that Theorem 18.13 transforms the problem to a linear programming problem. ∎

For surveys and further results, see Wilson [1972a], Knuth [1976], Itoga [1978, 1981], Roth [1982], Gale and Sotomayor [1985], Irving [1985], Gusfield [1987b,1988], Irving, Leather, and Gusfield [1987], Blair [1988], Gusfield and Irving [1989], Ng [1989], Knuth, Motwani, and Pittel [1990a,1990b], Ng and Hirschberg [1990], Ronn [1990], Roth and Sotomayor [1990], Khuller, Mitchell, and Vazirani [1991,1994], Tan [1991], Feder [1992], Roth, Rothblum, and Vande Vate [1993], Abeledo and Rothblum [1994], Feder, Megiddo, and Plotkin [1994,2000], Subramanian [1994], Abeledo and Blum [1996], Balinski and Ratier [1997], Teo and Sethuraman [1997, 1998], Teo, Sethuraman, and Tan [1999], Fleiner [2001a], and Aharoni and Fleiner [2002].

18.5h. Further notes

Perfect and Mirsky [1965] characterized which patterns can occur as the support of a doubly stochastic matrix. It is equivalent to characterizing matching-covered bipartite graphs (that is, bipartite graphs in which each edge belongs to at least one perfect matching).

Frank and Karzanov [1992] gave a polynomial-time combinatorial algorithm to determine the Euclidean distance of the perfect matching polytope of a bipartite graph to the origin.

Chapter 19

Bipartite edge cover and stable set

While matchings cover each vertex *at most* once, edge covers are required to cover each vertex *at least* once. Most edge cover results can be proved similarly to matching results, but in fact, they often can be reduced to matching results, by a method of Gallai.
In this chapter, graphs can be assumed to be simple.

19.1. Matchings, edge covers, and Gallai's theorem

Let $G = (V, E)$ be a graph. An *edge cover* is a subset F of E such that for each vertex v there exists an edge $e \in F$ satisfying $v \in e$. Note that an edge cover can exist only if G has no isolated vertices.

A *stable set* is a subset S of V such that no two vertices in S are adjacent. So for any $U \subseteq V$:

(19.1) $\quad S$ is a stable set $\iff$ $V \setminus S$ is a vertex cover.

Define:

(19.2)
$$\alpha(G) := \text{the maximum size of a stable set in } G,$$
$$\rho(G) := \text{the minimum size of an edge cover in } G.$$

These numbers are called the *stable set number* and the *edge cover number*, respectively.

It is not difficult to show that:

(19.3) $\quad \alpha(G) \leq \rho(G)$.

The triangle K_3 shows that strict inequality is possible. Recall that for the matching number $\nu(G)$ and the vertex cover number $\tau(G)$ we have

(19.4) $\quad \nu(G) \leq \tau(G)$.

In fact, equality in one of the relations (19.3) and (19.4) implies equality in the other, as Gallai [1959a] proved the following[35]:

[35] Gallai mentioned that he had formulated and proved this theorem in 1932 (cf. also Erdős [1982]), and that to his knowledge also D. Kőnig had known this theorem.

Theorem 19.1 (Gallai's theorem)**.** *For any graph $G = (V, E)$ without isolated vertices one has*

(19.5) $$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G).$$

Proof. The first equality follows directly from (19.1).

To see the second equality, let M be a maximum-size matching and let U be the set of vertices missed by M. For each vertex $v \in U$, choose an edge e_v containing v. Then $F = M \cup \{e_v \mid v \in U\}$ is an edge cover of size

(19.6) $$|F| = |M| + |U| = |M| + (|V| - 2|M|) = |V| - |M| = |V| - \nu(G).$$

So $\rho(G) \leq |V| - \nu(G)$.

To see the reverse inequality, let F be a minimum-size edge cover. Let M be an inclusionwise maximal matching contained in F. Let U be the set of vertices missed by M. Since U spans no edge in F, we have $|U| \leq |F \setminus M|$. Hence $|V| - 2|M| = |U| \leq |F \setminus M| = |F| - |M|$. This implies $\nu(G) \geq |M| \geq |V| - |F| = |V| - \rho(G)$. ∎

This proof method implies the following theorem (observed by Gallai [1959a] and Norman and Rabin [1959]):

Theorem 19.2. *Let $G = (V, E)$ be a graph without isolated vertices. Then every maximum-size matching is contained in a minimum-size edge cover, and every minimum-size edge cover contains a maximum-size matching.*

Proof. See above. ∎

Moreover, there is the following complexity result, observed by Norman and Rabin [1959]:

Theorem 19.3. *Let $G = (V, E)$ be an undirected graph with n vertices and m edges. If we have a maximum-size matching in G, we can find a minimum-size edge cover in time $O(m)$, and vice versa.*

Proof. See the proof of Gallai's theorem (Theorem 19.1). ∎

This gives:

Corollary 19.3a. *A minimum-size edge cover and a maximum-size stable set in a bipartite graph can be found in time $O(n^{1/2}m)$.*

Proof. By Theorems 16.4 and 19.3 and Corollary 16.6a. ∎

Short proof of Gallai's theorem. For any partition Π of V into edges and singletons, let $f(\Pi)$ be the number of edges in Π. So $f(\Pi) + |\Pi| = |V|$. Then $\nu(G)$ is equal to the maximum of $f(\Pi)$ over all such partitions, and $\rho(G)$ is equal to the minimum of $|\Pi|$ over all such partitions. Hence $\nu(G) + \rho(G) = |V|$.

19.2. The Kőnig-Rado edge cover theorem

Combination of Theorems 19.1 and 16.2 yields the following theorem, which Gallai [1958a,1958b] attributes to oral communication from D. Kőnig in 1932. In a different but equivalent form it was stated by Rado [1933] — see Section 19.5a. (Hoffman [1956b] called it a 'well-known theorem'.)

Theorem 19.4 (Kőnig-Rado edge cover theorem). *For any bipartite graph $G = (V, E)$ without isolated vertices one has*

(19.7) $\quad \alpha(G) = \rho(G).$

That is, the maximum size of a stable set in a bipartite graph is equal to the minimum size of an edge cover.

Proof. Directly from Theorems 19.1 and 16.2, as $\alpha(G) = |V| - \tau(G) = |V| - \nu(G) = \rho(G)$. ∎

By representing a bipartite graph as a partially ordered set, the Kőnig-Rado edge cover theorem can be derived also from Dilworth's decomposition theorem (Theorem 14.2).

19.3. Finding a minimum-weight edge cover

There is a straightforward reduction of the minimum-weight edge cover problem to the minimum-weight perfect matching problem. Indeed, let $G = (V, E)$ be a graph without isolated vertices, and let $w : E \to \mathbb{Q}_+$. Let $G' = (V', E')$ be the graph obtained from G by adding a disjoint copy $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ of G, and adding for each vertex v of G an edge $v\tilde{v}$ connecting v with its copy $\tilde{v}$. Let w' be the weight function on E' defined by:

(19.8) $\quad w'(e) := w'(\tilde{e}) := w(e)$ for each $e \in E$ (where $\tilde{e}$ is the copy of e); $w'(v\tilde{v}) := 2\mu(v)$ for each $v \in V$, where $\mu(v)$ is the minimum weight of the edges of G incident with v.

Then a minimum-weight perfect matching M in G' yields a minimum-weight edge cover F in G: replace any edge $v\tilde{v}$ in M by an edge e_v of minimum weight of G incident with v, and delete all edges in $M \cap \widetilde{E}$. Then $w(F) = \frac{1}{2}w'(M)$. Conversely, any edge cover F' of G gives by a reverse construction a perfect matching M' in G' with $w'(M') \leq 2w(F')$. Hence $w(F) = \frac{1}{2}w'(M) \leq \frac{1}{2}w'(M') \leq w(F')$. So F is a minimum-weight edge cover in G.

Note that if G is bipartite, then also G' is bipartite. Hence:

Corollary 19.4a. *A minimum-weight edge cover in a bipartite graph can be found in time $O(n(m + n \log n))$.*

Proof. From the above, using Theorem 17.3. ∎

19.4. Bipartite edge covers and total unimodularity

Similarly to Kőnig's matching theorem, also the Kőnig-Rado edge cover theorem (Theorem 19.4) can be derived from the total unimodularity of the $V \times E$ incidence matrix of a bipartite graph $G = (V, E)$. This follows by considering the LP-duality equation

$$(19.9) \qquad \min\{\mathbf{1}^\mathsf{T} x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\} = \max\{y^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, y^\mathsf{T} A \leq \mathbf{1}^\mathsf{T}\}.$$

More generally, we can derive the analogue of Egerváry's theorem:

Theorem 19.5. *Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{R}_+$ be a weight function on E. Then the minimum weight of an edge cover in G is equal to the maximum value of $y(V)$, where y ranges over all functions $y : V \to \mathbb{R}_+$ with $y_u + y_v \leq w_e$ for each edge $e = uv$ of G. If w is integer, we can restrict y to be integer.*

Proof. Again, let A be the $V \times E$ incidence matrix of G. Then the statement is equivalent to the statement that the minimum in

$$(19.10) \qquad \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\} = \max\{y^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, y^\mathsf{T} A \leq w^\mathsf{T}\}$$

has an integer optimum solution x. This fact follows from the total unimodularity of A. If w is integer, we can take also y integer. ∎

The integer part of this theorem can be formulated as follows. For any graph $G = (V, E)$ and $w \in \mathbb{Z}_+^E$, a *w-stable set* is a function $y \in \mathbb{Z}_+^V$ with $y_u + y_v \leq w_e$ for each edge $e = uv$. So if $w = \mathbf{1}$ and G has no isolated vertices, w-stable sets coincide with the incidence vectors of stable sets.

The *size* of a vector $y \in \mathbb{R}^V$ is equal to $y(V)$. Then:

Corollary 19.5a. *Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{Z}_+$ be a weight function on E. Then the minimum weight of an edge cover in G is equal to the maximum size of a w-stable set.*

Proof. Directly from Theorem 19.5. ∎

19.5. The edge cover and stable set polytope

Like in Sections 18.3 and 18.4, the total unimodularity of the incidence matrix of a bipartite graph yields descriptions of the edge cover and the stable set polytope for bipartite graphs.

The *edge cover polytope* $P_{\text{edge cover}}(G)$ of a graph is the convex hull of the incidence vectors of the edge covers in G. For any graph, each vector x in $P_{\text{edge cover}}(G)$ satisfies:

$$(19.11) \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \geq 1 & \text{for each } v \in V. \end{array}$$

Theorem 19.6. *If G is bipartite, the edge cover polytope is determined by* (19.11).

Proof. Directly from the total unimodularity of the constraint matrix in (19.11). ∎

This implication cannot be turned around, as is shown by the graph in Figure 18.1.

The *stable set polytope* $P_{\text{stable set}}(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the stable sets in G. For any graph G, each vector x in $P_{\text{stable set}}(G)$ satisfies:

(19.12) (i) $0 \leq x_v \leq 1$ for each $v \in V$,
(ii) $x_u + x_v \leq 1$ for each edge $e = uv \in E$.

Theorem 19.7. *The stable set polytope is determined by* (19.12) *if and only if G is bipartite.*

Proof. Sufficiency follows from the total unimodularity of the incidence matrix of a bipartite graph. Necessity follows from the fact that if C is an odd circuit in G, then defining $x_v := \frac{1}{2}$ for each $v \in V$, we obtain a vector x satisfying (19.12) but not belonging to the stable set polytope of G, since any stable set intersects C in at most $\frac{1}{2}|VC| - \frac{1}{2}$ vertices. ∎

In fact, there is an easy direct proof of sufficiency in Theorem 19.7. Let x satisfy (19.12) and let U and W be the colour classes of G. For any $\lambda \in [0, 1]$, define

(19.13) $$S_\lambda := \{u \in U \mid x_u > \lambda\} \cup \{w \in W \mid x_w > 1 - \lambda\}.$$

Then S_λ is a stable set, and

(19.14) $$x = \int_0^1 \chi^{S_\lambda} d\lambda.$$

This describes x as a convex combination of incidence vectors of stable sets.

19.5a. Some historical notes on bipartite edge covers

Gallai [1958a,1958b,1959a] wrote that the edge cover theorem (Theorem 19.4) was orally communicated to him by Kőnig in 1932. In the latter paper, Gallai also mentioned that he found Theorem 19.1 in 1932, and that, to his knowledge, also D. Kőnig knew this theorem. Together with Theorem 16.2 of Kőnig [1931] it implies Theorem 19.4.

The oldest written version of Theorem 19.4 seems to be the paper of Rado [1933] entitled *Bemerkungen zur Kombinatorik im Anschluß an Untersuchungen von Herrn D. Kőnig*[36]. The investigations referred to in the title are those of Kőnig [1916] on matchings in *regular* bipartite graphs.

[36] Remarks on combinatorics in connection to investigations of Mr D. Kőnig.

Rado formulated the edge cover theorem in terms of partitions:

> *Es seien* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ *endlich viele nicht leere, paarweise elementenfremde Mengen. Ebenso seien* $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ *endlich viele nicht leere, paarweise elementenfremde Mengen. Alle Mengen* $\mathcal{A}_\mu$ und $\mathcal{B}_\nu$ *seien Teilmengen einer Menge* $\mathcal{M}$. *Unter dieser Annahme gilt: Dann und nur dann sind die Mengen*
>
> (26) $\qquad \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$
>
> *durch* k *Elemente von* $\mathcal{M}$ *zu repräsentieren, wenn es unter den Mengen* (26) *keine* $k+1$ *zu einander fremde Mengen gibt.*[37]

The proof of Rado is based on a decomposition similar to that used by Frobenius (see Section 16.2a). The equivalence with Theorem 19.4 follows with the construction described in Section 16.7e. (A theorem similar to Rado's was published by Kreweras [1946].)

[37] *Let* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ *be finitely many nonempty, pairwise disjoint sets. Similarly, let* $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ *be finitely many nonempty, pairwise disjoint sets. All sets* $\mathcal{A}_\mu$ and $\mathcal{B}_\nu$ *are subsets of a set* $\mathcal{M}$. *Under this condition the following holds: The sets*

(26) $\qquad \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$

can be represented by k *elements of* $\mathcal{M}$, *if and only if there are no* $k+1$ *disjoint sets among the sets* (26).

Chapter 20

Bipartite edge-colouring

Edge-colouring means partitioning the edge set into matchings. While for general graphs, finding a minimum edge-colouring is NP-complete, another fundamental theorem of Kőnig gives a min-max relation for bipartite edge-colouring, and his proof method yields a polynomial-time algorithm. Also the capacitated case and the 'dual' problem of partitioning the edge set into edge covers are tractable for bipartite graphs.

20.1. Edge-colourings of bipartite graphs

For any graph $G = (V, E)$, an *edge-colouring* or *k-edge-colouring* is a partition $\Pi = (M_1, \ldots, M_k)$ of the edge set E into matchings. Each of the M_i is called a *colour*. If $e \in M_i$ we say that e *has* colour i.

The *edge-colouring number* $\chi(G)$ of G is the minimum number of colours in an edge-colouring of G.

Let $\Delta(G)$ denote the maximum degree of (the vertices of) G. Clearly,

(20.1) $\qquad \chi(G) \geq \Delta(G),$

since at each vertex v, the edges incident with v should have different colours. The triangle K_3 has strict inequality in (20.1). Kőnig [1916] showed that for bipartite graphs the two numbers are equal:

Theorem 20.1 (Kőnig's edge-colouring theorem)**.** *For any bipartite graph $G = (V, E)$,*

(20.2) $\qquad \chi(G) = \Delta(G).$

That is, the edge-colouring number of a bipartite graph is equal to its maximum degree.

Proof. Let $M_1, \ldots, M_{\Delta(G)}$ be a collection of disjoint matchings covering a maximum number of edges. If all edges are covered, we are done. So suppose that edge $e = uv$, say, is not covered. Then (since $\deg(u) \leq \Delta(G)$) some M_i misses u and (similarly) some M_j misses v. If $i = j$ we can extend M_i to $M_i \cup \{e\}$. If $i \neq j$, $M_i \cup M_j \cup \{e\}$ makes a bipartite graph of maximum degree at most two. Hence there exist matchings M and N with $M_i \cup M_j \cup \{e\} = M \cup N$.

So replacing M_i and M_j by M and N, increases the number of edges covered, contradicting our assumption. ∎

This proof, due to Kőnig [1916] (using a simplification of Skolem [1927]), also gives a polynomial-time algorithm to find a $\Delta(G)$-edge-colouring with $\Delta(G)$ colours. In fact, if G is simple, it gives an $O(nm)$ algorithm for edge-colouring. This bound can be achieved also for bipartite multigraphs using an appropriate data-structure — see Section 20.9a.

20.1a. Edge-colouring regular bipartite graphs

Kőnig's edge-colouring theorem is directly equivalent to the special case of regular bipartite graphs (since any bipartite graph of maximum degree Δ is a subgraph of a Δ-regular bipartite graph (Kőnig [1932])). Rizzi [1997,1998] gave the following very elegant short argument for the k-edge-colourability of k-regular bipartite graphs. (A similar proof in terms of common transversals of two partitions of a set into equally sized classes was given by Sperner [1927] — see Section 22.7d.)

Let G be a counterexample with fewest edges. So G has no perfect matching. Choose an edge $e = uv$. Then we can extend the graph $G - u - v$ to a k-regular bipartite graph H by adding at most $k - 1$ new edges. As H has fewer edges than G, H has a k-edge-colouring. Since less than k new edges have been added, there is a colour M that uses none of the new edges. Then $M \cup \{e\}$ is a perfect matching in G, a contradiction.

20.2. The capacitated case

Egerváry [1931] observed that the following capacitated version directly follows from Kőnig's edge-colouring theorem:

Corollary 20.1a. *Let $G = (V, E)$ be a bipartite graph and let $c : E \to \mathbb{Z}_+$ be a capacity function. Then the minimum size of a family of matchings such that each edge e is in at least c_e of them is equal to the maximum of $c(\delta(v))$ over all $v \in V$.*

Proof. Directly from Kőnig's edge-colouring theorem, by replacing each edge e by c_e parallel edges. ∎

This reduction being easy, it might not be satisfactory algorithmically. It would not yield a polynomial-time reduction for the following problem:

(20.3) given: a bipartite graph $G = (V, E)$ and a capacity function $c : E \to \mathbb{Z}_+$;
find: matchings $M_1, \ldots, M_k$ and nonnegative integers $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i \chi^{M_i} = c$ and such that $\sum_{i=1}^{k} \lambda_i$ is minimized.

However, there is an easy strongly polynomial-time algorithm for this problem: Let F be the subset of edges e of G with $c_e > 0$. Find a matching M in F covering all vertices v of G that maximize $c(\delta(v))$. Let $\lambda := \min\{c_e \mid e \in M\}$, and replace c by $c - \lambda\chi^M$. Next iterate this.

Since in each iteration the number of edges e with $c_e > 0$ decreases, there are at most $|E|$ iterations. Since a matching covering a given set R of vertices can be found in time $O(|R||E|)$, this gives an $O(nm^2)$ algorithm. However, by starting in each iteration with the matching left from the previous iteration, one can do better (Gonzalez and Sahni [1976]):

Theorem 20.2. *Problem* (20.3) *can be solved in time* $O(m^2)$.

Proof. We may assume that $c(\delta(v))$ is equal for all v, by duplicating G and connecting each vertex with its copy, giving the new edges appropriate capacities. We can also assume that $c_e > 0$ for each edge e.

First we find a perfect matching in G, which can be done in time $O(nm)$, since we can apply $O(n)$ matching-augmenting iterations to find a perfect matching.

In any further iteration, let M be the matching obtained in the previous iteration. Suppose that after resetting c, there exist α edges e in M with $c_e = 0$. Delete these edges. Then in α matching-augmenting steps we can obtain a perfect matching M' in the new graph. So the iteration takes $O(\alpha m)$ time. Since over all iterations the α add up to $|E|$, we have the time bound $O(m^2)$. ∎

20.3. Edge-colouring polyhedrally

Polyhedrally, edge-colouring can be studied with the help of the 'substar polytope' of an undirected graph $G = (V, E)$. Call a set F of edges of G a *substar* if $F \subseteq \delta(v)$ for some $v \in V$. The *substar polytope* $P_{\text{substar}}(G)$ of G is the convex hull of the incidence vectors of substars. So it is a polytope in $\mathbb{R}^E$.

Each vector x in the substar polytope trivially satisfies

(20.4) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(M) \leq 1$ for each matching M.

The following is direct from the description of the bipartite matching polytope (Corollary 18.1b) with the theory of antiblocking polyhedra:

Theorem 20.3. *The substar polytope of a bipartite graph is determined by* (20.4).

Proof. By Corollary 18.1b, the matching polytope is the antiblocking polyhedron of the substar polytope. Hence the substar polytope is the antiblocking polyhedron of the matching polytope (cf. Section 5.9), which is the content of the theorem. ∎

What Kőnig's edge-colouring theorem adds to it is:

Theorem 20.4. *System* (20.4) *is TDI.*

Proof. This is equivalent to Corollary 20.1a. ∎

Note that Kőnig's edge-colouring theorem also can be derived easily from the characterization of the matching polytope. For any bipartite graph $G = (V, E)$, the vector $\Delta(G)^{-1} \cdot \mathbf{1}$ belongs to the matching polytope (where $\mathbf{1}$ is the all-one vector in $\mathbb{R}^E$), and hence it is a convex combination of matchings. Each of these matchings should cover each maximum-degree vertex. So there exists a matching M covering all maximum-degree vertices. Hence $\Delta(G - M) = \Delta(G) - 1$, and we can apply induction.

Also, the integer decomposition property of the matching polytope is equivalent to Kőnig's edge-colouring theorem. (The integer decomposition property follows from the total unimodularity of the incidence matrix of G.)

20.4. Packing edge covers

A theorem 'dual' to Kőnig's edge-colouring theorem was shown by Gupta [1967,1978]. The edge-colouring number $\chi(G)$ of a graph G is the minimum number of matchings needed to cover the edges of a G. Dually, one can define the *edge cover packing number* $\xi(G)$ of a graph by:

(20.5) $\xi(G) :=$ the maximum number of disjoint edge covers in G.

So, in terms of colours, $\xi(G)$ is the maximum number of colours that can be used in colouring the edges of G in such a way that at each vertex all colours occur. Hence, if $\delta(G)$ denotes the minimum degree of G, then

(20.6) $\xi(G) \leq \delta(G)$.

The triangle K_3 again is an example having strict inequality. For bipartite graphs however Gupta [1967,1978] showed:

Theorem 20.5. *For any bipartite graph $G = (V, E)$:*

(20.7) $\xi(G) = \delta(G)$.

That is, the maximum number of disjoint edge covers is equal to the minimum degree.

Proof. We give a reduction to Kőnig's edge-colouring theorem (Theorem 20.1).

One may derive from G a bipartite graph H, each vertex of which has degree $\delta(G)$ or 1, by repeated application of the following procedure:

(20.8) for any vertex v of degree larger than $\delta(G)$, add a new vertex u, and replace one of the edges incident with v, $\{v, w\}$ say, by $\{u, w\}$.

So there is a one-to-one correspondence between the edges of the final graph H and the edges of G. Since H has maximum degree $\delta(G)$, by Theorem 20.1 the edges of H can be coloured with $\delta(G)$ colours such that no two edges of the same colour intersect. So at any vertex of H of degree $\delta(G)$, all colours occur. This gives a colouring of the edges of G with $\delta(G)$ colours such that at any vertex of G all colours occur. ▌

Gupta [1974,1978] gave the following common generalization of Theorems 20.1 on edge-colouring and 20.5 on disjoint edge covers:

Theorem 20.6. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then E can be partitioned into classes $E_1, \ldots, E_k$ such that each vertex v is covered by at least* $\min\{k, \deg_G(v)\}$ *of the E_i.*

Proof. Like in the proof of Theorem 20.5, split off edges from vertices of degree larger than k, until each vertex has degree at most k. Applying Kőnig's edge-colouring theorem to the final graph yields a partitioning of the original edge set as required. ▌

Call a set F of edges of a graph $G = (V, E)$ a *superstar* if $F \supseteq \delta(v)$ for some $v \in V$. The *superstar polytope* $P_{\text{superstar}}(G)$ of G is the convex hull of the incidence vectors of superstars in G. Consider

(20.9)
(i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(F) \geq 1$ for each edge cover F.

Theorem 20.7. *If G is bipartite, system* (20.9) *determines the superstar polytope and is TDI.*

Proof. With the theory of blocking polyhedra, Theorem 19.6 implies that the superstar polytope is determined by (20.9). Total dual integrality of (20.9) is equivalent to the capacitated version of Theorem 20.5. ▌

20.5. Balanced colours

McDiarmid [1972] and de Werra [1970,1972] showed the following generalization of Kőnig's edge-colouring theorem (in fact, it is a special case of a theorem of Folkman and Fulkerson [1969] (see Theorem 20.10 below), and also it is a consequence of the result in Dulmage and Mendelsohn [1969]):

Theorem 20.8. *Let $G = (V, E)$ be a bipartite graph and let $k \geq \Delta(G)$. Then E can be partitioned into matchings $M_1, \ldots, M_k$ such that*

$$\lfloor |E|/k \rfloor \leq |M_i| \leq \lceil |E|/k \rceil \tag{20.10}$$

for each $i = 1, \ldots, k$.

Proof. As $k \geq \Delta(G)$, by Kőnig's edge-colouring theorem, E can be partitioned into matchings $M_1, \ldots, M_k$ (possibly empty). Choose $M_1, \ldots, M_k$ such that

$$\sum_{i=1}^{k} |M_i|^2 \tag{20.11}$$

is minimized.

Suppose that (20.10) is violated. Then there exist M_i and M_j with $|M_i| \geq |M_j|+2$. Then $M_i \cup M_j$ has at least one component K containing more edges in M_i than in M_j. Let $M_i' := M_i \triangle K$ and $M_j' := M_j \triangle K$. Then $|M_i'|^2 + |M_j'|^2 = (|M_i|-1)^2 + (|M_j|+1)^2 = |M_i|^2 + |M_j|^2 - 2|M_i| + 2|M_j| + 2 < |M_i|^2 + |M_j|^2$. So replacing M_i and M_j by M_i' and M_j' decreases the sum (20.11), contradicting our minimality assumption. ∎

Related results can be found in Dulmage and Mendelsohn [1969], Folkman and Fulkerson [1969], Brualdi [1971b], and de Werra [1971,1976].

20.6. Packing perfect matchings

Packing perfect matchings seems less directly reducible to partitioning into matchings or edge covers. It can be handled with the following more general result of Folkman and Fulkerson [1969] on packing matchings of a fixed size p, which is proved by reduction to Menger's theorem:

Theorem 20.9. *Let $G = (V, E)$ be a bipartite graph and let $k, p \in \mathbb{Z}_+$. Then there exist k disjoint matchings of size p if and only if each subset X of V spans at least $k(p + |X| - |V|)$ edges.*

Proof. To see necessity, let $X \subseteq V$ and consider a matching M in G of size p. Since at most $|V| - |X|$ edges in M intersect $V \setminus X$, at least $|M| - (|V| - |X|) = p + |X| - |V|$ edges of M are spanned by X. So k disjoint matchings of size p have at least $k(p + |X| - |V|)$ edges spanned by X.

To see sufficiency, let U and W be the colour classes of G. Orient all edges from U to W. Moreover, add vertices s and t, and, for each $u \in U$, add k parallel arcs from s to u, and, for each $w \in W$, add k parallel arcs from w to t. Let D be the directed graph arising.

We show with Menger's theorem that D contains kp arc-disjoint $s-t$ paths. Consider any $s-t$ cut $\delta^{\text{out}}(Y)$, with $s \in Y$, $t \notin Y$. Let $X := (U \cap Y) \cup (W \setminus Y)$. Then

(20.12) $|\delta^{\text{out}}(Y)| = k|U \setminus Y| + k|W \cap Y| + |E[X]| = k(|V| - |X|) + |E[X]|$,

where $E[X]$ is the set of edges spanned by X. As $|E[X]| \geq k(p + |X| - |V|)$, it follows that $|\delta^{\text{out}}(Y)| \geq kp$.

So D contains kp arc-disjoint $s-t$ paths. The edges of G that belong to these paths form a subgraph of G with kp edges, of maximum degree at most k. So by Theorem 20.8, G has k disjoint matchings of size p. ∎

This implies the following theorem of Fulkerson [1964b] on the maximum number of disjoint perfect matchings (in fact equivalent to a result of Ore [1956], see Corollary 20.9b below):

Corollary 20.9a. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then G has k disjoint perfect matchings if and only if each subset X of V spans at least $k(|X| - \frac{1}{2}|V|)$ edges.*

Proof. Directly by taking $p := \frac{1}{2}|V|$ in Theorem 20.9. ∎

(Lebensold [1977] and Murty [1978] gave other proofs of this corollary.)

Note that, by Kőnig's edge-colouring theorem, a bipartite graph $G = (V, E)$ has k disjoint perfect matchings if and only if G has a k-factor. (A *k-factor* is a subset F of E with the graph (V, F) k-regular.)

So Corollary 20.9a is equivalent to the following result of Ore [1956]:

Corollary 20.9b. *A bipartite graph $G = (V, E)$ has a k-factor if and only if each subset X of V spans at least $k(|X| - \frac{1}{2}|V|)$ edges.*

Proof. Directly from Corollary 20.9a. ∎

20.6a. Polyhedral interpretation

We can interpret these results polyhedrally. In Theorem 18.8 we saw that for any bipartite graph $G = (V, E)$, the up hull of the perfect matching polytope of G,

(20.13) $P^{\uparrow}_{\text{perfect matching}}(G) = P_{\text{perfect matching}}(G) + \mathbb{R}^E_+$

is determined by the inequalities

(20.14) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(E[S]) \geq |S| - \frac{1}{2}|V|$ for each $S \subseteq V$.

Then Corollary 20.9a implies that for each $k \in \mathbb{Z}_+$, each integer vector $w \in k \cdot P^{\uparrow}_{\text{perfect matching}}(G)$ is the sum of k vectors in $P^{\uparrow}_{\text{perfect matching}}(G)$. In other words:

Corollary 20.9c. *$P^{\uparrow}_{\text{perfect matching}}(G)$ has the integer decomposition property.*

Proof. From Corollary 20.9a, by replacing each edge by $w(e)$ parallel edges. ∎

We can view this also in terms of the blocking polyhedron of $P^{\uparrow}_{\text{perfect matching}}(G)$, which is the polyhedron Q determined by

(20.15) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(M) \geq 1$ for each perfect matching M.

Since $P^{\uparrow}_{\text{perfect matching}}(G)$ is determined by (20.14), the theory of blocking polyhedra gives that Q is equal to the up hull of the convex hull of the vectors

$$(20.16)\qquad \tfrac{1}{|S|-\frac{1}{2}|V|}\chi^{E[S]}$$

where $S \subseteq V$ with $|S| > \frac{1}{2}|V|$.

So the minimum value of $\mathbf{1}^{\mathsf{T}}x$ over Q is equal to

$$(20.17)\qquad \min\{\tfrac{|E[S]|}{|S|-\frac{1}{2}|V|} \mid S \subseteq V, |S| > \tfrac{1}{2}|V|\}.$$

By LP-duality, this is equal to the maximum value of $\sum_M \lambda_M$, where M ranges over perfect matchings and where $\lambda_M \geq 0$ such that $\sum_M \lambda_M \chi^M \leq \mathbf{1}$. So Corollary 20.9a states: the maximum number of disjoint perfect matchings in a bipartite graph is equal to

$$(20.18)\qquad \lfloor \max\{\sum_M \lambda_M \mid \lambda_M \geq 0, \sum_M \lambda_M \chi^M \leq \mathbf{1}\}\rfloor.$$

As we can directly extend this to a weighted version, one has:

Corollary 20.9d. *System* (20.15) *has the integer rounding property.*

Proof. See above. ∎

20.6b. Extensions

The results of Sections 20.5 and 20.6 can be extended as follows, as was shown by Folkman and Fulkerson [1969]. It is based on the following theorem:

Theorem 20.10. *Let $G = (V, E)$ be a bipartite graph, let $k \geq \Delta(G)$, and let $p \geq |E|/k$. Then G has a k-edge-colouring in which l colours have size p if and only if G has l disjoint matchings of size p.*

Proof. Necessity being trivial, we show sufficiency. Let G have l disjoint matchings of size p. We must show that there exist l disjoint matchings of size p such that at each vertex v at most $k-l$ edges incident with v are in none of these matchings (since then the edges not contained in the matchings can be properly coloured by $k-l$ colours).

That is, by Theorem 20.8 it suffices to show that there exists a subset F of E such that

(20.19) (i) $\deg_F(v) \leq l$ and $\deg_{E\setminus F}(v) \leq k-l$ for each vertex v;
(ii) $|F| = lp$.

Let F be any subset of E satisfying (20.19)(i), with $|F| \leq lp$, and with $|F|$ as large as possible. Such an F exists, since by Theorem 20.8 we can k-edge-colour G such that each colour has size at most $\lceil |E|/k \rceil \leq p$. Any l of the colours gives F as required.

If $|F| = lp$ we are done, so assume that $|F| < lp$. Since G has l disjoint matchings of size p, E has a subset F' of size lp with $\deg_{F'}(v) \leq l$ for each vertex v. Choose F' with $F' \setminus F$ as small as possible.

Consider an orientation D of the graph $(V, F \triangle F')$, where each edge in $F \setminus F'$ is oriented from colour class U (say) to colour class W (say), and where each edge in $F' \setminus F$ is oriented from W to U. If D contains a directed circuit C, we can reduce $F' \setminus F$, by replacing F' by $F' \triangle C$. So D is acyclic, and hence we can partition $F \triangle F'$ into directed paths, where each path starts at a vertex v with $\deg_D^{\text{out}}(v) > \deg_D^{\text{in}}(v)$ and ends at a vertex v with $\deg_D^{\text{in}}(v) > \deg_D^{\text{out}}(v)$. As $|F'| > |F|$, at least one of these paths, P say, has more edges in F' than in F. Now replacing F by $F \triangle EP$ does not violate (20.19)(i), since $\deg_{F \triangle EP}(v) = \deg_F(v) + 1 \leq \deg_{F'}(v) \leq l$ if v is an end of P and $\deg_{F \triangle EP}(v) = \deg_F(v)$ for any other vertex v. As this increases $|F|$, it contradicts our maximality assumption. ∎

This implies the following result of Folkman and Fulkerson [1969], generalizing Theorems 20.8 and 20.9 (by taking $p_2 = 1$):

Corollary 20.10a. *Let $G = (V, E)$ be a bipartite graph and let $k_1, k_2, p_1, p_2 \in \mathbb{Z}_+$ be such that $k_1 + k_2 \geq \Delta(G)$, $k_1 p_1 + k_2 p_2 = |E|$, and $p_1 \geq p_2$. Then E can be partitioned into k_1 matchings of size p_1 and k_2 matchings of size p_2 if and only if each subset X of V spans at least $k_1(p_1 + |X| - |V|)$ edges.*

Proof. Necessity being easy, we prove sufficiency. By Theorem 20.9, G has k_1 disjoint matchings of size p_1. Let $k := k_1 + k_2$. Since $p_1 \geq p_2$, we have $p_1 \geq (p_1 k_1 + p_2 k_2)/k = |E|/k$. Hence, by Theorem 20.10, G has k_1 disjoint matchings of size p_1, such that the uncovered edges form a subgraph of maximum degree at most k_2. As this subgraph has $|E| - p_1 k_1 = p_2 k_2$ edges, by Theorem 20.8 we can split its edge set into k_2 matchings of size p_2. ∎

These results relate to simple b-matchings — see Corollary 21.29a.

20.7. Covering by perfect matchings

A series of results similar to those in Section 20.6 can be derived for covering by perfect matchings and for the down hull of the perfect matching polytope. Brualdi [1979] showed the covering analogue of Corollary 20.9a:

Theorem 20.11. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then E can be covered by k perfect matchings if and only if any vertex cover X spans at most $k(|X| - \frac{1}{2}|V|)$ edges.*

Proof. *Necessity.* Let G be covered by k perfect matchings and let X be a vertex cover. Each perfect matching contains $|V \setminus X|$ edges not spanned by X, and hence $\frac{1}{2}|V| - |V \setminus X| = |X| - \frac{1}{2}|V|$ edges spanned by X. This proves necessity.

Sufficiency. Assume that the condition holds. This implies that both colour classes of G have size $\frac{1}{2}|V|$, since each of them is a vertex cover X

spanning no edge, implying $|X| \geq \frac{1}{2}|V|$. It also implies that the maximum degree of G is at most k, since for each vertex v the set $U \cup \{v\}$ (where U is the colour class of G not containing v) spans at most k edges.

For each vertex v, let $b_v := k - \deg(v)$. Split each vertex v into b_v vertices, and replace any edge uv by $b_u b_v$ edges connecting the b_u copies of u with the b_v copies of v. This yields the bipartite graph H, with $k|V| - 2|E|$ vertices.

Now H has a perfect matching, as follows from Frobenius' theorem: if Y is a vertex cover in H, then the set X of vertices v of G for which all copies in H belong to Y, is a vertex cover in G. Now by the condition, X spans at most $k(|X| - \frac{1}{2}|V|)$ edges of G. Hence

$$(20.20) \qquad |Y| \geq \sum_{v \in X} (k - \deg(v)) = k|X| - |E| - |E[X]| \geq \tfrac{1}{2}k|V| - |E|.$$

So Y is not smaller than half the number of vertices of H. Therefore, by Frobenius' theorem, H has a perfect matching M.

For each edge e of G, add parallel edges to e as often as a copy of e occurs in M. We obtain a k-regular bipartite graph G'. By Kőnig's edge-colouring theorem, the edges of G' can be partitioned into k perfect matchings. This gives k perfect matchings in G covering E. ∎

(This proof method in fact consists of showing that G has a perfect b-matching — see Chapter 21.)

The result is equivalent to characterizing bipartite graphs that are k-regularizable. A graph $G = (V, E)$ is *k-regularizable* if we can replace each edge by a positive number of parallel edges so as to obtain a k-regular graph. Then:

Corollary 20.11a. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then G is k-regularizable if and only if any vertex cover X spans at most $k(|X| - \frac{1}{2}|V|)$ edges.*

Proof. Directly from Theorem 20.11. ∎

20.7a. Polyhedral interpretation

Again we can interpret Theorem 20.11 polyhedrally. In Theorem 18.9 we saw that for a bipartite graph $G = (V, E)$, the down hull of the perfect matching polytope of G,

$$(20.21) \qquad P^{\downarrow}_{\text{perfect matching}}(G) = (P_{\text{perfect matching}}(G) - \mathbb{R}^E_+) \cap \mathbb{R}^E_+$$

is determined by the inequalities

$$(20.22) \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(E[S]) \leq |S| - \frac{1}{2}|V| & \text{for each vertex cover } S. \end{array}$$

Then Theorem 20.11 implies that for each $k \in \mathbb{Z}_+$, each integer vector $w \in k \cdot P^{\downarrow}_{\text{perfect matching}}(G)$ is a sum of k integer vectors in $P^{\downarrow}_{\text{perfect matching}}(G)$. That is:

Corollary 20.11b. $P^{\downarrow}_{\text{perfect matching}}(G)$ *has the integer decomposition property.*

Proof. See above. ∎

We can view this result also in terms of the antiblocking polyhedron of $P^{\downarrow}_{\text{perfect matching}}(G)$, which is the polyhedron Q determined by

(20.23) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(M) \leq 1$ for each perfect matching M.

By the theory of antiblocking polyhedra, Q is equal to the down hull of the convex hull of the vectors

(20.24) $$\frac{1}{|S|-\frac{1}{2}|V|}\chi^{E[S]}$$

where S is a vertex cover with $|S| > \frac{1}{2}|V|$.

So the maximum value of $\mathbf{1}^{\mathsf{T}}x$ over Q is equal to

(20.25) $$\max\{\frac{|E[S]|}{|S|-\frac{1}{2}|V|} \mid S \text{ vertex cover, } |S| > \tfrac{1}{2}|V|\}.$$

By LP-duality, this is equal to the minimum value of $\sum_M \lambda_M$, where M ranges over perfect matchings and where $\lambda_M \geq 0$ with $\sum_M \lambda_M \chi^M \geq \mathbf{1}$. So Theorem 20.11 states: the minimum number of perfect matchings needed to cover all edges in a bipartite graph is equal to

(20.26) $$\lceil \min\{\sum_M \lambda_M | \lambda_M \geq 0, \sum_M \lambda_M \chi^M \geq \mathbf{1}\}\rceil.$$

As we can directly extend this to a weighted version, one has:

Corollary 20.11c. *The polyhedron determined by* (20.23) *has the integer rounding property.*

Proof. See above. ∎

20.8. The perfect matching lattice of a bipartite graph

The *perfect matching lattice* (often briefly the *matching lattice*) of a graph $G = (V, E)$ is the lattice generated by the incidence vectors of perfect matchings in G; that is,

(20.27) $$L_{\text{perfect matching}}(G) := \text{lattice}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

With the help of Kőnig's edge-colouring theorem, it is not difficult to characterize the perfect matching lattice of a bipartite graph (cf. Lovász [1985]). Recall that the *perfect matching space* of a graph G is the linear hull of the incidence vectors of the perfect matchings in G (cf. Section 18.5d).

Theorem 20.12. *The perfect matching lattice of a bipartite graph $G = (V, E)$ is equal to the set of integer vectors in the perfect matching space of G.*

Proof. Obviously, each vector in the perfect matching lattice is integer and belongs to the perfect matching space. To see the reverse inclusion, let x be an integer vector in the perfect matching space. So $x_e = 0$ for each edge covered by no perfect matching, and $x(\delta(u)) = x(\delta(v))$ for all $u, v \in V$. By adding to x incidence vectors of perfect matchings, we can assume that $x_e \geq 0$ for all $e \in E$.

Replace any edge e by x_e parallel copies. We obtain a k-regular bipartite graph H, with $k := x(\delta(v))$ for any $v \in V$. Hence, by Kőnig's edge-colouring theorem, H is k-edge-colourable. As each colour is a perfect matching in H, we can decompose x as a sum of k incidence vectors of perfect matchings in G. So x belongs to the perfect matching lattice of G. ∎

This gives a characterization of the perfect matching lattice for matching-covered bipartite graphs (which will be used in the characterization of the perfect matching lattice of an arbitrary graph in Chapter 38). A graph is called *matching-covered* if each edge belongs to a perfect matching.

Corollary 20.12a. *Let $G = (V, E)$ be a matching-covered bipartite graph and let $x \in \mathbb{Z}^E$ be such that $x(\delta(u)) = x(\delta(v))$ for any two vertices u and v. Then x belongs to the perfect matching lattice of G.*

Proof. Directly from Theorems 20.12 and 18.7. ∎

By lattice duality theory, Theorem 20.12 is equivalent to the following.

Corollary 20.12b. *Let $G = (V, E)$ be a bipartite graph and let $w \in \mathbb{R}^E$ be a weight function. Then each perfect matching has integer weight if and only if there exists a vector $b \in \mathbb{R}^V$ with $b(V) = 0$ and with $w_e - b_u - b_v$ integer for each edge $e = uv$ covered by at least one perfect matching.*

Proof. Sufficiency is easy, since if such a b exists, then, for each perfect matching M,

$$w(M) = b(V) + \sum_{e=uv \in M} (w_e - b_u - b_v) = \sum_{e=uv \in M} (w_e - b_u - b_v) \tag{20.28}$$

is an integer.

To see necessity, suppose that $w(M)$ is integer for each perfect matching M. Then (by definition of dual lattice) w belongs to the dual lattice of the perfect matching lattice. Theorem 20.12 implies that the dual lattice is the sum of $\mathbb{Z}^E$ and the linear space orthogonal to the perfect matching space. So $w = w' + w''$, where $w' \in \mathbb{Z}^E$ and w'' is orthogonal to the perfect matching space; that is, $w''(M) = 0$ for each perfect matching M. By Corollary 18.7a, there exists a vector $b \in \mathbb{R}^V$ with $b(V) = 0$ and with $w''_e = b_u + b_v$ for each edge $e = uv$ covered by at least one perfect matching. This is equivalent to the present Corollary. ∎

20.9. Further results and notes

20.9a. Some further edge-colouring algorithms

As mentioned, it is easy to implement an $O(nm)$-time algorithm for finding a $\Delta(G)$-edge-colouring in a simple bipartite graph G. Such an algorithm also exists if G has multiple edges:

Theorem 20.13. *The edges of a bipartite graph G can be coloured with $\Delta(G)$ colours in $O(nm)$ time.*

Proof. Let $\Delta := \Delta(G)$. We update a collection of disjoint matchings $M_1, \ldots, M_\Delta$ (the colours), each stored as a doubly linked list. For each edge e, we keep the i for which $e \in M_i$ ($i = 0$ if e is in no M_i). Initially we set $M_i := \emptyset$ for $i := 1, \ldots, \Delta$. We also store the colour classes U and W as lists.

The algorithm runs along all pairs of vertices $u \in U$ and $w \in W$. Fixing $u \in U$ and $w \in W$, make a list L of edges e connecting u and w (taking $O(\deg(u))$ time, by scanning $\delta(u)$); define $d(u, w) := |L|$; make a list I of $d(u, w)$ indices i for which M_i misses u (taking $O(\deg(u))$ time, by scanning $\delta(u)$); make a list J of $d(u, w)$ indices j for which M_j misses w (taking $O(\deg(w))$ time, by scanning $\delta(w)$); next, while there is an edge e_0 in L:

(20.29) choose $i \in I$ and $j \in J$;
if $i = j$, insert e_0 in M_i, delete e_0 from L, and delete i from I and J;
if $i \neq j$, make for each $v \in V$ a list T_v of edges in $M_i \cup M_j$ incident with v (taking $O(n)$ time, by scanning M_i and M_j);
identify the path component P in $M_i \cup M_j$ starting at u (taking $O(n)$ time, using the T_v);
for each edge e on P, if e is in M_i move e to M_j and if e is in M_j we move e to M_i (taking $O(n)$ time);
insert e_0 in M_j, delete e_0 from L, delete i from I, and delete j from J.

Fixing u and w, the preprocessing takes $O(\deg(u) + \deg(w))$ time, and each of the $d(u, w)$ iterations takes $O(n)$ time. As $\sum_{u \in U} \sum_{w \in W} (\deg(u) + \deg(w) + nd(u, w)) = 2nm$, we obtain an algorithm as required. ∎

From their linear-time perfect matching algorithm for regular bipartite graphs, Cole, Ost, and Schirra [2001] derived (using an idea of Gabow [1976c]):

Theorem 20.14. *A k-regular bipartite graph $G = (V, E)$ can be k-edge-coloured in time $O(m \log k)$.*

Proof. We describe a recursive algorithm, the case $k = 1$ being the basis.

If k is even, find an Eulerian orientation of G, let G' be the $\frac{1}{2}k$-regular graph consisting of all edges oriented from one colour class of G to the other, let G'' be the $\frac{1}{2}k$-regular graph consisting of the remaining edges, and recursively $\frac{1}{2}k$-edge-colour G' and G''. This gives a k-edge-colouring of G.

If k is odd and ≥ 3, find a perfect matching M in G, and recursively $(k-1)$-edge-colour $G - M$. With M, this gives a k-edge-colouring of G.

We show that the running time is $O(m \log k)$. The recursive step takes time $O(m)$, since finding an Eulerian orientation or finding a perfect matching takes $O(m)$ time (Corollary 16.11a). Moreover, in one or two recursive steps, the graph is split into two graphs with half the number of edges. Since $m \log_2 k = m + 2(\frac{1}{2}m \log_2(\frac{1}{2}k))$, the result follows. ∎

Corollary 20.14a. *The edges of a bipartite graph G can be coloured with $\Delta(G)$ colours in $O(m \log \Delta(G))$ time.*

Proof. Let $k := \Delta(G)$. First iteratively merge any two vertices in the same colour class of G if each of them has degree at most $\frac{1}{2}k$. The final graph H will have at most two vertices of degree at most $\frac{1}{2}k$, and moreover, $\Delta(H) = k$ and any k-edge-colouring of H yields a k-edge-colouring of G. Next make a copy H' of H, and join each vertex v of H by $k - \deg_H(v)$ parallel edges with its copy v' in H' (where $\deg_H(v)$ is the degree of v in H). This gives the k-regular bipartite graph G', with $|EG'| = O(|EG|)$.

By Theorem 20.14, we can find a k-edge-colouring of G' in $O(m \log k)$ time. This gives a k-edge-colouring of H and hence a k-edge-colouring of G. ∎

20.9b. Complexity survey for bipartite edge-colouring

	$O(nm)$	Kőnig [1916]
	$O(\sqrt{n}\, m\Delta)$	Hopcroft and Karp [1971,1973] (cf. Gabow and Kariv [1978])
*	$O(\tilde{m}^2)$	Gonzalez and Sahni [1976]
	$O(\sqrt{n}\, m \log \Delta)$	Gabow [1976c]
	$O(m\sqrt{n \log n})$	Gabow and Kariv [1978]
	$O(m\Delta \log n)$	Gabow and Kariv [1978]
	$O((m + n^2) \log \Delta)$	Gabow and Kariv [1978,1982]
	$O(m(\log n)^2 \log \Delta)$	Lev, Pippenger, and Valiant [1981]
	$O(m(\log m)^2)$	Gabow and Kariv [1982]
	$O(m \log m)$	Cole and Hopcroft [1982]
*	$O(n\tilde{m} \log \mu)$	Gabow and Kariv [1982]
	$O((m + n \log n \log^2 \Delta) \log \Delta)$	Cole and Hopcroft [1982]
	$O((m + n \log n \log \Delta) \log \Delta)$	Cole [1982]
	$O(n2^{2^{O(\Delta)}})$	Cole [1982]
	$O((m + n \log n) \log \Delta)$	R. Cole and K. Ost (cf. Ost [1995]), Kapoor and Rizzi [2000]
	$O(m\Delta)$	Schrijver [1999]
	$O(m \log \Delta + n \log n \log \Delta)$	Rizzi [2002]
*	$O(m \log \Delta)$	Cole, Ost, and Schirra [2001]

Here $\tilde{m}$ denotes the number of parallel classes of edges, μ the maximum size of a parallel class, and $\varDelta$ the maximum degree. As before, $*$ indicates an asymptotically best bound in the table.

Kapoor and Rizzi [2000] showed that a bipartite graph of maximum degree $\varDelta$ can be $\varDelta$-edge-coloured in time $T + O(m \log \varDelta)$, where T is the time needed to find a perfect matching in a k-regular bipartite graph with m edges and $k \leq \varDelta$. (So this is applied only once!)

20.9c. List-edge-colouring

An interesting extension of Kőnig's edge-colouring theorem was shown by Galvin [1995], which was the 'list-edge-colouring conjecture' for bipartite graphs (cf. Alon [1993], Häggkvist and Chetwynd [1992]). It implies the conjecture of J. Dinitz (1979) that the list-edge-colouring number of the complete bipartite graph $K_{n,n}$ equals n. (This is in fact a special case of the conjecture, formulated by V.G. Vizing in 1975, that the list-edge-colouring number of any graph is equal to its edge-colouring number (see Häggkvist and Chetwynd [1992]).) The proof of Galvin is based on the Gale-Shapley theorem on stable matchings (Theorem 18.11).

Let $G = (V, E)$ be a graph. Then G is *k-list-edge-colourable* if for each choice of finite sets L_e for $e \in E$ with $|L_e| = k$, we can choose $l_e \in L_e$ for $e \in E$ such that $l_e \neq l_f$ if e and f are incident. The smallest k for which G is k-list-edge-colourable is called the *list-edge-colouring number* of G.

Trivially, the list-edge-colouring number of G is at least the edge-colouring number of G, and hence at least the maximum degree $\varDelta(G)$ of G. Galvin [1995] showed:

Theorem 20.15. *The list-edge-colouring number of a bipartite graph is equal to its maximum degree.*

Proof. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, and with maximum degree $k := \varDelta(G)$. The theorem follows by applying the following statement to any $\varDelta(G)$-edge-colouring $\phi : E \to \{1, \ldots, \varDelta(G)\}$ of G.

(20.30) Let $\phi : E \to \mathbb{Z}$ be such that $\phi(e) \neq \phi(f)$ if e and f are incident. For each $e = uw \in E$ with $u \in U$ and $w \in W$, let L_e be a finite set satisfying

$$|L_e| > |\{f \in \delta(u) \mid \phi(f) < \phi(e)\}| + |\{f \in \delta(w) \mid \phi(f) > \phi(e)\}|.$$

Then there exist $l_e \in L_e$ ($e \in E$) such that $l_e \neq l_f$ if e and f are incident.

So it suffices to prove (20.30), which is done by induction on $|E|$. Choose $p \in \bigcup L_e$ and let $F := \{e \in E \mid p \in L_e\}$. Define for each $v \in V$ a total order $<_v$ on $\delta_F(v)$ by:

(20.31)
$$e \leq_v f \iff \phi(e) \geq \phi(f), \text{ if } v \in U,$$
$$e \leq_v f \iff \phi(e) \leq \phi(f), \text{ if } v \in W,$$

for $e, f \in \delta_F(v)$. By the Gale-Shapley theorem (Theorem 18.11), F contains a stable matching M. So M is a matching such that for each $e \in F$ there is an $f \in M$ with $e \leq_v f$ for some $v \in e$. Hence for each edge $e = uw \in F \setminus M$, with $u \in U$ and

$w \in W$: $\exists f \in M \cap \delta(u) : \phi(f) < \phi(e)$ or $\exists f \in M \cap \delta(w) : \phi(f) > \phi(e)$. So removing M from E and resetting $L_e := L_e \setminus \{p\}$ for each $e \in F \setminus M$, we can apply induction. ∎

(The proof by Slivnik [1996] is similar.) An extension of Galvin's theorem was given by Borodin, Kostochka, and Woodall [1997].

20.9d. Further notes

Edge-colouring relates to timetabling — see Appleby, Blake, and Newman [1960], Gotlieb [1963], Broder [1964], Cole [1964], Csima and Gotlieb [1964], Barraclough [1965], Duncan [1965], Almond [1966], Lions [1966b,1966a,1967], Welsh and Powell [1967], Yule [1967], Dempster [1968,1971], Wood [1968], de Werra [1970,1972], and McDiarmid [1972].

However, most practical timetabling problems require more than just bipartite edge-colouring, and are NP-complete. It is NP-complete to decide if a given partial edge-colouring in a bipartite graph can be extended to a minimum edge-colouring (Even, Itai, and Shamir [1975,1976]). This corresponds to a timetabling problem with 'time windows'. Moreover, the 3-dimensional analogue is NP-complete (Karp [1972b]): given three disjoint sets R, S, and T and a family $\mathcal{F}$ of triples $\{r, s, t\}$ with $r \in R$, $s \in S$, and $t \in T$, colour the sets in $\mathcal{F}$ with a minimum number of colours in such a way that sets of the same colour are disjoint.

Analogues of Kőnig's edge-colouring theorem, in terms of odd paths packing and covering, were given by de Werra [1986,1987]. The edge-colouring number of *almost bipartite graphs* (graphs which have a vertex whose deletion makes the graph bipartite) was characterized by Eggan and Plantholt [1986] and Reed [1999b].

Kőnig [1916] also proved an infinite extension of Theorem 20.1. We refer to Section 16.7h for some historical notes on the fundamental paper Kőnig [1916].

Sainte-Laguë [1923] mentioned (without proof and without reference to Kőnig's work) the result that each k-regular bipartite graph is k-edge-colourable.

Chapter 21

Bipartite b-matchings and transportation

The total unimodularity of the incidence matrix of a bipartite graph leads to general min-max relations, for b-matchings, b-edge covers, w-vertex covers, w-stable sets, and b-factors. The weighted versions of these problems relate to the classical transportation problem.

In this chapter, graphs can be assumed to be simple.

21.1. b-matchings and w-vertex covers

Let $G = (V, E)$ be a graph, with $V \times E$ incidence matrix A. We introduce the concepts of b-matching and w-vertex cover, which will turn out to be dual.

For $b : V \to \mathbb{Z}_+$, a b-*matching* is a function $x : E \to \mathbb{Z}_+$ such that for each vertex v of G:

(21.1) $$x(\delta(v)) \leq b_v,$$

where $\delta(v)$ is the set of edges incident with v. In other words, x is a b-matching if and only if x is an integer vector satisfying $x \geq \mathbf{0}$, $Ax \leq b$. So if $b = \mathbf{1}$, then b-matchings are precisely the incidence vectors of matchings.

For $w : E \to \mathbb{Z}_+$, a w-*vertex cover* is a function $y : V \to \mathbb{Z}_+$ such that for each edge $e = uv$ of G:

(21.2) $$y_u + y_v \geq w_e.$$

In other words, y is a w-vertex cover if and only if y is an integer vector satisfying $y \geq \mathbf{0}$, $y^{\mathsf{T}} A \geq w^{\mathsf{T}}$. So if $w = \mathbf{1}$, then $\{0, 1\}$-valued w-vertex covers are precisely the incidence vectors of vertex covers.

b-matchings and w-vertex covers are related by the following LP-duality equation:

(21.3) $$\max\{w^{\mathsf{T}} x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A \geq w^{\mathsf{T}}\}.$$

Since A is totally unimodular (Theorem 18.2), both optima are attained by integer vectors. In other words (where the w-*weight* of a vector x equals $w^{\mathsf{T}} x$ and the b-*weight* of a vector y equals $y^{\mathsf{T}} b$):

Theorem 21.1. *Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$ and $w : E \to \mathbb{Z}_+$. Then the maximum w-weight of a b-matching is equal to the minimum b-weight of a w-vertex cover.*

Proof. See above. ∎

Taking $b = \mathbf{1}$, we obtain Corollary 17.1a. For $w = \mathbf{1}$, we get the following min-max relation for maximum-size b-matching (again, the sum of the entries in a vector is called its *size*):

Corollary 21.1a. *Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Then the maximum size of a b-matching is equal to the minimum b-weight of a vertex cover.*

Proof. This is the special case $w = \mathbf{1}$ of Theorem 21.1. ∎

An alternative way of proving this is by derivation from Kőnig's matching theorem: Split each vertex v into b_v copies, and replace each edge uv by $b_u b_v$ edges connecting the b_u copies of u with the b_v copies of v. (This construction is due to Tutte [1954b].)

Corollary 21.1a implies a characterization of the existence of a perfect b-matching. A b-matching is called *perfect* if equality holds in (21.1) for each vertex v. So a b-matching is perfect if and only if it has size $\frac{1}{2}b(V)$. Hence:

Corollary 21.1b. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$. Then there exists a perfect b-matching if and only if $b(C) \geq \frac{1}{2}b(V)$ for each vertex cover C.*

Proof. Directly from Corollary 21.1a. ∎

21.2. The b-matching polytope and the w-vertex cover polyhedron

The total unimodularity of the incidence matrix also implies characterizations of the corresponding polyhedra.

The *b-matching polytope* is the convex hull of the b-matchings. For bipartite graphs it is determined by:

$$
\text{(21.4)} \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq b_v & \text{for each } v \in V. \end{array}
$$

Theorem 21.2. *The b-matching polytope of a bipartite graph $G = (V, E)$ is determined by* (21.4)*.*

Proof. Directly from the facts that system (21.4) amounts to $x \geq \mathbf{0}, Ax \leq b$ and that A is totally unimodular, where A is the $V \times E$ incidence matrix of G. By Theorem 5.20, the vertices of the polytope $\{x \geq \mathbf{0} \mid Ax \leq b\}$ are integer, hence they are b-matchings. ∎

This generalizes the sufficiency part of Corollary 18.1b.

Similarly, the *w-vertex cover polyhedron*, being the convex hull of the w-vertex covers, is, for bipartite graphs, determined by:

(21.5) (i) $y_v \geq 0$ for each $v \in V$,
(ii) $y_u + y_v \geq w_e$ for each $e = uv \in E$.

Theorem 21.3. *The w-vertex cover polyhedron of a bipartite graph is determined by* (21.5).

Proof. Directly from the total unimodularity of the incidence matrix of a bipartite graph. ∎

This generalizes the necessity part in Theorem 18.3.

21.3. Simple b-matchings and b-factors

In the context of b-matchings, call a vector x *simple* if it is $\{0,1\}$-valued. So a simple b-matching is the incidence vector of a set F of edges with $\deg_F(v) \leq b_v$ for each vertex v. We will identify the vector and the subset.

To characterize the maximum size of a simple b-matching, let, for any $X \subseteq V$, $E[X]$ denote the set of edges spanned by X.

Theorem 21.4. *The maximum size of a simple b-matching in a bipartite graph $G = (V, E)$ is equal to the minimum value of $b(V \setminus X) + |E[X]|$ taken over $X \subseteq V$.*

Proof. This can be reduced to the nonsimple case by replacing each edge uv by a path of length 3 connecting u and v (thus introducing two new vertices for each edge), and extending b by defining $b(s) := 1$ for each new vertex s. Then the maximum size of a simple b-matching in the original graph is equal to the maximum size of a b-matching in the new graph minus $|E|$, and we can apply Corollary 21.1a. ∎

(This construction is due to Tutte [1954b].)

The theorem can also be derived from the fact that both optima in the LP-duality equation:

(21.6)
$$\max\{\mathbf{1}^{\mathsf{T}}x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \leq b\}$$
$$= \min\{y^{\mathsf{T}}b + z^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^{\mathsf{T}}A + z^{\mathsf{T}} \geq \mathbf{1}^{\mathsf{T}}\}$$

have integer optimum solutions, since A (the incidence matrix of G) is totally unimodular.

Theorem 21.4 implies the following result of Ore [1956] (who formulated it in terms of directed graphs). A *b-factor* is a simple perfect b-matching. So it is a subset F of E with $\deg_F(v) = b_v$ for each $v \in V$ (again identifying a subset of E with its incidence vector in $\mathbb{R}^E$).

Corollary 21.4a. *Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Then G has a b-factor if and only if each subset X of V spans at least $b(X) - \frac{1}{2}b(V)$ edges.*

Proof. Directly from Theorem 21.4. ∎

If b is equal to a constant k, Theorem 21.4 amounts to (with the help of Kőnig's edge-colouring theorem):

Corollary 21.4b. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k matchings is equal to the minimum value of $k|V \setminus X| + |E[X]|$ taken over $X \subseteq V$.*

Proof. Apply Theorem 21.4 to $b_v := k$ for all $v \in V$. We obtain a formula for the maximum size of a subset F of E with $\deg_F(v) \leq k$ for all $v \in V$. By Theorem 20.1, this is the union of k matchings. ∎

A *k-factor* in a graph $G = (V, E)$ is a subset F of E with $\deg_F(v) = k$ for each $v \in V$. Then:

Corollary 21.4c. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then G has a k-factor if and only if each subset X of V spans at least $k(|X| - \frac{1}{2}|V|)$ edges.*

Proof. Directly from Corollary 21.4a. ∎

From this one can derive the result of Fulkerson [1964b] (Corollary 20.9a) that a bipartite graph has k disjoint perfect matchings if and only if each subset X of V spans at least $k(|X| - \frac{1}{2}|V|)$ edges.

By the total unimodularity of the incidence matrix of bipartite graphs, the *simple b-matching polytope* (the convex hull of the simple b-matchings) of a bipartite graph $G = (V, E)$ is determined by:

$$(21.7) \qquad \begin{array}{ll} 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ x(\delta(v)) \leq b_v & \text{for each } v \in V. \end{array}$$

Similarly, the following min-max relation for maximum-weight simple b-matching follows (Vogel [1963]):

Theorem 21.5. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$. Then the maximum weight $w^{\mathsf{T}}x$ of a simple b-matching x is equal to the minimum value of*

(21.8) $$\sum_{v \in V} y_v b_v + \sum_{e \in E} z_e$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^E$ with $y_u + y_v + z_e \geq w_e$ for each edge $e = uv$.

Proof. Directly from the LP-duality equation

(21.9) $$\begin{aligned} &\max\{w^\mathsf{T} x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \leq b\} \\ &= \min\{y^\mathsf{T} b + z^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^\mathsf{T} A + z^\mathsf{T} \geq w^\mathsf{T}\} \end{aligned}$$

(where A is the $V \times E$ incidence matrix of G), using the total unimodularity of A. ∎

Moreover:

Theorem 21.6. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a b-factor x is equal to the maximum value of*

(21.10) $$\sum_{v \in V} y_v b_v + \sum_{e \in E} z_e$$

where $y \in \mathbb{Z}^V$ and $z \in \mathbb{Z}_+^E$ with $y_u + y_v - z_e \leq w_e$ for each edge $e = uv$.

Proof. Directly from the LP-duality equation

(21.11) $$\begin{aligned} &\min\{w^\mathsf{T} x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax = b\} \\ &= \max\{y^\mathsf{T} b - z^\mathsf{T}\mathbf{1} \mid z \geq \mathbf{0}, y^\mathsf{T} A - z^\mathsf{T} \leq w^\mathsf{T}\} \end{aligned}$$

(where A is the $V \times E$ incidence matrix of G), using the total unimodularity of A. ∎

Notes. Hartvigsen [1999] gave a characterization of the convex hull of square-free simple 2-matching in a bipartite graph. (A 2-*matching* is a b-matching with $b = \mathbf{2}$. A simple 2-matching is *square-free* if it contains no circuit of length 4.) It implies that a maximum-weight square-free 2-matching in a bipartite graph can be found in strongly polynomial time.

21.4. Capacitated *b*-matchings

If we require that a b-matching x satisfies $x \leq c$ for some 'capacity' function $c : E \to \mathbb{Z}_+$, we speak of a *capacitated* b-matching. So simple b-matchings correspond to capacitated b-matchings for $c = \mathbf{1}$.

Theorem 21.7. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then the maximum size of a b-matching $x \leq c$ is equal to*

(21.12) $$\min_{X \subseteq V} b(V \setminus X) + c(E[X]).$$

Proof. The proof is similar to that of Theorem 21.4. Now we define $b(s) := c_e$ if s is a new vertex on the path connecting the end vertices of e. ∎

Alternatively, we can reduce this theorem to Theorem 21.4, by replacing each edge e by c_e parallel edges, or we can use total unimodularity similarly to (21.6).

Again we have the perfect case as direct consequence:

Corollary 21.7a. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then there exists a perfect b-matching $x \leq c$ if and only if*

(21.13) $$c(E[X]) \geq b(X) - \tfrac{1}{2}b(V)$$

for each $X \subseteq V$.

Proof. Directly from Theorem 21.7. ∎

Again, by the total unimodularity of the incidence matrix of bipartite graphs, the *c-capacitated b-matching polytope* (the convex hull of the b-matchings $x \leq c$) of a bipartite graph $G = (V, E)$ is determined by:

(21.14) $$\begin{aligned} 0 \leq x_e &\leq c_e && \text{for each } e \in E, \\ x(\delta(v)) &\leq b_v && \text{for each } v \in V. \end{aligned}$$

Similarly, the following min-max relation for maximum-weight capacitated b-matching follows:

Theorem 21.8. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $w, c \in \mathbb{Z}_+^E$. Then the maximum weight $w^\mathsf{T} x$ of a b-matching $x \leq c$ is equal to the minimum value of*

(21.15) $$\sum_{v \in V} y_v b_v + \sum_{e \in E} z_e c_e$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^E$ satisfy $y_u + y_v + z_e \geq w_e$ for each edge $e = uv$.

Proof. Directly from the LP-duality equation

(21.16) $$\begin{aligned} &\max\{w^\mathsf{T} x \mid \mathbf{0} \leq x \leq c, Ax \leq b\} \\ &= \min\{y^\mathsf{T} b + z^\mathsf{T} c \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^\mathsf{T} A + z^\mathsf{T} \geq w^\mathsf{T}\} \end{aligned}$$

(where A is the $V \times E$ incidence matrix of G), using the total unimodularity of A. ∎

21.5. Bipartite b-matching and w-vertex cover algorithmically

Algorithmically, optimization problems on b-matchings and w-vertex covers in bipartite graphs can be reduced to minimum-cost flow problems, and hence can be solved in strongly polynomial time.

Theorem 21.9. *Given a bipartite graph $G = (V, E)$, $b : V \to \mathbb{Z}_+$, $c : E \to \mathbb{Z}_+$, and $w : E \to \mathbb{Q}$, a b-matching $x \le c$ maximizing $w^\mathsf{T} x$ can be found in strongly polynomial time. Similarly, a perfect b-matching $x \le c$ minimizing $w^\mathsf{T} x$ can be found in strongly polynomial time.*

Proof. Let S and T be the colour classes of G, and orient the edges of G from S to T, giving the digraph D. Then b-matchings in G correspond to integer z-transshipments in D with $0 \le z(v) \le b(v)$ if $v \in T$ and $-b(v) \le z(v) \le 0$ if $v \in S$. Perfect b-matchings correspond to integer b'-transshipments, where $b'(v) := -b(v)$ if $v \in S$ and $b'(v) := b(v)$ if $v \in T$. Hence this theorem follows from Corollary 12.2d. ∎

Wagner [1958] (cf. Dantzig [1955]) observed that the capacitated version of the minimum-weight perfect b-matching problem can be reduced to the uncapacitated version, by a construction similar to that used in proving Theorem 21.4.

One similarly has for w-vertex covers:

Theorem 21.10. *Given a bipartite graph $G = (V, E)$, $b : V \to \mathbb{Q}_+$, $c : V \to \mathbb{Z}_+$, and $w : E \to \mathbb{Z}_+$, a w-vertex cover $y \le c$ minimizing $y^\mathsf{T} b$ can be found in strongly polynomial time.*

Proof. By reduction to Corollary 12.2e. ∎

Although these results suggest a symmetry between matchings and vertex covers, we mention here that the nonbipartite version of Theorem 21.9 holds true (Section 32.4), but that finding a maximum-size stable set in a nonbipartite graph is NP-complete (see Section 64.2).

21.6. Transportation

The minimum-weight perfect b-matching problem is close to the classical transportation problem. Given a bipartite graph $G = (V, E)$ and a vector $b \in \mathbb{R}^V_+$, a *b-transportation* is a vector $x \in \mathbb{R}^E_+$ with

$$(21.17) \qquad x(\delta(v)) = b_v$$

for each $v \in V$. So a b-transportation is a fractional version of a perfect b-matching. Integer b-transportations are exactly the perfect b-matchings.

The following characterization of the existence of a b-transportation was shown (in a much more general form) by Rado [1943] — compare Corollary 21.1b:

Theorem 21.11. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{R}^V_+$. Then there exists a b-transportation if and only if $b(C) \ge \frac{1}{2} b(V)$ for each vertex cover C.*

Proof. Since the inequalities $b(C) \geq \frac{1}{2}b(V)$ (for vertex covers C), define a rational polyhedral cone, we can assume that b is rational, and hence, by scaling, that b is integer. Then the theorem follows from Corollary 21.1b. ∎

Note that, trivially, there exists a b-transportation if and only if b belongs to the convex cone in $\mathbb{R}^V$ generated by the incidence vectors of the edges of G. So Theorem 21.11 characterizes this cone.

A negative cycle criterion follows directly from the corresponding criterion for transshipments. For any b-transportation x in a bipartite graph $G = (V, E)$ and any cost function $c : E \to \mathbb{R}$, make the directed graph $D_x = (V, A)$ as follows. Let U and W be the colour classes of G. For each edge $e = uv$ of G, with $u \in U$ and $v \in W$, let A have an arc (u, v) of cost c_e, and, if $x_e > 0$, an arc (v, u).of cost $-c_e$. Then (Tolstoĭ [1930]):

Theorem 21.12. *x is a minimum-cost b-transportation if and only if D_x has no negative-cost directed circuits.*

Proof. Directly from Theorem 12.3. ∎

Transportations in a complete bipartite graph can be formulated in terms of matrices. Fixing vectors $a \in \mathbb{R}^m_+$ and $b \in \mathbb{R}^n_+$, an $m \times n$ matrix $X = (x_{i,j})$ is called a *transportation* if

$$\text{(21.18)} \qquad \begin{array}{lll} \text{(i)} & x_{i,j} \geq 0 & i = 1, \ldots, m; j = 1, \ldots, n, \\ \text{(ii)} & \displaystyle\sum_{j=1}^{n} x_{i,j} = a_i & i = 1, \ldots, m, \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m} x_{i,j} = b_j & j = 1, \ldots, n. \end{array}$$

Clearly, a transportation exists if and only if $\sum_i a_i = \sum_j b_j$.

Given an $m \times n$ 'cost' matrix $C = (c_{i,j})$, the cost of a transportation $X = (x_{i,j})$ is defined as $\sum_{i,j} c_{i,j} x_{i,j}$. Then the *transportation problem* (also called the *Hitchcock-Koopmans transportation problem*) is:

(21.19) given: vectors $a \in \mathbb{Q}^m_+$, $b \in \mathbb{Q}^n_+$ and an $m \times n$ 'cost' matrix $C = (c_{i,j})$,
find: a minimum-cost transportation.

So it is equivalent to solving the LP problem of minimizing $\sum_{i,j} c_{i,j} x_{i,j}$ over (21.18). The transportation problem formed a major impulse to introduce linear programming. Hitchcock [1941] and Dantzig [1951a] showed that the simplex method applies to the transportation problem.

The transportation problem is also a special case of the minimum-cost b-transshipment problem, and hence can be solved with the methods of Chapter 12. In particular, it is solvable in strongly polynomial time.

Linear programming also yields a min-max relation, originally due to Hitchcock [1941] (also implicit in Kantorovich [1939]):

Theorem 21.13 (Hitchcock's theorem)**.** *The minimum cost of a transportation is equal to the maximum value of $y^{\mathsf{T}}a + z^{\mathsf{T}}b$, where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ such that $y_i + z_j \leq c_{i,j}$ for all i, j.*

Proof. This is LP-duality. ∎

(Hitchcock [1941] gave a direct proof.)

The transportation problem differs from the minimum-weight perfect b-matching problem in having a complete bipartite graph $K_{m,n}$ as underlying bipartite graph and in not requiring integrality of the output. This last however is not a restriction, as Dantzig [1951a] showed:

Theorem 21.14. *If a and b are integer, the transportation problem has an integer optimum solution x.*

Proof. Directly from the total unimodularity of the matrix underlying the system (21.18), which is the incidence matrix of the complete bipartite graph $K_{m,n}$. ∎

For a different proof, see the proof of Corollary 21.15a below.

Notes. Ford and Fulkerson [1955,1957b], Gleyzal [1955], Munkres [1957], and Egerváry [1958] described primal-dual methods for the transportation problem, and Ford and Fulkerson [1956a,1957a] extended it to the capacitated version.

If the a_i and b_j are small integers, the transportation problem can be reduced to the assignment problem, by 'splitting' each i into a_i or b_i copies. (This observation is due to Egerváry [1958], and in a different context to Tutte [1954b].)

21.6a. Reduction of transshipment to transportation

It is direct to transform a transportation problem to a transshipment problem. Orden [1955] observed a reverse reduction (similar to the reduction described in Section 16.7c). Indeed, let input $D = (V, A)$, $b \in \mathbb{R}^V$ and $k \in \mathbb{R}^A$ for the transshipment problem be given. Split each vertex v into two vertices v', v'' and replace each arc (u, v) by an arc (u', v''), with cost $k(u, v)$. Moreover, add arcs (v', v''), each with cost 0. Let $N := \sum_{v \in V} |b(v)|$. Define $b'(v') := -N$ and $b'(v'') := b(v) + N$. Then a minimum-cost b'-transshipment in the new structure gives a minimum-cost b-transshipment in the original structure. Since the new graph is bipartite with all edges oriented from one colour class to the other, we have a reduction to the transportation problem.

(Orden [1955] also gave an alternative reduction of the transshipment problem to the transportation problem. Let A' be the set of pairs (u, v) with $b_u < 0$ and $b_v > 0$ and with v is reachable in D from U. For each $(u, v) \in A'$, let $k'(u, v)$ be the length of a shortest $u - v$ path in D, taking k as length function. Then the (bipartite) transshipment problem for $D' := (V, A')$, b, and k' is equivalent to the original transshipment problem.)

Fulkerson [1960] gave the following reduction of the *capacitated* transshipment problem to the uncapacitated transportation problem. Let be given directed graph

$D = (V, A)$, $b \in \mathbb{R}^V$, a 'capacity' function $c \in \mathbb{R}^A$, and a 'cost' function $k \in \mathbb{R}^A$. Define $V' := V \cup A$ and $E' := \{\{a, v\} \mid a = (v, u) \text{ or } a = (u, v)\}$. Define $w(\{a, v\}) := k(a)$ if v is head of a, and $:= 0$ if v is tail of a. Let $b'(a) := c(a)$ and $b'(v) := b(v) + c(\delta^{\text{out}}(v))$. Then a minimum-cost b-transshipment subject to c corresponds to a minimum-cost b'-transportation. (More can be found in Wagner [1958].)

21.6b. The transportation polytope

Given $a \in \mathbb{R}^m_+$ and $b \in \mathbb{R}^n_+$, the *transportation polytope* is the set of all matrices $X = (x_{i,j})$ in $\mathbb{R}^{m \times n}$ satisfying (21.18). The transportation polytope was first studied by Hitchcock [1941]. The following result is due to Dantzig [1951a].

Theorem 21.15. *Let $X = (x_{i,j})$ belong to the transportation polytope. Then X is a vertex of the transportation polytope if and only if the set $F := \{ij \mid x_{i,j} > 0\}$ forms a forest in the complete bipartite graph $K_{m,n}$.*

Proof. If F contains a circuit $C = (i_0, j_1, i_1, j_2, i_2, \ldots, j_k, i_k)$, with $i_k = i_0$, define $Y = (y_{i,j})$ by: $y_{i,j} := 1$ if $(i, j) = (i_h, j_h)$ for some $h = 1, \ldots, k$, $y_{i,j} := -1$ if $(i, j) = (i_{h-1}, j_h)$ for some $h = 1, \ldots, k$, and $y_{i,j} := 0$ for all other (i, j). Then $X + \varepsilon Y$ belongs to the transportation polytope for any ε close enough to 0 (positive or negative), and hence X is not a vertex of the transportation polytope.

Conversely, if X is not a vertex of the transportation polytope, there exists a nonzero matrix $Y = (y_{i,j})$ such that $X + \varepsilon Y$ is in the transportation polytope for any ε close enough to 0 (positive or negative). Then Y satisfies $\sum_{j=1}^n y_{i,j} = 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m y_{i,j} = 0$ for $j = 1, \ldots, n$. Since Y is nonzero, the set $F' := \{ij \mid y_{i,j} \neq 0\}$ contains a circuit. Since $F' \subseteq F$, it implies that F contains a circuit. ∎

This gives:

Corollary 21.15a. *If a and b are integer vectors, the transportation polytope is an integer polyhedron.*

Proof. By Theorem 21.15, for any vertex $X = (x_{i,j})$ of the transportation polytope, the set of pairs (i, j) with $x_{i,j}$ not an integer is a forest. Hence, if it is nonempty, this forest has an end edge, say (i, j). Assume without loss of generality that i has degree 1 in this forest. Then $x_{i,j}$ is equal to a_i minus $\sum_{j' \neq j} x_{i,j'}$, which is an integer as a_i and each of the $x_{i,j'}$ $(j' \neq j)$ is an integer. ∎

The dimension of the transportation polytope is easy to determine (Koopmans and Reiter [1951], Dulmage and Mendelsohn [1962], Klee and Witzgall [1968]):

Theorem 21.16. *If $a > 0$ and $b > 0$, the dimension of the transportation polytope is equal to $(m - 1)(n - 1)$.*

Proof. Let $X = (x_{i,j})$ be a vector in the relative interior of the transportation polytope. So $x_{i,j} > 0$ for all i, j. For each (i, j) with $i \in \{1, \ldots, m - 1\}$ and $j \in \{1, \ldots, n - 1\}$, we can correct any small perturbation of $x_{i,j}$ by a unique change of the $x_{i,n}$ and $x_{m,j}$. So the dimension of the transportation polytope is $(m-1)(n-1)$. ∎

Notes. Balinski [1974] (cf. Balinski and Rispoli [1993]) showed the Hirsch conjecture for some classes of transportation polytopes. For counting and estimating the number of vertices of transportation polytopes, see Simonnard and Hadley [1959], Demuth [1961], Wintgen [1964], Szwarc and Wintgen [1965], Klee and Witzgall [1968], Bolker [1972], and Ahrens [1981]. For counting facets, see Klee and Witzgall [1968].

Given $C = (c_{i,j}) \in \mathbb{R}^{m\times n}$, the *dual transportation polyhedron* is the set of all vectors $(u; v) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfying[38]:

(21.20) $$\begin{aligned} &u_1 = 0 \\ &u_i + v_j \geq c_{i,j} \quad i = 1,\ldots,m; j = 1,\ldots,n. \end{aligned}$$

(The condition $u_1 = 0$ is added for normalization.) It is easy to see that the dimension of the dual transportation polyhedron is $m + n - 1$, and that $(u; v)$ satisfying (21.20) is a vertex of the dual transportation polyhedron if and only if the graph with vertex set $\{p_1,\ldots,p_m,q_1,\ldots,q_n\}$ and edge set $\{\{p_i, q_j\} \mid u_i + v_j = c_{i,j}\}$ is connected.

Balinski [1984] showed with the 'signature method' that the diameter of the dual transportation polyhedron is at most $(m-1)(n-1)$, thus proving the Hirsch conjecture for this class of polyhedra.

Balinski and Russakoff [1984] characterized vertices and higher-dimensional faces of dual transportation polyhedra. More can be found in Zhu [1963], Balinski [1983], and Kleinschmidt, Lee, and Schannath [1987].

21.7. b-edge covers and w-stable sets

Exchanging $\leq$ and $\geq$ appropriately in the definitions of b-matchings and w-vertex covers gives the b-edge covers and the w-stable sets. These concepts again turn out to be each others dual.

Let $G = (V, E)$ be a graph, with $V \times E$ incidence matrix A. For $b : V \to \mathbb{Z}_+$, a *b-edge cover* is a function $x : E \to \mathbb{Z}_+$ such that for each vertex v of G:

(21.21) $$x(\delta(v)) \geq b_v.$$

In other words, x is a b-edge cover if and only if x is an integer vector satisfying $x \geq \mathbf{0}$, $Ax \geq b$. So if $b = \mathbf{1}$, then $\{0, 1\}$-valued b-edge covers are precisely the incidence vectors of edge covers.

For $w : E \to \mathbb{Z}_+$, a *w-stable set* is a function $y : V \to \mathbb{Z}_+$ such that for each edge $e = uv$ of G:

(21.22) $$y_u + y_v \leq w_e.$$

[38] We write $(u; v)$ for $\binom{u}{v}$.

In other words, y is a w-stable set if and only if y is an integer vector satisfying $y \geq \mathbf{0}$, $y^{\mathsf{T}}A \leq w^{\mathsf{T}}$. So if $w = \mathbf{1}$, then $\{0,1\}$-valued w-stable sets are precisely the incidence vectors of stable sets.

In this case, b-edge covers and w-stable sets are related by the following LP-duality equation:

(21.23) $$\min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \geq b\} = \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \leq w^{\mathsf{T}}\}.$$

Since A is totally unimodular (Theorem 18.2), both optima are attained by integer vectors. This gives (where the *w-weight* of a vector x equals $w^{\mathsf{T}}x$ and the *b-weight* of a vector y equals $y^{\mathsf{T}}b$):

Theorem 21.17. *Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$ and $w : E \to \mathbb{Z}_+$. Then the minimum w-weight $w^{\mathsf{T}}x$ of a b-edge cover x is equal to the maximum b-weight of a w-stable set.*

Proof. See above. ∎

Taking $b = \mathbf{1}$, we obtain Corollary 19.5a. For $w = \mathbf{1}$, we get a min-max relation for minimum-size b-edge cover:

Corollary 21.17a. *Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Then the minimum size of a b-edge cover is equal to the maximum b-weight of a stable set.*

Proof. This is the special case $w = \mathbf{1}$ of Theorem 21.17. ∎

Again, an alternative way of proving this is by derivation from the Kőnig-Rado edge cover theorem (Theorem 19.4): Split each vertex v into b_v copies, replace each edge uv by $b_u b_v$ edges connecting the b_u copies of u with the b_v copies of v.

21.8. The b-edge cover and the w-stable set polyhedron

The total unimodularity of the incidence matrix of a bipartite graph also gives descriptions of the corresponding polyhedra.

The *b-edge cover polyhedron* is the convex hull of the b-edge covers. For bipartite graphs it is determined by:

(21.24)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(\delta(v)) \geq b_v$ for each $v \in V$.

Theorem 21.18. *The b-edge cover polyhedron of a bipartite graph $G = (V, E)$ is determined by* (21.24).

Proof. Directly from the facts that system (21.24) amounts to $x \geq \mathbf{0}, Ax \geq b$ and that A is totally unimodular. ∎

This extends Theorem 19.6 on the edge cover polytope.

Similarly, the *w-stable set polyhedron*, being the convex hull of the w-stable sets, is, for bipartite graphs, determined by:

(21.25) (i) $y_v \geq 0$ for each $v \in V$,
(ii) $y_u + y_v \leq w_e$ for each $e = uv \in E$.

Theorem 21.19. *The w-stable set polyhedron of a bipartite graph is determined by* (21.25)*.*

Proof. Directly from the total unimodularity of the incidence matrix of a bipartite graph. ∎

This generalizes the necessity part of Theorem 19.7.

21.9. Simple b-edge covers

Again, call a vector x *simple* if it is $\{0,1\}$-valued. Then a simple b-edge cover corresponds to a set F of edges with $\deg_F(v) \geq b_v$ for each $v \in V$. We will identify the vector and the set. Note that a simple b-edge cover can exist only if $b_v \leq \deg(v)$ for each vertex v.

It is easy to derive the following min-max relation for simple b-edge covers from Theorem 21.4 on the maximum size of a simple b-matching ($E[X]$ denote the set of edges spanned by X):

Theorem 21.20. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ with $b_v \leq \deg(v)$ for each vertex v. Then the minimum size of a simple b-edge cover in G is equal to the maximum value of $b(X) - |E[X]|$ taken over $X \subseteq V$.*

Proof. Define $b'(v) := \deg(v) - b(v)$ for each vertex v. Then a subset F of E is a simple b-edge cover if and only if $E \setminus F$ is a simple b'-matching. By Theorem 21.4, the maximum size of a simple b'-matching is equal to the minimum value of $b'(V \setminus X) + |E[X]|$ taken over $X \subseteq V$. Hence the minimum size of a simple b-edge cover is equal to the maximum value of

$$
\begin{aligned}
(21.26) \qquad |E| - b'(V \setminus X) - |E[X]| &= |E| - \sum_{v \in V \setminus X} (\deg(v) - b(v)) - |E[X]| \\
&= |E| - 2|E[V \setminus X]| - |\delta(X)| + b(V \setminus X) - |E[X]| \\
&= b(V \setminus X) - |E[V \setminus X]|,
\end{aligned}
$$

taken over $X \subseteq V$. ∎

Alternatively, the theorem follows from the fact that both optima in the LP-duality equation (where A is the $V \times E$ incidence matrix of G):

$$
\begin{aligned}
(21.27) \qquad &\min\{\mathbf{1}^{\mathsf{T}} x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \geq b\} \\
&= \max\{y^{\mathsf{T}} b - z^{\mathsf{T}} \mathbf{1} \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^{\mathsf{T}} A - z^{\mathsf{T}} \leq \mathbf{1}^{\mathsf{T}}\}
\end{aligned}
$$

have integer optimum solutions, since A is totally unimodular.

If b is equal to a constant k, Theorem 21.20 amounts to (with the help of the edge cover variant of Kőnig's edge-colouring theorem (Theorem 20.5)):

Corollary 21.20a. *Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_+$. Then the minimum size of the union of k disjoint edge covers is equal to the maximum value of $k|X| - |E[X]|$ taken over $X \subseteq V$.*

Proof. Apply Theorem 21.20 to $b_v := k$ for all $v \in V$. We obtain a formula for the maximum size of a subset F of E with $\deg_F(v) \geq k$ for all $v \in V$. By Theorem 20.5, F is the union of k disjoint edge covers. ∎

By the total unimodularity of the incidence matrix of bipartite graphs, the *simple b-edge cover polytope* (the convex hull of the simple b-edge covers) of a bipartite graph $G = (V, E)$ is determined by:

(21.28)
$$\begin{aligned} 0 \leq x_e \leq 1 &\quad \text{for each } e \in E, \\ x(\delta(v)) \geq b_v &\quad \text{for each } v \in V. \end{aligned}$$

LP-duality also gives a min-max formula for the minimum weight of simple b-edge covers:

Theorem 21.21. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^{\mathsf{T}}x$ of a simple b-edge cover x is equal to the maximum value of*

(21.29)
$$\sum_{v \in V} y_v b_v - \sum_{e \in E} z_e$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^E$ with $y_u + y_v - z_e \leq w_e$ for each edge $e = uv$.

Proof. Directly from the LP-duality equation

(21.30)
$$\begin{aligned} &\min\{w^{\mathsf{T}}x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \geq b\} \\ &= \max\{y^{\mathsf{T}}b - z^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^{\mathsf{T}}A - z^{\mathsf{T}} \leq w^{\mathsf{T}}\} \end{aligned}$$

(where A is the $V \times E$ incidence matrix of G), using the total unimodularity of A. ∎

21.10. Capacitated b-edge covers

If we require that a b-edge cover x satisfies $x \leq c$ for some 'capacity' function $c : E \to \mathbb{Z}_+$, we speak of a *capacitated* b-edge cover. So simple b-edge covers correspond to capacitated b-edge covers with $c = \mathbf{1}$.

Theorem 21.22. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ with $c(\delta(v)) \geq b_v$ for each $v \in V$. Then the minimum size of a b-edge cover $x \leq c$ is equal to*

(21.31) $$\max_{X \subseteq V} b(X) - c(E[X]).$$

Proof. The proof is similar to that of Theorem 21.20. ∎

Alternatively, we can reduce this theorem to Theorem 21.20, by replacing each edge e by c_e parallel edges, or we can use total unimodularity similarly to (21.27).

Theorem 21.23. *Let $G = (V, E)$ be a bipartite graph and let $b \in \mathbb{Z}_+^V$ and $c, w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a b-edge cover $x \leq c$ is equal to the maximum value of*

(21.32) $$\sum_{v \in V} y_v b_v - \sum_{e \in E} z_e c_e$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^E$ with $y_u + y_v - z_e \leq w_e$ for each edge $e = uv$.

Proof. Directly from the LP-duality equation

(21.33) $$\begin{aligned} &\min\{w^\mathsf{T} x \mid \mathbf{0} \leq x \leq c, Ax \geq b\} \\ &= \max\{y^\mathsf{T} b - z^\mathsf{T} c \mid y \geq \mathbf{0}, z \geq \mathbf{0}, y^\mathsf{T} A - z^\mathsf{T} \leq w^\mathsf{T}\} \end{aligned}$$

(where A is the $V \times E$ incidence matrix of G), using the total unimodularity of A. ∎

By the total unimodularity of the incidence matrix of G, the convex hull of b-edge covers $x \leq c$ of a bipartite graph G is determined by the inequalities

(21.34) $$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq c_e & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \geq b_v & \text{for each } v \in V. \end{array}$$

21.11. Relations between *b*-matchings and *b*-edge covers

Like for matchings and edge covers, there is also a close relation between maximum-size b-matchings and minimum-size b-edge covers, as was shown by Gallai [1959a]. This gives a connection between Corollaries 21.1a and 21.17a.

Let $G = (V, E)$ be an undirected graph without isolated vertices, and let $b \in \mathbb{Z}_+^V$. Define:

(21.35) $$\begin{aligned} &\nu_b(G) := \text{the maximum size of a } b\text{-matching}, \\ &\rho_b(G) := \text{the minimum size of a } b\text{-edge cover}. \end{aligned}$$

Theorem 21.24. *Let $G = (V, E)$ be an undirected graph without isolated vertices, and let $b \in \mathbb{Z}_+^V$. Then*

(21.36) $$\nu_b(G) + \rho_b(G) = b(V).$$

Proof. This can be reduced to Gallai's theorem (Theorem 19.1), by splitting each vertex v into b_v copies, and replacing each edge $e = uv$ by $b_u b_v$ edges connecting the b_u copies of u with the b_v copies of v. ∎

A direct proof of the previous theorem is given in the proof of the following theorem, also due to Gallai [1959a]:

Theorem 21.25. *Let $G = (V, E)$ be an undirected graph and let $b \in \mathbb{Z}_+^V$. Then for each maximum-size b-matching x there is a minimum-size b-edge cover y with $x \leq y$. Conversely, for each minimum-size b-edge cover y there is a maximum-size b-matching x with $x \leq y$.*

Proof. Let x be a maximum-size b-matching. For each vertex v of G, increase the value of x on some edge incident with v, by $b_v - x(\delta(v))$. We obtain a b-edge cover y satisfying

(21.37) $$y(E) = x(E) + \sum_{v \in V} (b_v - x(\delta(v))) = b(V) - x(E).$$

Conversely, let y be a minimum-size b-edge cover. For each vertex v of G, decrease the value of y on edges incident with v, by a total amount of $y(\delta(v)) - b_v$ (as long as $y \geq \mathbf{0}$). We obtain a b-matching x satisfying

(21.38) $$x(E) \geq y(E) - \sum_{v \in V} (y(\delta(v)) - b_v) = b(V) - y(E).$$

(21.37) and (21.38) imply that the y (x, respectively) obtained from x (y, respectively) is optimum, thus showing the theorem, and also showing (21.36). ∎

In a bipartite graph, a minimum-size b-edge cover and a maximum-weight stable set can be found in strongly polynomial time, by reduction to Theorem 21.9:

Corollary 21.25a. *Given a bipartite graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, a minimum-size b-edge cover and a maximum b-weight stable set can be found in strongly polynomial time.*

Proof. Since stable sets are exactly the complements of vertex covers, finding a maximum b-weight stable sets is directly reduced to finding a minimum b-weight vertex cover. The construction given in the proof of Theorem 21.25 implies that a maximum-size b-matching gives a minimum-size b-edge cover in polynomial time. So Theorem 21.9 gives the present corollary. ∎

Moreover, for the weighted case:

Theorem 21.26. *A minimum-weight capacitated b-edge cover in a bipartite graph can be found in strongly polynomial time.*

Proof. Directly from Corollary 12.2d, by orienting the edges from one colour class to the other. ∎

21.12. Upper and lower bounds

We finally consider upper *and* lower bounds. That is, for a graph $G = (V, E)$ and $a, b \in \mathbb{R}^V$ and $d, c \in \mathbb{R}^E$, we consider vectors $x \in \mathbb{R}^E$ satisfying:

(21.39) (i) $d_e \leq x_e \leq c_e$ for each $e \in E$,
(ii) $a_v \leq x(\delta(v)) \leq b_v$ for each $v \in V$,

If integer, x is both a b-matching and an a-edge cover.

The optimization problem can be reduced again to minimum-cost circulation, and hence:

Theorem 21.27. *Given $w : E \to \mathbb{Q}$, an integer vector x maximizing $w^{\mathsf{T}}x$ over* (21.39) *can be found in strongly polynomial time.*

Proof. This is a special case of Corollary 12.2d, by orienting the edges of G from one colour class to the other. ∎

Corresponding min-max and polyhedral characterizations directly follow from LP-duality and the total unimodularity of the incidence matrix of G. We formulate them for existence and optimum size of solutions of (21.39).

The following was formulated by Kellerer [1964]:

Theorem 21.28. *Let $G = (V, E)$ be a bipartite graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$ with $a \leq b$ and $d \leq c$. Then there exists an $x \in \mathbb{Z}^E$ satisfying* (21.39) *if and only if for each $X \subseteq V$ one has*

(21.40)
$$c(E[X]) - d(E[V \setminus X]) \geq \max\{a(S \cap X) - b(T \setminus X), a(T \cap X) - b(S \setminus X)\},$$

where S and T are the colour classes of G.

Proof. From Corollary 11.2i, by orienting all edges from S to T and taking $U := (S \setminus X) \cup (T \cap X)$. ∎

This theorem has several special cases. For $d = \mathbf{0}$ it implies the following result due to Fulkerson [1959a] (a generalization of Theorem 16.8):

Corollary 21.28a. *Let $G = (V, E)$ be a bipartite graph with colour classes S and T, let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$, and let $c \in \mathbb{Z}_+^E$. Then there is a vector $x \leq c$ that is both a b-matching and an a-edge cover if and only if there exist $y \in \mathbb{Z}_+^E$ and $z \in \mathbb{Z}_+^E$ with $y \leq c$ and $z \leq c$, such that*

(21.41)
$$y(\delta(v)) \leq b_v \text{ and } z(\delta(v)) \geq a_v \text{ for each } v \in S \text{ and}$$
$$y(\delta(v)) \geq a_v \text{ and } z(\delta(v)) \leq b_v \text{ for each } v \in T.$$

Proof. Note that (21.40) can be decomposed into two inequalities, one involving $a|S$ and $b|T$ only, the other involving $a|T$ and $b|S$ only[39]. This gives the present corollary. ∎

The special case $d = \mathbf{0}$, $c = \mathbf{1}$ is:

Corollary 21.28b. *Let $G = (V, E)$ be a bipartite graph with colour classes S and T and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then E has a subset F that is both a b-matching and an a-edge cover if and only if E has subsets F' and F'' such that F' contains at least a_v edges covering v if $v \in S$ and at most b_v edges covering v if $v \in T$, and F'' contains at least a_v edges covering v if $v \in T$ and at most b_v edges covering v if $v \in S$.*

Proof. Directly from Corollary 21.28a by taking $c = \mathbf{1}$. ∎

A min-max relation for such vectors can be derived from Hoffman's circulation theorem (Theorem 11.2):

Theorem 21.29. *Let $G = (V, E)$ be a bipartite graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$, such that there exists an $x \in \mathbb{Z}^E$ satisfying* (21.39). *Then the minimum size of such a vector x is equal to*

$$\text{(21.42)} \qquad \max_{Z \subseteq V}(a(Z) - c(E[Z])) + d(E[V \setminus Z]]),$$

while the maximum size of such a vector x is equal to

$$\text{(21.43)} \qquad \min_{Z \subseteq V}(c(E[V \setminus Z]) - d(E[Z]) + b(Z)).$$

For each integer value τ between (21.42) *and* (21.43) *there exists such a vector x of size τ.*

Proof. Choose $\tau \in \mathbb{Z}$. Make a directed graph $D = (V, A)$ as follows.

Let S and T be the colour classes of G. Orient each edge of G from S to T. Add new vertices s and t. For each $v \in S$, make an arc from s to v, with $d(s, v) := a_v$ and $c(s, v) := b_v$. For each $v \in T$, make an arc from v to t, with $d(v, t) := a_v$ and $c(v, t) := b_v$. Finally, make an arc from t to s with $d(t, s) := c(t, s) := \tau$.

It suffices to show that D has a circulation x satisfying $d \leq x \leq c$ if and only if τ is between (21.42) and (21.43). We do this by using Hoffman's circulation theorem. Choose a subset X of the vertex set of D. Consider Hoffman's condition:

$$\text{(21.44)} \qquad d(\delta^{\text{in}}(X)) \leq c(\delta^{\text{out}}(X)).$$

Since by assumption some vector x satisfying (21.39) exists, (21.44) holds if $s, t \in X$ or $s, t \notin X$ (as ignoring the bounds on (t, s) there is a circulation).

If $s \in X$ and $t \notin X$, we have

[39] $f|X$ denotes the restriction of a function f to a set X.

(21.45) $$d(\delta^{\text{in}}(X)) = \tau + d(E[(S \setminus X) \cup (T \cap X)])$$

and

(21.46) $$c(\delta^{\text{out}}(X)) = b(S \setminus X) + c(E[(S \cap X) \cup (T \setminus X)]) + b(T \cap X).$$

Hence (21.44) for such X is equivalent to

(21.47) $$\tau \le b(Z) + c(E[V \setminus Z]) - d(E[Z])$$

for all $Z \subseteq V$ (take $Z = (S \setminus X) \cup (T \cap X)$). That is, to τ being at most (21.43).

If $t \in X$ and $s \notin X$, we have

(21.48) $$d(\delta^{\text{in}}(X)) = a(S \cap X) + d(E[(S \setminus X) \cup (T \cap X)]) + a(T \setminus X)$$

and

(21.49) $$c(\delta^{\text{out}}(X)) = \tau + c(E[(S \cap X) \cup (T \setminus X)]).$$

Hence (21.44) for such X is equivalent to

(21.50) $$\tau \ge a(Z) - c(E[Z]) + d(E[V \setminus Z])$$

for all $Z \subseteq V$ (take $Z = (S \cap X) \cup (T \setminus X)$). That is, to τ being at least (21.42). ∎

A special case is the following theorem of Folkman and Fulkerson [1969]:

Corollary 21.29a. *Let $G = (V, E)$ be a bipartite graph, let $a, b \in \mathbb{Z}_+^V$, and let $\tau \in \mathbb{Z}_+$. Then E has a subset F with $a_v \le \deg_F(v) \le b_v$ for each $v \in V$ and with $|F| = \tau$ if and only if*

(21.51) $$|E[Z]| \ge \max\{a(Z) - \tau, \tau - b(V \setminus Z), a(S \cap Z) - b(T \setminus Z), a(T \cap Z) - b(S \setminus Z)\}$$

for each $Z \subseteq V$, where S and T are the colour classes of G.

Proof. Directly from Theorems 21.28 and 21.29. ∎

21.13. Further results and notes

21.13a. Complexity survey on weighted bipartite b-matching and transportation

Complexity survey for weighted b-matching in bipartite graphs ($*$ indicates an asymptotically best bound in the table):

$O(n^4 B)$	Munkres [1957]
$O(\beta \cdot \text{MF}(n, m, B))$	Ford and Fulkerson [1955,1957b]

$\gg$

continued

	$O(n^2 mB)$	Iri [1960]
*	$O(\beta \cdot \mathrm{SP}_+(n, m, W))$	Edmonds and Karp [1970]
	$O(nW \cdot \mathrm{MF}(n, m, B))$	Edmonds and Karp [1972]
*	$O(m \log B \cdot \mathrm{SP}_+(n, m, W))$	Edmonds and Karp [1972]
	$O(nm \log(nB))$	Dinits [1973a]
	$O(n \log \beta \cdot \mathrm{SP}_+(n, m, W))$	Lawler [1976b]
	$O(n \log W \cdot \mathrm{MF}(n, m, B))$	Röck [1980]
	$O(m^2 \log n \cdot \mathrm{MF}(n, m, B))$	Tardos [1985a]
*	$O(\beta^{3/4} m \log W)$	Gabow [1985b]
*	$O(\beta^{1/2} n^{1/3} m \log W)$	Gabow [1985b] *for simple graphs*
	$O(n^2 \log n \cdot \mathrm{SP}_+(n, m, W))$	Galil and Tardos [1986,1988]
	$O(nm \log(n^2/m) \log(nW))$	Goldberg and Tarjan [1987,1990]
	$O(n \log n(m + n \log n))$	Orlin [1988,1993]
*	$O((\beta^{1/2} m + \beta \log \beta) \log(nW))$	Gabow and Tarjan [1989]
*	$O(n_1 m + n_1^3 \log(n_1 W))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n_1 m \log(2 + \frac{n_1^2}{m} \log(n_1 W)))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n \log n(m + n_1 \log n_1))$	Kleinschmidt and Schannath [1995]

Here $B := \|b\|_\infty$, $\beta := \|b\|_1$, $W := \|w\|_\infty$ (assumed to be integer), and $n_1 := \min\{|S|, |T|\}$, where S and T are the colour classes of the bipartite graph. By $\mathrm{SP}_+(n, m, W)$ we denote the time required for solving a shortest path problem in a digraph with n vertices, m arcs, and nonnegative integer length function l with $\|l\|_\infty \leq W$. $\mathrm{MF}(n, m, B)$ denotes the time required to solve a maximum flow problem in a digraph with n vertices, m arcs, and integer capacity function c with $\|c\|_\infty \leq B$.

Complexity survey for the uncapacitated transportation problem:

	$O(n^4 B)$	Munkres [1957]
	$O(\beta \cdot \mathrm{MF}(n, n^2, B))$	Ford and Fulkerson [1955,1957b]
*	$O(n^3 \log(nB))$	Edmonds and Karp [1972], Dinits [1973a]
	$O(n^4 W)$	Edmonds and Karp [1972]
*	$O(\beta^{3/4} n^2 \log W)$	Gabow [1985b]
*	$O(\beta^{1/2} n^{7/3} \log W)$	Gabow [1985b]
	$O(n^4 \log n \cdot \mathrm{MF}(n, n^2, W))$	Tardos [1985a]
	$O(n^4 \log n)$	Galil and Tardos [1986,1988]
	$O(n^3 \log(nW))$	Goldberg and Tarjan [1987,1990]

≫

continued

	$O(n^3 \log n)$	Orlin [1988,1993]
*	$O((\beta^{1/2}n^2 + \beta \log \beta) \log(nW))$	Gabow and Tarjan [1989]
*	$O(n_1 n^2 + n_1^3 \log(n_1 W))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n_1 n^2 \log(2 + \frac{n_1^2}{n^2} \log(n_1 W)))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n_1^2 n \log^2 n)$	Tokuyama and Nakano [1992,1995]
*	$O(n_1 n^2 \log n)$	Kleinschmidt and Schannath [1995]

Complexity survey for weighted capacitated b-matching in bipartite graphs:

*	$O(n \max\{B, C\} \cdot \mathrm{SP}_+(n, m, W))$	Edmonds and Karp [1970]
	$O(nW \cdot \mathrm{MF}(n, m, \max\{B, C\}))$	Edmonds and Karp [1972]
*	$O(n \log \beta \cdot \mathrm{SP}_+(n, m, W))$	Lawler [1976b]
	$O(n \log W \cdot \mathrm{MF}(n, m, \max\{B, C\}))$	Röck [1980]
	$O(m^2 \log n \cdot \mathrm{MF}(n, m, \max\{B, C\}))$	Tardos [1985a]
	$O(\beta^{3/4} mC \log W)$	Gabow [1985b]
	$O(\beta^{1/2} n^{1/3} mC \log W)$	Gabow [1985b]
	$O(n^2 \log n \cdot \mathrm{SP}_+(n, m, W))$	Galil and Tardos [1986,1988]
*	$O(nm \log(n^2/m) \log(nW))$	Goldberg and Tarjan [1987,1990]
*	$O(m \log n \cdot \mathrm{SP}_+(n, m, W))$	Orlin [1988,1993]
*	$O(\beta^{1/2} mC \log(nW))$	Gabow and Tarjan [1988b,1989]
*	$O(n^{2/3} mC^{4/3} \log(nW))$	Gabow and Tarjan [1989]
*	$O((\beta^{1/2} m + \beta \log \beta) \log(nW))$	Gabow and Tarjan [1989]
*	$O((nm + \beta \log \beta) \log(nW))$	Gabow and Tarjan [1989]
*	$O(n_1 m + n_1^3 \log(n_1 W))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n_1 m \log(2 + \frac{n_1^2}{m} \log(n_1 W)))$	Ahuja, Orlin, Stein, and Tarjan [1994]

Here $C := \|c\|_\infty$.

Complexity survey for the capacitated transportation problem:

	$O(n^4 W)$	Edmonds and Karp [1972]
	$O(n^3 \log \max\{B, C\})$	Edmonds and Karp [1972]
	$O(n^4 \log W)$	Röck [1980]
	$O(n^4 \log n \cdot \mathrm{MF}(n, n^2, \max\{B, C\}))$	Tardos [1985a]
*	$O(n^2 B)$	Gabow [1985b]

≫

continued

	$O(\beta^{3/4}n^2C\log W)$	Gabow [1985b]
	$O(\beta^{1/2}n^{7/3}C\log W)$	Gabow [1985b]
*	$O(n^3\log(nW))$	Goldberg and Tarjan [1987,1990]
*	$O(n^4\log n)$	Galil and Tardos [1986,1988], Orlin [1988,1993]
*	$O(n^2\beta^{1/2}C\log(nW))$	Gabow and Tarjan [1988b,1989]
*	$O(n^{8/3}C^{4/3}\log(nW))$	Gabow and Tarjan [1989]
*	$O((\beta^{1/2}n^2+\beta\log\beta)\log(nW))$	Gabow and Tarjan [1989]
*	$O(n_1n^2+n_1^3\log(n_1W))$	Ahuja, Orlin, Stein, and Tarjan [1994]
*	$O(n_1n^2\log(2+\frac{n_1^2}{n^2}\log(n_1W)))$	Ahuja, Orlin, Stein, and Tarjan [1994]

Let $G=(V,E)$ be a bipartite graph, with colour classes S and T say. The existence of a perfect (capacitated) b-matching can be reduced quite directly to the problem of finding a maximum $s-t$ flow in the digraph obtained from G by adding two new vertices s and t, orienting each edge from S to T, and adding an arc (s,s') for each $s'\in S$, and adding an arc (t',t) for each $t'\in T$. Similarly, a maximum (capacitated) b-matching can be found.

It implies that if $\mathrm{MF}(n,m,C)$ is the running time of a maximum flow algorithm for inputs with n vertices, m arcs, and integer capacity function c with $\|c\|_\infty\le C$, then a maximum-size (capacitated) b-matching can be found in time $O(\mathrm{MF}(n,m,C))$, for bipartite graphs with n vertices, m edges and $b\in\mathbb{Z}^V$ satisfying $\|b\|_\infty\le C$ (and capacity function $c\in\mathbb{Z}^E$ satisfying $\|c\|_\infty\le C$).

In some cases, one can obtain better bounds, in particular if one of the colour classes is considerably smaller than the other. To this end, let $n_1:=\min\{|S|,|T|\}$. Implementing the shortest augmenting path rule described in Section 10.5, then gives an $O(n_1m^2)$ running time, since a shortest $s-t$ path has length at most $2n_1+1=O(n_1)$, implying that the number of iterations is bounded by n_1m.

Similarly, the blocking flow method of Dinits [1970] described in Section 10.6 can be performed in $O(n_1^2m)$ time, since the bound in Theorem 10.6 becomes $O(n_1m)$, while there are $O(n_1)$ blocking flow iterations. The method of Karzanov [1974] can be sharpened to $O(n_1^2n)$, as was shown by Gusfield, Martel, and Fernández-Baca [1987]. Ahuja, Orlin, Stein, and Tarjan [1994] gave a method taking the minimum of $O(n_1m+n_1^3)$, $O(n_1m+n_1^2\sqrt{m})$, $O(n_1m+n_1^2\sqrt{\log C})$, and $O(n_1m\log(2+\frac{n_1^2}{m}))$ time.

For the special case where $b_u=1$ for each u in the smaller colour class, Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975] gave an $O(n_1^{5/3}n)$ algorithm for finding a b-factor.

21.13b. The matchable set polytope

Let $G = (V, E)$ be a graph. A subset X of V is called *matchable*, if G has a matching M with $\bigcup M = X$; that is, if the subgraph $G[X]$ of G induced by X has a perfect matching.

The *matchable set polytope* of G is the convex hull of the incidence vectors of matchable sets. Theorem 21.11 implies a characterization of the matchable set polytope in case G is bipartite.

For any graph, each vector in the matchable set polytope trivially satisfies:

$$\text{(21.52)}\quad \begin{array}{lll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for each } v \in V, \\ \text{(ii)} & x(C) \leq \frac{1}{2}x(V) & \text{for each stable set } C. \end{array}$$

If G is bipartite, this set of inequalities determines the matchable set polytope, a result of Balas and Pulleyblank [1983]:

Theorem 21.30. *If G is bipartite, the matchable set polytope is determined by* (21.52)*.*

Proof. Let x satisfy (21.52). By Theorem 21.11, there exists an x-transportation $y \in \mathbb{R}^E_+$. That is, $x = Ay$, where A is the $V \times E$ incidence matrix of G.

As x satisfies (21.52)(i), y satisfies $y \geq \mathbf{0}$, $Ay \leq \mathbf{1}$. So, by Corollary 18.1b, y belongs to the matching polytope of G. So y is a convex combination of vectors χ^M, where M ranges over the matchings in G. Then x is a convex combination of the vectors χ^S, where S is matchable (that is, the set of vertices covered by some matching M). This follows from the fact that $A\chi^M = \chi^S$ if M is a matching and S is the set of vertices covered by M.

So x belongs to the matchable set polytope. ∎

It is easy to check that only for bipartite graphs the matchable set polytope is determined by (21.52).

Note that for bipartite graphs $G = (V, E)$, by Theorem 21.11, condition (21.52)(ii) is equivalent to x belonging to the convex cone generated by the incidence vectors (in $\mathbb{R}^V$) of edges, considered as subsets of V.

Qi [1987] gave an algorithm for the separation problem for the matchable set polytope of a bipartite graph. For more on the matchable set polytope, see Balas and Pulleyblank [1983] and Section 25.5d.

21.13c. Existence of matrices

If the bipartite graph is a complete bipartite graph, theorems on the existence of b-matchings and b-edge covers amount to theorems on the existence of matrices obeying prescribed bounds on the row and column sums. This gives the following theorem of Gale [1956,1957] and Ryser [1957]:

Theorem 21.31 (Gale-Ryser theorem)**.** *Let $a, b \in \mathbb{Z}^m_+$ and $a', b' \in \mathbb{Z}^n_+$ with $a \leq b$ and $a' \leq b'$ and satisfying $a_1 \geq a_2 \geq \cdots \geq a_m$ and $a'_1 \geq a'_2 \geq \cdots \geq a'_n$. Then there exists a $\{0, 1\}$-valued $m \times n$ matrix with ith row sum between a_i and b_i ($i = 1, \ldots, m$) and jth column sum between a'_j and b'_j ($j = 1, \ldots, n$) if and only if*

$$\text{(21.53)} \qquad \text{(i)} \sum_{i=1}^{k} a_i \leq \sum_{j=1}^{n} \min\{k, b'_j\} \textit{ for all } k = 1, \ldots, m,$$
$$\text{(ii)} \sum_{j=1}^{k} a'_j \leq \sum_{i=1}^{m} \min\{k, b_i\} \textit{ for all } k = 1, \ldots, n.$$

Proof. *Necessity.* Consider any inequality in (21.53)(i). The number of 1's in rows $1, \ldots, k$ is at least the left-hand side and at most the right-hand side. This proves necessity of the inequality. Necessity of the inequalities (ii) is shown similarly.

Sufficiency. This follows from Theorem 21.28 applied to the complete bipartite graph $G = K_{m,n}$. Then we must show that for each $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ one has:

$$\text{(21.54)} \qquad |I| \cdot |J| \geq \max\{a(I) - b'(\overline{J}), a'(J) - b(\overline{I})\},$$

where $\overline{I} := \{1, \ldots, m\} \setminus I$ and $\overline{J} := \{1, \ldots, n\} \setminus J$. By symmetry, it suffices to show

$$\text{(21.55)} \qquad |I| \cdot |J| \geq a(I) - b'(\overline{J}).$$

This follows from (21.53)(i), since

$$\text{(21.56)} \qquad a(I) \leq \sum_{i=1}^{|I|} \leq \sum_{j=1}^{n} \min\{|I|, b'_j\} \leq |J| \cdot |I| + b'(\overline{J})$$

for any $J \subseteq \{1, \ldots, n\}$. ∎

(Gale [1956,1957] proved this theorem for $a = \mathbf{0}$ and $b' = \infty$, and Ryser [1957] for $a = b$ and $a' = b'$.)

Corollary 21.28a due to Fulkerson [1959a], is equivalent to the following result extending the Gale-Ryser theorem:

Theorem 21.32. *Let $(c_{i,j})$ be a nonnegative $m \times n$ matrix and let $a, b \in \mathbb{Z}_+^m$ and $a', b' \in \mathbb{Z}_+^n$ with $a \leq b$ and $a' \leq b'$. Then there exists an integer $m \times n$ matrix $(x_{i,j})$ satisfying*

$$\text{(21.57)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_{i,j} \leq c_{i,j} & \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n, \\ \text{(ii)} & a_i \leq \sum_{j=1}^{n} x_{i,j} \leq b_i & \text{for all } i = 1, \ldots, m, \\ \text{(iii)} & a'_j \leq \sum_{i=1}^{m} x_{i,j} \leq b'_j & \text{for all } j = 1, \ldots, n, \end{array}$$

if and only if there exist an $m \times n$ matrix $(x'_{i,j})$ satisfying

$$\text{(21.58)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x'_{i,j} \leq c_{i,j} & \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n, \\ \text{(ii)} & \sum_{j=1}^{n} x'_{i,j} \leq b_i & \text{for all } i = 1, \ldots, m, \\ \text{(iii)} & a'_j \leq \sum_{i=1}^{m} x'_{i,j} & \text{for all } j = 1, \ldots, n, \end{array}$$

and an $m \times n$ matrix $(x''_{i,j})$ satisfying

$$\text{(21.59)}\quad \begin{array}{lll} \text{(i)} & 0 \leq x''_{i,j} \leq c_{i,j} & \text{for all } i = 1, \dots, m \text{ and } j = 1, \dots, n, \\ \text{(ii)} & a_i \leq \sum_{j=1}^{n} x''_{i,j} & \text{for all } i = 1, \dots, m, \\ \text{(iii)} & \sum_{i=1}^{m} x''_{i,j} \leq b'_j & \text{for all } j = 1, \dots, n. \end{array}$$

Proof. This is equivalent to Corollary 21.28a. ∎

21.13d. Further notes

Corollary 11.2c implies the following result of Hoffman [1956a]. Let $G = (V, E)$ be a bipartite graph and let $0 < \alpha < 1$. Then E has a subset F such that

$$\text{(21.60)}\qquad \lfloor \tfrac{\deg_E(v)}{\alpha} \rfloor \leq \deg_F(v) \leq \lceil \tfrac{\deg_E(v)}{\alpha} \rceil$$

for each vertex v.

Ikura and Nemhauser [1982] gave a strongly polynomial-time primal simplex algorithm for the maximum-weight stable set problem in bipartite graphs (the number of pivot steps is at most n^2; the method corresponds to a strongly polynomial-time dual simplex algorithm for the minimum-size b-edge cover problem, which is a special case of a minimum-flow problem). (An improvement was given by Armstrong and Jin [1996].) An interior-point method was described by Mizuno and Masuzawa [1989]. For more on capacitated b-matchings (in terms of matrices), see Anstee [1983].

We refer for further notes on algorithmic aspects of the transportation problem to Section 12.5d on the equivalent transshipment problem.

Heller [1963,1964] gave necessary and sufficient conditions for a linear program to be equivalent to a transportation problem. Katerinis [1987] and Enomoto, Ota, and Kano [1988] gave sufficient conditions for bipartite graphs to have a k-factor.

Goodman, Hedetniemi, and Tarjan [1976] gave a linear-time algorithm finding a maximum-weight simple b-matching in a tree.

Faster algorithms for transportation problems where the cost satisfies a quadrangle inequality where given by Karp and Li [1975] and Aggarwal, Bar-Noy, Khuller, Kravets, and Schieber [1995].

Variants of the transportation problem (minimax, bottleneck) were investigated by Szwarc [1966,1971], Hammer [1969,1971], Garfinkel and Rao [1971], Srinivasan and Thompson [1972a,1972b,1976], Derigs and Zimmermann [1979], Derigs [1982], Russell, Klingman, and Partow-Navid [1983], and Ahuja [1986]. Prager [1957b] and Kellerer [1961] gave a generalization.

Prager [1955] gave an extension to quadratic cost functions, i.e. given $b \in \mathbb{R}^m$, $d \in \mathbb{R}^n$, and $c_{i,j} \geq 0$, $q_{i,j} \geq 0$ ($i = 1, \dots, m$; $j = 1, \dots, n$):

$$\text{(21.61)}\quad \begin{array}{lll} \text{minimize} & \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{i,j} x_{i,j} + q_{i,j} x_{i,j}^2), & \\ \text{subject to} & \sum_{j=1}^{n} x_{i,j} = b_i & \text{for } i = 1, \dots, m, \\ & \sum_{i=1}^{m} x_{i,j} = d_j & \text{for } j = 1, \dots, n, \\ & x_{i,j} \geq 0 & \text{for } i = 1, \dots, m; j = 1, \dots, n. \end{array}$$

Among the books surveying transportation are Ford and Fulkerson [1962], Dantzig [1963], Murty [1976,1983], Bazaraa and Jarvis [1977], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Derigs [1988a], Nemhauser and Wolsey [1988], and Bazaraa, Jarvis, and Sherali [1990].

21.13e. Historical notes on the transportation and transshipment problems

Transportation can be considered as the special case of transshipment where all arcs are oriented from a source to a sink. By the techniques described in Section 21.6a, transshipment problems can be reduced conversely to transportation problems. This makes the history of the two problems intertwined. We should notice also that the transshipment problems studied by Kantorovich and Koopmans were in fact transportation problems, due to the fact that their cost functions are metrics.

Tolstoĭ

The first to study the transportation problem mathematically seems to be A.N. Tolstoĭ. In the collection *Transportation Planning, Volume I* of the National Commissariat of Transportation of the Soviet Union, Tolstoĭ [1930] published an article called *Methods of finding the minimal total kilometrage in cargo-transportation planning in space*. In it, Tolstoĭ described a number of approaches to solve the transportation problem, illuminated by applications to the transportation of salt, cement, and other cargo between sources and destination points along the railway network of the Soviet Union. He seems to be the first to give a negative cycle criterion for optimality. Moreover, a for that time large-scale instance of the transportation problem was solved to optimality.

First, Tolstoĭ considered the problem for the case where there are two sources. He observed that in that case one can order the destination points by the difference between the distances to the two sources. In that case, one source can provide the destinations starting from the beginning of the list, until the supply of that source has been used up. The other source supplies the remaining demands. Tolstoĭ observed that the list is independent of the supplies and demands, and hence

> such table is applicable for the whole life-time of factories, or sources of production.
> Using this table, one can immediately compose an optimal transportation plan every year, given quantities of output produced by these two factories and demands of the destination points.

Next, Tolstoĭ studied the transportation problem for the case where all sources and destinations are along one circular railway line. In this case, considering the negative cycle criterion yields directly the optimum solution. He calls this phenomenon 'circle dependency'.

Finally, Tolstoĭ combined the two methods into a heuristic to solve a concrete transportation problem coming from cargo transportation along the Soviet railway network. The problem has 10 sources and 68 sinks, and 155 links between sources and sinks (all other distances are taken infinite):

	Arkhangelsk	Yaroslavl'	Murom	Balakhonikha	Dzerzhinsk	Kishert'	Sverdlovsk	Artemovsk	Iledzhk	Dekonskaya	*demand:*
Agryz				709	1064	693					2
Aleksandrov					397			1180			4
Almaznaya								81		65	1.5
Alchevskaya								106		114	4
Baku								1554		1563	10
Barybino								985		968	2
Berendeevo		135			430						10
Bilimbai						200	59				1
Bobrinskaya								655		663	10
Bologoe		389						1398			1
Verkhov'e								678		661	1
Volovo								757		740	3
Vologda	634					1236					2
Voskresensk				427				1022		1005	1
V.Volochek		434						1353		1343	5
Galich	815	224				1056					0.5
Goroblagodatskaya						434	196				0.5
Zhlobin								882		890	8
Zverevo								227		235	5
Ivanovo					259						6
Inza				380	735					1272	2
Kagan								2445	2379		0.5
Kasimov			0								1
Kinel'				752		1208			454	1447	2
Kovylkino				355						1213	2
Kyshtym						421	159				3
Leningrad	1237	709						1667		1675	55
Likino			223		328						15
Liski								443		426	1
Lyuberdzhy			268		411					1074	1
Magnitogorskaya						932	678		818		1
Mauk						398	136				5
Moskva			288	378	405			1030		1022	141
Navashino			12	78							2
Nizhegol'								333		316	1
Nerekhta		50			349						5
Nechaevskaya			92								0.5
N.-Novgorod					32						25
Omsk						1159	904		1746		5
Orenburg									76		1.5
Penza				411				1040	883	1023	7
Perm'	1749					121					1
Petrozavodsk	1394										1
Poltoradzhk								1739	3085	1748	4
Pskov								1497		1505	10
Rostov/Don								287		296	20
Rostov/Yarosl		56			454						2
Rtishchevo								880		863	1
Savelovo		325						1206		1196	5
Samara				711					495	1406	7
San-Donato						416	157				1
Saratov								1072		1055	15
Sasovo				504				1096		1079	1
Slavyanoserbsk								119		115	1.1
Sonkovo		193						1337			0.5
Stalingrad								624		607	15.4
St.Russa		558						1507		1515	5
Tambov								783		766	4
Tashkent								3051	1775		3
Tula								840		848	8
Tyumen'						584	329				6
Khar'kov								251		259	60
Chelyabinsk						511	257		949		2
Chishmy				1123		773			889		0.5
Shchigry								566		549	4
Yudino				403	757	999					0.5
Yama								44		52	5
Yasinovataya								85		93	6
supply:	5	11.5	8.5	12	100	12	15	314	10	55	543

Table of distances (in kilometers) between sources and destinations, and of supplies and demands (in kilotons).

Tolstoĭ gave no distance for Kasimov. We have inserted a distance 0 to Murom, since from Tolstoĭ's solution it appears that Kasimov is connected only to Murom (by a waterway). Hence the distance is irrelevant.

Tolstoĭ's heuristic also makes use of insight into the geography of the Soviet Union. He goes along all sources (starting with the most remote source), where, for each source X, he lists those sinks for which X is the closest source or the second closest source. Based on the difference of the distances to the closest and second closest sources, he assigns cargo from X to the sinks, until the supply of X has been used up. In case Tolstoĭ foresees circle dependency, he deviates from this rule to avoid that a negative-length circuit would arise. No backtracking occurs.

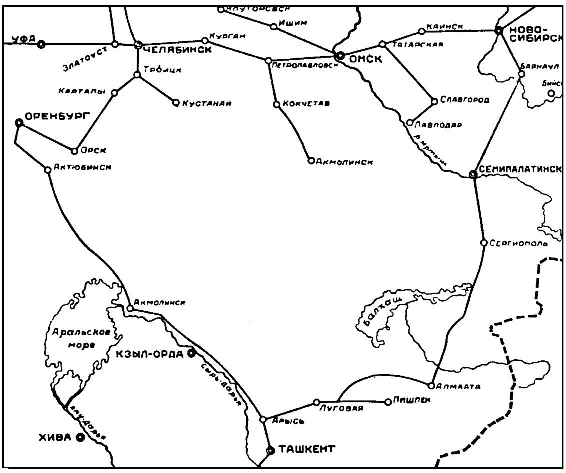


Figure 21.1
Figure from Tolstoĭ [1930] to illustrate a negative cycle.

In the following quotation, Tolstoĭ considers the cycles Dzerzhinsk-Rostov-Yaroslavl'-Leningrad-Artemovsk-Moscow-Dzerzhinsk and Dzerzhinsk-Nerekhta-Yaroslavl'-Leningrad-Artemovsk-Moscow-Dzerzhinsk. It is the sixth step in his method, after the transports from the factories in Iletsk, Sverdlovsk, Kishert', Balakhonikha, and Murom have been set:

> 6. The Dzerzhinsk factory produces 100,000 tons. It can forward its production only in the Northeastern direction, where it sets its boundaries in interdependency with the Yaroslavl' and Artemovsk (or Dekonskaya) factories.

	From Dzerzhinsk	From Yaroslavl'	Difference to Dzerzhinsk
Berendeevo	430 km	135 km	−295 km
Nerekhta	349 ,,	50 ,,	−299 ,,
Rostov	454 ,,	56 ,,	−398 ,,

	From Dzerzhinsk	From Artemovsk	Difference to Dzerzhinsk
Aleksandrov	397 km	1,180 km	+783 km
Moscow	405 ,,	1,030 ,,	+625 ,,

> The method of differences does not help to determine the boundary between the Dzerzhinsk and Yaroslavl' factories. Only the circle dependency, specified to be

an interdependency between the Dzerzhinsk, Yaroslavl' and Artemovsk factories, enables us to exactly determine how far the production of the Dzerzhinsk factory should be advanced in the Yaroslavl' direction.
Suppose we attach point Rostov to the Dzerzhinsk factory; then, by the circle dependency, we get:

Dzerzhinsk-Rostov	454 km	−398 km	Nerekhta	349 km	−299 km
Yaroslavl'- ,,	56 ,,		,,	50 ,,	
Yaroslavl'-Leningrad	709 ,,	+958 ,,	These points remain		
Artemovsk- ,,	1,667 ,,		unchanged because only the		
Artemovsk-Moscow	1,030 ,,	−625 ,,	quantity of production sent		
Dzerzhinsk- ,,	405 ,,		by each factory changes		
Total		−65 km			+34 km

Therefore, the attachment of Rostov to the Dzerzhinsk factory causes over-run in 65 km, and only Nerekhta gives a positive sum of differences and hence it is the last point supplied by the Dzerzhinsk factory in this direction.
As a result, the following points are attached to the Dzerzhinsk factory:

N. Novgorod	25,000 tons	
Ivanova	6,000 ,,	
Nerekhta	5,000 ,,	
Aleksandrov	4,000 ,,	
Berendeevo	10,000 ,,	
Likino	15,000 ,,	
Moscow	35,000 ,,	(remainder of factory's production)
Total	100,000 tons	

After 10 steps, when the transports from all 10 factories have been set, Tolstoĭ 'verifies' the solution by considering a number of cycles in the network, and he concludes that his solution is optimum:

> Thus, by use of successive applications of the method of differences, followed by a verification of the results by the circle dependency, we managed to compose the transportation plan which results in the minimum total kilometrage.

The objective value of Tolstoĭ's solution is 395,052 kiloton-kilometers. Solving the problem with modern linear programming tools (CPLEX) shows that Tolstoĭ's solution indeed is optimum. But it is unclear how sure Tolstoĭ could have been about his claim that his solution is optimum. Geographical insight probably has helped him in growing convinced of the optimality of his solution. On the other hand, it can be checked that there exist feasible solutions that have none of the negative-cost cycles considered by Tolstoĭ in their residual graph, but that are yet not optimum[40].

In the September 1939 issue of *Sotsialisticheskiĭ Transport*, Tolstoĭ [1939] published an article *Methods of removing irrational transportations in planning*, in which he again described his method of 'circle dependency', and applied it to the planning of driving empty cars and transporting heavy cargoes on the U.S.S.R. railway network. In this paper, Tolstoĭ restricted himself to sources and sinks arranged along a circular railway line, for which he gave his 'circle dependency' method:

[40] The maximum objective value of a feasible solution, whose residual graph contains no nonnegative-cost cycle of length 4, and none of the seven longer nonnegative-length cycles considered by Tolstoĭ (of lengths 6 and 8), is equal to 397,226.

> Before counting distances from cargo-senders to points of destination which form a circle dependency, it is necessary to attach points of destination to cargo-senders with complete distribution of waggons. In case of circle dependency determined by geographical location it can be done without special calculations. Then, by calculation of km in circle dependency, the initial attachment can be verified and if not correct, then it can be improved.

Tolstoĭ illustrated the method by the circuit Smolensk - Vitebsk - Velikiye-Luki - Zemtsy - Rzhev - Vyazma - Smolensk of the U.S.S.R. network. A negative-length directed circuit in the auxiliary directed graph gives an improvement, as in the following Table given by Tolstoĭ [1939]:

Source of cargoes	Amount km	Difference of distance	Amount of carriages
Vyazma-Smolensk	176	-37	$4-3=1$
Vitebsk ,,	139		$0+3=4$
Vitebsk-V. Luki	156	-37	$3-3=0$
Zemtsy ,,	119		$2+3=5$
Zemtsy-Rzhev	123	$+7$	$5-3=2$
Vyazma ,,	130		$1+3=4$
	Altogether . . .	-67	

Tolstoĭ then remarked:

> The negative total difference shows that the distribution was wrong and that there is an over-run of 67 km for every waggon which goes from upper cargo-senders.

According to Kantorovich [1987], there were some attempts to introduce Tolstoĭ's work by the appropriate department of the People's Commissariat of Transport. Tolstoĭ's method was also explained in the book *Planning Goods Transportation* by Pariĭskaya, Tolstoĭ, and Mots [1947].

Kantorovich

Apparently unaware (by that time) of the work of Tolstoĭ, L.V. Kantorovich studied a general class of problems, that includes the transportation problem. It formed a major impulse to the study of linear programming. In his memoirs, Kantorovich [1987] writes:

> Once some engineers from the veneer trust laboratory came to me for consultation with a quite skilful presentation of their problems. Different productivity is obtained for veneer-cutting machines for different types of materials; linked to this the output of production of this group of machines depended, it would seem, on the chance factor of which group of raw materials to which machine was assigned. How could this fact be used rationally?
> This question interested me, but nevertheless appeared to be quite particular and elementary, so I did not begin to study it by giving up everything else. I put this question for discussion at a meeting of the mathematics department, where there were such great specialists as Gyunter, Smirnov himself, Kuz'min, and Tartakovskii. Everyone listened but no one proposed a solution; they had already turned to someone earlier in individual order, apparently to Kuz'min. However, this question nevertheless kept me in suspense. This was the year of my marriage, so I was also distracted by this. In the summer or after the vacation

> concrete, to some extent similar, economic, engineering, and managerial situations started to come into my head, that also required the solving of a maximization problem in the presence of a series of linear constraints.
> In the simplest case of one or two variables such problems are easily solved—by going through all the possible extreme points and choosing the best. But, let us say in the veneer trust problem for five machines and eight types of materials such a search would already have required solving about a billion systems of linear equations and it was evident that this was not a realistic method. I constructed particular devices and was probably the first to report on this problem in 1938 at the October scientific session of the Herzen Institute, where in the main a number of problems were posed with some ideas for their solution.
> The universality of this class of problems, in conjunction with their difficulty, made me study them seriously and bring in my mathematical knowledge, in particular, some ideas from functional analysis.

In a footnote, Kantorovich's son V.L. Kantorovich adds:

> In L.V. Kantorovich's archives a manuscript from 1938 is preserved on "Some mathematical problems of the economics of industry, agriculture, and transport" that in content, apparently, corresponds to this report and where, in essence, the simplex method for the machine problem is described.

L.V. Kantorovich recalled that he created in January 1939 'a method of Lagrange (resolving) multipliers'.

> What became clear was both the solubility of these problems and the fact that they were widespread, so representatives of industry were invited to a discussion of my report at the university.

This meeting took place on 13 May 1939 at the Mathematical Section of the Institute of Mathematics and Mechanics of the Leningrad State University. A second meeting, which was devoted specifically to problems connected with construction, was held on 26 May 1939 at the Leningrad Institute for Engineers of Industrial Construction. These meetings provided the basis of the monograph *Mathematical Methods in the Organization and Planning of Production* (Kantorovich [1939]).

According to the Foreword by A.R. Marchenko to this monograph, Kantorovich's work was highly praised by mathematicians, and, in addition, at the special meeting industrial workers unanimously evinced great interest in the work.

The relevance was described by Kantorovich as follows:

> I want to emphasize again that the greater part of the problems of which I shall speak, relating to the organization and planning of production, are connected specifically with the Soviet system of economy and in the majority of cases do not arise in the economy of a capitalist society. There the choice of output is determined not by the plan but by the interests and profits of individual capitalists. The owner of the enterprise chooses for production those goods which at a given moment have the highest price, can most easily be sold, and therefore give the largest profit. The raw material used is not that of which there are huge supplies in the country, but that which the entrepreneur can buy most cheaply. The question of the maximum utilization of equipment is not raised; in any case, the majority of enterprises work at half capacity.
> In the USSR the situation is different. Everything is subordinated not to the interests and advantage of the individual enterprise, but to the task of fulfilling the state plan. The basic task of an enterprise is the fulfillment and overfulfillment of its plan, which is a part of the general state plan. Moreover, this not only means fulfillment of the plan in aggregate terms (i.e. total value of output, total tonnage, and so on), but the certain fulfillment of the plan for all kinds of output; that is, the fulfillment of the assortment plan (the fulfillment of the plan for each kind of output, the completeness of individual items of output, and so on).

In the monograph, Kantorovich outlined a new method to maximize a linear function under given linear constraints. One of the problems studied was a rudimentary form of a transportation problem:

(21.62) given: an $m \times n$ matrix $(a_{i,j})$;
find: an $m \times n$ matrix $(x_{i,j})$ such that:
(i) $x_{i,j} \geq 0$ for all i, j;
(ii) $\sum_{i=1}^{m} x_{i,j} = 1$ for each $j = 1, \ldots, n$;
(iii) $\sum_{j=1}^{n} a_{i,j} x_{i,j}$ is independent of i and is maximized.

Another problem studied by Kantorovich was 'Problem C' which can be stated as follows:

(21.63)
$$\begin{array}{lll} \text{maximize} & \lambda & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} x_{i,j} = 1 & (j = 1, \ldots, n) \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j,k} x_{i,j} = \lambda & (k = 1, \ldots, t) \\ & x_{i,j} \geq 0 & (i = 1, \ldots, m; j = 1, \ldots, n). \end{array}$$

The interpretation is: let there be n machines, which can do m jobs. Let there be one final product consisting of t parts. When machine i does job j, $a_{i,j,k}$ units of part k are produced ($k = 1, \ldots, t$). Now $x_{i,j}$ is the fraction of time machine i does job j. The number λ is the amount of the final product produced. 'Problem C' was later seen (by H.E. Scarf, upon a suggestion by Kantorovich — see Koopmans [1959]) to be equivalent to the general linear programming problem.

Kantorovich's method consists of determining dual variables ('resolving multipliers') and finding the corresponding primal solution. If the primal solution is not feasible, the dual solution is modified following prescribed rules. Kantorovich also indicated the role of the dual variables in sensitivity analysis, and he showed that a feasible primal solution for Problem C can be shown to be optimal by specifying optimal dual variables.

Kantorovich gave a wealth of practical applications of his methods, which he based mainly in the Soviet plan economy:

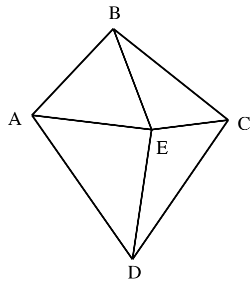
> Here are included, for instance, such questions as the distribution of work among individual machines of the enterprise or among mechanisms, the correct distribution of orders among enterprises, the correct distribution of different kinds of raw materials, fuel, and other factors. Both are clearly mentioned in the resolutions of the 18th Party Congress.

He described the applications to transportation:

> Let us first examine the following question. A number of freights (oil, grain, machines and so on) can be transported from one point to another by various methods; by railroads, by steamship; there can be mixed methods, in part by railroad, in part by automobile transportation, and so on. Moreover, depending on the kind of freight, the method of loading, the suitability of the transportation, and the efficiency of the different kinds of transportation is different. For example, it is particularly advantageous to carry oil by water transportation if oil tankers

are available, and so on. The solution of the problem of the distribution of a given freight flow over kinds of transportation, in order to complete the haulage plan in the shortest time, or within a given period with the least expenditure of fuel, is possible by our methods and leads to Problems A or C.

Let us mention still another problem of different character which, although it does not lead directly to questions A, B, and C, can still be solved by our methods. That is the choice of transportation routes.



Let there be several points A, B, C, D, E (Fig. 1) which are connected to one another by a railroad network. It is possible to make the shipments from B to D by the shortest route BED, but it is also possible to use other routes as well: namely, BCD, BAD. Let there also be given a schedule of freight shipments; that is, it is necessary to ship from A to B a certain number of carloads, from D to C a certain number, and so on. The problem consists of the following. There is given a maximum capacity for each route under the given conditions (it can of course change under new methods of operation in transportation). It is necessary to distribute the freight flows among the different routes in such a way as to complete the necessary shipments with a minimum expenditure of fuel, under the condition of minimizing the empty runs of freight cars and taking account of the maximum capacity of the routes. As was already shown, this problem can also be solved by our methods.

Kantorovich [1987] wrote in his memoirs:

The university immediately published my pamphlet, and it was sent to fifty People's Commissariats. It was distributed only in the Soviet Union, since in the days just before the start of the World War it came out in an edition of one thousand copies in all.

The number of responses was not very large. There was quite an interesting reference from the People's Commissariat of Transportation in which some optimization problems directed at decreasing the mileage of wagons was considered, and a good review of the pamphlet appeared in the journal *The Timber Industry.*

At the beginning of 1940 I published a purely mathematical version of this work in Doklady Akad. Nauk [76], expressed in terms of functional analysis and algebra. However, I did not even put in it a reference to my published pamphlet—taking into account the circumstances I did not want my practical work to be used outside the country.

In the spring of 1939 I gave some more reports—at the Polytechnic Institute and the House of Scientists, but several times met with the objection that the work used mathematical methods, and in the West the mathematical school in economics was an anti-Marxist school and mathematics in economics was a means for apologists of capitalism. This forced me when writing a pamphlet to avoid the term "economic" as much as possible and talk about the organization and planning of production; the role and meaning of the Lagrange multipliers had to be given somewhere in the outskirts of the second appendix and in the semi Aesopian language.

(Here reference [76] is Kantorovich [1940].) Kantorovich mentioned that the new area opened by his work played a definite role in forming the Leningrad Branch of the Mathematical Institute (LOMI), where he worked with M.K. Gavurin on this area. The problem that they studied occurred to them by itself, but they soon found out that railway workers were already studying the problem of planning haulage on railways, applied to questions of driving empty cars and transport of heavy cargoes.

Kantorovich and Gavurin wrote their method (the method of 'potentials') in a paper *Application of mathematical methods in questions of analysis of freight traffic* (Kantorovich and Gavurin [1949]), which was presented in January 1941 to the mathematics section of the Leningrad House of Scientists, but according to Kantorovich [1987]:

> The publication of this paper met with many difficulties. It had already been submitted to the journal *Railway Transport* in 1940, but because of the dread of mathematics already mentioned it was not printed then either in this or in any other journal, despite the support of Academicians A.N. Kolmogorov and V.N. Obraztsov, a well-known transport specialist and first-rank railway General.

Kantorovich [1987] said that he fortunately made an abstract version of the problem, Kantorovich [1942], in which he considered the following generalization of the transportation problem.

Let R be a compact metric space, with two measures μ and μ'. Let $\mathcal{B}$ be the collection of measurable sets in R. A *translocation (of masses)* is a function $\Psi : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_+$ such that for each $X \in \mathcal{B}$ the functions $\Psi(X, .)$ and $\Psi(., X)$ are measures and such that

(21.64) $$\Psi(X, R) = \mu(X) \text{ and } \Psi(R, X) = \mu'(X)$$

for each $X \in \mathcal{B}$.

Let a continuous function $r : R \times R \to \mathbb{R}_+$ be given. (The value $r(x, y)$ represents the work needed to transfer a unit mass from x to y.) Then the *work* of a *translocation* Ψ is by definition:

(21.65) $$\int_R \int_R r(x, y)\Psi(d\mu, d\mu').$$

Kantorovich argued that, if there exists a translocation, then there exists a *minimal* translocation, that is, a translocation Ψ minimizing (21.65).

He calls a translocation Ψ *potential* if there exists a function $p : R \to \mathbb{R}$ such that for all $x, y \in R$:

(21.66)
(i) $|p(x) - p(y)| \leq r(x, y)$;
(ii) $p(y) - p(x) = r(x, y)$ if $\Psi(U_x, U_y) > 0$ for any neighbourhoods U_x of x and U_y of y.

Kantorovich showed:

Theorem 21.33. *A translocation Ψ is minimal if and only if it is potential.*

This framework applies to the transportation problem (when $m = n$), by taking for R the space $\{1, \ldots, n\}$, with the discrete topology.

Kantorovich's proof of Theorem 21.33 is by a construction of a potential, that however only is correct if r satisfies the triangle inequality. Kantorovich remarked that his method is algorithmic:

> The theorem just demonstrated makes it easy for one to prove that a given mass translocation is or is not minimal. He has only to try and construct the potential in the way outlined above. If this construction turns out to be impossible, i.e. the given translocation is not minimal, he at least will find himself in the possession of the method how to lower the translocation work and eventually come to the minimal translocation.

Beside to a problem of leveling a land area, Kantorovich gave as application:

> Problem 1. *Location of consumption stations with respect to production stations.* Stations $A_1, A_2, \cdots, A_m$, attached to a network of railways deliver goods to an extent of $a_1, a_2, \cdots, a_m$ carriages per day respectively. These goods are consumed at stations $B_1, B_2, \cdots, B_n$ of the same network at a rate of $b_1, b_2, \cdots, b_n$ carriages per day respectively ($\sum a_i = \sum b_k$). Given the costs $r_{i,k}$ involved in moving one carriage from station A_i to station B_k, assign the consumption stations such places with respect to the production stations as would reduce the total transport expenses to a minimum.

As mentioned, Kantorovich's results remained unnoticed for some time by Western researchers. In a note introducing a reprint of the article of Kantorovich [1942], in *Management Science* in 1958, the following reassurance is given:

> It is to be noted, however, that the problem of determining an effective method of actually acquiring the solution to a specific problem is *not* solved in this paper. In the category of development of such methods we seem to be, currently, ahead of the Russians.

Kantorovich's method was elaborated by Kantorovich and Gavurin [1949], where moreover single- and multicommodity transportation models are studied, with applications to the railway network of the U.S.S.R.

Hitchcock

Independently, Hitchcock [1941] studied the transportation problem:

(21.67) given: an $m \times n$ matrix $C = (c_{i,j})$ and vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$;
find: an $m \times n$ matrix $X = (x_{i,j})$ such that:
(i) $x_{i,j} \geq 0$ for all i, j;
(ii) $\sum_{j=1}^{n} x_{i,j} = a_i$ for each $i = 1, \ldots, m$;
(iii) $\sum_{i=1}^{m} x_{i,j} = b_j$ for each $j = 1, \ldots, n$;
(iv) $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$ is as small as possible.

The interpretation of the problem is, in Hitchcock's words:

> When several factories supply a product to a number of cities we desire the least costly manner of distribution. Due to freight rates and other matters the cost of a ton of product to a particular city will vary according to which factory supplies it, and will also vary from city to city.

Hitchcock showed that the minimum is attained at a vertex of the feasible region, and he outlined a scheme for solving the transportation problem which has much in common with the simplex method for linear programming. It includes pivoting (eliminating and introducing basic variables) and the fact that nonnegativity of certain dual variables implies optimality. He showed that the *complementary slackness* conditions characterize optimality: $(x^*_{i,j})$ is an optimum vertex if and only if there exists a combination $\sum_{i,j} \lambda_{i,j} x_{i,j}$ of the left-hand sides of the constraints (ii) and (iii) such that $\lambda_{i,j} \geq c_{i,j}$ for all i, j and such that $\lambda_{i,j} = c_{i,j}$ if $x^*_{i,j} > 0$.

Hitchcock however seemed to have overlooked the possibility of cycling of his method, although he pointed at an example in which some dual variables are negative while yet the primal solution is optimum.

Hitchcock also gave a method to find an initial basic solution, now known as the *north-west rule*: set $x_{1,1} := \min\{a_1, b_1\}$; if the minimum is attained by a_1, reset $b_1 := b_1 - a_1$ and recursively find a basic solution $x_{i,j}$ satisfying $\sum_{j=1}^{n} x_{i,j} = a_i$ for each $i = 2, \ldots, m$ and $\sum_{i=2}^{m} x_{i,j} = b_j$ for each $j = 1, \ldots, n$; if the minimum is attained by b_1, proceed symmetrically. (The north-west rule was also described by Salvemini [1939] and Fréchet [1951] in a statistical context, namely in order to complete correlation tables given the marginal distributions.)

Koopmans

Also independently, Koopmans investigated transportation problems. In March 1942, Koopmans was appointed as a statistician on the staff of the British Merchant Shipping Mission, and later the Combined Shipping Adjustment Board (CSAB), a British-American agency dealing with merchant shipping problems during the Second World War (as they should go in convoys, under military protection). Influenced by his teacher J. Tinbergen (cf. Tinbergen [1934]) he was interested in tanker freights and capacities (cf. Koopmans [1939]). According to Koopmans' personal diary, in August 1942 while the Board was being organized, there was not much work for the statisticians,

> and I had a fairly good time working out exchange ratio's between cargoes for various routes, figuring how much could be carried monthly from one route if monthly shipments on another route were reduced by one unit.

At the Board he studied the assignment of ships to convoys so as to accomplish prescribed deliveries, while minimizing empty voyages (cf. Dorfman [1984]). According to the memoirs of his wife (Wanningen Koopmans [1995]), when Koopmans was with the Board,

> he had been appalled by the way the ships were routed. There was a lot of redundancy, no intensive planning. Often a ship returned home in ballast, when with a little effort it could have been rerouted to pick up a load elsewhere.

In his autobiography (published posthumously), Koopmans [1992] described how he came to the problem:

> My direct assignment was to help fit information about losses, deliveries from new construction, and employment of British-controlled and U.S-controlled ships into a unified statement. Even in this humble role I learned a great deal about the difficulties of organizing a large-scale effort under dual control—or rather in this case four-way control, military and civilian cutting across U.S. and U.K.

> controls. I did my study of optimal routing and the associated shadow costs of transportation on the various routes, expressed in ship days, in August 1942 when an impending redrawing of the lines of administrative control left me temporarily without urgent duties. My memorandum, cited below, was well received in a meeting of the Combined Shipping Adjustment Board (that I did not attend) as an explanation of the “paradoxes of shipping” which were always difficult to explain to higher authority. However, I have no knowledge of any systematic use of my ideas in the combined U.K.-U.S. shipping problems thereafter.

In the memorandum to the Board, Koopmans [1942] analyzed the sensitivity of the optimum shipments for small changes in the demands. In this memorandum, Koopmans did not give a method to find an optimum shipment. Further study led him to a ‘local search’ method for the transportation problem, stating that it leads to an optimum solution. According to Dorfman [1984], Koopmans found these results in 1943, but, due to wartime restrictions, published them only after the war (Koopmans [1948], Koopmans and Reiter [1949a,1949b,1951]). Koopmans [1948] wrote:

> Let us now for the purpose of argument (since no figures of war experience are available) assume that one particular organization is charged with carrying out a world dry-cargo transportation program corresponding to the actual cargo flows of 1925. How would that organization solve the problem of moving the empty ships economically from where they become available to where they are needed? It seems appropriate to apply a procedure of trial and error whereby one draws tentative lines on the map that link up the surplus areas with the deficit areas, trying to lay out flows of empty ships along these lines in such a way that a minimum of shipping is at any time tied up in empty movements.

The ‘trial and error’ method mentioned is one of local improvements, corresponding to finding a negative-cost directed circuit in the residual digraph. Koopmans’ *first theorem* is that it leads to an optimum solution:

> If, under the assumptions that have been stated, no improvement in the use of shipping is possible by small variations such as have been illustrated, then there is no—however thoroughgoing—rearrangement in the routing of empty ships that can achieve a greater economy of tonnage.

He illustrated the method by giving an optimum solution for a 3×12 transportation problem, with the following supplies and demands:

Net receipt of dry cargo in overseas trade, 1925

Unit: Millions of metric tons per annum

Harbour	Received	Dispatched	Net receipts
New York	23.5	32.7	−9.2
San Francisco	7.2	9.7	−2.5
St. Thomas	10.3	11.5	−1.2
Buenos Aires	7.0	9.6	−2.6
Antofagasta	1.4	4.6	−3.2
Rotterdam	126.4	130.5	− 4.1
Lisbon	37.5	17.0	20.5
Athens	28.3	14.4	13.9
Odessa	0.5	4.7	−4.2
Lagos	2.0	2.4	−0.4
Durban	2.1	4.3	−2.2
Bombay	5.0	8.9	−3.9
Singapore	3.6	6.8	−3.2
Yokohama	9.2	3.0	6.2
Sydney	2.8	6.7	−3.9
Total	266.8	266.8	0.0

Koopmans [1948] moreover claimed that there exist *potentials* $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ such that $c_{i,j} \geq p_i - q_j$ for all i, j and such that $c_{i,j} = p_i - q_j$ for each i, j for which $x_{i,j} > 0$.

The potentials give the *marginal costs* when modifying the input data. That is, if both a_i and b_j increase by 1, then the minimum cost increases by at least $p_i - q_j$. This is Koopmans' *second theorem.*

In the proof, Koopmans assumed that the cost function is symmetric and satisfies the triangle inequality. Moreover, he assumed that the graph of arcs having a positive transshipment value is weakly connected. The latter restriction was removed in a later paper by Koopmans and Reiter [1951]. In this paper, they investigated the economic implications of the model and the method:

> For the sake of definiteness we shall speak in terms of the transportation of cargoes on ocean-going ships. In considering only shipping we do not lose generality of application since ships may be "translated" into trucks, aircraft, or, in first approximation, trains, and ports into the various sorts of terminals. Such translation is possible because all the above examples involve particular types of movable transportation equipment.

They use the graph model, and in a footnote they remark:

> The cultural lag of economic thought in the application of mathematical methods is strikingly illustrated by the fact that linear graphs are making their entrance into transportation theory just about a century after they were first studied in relation to electrical networks, although organized transportation systems are much older than the study of electricity.

(For a review of Koopmans' research, see Scarf [1992].)

Robinson, 1950

Robinson [1950] might be the earliest reference stating clearly and generally that the absence of a negative-cost directed circuit in the residual digraph is necessary and sufficient for optimality. She mentioned that it can be 'verified directly', and observed that it gives an algorithm to find an optimum transportation. She concluded with:

> The number of steps in the iterative procedure depends on the "goodness" of the initial choice of X_0. The method does not seem to lend itself to machine calculation but may be efficient for hand computation with matrices of small order.

Linear programming and the simplex method

The breakthrough of general linear programming came at the end of the 1940s. In 1947, Dantzig formulated the linear programming problem and designed the *simplex method* for the linear programming problem, published in Dantzig [1951b]. The success of the method was enlarged by a simple tableau-form and a simple pivoting rule, and by the efficiency in practice. In another paper, Dantzig [1951a] described a direct implementation of the simplex method to the transportation problem (including an anti-cycling rule based on perturbation; variants were given by Charnes and Cooper [1954] and Eisemann [1956]).

The simplex method for transportation was described in terms of graphs by Koopmans and Reiter [1951], and Flood [1952,1953] aimed at giving a purely mathematical description of it. A continuous model of transportation was studied by Beckmann [1952].

Votaw and Orden [1952] reported on early computational results (on the SEAC), and claimed (without proof) that the simplex method is polynomial-time for the transportation problem (a statement refuted by Zadeh [1973a]):

> As to computation time, it should be noted that for moderate size problems, say $m \times n$ up to 500, the time of computation is of the same order of magnitude as the time required to type the initial data. The computation time on a sample computation in which m and n were both 10 was 3 minutes. The time of computation can be shown by study of the computing method and the code to be proportional to $(m+n)^3$.

Application to practice

The new ideas of applying linear programming to the transportation problem were quickly disseminated. Applications to routing empty boxcars over the U.S. railroads were given by Fox [1952] and Nerlove [1953]. Dantzig and Fulkerson [1954b,1954a] studied a rudimentary form of a minimum-cost circulation problem in order to determine the minimum number of tankers to meet a fixed schedule. Similarly, Bartlett [1957] and Bartlett and Charnes [1957] studied methods to determine the minimum railway stock to run a given schedule.

Applicability of linear programming to transportation to practice was also met with scepticism. At a Conference on Linear Programming in May 1954 in London, Land [1954] presented a study of applying linear programming to the problem of transporting coal for the British Coke Industry:

> The real crux of this piece of research is whether the saving in transport cost exceeds the cost of using linear programming.

In the discussion which followed, T. Whitwell of Powers Samas Accounting Machines Ltd remarked

> that in practice one could have one's ideas of a solution confirmed or, much more frequently, completely upset by taking a couple of managers out to lunch.

Gleyzal's primal-dual method for the transportation problem

Gleyzal [1955] published the following primal-dual method for the transportation problem (with integer data). Let $x_{i,j}$ be a feasible solution of the transportation problem. Transform $x_{i,j}$ such that the set $\{u_i v_j \mid x_{i,j} > 0\}$ contains no circuit, and transform $c_{i,j}$ such that $c_{i,j} = 0$ if $x_{i,j} > 0$. (These are easy by first cancelling circuits, and next redefining $c_{i,j}$.)

If $c_{i,j} \geq 0$ for all i, j we are done. Suppose that $c_{i_0,j_0} < 0$ for some i_0, j_0. Let $A := \{(u_i, v_j) \mid c_{i,j} \leq 0\} \cup \{(v_j, u_i) \mid x_{i,j} > 0\}$. If u_{i_0} is reachable in A from v_{j_0}, A contains a directed circuit C containing (u_{i_0}, v_{j_0}). Then we can reset $x_{i,j} := x_{i,j} - 1$ if (v_j, u_i) is in C and $x_{i,j} := x_{i,j} + 1$ if (u_i, v_j) is in C. This decreases $c^{\mathsf{T}} x$.

If u_{i_0} is not reachable in A from v_{j_0}, then for any vertex v let $r(v) := 1$ if v is reachable in A from v_{j_0} and $r(v) := 0$ otherwise. Reset $c_{i,j} := c_{i,j} - r(u_i) + r(v_j)$. This increases $\sum(c_{i,j} \mid c_{i,j} < 0)$, and hence the method terminates.

Munkres on the transportation problem

Munkres [1957] extended his variant of the Hungarian method for the assignment problem to the transportation problem. In graph terms, it amounts to the following.

Let $G = (V, E)$ be a complete bipartite graph, with colour classes U and W of size n, and let be given a weight function $w : E \to \mathbb{Z}_+$ and a function $b : V \to \mathbb{Z}_+$ with $b(U) = b(W)$. We must find a function $x : E \to \mathbb{Q}_+$ such that $\sum_{e\in\delta(v)} x_e = b_v$ for each vertex v and such that $\sum_e w_e x_e$ is minimized.

Let F be the set of edges e with $w_e = 0$ and let $H = (V, F)$. Suppose that we have found an $x : E \to \mathbb{Q}_+$ such that $x_e = 0$ if $e \notin F$ and such that $\sum_{e\in\delta(v)} x_e \leq b_v$ for each $v \in V$. Let U' and W' be the sets of vertices v in U and W for which strict inequality holds. If U', and hence W', are empty, x is an optimum solution. Otherwise, perform the following iteratively.

Orient each edge of H from U to W, and orient each edge e of H with $x_e > 0$ also from W to U (so they are two-way). Now determine the set R_M of vertices reachable by a directed path from U'.

Case 1: $R_M \cap W' \neq \emptyset$. Then D has a $U' - W'$ path, on which we can alternatingly increase and decrease the value of x_e, so as to make $\sum_e x_e$ larger.

Case 2: $R_M \cap W' = \emptyset$. So $w(uv) > 0$ for each $u \in U \cap R_M$ and $v \in W \setminus R_M$. Let h be the minimum of these $w(uv)$. Decrease $w(uv)$ by h if $u \in U \cap R_M, v \in W \setminus R_M$, and increase $w(uv)$ by h if $u \in U \setminus R_M, v \in W \cap R_M$.

This describes the iteration. Note that between any two occurrences of Case 1, only n times Case 2 can occur, since at each such iteration the set $R_M \cap W$ increases. Moreover, after Case 2 we can continue the previous search for R_M. So between any two Case 1-iterations, the Case 2-iterations take $O(n^2)$ time altogether.

Now Case 1 can occur at most $\sum_{v\in U} b_v$ times. So the algorithm is finite, and has running time $O(n^4 B)$ where $B := \max\{b_v \mid v \in V\}$. This specializes to the Hungarian method if $b_v = 1$ for all $v \in V$.

Further early methods

Also Ford and Fulkerson [1955,1957b] (cf. Ford and Fulkerson [1956c,1956d]) extended the Hungarian method to general transportation problems. Their method is essentially the same as that of Munkres [1957], except that successive occurrences of Case 1 iterations are combined to a maximum flow computation. A similar primal-dual method for the transportation problem was described by Egerváry [1958].

Ford and Fulkerson [1956a,1957a] extended the method of Ford and Fulkerson [1955,1957b] for the uncapacitated transportation problem to the *capacitated* transportation problem.

Orden [1955] showed the equivalence of the transshipment problem and the transportation problem. He also noted that the class of transportation problems covers the majority of the applications of linear programming which are in practical use or under active development. Also Prager [1957a] studied the transshipment problem by reduction to a transportation problem and by methods of elastostatics (cf. Kuhn [1957]).

Gallai [1957,1958a,1958b] studied the minimum-cost and the maximum-profit circulation problem, for which he gave min-max relations (see Section 12.5b). He also considered vertex capacities and demands. Beside combinatorial proofs based on potentials, Gallai gave proofs based on linear programming duality and total unimodularity.

A minimum-cost flow algorithm (in disguised form) was given by Ford and Fulkerson [1958b], to solve the 'dynamic flow' problem described in Section 12.5c. They described a method which essentially consists of repeatedly finding a zero-length $r-s$ path in the residual graph, making lengths nonnegative by translating the cost with the help of the current potential p. If no zero-length path exists, the potential is updated. (This is Routine I of Ford and Fulkerson [1958b].) The complexity of this was studied by Fulkerson [1958].

Yakovleva [1959] gave some implementations of the method of Kantorovich and Gavurin [1949]. The paper considers three cases of the problem in a digraph with demands (positive, negative, and zero) of vertices and costs of arcs: (i) noncapacitated case, (ii) capacitated case, and (iii) bipartite case (without zero demands). Two methods are developed for finding feasible potentials or improving the current flow. Time bounds are not indicated.

Among the other early algorithms for minimum-cost flow are successive shortest paths methods (Busacker and Gowen [1960], Iri [1960]), *out-of-kilter* methods (Minty [1960], Fulkerson [1961]), cycle-cancelling (Klein [1967]), and successive shortest paths maintaining potentials (Tomizawa [1971], Edmonds and Karp [1972]). An alternative method, which transforms the transportation problem to a nonlinear programming problem, with computational results, was given by Gerstenhaber [1958,1960].

Polynomial-time algorithms

Edmonds and Karp [1972] gave the first polynomial-time algorithm for the minimum-cost flow problem, based on capacity-scaling. They realized that in fact the method is only *weakly* polynomial; that is, the number of steps depends also on the size of the numbers in the input:

> Although it is comforting to know that the minimum-cost flow algorithm terminates, the bounds on the number of augmentations are most unfavorable. The scaling method of the next two sections is a variant of this algorithm in which the bound depends logarithmically, rather than linearly, on the capacities. A challenging open problem is to emulate the results of Section 1.2 for the maximum-value flow problem by giving a method for the minimum-cost flow problem having a bound on computation which is a polynomial in the number of nodes, and is independent of both costs and capacities.

Tarjan [1983] wrote: 'There is still much to be learned about the minimum cost flow problem'. Soon after, Edmonds and Karp's question was resolved by Tardos [1985a], by giving a strongly polynomial-time minimum-cost circulation algorithm. Her work has inspired a stream of further developments, part of which was discussed in Chapter 12.

Chapter 22

Transversals

The study of transversals of a family of sets is close to that of matchings in a bipartite graph, but with a shift in focus. While matchings are subsets of the edge set, transversals are subsets of one of the colour classes. This gives rise to a number of optimization and polyhedral problems and results that deserve special attention.
In this chapter we study transversals of one family of sets, while in the next chapter we go over to *common* transversals of two families of sets.

22.1. Transversals

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets. A set T is called a *transversal* of $\mathcal{A}$ if there exist distinct elements $a_1 \in A_1$, $\ldots$, $a_n \in A_n$ such that $T = \{a_1, \ldots, a_n\}$. So T is an *unordered* set with $|T| = n$. (Instead of 'transversal' one uses also the term *system of distinct representatives* or *SDR*.)

Transversals are closely related to matchings in bipartite graphs. In particular, the basic result on the existence of a transversal (Hall [1935]), is a consequence of Kőnig's matching theorem. This can be seen with the following basic construction of a bipartite graph $G = (V, E)$ associated with a family $\mathcal{A} = (A_1, \ldots, A_n)$ of subsets of a set S:

(22.1) $$V := \{1, \ldots, n\} \cup S,$$ $$E := \{\{i, s\} \mid i = 1, \ldots, n; s \in A_i\},$$

assuming that S is disjoint from $\{1, \ldots, n\}$ (which for our purposes can be done without loss of generality). So G has colour classes $\{1, \ldots, n\}$ and S. (This construction was given by Skolem [1917].)

Then trivially

(22.2) a set T is a transversal of $\mathcal{A}$ if and only if G has a matching M of size n such that T is the set of vertices in S covered by M.

So the existence of a transversal of $\mathcal{A}$ can be reduced to the existence of a matching in G of size n. Hence Kőnig's matching theorem applies to the existence of transversals.

It is convenient to introduce the following notation, for any family $(A_1, \ldots, A_n)$ of sets and any $I \subseteq \{1, \ldots, n\}$:

(22.3) $$A_I := \bigcup_{i \in I} A_i.$$

Theorem 22.1 (Hall's marriage theorem)**.** *A family $\mathcal{A} = (A_1, \ldots, A_n)$ of sets has a transversal if and only if*

(22.4) $$|A_I| \geq |I|$$

for each subset I of $\{1, \ldots, n\}$.

Proof. Necessity of the condition being easy, we prove sufficiency. Let G be the graph associated to $\mathcal{A}$ (as in (22.1)). Now the theorem is equivalent to Theorem 16.7 (taking $U := \{1, \ldots, n\}$). ∎

Condition (22.4) is called *Hall's condition*. The name 'marriage theorem' is due to Weyl [1949].

The polynomial-time algorithm given in Section 16.3 for finding a maximum matching in a bipartite graph directly yields a polynomial-time algorithm for finding a transversal of a family $(A_1, \ldots, A_n)$ of sets. In fact, Theorem 16.5 implies an $O(\sqrt{n}\, m)$ algorithm, where $m := \sum_i |A_i|$.

22.1a. Alternative proofs of Hall's marriage theorem

We give two alternative, direct proofs of the sufficiency of Hall's condition (22.4) for the existence of a transversal. Call a subset I of $\{1, \ldots, n\}$ *tight* if equality holds in (22.4).

If there is a $y \in A_n$ such that $A_1 \setminus \{y\}, \ldots, A_{n-1} \setminus \{y\}$ has a transversal, then we are done. Hence, we may assume that for each $y \in A_n$ there is a tight $I \subseteq \{1, \ldots, n-1\}$ with $y \in A_I$ (using induction).

The proof given by Easterfield [1946] (also by M. Hall [1948], Halmos and Vaughan [1950], and Mann and Ryser [1953]) continues as follows. Choose any such tight subset I. Without loss of generality, $I = \{1, \ldots, k\}$. By induction, $(A_1, \ldots, A_k)$ has a transversal, which must be $T := A_I$. Moreover, $(A_{k+1} \setminus T, \ldots, A_n \setminus T)$ has a transversal, Z say. This follows inductively, since for each $J \subseteq \{k+1, \ldots, n\}$,

(22.5) $$\Big|\bigcup_{i \in J}(A_i \setminus T)\Big| = \Big|\bigcup_{i \in I \cup J} A_i\Big| - |T| \geq |I| + |J| - |T| = |J|.$$

Then $T \cup Z$ is a transversal of $(A_1, \ldots, A_k, A_{k+1}, \ldots, A_n)$.

The proof due to Everett and Whaples [1949] continues slightly different. They noted that the collection of tight subsets of $\{1, \ldots, n\}$ is closed under taking intersections and unions. That is, if I and J are tight, then also $I \cap J$ and $I \cup J$ are tight, since

(22.6) $$|I| + |J| = |A_I| + |A_J| \geq |A_{I \cap J}| + |A_{I \cup J}| \geq |I \cap J| + |I \cup J| = |I| + |J|,$$

giving equality throughout. (In (22.6), the first inequality holds as $A_{I \cap J} \subseteq A_I \cap A_J$ and $A_{I \cup J} = A_I \cup A_J$.)

Since for each $y \in A_n$ there is a tight subset I of $\{1, \ldots, n-1\}$ with $y \in A_I$, it follows, by taking the union of them, that there is a tight subset I of $\{1, \ldots, n-1\}$ with $A_n \subseteq A_I$. For $J := I \cup \{n\}$ this gives the contradiction $|A_J| = |A_I| = |I| < |J|$.

The the closedness of tight subsets under intersections and unions was also noticed by Maak [1936] and Weyl [1949], who gave alternative proofs of a theorem of Rado and Hall's marriage theorem, respectively.

Edmonds [1967b] gave a linear-algebraic proof of Hall's marriage theorem (cf. Section 16.2b). Ford and Fulkerson [1958c] derived Hall's marriage theorem from the max-flow min-cut theorem.

22.2. Partial transversals

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets. A set T is called a *partial transversal* if it is a transversal of some subfamily $(A_{i_1}, \ldots, A_{i_k})$ of $(A_1, \ldots, A_n)$. (Instead of 'partial transversal' one uses also the term *partial system of distinct representatives* or *partial SDR.*)

Again, by the construction (22.1), we can study partial transversals with the help of bipartite matching theory. In particular, if G is the graph associated to a family $\mathcal{A}$ of subsets of a set S,

(22.7) a set T is a partial transversal of $\mathcal{A}$ if and only if G has a matching M such that T is the set of vertices in S covered by M.

This yields the following so-called defect form of Hall's marriage theorem, which is equivalent to Kőnig's matching theorem (cf. Ore [1955]):

Theorem 22.2 (defect form of Hall's marriage theorem)**.** *Let* $\mathcal{A} = (A_1, \ldots, A_n)$ *be a family of subsets of a set* S*. Then the maximum size of a partial transversal of* $\mathcal{A}$ *is equal to the minimum value of*

(22.8) $$|S \setminus X| + |\{i \mid A_i \cap X \neq \emptyset\}|,$$

where X *ranges over all subsets of* S*.*

Proof. Let G be the graph constructed in (22.1). The maximum size of a partial transversal of $\mathcal{A}$ is equal to the maximum size of a matching in G. By Kőnig's matching theorem, this is equal to the minimum size of a vertex cover of G. This minimum is attained by a vertex cover of form $(S \setminus X) \cup \{i \mid A_i \cap X \neq \emptyset\}$, which shows the theorem. ∎

An equivalent way of characterizing the maximum size of a partial transversal is:

Corollary 22.2a. *The maximum size of a partial transversal of* $\mathcal{A}$ *is equal to the minimum value of*

(22.9) $$\Big|\bigcup_{i \in I} A_i\Big| + n - |I|,$$

taken over $I \subseteq \{1, \ldots, n\}$*.*

Proof. Directly from Theorem 22.2, since we can assume that $S \setminus X = A_I$ where $I := \{i \mid A_i \cap X = \emptyset\}$. ▮

Note that it needs an argument to state that each partial transversal is a subset of a transversal, if a transversal exists. This was shown by Hoffman and Kuhn [1956b] (solving a problem of Mann and Ryser [1953]):

Theorem 22.3. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a system of sets having a transversal. Then each partial transversal is contained in a transversal.*

Proof. Directly from Theorem 16.8, using construction (22.1) (taking $R := \{1, \ldots, n\} \cup T$, where T is a partial transversal). ▮

One can generalize this to the case where the family need not have a transversal:

Theorem 22.4. *Let $\mathcal{A}$ be a family of sets. Then each partial transversal is contained in a maximum-size partial transversal.*

Proof. Again directly from Theorem 16.8, using construction (22.1). ▮

In other words, each inclusionwise maximal partial transversal is a maximum-size partial transversal. This is the basis of the fact that partial transversals form the independent sets of a matroid — see Chapter 39. It is equivalent to:

Corollary 22.4a (exchange property of transversals)**.** *Let $\mathcal{A}$ be a family of sets and let T and T' be partial transversals of $\mathcal{A}$, with $|T| < |T'|$. Then there exists an $s \in T' \setminus T$ such that $T \cup \{s\}$ is a partial transversal.*

Proof. To prove this, we can assume that each set in $\mathcal{A}$ is contained in $T \cup T'$. This implies that, if no s as required exists, T is an inclusionwise maximal partial transversal. However, as $|T'| > |T|$, this contradicts Theorem 22.4. ▮

Brualdi and Scrimger [1968] (extending a result of Mirsky and Perfect [1967]) observed:

Theorem 22.5. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets, let k be the maximum size of a partial transversal, and let $\mathcal{A}' = (A_1, \ldots, A_k)$ have a transversal. Then each maximum-size partial transversal of $\mathcal{A}$ is a transversal of $\mathcal{A}'$.*

Proof. Via construction (22.1) this follows from Corollary 16.8b. ▮

So when studying the collection of partial transversals of a certain collection $\mathcal{A}$ of sets, we can assume that $\mathcal{A}$ has a transversal.

22.3. Weighted transversals

Consider the problem of finding a minimum-weight transversal: given a family $\mathcal{A} = (A_1, \ldots, A_n)$ of subsets of a set S and a weight function $w : S \to \mathbb{Q}$, find a transversal T of $\mathcal{A}$ minimizing $w(T)$. This problem can be easily reduced to a minimum-weight perfect matching problem, implying that a minimum-weight transversal can be found in strongly polynomial time. In fact:

Theorem 22.6. *A minimum-weight transversal can be found in time $O(nm)$ where n is the number of sets and $m := \sum_i |A_i|$.*

Proof. Make the graph G as in (22.1) and define $w(\{i, s\}) := w(s)$ for each edge $\{i, s\}$ of G. Denote $R := \{1, \ldots, n\}$. Starting with $M = \emptyset$, we can apply the Hungarian method, to obtain an extreme matching of size n. The elements of S covered by M form a maximum-weight transversal. As each iteration of the Hungarian method takes $O(m)$ time, this gives the theorem. ∎

Note that in this algorithm, we grow a partial transversal until it is a (complete) transversal. In this respect it is a 'greedy method': we never backtrack. Again, this is a preview of the fact that transversals form a 'matroid' — see Chapter 39.

The method similarly solves the problem of finding a maximum-weight partial transversal:

Theorem 22.7. *A maximum-weight partial transversal can be found in time $O(rm)$, where r is the maximum size of a partial transversal and where $m := \sum_i |A_i|$.*

Proof. As above. ∎

22.4. Min-max relations for weighted transversals

We can also obtain a min-max relation for the minimum weight of a transversal:

Theorem 22.8. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set S having a transversal and let $w : S \to \mathbb{Z}$ be a weight function. Then the minimum weight of a transversal of $\mathcal{A}$ is equal to the maximum value of*

$$(22.10) \qquad y(S) + \sum_{i=1}^{n} \min_{s \in A_i} (w(s) - y(s))$$

taken over $y : S \to \mathbb{Z}_+$.

Proof. Let $t := |S|$. For $i = n+1, \ldots, t$, let $A_i := S$. Consider the bipartite graph $G = (V, E)$ defined by (22.1), for the family $(A_1, \ldots, A_t)$. Define a length function l on the edges of G as follows. For any edge $e = is$ of G, with $s \in A_i$, define $l_e := w(s)$ if $i \leq n$ and $l_e := 0$ otherwise. Then the minimum weight of a transversal of $(A_1, \ldots, A_n)$ is equal to the minimum length of a perfect matching in G. By Theorem 17.5 (a variant of Egerváry's theorem), the latter value is equal to the maximum value of $y(V)$ where $y \in \mathbb{Q}^V$ with $y(s) + y(i) \leq l(is)$ for each $i = 1, \ldots, t$ and $s \in A_i$. We can assume that the minimum of $y(s)$ over $s \in S$ is equal to 0 (since subtracting a constant to $y(s)$ for any $s \in S$ and adding it to $y(i)$ for any $i \in \{1, \ldots, t\}$ maintains the properties required for y). Then $y(i) = \min_{s \in A_i}(w(s) - y(s))$ if $i \leq n$ and $y(i) = 0$ if $i > n$. So $y(V)$ is equal to the value of (22.10). ∎

A min-max relation for the maximum weight of a partial transversal follows similarly:

Theorem 22.9. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ be a family of subsets of a set S and let $w : S \to \mathbb{Z}_+$ be a weight function. Then the maximum weight of a partial transversal of $\mathcal{A}$ is equal to the minimum value of*

$$(22.11) \qquad y(S) + \sum_{i=1}^{k} \max\{0, \max_{s \in A_i}(w(s) - y(s))\}$$

over functions $y : S \to \mathbb{Z}_+$.

Proof. Directly from Egerváry's theorem (Theorem 17.1), using construction (22.1). ∎

22.5. The transversal polytope

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set S. The *partial transversal polytope* $P_{\text{partial transversal}}(\mathcal{A})$ of $\mathcal{A}$ is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of the partial transversals of $\mathcal{A}$. That is,

$$(22.12) \qquad P_{\text{partial transversal}}(\mathcal{A}) = \text{conv.hull}\{\chi^T \mid T \text{ is a partial transversal of } \mathcal{A}\}.$$

It is easy to see that each vector x in the partial transversal polytope satisfies:

$$(22.13) \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for each } s \in S, \\ \text{(ii)} & x(S \setminus A_I) \leq n - |I| & \text{for each } I \subseteq \{1, \ldots, n\}. \end{array}$$

Corollary 22.9a. *System* (22.13) *determines the partial transversal polytope and is TDI.*

Proof. Consider a weight function $w : S \to \mathbb{Z}_+$. Let ω be the maximum weight of a partial transversal. By Theorem 22.9, there exists a function $y : S \to \mathbb{Z}_+$ such that

$$\omega = y(S) + \sum_{i=1}^{n} \max\{0, \max_{s \in A_i}(w(s) - y(s))\}. \tag{22.14}$$

For each $j \in \mathbb{Z}_+$, let I_j be the set of $i \in \{1, \ldots, n\}$ with

$$\max_{s \in A_i}(w(s) - y(s)) \leq j. \tag{22.15}$$

So $I_j = \{1, \ldots, n\}$ for j large enough.

Then

$$w - y \leq \sum_{j=0}^{\infty} \chi^{S \setminus A_{I_j}}, \tag{22.16}$$

since for $k := w(s) - y(s)$, we have for each $j < k$ there is no $i \in I_j$ with $s \in A_i$. Hence $s \in S \setminus A_{I_j}$ for all $j < k$. So y and the I_j give an integer feasible dual solution.

The fact that they are optimum follows from:

$$\begin{aligned} y(S) + \sum_{j=0}^{\infty}(n - |I_j|) &= y(S) + \sum_{j=0}^{\infty} \sum_{\substack{i = 1 \\ \max_{s \in A_i}(w(s) - y(s)) > j}}^{n} 1 \\ &= y(S) + \sum_{i=1}^{n} \sum_{\substack{j = 0 \\ \max_{s \in A_i}(w(s) - y(s)) > j}}^{\infty} 1 \\ &= y(S) + \sum_{i=1}^{n} \max\{0, \max_{s \in A_i}(w(s) - y(s))\} = \omega, \end{aligned} \tag{22.17}$$

by (22.14). ∎

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set S. The *transversal polytope* $P_{\text{transversal}}(\mathcal{A})$ of $\mathcal{A}$ is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of the transversals of $\mathcal{A}$. That is,

$$P_{\text{transversal}}(\mathcal{A}) = \text{conv.hull}\{\chi^T \mid T \text{ is a transversal of } \mathcal{A}\}. \tag{22.18}$$

It is easy to see that each vector x in the transversal polytope satisfies:

$$\begin{array}{rll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for each } s \in S, \\ \text{(ii)} & x(A_I) \geq |I| & \text{for each } I \subseteq \{1, \ldots, n\}, \\ \text{(iii)} & x(S) = n. & \end{array} \tag{22.19}$$

Corollary 22.9b. *System* (22.19) *determines the transversal polytope and is TDI.*

Proof. The transversal polytope is the facet of the partial transversal polytope determined by the equality $x(S) = n$. This is constraint (22.13)(ii) for $I = \emptyset$, set to equality. Now each inequality in (22.19) is a nonnegative integer combination of the inequalities in (22.13) and of $-x(S) \leq -n$ (since $-x(A_I) = x(S \setminus A_I) - x(S) \leq (n - |I|) - n = -|I|$). So using Theorem 5.25, the corollary follows. ∎

One may note that the number of facets of the matching polytope of a bipartite graph $G = (V, E)$ is at most $|V| + |E|$, while the number of facets of the closely related partial transversal polytope can be exponential in the size of the input (the family $\mathcal{A}$). In fact, the partial transversal polytope is a projection of the matching polytope of the corresponding graph. Thus we have an illustration of the phenomenon that projection can increase the number of facets dramatically, while this has no negative effect on the complexity of the corresponding optimization problem.

22.6. Packing and covering of transversals

The following min-max relation for the maximum number of disjoint transversals is an easy consequence of Hall's marriage theorem:

Theorem 22.10. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets and let k be a natural number. Then $\mathcal{A}$ has k disjoint transversals if and only if*

$$(22.20) \qquad |A_I| \geq k|I|$$

for each subset I of $\{1, \ldots, n\}$.

Proof. Replace each set A_i by k copies, yielding the family $\mathcal{A}'$. Then by Hall's marriage theorem and (22.20), $\mathcal{A}'$ has a transversal. This can be split into k transversals of $\mathcal{A}$. ∎

A generalization to disjoint partial transversals of prescribed sizes was given by Higgins [1959] (cf. Mirsky [1966], Mirsky and Perfect [1966]):

Theorem 22.11. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets and let $d_1, \ldots, d_k \in \{1, \ldots, n\}$. Then $\mathcal{A}$ has k disjoint partial transversals of sizes $d_1, \ldots, d_k$ respectively if and only if*

$$(22.21) \qquad |A_I| \geq \sum_{j=1}^{k} \max\{0, |I| - n + d_j\}$$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Necessity follows from the fact that if $T_1, \ldots, T_k$ are partial transversals as required, then

$$|A_I| \geq \sum_{j=1}^{k} |A_I \cap T_j| \geq \sum_{j=1}^{k} \max\{0, |I| - n + d_j\} \tag{22.22}$$

for each $I \subseteq \{1, \ldots, n\}$, since $|A_I \cap T_j| + (n - d_j) \geq |I|$.

To see sufficiency, let $B_1, \ldots, B_k$ be disjoint sets, disjoint also from all A_i, with $|B_j| = n - d_j$ for $j = 1, \ldots, k$. Define $A_{i,j} := A_i \cup B_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Then $\mathcal{A}$ has k disjoint partial transversals as required, if $(A_{i,j} \mid i = 1, \ldots, n; j = 1, \ldots, k)$ has a transversal. So it suffices to check Hall's condition (22.4) for the latter family. Take $K \subseteq \{1, \ldots, n\} \times \{1, \ldots, k\}$. Let $I := \{i \mid \exists j : (i, j) \in K\}$ and $J := \{j \mid \exists i : (i, j) \in K\}$. Then

$$\begin{aligned} |\bigcup_{(i,j)\in K} A_{i,j}| &= |\bigcup_{i \in I} A_i| + |\bigcup_{j \in J} B_j| \\ &\geq \sum_{j=1}^{k} \max\{0, |I| - n + d_j\} + \sum_{j \in J} (n - d_j) \geq \sum_{j \in J} |I| = |I| \cdot |J| \\ &\geq |K|. \end{aligned} \tag{22.23}$$

∎

(A proof based on total unimodularity was given by Hoffman [1976b].)

As to covering by partial transversals, Mirsky [1971b] (p. 51) mentioned that R. Rado proved in 1965:

Theorem 22.12. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set S and let k be a natural number. Then S can be covered by k partial transversals if and only if*

$$k \cdot |\{i \mid A_i \cap X \neq \emptyset\}| \geq |X| \tag{22.24}$$

for each subset X of S.

Proof. Let $\mathcal{A}'$ be the family obtained from $\mathcal{A}$ by taking each set k times. Then S can be covered by k partial transversals if and only if S is a partial transversal of $\mathcal{A}'$. By the defect form of Hall's marriage theorem (Theorem 22.2), this last is equivalent to the condition that

$$|S \setminus X| + k \cdot |\{i \mid A_i \cap X \neq \emptyset\}| \geq |S| \tag{22.25}$$

for each $X \subseteq S$. This is equivalent to (22.24). ∎

For covering by partial transversals of prescribed size, there is the following easy consequence of the exchange property of transversals (Corollary 22.4a):

Theorem 22.13. *Let $\mathcal{A}$ be a family of subsets of a set S and let $k \in \mathbb{Z}_+$. If S can be covered by k partial transversals, it can be covered by k partial transversals each of size $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$.*

Proof. Let $T_1, \ldots, T_k$ be partial transversals partitioning S. If $|T_i| \geq |T_j| + 2$ for some i, j, we can replace T_i and T_j by $T_i \setminus \{s\}$ and $T_j \cup \{s\}$ for some

$s \in T_i$. Repeating this, we finally achieve that $\big||T_i| - |T_j|\big| \leq 1$ for all i, j. Hence $\lfloor |S|/k \rfloor \leq |T_i| \leq \lceil |S|/k \rceil$ for all i. ∎

22.7. Further results and notes

22.7a. The capacitated case

Capacitated versions of the theorems on transversals can be derived straightforwardly from the previous results. First, Halmos and Vaughan [1950] showed the following generalized (but straightforwardly equivalent) version of Hall's marriage theorem:

Theorem 22.14. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets and let $b \in \mathbb{Z}_+^n$. Then there exist disjoint subsets $B_1, \ldots, B_n$ of $A_1, \ldots, A_n$ respectively with $|B_i| = b_i$ for $i = 1, \ldots, n$ if and only if*

(22.26) $\qquad |A_I| \geq b(I)$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Let $\mathcal{A}'$ be the family of sets obtained from $\mathcal{A}$ by repeating any A_i b_i times. Then the existence of the B_i is equivalent to the existence of a transversal of $\mathcal{A}'$. Moreover, (22.26) is equivalent to Hall's condition for $\mathcal{A}'$. ∎

This theorem concerns taking multiplicities on the sets in $\mathcal{A}$. If we put multiplicities on the elements of S, there is the following observation of R. Rado (as reported by Mirsky and Perfect [1966]):

Theorem 22.15. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets and let $r \in \mathbb{Z}_+$. Then there exist $x_i \in A_i$ $(i = 1, \ldots, n)$ such that no element occurs more than r times among the s_i if and only if*

(22.27) $\qquad |A_I| \geq |I|/r$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Let $\mathcal{A}'$ be the family of sets obtained from $\mathcal{A}$ by replacing any A_i by $A_i \times \{1, \ldots, r\}$. Then the existence of the required s_i is equivalent to the existence of a transversal of $\mathcal{A}'$. Moreover, (22.27) is equivalent to Hall's condition for $\mathcal{A}'$. ∎

These theorems are in fact direct consequences of the general Theorem 21.28. This theorem moreover gives the following result of Vogel [1961], which puts multiplicities both on the sets in $\mathcal{A}$ and on the elements of the underlying set S:

Theorem 22.16. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set S. Let $a \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+^S$. Then there exist subsets $B_1, \ldots, B_n$ of $A_1, \ldots, A_n$ respectively such that $|B_i| = a_i$ for $i = 1, \ldots, n$ and such that each $s \in S$ occurs in at most $b(s)$ of the B_i if and only if*

(22.28) $\qquad b(X) + \sum_{i \in I} |A_i \setminus X| \geq a(I)$

for each $X \subseteq S$ *and each* $I \subseteq \{1, \ldots, n\}$.

Proof. Consider the system

$$\text{(22.29)} \qquad \begin{array}{ll} 0 \leq x(i,s) \leq 1 & \text{for } i \in \{1,\ldots,n\} \text{ and } s \in A_i, \\ a_i \leq x(\delta(i)) \leq a_i & \text{for } i \in \{1,\ldots,n\}, \\ 0 \leq x(\delta(s)) \leq b_s & \text{for } s \in S, \end{array}$$

and apply Theorem 21.28. ∎

This has as special case (Vogel [1961]):

Corollary 22.16a. *Let* $\mathcal{A} = (A_1, \ldots, A_n)$ *be a family of subsets of a set* S *and let* $r, s \in \mathbb{Z}_+$. *Then there exist subsets* $B_1, \ldots, B_n$ *of* $A_1, \ldots, A_n$ *respectively such that* $|B_i| = s$ *for each* i *and such that each element belongs to at most* r *of the* B_i *if and only if*

$$\text{(22.30)} \qquad r|X| + \sum_{i \in I} |A_i \setminus X| \geq s|I|$$

for each $I \subseteq \{1, \ldots, n\}$ *and each* $X \subseteq S$.

Proof. This is a special case of Theorem 22.16. ∎

Similar methods apply to systems of *restricted* representatives, considered by Ford and Fulkerson [1958c]. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a collection of subsets of a set S and let $a, b \in \mathbb{Z}_+^S$ with $a \leq b$. A *system of restricted representatives* (or *SRR*) of $\mathcal{A}$ (with respect to a and b) is a sequence $(s_1, \ldots, s_n)$ such that

$$\text{(22.31)} \qquad \begin{array}{rl} \text{(i)} & s_i \in A_i \text{ for } i = 1, \ldots, n; \\ \text{(ii)} & a(s) \leq |\{i \mid s_i = s\}| \leq b(s) \text{ for } s \in S. \end{array}$$

Ford and Fulkerson [1958c] showed:

Theorem 22.17. $\mathcal{A}$ *has a system of restricted representatives if and only if*

$$\text{(22.32)} \qquad a(S - \bigcup_{i \notin I} A_i) \leq |I| \leq b(\bigcup_{i \in I} A_i)$$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Consider the system

$$\text{(22.33)} \qquad \begin{array}{ll} 0 \leq x(i,s) \leq \infty & \text{for } i \in \{1,\ldots,n\}, s \in A_i, \\ x(\delta(i)) = 1 & \text{for } i \in \{1,\ldots,n\}, \\ a_s \leq x(\delta(s)) \leq b_s & \text{for } s \in S, \end{array}$$

and apply Theorem 21.28. ∎

(For an alternative proof, see Mirsky [1968a].)

Considering both upper and lower bounds, the following theorem of Hoffman and Kuhn [1956a] follows from Hoffman's circulation theorem:

Theorem 22.18. *Let* $\mathcal{A} = (A_1, \ldots, A_n)$ *be a collection of subsets of a set* S, *let* $\mathcal{P} = (P_1, \ldots, P_m)$ *be a partition of* S, *and let* $a, b \in \mathbb{Z}_+^m$ *with* $a \leq b$. *Then* $\mathcal{A}$ *has a transversal* T *satisfying* $a_i \leq |T \cap P_i| \leq b_i$ *for each* $i = 1, \ldots, m$ *if and only if*

$$|P_I \cap A_J| \geq \max\{|J| - b(\overline{I}), |J| - n + a(I)\} \tag{22.34}$$

for all $I \subseteq \{1, \ldots, m\}$ *and* $J \subseteq \{1, \ldots, n\}$*, where* $\overline{I} := \{1, \ldots, n\} \setminus I$*.*

Proof. Make a directed graph as follows. Its vertex set is $\{r\} \cup \{u_1, \ldots, u_n\} \cup S \cup \{p_1, \ldots, p_m\} \cup \{t\}$, and there are arcs

$$\begin{aligned} &(r, u_i) \text{ for } i = 1, \ldots, n,\\ &(u_i, s) \text{ for } i = 1, \ldots, n \text{ and } s \in A_i,\\ &(s, p_j) \text{ for } j = 1, \ldots, m \text{ and } s \in P_j,\\ &(p_j, t) \text{ for } j = 1, \ldots, m. \end{aligned} \tag{22.35}$$

Put lower bound a_j and capacity b_j on each arc (p_j, t). On any other arc, put lower bound 0 and capacity 1. Then a transversal as required exists if and only if there is an integer $r-t$ flow of value n satisfying the lower bounds and capacities. Applying Corollary 11.2e gives the present theorem. ∎

(The proof of Hoffman and Kuhn [1956a] is based on the duality theorem of linear programming. Gale [1956,1957] and Fulkerson [1959a] derived the theorem from network flow theory. For further extensions, see Mirsky [1968b].)

22.7b. A theorem of Rado

Rado [1938] proved the following generalization (but also consequence) of Hall's marriage theorem:

Theorem 22.19. *Let $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ be sets. Then there exists an injection $f : A_1 \cup \cdots \cup A_n \to B_1 \cup \cdots \cup B_n$ such that $f[A_i] \subseteq B_i$ for $i = 1, \ldots, n$ if and only if each set obtained by intersections and unions of sets from $A_1, \ldots, A_n$ has size at most the size of the result of the same operations applied to $B_1, \ldots, B_n$.*

Proof. Let $A := A_1 \cup \cdots \cup A_n$. For each $s \in A$, define

$$C_s := \bigcap_{\substack{i\\ s \in A_i}} B_i. \tag{22.36}$$

Then for each subset S of A one has

$$|\bigcup_{s \in S} C_s| = |\bigcup_{s \in S} \bigcap_{\substack{i\\ s \in A_i}} B_i| \geq |\bigcup_{s \in S} \bigcap_{\substack{i\\ s \in A_i}} A_i| \geq |S|. \tag{22.37}$$

Hence, by Hall's marriage theorem, $(C_s \mid s \in A)$ has a transversal. This gives an injection $f : A \to B_1 \cup \cdots \cup B_n$ with $f(s) \in C_s$ for $s \in A$. This is as required. ∎

22.7c. Further notes

Shmushkovich [1939], de Bruijn [1943], Hall [1948], Henkin [1953], Tutte [1953], Mirsky [1967], Rado [1967a] (with H.A. Jung), Brualdi and Scrimger [1968], Folkman [1970], McCarthy [1973], Damerell and Milner [1974], Steffens [1974], Podewski and Steffens [1976], Nash-Williams [1978], Aharoni [1983c], and Aharoni, Nash-Williams, and Shelah [1983] considered extensions of Hall's marriage theorem to the infinite case. Perfect [1968] gave proofs of theorems on transversals with Menger's theorem.

For a 'very general theorem' see Brualdi [1969a]. For counting transversals, see Hall [1948], Rado [1967b], and Ostrand [1970].

Gale [1968] showed that for any family $\mathcal{A}$ of subsets of a finite set S and any total order $<$ on S, there is a transversal T of $\mathcal{A}$ such that for each transversal T' of $\mathcal{A}$ there exists a one-to-one function $\phi : T' \to T$ with $\phi(s) \geq s$ for each $s \in T'$. (Gale showed that this in fact characterizes matroids.)

The standard work on transversal theory is Mirsky [1971b]. Also Brualdi [1975] and Welsh [1976] provide surveys. Surveys on the relations between the theorems of Hall, Kőnig, Menger, and Dilworth were given by Jacobs [1969] and Reichmeider [1984].

22.7d. Historical notes on transversals

Results on transversals go back to the papers by Miller [1910] and Chapman [1912], who showed that if H is a subgroup of a finite group G, then the partitions of G into left cosets and into right cosets have a common transversal. This is an easy result, due to the fact that each component of the intersection graph of left and right cosets is a complete bipartite graph. This implies that any common partial transversal can be extended to a common (full) transversal (Chapman [1912]).

This result was extended by Scorza [1927] to: if H and K are subgroups of a finite group G, with $|H| = |K|$, then there exist $x_1, \ldots, x_m \in G$ with $x_1H \cup \cdots \cup x_mH = G = Kx_1 \cup \cdots \cup Kx_m$ and $m = |G|/|H|$. (Again this can be derived easily from the fact that each component of the intersection graph of left cosets of H and right cosets of K is a complete bipartite graph.)

As an extension of these results, in October 1926, van der Waerden [1927] presented the following theorem at the *Mathematisches Seminar* in Hamburg:

> *Es seien zwei Klasseneinteilungen einer endlichen Menge $\mathcal{M}$ gegeben. Die eine soll die Menge in μ zueinander fremde Klassen $\mathcal{A}_1, \ldots, \mathcal{A}_\mu$ zu je n Elementen zerlegen, die andere ebenfalls in μ fremde Klassen $\mathcal{B}_1, \ldots, \mathcal{B}_\mu$ zu je n Elementen. Dann gibt es ein System von Elementen $x_1, \ldots, x_\mu$, derart, daß jede A-Klasse und ebenso jede B-Klasse under den x_i durch ein Element vertreten wird.*[41]

The proof of van der Waerden is based on an augmenting path argument. Moreover, van der Waerden remarked that E. Artin had communicated orally to him that the result can be sharpened to the existence of n disjoint such common transversals.

In a note added in proof, van der Waerden observed that his theorem follows from Kőnig's theorem on the existence of a perfect matching in a regular bipartite graph:

> **Zusatz bei der Korrektur.** Ich bemerke jetzt, daß der hier bewiesene Satz mit einem Satz von Dénes Kőnig über reguläre Graphen äquivalent ist.[42]

---

[41] Let be given two partitions of a finite set $\mathcal{M}$. One of them should decompose the set into μ mutually disjoint classes $\mathcal{A}_1, \ldots, \mathcal{A}_\mu$ each of n elements, the other likewise in μ disjoint classes $\mathcal{B}_1, \ldots, \mathcal{B}_\mu$ each of n elements. Then there exists a system of elements $x_1, \ldots, x_\mu$ such that each $\mathcal{A}$-class and likewise each $\mathcal{B}$-class is represented by one element among the x_i.

[42] **Note added in proof.** I now notice that the theorem proved here is equivalent to a theorem of Dénes Kőnig on regular graphs.

Van der Waerden's article is followed by an article of Sperner [1927] (presented at the *Mathematisches Seminar* in Januari 1927) that gives a 'simple proof' of van der Waerden's result. We quote the full article (containing page references to van der Waerden's paper):

> Der auf S. 185 ff. bewiesene Satz gestattet auch folgenden einfachen Beweis.
> Der Satz lautete:
> Zwei beliebige Klasseneinteilungen von $m \cdot n$ Elementen in m Klassen zu je n Elementen haben immer ein gemeinsames Repräsentantensystem (vgl. S. 185).
> Der Satz ist evident für die Klassenzahl 1. Wir nehmen an, er sei bewiesen für die Klassenzahl m (und beliebiges n). Dan folgt für dieses m:
> 1. Die beiden Klasseneinteilungen haben sogar n verschiedene und zueinander fremde Repräsentantensysteme.
> Beweis wie auf S. 187 oben.
> 2. Streicht man daher in beiden Einteilungen dieselben k Elemente, wo $0 \leq k \leq n-1$, dan werden höchstens $n-1$ Repräsentantensysteme verletzt und wenigstens eins bleibt erhalten. Da man auch umgekehrt $m \cdot n - k$ Elemente durch k neue ergänzen kann, um diese nachher wieder zu streichen, so gilt:
> Zwei beliebige Klasseneinteilungen von $m \cdot n - k$ Elementen in m Klassen zu je höchstens n Elementen, wo $0 \leq k \leq n-1$, haben immer ein gemeinsames Repräsentantensystem.
> Nunmehr wenden wir vollständige Induktion an. Es seien zwei Klasseneinteilungen von $(m+1)\cdot n$ Elementen in $m+1$ Klassen zu je n Elementen gegeben. Dann greifen wir aus beiden Einteilungen je eine Klasse heraus, etwa die Klassen $\mathcal{A}$ und $\mathcal{B}$, die aber wenigstens 1 Element gemeinsam haben sollen, etwa A. Streichen wir dann in beiden Einteilungen die in $\mathcal{A}$ und $\mathcal{B}$ vorkommenden Elemente (also höchstens $2n-1$, aber wenigstens n Elemente), so bleiben zwei Klasseneinteilungen von $m \cdot n - k$ Elementen in je m Klassen zu je höchstens n Elementen übrig, wo $0 \leq k \leq n-1$. Zwei solche Einteilungen haben aber nach 2. ein gemeinsames Repräsentantensystem, das man sofort durch Hinzufügen von A zu einem gemeinsamen Repräsentantensysteme der beiden Einteilungen von $(m+1)n$ Elementen erweitert.[43]

[43] The theorem proved on p. 185 and following pages allows also the following simple proof.
The theorem reads:

Two arbitrary partitions of $m \cdot n$ elements into m classes of n elements each, always have a common system of representatives (cf. p. 185).

The theorem is evident for class number 1. We assume that it be proved for class number m (and arbitrary n). Then the following follows for this m:

1. Both partitions even have n different and disjoint systems of representatives.

Proof like on p. 187 above.

2. Therefore, if one cancels in both partitions the same k elements, where $0 \leq k \leq n-1$, then at most $n-1$ systems of representatives are injured and at least one is preserved. As one can also, reversely, complete $m \cdot n - k$ elements by k new ones, to cancel them after it again, the following therefore holds:

Two arbitrary partitions of $m \cdot n - k$ elements into m classes of at most n elements each, where $0 \leq k \leq n-1$, always have a common system of representatives.

Now we apply complete induction. Let be given two partitions of $(m+1)\cdot n$ elements into $m+1$ classes of n elements each. Then we select from each of the two partitions one class, say the classes $\mathcal{A}$ and $\mathcal{B}$, that however should have at least 1 element in common, say A. If we then cancel in both partitions the elements occurring in $\mathcal{A}$ and $\mathcal{B}$ (so at most $2n-1$, but at least n elements), two partitions of $m \cdot n - k$ elements into m classes of at most n elements each thus remain, where $0 \leq k \leq n-1$. Two such partitions have however, according to 2., a common system of representatives, that one extends, by adding A, to a common system of representatives of both partitions of $(m+1)n$ elements.

Hall

After having mentioned Kőnig's result on the existence of a common transversal for two partitions of a set where all classes have the same size, Hall [1935] said that he is 'concerned with a slightly different problem': to find a transversal

> for a finite collection of (arbitrarily overlapping) subsets of any given set of things. The solution, Theorem 1, is very simple.

Calling a transversal a 'C.D.R. (= complete system of distinct representatives)' and denoting a finite system $T_1, \ldots, T_m$ of subsets of a set S by '(1)', Hall formulated his theorem as follows:

> *In order that a C.D.R. of* (1) *shall exist, it is sufficient that for each* $k = 1, 2, \ldots, m$ *any selection of* k *of the sets* (1) *shall contain between them at least* k *elements of* S.

This result now is known as 'Hall's marriage theorem'.

In order to prove this theorem, Hall first showed the following lemma. Let $(A_1, \ldots, A_n)$ be a system of sets with at least one transversal and let R be the intersection of all transversals. Then there is an $I \subseteq \{1, \ldots, n\}$ with $A_I = R$ and $|I| = |R|$.

Hall proved this with the help of an alternating path argument. Having the lemma, the theorem is easy, by induction on n: we may assume that $(A_1, \ldots, A_{n-1})$ has a transversal; let R' be the intersection of all these transversals. So by the lemma, $R' = A_{I'}$ for some $I' \subseteq \{1, \ldots, n-1\}$ with $|I'| = |R'|$. Hence $A_n \not\subseteq R'$, since otherwise for $I := I' \cup \{n\}$ one has $\left|\bigcup_{i \in I} A_i\right| = |R'| < |I|$. Therefore, $(A_1, \ldots, A_{n-1})$ has a transversal not containing A_n as a subset, implying that $(A_1, \ldots, A_n)$ has a transversal.

Hall derived as a consequence that if $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are two partitions of a finite set S, then the two partitions have a common transversal if and only if for each subset I of $\{1, \ldots, n\}$, the set $\bigcup_{i \in I} A_i$ intersects at least $|I|$ sets among $B_1, \ldots, B_n$. Hall remarked that the theorem of Kőnig [1916] on the existence of a perfect matching in a regular bipartite graph follows as an immediate corollary, and that also a theorem of Rado [1933] can be derived (the Kőnig-Rado edge cover theorem — Theorem 19.4), but he did not observe that Hall's marriage theorem is equivalent to a theorem of Kőnig [1931] (Kőnig's matching theorem — Theorem 16.2).

As for *common* transversals, Maak [1936] showed that if $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ are partitions of a finite set S, then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if for each $I \subseteq \{1, \ldots, n\}$, the set $\bigcup_{i \in I} A_i$ contains at most $|I|$ of the sets B_i as a subset. This can be derived from Frobenius' theorem (Frobenius [1917]).

The basic characterization of common transversals of two arbitrary families of sets was given by Ford and Fulkerson [1958c] — see Section 23.1.

Shmushkovich [1939] and de Bruijn [1943] extended the results to the infinite case. Weyl [1949] introduced the name 'marriage theorem' for Hall's marriage theorem. Maak [1952] gave some historical notes.

Chapter 23

Common transversals

We consider sets that are transversals of two families of sets simultaneously. Again we denote, for any family $(A_1, \ldots, A_n)$ of sets and any $I \subseteq \{1, \ldots, n\}$,

$$A_I := \bigcup_{i \in I} A_i.$$

23.1. Common transversals

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. A set T is called a *common transversal* of $\mathcal{A}$ and $\mathcal{B}$ if T is a transversal of both $\mathcal{A}$ and $\mathcal{B}$. Similarly, T is called a *common partial transversal* of $\mathcal{A}$ and $\mathcal{B}$ if T is a partial transversal of both $\mathcal{A}$ and $\mathcal{B}$.

When considering two families $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ of subsets of a set S, it is helpful to construct the following directed graph $D = (V, A)$:

(23.1) $$\begin{aligned} V &:= \{a_1, \ldots, a_n\} \cup S \cup \{b_1, \ldots, b_m\}, \\ A &:= \{(a_i, s) \mid i = 1, \ldots, n; s \in A_i\} \cup \{(s, b_i) \mid i = 1, \ldots, m; s \in B_i\}, \end{aligned}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are distinct new elements, not in S.

Then one has, if $m = n$:

(23.2) a subset T of S is a common transversal of $\mathcal{A}$ and $\mathcal{B}$ if and only if D has n vertex-disjoint paths from $\{a_1, \ldots, a_n\}$ to $\{b_1, \ldots, b_n\}$ such that T is the set of vertices in S traversed by these paths.

A similar statement can be formulated with respect to common partial transversals.

With Menger's theorem, it yields the following characterization of the existence of a common transversal, due to Ford and Fulkerson [1958c]:

Theorem 23.1. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of sets. Then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if*

(23.3) $$|A_I \cap B_J| \geq |I| + |J| - n$$

for all $I, J \subseteq \{1, \ldots, n\}$.

Proof. To see necessity, let T be a common transversal. To prove (23.3), we can assume that $A_i \subseteq T$ and $B_j \subseteq T$ for all i, j. Then

$$|A_I \cap B_J| = |A_I| + |B_J| - |A_I \cup B_J| \geq |I| + |J| - |T| \geq |I| + |J| - n \tag{23.4}$$

for all $I, J \subseteq \{1, \ldots, n\}$.

To see sufficiency, make the digraph D associated to $\mathcal{A}, \mathcal{B}$ as in (23.1). Let $U := \{a_1, \ldots, a_n\}$ and $W := \{b_1, \ldots, b_n\}$. Then by (23.2), $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if D has n disjoint $U - W$ paths. By Menger's theorem, these paths exist if $|C| \geq n$ for each $C \subseteq U \cup S \cup W$ intersecting each $U - W$ path. To check this condition, let $I := \{i \mid a_i \notin C\}$ and $J := \{j \mid b_j \notin C\}$. Then

$$C \cap S \supseteq A_I \cap B_J, \tag{23.5}$$

since $A_I \cap B_J$ is equal to the set of vertices in S that are on a $U - W$ path not intersected by $C \cap (U \cup W)$. So (23.3) implies

$$|C \cap S| \geq |A_I \cap B_J| \geq |I| + |J| - n = (n - |C \cap U|) + (n - |C \cap W|) - n, \tag{23.6}$$

giving $|C| \geq n$. ∎

(For a direct derivation of this theorem from Hall's marriage theorem, see Perfect [1969c]. For a derivation from the Kőnig-Rado edge cover theorem, see Perfect [1980].)

This construction also implies, with Theorem 9.8, that a common transversal of two collections of n subsets of S can be found in time $O(n^{3/2}|S|)$ (cf. Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975]).

Perfect [1968] (cf. McDiarmid [1973]) strengthened Theorem 23.1 to a min-max relation for the maximum size of a common partial transversal:

Corollary 23.1a. *Let* $\mathcal{A} = (A_1, \ldots, A_n)$ *and* $\mathcal{B} = (B_1, \ldots, B_m)$ *be families of sets and let* $k \in \mathbb{Z}_+$. *Then* $\mathcal{A}$ *and* $\mathcal{B}$ *have a common partial transversal of size* k *if and only if*

$$|A_I \cap B_J| \geq |I| + |J| - n - m + k \tag{23.7}$$

for all $I \subseteq \{1, \ldots, n\}$ *and* $J \subseteq \{1, \ldots, m\}$.

Proof. We may assume that $m = n$ (if, say, $n < m$, add $m - n$ copies of $\emptyset$ to $\mathcal{A}$). Let X be a set disjoint from all A_i and B_i with $|X| = n - k$. Replace each A_i by $A'_i := A_i \cup X$ and each B_i by $B'_i := B_i \cup X$. Then $\mathcal{A}$ and $\mathcal{B}$ have a common partial transversal of size k if and only if $\mathcal{A}' = (A'_1, \ldots, A'_n)$ and $\mathcal{B}' = (B'_1, \ldots, B'_n)$ have a common transversal. Applying Theorem 23.1 to $\mathcal{A}'$ and $\mathcal{B}'$ gives this corollary. ∎

Generally, a common partial transversal of families $\mathcal{A}$ and $\mathcal{B}$ need not be contained in a common transversal, even not if a common transversal exists:

let $\mathcal{A} := (\{a\}, \{b, c\})$ and $\mathcal{B} := (\{b\}, \{a, c\})$. Then $\{c\}$ is a common partial transversal, while $\{a, b\}$ is the only common transversal.

The following result of Perfect [1968] and Welsh [1968] characterizes subsets contained in common transversals. It is a special case of a theorem of Ford and Fulkerson [1958c] (cf. Theorem 23.14), and will be derived from Theorem 23.1 with a method of Mirsky and Perfect [1968].

Corollary 23.1b. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S and let $X \subseteq S$. Then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal containing X if and only if*

$$|A_I \cap B_J| \geq |I| + |J| - n + |X \setminus (A_I \cup B_J)| \tag{23.8}$$

for all $I, J \subseteq \{1, \ldots, n\}$.

Proof. To see necessity, we can assume that there is a common transversal T containing each A_i, each B_j, and X. Then for all $I, J \subseteq \{1, \ldots, n\}$:

$$|A_I \cap B_J| = |A_I| + |B_J| - |A_I \cup B_J| \geq |I| + |J| + |X \setminus (A_I \cup B_J)| - n \tag{23.9}$$

since $|A_I \cup B_J| + |X \setminus (A_I \cup B_J)| \leq |T| = n$.

To see sufficiency, let $X = \{x_1, \ldots, x_k\}$ and let $x'_1, \ldots, x'_k$ be new elements. For each $i = 1, \ldots, n$, let A'_i be the set obtained from A_i by replacing any occurrence of x_j by x'_j. Then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal containing X if the families

$$\begin{aligned} &\mathcal{A}' := (A'_1, \ldots, A'_n, \{x_1\}, \ldots, \{x_k\}) \text{ and} \\ &\mathcal{B}' := (B_1, \ldots, B_n, \{x'_1\}, \ldots, \{x'_k\}) \end{aligned} \tag{23.10}$$

have a common transversal. So by Theorem 23.1 we must check condition (23.3) for $\mathcal{A}'$ and $\mathcal{B}'$. Let $I, J \subseteq \{1, \ldots, n\}$ and $I', J' \subseteq \{1, \ldots, k\}$. Define $Y := \{x_i \mid i \in I'\}$ and $Z := \{x_i \mid i \in J'\}$. Then

$$\begin{aligned} &\Big|\Big(\bigcup_{i \in I} A'_i \cup \bigcup_{i \in I'} \{x_i\}\Big) \cap \Big(\bigcup_{j \in J} B_j \cup \bigcup_{j \in J'} \{x'_j\}\Big)\Big| \\ &= |(A_I \cap B_J) \setminus X| + |A_I \cap Z| + |B_J \cap Y| \\ &= |(A_I \cap B_J) \setminus X| + |Z| - |Z \setminus A_I| + |Y| - |Y \setminus B_J| \\ &\geq |(A_I \cap B_J) \setminus X| + |Z| - |X \setminus A_I| + |Y| - |X \setminus B_J| \\ &= |A_I \cap B_J| - |X \setminus (A_I \cup B_J)| + |Y| + |Z| - |X| \\ &\geq |I| + |J| + |Y| + |Z| - |X| - n = |I| + |I'| + |J| + |J'| - n - k \end{aligned} \tag{23.11}$$

(the last inequality follows from (23.8)). ∎

23.2. Weighted common transversals

Consider the problem of finding a minimum-weight common transversal: given families $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ of subsets of a set S and a weight function $w : S \to \mathbb{Q}$, find a common transversal T of $\mathcal{A}$ and $\mathcal{B}$

minimizing $w(T)$. This problem can easily be solved by solving an associated minimum-cost flow problem.

Alternatively, it can be solved with the Hungarian method, as follows. For $s \in S$, introduce a copy s' of s. Let $S' := \{s' \mid s \in S\}$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be vertices. Make a bipartite graph G with colour classes $\{a_1, \ldots, a_n\} \cup S$ and $\{b_1, \ldots, b_n\} \cup S'$. Vertex a_i is connected with vertex $s' \in S'$ if $s \in A_i$. Vertex b_i is connected with vertex $s \in S$ if $s \in B_i$. Moreover, each $s \in S$ is connected with its copy $s' \in S'$. This describes all edges of G.

For any perfect matching M in G, the set of $s \in S$ with $\{s, s'\} \notin M$ is a common transversal of $\mathcal{A}$ and $\mathcal{B}$. Conversely, each common transversal can be obtained in this way from a perfect matching in G.

Therefore, a minimum-weight common transversal of $\mathcal{A}$ and $\mathcal{B}$ can be found by determining a maximum-weight perfect matching in G, taking weight $w(s)$ on any edge $\{s, s'\}$ and weight 0 on any other edge of G. So by Theorem 17.3 we can find a minimum-weight common transversal in time $O(k(m + k \log k))$, where

$$(23.12) \qquad k := n + |S| \text{ and } m := \sum_{i=1}^{n} (|A_i| + |B_i|).$$

Due to the special structure of G and its weight function one can sharpen this to:

Theorem 23.2. *A minimum-weight common transversal can be found in time $O(n(m + k \log k))$, with m and k as in* (23.12).

Proof. We may assume that $w(s) \geq 0$ for each $s \in S$ (we can add a constant to all weights). Then we can start the Hungarian method with the matching M consisting of all edges $\{s, s'\}$ with $s \in S$. This matching is extreme (that is, has maximum weight among all matchings of size $|M|$), and the Hungarian method requires only n iterations to obtain a maximum-weight perfect matching. ∎

Note that, unlike what happened in finding a minimum-weight transversal for *one* family of sets, in the algorithm above we do not grow a common partial transversal — we do backtrack.

We can also obtain a min-max relation for the minimum weight of a common transversal:

Theorem 23.3. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S and let $w : S \to \mathbb{Z}$ be a weight function. Then the minimum weight of a common transversal of $\mathcal{A}$ and $\mathcal{B}$ is equal to the maximum value of*

$$(23.13) \qquad \sum_{i=1}^{n} (\min_{s \in A_i} w_1(s) + \min_{s \in B_i} w_2(s)) + (w(S) - w_1(S) - w_2(S))$$

taken over $w_1, w_2 \in \mathbb{Z}^S$ *satisfying* $w_1 + w_2 \geq w$.

Proof. Consider the graph G above. By Theorem 17.5 (or by total unimodularity), the maximum weight of a perfect matching in G is equal to the minimum value of

$$(23.14) \qquad \sum_{i=1}^{n}(\lambda_i + \mu_i) + \sum_{s \in S}(w_1(s) + w_2(s))$$

taken over $\lambda, \mu \in \mathbb{Z}^n$ and $w_1, w_2 \in \mathbb{Z}^S$ satisfying

$$(23.15) \qquad \begin{array}{ll} \lambda_i + w_1(s) \geq 0 & \text{for } i = 1, \ldots, n \text{ and } s \in A_i, \\ w_1(s) + w_2(s) \geq w(s) & \text{for } s \in S, \\ \mu_i + w_2(s) \geq 0 & \text{for } i = 1, \ldots, n \text{ and } s \in B_i. \end{array}$$

We can assume that $\lambda_i = \max\{-w_1(s) \mid s \in A_i\}$ and $\mu_i = \max\{-w_2(s) \mid s \in B_i\}$ for each $i = 1, \ldots, n$.

Now the minimum weight of a common transversal is equal to $w(S)$ minus the maximum weight of a perfect matching in G. So it is equal to the maximum value of

$$(23.16) \qquad w(S) - \sum_{s \in S}(w_1(s) + w_2(s)) + \sum_{i=1}^{n}(\min_{s \in A_i} w_1(s) + \min_{s \in B_i} w_2(s)),$$

where $w_1, w_2 \in \mathbb{Z}^S$ satisfy $w_1 + w_2 \geq w$. This is equal to (23.13). ∎

23.3. Weighted common partial transversals

A maximum-weight common partial transversal can be found with the Hungarian method, like described at the beginning of Section 23.2. At any stage of the Hungarian method the current matching M is extreme (that is, it has optimum weight among all matchings of size $|M|$). So we can also apply it (like in Theorem 23.2) to find a maximum-weight common partial transversal of two families $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_l)$ of subsets of a set S. Taking

$$(23.17) \qquad n := k + l + |S| \text{ and } m := \sum_{i=1}^{k} |A_i| + \sum_{i=1}^{l} |B_i|,$$

we have:

Theorem 23.4. *A maximum-weight common partial transversal can be found in time* $O(\min\{k, l\}(m + n \log n))$.

Proof. As above. ∎

Note that, even if all weights are positive, a maximum-weight common partial transversal need not be a common transversal (a statement that is true if we delete ‘common’). To see this, let $\mathcal{A} = (\{a\}, \{b, c\})$, $\mathcal{B} = (\{b\}, \{a, c\})$, and $w(a) = w(b) = 1$, $w(c) = 3$. Then $\{c\}$ is the only maximum-weight common partial transversal, while $\{a, b\}$ is the only common transversal.

A min-max relation for the maximum weight of a common partial transversal can be derived from a min-max relation for the maximum weight of a matching in a bipartite graph, or from linear programming duality using total unimodularity, as we do in the proof below:

Theorem 23.5. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_l)$ be families of subsets of a set S and let $w : S \to \mathbb{Z}_+$ be a weight function. Then the maximum weight of a common partial transversal of $\mathcal{A}$ and $\mathcal{B}$ is equal to the minimum value of*

$$\text{(23.18)} \qquad \sum_{i=1}^{k} \max_{s \in A_i} w_1(s) + \sum_{i=1}^{l} \max_{s \in B_i} w_2(s) + (w - w_1 - w_2)(S)$$

where $w_1, w_2 \in \mathbb{Z}_+^S$ with $w_1 + w_2 \leq w$.

Proof. The maximum weight of a common partial transversal is equal to the maximum of $w^{\mathsf{T}} x$ where $x \in \mathbb{Z}^S$ such that there exist $y_1(i, s) \in \mathbb{Z}_+$ ($i = 1, \ldots, k$; $s \in A_i$) and $y_2(i, s) \in \mathbb{Z}_+$ ($i = 1, \ldots, l$; $s \in B_i$) satisfying

$$\text{(23.19)} \qquad \begin{array}{ll} \displaystyle\sum_{s \in A_i} y_1(i, s) \leq 1 & \text{for } i = 1, \ldots, k, \\ \displaystyle\sum_{s \in B_i} y_2(i, s) \leq 1 & \text{for } i = 1, \ldots, l, \\ x_s = \displaystyle\sum_{i, s \in A_i} y_1(i, s) & \text{for } s \in S, \\ x_s = \displaystyle\sum_{i, s \in B_i} y_2(i, s) & \text{for } s \in S, \\ 0 \leq x_s \leq 1 & \text{for } s \in S. \end{array}$$

By linear programming duality and the total unimodularity of the constraint matrix in (23.19), the maximum value is equal to the minimum value of

$$\text{(23.20)} \qquad \sum_{i=1}^{k} z_1(i) + \sum_{i=1}^{l} z_2(i) + \sum_{s \in S} u(s),$$

where $z_1, z_2 \in \mathbb{Z}_+^k$ and $u \in \mathbb{Z}_+^S$ satisfy

$$\text{(23.21)} \qquad \begin{array}{ll} z_1(i) \geq w_1(s) & \text{for } i = 1, \ldots, k \text{ and } s \in A_i, \\ z_2(i) \geq w_2(s) & \text{for } i = 1, \ldots, l \text{ and } s \in B_i, \\ w_1(s) + w_2(s) + u(s) \geq w(s) & \text{for } s \in S, \end{array}$$

for some $w_1, w_2 \in \mathbb{Z}^E$. We may assume that $w_1, w_2 \geq \mathbf{0}$, since replacing any negative $w_j(s)$ by 0 does not violate (23.21). We may assume that $w_1 + w_2 + u = w$, since $w \geq \mathbf{0}$, and hence we can decrease $w_1(s)$, $w_2(s)$ or $u(s)$ if $w_1(s) + w_2(s) + u(s) > w(s)$. This gives the theorem. ∎

By specializing w to the all-one function, Theorem 23.5 reduces to Corollary 23.1a on the maximum size of a common partial transversal. We can also derive an alternative min-max relation for the maximum weight of a common partial transversal, expressed in

(23.22) $$m(\mathcal{C}, w) := \text{maximum weight of a partial transversal of } \mathcal{C}$$

for any family $\mathcal{C}$ and weight function w (so we can plug in a min-max relation for $m(\mathcal{C}, w)$ to obtain a genuine min-max relation):

Corollary 23.5a. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_l)$ be families of subsets of a set S and let $w : S \to \mathbb{Z}_+$ be a weight function. Then the maximum weight of a common partial transversal of $\mathcal{A}$ and $\mathcal{B}$ is equal to the minimum value of $m(\mathcal{A}, w_1) + m(\mathcal{B}, w_2)$, taken over $w_1, w_2 \in \mathbb{Z}_+^S$ with $w_1 + w_2 = w$.*

Proof. Clearly, the maximum value here cannot be larger than the minimum value, since $w(T) = w_1(T) + w_2(T) \leq m(\mathcal{A}, w_1) + m(\mathcal{B}, w_2)$ for any maximum-weight common partial transversal T.

To see equality, consider w_1 and w_2 of Theorem 23.5, and let $w_2' := w - w_1$. Then for any partial transversal T_1 of $\mathcal{A}$ one has

(23.23) $$w_1(T_1) \leq \sum_{i=1}^{k} \max_{s \in A_i} w_1(s).$$

Moreover, for any partial transversal T_2 of $\mathcal{B}$ one has

(23.24) $$\begin{aligned} w_2'(T_2) &= w_2(T_2) + (w - w_1 - w_2)(T_2) \\ &\leq \sum_{i=1}^{k} \max_{s \in B_i} w_2(s) + (w - w_1 - w_2)(S). \end{aligned}$$

So by Theorem 23.5 we have that $m(\mathcal{A}, w_1) + m(\mathcal{B}, w_2')$ is not more than the maximum w-weight of a common partial transversal. ∎

The obvious generalization to common partial transversals of *three* families is not true: take

(23.25) $$\mathcal{A} = (\{a\}, \{b, c\}),\ \mathcal{B} = (\{b\}, \{a, c\}), \text{ and } \mathcal{C} = (\{c\}, \{a, b\}),$$

and $w(a) = w(b) = w(c) = 1$. Then the maximum weight of a common partial transversal is 1, but one cannot decompose w as $w = w_1 + w_2 + w_3$ with $m(\mathcal{A}, w_1) + m(\mathcal{B}, w_2) + m(\mathcal{C}, w_3) = 1$.

23.4. The common partial transversal polytope

Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_l)$ be families of subsets of a set S. The *common partial transversal polytope* $P_{\text{common partial transversal}}(\mathcal{A}, \mathcal{B})$ of

$\mathcal{A}$ and $\mathcal{B}$ is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of the common partial transversals of $\mathcal{A}$ and $\mathcal{B}$. That is,

(23.26) $$P_{\text{common partial transversal}}(\mathcal{A},\mathcal{B}) = \text{conv.hull}\{\chi^T \mid T \text{ is a common partial transversal of } \mathcal{A} \text{ and } \mathcal{B}\}.$$

It is easy to see that each vector x in the common partial transversal polytope satisfies:

(23.27) $$\begin{array}{rll} \text{(i)} & 0 \le x_s \le 1 & \text{for } s \in S, \\ \text{(ii)} & x(S \setminus A_I) \le k - |I| & \text{for } I \subseteq \{1,\ldots,k\}, \\ \text{(iii)} & x(S \setminus B_I) \le l - |I| & \text{for } I \subseteq \{1,\ldots,l\}. \end{array}$$

In fact, this fully determines the common partial transversal polytope:

Theorem 23.6. *The common partial transversal polytope is determined by* (23.27).

Proof. We must show that for any weight function $w \in \mathbb{Z}_+^S$, the maximum value of $w^\mathsf{T} x$ over (23.27) is equal to the maximum weight μ of any common partial transversal. By Corollary 23.5a, there exist weight functions $w_1, w_2 \in \mathbb{Z}^S$ with $w = w_1 + w_2$ and $\mu = m(\mathcal{A}, w_1) + m(\mathcal{B}, w_2)$. Now any x satisfying (23.27) belongs to the partial transversal polytopes of $\mathcal{A}$ and $\mathcal{B}$. So $w_1^\mathsf{T} x \le m(\mathcal{A}, w_1)$ and $w_2^\mathsf{T} x \le m(\mathcal{B}, w_2)$. Hence $w^\mathsf{T} x \le \mu$. ∎

Since (23.27) is the union of the systems that determine the partial transversal polytope of $\mathcal{A}$ and of $\mathcal{B}$, we have:

Corollary 23.6a. *Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S. Then*

(23.28) $$P_{\text{common partial transversal}}(\mathcal{A},\mathcal{B}) = P_{\text{partial transversal}}(\mathcal{A}) \cap P_{\text{partial transversal}}(\mathcal{B}).$$

Proof. Directly from Theorem 23.6 and Corollary 22.9a. ∎

Also:

Theorem 23.7. *System* (23.27) *is TDI.*

Proof. Directly from Corollaries 23.5a and 22.9a. ∎

Again one cannot make the obvious extension to three families of sets, by considering the families (23.25). In that case, the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ belongs to the intersection of the three partial transversal polytopes, but does not belong to the common partial transversal polytope.

23.5. The common transversal polytope

Similar results hold for the common transversal polytope. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S. The *common transversal polytope* $P_{\text{common transversal}}(\mathcal{A})$ of $\mathcal{A}$ and $\mathcal{B}$ is the convex hull of the incidence vectors (in $\mathbb{R}^S$) of the common transversals of $\mathcal{A}$ and $\mathcal{B}$. That is,

(23.29) $P_{\text{common transversal}}(\mathcal{A}, \mathcal{B}) = \text{conv.hull}\{\chi^T \mid T \text{ is a common transversal of } \mathcal{A} \text{ and } \mathcal{B}\}$.

It is easy to see that each vector x in the common transversal polytope satisfies:

(23.30)
(i) $0 \leq x_s \leq 1$ for $s \in S$,
(ii) $x(A_I) \geq |I|$ for $I \subseteq \{1, \ldots, n\}$,
(iii) $x(B_I) \geq |I|$ for $I \subseteq \{1, \ldots, n\}$,
(iv) $x(S) = n$.

Corollary 23.7a. *The common transversal polytope is determined by* (23.30).

Proof. The common transversal polytope is the facet of the common partial transversal polytope determined by the equality $x(S) = n$. So we must show that (23.30) implies (23.27), which is trivial, since if x satisfies (23.30), then $x(S \setminus A_I) = x(S) - x(A_I) \leq n - |I|$ and $x(S \setminus B_I) = x(S) - x(B_I) \leq n - |I|$ for any $I \subseteq \{1, \ldots, n\}$. ∎

Again this implies:

Corollary 23.7b. *Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S. Then*

(23.31) $P_{\text{common transversal}}(\mathcal{A}, \mathcal{B}) = P_{\text{transversal}}(\mathcal{A}) \cap P_{\text{transversal}}(\mathcal{B})$.

Proof. Directly from Corollaries 23.7a and 22.9b. ∎

In fact:

Theorem 23.8. *System* (23.30) *is TDI.*

Proof. This follows from Theorem 23.7, using Theorem 5.25. ∎

Weinberger [1976] proved the following conjecture of Fulkerson [1971a], which generalizes Theorem 18.8. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S. Then the up hull $P^{\uparrow}_{\text{common transversal}}(\mathcal{A}, \mathcal{B})$ of the common transversal polytope is determined by:

(23.32) $x(U) \geq n-$ maximum size of a common partial transversal contained in $S \setminus U$, for $U \subseteq S$. This will follow from Theorem 46.3 on polymatroids.

23.6. Packing and covering of common transversals

Fulkerson [1971b] and de Sousa [1971] detected that results on bipartite edge-colouring (or related results) imply characterizations of packings of common transversals. It was noticed by Brualdi [1971b] that the methods in fact yield more general results.

Basic is the following exchange property given by de Sousa [1971]:

Theorem 23.9. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ be families of subsets of a set S and let $k \in \mathbb{Z}_+$. Suppose that S can be covered by k partial transversals of $\mathcal{A}$ and that S can also be covered by k partial transversals of $\mathcal{B}$. Then S can be covered by k common partial transversals of $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let $T_1, \ldots, T_k$ be a partition of S into k partial transversals of $\mathcal{A}$. Since each T_i is a partial transversal of $\mathcal{A}$, it follows that each A_i has a subset A'_i such that $|A'_i| \leq k$ and such that $A'_1, \ldots, A'_n$ partition S. We can assume that $A'_i = A_i$ for each i, and hence that $\mathcal{A}$ is a partition of S into classes of size at most k.

Similarly, we can assume that $\mathcal{B}$ is a partition of S into classes of size at most k.

Now make a bipartite graph G, with colour classes $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$, connecting a_i and b_j by $|A_i \cap B_j|$ parallel edges. So G has maximum degree k, and hence, by Kőnig's edge-colouring theorem, the edges of G can be coloured with k colours. It implies that S can be partitioned as required. ∎

A consequence is a min-max formula for the minimum number of common partial transversals needed to cover S, stated by Brualdi [1971b]:

Corollary 23.9a. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S, each with union S. Then the minimum number of common partial transversals of $\mathcal{A}$ and $\mathcal{B}$ needed to cover S is equal to*

$$\left\lceil \max_{\substack{X \subseteq S \\ X \neq \emptyset}} \max\left\{\frac{|X|}{|\{i \mid A_i \cap X \neq \emptyset\}|}, \frac{|X|}{|\{i \mid B_i \cap X \neq \emptyset\}|}\right\} \right\rceil. \tag{23.33}$$

Proof. From Theorem 23.9, using Theorem 22.12. ∎

Theorem 23.9 also gives a variant of the exchange property (de Sousa [1971]):

Corollary 23.9b. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S and let $k \in \mathbb{Z}_+$. Suppose that S can be partitioned into k*

transversals of $\mathcal{A}$, and also can be partitioned into k transversals of $\mathcal{B}$. Then S can be partitioned into k common transversals of $\mathcal{A}$ and $\mathcal{B}$.

Proof. Directly from Theorem 23.9, since $|S| = nk$. ▮

This implies another variant (de Sousa [1971]):

Corollary 23.9c. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S and let $k \in \mathbb{Z}_+$. Suppose that S has a partition $(S_1, \ldots, S_n)$ with $|S_i| = k$ and $S_i \subseteq A_i$ for $i = 1, \ldots, n$. Suppose moreover that S has a partition $(Z_1, \ldots, Z_n)$ with $|Z_i| = k$ and $Z_i \subseteq B_i$ for $i = 1, \ldots, n$. Then S can be partitioned into common transversals of $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Note that if S has a partition $(S_1, \ldots, S_n)$ with $|S_i| = k$ and $S_i \subseteq A_i$ for $i = 1, \ldots, n$, then S can be partitioned into k transversals of $\mathcal{A}$. Similarly for $\mathcal{B}$. So the present corollary follows from Corollary 23.9b. ▮

This gives the following basic min-max relation for the maximum number of disjoint common transversals, given by Fulkerson [1971b] and de Sousa [1971]:

Corollary 23.9d. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of sets and let k be a natural number. Then $\mathcal{A}$ and $\mathcal{B}$ have k disjoint common transversals if and only if*

(23.34) $$|A_I \cap B_J| \geq k(|I| + |J| - n).$$

for all $I, J \subseteq \{1, \ldots, n\}$.

Proof. Necessity of (23.34) being easy, we show sufficiency.

Let $\mathcal{A}'$ arise by taking k copies of $\mathcal{A}$ and let $\mathcal{B}'$ arise from taking k copies of $\mathcal{B}$. Condition (23.34) implies that $\mathcal{A}'$ and $\mathcal{B}'$ have a common transversal, S say (by Theorem 23.1). Then we can partition S into subsets $A'_1, \ldots, A'_n$, with $A'_i \subseteq A_i$ and $|A'_i| = k$. Similarly, we can partition S into subsets $B'_1, \ldots, B'_n$, with $B'_i \subseteq B_i$ and $|B'_i| = k$. Then by Corollary 23.9c, S has a partition into k common transversals of $\mathcal{A}$ and $\mathcal{B}$. ▮

(Note that if $\mathcal{A}$ and $\mathcal{B}$ are partitions of a set, this corollary reduces to Corollary 20.9a.)

The following open problem, dealing with packing common transversals, was mentioned by Fulkerson [1971b]: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S and let $c \in \mathbb{Z}_+^S$. What is the maximum number k of common transversals $T_1, \ldots, T_k$ such that

(23.35) $$\chi^{T_1} + \cdots + \chi^{T_k} \leq c?$$

More generally than Corollary 23.9d, one has for disjoint common *partial* transversals of prescribed size (Fulkerson [1971b]):

Theorem 23.10. *Let $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_m)$ be families of sets and let $k, p \in \mathbb{Z}_+$. Then there exist k disjoint common partial transversals of size p if and only if*

$$|A_I \cap B_J| \geq k(|I| + |J| + p - n - m) \tag{23.36}$$

for all $I \subseteq \{1, \dots, n\}$ and $J \subseteq \{1, \dots, m\}$.

Proof. Construct $\mathcal{A}'$ and $\mathcal{B}'$ as in Corollary 23.9d. By Corollary 23.1a, (23.36) implies that $\mathcal{A}'$ and $\mathcal{B}'$ have a common partial transversal, T say, of size pk. Then each A_i has a subset A_i' such that $|A_i'| \leq k$ and such that $A_1', \dots, A_n'$ partition T. We can assume that $A_i' = A_i$ for each i, and hence that $\mathcal{A}$ is a partition of T into classes of size at most k.

Similarly, we can assume that $\mathcal{B}$ is a partition of T into classes of size at most k.

Now make a bipartite graph G, with colour classes $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$, connecting a_i and b_j by $|A_i \cap B_j|$ parallel edges. So G has kp edges and maximum degree k, and hence, by Theorem 20.8, the edges of G can be coloured with k colours, each of size p. It implies that T can be partitioned into common partial transversals of $\mathcal{A}$ and $\mathcal{B}$ of size p. ∎

Similarly to Theorem 23.9 one can prove the following exchange property:

Theorem 23.11. *Let $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_n)$ be families of subsets of a set S and let $k \in \mathbb{Z}_+$. Suppose that $\mathcal{A}$ has k disjoint transversals and that also $\mathcal{B}$ has k disjoint transversals. Then S has k disjoint subsets $S_1, \dots, S_k$ such that each S_i contains a transversal of $\mathcal{A}$ and contains a transversal of $\mathcal{B}$.*

Proof. As $\mathcal{A}$ has k disjoint transversals, there exist disjoint sets $A_1', \dots, A_n'$ with $A_i' \subseteq A_i$ and $|A_i'| = k$ for $i = 1, \dots, k$. For our purposes, we can assume that $A_i' = A_i$. Let Y be the union of the A_i. Similarly, we can assume that $B_1, \dots, B_n$ have size k each and partition some set Z.

Again, make a bipartite graph G, with colour classes $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$, connecting a_i and b_j by $|A_i \cap B_j|$ parallel edges. Then G has maximum degree at most k, and hence, by Kőnig's edge-colouring theorem (Theorem 20.1), G is k-edge-colourable. It gives a partition of $Y \cap Z$ into k classes each intersecting any A_i and B_i in at most one element. We can extend this partition to a partition of $Y \cup Z$ into classes each intersecting any A_i and any B_i in exactly one element. This is a partition as required. ∎

This implies another min-max relation:

Corollary 23.11a. *Let $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_n)$ be families of subsets of a set S. Then the maximum number k for which there exist disjoint subsets $S_1, \dots, S_k$ each containing a transversal of $\mathcal{A}$ and a transversal of $\mathcal{B}$ is equal to*

(23.37) $$\left\lfloor \min_{\emptyset \neq I \subseteq \{1,\ldots,n\}} \min\{\tfrac{|A_I|}{|I|}, \tfrac{|B_I|}{|I|}\} \right\rfloor.$$

Proof. Directly from Theorems 22.10 and 23.11. ∎

An analogue of Corollary 23.9d for covering by common transversals is:

Theorem 23.12. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of S and let $X \subseteq S$. Then X can be covered by k common transversals if and only if*

(23.38) $$k|A_I \cap B_J| \geq k(|I| + |J| - n) + |X \setminus (A_I \cup B_J)|$$

for all $I, J \subseteq \{1, \ldots, n\}$.

Proof. To see necessity, let $T_1, \ldots, T_k$ be common transversals covering X and let $I, J \subseteq \{1, \ldots, n\}$. Then

(23.39) $$\begin{aligned} k|A_I \cap B_J| &\geq \sum_{j=1}^{k} |A_I \cap B_J \cap T_j| \\ &= \sum_{j=1}^{k} (|A_I \cap T_j| + |B_J \cap T_j| - |T_j \cap (A_I \cup B_J)|) \\ &\geq \sum_{j=1}^{k} (|I| + |J| - |T_j \cap (A_I \cup B_J)|) \\ &= k(|I| + |J| - n) + \sum_{j=1}^{k} |T_j \setminus (A_I \cup B_J)| \\ &\geq k(|I| + |J| - n) + |X \setminus (A_I \cup B_J)|. \end{aligned}$$

To see sufficiency, make a directed graph D, with vertex set

(23.40) $$\{r\} \cup \{a_1, \ldots, a_n\} \cup S \cup S' \cup \{b_1, \ldots, b_n\},$$

where S' is a set consisting of, for each $s \in S$, a (new) copy s' of s, and with arcs, with demands and capacities, as follows:

(23.41) (r, a_i) with demand k and capacity k, for $i = 1, \ldots, n$,
(a_i, s) with demand 0 and capacity ∞ for $i = 1, \ldots, n$ and $s \in A_i$,
(s, s') with demand 1 (if $s \in X$) or 0 (if $s \notin X$) and capacity k, for $s \in S$,
(s', b_i) with demand 0 and capacity ∞, for $i = 1, \ldots, n$ and $s \in B_i$,
(b_i, r) with demand k and capacity k, for $i = 1, \ldots, n$.

Then by Hoffman's circulation theorem (Theorem 11.2), (23.38) implies the existence of a circulation f obeying the demands and capacities. Indeed, consider any set U of vertices of D. Let $I := \{i \mid a_i \in U\}$, $J := \{j \mid b_j \notin U\}$, $Y := U \cap S$ and $Z := \{s \in S \mid s' \notin U\}$. We can assume that the capacity of the arcs leaving U is finite, and hence, if $i \in I$, then $A_i \subseteq Y$ and if $j \in J$, then $B_j \subseteq Z$. That is, $A_I \subseteq Y$ and $B_J \subseteq Z$.

If $r \notin U$, then the total demand of the arcs entering U is equal to

$$k|I| + |X \setminus (Y \cup Z)| \tag{23.42}$$

and the total capacity of the arcs leaving U is equal to

$$k|Y \cap Z| + k(n - |J|). \tag{23.43}$$

Since $A_I \subseteq Y$ and $B_J \subseteq Z$, (23.38) implies that (23.42) is at most (23.43).

If $r \in U$, then the total demand of the arcs entering U is equal to

$$k|J| + |X \setminus (Y \cup Z)| \tag{23.44}$$

and the total capacity of the arcs leaving U is equal to

$$k|Y \cap Z| + k(n - |I|). \tag{23.45}$$

Since $A_I \subseteq Y$ and $B_J \subseteq Z$, (23.38) implies that (23.44) is at most (23.45).

So Hoffman's condition is satisfied, and hence there exists a circulation f.

Now f is at most k on any arc. Hence, by Corollary 11.2b, f is the sum of k $\{0,1\}$-valued circulations $f_1, \ldots, f_k$. For each circulation f_i, the set T_i of $s \in S$ with $f_i(s, s') = 1$ is a common transversal of $\mathcal{A}$ and $\mathcal{B}$. Moreover, since $f(s, s') \geq 1$ for each $s \in X$, these common transversals cover X. ∎

A covering theorem different from Theorem 23.12 is due to Brualdi [1971b]:

Theorem 23.13. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ be families of subsets of a set S. Suppose that S can be covered by k common partial transversals of $\mathcal{A}$ and $\mathcal{B}$. Then S can be covered by k common partial transversals each of size $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$.*

Proof. The assumption implies that each A_i contains a subset A'_i with $|A'_i| \leq k$, such that the A'_i partition S. For our purposes, we can assume that $A'_i = A_i$ for each i. Similarly, we can assume that $\mathcal{B}$ is a partition of S into classes of size at most k.

Again, make a bipartite graph G, with colour classes $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$, connecting a_i and b_j by $|A_i \cap B_j|$ parallel edges. Then G has maximum degree at most k, and hence, by Theorem 20.8, G is k-edge-colourable, where each colour has size $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$. This yields a partition of S into k common partial transversals as required. ∎

23.7. Further results and notes

23.7a. Capacitated common transversals

Recall the definition of system of restricted representatives: Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a collection of subsets of a set S and let $a, b \in \mathbb{Z}_+^S$ with $a \leq b$. A *system of restricted representatives* (or *SRR*) of $\mathcal{A}$ (with respect to a and b) is a sequence $(s_1, \ldots, s_n)$ such that

(23.46) (i) $s_i \in A_i$ for $i = 1, \ldots, n$;
(ii) $a(s) \leq |\{i \mid s_i = s\}| \leq b(s)$ for $s \in S$.

Ford and Fulkerson [1958c] derived the following characterization of the existence of a common system of restricted representatives from the max-flow min-cut theorem (we give the derivation from Corollary 23.1b based on splitting elements, due to Mirsky and Perfect [1968]):

Theorem 23.14. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a set S and let $a, b \in \mathbb{Z}_+^S$ with $a \leq b$. Then $\mathcal{A}$ and $\mathcal{B}$ have a common system of restricted representatives if and only if*

(23.47) $$b(A_I \cap B_J) \geq |I| + |J| - n + a(S \setminus (A_I \cup B_J))$$

for all $I, J \subseteq \{1, \ldots, n\}$.

Proof. Let for any $s \in S$, Z_s be a set of $b(s)$ (new) elements. Replace in each A_i and B_j, any occurrence of any $s \in S$ by the elements of Z_s. Choose from each Z_s, $a(s)$ elements, forming the set X. Then $\mathcal{A}$ and $\mathcal{B}$ have a common system of restricted representatives if and only if the new families have a common transversal containing X. Trivially, condition (23.47) is equivalent to condition (23.8) for the new families, and hence the theorem follows from Corollary 23.1b. ∎

(More can be found in Mirsky [1968b].)

23.7b. Exchange properties

Mirsky [1968a] showed the following exchange property of common transversals:

Theorem 23.15. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ be families of sets. Let $I', I'' \subseteq \{1, \ldots, n\}$ and $J', J'' \subseteq \{1, \ldots, m\}$. Suppose that $(A_i \mid i \in I')$ and $(B_j \mid j \in J')$ have a common transversal, and also that $(A_i \mid i \in I'')$ and $(B_j \mid j \in J'')$ have a common transversal. Then there exist I and J with $I' \subseteq I \subseteq I' \cup I''$ and $J'' \subseteq J \subseteq J' \cup J''$ such that $(A_i \mid i \in I)$ and $(B_j \mid j \in J)$ have a common transversal.*

Proof. Directly from Corollary 9.12a applied to the digraph defined in (23.1). ∎

This implies (Mirsky [1968a]):

Corollary 23.15a. *Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets and let $\mathcal{A}'$ and $\mathcal{B}'$ be subfamilies of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then there exist subfamilies $\mathcal{A}_0$ and $\mathcal{B}_0$ of $\mathcal{A}$ and $\mathcal{B}$*

respectively satisfying $\mathcal{A}' \subseteq \mathcal{A}_0$ *and* $\mathcal{B}' \subseteq \mathcal{B}_0$ *and having a common transversal if and only if* (i) $\mathcal{A}'$ *and some subfamily of* $\mathcal{B}$ *have a common transversal and* (ii) $\mathcal{B}'$ *and some subfamily of* $\mathcal{A}$ *have a common transversal.*

Proof. Directly from Theorem 23.15. ∎

23.7c. Common transversals of three families

It is NP-complete to test if *three* families of sets have a common transversal, even if each of the three families is a partition of S (E.L. Lawler — cf. Karp [1972b]).

Theorem 23.16. *Testing if three partitions have a common transversal is* NP-*complete.*

Proof. I. It suffices to show the NP-completeness of the following problem:

(23.48) given disjoint sets X, Y, Z with $|X| = |Y| \geq |Z|$ and a collection $\mathcal{C}$ of subsets U of $W := X \cup Y \cup Z$ with $|U \cap X| = |U \cap Y| = 1$ and $|U \cap Z| \leq 1$, decide if $\mathcal{C}$ contains a partition of W as subcollection.

To see this, first observe that we can assume that $|X| = |Y| = |Z|$. Indeed, we can extend Z by a set R of size $|X| - |Z|$ and replace each doubleton $\{x, y\}$ in $\mathcal{C}$ by all sets $\{x, y, w\}$ with $w \in R$. Then the new collection contains a partition if and only if the original collection contains one.

So we can assume that $|X| = |Y| = |Z|$. For $w \in W$, define $\mathcal{C}_w := \{C \in \mathcal{C} \mid w \in C\}$. Then the collection $\{\mathcal{C}_w \mid w \in W\}$ is the union of three partitions of $\mathcal{C}$. Moreover, these three partitions have a common transversal if and only if $\mathcal{C}$ contains a partition of W. So this reduces problem (23.48) to the problem of finding a common transversal of three partitions of a set.

II. So it suffices to show the NP-completeness of (23.48). We derive this from the NP-completeness of the (more general) *partition problem*: decide if a given collection $\mathcal{B}$ of subsets of a set Z contains a partition of Z as a subcollection (Corollary 4.1b).

Let $V := \{(B, z) \mid z \in B \in \mathcal{B}\}$. Make, for each $B \in \mathcal{B}$, an (arbitrary) directed circuit on $\{(B, z) \mid z \in B\}$. This makes the directed graph D on V (consisting of vertex-disjoint directed circuits). Define $X := V \times \{1\}$ and $Y := V \times \{2\}$. Let $\mathcal{C}$ be the collection of

(23.49) all triples $\{(B, z, 1), (B, z, 2), z\}$ for all $B \in \mathcal{B}$ and $z \in B$, and
all pairs $\{(B, z, 1), (B, z', 2)\}$, for all $B \in \mathcal{B}$ and $z, z' \in B$ such that D contains an arc from z to z'.

So each element of $X \cup Y$ is in precisely two sets in $\mathcal{C}$: a triple and a pair. Any partition $\mathcal{P} \subseteq \mathcal{C}$ of $X \cup Y \cup Z$ contains, for any $B \in \mathcal{B}$, either all triples containing B or all pairs containing B. (Here containing B means: containing (B, z, i) for some z, i.)

This implies that $\mathcal{C}$ contains a partition of $X \cup Y \cup Z$ if and only if $\mathcal{B}$ contains a partition of Z. ∎

As indicated in this proof, the problem of finding a common transversal of three partitions is equivalent to the 3-*dimensional matching problem*: given a partition

U, V, W of a finite set S and a collection $\mathcal{C}$ of subsets X of S satisfying $|X \cap U| = |X \cap V| = |X \cap W| = 1$, does $\mathcal{C}$ have a subcollection that partitions S?

The following *necessary* condition for the existence of a common transversal of three families $\mathcal{A} = (A_1, \ldots, A_n)$, $\mathcal{B} = (B_1, \ldots, B_n)$, and $\mathcal{C} = (C_1, \ldots, C_n)$ of sets is not sufficient: for all $I, J, K \subseteq \{1, \ldots, n\}$

$$(23.50) \qquad |A_I \cap B_J \cap C_K| \geq |I| + |J| + |K| - 2n.$$

(This would generalize condition (23.3).) To see this, consider $\mathcal{A} = (\{a\}, \{b, c\})$, $\mathcal{B} = (\{b\}, \{a, c\})$, $\mathcal{C} = (\{c\}, \{a, b\})$.

More on common transversals of more than two families is given by Brown [1976,1984], Dacić [1977,1979], Longyear [1977], and Zaverdinos [1981]. Woodall [1982] studied fractional transversals, and described a good characterization for the existence of a common fractional transversal for more than two families, based on linear programming.

23.7d. Further notes

Weinberger [1974b] observed that if the families $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ of subsets of a set S are uniform (that is, all sets have the same size) and regular (that is, each $s \in S$ is in the same number of sets), then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal.

Further work on common transversals (including extensions to the infinite case) is reported by Perfect [1969b], Brualdi [1970b,1971a], and Davies and McDiarmid [1976].

Part III

Nonbipartite Matching and Covering

Part III: Nonbipartite Matching and Covering

Nonbipartite matching is a highlight of combinatorial optimization, thanks to pioneering work of Tutte and Edmonds. In particular the 1965 papers of Edmonds on nonbipartite matching opened up areas that were not accessible with the 'classical' methods based on flows, linear programming, and total unimodularity found in the 1950s. The papers are pioneering in polyhedral combinatorics, giving the first nontrivial characterizations of combinatorially defined polytopes.

The techniques are highly self-refining, and extend to b-matchings, b-factors, T-joins, shortest paths in undirected graphs, and the Chinese postman problem. Nonbipartite matching also applies to practical problems where an optimal pairing has to be found, like in seat or room assignment, crew planning, and two-processor scheduling.

Chapters:

Chapter 24

Cardinality nonbipartite matching

In this chapter we consider maximum-cardinality matching, with as key results Tutte's characterization of the existence of a perfect matching (implying the Tutte-Berge formula for the maximum-size of a matching) and Edmonds' polynomial-time algorithm to find a maximum-size matching.
As in Section 16.1, we call a path P an *M-augmenting path* if P has odd length and connects two vertices not covered by M, and its edges are alternatingly out of and in M. By Theorem 16.1, a matching M has maximum size if and only if there is no M-augmenting path.
We say that a matching M *covers* a vertex v if v is incident with an edge in M. If M does not cover v, we say that M *misses* v.
In this chapter, graphs can be assumed to be simple.

24.1. Tutte's 1-factor theorem and the Tutte-Berge formula

A basic result of Tutte [1947b] characterizes graphs that have a perfect matching. Berge [1958a] observed that it implies a min-max formula for the maximum size of a matching in a graph, the Tutte-Berge formula.

Call a component of a graph *odd* if it has an odd number of vertices. For any graph G, let

(24.1) $\qquad o(G) :=$ number of odd components of G.

Let $\nu(G)$ denotes the maximum size of a matching. Then:

Theorem 24.1 (Tutte-Berge formula)**.** *For each graph $G = (V, E)$,*

(24.2) $$\nu(G) = \min_{U \subseteq V} \tfrac{1}{2}(|V| + |U| - o(G - U)).$$

Proof. To see $\leq$, we have for each $U \subseteq V$:

(24.3) $$\nu(G) \leq |U| + \nu(G - U) \leq |U| + \tfrac{1}{2}(|V \setminus U| - o(G - U)) \\ = \tfrac{1}{2}(|V| + |U| - o(G - U)).$$

We prove the reverse inequality by induction on $|V|$, the case $V = \emptyset$ being trivial. We can assume that G is connected, since otherwise we can apply induction to the components of G.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\nu(G-v) = \nu(G)-1$, and by induction there exists a subset U' of $V \setminus \{v\}$ with

(24.4) $$\nu(G-v) = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G-v-U')).$$

Then $U := U' \cup \{v\}$ gives equality in (24.2), since

(24.5) $$\nu(G) = \nu(G-v)+1 = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G-v-U')) + 1 \\ = \tfrac{1}{2}(|V| + |U| - o(G-U)).$$

So we can assume that there is no such v. In particular, $\nu(G) < \frac{1}{2}|V|$. We show that there exists a matching of size $\frac{1}{2}(|V|-1)$, which implies the theorem (taking $U := \emptyset$).

Indeed, suppose to the contrary that each maximum-size matching M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $\text{dist}(u,v)$ of u and v in G is as small as possible.

If $\text{dist}(u,v) = 1$, then u and v are adjacent, and hence we can augment M by the edge uv, contradicting the maximality of $|M|$. So $\text{dist}(u,v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u-v$ path. By assumption, there exists a maximum-size matching N missing t. Choose such an N with $|M \cap N|$ maximal.

By the minimality of $\text{dist}(u,v)$, N covers both u and v. Hence, as M and N cover the same number of vertices, there exists a vertex $x \neq t$ covered by M but not by N. Let $x \in e = xy \in M$. Then y is covered by some edge $f \in N$, since otherwise $N \cup \{e\}$ would be a matching larger than N. Replacing N by $(N \setminus \{f\}) \cup \{e\}$ would increase its intersection with M, contradicting the choice of N. ∎

(This proof is based on the proof of Lovász [1979b] of Edmonds' matching polytope theorem.)

The Tutte-Berge formula immediately implies Tutte's 1-factor theorem. A *perfect matching* (or 1-*factor*) is a matching covering all vertices of the graph.

Corollary 24.1a (Tutte's 1-factor theorem)**.** *A graph $G = (V,E)$ has a perfect matching if and only if $G-U$ has at most $|U|$ odd components, for each $U \subseteq V$.*

Proof. Directly from the Tutte-Berge formula (Theorem 24.1), since G has a perfect matching if and only if $\nu(G) \geq \frac{1}{2}|V|$. ∎

24.1a. Tutte's proof of his 1-factor theorem

The original proof of Tutte [1947b] of his 1-factor theorem (Corollary 24.1a), with a simplification of Maunsell [1952], and smoothed by Halton [1966] and Lovász [1975d], is as follows.

Suppose that there exist graphs $G = (V, E)$ satisfying the condition, but not having a perfect matching. Fixing V, take such a graph G with G simple and $|E|$ as large as possible. Let $U := \{v \in V \mid v \text{ is adjacent to every other vertex of } G\}$. We show that each component of $G - U$ is a complete graph.

Suppose to the contrary that there are distinct $a, b, c \notin U$ with $ab, bc \in E$ and $ac \notin E$. By the maximality of $|E|$, adding ac to E makes that G has a perfect matching (since the condition is maintained under adding edges). So G has a matching M missing precisely a and c. As $b \notin U$, there exists a vertex d with $bd \notin E$. Again by the maximality of $|E|$, G has a matching N missing precisely b and d. Now each component of $M \triangle N$ contains the same number of edges in M as in N — otherwise there would exist an M- or N-augmenting path, and hence a perfect matching in G, a contradiction. So the component P of $M \triangle N$ containing d is a path starting at d, with first edge in M and last edge in N, and hence ending at a or c; by symmetry we may assume that it ends at a. Moreover, P does not traverse b. Then extending P by the edge ab gives an N-augmenting path, and hence a perfect matching in G — a contradiction.

So each component of $G - U$ is a complete graph. Moreover, by the condition, $G - U$ has at most $|U|$ odd components. This implies that G has a perfect matching, contradicting our assumption.

More proofs were given by Gallai [1950,1963b], Edmonds [1965d], Balinski [1970], Anderson [1971], Brualdi [1971d], Hetyei [1972,1999], Mader [1973], and Lovász [1975a,1979b].

24.1b. Petersen's theorem

The following theorem of Petersen [1891] is a consequence of Tutte's 1-factor theorem (a graph is *cubic* if it is 3-regular):

Corollary 24.1b (Petersen's theorem)**.** *A bridgeless cubic graph has a perfect matching.*

Proof. Let $G = (V, E)$ be a bridgeless cubic graph. By Tutte's 1-factor theorem, we should show that $G - U$ has at most $|U|$ odd components, for each $U \subseteq V$.

Each odd component of $G - U$ is left by an odd number of edges (as G is cubic), and hence by at least three edges (as G is bridgeless). On the other hand, U is left by at most $3|U|$ edges, since G is cubic. Hence $G - U$ has at most $|U|$ odd components. ∎

24.2. Cardinality matching algorithm

The idea of finding an M-augmenting path to increase a matching M is fundamental in finding a maximum-size matching. However, the simple trick for bipartite graphs, of orienting the edges based on the colour classes of the graph, does not extend to the nonbipartite case. Yet one could try to find an M-augmenting path by finding an 'M-alternating walk', but such a walk can run into a loop that cannot simply be deleted. It was Edmonds [1965d] who found the trick to resolve this problem, namely by 'shrinking' the loop (for which he introduced the term 'blossom'). Then applying recursion to a smaller graph solves the problem[1].

Let $G = (V, E)$ be a graph, let M be a matching in G, and let X be the set of vertices missed by M. A walk $P = (v_0, v_1, \ldots, v_t)$ is called *M-alternating* if for each $i = 1, \ldots, t-1$ exactly one of the edges $v_{i-1}v_i$ and v_iv_{i+1} belongs to M. Note that one can find a shortest M-alternating $X-X$ walk of positive length, by considering the auxiliary directed graph $D = (V, A)$ with

(24.6) $$A := \{(u, v) \mid \exists x \in V : ux \in E, xv \in M\}.$$

Then each M-alternating $X-X$ walk of positive length yields a directed $X-N(X)$ path in D, and vice versa (where $N(X)$ denotes the set of neighbours of X).

An M-alternating walk $P = (v_0, v_1, \ldots, v_t)$ is called an *M-flower* if t is odd, $v_0, \ldots, v_{t-1}$ are distinct, $v_0 \in X$, and $v_t = v_i$ for some even $i < t$. Then the circuit $(v_i, v_{i+1}, \ldots, v_t)$ is called an *M-blossom* (associated with the M-flower).

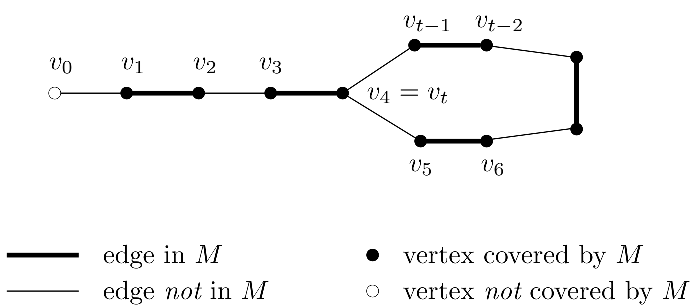


Figure 24.1
An M-flower

The core of the algorithm is the following observation. Let $G = (V, E)$ be a graph and let B be a subset of V. Denote by G/B the graph obtained by *contracting* (or *shrinking*) B to one new vertex, called B. That is, G/B has vertex set $(V \setminus B) \cup \{B\}$, and for each edge e of G an edge obtained from e by replacing any end vertex in B by the new vertex B. (We ignore loops that may arise.) We denote the new edge again by e. (So its ends are modified,

[1] The idea of applying shrinking recursively to matching problems was introduced by Petersen [1891], and was applied in an algorithmic way by Brahana [1917].

but not its name.) We say that the new edge is the *image* (or *projection*) of the original edge.

For any matching M, let M/B denote the set of edges in G/B that are images of edges in M not spanned by B. Obviously, if M intersects $\delta(B)$ in at most one edge, then M/B is a matching in G/B. In the following, we identify a blossom with its set of vertices.

Theorem 24.2. *Let B be an M-blossom in G. Then M is a maximum-size matching in G if and only if M/B is a maximum-size matching in G/B.*

Proof. Let $B = (v_i, v_{i+1}, \ldots, v_t)$.

First assume that M/B is not a maximum-size matching in G/B. Let P be an M/B-augmenting path in G/B. If P does not traverse vertex B of G/B, then P is also an M-augmenting path in G. If P traverses vertex B, we may assume that it enters B with some edge uB that is not in M/B. Then $uv_j \in E$ for some $j \in \{i, i+1, \ldots, t\}$.

(24.7) If j is odd, replace vertex B in P by $v_j, v_{j+1}, \ldots, v_t$.
If j is even, replace vertex B in P by $v_j, v_{j-1}, \ldots, v_i$.

In both cases we obtain an M-augmenting path in G. So M is not maximum-size.

Conversely, assume that M is not maximum-size. We may assume that $i = 0$, that is, $v_i \in X$, since replacing M by $M \triangle EQ$, where Q is the path $(v_0, v_1, \ldots, v_i)$, does not modify the theorem. Let $P = (u_0, u_1, \ldots, u_s)$ be an M-augmenting path in G. If P does not intersect B, then P is also an M/B-augmenting path in G/B. If P intersects B, we may assume that $u_0 \notin B$. (Otherwise replace P by its reverse.) Let u_j be the first vertex of P in B. Then $(u_0, u_1, \ldots, u_{j-1}, B)$ is an M/B-augmenting path in G/B. So M/B is not maximum-size. ∎

Another useful observation is:

Theorem 24.3. *Let $P = (v_0, v_1, \ldots, v_t)$ be a shortest M-alternating $X - X$ walk. Then either P is an M-augmenting path or $(v_0, v_1, \ldots, v_j)$ is an M-flower for some $j \le t$.*

Proof. Assume that P is not a path. Choose $i < j$ with $v_j = v_i$ and with j as small as possible. So $v_0, \ldots, v_{j-1}$ are all distinct.

If $j - i$ would be even, we can delete $v_{i+1}, \ldots, v_j$ from P so as to obtain a shorter M-alternating $X - X$ walk. So $j - i$ is odd. If j is even and i is odd, then $v_{i+1} = v_{j-1}$ (as it is the vertex matched to $v_i = v_j$), contradicting the minimality of j.

Hence j is odd and i is even, and therefore $(v_0, v_1, \ldots, v_j)$ is an M-flower. ∎

We now describe an algorithm (the *matching-augmenting algorithm*) for the following problem:

(24.8) given: a matching M;
find: an M-augmenting path, if any.

Denote the set of vertices missed by M by X.

(24.9) If there is no M-alternating $X-X$ walk of positive length, there is no M-augmenting path.
If there exists an M-alternating $X-X$ walk of positive length, choose a shortest one, $P = (v_0, v_1, \ldots, v_t)$ say.
Case 1: P is a path. Then output P.
Case 2: P is not a path. Choose j such that $(v_0, \ldots, v_j)$ is an M-flower, with M-blossom B. Apply the algorithm (recursively) to G/B and M/B, giving an M/B-augmenting path P in G/B. Expand P to an M-augmenting path in G (cf. (24.7)).

The correctness of this algorithm follows from Theorems 24.2 and 24.3. It gives a polynomial-time algorithm to find a maximum-size matching, which is a basic result of Edmonds [1965d].

Theorem 24.4. *Given a graph, a maximum-size matching can be found in time $O(n^2m)$.*

Proof. The algorithm directly follows from algorithm (24.9), since, starting with $M = \emptyset$, one can iteratively apply it to find an M-augmenting path P and replace M by $M \triangle EP$. It terminates if there is no M-augmenting path, whence M is a maximum-size matching.

By using (24.6), path P in (24.9) can be found in time $O(m)$. Moreover, the graph G/B can be constructed in time $O(m)$. Since the recursion has depth at most n, an M-augmenting path can be found in time $O(nm)$. Since the number of augmentations is at most $\frac{1}{2}n$, the time bound follows. ∎

This implies for perfect matchings:

Corollary 24.4a. *A perfect matching in a graph (if any) can be found in time $O(n^2m)$.*

Proof. Directly from Theorem 24.4, as a perfect matching is a maximum-size matching. ∎

24.2a. An $O(n^3)$ algorithm

The matching algorithm described above consists of a series of matching augmentations. Each matching augmentation itself consists of a series of two steps performed alternatingly:

(24.10) finding an M-alternating walk, and
shrinking an M-blossom,

until the M-alternating walk is simple, that is, is an M-augmenting path.

Each of these two steps can be done in time $O(m)$. Since there are at most n shrinkings and at most n matching augmentations, we obtain the $O(n^2m)$ time bound.

If we want to save time we must consider speeding up both the walk-finding step and the shrinking step. In a sense, our description above gives a brute-force polynomial-time method. The $O(m)$ time bound for shrinking gives us time to construct the shrunk graph completely, by copying all vertices that are not in the blossom, by introducing a new vertex for the shrunk blossom, and by introducing for each original edge its 'image' in the shrunk graph. The $O(m)$ time bound for finding an M-alternating walk gives us time to find, after any shrinking, a walk starting just from scratch.

In fact, we cannot do much better if we explicitly construct the shrunk graph. But if we modify the graph only locally, by shrinking the M-blossom B and removing loops and parallel edges, this can be done in time $O(|B|n)$. Since the sum of $|B|$ over all M-blossoms B is $O(n)$, this yields a time bound of $O(n^2)$ for shrinking.

To reduce the $O(m)$ time for walk-finding, we keep data from the previous walk-search for the next walk-search, with the help of an M-alternating forest, defined as follows.

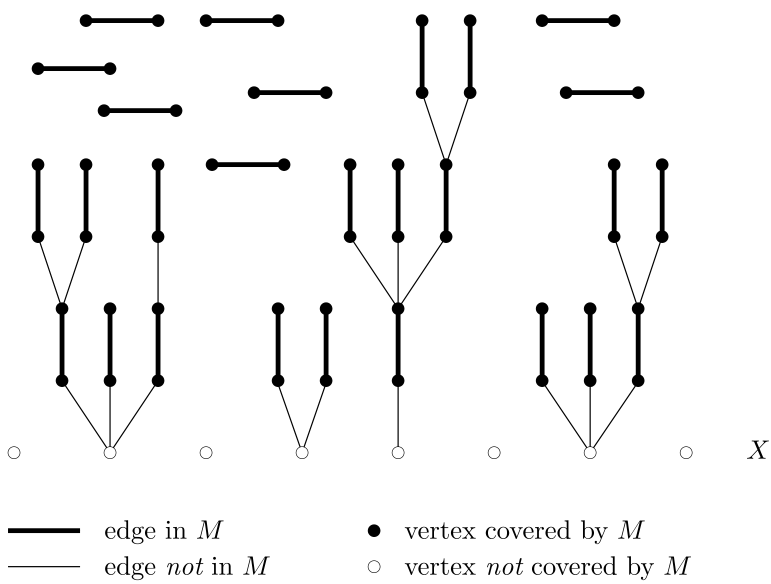


Figure 24.2
An M-alternating forest

Let $G = (V, E)$ be a simple graph and let M be a matching in G. Define X to be the set of vertices missed by M. An *M-alternating forest* is a subset F of E satisfying:

(24.11) F is a forest with $M \subseteq F$, each component of (V, F) contains either exactly one vertex in X or consists of one edge in M, and each path in F starting in X is M-alternating

(cf. Figure 24.2). For any M-alternating forest F, define

(24.12) $$\begin{aligned}\mathrm{even}(F) &:= \{v \in V \mid F \text{ contains an even-length } X - v \text{ path}\},\\ \mathrm{odd}(F) &:= \{v \in V \mid F \text{ contains an odd-length } X - v \text{ path}\},\\ \mathrm{free}(F) &:= \{v \in V \mid F \text{ contains no } X - v \text{ path}\}.\end{aligned}$$

Then each $u \in \mathrm{odd}(F)$ is incident with a unique edge in $F \setminus M$ and a unique edge in M. Moreover:

(24.13) if there is no edge connecting $\mathrm{even}(F)$ and $\mathrm{even}(F) \cup \mathrm{free}(F)$, then M is a maximum-size matching.

Indeed, if there is no such edge, $\mathrm{even}(F)$ is a stable set in $G - \mathrm{odd}(F)$. Hence, setting $U := \mathrm{odd}(F)$:

(24.14) $$o(G - U) \geq |\mathrm{even}(F)| = |X| + |\mathrm{odd}(F)| = (|V| - 2|M|) + |U|,$$

and hence M has maximum size by (24.2).

Now algorithmically, we keep, next to E and M, an M-alternating forest F. We keep the set of vertices by a doubly linked list. We keep for each vertex v, the edges in E, M, and F, incident with v as doubly linked lists. We also keep the incidence functions $\chi^{\mathrm{even}(F)}$ and $\chi^{\mathrm{odd}(F)}$. Moreover, we keep for each vertex v of G one edge $e_v = vu$ with $u \in \mathrm{even}(F)$, if such an edge exists.

Initially, $F := M$ and for each $v \in V$ we select an edge $e_v = vu$ with $u \in X$ (if any). The iteration is:

(24.15) Find a vertex $v \in \mathrm{even}(F) \cup \mathrm{free}(F)$ for which $e_v = vu$ exists.
Case 1: $v \in \mathrm{free}(F)$. Add uv to F. Let vw be the edge in M incident with v. For each edge wx incident with w, set $e_x := wx$.
Case 2: $v \in \mathrm{even}(F)$. Find the $X - u$ and $X - v$ paths P and Q in F.
Case 2a: P and Q are disjoint. Then P and Q form with uv an M-augmenting path.
Case 2b: P and Q are not disjoint. Then P and Q contain an M-blossom B. For each edge bx with $b \in B$ and $x \notin B$, set $e_x := Bx$. Replace G by G/B and remove all loops and parallel edges from E, M, and F.

The number of iterations is at most $|V|$, since, in each iteration, $|V| + |\mathrm{free}(F)|$ decreases by at least 2 (one of these terms decreases by at least 2 and the other does not change). We end up either with a matching augmentation or with the situation that there is no edge connecting $\mathrm{even}(F)$ and $\mathrm{even}(F) \cup \mathrm{free}(F)$, in which case M has maximum size by (24.13).

It is easy to update the data structure in Case 1 in time $O(n)$. In Case 2, the paths P and Q can be found in time $O(n)$, and hence in Case 2a, the M-augmenting path is found in time $O(n)$.

Finally, the data structure in Case 2b can be updated in $O(|B|n)$ time[2]. Also a matching augmentation in G/B can be transformed to a matching augmentation in G in time $O(|B|n)$. Since $|B|$ is bounded by twice the decrease in the number of vertices of the graph, this takes time $O(n^2)$ overall.

Hence a matching augmentation can be found in time $O(n^2)$, and therefore:

Theorem 24.5. *A maximum-size matching can be found in time $O(n^3)$.*

Proof. From the above. ∎

The first $O(n^3)$-time cardinality matching algorithm was published by Balinski [1969], and consists of a depth-first strategy to find an M-alternating forest, replacing shrinking by a clever labeling technique.

Bottleneck in a further speedup is storing the shrinking. With the disjoint set union data structure of Tarjan [1975] one can obtain an $O(nm\alpha(m,n))$-time algorithm (Gabow [1976a]). A special set union data structure of Gabow and Tarjan [1983,1985] gives an $O(nm)$-time algorithm. An $O(\sqrt{n}\,m)$-time algorithm was announced (with partial proof) by Micali and Vazirani [1980]. A proof was given by Blum [1990], Vazirani [1990,1994], and Gabow and Tarjan [1991] (cf. Peterson and Loui [1988]).

24.3. Matchings covering given vertices

Brualdi [1971d] derived from Tutte's 1-factor theorem the following extension of the Tutte-Berge formula:

Theorem 24.6. *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the maximum size of a subset S of T for which there is a matching covering S is equal to the minimum value of*

$$|T| + |U| - o_T(G - U) \tag{24.16}$$

over $U \subseteq V$. Here $o_T(G-U)$ denotes the number of odd components of $G-U$ contained in T.

Proof. For any matching M in G and any $U \subseteq V$, at most $|U|$ odd components of $G - U$ can be covered completely by M. So M misses at least $o_T(G-U) - |U|$ vertices in T. This shows that the minimum is not less than the maximum.

To see equality, let μ be equal to the minimum. Let C be a set disjoint from V with $|C| = |V|$ and let $C' \subseteq C$ with $|C'| = |T| - \mu$. Make a new graph H by extending G by C, in such a way that C is a clique, each vertex in C'

[2] For each $Z \in \{E, M, F\}$, we scan the vertices b in B, and for $b \in B$ we scan the Z-neighbours w of b. If w does not belong to B and was not met as a Z-neighbour of an earlier scanned vertex in B, we replace bw by Bw in Z. Otherwise, we delete bw from Z.

is adjacent to each vertex in V, and each vertex in $C \setminus C'$ is adjacent to each vertex in $V \setminus T$.

If H has a perfect matching M, then M contains at most $|C'| = |T| - \mu$ edges connecting T and C (since T is not connected to $C \setminus C'$). Hence at least μ vertices in T are covered by edges in M spanned by V, as required.

So we may assume that H has no perfect matching. Then by Tutte's 1-factor theorem, there is a set W of vertices of H such that $H - W$ has at least $|W| + 2$ odd components (since $|V| + |C|$ is even).

If $C' \not\subseteq W$, then $H - W$ has only one component (since each vertex in C' is adjacent to every other vertex), a contradiction. If $C \subseteq W$, then $H - W$ has at most $|V|$ components, while $|W| + 2 \geq |C| + 2 = |V| + 2$, a contradiction. So $C' \subseteq W$ and $C \setminus C' \not\subseteq W$. Then at most one component of $H - W$ is not contained in T (since $C \setminus C'$ is a clique and each vertex in $C \setminus C'$ is adjacent to each vertex in $V \setminus T$). Let $U := W \cap V$. Then

$$o_T(G - U) = o_T(H - W) \geq o(H - W) - 1 > |W| \geq |C'| + |U| = |T| - \mu + |U|, \tag{24.17}$$

contradicting the definition of μ. ∎

(This theorem was also given by Las Vergnas [1975b].)

A consequence is a result of Lovász [1970c] on sets of vertices covered by matchings:

Corollary 24.6a. *Let $G = (V, E)$ be a graph and let T be a subset of V. Then G has a matching covering T if and only if T contains at most $|U|$ odd components of $G - U$, for each $U \subseteq V$.*

Proof. Directly from Theorem 24.6. ∎

(This theorem was also given by McCarthy [1975].)

24.4. Further results and notes

24.4a. Complexity survey for cardinality nonbipartite matching

$O(n^2m)$	Edmonds [1965d] (cf. Witzgall and Zahn [1965])
$O(n^3)$	Balinski [1969] (also Gabow [1973,1976a], Karzanov [1976], Lawler [1976b])
$O(nm\alpha(m,n))$	Gabow [1976a]
$O(n^{5/2})$	Even and Kariv [1975], Kariv [1976] (also Bartnik [1978])
$O(\sqrt{n}\, m \log n)$	Even and Kariv [1975], Kariv [1976]

≫

continued

	$O(\sqrt{n}\,m \log\log n)$	Kariv [1976]
	$O(\sqrt{n}\,m + n^{1.5+\varepsilon})$	Kariv [1976] for each $\varepsilon > 0$
	$O(\sqrt{n}\,m)$	announced by Micali and Vazirani [1980], full proof in Blum [1990], Vazirani [1990,1994], and Gabow and Tarjan [1991](cf. Gabow and Tarjan [1983,1985])
$*$	$O(\sqrt{n}\,m \log_n \frac{n^2}{m})$	Goldberg and Karzanov [1995]

Here $*$ indicates an asymptotically best bound in the table. (Kameda and Munro [1974] claim to give an $O(nm)$-time cardinality matching algorithm, but the proof contains some errors which I could not resolve.)

Gabow and Tarjan [1988a] observed that the method of Micali and Vazirani [1980] also implies that one can find, for given k, a matching of size at least $\nu(G) - \frac{n}{k}$ in time $O(km)$. They derived that a maximum-size matching M minimizing $\max_{e\in M} w(e)$ can be found in time $O(\sqrt{n \log n}\, m)$. (the 'bottleneck matching problem').

Mulmuley, Vazirani, and Vazirani [1987a,1987b] showed that 'matching is as easy as matrix inversion', which is especially of interest for the parallel complexity.

24.4b. The Edmonds-Gallai decomposition of a graph

There is a canonical set U that attains the minimum in (24.2). It has the property that the odd components of $G-U$ cover an inclusionwise minimal set of vertices, and is given by the *Edmonds-Gallai decomposition*, independently found by Edmonds [1965d] and Gallai [1963a,1964].

Let $G = (V, E)$ be a graph. The Edmonds-Gallai decomposition of G is the partition of V into $D(G)$, $A(G)$, and $C(G)$ defined as follows (recall that $N(U) := \{v \in V \setminus U \mid \exists u \in U\colon uv \in E\}$):

(24.18)
$$D(G) := \{v \in V \mid \text{there exists a maximum-size matching missing } v\},$$
$$A(G) := N(D(G)),$$
$$C(G) := V \setminus (D(G) \cup A(G)).$$

It yields a 'canonical' certificate of maximality of a matching:

Theorem 24.7. *$U := A(G)$ attains the minimum in* (24.2)*, $D(G)$ is the union of the odd components of $G-U$, and (hence) $C(G)$ is the union of the even components of $G-U$.*

Proof. *Case 1: $D(G)$ is a stable set.* Let M be a maximum-size matching and let X be the set of vertices missed by M. Then each vertex v in $A(G)$ is contained in an edge $uv \in M$ (as $v \notin D(G)$). We show that $u \in D(G)$. Assume that $u \notin D(G)$.

Since $v \in A(G) = N(D(G))$, there is an edge vw with $w \in D(G)$. Let N be a matching missing w. Then $M \triangle N$ contains a path component starting at a vertex in X and ending at w. Let $(v_0, v_1, \ldots, v_t)$ be this path, with $v_0 \in X$ and $v_t = w$. Then t is even and $v_i \in D(G)$ for each even i (because $M \triangle \{v_0v_1, v_2v_3, \ldots, v_{i-1}v_i\}$ is a maximum-size matching missing v_i). Hence, assuming $u \notin D(G)$, the edge vu is not on P. So extending P by wv and vu gives a path Q. Then $M \triangle Q$ is a maximum-size matching missing u. So $u \in D(G)$.

As this is true for any $v \in A(G)$, we see that part of M matches $A(G)$ and $D(G) \setminus X$. Hence

(24.19) $\quad o(G-U) \geq |D(G)| = |X| + |A(G)| = |V| - 2|M| + |U|.$

So U attains the minimum in (24.2), and moreover $o(G-U) = |D(G)|$, that is, $D(G)$ is the union of the odd components of $G-U$.

Case 2: $D(G)$ spans some edge $e = uv$. Let M and N be maximum-size matchings missing u and v, respectively. Then $M \cup N$ contains a path component P starting at u. If it does not end at v, then $P \cup \{e\}$ forms an N-augmenting path, contradicting the maximality of N. So P ends at v, and hence $P \cup \{e\}$ gives an M-blossom B.

Let $G' := G/B$ and $M' := M/B$ and let X' be the set of vertices of G' missed by M'. By Theorem 24.2, $|M'| = \nu(G')$. Then

(24.20) $\quad D(G') = (D(G) \setminus B) \cup \{B\},$

since $B \in D(G')$ and since for each $v \in V \setminus B$:

(24.21) $\quad v \in D(G') \iff G'$ has an even-length M'-alternating $X' - v$ path $\iff G$ has an even-length M-alternating $X - v$ path $\iff v \in D(G)$.

This proves (24.20), which implies that $A(G') = A(G)$ and $C(G') = C(G)$. By induction, $D(G')$ is the union of the odd components of $G' - U$. Hence $D(G)$ is the union of the odd components of $G - U$ (since $B \subseteq D(G)$ by (24.20)). Also by induction, $|M'| = \frac{1}{2}(|V'| + |U| - o(G' - U))$. Hence $|M| = \frac{1}{2}(|V| + |U| - o(G-U))$, since $|V| - 2|M| = |V'| - 2|M'|$. ∎

So $U = A(G)$ is the unique set attaining the minimum in (24.2) for which the union of the odd components of $G - U$ is inclusionwise minimal.

Note that:

(24.22) for any U attaining the minimum in (24.2), each maximum-size matching M has exactly $\lfloor \frac{1}{2}|K| \rfloor$ edges contained in any component K of $G - U$, and each edge of M intersecting U also intersects some odd component of $G - U$.

This implies the following. Call a graph $G = (V, E)$ *factor-critical* if $G - v$ has a perfect matching for each vertex v.

Corollary 24.7a. *Let $G = (V, E)$ be a graph. Then each component K of $G[D(G)]$ is factor-critical.*

Proof. Directly from Theorem 24.7 and (24.22): if $v \in K$, then $v \in D(G)$, and hence $G - v$ has a maximum-size matching M missing v. By (24.22), M has $\lfloor \frac{1}{2}|K| \rfloor$ edges contained in K. So $K - v$ has a perfect matching. ∎

The Edmonds-Gallai decomposition can be found in polynomial time, since the set $D(G)$ of vertices missed by at least one maximum-size matching can be determined in polynomial time (with the cardinality matching algorithm). In fact, with the alternating forest approach of Section 24.2a one can find the Edmonds-Gallai decomposition in time $O(n^3)$. If we have a maximum-size matching, it takes $O(n^2)$ time.

24.4c. Strengthening of Tutte's 1-factor theorem

Tutte's 1-factor theorem can be (self-)refined as follows (this theorem also can be derived from Theorem 24.7 and Corollary 24.7a; we give a direct derivation from Tutte's 1-factor theorem):

Theorem 24.8. *A graph $G = (V, E)$ has a perfect matching if and only if for each $U \subseteq V$, the graph $G - U$ has at most $|U|$ factor-critical components.*

Proof. Necessity is easy, since each factor-critical component is odd. To see sufficiency, let the condition be satisfied, and suppose that G has no perfect matching. By Tutte's 1-factor theorem, there is a subset U of V such that $G - U$ has more than $|U|$ odd components. Choose an inclusionwise maximal such set U.

By the condition, at least one component K of $G-U$ is not factor-critical. That is, K contains a vertex v such that $K - v$ has no perfect matching. Then by Tutte's 1-factor theorem, there exists a subset U' of $K - v$ such that $K - v - U'$ has more than $|U'|$ odd components, and hence at least $|U'|+2$ odd components (since $K-v$ has an even number of vertices). Now define $U'' := U \cup U' \cup \{v\}$. Then $G - U''$ has more than $|U''|$ odd components. As $U'' \supset U$, this contradicts the maximality of U. ∎

24.4d. Ear-decomposition of factor-critical graphs

As mentioned, a graph $G = (V, E)$ is *factor-critical* if, for each $v \in V$, the graph $G - v$ has a perfect matching. Lovász [1972b] showed that all factor-critical graphs can be constructed by 'odd ear-decompositions' in the following sense. We say that graph H arises by *adding an odd ear* from G, if H arises from G by adding an odd-length path at two (not necessarily distinct) vertices of G. That is, if there is a path or circuit $(v_0, v_1, \ldots, v_t)$ in H with t odd, $v_1, \ldots, v_{t-1}$ each having degree 2, and $G = H - \{v_1, \ldots, v_{t-1}\}$.

It is easy to see that if H arises by adding an odd ear to a factor-critical graph G, then H is again factor-critical. Now each factor-critical graph arises in this way from the one-vertex graph:

Theorem 24.9. *A graph G is factor-critical if and only if there exists a series of graphs $G_0, \ldots, G_k$ with G_0 being a one-vertex graph, $G_k = G$, and G_i arising by adding an odd ear to G_{i-1} ($i = 1, \ldots, k$).*

Proof. For sufficiency, see above. To see necessity, fix, for each vertex v of G, a perfect matching M_v of $G-v$. Choose a vertex u of G. Let H be a maximal subgraph of G such that

(24.23) (i) H arises by a series of odd ear addings from the one-vertex graph on u;

(ii) for each edge $e \in M_u$, if e intersects VH, then $e \in EH$.

Such a graph trivially exists, as the one-vertex graph on u satisfies (24.23).

If $EH = EG$ we are done, so assume $EH \neq EG$. As G is factor-critical, G is connected, and hence there is an edge $e = vw \in EG \setminus EH$ with $v \in VH$. Consider $M_w \cup M_u$. One of its components is an even-length $w-u$ path $P = (v_1, \ldots, v_t)$ with $v_1 = w$ and $v_t = u$. So $v_t \in VH$. Let j be the smallest index with $v_j \in VH$. Then j is odd, since otherwise $v_{j-1}v_j \in M_u$ with $v_{j-1} \notin VH$ and $v_j \in VH$, contradicting (24.23)(ii).

Let Q be the path $(v, v_1, \ldots, v_j)$. Then $H \cup Q$ arises by adding an odd ear to H, and moreover, it satisfies (24.23)(ii) again, contradicting the maximality of H. ∎

(This is the original proof of Lovász [1972b].)

As a consequence we have a recursive characterization of factor-critical graphs:

Corollary 24.9a. *Let $G = (V, E)$ be a graph with $|V| \geq 2$. Then G is factor-critical if and only if G has an odd circuit C with G/C factor-critical.*

Proof. To see sufficiency, let C be an odd circuit with G/C factor-critical. We show that G is factor-critical. Choose $v \in V$. If $v \in C$, let M' be a perfect matching of $G[C \setminus \{v\}]$. Since G/C is factor-critical, $G - C$ has a perfect matching M''. Then $M \cup M''$ is a perfect matching of $G - v$.

If $v \notin C$, let M'' be a perfect matching of $(G/C) - v$. In G this gives a matching covering all vertices in $V \setminus (C \cup \{v\})$ and exactly one vertex, u say, in C. Let M' be a perfect matching in $G[C \setminus \{u\}]$. Then $M' \cup M''$ is a perfect matching of $G - v$. This shows sufficiency.

Necessity is shown with Theorem 24.9. Let G be factor-critical. Consider an odd ear-decomposition of G, and let C be the first odd ear. Then the remaining ears form an odd ear-decomposition of G/C, and hence G/C is factor-critical. ∎

(Related results were given by Cornuéjols and Pulleyblank [1983].)

24.4e. Ear-decomposition of matching-covered graphs

A graph $G = (V, E)$ is called *matching-covered* if each edge of G belongs to a perfect matching of G. Matching-covered graphs can be constructed similarly to factor-critical graphs, but now starting from an even circuit (however, the decomposition does not characterize matching-covered graphs). This will be used in proving Theorem 29.11 on the maximum size of a join.

Theorem 24.10. *For each connected matching-covered graph G with at least four vertices there exists a series of graphs $G_0, \ldots, G_k$ with G_0 being an even circuit, $G_k = G$, and G_i arising by adding an odd ear to G_{i-1} ($i = 1, \ldots, k$).*

Proof. For each edge e of G, fix a perfect matching M_e of G containing e. Fix a perfect matching M of G. One easily checks that G contains an M-alternating even circuit C. Let H be a maximal subgraph of G such that

(24.24) (i) H arises by a series of odd ear addings from C;
(ii) for each edge $e \in M$, if e intersects VH, then $e \in EH$.

Such a graph trivially exists, as C satisfies (24.24).

If $EH = EG$ we are done, so assume $EH \neq EG$. As G is connected, there is an edge $e \in EG \setminus EH$ intersecting VH. Consider $M_e \cup M$. Then the component of $M_e \cup M$ containing e gives an odd ear that can be added to H, contradicting the maximality of H. ∎

A direct algorithmic proof was given by Little and Rendl [1989]. Little [1974] showed that in a matching-covered graph, any two edges belong to a circuit that is in the symmetric difference of two perfect matchings. Carvalho, Lucchesi, and Murty [1999] gave more results on ear-decompositions of matching-covered graphs.

24.4f. Barriers in matching-covered graphs

A *barrier* in a graph $G = (V, E)$ is a subset B of V such that $G - B$ has $|B|$ odd components. Note that if B is a barrier in a connected matching-covered graph G, then B is a stable set and each component of $G - B$ is odd.

Lovász and Plummer [1975,1986] showed:

Theorem 24.11. *Let B and C be barriers in a connected matching-covered graph $G = (V, E)$ with $B \cap C \neq \emptyset$. Then $B \cap C$ and $B \cup C$ are barriers again.*

Proof. We first show:

(24.25) if B and C are distinct barriers with $B \cap C \neq \emptyset$, then there exists a nonempty set D with $D \subseteq B \setminus C$ or $D \subseteq C \setminus B$ such that $B \triangle D$ and $C \triangle D$ are barriers again.

As B and C are stable sets, there is a path from $B \cap C$ to $B \triangle C$. Consider a shortest such path, say it runs from $B \cap C$ to $C \setminus B$. It implies that $G - B$ has a component K with a neighbour in $B \cap C$ and intersecting $C \setminus B$. Define $D := K \cap C$. We show that $B \cup D$ and $C \setminus D$ are barriers again.

Fix an edge e connecting $B \cap C$ and K. Let L be the component of $G - C$ incident with e. Let M be a perfect matching containing e. As e connects $K \cap L$ and $B \cap C$, all other edges in M incident with K are contained in K. So if some edge $f \in M$ leaves $K \cap L'$ for some component L' of $G - C$, and $f \neq e$, then f does not leave K. Hence f leaves L', implying $L' \neq L$ (otherwise, L is left by two edges in M). It also implies that f connects $K \cap L'$ and $K \cap C$ and that f is the only edge in M leaving $K \cap L'$. Moreover, each vertex in D is covered by an edge in M, and hence it is such an edge f. Hence the number of components L' of $G - C$ with $K \cap L'$ odd is equal to $|D| + 1$.

Now $B \cup D$ is a barrier, since $G[K \setminus D]$ has $|D|+1$ odd components. So $G-(B \cup D)$ has at least $|B| + |D|$ odd components, and hence $B \cup D$ is a barrier.

Hence, as G is matching-covered, each component of $G - B - D$ is odd. So each component of $G[K \setminus D]$ is odd, and therefore $G[K \setminus D]$ has exactly $|D| + 1$ components. So all but at most $|D| + 1$ components of $G - C$ are also components of $G - (C \setminus D)$. Hence the number of odd components of $G - (C \setminus D)$ is at least $|C| - |D| - 1$, and hence, by parity, at least $|C \setminus D|$. So $C \setminus D$ is a barrier. This proves (24.25).

Now to prove that $B \cup C$ is a barrier, we can assume that we have chosen B and C inclusionwise maximal barriers contained in $B \cup C$. Then $B = C$ by (24.25).

Similarly, to prove that $B \cap C$ is a barrier, we can assume that we have chosen B and C inclusionwise minimal barriers containing $B \cap C$. Again we have $B = C$ by (24.25). ∎

This has the following consequence due to Lovász [1972e] (cf. Kotzig [1960] (Theorem 31)):

Corollary 24.11a. *Any two distinct maximal barriers in a connected matching-covered graph are disjoint.*

Proof. Directly from Theorem 24.11. ∎

Since each singleton is a barrier, Corollary 24.11a implies that the maximal barriers in a connected matching-covered graph partition the vertex set of G. This gives the result of Kotzig [1959b] (Theorem 11):

Corollary 24.11b. *Let $G = (V, E)$ be a connected matching-covered graph. For $u, v \in V$ define $u \sim v$ by:*

(24.26) $\qquad u \sim v$ *if and only if* $G - u - v$ *has no perfect matching.*

Then $\sim$ is an equivalence relation.

Proof. Note that $u \sim v$ if and only if $\{u, v\}$ is contained in some barrier. So the corollary follows directly from Corollary 24.11a. ∎

For much more on barriers in matching-covered graphs, see Lovász and Plummer [1986].

24.4g. Two-processor scheduling

The following problem was considered by Fujii, Kasami, and Ninomiya [1969]. Suppose that we have to carry out certain jobs, where some of the jobs have to be done before other. We can represent this by a partially ordered set $(V, \leq)$ where V is the set of jobs and $x < y$ indicates that job x has to be done before job y. Each job takes one time-unit, say one hour.

Suppose now that there are two workers, each of which can do one job at a time. Alternatively, suppose that you have one machine, that can do at each moment two jobs simultaneously (a *two-processor*).

We wish to do all jobs within a minimum total time span. This problem can be solved with the matching algorithm as follows. Make a graph $G = (V, E)$, with vertex set V (the set of jobs) and with edge set

(24.27) $\qquad E := \{\{u, v\} \mid u \not\leq v \text{ and } v \not\leq u\}.$

(So (V, E) is the complementary graph of the 'comparability graph' associated with $(V, \leq)$.)

Consider now a possible schedule of the jobs. That is, we have a sequence $p_1, \ldots, p_t$, where each p_i is either a singleton vertex or an edge of G such that $p_1, \ldots, p_t$ partition V and such that if $u < v$ and $u \in p_i$ and $v \in p_j$, then $i < j$.

Now the pairs in this list should form a matching M in G. Hence $t = |V| - |M|$. In particular, t cannot be smaller than $|V| - \nu(G)$, where $\nu(G)$ is the matching number of G.

Fujii, Kasami, and Ninomiya [1969] showed that in fact one can always make a schedule with $t = |V| - \nu(G)$. For that it is sufficient to show:

Theorem 24.12. *G contains a maximum-size matching $M = \{e_1, \ldots, e_t\}$ such that if $u \in e_i$ and $v \in e_j$ with $u < v$, then $i < j$.*

Proof. The proof is by induction on $|V|$. Let M be a maximum-size matching in G. We may assume that M is a perfect matching, since otherwise we can delete all vertices missed by M, and apply induction.

Let $V^{\min}$ be the set of minimal elements of $(V \leq)$. If $V^{\min}$ contains an edge $uv \in M$ as a subset, we can delete u and v from V, and apply induction. So we may assume that each $s \in V^{\min}$ is contained in an edge $st \in M$ with $t \notin V^{\min}$. Choose an edge $st \in M$ with $s \in V^{\min}$ and with the height of t as small as possible. (The *height* of an element t is the maximum size of a chain in $(V, \leq)$ with maximum element t.) As $t \notin V^{\min}$ there exists an $s't' \in M$ with $s' \in V^{\min}$ and $s' < t$.

Now clearly ss' is an edge of G, as s and s' are minimal elements. Moreover, tt' is an edge of G. For if $t < t'$, then $s' < t < t'$, contradicting the fact that $s't' \in E$; and if $t' < t$, then t' would have smaller height than t.

So replacing st and $s't'$ in M by ss' and tt', we have $ss' \subseteq V^{\min}$, and so by deleting s and s' from V we can apply induction as before. ∎

The theorem implies that there is a linear extension $\preceq$ of $\leq$ and a maximum-size matching M in G such that if $uv \in M$, then u and v are neighbouring in $\preceq$.

Coffman and Graham [1972] gave a direct, $O(n^2)$-time algorithm. (Muntz and Coffman [1969] gave an algorithm for the two-processor scheduling problem if jobs may be interrupted and continued later.) This was improved to $O(m + n\alpha(m, n))$ by Gabow [1982] and to $O(m + n)$ by Gabow and Tarjan [1983,1985].

24.4h. The Tutte matrix and an algebraic matching algorithm

Tutte [1947b] observed the following. Let $G = (V, E)$ be a graph. Choose for each edge e an indeterminate x_e. Let M be a skew-symmetric[3] $V \times V$ matrix with $M_{u,v} = \pm x_e$ if $e = \{u, v\} \in E$, and $M_{u,v} = 0$ otherwise (including $u = v$) (the *Tutte matrix*). Then the rank of M is equal to twice the matching number of G.

Lovász [1979c] showed that substituting random integers for the x_e, gives an efficient randomized algorithm for finding the matching number of G. This idea was extended by Geelen [2000], who proved the following:

(24.28) Let M' arise from M by substituting the x_e by integers from $\{1, \ldots, n\}$, where $n := |V|$. If $\text{rank}(M') < \text{rank}(M)$, then there is an edge e of G and a number $b \in \{1, \ldots, n\}$ such that for the matrix M'' arising from

[3] A matrix M is *skew-symmetric* if $M^{\mathsf{T}} = -M$.

M' by resetting the $\pm x_e$ entries to $\pm b$, we have $\text{rank}(M'') > \text{rank}(M')$, or $\text{rank}(M'') = \text{rank}(M')$ and $D(M'') \supset D(M')$.

Here $D(A)$ denotes the set of $v \in V$ such that the $V \setminus \{v\} \times V \setminus \{v\}$ submatrix of A has the same rank as A.

(24.28) implies a polynomial-time algorithm to compute the matching number of G (and hence to find a maximum-size matching in G): start with an arbitrary matrix M' obtained by substituting the x_e by numbers in $\{1, \ldots, n\}$, and iteratively try to reset an entry to another number from $\{1, \ldots, n\}$, as long as it either increases the rank of M', or maintains the rank and increases $D(M')$. The final matrix has rank equal to the matching number of G.

L. Lovász (cf. Geelen [1995]) extended Tutte's result to the rank of any (not necessarily principal) submatrix of M. Geelen [1995] described the corresponding system of linear inequalities and proved its total dual integrality, generalizing Edmonds' matching polytope theorem.

24.4i. Further notes

Biedl, Bose, Demaine, and Lubiw [1999,2001] gave an $O(n \log^4 n)$ time algorithm to find a perfect matching in cubic bridgeless graphs (linear-time if the graph is moreover planar). Biedl [2001] gave a linear-time reduction of the general matching problem to the matching problem for cubic graphs.

Lower bounds on the maximum size of a matching were given by Nishizeki and Baybars [1979] for planar graphs and by Biedl, Demaine, Duncan, Fleischer, and Kobourov [2001] for several other classes of graphs.

Fulkerson, Hoffman, and McAndrew [1965] showed that any regular graph with an even number of vertices and with the property that each two vertex-disjoint odd circuits are connected by an edge, has a perfect matching (cf. Mahmoodian [1977], Berge [1978b,1981]). Other sufficient conditions were given by Anderson [1972], Sumner [1974a], Las Vergnas [1975a], and Chartrand, Goldsmith, and Schuster [1979].

Plesník [1972] showed that in a k-regular $(k-1)$-edge-connected graph with an even number of vertices, there is a perfect matching not containing $k-1$ prescribed edges (cf. Chartrand and Nebeský [1979]). For $k = 3$ this was proved by Schönberger [1934]. For general k, it can also be derived from Edmonds' perfect matching polytope theorem (Theorem 25.1 below). See also Plesník [1979].

Further studies on the structure of *matching-covered graphs* (graphs in which each edge belongs to a perfect matching) were made by Kotzig [1959a,1959b,1960], Hetyei [1964], Lovász [1970d,1972f,1972d,1972e,1983a], Little, Grant, and Holton [1975], Lovász and Plummer [1975], Gabow [1979], Edmonds, Lovász, and Pulleyblank [1982], Naddef [1982], and Szigeti [1998b].

Gabow, Kaplan, and Tarjan [1999,2001] gave fast algorithms to test if a given perfect matching is unique, to find it, and if it not unique to find another perfect matching.

Sumner [1974b,1976] studied sets U with $o(G-U) > |U|$. Weinstein [1963,1974] and Bollobás and Eldridge [1976] related the matching number to the minimum and maximum degree and the connectivity. Chvátal and Hanson [1976] evaluated the maximum number $f(n, b, d)$ of edges of a graph with n vertices having no vertex of degree $> d$ and no matching of size $> b$.

Implementing cardinality matching algorithms were studied by Burkard and Derigs [1980], Crocker [1993], and Mattingly and Ritchey [1993]. A simulated annealing approach was described by Sasaki and Hajek [1988].

Books covering nonbipartite matching algorithms include Christofides [1975], Lawler [1976b], Minieka [1978], Papadimitriou and Steiglitz [1982], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Derigs [1988a], Nemhauser and Wolsey [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000]. Surveys on matching algorithms were given by Galil [1983,1986a,1986b].

Motwani [1989,1994] investigated the expected running time of matching algorithms.

Gallai [1950], Tutte [1950], Kaluza [1953], Steffens [1977], and Aharoni [1984a, 1984c,1984d,1988] gave extensions to infinite graphs. The Edmonds-Gallai decomposition was extended to locally finite graphs by Bry and Las Vergnas [1982] (cf. Steffens [1985]).

The behaviour of a greedy heuristic for finding a large matching was investigated by Dyer and Frieze [1991], Dyer, Frieze, and Pittel [1993], and Aronson, Dyer, Frieze, and Suen [1994].

The standard work on matching theory is Lovász and Plummer [1986]. Other books discussing nonbipartite matching include Berge [1973b], Bondy and Murty [1976], Bollobás [1978,1979], Tutte [1984], and Diestel [1997]. Survey articles on matchings were given by Akiyama and Kano [1985b] and Lovász and Plummer [1986], Gerards [1995a], Pulleyblank [1995], and Cunningham [2002].

24.4j. Historical notes on nonbipartite matching

Petersen and Sylvester

Petersen [1891] was among the first to study perfect matchings (1-factors) in graphs, introducing several basic concepts and methods, like factors and alternating paths. He was motivated by finite basis theorems in invariant theory, especially by the question which polynomials form a finite basis. Petersen cooperated with J.J. Sylvester, who did similar studies, leading to an intensive correspondence on the topic in the years 1889-1890 — see Sabidussi [1992] (unfortunately, the letters of Petersen to Sylvester were not found).

In particular, they considered homogeneous polynomials of the form

$$\prod_{i<j}(x_i - x_j)^{r_{i,j}}, \tag{24.29}$$

and were interested in conditions under which such a polynomial can be factorized into other homogeneous polynomials of the same form. This is equivalent to characterizing the existence of k-factors in regular graphs. (Graph terms like 'factor' and 'degree' introduced by Petersen are motivated by this interpretation.)

In a letter of 18 October 1889, Sylvester expressed to Petersen the conjecture that each graph of minimum degree at least two has a 2-factor. He had checked it for graphs with up to 7 vertices, and said that he had 'not much doubt of being able to establish the proof for all values of n by the same process which has been successful for the earlier numbers'. Sylvester considered this as the most important theorem discovered hitherto in the science of chemical graphology, a field initiated by Sylvester [1878].

Two days later, Sylvester wrote a letter in which he restricted his conjecture to the case of regular graphs, and he was more doubtful on whether it is true. After a reply of Petersen, Sylvester gave in a letter of 27 October 1889 an example of a graph with 7 vertices, with degrees 2 and 3, not having a 2-factor. In this letter, Sylvester also remarked that as a consequence of his conjecture, each regular graph of odd order has a 2-factorization.

Then, in a letter of 8 November 1889, Sylvester observed that there is a cubic graph on 10 vertices that has no factorization (Figure 24.3). (A graph is *cubic* if it

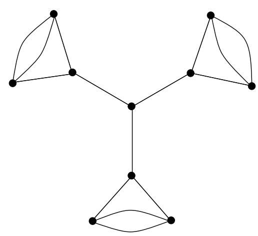


Figure 24.3
Sylvester's graph

is 3-regular.)

Subsequently, on 16 November 1889, Sylvester wrote to Petersen:

> Thanks for your interesting note—I also have a proof of the 'theorem of Ablation' for even equifrequencies.

Apparently, Petersen had written about his theorem that each regular graph of even degree has a 2-factorization, for which Sylvester also said to have a proof.

Next follows correspondence on the proofs the two have, with a lot of mutual misunderstanding. However, after hearing Petersen's proof at a visit of Petersen to Sylvester, at the end of December 1889, Sylvester became convinced of the correctness of Petersen's proof, and found it 'a very beautiful method'. On the other hand, Petersen remained very sceptical about Sylvester's proof, which Sylvester said was by induction on the number of vertices. They decided to publish their proofs separately. However, Sylvester did not publish on the topic; Petersen's proof appeared in the paper Petersen [1891].

Petersen's 1891 paper

In this paper, Petersen first observed that Gordan's finite basis theorem implies that for each n there exists a finite set $\mathcal{G}$ of regular graphs on n vertices (of nonzero degree) with the property that each regular graph on n vertices contains at least one graph in $\mathcal{G}$ as spanning subgraph (factor). (This result can also be proved by elementary means.) Petersen next puts as his goal to characterize all *primitive* graphs, that is, all regular graphs that have no other factors than itself and the 0-regular subgraph.

First, Petersen observed that a 2-regular graph is primitive if and only if at least one of its components is odd. Next, he showed that each 4-regular graph has a 2-factor. To this end, he made an Eulerian tour along all edges, colouring them alternatingly blue and red. The blue edges then form a 2-factor. He observed that similarly one can show more generally that each $2k$-regular graph with an even number of edges has a k-factor.

Next, Petersen showed that each $2k$-regular graph has a 2-factorization. His proof is by observing that the existence of a 2-factorization is invariant under replacing any two disjoint edges ab and cd by ac and bd (by using the result that each 4-regular graph has a 2-factorization).

This solves the factorization problem for k-regular graphs with k even. Petersen next considered the case of odd k. He gave an example of primitive k-regular graphs for arbitrary odd k. He showed that each k-regular graph on n vertices with $k > \frac{1}{2}n + 1$ has a perfect matching. To this end, he considered a matching M and observed that

(24.30) M has maximum size if and only if there is no M-augmenting path.

To formulate this, Petersen coloured the edges in M red, and all other edges blue. A *Wechselweg* (alternating path) is a path coloured alternatingly red and blue. Let $2n$ be the number of vertices of the graph and let α be the size of the matching (thus it misses $2n - 2\alpha$ vertices). Then:

> Wir sahen oben, dass α grösser gemacht werden konnte, wenn wir zwischen zwei von den $2n - 2\alpha$ Punkten einen Wechselweg *cabd* finden konnten; dasselbe gilt wenn wir zwischen zwei von den $2n - 2\alpha$ Punkten überhaupt einen Wechselweg finden können, denn verändert man die Farben der Seiten eines solches Weges, so wird die Anzahl der rothen Linien um eins vergrössert. Man beweist leicht, dass diese Bedingung auch notwendig is.[4]

This brought Petersen to propose an algorithm to find a 1-factor:

> Indem wir die α Linien aufs Geradewohl ausnehmen und dann mittelst Wechselwege α zu vergrössern suchen, können wir untersuchen, ob ein gegebener *graph* primitiv ist oder nicht;[5]

Petersen however preferred a direct characterization:

> es entsteht aber die Frage, ob die primitiven *graphs* sich nicht durch einfache Kennzeichen von den zerlegbaren scheiden.[6]

He conjectured:

---

[4] We saw above that α can be increased, if we could find an alternating path *cabd* between two of the $2n-2\alpha$ points; the same holds if we can find an alternating path at all between two of the $2n-2\alpha$ points, because if one changes the colours of the edges in such a path, then the number of red edges increases by one. One easily proves that this condition is also necessary.

[5] While we select the α edges arbitrarily and then try to increase α by alternating paths, we can investigate if a given *graph* is primitive or not;

[6] the question however arises if the primitive *graphs* are not distinguished from the factorizable by simple characteristics.

> Es spricht etwas dafür, dass ein primitiver *graph Blätter* haben muss, indem ein Blatt ein solcher Theil des *graphs* ist, der nur durch eine einzelne Linie mit dem übrigen Theil in Verbindung steht. Ich habe daher versucht dieses zu beweisen, habe aber die Schwierigkeiten so gross gefunden, dass ich die Untersuchung auf den *graph* dritten Grades beschränkt habe.[7]

Petersen [1891] described the cubic graph on 10 vertices found by Sylvester that has no 1-factor (Figure 24.3), which he called *Sylvester's graph*.

Petersen showed that each primitive cubic graph has at least three leaves. As mentioned, a *leaf* is a subset U of the vertices with $|\delta(U)| = 1$. (A graph is *cubic* if it is 3-regular.)

Again, Petersen showed his theorem with the help of studying alternating paths. Those edges that can be traversed in both directions by alternating paths starting at a 'free' vertex are called 'zweipfeilig' (two-arrow as adjective). He then reduced the problem by shrinking and stated:

> Wir ziehen jetzt jedes zweipfeiliges System in einen Punkt zusammen;[8]

Proofs and extensions of Petersen's theorem

Brahana [1917] gave a shorter proof of Petersen's theorem. He restricted the concept of leaf to a *minimal* set of vertices connected by only one edge to the remainder of the graph. (In fact, also Petersen's proof is valid for this restricted interpretation of leaf.)

Brahana's method is again based on augmenting paths and shrinking. Moreover, he used a reduction to smaller graphs by deleting two adjacent vertices u and v and connecting the two further vertices adjacent to u and v by new edges. This can be done in such a way that the number of leaves remains at most 2.

In fact, part of Brahana's method is algorithmic, and can be considered as a specialization of Edmonds' cardinality matching algorithm. Brahana needs to find a 1-factor, given a matching M of size $\frac{1}{2}n - 1$ (where n is the number of vertices). He described a depth-first method to find an M-augmenting path starting from a vertex missed by M. If it runs into a loop (a 'bicursal circuit'), it can be removed by shrinking:

> We continue this shrinking process as long as there are such bicursal circuits.

Also Errera [1921,1922], Frink [1925], Schönberger [1934], Kőnig [1936], and Baebler [1954] gave alternative proofs of Petersen's theorem (see also Sainte-Laguë [1926b]). The proof of Frink is 'by induction, no shrinking or counting processes being used.' He overlooked however some complications (in relation to the construction of a new 2-connected graph in the proof of his 'Theorem II') — they were resolved by Kőnig [1936]. The proof yields a polynomial-time algorithm to find a perfect matching in a 2-connected cubic graph.

Schönberger [1934] showed that in any 2-connected cubic graph each edge is in a perfect matching, and (more generally) for any two prescribed edges there is a perfect matching not containing these edges.

---

[7] Something speaks for it that a primitive *graph* must have *leaves*, while a leaf is such a part of the *graph* that is in connection with the remaining part only by one single edge. I therefore have tried to prove this, but have found the difficulties that big, that I have restricted the investigation to the *graph* of third degree.

[8] We now contract each two-arrow system to one point;

Baebler [1937] showed that each k-regular l-edge-connected graph, with k odd and l even, has an l-factor. His proof is based on shrinking.

Tutte

Tutte [1947b] characterized the graphs that have a perfect matching. His proof is essentially that given in Section 24.1a, defining a graph to be 'hyperprime' if it has no perfect matching, but adding any edge creates a perfect matching. He used 'pfaffians' in order to show that, in a hyperprime graph, each component of the subgraph induced by the set of vertices that are not adjacent to all other vertices, is complete. A combinatorial proof of this fact was given by Maunsell [1952].

Tutte's theorem was extended to arbitrary l-factors ($l \in \mathbb{Z}_+$) by Belck [1950] (see Chapter 33); the proof is by extension of Tutte's method. This in turn was generalized by Tutte [1952] to b-factors where $b \in \mathbb{Z}_+^V$. As an 'allied problem', Tutte [1952] considered perfect b-matchings, that is, functions $f \in \mathbb{Z}_+^E$ with $f(\delta(v)) = b(v)$ for each vertex v. The proof is by reduction to the b-factor case, by replacing each edge by several parallel edges.

Then in Tutte [1954b] it is realized that the b-factor and b-matching theorems can be reduced to the case $b = \mathbf{1}$ by splitting vertices and by the construction given in the proof of Theorem 32.1.

Gallai [1950] gave a short proof of Tutte's 1-factor theorem. He showed the following. Let G be a graph without a perfect matching, let M be a maximum-size matching in G, and let v be a vertex missed by M. Let U be the set of vertices u for which there is an M-alternating $v-u$ path of odd length. Then $G-U$ has more than $|U|$ odd components. Gallai [1950] also gave several characterizations for the existence of l-factors in regular graphs, and he considered the infinite case.

Also Tutte [1950] and Kaluza [1953] gave extensions to the infinite case. The main theorem of Ore [1957] is an alternative characterization of the existence of a b-factor. Berge [1958a] extended Tutte's 1-factor theorem to a min-max relation for the maximum size of a matching, the Tutte-Berge formula.

Kotzig [1959a,1959b,1960] studied the structure of matching-covered graphs, leading to a decomposition of any graph (cf. Ore [1959]).

Augmenting paths

Like Petersen, Berge [1957] observed that a matching M is maximum if and only if there is no M-augmenting path, and he suggested the following procedure for solving the cardinality matching problem:

> Construct a maximal matching V, and determine whether there exists an alternating chain W connecting two neutral points. (The procedure is known.) If such a chain exists, change V into $(V \setminus W) \cup (W \setminus V)$, and look again for a new alternating chain; if such a chain does not exist, V is maximum.

In Berge [1958b], a depth-first search approach to finding an augmenting path is sketched, however without shrinking, and not leading to a polynomial-time algorithm.

Also Norman and Rabin [1958,1959] found the augmenting path criterion for maximality of a matching (and similarly, for minimality of an edge cover):

> These results immediately lead to algorithms for a minimum cover and a maximum matching respectively.

Edmonds [1962] and Ray-Chaudhuri [1963] extended the augmenting path criterion to arbitrary hypergraphs.

Edmonds

Edmonds observed that Berge's proposal for finding an augmenting path (quoted above) does not lead to a polynomial-time algorithm. In his personal recollections, Edmonds [1991] stated:

> It is really hard for anyone to see that it isn't easy that when you've got a matching in a graph and you are starting at a deficient node, that you cannot just grow a tree looking for a Berge augmenting path.

Edmonds [1965d] argued:

> Berge proposed searching for augmenting paths as an algorithm for maximum matching. In fact, he proposed to trace out an alternating path from an exposed vertex until it must stop and, then, if it is not augmenting, to back up a little and try again, thereby exhausting possibilities.
> His idea is an important improvement over the completely naive algorithm. However, depending on what further directions are given, the task can still be one of exponential order, requiring an equally large memory to know when it is done.

In the summer of 1963, at a Workshop at the RAND Corporation, Edmonds discovered that shrinking leads to a polynomial-time algorithm to find a maximum-size matching in any graph. The result was described in the paper Edmonds [1965d] (received 22 November 1963), in which paper he also described his views on algorithms and complexity:

> For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance". This is roughly the meaning I want — in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good".
> I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.
> There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

Moreover:

> For practical purposes the difference between algebraic and exponential order is often more crucial than the difference between finite and non-finite.

Edmonds described his algorithm, in terms of paths, trees, flowers, and blossoms, and concluded that the 'order of difficulty' is n^4 (more precisely, it is $O(n^2m)$).

In this paper, Edmonds also introduced the decomposition of any graph which is now called the Edmonds-Gallai decomposition. Also in 1963, Gallai submitted a paper (Gallai [1963a]), in which this decomposition is described implicitly, which was made more explicit in Gallai [1964].

In the Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems in March 1964 in Yorktown Heights, New York, at the end of Gomory [1966], the following discussion is reported:

J. EDMONDS: I have a comment on the polyhedral approach to complete analysis, supplementing Professor Kuhn's remarks. I do not believe there is any reason for taking as a measure of the algorithmic difficulty of a class of combinatorial extremum problems the number of faces in the associated polyhedra. For example, consider the generalization of the assignment problem from bipartite graphs to arbitrary graphs. Unlike the case of bipartite graphs, the number of faces in the associated polyhedron increases exponentially with the size of the graph. On the other hand, there is an algorithm for this generalized assignment problem which has an upper bound on the work involved just as good as the upper bound for the bipartite assignment problem.
H.W. KUHN: I could not agree with you more. That is shown by the unreasonable effectiveness of the Norman-Rabin scheme for solving this problem. Their result is unreasonable only in the sense that the number of faces of the polyhedron suggests that it ought to be a harder problem than it actually turned out to be. It is not impossible that some day we will have a practical combinatorial algorithm for this problem.
J. EDMONDS: Actually, the amount of work in carrying out the Norman-Rabin scheme generally increases exponentially with the size of the graph.
The algorithm I had in mind is one I introduced in a paper submitted to the Canadian Journal of Mathematics (see Edmonds, 1965). This algorithm depends crucially on what amounts to knowing all the bounding inequalities of the associated convex polyhedron—and, as I said, there are many of them. The point is that the inequalities are known by an easily verifiable characterization rather than by exhaustive listing—so their number is not important.

Chapter 25

The matching polytope

As a by-product of his weighted matching algorithm (to be discussed in Chapter 26), Edmonds obtained a characterization of the matching polytope in terms of defining inequalities. It forms the first class of polytopes whose characterization does not simply follow just from total unimodularity, and its description was a breakthrough in polyhedral combinatorics.

25.1. The perfect matching polytope

The *perfect matching polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the perfect matchings in G. It is denoted by $P_{\text{perfect matching}}(G)$:

(25.1) $$P_{\text{perfect matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

So $P_{\text{perfect matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Consider the following set of linear inequalities for $x \in \mathbb{R}^E$:

(25.2)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) = 1 & \text{for each } v \in V, \\ \text{(iii)} & x(\delta(U)) \geq 1 & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

In Section 18.1 we saw that if G is bipartite, the perfect matching polytope is fully determined by the inequalities (25.2)(i) and (ii). These inequalities are not enough for, say, K_3: taking $x_e := \frac{1}{2}$ for each edge e of K_3 gives a vector x satisfying (25.2)(i) and (ii) but not belonging to the perfect matching polytope of K_3 (as it is empty).

Edmonds [1965b] showed that for general graphs, adding (25.2)(iii) is enough. It is clear that for any perfect matching M in G, the incidence vector χ^M satisfies (25.2). So $P_{\text{perfect matching}}(G)$ is contained in the polytope determined by (25.2). The essence of Edmonds' theorem is that one needs no more inequalities.

Theorem 25.1 (Edmonds' perfect matching polytope theorem)**.** *The perfect matching polytope of any graph $G = (V, E)$ is determined by* (25.2).

Proof. Clearly, the perfect matching polytope is contained in the polytope Q determined by (25.2). Suppose that the converse inclusion does not hold. So we can choose a vertex x of Q that is not in the perfect matching polytope.

We may assume that we have chosen this counterexample such that $|V|+|E|$ is as small as possible. Hence $0 < x_e < 1$ for all $e \in E$ (otherwise, if $x_e = 0$, we can delete e, and if $x_e = 1$, we can delete e and its ends). So each degree of G is at least 2, and hence $|E| \geq |V|$. If $|E| = |V|$, each degree is 2, in which case the theorem is trivially true. So $|E| > |V|$. Note also that $|V|$ is even, since otherwise $Q = \emptyset$ (consider $U := V$ in (25.2)(iii)).

As x is a vertex, there are $|E|$ linearly independent constraints among (25.2) satisfied with equality. Since $|E| > |V|$, there is an odd subset U of V with $3 \leq |U| \leq |V| - 3$ and $x(\delta(U)) = 1$.

Consider the projections x' and x'' of x to the edge sets of the graphs $G/\overline{U}$ and G/U, respectively (where $\overline{U} := V \setminus U$). Here we keep parallel edges. Then x' and x'' satisfy (25.2) for $G/\overline{U}$ and G/U, respectively, and hence belong to the perfect matching polytopes of $G/\overline{U}$ and G/U, by the minimality of $|V| + |E|$.

So $G/\overline{U}$ has perfect matchings $M'_1, \ldots, M'_k$ and G/U has perfect matchings $M''_1, \ldots, M''_k$ with

$$(25.3) \qquad x' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M'_i} \text{ and } x'' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M''_i}.$$

(Note that x is rational as it is a vertex of Q.)

Now for each $e \in \delta(U)$, the number of i with $e \in M'_i$ is equal to $kx'(e) = kx(e) = kx''(e)$, which is equal to the number of i with $e \in M''_i$. Hence we can assume that, for each $i = 1, \ldots, k$, M'_i and M''_i have an edge in $\delta(U)$ in common. So $M_i := M'_i \cup M''_i$ is a perfect matching of G. Then

$$(25.4) \qquad x = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i}.$$

Hence x belongs to the perfect matching polytope of G. ∎

Notes. This proof was given by Aráoz, Cunningham, Edmonds, and Green-Krótki [1983] and Schrijver [1983c], with ideas of Seymour [1979a]. For other proofs, see Balinski [1972], Hoffman and Oppenheim [1978], and Lovász [1979b]. A proof can also be derived from Edmonds' weighted matching algorithm (Chapter 26).

25.2. The matching polytope

The characterization of the perfect matching polytope implies Edmonds' matching polytope theorem. It characterizes the *matching polytope* of a graph $G = (V, E)$, denoted by $P_{\text{matching}}(G)$, which is the convex hull of the incidence vectors of the matchings in G:

(25.5) $P_{\text{matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}$.

Again, $P_{\text{matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Corollary 25.1a (Edmonds' matching polytope theorem)**.** *For any graph $G = (V, E)$, the matching polytope is determined by:*

(25.6)
$$\begin{array}{rlll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Proof. Clearly, each vector x in the matching polytope satisfies (25.6). To see that the inequalities (25.6) are enough, let x satisfy (25.6). Make a copy $G' = (V', E')$ of G, and add edges vv' for each vertex $v \in V$, where v' is the copy of v in V'. This makes the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$.

Define $\tilde{x}_e := \tilde{x}_{e'} := x_e$ for each $e \in E$, where e' is the copy of e in E', and $\tilde{x}(vv') := 1 - x(\delta(v))$ for each $v \in V$. Then by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, since $\tilde{x}$ satisfies (25.2) with respect to $\widetilde{G}$.

Indeed, for each $v \in V$ one has $\tilde{x}(\tilde{\delta}(v)) = \tilde{x}(\tilde{\delta}(v')) = 1$ (where $\tilde{\delta} := \delta_{\widetilde{G}}$). Moreover, consider any odd subset U of $\widetilde{V} = V \cup V'$, say $U = W \cup X'$ with $W, X \subseteq V$. Then $\tilde{x}(\tilde{\delta}(U)) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W'))$. So we may assume that $W \cap X = \emptyset$, and by symmetry we may assume that W is odd, and hence that $X = \emptyset$. So it suffices to show that for any odd $U \subseteq V$ one has $\tilde{x}(\tilde{\delta}(U)) \geq 1$. Now

(25.7)
$$\tilde{x}(\tilde{\delta}(U)) + 2\tilde{x}(\widetilde{E}[U]) = \sum_{v \in U} \tilde{x}(\tilde{\delta}(v)) = |U|,$$

and hence

(25.8)
$$\tilde{x}(\tilde{\delta}(U)) = |U| - 2\tilde{x}(\widetilde{E}[U]) \geq |U| - 2\lfloor \tfrac{1}{2}|U| \rfloor = 1.$$

So by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, and hence x belongs to the matching polytope of G. ∎

25.3. Total dual integrality: the Cunningham-Marsh formula

With linear programming duality one can derive from Corollary 25.1a a min-max relation for the maximum weight of a matching:

Corollary 25.1b. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{R}^E_+$ be a weight function. Then the maximum weight of a matching is equal to the minimum value of*

(25.9) $$\sum_{v\in V} y_v + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor,$$

where $y \in \mathbb{R}^V_+$ *and* $z \in \mathbb{R}^{\mathcal{P}_{\text{odd}}(V)}_+$ *satisfy*

(25.10) $$\sum_{v\in V} y_v \chi^{\delta(v)} + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \geq w.$$

Proof. Directly with LP-duality from Corollary 25.1a. ∎

The constraints (25.6) determining the matching polytope in fact are totally dual integral, as was shown by Cunningham and Marsh [1978]. This implies that a stronger min-max relation holds than obtained by linear programming duality from the matching polytope inequalities: if w is integer-valued, then in Corollary 25.1b we can restrict y and z to integer vectors:

Theorem 25.2 (Cunningham-Marsh formula)**.** *In Corollary* 25.1b*, if w is integer, we can take y and z integer. We can take z moreover such that the collection $\{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$ is laminar.*[9]

Proof. We prove the theorem by induction on $|E| + w(E)$. If $w(e) = 0$ for some $e \in E$, we can delete e and apply induction. So we may assume that $w(e) \geq 1$ for each $e \in E$.

First assume that there exists a vertex u of G covered by every maximum-weight matching. Let $w' := w - \chi^{\delta(u)}$. By induction, there exist integer y'_v, z'_U that are optimum with respect to w'. Now increasing y'_u by 1, gives y_v, z_U as required for w, since the maximum of $w'(M)$ over all matchings M is strictly less than the maximum of $w(M)$ over all matchings M, as each maximum-weight matching M contains an edge e incident with u.

So we may assume that for each vertex v there exists a maximum-weight matching missing v. Hence if $y \in \mathbb{R}^V_+$ and $z \in \mathbb{R}^{\mathcal{P}_{\text{odd}}(V)}_+$ satisfying (25.10) attain the minimum of (25.9), then $y = \mathbf{0}$. (If $y_u > 0$, then each maximum-weight matching covers u, by complementary slackness.)

Now choose z attaining the minimum (with $y = \mathbf{0}$) such that

(25.11) $$\sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor^2$$

is as large as possible. Let $\mathcal{F} := \{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$. Then $\mathcal{F}$ is laminar. For suppose not. Let $U, W \in \mathcal{F}$ with $U \cap W \neq \emptyset$ and $U \not\subseteq W \not\subseteq U$. Then $|U \cap W|$ is odd. To see this, choose $v \in U \cap W$. Then there is a maximum-weight matching M missing v. Since $z_U > 0$, $E[U]$ contains $\lfloor \frac{1}{2}|U| \rfloor$ edges in M, and hence each vertex in $U \setminus \{v\}$ is covered by an edge in M contained in U. Similarly, each vertex in $W \setminus \{v\}$ is covered by an edge in M contained in

[9] A collection $\mathcal{F}$ of sets is called *laminar* if $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$ for all $U, W \in \mathcal{F}$.

W. Hence each vertex in $(U \cap W) \setminus \{v\}$ is covered by an edge in M contained in $U \cap W$. So $|(U \cap W) \setminus \{v\}|$ is even, and hence $|U \cap W|$ is odd.

Now let $\alpha := \min\{z_U, z_W\}$, and decrease z_U and z_W by α and increase $z_{U\cap W}$ and $z_{U\cup W}$ by α. This resetting maintains (25.10), does not change (25.9), but increases (25.11), contradicting our assumption.

This shows that $\mathcal{F}$ is laminar. Now suppose that z is not integer-valued, and let U be an inclusionwise maximal set in $\mathcal{F}$ with $z_U \notin \mathbb{Z}$. Let $U_1, \ldots, U_k$ be the inclusionwise maximal sets in $\mathcal{F}$ properly contained in U (possibly $k = 0$). As $\mathcal{F}$ is laminar, the U_i are disjoint. Let $\alpha := z_U - \lfloor z_U \rfloor$. Then decreasing z_U by α and increasing each z_{U_i} by α would maintain (25.10) (by the integrality of w), but would strictly decrease (25.9) (since $\sum_{i=1}^{k} \lfloor \frac{1}{2}|U_i| \rfloor < \lfloor \frac{1}{2}|U| \rfloor$). This contradicts the minimality of (25.9). ∎

(This proof follows the method given by Schrijver and Seymour [1977]. Other proofs were given by Hoffman and Oppenheim [1978], Schrijver [1983a,1983c], and Cook [1985].)

Note that the Cunningham-Marsh formula has the Tutte-Berge formula (Corollary 24.1) as special case. The previous theorem is equivalent to:

Corollary 25.2a. *System* (25.6) *is totally dual integral.*

Proof. This follows from Theorem 25.2. ∎

25.3a. Direct proof of the Cunningham-Marsh formula

We give a direct proof of the Cunningham-Marsh formula, as given in Schrijver [1983a] (generalizing the proof of Lovász [1979b] of Edmonds' matching polytope theorem). It does not use Edmonds' matching polytope theorem, which rather follows as a consequence.

Let $G = (V, E)$ be a graph. For each weight function $w \in \mathbb{Z}_+^E$, let ν_w denote the maximum weight of a matching. We must show that for each $w \in \mathbb{Z}_+^E$ there exist $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}_{\text{odd}}(V)}$ such that

(25.12) $$\sum_{v\in V} y_v + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor \le \nu_w$$

and

(25.13) $$\sum_{v\in V} y_v \chi^{\delta(v)} + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \ge w.$$

Suppose that G and w contradict this, with $|V|+|E|+w(E)$ as small as possible. Then G is connected (otherwise one of the components of G will form a smaller counterexample) and $w(e) \ge 1$ for each edge e (otherwise we can delete e). Now there are two cases.

Case 1: There is a vertex u covered by every maximum-weight matching. In this case, let $w' := w - \chi^{\delta(u)}$. Then $\nu_{w'} = \nu_w - 1$. Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing y'_u by 1 gives y and z satisfying (25.12) and (25.13) with respect to w.

Case 2: No vertex is covered by every maximum-weight matching. Now let w' arise from w by decreasing all weights by 1. Let M be a matching with $w'(M) = \nu_{w'}$ and with $|M|$ as large as possible.

Then M does not cover all vertices, as, otherwise, for any matching N of maximum w-weight not covering all vertices:

(25.14) $$w'(N) = w(N) - |N| > w(N) - |M| \geq w(M) - |M| = w'(M) = \nu_{w'},$$

contradicting the definition of $\nu_{w'}$.

Suppose that M covers all but one vertex (in particular, $|V|$ is odd). Then

(25.15) $$\nu_w \geq w(M) = w'(M) + |M| = \nu_{w'} + \lfloor \tfrac{1}{2}|V| \rfloor.$$

Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing z'_V by 1 gives y and z satisfying (25.12) and (25.13) with respect to w (by (25.15)), a contradiction.

So we know that M leaves at least two vertices in V uncovered. Let u and v be not covered by M. We can assume that we have chosen M, u, v under the additional condition that the distance $d(u, v)$ of u and v in G is as small as possible. Then $d(u, v) > 1$, since otherwise we could augment M by edge $\{u, v\}$, thereby increasing $|M|$ while not decreasing $w'(M)$. Let t be an internal vertex of a shortest $u - v$ path. Let N be a matching not covering t, with $w(N) = \nu_w$.

Let P be the component of $M \cup N$ containing t. Then P forms a path starting at t and not covering both u and v (as t is not covered by N and u and v are not covered by M). We can assume that P does not cover u. Now the symmetric differences $M' := M \triangle P$ and $N' := N \triangle P$ are matchings again, and $|M'| \leq |M|$ (as M covers t), implying

(25.16) $$\begin{aligned} w'(M') - w'(M) &= w(M') - |M'| - w(M) + |M| \geq w(M') - w(M) \\ &= w(N) - w(N') = \nu_w - w(N') \geq 0. \end{aligned}$$

So $w'(M') \geq w'(M) = \nu_{w'}$ and hence we have equality throughout. So $w(M') = w(M)$, $w'(M') = w'(M)$, and $|M'| = |M|$. However, M' does not cover t and u while $d(u, t) < d(u, v)$, contradicting our choice of M, u, v.

25.4. On the total dual integrality of the perfect matching constraints

System (25.2) determining the perfect matching polytope is generally not totally dual integral. Indeed, consider the complete graph $G = K_4$ on four vertices, with $w(e) := 1$ for each edge e; then the maximum weight of a perfect matching is 2, while the dual of optimizing $w^{\mathsf{T}}x$ subject to (25.2) is attained only by taking $y(\{v\}) = \frac{1}{2}$ for each vertex v.

However, consider the following system, again determining the perfect matching polytope (by Corollary 25.1a):

(25.17)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E; \\ \text{(ii)} & x(\delta(v)) = 1 & \text{for each } v \in V; \\ \text{(iii)} & x(E[U]) \leq \lfloor \tfrac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Corollary 25.2b. *System* (25.17) *is totally dual integral.*

Proof. Directly from Corollary 25.2a, with Theorem 5.25. ∎

This implies a result stated by Edmonds and Johnson [1970]:

Corollary 25.2c. *The perfect matching inequalities* (25.2) *form a totally dual half-integral system.*

Proof. Let $w \in \mathbb{Z}^E$, and minimize $w^\mathsf{T} x$ subject to (25.2). As it is the same as minimizing $w^\mathsf{T} x$ subject to (25.17), by Corollary 25.2b there is an optimum dual solution $y \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\mathrm{odd}}(V)}$. Since $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is half of the sum of the inequalities $x(\delta(v)) = 1$ $(v \in U)$ and $-x(\delta(U)) \leq -1$, we obtain the total dual half-integrality of (25.2). ∎

This can be strengthened to (Barahona and Cunningham [1989]):

Corollary 25.2d. *If $w \in \mathbb{Z}^E$ and $w(C)$ is even for each circuit C, then the problem of minimizing $w^\mathsf{T} x$ subject to* (25.2) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset T of V with $\{e \in E \mid w(e) \text{ is odd}\} = \delta(T)$. Now replace w by $\tilde{w} := w + \sum_{v \in T} \chi^{\delta(v)}$. Then $\tilde{w}(e)$ is an even integer for each edge e. Hence by Corollary 25.2c there is an optimum dual solution $\tilde{y} \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\mathrm{odd}}(V)}$ for the problem of minimizing $\tilde{w}^\mathsf{T} x$ subject to (25.2). Now setting $y_v := \tilde{y}_v - 1$ if $v \in T$ and $y_v := \tilde{y}_v$ if $v \notin T$ gives an integer optimum dual solution for w. ∎

25.5. Further results and notes

25.5a. Adjacency and diameter of the matching polytope

Balinski and Russakoff [1974] and Chvátal [1975a] characterized adjacency on the matching polytope:

Theorem 25.3. *Let M and N be distinct matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the matching polytope if and only if $M \triangle N$ is a path or circuit.*

Proof. To see necessity, let χ^M and χ^N be adjacent. Let P be any nontrivial component of $M \triangle N$ and let $M' := M \triangle P$ and $N' := N \triangle P$. So M' and N' are matchings again. Then

$$(25.18) \qquad \tfrac{1}{2}(\chi^M + \chi^N) = \tfrac{1}{2}(\chi^{M'} + \chi^{N'}).$$

As χ^M and χ^N are adjacent, it follows that $\{M', N'\} = \{M, N\}$. So $M' = N$ and $N' = M$, and therefore $M \triangle N = P$.

To see sufficiency, let $P := M \triangle N$ be a path or circuit. Suppose that χ^M and χ^N are not adjacent. Then there exists a matching $L \neq M, N$ that belongs to the smallest face of the matching polytope containing $x := \frac{1}{2}(\chi^M + \chi^N)$. As $x_e = 0$ for each edge $e \notin M \cup N$ and $x_e = 1$ for each edge $e \in M \cap N$, we know that $M \cap N \subseteq L \subseteq M \cup N$. Moreover, $x(\delta(v)) = 1$ for each vertex v covered both by M and by N. Hence each vertex v covered both by M and by N is covered by L. As P is a path or a circuit, it follows that $L = M$ or $L = N$, a contradiction. ∎

This has as consequence for the diameter:

Corollary 25.3a. *The diameter of the matching polytope of any graph $G = (V, E)$ is equal to the maximum size $\nu(G)$ of the matchings.*

Proof. First, by Theorem 25.3, for any two matchings M and N, the distance of χ^M and χ^N is at most the number of nontrivial components of $M \triangle N$. Since each such component contains at least one edge and since these edges are pairwise disjoint, this number is at most $\nu(G)$. So the diameter is at most $\nu(G)$.

Equality follows from the fact that $\emptyset$ and any matching M have distance $|M|$. This follows from the fact that if M and N are adjacent, then $\big||M| - |N|\big| \leq 1$ by Theorem 25.3. ∎

Another direct consequence concerns adjacency on the *perfect* matching polytope:

Corollary 25.3b. *Let M and N be perfect matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the perfect matching polytope if and only if $M \triangle N$ is a circuit.*

Proof. Directly from Theorem 25.3. ∎

This in turn implies for the diameter of the perfect matching polytope:

Corollary 25.3c. *The perfect matching polytope of a graph $G = (V, E)$ has diameter at most $\frac{1}{2}|V|$ ($\frac{1}{4}|V|$ if G is simple).*

Proof. For any two perfect matching M, N, the symmetric difference has at most $\frac{1}{2}|V|$ components (each being a circuit). Hence Corollary 25.3b implies that χ^M and χ^N have distance at most $\frac{1}{2}|V|$.

If G is simple the bounds can be sharpened to $\frac{1}{4}|V|$, as each even circuit has at least four vertices. ∎

Padberg and Rao [1974] showed that if G is a complete graph with an even number $2n$ of vertices, then $P_{\text{perfect matching}}(G)$ has diameter at most 2. (This can be derived from Theorem 18.5, since any two perfect matchings belong to some $K_{n,n}$-subgraph of G, which subgraph gives a face of $P_{\text{perfect matching}}(G)$.)

25.5b. Facets of the matching polytope

Pulleyblank and Edmonds [1974] (cf. Pulleyblank [1973]) characterized which of the inequalities (25.6) give a facet of the matching polytope:

Let $G = (V, E)$ be a graph. Define

(25.19) $I := \{v \in V \mid \deg_G(v) \geq 3$, or $\deg_G(v) = 2$ and v is contained in no triangle, or $\deg_G(v) = 1$ and the neighbour of v also has degree $1\}$,
$\mathcal{T} := \{U \subseteq V \mid |U| \geq 3$, $G[U]$ is factor-critical and 2-vertex-connected$\}$.

(Recall that graph G is *factor-critical* if, for each vertex v of G, $G - v$ has a perfect matching.)

Consider the system

(25.20)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for } v \in I, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for } U \in \mathcal{T}. \end{array}$$

We first show:

Theorem 25.4. *Each inequality in* (25.6) *is a nonnegative integer combination of inequalities* (25.20)*.*

Proof. First consider a vertex $v \notin I$. If $\deg_G(v) = 1$, let u be the neighbour of v. Then $u \in I$ and

(25.21)
$$x(\delta(v)) = x(\delta(u)) - \sum_{e \in \delta(u) - \delta(v)} x_e.$$

If $\deg_G(v) = 2$ and v is contained in a triangle $G[U]$, then $x(\delta(v)) = x(E[U]) - x_e$, where e is the edge in $E[U]$ not incident with v.

Next consider a subset U of V with $|U|$ odd and $|U| \geq 3$. We show that $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), by induction on $|U|$. If $U \in \mathcal{T}$ we are done. So assume that $U \notin \mathcal{T}$. Let $H := G[U]$. If H is not factor-critical, there is a vertex v such that $H - v$ has no perfect matching. Let $U' = U \setminus \{v\}$. Then $x(E[U']) \leq \lfloor \frac{1}{2}|U| \rfloor - 1$ for the incidence vector x of any matching, and hence also for each vector x in the matching polytope. By the total dual integrality of the matching constraints (Corollary 25.2a), this constraint is a sum of constraints (25.6), and hence, by induction, of constraints (25.20). So $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), as $E[U] \subseteq E[U'] \cup \delta(v)$.

If H is factor-critical, it has a cut vertex v. Let $K_1, \ldots, K_t$ be the components of $H - v$ and let $U_i := K_i \cup \{v\}$ for each i. As H is factor-critical, each $|U_i|$ is odd. Hence $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of the constraints $x(E[U_i]) \leq \lfloor \frac{1}{2}|U_i| \rfloor$. ∎

This implies that (25.20) is sufficient:

Corollary 25.4a. (25.20) *determines the matching polytope.*

Proof. Directly from Corollary 25.1a and Theorem 25.4. ∎

Another consequence is the result of Cunningham and Marsh [1978] that the irredundant system still is totally dual integral:

Corollary 25.4b. (25.20) *is TDI.*

Proof. Directly from Theorem 25.4, using the total dual integrality of system (25.6). ∎

(For a short proof of this result, see Cook [1985].)

Next we show that each inequality in (25.20) determines a facet. To this end, we first show:

Lemma 25.5α. *Let $G = (V, E)$ be a 2-vertex-connected factor-critical graph and let W be a proper subset of V with $|W|$ odd and ≥ 3. Then G has a matching of size $\lfloor \frac{1}{2}|V| \rfloor$ containing less than $\lfloor \frac{1}{2}|W| \rfloor$ edges in $E[W]$.*

Proof. Choose a vertex $v \in W$ that is adjacent to at least one vertex in $V \setminus W$. If v has no neighbour in W, choose $u \in W \setminus \{v\}$ and let M be a perfect matching in $G - u$. This matching has the required properties.

So we may assume that v has a neighbour in W. Make from G a graph G', by splitting v into two vertices v' and v'', where v' is adjacent to all vertices in W adjacent to v and where v'' is adjacent to all vertices in $V \setminus W$ adjacent to v.

If G' has a perfect matching M', then deleting the edge in M' covering v', and identifying v' and v'', gives a matching M in G with $|M| = \lfloor \frac{1}{2}|V| \rfloor$, but with $|M \cap E[W]| < \lfloor \frac{1}{2}|W| \rfloor$.

So we can assume that G' has no perfect matching. Then by Tutte's 1-factor theorem, there is a subset U of VG' such that $G' - U$ has more than $|U|$ odd components. Since the graph $G' \cup \{v'v''\}$ has a perfect matching[10] (as G is factor-critical), we know that $v', v'' \notin U$.

If $U = \emptyset$, G' has an odd component, contradicting the fact that G' is connected (since G is 2-vertex-connected) and has an even number of vertices. So $U \neq \emptyset$. Choose $u \in U$, and let M be a perfect matching in $G - u$. Then M yields a matching M' in G' missing u and exactly one of v', v''. So $G' \cup \{uv'\}$ or $G' \cup \{uv''\}$ has a perfect matching, contradicting the fact that $u \in U$ and $G' - U$ has more than $|U|$ odd components. ∎

This lemma is used in proving:

Theorem 25.5. *Each inequality in* (25.20) *determines a facet.*

Proof. We clearly cannot delete any inequality $x_e \geq 0$, since otherwise the vector x defined by $x_e := -1$ and $x_{e'} := 0$ for each $e' \neq e$ would be a solution. So it determines a facet.

Consider next an inequality

$$(25.22) \qquad x(\delta(v)) \leq 1$$

for some $v \in I$. Let F be the set of vectors x in the matching polytope satisfying $x(\delta(v)) = 1$. Suppose that F is not a facet. Then there is a facet F' with $F' \supset F$. So F' is determined by one of the inequalities (25.20).

[10] By $G' \cup \{uv\}$ we denote the graph obtained from G' by adding edge uv.

If F' is determined by $x_e = 0$ for some $e \in E$, choose a matching M with $e \in M$ and covering v (the existence of such a matching follows from the definition of I). Then $\chi^M \in F \setminus F'$, a contradiction.

If F' is determined by $x(\delta(u)) = 1$ for some $u \in I$, then $u \neq v$ (since $F' \neq F$) and there is an edge e incident with u but not with v (since $u, v \in I$). Hence for matching $M := \{e\}$ we have $\chi^M \in F \setminus F'$, a contradiction.

If F' is determined by $x(E[U]) = \lfloor \frac{1}{2}|U| \rfloor$ for some $U \in \mathcal{T}$, then $\delta(u) \subseteq E[U]$ and $\lfloor \frac{1}{2}|U| \rfloor = 1$ (since $\chi^M \in F \subseteq F'$ for $M = \{e\}$, for each $e \in \delta(v)$). So $|U| = 3$. Since $F' \neq F$, U determines a triangle, contradicting the fact that $v \in I$.

Finally consider an inequality

$$(25.23) \qquad x(E[U]) \leq \lfloor \tfrac{1}{2}|U| \rfloor$$

for some $U \in \mathcal{T}$. Let F be the set of vectors x in the matching polytope satisfying $x(E[U]) = \lfloor \frac{1}{2}|U| \rfloor$.

Suppose that F is not a facet, and let F' be a facet with $F' \supset F$.

First assume that F' is determined by $x_e = 0$ for some $e \in E$. If e is not spanned by U, there is a $v \in U$ such that $U \setminus \{v\}$ is not intersected by e. Let M be a perfect matching of $G[U] - v$. Then $\chi^{M \cup \{e\}} \in F \setminus F'$, a contradiction. If e is spanned by U, choose $v \in e$ and let M be a perfect matching of $G[U] - v$. Let $f \in M$ intersect e, and define $M' := (M \setminus \{f\}) \cup \{e\}$. Then $\chi^{M'} \in F \setminus F'$, a contradiction.

Next assume that F' is determined by $x(\delta(v)) = 1$ for some $v \in I$. Then, as $G[U]$ is factor-critical, there is a matching M with $|M \cap E[U]| = \lfloor \frac{1}{2}|U| \rfloor$ and $M \cap \delta(v) = \emptyset$. So $\chi^M \in F \setminus F'$, a contradiction.

Finally assume that F' is determined by $x(E[U']) = \lfloor \frac{1}{2}|U'| \rfloor$ for some $U' \in \mathcal{T}$. If $U' \not\subseteq U$, there is a matching M with $|M \cap E[U]| = \lfloor \frac{1}{2}|U| \rfloor$ missing at least two vertices in U' and hence $|M \cap E[U']| < \lfloor \frac{1}{2}|U'| \rfloor$. Then $\chi^M \in F \setminus F'$, a contradiction.

So $U' \subset U$. By Lemma 25.5α, $G[U]$ has a matching M of size $\lfloor \frac{1}{2}|U| \rfloor$ such that less than $\lfloor \frac{1}{2}|U'| \rfloor$ edges in M are spanned by U'. Then $\chi^M \in F \setminus F'$, a contradiction. ∎

(This proof is due to L. Lovász (cf. Cornuéjols and Pulleyblank [1982]). For another proof, see Cook [1985]. See also Giles [1978b].)

Edmonds, Lovász, and Pulleyblank [1982] gave an irredundant system of linear inequalities describing the *perfect* matching polytope. More on the combinatorial structure of the (perfect) matching polytope is given by Naddef and Pulleyblank [1981a].

25.5c. Polynomial-time solvability with the ellipsoid method

In Chapter 26 we shall describe Edmonds' strongly polynomial-time algorithm for the weighted matching problem. This algorithm gives as a by-product the inequalities describing the perfect matching polytope, as we shall see in Section 26.3b.

It turns out that conversely one can derive the strong polynomial-time solvability of the weighted matching problem from the description of the perfect matching polytope (albeit that the method is impractical).

Indeed, the weighted perfect matching problem is equivalent to the optimization problem over the perfect matching polytope. So, by the ellipsoid method, there exists a polynomial-time weighted perfect matching algorithm if and only if there exists a polynomial-time separation algorithm for the perfect matching polytope.

Such a polynomial-time algorithm indeed exists (and would follow conversely also with the ellipsoid method from the polynomial-time solvability of the weighted matching problem). A direct proof was given by Padberg and Rao [1982], and is as follows.

The separation problem for the perfect matching polytope is: given a graph $G = (V, E)$ and a vector $x \in \mathbb{R}_+^E$, decide if x belongs to the perfect matching polytope, and if not, find a separating hyperplane. To answer this question we can first check the constraints (25.2)(i)(ii) in polynomial time. If one of them is violated, it gives a separating hyperplane. If each of them is satisfied, we should check if $x(\delta(U)) < 1$ for some odd subset U of V. Considering x as a capacity function, we should find an odd cut of capacity less than 1. Here an *odd cut* is a cut $\delta(U)$ with $|U|$ odd.

Such a cut can be found in strongly polynomial time. For a graph $G = (V, E)$ and a tree $T = (V, F)$, a *fundamental cut determined by* T is a cut $\delta_E(W_f)$, where $f \in F$ and W_f is one of the components of $T - f$. Then:

Theorem 25.6. *Let $G = (V, E)$ be a graph with $|V|$ even, let $c \in \mathbb{R}_+^E$ be a capacity function, and let $T = (V, F)$ be a Gomory-Hu tree for G and c. Then one of the fundamental cuts determined by T is a minimum-capacity odd cut in G.*

Proof. For each $f \in F$, choose W_f as one of the two components of $T - f$. Let $\delta_G(U)$ be a minimum-capacity odd cut of G. Then U or $V \setminus U$ is equal to the symmetric difference of the W_f over $f \in \delta_F(U)$. Hence $|W_f|$ is odd for at least one $f \in \delta_F(U)$. So $\delta_G(W_f)$ is an odd cut. Let $f = uv$. As $\delta_G(W_f)$ is a minimum-capacity $u - v$ cut and as $\delta_G(U)$ is a $u - v$ cut, we have $c(\delta_G(W_f)) \leq c(\delta_G(U))$. So $\delta_G(W_f)$ is a minimum-capacity odd cut. ▮

This gives algorithmically:

Corollary 25.6a. *A minimum-capacity odd cut can be found in strongly polynomial time.*

Proof. This follows from Theorem 25.6, since a Gomory-Hu tree can be found in strongly polynomial time, by Corollary 15.15a. ▮

As the separation problem for the perfect matching polytope can be reduced to finding a minimum-capacity odd cut, this implies:

Corollary 25.6b. *The separation problem for the perfect matching polytope can be solved in strongly polynomial time.*

Proof. See above. ▮

Corollary 25.6c. *A minimum-weight perfect matching can be found in strongly polynomial time.*

Proof. This follows from Corollary 25.6b, with Theorem 5.11. ▮

25.5d. The matchable set polytope

Let $G = (V, E)$ be a graph. A subset U of V is called *matchable* if the graph $G[U]$ has a perfect matching. The *matchable set polytope* of G is the convex hull (in $\mathbb{R}^V$) of the incidence vectors of matchable sets.

Balas and Pulleyblank [1989] characterized the matchable set polytope as follows (where $N(U)$ is the set of neighbours of U and $o(G[U])$ is the number of odd components of $G[U]$):

Theorem 25.7. *The matchable set polytope of a graph $G = (V, E)$ is determined by:*

$$
\text{(25.24)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & x(U) - x(N(U)) \leq |U| - o(G[U]) & \text{for } U \subseteq V. \end{array}
$$

Proof. Each vector in the matchable set polytope of G satisfies (25.24), since the incidence vector of any matchable set satisfies (25.24), since if any odd component K of $G[U]$ is covered by a matching M, then M has an edge connecting K and $N(U)$.

To see the reverse, choose a counterexample with $|V| + |E|$ minimal, and let x be a vertex of the polytope determined by (25.24) that is not in the matchable set polytope.

Then $x_v > 0$ for each vertex v, since otherwise we can obtain a smaller counterexample by deleting v. Moreover, there exists at least one vertex v with $x_v < 1$, since otherwise $x = \chi^V$, while V is matchable (as follows from Tutte's theorem, using (25.24)(ii)).

Hence, since x is a vertex of the polytope determined by (25.24), at least one constraint in (25.24)(ii) is attained with equality for some U with $o(G[U]) \geq 1$ (for any other U, (ii) follows from (i)).

Choose such a U with U inclusionwise minimal. Let $\mathcal{K}$ be the collection of components of $G[U]$. Then

$$
\text{(25.25)} \qquad G[K] \text{ is factor-critical for each } K \in \mathcal{K}.
$$

Otherwise, if K is even, then

$$
\text{(25.26)} \qquad \begin{aligned} & x(U \setminus K) - x(N(U \setminus K)) \geq x(U) - x(K) - x(N(U)) \\ & \geq x(U) - |K| - x(N(U)) = |U| - o(G[U]) - |K| \\ & = |U \setminus K| - o(G[U \setminus K]), \end{aligned}
$$

contradicting the minimality of U.

So K is odd. If $G[K]$ is not factor-critical, then by Tutte's 1-factor theorem, K has a nonempty subset C with $o(G[K] - C) \geq |C| + 1$. Then

$$
\text{(25.27)} \qquad \begin{aligned} & x(U \setminus C) - x(N(U \setminus C)) \geq x(U) - 2x(C) - x(N(U)) \\ & = |U| - o(G[U]) - 2x(C) \geq |U| - o(G[U]) - 2|C| \\ & = |U \setminus C| - o(G[U]) - |C| \geq |U \setminus C| - o(G[U]) - o(G[K \setminus C]) + 1 \\ & = |U \setminus C| - o(G[U \setminus C]). \end{aligned}
$$

So we have equality by (25.24)(ii), contradicting the minimality of U. This shows (25.25).

Let $S := U \cup N(U)$. Let $G' := G - S$ and let x' be the restriction of x to $V \setminus S$. Then x' satisfies (25.24) with respect to G'. Indeed, (i) is trivial. To see (ii), choose a subset $U' \subseteq V \setminus S$. Then (since no edge connects U and U'):

(25.28)
$$\begin{aligned}&x'(U') - x'(N_{G'}(U')) = x(U') - x(N(U') \setminus S)\\&= x(U \cup U') - x(N(U \cup U')) - (x(U) - x(N(U)))\\&\leq |U| + |U'| - o(G[U \cup U']) - (|U| - o(G[U])) = |U'| - o(G'[U']),\end{aligned}$$

as required.

Hence, by the minimality of G, x' belongs to the matchable set polytope of G'. Hence we are done if we have shown that the restriction of x to $G[S]$ belongs to the matchable set polytope of G.

Let H be the bipartite graph obtained from $G[S]$ by deleting all edges spanned by $N(U)$ and by contracting each $K \in \mathcal{K}$ to one vertex, u_K say. Define y on the vertices of H by: $y(v) := x(v)$ if $v \in N(U)$ and $y(u_K) := x(K) - |K| + 1$ for $K \in \mathcal{K}$.

Then y belongs to the matchable set polytope of H. To see this, we apply Theorem 21.30. Trivially $0 \leq y(v) \leq 1$ for each $v \in N(U)$. Moreover, $y(u_K) \geq 0$ for each $K \in \mathcal{K}$, since otherwise $x(K) < |K| - 1$ implying

(25.29)
$$\begin{aligned}&x(U \setminus K) - x(N(U \setminus K)) \geq x(U) - x(K) - x(N(U))\\&= |U| - o(G[U]) - x(K) > |U| - o(G[U]) - |K| + 1\\&= |U \setminus K| - o(G[U \setminus K]),\end{aligned}$$

contradicting (25.24)(ii). The inequality $y(u_K) \leq 1$ follows from the fact that $x(K) \leq |K|$.

Now

(25.30)
$$\sum_{K \in \mathcal{K}} y(u_K) = x(U) - |U| + |\mathcal{K}| = x(N(U)) = \sum_{v \in N(U)} y(v).$$

This implies, by Theorem 21.30, that if y is not in the matchable set polytope of H, then there exists a subcollection $\mathcal{L}$ of $\mathcal{K}$ with

(25.31)
$$y(N(U')) < \sum_{K \in \mathcal{L}} y(u_K),$$

where $U' := \bigcup \mathcal{L}$. However, by (25.24) we have

(25.32)
$$\begin{aligned}&\sum_{K \in \mathcal{L}} y(u_K) = \sum_{K \in \mathcal{L}} (x(K) - |K| + 1) = x(U') - |U'| + |\mathcal{L}|\\&= x(U') - |U'| + o(G[U']) \leq x(N(U')) = y(N(U')).\end{aligned}$$

So y belongs to the matchable set polytope of H. Assuming that the restriction of x to S does not belong to the matchable set polytope of $G[S]$, there exists a vector $w \in \mathbb{R}^V$ with $w^{\mathsf{T}}x > w(Y)$ for each matchable set Y of $G[S]$ and with $w(v) = 0$ if $v \notin S$. For each $K \in \mathcal{K}$, let $v_K \in K$ minimize $w(v)$ over K. Define w' on the vertices of H by: $w'(v) := w(v)$ for $v \in N(U)$ and $w'(u_K) := w(v_K)$ for $K \in \mathcal{K}$. Since y belongs to the matchable set polytope of H, H has a matchable set Y' satisfying $w'(Y') \geq w'^{\mathsf{T}}y$. Let Y be the union of Y', of all K with $u_K \in Y'$, and of all $K \setminus \{v_K\}$. Since each $G[K]$ is factor-critical, Y is matchable. Moreover,

(25.33)
$$\begin{aligned}w(Y) &= w'(Y') + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\}) \geq w'^{\mathsf{T}}y + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\})\\&= \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} w(v_K)(x(K) - |K| + 1) + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\})\\&\geq \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} \sum_{v \in K} (w(v) - w(v_K) + w(v_K)x(v))\\&\geq \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} \sum_{v \in K} w(v)x(v) = w^{\mathsf{T}}x\end{aligned}$$

(the last inequality follows from $(w(v) - w(v_K))(1 - x(v)) \geq 0$), contradicting our assumption. ∎

Cunningham and Green-Krótki [1994] gave a combinatorial, polynomial-time separation algorithm for the matchable set polytope that implies a proof of Theorem 25.7. A combinatorial, strongly polynomial-time algorithm was given by Cunningham and Geelen [1996,1997]. Qi [1987] characterized adjacency of vertices on the matchable set polytope. Related work can be found in Barahona and Mahjoub [1994a].

25.5e. Further notes

We postpone a discussion of the dimension of the perfect matching polytope to Chapter 37.

Note that Edmonds' matching polytope theorem gives the linear inequalities determining the convex hull of all *symmetric* permutation matrices.

Hoffman and Oppenheim [1978] showed that for each graph $G = (V, E)$ and for each vertex x of the matching polytope of G, there exist $|E|$ linearly independent constraints among (25.6) satisfied by x with equality and yielding a matrix of determinant ± 1. This also implies the total dual integrality of the constraints (25.6).

Unlike in the bipartite case, the convex hull of incidence vectors of edge sets containing a perfect matching is not determined by linear inequalities with $0, 1$ coefficients (in the left-hand side), as was shown by Cunningham and Green-Krótki [1986]. They showed that for each integer $n > 0$ there exists a graph $G = (V, E)$ with $|V| = 2n+4$ such that the convex hull of the incidence vectors of supersets of perfect matchings has facet-inducing inequalities with coefficient set $\{0, 1, \ldots, n\}$. They also showed that for odd n a similar result holds for subsets of perfect matchings. So the polyhedra $P^{\uparrow}_{\text{perfect matching}}(G)$ and $P^{\downarrow}_{\text{perfect matching}}(G)$ are not determined by $0, 1$ inequalities.

Naddef and Pulleyblank [1981b] observed that Edmonds' perfect matching polytope theorem implies that any $(k-1)$-edge connected k-regular graph $G = (V, E)$ with an even number of vertices, is matching-covered. (This can be seen by showing that the all-$\frac{1}{k}$ vector in $\mathbb{R}^E$ belongs to the perfect matching polytope.)

Rispoli [1992] noticed that the 'monotonic diameter' of the perfect matching polytope of K_n is equal to $\lfloor \frac{n}{4} \rfloor$. So for any weight function w there is a polytopal path with monotonically increasing $w^{\mathsf{T}} x$ and leading from any vertex to a vertex maximizing $w^{\mathsf{T}} x$, of length at most $\lfloor \frac{n}{4} \rfloor$.

Chapter 26

Weighted nonbipartite matching algorithmically

In the previous chapter we gave good characterizations for the maximum-weight matching problem. In the present chapter we go over to the algorithmic side, and describe Edmonds' strongly polynomial-time algorithm for finding a minimum-weight perfect matching in any graph. It implies a strongly polynomial-time algorithm for finding a maximum-weight matching.

In this chapter, graphs can be assumed to be simple.

26.1. Introduction and preliminaries

As an extension of the cardinality matching algorithm, Edmonds [1965b] proved that also a maximum-*weight* matching can be found in strongly polynomial time. Equivalently, a minimum-weight perfect matching can be found in strongly polynomial time.

Like the cardinality matching algorithm, the weighted matching algorithm is based on shrinking sets of vertices. Unlike the cardinality matching algorithm however, for weighted matchings one has, at times, to 'deshrink' sets of vertices (the reverse operation of shrinking). For this purpose we have to keep track of the shrinking history throughout the iterations.

Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}^E$ be a weight function. We describe a strongly polynomial-time algorithm to find a minimum-weight perfect matching in G. We can assume that G has at least one perfect matching and that $w \geq \mathbf{0}$.

The algorithm is 'primal-dual'. The 'vehicle' carrying us to a minimum-weight perfect matching is a pair of a laminar[11] collection Ω of odd-size subsets of V and a function $\pi : \Omega \to \mathbb{Q}$ satisfying:

$$\text{(26.1)}\qquad \begin{array}{lll} \text{(i)} & \pi(U) \geq 0 & \text{if } U \in \Omega \text{ and } |U| \geq 3, \\ \text{(ii)} & \displaystyle\sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U) \leq w(e) & \text{for each } e \in E. \end{array}$$

[11] A collection Ω of sets is called *laminar* if $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$ for any $U, W \in \Omega$.

Condition (26.1) implies

(26.2) $$w(M) \geq \sum_{U \in \Omega} \pi(U)$$

for each perfect matching M in G, since

(26.3) $$w(M) = \sum_{e \in M} w(e) \geq \sum_{e \in M} \sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U) = \sum_{U \in \Omega} \pi(U)|M \cap \delta(U)| \geq \sum_{U \in \Omega} \pi(U).$$

Hence M is a minimum-weight perfect matching if equality holds throughout in (26.3).

Notation. Let be given Ω and $\pi : \Omega \to \mathbb{Q}$. Define for any edge e:

(26.4) $$w_\pi(e) := w(e) - \sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U).$$

So (26.1)(ii) says that $w_\pi(e) \geq 0$ for each $e \in E$. Let E_π denote the set of edges e with $w_\pi(e) = 0$, and let $G_\pi = (V, E_\pi)$.

Throughout the algorithm we will have that $\{v\} \in \Omega$ for each $v \in V$. Hence, as Ω is laminar, the collection $\Omega^{\max}$ of inclusionwise maximal sets in Ω is a partition of V.

By G' we denote the graph obtained from G_π by shrinking all sets in $\Omega^{\max}$:

(26.5) $$G' := G_\pi / \Omega^{\max}.$$

(So G' depends on Ω and π.) The vertex set of G' is $\Omega^{\max}$, with two distinct elements $U, U' \in \Omega^{\max}$ adjacent if and only if G_π has an edge connecting U and U'. We denote any edge of G' by the original edge in G.

Finally, for $U \in \Omega$ with $|U| \geq 3$, we denote by H_U the graph obtained from $G_\pi[U]$ by contracting each inclusionwise maximal proper subset of U that belongs to Ω.

26.2. Weighted matching algorithm

We keep a laminar collection Ω of odd-size subsets of V, a function $\pi : \Omega \to \mathbb{Q}$ satisfying (26.1), a matching M in G', and for each $U \in \Omega$ with $|U| \geq 3$, a Hamiltonian circuit C_U in H_U. We assume that G is simple and has at least one perfect matching.

Initially, we set $\Omega := \{\{v\} \mid v \in V\}$, $\pi(\{v\}) := 0$ for each $v \in V$, and $M := \emptyset$. The iteration is as follows. Let X be the set of vertices of G' missed by M. (In the algorithm, 'positive length' means: having at least one edge.)

(26.6) **Case 1: G' has an M-alternating $X - X$ walk of positive length.** Choose a shortest such walk P. If P is a path, it is an M-augmenting path in G'. Reset $M := M \triangle EP$ (*matching augmentation*) and iterate.
If P is not a path, it contains an M-flower (Theorem 24.3). Let C be the circuit in it. Add $U := \bigcup VC$ to Ω (*shrinking*), set $\pi(U) := 0$, $M := M \setminus EC$, and $C_U := C$, and iterate.
Case 2: G' has no M-alternating $X - X$ walk of positive length. Let $\mathcal{S}$ be the set of vertices U of G' for which G' has an odd-length M-alternating $X - U$ walk and let $\mathcal{T}$ be the set of vertices U of G' for which G' has an even-length M-alternating $X - U$ walk. Reset $\pi(U) := \pi(U) + \alpha$ if $U \in \mathcal{T}$ and $\pi(U) := \pi(U) - \alpha$ if $U \in \mathcal{S}$, where α is the largest value maintaining (26.1). If after this resetting $\pi(U) = 0$ for some $U \in \mathcal{S}$ with $|U| \geq 3$, delete U from Ω (*deshrinking*), extend M by the perfect matching of $C_U - v$, where v is the vertex of C_U covered by M, and iterate.

In Case 2, α is bounded, since $|\mathcal{T}| > |\mathcal{S}|$ if M is not perfect and since by (26.3), $\sum_{U \in \Omega} \pi(U)$ is bounded (as there exists at least one perfect matching by assumption).

The iterations stop if M is a perfect matching in G', and then we are done: using the C_U we can expand M to a perfect matching N in G with $w_\pi(N) = 0$ and $|N \cap \delta(U)| = 1$ for each $U \in \Omega$. Then N has equality throughout in (26.3), and hence it is a minimum-weight perfect matching.

As for estimating the number of iterations, it is good to observe that the laminarity of Ω implies (cf. Theorem 3.5)

(26.7) $|\Omega| \leq 2|V|$,

assuming $V \neq \emptyset$.

Theorem 26.1. *There are at most $2|V|^2$ iterations.*

Proof. There are at most $\frac{1}{2}|V|$ matching augmentations, since at each matching augmentation the size of X decreases by 2, and remains unchanged in any other iteration.

The further proof is based on the following observation:

(26.8) Any set U added to Ω ('shrinking') will not be removed from Ω ('deshrinking') before the next matching augmentation.

Indeed, after shrinking U, there exists an even-length M-alternating $X - U$ path. Until the next matching augmentation, this remains the case, or U is swallowed by a larger set that is shrunk. So U is not in $\mathcal{S}$ before the next matching augmentation, proving (26.8).

Consider any sequence of iterations between two consecutive matching augmentations. By (26.8), the number of deshrinkings is not more than the size of Ω at the start of the sequence. Similarly by (26.8), the number of shrinkings is not more than the size of Ω at the end of the sequence. So, by (26.7), both the number of shrinkings and the number of deshrinkings are at most $2|V|$.

If in Case 2 we do not deshrink, then there is an edge e connecting a vertex $U \in \mathcal{T}$ with a vertex $W \notin \mathcal{S}$ of G' for which $w_\pi(e)$ has decreased to 0. If $W \notin \mathcal{T}$, then after resetting π, $W \in \mathcal{S}$, and hence the number of vertices of G' not in $\mathcal{S} \cup \mathcal{T}$ decreases. If $W \in \mathcal{T}$, then, in the next iteration, Case 1 applies. So the number of Case 2 iterations in which we do not deshrink is at most $|V|$. This proves the theorem. ∎

This gives the theorem of Edmonds [1965b]:

Corollary 26.1a. *A minimum-weight perfect matching can be found in time $O(n^2m)$.*

Proof. By Theorem 26.1, since each iteration can be performed in time $O(m)$. ∎

This implies that also a maximum-weight matching can be found in time $O(n^2m)$:

Corollary 26.1b. *A maximum-weight matching can be found in time $O(n^2m)$.*

Proof. Let $G = (V, E)$ be a graph with weight function $w \in \mathbb{Q}^E$. Extend G as follows. Make copies G' and w' of G and w. Connect each $v \in V$ to its copy in V', by an edge of weight 0. Let M be a maximum-weight perfect matching in the extended graph. The restriction of M to the original edges is a maximum-weight matching in G. ∎

Notes. In fact, a bound of $\frac{3}{2}|V|$ can be shown in (26.7) (as the size of any set in Ω is odd), implying a bound of $|V|^2$ on the number of iterations in Theorem 26.1.

26.2a. An $O(n^3)$ algorithm

In the above description, we estimated the time required for any iteration by $O(m)$. This leaves time to find the walk in each iteration just from scratch, and to construct the graph $G' = G_\pi/\Omega$ from scratch, after any shrinking or deshrinking step.

Like in the cardinality case, we can speed this up (i) by using the result of the previous walk-search in the next walk-search, and (ii) by constructing the graph G' only in an implicit way. In this way we can reduce the time per iteration from $O(m)$ to $O(n)$ on average, leading to an overall time bound of $O(n^3)$.

Again we use M-alternating forests to reach this goal. Thus, next to Ω, π, M, and the C_U, we keep an M-alternating forest F in $G' := G_\pi/\Omega^{\max}$.

We do not keep the graph G'. Instead, we keep for each pair Y, Z of disjoint sets in Ω an edge e_{YZ} of G connecting Y and Z and minimizing $w_\pi(e_{YZ})$. We take e_{YZ} void if no such edge exists. We keep the e_{YZ} as lists: for each $Y \in \Omega$ we have a list containing the e_{YZ}.

Moreover, for each $Y \in \Omega$ we keep an edge e_Y with $e_Y = e_{YZ}$ for some $Z \in \text{even}(F)$ and with $w_\pi(e_{YZ})$ minimal. Again, if no such e_{YZ} exists, e_Y is void.

Finally, for each $v \in V$ we keep

(26.9) $$p(v) := \sum_{\substack{U \in \Omega \\ v \in U}} \pi(U).$$

Initially, we set $\Omega := \{\{v\} \mid v \in V\}$, $\pi(\{v\}) := 0$ and $p(v) := 0$ for each $v \in V$, and $M := \emptyset$, $F := \emptyset$. The e_{YZ} and e_Y are easily set.

Next we apply the following iteratively:

(26.10) Reset $\pi(U) := \pi(U) - \alpha$ for $U \in \text{odd}(F)$ and $\pi(U) := \pi(U) + \alpha$ for $U \in \text{even}(F)$, where α is the largest value maintaining (26.1). Update p accordingly. After that, at least one of the following three cases applies.
Case 1: $w_\pi(e_U) = 0$ for some $U \in \text{free}(F)$. Extend F by e_U and update the e_Y (*forest augmentation*).
Case 2: $w_\pi(e_U) = 0$ for some $U \in \text{even}(F)$. Let e_U connect vertices U and W in $\text{even}(F)$. Let P and Q be the $X - U$ and the $X - W$ path in $(\Omega^{\max}, F)$, respectively.
Case 2a: Paths P and Q are disjoint. Then P and Q form with e_U an M-augmenting path, yielding a matching M' in G' with $|M'| = |M| + 1$. Reset $M := M'$, $F := M'$, and update the e_Y (*matching augmentation*).
Case 2b: Paths P and Q intersect. Then they contain (with e_U) an M-blossom B. Let T be the union of the sets (in $\Omega^{\max}$) forming the vertices of B. Add T to Ω, setting $C_T := B$ and $\pi(T) := 0$. Reset $F := F \setminus EB$ and $M \setminus EB$, and update the e_{YZ} and e_Y (*shrinking*).
Case 3: $\pi(U) = 0$ for some $U \in \text{odd}(F)$ with $|U| \geq 3$. Let v be the vertex in C_U covered by an edge in M and let u be the vertex in C_U covered by an edge in $F \setminus M$. Let P be the even-length $u - v$ path in C_U and let N be the matching in $C_U - v$. Delete U from Ω, reset $F := F \cup EP \cup N$ and $M := M \cup N$, and update the e_{YZ} and e_Y (*deshrinking*).

(In updating F and M, we update them as graphs on $\Omega^{\max}$.)

The number of iterations between any two matching augmentations is at most $|V|$, as may be proved similarly to the proof of Theorem 26.1 (replacing $\mathcal{S}$ by $\text{odd}(F)$ and $\mathcal{T}$ by $\text{even}(F)$).

In the iteration (26.10), we can find the value α in $O(n)$ time, as it is the minimum of $w_\pi(e_U)$ over $U \in \text{free}(F)$, of $\frac{1}{2}w_\pi(e_U)$ over $U \in \text{even}(F)$, and of $\pi(U)$ over $U \in \text{odd}(F)$ with $|U| \geq 3$. So we can update π and p in $O(n)$ time. Also F and M can be updated in $O(n)$ time (as they have $O(n)$ edges).

Note that each time we need the value of $w_\pi(e)$ for some edge e (when determining α or the e_{YZ} and e_Y), then e connects two disjoint sets in $\Omega^{\max}$, and hence $w_\pi(e) = w(e) - p(u) - p(v)$. Note also that the resetting of π on $\Omega^{\max}$ changes no e_{YZ} and e_Y.

In Case 1, Ω, π, p, and the e_{YZ} are unchanged. The set $U \in \Omega^{\max}$ is moved from free(F) to odd(F), and a set $W \in \Omega^{\max}$ (the mate of U in M) is moved from free(F) to even(F). To update the e_Y, it suffices to scan the list of the e_{WZ}. This can be done in $O(n)$ time.

In Case 2a, Ω, π, p, and the e_{YZ} are unchanged. Since (in the new situation) $F = M$, we delete from even(F) and odd(F) all sets in $\Omega^{\max}$ covered by M. We can find the e_Y by scanning all e_{YZ}. We have $O(n^2)$ time for this, since there are only $\frac{1}{2}|V|$ matching augmentations.

In Case 2b, set T is inserted into $\Omega^{\max}$ and into even(F), and the sets in VB are removed from even(F) and odd(F). We need to find the e_{TZ}, which can be done by scanning the e_{YZ} for each $Y \in VB$. At the same time, the e_Z can be updated. This can be done in $O(|VB|n)$ time.

In Case 3, set U is removed from $\Omega^{\max}$ and from odd(F), and the sets in VC_U become members of $\Omega^{\max}$ and are inserted into even(F) or odd(F). This modifies no e_{YZ} (except that all e_{UZ} disappear). By scanning the e_{YZ} for each $Y \in VC_U$, we can update the e_Z. This can be done in $O(|VC_U|n)$ time.

Now, between any two matching augmentations, the sum of the $|VC_U|$ over the U added or removed is $O(n)$, since any set added will not be removed before the next matching augmentation (cf. (26.8)). So between any two matching augmentations, the iterations can be done in $O(n^2)$ time.

This gives the result of Gabow [1973] and Lawler [1976b]:

Theorem 26.2. *A minimum-weight perfect matching can be found in $O(n^3)$ time.*

Proof. See above. ∎

Several ingredients in this method can be implemented so as to require only $O(m)$ time between any two matching augmentations. However, reducing the time needed to administer Ω requires additional data structure — see the references in Section 26.3a.

26.3. Further results and notes

26.3a. Complexity survey for weighted nonbipartite matching

Complexity survey for weighted nonbipartite matching ($*$ indicates an asymptotically best bound in the table):

$O(n^4)$	Edmonds [1965b]
$O(n^3)$	Gabow [1973], Lawler [1976b]
$O(nm \log n)$	Galil, Micali, and Gabow [1982, 1986] (cf. Ball and Derigs [1983])
$O(n(m \log\log\log_{m/n} n + n \log n))$	Gabow, Galil, and Spencer [1984, 1989]
$O(n^{3/4} m \log W)$	Gabow [1985a,1985b]

$\gg$

continued

*	$O(n(m + n \log n))$	Gabow [1990]
*	$O(m \log(nW)\sqrt{n\alpha(m,n) \log n})$	Gabow and Tarjan [1991]

Here W is the maximum absolute value of the weights, assuming they are integer.

Cunningham and Marsh [1978] gave a *primal* algorithm for weighted nonbipartite matching that takes $O(n^2m)$ time (where, throughout the algorithm, there is a perfect matching at hand, the weight of which is improved iteratively). They state that it can be improved to $O(n^3)$. Derigs [1981] gave a shortest augmenting path method of running time $O(n^3)$. In Derigs [1988b] an $O(\min\{n^3, nm \log n\})$ algorithm is given based on successive improvement of a perfect matching by choosing an improving alternating circuit.

26.3b. Derivation of the matching polytope characterization from the algorithm

Edmonds' weighted matching algorithm directly yields the description of the perfect matching polytope. Indeed, one can derive from Edmonds' algorithm the following. Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}^E$ be a weight function. Then:

(26.11) the minimum weight of a perfect matching is equal to the maximum value of $\sum_{U \in \mathcal{P}_{\text{odd}}(V)} \pi(U)$ where π ranges over all functions $\pi : \mathcal{P}_{\text{odd}}(V) \to \mathbb{Q}$ satisfying (26.1),

where $\mathcal{P}_{\text{odd}}(V)$ denotes the collection of odd-size subsets of V.

To see this, we may assume that w is nonnegative: if μ is the minimum value of $w(e)$ over all edges e, decreasing each $w(e)$ by μ decreases both the maximum and the minimum by $\frac{1}{2}|V|\mu$.

That the minimum is not smaller than the maximum follows from (26.3). Equality follows from the fact that in the algorithm the final perfect matching and the final function π have equality throughout in (26.1). This shows (26.11).

It implies Edmonds' perfect matching polytope theorem: the perfect matching polytope of any graph $G = (V, E)$ is determined by (25.2). Indeed, by (weak) LP-duality, for any weight function $w \in \mathbb{Q}^E$, the minimum weight of a perfect matching is equal to the minimum of $w^{\mathsf{T}}x$ taken over the polytope determined by (25.2). Hence the two polytopes coincide.

26.3c. Further notes

Weber [1981] and Derigs [1985a] analyzed the sensitivity of minimum-weight perfect matchings to changing edge weights. White [1974] studied the maximum weight of a matching of size k, as a function of k.

An outstanding open problem is to formulate the weighted matching problem as a linear programming problem of size polynomial in the size of the graph, by extending the set of variables. That is, is the matching polytope of a graph $G = (V, E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with A and b having size polynomial in $|V| + |E|$?

Yannakakis [1988,1991] showed that this is not possible in a symmetric fashion. (That is, for $G = K_n$ there is not a system $Ax \leq b$ which is invariant under each permutation of the vertex set.) For further partial results, see Yannakakis [1988, 1991], Gerards [1991], and Barahona [1993a,1993b].

Gabow, Kaplan, and Tarjan [1999,2001] gave fast algorithms to test uniqueness of a minimum-weight perfect matching.

For heuristics and fast approximation methods for the weighted matching problem if the weight function satisfies the triangle inequality (including matching points in Euclidean space), see Papadimitriou [1977b], Avis [1978,1981,1983], Supowit, Plaisted, and Reingold [1980], Iri, Murota, and Matsui [1981,1983], Reingold and Tarjan [1981], Bartholdi and Platzman [1983], Reingold and Supowit [1983], Supowit and Reingold [1983], Supowit, Reingold, and Plaisted [1983], Plaisted [1984], Grigoriadis and Kalantari [1986,1988], Grigoriadis, Kalantari, and Lai [1986], Imai [1986], Weber and Liebling [1986], Avis, Davis, and Steele [1988], Vaidya [1988, 1989a,1989b], Kalyanasundaram and Pruhs [1991,1993], Marcotte and Suri [1991], Goemans and Williamson [1992,1995a], Osiakwan and Akl [1994], Williamson and Goemans [1994], Jünger and Pulleyblank [1995], Arora [1997,1998], Varadarajan [1998], and Varadarajan and Agarwal [1999].

For studies of implementing weighted matching algorithms, see Cunningham and Marsh [1978], Burkard and Derigs [1980], Derigs [1981,1986a,1986b,1988b], Lessard, Rousseau, and Minoux [1989], Derigs and Metz [1991], Applegate and Cook [1993], and Cook and Rohe [1999].

Grötschel and Holland [1985] report on implementing a cutting plane algorithm for the weighted matching problem based on the simplex method (cf. Derigs and Metz [1991]). For an alternative approach, see Lessard, Rousseau, and Minoux [1989]. Derigs and Metz [1986b] showed how solving the matching problem fractionally can help in finding a shortest augmenting path.

Megiddo and Tamir [1978] gave an $O(n \log n)$ algorithm to find a maximum-weight matching in a graph $G = (V, E)$, if each weight $w(uv)$ is equal to $a(u) + b(v)$ for $u < v$, where the vertices are ordered by $<$ and where $a, b : V \to \mathbb{Q}$.

For weighted matching problems with side constraints, see Ball, Derigs, Hilbrand, and Metz [1990].

For a survey on weighted matching algorithms, see Galil [1983,1986a,1986b]. Books covering weighted nonbipartite matching algorithms include Christofides [1975], Lawler [1976b], Minieka [1978], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Derigs [1988a], Nemhauser and Wolsey [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], and Korte and Vygen [2000].

Chapter 27

Nonbipartite edge cover

Edge cover is closely related to matching, through a construction described by Gallai. In this chapter we derive basic results on edge covers (min-max relation, polyhedral characterization, strongly polynomial-time algorithm) from the results on matchings given in the previous chapters.

In this chapter, graphs can be assumed to be loopless.

27.1. Minimum-size edge cover

With Gallai's theorem, the Tutte-Berge formula implies a formula for the edge cover number $\rho(G)$ (where $o(G[U])$ denotes the number of odd components of $G[U]$):

Theorem 27.1. *Let $G = (V, E)$ be a graph without isolated vertices. Then*

(27.1) $$\rho(G) = \max_{U \subseteq V} \frac{|U| + o(G[U])}{2}.$$

Proof. By Gallai's theorem (Theorem 19.1) and the Tutte-Berge formula (Theorem 24.1),

(27.2) $$\rho(G) = |V| - \nu(G) = |V| - \min_{U \subseteq V} \frac{|V| + |U| - o(G - U)}{2}$$
$$= \max_{U \subseteq V} \frac{|U| + o(G[U])}{2}.$$ ∎

This min-max relation is equivalent to: $\rho(G)$ is equal to the maximum value of

(27.3) $$\sum_{U \in \mathcal{U}} \lceil \tfrac{1}{2}|U| \rceil,$$

where $\mathcal{U}$ is a collection of disjoint odd subsets of V such that no edge of G connects two distinct sets in $\mathcal{U}$.

By the method of Gallai's theorem, one can derive a minimum-size edge cover from a maximum-size matching M, just by adding for each vertex v

missed by M, an arbitrary edge incident with v. Hence a minimum-size edge cover can be found in polynomial time.

One can reduce the problem of finding a minimum-weight edge cover to that of finding a minimum-weight perfect matching, as described in Section 19.3. It gives the following result of Edmonds and Johnson [1970]:

Theorem 27.2. *A minimum-weight edge cover can be found in $O(n^3)$ time.*

Proof. From Corollary 26.1b, with the method of Section 19.3. ∎

27.2. The edge cover polytope and total dual integrality

The *edge cover polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of edge covers. We will show that the edge cover polytope is determined by

(27.4) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(E[U] \cup \delta(U)) \geq \lceil \frac{1}{2}|U| \rceil$ for each $U \subseteq V$ with $|U|$ odd,

and moreover, that this system is totally dual integral. The latter statement will be derived from the Cunningham-Marsh formula (Theorem 25.2), and is equivalent to:

Theorem 27.3. *Let $G = (V, E)$ be a graph without isolated vertices and let $w \in \mathbb{Z}_+^E$ be a weight function. Then the minimum weight of an edge cover is equal to the maximum value of*

(27.5) $$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lceil \tfrac{1}{2}|U| \rceil,$$

where $z_U \in \mathbb{Z}_+$ for each $U \in \mathcal{P}_{\text{odd}}(V)$ such that

(27.6) $$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U] \cup \delta(U)} \leq w.$$

Proof. We first show:

(27.7) in the Cunningham-Marsh formula one can assume that for each $v \in V$ there is an edge $e \in \delta(v)$ with $y_v + \sum_{U \ni v} z_U \leq w(e)$.

Indeed, by Theorem 25.2 we can take y, z such that $\mathcal{F} := \{U \mid z_U > 0\}$ is laminar. Now choose $v \in V$. Suppose that $y_v + \sum_{U \ni v} z_U > w(e)$ for each edge $e \in \delta(v)$. If no set in $\mathcal{F}$ covers v, then reducing y_v by 1 would maintain the conditions, contradicting the fact that y, z attain the minimum in the Cunningham-Marsh formula.

So some $T \in \mathcal{F}$ covers v. Choose an inclusionwise minimal set $T \in \mathcal{F}$ covering v. As $\mathcal{F}$ is laminar, $U \supseteq T$ for each $U \in \mathcal{F}$ containing v. Then for each edge $e = uv$ with $v \in e \subseteq T$ one has for each $U \in \mathcal{F}$: if $v \in U$, then $e \subseteq U$. So for each such edge $e = uv$,

$$y_u + y_v + \sum_{U \supseteq e} z_U \geq y_v + \sum_{U \ni v} z_U > w(e). \tag{27.8}$$

Hence, if we choose $s \in T \setminus \{v\}$, then decreasing z_T by 1 and increasing y_s and $z_{T\setminus\{v,s\}}$ by 1, gives again an optimum solution. Iterating this for all v, gives a solution as in (27.7).

We next show the theorem. For each vertex v, let e_v be an edge incident with v of minimum weight and let $\mu(v) := w(e_v)$. For each edge $e = uv$, define $w'(e) := \mu(u) + \mu(v) - w(e)$.

By the Cunningham-Marsh formula, there exists a matching M and $y_v \in \mathbb{Z}_+$ $(v \in V)$ and $z'_U \in \mathbb{Z}_+$ $(U \in \mathcal{P}_{\text{odd}}(V))$ such that

(27.9) (i) $y_u + y_v + \sum_{U \supseteq e} z'_U \geq w'(e)$ for each edge $e = uv$;

(ii) $w'(M) = \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z'_U \lfloor \frac{1}{2}|U| \rfloor$.

We may assume that $z'_U = 0$ if $|U| = 1$. By (27.7) we may assume that for each $v \in V$:

$$y_v + \sum_{T \ni v} z'_T \leq w'(e) \tag{27.10}$$

for some edge e incident with v.

Let F be the edge cover obtained from M by adding the edge e_v for each vertex v missed by M. For each $U \in \mathcal{P}_{\text{odd}}(V)$, define:

$$z_U := \begin{cases} \mu(v) - y_v - \sum_{T \ni v} z'_T & \text{if } U = \{v\}, \\ z'_U & \text{if } |U| \geq 3. \end{cases} \tag{27.11}$$

Clearly $z_U \geq 0$ if $|U| \geq 3$. If $U = \{v\}$, then let $e = uv \in \delta(v)$ satisfy satisfying (27.10). Hence

$$z_{\{v\}} = \mu(v) - y_v - \sum_{T \ni v} z'_T \geq \mu(v) - w'(e) = w(e) - \mu(u) \geq 0. \tag{27.12}$$

So z is nonnegative.

Now for each edge $e = uv$ one has:

$$\begin{aligned} \sum_{U \cap e \neq \emptyset} z_U &= z_{\{u\}} + z_{\{v\}} + \sum_{U \cap e \neq \emptyset} z'_U \\ &= \mu(u) - y_u - \sum_{U \ni u} z'_U + \mu(v) - y_v - \sum_{U \ni v} z'_U + \sum_{U \cap e \neq \emptyset} z'_U \\ &= \mu(u) + \mu(v) - y_u - y_v - \sum_{U \supseteq e} z'_U \leq \mu(u) + \mu(v) - w'(e) \\ &= w(e). \end{aligned} \tag{27.13}$$

Moreover,

$$(27.14)\qquad \sum_U z_U \lceil \tfrac{1}{2}|U| \rceil = \sum_{v\in V} (\mu(v) - y_v - \sum_{U \ni v} z'_U) + \sum_U z'_U \lceil \tfrac{1}{2}|U| \rceil$$
$$= \sum_{v \in V} \mu(v) - \sum_{v\in V} y_v - \sum_U z'_U \lfloor \tfrac{1}{2}|U| \rfloor = \sum_{v\in V} \mu(v) - w'(M) = w(F).$$

∎

(The idea of using w' was given by J.F. Geelen.)

Equivalently, we can state:

Corollary 27.3a. *System* (27.4) *determines the edge cover polytope and is TDI.*

Proof. This is equivalent to Theorem 27.3. ∎

27.3. Further notes on edge covers

27.3a. Further notes

Inspired by Edmonds' algorithm for maximum-weight matching, White [1967] and Murty and Perin [1982] described minimum-weight edge cover algorithms based on blossoms.

White and Gillenson [1975] and Murty and Perin [1982] described a blossom-type algorithm to find a minimum-weight edge cover of given size k. Also White [1971] considered the problem of finding a minimum-weight edge cover of a given size, by parametrizing the weight function.

In fact, the convex hull of incidence vectors of edge covers F with $k \leq |F| \leq l$ is equal to the edge cover polytope intersected with $\{x \in \mathbb{R}^E \mid k \leq x(E) \leq l\}$. This can be proved similarly to the proof of Corollary 18.10a.

Hurkens [1991] characterized adjacency on the edge cover polytope and derived that its diameter is equal to $|E| - \rho(G)$. (This turns out to be harder to prove than the corresponding results for the matching polytope given in Section 25.5a.)

27.3b. Historical notes on edge covers

The nonbipartite edge cover problem was considered by Gallai [1959a] and Norman and Rabin [1959]. The latter were motivated by a problem of Roth [1958] related to minimizing the number of switches in a switching systems, for which they considered the problem of finding a minimum cover for a cubical complex.

Norman and Rabin [1959] showed that an edge cover F in a graph has minimum size if and only if there is no path P such that the end vertices of P are covered more than once by F, while all intermediate vertices are covered exactly once by F, and such that the edges of P are alternatingly in and out F, with the first and last edge in F. (Thus $F \triangle P$ is an edge cover of smaller size than F.)

Chapter 28

Edge-colouring

Edge-colouring means covering the edge set by matchings. The problem goes back to Tait [1878b], who showed that the four-colour conjecture is equivalent to the 3-edge-colourability of any bridgeless cubic planar graph. Nonbipartite edge-colouring is less tractable than in the special case of bipartite graphs. No tight min-max relation is known and finding a minimum edge-colouring is NP-complete. In this chapter we prove Vizing's theorem, which gives an almost tight min-max relation. Moreover, we consider the 'fractional' edge-colouring number, which approximates the edge-colouring number. It can be characterized and computed with the help of matching results. We also consider the related problem of packing edge covers.

28.1. Vizing's theorem for simple graphs

We recall some definitions and notation. Let $G = (V, E)$ be a graph. An *edge-colouring* is a partition of E into matchings. Each matching in an edge-colouring is called a *colour* or an *edge-colour*. A *k-edge-colouring* is an edge-colouring with k colours. G is *k-edge-colourable* if a k-edge-colouring exists. The smallest k for which G is k-edge-colourable is called the *edge-colouring number* of G, denoted by $\chi'(G)$. Since an edge-colouring of G is a vertex-colouring of the line-graph $L(G)$ of G, we have that $\chi'(G) = \chi(L(G))$.

Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. We saw that $\chi'(G) = \Delta(G)$ if G is bipartite (Kőnig's edge-colouring theorem (Theorem 20.1)). On the other hand, $\chi'(G) > \Delta(G)$ if $G = K_3$. It was proved by Holyer [1981] that deciding if $\chi(G) \leq 3$ is NP-complete.

Nevertheless, $\Delta(G)$ is a good estimate of the edge-colouring number as Vizing [1964,1965a] showed the following (our proof roots in Ehrenfeucht, Faber, and Kierstead [1984]):

Theorem 28.1 (Vizing's theorem for simple graphs)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ *for any simple graph* G*.*

Proof. The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + 1$. To prove this inductively, it suffices to show for any simple graph G:

(28.1) Let v be a vertex such that v and all its neighbours have degree at most k, while at most one neighbour has degree precisely k. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable.

We prove (28.1) by induction on k, the case $k = 0$ being trivial. We can assume that each neighbour u of v has degree $k-1$, except for one neighbour having degree exactly k, since otherwise we can add a new vertex w and an edge uw without violating the condition in (28.1).

Consider any k-edge-colouring of $G - v$. For $i = 1, \ldots, k$, let X_i be the set of neighbours of v that are missed by colour i. Choose the colouring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized.

First assume that $|X_i| \neq 1$ for all i. Since all but one neighbour of v is in precisely two of the X_i, and one neighbour is in precisely one X_i, we have

$$(28.2) \qquad \sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 < 2k.$$

Hence there exist i, j with $|X_i| < 2$ and $|X_j|$ odd. So $|X_i| = 0$ and $|X_j| \geq 3$. Consider the subgraph H made by all edges of colours i and j, and consider a component of H containing a vertex in X_j. This component is a path P starting in X_j. Exchanging colours i and j on P reduces $|X_i|^2 + |X_j|^2$, contradicting our minimality assumption.

So we can assume that $|X_k| = 1$, say $X_k = \{u\}$. Let G' be the graph obtained from G by deleting edge vu and deleting all edges of colour k. So $G' - v$ is $(k-1)$-edge-coloured. Moreover, in G', vertex v and all its neighbours have degree at most $k - 1$, and at most one neighbour has degree $k - 1$. So by the induction hypothesis, G' is $(k - 1)$-edge-colourable. Restoring colour k, and giving edge vu colour k, gives a k-edge-colouring of G. ∎

Notes. This theorem was also announced in an abstract of Gupta [1966].

The above proof implies the stronger result of Fournier [1973] that a simple graph G is $\Delta(G)$-edge-colourable if the maximum-degree vertices span no circuit (since this last condition implies that the maximum-degree vertices induce a forest as subgraph, and hence there exists a maximum-degree vertex v with at most one neighbour that has maximum degree).

Petersen [1898] gave the example of the (now-called) Petersen graph (Figure 28.1) which is 2-connected and cubic but not 3-edge-colourable. It was conjectured by Tutte [1966] that each 2-connected cubic graph without Petersen graph minor, is 3-edge-colourable. This conjecture was proved (using the 4-colour theorem) by the combined efforts of Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Complexity. The proof gives a polynomial-time algorithm to find a $(\Delta+1)$-edge-colouring of a simple graph, in fact, $O(\Delta n^2)$-time. As we can assume that $\Delta n = O(m)$ (since we can merge vertices of degree at most $\frac{1}{2}\Delta$), this implies an $O(nm)$-time algorithm.

Gabow, Nishizeki, Kariv, Leven, and Terada [1985] gave algorithms finding a $(\Delta + 1)$-edge-colouring of a simple graph G of maximum degree Δ, with running

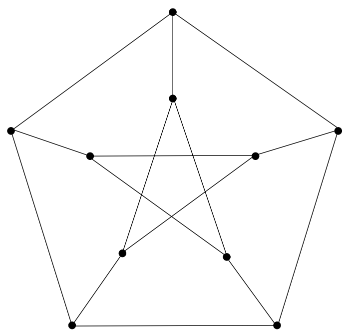


Figure 28.1
The Petersen graph

times $O(m\Delta \log n)$ and $O(m\sqrt{n \log n})$ (improving $O(nm)$ of Terada and Nishizeki [1982]).

28.2. Vizing's theorem for general graphs

In Theorem 28.1 we cannot delete the condition that G be simple: the graph G obtained from K_3 by replacing each edge by two parallel edges, has $\chi'(G) = 6$ and $\Delta(G) = 4$. However, Vizing's theorem can be extended so as to take also the nonsimple case into account. For any graph $G = (V, E)$ and $u, v \in V$, let $\mu(u, v)$ denote the number of edges connecting u and v, called the *multiplicity* of $\{u, v\}$. Let $\mu(G)$ denote the maximum of $\mu(u, v)$ over all distinct $u, v \in V$. Then Vizing [1964,1965a] showed (again, our proof roots in Ehrenfeucht, Faber, and Kierstead [1984]):

Theorem 28.2 (Vizing's theorem). *$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ for any graph G.*

Proof. The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + \mu(G)$. To prove this inductively, it suffices to show for any graph G:

(28.3) Let v be a vertex of degree at most k such that each neighbour u of v satisfies $\deg(u) + \mu(u, v) \leq k + 1$, with equality for at most one neighbour. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable.

We prove (28.3) by induction on k, the case $k = 0$ being trivial. We can assume that for each vertex u in $N(v)$ (the set of neighbours of v) we have $\deg(u) + \mu(u, v) = k$, except for one satisfying $\deg(u) + \mu(u, v) = k + 1$, since otherwise we can add a new vertex w and an edge uw without violating the condition in (28.3).

Consider any k-edge-colouring of $G - v$. For $i = 1, \ldots, k$, let X_i be the set of neighbours of v that are missed by colour i. Choose the colouring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized.

First assume that $|X_i| \neq 1$ for all i. As each $u \in N(v)$ is in precisely $2\mu(u,v)$ of the X_i, except for one $u \in N(v)$ being in $2\mu(u,v) - 1$ of the X_i, we know

$$\sum_{i=1}^{k} |X_i| = -1 + 2 \sum_{u \in N(v)} \mu(u,v) = 2\deg(v) - 1 < 2k. \tag{28.4}$$

Hence there exist i, j with $|X_i| < 2$ and $|X_j|$ odd. So $|X_i| = 0$ and $|X_j| \geq 3$. Consider the subgraph H made by all edges of colours i and j, and consider a component of H containing a vertex in X_j. This component is a path P starting in X_j. Exchanging colours i and j on P reduces $|X_i|^2 + |X_j|^2$, contradicting our minimality assumption.

So we can assume that $|X_k| = 1$, say $X_k := \{u\}$. Let G' be the graph obtained from G by deleting one of the edges vu and deleting all edges of colour k. So $G' - v$ is $(k-1)$-edge-coloured. Moreover, in G', vertex v has degree at most $k-1$ and each neighbour w of v satisfies $\deg_{G'}(w) + \mu_{G'}(w,v) \leq k$, with equality for at most one neighbour. So by the induction hypothesis, G' is $(k-1)$-edge-colourable. Restoring colour k, and giving the deleted edge vu colour k, gives a k-edge-colouring of G. ∎

Notes. The proof of Theorem 28.2 in fact implies that the edge-colouring number of a graph G is at most

$$\max_{u \in V} (\deg(u) + \max\{1, \max_{\substack{v \in V \\ \deg(v) \geq \deg(u)}} \mu(u,v)\}), \tag{28.5}$$

where $\mu(u,v)$ is the number of edges connecting u and v (cf. Ore [1967]).

Other proofs of Vizing's theorem were given by Ore [1967], Fournier [1973], Berge and Fournier [1991], Misra and Gries [1992], Rao and Dijkstra [1992], and Chew [1997b].

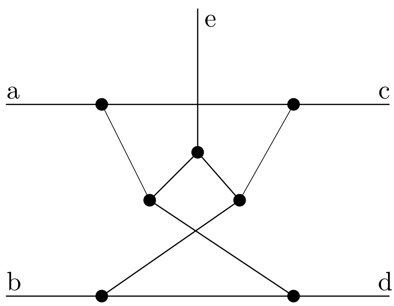
28.3. NP-completeness of edge-colouring

Vizing's theorem gives us a close approximation to the edge-colouring number of a simple graph. The error is at most 1. However, it turns out to be NP-complete to determine the edge-colouring number precisely, even for cubic graphs, which was shown by Holyer [1981]:

Theorem 28.3. *It is* NP*-complete to decide if a given cubic graph is* 3*-edge-colourable.*

Proof. We show that the 3-satisfiability problem (3-SAT) can be reduced to the edge-colouring problem of graphs of maximum degree 3. One easily reduces this last problem to the edge-colouring problem for cubic graphs (by deleting iteratively all vertices of degree ≤ 1, next making a copy of the graph left, and adding an edge between each degree-2 vertex and its copy).

Consider the graph fragment, called the *inverting component*, given by the left-hand picture of Figure 28.2, where the right-hand picture gives its symbolic representation if we take it as part of larger graphs. The pairs a, b and c, d are called the *output pairs*.



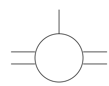


Figure 28.2

The *inverting component* and its symbolic representation.

This graph fragment has the property that a 3-colouring of the edges a, b, c, d, and e is extendible to a 3-edge-colouring of the fragment if and only if either a and b have the same colour while c, d, and e have three distinct colours, or c and d have the same colour while a, b, and e have three distinct colours.

Consider now an instance of the 3-satisfiability problem. From the inverting component we build larger graph fragments. A *splitting component* is given in Figure 28.3(a). For each variable u, occurring k times, as u or $\neg u$, we introduce a fragment Γ_u by concatenating $k-2$ splitting components. So Γ_u has k output pairs, and it has the property that in any colouring either all output pairs are monochromatic, or they all are nonmonochromatic.

For each clause C we introduce a component Δ_C given by Figure 28.3(b). If a variable u occurs in a clause C as u, we connect one of the output pairs of Γ_u with one of the output pairs of Δ_C. If a variable u occurs in a clause C as $\neg u$, we connect one of the output pairs of Γ_u with one side of an inverting component, and connect the other side of this inverting component with one of the output pairs of Δ_C.

In this way we can match up all output pairs of the Γ_u and those of the Δ_C. Deleting all loose ends, we obtain a graph G of maximum degree 3. Now, given the properties of the fragments, one easily checks that the input of the 3-satisfiability problem is satisfiable if and only if G is 3-edge-colourable. ∎

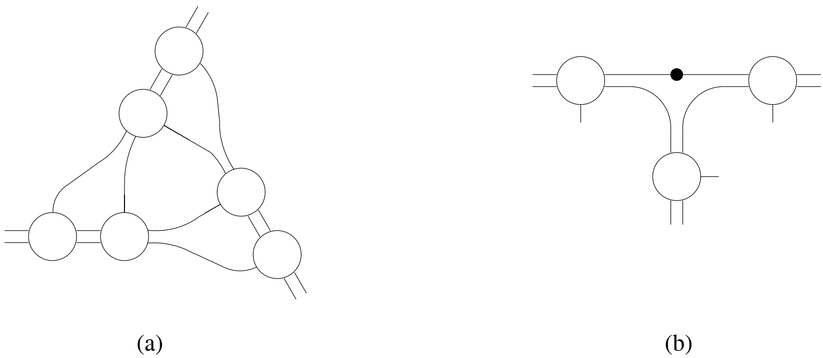


Figure 28.3
Fragment (a) (the *splitting component*) has the property that for any 3-edge-colouring either all three output pairs are monochromatic or all are nonmonochromatic.
Fragment (b) has the property that a colouring of the output edges is extendible to a 3-edge-colouring of the fragment if and only if at least one of the output pairs is monochromatic.

Leven and Galil [1983] showed more generally that for each k, finding the edge-colouring number of a k-regular graph is NP-complete. (This does not seem to follow from the case $k = 3$.)

28.4. Nowhere-zero flows and edge-colouring

Let $D = (V, A)$ be a directed graph and let Γ be an additive abelian group. A *flow over* Γ is a function $f : A \to \Gamma$ such that for each $v \in V$:

(28.6) $$f(\delta^{\text{in}}(v)) = f(\delta^{\text{out}}(v)).$$

The flow is called *nowhere-zero* if all values of f are nonzero.

If G is an undirected graph, then a *flow over* Γ is a flow over Γ in some orientation of G. We say that an undirected graph G has a nowhere-zero flow over Γ if G has an orientation having a nowhere-zero flow over Γ.

Colouring the edges of an undirected graph is related to the problem of finding a nowhere-zero flow over a finite abelian group in the graph. This might be illustrated best by the following easy fact:

(28.7) a cubic graph G is 3-edge-colourable $\Longleftrightarrow$ G has a nowhere-zero flow over $\mathrm{GF}(4)$.

Since $-x = x$ for each $x \in \mathrm{GF}(4)$, the orientation is irrelevant in this case.

Statement (28.7) implies that the four-colour theorem is equivalent to:

(28.8) each bridgeless cubic planar graph has a nowhere-zero flow over GF(4)

(since the four-colour theorem is equivalent to each bridgeless cubic planar graph being 3-edge-colourable (Tait [1878b])).

In studying nowhere-zero flows, the following theorem shows that for the existence of a nowhere-zero flow, only the size of the group is relevant (the equivalence (i)⇔(ii) was shown by Tutte [1947a], the equivalence (i)⇔(iii) by Tutte [1949], and the equivalence (iii)⇔(iv) by Minty [1967]):

Theorem 28.4. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}$ with $k \geq 1$. Then the following are equivalent:*

(28.9) (i) *G has a nowhere-zero flow over some abelian group with precisely k elements;*
(ii) *G has a nowhere-zero flow over each abelian group with at least k elements;*
(iii) *G has a flow over $\mathbb{Z}$ taking values in the interval $[1, k-1]$ only;*
(iv) *G has an orientation $D = (V, A)$ with $d_A^{\text{in}}(U) \geq \frac{1}{k} d_E(U)$ for each $U \subseteq V$.*

Proof. The implication (ii)⇒(i) is trivial, while the implication (iii)⇒(i) is easy, by considering the integer values of (iii) as values in the group of integers mod k.

For any graph $G = (V, E)$ and any finite abelian group Γ, let $\phi_\Gamma(G)$ denote the number of nowhere-zero flows over Γ in G. Then for any nonloop edge e of G one has (where G/e is the graph obtained from G by contracting e):

(28.10) $$\phi_\Gamma(G) = \phi_\Gamma(G/e) - \phi_\Gamma(G - e).$$

Moreover, if each edge of G is a loop, then:

(28.11) $$\phi_\Gamma(G) = (|\Gamma| - 1)^{|E|}.$$

This proves that if Γ and Γ' are finite abelian groups with $|\Gamma| = |\Gamma'|$, then $\phi_\Gamma(G) = \phi_{\Gamma'}(G)$. Hence G has a nowhere-zero flow over Γ if and only if G has a nowhere-zero flow over Γ'. Therefore:

(28.12) if G has a nowhere-zero flow over some abelian group of size k, then it has one over each abelian group of size k.

We now consider (i)⇒(iii). By (28.12), (i) implies that G has a nowhere-zero flow over the group of integers mod k. This implies that there is an orientation $D = (V, A)$ of G and a function $f : A \to \{1, \ldots, k-1\}$ such that for each $v \in V$:

(28.13) $$f(\delta^{\text{in}}(v)) \equiv f(\delta^{\text{out}}(v)) \pmod{k}.$$

We choose the orientation D and the function f such that the sum

(28.14) $$\sum_{v \in V} |f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v))|$$

is minimized. If the sum is 0, we are done. So assume that the sum is nonzero. Define

(28.15) $$U_+ := \{v \in V \mid f(\delta^{\text{in}}(v)) > f(\delta^{\text{out}}(v))\} \text{ and}$$
$$U_- := \{v \in V \mid f(\delta^{\text{in}}(v)) < f(\delta^{\text{out}}(v))\}.$$

Necessarily, there is a directed path P in D from U_- to U_+ (Theorem 11.1). Now reverse the orientation of each arc a on P to its reverse a^{-1}, and define $f(a^{-1}) := k - f(a)$. This maintains (28.13) but reduces the sum (28.14), a contradiction.

This proves (i)⇒(iii), and hence (i)⇔(iii). Since (iii) is maintained if we increase k, also (i) is maintained if we increase k. So with (28.12), (i) implies (ii) if (ii) is restricted to finite groups. Since each infinite abelian group has $\mathbb{Z}$ as subgroup or has arbitrarily large finite subgroups, (iii)⇒(ii) also follows for infinite groups.

The equivalence of (iii) and (iv) follows directly from Hoffman's circulation theorem (Theorem 11.2). ∎

This theorem implies that in studying the existence of nowhere-zero flows, we can restrict ourselves to the group $\mathbb{Z}_k$ with elements $0, \ldots, k-1$ and addition mod k. A *nowhere-zero k-flow* is a nowhere-zero flow over $\mathbb{Z}_k$.

It is easy to characterize the graphs having a nowhere-zero 2-flow: they are precisely the Eulerian graphs. As to larger values of k there are the following three famous conjectures of Tutte. The 5-*flow conjecture* (Tutte [1954a]):

(28.16) (?) each bridgeless graph has a nowhere-zero 5-flow, (?)

the 4-*flow conjecture* (Tutte [1966]):

(28.17) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow, (?)

and the 3-*flow conjecture* (W.T. Tutte, 1972 (cf. Bondy and Murty [1976], Unsolved problem 48)):

(28.18) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

For planar graphs this is equivalent to the theorem of Grötzsch [1958] that each loopless triangle-free planar graph is 3-vertex-colourable.

It may be seen that a cubic graph G has a nowhere-zero 3-flow if and only if G is bipartite. This follows from the fact that the existence of such a flow implies that G has an orientation such that in each vertex the indegree and outdegree differ by a multiple of 3. Hence, one of them is 3, the other 0. Hence each arc is oriented from a source to a sink, and so G is bipartite. The reverse implication is easy, by orienting each edge from one colour class to the other.

Jaeger [1979] showed that each 4-edge-connected graph has a nowhere-zero 4-flow: a 4-edge-connected graph $G = (V, E)$ has two edge-disjoint spanning trees T_1 and T_2 (by Corollary 51.1a). For $i = 1, 2$, let C_i be the symmetric difference of all fundamental circuits of T_i. Then C_1 and C_2 are cycles covering E. This gives a nowhere-zero 4-flow.

Jaeger [1988] proposed a weakened version of the 3-flow conjecture, the *weak 3-flow conjecture*:

(28.19) (?) there exists a number k such that each k-edge-connected graph has a nowhere-zero 3-flow. (?)

By (28.8), the 4-flow conjecture implies the four-colour theorem. For cubic graphs, (28.17) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

One should note that having a nowhere-zero 4-flow is equivalent to the existence of two cycles covering the edge set. In other words, there exist two disjoint T-joins, where T is the set of odd-degree vertices (see Chapter 29).

It was proved by Seymour [1981b] that each bridgeless graph has a nowhere-zero 6-flow. (Inspired by Seymour's method, Younger [1983] gave a polynomial-time algorithmic proof.)

Seymour's theorem improves an earlier result of Jaeger [1976,1979] that each bridgeless graph has a nowhere-zero 8-flow. This is equivalent to: each bridgeless graph contains three cycles covering all edges.

Jaeger [1984] offered a conjecture, the *circular flow conjecture*, that implies both the 3-flow and the 5-flow conjecture:

(28.20) (?) for each $k \geq 1$, each $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k + 1$. (?)

For $k = 1$, this is equivalent to the 3-flow conjecture. For $k = 2$, it implies the 5-flow conjecture: Let $G = (V, E)$ be a 3-edge-connected graph, and replace each edge by 3 parallel edges. The new graph, H say, is 9-edge-connected. If (28.20) is true for $k = 2$, H has an orientation such that in each vertex, the indegree and the outdegree differ by a multiple of 5. This can easily be transformed to a nowhere-zero 5-flow in G.[12]

More on the 3-flow conjecture can be found in Fan [1993] and Kochol [2001]. Jaeger [1979,1988] and Seymour [1995a] gave surveys on nowhere-zero flows, and a book on this topic was written by Zhang [1997b]. We continue discussing nowhere-zero flows in Section 38.8.

[12] Orient any edge e of G in the direction of the majority of the direction of the three parallel edges in H made from e, with flow equal to 3 if all three edges have the same orientation, and 1 otherwise.

28.5. Fractional edge-colouring

Determining the edge-colouring number of a graph is NP-complete, but with matching techniques one can determine a fractional version of it in polynomial time.

Let $G = (V, E)$ be a graph. The *fractional edge-colouring number* $\chi'^*(G)$ of G is defined as

$$\chi'^*(G) := \min\{\sum_{M\in\mathcal{M}} \lambda_M \mid \lambda \in \mathbb{R}_+^{\mathcal{M}}, \sum_{M\in\mathcal{M}} \lambda_M \chi^M = \mathbf{1}\}, \tag{28.21}$$

where $\mathcal{M}$ denotes the collection of all matchings in G.

So if we require the λ_M to be integer, this would define the edge-colouring number of G. Therefore, we have

$$\chi'^*(G) \leq \chi'(G). \tag{28.22}$$

The Petersen graph is an example of a graph G with $\chi'^*(G) = 3$ and $\chi'(G) = 4$. In Section 28.7 we shall see that $\chi'^*(G)$ can be computed in polynomial time.

$\chi'^*(G)$ can be characterized as follows. For any natural number $k \geq 1$, let G_k be the graph obtained from G by replacing each edge by k parallel edges. Then

$$\chi'^*(G) = \min_{k\geq 1} \frac{\chi'(G_k)}{k}. \tag{28.23}$$

This follows from the fact that the minimum in (28.21) is attained by rational λ_M. Then the minimum in (28.23) is attained by $k :=$ the l.c.m. of the denominators of the λ_M.

From Edmonds' matching polytope theorem (Corollary 25.1a), a characterization of the fractional edge-colouring number follows:

Theorem 28.5. *The fractional edge-colouring number $\chi'^*(G)$ satisfies:*

$$\chi'^*(G) = \max\{\Delta(G), \max_{U\subseteq V, |U|\geq 3} \frac{|E[U]|}{\lfloor \frac{1}{2}|U|\rfloor}\}. \tag{28.24}$$

Proof. Let μ be equal to the maximum in (28.24). Then $\chi'^*(G) \geq \mu$, since if λ_M attains minimum (28.21) and if vertex v has maximum degree, then

$$\begin{aligned}\chi'^*(G) &= \sum_M \lambda_M \geq \sum_M \lambda_M |M \cap \delta(v)| = \sum_{e\in\delta(v)} \sum_{M\ni e} \lambda_M \\ &= \sum_{e\in\delta(v)} 1 = \Delta(G).\end{aligned} \tag{28.25}$$

Moreover, for each $U \subseteq V$ with $|U| \geq 3$,

$$(28.26) \qquad \chi'^*(G) = \sum_M \lambda_M \geq \sum_M \lambda_M \frac{|M \cap E[U]|}{\lfloor \frac{1}{2}|U| \rfloor} = \frac{1}{\lfloor \frac{1}{2}|U| \rfloor} \sum_{e \in E[U]} \sum_{M \ni e} \lambda_M = \frac{|E[U]|}{\lfloor \frac{1}{2}|U| \rfloor}.$$

To see that $\chi'^*(G) = \mu$, let x be the all-$\frac{1}{\mu}$ vector in $\mathbb{R}^E$. Then $x(\delta(v)) \leq 1$ for each $v \in V$ and $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ for each $U \subseteq V$ with $|U| \geq 3$. Hence x belongs to the matching polytope of G. So x is a convex combination of incidence vectors of matchings. Therefore $\mathbf{1} = \mu \cdot x = \sum_M \lambda_M \chi^M$ for some $\lambda_M \geq 0$ with $\sum_M \lambda_M = \mu$, showing that $\chi'^*(G) \leq \mu$. ∎

This implies for regular graphs:

Corollary 28.5a. *Let $G = (V, E)$ be a k-regular graph. Then $\chi'^*(G) = k$ if and only if $|\delta(U)| \geq k$ for each odd subset U of V.*

Proof. By Theorem 28.5, $\chi'^*(G) = k$ if and only if $|E[U]| \leq k\lfloor \frac{1}{2}|U| \rfloor$ for each subset U of V. This last is equivalent to $|\delta(U)| \geq k$ for each odd subset U of V. ∎

Call a graph $G = (V, E)$ a *k-graph* if G is regular of degree k and if $|\delta(U)| \geq k$ for each odd subset U of V. So by Corollary 28.5a, a k-regular graph G is a k-graph if and only if $\chi'^*(G) = k$.

28.6. Conjectures

Seymour [1979a] conjectures that

$$(28.27) \qquad (?) \ \lceil \chi'^*(G) \rceil = \lceil \tfrac{1}{2} \chi'(G_2) \rceil \ (?)$$

for each graph G, where G_2 arises from G by replacing each edge by two parallel edges. Conjecture (28.27) is equivalent to the conjecture that, for each k,

(28.28) (?) for each k-graph G one has $\chi'(G_2) = 2k$ (?);

equivalently, for each k-graph G, the minimum (28.21) for $\chi'^*(G)$ is attained by half-integer λ_M. In other words, it is conjectured that any k-graph has $2k$ perfect matchings covering each edge exactly twice. (The equivalence of (28.27) and (28.28) can be seen as follows. The implication (28.27)⇒(28.28) is easy. To see the reverse implication, let G be any graph and define $k := \lceil \chi'^*(G) \rceil$. Make a disjoint copy G' of G, and connect each vertex v of G by $k - \deg_G(v)$ parallel edges to its copy v' in G'. This makes a k-regular graph H with $\chi'^*(H) = k$. So H is a k-graph, and hence by (28.28), $\chi'(H_2) = 2k$. Hence $\chi'(G_2) \leq 2k$, implying (28.27).)

Seymour called (28.28) the *generalized Fulkerson conjecture*, as it generalizes the special case $k = 3$ asked (but not conjectured) by Fulkerson [1971a]. This special case is called the 'Fulkerson conjecture'[13]. (By Corollary 28.5a, a cubic graph G has $\chi'^*(G) = 3$ if and only if G is bridgeless.) For a partial result, see Corollary 38.11e.

Berge [1979a] conjectured that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. This would follow from the Fulkerson conjecture.

A conjecture of Gol'dberg [1973] (and also of Seymour [1979a]) is that for each (possibly nonsimple) graph G one has

(28.29) (?) $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \chi'^*(G) \rceil\}$. (?)

(An equivalent conjecture was stated by Andersen [1977].)

As $\chi'(G) \geq \max\{\Delta(G), \lceil \chi'^*(G) \rceil\}$, validity of (28.29) would yield a tight (gap 1) bound for $\chi'(G)$ also for nonsimple graphs. In particular, if $\Delta(G) < \chi'^*(G)$, we would have equality in (28.29). Seymour [1979a] mentioned that he has shown that $\chi'(G) \leq \lceil \chi'^*(G) \rceil + 1$ for graphs G with $\chi'^*(G) \leq 6$.

Conjecture (28.29) would generalize Theorem 28.2 due to Vizing. For let $\mu(G)$ again denote the maximum multiplicity of any edge of $G = (V, E)$. Then for any subset U of V,

(28.30) $$|E[U]| \leq \tfrac{1}{2}\Delta(G[U])|U| \leq (\Delta(G[U]) + \mu(G))\tfrac{1}{2}(|U| - 1) \\ \leq (\Delta(G) + \mu(G))\lfloor \tfrac{1}{2}|U| \rfloor$$

(The second inequality follows from $\Delta(G[U]) \leq \mu(G)(|U| - 1)$.) So with Theorem 28.5 we know that $\chi'^*(G) \leq \Delta(G) + \mu(G)$.

A well-known equivalent form of the four-colour theorem is that each bridgeless cubic planar graph is 3-edge-colourable. This equivalence was discovered by Tait [1878b]. Seymour [1981c] conjectures the following generalization:

(28.31) (?) each planar k-graph is k-edge-colourable. (?)

This was proved for $k = 4$ and $k = 5$ by Guenin [2002b].

A consequence of the 4-flow conjecture of Tutte [1966] is:

(28.32) each bridgeless cubic graph without Petersen graph minor is 3-edge-colourable.

This was proved jointly by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

(28.31) and (28.32) made Lovász [1987] conjecture:

(28.33) (?) each k-graph without Petersen graph minor is k-edge-colourable. (?)

[13] Seymour [1979a] says that it was first conjectured by C. Berge, but that it is usually called Fulkerson's conjecture because the latter put it into print.

(An equivalent conjecture was given by Rizzi [1997,1999].) This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base (cf. Section 5.18). It relates to Lovász's work on the perfect matching lattice — see Chapter 38.

Goddyn [1993] noted that (28.33) would not yield a full characterization, since also the perfect matchings of the Petersen graph form a Hilbert base. (This is due to the fact that the all-one vector does not belong to the perfect matching lattice of the Petersen graph.)

Notes. Seymour [1979a] conjectured that if $k \geq 4$, any k-graph $G = (V, E)$ has a perfect matching M such that $G-M$ is a $(k-1)$-graph. However, this was disproved by Rizzi [1997,1999], who showed that for any $k \geq 3$, there exists a k-graph in which any two perfect matchings intersect. Hence, for any $k \geq 3$ there exists a k-graph that cannot be decomposed into a k_1- and a k_2-graph for any $k_1, k_2 \geq 1$ with $k_1 + k_2 = k$.

Nishizeki and Kashiwagi [1990] showed that

(28.34) $$\chi'(G) \leq \max\{\tfrac{11}{10}\Delta(G) + \tfrac{4}{5}, \lceil \chi'^*(G) \rceil\},$$

and they gave a polynomial-time algorithm finding an edge-colouring fulfilling this bound. (This improves earlier results of Andersen [1977], Goldberg [1984], and Hochbaum, Nishizeki, and Shmoys [1986].)

Marcotte [1986,1990a,1990b,2001], Seymour [1990a], Lee and Leung [1993], and Caprara and Rizzi [1998] gave other partial results on conjecture (28.29).

28.7. Edge-colouring polyhedrally

Let $G = (V, E)$ be a graph and let Q be the polytope determined by

(28.35) $$\begin{array}{ll} x_e \geq 0 & (e \in E), \\ x(M) \leq 1 & (M \text{ matching}). \end{array}$$

So Q is the antiblocking polyhedron of the matching polytope. By the description of the matching polytope and by the theory of antiblocking polyhedra, Q is equal to the convex hull of the following set of vectors:

(28.36) $$\begin{array}{ll} \chi^S & S \text{ substar}, \\ \frac{1}{\lfloor \frac{1}{2}|\bigcup F| \rfloor}\chi^F & \text{for nonempty } F \subseteq E. \end{array}$$

Here a *substar* is any set S of edges with $S \subseteq \delta(v)$ for some $v \in V$. By $\bigcup F$ we denote the set of vertices covered by F.

Now the fractional edge-colouring number $\chi'^*(G)$ is equal to the maximum value of $\mathbf{1}^{\mathsf{T}}x$ over Q (by LP-duality). The ellipsoid method then gives:

Theorem 28.6. *The fractional edge-colouring number of a graph can be determined in polynomial time.*

Proof. The separation problem over Q is equivalent to the weighted matching problem, and hence is solvable in polynomial time. Therefore, with the ellipsoid method, also the optimization problem over Q is solvable in polynomial time. This gives the fractional edge-colouring number. ∎

For any weight function $w \in \mathbb{R}_+^E$, the maximum of $w^\mathsf{T} x$ where x ranges over the vectors (28.36), is equal to the minimum value of $\sum_M \lambda_M$ where $\lambda_M \geq 0$ for $M \in \mathcal{M}$ such that $\sum_M \lambda_M \chi^M = w$. Thus we have a min-max relation for the 'weighted fractional edge-colouring number'.

We should note that (generally) the matching polytope does not have the integer decomposition property, and (equivalently) that system (28.35) does not have the integer rounding property. Indeed, for the Petersen graph, the maximum of $\mathbf{1}^\mathsf{T} x$ over (28.35) is equal to 3. So it has a fractional optimum dual solution of value 3. However, there is no integer optimum dual solution, since the edges of the Petersen graph cannot be decomposed into three matchings.

28.8. Packing edge covers

The results on edge-colouring (which is essentially covering by matchings), can be dualized to packing edge covers, as observed by Gupta [1974] (where $\delta(G)$ denotes the minimum degree of G):

Theorem 28.7. *A simple graph $G = (V, E)$ has $\delta(G) - 1$ disjoint edge covers.*

Proof. Make an auxiliary graph H as follows. For each $v \in V$, do the following. Make $\deg_G(v) - \delta(G)$ new vertices, and reconnect $\deg_G(v) - \delta(G)$ of the edges incident with v with the new vertices, in such a way that v has degree $\delta(G)$, while each new vertex has degree 1.

Then H has maximum degree $\delta(G)$ and there is a one-to-one mapping between the edges of G and those of H. By Vizing's theorem for simple graphs (Theorem 28.1), H has matchings $M_1, \ldots, M_{\delta(G)+1}$ partitioning E. We denote the corresponding edge sets in G also by M_i.

Then each vertex v of G is covered by all but at most one of the matchings $M_1, \ldots, M_{\delta(G)+1}$. Let U be the set of vertices of G missed by one of $M_1, \ldots, M_{\delta(G)-1}$. Then each vertex in U is covered by both $M_{\delta(G)}$ and $M_{\delta(G)+1}$. So $M_{\delta(G)} \cup M_{\delta(G)+1}$ forms a graph on V where each vertex in U has degree at least 2. Hence we can orient the edges in $M_{\delta(G)} \cup M_{\delta(G)+1}$ such that each vertex in U is head of at least one of the oriented edges.

Now for each $i = 1, \ldots, \delta(G) - 1$, add to M_i all edges in $M_{\delta(G)} \cup M_{\delta(G)+1}$ that are oriented towards a vertex missed by M_i. This gives $\delta(G) - 1$ disjoint edge covers. ∎

This can be formulated in terms of the *edge cover packing number* $\xi(G)$ of G, which is the maximum number of disjoint edge covers in G. Then, if G is simple,

$$(28.37) \qquad \xi(G) \geq \delta(G) - 1.$$

Gupta [1974] showed more generally for not necessarily simple graphs (where $\mu(G)$ denotes the maximum multiplicity of the edges of G):

Theorem 28.8. *Any graph G has $\delta(G) - \mu(G)$ disjoint edge covers.*

Proof. Let $\delta := \delta(G)$ and $\mu := \mu(G)$. Make an auxiliary graph H as follows. For each $v \in V$, do the following. Make $\deg_G(v) - \delta$ new vertices, and reconnect $\deg_G(v) - \delta$ of the edges incident with v with the new vertices, in such a way that v has degree δ, while each new vertex has degree 1.

Then H has maximum degree $\delta(G)$, and there is a one-to-one mapping between the edges of G and those of H. By Vizing's theorem (Theorem 28.2), H has matchings $M_1, \ldots, M_{\delta+\mu}$ partitioning E. We denote the corresponding edge sets in G also by M_i. Let

$$(28.38) \qquad F := M_{\delta-\mu+1} \cup \cdots \cup M_{\delta+\mu}.$$

Orient the edges in F such that each vertex v is entered by at least $\lfloor \frac{1}{2} \deg_F(v) \rfloor$ of the edges incident with v.

Consider any vertex v, and let v be missed by α of the $M_1, \ldots, M_{\delta-\mu}$. Let k be the number of $M_{\delta-\mu+1}, \ldots, M_{\delta+\mu}$ covering v. As v is covered by at least δ of the $M_1, \ldots, M_{\delta+\mu}$, we know $k + (\delta - \mu - \alpha) \geq \delta$, that is, $k \geq \mu + \alpha$. Since $k \leq 2\mu$, it follows that $\deg_F(v) \geq k = 2k - k \geq 2(\mu + \alpha) - 2\mu = 2\alpha$. Hence v is entered by at least α edges.

So for each $i = 1, \ldots, \delta - \mu$, if v is missed by M_i, then we can extend M_i by an edge in F oriented towards v. Doing this for each vertex v, we obtain $\delta - \mu$ disjoint edge covers. ∎

Equivalently, for any graph,

$$(28.39) \qquad \xi(G) \geq \delta(G) - \mu(G).$$

Gupta [1974] announced (without proof) and Fournier [1977] showed that for any graph $G = (V, E)$ and any $k \in \mathbb{Z}_+$, E can be partitioned into classes $E_1, \ldots, E_k$ such that each vertex v is covered by at least

$$(28.40) \qquad \min\{k, \deg(v), \max\{k, \deg(v)\} - \mu(v)\}$$

of the E_i, where $\mu(v)$ denotes the maximum multiplicity of the edges incident with v.

28.9. Further results and notes

28.9a. Shannon's theorem

Shannon [1949] gave the following upper bound on the edge-colouring number that can be better than Vizing's bound if G is not simple. As Vizing [1965a] observed, the bound can be derived from Vizing's theorem, as below.

Theorem 28.9. *The edge-colouring number $\chi'(G)$ of a graph $G = (V, E)$ is at most $\lfloor \frac{3}{2}\Delta(G) \rfloor$.*

Proof. Let G be a counterexample with a minimum number of edges. Define $k := \lfloor \frac{3}{2}\Delta(G) \rfloor$. So $\chi'(G) > k$ and by Vizing's theorem (Theorem 28.2), $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum edge-multiplicity of G. Hence $\mu(G) > \frac{1}{2}\Delta(G)$.

Let u and v be vertices connected by $\mu(G)$ parallel edges. Choose one such edge, e say. By the minimality of G, $\chi'(G - e) \leq k$. Consider a k-edge-colouring of $G - e$. Let I_u and I_v be the sets of colours covering u and v respectively. Then $|I_u \cap I_v| \geq \mu(G) - 1$, since $\mu(G) - 1$ edges of $G - e$ connect u and v. Moreover, $|I_u| \leq \Delta(G) - 1$, since u has degree less than $\Delta(G)$ in $G - e$; similarly, $|I_v| \leq \Delta(G) - 1$. So

(28.41) $$\begin{aligned} |I_u \cup I_v| &= |I_u| + |I_v| - |I_u \cap I_v| \leq 2(\Delta(G) - 1) - (\mu(G) - 1) \\ &= 2\Delta(G) - \mu(G) - 1 < \tfrac{3}{2}\Delta(G) - 1 < k \end{aligned}$$

(since $\mu(G) > \frac{1}{2}\Delta(G)$), and hence at least one colour does not occur in $I_u \cup I_v$. This colour can be given to edge e to obtain a k-edge-colouring of G. ∎

The bound in this theorem is sharp, as is shown by a graph H on three vertices u, v, and w, with $\lceil \frac{1}{2}\Delta \rceil$ parallel arcs connecting u and v, $\lfloor \frac{1}{2}\Delta \rfloor$ parallel arcs connecting u and w, and $\lfloor \frac{1}{2}\Delta \rfloor$ parallel arcs connecting v and w. Then $\Delta(H) = \Delta$ and $\chi'(H) = \lfloor \frac{3}{2}\Delta \rfloor$.

Vizing [1965a] showed that any graph G with $\Delta(G) \geq 4$ and $\chi'(G) = \lfloor \frac{3}{2}\Delta(G) \rfloor$ contains this graph H as a subgraph.

The case $\Delta(G)$ even in Theorem 28.9 can be proved simpler as follows. We may assume that each degree of G is even (we can pair up the odd-degree vertices by new edges). Let $k := \frac{1}{2}\Delta(G)$. Make an Eulerian orientation of G. Split each vertex v into two vertices v' and v'', and replace any edge oriented from u to v, by an edge connecting u' and v''. In this way we obtain a bipartite graph H, of maximum degree k. Hence, by Kőnig's edge-colouring theorem, H has a k-edge-colouring. This yields a decomposition of the edges of G into classes $E_1, \ldots, E_k$ such that each graph $G_i = (V, E_i)$ has maximum degree 2. Hence each G_i is 3-edge-colourable, and therefore G is $3k$-colourable.

28.9b. Further notes

For simple *planar* graphs, if $\Delta(G) \geq 7$, then $\chi'(G) = \Delta(G)$ (for $\Delta \geq 8$, this was proved by Vizing [1965b], and for $\Delta = 7$ by Sanders and Zhao [2001] and Zhang [2000]). For $2 \leq \Delta \leq 5$ there exist simple planar graphs of maximum degree Δ with $\chi'(G) = \Delta + 1$. This is unknown for $\Delta = 6$ (and constitutes a question of Vizing

[1968]). For $\Delta \geq 8$, polynomial-time algorithms finding a Δ-edge-colouring of a simple planar graph were given by Terada and Nishizeki [1982] ($O(n^2)$), Chrobak and Yung [1989] ($O(n)$ if $\Delta \geq 19$), and Chrobak and Nishizeki [1990] ($O(n \log n)$ if $\Delta \geq 9$).

Kotzig [1957] showed the following theorem:

Theorem 28.10. *Let $G = (V, E)$ be a connected cubic graph with an even number of edges. Then G is 3-edge-colourable if and only if the line graph $L(G)$ of G is 4-edge-colourable.*

Proof. I. First assume that $L(G)$ is 4-edge-colourable, say with colours 0, 1, 2, and 3. We colour the edges of G with colours labeled by the three partitions of $\{0, 1, 2, 3\}$ into pairs. Consider an edge $e = uv$ of G. Let f_1 and f_2 be the two other edges incident with u and let g_1 and g_2 be the two other edges incident with v. Let i_1 and i_2 be the colours of the edges ef_1 and ef_2 of $L(G)$ and let j_1 and j_2 be the colours of the edges eg_1 and eg_2 of $L(G)$. Give e the colour labeled by the partition of $\{0, 1, 2, 3\}$ into the pairs $\{i_1, i_2\}$ and $\{j_1, j_2\}$. This gives a 3-edge-colouring of G.

II. Conversely, assume that G is 3-edge-colourable. We first show that $L(G)$ has a perfect matching. Indeed, there is a subset M of the edge set of $L(G)$ such that each vertex of $L(G)$ is covered an odd number of times. To see this, choose an arbitrary partition Π of the vertices of $L(G)$ into pairs, and for each pair $\{e, f\} \in \Pi$, we choose an $e - f$ path $P_{e,f}$ in $L(G)$. Then the symmetric difference of all these paths is a subset M as required.

Now choose such an M with $|M|$ as small as possible. We claim that each vertex of $L(G)$ is covered exactly once by M; that is, M is a perfect matching in $L(G)$. Suppose that vertex e of $L(G)$ is covered by three edges in M, say ee_1, ee_2, and ee_3. We can assume that e, e_1 and e_2 are pairwise adjacent in $L(G)$. Hence, replacing M by $M \triangle \{ee_1, ee_2, e_1e_2\}$, gives a subset M' covering each vertex an odd number of times, however with $|M'| < |M|$. This contradicts our assumption.

So M is a perfect matching in $L(G)$, forming our first colour 0. Let G be edge-coloured with colours 1, 2, and 3. Consider an edge e_1e_2 of $L(G)$ not having colour 0. Let e_0 be the third edge of G incident with the common vertex of e_1 and e_2. If e_0e_1 has colour 0, give e_1e_2 the colour of edge e_1. If e_0e_2 has colour 0, give e_1e_2 the colour of edge e_2. If neither e_0e_1 nor e_0e_2 has colour 0, give e_1e_2 the colour of edge e_0. It is straightforward to check that this gives a 4-edge-colouring of $L(G)$. ∎

For more on edge-colouring cubic graphs, see Kotzig [1975,1977].

McDiarmid [1972] observed that in any graph $G = (V, E)$, if $p \geq \chi'(G)$, then there is a p-edge-colouring with $\lfloor |E|/p \rfloor \leq |M| \leq \lceil |E|/p \rceil$ for each colour M. This can be proved in the same way as Theorem 20.8.

Meredith [1973] gave k-regular non-Hamiltonian non-k-edge colourable graphs with an even number of vertices, for each $k \geq 3$ (cf. Isaacs [1975]). Johnson [1966a] gave a short proof that any cubic graph is 4-edge-colourable.

Vizing [1965a] asked if the minimum number of colours of the edges of a graph can be obtained from any edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

Marcotte and Seymour [1990] observed that the following is a necessary condition for extending a partial k-edge colouring a graph $G = (V, E)$ to a complete k-edge colouring:

(28.42) $$|F| \leq \sum_{i=1}^{k} \mu_i(F) \text{ for each } F \subseteq E,$$

where $\mu_i(F)$ is the maximum size of a matching $M \subseteq F$ not covering any vertex covered by the ith colour. They studied graphs where this condition is sufficient as well.

Vizing [1965a] showed that if G is nonsimple and $\Delta(G) = 2\mu(G) + 1$, then $\chi'(G) \leq 3\mu(G)$ (where $\mu(G)$ is the maximum edge-multiplicity of G).

The *list-edge-colouring number* $\chi^l(G)$ of a graph $G = (V, E)$ is the minimum number k such that for each choice of sets L_e for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges e, f one has $l_e \neq l_f$. Vizing [1976] conjectures that $\chi^l(G)$ is equal to the edge-colouring number of G, for each graph G (see Häggkvist and Chetwynd [1992]).

The *total colouring number* of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint. Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph G is at most $\Delta(G) + 2$. Molloy and Reed [1998] showed that there exists a constant C such that the total colouring number of any simple graph is at most $\Delta(G) + C$. A polynomial-time algorithm finding a total colouring with $\Delta(G)$+poly($\log \Delta$) colours is given by Hind, Molloy, and Reed [1999].

More generally, Vizing [1968] conjectures that the total colouring number of a graph G is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum size of a parallel class of edges. Partial results have been found by Kostochka [1977], Hind [1990, 1994], Kilakos and Reed [1992], and McDiarmid and Reed [1993].

For extensions of Vizing's theorem, see Vizing [1965b], Fournier [1973], Jakobsen [1973], Gol'dberg [1974], Fiorini [1975], Hilton [1975], Jakobsen [1975], Andersen [1977], Kierstead and Schmerl [1983], Kostochka [1983], Ehrenfeucht, Faber, and Kierstead [1984], Goldberg [1984], Hilton and Jackson [1987], Berge [1990], and Chew [1997a]. The fractional edge-colouring number $\chi'^*(G)$ was studied by Hilton [1975] and Stahl [1979]. A computational study based on fractional edge-colouring was made by Nemhauser and Park [1991]. Equitable edge-colourings were investigated by de Werra [1981].

Generalizations of edge-colouring were studied by Hakimi and Kariv [1986] and Nakano, Nishizeki, and Saito [1988,1990]. Fiorini and Wilson [1977,1978], Fiorini [1978], Jensen and Toft [1995], Nakano, Zhou, and Nishizeki [1995], and Zhou and Nishizeki [2000] gave surveys on edge-colouring and extensions.

28.9c. Historical notes on edge-colouring

Historically, studying edge-colouring was motivated by the equivalence of the four-colour conjecture with the 3-edge-colourability of planar bridgeless cubic graphs. The four-colour conjecture was raised in 1852 by F. Guthrie. An early attempt to prove the conjecture by Kempe [1879,1880] was shown by Heawood in 1890 to contain an error — see below.

Also Tait [1878a] studied the four-colour problem. He claimed without proof that each triangulated planar graph has two disjoint sets of edges such that each triangular face is incident with exactly one edge in each of these sets. From this he derived (correctly) that each loopless planar graph is 4-vertex-colourable. He also observed that his claim implies that each planar bridgeless cubic graph is 3-edge-colourable.

In a note following a note of Guthrie [1878] (describing the very early history of the four-colour problem, which note itself was a reaction to the article of Tait [1878a]), Tait [1878b] remarked that in his earlier paper

> I gave a series of proofs of the theorem that four colours suffice for a map. All of these were long, and I felt that, while more than sufficient to prove the truth of the theorem, they gave little insight into its real nature and bearings. A somewhat similar remark may, I think, be made about Mr Kempe's proof.

He therefore withdrew the former paper, and replaced it by the present note, in which, without proof, the following 'elementary theorem' is formulated:

> if an even number of points be joined, so that three (and only three) lines meet in each, these lines may be coloured with *three* colours only, so that no two conterminous lines shall have the same colour. (When an odd number of the points forms a group, connected by *one* line only with the rest, the theorem is not true.)

Tait next gave the now well-known construction of deriving 3-edge-colourability of bridgeless planar cubic graphs from the 4-vertex-colourability of planar loopless graphs. At that time, the error in Kempe's proof of the four-colour conjecture was not yet detected.

But Tait also said:

> The proof of the elementary theorem is given easily by induction; and then the proof that four colours suffice for a map follows almost immediately from the theorem, by an inversion of the demonstration just given.

Tait [1880] claimed that in Tait [1878b] the 3-edge-colourability of bridgeless planar cubic graphs had been shown:

> *If* 2n *points be joined by* 3n *lines, so that three lines, and three only, meet at each point, these lines can be divided (usually in many different ways) into three groups of* n *each, such that one of each group ends at each of the points.*

While Tait did not mention it explicitly, he restricted himself to *planar* cubic graphs, since he considered them equivalently as the skeletons of polytopes[14]. Also the figures given in Tait [1880] are planar (and also those in Tait [1884], where similar claims are made).

The validity of Kempe's proof of the four-colour conjecture was accepted until Heawood [1890] discovered a fatal error in Kempe's proof, and showed that it in fact gives a five-colour theorem for planar graphs. The error in his proof was acknowledged by Kempe [1889]. (For an account of the early history of the four-colour problem, see Biggs, Lloyd, and Wilson [1976].)

After that, the problem of the 3-edge-colourability of bridgeless planar cubic graphs was open again. At several occasions, this problem was advanced (cf. Goursat [1894]). It was only resolved in 1977 when Appel and Haken proved the four-colour theorem.

Petersen [1898] asserted that Tait had claimed to have proved the 3-edge-colourability of any (also nonplanar) bridgeless cubic graph. It motivated him to present, as a counterexample, the now-called *Petersen graph*, which is a bridgeless cubic graph that is not 3-edge-colourable:

[14] Tait's polytopes are 3-dimensional, since each vertex has degree 3.

j'ai réussi à construire un graphe où le théorème de Tait ne s'applique pas.[15]

Petersen [1898] drew the Petersen graph as follows:

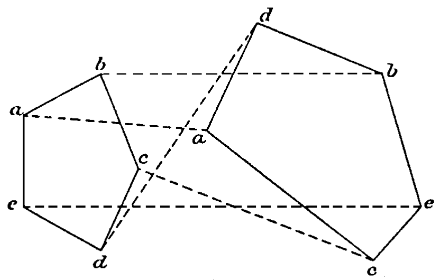


Figure 28.4

For another purpose, the Petersen graph was given earlier by Kempe [1886], who represented it as follows:

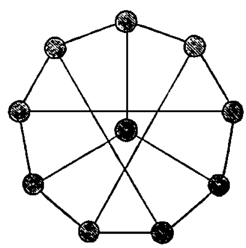


Figure 28.5

Sainte-Laguë [1926a] introduced the term *class* for the edge-colouring number of a graph. He noted (without exact argumentation) that Petersen's theorem on the existence of a perfect matching in a bridgeless cubic graph implies that each cubic graph has edge-colouring number 3 or 4.

[15] I have succeeded in constructing a graph where the theorem of Tait does not apply.

Chapter 29

T-joins, undirected shortest paths, and the Chinese postman

The methods for weighted matching also apply to shortest paths in undirected graphs (provided that each circuit has nonnegative length) and to the Chinese postman problem — more generally, to *T-joins*.

29.1. T-joins

Let $G = (V, E)$ be a graph and let $T \subseteq V$. A subset J of E is called a *T-join* if T is equal to the set of vertices of odd degree in the graph (V, J). So if a T-join exists, then $|T|$ is even. More precisely,

(29.1) G has a T-join if and only if $|K \cap T|$ is even for each component K of G.

T-joins are close to matchings. In fact, from Corollary 26.1a it can be derived that a shortest T-join can be found in strongly polynomial time. To see this, one should observe the following elementary graph-theoretical fact representing T-joins as sets of paths:

(29.2) each T-join is the edge-disjoint union of circuits and $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices in T;
the symmetric difference of a set of circuits and $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices in T is a T-join.

This is used in showing that a shortest T-join can be found in strongly polynomial time:

Theorem 29.1. *Given a graph $G = (V, E)$, a length function $l \in \mathbb{Q}^E$, and a subset T of V, a shortest T-join can be found in strongly polynomial time.*

Proof. First we dispose of negative lengths. Let N be the set of edges e with $l(e) < 0$, let U be the set of vertices v with $\deg_N(v)$ odd, let $T' := T \triangle U$, and let $l' \in \mathbb{Q}_+^E$ be defined by $l'(e) := |l(e)|$ for $e \in E$.

Then, if J' is a T'-join minimizing $l'(J')$, the set $J := J' \triangle N$ is a T-join minimizing $l(J)$. To see this, let $\widetilde{J}$ be any T-join. Then $\widetilde{J} \triangle N$ is a T'-join, and hence $l'(J') \le l'(\widetilde{J} \triangle N)$. Therefore, (29.3) $$l(J) = l'(J') + l(N) \leq l'(\widetilde{J} \triangle N) + l(N) = l(\widetilde{J}).$$

So we can assume $l \geq \mathbf{0}$. Now consider the complete graph K_T with vertex set T. For each edge st of K_T, determine a path P_{st} in G of minimum length, say, $w(st)$. Find a perfect matching M in K_T minimizing $w(M)$. Then the symmetric difference of the paths P_{st} for $st \in M$ is a shortest T-join in G. This follows directly from (29.2). ∎

(This method is due to Edmonds [1965e].)

Note that a T-join J has minimum length if and only if $l(C \cap J) \leq \frac{1}{2} l(C)$ for each circuit C. (This was observed essentially by Guan [1960].)

Theorem 29.1 implies that also a *longest* T-join can be found in strongly polynomial time:

Corollary 29.1a. *Given a graph $G = (V, E)$, a length function $l \in \mathbb{Q}^E$, and a subset T of V, a longest T-join can be found in strongly polynomial time.*

Proof. Apply Theorem 29.1 to $-l$. ∎

An application is finding a maximum-capacity cut in a planar graph $G = (V, E)$ (Orlova and Dorfman [1972][16], Hadlock [1975]): it amounts to finding a maximum-capacity $\emptyset$-join in the planar dual graph. (Barahona [1990] gave an $O(n^{3/2} \log n)$ time bound.)

Another consequence is:

Corollary 29.1b. *Given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{Q}$, one can check if there is a negative-length circuit in strongly polynomial time.*

Proof. There is a negative-length circuit if and only if there exists an $\emptyset$-join J with $l(J) < 0$. So Theorem 29.1 gives the corollary. ∎

Complexity. With Dijkstra's shortest path method (Theorem 7.3) one derives from Theorem 26.2 that a shortest T-join can be found in $O(n^3)$ time. Generally, one has an $O(\text{APSP}_+(n, m, L) + \text{WM}(n, n^2, nL))$-time algorithm, where L is the maximum absolute value of the lengths of the edges in G (assuming they are integer), where $\text{APSP}_+(n, m, L)$ is the time in which the all-pairs shortest paths problem can be solved, in an undirected graph, with n vertices and m edges and with nonnegative integer lengths at most L, and where $\text{WM}(n, m, W)$ is the time in which a minimum-weight perfect matching can be found, in a graph with n vertices and m edges and with integer weights at most W in absolute value.

[16] Orlova and Dorfman observed that finding a maximum-size cut in a planar graph amounts to finding shortest paths connecting the odd-degree vertices in the dual graph, but described a branch-and-bound method for it, and did not state that it can be solved in polynomial time by matching techniques.

29.2. The shortest path problem for undirected graphs

In Chapter 8 we saw that a shortest path in a directed graph without negative-length directed circuits, can be found in strongly polynomial time. It implies a strongly polynomial-time shortest path algorithm for undirected graphs, provided that all lengths are nonnegative. This, because the reduction replaces each undirected edge uv by two directed edges (u, v) and (v, u) — which would create a negative-length directed circuit if uv has negative length.

However, Theorem 29.1 implies that one can find (in strongly polynomial-time) a shortest path in undirected graphs even if there are negative-length edges, provided that all circuits have nonnegative length:

Corollary 29.1c. *Given a graph $G = (V, E)$, $s, t \in V$, and a length function $l \in \mathbb{Q}^E$ such that each circuit has nonnegative length, a shortest $s - t$ path can be found in strongly polynomial time.*

Proof. Define $T := \{s, t\}$ and apply Theorem 29.1. By observation (29.2), a shortest T-join J can be partitioned into an $s - t$ path and a number of circuits. Since by assumption any circuit has nonnegative length, we can delete the circuits from J. ∎

Complexity. Since by Gabow [1990] the weighted matching problem is solvable in $O(n(m + n \log n))$ time, a shortest path in an undirected graph, without negative-length circuits, can be found in $O(n(m + n \log n))$ time. This can be derived as follows: If we want to find a shortest $s - t$ path, add to each vertex v a 'copy' v', for each edge uv add edges uv', $u'v$, and $u'v'$ (each with the same length as uv), and for each vertex v add an edge vv', of length 0. Let G' be the graph obtained. Then a minimum-weight perfect matching in $G' - s' - t'$ gives a shortest $s - t$ path in G.

Gabow [1983a] gave an $O(n \min\{m \log n, n^2\})$-time algorithm for the all-pairs shortest paths problem in undirected graphs.

29.3. The Chinese postman problem

Call a walk $C = (v_0, e_1, v_1, \ldots, e_t, v_t)$ in a graph G a *Chinese postman tour* if $v_t = v_0$ and each edge of G occurs at least once in C. The *Chinese postman problem*, first studied by Guan [1960] (and named by Edmonds [1965e]), is:

(29.4) given: a connected graph $G = (V, E)$ and a length function $l \in \mathbb{Q}^E_+$,
find: a shortest Chinese postman tour C.

By Euler's theorem, if each vertex has even degree, there is an Eulerian tour, that is, a walk traversing each edge *exactly* once. So in that case, any Eulerian tour is a shortest Chinese postman tour.

But if not all degrees are even, certain edges have to be traversed more than once. These edges form in fact a shortest T-join for $T := \{v \mid \deg_G(v)$ odd$\}$. This is the base of the following consequence of Theorem 29.1:

Corollary 29.1d. *The Chinese postman problem can be solved in strongly polynomial time.*

Proof. Let $T := \{v \mid \deg_G(v) \text{ odd}\}$. Find a shortest T-join J. Add to each edge e in J an edge e' parallel to e. This gives the Eulerian graph G'. Then any Eulerian tour in G' gives a shortest Chinese postman tour (by identifying any new edge e' with its parallel e).

To see this, note that obviously the Eulerian tour gives a Chinese postman tour C of length $l(E) + l(J)$. Suppose that there is a shorter tour C'. Let J' be the set of edges traversed an even number of times by C'. Then J' is a T-join, and so $l(J') \geq l(J)$. Hence $l(C') \geq l(E) + l(J') \geq l(E) + l(J) = l(C)$. ∎

Observe that a postman never has to traverse any street more than twice.

Complexity. The above gives an $O(n^3)$-time algorithm for the Chinese postman problem (more precisely, $O(k(m + n \log n) + k^3 + m)$, where k is the number of vertices of odd degree).

29.4. T-joins and T-cuts

There is an interesting min-max relation for the minimum size of T-joins. To this end, call, for any graph $G = (V, E)$ and any $T \subseteq V$, a subset C of E a *T-cut* if $C = \delta(U)$ for some $U \subseteq V$ with $|U \cap T|$ odd.

Trivially, each T-cut intersects each T-join. Moreover, each edge set C intersecting each T-join contains a T-cut (since otherwise each component of $(V, E \setminus C)$ has an even number of vertices in T, and hence there is a T-join disjoint from C). So the inclusionwise minimal T-cuts are exactly the inclusionwise minimal edge sets intersecting all T-joins. Hence the inclusionwise minimal T-joins are exactly the inclusionwise minimal edge sets intersecting all T-cuts.

Call a family $\mathcal{F}$ of cuts in $G = (V, E)$ *cross-free* if $\mathcal{F} = \{\delta(U) \mid U \in \mathcal{C}\}$ for some cross-free collection $\mathcal{C}$ of subsets of V; that is, a collection $\mathcal{C}$ with

(29.5) $\quad U \subseteq W$ or $W \subseteq U$ or $U \cap W = \emptyset$ or $U \cup W = V$

for all $U, W \in \mathcal{C}$.

The following min-max relation for minimum-size T-joins in bipartite graphs was proved by Seymour [1981d] — we give the simple proof due to Sebő [1987]:

Theorem 29.2. *Let $G = (V, E)$ be a bipartite graph and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. The maximum is attained by a cross-free family of cuts.*

Proof. We may assume $T \neq \emptyset$. Let J be a minimum-size T-join in G. Define a length function $l : E \to \{+1, -1\}$ by: $l(e) := -1$ if $e \in J$ and $l(e) := +1$ if $e \notin J$. Then every circuit C has nonnegative length, since $J \triangle C$ is again a T-join, and hence $|J \triangle C| \geq |J|$, implying $l(C) = |C \setminus J| - |C \cap J| \geq 0$.

Let P be a minimum-length walk in G traversing no edge more than once. Choose P such that it traverses a minimum number of edges. So P is a path (as we can delete any circuit occurring in P). Let t be an end vertex of P and let f be the last edge of P.

Then $f \in J$, since otherwise we could make the walk shorter by deleting f from P. Moreover, $\deg_J(t) = 1$, as if J has another edge, e say, incident with t, then extending P by e would make the walk shorter.

We next show:

(29.6) Each circuit C traversing t and not traversing f has positive length.

If C has only vertex t in common with P, let e be the first edge of C. So $l(e) = 1$. Consider the walk $P' := P \cup (C - e)$. Then $l(P') \geq l(P)$ and hence $l(C - e) \geq 0$. So $l(C) > 0$.

If C has another vertex in common with P, let u be the last vertex on P with $u \neq t$ and traversed by C. Let P' be the $u - t$ part of P. Split C into two $u - t$ paths C' and C''. By the minimality of $|P|$, $l(P') < 0$. Hence, as $C' \cup P'$ and $C'' \cup P'$ are circuits, $l(C') > 0$ and $l(C'') > 0$. This implies $l(C) > 0$.

Now shrink $\{t\} \cup N(t)$ to one new vertex v_0, giving the graph G'. If $|T \cap (\{t\} \cup N(t))|$ is odd, let $T' := (T \setminus (\{t\} \cup N(t))) \cup \{v_0\}$, and otherwise let $T' := T \setminus (\{t\} \cup N(t))$. Let $J' := J \setminus \{f\}$.

Then J' is a minimum-size T'-join in G'. For suppose to the contrary that G' contains a circuit C' with $|C' \setminus J'| < |C' \cap J'|$. If C' comes from a circuit C in G not traversing t, we would have $|C \setminus J| < |C \cap J|$, a contradiction. So C' comes from a circuit C in G traversing t.

If C traverses f, then $|C' \setminus J'| - |C' \cap J'| = |C \setminus J| - |C \cap J| \geq 0$, a contradiction. If C does not traverse f, then, by (29.6), $l(C) > 0$, and hence $l(C) \geq 2$. So $|C' \setminus J'| = |C \setminus J| - 2 \geq |C \cap J| = |C' \cap J'|$, again a contradiction.

Now by induction (on $|V| + |T|$), G' has disjoint cross-free T'-cuts $D_1, \ldots, D_{|J'|}$. With the T-cut $\delta(t)$ this gives $|J|$ disjoint cross-free T-cuts in G. ∎

(For another, algorithmic proof, see Barahona [1990].)

This implies for not necessarily bipartite graphs (Lovász [1975a]):

Corollary 29.2a. *Let $G = (V, E)$ and let $T \subseteq V$ with $|T|$ even. Then the minimum size of a T-join is equal to half of the maximum number of T-cuts covering each edge at most twice. The maximum is attained by a cross-free family of cuts.*

Proof. Replace each edge of G by a path of length two, thus obtaining the bipartite graph G'. Applying Theorem 29.2 to G' gives the corollary. ∎

In general it is not true that the minimum size of a T-cut is equal to the maximum number of disjoint T-joins — see Section 29.11c.

Notes. Frank, Tardos, and Sebő [1984] sharpened Theorem 29.2 to the following. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum of

(29.7) $$\sum_{S \in \Pi} q_T(S),$$

where Π ranges over all partitions of U. Here $q_T(S)$ denotes the number of components K of $G - S$ with $|K \cap T|$ odd. If G is arbitrary one takes the maximum of $\frac{1}{2}\sum_{S\in\Pi} q_T(S)$ over all partitions Π of V. (For extensions, see Kostochka [1994].)

29.5. The up hull of the T-join polytope

The last corollary implies a polyhedral result due to Edmonds and Johnson [1973] (also stated by Seymour [1979b]). Let $G = (V, E)$ be a graph and let $T \subseteq V$. The *T-join polytope*, denoted by $P_{T\text{-join}}(G)$, is the convex hull of the incidence vectors of T-joins. So it is a polytope in $\mathbb{R}^E$.

We first consider the 'up hull' of $P_{T\text{-join}}(G)$, that is,

(29.8) $$P^{\uparrow}_{T\text{-join}}(G) := P_{T\text{-join}}(G) + \mathbb{R}^E_+,$$

which turns out to be determined by the system:

(29.9)
$$\begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } T\text{-cut } C. \end{array}$$

Corollary 29.2b. *The polyhedron $P^{\uparrow}_{T\text{-join}}(G)$ is determined by* (29.9)*.*

Proof. It is easy to see that $P^{\uparrow}_{T\text{-join}}(G)$ is contained in the polyhedron determined by (29.9). If the converse inclusion does not hold, there is a weight function $w \in \mathbb{Q}^E$ with $w > \mathbf{0}$ such that the minimum value of $w^{\mathsf{T}}x$ subject to (29.9) is less than the minimum weight α of any T-join. We may assume that each $w(e)$ is an even integer.

We make a new graph $G' = (V', E')$ as follows. Replace each edge $e = uv$ of G by a path from u to v of length $w(e)$. Then α is equal to the minimum size of a T-join in G'. Hence by Theorem 29.2, G' has α disjoint T-cuts. This gives a family of α T-cuts in G such that each edge e of G is in at most $w(e)$ of these T-cuts. Let y_C be the number of times that T-cut C occurs in this list. Then the y_C give a feasible dual solution to the problem of minimizing $w^{\mathsf{T}}x$ over (29.9), with value $\sum_C y_C = \alpha$. This contradicts our assumption that the minimum value of $w^{\mathsf{T}}x$ subject to (29.9) is less than α. ∎

(Gastou and Johnson [1986] gave a proof based on binary groups.)

By adding $x_e \leq 1$ for each $e \in E$ we obtain from (29.9) the system

$$
\text{(29.10)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } T\text{-cut } C. \end{array}
$$

Corollary 29.2c. *The convex hull of the incidence vectors of edge sets containing a T-join as a subset is determined by* (29.10).

Proof. Directly from Corollary 29.2b, with Theorem 5.19. ∎

These systems are totally dual half-integral:

Corollary 29.2d. *Systems* (29.9) *and* (29.10) *are totally dual half-integral.*

Proof. This follows from the proof of Corollary 29.2b, observing that the y_C are integer if each w_e is an even integer. ∎

Generally these systems are not TDI, as is shown by taking $G = K_4$ and $T = V$ — see Section 29.11b.

Barahona [2002] gave an $O(n^6)$-time algorithm to decompose a vector in the up hull of the T-join polytope as a convex combination of incidence vectors of T-joins, added with a nonnegative vector.

29.6. The T-join polytope

In the previous section we considered the up hull of the T-join polytope. We can derive from it an inequality system determining the T-join polytope itself. Consider the following system of linear inequalities for $x \in \mathbb{R}^E$:

$$
\text{(29.11)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V, F \subseteq \delta(U), \\ & & |U \cap T| + |F| \text{ odd}). \end{array}
$$

Corollary 29.2e. *The T-join polytope is determined by* (29.11).

Proof. First, the incidence vector x of any T-join J satisfies (29.11). Indeed, if $U \subseteq V$, then $|\delta(U) \cap J| \equiv |U \cap T| \pmod 2$. Hence if $F \subseteq \delta(U)$ with $|U \cap T| + |F|$ odd, then $|\delta(U) \cap J| + |F|$ is odd, and hence if $x(F) = |F|$ one has $x(\delta(U) \setminus F) \geq 1$. This shows (29.11). So the T-join polytope is contained in the polytope determined by (29.11).

To see the reverse inclusion, choose a weight function $w \in \mathbb{Q}^E$. We show that the minimum value of $w^{\mathsf{T}}x$ subject to (29.11) is equal to $w(J)$ for some T-join J.

Define

(29.12) $\quad N := \{e \mid w(e) < 0\}$ and $T' := T \triangle \{v \mid \deg_N(v) \text{ odd}\}$.

Let $w'(e) := |w(e)|$ for each $e \in E$. Let J' be a T'-join minimizing $w'(J')$. By Corollary 29.2c, there exist λ_U for $U \subseteq V$ with $|U \cap T'|$ odd, such that

(29.13) (i) $\lambda_U \geq 0$ for each U with $|U \cap T'|$ odd, with equality if $|J' \cap \delta(U)| > 1$,

(ii) $\displaystyle\sum_{\substack{U \\ e \in \delta(U)}} \lambda_U \leq w'(e)$ for each $e \in E$, with equality if $e \in J'$.

Define $\mu, \nu : E \to \mathbb{R}_+$ by the conditions that $\mu(e)\nu(e) = 0$ for each $e \in E$ and that

(29.14) $$\nu - \mu + \sum_U \lambda_U(\chi^{\delta(U)\setminus N} - \chi^{\delta(U)\cap N}) = w.$$

So the $\nu(e)$, $\mu(e)$, and λ_U give a feasible dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.11) (taking $F := \delta(U) \cap N$).

Let $J := J' \triangle N$. So J is a T-join. We show that J, $\mu(e)$, $\nu(e)$, λ_U satisfy the complementary slackness conditions, thus finishing our proof.

First we show that if $e \in J$, then $\nu(e) = 0$. Indeed, if $e \in J \setminus N$, then $e \in J'$, and hence

(29.15) $$\sum_{U, e \in \delta(U)\setminus N} \lambda_U - \sum_{U, e \in \delta(U)\cap N} \lambda_U$$

is equal to $w'(e) = w(e)$ by (29.13)(ii), and hence $\nu(e) = 0$. If $e \in J \cap N$, then (29.15) is at least $-w'(e) = w(e)$ by (29.13)(ii), and hence $\nu(e) = 0$.

Second we show that if $e \notin J$, then $\mu(e) = 0$. If $e \notin J \cup N$, then (29.15) is at most $w'(e) = w(e)$ by (29.13)(ii), implying $\mu(e) = 0$. If $e \in N \setminus J$, then $e \in J'$, and hence (29.15) is equal to $-w'(e) = w(e)$ by (29.13)(ii), implying $\mu(e) = 0$.

Finally if $\lambda_U > 0$, then (as $J = J' \triangle N$ and $|J' \cap \delta(U)| = 1$ by (29.13)(i))

(29.16) $$\begin{aligned} &|J \cap (\delta(U) \setminus N)| - |J \cap (\delta(U) \cap N)| \\ &= |(J' \setminus N) \cap \delta(U)| - |(N \setminus J') \cap \delta(U)| \\ &= |J' \cap \delta(U)| - |N \cap \delta(U)| = 1 - |\delta(U) \cap N|. \end{aligned}$$ ∎

In Section 29.11b we show that (29.11) is TDI if and only if G is series-parallel.

29.7. Sums of circuits

Given a graph $G = (V, E)$, the *circuit cone* is the cone in $\mathbb{R}^E$ generated by the incidence vectors of circuits. Seymour [1979b] showed that this cone is determined by:

$$\text{(29.17)} \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(D) \geq 2x_e & \text{for each cut } D \text{ and } e \in D. \end{array}$$

As J. Edmonds (cf. Seymour [1979b]) pointed out, this can be derived from (essentially) matching theory:

Corollary 29.2f. *The circuit cone is determined by* (29.17)*.*

Proof. Since the incidence vector x of any circuit satisfies (29.17), the circuit cone is contained in the cone determined by (29.17).

To see the converse inclusion, let x satisfy (29.17). To show that x belongs to the circuit cone, we may assume (by scaling) that $x(E) \leq 1$. It suffices to show that x belongs to the $\emptyset$-join polytope of G. Hence, by Corollary 29.2e, it suffices to show that $x(\delta(U)) - 2x(F) \geq 1 - |F|$ for each $U \subseteq V$ and $F \subseteq \delta(U)$ with $|F|$ odd. If $|F| = 1$, this follows from (29.17)(ii). If $|F| \geq 3$, then $x(\delta(U)) - 2x(F) \geq -x(E) \geq -1 \geq 1 - |F|$. ∎

(This proof is due to Aráoz, Cunningham, Edmonds, and Green-Krótki [1983]. Hoffman and Lee [1986] gave a 'different (but not shorter) proof'. Coullard and Pulleyblank [1989] gave a short elementary proof, together with decomposition results.)

Seymour [1979b] in fact characterized when a box has a nonempty intersection with the circuit cone:

Corollary 29.2g. *Let $G = (V, E)$ be a graph and let $l, u \in \mathbb{R}^E_+$ satisfying $l \leq u$. Then there exists an x in the circuit cone of G with $l \leq x \leq u$ if and only if*

$$\text{(29.18)} \qquad u(D \setminus \{e\}) \geq l(e) \textit{ for each cut } D \textit{ and each } e \in D.$$

Proof. Necessity being trivial, we show sufficiency. Choose a counterexample with $\sum_{e \in E}(u_e - l_e)$ minimum. Suppose that $u_e > l_e$ for some edge e. Then there exist a cut D and $e \in D$ with $u(D \setminus \{e\}) = l(e)$ and there exist a cut D' containing e, and $f \in D' \setminus \{e\}$ with $u(D' \setminus \{f\}) = l(f)$. Then $f \notin D$, since otherwise $e, f \in D \cap D'$, implying

$$\text{(29.19)} \qquad \begin{aligned} u(D \triangle D') &\leq u(D \setminus \{e, f\}) + u(D' \setminus \{e, f\}) \\ &= l(e) - u(f) + l(f) - u(e) < 0. \end{aligned}$$

Hence the cut $D \triangle D'$ satisfies

$$\text{(29.20)} \qquad \begin{aligned} u(D \triangle D' \setminus \{f\}) &\leq u(D \setminus \{e\}) + u(D' \setminus \{e, f\}) = l(e) - u(e) + l(f) \\ &< l(f), \end{aligned}$$

contradicting (29.18).

So $u_e = l_e$ for each edge e, and hence the corollary follows from Corollary 29.2f. ∎

Let $G = (V, E)$ be a graph. A function $l : E \to \mathbb{R}$ is called *conservative* if $l(C) \geq 0$ for each circuit C. The conservative functions form a polyhedral convex cone, and Corollary 29.2f gives functions that generate this cone:

Corollary 29.2h. *The cone of conservative functions is generated by the nonnegative functions and by the functions l for which there is a subset U of V and an edge $e \in \delta(U)$ such that*

(29.21) $$l = \chi^{\delta(U)\setminus\{e\}} - \chi^e.$$

Proof. Directly by polarity (cf. Section 5.7) from Corollary 29.2f. ∎

In Section 29.11b we show that system (29.17) is TDI if and only if G is series-parallel.

29.8. Integer sums of circuits

Seymour [1979b] gave the following characterization of integer sums of circuits in planar graphs. It is equivalent to saying that the incidence vectors of circuits in a planar graph form a Hilbert base. (We follow a proof suggested by A.V. Karzanov, which starts like Seymour's proof but does not use the four-colour theorem.)

Theorem 29.3. *Let $G = (V, E)$ be a planar graph and let $x \in \mathbb{R}^E$. Then x is a nonnegative integer combination of incidence vectors of circuits if and only if x is an integer vector in the circuit cone with $x(\delta(v))$ even for each vertex v.*

Proof. Necessity being easy, we show sufficiency. Consider a counterexample with

(29.22) $$|V| + \sum_{e\in E}(x(e) + 1)^2$$

minimal. Then G is connected (otherwise one of the components forms a counterexample with (29.22) smaller), $x(e) \geq 1$ for each $e \in E$ (otherwise we can delete e), and each vertex v has degree at least 3 (the degree is at least 2 by (29.17)(ii); if it is precisely 2, then x has the same value on the two edges incident with v (by (29.17)(ii)), and hence we can replace them by one edge).

Consider any edge e_0 with $x(e_0) \geq 2$ and $x(e_0)$ minimal. Let e_0 connect vertices p and q say. Let G' be the graph obtained from G by adding a new (parallel) edge f between p and q. Define $x'(e_0) := x(e_0) - 1$, $x'(f) := 1$, and

$x'(e) := x(e)$ for all other edges e of G'. Then condition (29.17) is maintained, but sum (29.22) decreases. So x' is a sum of circuits[17] in G'. If none of these circuits consist of e_0 and f, then x is a sum of circuits in G, a contradiction. So $\{e_0, f\}$ is one of the circuits. Therefore, in G, the vector

(29.23) $\qquad y := x - 2\chi^{e_0}$

is a sum of circuits, say

(29.24) $\qquad y = \sum_{C \in \mathcal{C}} \lambda_C \chi^{EC},$

where $\mathcal{C}$ is a collection of circuits and where the λ_C are positive integers. Let $\mathcal{C}_0$ be the collection of circuits in $\mathcal{C}$ traversing e_0, and let $\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0$.

We construct a directed graph $D = (V, A)$. We say that a circuit C *generates* a pair (u, v) of distinct vertices if C traverses both u and v, in such a way that if C traverses e_0, then C traverses p, q, u, v cyclically in this order (possibly $u = q$ or $v = p$). The arc set A of D consists of all pairs (u, v) generated by at least one $C \in \mathcal{C}$. Then:

(29.25) $\qquad D$ contains a directed path from p to q.

For suppose not. Let U be the set of vertices reachable in D from p. So $q \notin U$, and no arc of D leaves U. Hence no $C \in \mathcal{C}_1$ intersects $\delta_E(U)$, and each $C \in \mathcal{C}_0$ intersects $\delta_E(U)$ precisely twice: once in e_0 and once elsewhere. So

(29.26) $\qquad x(\delta_E(U) \setminus \{e_0\}) = y(\delta_E(U) \setminus \{e_0\}) = y(e_0) < x(e_0),$

contradicting (29.17). This shows (29.25).

Now choose a shortest directed $p-q$ path P in D, say $P = (v_0, v_1, \ldots, v_k)$, with $v_0 = p$ and $v_k = q$. Let $\mathcal{C}'$ be an inclusionwise minimal subcollection of $\mathcal{C}$ with the property that each arc of P is generated by some C in $\mathcal{C}'$. Define $\mathcal{C}'_0 := \mathcal{C}' \cap \mathcal{C}_0$, and

(29.27) $\qquad z := 2\chi^{e_0} + \sum_{C \in \mathcal{C}'} \chi^{EC}.$

We show:

(29.28) $\qquad z = x$, $\mathcal{C}' = \mathcal{C}$, and $\lambda_C = 1$ for each $C \in \mathcal{C}$.

It suffices to show that $z = x$. Suppose $z \neq x$. Then, since $z \leq x$, by the minimality of (29.22), z is a sum of circuits. To see this, it suffices to show that (29.17) is satisfied by z. To this end, let $U \subseteq V$ and $e \in \delta(U)$. If $e \neq e_0$, then (29.17)(ii) follows since $z - 2\chi^{e_0}$ is a sum of circuits. If $e = e_0$, then we can assume that $p \in U$, $q \notin U$. Hence some arc (v_{i-1}, v_i) leaves U. Let $C' \in \mathcal{C}'$ generate (v_{i-1}, v_i). Then C' has at least two edges in $\delta(U)$, and at least four if $C' \in \mathcal{C}'_0$. Moreover, any $C \in \mathcal{C}'_0$ has at least two edges in $\delta(U)$. Hence

[17] By a 'sum of circuits' we mean a sum of incidence vectors of circuits.

(29.29) $$z(\delta(U) \setminus \{e_0\}) = \sum_{C \in \mathcal{C}'} |EC \cap \delta(U) \setminus \{e_0\}| \geq |\mathcal{C}'_0| + 2 = z(e_0).$$

So (29.17) is satisfied by z. Hence z is a sum of circuits. But then also x is a sum of circuits, since

(29.30) $$\begin{aligned} x - z = y - \sum_{C \in \mathcal{C}'} \chi^{EC} &= \sum_{C \in \mathcal{C}} \lambda_C \chi^{EC} - \sum_{C \in \mathcal{C}'} \chi^{EC} \\ &= \sum_{C \in \mathcal{C} \setminus \mathcal{C}'} \lambda_C \chi^{EC} + \sum_{C \in \mathcal{C}'} (\lambda_C - 1) \chi^{EC}. \end{aligned}$$

This contradicts our assumption, proving (29.28).

Then:

(29.31) each vertex v is traversed by at most two circuits in $\mathcal{C}_1$.

Otherwise, there exist three arcs on P generated by circuits in $\mathcal{C}_1$ traversing v. Hence there exist arcs (v_{i-1}, v_i) and (v_{j-1}, v_j) on P generated by circuits C and C' in $\mathcal{C}_1$ traversing v, with $i < j - 1$. This implies that we can make P shorter (by replacing $v_i, v_{i+1}, \ldots, v_{j-1}$ by v), a contradiction. This shows (29.31).

Consider now any vertex $v \neq p, q$ and any $f \in \delta(v)$ with $x(f) \geq 2$. By the choice of e_0 we know $x(f) \geq x(e_0)$. Hence, using (29.31),

(29.32) $$2x(f) \leq x(\delta(v)) \leq 2|\mathcal{C}_0| + 4 = 2(y(e_0) + 2) = 2x(e_0) \leq 2x(f).$$

So we have equality throughout. In particular, v is traversed by precisely two circuits in $\mathcal{C}_1$, and $x(f) = x(e_0)$.

It follows that, for any $i = 1, \ldots, k-1$, the arcs (v_{i-1}, v_i) and (v_i, v_{i+1}) are generated by circuits in $\mathcal{C}_1$ (by taking $v = v_i$). Trivially, if $k = 1$, the arc (v_0, v_1) is not generated by any circuit in $\mathcal{C}_0$, and hence by some circuit in $\mathcal{C}_1$. Therefore, by the minimality of $\mathcal{C}$, $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_1 = \mathcal{C}$. Hence $y(e_0) = 0$, and so $x(e_0) = 2$. Therefore, $x(e) \in \{1, 2\}$ for each edge e.

Since each vertex $v \neq p, q$ is traversed by precisely two circuits in $\mathcal{C}$, we know that v is incident with at most one edge e with $x(e) = 2$. Since any e with $x(e) = 2$ can play the role of e_0, this also holds for $v \in \{p, q\}$. So

(29.33) the edges e with $x(e) = 2$ form a matching M in G.

Consider the path P above. Let arc (v_{i-1}, v_i) be generated by circuit $C_i \in \mathcal{C}$, for $i = 1, \ldots, k$. By the minimality of k, C_i and C_j are vertex-disjoint if $j > i + 1$. Let D_1 be the union of the EC_i for odd i, and let D_2 be the union of the EC_i for even i. So (for each $i = 1, 2$) D_i consists of vertex-disjoint circuits, and $D_1 \cap D_2 = M \setminus \{e_0\}$.

This is used in proving:

(29.34) each nonempty cut D contained in M is odd.

Indeed, by symmetry we may assume that $e_0 \in D$. Then $D \setminus \{e_0\} = D \cap D_1$ (since $D \setminus \{e_0\} \subseteq M \setminus \{e_0\} \subseteq D_1$ and since $e_0 \notin D_1$). Moreover, $|D \cap D_1|$ is even, since D_1 is a disjoint union of circuits.

This proves (29.34), which implies that

(29.35) $G - M$ has at most two components,

since if K and L are components with $K \cup L \neq V$, then at least one of $\delta_E(K)$, $\delta_E(L)$, and $\delta_E(K \cup L)$ is nonempty and even, contradicting (29.34).

Moreover:

(29.36) M forms a cut in G.

Otherwise, M has an edge spanned by a component of $G-M$. Hence G has a circuit C with $|C \cap M| = 1$ By symmetry, we may assume that $C \cap M = \{e_0\}$. Then $C \triangle D_1$ and $C \triangle D_2$ form cycles whose incidence vectors add up to x. Hence x is a sum of circuits, a contradiction. So we have (29.36).

Now let K_1 and K_2 be the components of $G - M$. They are connected Eulerian graphs. Since M forms a cut, we can assume that the attachments of M at K_1 and at K_2 are at the outer boundaries B_1 of K_1 and B_2 of K_2. By the planarity of G, the attachments of M occur in the same order on B_1 as on B_2. So $\chi^{EB_1} + \chi^{EB_2} + 2\chi^M$ is a sum of circuits. Since $EK_1 \setminus EB_1$ and $EK_2 \setminus EB_2$ are cycles, this gives x as a sum of circuits. ∎

(The proof of Seymour [1979b] of Theorem 29.3 uses the four-colour theorem. Fleischner and Frank [1990] showed that a method of Fleischner [1980] gives a proof not using the four-colour theorem. Also Alspach and Zhang [1993] gave a proof not using the four-colour theorem.)

In Theorem 29.3 we cannot delete the planarity condition, as is shown by the Petersen graph: fix a perfect matching M, and set $x_e := 2$ if $e \in M$ and $x_e := 1$ if $e \notin M$. Alspach, Goddyn, and Zhang [1994] (extending Alspach and Zhang [1993]) proved that the Petersen graph is the critical example:

Theorem 29.4. *For any graph $G = (V, E)$, the following are equivalent:*

(29.37) (i) *each integer vector x in the circuit cone with $x(\delta(v))$ even for each vertex v is a nonnegative integer combination of incidence vectors of circuits;*
(ii) *G has no Petersen graph minor.*

(This was generalized to binary matroids by Fu and Goddyn [1999] — see Section 81.9.)

Seymour [1979b] conjectures that each *even* integer vector x in the circuit cone is a nonnegative integer combination of incidence vectors of circuits. A special case of this is the *circuit double cover conjecture* (it was asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them. Thus Theorem 29.4 implies that the circuit double cover conjecture is true for graphs without Petersen graph minor.

It has been proved that for any even integer $k \geq 4$, each bridgeless graph has circuits such that each edge is covered by precisely k of them. (For $k = 6$ by Jaeger [1979] and for $k = 4$ by Fan [1992] — hence any even $k \geq 4$ follows.)

This relates to the 4-*flow conjecture* of Tutte [1966], which generalizes the four-colour theorem:

(29.38) (?) The edges of any bridgeless graph without Petersen graph minor can be covered by two Eulerian subgraphs. (?)

(It is called the 4-flow conjecture, since it is equivalent to saying that for each bridgeless graph $G = (V, E)$ without Petersen graph minor, there is an orientation $D = (V, A)$ of G and a function $f : A \to \{1, 2, 3\}$ with $f(\delta^{\text{in}}(v)) = f(\delta^{\text{out}}(v))$ for each $v \in V$ — see Section 28.4.)

Conjecture (29.38) was proved for 4-edge-connected graphs by Jaeger [1979], and for cubic graphs jointly by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

(29.38) is equivalent to:

(29.39) (?) Any bridgeless graph without Petersen graph minor has two disjoint T-joins, where T is the set of vertices of odd degree (?)

(since J is a T-join if and only if $E \setminus T$ yields an Eulerian graph).

It is NP-complete to decide if a graph has two disjoint T-joins, since for cubic graphs it is equivalent to 3-edge-colourability (cf. Theorem 28.3).

Related work can be found in Zhang [1993c]. Surveys on the circuit double cover conjecture were given by Jaeger [1985], Jackson [1993], and Zhang [1993a,1993b,1997b], and on integer decomposition of the circuit cone (and more general decompositions) by Goddyn [1993].

29.9. The T-cut polytope

The T-*cut polytope* $P_{T\text{-cut}}(G)$ — the convex hull of the incidence vectors of T-cuts — is a 'hard' polytope, even if $|T| = 2$, since finding a maximum cut separating two given vertices in a graph is NP-complete. However, the up hull of the T-cut polytope:

(29.40) $$P^{\uparrow}_{T\text{-cut}}(G) := P_{T\text{-cut}}(G) + \mathbb{R}^E_+$$

is tractable, as follows directly with the theory of blocking polyhedra from the results above on the up hull of the T-join polytope, and is determined by:

(29.41)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(J) \geq 1$ for each T-join J.

Theorem 29.5. *The up hull $P^{\uparrow}_{T\text{-cut}}(G)$ of the T-cut polytope of G is determined by* (29.41).

Proof. Directly with the theory of blocking polyhedra from Corollary 29.2b. ∎

This implies that the following system:

(29.42) (i) $0 \le x_e \le 1$ for each $e \in E$,
(ii) $x(J) \ge 1$ for each T-join J,

describes a convex hull as follows.

Corollary 29.5a. *The convex hull of the incidence vectors of edge sets containing a T-cut is determined by* (29.42).

Proof. Directly from Theorem 29.5 with Theorem 5.19. ∎

(For a direct derivation from Edmonds' perfect matching polytope theorem, see Seymour [1979a].)

In general, (29.41) is not TDI, not even totally dual half-integral (Seymour [1979a]). Seymour [1977b] characterized pairs of G, T for which (29.41) is TDI — see Section 29.11c.

Rizzi [1997] showed that the minimal TDI-system for the up hull of the T-cut polytope can have arbitrarily large coefficients and right-hand sides.

29.10. Finding a minimum-capacity T-cut

Like in Section 25.5c we can find a minimum-capacity T-cut by constructing a Gomory-Hu tree (for a graph $G = (V, E)$ and a tree $H = (V, F)$, a *fundamental cut* is a cut $\delta_E(W_f)$, where $f \in F$ and W_f is a component of $H - f$):

Theorem 29.6. *Let $G = (V, E)$ be a graph and let $T \subseteq V$ with $|T|$ even. Let $c \in \mathbb{R}^E_+$ be a capacity function and let $H = (V, F)$ be a Gomory-Hu tree. Then one of the fundamental cuts of H is a minimum-capacity T-cut in G.*

Proof. For each $f \in F$, choose W_f to be one of the two components of $H - f$. Let $\delta_G(U)$ be a minimum-capacity T-cut of G. So $|U \cap T|$ is odd.

Then U or $V \setminus U$ is equal to the symmetric difference of the W_f over $f \in \delta_F(U)$. Hence $|W_f \cap T|$ is odd for at least one $f \in \delta_F(U)$. So $\delta_G(W_f)$ is a T-cut.

Let $f = uv$. As $\delta_G(W_f)$ is a minimum-capacity $u-v$ cut and as $\delta_G(U)$ is a $u-v$ cut, we have $c(\delta_G(W_f)) \le c(\delta_G(U))$. So $\delta_G(W_f)$ is a minimum-capacity T-cut. ∎

This gives algorithmically (Padberg and Rao [1982]):

Corollary 29.6a. *A minimum-capacity T-cut can be found in strongly polynomial time.*

Proof. This follows from Theorem 29.6, since a Gomory-Hu tree can be found in strongly polynomial time, by Corollary 15.15a. ∎

Barahona and Conforti [1987] showed that a cut $\delta(U)$ with $T \cap U$ and $T \setminus U$ even and nonempty, and of minimum capacity, can be found in strongly polynomial time.

Barahona [2002] gave a combinatorial strongly polynomial-time algorithm to solve the dual of maximizing $c^{\mathsf{T}}x$ over (29.41) (yielding a fractional packing of T-joins).

29.11. Further results and notes

29.11a. Minimum-mean length circuit

Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Q}^E$ be a length function. The *mean length* of a circuit C is equal to $l(C)/|C|$. Barahona [1993b] showed (using an argument of Cunningham [1985c]) that a minimum-mean length circuit in an undirected graph can be found in strongly polynomial time, by solving at most m Chinese postman problems.

Theorem 29.7. *A minimum-mean length circuit in an undirected graph can be found in strongly polynomial time.*

Proof. Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Q}^E$ be a length function. Note that by adding a constant γ to all edge-lengths, the collection of minimum-mean length circuits does not change (as the mean length of any circuit increases by exactly γ). So we can assume that there exists a circuit C with $l(C) < 0$.

The algorithm is as follows:

(29.43) Find a minimum-length $\emptyset$-join J.
If $l(J) = 0$, output a circuit of length 0, and stop.
If $l(J) < 0$, add $\gamma := -l(J)/|J|$ to all edge-lengths, and iterate.

We first show that the algorithm stops; in fact, in at most $|E| + 1$ iterations. To this end, consider two subsequent iterations. Let l and l' be two subsequent length functions and let J and J' be the shortest $\emptyset$-joins found. So $l'(e) = l(e) - l(J)/|J|$ for all $e \in E$. If $l'(J') < 0$, then $|J'| < |J|$, since

$$(29.44) \qquad 0 > l'(J') = l(J') - \frac{l(J)}{|J|}|J'| \geq l(J) - \frac{l(J)}{|J|}|J'| = l(J)(1 - \frac{|J'|}{|J|})$$

(note that $l(J) < 0$ and $l(J') \geq l(J)$). This shows that the algorithm stops after at most $|E| + 1$ iterations.

As throughout the iterations, the collection of minimum-mean length circuits is invariant, a minimum-mean length circuit for the final length function, is also a minimum-mean length circuit for the initial length function. Hence the output is correct.

Finally, for the 0-length circuit C in the final iteration we can take any circuit contained in the $\emptyset$-join J found in the one but last iteration (as J has length 0 in the last iteration). ∎

Barahona [1993b] also showed that, conversely, the minimum-length T-join problem can be solved by solving $O(m^2 \log n)$ minimum-mean length circuit problems, as follows. Let $l \in \mathbb{Q}^E$ be a length function. Start with any T-join J. Find a minimum-mean length circuit C for the length function l' given by: $l'(e) := -l(e)$ if $e \in T$ and $l'(e) := l(e)$ otherwise. If $l'(C) \geq 0$, then J is a T-join minimizing $l(J)$. Otherwise, reset $T := T \triangle C$, and iterate.

(We note here that Guan [1960] proposed to find a circuit C minimizing $l'(C)$ and iteratively reset T as above, until $l'(C) \geq 0$. It is however NP-complete to find such a circuit, and moreover, no polynomial upper bound on the number of iterations is known.)

Barahona [1993b] also observed that the minimum-mean length circuit problem can be solved by solving a 'compact' linear programming problem (that is, one in which the number of variables and constraints is bounded by a polynomial in the size of the graph).

This follows from the fact that, for any graph $G = (V, E)$, the convex hull of

(29.45) $$\{\frac{1}{|C|}\chi^C \mid C \text{ circuit}\}$$

(where χ^C is the incidence vector of C in $\mathbb{R}^E$) consists of all vectors x in the circuit cone of G satisfying $\mathbf{1}^\mathsf{T} x = 1$; moreover, by Corollary 29.2f, x belongs to the circuit cone of G if and only if for each edge $e = st$ there exists an $s-t$ flow $y \leq x$ in $G-e$ of value x_e. Here the flow is described on the directed graph obtained from $G - e$ by replacing each edge uv by two arcs (u, v) and (v, u). As the flows are determined by flow conservation constraints (next to the negativity and capacity constraints), this yields a compact linear program.

A minimum mean-weight circuit therefore can be found in polynomial time with any polynomial-time LP-algorithm.

29.11b. Packing T-cuts

System (29.9) generally is not TDI, as is shown by taking $G = K_4$ and $T = VK_4$. This example is the critical example, since Seymour [1977b] showed that if system (29.9) is not TDI, then G, T contains K_4, VK_4 as a 'minor' — see Corollary 29.9b below. To prove this, we follow the approach of Frank and Szigeti [1994] using the results of Sebő [1988b].

Each polyhedron is determined by a TDI-system, albeit not necessarily the minimal system defining the polyhedron. Sebő [1988b] showed that system (29.9) can be extended as follows to a TDI-system defining the up hull of the T-join polytope.

Let $G = (V, E)$ be a graph and let T be an even-size subset of V. Call a set B of edges a *T-border* if there exists a partition $\mathcal{P} = (U_1, \ldots, U_k)$ of V such that

$|U_i \cap T|$ is odd for each i and such that B is equal to the set of edges connecting distinct classes of $\mathcal{P}$. The *value* $\mathrm{val}(B)$ of the T-border B is, by definition, half of the number of components K of $G - B$ with $|K \cap T|$ odd. (This is at least $\frac{1}{2}k$.) So a T-border is a T-cut if and only if $\mathrm{val}(B) = 1$. Moreover, each T-join intersects any T-border B in at least $\mathrm{val}(B)$ edges. Hence the minimum size of any T-join is at least the maximum total value of any packing of T-borders. (The *total value* of a collection of T-borders is the sum of the values of the T-borders in the collection.) Sebő [1988b] showed that the minimum and maximum are equal:

Theorem 29.8. *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum total value of a packing of T-borders.*

Proof. Choose a counterexample with $|V|$ as small as possible. Then G is connected.

By Corollary 29.2a, it suffices to show that the maximum total value of a packing of T-borders is at least half of the maximum size of a 2-packing of T-cuts[18] Choose a maximum-size 2-packing of T-cuts $\delta(U_1), \ldots, \delta(U_k)$, which by Corollary 29.2a we may assume to be cross-free. We must find a packing of T-borders of total value $\frac{1}{2}k$.

We choose the U_i such that

(29.46) $$\sum_{i=1}^{k} |U_i|$$

is as small as possible. In particular, $|U_i| \leq |V \setminus U_i|$ for each i.

For each such 2-packing we have

(29.47) $\delta(U_i) \neq \delta(U_j)$ if $i \neq j$,

since otherwise we can contract the edges in $\delta(U_i)$ to obtain G', T' and apply induction. We obtain a packing of T'-borders in G', of total value $\frac{1}{2}(k-2)$. Together with the T-border $B := \delta(U_i)$ this gives a packing of T-borders in G of total value $\frac{1}{2}k$. This shows (29.47).

We next show

(29.48) $|U_i| = 1$ for each i.

Suppose not. Choose an inclusionwise minimal set U_i with $|U_i| > 1$. So for any j, if $U_j \subset U_i$, then $U_j = \{t\}$ for some $t \in T \cap U_i$. Moreover, for each $t \in T \cap U_i$, there is a j with $U_j = \{t\}$, since otherwise we could reset $U_i := \{t\}$, contradicting the minimality of the sum (29.46). Then $U_i \subseteq T$, since otherwise we can replace U_i by $T \cap U_i$, again contradicting the minimality of the sum (29.46). It follows that the union of the $\delta(t)$ for $t \in U_i$ forms a T-border B of value $\frac{1}{2}(|U_i| + 1)$. Contracting the edges in B gives G', T' say. Applying induction to G', T' (in which there exists a 2-packing of T'-cuts of size $k - (|U_i| + 1)$), gives a packing of T'-borders in G' of total value $\frac{1}{2}(k - |U_i| - 1)$. Adding B, gives a packing of T-borders in G of total value $\frac{1}{2}k$.

So we can assume that $|U_i| = 1$ for each i. Then the union of the $\delta(U_i)$ for $i = 2, \ldots, k$ forms a T-border of value $\frac{1}{2}k$. ∎

This theorem bears upon the system

[18] A 2-*packing* is a family of sets such that no element is in more than two of them.

(29.49) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(B) \geq \text{val}(B)$ for each T-border B.

Since each inequality (29.9)(ii) occurs among (29.49), and since, conversely, each inequality (29.49)(ii) is a half-integer sum of inequalities (29.9)(ii), the two systems (29.9) and (29.49) define the same polyhedron — namely $P^{\uparrow}_{T\text{-join}}(G)$. In fact:

Corollary 29.8a. *System* (29.49) *is TDI.*

Proof. For any weight function $w \in \mathbb{Z}_+^E$ we can replace any edge $e = uv$ by a $u-v$ path of length $w(e)$, contracting e if $w(e) = 0$. Applying Theorem 29.8 to the new graph gives an integer optimum dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.49). ∎

We next use Theorem 29.8 to show that system (29.9) is TDI if G, T contains no K_4, VK_4 as a 'minor'. We follow the line of proof given by Frank and Szigeti [1994]. We first prove the following.

Call a graph $G = (V, E)$ *bicritical* if $G - u - v$ has a perfect matching for each pair of distinct vertices u and v. Call a graph $G = (V, E)$ *oddly contractible* to K_4 if V can be partitioned into four odd sets V_1, V_2, V_3, V_4 such that $G[V_i \cup V_j]$ is connected for all i, j (also if $i = j$). The following result is due to A. Sebő (cf. Frank and Szigeti [1994]):

Theorem 29.9. *A bicritical graph with at least four vertices is oddly contractible to K_4.*

Proof. Let $G = (V, E)$ be a bicritical graph with $|V| \geq 4$. This immediately implies that G is connected and has a perfect matching M. Moreover,

(29.50) for all $u, v \in V$ with $u \neq v$ there is an odd-length M-alternating $u-v$ path $P_{u,v}$ with first and last edge not in M.

To see this, first assume that $uv \in M$. Then there is a perfect matching N not containing uv (since there exists an edge uw with $w \neq v$ (by the connectedness of G), and hence the perfect matching of $G = \{u, w\}$ together with uw forms a perfect matching). Let C be the circuit in $M \cup N$ containing uv. Then $C - uv$ is a path as required in (29.50).

If $uv \notin M$, let u' and v' be such that $uu' \in M$ and $vv' \in M$. Let N be a perfect matching in $G - u' - v'$. Then $(M \cup N) \setminus \{uu', vv'\}$ contains a $u-v$ path as required. This shows (29.50).

Now (29.50) implies:

(29.51) there exists an odd-length M-alternating circuit $C = (v_0, v_1, \ldots, v_t)$.

(So t is odd, and $v_i v_{i+1} \in M$ if and only if i is odd.) To see (29.51), choose edges $uv \in M$ and $vw \notin M$. Then $P_{u,w}$ does not traverse v (otherwise uv is on $P_{u,w}$). So $C := EP_{u,w} \cup \{uv, vw\}$ is a circuit as required in (29.51).

Let w be such that $wv_0 \in M$. Let K be the component of $G - VC$ containing w. So $N(K) \subseteq VC$. We first show that $|N(K)| \geq 3$. Indeed, first we have $v_0 \in N(K)$. Let s be the first vertex in P_{w,v_1} contained in VC. Then $s \neq v_0$, since otherwise $v_0 w \in EP_{w,v_1}$. Let s' be such that $ss' \in M$. So $s' \neq v_0$. Let r be the first vertex in

$P_{w,s'}$ contained in VC. Again $r \neq v_0$. Moreover, $r \neq s$, since otherwise $ss' \in EP_{w,s'}$ (implying that the last edge of $P_{w,s'}$ is in M, a contradiction). As $v_0, s, r \in N(K)$, we have $|N(K)| \geq 3$.

As K is the union of w with a number of edges in M, $|K|$ is odd. Similarly, any other component of $G - VC$ is even. As C is an odd circuit, VC can be partitioned into three paths with an odd number of vertices, each containing a neighbour of K. Hence G is oddly contractible to K_4. ∎

We define deletion, contraction, and minor for pairs G, T. Let $G = (V, E)$ be a graph, $T \subseteq V$, and $e = uv \in E$. We say that $G - e, T$ arises from G, T by *deleting* e. Let G/e be the graph obtained from G by contracting e. Denote the new vertex to which e is contracted by v^e. Define $T' := T \setminus \{u, v\}$ if $|T \cap \{u, v\}|$ is even, and $T' := (T \setminus \{u, v\}) \cup \{v^e\}$ if $|T \cap \{u, v\}|$ is odd. Then we say that $G/e, T'$ arises from G, T by *contracting* e.

We say that the pair G', T' is a *minor* of the pair G, T if G', T' arises from G, T by a series of deletions and contractions of edges, and of deletions of vertices not in T. Then the following is a special case of a more general hypergraph theorem of Seymour [1977b] (Theorem 80.1):

Corollary 29.9a. *Let $G = (V, E)$ be a graph and let $T \subseteq V$ with $|T|$ even, such that K_4, VK_4 is not a minor of G, T. Then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts.*

Proof. By Theorem 29.8, the minimum size of a T-join is equal to the maximum total value of a packing of T-borders. Consider such an optimum packing, with the number of T-borders as large as possible. If each T-border is a T-cut, we are done. So assume that one of the T-borders, B say, has value at least 2. Let $\mathcal{P} = (U_1, \ldots, U_k)$ be a partition of V with $|U_i \cap T|$ odd for each i and such that B is the union of the $\delta(U_i)$.

Let $G' = (V', E'), T'$ be obtained from G, T by contracting each U_i to one vertex. So $T' = V'$. As G', T' contains no K_4, VK_4 as a minor, G is not bicritical, by Theorem 29.9. Hence there are distinct $u, v \in V'$ such that $G' - u - v$ has no perfect matching. By Tutte's 1-factor theorem this implies that there is a subset U of V' with $u, v \in U$ and with $o(G' - U) \geq |U|$. Take such a U with $|U|$ maximal. Then each component of $G' - U$ is odd. (Otherwise, we can add an element of some even component to U, contradicting the maximality of $|U|$.)

For each component K of $G' - U$, the set of edges of G' incident with K form a V'-border in G' of value $\frac{1}{2}(|K| + 1)$. So G' has a packing of V'-borders of total value $|V' \setminus U| + o(G' - U) \geq |V'| = k$. Since $|U| \geq 2$ (as $u, v \in U$), we have $o(G' - U) \geq 2$, so there are at least two such components. Hence the packing contains at least two V'-borders. Decontracting the U_i gives a decomposition of B into a packing of at least two T-borders, of total value k. This contradicts the maximality of the number of T-borders in the original packing. ∎

This can be formulated equivalently in terms of total dual integrality. Note that total dual integrality of system (29.9) is closed under taking minors: deletion of an edge e corresponds to intersection with the hyperplane $H := \{x \mid x_e = 0\}$, while contracting e corresponds to projecting on H. Hence total dual integrality of (29.9) can be characterized by forbidden minors; in fact, there is only one forbidden minor:

Corollary 29.9b. *System* (29.9) *is totally dual integral if and only if* G, T *has no minor* K_4, VK_4.

Proof. To see necessity, it suffices to show that if $G = K_4$ and $T = VK_4$, then (29.9) is not TDI. Taking $w_e := 1$ for each $e \in EG$, the minimum weight of a T-join equals 2, while each two T-cuts intersect, implying that there is no integer optimum dual solution.

To see sufficiency, let K_4, VK_4 not be a minor of $G = (V, E), T$. Let $w \in \mathbb{Z}_+^E$. Let G', T' arise from G, T by replacing each edge e by a path of length w_e, contracting e if $w_e = 0$. Then also G', T' has no minor K_4, VK_4. Moreover, the minimum weight k of a T-join in G is equal to the minimum size of a T'-join in G'. By Corollary 29.9a, G' contains a T'-cut packing of size k. So G contains k T-cuts such that each edge e of G is in at most $w(e)$ of them. This gives an integer optimum dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.9). ∎

This implies a characterization of series-parallel graphs:

Corollary 29.9c. *The following are equivalent for any graph* $G = (V, E)$*:*

(29.52)
(i) *G is series-parallel;*
(ii) (29.9) *is TDI for each choice of T;*
(iii) (29.11) *is TDI for each choice of T;*
(iv) (29.11) *is TDI for some choice of T;*
(v) (29.17) *is TDI.*

Proof. The equivalence of (i) and (ii) follows from Corollary 29.9b, since a graph G is series-parallel if and only if G has no K_4 minor. The implication (iii)⇒(iv) is direct.

We next show (v)⇒(ii). Let (29.17) be TDI. Choose $T \subseteq V$ and $w \in \mathbb{Z}_+^E$. Let J be a T-join minimizing $w(J)$. Define $\tilde{w}(e) := w(e)$ if $e \in E \setminus J$ and $\tilde{w}(e) := -w(e)$ if $e \in J$. Then $\emptyset$ is a $\tilde{w}$-minimal $\emptyset$-join. Since (29.17) is TDI, there exist $\lambda_{U,e} \in \mathbb{Z}_+$ for $U \subseteq V$ and $e \in \delta(U)$ with

(29.53) $$\tilde{w} \geq \sum_{U,e} \lambda_{U,e}(\chi^{\delta(U)\setminus\{e\}} - \chi^e).$$

Choose the $\lambda_{U,e}$ such that $\sum_{U,e} \lambda_{U,e}$ is minimized. Then

(29.54) if $\lambda_{U,e} \geq 1$ and $\lambda_{U',e'} \geq 1$, then $e' \notin \delta(U) \setminus \{e\}$.

Otherwise, if $e \in \delta(U')$, then

(29.55) $$(\chi^{\delta(U)\setminus\{e\}} - \chi^e) + (\chi^{\delta(U')\setminus\{e'\}} - \chi^{e'})$$

is nonnegative, and hence we can decrease $\lambda_{U,e}$ and $\lambda_{U',e'}$ by 1, without violating (29.53), contradicting our minimality assumption.

If $e \notin \delta(U')$, then $e \in \delta(U \triangle U')$. Also, (29.55) is at least

(29.56) $$\chi^{\delta(U\triangle U')\setminus\{e\}} - \chi^e,$$

and hence we can decrease $\lambda_{U,e}$ and $\lambda_{U',e'}$ by 1, and increase $\lambda_{U\triangle U',e}$ by 1, without violating (29.53), again contradicting our minimality assumption.

This shows (29.54). So there are no two $\lambda_{U,e} \geq 1$ and $\lambda_{U',e'} \geq 1$ such that the vectors $\chi^{\delta(U)\setminus\{e\}} - \chi^e$ and $\chi^{\delta(U')\setminus\{e'\}} - \chi^{e'}$ have opposite signs in some position. The minimality of $\sum \lambda_{U,e}$ then implies that $\sum \lambda_{U,e} = -\tilde{w}(J) = w(J)$ and that $J \cap \delta(U) = \{e\}$ for each U, e with $\lambda_{U,e} \geq 1$. So each such $\delta(U)$ is a T-cut. Moreover,

(29.57) $$w \geq \sum_{U,e} \lambda_{U,e} \chi^{\delta(U)}.$$

So we have an integral dual solution for the problem of minimizing $w^{\mathsf{T}} x$ over (29.9). This proves (v)⇒(ii).

We next show the reverse implication (ii)⇒(v). Let (29.9) be TDI for each choice of T. To prove that (29.17) is TDI, choose $w \in \mathbb{Z}^E$, such that minimizing $w^{\mathsf{T}} x$ over (29.17) is finite — that is (as (29.17) determines the circuit cone) $w(C) \geq 0$ for each circuit C.

Define $J := \{e \in E \mid w(e) < 0\}$ and $T := \{v \in V \mid \deg_J(v) \text{ is odd}\}$. Moreover, $\tilde{w}(e) := |w(e)|$ for $e \in E$. Then J is a T-join minimizing $\tilde{w}(J)$ (as $w(C) \geq 0$ for each circuit C). Hence, as (29.9) is TDI, there exist $\lambda_U \in \mathbb{Z}_+$ for U with $T \cap U$ odd, such that

(29.58) $$\sum_U \lambda_U \chi^{\delta(U)} \leq \tilde{w} \text{ and } \sum_U \lambda_U = \tilde{w}(J).$$

For each U with $\lambda_U \geq 1$ one has $|J \cap \delta(U)| = 1$; let e_U be the edge in $J \cap \delta(U)$. Then

(29.59) $$w \geq \sum_U \lambda_U (\chi^{\delta(U)\setminus e_U} - \chi^{e_U}),$$

proving total dual integrality of (29.17).

Finally we show (v)⇔(iii)⇔(iv). Consider any $T \subseteq V$ and any vertex χ^J of the T-join polytope, determined by T-join J. Total dual integrality of (29.11) in χ^J means that the following system is TDI:

(29.60)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E \setminus J, \\ x_e \leq 1 & \text{for each } e \in J, \\ x(H) - x(F) \geq 1 - |F|, & \text{for each } U \subseteq V \text{ and partition } F, H \text{ of} \\ & \delta(U) \text{ with } |U \cap T| + |F| \text{ odd and} \\ & |H \cap J| + |F \setminus J| = 1. \end{array}$$

The condition $|H \cap J| + |F \setminus J| = 1$ implies that there exists an edge $e \in \delta(U)$ with $F = (\delta(U) \cap J) \triangle \{e\}$ and $H = (\delta(U) \setminus J) \triangle \{e\}$.

Setting $\tilde{x}_e := 1 - x_e$ if $e \in J$ and $\tilde{x}_e := x_e$ if $e \in E \setminus J$, (29.60) is equivalent to:

(29.61)
$$\begin{array}{l} \tilde{x}_e \geq 0 \text{ for } e \in E, \\ \tilde{x}(H \setminus J) + |H \cap J| - \tilde{x}(H \cap J) - \tilde{x}(F \setminus J) - |F \cap J| + \tilde{x}(F \cap J) \geq 1 - |F| \end{array}$$

for each U, F, H as described in (29.60). The second line in (29.61) is equivalent to:

(29.62) $$\tilde{x}(H \triangle (J \cap \delta(U))) - \tilde{x}(F \triangle (J \cap \delta(U))) \geq 1 - |F \triangle (J \cap \delta(U))|.$$

and hence to

(29.63) $$\tilde{x}(\delta(U) \setminus \{e\}) - \tilde{x}_e \geq 0,$$

where $\{e\} := F \triangle (J \cap \delta(U))$. As this equivalence holds for any fixed T, this proves both (iv)⇒(v) and (v)⇒(iii). ∎

(Korach [1982] gave an algorithmic proof of this corollary.)

Sebő [1988b] also characterized the minimal TDI-system for the polyhedron $P^{\uparrow}_{T\text{-join}}(G)$. Call a T-border B *reduced* if $B = \delta(U_1) \cup \cdots \cup \delta(U_k)$ for some partition $\mathcal{P} = (U_1, \ldots, U_k)$ of V such that $|U_i \cap T|$ is odd and $G[U_i]$ is connected for each i and such that the graph obtained by contracting each U_i to one vertex is bicritical. Then the following is a minimal TDI-system for connected graphs:

(29.64) (i) $x_e \geq 0$ for each edge e for which $\{e\}$ is not a T-cut,
(ii) $x(B) \geq \text{val}(B)$ for each reduced T-border B.

Sebő [1993c] showed that for each fixed k, the problem of finding a maximum integer packing of T-cuts subject to a capacity constraint is polynomial-time solvable if $|T| = k$. The method uses that integer linear programming is polynomial-time solvable in fixed dimension (Lenstra [1983]).

29.11c. Packing T-joins

In the previous section we considered packing T-cuts, which relates to the total dual integrality of system (29.9). We now consider packing T-joins, which relates to the total dual integrality of system (29.41).

System (29.41) generally is not TDI. Indeed, let G be the graph $K_{2,3}$ and let $T_0 := VK_{2,3} \setminus \{v_0\}$, where v_0 is one of the two vertices of degree 3 in $K_{2,3}$. Then the minimum size of a T_0-cut in $K_{2,3}$ is equal to 2, while there are no two disjoint T_0-joins. This again is the critical example, as follows again from a more general hypergraph theorem of Seymour [1977b] (Theorem 80.1). For this special case, we follow the line of proof given by Codato, Conforti, and Serafini [1996].

Theorem 29.10. *Let $G = (V, E)$ be a graph and let $T \subseteq V$, such that $K_{2,3}, T_0$ is not a minor of G, T. Then the minimum size of a T-cut is equal to the maximum number of disjoint T-joins.*

Proof. Let G, T form a counterexample, with $|V| + |E|$ as small as possible. Let k be the minimum size of a T-cut. Then trivially G is connected. Moreover:

(29.65) any T-cut C of size k satisfies $C = \delta(t)$ for some $t \in T$.

Indeed, let $C = \delta(U)$ for $U \subseteq V$ with $|U \cap T|$ odd and $|C| = k$. Assume that $1 < |U| < |V| - 1$. Then $G[U]$ is connected, since otherwise there would exist a T-cut smaller than k. Similarly, $G - U$ is connected.

Now contract U to one vertex v', yielding minor G', T' of G, T. The minimum size of a T'-cut in G' equals k. As $|VG'| < |VG|$, we know that G' has k disjoint T'-joins. Each of them intersects $\delta_{G'}(v')$ in exactly one edge (as it is a T'-cut of size k).

We can contract $V \setminus U$ to one vertex v'', yielding minor G'', T'' of G, T. Again, G'' has k disjoint T''-joins, each intersecting $\delta_{G''}(v'')$ in exactly one edge.

Using the one-to-one correspondence between $\delta_{G'}(v')$ and $\delta_{G''}(v'')$, we can glue the two collections of joins together, to obtain k disjoint T-joins in G, contradicting our assumption. This gives (29.65).

Let $T' := \{t \in T \mid \deg(t) = k\}$. Then (29.65) implies that

(29.66) each edge of G intersects T'.

Otherwise we could delete the edge without decreasing the minimum size of a T-cut, by (29.65). This would give a smaller counterexample, contradicting our assumption.

We next have:

(29.67) $|V \setminus T'| \geq 2$.

For suppose $|V \setminus T'| \leq 1$. We know that G has $k-1$ disjoint T-joins (by the minimality of $|V| + |E|$ — otherwise deleting any edge would give a smaller counterexample). Let F be the union of these T-joins. Then $\deg_F(v)$ is even if $v \notin T$ while $\deg_F(v) \equiv k-1 \pmod 2$ if $v \in T$. Hence $\deg_{E\setminus F}(v)$ is odd for each $v \in T'$. As $|V \setminus T'| \leq 1$ it follows that $E \setminus F$ is a T-join, and hence G would have k disjoint T-joins. This contradicts our assumption, and proves (29.67).

Then

(29.68) there is no subset U of T' with $|U| \leq 2$ and $G - U$ disconnected.

Suppose not. If $|U| = 1$, let $U = \{t\}$ for $t \in T'$. Then $|K \cap T|$ is odd for some component K of $G - t$. As $G - t$ is disconnected, $|\delta(K)| < \deg(t) = k$, contradicting the fact that $\delta(K)$ is a T-cut.

If $|U| = 2$, let $U = \{t, t'\}$ for $t, t' \in T'$. Choose a component K of $G - U$ not contained in T'. Let l (l', respectively) be the number of edges connecting K and t (K and t', respectively). If $|K \cap T|$ is odd, then $l + l' = |\delta(K)| > k$ and hence $|\delta(K \cup \{t, t'\})| \leq (k - l) + (k - l') < k$, a contradiction. If $|K \cap T|$ is even, then $l' + (k - l) = |\delta(K \cup \{t\})| > k$, and similarly $l + (k - l') > k$, a contradiction. This proves (29.68).

Now choose $u \in V \setminus T'$. As $N(u) \subseteq T'$ (by (29.66)), by (29.68) we know $|N(u)| \geq 3$. Choose a component K of $G' := G - (\{u\} \cup N(u))$, with $|N(K)|$ as small as possible. (K exists by (29.67).) If possible, we take K such that moreover $|K \cap T|$ is odd.

Again by (29.68), $|N(K)| \geq 3$. Choose $t_1, t_2, t_3 \in N(K)$. Then

(29.69) for any component $L \neq K$ of G' with $N(L) = \{t_1, t_2, t_3\}$ one has $|L \cap T|$ even.

For suppose that $|L \cap T|$ is odd. By the minimality of $|N(K)|$, we know $N(K) = \{t_1, t_2, t_3\}$. Moreover, $|K \cap T|$ is odd. Let k_i be the number of edges connecting K and t_i and let l_i be the number of edges connecting L and t_i, for $i = 1, 2, 3$. Then $k_1 + k_2 + k_3 = |\delta(K)| \geq k$, and similarly $l_1 + l_2 + l_3 \geq k$. This gives the contradiction

(29.70) $k < |\delta(K \cup L \cup \{t_1, t_2, t_3\})| \leq (k - k_1 - l_1) + (k - k_2 - l_2) + (k - k_3 - l_3) \leq k$

(the first inequality follows from (29.65)). This shows (29.69).

Now contract the union of $\{u\} \cup (N(u) \setminus \{t_1, t_2, t_3\})$ and all components $L \neq K$ of G' with $N(L) \neq \{t_1, t_2, t_3\}$ to one vertex u'. Moreover, contract the union of $\{t_1\}$ and all components $L \neq K$ of G' with $N(L) = \{t_1, t_2, t_3\}$ to one vertex t_1'. Finally contract K to one vertex u''. This gives minor G'', T'' of G, T.

So G'' has vertices u', u'', t_1', t_2, t_3, with each of u', u'' adjacent to each of t_1', t_2, t_3 (possibly there are more adjacencies). Each of t_1', t_2, t_3 belongs to T''. As $|T''|$ is even, exactly one of u', u'' belongs to T''. Hence G, T has minor $K_{2,3}, T_0$, a contradiction. ∎

This implies the characterization:

Corollary 29.10a. *System* (29.41) *is TDI if and only if* $K_{2,3}, T_0$ *is not a minor of* G, T.

Proof. Necessity follows from the fact that total dual integrality of (29.41) is maintained under taking minors (contraction of an edge e corresponds to intersecting the polytope with the hyperplane $x_e = 0$, and deletion of e corresponds to projecting on it), while the minimum size of a T_0-cut in $K_{2,3}$ is 2, and $K_{2,3}$ has no two disjoint T_0-joins.

To see sufficiency, let $w \in \mathbb{Z}_+^E$. Replace any edge $e = uv$ of G by $w(e)$ parallel edges connecting u and v, yielding the graph G'. Then the minimum weight of a T-cut in G is equal to the minimum size of a T-cut in G'. By Theorem 29.10, this minimum size is equal to the maximum number of disjoint T-joins in G'. These T-joins give an integer optimum dual solution to the problem of minimizing $w^\mathsf{T} x$ subject to (29.41). ∎

Generally, system (29.41) is not totally dual half-integral, as is shown by the following example of Seymour [1979a]. Let $G' = (V', E')$ be a connected bridgeless cubic graph with $\chi'(G) = 4$ and with an even number of edges. (For instance, G' is the Petersen graph with one vertex replaced by a triangle (in such a way that the three vertices adjacent to it in the Petersen graph, now each are adjacent to one of the vertices in the triangle).)

Let $G = (V, E)$ be obtained from G' by replacing each edge by a path of length 2. So $|V|$ is even.

Then trivially the minimum size of a V-cut is equal to 2. However, the maximum number of V-joins covering each edge at most twice is equal to 3. For suppose that there exist four V-joins $J_1, \ldots, J_4$ covering each edge at most twice. Since each edge of G is incident with a vertex of degree two, each edge of G is covered exactly twice by the J_i. For $i = 1, 2, 3$, let $C_i := J_i \triangle J_4$. Then each C_i is a vertex-disjoint union of circuits, and each edge of G is in exactly two of the C_i. Then the complements of the C_i form edge-disjoint V'-joins in G. This would yield a 3-edge-colouring of G' — a contradiction.

If we replace each edge of G by two parallel edges, thus obtaining an Eulerian graph, the minimum size of a V-cut equals 4, whereas the maximum number of disjoint V-joins is 3.

If Seymour's 'generalized Fulkerson conjecture' (see Section 28.5) is true, there exists a $\frac{1}{4}$-integer packing (that is, the minimum size of a T-cut is equal to one quarter of the maximum size of a 4-packing of T-joins); in other words, the total dual quarter-integrality of the T-join constraints (29.41) follows — we give the proof of Seymour [1979a] of this derivation.

Proof that the generalized Fulkerson conjecture implies the total dual quarter-integrality of the T-join constraints. Let $G = (V, E)$ be a graph and let $T \subseteq V$. Let k be the minimum size of a T-cut. We must show that the generalized Fulkerson conjecture implies:

(29.71) there exist T-joins $J_1, \ldots, J_{4k}$ covering each edge of G at most four times.

First assume that $T = V$. We show:

(29.72) if each vertex of G has even degree, then there exist V-joins $J_1, \ldots, J_{2k}$ covering each edge of G at most twice.

To see this, assume that each vertex of G has even degree. So k is even. If $k \leq 2$, (29.72) is trivial. (If $k = 2$ there exists a V-join J; then the complement $E \setminus J$ is a V-join again.) So we can assume that $k \geq 4$.

For each $v \in V$, let G_v be a $(k-1)$-edge-connected graph with $\deg_G(v) + 1$ vertices, one of degree k and all other vertices of degree $k - 1$. (Such graphs G_v exist: If $k = 4$, take any cubic 3-edge-connected graph on $\deg_G(v) + 2$ vertices (for instance, by taking a circuit on $\deg_G(v) + 2$ vertices, and making opposite vertices adjacent), and contract an arbitrary edge of it. If $k \geq 6$, add a Hamiltonian circuit to the graph for the case $k - 2$.)

We take the G_v vertex-disjoint. Now transform G to a graph H, by replacing each vertex v by G_v, and making each edge of G which was incident with v, incident instead with one of the $\deg_G(v)$ vertices of G_v of degree $k - 1$, in such a way that the resulting graph H is k-regular.

We show that H is a k-graph, by showing

(29.73) $|\delta_H(U)| \geq k$ for each $U \subseteq VH$ with $|U|$ odd.

To see this, assume $|\delta_H(U)| < k$. Observe that $|\delta_H(U)|$ is even, as k is even and H is k-regular. Hence $|\delta_H(U)| \leq k - 2$. Since each G_v is $(k-1)$-edge-connected, for each $v \in V$ we know that either $VG_v \subseteq U$ or $VG_v \cap U = \emptyset$. Define

(29.74) $X := \{v \in V \mid VG_v \subseteq U\}$.

Then $|\delta_H(U)| = |\delta_G(X)|$. Moreover, $|X|$ is odd as $|VG_v|$ is odd for each $v \in V$. Therefore $|\delta_G(X)| \geq k$ and hence $|\delta_H(U)| \geq k$. This shows (29.73).

Then by the generalized Fulkerson conjecture, there exist perfect matchings $M_1, \ldots, M_{2k}$ in H covering each edge of H exactly twice. Projecting these matchings to the original edges of G, gives V-joins as required in (29.72).

Now, for $T = V$, (29.71) follows from (29.72) by replacing each edge of G by two parallel edges. The case of general T can be reduced to the case $T = V$ as follows. Let T be arbitrary. For each vertex $v \in V \setminus T$, make a new vertex v', connected by k parallel edges with v. This gives the graph $G' = (V', E')$. Then the minimum size of a V'-cut in G' is equal to k. Hence by (29.71) there exist V'-joins $J'_1, \ldots, J'_{4k}$ in G' covering each edge of G' at most four times. Restricting the J'_i to the edges of G, gives V-joins in G as required.

Cohen and Lucchesi [1997] showed that conjecture (29.72) is equivalent to: if all T-cuts have the same parity, then the maximum size of a 2-packing of T-joins is equal to twice the minimum size of a T-cut. They also showed that this is true if $|T| \leq 8$; more strongly, that if $|T| \leq 8$ and all T-cuts have the same parity, then the maximum number of disjoint T-joins is equal to the minimum size of a T-cut.

29.11d. Maximum joins

Let $G = (V, E)$ be a graph. Call a subset J of E a *join* if $|J \cap C| \leq \frac{1}{2}|C|$ for each circuit C; that is, $|J \triangle C| \geq |C|$ for each circuit C. This can be expressed in terms of the length function $l_J : E \to \{-1, +1\}$, defined by

(29.75) $$l_J(e) := \begin{cases} -1 & \text{if } e \in J, \\ +1 & \text{if } e \notin J. \end{cases}$$

So

(29.76) $\quad l_J(F) = |F \triangle J| - |J|$

for each $F \subseteq E$. Then a set J is a join if and only if $l_J(C) \geq 0$ for each circuit C. Note also that

(29.77) $\quad$ a set J is a join if and only if it is a minimum-size T-join for $T := \{v \in V \mid \deg_J(v) \text{ odd}\}$.

Frank [1990b,1993b] gave a min-max relation for the maximum size of a join. By Corollary 29.2a and (29.77), the maximum size of a join is equal to the maximum size of a fractional packing of T-cuts, taken over $T \subseteq V$ with $|T \cap K|$ even for each component K of G. This, however, is not a min-max relation.

A min-max relation can be described in terms of ear-decomposition. Let $G = (V, E)$ be an undirected graph. An *ear* of G is a path or circuit P in G, of length ≥ 1, such that all internal vertices of P have degree 2 in G. The path may consist of a single edge — so any edge of G is an ear.

If I is the set of internal vertices of an ear P, we say that G arises from $G - I$ by *adding ear*. An *ear-decomposition* of G is a series of graphs $G_0, G_1, \ldots, G_k$, where $G_0 = K_1$, $G_k = G$, and G_i arises from G_{i-1} by adding an ear ($i = 1, \ldots, k$).

A graph $G = (V, E)$ has an ear-decomposition if and only if G is 2-edge-connected (see Theorem 15.17). Moreover, the number of ears in any ear-decomposition is equal to $|E| - |V| + 1$. Then the min-max relation for maximum-size join in 2-connected graphs is formulated as:

Theorem 29.11. *Let $G = (V, E)$ be a 2-edge-connected graph. Then the maximum size of a join is equal to the minimum value of*

(29.78) $$\sum_{i=1}^{k} \lfloor \tfrac{1}{2} |EP_i| \rfloor$$

taken over all ear-decompositions $(P_1, \ldots, P_k)$ of G.

Proof. We first show that the maximum is not more than the minimum. Let J be a join in G and let $\Pi = (P_1, \ldots, P_k)$ be an ear-decomposition of G. Let $G' = (V', E')$ be the graph made by $P_1, \ldots, P_{k-1}$ and let $J' := J \cap E'$. By induction we know

(29.79) $$|J'| \leq \sum_{i=1}^{k-1} \lfloor \tfrac{1}{2} |EP_i| \rfloor.$$

If $|J \cap EP_k| \leq \lfloor \frac{1}{2}|EP_k| \rfloor$ we are done. So assume that $|J \cap EP_k| > \lfloor \frac{1}{2}|EP_k| \rfloor$; that is, $l_J(P_k) < 0$. Let u and v be the end vertices of P_k. Let Q be a $u - v$ path in G' minimizing $l_{J'}(Q)$. So $l_J(P_k) + l_J(Q) \geq 0$ (since J is a maximum-size join). Since $l_{J'}(Q) = |J' \triangle EQ| - |J'|$, Q minimizes $|J' \triangle EQ|$.

Then $J'' := J' \triangle EQ$ is again a join in G', since for any circuit C in G':

(29.80) $\quad |J'' \triangle C| = |J' \triangle (EQ \triangle C)| \geq |J' \triangle EQ| = |J''|$

(since Q minimizes $|J' \triangle EQ|$). Moreover,

(29.81) $$\begin{aligned} &|J''| - |J'| = |J' \triangle EQ| - |J'| = l_J(Q) \geq -l_J(P_k) \\ &= |J \cap EP_k| - |EP_k \setminus J| \geq |J \cap EP_k| - \lfloor \tfrac{1}{2} |EP_k| \rfloor. \end{aligned}$$

Hence, by induction applied to J'',

$$|J| = |J'| + |J \cap EP_k| \leq |J''| + \lfloor \tfrac{1}{2}|EP_k| \rfloor \leq \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|EP_i| \rfloor. \tag{29.82}$$

This shows that the maximum is not more than the minimum. To see equality, for any graph G let $\beta(G)$ be the maximum size of a join in G. For any ear-decomposition $\Pi = (P_1, \ldots, P_k)$, let $\sigma(\Pi) := \sum_{i=1}^{k} \lfloor \frac{1}{2}|EP_i| \rfloor$. Let $\pi(G)$ be the minimum of $\sigma(\Pi)$ over all ear-decompositions Π of G. So we must prove $\beta(G) = \pi(G)$. Call an ear-decomposition Π *optimum* if it minimizes $\sigma(\Pi)$.

We first show:

(29.83) Let $U \subseteq V$ with $G[U]$ 2-edge-connected. Then $\pi(G) \leq \pi(G[U]) + \pi(G/U)$.

To see this, first observe that if $G[U]$ has a Hamiltonian circuit C, then an optimum ear-decomposition Π' of $G[U]$ is obtained by first taking C, and next adding the remaining edges as ears. Now in any optimum ear-decomposition Π'' of G/U, we can insert Π' at the first ear of Π'' containing the vertex into which U is contracted (by splitting C appropriately). In this way we obtain an ear-decomposition Π of G with $\sigma(\Pi) \leq \sigma(\Pi') + \sigma(\Pi'')$.

If $G[U]$ has no Hamiltonian circuit, let Π' be an optimum ear-decomposition of $G[U]$. Let C be its first ear. By the above, $\pi(G) \leq \pi(G[VC]) + \pi(G/VC)$. Also, by induction, $\pi(G/VC) \leq \pi((G[U])/VC) + \pi(G/U)$. As C is the first ear of Π', we have $\pi(G[VC]) + \pi((G[U])/VC) = \pi(G[U])$. Combining, we get $\pi(G) \leq \pi(G[U]) + \pi(G/U)$, showing (29.83).

Next we state:

(29.84) if G is factor-critical, then $\pi(G) \leq \lfloor \frac{1}{2}|VG| \rfloor$.

This follows directly from Theorem 24.9, since $\lfloor \frac{1}{2}|EP_i| \rfloor$ is at most $\frac{1}{2}$ the number of internal vertices of P_i.

In particular, it follows that if G is factor-critical, then $\beta(G) = \pi(G)$, as G has a join of size $\lfloor \frac{1}{2}|VG| \rfloor$, namely a matching. So we can assume that G is not factor-critical.

A graph G is called *matching-covered* if each edge of G is contained in a perfect matching. By Theorem 24.10,

(29.85) if G is matching-covered and 2-edge-connected, then $\pi(G) \leq \frac{1}{2}|VG|$.

For any subset W of V let H_W be the graph obtained from $G[W \cup N(W)]$ by deleting all edges in $N(W)$ and contracting all edges in W. (H_W may have parallel edges.) So H_W is a bipartite graph with colour classes $N(W)$ and $\kappa(W) :=$ the set of components of $G[W]$.

(29.86) There is a nonempty subset W of V such that each component of $G[W]$ is factor-critical and such that H_W is 2-edge-connected and matching-covered.

To see this, we first observe that there is a nonempty subset X of V such that each component of $G[X]$ is factor-critical and such that H_X has a matching M covering $N(X)$. Indeed, if G has no perfect matching, then we can take $X := D(G)$ (= the set of vertices v for which G has a maximum-size matching missing v). By Corollary

24.7a, X has the required properties. If G has a perfect matching, call it M. Choose $u \in V$, and let $X := D(G-u)$. Then X has the required properties (note that the vertex matched in M to u belongs to $D(G-u)$).

Having X and M, orient the edges in M in the direction from $\kappa(X)$ to $N(X)$, and all other edges of H_X in the direction from $N(X)$ to $\kappa(X)$. This gives a directed graph, that has (like any directed graph) a strong component L such that no arc enters L. Let W be the union of those components of $G[X]$ whose contraction belong to L. Since no arc leaves L, for any edge $e = uv \in M$, if $u \in N(X)$ and $u \in L$, then $v \in W$. Conversely, if $v \in W$, then $u \in L$. For let $v \in K \in L$. As G is 2-edge-connected, there exists an edge $f \neq e$ leaving K. As $K \in L$ and no arc enters L, both ends of f belong to L. As L is strongly connected, f belongs to a directed circuit. Necessarily, e is in this directed circuit. So both ends of e are in L.

Hence the edges of M intersecting W, form a perfect matching M' in H_W, and so $|N(W)| = |\kappa(W)|$. Moreover, consider any edge e of H_W not in M. In H_X, e is oriented from $N(W)$ to $\kappa(W)$, and hence, as L is a strong component, it is contained in a directed circuit. This directed circuit forms an M'-alternating circuit in H_W, implying that e belongs to a perfect matching in H_W. So H_W is matching-covered. Finally H_W is 2-edge-connected, as it has a strongly connected orientation, since L is a strong component. This shows (29.86).

Define $U := W \cup N(W)$. Then (29.83), (29.84), (29.85), and (29.86) imply

(29.87)
$$\pi(G) \leq \pi(G/U) + \pi(G[U]) \leq \pi(G/U) + \pi(H_W) + \sum_{K \in \kappa(W)} \pi(G[K])$$
$$\leq \pi(G/U) + \tfrac{1}{2}|VH_W| + \sum_{K \in \kappa(W)} \lfloor \tfrac{1}{2}|K| \rfloor \leq \pi(G/U) + \tfrac{1}{2}|U|.$$

On the other hand, we have

(29.88)
$$\beta(G) \geq \beta(G/U) + \tfrac{1}{2}|U|.$$

Indeed, let $G' := G/N(W)$. Then trivially, $\beta(G) \geq \beta(G')$. The contracted $N(W)$ forms a cut vertex v_0 in G', and so $\beta(G')$ is equal to the sum of the $\beta(G'[K \cup \{v_0\}])$ over all components K of $G - v_0$. Now for each component K of $G[W]$ we have $\beta(G'[K \cup \{v_0\}]) \geq \frac{1}{2}(|K|+1)$, since $G'[K \cup \{v_0\}]$ has a perfect matching (as K is factor-critical), which is a join. Since $G[W]$ has $|N(W)|$ components, this proves (29.88).

Hence the theorem follows by induction. ∎

The proof gives a polynomial-time algorithm to find a maximum-size join and an ear-decomposition minimizing (29.78).

In Section 24.4d we saw that a graph is factor-critical if and only if it has an ear-decomposition with odd ears only. This can be generalized to (where G/F arises from G by contracting all edges in F):

Theorem 29.12. *Let $G = (V, E)$ be a 2-edge-connected graph. Then the minimum number of even ears in an ear-decomposition of G is equal to the minimum size of a subset F of E with G/F factor-critical.*

Proof. First let $P_1, \ldots, P_k$ be an ear-decomposition of G. Choose one edge from each even ear. This gives a set F with G/F factor-critical, by Theorem 24.9.

Conversely, let $F \subseteq E$ with G/F factor-critical and $|F|$ minimum. By Theorem 24.9, G/F has an ear-decomposition $(P_1, \ldots, P_k)$ with odd ears only. Then we can partition F into $F_1, \ldots, F_k$ such that $P_1 \cup F_1, \ldots, P_k \cup F_k$ is an ear-decomposition of G. This ear-decomposition has at most $|F|$ even ears. ∎

We can derive from this a characterization of the maximum size of a join in any graph:

Corollary 29.12a. *Let $G = (V, E)$ be a connected graph. Then the maximum size $\beta(G)$ of a join is equal to*

$$(29.89) \qquad \tfrac{1}{2}(\phi(G) + |V| - 1).$$

where $\phi(G)$ is the minimum size of a subset F of E with G/F factor-critical.

Proof. If G has a cut edge e, the corollary follows by applying induction to G/e, since $\beta(G) = \beta(G/e) + 1$ and $\phi(G) = \phi(G/e) + 1$.

So we can assume that G is 2-edge-connected, and then the corollary follows from Theorem 29.11, with Theorem 29.12. Note that

$$(29.90) \qquad \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|EP_i| \rfloor = \tfrac{1}{2}(\text{number of even ears} + \sum_{i=1}^{k}(|EP_i| - 1)) \\ = \tfrac{1}{2}(\text{number of even ears} + |V| - 1).$$ ∎

For 2-edge-connected bipartite graphs we have:

Corollary 29.12b. *Let $G = (V, E)$ be a 2-edge-connected bipartite graph, with colour classes U and W. Then the maximum size of a join is equal to the minimum number of edges oriented towards U in any strongly connected orientation of G.*

Proof. To see that the maximum is not more than the minimum, consider any strongly connected orientation of G, yielding the directed graph D. By Theorem 6.9, D has an ear-decomposition $(P_1, \ldots, P_k)$. Any ear P_i contains at least $\lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. So the sum (29.78) is at most the total number of edges oriented towards U. Hence by Theorem 29.11, the maximum is not more than the minimum.

To see equality, consider an ear-decomposition $P_1, \ldots, P_k$ of G minimizing (29.78). In any ear P_i, we can orient the edges so as to obtain a directed path, with exactly $\lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. This gives a strongly connected orientation with $\sum_i \lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. So Theorem 29.11 gives equality. ∎

We can derive some more min-max relations for bipartite graphs. Seymour [1981d] observed that Theorem 29.2 is equivalent to:

Theorem 29.13. *Let $G = (V, E)$ be bipartite and let $J \subseteq E$. Then J is a join if and only if there exist $|J|$ disjoint cuts each intersecting J in exactly one edge.*

Proof. By Theorem 29.2, using (29.77). ∎

This implies a max-max relation for the maximum size of a join in bipartite graphs:

Corollary 29.13a. *Let G be bipartite. Then the maximum size of a join is equal to the maximum number of disjoint nonempty cuts.*

Proof. Directly from Theorem 29.13. ∎

Hence, with Corollary 29.12b, a result of D.H. Younger follows (cf. Frank [1993b]):

Corollary 29.13b. *Let G be a 2-edge-connected bipartite graph, with colour classes U and W. Then the minimum number of edges oriented towards U in any strongly connected orientation of G is equal to the maximum number of disjoint nonempty cuts in G.*

Proof. From Corollaries 29.13a and 29.12b. ∎

Frank, Tardos, and Sebő [1984] showed the following. Let G be a 2-edge-connected bipartite graph, with colour classes U and W. Then the minimum number of edges oriented towards U in any strongly connected orientation of G is equal to the maximum value of

$$\sum_{S \in \Pi} \kappa(G - S), \tag{29.91}$$

ranging over all partitions Π of U, where $\kappa(H)$ denotes the number of components of H.

For an extension, see Kostochka [1994]. Szigeti [1996] gave a weighted version, based on matroids. Fraenkel and Loebl [1995] showed that it is NP-complete to find the maximum size of a subset J of the edge set E of a graph G with $l_J(C) < \frac{1}{2}|EC|$ for each circuit C (even if G is planar and bipartite). Connected joins were investigated by Sebő and Tannier [2001].

29.11e. Odd paths

We saw in Section 29.2 that the problem of finding a shortest $s-t$ path in an undirected graph $G = (V, E)$, with length function $l : E \to \mathbb{Q}$ can be solved in polynomial time, if each circuit has nonnegative length. This is by reduction to the weighted matching problem.

As J. Edmonds (cf. Grötschel and Pulleyblank [1981]) observed, another problem reducible to the weighted matching problem is: given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{Q}_+$, find a shortest odd $s-t$ path. Here a path is *odd* if it has an odd number of edges.

This reduction is as follows: make a copy $G' = (V', E')$ of G, and a copy $l' : E' \to \mathbb{Q}_+$ of l, add edges vv' for each $v \in V$ (where v' is the copy of v), each of length 0. Call the extended graph H. Then a minimum-length odd $s-t$ path in G can be found by finding a minimum-length perfect matching M in $H - s' - t'$: let N be the perfect matching $\{vv' \mid v \in V\}$ in H; then the component of $M \cup N$ containing s and t gives a shortest odd $s-t$ path in G.

Next consider the following polyhedron Q in $\mathbb{R}^E$:

(29.92) $$Q := \text{conv.hull}\{\chi^P \mid P \text{ odd } s-t \text{ path}\} + \mathbb{R}_+^E$$

and its blocking polyhedron

(29.93) $$B(Q) = \{x \in \mathbb{R}_+^E \mid x(P) \geq 1 \text{ for each odd } s-t \text{ path } P\}.$$

By the above method, one can optimize over Q in polynomial time. Hence, with the ellipsoid method, one can decide if a given $x \in \mathbb{Q}^E$ belongs to Q or not, and if not, find a separating facet. This also implies that for given capacity function $c : E \to \mathbb{Q}_+$, one can find in polynomial time a fractional packing of odd $s-t$ paths subject to c, of maximum value (by minimizing $c^\mathsf{T}x$ over $B(Q)$).

Schrijver and Seymour [1994] considered the problem (raised by Grötschel [1984]) of finding an explicit system of inequalities describing Q; equivalently, of describing the vertices of $B(Q)$.

Call a subset F of E *odd-blocking* if each odd $s-t$ path contains an edge in F. For each $F \subseteq E$, define $h_F \in \mathbb{Z}_+^E$ as follows, where $e = uv \in E$ and $W_F := \{s,t\} \cup \{v \in V \mid v \text{ is incident with at least one edge in } E \setminus F\}$:

(29.94) $$h_F(e) := \begin{cases} 2 & \text{if } u,v \in W_F \text{ and } e \in F, \\ 1 & \text{if exactly one of } u,v \text{ belongs to } W_F, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

(29.95) $$h_F = \sum_{v \in W_F} \chi^{\delta(v) \cap F}.$$

In particular, $h_F(e) = 0$ if $e \notin F$.

Note that for each $x \in \mathbb{Z}_+^E$ one has:

(29.96) $h_F^\mathsf{T}x \geq 1$ for each odd-blocking F $\iff$ there exists an odd $s-t$ path P with $\chi^P \leq x$ $\iff$ $h_F^\mathsf{T}x \geq 2$ for each odd-blocking F.

Then Schrijver and Seymour [1994] proved:

(29.97) Let $l : E \to \mathbb{Z}_+$ be a length function such that each circuit and each $s-t$ path has even length. Then the minimum length of an odd $s-t$ path is equal to the maximum value of $2k$ for which there exist odd-blocking sets $F_1, \ldots, F_k$ with $h_{F_1} + \cdots + h_{F_k} \leq l$.

This implies:

(29.98) Let $l : E \to \mathbb{Z}_+$ be a length function. Then the minimum length of an odd $s-t$ path is equal to the maximum value of k for which there exist odd-blocking $F_1, \ldots, F_k$ with $\frac{1}{2}h_{F_1} + \cdots + \frac{1}{2}h_{F_k} \leq l$.

This can be formulated in terms of LP-duality. Let $\mathcal{F}$ be the collection of odd-blocking sets and let H be the $\mathcal{F} \times E$ matrix whose Fth row equals h_F (for $F \in \mathcal{F}$). Then (29.98) states that for $l : E \to \mathbb{Z}_+$:

(29.99) $$\min\{l^\mathsf{T}x \mid x \in \mathbb{Z}_+^E, (\tfrac{1}{2}H)x \geq \mathbf{1}\} = \max\{y^\mathsf{T}\mathbf{1} \mid y \in \mathbb{Z}_+^{\mathcal{F}}, y^\mathsf{T}(\tfrac{1}{2}H) \leq l^\mathsf{T}\}.$$

Equivalently, the system

(29.100) $$\begin{array}{ll} x_e \geq 0 & e \in E, \\ \frac{1}{2}h_F^\mathsf{T}x \geq 1 & F \text{ odd-blocking,} \end{array}$$

determines Q and is TDI. Hence:

(29.101) each vertex of $B(Q)$ is equal to $\frac{1}{2}h_F$ for some odd-blocking $F \subseteq E$

(this implies the conjecture of W.J. Cook and A. Sebő that the vertices of $B(Q)$ are half-integer).

Minimizing $c^{\mathsf{T}}x$ over $B(Q)$ then gives the following. Let $G = (V, E)$ be an undirected graph, let $s, t \in V$, and let $c : E \to \mathbb{R}_+$. Then the maximum value of a fractional packing of odd $s-t$ paths subject to c is equal to the minimum value of

(29.102) $$\frac{1}{2} \sum_{v \in W_F} c(\delta(v) \cap F),$$

taken over odd-blocking $F \subseteq E$.

L. Lovász asked for the complexity of the following combination of two of the problems above: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$, such that each circuit has nonnegative length, find a shortest odd $s-t$ path.

29.11f. Further notes

Complexity survey for all-pairs shortest paths in undirected graphs without negative-length circuits ($*$ indicates an asymptotically best bound in the table):

$*$	$O(nm \log n)$	Gabow [1983a]
$*$	$O(n^3)$	Gabow [1983a]

(The algorithm proposed by Bernstein [1984] fails (for instance, for a graph with four vertices).)

Karzanov [1986] gave an $O(|T|m \log n + |T|^3 \log |T|)$-time algorithm to find a shortest T-join and a maximum fractional packing of T-cuts.

It is easy to see that the vertices of $P^{\uparrow}_{T\text{-join}}(G)$ are the incidence vectors of the inclusionwise minimal T-joins (that is, those T-joins that are a forest). Indeed, consider a T-join J. If J contains another T-join J' as subset, then $\chi^{J'} \leq \chi^J$, and hence χ^J is not a vertex of $P^{\uparrow}_{T\text{-join}}(G)$. Conversely, if χ^J is not a vertex, then $\chi^J \geq x$ for some convex combination x of incidence vectors T-joins. Each of these T-joins J' satisfies $\chi^{J'} \leq \chi^J$, and hence $J' \subseteq J$.

Similarly, an inequality $x(C) \geq 1$ for a T-cut C determines a facet if and only if C is an inclusionwise minimal T-cut.

Giles [1981] showed that two inclusionwise minimal T-joins J and J' give adjacent vertices of the polyhedron $P^{\uparrow}_{T\text{-join}}(G)$ if and only if $J \cup J'$ contains exactly one circuit. It implies that the distance of J and J' in $P^{\uparrow}_{T\text{-join}}(G)$ is at most $|J \setminus J'|$ — this implies the Hirsch conjecture for $P^{\uparrow}_{T\text{-join}}(G)$.

Gerards [1992b] showed the following. For any graph H, an *odd-H* is a subdivision of H such that each odd circuit of H becomes an odd circuit in the subdivision. In other words, the edges of H that become an even-length path form a cut in H. The *prism* is the complement of the 6-circuit C_6. Let $G = (V, E)$ be a graph not containing an odd-K_4 or an odd-prism as subgraph. Then for each $T \subseteq V$, the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. This generalizes Corollary 29.9c and Theorem 29.2.

Call a graph $G = (V, E)$ a *Seymour graph* if for each subset T of V for which there exists a T-join, the minimum-size of a T-join is equal to the maximum number of disjoint T-cuts. Ageev, Kostochka, and Szigeti [1995,1997] showed that G is a Seymour graph if and only if for each length function $l \in \mathbb{Z}^E$ with $l(C) \geq 0$ for each circuit C, and for each pair of circuits C_1 and C_2 with $l(C_1) = 0$ and $l(C_2) = 0$, the graph formed by $C_1 \cup C_2$ is neither an odd-K_4 nor an odd-prism. (Here sufficiency was proved by A. Sebő.)

Seymour [1981d] characterized for which pairs G, T with $|T| = 4$, the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. In fact, let $T = \{t_1, t_2, t_3, t_4\}$ and let $k \in \mathbb{Z}_+$. Then there is a packing of T-cuts of size k if and only if

(29.103)
$$\begin{aligned} &\text{dist}(t_1, t_2) + \text{dist}(t_3, t_4) \geq k, \\ &\text{dist}(t_1, t_3) + \text{dist}(t_2, t_4) \geq k, \\ &\text{dist}(t_1, t_4) + \text{dist}(t_2, t_3) \geq k, \end{aligned}$$

such that if equality holds in each of these inequalities, then $\text{dist}(t_1, t_2) + \text{dist}(t_1, t_3) + \text{dist}(t_2, t_3)$ is even.

Korach [1982] characterized such pairs for $|T| = 6$, and gave a polynomial-time algorithm recognizing them.

The existence of T-joins satisfying given upper bounds on the degrees can be characterized by reduction to Tutte's 1-factor theorem (cf. Ning [1987]).

Middendorf and Pfeiffer [1990b,1993] showed that it is NP-complete to decide, for given planar graph $G = (V, E)$ and $T \subseteq V$, if the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. As a minimum-size T-join can be found in polynomial time, it follows that it is NP-complete to determine a maximum packing of T-cuts. (Related results are given by Korach and Penn [1992], Korach [1994], and Granot and Penn [1995].)

The *directed* Chinese postman problem can be solved as a minimum-cost circulation problem (see Section 12.5b). The *mixed* Chinese postman problem (with directed and undirected edges) however is NP-complete (Papadimitriou [1976]). Guan [1984] derived from this that the *windy* (or *asymmetric*) *postman problem* (where the length of an edge may depend on the direction in which it is traversed) is NP-complete.

Edmonds and Johnson [1973] showed that the mixed Chinese postman problem in which each vertex has even total degree is polynomial-time solvable. (The *total degree* of a vertex v is the total number of edges (directed and undirected) incident with v.) Similarly, Guan and Pulleyblank [1985] and Win [1989] showed that the windy postman problem is solvable in polynomial time if the graph is Eulerian (by reduction to a minimum-cost circulation problem). More on the windy postman can be found in Grötschel and Win [1992], Pearn and Li [1994], and Raghavachari and Veerasamy [1999b].

For approximation algorithms for the mixed postman problem, see Frederickson [1979] and Raghavachari and Veerasamy [1998,1999a]. Further work on the mixed postman problem is reported in Kappauf and Koehler [1979], Minieka [1979], Brucker [1981], Christofides, Benavent, Campos, Corberán, and Mota [1984], Ralphs [1993], and Nobert and Picard [1996].

An extension of the Edmonds-Gallai decomposition to T-joins was given by Sebő [1990b] (cf. Sebő [1986,1997]). Goemans and Williamson [1992,1995a] gave a fast 2-approximative algorithm for finding a shortest T-join.

Benczúr and Fülöp [2000] give fast algorithms for finding minimum-size T-cut, with generalization to directed graphs.

Tobin [1975] studied finding a negative-length circuit with Edmonds' algorithm. For more on packing T-joins, see Rizzi [1997]. For surveys on T-joins and T-cuts, see Sebő [1988a] and Frank [1996a].

29.11g. On the history of the Chinese postman problem

In a paper in Chinese in *Acta Mathematica Sinica*, entitled (in translation) 'Graphic programming using odd or even points', Guan [1960] introduced the problem of finding a shortest postman route:

> When the author was plotting a diagram for a mailman's route, he discovered the following problem: "A mailman has to cover his assigned segment before returning to the post office. The problem is to find the shortest walking distance for the mailman."

(In a footnote it is mentioned that 'In postal service, a mailman's route is called a segment'.) Next:

> This problem can be reduced to the following: "Given a connected graph in the plane, we are to draw a continuous graph (repetition permitted) from a given point and back minimizing the number of repeated arcs."

So Guan restricted himself to planar graphs. He observed that a postman never has to traverse any edge more than twice. Hence the problem amounts to finding a minimum-length set J of edges such that adding a parallel edge to each of them, gives an Eulerian graph. He next gave an algorithm, which consist of starting with any such set J, and next iteratively improving it by finding a circuit C such that the length of $J \cap C$ is larger than half of the length of C, and replacing J by $J \triangle C$. As in each iteration the length of J decreases, the method finds a shortest route after a finite number of steps.

In a review in Mathematical Reviews of the article of Guan [1960], Fulkerson [1964a] observed:

> Unfortunately, the construction involves examining all simple cycles to see whether the minimality test is met or not, and this is easier said than done.

Therefore, Edmonds [1965e] announced a better method in an abstract for the 27th National Meeting of the Operations Research Society of America (May 1965 in Boston):

> We present an algorithm which does not involve examining simple cycles. It is "good" in the sense that the amount of work in applying it is at worst moderately algebraic, relative to the size of the graph, rather than exponential. It combines two earlier known algorithms: (1) the well-known "shortest path" algorithm, (2) a recent algorithm for "maximum matching".

The name of the problem seems to occur first in the title of this abstract: 'The Chinese Postman's Problem' (where 'The Chinese's Postman Problem' would be more appropriate).

Chapter 30

2-matchings, 2-covers, and 2-factors

The results on matchings are strongly self-refining, as was pointed out by Tutte [1952,1954b] and Edmonds and Johnson [1970,1973]. In this chapter we see a first instance of this phenomenon. By splitting vertices, results on 2-matchings can be derived from those on ordinary matchings. 2-matchings are of interest for the traveling salesman problem.

30.1. 2-matchings and 2-vertex covers

Let $G = (V, E)$ be an undirected graph. A 2-*matching* is a vector $x \in \mathbb{Z}_+^E$ satisfying $x(\delta(v)) \leq 2$ for each vertex v. A 2-*vertex cover* is a vector $y \in \mathbb{Z}_+^V$ such that $y_u + y_v \geq 2$ for each edge uv of G. Defining the *size* of a vector as the sum of its entries, we denote:

(30.1) $\nu_2(G) :=$ the maximum size of a 2-matching in G,
$\tau_2(G) :=$ the minimum size of a 2-vertex cover in G.

Note that

(30.2) $\tau_2(G) = \min\{|V \setminus S| + |N(S)| \mid S \subseteq V, S \text{ stable set}\}$,

since for a minimum-size 2-vertex cover y, the set $S := \{v \in V \mid y_v = 0\}$ is a stable set, while $N(S) = \{v \in V \mid y_v = 2\}$, and since $\chi^{V \setminus S} + \chi^{N(S)}$ is a 2-vertex cover for each stable set S.

Note also that

(30.3) $\nu(G) \leq \frac{1}{2}\nu_2(G) \leq \frac{1}{2}\tau_2(G) \leq \tau(G)$.

The following is a special case of a theorem of Gallai [1957,1958a,1958b] (cf. Theorem 31.7), and can be derived from Kőnig's matching theorem.

Theorem 30.1. *$\nu_2(G) = \tau_2(G)$ for any graph G. That is, the maximum size of a 2-matching is equal to the minimum size of a 2-vertex cover.*

Proof. Make for each vertex v of G a new vertex v', and replace each edge uv of G by two edges $u'v$ and uv'. This makes the bipartite graph H. By Kőnig's matching theorem (Theorem 16.2), H has a vertex cover C and a matching M with $|C| = |M|$. For any edge $e = uv$ of G let $x_e := |\{u'v, uv'\} \cap M|$ and for any vertex v of G let $y_v := |\{v, v'\} \cap C|$. Then x is a 2-matching and y is a 2-vertex cover with $x(E) = |M| = |C| = y(V)$. ∎

This construction was given by Nemhauser and Trotter [1975]. It also yields a polynomial-time reduction of the problems of finding a maximum-size 2-matching and a minimum-size 2-vertex cover to the problems of finding a minimum-size matching and a maximum-size vertex cover in a bipartite graph — hence these problems are polynomial-time solvable.

Call a 2-matching x *perfect* if $x(\delta(v)) = 2$ for each vertex v. So a 2-matching x is perfect if and only if $x(E) = |V|$. Theorem 30.1 implies a characterization of the existence of a perfect 2-matching (Tutte [1952]):

Corollary 30.1a. *Let $G = (V, E)$ be a graph. Then G has a perfect 2-matching if and only if $|N(S)| \geq |S|$ for each stable set S.*

Proof. Directly from Theorem 30.1, since G has a perfect 2-matching $\iff$ $\nu_2(G) \geq |V| \iff \tau_2(G) \geq |V|$. With (30.2), this last is equivalent to the condition of the present corollary. ∎

As finding a perfect 2-matching can be reduced to finding a maximum-size 2-matching, it is polynomial-time solvable.

30.2. Fractional matchings and vertex covers

Any vector $x \in \mathbb{R}^E$ satisfying

(30.4)
$$\begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for } v \in V, \end{array}$$

is called a *fractional matching*. The maximum size $x(E)$ of a fractional matching is called the *fractional matching number*, denoted by $\nu^*(G)$. By linear programming duality, $\nu^*(G)$ is equal to the *fractional vertex cover number* $\tau^*(G)$ — the minimum size of a *fractional vertex cover*, which is any solution $y \in \mathbb{R}^V$ of

(30.5)
$$\begin{array}{lll} \text{(i)} & 0 \leq y_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & y_u + y_v \geq 1 & \text{for } uv \in E. \end{array}$$

The equality $\nu^*(G) = \tau^*(G)$ also follows from Theorem 30.1, since trivially

(30.6)
$$\tfrac{1}{2}\nu_2(G) \leq \nu^*(G) \leq \tau^*(G) \leq \tfrac{1}{2}\tau_2(G).$$

(An extension to infinite graphs was given by Aharoni and Ziv [1990].)

30.3. The fractional matching polytope

Let $G = (V, E)$ be a graph. The *fractional matching polytope* of G is the polytope determined by (30.4). Balinski [1965] showed:

Theorem 30.2. *Each vertex of the fractional matching polytope of G is half-integer.*

Proof. Let x be a vertex of the fractional matching polytope. We can assume that $x_e > 0$ for each edge e, since if $x_e = 0$ we can apply induction to $G - e$. Hence we can assume also that $x_e < 1$ for each edge e; equivalently, that each vertex of G has degree at least two.

As x is a vertex, there are $|E|$ constraints among (30.4)(ii) satisfied with equality. So $|E| \leq |V|$, implying that G is 2-regular. Then $x_e = \frac{1}{2}$ for each $e \in E$, as it is a solution to setting (30.4)(ii) to equality, and as the solution must be unique (as x is a vertex). ∎

Balinski [1965] also observed that the support of any vertex x of the fractional matching polytope can be partitioned into a matching M, with $x_e = 1$ for $e \in M$, and a set of odd circuits, vertex-disjoint and disjoint from M, with $x_e = \frac{1}{2}$ for each edge e in any of the odd circuits.

30.4. The 2-matching polytope

The 2-*matching polytope* of G is the convex hull of the 2-matchings in G. Theorem 30.2 implies a characterization of the 2-matching polytope (Edmonds [1965b]):

Corollary 30.2a. *The* 2*-matching polytope is determined by:*

(30.7)
(i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \leq 2$ for $v \in V$.

Proof. Directly from Theorem 30.2, since it implies that the vertices of the polytope determined by (30.7) are integer, and hence are 2-matchings. ∎

Given a graph $G = (V, E)$, the *perfect* 2*-matching polytope* of G is the convex hull of the perfect 2-matchings in G. As the perfect 2-matching polytope is a face of the 2-matching polytope (if nonempty), Corollary 30.2a implies (Edmonds [1965b]):

Corollary 30.2b. *The perfect* 2*-matching polytope is determined by*

(30.8)
(i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) = 2$ for $v \in V$.

Proof. Directly from Corollary 30.2a. ∎

Pulleyblank [1987] related the vertices of the 2-matching polytope with the Edmonds-Gallai decomposition of the graph.

Similar results as for fractional matchings and 2-matchings hold for fractional vertex covers and 2-vertex covers. We discuss them in Section 64.6.

30.5. The weighted 2-matching problem

Given a graph $G = (V, E)$ and a weight function $w \in \mathbb{Q}^E$, the *weight* of a 2-matching x is $w^\mathsf{T} x$. The weighted 2-matching problem is strongly polynomial-time solvable:

Theorem 30.3. *A maximum-weight 2-matching can be found in time $O(n^3)$.*

Proof. Make the bipartite graph H as in the proof of Theorem 30.1, with weight function $w'(u'v) := w'(uv') := w(uv)$ for each edge uv of G. Then a maximum-weight matching in the new graph gives a maximum-weight 2-matching in the original graph. So Theorem 17.4 gives the present theorem. ∎

One can derive similarly from Egerváry's theorem a characterization of the maximum weight of a 2-matching, given by Gallai [1957,1958a,1958b]. Given $w : E \to \mathbb{Z}_+$, call a vector $y : V \to \mathbb{Z}_+$ a *w-vertex cover* if $y_u + y_v \geq w(e)$ for each edge $e = uv$.

Theorem 30.4. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Z}_+^E$. Then the maximum weight $w^\mathsf{T} x$ of a 2-matching x is equal to the minimum size of a $2w$-vertex cover.*

Proof. It is easy to see that the maximum cannot be larger than the minimum. To see equality, make the bipartite graph H as in the proof of Theorem 30.1, with weight $w'(u'v) := w'(uv') := w(uv)$ for each edge uv of G. Then the maximum w-weight of a 2-matching in G is equal to the maximum w'-weight of a matching in H. By Theorem 17.1, the latter is equal to the minimum of $y'(V \cup V')$ where $y' : V \cup V' \to \mathbb{Z}_+$ with $y'(u) + y'(v') \geq w(uv)$ and $y'(u') + y'(v) \geq w(uv)$ for each edge uv of G. Defining $y_v := y'_v + y'_{v'}$ for each $v \in V$, we obtain y as required. ∎

System (30.7) is generally not totally dual integral: if $G = (V, E)$ is the complete graph K_3 on three vertices, and $w(e) := 1$ for each $e \in E$, then the maximum weight of a 2-matching is equal to 3, while there is no integer dual solution of odd value (when considering the dual of maximizing $w^\mathsf{T} x$ subject to (30.7)).

However, *half*-integrality holds:

Corollary 30.4a. *System* (30.7) *is totally dual half-integral.*

Proof. This is equivalent to Theorem 30.4. ∎

Pulleyblank [1973,1980] showed that (30.7) can be extended to a TDI system as follows:

Corollary 30.4b. *The following system is totally dual integral:*

$$\text{(30.9)} \qquad \begin{array}{rlll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2 & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq |U| & \text{for } U \subseteq V. \end{array}$$

Proof. Choose $w \in \mathbb{Z}_+^E$. By Corollary 30.4a, the problem of maximizing $w^\mathsf{T} x$ over (30.7) has an optimum dual solution $y \in \frac{1}{2}\mathbb{Z}_+^V$. Let $y'_v := \lfloor y_v \rfloor$ and $T := \{v \in V \mid y_v \notin \mathbb{Z}\}$. Let $z_T := 1$ and $z_U := 0$ for each $U \subseteq V$ with $U \neq T$. Then y', z is an integer optimum dual solution of the problem of maximizing $w^\mathsf{T} x$ over (30.9). ∎

Corollary 30.4a gives the total dual half-integrality of the perfect 2-matching constraints (30.8):

Corollary 30.4c. *System* (30.8) *is totally dual half-integral.*

Proof. Directly from Corollary 30.4a. ∎

More strongly, one has:

Corollary 30.4d. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^\mathsf{T} x$ subject to* (30.8) *has an integer optimum dual solution.*

Proof. As $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 30.4c there is an integer optimum dual solution y'_v $(v \in V)$ for the problem of minimizing $w'^\mathsf{T} x$ subject to (30.8). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution y for w. ∎

30.5a. Maximum-size 2-matchings and maximum-size matchings

Uhry [1975] gave the following relation between maximum-size 2-matchings and maximum-size matchings:

Theorem 30.5. *For each maximum-size* 2*-matching* x *in a graph* G*, there exists a maximum-size matching* M *missing each vertex* v *with* $x(\delta(v)) = 0$*.*

Proof. Let x be a maximum-size 2-matching in G and let M be a maximum-size matching covering a minimum number of vertices v with $x(\delta(v)) = 0$. Suppose that M covers a vertex u with $x(\delta(u)) = 0$. To prove the theorem, we can assume that x has inclusionwise minimal support. This implies that the edges e with $x_e = 1$ form a collection of vertex-disjoint odd circuits.

Let N be the matching consisting of those edges e with $x_e = 2$. Let P be the component of $M \cup N$ containing u. Then P is a path starting at u, and ending at, say, w. If P has even length, then $M \triangle P$ is a maximum-size matching covering fewer vertices v with $x(\delta(v)) = 0$ than M does — a contradiction. So P has odd length, and hence, since x is a maximum-size 2-matching, w belongs to the vertex set of some odd circuit C consisting of edges e with $x_e = 1$. However, in that case we can augment x, by redefining $x_e := 0$ if $e \in P \cap N$, $x_e := 2$ if $e \in P \cap M$, and $x_e := 0$ or 2 alternatingly on the edges of C. ∎

Uhry [1975] (cf. Pulleyblank [1987]) related maximum-size 2-matchings and maximum-size matchings further by:

Theorem 30.6. *Let* x *be a maximum-size* 2*-matching with the set* $\{e \mid x_e = 1\}$ *inclusionwise minimal. Then the support of* x *contains a maximum-size matching* M *of* G*.*

Proof. As the set $F := \{e \mid x_e = 1\}$ is inclusionwise minimal, it forms a collection $\mathcal{C}$ of vertex-disjoint odd circuits. So $x(\delta(v)) = 0$ or 2 for each vertex v. By Theorem 30.5, we can assume that $x(\delta(v)) = 2$ for each $v \in V$, since deleting all vertices v with $x(\delta(v)) = 0$ does not decrease the maximum size of a matching.

Let M be a maximum-size matching containing a minimum number of edges e with $x_e = 0$. Let N be the matching consisting of those edges e with $x_e = 2$. Consider any component P of $M \cup N$. Then P is not a circuit or an even path of positive length, since otherwise $M \triangle P$ is a maximum-size matching having fewer edges e with $x_e = 0$ than M has — a contradiction. So if P is not a singleton, it is a path of odd length; let it connect vertices u and w. Since P is not M-augmenting, both u and w are vertices on odd circuits in $\mathcal{C}$, say on C_u and C_w respectively. If $C_u \neq C_w$, we can modify x so as to decrease the set of edges e with $x_e = 1$. So $C_u = C_w$.

It follows that each $C \in \mathcal{C}$ contains an even number of vertices covered by M, and hence an odd number of vertices missed by M. Hence

(30.10) $$2|M| \leq |V| - |\mathcal{C}| = 2|N| + \sum_{C \in \mathcal{C}} (|C| - 1).$$

Therefore, by augmenting N with a matching of size $\frac{1}{2}(|C| - 1)$ contained in C, for each circuit $C \in \mathcal{C}$, we obtain a matching M' with $|M'| \geq |M|$ contained in the support of x. ∎

(Theorem 30.6 was generalized in (30.88).) Related results were obtained by Balas [1981].

Mühlbacher, Steinparz, and Tinhofer [1984] showed that if x is a vertex of the 2-matching polytope maximizing $|\{e \in E \mid x_e = 2\}|$, then the vector $(i_3(x), i_5(x), \ldots)$ is lexicographically maximal, where $i_k(x)$ is the number of circuits in the support of x of size k. For related work, see Mühlbacher [1979] and Hell and Kirkpatrick [1981].

30.6. Simple 2-matchings and 2-factors

Call a 2-matching x *simple* if x is a 0,1 vector. So we can identify simple 2-matchings with subsets F of E satisfying $\deg_F(v) \leq 2$ for each $v \in V$.

A construction of Tutte [1954b] gives the following characterization of the maximum size of a simple 2-matching, with the help of the Tutte-Berge formula ($E[K,S]$ denotes the set of edges connecting K and S):

Theorem 30.7. *Let $G = (V,E)$ be a graph. The maximum size of a simple 2-matching is equal to the minimum value of*

$$(30.11) \qquad |V| + |U| - |S| + \sum_K \lfloor \tfrac{1}{2}|E[K,S]| \rfloor,$$

where U and S are disjoint subsets of V, with S a stable set, and where K ranges over the components of $G - U - S$.

Proof. To see that the maximum is not more than the minimum, let F be a simple 2-matching and let U and S be disjoint subsets of V, with S a stable set. Then F has at most $2|U|$ edges incident with U. Moreover, for each component K of $G-U-S$, the number of edges in F spanned by $K \cup S$ is at most $|K| + \lfloor \frac{1}{2}|E[K,S]| \rfloor$, since

$$(30.12) \qquad \begin{aligned} &2|F \cap E[K \cup S]| = 2|F \cap E[K]| + 2|F \cap E[K,S]| \\ &\leq 2|F \cap E[K]| + |F \cap E[K,S]| + |E[K,S]| \leq 2|K| + |E[K,S]|. \end{aligned}$$

Hence

$$(30.13) \qquad |F| \leq 2|U| + \sum_K (|K| + \lfloor \tfrac{1}{2}|E[K,S]| \rfloor)$$

(where K ranges over the components of $G-U-S$), giving that F is at most (30.11).

To see the reverse inequality, make a graph $G' = (V', E')$ as follows. For each vertex v of G, introduce vertices v' and v'' of G'. For each edge $e = uv$ of G, introduce vertices $p_{e,u}$ and $p_{e,v}$ and edges

$$(30.14) \qquad u'p_{e,u}, u'p_{e,u}, p_{e,u}p_{e,v}, v'p_{e,v}, v''p_{e,v}.$$

This defines all vertices and edges of G'.

Now:

$$(30.15) \qquad \nu_2^{\mathrm{s}}(G) = \nu(G') - |E|,$$

where $\nu(G')$ denotes the maximum size of a matching in G' and $\nu_2^{\rm s}(G)$ denotes the maximum size of a simple 2-matching in G. In this proof we only need $\geq$ in (30.15). This inequality holds as there is a maximum-size matching M in G' with the property that for each edge $e = uv$ of G, both vertices $p_{e,u}$ and $p_{e,v}$ of G' are covered by M. Then the edges e of G for which edge $p_{e,u}p_{e,v}$ does not belong to M, form a simple 2-matching N in G with $|N| = |M| - |E|$. So we have $\geq$ in (30.15).

By the Tutte-Berge formula (Theorem 24.1), there is a subset X of V' such that the number $o(G' - X)$ of odd components of $G' - X$ is at least $|V'| - 2\nu(G') + |X|$. We take X inclusionwise minimal with this property.

Then for each $v \in V$, if one of v', v'' does not belong to X, then both do not belong to X. For suppose $v' \in X$ and $v'' \notin X$. As v' and v'' have the same set of neighbours in G', removing v' from X, decreases X by 1 and decreases the number of odd components of $o(G' - X)$ by at most one. So we would obtain a smaller set X as required, contradicting the minimality assumption.

Consider any vertex v of G and any edge $e = uv$ of G with $p_{e,v} \in X$. Then the three neighbours of $p_{e,v}$ in G' belong to three different odd components of $G' - X$. (Otherwise, removing $p_{e,v}$ from X decreases X by 1, and decreases $o(G' - X)$ be at most 1, contradicting the minimality of X.) Hence $p_{e,u}, v', v'' \notin X$, and moreover $p_{f,v} \in X$ for each edge f of G incident with v.

Let U be the set of $v \in V$ for which $v', v'' \in X$ and let S be the set of $v \in V$ for which $p_{e,v} \in X$ for each edge e of G incident with v. So U and S are disjoint, and S is a stable set.

Then $|X| = 2|U| + |\delta(S)|$. Let κ denote the number of components K of $G - U - S$ with $|E[K, S]|$ odd. Then

(30.16) $$o(G' - X) = 2|S| + |E[U, S]| + \kappa.$$

Hence we have

(30.17) $$\begin{aligned} \nu_2^{\rm s}(G) &\geq \nu(G') - |E| \geq \tfrac{1}{2}(|V'| + |X| - o(G' - X)) - |E| \\ &= |V| + |U| + \tfrac{1}{2}|\delta(S)| - |S| - \tfrac{1}{2}|E[U, S]| - \tfrac{1}{2}\kappa \\ &= |V| + |U| - |S| + \sum_K \lfloor \tfrac{1}{2}|E[K, S]| \rfloor \end{aligned}$$

(where K ranges over the components of $G - U - S$), as required. ∎

A 2-*factor* is a simple perfect 2-matching. Equivalently, it is a subset F of E with $\deg_F(v) = 2$ for each $v \in V$.

Theorem 30.7 implies the following result of Belck [1950] (also Gallai [1950] announced a characterization of the existence of a 2-factor):

Corollary 30.7a. *A graph $G = (V, E)$ has a 2-factor if and only if*

(30.18) $$|S| \leq |U| + \sum_K \lfloor \tfrac{1}{2}|E[K, S]| \rfloor$$

for each pair of disjoint subsets U, S of V, with S a stable set, where K ranges over the components of $G - U - S$.

Proof. Directly from Theorem 30.7. ∎

This implies a classical result of Petersen [1891]:

Corollary 30.7b. *Each $2k$-regular graph has a 2-factor.*

Proof. Let $G = (V, E)$ be $2k$-regular. We check (30.18). Let U and S be disjoint subsets of V, with S a stable set. Let l be the number of components K of $G - U - S$ with $|E[K, S]|$ odd. Then for each such component K we have $|E[K, U]| \geq 1$ (since G is Eulerian). Hence $|E[U, S]| \leq 2k|U| - l$. Therefore,

$$\begin{aligned}(30.19)\qquad 2k|S| &= |\delta(S)| = |E[U, S]| + \sum_K |E[K, S]| \\ &\leq 2k|U| - l + \sum_K |E[K, S]| = 2k|U| + \sum_K 2\lfloor \tfrac{1}{2}|E[K, S]|\rfloor \\ &\leq 2k(|U| + \sum_K \lfloor \tfrac{1}{2}|E[K, S]|\rfloor)\end{aligned}$$

(where K ranges over the components of $G - U - S$), and (30.18) follows. ∎

The construction above gives also a reduction of finding a maximum-weight simple 2-matching to finding a maximum-weight matching — hence it can be done in strongly polynomial time. This implies that also a minimum-weight 2-factor can be found in strongly polynomial time.

(Grötschel and Holland [1987] gave computational results on a cutting plane method to find a minimum-weight 2-factor.)

30.7. The simple 2-matching polytope and the 2-factor polytope

Given a graph $G = (V, E)$, the *simple 2-matching polytope* is the convex hull of the simple 2-matchings in G. It can be characterized as follows (Edmonds [1965b]):

Theorem 30.8. *The simple 2-matching polytope is determined by*

$$(30.20)\qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq |U| + \lfloor \tfrac{1}{2}|F|\rfloor & (U \subseteq V, F \subseteq \delta(U), \\ & & F \text{ matching}, |F| \text{ odd}). \end{array}$$

Proof. It is easy to show that each simple 2-matching x satisfies (30.20). Condition (iii) follows from

(30.21) $\qquad x(E[U]) + x(F) \leq x(E[U]) + \frac{1}{2}x(\delta(U)) + \frac{1}{2}x(F) \leq |U| + \frac{1}{2}|F|$

if x is a simple 2-matching.

To show that (30.20) is enough to determine the simple 2-matching polytope, we first show that (30.20) implies an extended version of (30.20)(iii), where we delete the condition that F be a matching. This can be seen by induction on $|F|$. Indeed, suppose that F contains edges f_1, f_2 incident with a vertex v. Let $F' := F \setminus \{f_1, f_2\}$. Then, if $v \in U$, setting $U' := U \setminus \{v\}$:

(30.22) $\qquad x(E[U]) + x(F) \leq x(E[U']) + x(F') + x(\delta(v)) \leq |U'| + \frac{1}{2}|F'| + 2$
$\qquad\qquad = |U| + \frac{1}{2}|F|$.

If $v \notin U$, setting $U' := U \cup \{v\}$:

(30.23) $\qquad x(E[U]) + x(F) \leq x(E[U']) + x(F') \leq |U'| + \frac{1}{2}|F'| = |U| + \frac{1}{2}|F|$.

So we can delete in (iii) the requirement that F be a matching.

We now prove that the conditions determine the simple 2-matching polytope. Let $G' = (V', E')$ be as in the proof of Theorem 30.7. Let x satisfy (30.20). Define $x' \in \mathbb{R}^{E'}$ by

(30.24) $\qquad x'(u'p_{e,u}) := x'(u''p_{e,u}) := x'(v'p_{e,v}) := x'(v''p_{e,v}) := \frac{1}{2}x_e$ and
$\qquad\qquad x'(p_{e,u}p_{e,v}) := 1 - x_e$,

for any edge $e = uv$ of G. We show that x' belongs to the matching polytope of G'.

That is, by Edmonds' matching polytope theorem (Corollary 25.1a), we should check

(30.25) $\qquad$ (i) $x'(e') \geq 0$ for $e' \in E'$,
$\qquad\qquad$ (ii) $x'(\delta'(v')) \leq 1$ for $v' \in V'$,
$\qquad\qquad$ (iii) $x'(E'[Y]) \leq \lfloor \frac{1}{2}|Y| \rfloor$ for $Y \subseteq V'$ with $|Y|$ odd,

where $\delta' := \delta_{G'}$ and where $E'[Y]$ is the set of edges in E' spanned by Y.

Trivially we have (30.25)(i) and (ii) by (30.20)(i) and (ii). To prove (30.25)(iii), let Y violate (30.25)(iii). We first show that if one of v', v'' belongs to Y, then both belong to Y. For suppose that $v' \in Y$ and $v'' \notin Y$. Let $Y_1 := Y \setminus \{v'\}$ and $Y_2 := Y \cup \{v''\}$. Then

(30.26) $\qquad x'(E'[Y]) = \frac{1}{2}(x'(E'[Y_1]) + x'(E'[Y_2])) \leq x'(E'[Y_1]) + \frac{1}{2}x'(\delta'(Y_1))$
$$= \frac{1}{2}\sum_{u \in Y_1} x'(\delta'(u)) \leq \frac{1}{2}|Y_1| = \lfloor \tfrac{1}{2}|Y| \rfloor,$$

a contradiction.

We choose Y with $|Y| + |\delta'(Y)|$ minimal. Then:

(30.27) $\qquad$ (i) if $u', v' \in Y$, then $p_{e,u} \in Y$ and $p_{e,v} \in Y$,
$\qquad\qquad$ (ii) if $p_{e,u} \in Y$, then $u' \in Y$.

To see (30.27)(i), first suppose that $u', v' \in Y$ and $p_{e,u} \notin Y$. Define $Y' := Y \cup \{p_{e,u}, p_{e,v}\}$. Then $|Y'| + |\delta'(Y')| < |Y| + |\delta'(Y)|$, and hence Y' satisfies inequality (30.25)(iii). Therefore, (30.28) $$x'(E'[Y]) \leq x'(E'[Y']) - x'(\delta'(p_{e,u})) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor - 1 \leq \lfloor \tfrac{1}{2}|Y| \rfloor.$$

This contradicts our assumption that Y violates (30.25)(iii).

To see (30.27)(ii), let $p_{e,u} \in Y$ and $u' \notin Y$. Define $Y' := Y \setminus \{p_{e,u}, p_{e,v}\}$. Again $|Y'| + |\delta'(Y')| < |Y| + |\delta'(Y)|$, and hence Y' satisfies inequality (30.25)(iii). If $p_{e,v} \notin Y$, then

(30.29) $$x'(E'[Y]) = x'(E'[Y']) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor \leq \lfloor \tfrac{1}{2}|Y| \rfloor.$$

If $p_{e,v} \in Y$, then

(30.30) $$x'(E'[Y]) \leq x'(E'[Y']) + x'(\delta'(p_{e,v})) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor + 1 = \lfloor \tfrac{1}{2}|Y| \rfloor.$$

Both (30.29) and (30.30) contradict our assumption that Y does not satisfy (30.25)(iii). This proves (30.27).

Let $U := \{v \in V \mid v', v'' \in Y\}$ and let F be the set of those edges $e = uv$ in $\delta(U)$ with $u \in U$, $v \notin U$, and $p_{e,u} \in Y$. Then $x'(E'[Y]) = x(E[U]) + |E[U]| + x(F)$ and $|Y| = 2|U| + 2|E[U]| + |F|$. Hence (30.20)(iii) implies (30.25)(iii).

So x' is a convex combination of incidence vectors of matchings in G'. Each such vector y satisfies $y(\delta'(v')) = 1$ for each vertex $v' = p_{e,u}$ (as x' satisfies this equality). Hence each such matching corresponds to a simple 2-matching in G, and we obtain x as convex combination of simple 2-matchings in G. ∎

Given a graph $G = (V, E)$, the 2-*factor polytope* is the convex hull of (the incidence vectors of) 2-factors in G. Then:

Corollary 30.8a. *The* 2*-factor polytope is determined by*

(30.31)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) = 2 & (v \in V), \\ \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V,\ F \subseteq \delta(U), \\ & & F \text{ matching, } |F| \text{ odd}). \end{array}$$

Proof. Directly from Theorem 30.8, since (30.31)(ii) implies $x(E[U]) = |U| - \frac{1}{2}x(\delta(U))$. ∎

Notes. Grötschel [1977a] characterized the facets of the simple 2-matching polytope and of the 2-factor polytope of the complete graph K_n. Rispoli and Cosares [1998] showed that the diameter of the 2-factor polytope of a complete graph is at most 6. Rispoli [1994] showed that the 'monotonic diameter' of the 2-factor polytope is equal to $\lfloor \frac{1}{2}n \rfloor$ if $n \geq 5$ and $n \neq 8, 9$, and to $\lfloor \frac{1}{2}n \rfloor - 1$ if $n = 3, 4, 8, 9$.

Boyd and Carr [1999] showed that if $G = (V, E)$ is a complete graph and $l : E \to \mathbb{R}_+$ satisfies the triangle inequality, then the minimum value of $l^{\mathsf{T}}x$ over (30.31) is at most $\frac{4}{3}$ times the minimum value of $l^{\mathsf{T}}x$ over (30.31)(i)(ii). They also show that the factor $\frac{4}{3}$ is best possible.

30.8. Total dual integrality

Consider the system

(30.32)
(i) $0 \le x_e \le 1$ $(e \in E)$,
(ii) $x(\delta(v)) \le 2$ $(v \in V)$,
(iii) $x(E[U]) + x(F) \le |U| + \lfloor \frac{1}{2}|F| \rfloor$ $(U \subseteq V, F \subseteq \delta(U),$ F matching).

(So $|F|$ is not required to be odd.)

It is a special case of Theorem 32.3 (cf. Cook [1983b]) that system (30.32) is TDI (the restriction in (30.32) that F is a matching follows from (30.22) and (30.23)). This implies that (30.31) is totally dual half-integral. This also gives:

(30.33) Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^\mathsf{T} x$ subject to (30.31) has an integer optimum dual solution.

To see this, notice that if $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence there is an integer optimum dual solution y'_v $(v \in V)$, for the problem of minimizing ${w'}^\mathsf{T} x$ subject to (30.31). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w.

30.9. 2-edge covers and 2-stable sets

Let $G = (V, E)$ be an undirected graph. A 2-*edge cover* is a vector $x \in \mathbb{Z}_+^E$ satisfying $x(\delta(v)) \ge 2$ for each vertex v. A 2-*stable set* is a vector $y \in \mathbb{Z}_+^V$ such that $y_u + y_v \le 2$ for each edge uv of G. Defining the *size* of a vector as the sum of its entries, we denote:

(30.34)
$\rho_2(G) :=$ the minimum size of a 2-edge cover in G,
$\alpha_2(G) :=$ the maximum size of a 2-stable set in G.

Note that if G has no isolated vertices, then:

(30.35) $\alpha_2(G) = \max\{|V| + |U| - |N(U)| \bigm| U \subseteq V, U \text{ stable set}\}$

and that

(30.36) $\alpha(G) \le \frac{1}{2}\alpha_2(G) \le \frac{1}{2}\rho_2(G) \le \rho(G)$.

Gallai's theorem (Theorem 19.1) can be extended to 2-matchings and 2-stable sets, which was published also in Gallai [1959a]:

Theorem 30.9. *For any graph $G = (V, E)$ without isolated vertices:*

(30.37) $\alpha_2(G) + \tau_2(G) = \nu_2(G) + \rho_2(G) = 2|V|$.

Proof. Let x be a minimum-size 2-vertex cover. Then $x_v \leq 2$ for each vertex v. Define $y_v := 2 - x_v$ for each vertex v. Then y is a 2-stable set, and hence $\alpha_2(G) \geq y(V) = 2|V| - x(V) = 2|V| - \tau_2(G)$.

Conversely, let y be a maximum-size 2-stable set. Then $y_v \leq 2$ for each vertex v. Define $x_v := 2 - y_v$ for each vertex v. Then x is a 2-vertex cover, and hence $\tau_2(G) \leq x(V) = 2|V| - y(V) = 2|V| - \alpha_2(G)$. This shows that $\alpha_2(G) + \tau_2(G) = 2|V|$.

To see that $\nu_2(G) + \rho_2(G) = 2|V|$, let x be a minimum-size 2-edge cover. For each $v \in V$, reduce $x(\delta(v))$ by $x(\delta(v)) - 2$, by reducing x_e on edges $e \in \delta(v)$. We obtain a 2-matching y of size

(30.38) $$y(E) \geq x(E) - \sum_{v\in V}(x(\delta(v)) - 2) = 2|V| - x(E) = 2|V| - \rho_2(G).$$

Hence $\nu_2(G) \geq 2|V| - \rho_2(G)$.

Conversely, let y be a maximum-size 2-matching. For each $v \in V$, increase $y(\delta(v))$ by $2 - y(\delta(v))$, by increasing y_e on edges $e \in \delta(v)$. We obtain a 2-edge cover x of size

(30.39) $$x(E) \leq y(E) + \sum_{v\in V}(2 - y(\delta(v))) = 2|V| - y(E) = 2|V| - \nu_2(G).$$

Hence $\rho_2(G) \leq 2|V| - \nu_2(G)$. ∎

This implies the following, which is a special case of a theorem of Gallai [1957,1958a,1958b] (cf. Theorem 30.11) (and can be derived alternatively from the Kőnig-Rado edge cover theorem):

Corollary 30.9a. $\alpha_2(G) = \rho_2(G)$ *for any graph G without isolated vertices. That is, the maximum size of a 2-stable set is equal to the minimum size of a 2-edge cover.*

Proof. Directly from Theorems 30.1 and 30.9. ∎

These reductions also imply the polynomial-time solvability of the problems of finding a minimum-size 2-edge cover and a maximum-size 2-stable set.

30.10. Fractional edge covers and stable sets

Any vector $x \in \mathbb{R}^E$ satisfying

(30.40)
(i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 1$ for $v \in V$,

is called a *fractional edge cover*. The minimum size $x(E)$ of a fractional edge cover is called the *fractional edge cover number* and is denoted by $\rho^*(G)$. By linear programming duality, $\rho^*(G)$ is equal to the *fractional stable set number* $\alpha^*(G)$ — the maximum size of a *fractional stable set*, which is any solution $y \in \mathbb{R}^V$ of

(30.41) (i) $0 \leq y_v \leq 1$ for $v \in V$,
(ii) $y_u + y_v \leq 1$ for $uv \in E$.

The equality $\rho^*(G) = \alpha^*(G)$ also follows from Corollary 30.9a, since trivially

(30.42) $\frac{1}{2}\rho_2(G) \geq \rho^*(G) \geq \alpha^*(G) \geq \frac{1}{2}\alpha_2(G)$.

30.11. The fractional edge cover polyhedron

Let $G = (V, E)$ be a graph. The *fractional edge cover polyhedron* of G is the polyhedron determined by (30.40). Balinski [1965] showed:

Theorem 30.10. *Each vertex of the fractional edge cover polyhedron of G is half-integer.*

Proof. Let x be a vertex of the fractional edge cover polyhedron. We can assume that $x_e > 0$ for each edge e, since if $x_e = 0$ we can apply induction to $G - e$. Moreover, we can assume that G is connected and has at least three vertices.

As x is a vertex, there are $|E|$ constraints among (30.40)(ii) satisfied with equality. Define $U := \{v \mid x(\delta(v)) = 1\}$. So $|E| \leq |V|$. If there exists an end vertex v in U, with neighbour u say, then $u \in U$ and there is no other edge incident with u (otherwise it would have $x_e = 0$), implying the theorem. So no such end vertex exists.

If G is a tree, then there is at most one vertex w with $x(\delta(w)) \neq 1$, implying the existence of an end vertex v and a neighbour u of v with $u, v \in U$.

So G is not a tree, and hence $|E| = |V|$ and $U = V$. Since G has no end vertex, G is a circuit. Then $\frac{1}{2} \cdot \mathbf{1}$ satisfies all constraints that x satisfies. So $x = \frac{1}{2} \cdot \mathbf{1}$, as x is a vertex. ∎

30.12. The 2-edge cover polyhedron

Theorem 30.10 implies a characterization of the 2-*edge cover polyhedron* of G, which is, by definition, the convex hull of the 2-edge covers in G:

Corollary 30.10a. *The 2-edge cover polyhedron is determined by*

(30.43) (i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 2$ for $v \in V$.

Proof. Directly from Theorem 30.10, since it implies that the vertices of the polyhedron determined by (30.43) are integer, and hence 2-edge covers. ∎

Similar results as for fractional edge covers and 2-edge covers hold for fractional stable sets and 2-stable sets. We discuss them in Section 64.5.

30.13. Total dual integrality of the 2-edge cover constraints

Finding a minimum-weight 2-edge cover is easily reduced to the minimum-weight edge cover problem, by splitting vertices. Gallai [1957,1958a,1958b] characterized the minimum weight as follows. Given $w : E \to \mathbb{Z}_+$, a *w-stable set* is a function $y : V \to \mathbb{Z}_+$ with $y_u + y_v \leq w(e)$ for each edge $e = uv$.

Theorem 30.11. *Let $G = (V, E)$ be a graph without isolated vertices and let $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a 2-edge cover x is equal to the maximum size of a $2w$-stable set.*

Proof. From Egerváry's theorem (Theorem 17.1). ∎

This is equivalent to the following result:

Corollary 30.11a. *System* (30.43) *is totally dual half-integral.*

Proof. Choose $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a 2-edge cover is equal to

(30.44) $\max\{2y(V) \mid y \in \frac{1}{2}\mathbb{Z}_+^V, y_u + y_v \leq w(e) \text{ for each } e = uv \in E\}$,

by Theorem 30.11. ∎

System (30.43) can be extended to a TDI system as follows:

Corollary 30.11b. *The following system is totally dual integral:*

(30.45) (i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 2$ for $v \in V$,
(ii) $x(E[U] \cup \delta(U)) \geq |U|$ for $U \subseteq V$.

Proof. Choose $w \in \mathbb{Z}_+^E$. By Corollary 30.11a, the problem of minimizing $w^\mathsf{T} x$ over (30.43) has an optimum dual solution $y \in \frac{1}{2}\mathbb{Z}_+^V$. Define $y'_v := \lfloor y_v \rfloor$ for $v \in V$, and $T := \{v \in V \mid y_v \notin \mathbb{Z}\}$. Define $z_T := 1$ and $z_U := 0$ for each $U \subseteq V$ with $U \neq T$. Then y', z is an integer optimum dual solution for the problem of minimizing $w^\mathsf{T} x$ over (30.45). ∎

30.14. Simple 2-edge covers

Call a 2-edge cover x *simple* if x is a 0,1 vector. Thus we can identify simple 2-edge covers with subsets F of E satisfying $\deg_F(v) \geq 2$ for each $v \in V$. A 2-edge cover exists if and only if all degrees are at least 2. Define

(30.46) $\nu_2^{\mathrm{s}}(G) :=$ the maximum size of a simple 2-matching,
$\rho_2^{\mathrm{s}}(G) :=$ the minimum size of a simple 2-edge cover.

Again there is a relation between $\nu_2^{\mathrm{s}}(G)$ and $\rho_2^{\mathrm{s}}(G)$ similar to Gallai's theorem (Theorem 19.1):

Theorem 30.12. *For any graph $G = (V, E)$ of minimum degree at least 2 one has:*

(30.47) $$\nu_2^{\mathrm{s}}(G) + \rho_2^{\mathrm{s}}(G) = 2|V|.$$

Proof. Let M be a maximum-size simple 2-matching. For each $v \in V$, add to M $2 - \deg_M(v)$ edges incident with v. We can do this in such a way that we obtain a simple 2-edge cover F with

(30.48) $$|F| \leq |M| + \sum_{v \in V}(2 - \deg_M(v)) = 2|V| - |M|.$$

So $\rho_2^{\mathrm{s}}(G) \leq 2|V| - |M| = 2|V| - \nu_2^{\mathrm{s}}(G)$.

To see the reverse inequality, let F be a minimum-size simple 2-edge cover. For each $v \in V$, delete from F $\deg_F(v) - 2$ edges incident with v. We obtain a simple 2-matching M with

(30.49) $$|M| \geq |F| - \sum_{v \in V}(\deg_F(v) - 2) = 2|V| - |F|.$$

So $\nu_2^{\mathrm{s}}(G) \geq 2|V| - |F| = 2|V| - \rho_2^{\mathrm{s}}(G)$, which shows (30.47). ∎

This implies a min-max relation for minimum-size simple 2-edge cover:

Corollary 30.12a. *Let $G = (V, E)$ be a graph of minimum degree at least 2. Then the minimum size of a simple 2-edge cover is equal to the maximum value of*

(30.50) $$|V| - |U| + |S| - \sum_K \lfloor \tfrac{1}{2}|E[K, S]| \rfloor,$$

where U and S are disjoint subsets of V, with S a stable set, and where K ranges over the components of $G - U - S$.

Proof. Directly from Theorems 30.7 and 30.12. ∎

These reductions also imply the polynomial-time solvability of the problem of finding a minimum-size simple 2-edge cover.

Given a graph $G = (V, E)$, the *simple 2-edge cover polytope* is the convex hull of the simple 2-edge covers in G. A special case of Theorem 34.9 below is that the simple 2-edge cover polytope is determined by

(30.51)
$$\begin{array}{rlll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \geq |U| + \lceil \frac{1}{2}|F| \rceil & (U \subseteq V, F \subseteq \delta(U), \\ & & |F| \text{ odd}). \end{array}$$

We refer to Theorem 34.10 for the total dual integrality of the following system (Cook [1983b]):

(30.52)
$$\begin{array}{rlll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \geq |U| + \lceil \frac{1}{2}|F| \rceil & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Theorem 34.11 implies that a minimum-weight simple 2-edge-cover can be found in strongly polynomial time.

30.15. Graphs with $\nu(G) = \tau(G)$ and $\alpha(G) = \rho(G)$

Kőnig's matching theorem states that the matching number $\nu(G)$ is equal to the vertex cover number $\tau(G)$ for each bipartite graph G. A graph G therefore is said to have the *Kőnig property* if $\nu(G) = \tau(G)$. Deming [1979b] and Sterboul [1979] characterized the class of graphs with the Kőnig property.

Note that by Gallai's theorem (Theorem 19.1), for any graph G without isolated vertices:

(30.53) $$\nu(G) = \tau(G) \iff \alpha(G) = \rho(G)$$

(where $\alpha(G)$ and $\rho(G)$ denote the stable set and edge cover number of G, respectively).

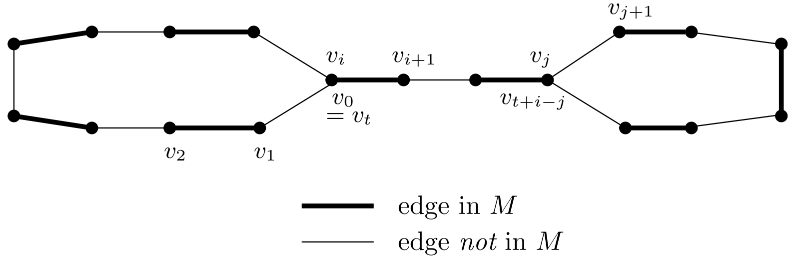


Figure 30.1
An M-posy
The two circuits may intersect.

To characterize graphs G with $\nu(G) = \tau(G)$, Sterboul defined, for any graph $G = (V, E)$ and any matching M in G, an *M-posy* to be an even-length M-alternating closed walk $(v_0, v_1, \ldots, v_t)$, with $v_{i-1}v_i \in M$ if i is even, such that there exist $i < j$ with i odd and j even, $v_1, \ldots, v_j$ all distinct, $v_{j+1}, \ldots, v_t$ all distinct, and

(30.54) $\qquad v_i = v_t, v_{i+1} = v_{t-1}, \ldots, v_j = v_{t+i-j}.$

Lemma 30.13α. *If there exists an even-length M-alternating closed walk $C = (v_0, v_1, \ldots, v_t)$ with $v_i = v_j$ for i, j of different parity, then there exists an M-posy.*

Proof. Let C be a shortest such closed walk, covering a minimum number of edges (in this order of priority). Then

(30.55) $\qquad$ there exist no three distinct $h, i, k \geq 1$ with $v_h = v_i = v_k$,

since otherwise we may assume that h and i have the same parity. Leaving out one of the $v_h - v_i$ parts of C gives a shorter such closed walk.

Choose h, i of different parity with $v_h = v_i$ and with $|h - i|$ minimal. We may assume that $h = 0$ and that $v_0v_1 \notin M$. Choose $j, k \geq i$ of different parity with $v_j = v_k$ and $j < k$, and with $k - j$ minimal. (Such j, k exist, as $v_i = v_t$.) Then j is even and k is odd, since otherwise $v_{j+1} = v_{k-1}$ (as it is the vertex matched to $v_j = v_k$). Moreover, $j - i = t - k$ and

(30.56) $\qquad v_i = v_t, v_{i+1} = v_{t-1}, \ldots, v_j = v_k.$

Otherwise, resetting the $v_k - v_t$ part of C to the $v_j - v_i$ part of C^{-1} or conversely, we obtain again a shortest such closed walk, however covering a fewer number of edges, a contradiction.

Then C is an M-posy, since $v_1, \ldots, v_j$ are all distinct and $v_{j+1}, \ldots, v_t$ are all distinct. If say $v_a = v_b$ with $1 \leq a < b \leq j$, then $b \leq i$ (since otherwise $v_a = v_b = v_l$ for some $l > b$, contradicting (30.55)). So by the minimality of $|h - i|$, $a \equiv b \pmod 2$. Hence, deleting the $v_a - v_b$ part from C gives a shortest such walk, a contradiction. ∎

This is used in proving:

Theorem 30.13. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

(30.57)
- (i) *G has the Kőnig property, that is $\nu(G) = \tau(G)$;*
- (ii) *for some maximum-size matching M there is no M-flower and no M-posy;*
- (iii) *for each maximum-size matching M there is no M-flower and no M-posy.*

Proof. The implication (iii)$\Rightarrow$(ii) is trivial, and the implication (i)$\Rightarrow$(iii) is easy: suppose $\nu(G) = \tau(G)$, let M be a maximum-size matching and let U

be a minimum-size vertex cover. Then each edge in M has exactly one vertex in U. Suppose that $P = (v_0, \ldots, v_t)$ is an M-flower or an M-posy. Then for each odd k, exactly one of v_k and v_{k+1} belongs to U, while for each even k at least one of v_k and v_{k+1} belongs to U. If $v_t \notin U$, then $v_k \in U$ for each even k. Since $v_t = v_j$ for some even j, it follows that $v_t \in U$. If $v_0 \notin U$, then $v_k \in U$ for each odd k. Since $v_j \in U$ and j is even, we have $v_0 \in U$. So v_0 is covered by M, and hence P is an M-posy. So $v_0 = v_i$ for some odd i. So $v_i \in U$ for some odd i, a contradiction.

It remains to prove (ii)⇒(i). Let M be a maximum-size matching in G and let X be the set of vertices missed by M. Then there is no M-alternating $X - X$ walk (since M has maximum size and since there is no M-flower (cf. Theorem 24.3)). Let U be the set of vertices v for which there is an M-alternating $X - v$ walk and let Z be the set of vertices v for which there exists an odd-length M-alternating $X - v$ walk. Then Z intersects each edge intersecting U, while $|Z|$ is equal to the number of edges in M contained in U.

So we can apply induction to $G - U$ if $U \neq \emptyset$. Hence we may assume that $U = \emptyset$. Equivalently, $X = \emptyset$, that is, M is a perfect matching. Choose $e = uv \in M$. By Lemma 30.13α, $G - u$ has no $M \setminus \{e\}$-flower or $G - v$ has no $M \setminus \{e\}$-flower. By symmetry, we may assume that $G - v$ has no $M \setminus \{e\}$-flower. Since G has no M-posy, $G - v$ has no $M \setminus \{e\}$-posy. Hence, by induction:

(30.58) $$\nu(G) = \nu(G - v) + 1 = \tau(G - v) + 1 \geq \tau(G).$$

Hence $\nu(G) = \tau(G)$. ∎

This implies a characterization due to Lovász [1974] ((i)⇔(ii) below) and Lovász and Plummer [1986] ((i)⇔(iii) below), based on the minimum size $\tau_2(G)$ of a 2-vertex cover studied in Section 30.1:

Corollary 30.13a. *For any graph G, the following are equivalent:*

(30.59)
(i) $\nu(G) = \tau(G)$,
(ii) $\tau_2(G) = 2\tau(G)$,
(iii) *the edges e for which there exists a maximum-size 2-matching x with $x_e \geq 1$, form a bipartite graph.*

Proof. The implication (i)⇒(ii) follows from (30.3). To see (ii)⇒(iii), let U be a minimum-size vertex cover and let x be a maximum-size 2-matching. Then, using Theorem 30.1,

(30.60) $$\tau_2(G) = \nu_2(G) = \sum_{e \in E} x_e \leq \sum_{e \in E} x_e |e \cap U| = \sum_{v \in U} x(\delta(v)) \leq 2|U| \\ = 2\tau(G),$$

and hence we have equality throughout. So $e \in \delta(U)$ if $x_e \geq 1$. As this is true for each maximum-size 2-matching x, we have (iii).

We finally show (iii)⇒(i), which we derive from Theorem 30.13. Suppose that (iii) holds, and let M be a maximum-size matching. If there would exist any M-flower or M-posy, then we can find a 2-matching of size at least $2|M|$ such that M and the support of the 2-matching contains an odd circuit. For an M-flower this is trivial. For an M-posy $(v_0, \ldots, v_t)$, let

$$(30.61) \qquad x := 2\chi^M - \sum_{h=1}^{t}(-1)^h \chi^{v_{h-1}v_h}.$$

Then x is a 2-matching of size $2|M|$. However, the support of x together with M contains an odd circuit. This contradicts (iii). ∎

Note that characterization (iii) can be checked in polynomial time. By Theorem 30.9 and its proof method, we know that (i), (ii), and (iii) are also equivalent to each of:

(30.62)
(iv) $\alpha(G) = \rho(G)$,
(v) $\alpha_2(G) = 2\alpha(G)$,
(vi) the edges e for which there exists a minimum-size 2-edge cover x with $x_e \geq 1$, form a bipartite graph.

More on the Kőnig property can be found in Korach [1982], Bourjolly, Hammer, and Simeone [1984], and Bourjolly and Pulleyblank [1989], and related results in Tipnis and Trotter [1989].

30.16. Excluding triangles

Let $G = (V, E)$ be a graph. Call a 2-matching x *triangle-free* if $x(ET) \leq 2$ for each triangle T in G. (A *triangle* is a subgraph isomorphic to K_3.) The *triangle-free 2-matching polytope* is the convex hull of the triangle-free 2-matchings.

In order to characterize the triangle-free 2-matching polytope, Cornuéjols and Pulleyblank [1980a] (cf. Cook [1983b], Cook and Pulleyblank [1987]) showed the following:

Theorem 30.14. *Let $G = (V, E)$ be a simple graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following system is totally dual integral:*

$$(30.63) \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & \tfrac{1}{2}x(\delta(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(ET) \leq 2 & \text{for each } T \in \mathcal{T}. \end{array}$$

Proof. Let $w \in \mathbb{Z}_+^E$ and consider the problem dual to maximizing $w^{\mathsf{T}}x$ over (30.63):

(30.64) $$\begin{aligned}&\text{minimize } \sum_{v\in V} y_v + 2\sum_{T\in\mathcal{T}} z_T\\ &\text{subject to } \tfrac{1}{2}\sum_{v\in V} y_v\chi^{\delta(v)} + \sum_{T\in\mathcal{T}} z_T\chi^{ET} \geq w,\end{aligned}$$

with $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{T}}$. We must show that there exists an integer optimum solution y, z. We take a counterexample with $|E|+w(E)$ minimal. This implies that G is connected. Moreover, $w(e) \geq 1$ for each edge e, since otherwise we could delete e.

On the other hand, $w(e) \leq 2$ for each edge e. To see this, let y, z be any optimum solution. If $y_u \geq 2$ for some vertex u, we can reset $w(e) := w(e) - 1$ for each $e \in \delta(u)$. By resetting, the optimum value decreases by at least 2 (since resetting $y_u := y_u - 2$ gives a feasible solution for the new w, with objective value 1 less than the original objective value). By the minimality of G, w, for the new w there is an integer optimum solution y, z. Resetting $y_u := y_u + 2$ then gives an integer optimum solution for the original w.

So we can assume that $y_v < 2$ for each vertex v, and similarly, that $z_T < 1$ for each $T \in \mathcal{T}$.

Choose an optimum solution y, z with $\sum_{T\in\mathcal{T}} z_T$ minimal. Let $\mathcal{T}_+ := \{T \in \mathcal{T} \mid z_T > 0\}$. Then:

(30.65) no two triangles in $\mathcal{T}_+$ have an edge in common.

For suppose that $T_1, T_2 \in \mathcal{T}_+$ have $ET_1 \cap ET_2 = \{e\}$, say $e = v_1v_2$. Resetting $z_{T_i} := z_{T_i} - \varepsilon$ and $y_{v_i} := y_{v_i} + 2\varepsilon$ for $i = 1, 2$, for $\varepsilon > 0$ small enough, gives again an optimum solution. However, $\sum_{T\in\mathcal{T}} z_T$ decreases, contradicting our assumption. This proves (30.65).

This implies that $w(e) \leq 2$ for each edge e, since $y_v < 2$ and $z_T < 1$.

Next:

(30.66) for any $T \in \mathcal{T}_+$ and any $v \in VT$ one has either $0 < y_v < 1$ for each $v \in VT$ and $w(e) = 1$ for each $e \in ET$, or $1 < y_v < 2$ for each $v \in VT$ and $w(e) = 2$ for each $e \in ET$.

Let $VT = \{v_1, v_2, v_3\}$. First assume that $\frac{1}{2}y_{v_1} + \frac{1}{2}y_{v_2} + z_T > w(v_1v_2)$. Then after resetting $y_{v_3} := y_{v_3} + 2\varepsilon$ and $z_T := z_T - \varepsilon$ we obtain again an optimum solution, for $\varepsilon > 0$ small enough. However, $\sum_{T\in\mathcal{T}} z_T$ decreases, contradicting our assumption. So $\frac{1}{2}y_{v_1} + \frac{1}{2}y_{v_2} + z_T = w(v_1v_2)$, and similarly for any other pair from v_1, v_2, v_3. This implies

(30.67) $$y_{v_1} = w(v_1v_2) + w(v_1v_3) - w(v_2v_3) - z_T,$$

and similarly for v_2 and v_3. So if $w(e) = 1$ for each $e \in ET$, then $0 < y_v < 1$ for each $v \in VT$. Similarly, if $w(e) = 2$ for each $e \in ET$, then $1 < y_v < 2$ for each $v \in VT$. If not all three edges of T have the same weight, (30.67) implies that there is a vertex v in T with $y_v > 2$ or $y_v < 0$, a contradiction. This proves (30.66).

Now consider resetting

$$\begin{array}{llll} (30.68) \quad & y_v & := y_v - \varepsilon & \text{if } 0 < y_v < 1, \\ & y_v & := y_v + \varepsilon & \text{if } 1 < y_v < 2, \\ & z_T & := z_T + \varepsilon & \text{if } T \in \mathcal{T}_+ \text{ and } w(e) = 1 \text{ for each edge } e \text{ in } T, \\ & z_T & := z_T - \varepsilon & \text{if } T \in \mathcal{T}_+ \text{ and } w(e) = 2 \text{ for each edge } e \text{ in } T. \end{array}$$

If we choose ε close enough to 0 (positive or negative), we obtain again a feasible solution of (30.64), by (30.65) and (30.66), using the integrality of w. Moreover, the objective value changes linearly in ε. However, as y, z is an optimum solution, the objective value cannot decrease. Hence there is no change in the objective value at all. That is, for any ε close enough to 0, we obtain again an optimum solution. Therefore, by choosing ε appropriately, we can decrease the number of noninteger values of y_v, z_T. ∎

This theorem implies (in fact, is equivalent to) the following TDI result:

Corollary 30.14a. *Let $G = (V, E)$ be a simple graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following system is totally dual integral:*

$$\begin{array}{llll} (30.69) \quad & \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ & \text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\ & \text{(iii)} & x(ET) \leq 2 & \text{for each } T \in \mathcal{T}, \\ & \text{(iv)} & x(E[U]) \leq |U| & \text{for each } U \subseteq V. \end{array}$$

Proof. Let $w \in \mathbb{Z}_+^E$. Let μ be the maximum value of $w^\mathsf{T} x$ over (30.69), This is equal to the maximum value of $w^\mathsf{T} x$ over (30.63) (since (30.69)(iv) follows from (i) and (ii)).

Consider an integer optimum solution y_v ($v \in V$), z_T ($T \in \mathcal{T}$) of the problem dual to maximizing $w^\mathsf{T} x$ over (30.63). Define $y'_v := \lfloor \frac{1}{2} y_v \rfloor$ for $v \in V$ and $T := \{v \in V \mid \frac{1}{2} y_v \notin \mathbb{Z}\}$. Define $a_U := 1$ if $U = T$ and $a_U := 0$ for any other subset U of V.

Then y', a, z is an integer feasible solution of the problem dual to maximizing $w^\mathsf{T} x$ over (30.69), as w is integer. Moreover, it is optimum, since

$$(30.70) \quad \sum_{v \in V} 2y'_v + \sum_{U \subseteq V} a_U |U| + \sum_{T \in \mathcal{T}} 2z_T = \sum_{v \in V} y_v + \sum_{T \in \mathcal{T}} 2z_T = \mu. \qquad ∎$$

The theorem implies the following characterization of the triangle-free 2-matching polytope, given by Cornuéjols and Pulleyblank [1980a] and J.F. Maurras (cf. Cornuéjols and Pulleyblank [1980b]):

Corollary 30.14b. *Let $G = (V, E)$ be a graph. The triangle-free 2-matching polytope is determined by:*

$$\begin{array}{llll} (30.71) \quad & \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ & \text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\ & \text{(iii)} & x(ET) \leq 2 & \text{for each triangle } T \text{ in } G. \end{array}$$

Proof. Theorem 30.14 implies that the polytope determined by (30.63) is integer (as the right-hand sides are integer). Since (30.71) determines the same polytope, the corollary follows. ∎

In fact, there is a sharper consequence, where we just consider an arbitrary subcollection $\mathcal{T}$ of the triangles:

Corollary 30.14c. *Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following inequalities determine an integer polytope:*

$$\text{(30.72)} \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\ \text{(iii)} & x(ET) \leq 2 & \text{for each triangle } T \in \mathcal{T}. \end{array}$$

Proof. Similar to the proof of the previous corollary. ∎

Cornuéjols and Pulleyblank [1980a] also showed that the inequalities (30.72)(i) and (iii) are all necessary, while (ii) is necessary unless $\deg_G(v) = 2$ and v is in a triangle in $\mathcal{T}$ (assuming that G is connected and has at least three vertices). They also gave a polynomial-time algorithm to find a maximum-weight triangle-free 2-matching.

Moreover, they showed the following. A *triangle cluster* is a graph defined recursively as follows: any one-vertex graph is a triangle cluster; if G is a triangle cluster and v is a vertex of G, then by introducing two new vertices u, u' and adding edges vu, vu' and uu', we obtain again a triangle cluster.

For any graph G, let $\beta(G)$ denote the number of components of G that are triangle clusters. This is used in the following min-max relation for maximum-size triangle-free 2-matching (Cornuéjols and Pulleyblank [1980a]):

Theorem 30.15. *The maximum size of a triangle-free 2-matching in a graph $G = (V, E)$ is equal to the minimum value of $|V| + |U| - \beta(G - U)$ taken over $U \subseteq V$.*

Proof. To see that the maximum is not more than the minimum, let x be a maximum-size triangle-free 2-matching in G. Let $U \subseteq V$ and let W be the set of vertices of $G - U$ that are in triangle cluster components. Consider any component K of $G - U$ that is a triangle cluster. Then the edges of K can be partitioned into $\frac{1}{2}(|K| - 1)$ triangles. Hence $x(E[K]) \leq |K| - 1$, and therefore

$$\text{(30.73)} \qquad \sum_{v \in K} x(\delta(v)) = 2x(E[K]) + x(\delta(K)) \leq 2(|K| - 1) + x(\delta(K)).$$

Summing over all components K that are triangle cluster, we see that

$$\text{(30.74)} \qquad \sum_{v \in W} x(\delta(v)) \leq 2|W| - 2\beta(G - U) + x(\delta(W)).$$

Moreover,

(30.75) $$x(\delta(W)) \leq x(\delta(U)) \leq \sum_{v\in U} x(\delta(v)) \leq 2|U|.$$

This implies

(30.76) $$\begin{aligned}2x(E) &= \sum_{v\in W} x(\delta(v)) + \sum_{v\in V\setminus W} x(\delta(v)) \\ &\leq 2|W| - 2\beta(G-U) + 2|U| + 2|V\setminus W| \\ &= 2(|V| + |U| - \beta(G-U)).\end{aligned}$$

This shows that the maximum is not more than the minimum.

To see the reverse inequality, let $\mathcal{T}$ denote the set of triangles in G. By Theorem 30.14, the maximum size of a triangle-free 2-matching is equal to the minimum value of

(30.77) $$\sum_{v\in V} y_v + 2\sum_{T\in\mathcal{T}} z_T$$

where $y_v \in \mathbb{Z}_+$ (for $v \in V$) and $z_T \in \mathbb{Z}_+$ (for $T \in \mathcal{T}$) such that

(30.78) $$\tfrac{1}{2}\sum_{v\in V} y_v\chi^{\delta(v)} + \sum_{T\in\mathcal{T}} z_T\chi^{ET} \geq \mathbf{1}.$$

Choose y, z attaining this minimum, with

(30.79) $$\sum_{T\in\mathcal{T}} z_T \text{ as small as possible.}$$

Clearly, $y_v \leq 2$ for each $v \in V$ and $z_T \leq 1$ for each $T \in \mathcal{T}$. Let $\mathcal{T}_+ := \{T \in \mathcal{T} \mid z_T = 1\}$.

Then we have:

(30.80) if $T \in \mathcal{T}_+$ and $v \in T$, then $y_v = 0$.

Indeed, suppose $y_v \geq 1$, and let u and u' be the two other vertices in T. Then resetting $z_T := 0$, $y_u := y_u + 1$, and $y_{u'} := y_{u'} + 1$, we obtain y, z again attaining the minimum value (30.77), contradicting our minimality assumption (30.79). This shows (30.80).

Let F be the set of edges contained in some $T \in \mathcal{T}_+$. Then

(30.81) each component of the graph (V, F) is a triangle cluster.

If not, there exist distinct $T_1, \ldots, T_k \in \mathcal{T}_+$ and distinct $v_1, \ldots, v_k \in V$, such that, taking $v_0 := v_k$,

(30.82) $$v_{i-1}v_i \in T_i$$

for $i = 1, \ldots, k$, and such that $k > 1$. Then resetting $z_{T_i} := 0$ and $y_{v_i} := 2$ for $i = 1, \ldots, k$, we obtain y, z again attaining the minimum value (30.77), contradicting our minimality assumption (30.79). This shows (30.81).

Now let $W := \{v \in V \mid y_v = 0\}$. Then each edge contained in W is contained in some $T \in \mathcal{T}_+$, and hence, by (30.81), each component of $G[W]$

is a triangle cluster. Let k be the number of components of $G[W]$. Then $\sum_{T\in\mathcal{T}} z_T = \frac{1}{2}(|W|-k)$.

Define $U := N(W)$. Then $y_v = 2$ for each $v \in U$, since each edge e connecting W and U should satisfy (30.78). Therefore, (30.77) is at least

(30.83) $$|V|-|W|+|U|+2\cdot\tfrac{1}{2}(|W|-k) = |V|+|U|-k \geq |V|+|U|-\beta(G-U),$$

proving the theorem. ∎

This characterizes the existence of a triangle-free perfect 2-matching:

Corollary 30.15a. *A graph $G = (V,E)$ has a triangle-free perfect 2-matching if and only if $G - U$ has at most $|U|$ components that are triangle clusters, for each $U \subseteq V$.*

Proof. Directly from Theorem 30.15. ∎

Cornuéjols and Pulleyblank [1980b] gave a polynomial-time algorithm to find a triangle-free perfect b-matching. Cook [1983b] and Cook and Pulleyblank [1987] characterized the facets and the minimal TDI-system for the triangle-free 2-matching polytope.

30.16a. Excluding higher polygons

Cornuéjols and Pulleyblank [1983] considered excluding higher polygons. For any collection P of graphs, call a graph G *P-critical* if $G \notin P$ while $G - v \in P$ for each vertex v of G. Let P_k be the collection of graphs that have a perfect 2-matching in which each circuit has length larger than k. Then for each k and each graph $G = (V,E)$:

(30.84) If G is P_k-critical, then G is factor-critical,

and

(30.85) V can be partitioned into edges and subsets U with $G[U]$ P_k-critical if and only if for each $S \subseteq V$, the graph $G-S$ has at most $|S|$ P_k-critical components.

This generalizes Theorem 24.8 and (30.86) below.

Corollary 30.14b does not extend to 2-matchings excluding triangles and pentagons, as is shown by the example given in Figure 30.2. (The sum of the values is at most 4 on each pentagon, but it does not belong to the convex hull of the 2-matchings without pentagons, since the sum of the values is equal to $\frac{20}{3}$, but there is no pentagon-free 2-matching of size ≥ 7.)

30.16b. Packing edges and factor-critical subgraphs

Cornuéjols, Hartvigsen, and Pulleyblank [1982] and Cornuéjols and Hartvigsen [1986] discovered an interesting direction of extensions of the results on matchings. Let $G = (V,E)$ be a graph. Call a subset U of V *factor-critical* if $G[U]$ is factor-critical; that is, if for each $v \in U$, the set $U \setminus \{v\}$ is matchable.

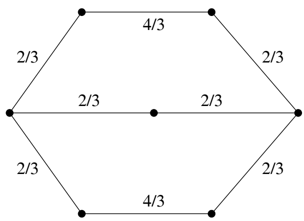


Figure 30.2

Let $\mathcal{F}$ be a collection of factor-critical subsets of V. An *$\mathcal{F}$-matching* is a collection of disjoint subsets from $E \cup \mathcal{F}$. It is *perfect* if it covers V. Call a subset U of V *$\mathcal{F}$-critical* if $G[U]$ has no perfect $\mathcal{F}$-matching but for each $v \in U$, the graph $G[U] - v$ has one. Cornuéjols, Hartvigsen, and Pulleyblank [1982] showed that

(30.86) if U is $\mathcal{F}$-critical, then U is factor-critical.

Then Cornuéjols and Hartvigsen [1986] proved the following extension of Tutte's 1-factor theorem (Theorem 24.1a):

(30.87) G has a perfect $\mathcal{F}$-matching if and only if for each $U \subseteq V$, the graph $G - U$ has at most $|U|$ $\mathcal{F}$-critical components.

Call an $\mathcal{F}$-matching $\mathcal{M}$ *maximum* if it maximizes $\sum_{U \in \mathcal{M}} |U|$. Cornuéjols and Hartvigsen [1986] also showed:

(30.88) Let $\mathcal{M}$ be a maximum $\mathcal{F}$-matching containing a minimum number of sets in $\mathcal{F}$. Let M be a matching containing $\mathcal{M} \cap E$ and having $\lfloor \frac{1}{2}|U| \rfloor$ edges in any $U \in \mathcal{M} \cap \mathcal{F}$. Then M is a maximum-size matching in G.

They also described an extension of the Edmonds-Gallai decomposition theorem. Cornuéjols, Hartvigsen, and Pulleyblank [1982] gave a polynomial-time algorithm to find a maximum $\mathcal{F}$-matching. Related results were obtained by Kirkpatrick and Hell [1978,1983] and Hell and Kirkpatrick [1984,1986].

30.16c. 2-factors without short circuits

Hartvigsen [1984] showed that a maximum size simple 2-matching without triangles can be found in polynomial time. He also gave good characterization for the existence of a 2-factor without triangles.

On the other hand, Cornuéjols and Pulleyblank [1980a] showed with a method of C.H. Papadimitriou that the problem of finding a 2-factor without circuits of length at most 5, is NP-complete. The complexity of deciding if a 2-factor exists without circuits of length at most 4 is not known.

Vornberger [1980] showed the NP-completeness of finding a maximum-weight 2-factor without circuits of length at most 4. The complexity status of finding a maximum-weight 2-factor without circuits of length at most 3 is unknown. Hell, Kirkpatrick, Kratochvíl, and Kříž [1988] and Cunningham and Wang [2000] give related results.

Chapter 31

b-matchings

b-matchings form an extension of 2-matchings and can be handled again by applying splitting techniques to ordinary matchings.

31.1. *b*-matchings

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. A *b-matching* is a function $x \in \mathbb{Z}_+^E$ satisfying

(31.1) $\quad x(\delta(v)) \leq b(v)$

for each $v \in V$. This is equivalent to: $Mx \leq b$, where M is the $V \times E$ incidence matrix of G.

In (31.1), we count multiplicities: if e is a loop at v, then x_e is added twice at v. (This is consistent with our definition of $\delta(v)$ as a *family* of edges, in which each loop at v occurs twice.)

It is convenient to consider the graph G_b arising from G by splitting each vertex v into $b(v)$ copies, and by replacing any edge uv by $b(u)b(v)$ edges connecting the $b(u)$ copies of u with the $b(v)$ copies of v. More formally, $G_b = (V_b, E_b)$, where

(31.2) $$\begin{aligned} V_b &:= \{q_{v,i} \mid v \in V, 1 \leq i \leq b(v)\},\\ E_b &:= \{q_{u,j}q_{v,i} \mid uv \in E, 1 \leq j \leq b(u), 1 \leq i \leq b(v), q_{u,j} \neq q_{v,i}\}. \end{aligned}$$

The condition $q_{u,j} \neq q_{v,i}$ is relevant only if $u = v$, that is, if there is a loop at u.

This construction was given by Tutte [1954b], and yields a min-max relation for maximum-size b-matching (where again the *size* of a vector is the sum of its components):

Theorem 31.1. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a b-matching is equal to the minimum value of*

(31.3) $$b(U) + \sum_K \lfloor \tfrac{1}{2} b(K) \rfloor$$

taken over $U \subseteq V$, where K ranges over the components of $G - U$ spanning at least one edge[19].

Proof. To see that the maximum is not more than the minimum, consider a b-matching x and a subset U of V. Then the sum of x_e over the edges e intersecting U is at most $b(U)$. The sum of x_e over the edges e contained in some component K of $G - U$ is at most $\lfloor \frac{1}{2} b(K) \rfloor$.

Equality is derived from the Tutte-Berge formula (Theorem 24.1). Let G_b be the graph described in (31.2). Then the maximum size of a b-matching in G is equal to the maximum size of a matching in G_b. By the Tutte-Berge formula, this is equal to the minimum value of

$$\text{(31.4)} \qquad \tfrac{1}{2}(|V_b| + |U'| - o(G_b - U'))$$

over $U' \subseteq V_b$ (where $o(H)$ denotes the number of odd components of a graph H).

Let U' attain this minimum. We may assume that if U' misses at least one copy of some vertex v of G, it misses all copies of v (since deleting all copies does not increase (31.4)). Hence there is a subset U of V such that U' is equal to the set of copies of vertices in U. We take $v \in U$ if $b(v) = 0$.

Let I_U be the set of isolated (hence loopless) vertices of $G - U$. Then $o(G_b - U')$ is equal to $b(I_U)$ plus the number of components K of $G - U$ that span at least one edge and have $b(K)$ odd. Setting k to the number of such components, (31.4) is equal to

$$\text{(31.5)} \qquad \begin{aligned} &\tfrac{1}{2}(b(V) + b(U) - o(G_b - U')) = b(U) + \tfrac{1}{2}(b(V \setminus U) - o(G_b - U')) \\ &= b(U) + \tfrac{1}{2}(b(V \setminus U) - b(I_U) - k), \end{aligned}$$

which is equal to (31.3). ∎

This theorem directly gives a characterization of the existence of a *perfect b-matching*, that is a b-matching having equality in (31.1) for each $v \in V$. This characterization is due to Tutte [1952]. By I_U we denote the set of isolated, loopless vertices of $G - U$.

Corollary 31.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then there exists a perfect b-matching if and only if for each $U \subseteq V$, $G - U - I_U$ has at most $b(U) - b(I_U)$ components K with $b(K)$ odd.*

Proof. Directly from Theorem 31.1, by observing that a perfect b-matching exists if and only if the minimum value of (31.3) is at least $\frac{1}{2}b(V)$. ∎

31.2. The b-matching polytope

By a similar construction we can derive a characterization of the b-matching polytope. Given a graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, the *b-matching polytope* is

[19] So K may consist of one vertex with a loop attached.

the convex hull of the b-matchings. The inequalities describing the b-matching polytope were announced by Edmonds [1965b] (cf. Pulleyblank [1973], Edmonds [1975]):

Theorem 31.2. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the b-matching polytope is determined by the inequalities*

$$(31.6)\qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq b(v) & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor & \text{for } U \subseteq V \text{ with } b(U) \text{ odd.} \end{array}$$

Proof. The inequalities (31.6) are trivially valid for the vectors in the b-matching polytope. To see that they determine the b-matching polytope, let x satisfy (31.6). We may assume that $b \geq \mathbf{1}$.

Again consider the graph $G_b = (V_b, E_b)$ obtained by splitting each vertex v into $b(v)$ copies (cf. (31.2)). For any edge $e' = u'v'$ of G_b, with u' and v' copies of u and v in G, define $x'(e') := x_e/b(u)b(v)$, where $e := uv$. We show that x' belongs to the matching polytope of G_b, which implies the theorem.

By Edmonds' matching polytope theorem, it suffices to show that x' satisfies:

$$(31.7)\qquad \begin{array}{rll} \text{(i)} & x'(e') \geq 0 & \text{for each edge } e' \in E_b, \\ \text{(ii)} & x'(\delta'(u')) \leq 1 & \text{for each vertex } u' \in V_b, \\ \text{(iii)} & x'(E'[U']) \leq \lfloor \frac{1}{2} |U'| \rfloor & \text{for each } U' \subseteq V_b \text{ with } |U'| \text{ odd.} \end{array}$$

Clearly (i) holds. To see (31.7)(ii), let u' be a vertex of G_b, being a copy of vertex u of G. Then

$$(31.8)\qquad x'(\delta'(u')) = x(\delta(u))/b(u) \leq 1,$$

since for any edge $e = uv$ of G one has that

$$(31.9)\qquad \sum_{v'} x'(u'v') = \sum_{v'} x(uv)/b(u)b(v) = x(uv)/b(u),$$

where v' ranges over the copies of v in G_b. So summing over all neighbours v' of u' gives $x(\delta(u))/b(u)$.

To see (31.7)(iii), choose $U' \subseteq V_b$ with $|U'|$ odd. Note that x satisfies (31.6)(iii) for *all* subsets U of V, since if $b(U)$ is even, then $x(E[U]) \leq \frac{1}{2}\sum_{v \in U} x(\delta(v)) \leq \frac{1}{2} b(U)$ by (31.6)(ii).

For any vertex v of G let B_v denote the set of copies of v in G_b. We show (31.7)(iii) by induction on the number of $v \in V$ for which U' 'splits' B_v, that is, for which

$$(31.10)\qquad B_v \cap U' \neq \emptyset \text{ and } B_v \not\subseteq U'.$$

If this number is 0, (31.7)(iii) follows from (31.6)(iii). If this number is nonzero, choose a vertex v satisfying (31.10). Let $U_1 := U' \setminus B_v$ and $U_2 := U' \cup B_v$. So by induction we know

(31.11) $\quad x'(E'[U_1]) \leq \frac{1}{2}|U_1|$ and $x'(E'[U_2]) \leq \frac{1}{2}|U_2|$.

Moreover, (31.8) implies:

(31.12) $$x'(E'[U_1]) + x'(E'[U_2]) \leq \sum_{u' \in U_1} x'(\delta'(u')) \leq |U_1|.$$

(This uses the fact that $B_v = U_2 \setminus U_1$ is a stable set in G_b.) Now define $\lambda := |B_v \cap U'|/b(v)$ and $\mu := |B_v \setminus U'|/b(v)$. So $\lambda + \mu = 1$ and

(31.13) $$x'(E'[U']) = \lambda x'(E'[U_2]) + \mu x'(E'[U_1]).$$

If $\lambda \leq \frac{1}{2}$, then, by (31.11) and (31.12):

(31.14) $$\begin{aligned} x'(E'[U']) &= (\mu - \lambda)x'(E'[U_1]) + \lambda(x'(E'[U_1]) + x'(E'[U_2])) \\ &\leq \tfrac{1}{2}(\mu - \lambda)|U_1| + \lambda|U_1| = \tfrac{1}{2}|U_1| \leq \lfloor \tfrac{1}{2}|U'| \rfloor. \end{aligned}$$

(The last inequality holds as $U_1 \subset U'$.)

If $\lambda > \frac{1}{2}$, then, by (31.11) and (31.12):

(31.15) $$\begin{aligned} x'(E'[U']) &= (\lambda - \mu)x'(E'[U_2]) + \mu(x'(E'[U_1]) + x'(E'[U_2])) \\ &\leq (\lambda - \mu)\tfrac{1}{2}|U_2| + \mu|U_1| = \tfrac{1}{2}|U_1| + \tfrac{1}{2}(\lambda - \mu)|U_2 \setminus U_1| \\ &= \tfrac{1}{2}|U_1| + \tfrac{1}{2}(\lambda - \mu)b_v = \tfrac{1}{2}|U_1| + \tfrac{1}{2}(|B_v \cap U'| - |B_v \setminus U'|) \\ &\leq \tfrac{1}{2}|U_1| + \tfrac{1}{2}(|B_v \cap U'| - 1) \leq \lfloor \tfrac{1}{2}|U'| \rfloor. \end{aligned}$$

(The last inequality holds as $U' = U_1 \cup (B_v \cap U')$.)

Thus we have (31.7)(iii). ∎

(This theorem follows also from the proof of the total dual integrality of the constraints (31.17) in Theorem 31.3 below.)

Given a graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, the *perfect b-matching polytope* is the convex hull of the perfect b-matchings in G. As it is a face of the b-matching polytope (if nonempty), the previous theorem implies:

Corollary 31.2a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the perfect b-matching polytope is determined by the inequalities*

(31.16)

(i)	$x_e \geq 0$	for $e \in E$,
(ii)	$x(\delta(v)) = b(v)$	for $v \in V$,
(iii)	$x(\delta(U)) \geq 1$	for $U \subseteq V$ with $b(U)$ odd.

Proof. Directly from Theorem 31.2. ∎

(For a direct proof of this Corollary also based on considering the graph G_b obtained from G by splitting each vertex v into $b(v)$ copies, see Aráoz, Cunningham, Edmonds, and Green-Krótki [1983].)

Hurkens [1988] characterized adjacency on the b-matching polytope and showed that the diameter of the b-matching polytope is equal to the maximum size of a b-matching.

31.3. Total dual integrality

System (31.6) generally is not totally dual integral: if $G = (V, E)$ is the complete graph K_3 on three vertices, and $b(v) := 2$ for each $v \in V$ and $w(e) := 1$ for each $e \in E$, then the maximum weight of a b-matching is equal to 3, while there is no integer dual solution of odd value (when considering the dual of optimizing $w^{\mathsf{T}}x$ subject to (31.6)).

However, if we extend (31.6)(iii) to *all* subsets U of V, the system is totally dual integral, as was shown by Pulleyblank [1980]. So the system becomes:

(31.17)
$$\begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq b(v) & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \tfrac{1}{2} b(U) \rfloor & \text{for } U \subseteq V. \end{array}$$

It is equivalent to the following result:

Theorem 31.3. *Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and let $w \in \mathbb{Z}_+^E$. Then the maximum weight $w^{\mathsf{T}}x$ of a b-matching x is equal to the minimum value of*

(31.18)
$$\sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z(U) \lfloor \tfrac{1}{2} b(U) \rfloor,$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfy

(31.19)
$$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \subseteq V} \chi^{E[U]} \geq w.$$

Proof. By Theorem 31.2 and LP-duality, the maximum weight of a b-matching is equal to the minimum of (31.18) over $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{P}(V)}$ satisfying (31.19). Suppose that this minimum is strictly smaller than if we restrict y and z to integer-valued functions. Then there exists a $t \in \mathbb{Z}_+$ such that the minimum with y and z restricted to values in $2^{-t}\mathbb{Z}_+$ is strictly smaller than when restricting y and z to values in $\mathbb{Z}_+$, because we can slightly increase any value of y_v and $z(U)$ to a dyadic vector. Choose t with this property as small as possible. By replacing w by $2^{t-1}w$, we may assume that $t = 1$.

It therefore is enough to show that for each $y \in \frac{1}{2}\mathbb{Z}_+^V$ and $z \in \frac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)}$ satisfying (31.19), there exist $y' \in \mathbb{Z}_+^V$ and $z' \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfying (31.19) such that

(31.20)
$$\sum_{v \in V} y'_v(v) + \sum_{U \subseteq V} z'(U) \lfloor \tfrac{1}{2} b(U) \rfloor \leq \sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z(U) \lfloor \tfrac{1}{2} b(U) \rfloor.$$

We show this by induction on $w(E)$. More precisely, we consider a counterexample $y \in \frac{1}{2}\mathbb{Z}_+^V$ and $z \in \frac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)}$ with smallest $w(E)$. Then necessarily

(31.21)
$$y \in \{0, \tfrac{1}{2}\}^V \text{ and } z \in \{0, \tfrac{1}{2}\}^{\mathcal{P}(V)},$$

since if $y_v \geq 1$ for some vertex v we can reduce $w(e)$ by 1 for each $e \in \delta(v)$ and reduce y_v by 1, to obtain a counterexample with smaller $w(E)$. Similarly, if $z(U) \geq 1$ for some $U \subseteq V$ we can reduce $w(e)$ by 1 for each $e \in E[U]$ and reduce $z(U)$ by 1, to obtain a counterexample with smaller $w(E)$.

Put on $y \in \{0, \frac{1}{2}\}^V$ and $z \in \{0, \frac{1}{2}\}^{\mathcal{P}(V)}$ the additional requirements that, first, $y(V)$ is as large as possible, and, second, that

(31.22) $$\sum_{U \subseteq V} z(U)|U||V \setminus U|$$

is as small as possible.

Let $S := \{v \in V \mid y_v = \frac{1}{2}\}$ and $\mathcal{F} := \{U \subseteq V \mid z(U) = \frac{1}{2}\}$. We first show that $\mathcal{F}$ is laminar; that is,

(31.23) if $U, W \in \mathcal{F}$, then $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$.

Indeed, suppose that $U \cap W \neq \emptyset$, $U \not\subseteq W$, and $W \not\subseteq U$ for some $U, W \in \mathcal{F}$.

If $b(U \cap W)$ is odd, then decreasing $z(U)$ and $z(W)$ by $\frac{1}{2}$, and increasing $z(U \cap W)$ and $z(U \cup W)$ by $\frac{1}{2}$, would not increase (31.18) (since $\lfloor\frac{1}{2}b(U \cap W)\rfloor + \lfloor\frac{1}{2}b(U \cup W)\rfloor \leq \lfloor\frac{1}{2}b(U)\rfloor + \lfloor\frac{1}{2}b(W)\rfloor$), would maintain (31.19) (since $\chi^{E[U\cap W]} + \chi^{E[U\cup W]} \geq \chi^{E[U]} + \chi^{E[W]}$), would leave $y(V)$ unchanged, but would decrease (31.22), contradicting the minimality of (31.22).

If $b(U \cap W)$ is even, then resetting

(31.24) $$z(U) := z(U) - \tfrac{1}{2},\ z(W) := z(W) - \tfrac{1}{2},\ z(U \setminus W) := z(U \setminus W) + \tfrac{1}{2},$$
$$z(W \setminus U) := z(W \setminus U) + \tfrac{1}{2}, \text{ and } y_v := y_v + \tfrac{1}{2} \text{ for each } v \in U \cap W,$$

would not increase (31.18) (since $\lfloor\frac{1}{2}b(U \setminus W)\rfloor + \lfloor\frac{1}{2}b(W \setminus U)\rfloor + b(U \cap W) \leq \lfloor\frac{1}{2}b(U)\rfloor + \lfloor\frac{1}{2}b(W)\rfloor$), would maintain (31.19) (since $\chi^{E[U\setminus W]} + \chi^{E[W\setminus U]} + \sum_{v\in U\cap W} \chi^{\delta(v)} \geq \chi^{E[U]} + \chi^{E[W]}$), but would increase $y(V)$, contradicting the maximality of $y(V)$.

This shows (31.23). Suppose $\mathcal{F} \neq \emptyset$. Then choose an inclusionwise minimal set $U \in \mathcal{F}$ with the property that there exist an even number of sets $W \in \mathcal{F}$ with $W \supset U$. Let $U_1, \ldots, U_k$ be the inclusionwise maximal proper subsets of U with $U_i \in \mathcal{F}$ (possibly $k = 0$). By the choice of U, none of the U_i contain properly a set in $\mathcal{F}$. Then

(31.25) $$\lfloor\tfrac{1}{2}b(U)\rfloor + \sum_{i=1}^{k} \lfloor\tfrac{1}{2}b(U_i)\rfloor \geq b(U \cap S) + \sum_{i=1}^{k} 2\lfloor\tfrac{1}{2}b(U_i \setminus S)\rfloor$$

or

(31.26) $$\lfloor\tfrac{1}{2}b(U)\rfloor + \sum_{i=1}^{k} \lfloor\tfrac{1}{2}b(U_i)\rfloor \geq b(U \setminus S) + \sum_{i=1}^{k} 2\lfloor\tfrac{1}{2}b(U_i \cap S)\rfloor,$$

as

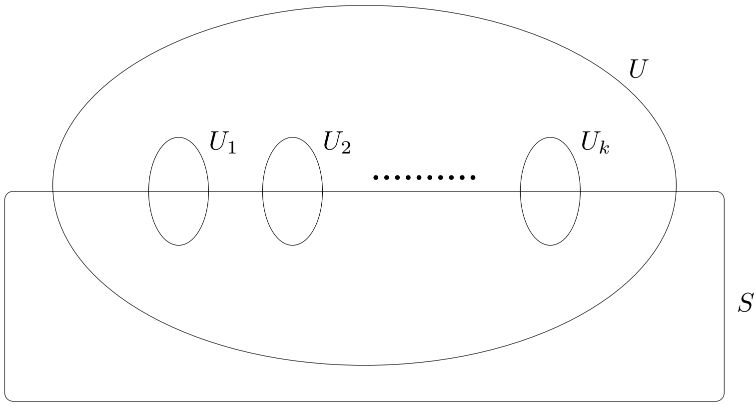


Figure 31.1

(31.27)
$$b(U) + 2\sum_{i=1}^{k}\lfloor \tfrac{1}{2}b(U_i)\rfloor$$
$$\geq b(U\cap S) + b(U\setminus S) + 2\sum_{i=1}^{k}\lfloor \tfrac{1}{2}b(U_i\setminus S)\rfloor + 2\sum_{i=1}^{k}\lfloor \tfrac{1}{2}b(U_i\cap S)\rfloor .$$

If (31.25) holds, then resetting $y_v := y_v + \frac{1}{2}$ for each $v \in U \cap S$, $z(U) := z(U) - \frac{1}{2}$, and $z(U_i) := z(U_i) - \frac{1}{2}$, $z(U_i \setminus S) := z(U_i \setminus S) + 1$ for each $i = 1, \ldots, k$ would not increase (31.18) (by (31.25)) and would maintain (31.19): on edges not spanned by U, the left-hand side of (31.19) does not decrease; on edges spanned by U the contribution of the nonmodified variables is integer, and

(31.28)
$$\lfloor \tfrac{1}{2}\Big(\sum_{v\in U\cap S}\chi^{\delta(v)} + \chi^{E[U]} + \sum_{i=1}^{k}\chi^{E[U_i]}\Big)\rfloor \leq \sum_{v\in U\cap S}\chi^{\delta(v)} + \sum_{i=1}^{k}\chi^{E[U_i\setminus S]}.$$

By the maximality of $y(V)$ it follows that $U \cap S = \emptyset$. Hence, after resetting we have $z(U_i) = 1$ for each $i = 1, \ldots, k$. If $k > 0$ we contradict (31.21). So $k = 0$, and therefore (as $z(U)$ decreases) (31.18) decreases, contradicting the minimality of (31.18).

If (31.26) holds, then resetting $y_v := y_v + \frac{1}{2}$ for each $v \in U \setminus S$, $z(U) := z(U) - \frac{1}{2}$, and $z(U_i) := z(U_i) - \frac{1}{2}$, $z(U_i \cap S) := z(U_i \cap S) + 1$ for each $i = 1, \ldots, k$ would not increase (31.18) (by (31.26)) and would maintain (31.19), since now

(31.29)
$$\lfloor \tfrac{1}{2}\Big(\sum_{v\in U\cap S}\chi^{\delta(v)} + \chi^{E[U]} + \sum_{i=1}^{k}\chi^{E[U_i]}\Big)\rfloor \leq \tfrac{1}{2}\sum_{v\in U}\chi^{\delta(v)} + \sum_{i=1}^{k}\chi^{E[U_i\cap S]}.$$

By the maximality of $y(V)$ it follows that $U \setminus S = \emptyset$, that is, $U \subseteq S$. Hence, after resetting we have $z(U_i) = 1$ for each $i = 1, \ldots, k$. If $k > 0$ we again contradict (31.21). So $k = 0$, and therefore (as $z(U)$ decreases) (31.18) decreases, again contradicting the minimality of (31.18).

So $\mathcal{F} = \emptyset$. Now setting $z'_S := 1$ and $y' := \mathbf{0}$ gives (31.20). ∎

(This is the proof method followed by Schrijver and Seymour [1977]. For a related proof, see Hoffman and Oppenheim [1978]. See also Cook [1983b].)

This theorem can be formulated equivalently in terms of total dual integrality:

Corollary 31.3a. *System* (31.17) *is TDI.*

Proof. Directly from Theorem 31.3. ∎

If we restrict the subsets U to odd-size subsets, the system is totally dual half-integral — a result stated by Pulleyblank [1973] and Edmonds [1975]:

Corollary 31.3b. *System* (31.6) *is totally dual half-integral.*

Proof. This follows from Corollary 31.3a, by using the fact that inequality (31.17)(iii) for $|U|$ even, is a half-integer sum of inequalities (31.6)(i) and (ii). ∎

Next considering the perfect b-matching polytope, generally (31.16) is not TDI. However:

Corollary 31.3c. *System* (31.16) *with* (31.16)(iii) *replaced by* (31.17)(iii) *is TDI.*

Proof. Directly from Corollary 31.3a with Theorem 5.25. ∎

This implies for the original system (Edmonds and Johnson [1970]):

Corollary 31.3d. *System* (31.16) *is totally dual half-integral.*

Proof. Consider an inequality $x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor$ in (31.17). If $b(U)$ is odd, this inequality is half of the sum of the inequalities $x(\delta(v)) \leq b(v)$ for $v \in U$ and of $-x(\delta(U)) \leq -1$. If $b(U)$ is even, this inequality is half of the sum of the inequalities $x(\delta(v)) \leq b(v)$ for $v \in U$ and of $-x_e \leq 0$ for $e \in \delta(U)$. ∎

In fact (Barahona and Cunningham [1989]):

Corollary 31.3e. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}} x$ subject to* (31.16) *has an integer optimum dual solution.*

Proof. As $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 31.3d there is an integer optimum dual solution y'_v ($v \in V$), z_U ($U \subseteq V$, $b(U)$ odd) for the problem of minimizing $w'^{\mathsf{T}}x$ subject to (31.16). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

31.4. The weighted b-matching problem

We now consider the problem of finding a maximum-weight b-matching. Here, for a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, the *weight* of a b-matching x is $w^{\mathsf{T}}x$.

It should be noted that the method of reducing a b-matching problem to a matching problem by replacing each vertex v by $b(v)$ copies, does not yield a polynomial-time algorithm for the weighted b-matching problem. W.H. Cunningham and A.B. Marsh, III (with suggestions of W.R. Pulleyblank, K. Truemper, and M.R. Rao — cf. Marsh [1979]) and Gabow [1983a] gave polynomial-time algorithms for the weighted b-matching problem. Padberg and Rao [1982] showed, with a method similar to that described in Section 25.5c, that one can test the constraints (31.16) in polynomial time, thus yielding the polynomial-time solvability of the maximum-weight b-matching problem (with the ellipsoid method).

Gerards [1995a] attributed the following method, leading to a strongly polynomial-time algorithm, to J. Edmonds. It extends a similar approach of Anstee [1987], and amounts to reducing the b-matching problem to a bipartite b-matching problem and a nonbipartite **1**-matching problem.

First there is the following observation.

Lemma 31.4α. *Let $G = (V, E)$ be a graph and let $b, b' \in \mathbb{Z}_+^V$ with $\|b - b'\|_1 = 1$. Let x be a b-matching and let x' be a b'-matching. Then there exists a $y \in \mathbb{Z}^E$ such that $\|y\|_\infty \leq 2$ and such that $x + y$ is a b'-matching and $x' - y$ is a b-matching.*

Proof. By symmetry we may assume that there exists a $u \in V$ such that $b'(u) = b(u) + 1$ and $b'(v) = b(v)$ if $v \neq u$. Hence x is a b'-matching. If x' is a b'-matching, we are done (taking $y = \mathbf{0}$). So we may assume that x' is not a b-matching, that is, $x'_u = b'(u)$. Then there exists a walk $P = (v_0, e_1, v_1, \ldots, e_t, v_t)$ in G such that

(31.30) (i) $v_0 = u$, $x'_{e_i} > x_{e_i}$ if i is odd, $x'_{e_i} < x_{e_i}$ if i is even, and each edge e is traversed at most $|x'_e - x_e|$ times,
(ii) $x'(\delta(v_t)) < x(\delta(v_t))$ if t is even, and $x'(\delta(v_t)) > x(\delta(v_t))$ if t is odd (if $v_t = v_0$ and t is odd, then $x'(\delta(v_t)) \geq x(\delta(v_t)) + 2$).

The existence of such a path follows by taking a longest path satisfying (31.30)(i).

We now assume that P is a shortest path satisfying (31.30). Then no vertex is traversed more than twice (otherwise we can shortcut P), hence no edge is traversed more than twice. Let y_e be the number of times P traverses e, if $x'_e \geq x_e$, and let y_e be *minus* the number of times P traverses e, if $x'_e < x_e$. Then $x + y$ is a b'-matching, $x' - y$ is a b-matching, and $\|y\|_\infty \leq 2$. ∎

This implies a sensitivity result for maximum-weight b-matchings if we vary b:

Lemma 31.4β. *Let $G = (V, E)$, let $b, b' \in \mathbb{Z}_+^V$ and let a weight function $w \in \mathbb{R}^E$ be given. Then for any maximum-weight b-matching x there exists a maximum-weight b'-matching x' satisfying*

(31.31) $\qquad \|x - x'\|_\infty \leq 2\|b - b'\|_1.$

Proof. We may assume that $\|b - b'\|_1 = 1$. Let x be a maximum-weight b-matching and let x' be a maximum-weight b'-matching. By Lemma 31.4α, we know that there exists an integer vector y with $x + y$ a b'-matching, $x' - y$ a b-matching, and $\|y\|_\infty \leq 2$. Since $x' - y$ is a b-matching and since x is a maximum-weight b-matching, we have $w^\mathsf{T} x \geq w^\mathsf{T}(x' - y)$, and hence $w^\mathsf{T}(x+y) \geq w^\mathsf{T} x'$. Since x' is a maximum-weight b'-matching, it follows that $x'' := x + y$ is a maximum-weight b'-matching with $\|x'' - x\|_\infty = \|y\|_\infty \leq 2$. ∎

This is used in showing the strong polynomial-time solvability of the weighted b-matching problem:

Theorem 31.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight b-matching can be found in strongly polynomial time.*

Proof. I. First consider the case that b is even. Make a bipartite graph H as follows. Make a new vertex v' for each $v \in V$. Let H have edges $u'v$ and uv' for each edge uv of G. Define $\tilde{b}(v) := \tilde{b}(v') := \frac{1}{2}b(v)$ for each $v \in V$. Define a weight $\tilde{w}(u'v) := \tilde{w}(uv') := w(uv)$ for each edge uv of G.

Find a maximum-weight $\tilde{b}$-matching $\tilde{x}$ in H. This can be done in strongly polynomial time by Theorem 21.9. Defining $x(uv) := \tilde{x}(u'v) + \tilde{x}(uv')$ for each edge uv of G, gives a maximum-weight b-matching x in G. Indeed, if there would be a b-matching in G of larger weight than that of x, then there is a half-integer $\tilde{b}$-matching in H of larger weight than that of $\tilde{x}$. This contradicts the fact that in a bipartite graph a maximum-weight b-matching is also a maximum-weight fractional b-matching (by Theorem 21.1).

II. Next consider the case of arbitrary b. Define $b' := 2\lfloor \frac{1}{2}b \rfloor$. Since b' is even, by part I of this proof we can find a maximum-weight b'-matching x' in G in strongly polynomial time. Now b arises from b' by at most $|V|$ resettings of b' to $b' + \chi^u$ for some $u \in V$. So it suffices to give a strongly polynomial-time method to obtain a maximum-weight b'-matching from a maximum-weight b-matching x, where $b' = b + \chi^u$ for some $u \in U$.

To this end, define

$$(31.32) \qquad z := \max\{\mathbf{0}, x - \mathbf{2}\} \text{ and } b'' := \min\{b' - Mz, M\mathbf{4}\}$$

(taking the maximum componentwise), where M is the $V \times E$ incidence matrix of G. ($\mathbf{0}$, $\mathbf{2}$, and $\mathbf{4}$ denote the all-0, all-2, and all-4 vector.)

Now we can find a maximum-weight b''-matching x'' in strongly polynomial time. This follows from the fact that $b''(v) \leq 4\deg(v)$ for each vertex v. So we can consider the graph $G_{b''}$ obtained by splitting each vertex v of G into $b''(v)$ copies, and replacing any edge uv by $b''(u)b''(v)$ edges connecting the $b''(u)$ copies of u by the $b''(v)$ copies of v. Then a maximum-weight matching in $G_{b''}$ gives a maximum-weight b''-matching x'' in G''.

Then $x''+z$ is a b'-matching, since $x''+z \geq \mathbf{0}$ and $M(x''+z) \leq b''+Mz \leq b'$. Moreover, $x''+z$ is a maximum-weight b'-matching, since by Lemma 31.4β, there exists a maximum-weight b'-matching x' satisfying $x - \mathbf{2} \leq x' \leq x + \mathbf{2}$. Then $x'-z$ is a b''-matching (since $x'-z \leq \mathbf{4}$), and hence $w^\mathsf{T} x'' \geq w^\mathsf{T}(x'-z)$. Therefore $w^\mathsf{T}(x'' + z) \geq w^\mathsf{T} x'$. ∎

Elaboration of this method gives an $O(n^2 m(n^2+m\log n))$-time algorithm. A similar approach of Anstee [1987] gives $O((m+n\log n)n\log\|b\|_\infty + n^2 m)$- and $O(n^2\log n(m+n\log n))$-time algorithms.

For weighted *perfect* b-matching, a similar result follows:

Corollary 31.4a. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight perfect b-matching can be found in strongly polynomial time.*

Proof. By flipping signs, it suffices to describe a method finding a maximum-weight perfect b-matching in strongly polynomial time.

We can increase each weight by a constant $C := BW + W$, where $W := \|w\|_\infty + 1$ and $B := \|b\|_1$. So each weight becomes $\geq C - W$ and $\leq C + W$. Then each perfect b-matching has weight at least $\frac{1}{2}B(C-W) = \frac{1}{2}B^2W$, while each nonperfect b-matching has weight at most

$$(31.33) \qquad \begin{aligned} &(\tfrac{1}{2}B - 1)(C + W) = \tfrac{1}{2}BC + \tfrac{1}{2}BW - C - W \\ &= \tfrac{1}{2}B^2W + \tfrac{1}{2}BW + \tfrac{1}{2}BW - BW - W - W < \tfrac{1}{2}B^2W. \end{aligned}$$

So each maximum-weight b-matching is perfect. Therefore, Theorem 31.4 applies. (Alternatively, we could repeat the above reduction process.) ∎

31.5. If b is even

The results on b-matchings can be simplified if b is even. In that case, the proofs can be reduced to the bipartite case. The maximum size of a $2b$-

matching is equal to the minimum weight of a 2-vertex cover, taking b as weight:

Theorem 31.5. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a $2b$-matching is equal to the minimum value of $y^{\mathsf{T}} b$ taken over 2-vertex covers y; equivalently, the minimum value of*

$$(31.34) \qquad b(V) + b(N(S)) - b(S),$$

taken over stable sets S.

Proof. Make a bipartite graph H as follows. Make a new vertex v' for each $v \in V$, and let $V' := \{v' \mid v \in V\}$. H has vertex set $V \cup V'$ and edges all $u'v$ and uv' for $uv \in E$.

Define $b' : V \cup V' \to \mathbb{Z}_+$ by $b'(v) := b'(v') := b(v)$ for all $v \in V$. Then the maximum size of a $2b$-matching in G is equal to the maximum size of a b'-matching in H. By Corollary 21.1a, this is equal to the minimum b'-weight of a vertex cover in H, which is equal to the minimum of $y^{\mathsf{T}} b$ over 2-vertex covers y. ∎

It implies the following characterization of the existence of perfect b-matchings for even b:

Corollary 31.5a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with b even. Then there exists a perfect b-matching if and only if $b(N(S)) \geq b(S)$ for each stable set S of G.*

Proof. Directly from Theorem 31.5. ∎

This can also be derived directly from Corollary 31.1a. The following two theorems can be derived from the bipartite case in a way similar to the proof of Theorem 31.5, but they also are special cases of results in this chapter.

First we have a characterization of the $2b$-matching polytope:

Theorem 31.6. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the $2b$-matching polytope is determined by*

$$(31.35) \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2b(v) & \text{for each } v \in V. \end{array}$$

Proof. This is a special case of Theorem 31.2. ∎

Second, we mention a result of Gallai [1957,1958a,1958b]. For a graph $G = (V, E)$ and $w : E \to \mathbb{Z}_+$, a *w-vertex cover* is a function $y : V \to \mathbb{Z}_+$ satisfying $y_u + y_v \geq w(uv)$ for each edge uv.

Theorem 31.7. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Z}_+^E$ and $b \in \mathbb{Z}_+^V$. Then the maximum weight $w^\mathsf{T} x$ of a $2b$-matching x is equal to the minimum value of $y^\mathsf{T} b$ taken over $2w$-vertex covers y.*

Proof. This follows from Theorem 31.3. ∎

31.6. If b is constant

The results on b-matchings can be specialized to 'k-matchings'. Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. A *k-matching* is a function $x \in \mathbb{Z}_+^E$ with $x(\delta(v)) \leq k$ for each vertex v. Thus if we identify k with the all-k vector in $\mathbb{Z}_+^V$, we have a k-matching as before. Therefore, Theorem 31.1 gives a min-max relation for maximum-size k-matching:

Theorem 31.8. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the maximum size of a k-matching is equal to the minimum value of*

$$k|U| + \sum_K \lfloor \tfrac{1}{2} k |K| \rfloor, \tag{31.36}$$

taken over $U \subseteq V$, where K ranges over the components of $G - U$ spanning at least one edge.

Proof. Directly from Theorem 31.1. ∎

Note that it follows that if k is even, we need not round, and hence the maximum size of a k-matching is equal to $\frac{1}{2}k$ times the maximum-size of a 2-matching. This maximum size is described in Theorem 30.1.

Again, a k-matching x is *perfect* if $x_v = k$ for each vertex v. In characterizing the existence, it is convenient to distinguish between the cases of k odd and k even. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Corollary 31.8a. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$ be odd. Then G has a perfect k-matching if and only if for each $U \subseteq V$, $G - U - I_U$ has at most $k(|U| - |I_U|)$ odd components K.*

Proof. Directly from Corollary 31.1a. ∎

For even k, there is the following result due to Tutte [1952]:

Corollary 31.8b. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$ be even. Then G has a perfect k-matching if and only if $|N(S)| \geq |S|$ for each stable set S.*

Proof. Directly from Corollary 31.5a. ∎

So if k is even, there exists a perfect k-matching if and only if there exists a perfect 2-matching.

We also give the characterization of the *k-matching polytope* (the convex hull of k-matchings):

Theorem 31.9. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the k-matching polytope is determined by*

(31.37)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq k & \text{for each } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}k|U| \rfloor & \text{for each } U \subseteq V \text{ with } k|U| \text{ odd.} \end{array}$$

Proof. This is a special case of Theorem 31.2. ∎

31.7. Further results and notes

31.7a. Complexity survey for the *b*-matching problem

Complexity survey for the maximum-weight b-matching problem:

*	$O(n^2 B)$	Pulleyblank [1973]
	$O(n^2 m \log B)$	W.H. Cunningham and A.B. Marsh, III (cf. Marsh [1979])
*	$O(m^2 \log n \log B)$	Gabow [1983a]
*	$O(n^2 m + n \log B(m + n \log n))$	Anstee [1987]
*	$O(n^2 \log n(m + n \log n))$	Anstee [1987]

Here $B := \|b\|_\infty$, and $*$ indicates an asymptotically best bound in the table.

Johnson [1965] extended Edmonds' matching algorithm to an algorithm (not based on splitting vertices) finding a maximum-size b-matching, with running time polynomially bounded in n, m, and B. Gabow [1983a] gave an $O(nm \log n)$-time algorithm to find a maximum-size b-matching.

31.7b. Facets and minimal systems for the *b*-matching polytope

Edmonds and Pulleyblank (see Pulleyblank [1973]) described the facets of the b-matching polytope. Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Call G *b-critical* if for each $u \in V$ there exists a b-matching x such that $x(\delta(u)) = b(u) - 1$ and $x(\delta(v)) = b(v)$ for each $v \neq u$.

Let G be simple and connected with at least three vertices and let $b > \mathbf{0}$. Then an inequality $x(\delta(v)) \leq b(v)$ determines a facet of the b-matching polytope if and only if $b(N(v)) > b(v)$, and if $b(N(v)) = b(v) + 1$, then $E[N(v)] \neq \emptyset$.

Moreover, an inequality $x(E[U]) \leq \lfloor \frac{1}{2}b(U) \rfloor$ determines a facet if and only if $G[U]$ is b-critical and has no cut vertex v with $b(v) = 1$.

Unlike in the matching case, the facet-inducing inequalities do not form a totally dual integral system. The minimal TDI-system for the b-matching polytope was characterized by Cook [1983a] and Pulleyblank [1981]. To describe this, call a graph $G = (V, E)$ *b-bicritical* if G is connected and for each $u \in V$ there is a b-matching x with $x(\delta(u)) = b(u) - 2$ and $x(\delta(v)) = b(v)$ for each $v \neq u$. Then a minimal TDI-system for the b-matching polytope (if G is simple and connected and has at least three vertices and if $b > \mathbf{0}$) is obtained by adding the following to the facet-inducing inequalities:

(31.38) $x(E[U]) \leq \frac{1}{2}b(U)$ for each $U \subseteq V$ with $|U| \geq 3$, $G[U]$ b-bicritical and $b(v) \geq 2$ for each $v \in N(U)$.

(The facets of the 2-matching polytope of a complete graph were also given by Grötschel [1977b].)

The vertices of the 2-matching polytope are characterized by:

Theorem 31.10. *Let $G = (V, E)$ be a graph. Then a 2-matching x is a vertex of the 2-matching polytope P if and only if the edges e with $x_e = 1$ form vertex-disjoint odd circuits.*

Proof. Let x be a 2-matching. Define $F := \{e \in E \mid x_e = 1\}$. Clearly, $\deg_F(v) \leq 2$ for each $v \in V$. So F forms a vertex-disjoint set of paths and circuits.

To see necessity in the theorem, let x be a vertex of P. Suppose that K is a component of F that forms a path or an even circuit. Then we can split K into matchings M and N. Then both $x + \chi^M - \chi^N$ and $x - \chi^M + \chi^N$ belong to P, contradicting the fact that x is a vertex of P.

To see sufficiency, suppose that x is not a vertex of P. Then there exists a nonzero vector y such that $x + y$ and $x - y$ belong to P. If $x_e = 0$ or $x_e = 2$, then $y_e = 0$, as $0 \leq x_e \pm y_e \leq 2$. If e and f are two edges in F incident with a vertex v, then $y_e = -y_f$, since $(x_e + x_f) \pm (y_e + y_f) \leq 2$. Hence, if each component of F is an odd circuit, we have $y = \mathbf{0}$, contradicting our assumption. ∎

31.7c. Regularizable graphs

A graph $G = (V, E)$ is called *regularizable* if there exists a k and a perfect k-matching x with $x \geq \mathbf{1}$. So we obtain a k-regular graph by replacing each edge e by x_e parallel edges. Berge [1978c] characterized regularizability as follows:

Theorem 31.11. *Let $G = (V, E)$ be connected and nonbipartite. Then G is regularizable if and only if $|N(U)| > |U|$ for each nonempty stable set U.*

Proof. Necessity being easy, we show sufficiency. Make a bipartite graph H by making for each vertex v a copy v', and replacing any edge uv by two edges uv' and $u'v$. Then every edge of H belongs to some perfect matching of H. To see this, suppose that edge uv' belongs to no perfect matching. Then by Frobenius' theorem (Corollary 16.2a), there exists a subset X of $V \setminus \{u\}$ such that X has less than $|X|$ neighbours in $V' \setminus \{v'\}$ (in the graph H; here $V' := \{v' \mid v \in V\}$). That is, defining $N'(X) := \cup_{u \in X} N_G(u)$,

(31.39) $|N'(X) \setminus \{v\}| < |X|$.

Let $U := X \setminus N'(X)$. Then U is a stable set. Moreover, $N(U) \subseteq N'(X) \setminus X$. By (31.39), $|N'(X)| \leq |X|$, and therefore $|N(U)| \leq |U|$. So by the condition given in the theorem, $U = \emptyset$; that is, $X \subseteq N'(X)$, and so, by (31.39), $X = N'(X)$. However, as G is connected and nonbipartite, H is connected. This contradicts the fact that $X = N'(X)$ and $X \neq V$.

So each edge of H belongs to a perfect matching. Hence each edge of G belongs to a perfect 2-matching. Adding up these perfect 2-matchings gives a perfect k-matching $x \geq \mathbf{1}$ for some k. ∎

Berge [1978b] remarked that this theorem is equivalent to: a connected nonbipartite graph G is regularizable if and only if the only 2-vertex cover of size $\tau_2(G)$ is the all-1 vector (this follows with (30.2)).

With the help of b-matchings, one can also characterize k-regularizable graphs — graphs that can become k-regular by adding edges parallel to existing edges. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Theorem 31.12. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then G is k-regularizable if and only if for each $U \subseteq V$, $G - U - I_U$ has at most*

(31.40) $$k|U| - k|I_U| - 2|E[U]| - |\delta(U \cup I_U)|$$

components K with $k|K| + |\delta(K)|$ odd.

Proof. From Corollary 31.1a applied to $b : V \to \mathbb{Z}_+$ defined by $b(v) := k - \deg(v)$ for $v \in V$.

Note that the condition implies $b(v) \geq 0$ for each vertex v. For suppose $\deg(v) > k$. If $k = 0$, then (31.40) is negative for $U := V$, a contradiction. So $k > 0$. Taking $U := \{v\}$, the condition implies that $k - k|I_U| - 2|E[U]| - |\delta(\{v\} \cup I_U)| \geq 0$. As $|\delta(v)| > k$, it follows that $I_U \neq \emptyset$, hence $|I_U| = 1$, say $I_U = \{w\}$. So $|E[U]| = 0$, that is, v is loopless. Moreover, $\delta(U \cup I_U) = \emptyset$, that is, $\{v, w\}$ is a component of G. But then the nonnegativity of (31.40) for $U' := \{v, w\}$ implies $2k \geq 2|E[U']| \geq 2\deg(v)$ (as v is loopless), a contradiction. ∎

See also Berge [1978b,1978d,1981].

31.7d. Further notes

Hoffman and Oppenheim [1978] showed that system (31.17) is 'locally strongly modular'; that is, each vertex of the b-matching polytope is determined by a linearly independent set of inequalities among (31.17) (set to equality), where the matrix in the system has determinant ± 1.

Johnson [1965] characterized the vertices of the fractional b-matching polytope. Koch [1979] studied bases (in the sense of the simplex method) for the linear programming problem of finding a maximum-weight b-matching.

Padberg and Wolsey [1984] described a strongly polynomial-time algorithm to find for any vector x the largest λ such that $\lambda \cdot x$ belongs to the b-matching polytope, and to describe $\lambda \cdot x$ as a convex combination of b-matchings.

b-matching algorithms are studied in the books by Gondran and Minoux [1984] and Derigs [1988a].

Chapter 32

Capacitated b-matchings

In the previous chapter we studied b-matchings, without upper bound given on the values of the edges. In this chapter we refine the results to the case where each edge has a prescribed 'capacity' that bounds the value on the edge. This can be reduced to uncapacitated b-matching.

32.1. Capacitated b-matchings

The capacitated b-matching problem considers b-matchings x satisfying a prescribed capacity constraint $x \leq c$. By a construction of Tutte [1954b], results on capacitated b-matchings can be derived from the results for the uncapacitated case as follows. Denote

(32.1) $$E[X,Y] := \{e \in E \mid \exists x \in X, y \in Y : e = \{x,y\}\}.$$

Theorem 32.1. *Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ with $c > \mathbf{0}$. Then the maximum size of a b-matching $x \leq c$ is equal to the minimum value of*

(32.2) $$b(U) + c(E[W]) + \sum_K \lfloor \tfrac{1}{2}(b(K) + c(E[K,W])) \rfloor,$$

taken over disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. To see that the maximum is not more than the minimum, let x be a b-matching with $x \leq c$ and let U, W be disjoint subsets of V. Then $x(E[U] \cup \delta(U)) \leq b(U)$ and $x(E[W]) \leq c(E[W])$. Consider next a component K of $G - U - W$. Then $2x(E[K]) + x(E[K,W]) \leq b(K)$ and $x(E[K,W]) \leq c(E[K,W])$. Hence $x(E[K] \cup E[K,W]) \leq \lfloor \frac{1}{2}(b(K) + c(E[K,W])) \rfloor$, and the inequality follows.

The reverse inequality is proved by reduction to Theorem 31.1. Make a new graph $G' = (V', E')$ by replacing each edge of G by a path of length three. That is, for each edge $e = uv$ introduce two new vertices $p_{e,u}$ and $p_{e,v}$ and three edges: $up_{e,u}$, $p_{e,u}p_{e,v}$, and $p_{e,v}v$.

Define $b' \in \mathbb{Z}_+^{V'}$ by $b'(v) := b(v)$ if $v \in V$ and $b'(p_{e,v}) := c(e)$ for any new vertex $p_{e,v}$. Then the maximum size of a b-matching x in G with $x \leq c$ is equal to the maximum size of a b'-matching in G', minus $c(E)$. By Theorem 31.1, there exists a subset U' of V' such that the maximum size of a b'-matching in G' equals

(32.3) $$b'(U') + \sum_{K'} \lfloor \tfrac{1}{2} b'(K') \rfloor,$$

where K' ranges over the components of $G' - U'$ with $|K'| \geq 2$. (Note that G' has no loops.) We choose U' with $|U'|$ as small as possible.

Let $U := V \cap U'$ and let W be the set of isolated vertices of $G' - U'$ that belong to V. We show that (32.2) is at most (32.3) minus $c(E)$, which proves the theorem.

First observe that

(32.4) if $p_{e,v} \in U'$, then $v \in W$.

Otherwise, deleting $p_{e,v}$ from U' does not increase (32.3), contradicting the minimality of $|U'|$. (Here we use that $p_{e,v}$ has degree 2 and that $b'(p_{e,v}) > 0$, that is, $c(e) > 0$. Then $b'(U')$ decreases by $c(e)$ while the sum in (32.3) increases by at most $\lfloor \frac{1}{2} c(e) + 1 \rfloor$, which is at most $c(e)$.)

Hence

(32.5) $$\begin{aligned} b'(U') &= b(U) + b'(U' \setminus V) = b(U) + \sum_{v \in W} c(\delta(v)) \\ &= b(U) + 2c(E[W]) + c(\delta(W)). \end{aligned}$$

Consider a component K' of $G' - U'$ with $|K'| \geq 2$. If K' does not intersect V, then it is equal to $\{p_{e,u}, p_{e,v}\}$ for some edge $e = uv$ of G with $u, v \in U$. So $b'(K') = 2c(e)$. If K' intersect V, let $K := K' \cap V$. Then K is a component of $G - U - W$. Indeed, any edge spanned by K gives a path of length 3 in K' (by (32.4)), and any path in K' between vertices in K gives a path in K. Any edge of G leaving K gives a path of length 3 in G' connecting K' and $U \cup W$. So

(32.6) $$K' = K \cup \{p_{e,u} \mid e = uv \in E, u \in K\} \cup \{p_{e,v} \mid e = uv \in E, u \in K, v \in U\}.$$

Hence

(32.7) $$b'(K') = b(K) + c(E[K, W]) + 2c(E[K]) + 2c(E[K, U]).$$

Therefore, (32.3) is equal to

(32.8) $$\begin{aligned} &b(U) + 2c(E[W]) + c(\delta(W)) + c(E[U]) \\ &+ \sum_{K} \big(\lfloor \tfrac{1}{2}(b(K) + c(E[K, W])) \rfloor + c(E[K]) + c(E[K, U]) \big), \end{aligned}$$

where K ranges over the components of $G - U - W$. Since

(32.9) $$c(E) = c(E[W]) + c(\delta(W)) + c(E[U]) + \sum_K (c(E[K]) + c(E[K,U])),$$

(32.3) minus $c(E)$ is equal to (32.2). ∎

This implies for *perfect* b-matchings:

Corollary 32.1a. *Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ with $c > \mathbf{0}$. Then G has a perfect b-matching $x \leq c$ if and only if for each partition T, U, W of V, $G[T]$ has at most*

(32.10) $$b(U) - b(W) + 2c(E[W]) + c(E[T,W])$$

components K with $b(K) + c(E[K,W])$ odd.

Proof. Directly from Theorem 32.1. ∎

32.2. The capacitated b-matching polytope

Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The *c-capacitated b-matching polytope* is the convex hull of the b-matchings x satisfying $x \leq c$. A description of this polytope follows again from that for the uncapacitated b-matching polytope.

Theorem 32.2. *Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The c-capacitated b-matching polytope is determined by*

(32.11)
$$\begin{array}{rlll}
\text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\
\text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\
\text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + c(F)) \rfloor & (U \subseteq V, F \subseteq \delta(U), \\
 & & b(U) + c(F) \text{ odd}).
\end{array}$$

Proof. It is easy to show that each b-matching $x \leq c$ satisfies (32.11). To show that the inequalities (32.11) completely determine the c-capacitated b-matching polytope, let $x \in \mathbb{R}^E$ satisfy (32.11). Let $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ be as in the proof of Theorem 32.1. Define $x' \in \mathbb{R}^{E'}$ by $x'(up_{e,u}) := x'(vp_{e,v}) := x_e$ and $x'(p_{e,u}p_{e,v}) := c(e) - x_e$, for any edge $e = uv$ of G. We show that x' belongs to the b'-matching polytope with respect to G'.

By Theorem 31.2, it suffices to check the constraints (31.6) for x' with respect to G' and b'. That is, we should check (where $\delta' := \delta_{G'}$ and $E'[U']$ is the set of edges in E' spanned by U'):

(32.12)
$$\begin{array}{rll}
\text{(i)} & x'(e') \geq 0 & (e' \in E'), \\
\text{(ii)} & x'(\delta'(v')) \leq b'(v') & (v' \in V'), \\
\text{(iii)} & x'(E'[U']) \leq \lfloor \frac{1}{2} b'(U') \rfloor & (U' \subseteq V' \text{ with } b'(U') \text{ odd}).
\end{array}$$

Trivially we have (32.12)(i) by (32.11)(i). Moreover, for each vertex $v \in V$ one has $x'(\delta'(v)) \leq b'(v)$ by (32.11)(ii). For any vertex $p_{e,u}$ of G', with $e = uv \in E$, one has $x'(\delta'(p_{e,u})) = c(e) = b'(p_{e,u})$.

To prove (32.12)(iii), we first show that it suffices to prove it for those $U' \subseteq V'$ satisfying for each edge $e = uv \in E$:

(32.13) (i) if $u, v \in U'$, then $p_{e,u} \in U'$ and $p_{e,v} \in U'$,
(ii) if $p_{e,u} \in U'$, then $u \in U'$.

To see (32.13)(i), first let $u, v \in U'$ and $p_{e,u} \notin U'$. Define $U'' := U' \cup \{p_{e,u}, p_{e,v}\}$. Then

(32.14) $$x'(E'[U']) \leq x'(E'[U'']) - x'(\delta'(p_{e,u})) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor - b'(p_{e,u}) \leq \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

To see (32.13)(ii), let $p_{e,u} \in U'$ and $u \notin U'$. Define $U'' := U' \setminus \{p_{e,u}, p_{e,v}\}$. If $p_{e,v} \notin U'$, then

(32.15) $$x'(E'[U']) = x'(E'[U'']) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor \leq \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

If $p_{e,v} \in U'$, then

(32.16) $$x'(E'[U']) = x'(E'[U'']) + x'(\delta'(p_{e,v})) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor + b'(p_{e,v}) = \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

This proves that we may assume (32.13) (as repeated application of these modifications gives finally (32.13)). Let $U := U' \cap V$ and let F be the set of those edges $e = uv$ in $\delta(U)$ with $u \in U$, $v \notin U$, and $p_{e,u} \in U'$. Then $x'(E'[U']) = x(E[U]) + c(E[U]) + x(F)$ and $b'(U') = b(U) + 2c(E[U]) + c(F)$. Hence (32.11)(iii) implies (32.12)(iii).

So x' is a convex combination of b'-matchings in G'. Each such b'-matching y satisfies $y(\delta'(v')) = b'(v')$ for each 'new' vertex $v' = p_{e,u}$ (as x' satisfies this equality). Hence each such b'-matching corresponds to a b-matching subject to c in G, and we obtain x as convex combination of b-matchings subject to c in G. ∎

Similarly, the *c-capacitated perfect b-matching polytope* is the convex hull of the perfect b-matchings x satisfying $x \leq c$. Theorem 32.2 implies the following (announced by Edmonds and Johnson [1970] (cf. Green-Krótki [1980], Aráoz, Cunningham, Edmonds, and Green-Krótki [1983])):

Corollary 32.2a. *The c-capacitated perfect b-matching polytope is determined by*

(32.17)
(i)	$0 \leq x_e \leq c(e)$	$(e \in E)$,
(ii)	$x(\delta(v)) = b(v)$	$(v \in V)$,
(iii)	$x(\delta(U) \setminus F) - x(F) \geq 1 - c(F)$	$(U \subseteq V, F \subseteq \delta(U),$ $b(U) + c(F)$ odd$)$.

Proof. Directly from Theorem 32.2, as (32.17)(ii) implies that $x(E[U]) = \frac{1}{2} b(U) - \frac{1}{2} x(\delta(U))$. ∎

32.3. Total dual integrality

System (32.11) generally is not TDI (cf. the example in Section 30.5). To obtain a TDI-system, one should delete the restriction in (32.11)(iii) that $b(U) + c(F)$ is odd. Thus we obtain:

(32.18)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + c(F)) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Theorem 32.3. *System* (32.18) *is TDI.*

Proof. Let $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ be as in the proof of Theorem 32.1. By Corollary 31.3a, the following system is TDI:

(32.19)
$$\begin{array}{rll} \text{(i)} & x'(e') \geq 0 & (e' \in E'), \\ \text{(ii)} & x'(\delta'(v')) \leq b'(v') & (v' \in V'), \\ \text{(iii)} & x'(E'[U']) \leq \lfloor \frac{1}{2} b'(U') \rfloor & (U' \subseteq V'). \end{array}$$

Since setting inequalities to equalities maintains total dual integrality (Theorem 5.25), the following system is TDI:

(32.20)
$$\begin{array}{rll} \text{(i)} & x'(e') \geq 0 & (e' \in E'), \\ \text{(ii)} & x'(\delta'(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x'(up_{e,u}) + x'(p_{e,u}p_{e,v}) = c(e) & (u \in e = uv \in E), \\ \text{(iv)} & x'(E'[U']) \leq \lfloor \frac{1}{2} b'(U') \rfloor & (U' \subseteq V'). \end{array}$$

The inequalities (32.14), (32.15), and (32.16) show that in (32.20)(iv) we may restrict the U' to those satisfying (32.13). So U' is determined by $U := U' \cap V$ and $F := \{e = uv \in E \mid u, p_{e,u} \in U', v \notin U'\}$.

Moreover, with Theorem 5.27 we can eliminate the variables $x'(up_{e,u})$ for $e \in E$ and $u \in e$ with the equalities (32.20)(iii). That is, we replace $x'(up_{e,u})$ by $c(e) - y_e$, where we set $y_e := x'(p_{e,u}p_{e,v})$ for $e = uv \in E$. Then:

(32.21)
$$x'(E'[U']) = y(E[U]) + 2(c(E[U]) - y(E[U])) + c(F) - y(F) \text{ and}$$
$$b'(U') = b(U) + 2c(E[U]) + c(F).$$

Hence the system becomes:

(32.22)
$$\begin{array}{rll} \text{(i)} & y_e \geq 0 & (e \in E), \\ \text{(ii)} & y_e \leq c(e) & (e \in E), \\ \text{(iii)} & -y(\delta(v)) \leq b(v) - c(\delta(v)) & (v \in V), \\ \text{(iv)} & -y(E[U]) - y(F) \leq \lfloor \frac{1}{2} b(U) + c(F) \rfloor - c(E[U]) - c(F) & \\ & & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Setting y_e to $c(e) - x_e$, the system becomes (32.18) and remains TDI. ∎

32.4. The weighted capacitated b-matching problem

By the construction given in the proof of Theorem 32.1, the weighted capacitated b-matching problem can easily be reduced to the uncapacitated variant:

Theorem 32.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight b-matching $x \leq c$ can be found in strongly polynomial time.*

Proof. We may assume that $w \geq \mathbf{0}$. Make $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ as in the proof of Theorem 32.1. Moreover, define a weight function w' on the edges of G' by $w'(up_{e,u}) := w'(p_{e,u}p_{e,v}) := w'(p_{e,v}v) := w(e)$ for any edge $e = uv$ of G.

Let x' be a maximum-weight b'-matching in G'. Then we may assume that for each edge $e = uv$ of G one has $x'(up_{e,u}) = c(e) - x'(p_{e,u}p_{e,v}) = x'(p_{e,v}v)$. (This follows from the fact that we can assume that $x'(up_{e,u}) = x'(p_{e,v}v)$, since if say $x'(up_{e,u}) = x'(p_{e,v}v) + \tau$ with $\tau > 0$, we can decrease $x'(up_{e,u})$ by τ and increase $x'(p_{e,u}p_{e,v})$ by τ. Next we can reset $x'(p_{e,u}p_{e,v}) := c(e) - x'(up_{e,u})$.)

Now define $x_e := x'(up_{e,u})$ for each edge $e = uv$ of G. Then x is a maximum-weight b-matching with $x \leq c$. ∎

Similarly, for the weighted capacitated perfect b-matching problem:

Theorem 32.5. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight perfect b-matching $x \leq c$ can be found in strongly polynomial time.*

Proof. As in the previous proof replace each edge by a path of length three, yielding the graph G', and define b', and w' similarly. Let x' be a maximum-weight perfect b'-matching in G'. Then for each edge $e = uv$ of G one has $x'(up_{e,u}) = c(e) - x'(p_{e,u}p_{e,v}) = x'(p_{e,v}v)$. Defining $x_e := x'(up_{e,u})$ for each edge $e = uv$ of G, gives a maximum-weight b-matching $x \leq c$. ∎

32.4a. Further notes

Cook [1983b] and Cook and Pulleyblank [1987] determined the facets and the minimal TDI-system for the capacitated b-matching polytope.

Johnson [1965] extended Edmonds' matching algorithm to an algorithm (not based on reduction to matching) that finds a maximum-size capacitated b-matching, with running time bounded by a polynomial in n, m, and $\|b\|_\infty$. Gabow [1983a] gave an $O(nm \log n)$-time algorithm for this.

Cunningham and Green-Krótki [1991] showed the following. Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then the convex hull of the integer vectors $y \leq b$ for which there is a perfect y-matching $x \leq c$ is determined by the inequalities

$$(32.23) \qquad \begin{aligned} &\mathbf{0} \leq y \leq b, \\ &y(\bigcup_{i=0}^{k} A_i) - y(B) \leq \sum_{i=1}^{k}(b(A_i) - 1) + c(E[A_0]) + c(E[A_0, V \setminus B]), \end{aligned}$$

where A_0 and B are disjoint subsets of V and where $A_1, \ldots, A_k$ are some of the components of $G - A_0 - B$ such that $b(A_i) + c(E[A_0, A_i])$ is odd for each $i = 1, \ldots, k$.

This characterizes the convex hull of degree-sequences of capacitated b-matchings, where the *degree-sequence* of $x \in \mathbb{Z}^E$ is the vector $y \in \mathbb{Z}^E$ defined by $y_v = x(\delta(v))$ for $v \in V$.

This generalizes the results of Balas and Pulleyblank [1989] on the matchable set polytope (Section 25.5d) and of Koren [1973] on the convex hull of degree-sequences of simple graphs (Section 33.6c below). See also Cunningham and Green-Krótki [1994] and Cunningham and Zhang [1992].

Chapter 33

Simple b-matchings and b-factors

A special case of capacitated b-matchings is obtained when we take capacity 1 on every edge. So the b-matching takes values 0 and 1 only. Such a b-matching is called *simple*. A simple b-matching is the incidence vector of some set of edges. If the b-matching is simple and perfect it is called a *b-factor*.
In this chapter we derive results on simple b-matchings and b-factors in a straightforward way from those on capacitated b-matchings obtained in the previous chapter.

33.1. Simple b-matchings and b-factors

Call a b-matching x *simple* if x is a 0,1 vector. We can identify simple b-matchings with subsets F of E with $\deg_F(v) \leq b(v)$ for each $v \in V$.

Simple b-matchings are special cases of capacitated b-matchings, namely by taking capacity function $c = \mathbf{1}$. Hence a min-max relation for maximum-size simple b-matching follows from the more general capacitated version:

Theorem 33.1. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a simple b-matching is equal to the minimum value of*

$$(33.1) \qquad b(U) + |E[W]| + \sum_K \lfloor \tfrac{1}{2}(b(K) + |E[K, W]|) \rfloor,$$

taken over all disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. The theorem is the special case $c = \mathbf{1}$ of Theorem 32.1. ∎

A *b-factor* is a simple perfect b-matching. In other words, it is a subset F of E with $\deg_F(v) = b(v)$ for each $v \in V$. The existence of a b-factor was characterized by Tutte [1952,1974] (cf. Ore [1957]):

Corollary 33.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then G has a b-factor if and only if for each partition T, U, W of V, the graph $G[T]$ has at most*

(33.2) $$b(U) - b(W) + 2|E[W]| + |E[T, W]|$$

components K with $b(K) + |E[K, W]|$ odd.

Proof. Directly from Theorem 33.1 (or Corollary 32.1a). ∎

(An algorithmic proof was given by Anstee [1985], yielding an $O(n^3)$-time algorithm to find a b-factor. Tutte [1981] gave another proof and a sharpening.)

33.2. The simple b-matching polytope and the b-factor polytope

Given a graph $G = (V, E)$ and a vector $b \in \mathbb{Z}_+^V$, the *simple b-matching polytope* is the convex hull of the simple b-matchings in G. It can be characterized by (Edmonds [1965b]):

Theorem 33.2. *The simple b-matching polytope is determined by*

(33.3)
$$\begin{array}{rll} \text{(i)} & 0 \le x_e \le 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \le b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \le \lfloor \tfrac{1}{2}(b(U) + |F|) \rfloor & (U \subseteq V, F \subseteq \delta(U), \\ & & b(U) + |F| \text{ odd}). \end{array}$$

Proof. The theorem is a special case of Theorem 32.2. ∎

Given a graph $G = (V, E)$ and a vector $b \in \mathbb{Z}_+^V$, the *b-factor polytope* is the convex hull of (the incidence vectors of) b-factors in G. As it is a face of the simple b-matching polytope (if nonempty), we have:

Corollary 33.2a. *The b-factor polytope is determined by*

(33.4)
$$\begin{array}{rll} \text{(i)} & 0 \le x_e \le 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) = b(v) & (v \in V), \\ \text{(iii)} & x(\delta(U) \setminus F) - x(F) \ge 1 - |F| & (U \subseteq V, F \subseteq \delta(U), \\ & & b(U) + |F| \text{ odd}). \end{array}$$

Proof. Directly from Theorem 33.2. ∎

33.3. Total dual integrality

Consider the system (extending (33.3)):

(33.5)
$$\begin{array}{rlr} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + |F|) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

A special case of Theorem 32.3 is (cf. Cook [1983b]):

Theorem 33.3. *System* (33.5) *is TDI.*

Proof. Directly from Theorem 32.3. ∎

It implies for the b-factor polytope:

Corollary 33.3a. *System* (33.4) *is totally dual half-integral.*

Proof. By Theorems 33.3 and 5.25, the system obtained from (33.5) by setting (33.5)(ii) to equality, is TDI. Then each inequality (33.5) is a half-integer sum of inequalities (33.4), and the theorem follows. ∎

This can be extended to:

Corollary 33.3b. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}}x$ subject to* (33.4) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 33.3a there is an integer optimum dual solution y'_v ($v \in V$), z_U ($U \subseteq V$, $b(U)$ odd) for the problem of minimizing $w'^{\mathsf{T}}x$ subject to (33.4). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

33.4. The weighted simple b-matching and b-factor problem

Also algorithmic results can be derived from the general capacity case, but some arguments can be simplified. While finding a minimum-weight b-factor can be reduced to finding a minimum-weight perfect b-matching, there is a more direct construction, since we can assume that b is not too large. We give the precise arguments in the proofs below.

Theorem 33.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight simple b-matching can be found in strongly polynomial time.*

Proof. We may assume that $b(v) \leq \deg_G(v)$ for each $v \in V$, since replacing $b(v)$ by $\min\{b(v), \deg_G(v)\}$ for each v does not change the problem.

Now the techniques described in Chapters 31 and 32 (replacing each vertex by $b(v)$ vertices, and next each edge by a path of length three), yield a strongly polynomial reduction to the maximum-weight matching problem. ∎

So a maximum-size simple b-matching and a b-factor (if any) can be found in polynomial time.

A similar construction applies to the weighted b-factor problem:

Theorem 33.5. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight b-factor can be found in strongly polynomial time.*

Proof. We may assume that $b(v) \leq \deg_G(v)$ for each $v \in V$, since otherwise there is no b-factor. Now the reduction techniques described in Chapters 31 and 32 yield a strongly polynomial reduction to the minimum-weight perfect matching problem. ∎

33.5. If b is constant

Again we can specialize the results above to k-matchings and k-factors, for $k \in \mathbb{Z}_+$. First we have for the maximum size of a simple k-matching:

Theorem 33.6. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. The maximum size of a simple k-matching is equal to the minimum value of*

$$(33.6) \qquad k|U| + |E[W]| + \sum_K \lfloor \tfrac{1}{2}(k|K| + |E[K, W]|) \rfloor,$$

taken over all disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. Directly from Theorem 33.1. ∎

A *k-factor* is a simple perfect k-matching. In other words, it is a subset F of E with (V, F) k-regular. Theorem 33.6 implies a classical theorem of Belck [1950]:

Corollary 33.6a. *A graph $G = (V, E)$ has a k-factor if and only if for each partition T, U, W of V, $G[T]$ has at most*

$$(33.7) \qquad k(|U| - |W|) + 2|E[W]| + |E[T, W]|$$

components K with $k|K| + |E[K, W]|$ odd.

Proof. Directly from Theorem 33.6. ∎

Petersen [1891] showed that the following is easy:

Theorem 33.7. *Each connected $2k$-regular graph G with an even number of edges has a k-factor.*

Proof. Make an Eulerian tour in G, and colour the edges alternatingly red and blue. Then the red edges form a k-factor. ∎

33.6. Further results and notes

33.6a. Complexity results

Urquhart [1967] gave an $O(b(V)n^3)$-time algorithm for finding a maximum-weight simple b-matching. This was improved by Gabow [1983a] to $O(b(V)m \log n)$ (by reduction to the $O(nm \log n)$-time algorithm of Galil, Micali, and Gabow [1982, 1986] for maximum-weight matching) and to $O(b(V)n^2)$. For maximum-*size* simple b-matching, Gabow [1983a] gave algorithms of running time $O(\sqrt{b(V)}\,m)$ (by reduction to Micali and Vazirani [1980]) and to $O(nm \log n)$.

33.6b. Degree-sequences

A sequence $d_1, \ldots, d_n$ is called a *degree-sequence* of a graph $G = (V, E)$ if we can order the vertices as $v_1, \ldots, v_n$ such that $\deg_G(v_i) = d_i$ for $i = 1, \ldots, n$.

From Corollary 33.1a one can derive the characterization of degree-sequences of simple graphs due to Erdős and Gallai [1960]: there exists a simple graph with degrees $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\text{(33.8)} \qquad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

for $k = 1, \ldots, n$.

Havel [1955] gave the following recursive algorithm to decide if a sequence is the degree-sequence of a simple graph. A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree-sequence of a simple graph if and only if $0 \leq d_n \leq n-1$ and $d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1}$ is the degree sequence of a simple graph.

Koren [1973] showed that the convex hull of degree-sequences of simple graphs on a finite vertex set V is determined by:

$$\text{(33.9)} \qquad \begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for each } v \in V, \\ \text{(ii)} & x(U) - x(W) \leq |U|(|V| - |W| - 1) & \text{for disjoint } U, W \subseteq V. \end{array}$$

If the graph need not be simple (but yet is loopless), condition (33.8) can be replaced by $\sum_{i=2}^{n} d_i \geq d_1$, as can be shown easily (cf. Hakimi [1962a]). Related work was done by Peled and Srinivasan [1989], who showed that system (33.9) is totally dual integral and characterized vertices, facets, and adjacency on the polytope determined by (33.9).

Kundu [1973] showed that if both sequences $d_1 \geq \cdots \geq d_n \geq k$ and $d_1 - k \geq \cdots \geq d_n - k \geq 0$ are realizable (as degree-sequence of a simple graph), then the first sequence is realizable by a graph with a k-factor (answering a question of Grünbaum [1970]). See also Edmonds [1964] and Cai, Deng, and Zang [2000].

33.6c. Further notes

Cook [1983b] and Cook and Pulleyblank [1987] determined the facets and the minimal TDI-system for the simple b-matching polytope. Hausmann [1978a,1981] characterized adjacency on the simple b-matching polytope.

Lovász [1972f] extended the Edmonds-Gallai decomposition to b-factors (cf. Lovász [1972e] and Graver and Jurkat [1980]). For a sharpening of Corollary 33.1a by specializing T, U, W, see Tutte [1974,1978].

Fulkerson, Hoffman, and McAndrew [1965] showed the following. Let $G = (V, E)$ be a graph such that any two odd circuits have a vertex in common or are connected by an edge. Let $b \in \mathbb{Z}_+^V$. Then G has a b-factor if and only if $b(V)$ is even and

$$b(U) + 2|E[W]| + |E[T, W]| \geq b(W) \tag{33.10}$$

for each partition T, U, W of V (cf. Mahmoodian [1977]).

Baebler [1937] showed that any k-regular l-connected graph has an l-factor if k is odd and l is even. Era [1985] proved the following conjecture of Akiyama [1982]: for each k there exists a t such that for each r-regular graph $G = (V, E)$ with $r \geq t$, E can be partitioned into $E_1, \ldots, E_s$ with for each $i = 1, \ldots, s$ one has $k \leq \deg_{E_i}(v) \leq k + 1$ for each vertex v.

Katerinis [1985] showed that if k', k, k'' are odd natural numbers with $k' \leq k \leq k''$, then any graph G having a k'-factor and a k''-factor, also has a k-factor. Related results are reported in Enomoto, Jackson, Katerinis, and Saito [1985].

Goldman [1964] studied augmenting paths for simple b-matchings by reduction to $\mathbf{1}$-matchings. More on b-matchings and b-factors can be found in Bollobás [1978], Tutte [1984], and Bollobás, Saito, and Wormald [1985].

Chapter 34

b-edge covers

The covering analogue of a b-matching is the b-edge cover. It is not difficult to derive min-max relations, polyhedral characterizations, and algorithms for b-edge covers from those for b-matchings.

34.1. *b*-edge covers

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. A *b-edge cover* is a function $x \in \mathbb{Z}_+^E$ satisfying

(34.1) $$x(\delta(v)) \geq b(v)$$

for each $v \in V$.

There is a direct analogue of Gallai's theorem (Theorem 19.1), also given in Gallai [1959a], relating maximum-size b-matchings and minimum-size b-edge covers:

Theorem 34.1. *Let $G = (V, E)$ be a graph without isolated vertices and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a b-matching plus the minimum size of a b-edge cover is equal to $b(V)$.*

Proof. Let x be a minimum-size b-edge cover. For any $v \in V$, reduce $x(\delta(v))$ by $x(\delta(v)) - b(v)$, by reducing x_e on edges $e \in \delta(v)$. We obtain a b-matching y of size

(34.2) $$y(E) \geq x(E) - \sum_{v \in V} (x(\delta(v)) - b(v)) = b(V) - x(E).$$

Hence the maximum-size of a b-matching is at least $b(V) - x(E)$.

Conversely, let y be a maximum-size b-matching. For any $v \in V$, increase $y(\delta(v))$ by $b(v) - y(\delta(v))$, by increasing y_e on edges $e \in \delta(v)$. We obtain a b-edge cover x of size

(34.3) $$x(E) \leq y(E) + \sum_{v \in V} (b(v) - y(\delta(v))) = b(V) - y(E).$$

Hence the minimum-size of a b-edge cover is at most $b(V) - y(E)$. ∎

(An alternative way of proving this is by applying Gallai's theorem for the case $b = \mathbf{1}$ directly to the graph G_b described in (31.2), obtained from G by splitting any vertex v into $b(v)$ vertices.)

With Theorem 34.1, we can derive a min-max relation for minimum-size b-edge cover from that for maximum-size b-matching. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Corollary 34.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the minimum size of a b-edge cover is equal to the maximum value of*

$$(34.4) \qquad b(I_U) + \sum_K \lceil \tfrac{1}{2} b(K) \rceil,$$

taken over $U \subseteq V$, where K ranges over the components of $G - U - I_U$.

Proof. Directly from Theorems 34.1 and 31.1. ∎

The construction in the proof of Theorem 34.1 also implies that a minimum-size b-edge cover can be found in polynomial time.

34.2. The b-edge cover polyhedron

Given a graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, the *b-edge cover polyhedron* is the convex hull of the b-edge covers. The inequalities describing the b-edge cover polyhedron can be easily derived from the description of the edge cover polytope, similar to Theorem 31.2.

Theorem 34.2. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the b-edge cover polyhedron is determined by the inequalities*

$$(34.5) \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U] \cup \delta(U)) \geq \lceil \tfrac{1}{2} b(U) \rceil & (U \subseteq V, b(U) \text{ odd}). \end{array}$$

Proof. Similar to the proof of Theorem 31.2, by construction of G_b and reduction to the description of the edge cover polytope (Corollary 27.3a). The theorem also follows from Theorem 34.3 below. ∎

34.3. Total dual integrality

The constraints (34.5) are totally dual integral if we delete the parity condition in (34.5)(iii):

$$\begin{array}{llll} \text{(34.6)} & \text{(i)} & x_e \geq 0 & (e \in E), \\ & \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ & \text{(iii)} & x(E[U] \cup \delta(U)) \geq \lceil \tfrac{1}{2} b(U) \rceil & (U \subseteq V). \end{array}$$

It is equivalent to the following:

Theorem 34.3. *Let $G = (V, E)$ be a graph, $b \in \mathbb{Z}_+^V$, and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^{\mathsf{T}} x$ of a b-edge cover x is equal to the maximum value of*

$$\text{(34.7)} \qquad \sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z_U \lceil \tfrac{1}{2} b(U) \rceil,$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfy

$$\text{(34.8)} \qquad \sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \subseteq V} z_U \chi^{E[U] \cup \delta(U)} \leq w.$$

Proof. We derive this from Theorem 32.3. Define $B := b(V) + 1$. Then the minimum is attained by a b-edge cover $x < B \cdot \mathbf{1}$. So adding $x_e \leq B$ for $e \in E$ as inequalities to (34.6) does not make it TDI if it wasn't. Let $\tilde{b}(v) := B \cdot \deg(v) - b(v)$ for each $v \in V$. Then by Theorem 32.3, the following system is TDI:

$$\begin{array}{lll} \text{(34.9)} & 0 \leq \tilde{x}_e \leq B & (e \in E), \\ & \tilde{x}(\delta(v)) \leq \tilde{b}(v) & (v \in V), \\ & \tilde{x}(E[U] \cup F) \leq \lfloor \tfrac{1}{2}(\tilde{b}(U) + B|F|) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Hence also the following system is TDI (by resetting $x_e = B - \tilde{x}_e$ for each $e \in E$):

$$\begin{array}{lll} \text{(34.10)} & 0 \leq x_e \leq B & (e \in E), \\ & x(\delta(v)) \geq b(v) & (v \in V), \\ & x(E[U] \cup F) \geq \lceil \tfrac{1}{2}(b(U) - B|\delta(U) \setminus F|) \rceil & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Now we can restrict ourselves in the last set of inequalities to those with $F = \delta(U)$, as otherwise the right-hand side is negative. So we have system (34.6) added with the superfluous inequalities $x_e \leq B$ for $e \in E$. ∎

Equivalently, in TDI terms:

Corollary 34.3a. *System* (34.6) *is totally dual integral.*

Proof. Directly from Theorem 34.3. ∎

34.4. The weighted *b*-edge cover problem

A minimum-weight b-edge cover can be found in strongly polynomial time, by reduction to maximum-weight b-matching:

Theorem 34.4. *For any graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and weight function $w \in \mathbb{Q}^E$, a minimum-weight b-edge cover can be found in strongly polynomial time.*

Proof. Define $B := \|b\|_\infty$. Then we can assume that a minimum-weight b-edge cover x satisfies $x_e \leq B$ for each $e \in E$. Define $\tilde{b}(v) := B \cdot \deg(v) - b(v)$ for each $v \in V$. By Theorem 32.4, we can find a maximum-weight $\tilde{b}$-matching x in strongly polynomial time. Defining $x_e := B - \tilde{x}_e$ for each e then gives a minimum-weight b-edge cover. ∎

34.5. If b is even

The results can be simplified if b is even. In that case, the proofs can be reduced to the bipartite case.

Minimum-size $2b$-edge cover relates to maximum-weight 2-stable set, taking b as weight. Here a 2-*stable set* is a function $y \in \mathbb{Z}_+^V$ with $y_u + y_v \leq 2$ for each edge uv.

Theorem 34.5. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the minimum size of a $2b$-edge cover is equal to the maximum value of $y^\mathsf{T} b$ where y is a 2-stable set; equivalently, to the maximum value of*

(34.11) $$b(V) + b(S) - b(N(S)),$$

taken over stable sets S.

Proof. Similar to the proof of Theorem 31.5. (Alternatively, the present theorem can be derived with Theorem 34.1 from Theorem 31.7.) ∎

For a graph $G = (V, E)$ and $w : E \to \mathbb{Z}_+$, a w-*stable set* is a function $y : V \to \mathbb{Z}_+$ satisfying $y_u + y_v \leq w(uv)$ for each edge uv. Gallai [1957,1958a, 1958b] showed:

Theorem 34.6. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a $2b$-edge cover is equal to the maximum value of $y^\mathsf{T} b$ where y is a $2w$-stable set.*

Proof. This follows from Theorem 34.3. ∎

34.6. If b is constant

The above results can also be specialized to k-edge covers, for $k \in \mathbb{Z}_+$. That is, b is constant.

Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. A k-*edge cover* is a function $x \in \mathbb{Z}_+^E$ with $x(\delta(v)) \geq k$ for each vertex v. Thus if we identify k with the all-k

vector in $\mathbb{Z}_+^V$, we have a k-edge cover as before. Therefore, Corollary 34.1a gives the following, where I_U denotes the set of isolated (hence loopless) vertices of $G - U$:

Theorem 34.7. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a k-edge cover is equal to the maximum value of*

$$(34.12) \qquad k|I_U| + \sum_K \lceil \tfrac{1}{2} k|K| \rceil,$$

over $U \subseteq V$, where K ranges over the components of $G - U - I_U$.

Proof. Directly from Corollary 34.1a. ∎

Note that it follows that if k is even, we need not round, and hence the minimum size of a k-edge cover is equal to $\frac{1}{2}k$ times the minimum-size of a 2-edge cover.

34.7. Capacitated b-edge covers

The capacitated b-edge cover problem considers b-edge covers x satisfying a prescribed capacity constraint $x \leq c$. Results on capacitated b-edge covers can be easily derived from the results on capacitated b-matchings.

For minimum-size capacitated b-edge cover, one has:

Theorem 34.8. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then the minimum size of a b-edge cover $x \leq c$ is equal to the maximum value of*

$$(34.13) \qquad b(U) - c(E[U]) + \sum_K \lceil \tfrac{1}{2}(b(K) - c(E[K, U])) \rceil,$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. Define $b'(v) := c(\delta(v)) - b(v)$ for each $v \in V$. Then by Theorem 32.1,

$$\begin{aligned}(34.14) \qquad & \text{minimum size of a } b\text{-edge cover } x \leq c \\ & = c(E) - \text{maximum size of a } b'\text{-matching } x' \leq c \\ & = c(E) - \min_{T,U,W} (b'(U) + c(E[W]) \\ & \quad + \sum_K \lfloor \tfrac{1}{2}(b'(K) + c(E[K, W])) \rfloor) \\ & = \max_{T,U,W} c(E) - 2c(E[U]) - c(\delta(U)) + b(U) - c(E[W]) \\ & \quad - \sum_K \lfloor \tfrac{1}{2}(2c(E[K]) + c(\delta(K)) - b(K) + c(E[K, W])) \rfloor \\ & = \max_{T,U,W} b(U) - c(E[U]) + \sum_K \lceil \tfrac{1}{2}(b(K) - c(E[K, U])) \rceil\end{aligned}$$

(since $c(E) = c(E[U]) + c(\delta(U)) + c(E[W]) + c(E[T, W])$), where T, U, W range over partitions of V and where K ranges over the components of $G[T]$. ∎

This reduction also implies that a minimum-size b-edge cover $x \leq c$ can be found in strongly polynomial time.

Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The *c-capacitated b-edge cover polytope* is the convex hull of the b-edge covers x satisfying $x \leq c$. The description of the inequalities follows again from that for the capacitated b-matching polytope.

Theorem 34.9. *The c-capacitated b-edge cover polytope is determined by*

$$
(34.15)\qquad
\begin{array}{rll}
\text{(i)} & 0 \leq x_e \leq c(e) & (e \in E),\\
\text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V),\\
\text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \frac{1}{2}(b(U) - c(F)) \rceil & \\
& \qquad\qquad (U \subseteq V, F \subseteq \delta(U),\ b(U) - c(F) \text{ odd}). &
\end{array}
$$

Proof. From Theorem 32.2, by setting $\tilde{b}(v) := c(\delta(v)) - b(v)$ and $\tilde{x}_e := c(e) - x_e$. ∎

By deleting the parity condition in (34.15)(iii), the system becomes totally dual integral:

Theorem 34.10. *The following system is TDI:*

$$
(34.16)\qquad
\begin{array}{rll}
\text{(i)} & 0 \leq x_e \leq c(e) & (e \in E),\\
\text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V),\\
\text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \frac{1}{2}(b(U) - c(F)) \rceil & \\
& \qquad\qquad (U \subseteq V, F \subseteq \delta(U)). &
\end{array}
$$

Proof. From Theorem 32.3, with the substitutions as given in the proof of the previous theorem. ∎

The weighted capacitated b-edge cover problem can easily be reduced to the uncapacitated variant:

Theorem 34.11. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight b-edge cover $x \leq c$ can be found in strongly polynomial time.*

Proof. From Theorem 32.4, with the construction given in the proof of Theorem 34.8. ∎

Agarwal, Sharma, and Mittal [1982] showed that a minimum-weight b-edge cover $x \leq c$ can be obtained from a minimum-weight 'fractional' b-edge cover $x' \leq c$ with the help of a minimum-weight $\mathbf{1}$-edge cover algorithm.

34.8. Simple b-edge covers

Call a b-edge cover x *simple* if x is a 0,1 vector. Thus we can identify simple b-edge covers with subsets F of E such that $\deg_F(v) \geq b(v)$ for each $v \in V$.

So defining $\tilde{b}(v) := \deg_G(v) - b(v)$ for $v \in V$, a vector x is a simple b-edge cover if and only if $\mathbf{1} - x$ is a simple $\tilde{b}$-matching. This reduces simple b-edge cover problems to simple $\tilde{b}$-matching problems. With this reduction, Theorem 33.1 gives:

Theorem 34.12. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with $b(v) \leq \deg(v)$ for each $v \in V$. Then the minimum size of a simple b-edge cover is equal to the maximum value of*

$$b(U) - |E[U]| + \sum_K \lceil \tfrac{1}{2}(b(K) - |E[K, U]|) \rceil, \tag{34.17}$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. From Theorem 33.1 applied to $\tilde{b}$. ∎

The *simple b-edge cover polytope* is the convex hull of the simple b-edge covers in G.

Theorem 34.13. *The simple b-edge cover polytope is determined by*

$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \tfrac{1}{2}(b(U) - |F|) \rceil & \\ & & (U \subseteq V, F \subseteq \delta(U),\ b(U) - |F| \text{ odd}). \end{array} \tag{34.18}$$

Proof. This is a special case of Theorem 34.9. ∎

Again the system is TDI:

Theorem 34.14. *System* (34.18) *is totally dual integral after deleting the parity condition in* (iii).

Proof. The theorem is a special case of Theorem 34.10. ∎

Simple b-matchings are special cases of capacitated b-matchings, namely by taking the capacity function $c = \mathbf{1}$. Hence a minimum-weight simple b-edge cover can be found in strongly polynomial time:

Theorem 34.15. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight simple b-edge cover can be found in strongly polynomial time.*

Proof. The theorem is a special case of Theorem 34.11. ∎

We can specialize these results to k-edge covers, for $k \in \mathbb{Z}_+$. A *simple k-edge cover* is a set of edges covering each vertex at least k times. Thus it corresponds to subgraphs of minimum degree at least k. A min-max relation for minimum-size simple k-edge cover reads:

Theorem 34.16. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a simple k-edge cover is equal to the maximum value of*

$$k|U| - |E[U]| + \sum_K \lceil \tfrac{1}{2}(k|K| - |E[K, U]|) \rceil, \tag{34.19}$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. This is a special case of Theorem 34.12. ∎

34.8a. Simple b-edge covers and b-matchings

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with $b(v) \leq \deg_G(v)$ for each $v \in V$. Define

$$\begin{aligned} \nu^{\mathrm{s}}(b) &:= \text{the maximum size of a simple } b\text{-matching}, \\ \rho^{\mathrm{s}}(b) &:= \text{the minimum size of a simple } b\text{-edge cover}. \end{aligned} \tag{34.20}$$

Similar to Theorem 34.1, there is a relation between $\nu^{\mathrm{s}}(b)$ and $\rho^{\mathrm{s}}(b)$, generalizing Gallai's theorem (Theorem 19.1):

$$\nu^{\mathrm{s}}(b) + \rho^{\mathrm{s}}(b) = b(V). \tag{34.21}$$

To see this, let M be a maximum-size simple b-matching. For each $v \in V$, add to M $b(v) - \deg_M(v)$ edges incident with v. We can do this in such a way that we obtain a simple b-edge cover F with $|F| \leq |M| + \sum_{v \in V}(b(v) - \deg_M(v)) = b(V) - |M|$. So $\rho^{\mathrm{s}}(b) \leq b(V) - |M| = b(V) - \nu^{\mathrm{s}}(b)$.

To see the reverse inequality, let F be a minimum-size simple b-edge cover. For each $v \in V$, delete from F $\deg_F(v) - b(v)$ edges incident with v. We obtain a simple b-matching M with $|M| \geq |F| - \sum_{v \in V}(\deg_F(v) - b(v)) = b(V) - |F|$. So $\nu^{\mathrm{s}}(b) \geq b(V) - |F| = b(V) - \rho^{\mathrm{s}}(b)$, which shows (34.21).

There is a second relation between simple b-matchings and simple b-edge covers. Define $\tilde{b}(v) := \deg_G(v) - b(v)$ for each $v \in V$. Then trivially (by complementing),

$$\nu^{\mathrm{s}}(b) + \rho^{\mathrm{s}}(\tilde{b}) = |E|. \tag{34.22}$$

(34.21) implies

$$b(V) - 2\nu^{\mathrm{s}}(b) = \rho^{\mathrm{s}}(b) - \nu^{\mathrm{s}}(b) = 2\rho^{\mathrm{s}}(b) - b(V), \tag{34.23}$$

and (34.22) implies

$$\rho^{\mathrm{s}}(b) - \nu^{\mathrm{s}}(b) = \rho^{\mathrm{s}}(\tilde{b}) - \nu^{\mathrm{s}}(\tilde{b}). \tag{34.24}$$

Hence

$$b(V) - 2\nu^{\mathrm{s}}(b) = \tilde{b}(V) - 2\nu^{\mathrm{s}}(\tilde{b}) = 2\rho^{\mathrm{s}}(b) - b(V) = 2\rho^{\mathrm{s}}(\tilde{b}) - \tilde{b}(V). \tag{34.25}$$

So the 'deficiency' of a maximum-size b-matching is equal to the 'surplus' of a minimum-size b-edge cover, and this parameter is invariant under replacing b by $\tilde{b} = \deg_G - b$.

34.8b. Capacitated *b*-edge covers and *b*-matchings

The results of the previous section hold more generally for capacitated b-matchings. Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and let $c \in \mathbb{Z}_+^E$ with $b(v) \leq c(\delta(v))$ for each $v \in V$. Define

(34.26) $\nu^c(b) :=$ the maximum size of a b-matching $x \leq c$,
$\rho^c(b) :=$ the minimum size of a b-edge cover $x \leq c$.

Then:

(34.27) $\nu^c(b) + \rho^c(b) = b(V)$.

To see this, consider a maximum-size b-matching $x \leq c$. We can increase x to obtain a b-edge cover $y \leq c$, in such a way that $y(E) \leq x(E) + \sum_{v \in V}(b(v) - x(\delta(v))) = b(V) - x(E)$. So $\rho^c(b) \leq b(V) - x(E) = b(V) - \nu^c(b)$.

To see the reverse inequality, consider a minimum-size b-edge cover $y \leq c$. We can decrease y to obtain a b-matching $x \leq y$ such that $x(E) \geq y(E) - \sum_{v \in V}(y(\delta(v)) - b(v)) = b(V) - y(E)$. So $\nu^c(b) \geq b(V) - y(E) = b(V) - \rho^c(b)$, which shows (34.27).

Again, there is a second relation between capacitated b-matchings and capacitated b-edge covers. Define $\tilde{b}(v) := c(\delta(v)) - b(v)$ for each $v \in V$. Then trivially (by replacing x by $c - x$),

(34.28) $\nu^c(b) + \rho^c(\tilde{b}) = c(E)$.

Combining (34.27) and (34.28) gives as in (34.25):

(34.29) $b(V) - 2\nu^c(b) = \tilde{b}(V) - 2\nu^c(\tilde{b}) = 2\rho^c(b) - b(V) = 2\rho^c(\tilde{b}) - \tilde{b}(V)$.

So the 'deficiency' of a maximum-size b-matching $x \leq c$ is equal to the 'surplus' of a minimum-size b-edge cover $y \leq c$, and this parameter is invariant under replacing b by $\tilde{b} := c \circ \delta - b$.

Chapter 35

Upper and lower bounds

In the previous chapters we considered nonnegative integer functions satisfying certain lower *or* upper bounds. We now turn over to the more general case where we put both upper *and* lower bounds. We also relax the condition that the functions be nonnegative. Again, the results can be proved by refining the results of previous chapters — thus all results are obtained essentially by reduction to the fundamental results of Tutte and Edmonds.

35.1. Upper and lower bounds

Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$. We will consider functions $x \in \mathbb{Z}^E$ satisfying

(35.1) (i) $d(e) \leq x_e \leq c(e)$ for all $e \in E$,
(ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for all $v \in V$.

The existence of such a function is characterized in the following theorem. (As usual, $E[X, Y]$ denotes the set of edges xy in E with $x \in X$ and $y \in Y$.)

Theorem 35.1. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ with $a \leq b$ and $d, c \in \mathbb{Z}^E$ with $d < c$. Then there exists an $x \in \mathbb{Z}^E$ satisfying* (35.1) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and*

(35.2) $$b(K) + c(E[K, W]) + d(E[K, U])$$

odd is at most

(35.3) $$b(U) - 2d(E[U]) - d(E[T, U]) - a(W) + 2c(E[W]) + c(E[T, W]).$$

Proof. To see necessity, consider a component K of $G[T]$ with $b(K) = a(K)$. Then

(35.4) $$2x(E[K]) = b(K) - x(\delta(K)) = b(K) - x(E[K, U]) - x(E[K, W]).$$

Hence, if (35.2) is odd, we have $x(E[K, U]) \geq d(E[K, U]) + 1$ or $x(E[K, W]) \leq c(E[K, W]) - 1$. So $x(E[T, U]) - d(E[T, U]) + c(E[T, W]) - x(E[T, W])$ is at least the number of such components. On the other hand, (35.5) $$x(E[T,U]) - x(E[T,W]) = x(\delta(U)) - x(\delta(W)) \\ \leq b(U) - 2d(E[U]) - a(W) + 2c(E[W]).$$

This proves necessity.

To see sufficiency, we may assume that $d = \mathbf{0}$, since the theorem is invariant under replacing $a(v)$ by $a(v) - d(\delta(v))$ and $b(v)$ by $b(v) - d(\delta(v))$ for each v, and c by $c - d$ and d by $\mathbf{0}$. (It does not change the parity of (35.2) and does not change (35.3).)

We show sufficiency by application of Corollary 32.1a. Define

(35.6) $$R := \{v \in V \mid a(v) < b(v)\}.$$

Extend G to a graph $G' = (V', E')$, and define $b' \in \mathbb{Z}_+^{V'}$ and $c' \in \mathbb{Z}_+^{E'}$, as follows. For each $v \in V$, let $b'(v) := b(v)$ and for each $e \in E$, let $c'(e) := c(e)$. Introduce a new vertex v_0, with $b'(v_0) := b(V)$, and a loop v_0v_0 at v_0, with $c'(v_0v_0) := \infty$. Moreover, for each $v \in R$ introduce an edge vv_0 with $c'(vv_0) := b(v) - a(v)$.

Now a function x as required exists if and only if there exists a perfect b'-matching $x' \leq c'$ in G'. So it suffices to test the constraints given by Corollary 32.1a for G', b', and c'. Assuming x' does *not* exist, we can partition V' into T', U', and W' such that $G'[T']$ has more than $b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W'])$ components K' with $b'(K') + c'(E'[K', W'])$ odd. By parity, the excess is at least 2. (This follows from the fact that $b'(V') = 2b(V)$ is even.)

Let $T := T' \setminus \{v_0\}$, $U := U' \setminus \{v_0\}$, and $W := W' \setminus \{v_0\}$.

First assume that $v_0 \in U'$; so $T' = T$ and $W' = W$. Then the number of components K of $G'[T'] = G[T]$ with $b(K) + c(E[K,W])$ odd is trivially at most $b(T) + c(E[T,W])$, and hence at most

(35.7) $$b(U) + b(V) - b(W) + 2c(E[W]) + c(E[T,W]) \\ = b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W']),$$

a contradiction.

Second assume that $v_0 \in W'$. Then $c'(E'W') = \infty$, which is again a contradiction.

Hence we may assume that $v_0 \in T'$; so $U' = U$ and $W' = W$. Then $G'[T']$ has exactly one component containing v_0. All other components K are components of $G[T]$ that are disjoint from R (since no vertex in K is adjacent to v_0). So $G[T]$ has more than

(35.8) $$b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W']) \\ = b(U) - a(W) + 2c(E[W]) + c(E[T,W])$$

components K contained in $V \setminus R$ with $b(K) + c(E[K,W])$ odd. This contradicts the condition of the theorem. ∎

By taking $d = \mathbf{0}$ and $c = \boldsymbol{\infty}$ we obtain as special case (where again I_U denotes the set of isolated (hence loopless) vertices of $G - U$):

Corollary 35.1a. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \le b$. Then there exists a function $x \in \mathbb{Z}_+^E$ satisfying*

(35.9) $\qquad a(v) \le x(\delta(v)) \le b(v)$

for each $v \in V$ if and only if for each $U \subseteq V$, $G - U - I_U$ has at most $b(U) - a(I_U)$ components K with $b(K)$ odd and $a(K) = b(K)$.

Proof. We show sufficiency. Suppose that no such x exists. By Theorem 35.1 (for $d = \mathbf{0}$, $c = \infty$), there exists a partition T, U, W of V with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ such that the number of components K of $G[T]$ with $b(K) = a(K)$ and $b(K)$ odd, is more than $b(U) - a(W)$. We may assume that each component K of $G[T]$ spans at least one edge: otherwise, if $K = \{v\}$, moving v from T to W, decreases the number of such components by at most 1, while $b(U) - a(W)$ decreases by at least 1 (since $b(v) = a(v)$ and $b(v)$ is odd).

So we can assume that $W = I_U$, in which case we have a contradiction with the condition in the present corollary. ∎

Another special case, for $d = \mathbf{0}$, and $c = \mathbf{1}$, is the characterization of Lovász [1970c] of the existence of subgraphs with prescribed degrees:

Corollary 35.1b. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \le b$. Then E has a subset F such that*

(35.10) $\qquad a(v) \le \deg_F(v) \le b(v)$

for each $v \in V$ if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and $b(K) + |E[K, W]|$ odd is at most $b(U) - a(W) + 2|E[W]| + |E[T, W]|$.

Proof. This is the case $d = \mathbf{0}$, $c = \mathbf{1}$ of Theorem 35.1. ∎

The construction described in the proof of Theorem 35.1 also implies:

Theorem 35.2. *Given a graph $G = (V, E)$, $a, b \in \mathbb{Z}^V$, $d, c \in \mathbb{Z}^E$, and $w \in \mathbb{Q}^E$, a vector $x \in \mathbb{Z}^E$ satisfying $d \le x \le c$ and $a(v) \le x(\delta(v)) \le b(v)$ for each $v \in V$, and minimizing $w^{\mathsf{T}} x$, can be found in strongly polynomial time.*

Proof. The construction in the proof of Theorem 35.1 reduces this to Theorem 32.4. ∎

35.2. Convex hull

We now characterize the convex hull of the functions $x \in \mathbb{Z}^E$ satisfying (35.1):

Theorem 35.3. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$ with $a \le b$ and $d \le c$. Then the convex hull of the vectors $x \in \mathbb{Z}^E$ satisfying* (35.1) *is determined by* (35.1) *together with the inequalities*

$$\text{(35.11)} \qquad x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W)) \leq \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor,$$

where U and W are two disjoint subsets of V and where F and H partition $\delta(U \cup W)$, with $b(U) - a(W) + c(F) - d(H)$ odd.

Proof. Necessity of (35.11) follows by adding up the following inequalities, each implied by (35.1):

$$\text{(35.12)} \qquad \begin{aligned} &x(E[U]) + \tfrac{1}{2}x(\delta(U)) \leq \tfrac{1}{2}b(U),\\ &-x(E[W]) - \tfrac{1}{2}x(\delta(W)) \leq -\tfrac{1}{2}a(W),\\ &\tfrac{1}{2}x(F) \leq \tfrac{1}{2}c(F),\\ &-\tfrac{1}{2}x(H) \leq -\tfrac{1}{2}d(H). \end{aligned}$$

The left-hand sides add up to the left-hand side of (35.11), and the right-hand side to the unrounded right-hand side of (35.11).

To see sufficiency of (35.11), we may assume that $d = \mathbf{0}$. Indeed, the theorem is invariant under resetting $a(v) := a(v) - d(\delta(v))$, and $b(v) := b(v) - d(\delta(v))$ for all $v \in V$, and $c := c - d$ and $d := \mathbf{0}$. Then, as above, we can reduce the theorem to Corollary 32.2a characterizing the convex hull of capacitated b-matchings.

Let x satisfy (35.1) and (35.11). Let R, G', b', and c' be as in the proof of Theorem 35.1. Define $x'(e) := x_e$ for each $e \in E$, $x'(vv_0) := b(v) - x(\delta(v))$ for each $v \in R$, and $x'(v_0v_0) := 2x(E)$.

We show that x' belongs to the c'-capacitated perfect b'-matching polytope (with respect to G'). This implies that x belongs to the convex hull of vectors $x \in \mathbb{Z}_+^E$ satisfying (35.1).

By Corollary 32.2a, it suffices to check

$$\text{(35.13)} \qquad \begin{aligned} &\text{(i)} \quad 0 \leq x'(e') \leq c'(e') \quad (e' \in E'),\\ &\text{(ii)} \quad x'(\delta'(v')) = b'(v') \quad (v' \in V'),\\ &\text{(iii)} \quad x'(\delta'(U') \setminus F') - x'(F') \geq 1 - c'(F')\\ &\qquad\qquad (U' \subseteq V',\ F' \subseteq \delta'(U') \text{ with } b'(U') + c'(F') \text{ odd}). \end{aligned}$$

(35.13)(i) and (ii) are direct. To see (35.13)(iii), let $U' \subseteq V'$ and $F' \subseteq \delta'(U')$ with $b'(U')+c'(F')$ odd. We may assume that $v_0 \in U'$ (as we can replace U' by its complement, since $b'(V') = 2b(V)$ is even). Let $W := \{v \in V \mid vv_0 \in F\}$ and $U := V \setminus (U' \cup W)$. Let $F := F' \cap E$ and $H := \delta_E(U \cup W) \setminus F$.

Now $b'(U') = b(V) + b(V \setminus (U \cup W))$ and $c'(F') = c(F) + (b - a)(W)$. So $b'(U') + c'(F')$ odd implies that $b(U) - a(W) + c(F)$ is odd. So by (35.11),

$$\text{(35.14)} \qquad 2x(E[U]) - 2x(E[W]) + 2x(F \cap \delta(U)) - 2x(H \cap \delta(W)) \leq b(U) - a(W) + c(F) - 1.$$

Hence

$$\text{(35.15)} \qquad x'(\delta'(U') \setminus F') - x'(F') = x(H) + \sum_{v \in U}(b(v) - x(\delta(v))) - x(F) - \sum_{v \in W}(b(v) - x(\delta(v)))$$

$$
\begin{aligned}
&= x(H) + b(U) - 2x(E[U]) - x(\delta(U)) - x(F) - b(W) \\
&+ 2x(E[W]) + x(\delta(W)) = -2x(E[U]) + 2x(E[W]) - 2x(F \cap \delta(U)) \\
&+ 2x(H \cap \delta(W)) + (b(U) - b(W)) \\
&\geq 1 - b(U) + a(W) - c(F) + (b(U) - b(W)) \\
&= 1 - c(F) - b(W) + a(W) = 1 - c'(F'),
\end{aligned}
$$

proving (35.13)(iii). ∎

The special case $d = \mathbf{0}$, $c = \boldsymbol{\infty}$ was mentioned by Schrijver and Seymour [1977]:

Corollary 35.3a. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$. Then the convex hull of those $x \in \mathbb{Z}^E$ satisfying*

(35.16) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for each $v \in V$,

is determined by (35.16) *together with the inequalities:*

(35.17) $x(E[U]) - x(E[W]) - x(\delta(W) \setminus \delta(U)) \leq \lfloor \frac{1}{2}(b(U) - a(W)) \rfloor$,

where U and W are disjoint subsets of V with $b(U) - a(W)$ odd.

Proof. This is a special case of Theorem 35.3. ∎

Similarly, we can characterize the convex hull of subgraphs with prescribed bounds on the degrees:

Corollary 35.3b. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then the convex hull of the incidence vectors of subsets F of E satisfying*

(35.18) $a(v) \leq \deg_F(v) \leq b(v)$

for each $v \in V$, is determined by

(35.19) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for each $v \in V$,

together with the inequalities

(35.20) $x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W))$
$\leq \lfloor \frac{1}{2}(b(U) - a(W) + |F|) \rfloor$,

where U and W are disjoint subsets of V and where F and H partition $\delta(U \cup W)$, with $b(U) - a(W) + |F|$ odd.

Proof. Again this is a special case of Theorem 35.3. ∎

We note that for the $V \times E$ incidence matrix M of any graph $G = (V, E)$, any $a, b \in \mathbb{Z}^V$, $d, c \in \mathbb{Z}^E$, and any $k, l \in \mathbb{Z}$ one has:

$$\text{(35.21)}\qquad \begin{aligned}&\text{conv.hull}\{x \in \mathbb{Z}^E \mid d \le x \le c, a \le Mx \le b, k \le x(E) \le l\}\\ &= \text{conv.hull}\{x \in \mathbb{Z}^E \mid d \le x \le c, a \le Mx \le b\}\\ &\cap \{x \in \mathbb{R}^E \mid k \le x(E) \le l\}.\end{aligned}$$

This can be proved similarly to Corollary 18.10a.

35.3. Total dual integrality

System (35.1) together with the inequalities (35.11) generally is not TDI (cf. the example in Section 30.5). To obtain a totally dual integral system we should delete the restriction in (35.11) that $b(U) - a(W) + c(F) - d(H)$ be odd. Thus we obtain the system:

$$\text{(35.22)}\qquad \begin{array}{rll}\text{(i)} & d(e) \le x_e \le c(e) & (e \in E),\\ \text{(ii)} & a(v) \le x(\delta(v)) \le b(v) & (v \in V),\\ \text{(iii)} & \multicolumn{2}{l}{x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W))}\\ & \quad \le \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor & \\ & & (U, W \subseteq V, U \cap W = \emptyset,\ F, H\\ & & \text{partition } \delta(U \cup W)).\end{array}$$

Theorem 35.4. *System* (35.22) *is totally dual integral.*

Proof. Again we may assume $d = \mathbf{0}$. Let R, G', b', and c' be as in the proof of Theorem 35.1. By Theorem 32.3, the following system, in the variable $x' \in \mathbb{R}^{E'}$, is TDI (where $\delta' := \delta_{G'}$ and $E'[U']$ is the set of edges in E' spanned by U'):

$$\text{(35.23)}\qquad \begin{array}{rll}\text{(i)} & 0 \le x'(e') \le c'(e') & (e' \in E'),\\ \text{(ii)} & x'(\delta'(v')) = b'(v') & (v' \in V'),\\ \text{(iii)} & \multicolumn{2}{l}{x'(E'[U']) + x'(F') \le \lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor}\\ & & (U' \subseteq V', F' \subseteq \delta'(U')).\end{array}$$

Adding the equality

$$\text{(35.24)}\qquad x'(v_0v_0) - x'(E) = 0$$

to (35.23) maintains total dual integrality (since (35.24) is valid for each vector x' satisfying (35.23)).

We can restrict the inequalities (35.23)(iii) to those with $v_0 \notin U'$. To see this, assume $v_0 \in U'$. Define $U := U' \cap V$ and $U'' := V \setminus U'$. Then

$$\text{(35.25)}\qquad \begin{aligned}x'(E'[U']) &= x'(v_0v_0) + x'(E'[U]) + \sum_{v \in U \cap R} x'(vv_0)\\ &= x'(E) + x'(E'[U]) + \sum_{v \in U \cap R} x'(vv_0) = x'(E'[U'']) + \sum_{v \in U} x'(\delta'(v))\end{aligned}$$

and

$$\lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor = \lfloor \tfrac{1}{2}(b'(U'') + 2b'(U) + c'(F')) \rfloor = \lfloor \tfrac{1}{2}(b'(U'') + c'(F')) \rfloor + \sum_{v \in U} b'(v), \tag{35.26}$$

since $b'(U') = b'(U) + b'(v_0) = b'(U) + b'(V)$. So the inequality

$$x'(E'[U']) + x'(F') \leq \lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor \tag{35.27}$$

is a sum of

$$x'(E'[U'']) + x'(F') \leq \lfloor \tfrac{1}{2}(b'(U'') + c'(F')) \rfloor \tag{35.28}$$

and of $x'(\delta'(v)) = b'(v)$ for $v \in U$. So we can assume that $v_0 \notin U'$.

Now adding an integer multiple of a valid equality to another constraint, maintains total dual integrality. So using (35.23)(ii) we can replace (35.23)(i) by:

$$\begin{array}{ll} 0 \leq x'(e) \leq c(e) & (e \in E), \\ a(v) \leq x'(\delta(v)) \leq b(v) & (v \in V), \end{array} \tag{35.29}$$

since for $v \in R$, subtracting $x'(\delta'(v)) = b(v)$ from $0 \leq x'(vv_0) \leq b(v) - a(v)$ gives $-b(v) \leq -x'(\delta(v)) \leq -a(v)$.

For $U' \subseteq V$ and $F' \subseteq \delta'(U')$, let $W := \{v \in V \mid vv_0 \in F'\}$ and $F := F' \cap E$. As

$$\begin{aligned} & x'(E'[U']) + x'(F') - \sum_{v \in W} x'(\delta'(v)) \\ & = x'(E'[U']) + x'(F') - 2x'(E[W]) - x'(\delta(W)) \\ & = x'(E'[U' \setminus W]) + x'(F') - x'(E[W]) - x'(\delta(W) \setminus \delta(U' \setminus W)) \end{aligned} \tag{35.30}$$

and

$$\lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor - b(W) = \lfloor \tfrac{1}{2}(b(U' \setminus W) - a(W) + c(F)) \rfloor, \tag{35.31}$$

we can replace (35.23)(iii) by (taking $U := U' \setminus W$):

$$\begin{aligned} & x'(E'[U]) + x'(F) - x'(E[W]) - x'(\delta(W) \setminus \delta(U)) \leq \tfrac{1}{2}\lfloor b(U) - a(W) \\ & +c(F) \rfloor \text{ for } U, W \subseteq V \text{ with } U \cap W = \emptyset \text{ and for } F \subseteq \delta(U \cup W). \end{aligned} \tag{35.32}$$

Each of the variables $x'(vv_0)$ $(v \in R)$ and $x'(v_0v_0)$ occurs exactly once in the system, in an equality constraint, with coefficient 1. So we can delete these variables maintaining total dual integrality (Theorem 5.27), and we obtain system (35.22). ∎

As special cases one can derive the total dual integrality of the systems corresponding to $d = \mathbf{0}$, $c = \boldsymbol{\infty}$ and to $d = \mathbf{0}$, $c = \mathbf{1}$ (the subgraph polytope).

35.4. Further results and notes

35.4a. Further results on subgraphs with prescribed degrees

Corollary 35.1b of Lovász [1970c] implies the following. Let $G = (V, E)$ be a graph and let $b, b' \in \mathbb{Z}_+^V$ with $b + b' > \deg_G$. Then E can be partitioned into a simple b-matching and a simple b'-matching if and only if

(35.33) $\quad |E[U, W]| \leq b(U) + b'(W)$

for each pair of disjoint subsets U and W of V.

This corresponds to the case $a < b$ in Corollary 35.1b, by taking $a := \deg_G -b'$. Then there are no components K with $a(K) = b(K)$.

The condition can be equivalently described as:

(35.34) $$\sum_{v \notin U} \max\{0, a(v) - \deg_{G-U}(v)\} \leq b(u)$$

for $U \subseteq V$.

This implies the following result of Lovász [1970c]:

(35.35) Let $G = (V, E)$ be a graph of maximum degree k and let $k_1, k_2 \geq 0$ with $k_1 + k_2 = k + 1$. Then E can be partitioned into a simple k_1-matching and a simple k_2-matching

(since $|E[U, W]| \leq (k_1 + k_2) \min\{|U|, |W|\} \leq k_1|U| + k_2|W|$). A special case is a result noted by Tutte [1978]: for all $0 \leq r \leq k$, each k-regular graph has a subgraph in which each degree belongs to $\{r, r + 1\}$.

Thomassen [1981a] gave the following short direct proof of (35.35). In fact he proved the following extension of (35.35):

(35.36) Let $G = (V, E)$ be a graph in which each vertex has degree k or $k + 1$ and let $1 \leq k' < k$. Then G has a subgraph $G' = (V, E')$ in which each vertex has degree k' or $k' + 1$.

Note that (35.35) follows from this by embedding G into a k-regular graph.

To prove (35.36), it suffices to prove the case $k' = k - 1$. Let U be the set of vertices of degree $k + 1$ in G. We can assume that deleting any edge of G results in a vertex of degree less than k. Hence no two distinct vertices in U are adjacent. There may be loops at the vertices in U; let W be the set of those vertices in U that are not incident with a loop. Since each vertex in W has degree $k+1$ and each vertex in $V \setminus U$ has degree k, by Hall's marriage theorem, G contains a matching M connecting W to $V \setminus U$. Now deleting the edges in M and deleting, for each vertex $v \in U \setminus W$, one of the loops attached at v, gives a graph G' as required.

A 'dual' consequence was noted by Gupta [1978]:

(35.37) Let $G = (V, E)$ be a graph of minimum degree δ and let $\delta_1, \delta_2 \geq 0$ with $\delta_1 + \delta_2 = \delta - 1$. Then E can be partitioned into E_1 and E_2 such that $G_i = (V, E_i)$ has minimum degree at least δ_i for $i = 1, 2$.

Gupta [1978] mentioned that the following direct derivation from Theorem 20.6 was shown to him by C. Berge:

Apply induction on δ_1, the case $\delta_1 = 0$ being trivial. If $\delta_1 > 0$, by the induction hypothesis E can be partitioned into E_1 and E_2 such that $\delta(G_1) \geq \delta_1 - 1$ and

$\delta(G_2) \geq \delta_2 + 1 = \delta - \delta_1$. (Here $G_i = (V, E_i)$ for $i = 1, 2$.) We choose this partition with $|E_2|$ minimal.

Let S be the set of vertices v with $\deg_{E_1}(v) = \delta_1 - 1$. By the minimality of $|E_2|$, S spans no edge of E_2. Let $F := \delta(S) \cap E_2$. So $\deg_F(v) = \deg_{E_2}(v) = \deg_G(v) - \deg_{E_1}(v) \geq \delta - \delta_1 + 1$ for each $v \in S$. Let $p := \delta - \delta_1 + 1 = \delta_2 + 2$. Now by Theorem 20.6, F can be partitioned into $F_1, \ldots, F_p$ such that each vertex v is covered by at least $\min\{p, \deg_F(v)\}$ of the F_i. Then replacing E_1 by $E_1 \cup F_1$ and E_2 by $E_2 \setminus F_1$ gives a partition as required. Indeed, if $\deg_F(v) \geq p$, then $\deg_{F_i}(v) \geq 1$ for each i, implying

(35.38) $$\deg_{E_1 \cup F_1}(v) = \deg_{E_1}(v) + \deg_{F_1}(v) \geq (\delta_1 - 1) + 1 = \delta_1$$

and

(35.39) $$\deg_{E_2 \setminus F_1}(v) \geq \sum_{i=2}^{p} \deg_{F_i}(v) \geq p - 1 = \delta_2 + 1.$$

If $\deg_F(v) < p$, then $v \notin S$ and $\deg_{F_1}(v) \leq 1$, and hence

(35.40) $$\deg_{E_1 \cup F_1}(v) \geq \deg_{E_1}(v) \geq \delta_1$$

and

(35.41) $$\deg_{E_2 \setminus F_1}(v) \geq \deg_{E_2}(v) - 1 \geq (\delta_2 + 1) - 1 = \delta_2.$$

This proves (35.37).

Las Vergnas [1978] showed that if $a \leq \mathbf{1} \leq b$ holds, a simpler condition can be formulated in Corollary 35.1b:

(35.42) for each $U \subseteq V$, the number of odd components K of $G - U$ with $|K| = 1$ and $a(K) = 1$, or with $|K| \geq 3$ and $a(K) = b(K)$ is at most $b(U)$.

Anstee [1985] gave a proof of Lovász's theorem, with an $O(n^3)$-time algorithm to find the subgraph. Heinrich, Hell, Kirkpatrick, and Liu [1990] gave a simplified proof of Lovász's theorem for $a < b$, implying an $O(\sqrt{a(V)}\,m)$-time algorithm.

Lovász [1970c] also characterized the minimum deviation that subsets can have from prescribed lower and upper bounds on the degrees. In fact, he showed the following (where $\alpha_+ := \max\{0, \alpha\}$ for any $\alpha \in \mathbb{R}$): Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then the minimum of

(35.43) $$\sum_{v \in V} ((a(v) - \deg_F(v))_+ + (\deg_F(v) - b(v))_+)$$

over $F \subseteq E$ is equal to the maximum value of

(35.44) $a(W) - b(U) - 2|E[W]| - |E[T, W]| +$number of components K of $G[T]$ with $a(K) = b(K)$ and with $a(K) + |E[K, W]|$ odd,

taken over all partitions T, U, W of V.

Let $B : V \to \mathcal{P}(\mathbb{Z}_+)$. The B-matching problem asks for a subgraph H of G such that $\deg_H(v) \in B(v)$ for each $v \in V$. In general, this is NP-complete, even when $B(v) \in \{\{1\}, \{0, 3\}\}$ for each $v \in V$ (Lovász [1972f]).

If $|\mathbb{Z}_+ \setminus B(v)| = 1$ for each vertex v, Lovász [1973a] gave a characterization. Lovász [1972f] investigated the case where $\mathbb{Z}_+ \setminus B(v)$ contains no two consecutive integers, for which Cornuéjols [1988] gave a polynomial-time algorithm, and Sebő [1993b] a good characterization.

For algorithms to find subgraphs of minimum deviation see Hell and Kirkpatrick [1993]. Other work on subgraphs with prescribed degrees includes Berge and Las Vergnas [1978], Shiloach [1981], Kano and Saito [1983], Akiyama and Kano [1985a], Kano [1985,1986], Anstee [1990], Cai [1991], and Li and Cai [1998]. A survey is given by Akiyama and Kano [1985b].

35.4b. Odd walks

Let $G = (V, E)$ be an undirected graph, let $s, t \in V$, and let $l : E \to \mathbb{Q}$. Call a walk *odd* if it has an odd number of edges. Then a shortest odd $s-t$ walk without repeated edges can be found as follows. For each edge e of G, set $d(e) := 0$ and $c(e) := 1$, and add an edge $\tilde{e}$ parallel to e, of length $l(\tilde{e}) := -l(e)$, and define $d(\tilde{e}) := -1$, $c(\tilde{e}) := 0$. Let M be the $V \times E'$ incidence matrix of the extended graph $G' = (V, E')$. Define $b : V \to \mathbb{Z}$ by $b(s) := b(t) := 1$ and $b(v) := 0$ for each $v \in V \setminus \{s, t\}$. Then a shortest odd $s-t$ walk without repeated edges can be found by finding an $x \in \mathbb{Z}^{E'}$ satisfying $d \leq x \leq c$ and $Mx = b$ and minimizing $l^{\mathsf{T}} x$.

So by Theorem 35.2, this can be solved in strongly polynomial time. Better running times were given by Goldberg and Karzanov [1994,1996]: $O(m)$ for finding such an odd $s-t$ walk, $O(nm \log n)$ and $O(nm\sqrt{\log L})$ for finding a shortest such odd $s-t$ walk, strengthened to $O(m \log n)$ and $O(m\sqrt{\log L})$ for nonnegative lengths. (L is the maximum absolute value of the lengths, assuming they are integer.)

Chapter 36

Bidirected graphs

In the previous chapter we considered integer solutions of $d \le x \le c$, $a \le Mx \le b$ where M is the incidence matrix of an undirected graph. Earlier, in Chapter 12, we considered the same problem if M is the incidence matrix of a directed graph.
Edmonds and Johnson [1970] showed that M can more generally be the incidence matrix of a 'bidirected' graph — a structure that comprises both undirected and directed graphs. That is, M has entries 0, ± 1, and ± 2, such that the sum of the absolute values of the entries in any column is equal to 2. The results are obtained by a simple reduction to the undirected case, although the elaboration takes some effort.
The results could be formulated just in terms of matrices, but the graph-theoretic interpretation is helpful in formulating, visualizing, and proving the results.

36.1. Bidirected graphs

A *bidirected graph* is a triple $G = (V, E, \sigma)$, where (V, E) is an undirected graph and where σ assigns to each $e \in E$ and $v \in e$ a 'sign' $\sigma_{e,v} \in \{+1, -1\}$.

If e is a loop, that is, e is a family $\{v, v\}$, we may assign different signs to the two occurrences of v. However, in the problems discussed in this chapter, loops where the signs are different are irrelevant. So we assume that the signs in a loop are the same, either both $+1$, or both -1.

Clearly, undirected graphs and directed graphs can be considered as special cases of bidirected graphs. Graph terminology for the graph (V, E) extends in an obvious way to the bidirected graph (V, E, σ).

Let $G = (V, E, \sigma)$ be a bidirected graph. The edges e for which $\sigma_{e,v} = 1$ for each $v \in e$ are called the *positive edges*, those with $\sigma_{e,v} = -1$ for each $v \in e$ the *negative edges*, and the remaining edges are called the *directed edges*. The $V \times E$ *incidence matrix* of G is the $V \times E$ matrix M defined by:

(36.1) $$M_{v,e} := \sigma_{e,v} \text{ if } e \text{ is not a loop,}$$
$$M_{v,e} := 2\sigma_{e,v} \text{ if } e \text{ is a loop,}$$

setting $\sigma_{e,v} := 0$ if $v \notin e$. It follows that an integer matrix M is the $V \times E$ incidence matrix of a bidirected graph if and only if the sum of the absolute values of the entries in any column of M is equal to 2.

For vectors $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$, we consider integer solutions $x \in \mathbb{R}^E$ of

(36.2) (i) $d \leq x \leq c$,
(ii) $a \leq Mx \leq b$.

The related existence and optimization problems can be reduced as follows to the case where G is just an undirected graph. First, we can assume that G has no negative edges, since multiplying the corresponding column of M by -1 gives an equivalent problem. Next, any directed edge $f = su$, with $\sigma_{f,s} = -1$ and $\sigma_{f,u} = +1$, can be handled as follows.

(36.3) Extend G by a new vertex t and replace edge e by two new positive edges st and tu. This makes the bidirected graph $G' = (V', E')$. Define $a', b' \in \mathbb{Z}^{V'}$ by $a'(v) := a(v)$ and $b'(v) := b(v)$ for $v \in V$ and $a'(t) := b'(t) := 0$. Define $d', c' \in \mathbb{Z}^{E'}$ by $d'(e) := d(e)$ and $c'(e) := c(e)$ for $e \in E \setminus \{f\}$, and $d'(st) := -\infty$, $c'(st) := \infty$, $d'(tu) := d(f)$, and $c'(tu) := c(f)$. Let M' be the $V' \times E'$ incidence matrix of G'.

Then there is a one-to-one relation between (integer) solutions of (36.2) and those for the system corresponding to G', M', a', b', c', d': just define $x(tu) := x_f$ and $x(st) := -x_f$.

Algorithmically, this gives a direct reduction to the undirected case:

Theorem 36.1. *For $w \in \mathbb{Q}^E$, an integer vector x maximizing $w^{\mathsf{T}}x$ over* (36.2) *can be found in strongly polynomial time.*

Proof. By multiplying columns of M by -1, we can assume that G has no negative edges. Next, apply (36.3) to each directed edge. This reduces the problem to one on a bidirected graph with all edges positive, that is, on an undirected graph. Hence, the theorem follows from Theorem 35.2. ∎

We next consider characterizations. Let $G = (V, E, \sigma)$ be a bidirected graph. For any $T \subseteq V$, $G[T]$ denotes the bidirected subgraph of G induced by T (that is, $G[T] = (T, E[T], \sigma')$, where σ' is the restriction of σ to pairs e, v with $e \in E[T]$). We set for $U \subseteq V$:

(36.4) $\delta(U) := \delta_E(U)$.

For disjoint $X, Y \subseteq V$, we denote:

(36.5) $E[X, Y^+] := \{e \in \delta(X) \mid \exists v \in Y : \sigma_{e,v} = +1\}$,
$E[X, Y^-] := \{e \in \delta(X) \mid \exists v \in Y : \sigma_{e,v} = -1\}$.

For any vector z, let z_+ be the vector obtained from z by setting each negative entry to 0. Similarly, let z_- be the vector obtained from z by setting each positive entry to 0. So $z = z_+ + z_-$.

In the following theorem the condition that $d < c$ is not really a restriction: if $d_e = c_e$ we know that $x_e = d_e$ and hence we can dispose of e by contracting it appropriately. But if we delete the condition $d < c$, the formulation of the theorem would be more complicated.

Theorem 36.2. *Let $a \leq b$ and $d < c$. Then there exists an integer vector x satisfying* (36.2) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and*

(36.6) $$b(K)+c(E[K,W^+])+c(E[K,U^-])+d(E[K,U^+])+d(E[K,W^-])$$

odd is at most

(36.7) $$y_+^{\mathsf{T}} b + y_-^{\mathsf{T}} a - (y^{\mathsf{T}} M)_- c - (y^{\mathsf{T}} M)_+ d,$$

where $y := \chi^U - \chi^W$.

Proof. The validity of the theorem is invariant under multiplying a row v of M by -1 and replacing $b(v)$ by $-a(v)$ and $a(v)$ by $-b(v)$ (then if $v \in U$ we move v to W, and if $v \in W$ we move v to U). Similarly, the validity is invariant under multiplying a column e by -1 and replacing $c(e)$ by $-d(e)$ and $d(e)$ by $-c(e)$.

Hence, to see necessity, we can assume that $W = \emptyset$ and $E[T, U^-] = \emptyset$. So $y_+ = y$ and $y_- = \mathbf{0}$ and $E[T, U^+] = \delta(T)$. Then

(36.8) $$\begin{aligned} (x-d)(\delta(T)) &= (x-d)(E[T,U^+]) \leq (y^{\mathsf{T}} M)_+(x-d) \\ &\leq (y^{\mathsf{T}} M)_+(x-d) - (y^{\mathsf{T}} M)_-(c-x) \\ &= y^{\mathsf{T}} Mx - (y^{\mathsf{T}} M)_+ d - (y^{\mathsf{T}} M)_- c \\ &= y_+^{\mathsf{T}} Mx + y_-^{\mathsf{T}} Mx - (y^{\mathsf{T}} M)_+ d - (y^{\mathsf{T}} M)_- c \\ &\leq y_+^{\mathsf{T}} b + y_-^{\mathsf{T}} a - (y^{\mathsf{T}} M)_+ d - (y^{\mathsf{T}} M)_- c. \end{aligned}$$

On the other hand, for each component K of $G[T]$ one has $(x-d)(\delta(K)) \geq 0$. Moreover, if $b(K) = a(K)$ and (36.6) is odd, then $(x-d)(\delta(K))$ is odd, since

(36.9) $$(x-d)(\delta(K)) \equiv (x-d)(\delta(K)) + 2x(E[K]) \equiv b(K) + d(\delta(K)) \pmod 2$$

So $(x-d)(\delta(T))$ is not less than the number of components K of $G[T]$ with $a(K) = b(K)$ and (36.6) odd, showing necessity of the condition.

To show sufficiency, we can assume that G has no negative edges, since we can multiply columns of M by -1. We show sufficiency by induction on the number of directed edges. If this number is 0, the theorem reduces to Theorem 35.1. So we can assume that there is an edge $f = su$ with $\sigma_{f,s} = -1$ and $\sigma_{f,u} = +1$. Then we apply construction (36.3), to obtain $G' = (V', E'), M', a', b', d', c'$.

Now there exists an integer vector x satisfying $d \leq x \leq c$ and $a \leq Mx \leq b$ if there exists an integer vector x' satisfying $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. So we can assume that no such x' exists. By induction (as G' has fewer directed edges than G), we know that V' can be partitioned into T', U', and W' such that the number of components K' of $G'[T']$ with $b'(K') = a'(K')$ and

$$(36.10) \qquad b'(K') + c'(E'[K', W'^+]) + c'(E'[K', U'^-]) + d'(E'[K', U'^+]) + d'(E'[K', U'^-])$$

odd, is more than

$$(36.11) \qquad y'^{\mathsf{T}}_+ b' + y'^{\mathsf{T}}_- a' - (y'^{\mathsf{T}} M')_- c' - (y'^{\mathsf{T}} M')_+ d',$$

where $y' := \chi^{U'} - \chi^{W'}$.

Since $c'(st) = \infty$ we know that $(y'^{\mathsf{T}} M')_{st} \geq 0$, that is, $y'_s + y'_t \geq 0$. Similarly, since $d'(st) = -\infty$ we know that $(y'^{\mathsf{T}} M')_{st} \leq 0$, that is, $y'_s + y'_t \leq 0$. So $y'_s = -y'_t$, and hence either $s \in U'$, $t \in W'$, or $s \in W'$, $t \in U'$, or $s, t \in T'$.

Let $U := U' \cap V$, $W := W' \cap V$, and $T := T' \cap V$. Then for any component K' of $G'[T']$ with $b'(K') = a'(K')$ and (36.10) odd, $K := K' \cap V$ is a component of $G[T]$ with $b(K) = a(K)$ and (36.6) odd. Moreover, (36.11) is equal to (36.7). Hence we have a contradiction with the condition given in the theorem. ∎

36.2. Convex hull

Also the convex hull of the integer solutions of (36.2) can be characterized (where we do not assume $d < c$):[20]

Theorem 36.3. *The convex hull of the integer solutions of* (36.2) *is determined by*

$$(36.12) \qquad \begin{array}{rl} \text{(i)} & d \leq x \leq c, \\ \text{(ii)} & a \leq Mx \leq b, \\ \text{(iii)} & \frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \\ & \quad \leq \lfloor \frac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor \\ & \qquad\qquad \text{for } U, W \subseteq V \text{ with } U \cap W = \emptyset, \\ & \qquad\qquad \text{and for partitions } F, H \text{ of } \delta(U \cup W) \\ & \qquad\qquad \text{with } b(U) - a(W) + c(F) - d(H) \text{ odd.} \end{array}$$

Proof. Necessity of (36.12) follows from the facts that $\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)$ is an integer vector and that for each vector x satisfying (36.2) one has $\chi^U Mx \leq \chi^U b = b(U)$, $\chi^W Mx \geq \chi^W a = a(W)$, $\chi^F x \leq \chi^F c = c(F)$, and $\chi^H x \geq \chi^H d = d(H)$.

Again, to show sufficiency, we can assume that G has no negative edges, and we apply induction on the number of directed edges. If this number is 0, the theorem reduces to Theorem 35.3.

So we can assume that there is a directed edge $f = su$. Again, construct $G' = (V', E')$, M', a', b', d', c', as in (36.3). By induction we know that the theorem holds for the new structure.

[20] In order to reduce notation, in this chapter we take incidence vectors χ^U, χ^W, χ^F, and χ^H as *row* vectors.

Let $x \in \mathbb{R}^E$ satisfy (36.12) for G, a, b, c, d. Define $x' \in \mathbb{R}^{E'}$ by $x'(e) := x(e)$ for each $e \in E \setminus \{f\}$, and $x'(st) := -x(f)$ and $x'(tu) := x(f)$. Now it suffices to show that x' satisfies the inequalities for G', a', b', c', d' (since of x' is a convex combination of integer solutions, also x is).

So let U' and W' be disjoint subsets of V' and let F' and H' partition $\delta'(U' \cup W')$, with $b'(U') - a'(W') + c'(F') - d'(H')$ odd. Since $c'(st) = \infty$ and $d'(st) = -\infty$, we know that $st \notin \delta'(U' \cup W')$.

Let $U := U' \cap V$ and $W := W' \cap V$. Moreover, let F and H arise from F' and H' by replacing any occurrence of tu by f. Then

(36.13) $$\tfrac{1}{2}((\chi^{U'} - \chi^{W'})M' + \chi^{F'} - \chi^{H'})x' = \tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x$$

since $x'(F') = x(F)$ and $x'(H') = x(H)$, and moreover, $\chi^{U'}M'x' = \chi^U Mx$ and $\chi^{W'}M'x' = \chi^W Mx$ (as $\chi^t M'x' = x'(st) + x'(tu) = 0$).

Also we have

(36.14) $$\begin{aligned} &\lfloor \tfrac{1}{2}(b'(U') - a'(W') + c'(F') - d'(H')) \rfloor \\ &= \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor, \end{aligned}$$

as $a'(t) = b'(t) = 0$. Hence (36.12) gives the required inequality for U', W', F', H'. ∎

The special case $a = b$, $d = \mathbf{0}$ was announced by Edmonds and Johnson [1970] and elaborated by Aráoz, Cunningham, Edmonds, and Green-Krótki [1983]. It amounts to, for $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$:

(36.15) $$\mathbf{0} \leq x \leq c, Mx = b.$$

Then:

Corollary 36.3a. *The convex hull of the integer solutions of* (36.15) *is determined by* (36.15) *together with the constraints*

(36.16) $$x(\delta(U) \setminus F) - x(F) \geq 1 - c(F)$$

where $U \subseteq V$ and $F \subseteq \delta(U)$ with $b(U) + c(F)$ odd.

Proof. Directly from Theorem 36.3, by replacing Mx by b in (36.12)(iii). ∎

For undirected graphs we obtain a characterization of the capacitated perfect b-matching polytope as special case — cf. Corollary 32.2a.

36.3. Total dual integrality

System (36.12) generally is not totally dual integral (cf. the example in Section 30.5). However, if we delete the parity condition in (36.12)(iii):

(36.17)
(i) $d \leq x \leq c$,
(ii) $a \leq Mx \leq b$,
(iii) $\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x$
$\leq \lfloor \frac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor$
for $U, W \subseteq V$ with $U \cap W = \emptyset$ and for partition F, H of $\delta(U \cup W)$,

we obtain a totally dual integral system:

Theorem 36.4. *System* (36.17) *is totally dual integral.*

Proof. Again we can assume that there are no negative edges, and apply induction on the number of directed edges of G. If there is no directed edge, the theorem reduces to Theorem 35.4. If there is a directed edge $f = su$, we again construct $G' = (V', E')$, M', a', b', d', c' as in (36.3).

Let Σ and Σ' be the systems for G, a, b, c, d, and for G', a', b', c', d', respectively. By induction we know that Σ' is totally dual integral. Now the constraint $x'(st) + x'(tu) = 0$ belongs to Σ'. This implies the total dual integrality of the system Σ'' obtained from Σ' by adding an integer multiple of $x'(st)+x'(tu) = 0$ to any other constraint of Σ' so as to make the coefficient of the variable $x'(st)$ equal to 0.

Now deleting the constraints $x'(st) + x'(tu) = 1$ and the variable $x'(st)$ from Σ'', identifying $x'(e) = x(e)$ for all $e \in E \setminus \{f\}$, and identifying $x'(tu) = x(f)$, gives again a totally dual integral system. We show that it is system Σ.

Indeed, $a' \leq M'x' \leq b'$ becomes $a \leq Mx \leq b$. Similarly, for each $e \in E \setminus \{f\}$, $d'(e) \leq x'(e) \leq c'(e)$ becomes $d(e) \leq x_e \leq c(e)$, and $d'(tu) \leq x'(tu) \leq c'(tu)$ becomes $d(f) \leq x(f) \leq c(f)$, while $d'(st) \leq x'(st) \leq c'(st)$ is void (as the bounds are $-\infty$ and $+\infty$).

Consider next the following inequality of Σ':

(36.18) $$\frac{1}{2}((\chi^{U'} - \chi^{W'})M + \chi^{F'} - \chi^{H'})x' \leq \lfloor \frac{1}{2}(b'(U') - a'(W') + c'(F') - d'(H')) \rfloor,$$

where U' and W' are disjoint subsets of V' and where F' and H' partition $\delta'(U' \cup W')$.

Since $c'(st) = \infty$, $d'(s,t) = -\infty$, we know that $st \notin \delta'(U' \cup W')$. Consider the coefficient of $x'(st)$ in (36.18). If this coefficient is 0, (36.18) reduces to (36.17)(iii). If this coefficient is positive, then $s, t \in U'$. Set $U'' := U' \setminus \{t\}$ and $W'' := W' \cup \{t\}$. Then in Σ'', (36.18) becomes (by subtracting $x'(st) + x'(tu) = 0$):

(36.19) $$\frac{1}{2}((\chi^{U''} - \chi^{W''})M + \chi^{F'} - \chi^{H'})x' \leq \lfloor \frac{1}{2}(b'(U'') - a'(W'') + c'(F') - d'(H')) \rfloor$$

(since $b'(t) = a'(t) = 0$). In (36.19), the coefficient of $x'(st)$ is 0, and hence (36.19) reduces to (36.17)(iii).

We proceed similarly if the coefficient of $x'(st)$ in (36.18) is negative. ∎

A consequence is the total dual half-integrality of the original system:

Corollary 36.4a. *System* (36.12) *is totally dual half-integral.*

Proof. This follows from the fact that each inequality in (36.17) is a half-integer nonnegative combination of inequalities in (36.12). ∎

A special case is the total dual half-integrality of

$$(36.20)\qquad \begin{array}{rll} \text{(i)} & x \geq \mathbf{0}, & \\ \text{(ii)} & Mx = b, & \\ \text{(iii)} & x(\delta(U)) \geq 1 & \text{for each } U \subseteq V \text{ with } b(U) \text{ odd} \end{array}$$

(Edmonds and Johnson [1970]):

Corollary 36.4b. *System* (36.20) *is totally dual half-integral.*

Proof. This is a special case of Corollary 36.4a. ∎

From this one can derive (Barahona and Cunningham [1989]):

Corollary 36.4c. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}}x$ subject to* (36.20) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} M_v^{\mathsf{T}}$, where M_v denotes row v of M. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 36.4b there is an integer optimum dual solution y'_v $(v \in V)$, z_U $(U \subseteq V,\ b(U)$ odd) for the problem of minimizing ${w'}^{\mathsf{T}}x$ subject to (36.20). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

36.4. Including parity conditions

We are not yet at the end of our self-refining trip. As was observed by Edmonds and Johnson [1973], the results can be generalized even further by including parity constraints. This can be reduced to the previous case by adding loops at the vertices at which there is a parity constraint.

Let $G = (V, E, \sigma)$ be a bidirected graph and let M be the $V \times E$ incidence matrix of G. (For definitions and terminology, see Section 36.1.) Let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$ and let S^{odd} and S^{even} be two disjoint subsets of V. We consider integer solutions x of:

(36.21)
(i) $d \leq x \leq c$,
(ii) $a \leq Mx \leq b$,
(iii) $(Mx)_v$ is odd if $v \in S^{\text{odd}}$,
(iv) $(Mx)_v$ is even if $v \in S^{\text{even}}$.

The problem of finding a maximum-weight integer vector x satisfying (36.21) can be easily reduced to the special case without parity constraints, discussed in the previous chapter:

Theorem 36.5. *For any $w \in \mathbb{Q}^E$, an integer vector x maximizing $w^\mathsf{T} x$ over* (36.21) *can be found in strongly polynomial time.*

Proof. The condition $(Mx)_v$ is odd, can be replaced by $1 \leq (Mx)_v + 2z_v \leq 1$, where z_v is a new integer variable (bounded by $-\infty$ and ∞). Similarly, for the even case. This gives a reduction to the problem of Theorem 36.1, which implies the present theorem. ∎

We next characterize the existence of an integer vector x satisfying (36.21). To this end we make the following assumptions, which can easily be satisfied:

(36.22)
(i) $a(v)$ and $b(v)$ are odd (if finite) for each $v \in S^{\text{odd}}$,
(ii) $a(v)$ and $b(v)$ are even (if finite) for each $v \in S^{\text{even}}$,
(iii) if $a(v) = b(v)$, then $v \in S^{\text{odd}} \cup S^{\text{even}}$.

Define $S := S^{\text{odd}} \cup S^{\text{even}}$. Moreover, for any vector z, again let z_+ arise from z by replacing any negative component by 0, and let z_- arise from z by replacing any positive component by 0. So $z = z_+ + z_-$.

Theorem 36.6. *Assume* (36.22) *and that $d < c$. Then there exists an integer vector $x \in \mathbb{Z}^E$ satisfying* (36.21) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ contained in $S^{\text{odd}} \cup S^{\text{even}}$ and with*

(36.23) $$|K \cap S^{\text{odd}}| + c(E[K, W^+]) + c(E[K, U^-]) + d(E[K, U^+]) + d(E[K, W^-])$$

odd is at most

(36.24) $$y_+^\mathsf{T} b + y_-^\mathsf{T} a - (y^\mathsf{T} M)_- c - (y^\mathsf{T} M)_+ d,$$

where $y := \chi^U - \chi^W$.

Proof. Define $L := \{v \in S \mid a(v) < b(v)\}$, $L^{\text{odd}} := L \cap S^{\text{odd}}$, and $L^{\text{even}} := L \cap S^{\text{even}}$.

Extend the bidirected graph G by a loop l at any vertex $v \in L$, where l has two positive ends at v. This makes the bidirected graph $G' = (V, E', \sigma')$, with $V \times E'$ incidence matrix M'. Define $a'(v) := a(v)$ and $b'(v) := b(v)$ for each $v \in V \setminus L$. Moreover, $a'(v) := b'(v) := 1$ for $v \in L^{\text{odd}}$ and $a'(v) := b'(v) := 0$

for $v \in L^{\mathrm{even}}$. Define $d'(e) := d(e)$ and $c'(e) := c(e)$ for each $e \in E$. For each loop l at $v \in L$, define $d'(l) := \frac{1}{2}(b'(v) - b(v))$ and $c'(l) := \frac{1}{2}(a'(v) - a(v))$.

Now there exist an integer vector x satisfying (36.21) if and only if there exists an integer vector $x' \in \mathbb{Z}^{E'}$ satisfying $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. So we should show that the conditions given in the present theorem imply those given in Theorem 36.2 (for the modified structure). (Since in Theorem 36.2 the condition $d < c$ is required, we had to exclude loops at vertices in $S \setminus L$.)

To this end, let T, U, W partition V. Then any component K of $G'[T]$ with $b'(K) = a'(K)$ and

(36.25) $$b'(K) + c'(E'[K, W^+]) + c'(E'[K, U^-]) + d'(E'[K, U^+]) + d'(E[K, W^-])$$

odd, is a component of $G[T]$ contained in S, with $|K \cap S^{\mathrm{odd}}| + c(E[K, W^+]) + c(E[K, U^-]) + d(E[K, U^+]) + d(E[K, W^-])$ odd (note that $a'(v) = b'(v) \iff v \in S$, and that $b'(K) \equiv |K \cap S^{\mathrm{odd}}| \bmod 2$). Moreover, for $y := \chi^U - \chi^W$ one has

(36.26) $$\begin{aligned} &y_+^{\mathsf{T}} b' + y_-^{\mathsf{T}} a' - (y^{\mathsf{T}} M')_- c' - (y^{\mathsf{T}} M')_+ d' \\ &= y_+^{\mathsf{T}} b + y_-^{\mathsf{T}} a - (y^{\mathsf{T}} M)_- c - (y^{\mathsf{T}} M)_+ d, \end{aligned}$$

since

(36.27) $$\begin{aligned} &y_+^{\mathsf{T}} b' = b'(U) = b(U \setminus L) + |U \cap L^{\mathrm{odd}}|, \\ &y_-^{\mathsf{T}} a' = -a'(W) = -a(W \setminus L) - |W \cap L^{\mathrm{odd}}|, \\ &(y^{\mathsf{T}} M')_- c' = (y^{\mathsf{T}} M)_- c - 2(\tfrac{1}{2}(a'(W \cap L) - a(W \cap L))) \\ &= (y^{\mathsf{T}} M)_- c - |W \cap L^{\mathrm{odd}}| + a(W \cap L), \\ &(y^{\mathsf{T}} M')_+ d' = (y^{\mathsf{T}} M)_+ d + 2(\tfrac{1}{2}(b'(U \cap L) - b(U \cap L))) \\ &= (y^{\mathsf{T}} M)_+ d + |U \cap L^{\mathrm{odd}}| - b(U \cap L). \end{aligned}$$ ∎

A special case is the following result on orientations by Frank, Tardos, and Sebő [1984].

Corollary 36.6a. *Let $G = (V, E)$ be an undirected graph and let $l, u \in \mathbb{Z}_+^V$ be such that $l(v) \equiv u(v)$ (mod 2) for each $v \in V$. Then G has an orientation $D = (V, A)$ such that*

(36.28) $$l(v) \leq \deg_D^{\mathrm{out}}(v) \leq u(v) \textit{ and } \deg_D^{\mathrm{out}}(v) \equiv u(v) \pmod 2$$

for each $v \in V$ if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $u(K) + |E[K]| + |E[K, U]|$ odd is at most

(36.29) $$u(U) - l(W) - |E[U]| + |E[W]| + |\delta(W)|.$$

Proof. Let $D' = (V, A')$ be an arbitrary orientation of G. Let $\delta^{\mathrm{out}}(U) := \delta_{D'}^{\mathrm{out}}(U)$ and $\delta^{\mathrm{in}}(U) := \delta_{D'}^{\mathrm{in}}(U)$ for any $U \subseteq V$.

Then G has an orientation as required in the theorem if and only if there exists a vector $x \in \mathbb{Z}^{A'}$ with $\mathbf{0} \leq x \leq \mathbf{1}$ and

(36.30) $\qquad l(v) \leq x(\delta^{\text{in}}(v)) + |\delta^{\text{out}}(v)| - x(\delta^{\text{out}}(v)) \leq u(v)$

and

(36.31) $\qquad x(\delta^{\text{in}}(v)) + |\delta^{\text{out}}(v)| - x(\delta^{\text{out}}(v)) \equiv u(v) \pmod 2$

for each $v \in V$. (This can be seen by reversing the orientation if and only if $x_a = 1$.)

Define for each $v \in V$,

(36.32) $\qquad a(v) := l(v) - |\delta^{\text{out}}(v)|$ and $b(v) := u(v) - |\delta^{\text{out}}(v)|$.

Moreover, let $d, c \in \mathbb{Z}^{A'}$ with $d = \mathbf{0}$ and $c = \mathbf{1}$. Let M be the $V \times A'$ incidence matrix of D' (such that $M_{v,a} = -1$ if a leaves v and $M_{v,a} = +1$ if a enters v). Let S^{odd} and S^{even} be the sets of vertices v with $b(v)$ odd and even, respectively.

Then the existence of an orientation as required is equivalent the existence of an integer vector x satisfying (36.21). Hence, by Theorem 36.6, it is equivalent to the condition that for each partition T, U, W of V the number of components K of $G[T]$ with (for the bidirected graph $G = (V, E, \sigma)$ obtained from M):

(36.33) $\qquad b(K) + |E[K, W^+]| + |E[K, U^-]|$

odd is at most

(36.34) $$u(U) - \sum_{v \in U} |\delta^{\text{out}}(v)| - l(W) + \sum_{v \in W} |\delta^{\text{out}}(v)| + |\delta^{\text{out}}(U)| + |\delta^{\text{in}}(W)|.$$

Now (36.33) is equal to

(36.35) $$\begin{aligned} & u(K) - \sum_{v \in K} |\delta^{\text{out}}(v)| + |E[K, W^+]| + |E[K, U^-]| \\ & = u(K) - |E[K]| + |\delta^{\text{out}}(K)| + |E[K, W^+]| + |E[K, U^-]| \\ & \equiv u(K) - |E[K]| + |\delta^{\text{out}}(K)| + |E[K, W^+]| + 2|E[K, U^+]| \\ & + |E[K, U^-]| \equiv u(K) + |E[K]| + |E[K, U]| \pmod 2, \end{aligned}$$

since $|\delta^{\text{out}}(K)| = |E[K, U^+]| + |E[K, W^+]|$ and $|E[K, U]| = |E[K, U^+]| + |E[K, U^-]|$. Moreover, (36.34) is equal to (36.29), proving the corollary. ∎

One can similarly derive the following two further orientation results of Frank, Tardos, and Sebő [1984].

Corollary 36.6b. *Let $G = (V, E)$ be an undirected graph and let $u \in \mathbb{Z}_+^V$. Then G has an orientation $D = (V, A)$ such that*

(36.36) $\qquad \deg_D^{\text{out}}(v) \leq u(v)$ *and* $\deg_D^{\text{out}}(v) \equiv u(v)$ *(mod 2)*

for each $v \in V$ if and only if for each $U \subseteq V$ the number of components K of $G - U$ with $u(K) + |E[K]| + |\delta(K)|$ odd is at most $u(U) - |E[U]|$.

Proof. Similar to the proof of Corollary 36.6a. ∎

Corollary 36.6c. *Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Z}_+^V$. Then G has an orientation $D = (V, A)$ such that*

(36.37) $\quad \deg_D^{\text{out}}(v) \geq l(v)$ *and* $\deg_D^{\text{out}}(v) \equiv l(v)$ *(mod 2)*

for each $v \in V$ if and only if for each $U \subseteq V$ the number of components K of $G - U$ with $l(K) + |E[K]| + |\delta(K)|$ odd is at most $|E[U]| + |\delta(U)| - l(U)$.

Proof. Similar to the proof of Corollary 36.6a. ∎

36.5. Convex hull

The convex hull of the integer solutions of (36.21) is characterized by:

Theorem 36.7. *Assuming* (36.22), *the convex hull of the integer solutions of* (36.21) *is determined by* (36.21)(i) *and* (ii), *together with the constraints*

(36.38) $\quad \frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \leq \lfloor \frac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor,$

where U and W are disjoint subsets of $V \setminus S$ and where F and H partition $\delta(U \cup W \cup R)$ for some $R \subseteq S$ with $|R \cap S^{\text{odd}}| + b(U) - a(W) + c(F) - d(H)$ odd.

Proof. To see necessity of (36.38), let x be an integer vector satisfying (36.21), and choose U, W, R, F and H as described in the theorem. As x satisfies $d \leq x \leq c$ and $a \leq Mx \leq b$ one directly has $((\chi^U - \chi^W)M + \chi^F - \chi^H)x \leq b(U) - a(W) + c(F) - d(H)$. So it suffices to show that strict inequality holds. Now $(\chi^U + \chi^W + \chi^R)M + \chi^F + \chi^H$ is an even vector. So (using (36.21)(iii) and (iv))

(36.39) $$\begin{aligned}&((\chi^U - \chi^W)M + \chi^F - \chi^H)x \equiv \chi^R Mx \equiv |R \cap S^{\text{odd}}| \\ &\not\equiv b(U) - a(W) + c(F) - d(H) \pmod 2\end{aligned}$$

This shows strict inequality.

We next show that (36.38) determines the convex hull, by reduction to Theorem 36.3. Let L, L^{odd}, L^{even}, $G' = (V, E')$, M', a', b', d', c' be as in the proof of Theorem 36.6. Let $x \in \mathbb{R}^E$ satisfy (36.21)(i) and (ii) and all constraints (36.38). Define $x \in \mathbb{R}^{E'}$ by $x'(e) := x(e)$ for each $e \in E$, and $x'(l) := a'(v) - x(\delta(v))$ for the loop l at any $v \in L$. Then $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. It suffices to show that x' is a convex combination of integer solutions of this system. By Theorem 36.3, it suffices to check condition (36.12)(iii) for G', x'.

Let U' and W' be disjoint subsets of V and let F and H partition $\delta'(U' \cup W')$, with $b'(U') - a'(W') + c'(F) - d'(H)$ odd. Define $U := U' \setminus S$, $W := W' \setminus S$, and $R := (U' \cup W') \cap S$. Then $|R \cap S^{\text{odd}}| + b(U) - a(W) + c(F) - d(H)$ is odd, since $|R \cap S^{\text{odd}}| \equiv b'(U' \cap S) - a'(W' \cap S)$ (mod 2). Moreover,

$$\begin{aligned}(36.40)\qquad &\chi^{U'}M'x' = \chi^U Mx + b(U' \cap (S \setminus L)) + |U' \cap L^{\text{odd}}|,\\ &\chi^{W'}M'x' = \chi^W Mx + a(W' \cap (S \setminus L)) + |W' \cap L^{\text{odd}}|,\\ &b'(U') = b(U) + b(U' \cap (S \setminus L)) + |U' \cap L^{\text{odd}}|, \text{ and}\\ &a'(W') = a(W) + a(W' \cap (S \setminus L)) + |W' \cap L^{\text{odd}}|.\end{aligned}$$

Hence (36.38) for x implies (36.12)(iii) for x'. ∎

36.5a. Convex hull of vertex-disjoint circuits

Green-Krótki [1980] and Aráoz, Cunningham, Edmonds, and Green-Krótki [1983] showed that the previous theorem implies a characterization of the convex hull of disjoint sets of circuits:

Corollary 36.7a. *Let $G = (V, E)$ be a graph. Then the convex hull of the vectors χ^F where F is the edge set of the union of a number of vertex-disjoint circuits is given by:*

$$\begin{array}{llll}(36.41) & \text{(i)} & 0 \leq x_e \leq 1 & (e \in E),\\ & \text{(ii)} & x(\delta(v)) \leq 2 & (v \in V),\\ & \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V,\ F \subseteq \delta(U), |F| \text{ odd}).\end{array}$$

Proof. This follows directly from Theorem 36.7, since x is an incidence vector χ^F of the edge set of a vertex-disjoint union of disjoint circuits if and only if (36.41)(i) and (ii) are satisfied, together with: $x(\delta(v))$ even for each $v \in V$. So we can take $a = \mathbf{0}$, $b = \mathbf{2}$, $d = \mathbf{0}$, $c = \mathbf{1}$, $S^{\text{even}} = V$, and $S^{\text{odd}} = \emptyset$. In particular, U and W are empty in (36.38). ∎

Note that Corollaries 29.2e and 36.7a imply that the polytope described in Corollary 36.7a is obtained from the $\emptyset$-join polytope by adding the constraint (36.41)(ii).

This has as consequence Corollary 29.2f (due to Seymour [1979b]) characterizing the circuit cone. Given a graph $G = (V, E)$, the *circuit cone* is the cone in $\mathbb{R}^E$ generated by the incidence vectors of circuits. This cone is determined by:

$$\begin{array}{llll}(36.42) & \text{(i)} & x_e \geq 0 & \text{for each } e \in E,\\ & \text{(ii)} & x(D) \geq 2x_e & \text{for each cut } D \text{ and } e \in D.\end{array}$$

To prove this, we may assume (by scaling) that $x(E) \leq 1$. Then (36.42)(ii) implies (36.41)(iii), and hence the characterization follows from Corollary 36.7a.

36.6. Total dual integrality

We finally show that the system given by (36.21)(i) and (ii) and (36.38) after deleting the parity constraint on R, is TDI:

Theorem 36.8. *Assuming* (36.22), *the following system is TDI (setting $T := V \setminus S$):*

(36.43)
$$\begin{array}{rl} \text{(i)} & d \leq x \leq c, \\ \text{(ii)} & \frac{1}{2}a_v \leq \frac{1}{2}(Mx)_v \leq \frac{1}{2}b_v, \text{ for } v \in S, \\ \text{(iii)} & a_v \leq (Mx)_v \leq b_v, \text{ for } v \in T, \\ \text{(iv)} & \frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \\ & \qquad \leq \frac{1}{2}(b(U) - a(W) + c(F) - d(H) - \varepsilon), \end{array}$$

where U and W are disjoint subsets of T, where F and H partition $\delta(U \cup W \cup R)$ for some $R \subseteq S$, and where $\varepsilon \in \{0,1\}$ such that $\varepsilon \equiv |R \cap S^{\text{odd}}| + b(U) - a(W) + c(F) - d(H)$ (mod 2).

Proof. The partition of V into S and T induces a partition of M, a, b into M_S, a_S, b_S and M_T, a_T, b_T. By Theorem 36.4, the system

(36.44)
$$\begin{array}{rl} \text{(i)} & d \leq x \leq c, \\ \text{(ii)} & 0 \leq z \leq \frac{1}{2}(b_S - a_S), \\ \text{(iii)} & M_S x + 2z = b_S, \\ \text{(iv)} & a_T \leq M_T x \leq b_T \end{array}$$

becomes TDI by adding the inequalities

(36.45)
$$\begin{aligned} &\tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + z(U \cap S) - z(W \cap S) \\ &\leq \lfloor \tfrac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H)) \rfloor, \end{aligned}$$

for disjoint subsets U, W of V and partitions F, H of $\delta(U \cup W)$. (36.44) is equivalent to:

(36.46)
$$\begin{aligned} &\tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + z(U \cap S) - z(W \cap S) \\ &\leq \tfrac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) - \varepsilon), \end{aligned}$$

where $\varepsilon \in \{0,1\}$ and

(36.47)
$$\varepsilon \equiv b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) \pmod 2.$$

Substituting $z := \frac{1}{2}(b_S - M_S x)$ in (36.44)(ii) gives (36.43)(ii), and in (36.46) gives

(36.48)
$$\begin{aligned} &\tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + \tfrac{1}{2}b(U \cap S) - \tfrac{1}{2}\chi^{U \cap S}M_S x \\ &- \tfrac{1}{2}b(W \cap S) + \tfrac{1}{2}\chi^{W \cap S}M_S x \\ &\leq \tfrac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - c(H) - \varepsilon). \end{aligned}$$

Equivalently:

(36.49)
$$\begin{aligned} &\tfrac{1}{2}((\chi^{U \cap T} - \chi^{W \cap T})M + \chi^F - \chi^H)x \\ &\leq \tfrac{1}{2}(b(U \cap T) - a(W \cap T) + c(F) - d(H) - \varepsilon). \end{aligned}$$

This is equivalent to (36.43)(iv), and total dual integrality is maintained by Theorem 5.27. Note that

(36.50)
$$\begin{aligned} \varepsilon &\equiv b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) \\ &\equiv b(U \cap T) - a(W \cap T) + c(F) - d(H) + b(U \cap S) + b(W \cap S) \\ &\equiv b(U \cap T) - a(W \cap T) + c(F) - d(H) + |R \cap S^{\text{odd}}| \pmod 2, \end{aligned}$$

where $R := (U \cup W) \cap S$. ∎

We remark that the coefficients of the inequalities in (36.43) generally are not all integer.

36.7. Further results and notes

36.7a. The Chvátal rank

The results on the convex hull in this chapter (and in previous chapters) can be interpreted in terms of the so-called 'Chvátal rank' of a system of inequalities or of a matrix. (This relates to the cutting planes reviewed in Section 5.21.)

For any polyhedron P, let P_{I} denote the *integer hull* of P, that is, the convex hull of the integer vectors in P. If P is a rational polyhedron, then P_{I} is again a rational polyhedron. This polyhedron can be approached as follows.

Define for any polyhedron P, the set P' by:

(36.51) $$P' := \bigcap_{H \supseteq P} H_{\mathrm{I}},$$

where H ranges over all rational affine halfspaces containing P as a subset. Here an *affine halfspace* is a set of the form

(36.52) $$H = \{x \in \mathbb{R}^n \mid w^{\mathsf{T}} x \leq \alpha\}$$

for some nonzero $w \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$. It is *rational* if w and α are rational. So trivially (since $P \subseteq H \Rightarrow P_{\mathrm{I}} \subseteq H_{\mathrm{I}}$):

(36.53) $$P \supseteq P' \supseteq P_{\mathrm{I}}.$$

Note that if H is as in (36.52) and w is integer, with relatively prime components, then

(36.54) $$H_{\mathrm{I}} = \{x \in \mathbb{R}^n \mid w^{\mathsf{T}} x \leq \lfloor \alpha \rfloor\}.$$

So P' arises from P by adding a 'first round of cuts'. Observe that if $P = \{x \mid Mx \leq b\}$ for some rational $m \times n$ matrix M and some vector $b \in \mathbb{Q}^m$, then in (36.51) we can restrict the affine hyperplanes H to those for which there exists a vector $y \in \mathbb{Q}_+^m$ with $y^{\mathsf{T}} M$ integer and nonzero and

(36.55) $$H = \{x \mid (y^{\mathsf{T}} M)x \leq y^{\mathsf{T}} b\}$$

(by Farkas' lemma).

It can be shown that P' is a rational polyhedron again. To P' we can apply this operation again, and obtain $P'' = (P')'$. We thus obtain a series of polyhedra P, P', P'',..., $P^{(t)}$, ..., satisfying

(36.56) $$P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P^{(t)} \supseteq \cdots P_I.$$

Now Chvátal [1973a] (cf. Schrijver [1980b]) showed that for each polyhedron P there is a finite t with $P^{(t)} = P_{\mathrm{I}}$. The smallest such t is called the *Chvátal rank* of P.

It can be proved more strongly (Cook, Gerards, Schrijver, and Tardos [1986]) that for each rational matrix M there is a finite value t such that the polyhedron $P := \{x \mid Mx \leq b\}$ has Chvátal rank at most t, for each integer vector b (of appropriate dimension). The smallest such t is called the *Chvátal rank* of M. So each totally unimodular matrix has Chvátal rank 0.

The *strong Chvátal rank* of M is, by definition, the Chvátal rank of the matrix

$$\begin{pmatrix} I \\ -I \\ M \\ -M \end{pmatrix}. \tag{36.57}$$

So the strong Chvátal rank of M is the smallest t such that for all integer vectors d, c, a, b the polyhedron $\{x \mid d \leq x \leq c, a \leq Mx \leq b\}$ has Chvátal rank at most t. So M is totally unimodular if and only if M is integer and has strong Chvátal rank 0 (this is the Hoffman-Kruskal theorem).

Theorem 36.3 implies that the $V \times E$ incidence matrix of a bidirected graph has strong Chvátal rank at most 1. (Matrices of strong Chvátal rank at most 1 are said in Gerards and Schrijver [1986] to have the *Edmonds-Johnson property*.)

Theorem 36.9. *The $V \times E$ incidence matrix of a bidirected graph has strong Chvátal rank at most* 1.

Proof. We must show that for each integer d, c, a, b, one has $P' = P_{\mathrm{I}}$ for $P := \{x \mid d \leq x \leq c, a \leq Mx \leq b\}$. This follows from

$$\begin{aligned} P' &\subseteq \{x \in P \mid \forall y \in \{0, \tfrac{1}{2}\}^n : y^{\mathsf{T}}M \in \mathbb{Z}^n \Rightarrow y^{\mathsf{T}}Mx \leq \lfloor y^{\mathsf{T}}b \rfloor\} \\ &= P_{\mathrm{I}} \subseteq P', \end{aligned} \tag{36.58}$$

where the equality follows from Theorem 36.3. ∎

It is generally not true that also the *transpose* M^{T} of these matrices have Chvátal rank at most 1, as is shown by the incidence matrix M of the complete graph K_4. In Section 68.6c we shall study the Chvátal rank of such matrices M^{T}.

36.7b. Further notes

Gabow [1983a] gave an $O(m^{\frac{3}{2}})$-time algorithm for finding a maximum $s-t$ flow in a bidirected graph with unit capacities. Moreover, he gave $O(m^2 \log n)$- and $O(n^2 m)$-time algorithms for finding a minimum-cost bidirected $s-t$ flow of given value, with unit capacities.

Chapter 37

The dimension of the perfect matching polytope

In this chapter the dimension of the perfect matching polytope is characterized. It implies a characterization of the dimension of the perfect matching space — the linear space spanned by the incidence vectors of perfect matchings.
The basis of determining the dimension is formed by the matching-covered graphs without nontrivial tight cuts. For such graphs, there is an easy formula for the dimension.
Key result (needed in characterizing the perfect matching lattice in the next chapter) is a characterization of Lovász of the matching-covered graphs without nontrivial tight cuts: the 'braces' and the 'bricks'.

37.1. The dimension of the perfect matching polytope

Naddef [1982] gave a min-max formula for the dimension of the perfect matching polytope. By the work of Edmonds, Lovász, and Pulleyblank [1982], it is equivalent to the following.

Let $G = (V, E)$ be a graph and let E_0 be the set of edges covered by at least one perfect matching. Defining $G_0 := (V, E_0)$, one trivially has:

(37.1) $$\dim(P_{\text{perfect matching}}(G)) = \dim(P_{\text{perfect matching}}(G_0)).$$

So when investigating the dimension of the perfect matching polytope, we can confine ourselves to *matching-covered* graphs, that is, to graphs in which each edge is contained in at least one perfect matching.

A further reduction can be obtained by considering tight cuts. A cut C is called *odd* if $C = \delta(U)$ for some $U \subseteq V$ with $|U|$ odd. A cut C is called *tight* if it is odd and each perfect matching intersects C in exactly one edge.

Let $G = (V, E)$ be a graph and let $U \subseteq V$. Recall that G/U denotes the graph obtained from G by contracting U to one vertex, which vertex we will call U. In the obvious way, we will consider the edge set of G/U as a subset of the edge set of G. Hence, for any $x \in \mathbb{R}^E$, we can speak of the *projection* of x to the edges of G/U.

Theorem 37.1. *Let $G = (V, E)$ be a matching-covered graph and let $\delta(U)$ be a tight cut. Define $G_1 := G/U$ and $G_2 := G/\overline{U}$ (where $\overline{U} := V \setminus U$). Then*

$$(37.2) \qquad \begin{aligned} &\dim(P_{\text{perfect matching}}(G)) = \\ &\dim(P_{\text{perfect matching}}(G_1)) + \dim(P_{\text{perfect matching}}(G_2)) - |\delta(U)| + 1. \end{aligned}$$

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then a vector $x \in \mathbb{R}^E$ belongs to the perfect matching polytope of G if and only if its projections to E_1 and E_2 belong to the perfect matching polytopes of G_1 and G_2 respectively. Moreover, since G is matching-covered and since $\delta(U)$ is tight, the projection of $P_{\text{perfect matching}}(G)$ on $\delta(U)$ has dimension equal to $|\delta(U)| - 1$. ∎

This theorem gives a reduction if there exists a nontrivial tight cut. (A cut C is called *nontrivial* if $C = \delta(U)$ for some U with $1 < |U| < |U| - 1$.) Then:

Theorem 37.2. *Let $G = (V, E)$ be a matching-covered graph without any nontrivial tight cut and with at least one perfect matching. Then*

$$(37.3) \qquad \dim(P_{\text{perfect matching}}(G)) = |E| - |V| + k,$$

where k is the number of bipartite components of G.

Proof. We may assume that G is connected. If G is bipartite, the result follows from Theorem 18.6. If G is nonbipartite, consider a vector x in the relative interior of the perfect matching polytope of G. Since G is matching-covered, we know that $x_e > 0$ for each edge e, and since G has no nontrivial tight cut, we know that $x(C) > 1$ for each nontrivial odd cut C. Hence the only constraints in (25.2) satisfied by x with equality are the constraints $x(\delta(v)) = 1$ for $v \in V$. So $\dim(P_{\text{perfect matching}}(G)) \geq |E| - |V|$.

To see equality, we show that the constraints $x(\delta(v)) = 1$ are independent. For let $u \in V$, and choose an odd-length $u-u$ walk $(u, e_1, \ldots, e_t, u)$. For each $e \in E$, let x_e be the number of odd i with $e = e_i$, minus the number of even i with $e = e_i$. Then $x(\delta(u)) = 2$ and $x(\delta(v)) = 0$ for all $v \neq u$. ∎

Theorems 37.1 and 37.2 describe the decomposition of the dimension problem. We now aggregate these results.

For any cut C, any set U with $C = \delta(U)$ is called a *shore* of C. Two cuts C and C' are called *cross-free* if they have shores U and U' that are disjoint. A collection $\mathcal{F}$ of cuts is *cross-free* if each two cuts in $\mathcal{F}$ are cross-free.

Let $\mathcal{F}$ be a cross-free collection of nontrivial cuts. An *$\mathcal{F}$-contraction* of G is a graph obtained from G by choosing a $U_0 \subseteq V$ with $\delta(U_0) \in \mathcal{F}$, contracting U_0, and contracting each maximal proper subset U of $V \setminus U_0$ with $\delta(U) \in \mathcal{F}$.

One easily checks that, if G is connected, there exist precisely $|\mathcal{F}| + 1$ $\mathcal{F}$-contractions. Let $\text{nonbip}_G(\mathcal{F})$ denote the number of $\mathcal{F}$-contractions that are nonbipartite.

Corollary 37.2a. *Let $G = (V, E)$ be a connected matching-covered graph with $|V| \geq 2$. Let $\mathcal{F}$ be any inclusionwise maximal cross-free collection of nontrivial tight cuts. Then*

(37.4) $$\dim(P_{\text{perfect matching}}(G)) = |E| - |V| - \text{nonbip}_G(\mathcal{F}) + 1.$$

Proof. The corollary follows directly by induction from Theorems 37.1 and 37.2, as follows.

If $\mathcal{F} = \emptyset$, then (37.4) follows from Theorem 37.2. If $\mathcal{F} \neq \emptyset$, choose a cut $\delta(U) \in \mathcal{F}$. Let $G_1 := G/U$ and $G_2 := G/\overline{U}$ (where $\overline{U} := V \setminus U$). Then G_1 and G_2 are connected and matching-covered again.

Let $\mathcal{F}_1$ be the set of cuts in $\mathcal{F}$ that have a shore properly contained in $V \setminus U$ and let $\mathcal{F}_2$ be the set of cuts in $\mathcal{F}$ that have a shore properly contained in U.

Then $\mathcal{F}_1$ forms an inclusionwise maximal cross-free collection of nontrivial tight cuts in G_1. So inductively

(37.5) $$\dim(P_{\text{perfect matching}}(G_1)) = |EG_1| - |VG_1| - \text{nonbip}_{G_1}(\mathcal{F}_1) + 1.$$

Similarly,

(37.6) $$\dim(P_{\text{perfect matching}}(G_2)) = |EG_2| - |VG_2| - \text{nonbip}_{G_2}(\mathcal{F}_2) + 1.$$

Now each $\mathcal{F}$-contraction of G is an $\mathcal{F}_i$ contraction of G_i for exactly one $i \in \{1, 2\}$. Hence

(37.7) $$\text{nonbip}_G(\mathcal{F}) = \text{nonbip}_{G_1}(\mathcal{F}_1) + \text{nonbip}_{G_2}(\mathcal{F}_2).$$

Since moreover $|EG| = |EG_1| + |EG_2| - |\delta(U)|$ and $|VG_1| + |VG_2| = |VG| + 2$, we obtain (37.4) with Theorem 37.1. ∎

37.2. The perfect matching space

We derive from Corollary 37.2a a characterization of the perfect matching space and its dimension. The *perfect matching space* of a graph $G = (V, E)$ is the linear hull of the incidence vectors of perfect matchings; that is,

(37.8) $$S_{\text{perfect matching}}(G) := \text{lin.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

(Here lin.hull denotes linear hull.)

Corollary 37.2a directly gives for the dimension of the perfect matching space:

Corollary 37.2b. *Let $G = (V, E)$ be a connected matching-covered graph with $|V| \geq 2$. Let $\mathcal{F}$ be any inclusionwise maximal cross-free collection of nontrivial tight cuts. Then*

(37.9) $$\dim(S_{\text{perfect matching}}(G)) = |E| - |V| - \text{nonbip}_G(\mathcal{F}) + 2.$$

Proof. The dimension of the perfect matching space is 1 more than the dimension of the perfect matching polytope. So the corollary follows from Corollary 37.2a. ∎

With the help of the description of the perfect matching polytope we can similarly describe the perfect matching space in terms of equations:

Theorem 37.3. *The perfect matching space of a graph $G = (V, E)$ is equal to the set of vectors $x \in \mathbb{R}^E$ satisfying*

(37.10) (i) $x_e = 0$ if e is contained in no perfect matching,
(ii) $x(C) = x(\delta(v))$ for each tight cut C and each vertex v.

Proof. Condition (37.10) clearly is necessary for each vector x in the perfect matching space. To see sufficiency, let $x \in \mathbb{R}^E$ satisfy (37.10). We can assume that G has at least one perfect matching.

By adding sufficiently many incidence vectors of perfect matchings to x, we can achieve that $x_e \geq 0$ for each edge e, and $x_e > 0$ for at least one edge e, and $x(C) \geq x(\delta(v))$ for each odd cut C and each vertex v. By scaling we can achieve that $x(\delta(v)) = 1$ for each $v \in V$. Then x belongs to the perfect matching polytope of G, and hence to the perfect matching space. ∎

37.3. The brick decomposition

For any inclusionwise maximal cross-free collection $\mathcal{F}$ of nontrivial tight cuts, the family of $\mathcal{F}$-contractions is called a *brick decomposition*. (We note here that it does not mean that each $\mathcal{F}$-contraction is a brick as defined in Section 37.6.)

Lovász [1987] showed that a brick decomposition is a unique family of graphs (up to isomorphism), independently of the maximal cross-free collection of tight cuts chosen:

Theorem 37.4. *All brick decompositions of a matching-covered graph $G = (V, E)$ are the same (up to isomorphism).*

Proof. By induction on $|V|$. Consider two maximal cross-free collection $\mathcal{F}$ and $\mathcal{F}'$ of nontrivial tight cuts.

Case 1: $\mathcal{F}$ and $\mathcal{F}'$ have a common member $\delta(U)$. By induction, the result of two decompositions of $G/\overline{U}$ is the same (where $\overline{U} := V \setminus U$). Similarly, the result of two decompositions of G/U is the same. The theorem follows.

Case 2: There exist $C \in \mathcal{F}$ and $C' \in \mathcal{F}'$ with C and C' cross-free. Let $\mathcal{F}''$ be a maximal cross-free collection of nontrivial tight cuts containing C and C'. By Case 1, the decompositions of G by $\mathcal{F}$ and $\mathcal{F}''$ result in the same family of graphs. Similarly, the decompositions of G by $\mathcal{F}'$ and $\mathcal{F}''$ result in the same family of graphs. The theorem follows.

Case 3: There exist $C = \delta(U) \in \mathcal{F}$ and $C' = \delta(U') \in \mathcal{F}'$ with $|U \cap U'|$ odd and at least 3. Then trivially $C'' := \delta(U \cap U')$ is tight again. Let $\mathcal{F}''$ be a maximal cross-free collection of nontrivial tight cuts containing C''. By Case 2, the decompositions of G by $\mathcal{F}$ and $\mathcal{F}''$ result in the same family of graphs. Similarly, the decompositions of G by $\mathcal{F}'$ and $\mathcal{F}''$ result in the same family of graphs. Again, the theorem follows.

Case 4: None of the previous cases applies. Let $C = \delta(U) \in \mathcal{F}$ and $C' = \delta(U') \in \mathcal{F}'$. Then $\mathcal{F} = \{C\}$ and $\mathcal{F}' = \{C'\}$. For suppose that say $\mathcal{F}$ contains another cut $C'' = \delta(U'')$. We can assume that $U \subseteq U''$ and that $U'' \cap U'$ is odd. So $|U'' \cap U'| = 1$ (as Case 3 does not apply), and therefore $|U \cap U'| = 1$ (as Case 2 does not apply). However, $U \cup U'$ is odd and disjoint from $U'' \setminus U$, implying that $U \cup U'$ is at most $|V| - 2$, and so Case 3 applies, a contradiction.

So $\mathcal{F} = \{C\}$ and $\mathcal{F}' = \{C'\}$. We can now assume that $U \cap U'$ is odd. Since Case 3 does not apply, $|U \cap U'| = 1$ and $|U \cup U'| = |V| - 1$. Let $U \cap U' = \{u\}$ and $U' \cup U = V \setminus \{v\}$.

Now $\{u, v\}$ is a 2-vertex-cut in G, separating $U \setminus \{u\}$ and $U' \setminus \{u\}$. For suppose that there is an edge e connecting $U \setminus \{u\}$ and $U' \setminus \{u\}$. Let M be a perfect matching containing e. Let f be the edge in M covering u. Then f leaves at least one of U and U'. Since e leaves both U and U', this contradicts the fact that U and U' give tight cuts.

As G has no cut vertices (as G is matching-covered), this implies that $G/\overline{U}$ and G/U' are isomorphic graphs, and similarly that G/U and $G/\overline{U}$ are isomorphic. The theorem follows. ∎

37.4. The brick decomposition of a bipartite graph

All graphs in the brick decomposition of a bipartite graph are bipartite:

Theorem 37.5. *Let G be a matching-covered graph and let $\mathcal{F}$ be an inclusionwise maximal cross-free collection of nontrivial tight cuts. Then G is bipartite if and only if each $\mathcal{F}$-contraction is bipartite.*

Proof. It suffices to prove that for any nontrivial tight cut $\delta(U)$:

(37.11) $\quad G$ is bipartite if and only if G/U and $G/\overline{U}$ are bipartite

(where $\overline{U} := VG \setminus U$). Sufficiency in (37.11) is direct (actually, it holds for any cut). To see necessity in (37.11), note that, since G is matching-covered, U has neighbours only in the largest colour class of the bipartite graph $G - U$. So G/U is bipartite, and similarly, $G/\overline{U}$ is bipartite. ∎

37.5. Braces

A bipartite graph $G = (V, E)$, with colour classes U and W, is called a *brace* if G is matching-covered with $|V| \geq 4$ and for all distinct $u, u' \in U$ and $w, w' \in W$, the graph $G - u - u' - w - w'$ has a perfect matching.

By Hall's marriage theorem (Theorem 22.1), a connected bipartite graph $G = (V, E)$ with equal-sized colour classes U and W is a brace if and only if for each subset X of U with $1 \leq |X| \leq |U| - 2$ one has

$$|N(X)| \geq |X| + 2. \tag{37.12}$$

Theorem 37.6. *Each tight cut in a brace is trivial.*

Proof. Let $G = (V, E)$ be a brace with colour classes U and W, and suppose that $\delta(T)$ is a nontrivial tight cut. As $|T|$ is odd, by symmetry we can assume that $|U \cap T| < |W \cap T|$.

Then $|U \cap T| = |W \cap T| - 1$, since there exists a perfect matching intersecting $\delta(T)$ in exactly one edge. Since $\delta(T)$ is nontrivial, $1 \leq |U \cap T| \leq |U| - 2$.

Moreover, there is no edge e connecting $U \cap T$ and $W \setminus T$. Otherwise this e would be contained in a perfect matching M. This perfect matching also contains an edge connecting $U \setminus T$ and $W \cap T$, contradicting the tightness of $\delta(T)$.

So $N(U \cap T) \subseteq W \cap T$, and hence $|N(U \cap T)| \leq |U \cap T| + 1$, contradicting (37.12). ∎

37.6. Bricks

A graph G is called a *brick* if G is 3-connected and bicritical, and has at least four vertices. (A graph G is called *bicritical* if $G - u - v$ has a perfect matching for any two distinct vertices u, v.)

The following key result was shown by Edmonds, Lovász, and Pulleyblank [1982]:

Theorem 37.7. *Each tight cut in a brick is trivial.*

Proof. Let $G = (V, E)$ be a brick, and suppose that it has a nontrivial tight cut C_0. Let $\mathcal{C}$ be the collection of odd cuts in G.

For any $b \in \mathbb{Q}^V$, consider the linear program

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{e=uv \in E} (b(u) + b(v))x_e & \\ \text{subject to} & x(C) \geq 1 & (C \in \mathcal{C}), \\ & x_e \geq 0 & (e \in E). \end{array} \tag{37.13}$$

and its dual

(37.14) maximize $\sum_{C\in\mathcal{C}} y(C)$

subject to $\sum_{C\ni e} y(C) \leq b(u)+b(v) \quad (e=uv\in E),$

$y(C)\geq 0 \quad (C\in\mathcal{C}).$

We first show:

(37.15) there exist $y\in\mathbb{Q}_+^{\mathcal{C}}$ and $b\in\mathbb{Q}_+^V$ such that $\sum_{C\ni e} y(C)\leq b(u)+b(v)$ for each edge $e=uv$, and such that $y(\mathcal{C})=b(V)$ and $y(C_0)>0$.

To prove this, define $w=\chi^{C_0}$ (the incidence vector of C_0 in $\mathbb{R}^E$). As C_0 is tight, the maximum of $w(M)$ over perfect matchings M is equal to 1. Hence, by Edmonds' perfect matching polytope theorem (Theorem 25.1) and by linear programming duality, there exists a vector $z\in\mathbb{Q}^{\mathcal{C}}$ such that

(37.16) (i) $\sum_{C\ni e} z(C)\leq -w(e)$ for each edge e,

(ii) $z(C)\geq 0$ if C is nontrivial,

(iii) $z(\mathcal{C})=-1$.

For $v\in V$, define $b(v):=-z(\delta(v))$ if $z(\delta(v))<0$, and $b(v):=0$ otherwise. For $C\in\mathcal{C}$, define $y(C):=z(C)$ if $z(C)>0$, and $y(C):=0$ otherwise. Then (37.16) implies:

(37.17) (i) $b(u)+b(v)\geq \sum_{C\ni e} y(C)+w(e)$ for each edge $e=uv$,

(ii) $b\geq \mathbf{0}$, $y\geq\mathbf{0}$,

(iii) $b(V)=y(\mathcal{C})+1$.

So resetting $y(C_0):=y(C_0)+1$ gives b and y as required in (37.15), proving (37.15).

This implies:

(37.18) for some vector $b\in\mathbb{Z}_+^V$ there exists an integer optimum solution $y\in\mathbb{Z}_+^{\mathcal{C}}$ of (37.14) such that $y(C_0)\geq 1$.

Indeed, in (37.15) we can assume (by scaling) that b and y are integer. Then by the properties described in (37.15), y is a feasible solution of (37.14). Since the maximum in (37.13) is at least $b(V)$ (as any perfect matching M satisfies $w(M)=b(V)$), and since $y(\mathcal{C})=b(V)$, we know that y is an optimum solution of (37.14). This proves (37.18).

Now fix a b as in (37.18), with $b(v)$ minimal. Then

(37.19) for any optimum solution y of (37.14) and any $C\in\mathcal{C}$ one has that if $y(C)>0$, then C is tight.

Indeed, any perfect matching M attains the maximum (37.13) (as the maximum value equals $b(V)$). So if $y(C)>0$, by complementary slackness, $|M\cap C|=1$. This shows (37.19).

Call a vector $y \in \mathbb{R}^{\mathcal{C}}_{+}$ *laminar* if the collection $\{C \in \mathcal{C} \mid y(C) > 0\}$ is laminar. Then:

(37.20) there exists a laminar integer optimum solution of (37.14) such that $y(C) \geq 1$ for at least one nontrivial tight cut C.

To see this, choose an integer optimum solution y of (37.14) such that $y(C) \geq 1$ for at least one nontrivial tight cut C, with

(37.21) $$\sum_{C \in \mathcal{C}} y(C)s(C)$$

minimized, where $s(C)$ denotes the number of pairs of vertices separated by C. We show that y is laminar.

Suppose to the contrary that C and C' cross, with $y(C) > 0$ and $y(C') > 0$. We can choose $U', U'' \subseteq V$ such that $C = \delta(U)$, $C' = \delta(U')$, and $|U \cap U'|$ is odd. Let $D := \delta(U \cap U')$ and $D' := \delta(U \cup U')$. Let $\varepsilon := \min\{y(C), y(C')\}$. Decrease $y(C)$ and $y(C')$ by ε, and increase $y(D)$ and $y(D')$ by ε. Then we obtain again a feasible solution of (37.14), while (37.21) is smaller. So both D and D' are trivial. Hence $U \cap U' = \{u\}$ and $U \cup U' = V \setminus \{v\}$ for some vertices u and v. As G is 3-connected, there is an edge e connecting $U \setminus U'$ and $U' \setminus U$. Since G is matching-covered, there is a perfect matching M containing e. So $e \in C \cap C'$. As C and C' are tight, e is the only edge of M intersecting $C \cup C'$. Hence no edge of M intersects $D = \delta(U \cap U')$, a contradiction. This proves (37.20).

Fix y satisfying (37.20). We note that the first set of constraints in (37.14) gives:

(37.22) if $e = uv \in C$ and $y(C) > 0$, then $b(u) > 0$ or $b(v) > 0$.

Moreover,

(37.23) for each $u \in V$, $b(u) = 0$ or $y(\delta(u)) = 0$.

Otherwise, decreasing $b(u)$ and $y(\delta(u))$ by 1 would give b and y with smaller $b(V)$.

We also show:

(37.24) if $y(\delta(U)) > 0$, then $G[U]$ is connected.

If not, let K be an odd component of $G[U]$ and let e be an edge in $\delta(U)$ not incident with K. Let M be a perfect matching containing e. Then M intersects $\delta(U)$ in more than one edge (since K is odd), while $\delta(U)$ is tight since $y(\delta(U)) > 0$. This contradiction proves (37.24).

Now choose an odd cut $C = \delta(U)$ with $y(C) > 0$, an edge $e_0 = u_0v \in C$ with $u_0 \in U$ and $b(u_0) > 0$, such that $|U|$ is as small as possible. (Such U, e_0, u_0 exist by (37.22).)

By (37.23), $|U| > 1$. Let $U_1, \ldots, U_k$ be the maximal proper subsets of U with $y(\delta(U_i)) > 0$. By (37.20), the U_i are pairwise disjoint. Note that $u_0 \notin U_1 \cup \cdots \cup U_k$, by the minimality of $|U|$.

Define

(37.25) $U' := U \setminus (U_1 \cup \cdots \cup U_k)$, $U_+ := \{u \in U' \mid b(u) > 0\}$, and $U_0 := U' \setminus U_+$.

Then

(37.26) there is no edge joining distinct sets among $U_0, U_1, \ldots, U_k$.

Directly from (37.22) and the minimality of $|U|$.

Moreover,

(37.27) there is no edge $e = uv$ with $u \in U_+$ and $v \in U'$.

For suppose that such an edge e exists. Then there is a perfect matching containing e. Hence, by complementary slackness, we have equality in the corresponding constraint of (37.14). As $b(u)+b(v) > 0$, we know that $y(C) > 0$ for some C with $e \in C$. Then $C = \delta(S)$ for some $S \subseteq U$. This contradicts the definition of the U_i, proving (37.27).

As $G[U]$ is connected (by (37.24)), it follows that $U_0 = \emptyset$. Next

(37.28) $|U_+| = k + 1$.

For consider any perfect matching M containing edge e_0. Then M intersects any $\delta(U_i)$ in exactly one edge (as each $\delta(U_i)$ is tight, by (37.19)) and it also intersects $\delta(U)$ in exactly one edge, namely e_0. Since $|M \cap \delta(U)| = 1$, we know with (37.26) that the edge in $M \cap \delta(U_i)$ connects U_i and U_+. Moreover, no edge in M connects two vertices in U_+ (by (37.27)). Hence we have (37.28).

(37.29) No edge connects any U_i with $V \setminus U$.

Otherwise, the same counting as for proving (37.28) gives $|U_+| = k$, a contradiction.

As $|U| > 1$ we know $k > 0$. Choose $s, t \in U_+$. As G is bicritical, $G - s - t$ has a perfect matching M. Then M intersects each $\delta(U_i)$ at least once, and hence (by (37.29)) $|U_+ \setminus \{s, t\}| \geq k$, a contradiction. ∎

37.7. Matching-covered graphs without nontrivial tight cuts

The foregoing is used in obtaining the following basic result of Lovász [1987]:

Theorem 37.8. *Let $G = (V, E)$ be a connected graph with at least four vertices. Then G is matching-covered without nontrivial tight cuts if and only if G is a brick or a brace.*

Proof. If G is a brick or a brace, then trivially G is matching-covered. Moreover, Theorems 37.6 and 37.7 show that braces and bricks have no nontrivial tight cuts.

Conversely, assume that G is matching-covered and has no nontrivial tight cuts.

Case 1: G is not bicritical. We show that G is a brace. As G is not bicritical, by Tutte's 1-factor theorem (Theorem 24.1a) there exists a subset U of V such that $G - U$ has $|U|$ odd components, with $|U| \geq 2$. As G is matching-covered, U is a stable set, and $G - U$ has no even components. For each component K of $G - U$, $\delta(K)$ is tight, and hence trivial, that is $|K| = 1$. So G is bipartite, and U is one of its colour classes. If G is not a brace, there exists a subset X of U with $1 \leq |X| \leq |U| - 2$ and $|N(X)| \leq |X| + 1$. Let $Y \subseteq V \setminus U$ with $N(X) \subseteq Y$ and $|Y| = |X| + 1$. Then $\delta(X \cup Y)$ is a nontrivial tight cut, a contradiction.

Case 2: G is bicritical. We show that G is a brick. So we must show that G is 3-connected. As G is matching-covered, G is trivially 2-connected. Suppose that $\{u, v\}$ is a 2-vertex-cut. Let K be a component of $G - u - v$. As $G - u - v$ has a perfect matching, $|K|$ is even. Then $\delta(K \cup \{u\})$ is a nontrivial cut which is tight, since the intersection of $\delta(K \cup \{u\})$ with any perfect matching M is odd and at most 2 (as each edge in the intersection is incident with u or v). ∎

Chapter 38

The perfect matching lattice

This chapter is devoted to giving a proof of the deep theorem of Lovász [1987] characterizing the perfect matching lattice of a graph — the lattice generated by the incidence vectors of perfect matchings.

We summarize concepts and results from previous chapters that we need in the proof. Let $G = (V, E)$ be a graph. The following notions will be used:

- A cut C in G is *tight* if each perfect matching intersects C in exactly one edge.
- A cut C is *trivial* if $C = \delta(v)$ for some vertex v.
- G is *matching-covered* if each edge is contained in a perfect matching.
- G is *bicritical* if for each two distinct vertices u and v, the graph $G-u-v$ has a perfect matching.
- G is a *brick* if it is 3-connected and bicritical and has at least 4 vertices.
- A subset B of V is a *barrier* if $G-B$ has at least $|B|$ odd components. A *maximal barrier* is an inclusionwise maximal barrier. A *nontrivial barrier* is a barrier B with $|B| \geq 2$.

Moreover, the following results will be used:

- the perfect matching lattice of a bipartite graph is equal to the set of integer vectors in the perfect matching space (this is an easy consequence of Kőnig's edge-colouring theorem, see Theorem 20.12).
- Any two distinct inclusionwise maximal barriers in a connected matching-covered graph are disjoint (Corollary 24.11a).
- A graph is a brick if and only if it is nonbipartite and matching-covered and has no nontrivial tight cuts (a consequence of Theorem 37.8).
- A graph is bicritical if and only if it has no nontrivial barrier (a consequence of Tutte's 1-factor theorem (Corollary 24.1a)).

Throughout this chapter, $\overline{U}$ denotes the complement of U.

38.1. The perfect matching lattice

The *perfect matching lattice* (usually briefly the *matching lattice*) of a graph $G = (V, E)$ is the lattice generated by the incidence vectors of perfect matchings in G; that is,

(38.1) $$L_{\text{perfect matching}}(G) := \text{lattice}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

So it is a sublattice of $\mathbb{Z}^E$ and is contained in the perfect matching space of G.

In Section 20.8 we saw that the perfect matching lattice of a *bipartite* graph $G = (V, E)$ is equal to the intersection of $\mathbb{Z}^E$ with the perfect matching space of G. This characterization does not hold in general for nonbipartite graphs, as is shown by the Petersen graph. However, as was proved by Lovász [1987], any graph for which the characterization does not hold, contains the Petersen graph in some sense. In particular, for any graph without Petersen graph minor, the characterization remains valid.

In analyzing the perfect matching lattice of G, two initial observations are of interest:

- We can assume that G is matching-covered, since any edge contained in no perfect matching can be deleted;
- If G has a nontrivial tight cut, we can reduce the analysis by considering the two graphs obtained by contracting either of the shores of the cut.

So we can focus the investigations on nonbipartite matching-covered graphs without nontrivial tight cuts; that is, by Theorem 37.8, on bricks.

38.2. The perfect matching lattice of the Petersen graph

We will need a characterization of the perfect matching lattice of the Petersen graph, which is easy to prove:

Theorem 38.1. *Let G be the Petersen graph and let C be a 5-circuit in G. Then the perfect matching lattice consists of all integer vectors x in the perfect matching space with $x(EC)$ even.*

Proof. Inspection of the Petersen graph (cf. Figure 38.1) shows that each edge of G is contained in exactly two perfect matchings, that (hence) G has exactly six perfect matchings, that any two perfect matchings intersect each other in exactly one edge, and that each perfect matching intersects EC in an even number of edges.

Let $M_0 := \delta(VC)$ (the set of edges intersecting VC in one vertex). Then M_0 is a perfect matching of G. Let $M_1, \ldots, M_5$ be the five other perfect matchings of G. So each of the M_i intersects M_0 in one edge.

By adding appropriate integer multiples of $\chi^{M_1}, \ldots, \chi^{M_5}$ to x we can achieve that $x_e = 0$ for each $e \in M_0$. As x is in the perfect matching space, we know that there exists a number t such that $x(\delta(v)) = t$ for each vertex v. Hence, as $|EC|$ is odd, $x_e = \frac{1}{2}t$ for all $e \in EC$; similarly, for each edge e in the 5-circuit vertex-disjoint from C one has $x_e = \frac{1}{2}t$. As $x(EC)$ is even, we know that $\frac{5}{2}t$ is even, hence $\frac{1}{2}t$ is even. Now the vector

(38.2) $$y := \chi^{M_1} + \cdots \chi^{M_5} - \chi^{M_0}$$

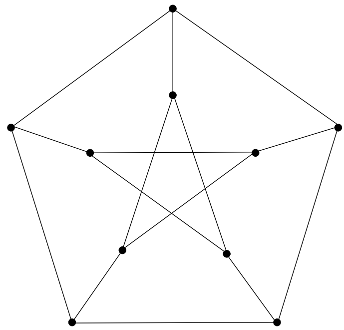


Figure 38.1
The Petersen graph

satisfies $y_e = 0$ for $e \in M_0$ and $y_e = 2$ for $e \notin M_0$. Hence x is an integer multiple of y, proving that x belongs to the perfect matching lattice of G. ∎

38.3. A further fact on the Petersen graph

In the proof of the characterization of the perfect matching lattice, we need a further, technical fact on the Petersen graph.

Let $G = (V, E)$ be a graph and let $b : V \to \mathbb{Z}_+$. Recall that a *b-factor* is a subset F of E with $\deg_F(v) = b(v)$ for each $v \in V$.

Theorem 38.2. *Let $G = (V, E)$ be the Petersen graph and let C be a 5-circuit in G. Let $b : V \to \mathbb{Z}_+$ be such that*

- *either there exists a $u \in V$ with $b(u) = 3$ and $b(v) = 1$ for all $v \neq u$,*
- *or there exist distinct $u, u' \in V$ with $b(u), b(u') \in \{0, 2\}$ and $b(v) = 1$ for all $v \neq u, u'$, such that if $b(u) = b(u') = 0$, then u and u' are nonadjacent.*

Then there exist b-factors F and F' such that $|F \cap EC|$ and $|F' \cap EC|$ have different parities.

Proof. By induction on $b(V)$. If $b(u) = b(u') = 0$ for some distinct $u, u' \in V$, then u and u' are nonadjacent. Let x be the common neighbour of u and u' and let y be the neighbour of x distinct from u and u'. Then $G - x - N(x)$ forms a 6-circuit (by inspection — cf. Figure 38.1), D say. Split ED into two matchings, M and M'. Adding edge xy to M and M' gives b-factors as required since EC intersects ED in an odd number of edges (as $ED = EG \setminus \delta(N(x))$, and $|EC|$ is odd and $|EC \cap \delta(N(x))|$ is even).

If $b(u') = 2$, choose a neighbour u'' of u' with u'' different from and nonadjacent to u. Define $b'(u'') := 0$, $b'(u) := b(u)$, and $b'(v) := 1$ for all other vertices. By induction, there exist b'-factors F and F' such that F and

F' intersect EC in different parities. Adding edge $u'u''$ to F and F' gives b-factors as required.

If $b(u) = 3$ for some $u \in V$, we can choose any neighbour u' of u, define $b'(u) := 2$, $b'(u') := 0$, and $b'(v) := 1$ for all $v \neq u, u'$, and apply induction as above. ∎

38.4. Various useful observations

In this section we prove a few easy facts that turn out to be useful.

Let $G = (V, E)$ be a graph and let $U \subseteq V$. Recall that G/U denotes the graph obtained from G by contracting U to one vertex, which vertex we will call U. In the obvious way, we will consider the edge set of G/U as a subset of the edge set of G. Hence, for any $x \in \mathbb{R}^E$, we can speak of the *projection* of x to the edges of G/U.

We now characterize when G/U is a brick if G is a brick:

Theorem 38.3. *Let $G = (V, E)$ be a brick and let $U \subseteq V$. Then G/U is a brick if and only if $G - U$ is 2-connected and factor-critical.*

Proof. Necessity being easy, we prove sufficiency.

First, let $G - U$ be 2-connected. Then G/U is 3-connected, for suppose that vertices u and u' of G/U form a 2-vertex-cut of G/U. If both u and u' are different from vertex U of G/U, then u, u' would also form a 2-vertex-cut of G, contradicting the 3-connectivity of G. If, say, u' is equal to vertex U of G/U, then u is a cut vertex of $G - U$, contradicting the 2-connectivity of $G - U$.

Second, let $G - U$ be factor-critical. To see that G/U is bicritical, let B be a nontrivial barrier of G/U. If B does not contain vertex U of G/U, then B would also be a nontrivial barrier of G, contradicting the bicriticality of G. If B contains vertex U, then $G - U$ is not factor-critical. ∎

Maximal barriers leave factor-critical components:

Theorem 38.4. *Let $G = (V, E)$ be a graph with a perfect matching and let B be a maximal barrier. Then each component K of $G - B$ is factor-critical.*

Proof. Suppose not. Then K has a nonempty subset B' such that $(G[K]) - B'$ has at least $|B'| + 1$ odd components. Hence $B \cup B'$ is a barrier of G, contradicting the maximality of B. ∎

We note that

(38.3) if $B_1, \ldots, B_k$ are the maximal nontrivial barriers of a graph $G = (V, E)$, having a perfect matching, then for each $u \in V \setminus (B_1 \cup \cdots \cup B_k)$, the graph $G - u$ is factor-critical.

In bicritical graphs, nonempty stable sets have many neighbours (a *neighbour* of S is a vertex not in S adjacent to at least one vertex in S):

Theorem 38.5. *Let $G = (V, E)$ be bicritical with $|V| \geq 4$. Then any nonempty stable set S has at least $|S| + 2$ neighbours.*

Proof. Suppose that $|N(S)| \leq |S| + 1$. Since $|V \setminus S| \geq |S|$ (as G has a perfect matching), we know $|V \setminus S| \geq 2$. Hence we can choose two vertices $v, v' \in V \setminus S$ such that $|N(S) \setminus \{v, v'\}| < |S|$. This however contradicts the fact that $G - v - v'$ has a perfect matching, since each vertex in S should be matched to a vertex in $N(S)$. ∎

It will also be useful to make the following observation:

Theorem 38.6. *Let $G = (V, E)$ be a graph and let U be an odd subset of V, such that for each edge $e \in \delta(U)$ there is a perfect matching M_e with $M_e \cap \delta(U) = \{e\}$. Define $G_1 := G/\overline{U}$ and $G_2 := G/U$, and let $x \in \mathbb{Z}^E$. If, for each $i = 1, 2$, the projection of x to EG_i belongs to the perfect matching lattice of G_i, then x belongs to the perfect matching lattice of G.*

Proof. Let x' and x'' be the projections of x to EG_1 and to EG_2, respectively. Since x' belongs to the perfect matching lattice of G_1, there exist perfect matchings $M'_1, \ldots, M'_{k'}$ and $N'_1, \ldots, N'_{l'}$ of G_1 such that

(38.4) $$x' = \sum_{i=1}^{k'} \chi^{M'_i} - \sum_{j=1}^{l'} \chi^{N'_j}.$$

Similarly, there exist perfect matchings $M''_1, \ldots, M''_{k''}$ and $N''_1, \ldots, N''_{l''}$ of G_2 such that

(38.5) $$x'' = \sum_{i=1}^{k''} \chi^{M''_i} - \sum_{j=1}^{l''} \chi^{N''_j}.$$

Consider any $e \in \delta(U)$. Then $x'_e = x_e = x''_e$. Hence, using the projections of M_e to EG_1 and to EG_2, we can assume that

(38.6) $$\begin{aligned} &|\{i = 1, \ldots, k' \mid e \in M'_i\}| = |\{i = 1, \ldots, k'' \mid e \in M''_i\}| \text{ and} \\ &|\{j = 1, \ldots, l' \mid e \in N'_j\}| = |\{j = 1, \ldots, l'' \mid e \in N''_j\}|, \end{aligned}$$

since we can add the projection of M_e to EG_1 to both sums in (38.4), if the number of i with $e \in M'_i$ is less than the number of i with $e \in M''_i$; similarly, if it would be more.

We can do this for each $e \in \delta(U)$, to obtain (38.6) for each $e \in \delta(U)$. It implies that $k' = k''$ and $l' = l''$. It also implies that we can 'match' the M'_i and M''_i in common edges in $\delta(U)$. That is, by permuting indices, we can assume that M'_i and M''_i have an edge in $\delta(U)$ in common, for each $i = 1, \ldots, k'$. In other words, each $M'_i \cup M''_i$ is a perfect matching of G. Similarly, we can assume that each $N'_j \cup N''_j$ is a perfect matching of G. Then

(38.7) $$x = \sum_{i=1}^{k'} \chi^{M_i' \cup M_i''} - \sum_{j=1}^{l'} \chi^{N_j' \cup N_j''}.$$

So x belongs to the perfect matching lattice of G. ∎

38.5. Simple barriers

In this section, we fix a brick $G = (V, E)$ and an edge e such that $G - e$ is matching-covered, and study barriers of $G - e$. In particular we focus on 'simple' barriers of $G - e$. They play an important role in the proof of the characterization of the perfect matching lattice.

For any $B \subseteq V$, let $I(B)$ denote the set of isolated vertices of $G - e - B$ and let $K(B)$ denote the set of nonisolated vertices of $G - e - B$. Then B is called a *simple barrier* of $G - e$ if $|I(B)| = |B| - 1$. So a simple barrier is a barrier of $G - e$, and hence a stable set (as $G - e$ is matching-covered). Note that each singleton is a simple barrier.

For any simple barrier B of $G-e$, $K(B)$ is an odd component of $G-e-B$, since $G-e$ is matching-covered and connected. (Trivially, $|K(B)|$ is odd, since $|V|$ is even and $|I(B)| = |B| - 1$. If $K(B)$ would not be connected, let L be an odd component of $K(B)$ and let f be an edge connecting $K(B) \setminus L$ and B. Let M be a perfect matching of $G - e$ containing f. Necessarily some edge in M leaves L. But then more that one edge in M connects $K(B)$ and B, and also each vertex in $I(B)$ is matched to B, while $|I(B)| = |B| - 1$, a contradiction.)

Since a barrier B of $G - e$ with $|B| \geq 2$ is not a barrier of G (since G is bicritical), e necessarily connects two odd components of $G - e - B$. If B is a simple barrier of $G - e$ with $|B| \geq 2$, then e connects $K(B)$ with some vertex $v_1 \in I(B)$. (G has a perfect matching M intersecting $\delta(K(B))$ in at least three edges, and hence M contains an edge connecting $K(B)$ and $I(B)$. This edge must be e.)

Then the perfect matchings M of G are of two types:

(38.8) M does not contain e, in which case M matches B with the components of $G - e - B$,
or M contains e, in which case two of the edges in M leaving B are incident with $K(B)$, and the other edges in M leaving B are incident with $I(B) \setminus \{v_1\}$.

We now give some further easy properties of simple barriers. Recall that a subset U of the vertex set V of a graph G is called *matchable* if $G[U]$ has a perfect matching.

Theorem 38.7. *Let $G = (V, E)$ be a brick, let $e \in E$ be such that $G - e$ is matching-covered and let B be a simple barrier of $G - e$. Let $e = v_1v_2$ with $v_1 \in B \cup I(B)$ and $v_2 \in K(B)$. Then:*

(38.9)
(i) *if* $|B| \geq 2$, *then* $v_1 \in I(B)$;
(ii) *for any* $u \in B$, *the set* $(B-u) \cup I(B)$ *is matchable;*
(iii) *for any distinct* $u, u' \in B$, *the set* $(B-u-u') \cup (I(B)-v_1)$ *is matchable;*
(iv) $G-e/K(B)$ *is matching-covered;*
(v) $G[B \cup I(B)]$ *is connected;*
(vi) *any cut vertex* v *of* $G[B \cup I(B)]$ *belongs to* $I(B)$;
(vii) *if* $Y \subseteq I(B)$ *and* $G[B \cup I(B)] - Y$ *has at least* $|Y|+1$ *components, then it contains precisely* $|Y|+1$ *components and any component of* $G[B \cup I(B)] - Y$ *not containing* v_1 *consists of a singleton vertex in* B.

Proof. Since all assertions are trivial if $|B| = 1$, we can assume that $|B| \geq 2$. We saw above that then $v_1 \in I(B)$, proving (i).

(ii) follows from the fact that $G-u-v_2$ has a perfect matching. Similarly, (iii) follows from the fact that $G-u-u'$ has a perfect matching, necessarily containing e. (iv) is directly implied by the fact that $G-e$ is matching-covered, and (v) follows from (ii).

To see (vi), assume that $v \in B$. Choose a component K of $G[B \cup I(B)] - v$ not containing v_1. Since $(B-v) \cup I(B)$ is matchable by (ii), $|K \cap B| = |K \cap I(B)|$. Choose $v' \in K \cap B$. Then $(B-v-v') \cup (I(B)-v_1)$ is matchable by (iii). However, $|K \cap B \setminus \{v'\}| < |K \cap I(B)|$, a contradiction. This proves (vi).

To prove (vii), let α be the number of components of $G[B \cup I(B)] - Y$ containing v_1, let β be the number of other components intersecting $I(B)$, and let γ be the number of other components (hence each consisting of a singleton vertex in B). So $\alpha + \beta + \gamma \geq |Y| + 1$. Now by Theorem 38.5, each component K satisfies

$$(38.10) \qquad |K \cap B| \geq |K \cap I(B)| + 1.$$

Indeed, if $K \cap I(B) = \emptyset$, this is trivial. If $K \cap I(B) \neq \emptyset$, then by Theorem 38.5, $|K \cap I(B)| + 2 \leq N(K \cap I(B))| \leq |K \cap B| + 1$, as $N(K \cap I(B)) \subseteq (K \cap B) \cup \{v_2\}$. This proves (38.10).

Moreover,

$$(38.11) \qquad \text{if } v_1 \notin K \text{ and } K \cap I(B) \neq \emptyset, \text{ then } |K \cap B| \geq |K \cap I(B)| + 2,$$

since then $N(K \cap I(B)) \subseteq K \cap B$.

(38.10) and (38.11) imply

$$(38.12) \qquad \begin{aligned} \alpha + 2\beta + \gamma &\leq \sum_K (|K \cap B| - |K \cap I(B)|) = |B| - |I(B) \setminus Y| \\ &= |Y| + 1 \leq \alpha + \beta + \gamma, \end{aligned}$$

where K ranges over the components of $G[B \cup I(B)] - Y$. Hence $\beta = 0$, and (vii) follows. ∎

We next consider the case where v_2 is a cut vertex of $G[K(B)]$.

Theorem 38.8. *Let $G = (V, E)$ be a brick and let $e = v_1v_2$ be an edge such that $G - e$ is matching-covered. Let B be a simple barrier of $G - e$ with $v_1 \in I(B)$ and $v_2 \in K(B)$. Let Z be a union of components of $G[K(B)] - v_2$, with $Z \neq K(B) - v_2$. Then $G/(Z \cup \{v_2\})$ is matching-covered and has exactly one brick in its brick decomposition.*

Proof. Define $U := Z \cup \{v_2\}$ and $L := K(B) \setminus U$. Note that L is matchable, since $G - v - v'$ has a perfect matching for some $v, v' \in B$ (necessarily containing e and containing no edge connecting $K(B)$ and B). So $|L|$ is even.

We first show that

(38.13) G/U is matching-covered.

Consider first any perfect matching M of $G - e$. Then M has exactly one edge leaving $B \cup I(B)$. Hence M has exactly one edge leaving U (since if there were at least three, then at least two of them should leave $B \cup I(B)$). So M gives a perfect matching in G/U. Since $G - e$ is matching-covered, this implies that each edge of G/U except the image of e is contained in a perfect matching of G/U.

As $L \neq \emptyset$ and $|L|$ is even, G has a perfect matching M with at least three edges leaving $L \cup \{v_2\}$. So it contains at least two edges connecting L and B. Hence M contains e, and all other edges leaving $B \cup I(B)$ connect it with L. So the image of M is a perfect matching in G/U containing the image of e. This shows (38.13).

To see that G/U has only one brick in its brick decomposition, choose a counterexample with $|B|$ as small as possible. This implies:

(38.14) $|N(X) \cap I(B)| > |X|$ for each nonempty subset X of $B \setminus N(L)$.

Assume that this is not the case. Since $|B \cap N(L)| \geq 2$ (as G is 3-connected), we know $|X| \leq |B| - 2$, and so $|N(X) \cap I(B)| \leq |B| - 2 = |I(B)| - 1$, implying $I(B) \not\subseteq N(X)$. Each neighbour of $I(B) \setminus N(X)$ belongs to $(B \setminus X) \cup \{v_2\}$, as there is no edge connecting X and $I(B) \setminus N(X)$. So, using Theorem 38.5,

(38.15)
$$|B| - |X| = |B \setminus X| \geq |N(I(B) \setminus N(X))| - 1 \geq |I(B) \setminus N(X)| + 1$$
$$= |B| - |N(X) \cap I(B)|,$$

implying $|N(X) \cap I(B)| = |X|$ and $v_1 \notin N(X)$. Define $B' := B \setminus X$. Then B' is a simple barrier of $G - e$ again, with $I(B') = I(B) \setminus N(X)$ and $K(B') = K(B) \cup N(X) \cup X$.

Let S be the union of X, $N(X) \cap I(B)$, and the contracted vertex U of G/U. Then each perfect matching of G/U has exactly one edge leaving S (as X is matched to $(I(B) \cap N(X)) \cup \{U\}$ in G/U, since $X \cap N(L) = \emptyset$). So S determines a tight cut in G/U. As $G/\overline{S}$ is bipartite, it suffices to show that the brick decomposition of $G/U/S$ contains exactly one brick.

Since $X \cap N(L) = \emptyset$, L is a union of components of $G[K(B')] - v_2$. Then

(38.16) $G/U/S = G/(K(B') \setminus L) \cup \{v_2\}.$

Hence, by the minimality of B, G/U has a brick decomposition with exactly one brick. This shows (38.14).

We finally derive from (38.14) that G/U has only one brick in its brick decomposition, in fact, that it *is* a brick — equivalently that $G - U$ is 2-connected and factor-critical (Theorem 38.3).

Assume that $G-U$ is not 2-connected, and let v be a cut vertex of $G-U$. Then each component of $G - U - v$ intersects $B \cup I(B)$ as G is 3-connected. Hence v is a cut vertex of $G[B \cup I(B)]$. So Theorem 38.7(vi) applies. In particular, $v \in I(B)$.

Since each component of $G[L]$ is adjacent to at least two vertices in B (since G is 3-connected), we know by Theorem 38.7(vii) that all vertices in B adjacent to L belong to the same component of $G[B \cup I(B)] - v$ as v_1. Any other component consists of one vertex, w say, in B. But then this contradicts (38.14), taking $X = \{w\}$. So $G - U$ is 2-connected.

To show that $G - U$ is factor-critical, suppose to the contrary that there exists a nonempty subset Y of $\overline{U}$ such that $G - U - Y$ has at least $|Y| + 1$ odd components.

Then $Y \subseteq I(B)$. Otherwise choose $v \in Y \setminus I(B)$. So $v \in L \cup B$. Then $G - U - v$ has no perfect matching. However, as G is bicritical, $G - v - v_2$ has a perfect matching M. Then the restriction of M to $\overline{U}$ is a perfect matching of $G - U - v$, a contradiction. So $Y \subseteq I(B)$.

Each component of $G-U-Y$ containing a component of L has at least two elements in B (since G is 3-connected). So $G[B \cup I(B)] - Y$ has precisely $|Y|+1$ components. Hence it has $|Y|$ singleton components in B, without neighbours in L (by Theorem 38.7(vii)). Let X be the union of these components. Each neighbour y of any $x \in X$ with $y \notin U$ belongs to Y. So $|X| \geq |Y| \geq |N(X) \cap I(B)|$, contradicting (38.14). ∎

We next consider *pairs* of simple barriers B_1, B_2. The following auxiliary theorem is of special interest for disjoint simple barriers B_1 and B_2 of $G - e$ where B_2 intersects $I(B_1)$.

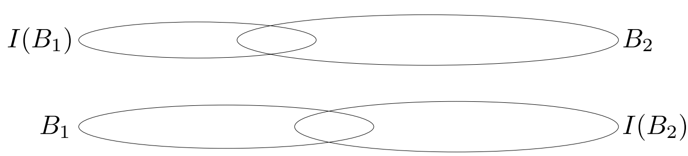


Figure 38.2

Theorem 38.9. *Let $G = (V, E)$ be a brick and let $e = v_1v_2 \in E$ be such that $G - e$ is matching-covered. Let B_1 and B_2 be disjoint simple barriers of $G - e$ with $v_1 \in I(B_1)$ and $v_2 \in I(B_2)$. Then*

(38.17) (i) $I(B_1) \cap I(B_2) = \emptyset$;
(ii) $B_1 \cup I(B_2)$ *and* $B_2 \cup I(B_1)$ *are stable sets;*
(iii) $|B_1 \cap I(B_2)| = |B_2 \cap I(B_1)|$;
(iv) $B_2 \setminus I(B_1)$ *is again a simple barrier of* $G - e$, *with* $I(B_2 \setminus I(B_1)) = I(B_2) \setminus B_1$.

Proof. (i) follows from the fact that all neighbours of any $u \in I(B_1) \cap I(B_2)$ belong to $B_1 \cap B_2 = \emptyset$. Since $N(I(B_2)) \subseteq B_2 \cup \{v_1\}$, which is disjoint from B_1, we have that $B_1 \cup I(B_2)$ is a stable set. Similarly, $B_2 \cup I(B_1)$ is a stable set, implying (ii).

Since $I(B_2) \setminus B_1 \subseteq I(B_2 \setminus I(B_1))$, we know

(38.18)
$$\begin{aligned}&|I(B_2)| - |I(B_2) \cap B_1| = |I(B_2) \setminus B_1| \leq |I(B_2 \setminus I(B_1))|\\ &\leq |B_2 \setminus I(B_1)| - 1 = |B_2| - 1 - |B_2 \cap I(B_1)|\\ &= |I(B_2)| - |B_2 \cap I(B_1)|.\end{aligned}$$

So $|B_2 \cap I(B_1)| \leq |B_1 \cap I(B_2)|$, and hence by symmetry $|B_2 \cap I(B_1)| = |B_1 \cap I(B_2)|$, and we have equality throughout in (38.18). This gives (iii) and (iv). ∎

The last auxiliary theorem in this section reads:

Theorem 38.10. *Let G be a brick and let $e = v_1v_2$ be an edge of G with $G-e$ matching-covered. Let B_1 and B_2 be simple barriers of $G-e$, and define $J_i := B_i \cup I(B_i)$ for $i = 1, 2$, with $v_1 \in J_1$ and $v_2 \in J_2$, and $X := V \setminus (J_1 \cup J_2)$. Assume that $J_1 \cap J_2 = \emptyset$, and that, for each $u \in X$, $G-e-u$ is factor-critical and $G-u/J_1$ and $G-u/J_2$ are 2-connected. Then if $G-e$ has a 2-vertex-cut separating J_1 and J_2, it has a 2-vertex-cut $\{u, u'\}$ separating J_1 and J_2 such that for some component K of $G-e-u-u'$, both $G/(K \cup \{u\})$ and $G/\overline{K \cup \{u\}}$ are bricks.*[21]

Proof. Note that if $\{u, u'\}$ is $J_1 - J_2$ separating in $G-e$ (which by definition implies that $u, u' \notin J_1 \cup J_2$), then $G-e-u-u'$ has a perfect matching (by the assumption in the theorem). Moreover, since G is 3-connected, $G-u-u'$ is connected. Hence $G-e-u-u'$ has exactly two components, one containing J_1 and one containing J_2. We will apply Theorem 38.3.

We first show:

(38.19) Let $\{u, u'\}$ be $J_1 - J_2$ separating in $G-e$ and let K be a component of $G-e-u-u'$. Then the graph $G[K \cup \{u\}]$ is factor-critical.

By symmetry, we may assume that $J_1 \subseteq K$. Define $S := K \cup \{u\}$. Choose a vertex $v \in S$. We prove that $G[S] - v$ has a perfect matching. If $v = u$, then $G[S] - v = G[K]$ has a perfect matching (as K is a component of $G-e-u-u'$). So let $v \neq u$. As $G-e-u'$ is factor-critical by the assumption

[21] It is important to note that it is not concluded that also $G/(K \cup \{u'\})$ and $G/\overline{K \cup \{u'\}}$ are bricks.

in the theorem, $G - e - u' - v$ has a perfect matching M. Since $|K|$ is even, the edge in M incident with u, connects u with K. So M contains a matching spanning $S \setminus \{v\}$. This proves (38.19).

In order to prove that $G[K \cup \{u\}]$ is 2-connected, we need a special kind of 2-vertex-cut and a special order of the components:

(38.20) there exist a pair u, u' separating J_1 and J_2 in $G-e$ and components $K \supseteq J_1$ and $L \supseteq J_2$ of $G - e - u - u'$ such that for each $v \in K \setminus J_1$, $\{u', v\}$ does not separate J_1 and J_2 in $G-e$ and for each $v \in L \setminus J_2$, $\{u, v\}$ does not separate J_1 and J_2 in $G - e$.

To prove this, let $\{u, u'\}$ be a 2-vertex-cut separating J_1 and J_2 in $G-e$. Let K and L be the components of $G-e-u-u'$ containing J_1 and J_2, respectively. We choose u and u' such that L is minimal. Then by the minimality of L, for each $v \in L \setminus J_2$, neither $\{u, v\}$ nor $\{u', v\}$ separates J_1 and J_2 in $G - e$.

If (38.20) does not hold, then there exist $v, v' \in K \setminus J_2$ such that $\{u, v\}$ and $\{u', v'\}$ are vertex-cuts in $G-e$, each separating J_1 and J_2. Let Y be the component of $G-e-u-v$ not containing u'. Since $N_{G-e}(Y) \subseteq \{u, v\}$, we know that $v_1 \in Y$ and hence $J_1 \subseteq Y$. Let Y' be the component of $G-e-u'-v'$ not containing u. Again $J_1 \subseteq Y'$. Hence $J_1 \subseteq Y \cap Y'$. Now $N_{G-e}(Y \cap Y') \subseteq \{v, v'\}$ (since $N_{G-e}(Y \cap Y') \subseteq N_{G-e}(Y) \cup N_{G-e}(Y') \subseteq \{u, u', v, v'\}$; but u' is not a neighbour of $Y \cap Y'$ since u' is not in component Y of $G-e-u-v$; similarly for u). This implies $v \neq v'$. Hence $v' \in Y$.

Let A be the component of $G - e - u - v$ different from Y, and let A' be the component of $G - e - u' - v'$ different from Y'. Then $Y' \cap A = \emptyset$. Indeed, $N(Y' \cap A) \subseteq N(Y') \cup N(A) \subseteq \{u, v, u', v'\}$. Moreover, $u, v' \notin N(Y' \cap A)$, since $u \in A'$ and $v' \in Y$. So $|N(Y' \cap A)| \leq 2$, implying $Y' \cap A = \emptyset$ by the 3-connectivity of G.

Similarly, $K \cap A \cap A' = \emptyset$. Indeed, $N(K \cap A \cap A') \subseteq N(K) \cup N(A) \cup N(A') = \{u, v, u', v'\}$. Moreover, $v, v' \notin N(K \cap A \cap A')$, since $v \in Y'$ and $v' \in Y$. So $|N(K \cap A \cap A')| \leq 2$, implying $K \cap A \cap A' = \emptyset$.

So $K \cap A$ intersects neither Y' nor A', hence $K \cap A \subseteq \{u', v'\}$. However, $u' \notin K$ and $v' \notin A$. So $K \cap A = \emptyset$. Hence $K \subseteq Y \cup \{v\}$. So $Y = K \setminus \{v\}$, implying that $|Y|$ is odd, a contradiction (since $G - e - u - v$ has a perfect matching). This proves (38.20).

Let u, u' be as in (38.20). By symmetry, it suffices to show:

(38.21) $G[K \cup \{u\}]$ is 2-connected.

Let $S := K \cup \{u\}$. Suppose that there exists a $v \in S$ with $G[S \setminus \{v\}]$ disconnected. Let Z be a component of $G[S \setminus \{v\}]$ not containing v_1, and let Y be any other component. If $u \notin Z$, then $N(Z) \subseteq \{v, u'\}$, contradicting the 3-connectivity of G. So $u \in Z$.

So $u \notin Y$, and hence $N(Y) \subseteq \{u', v, v_2\}$, implying by the 3-connectivity of G, that $v_1 \in Y$. So $N_{G-e}(Y) = \{u', v\}$. If $v \notin J_1$, then $J_1 \subseteq Y$ (as $G[J_1]$ is connected), implying that $\{u', v\}$ is $J_1 - J_2$ separating in $G-e$, contradicting the condition in (38.20). So $v \in J_1$.

If $Y \not\subseteq J_1$, then $Y \setminus J_1$ has only two neighbours in G/J_1: J_1 and u', contradicting the fact that $G - u'/J_1$ is 2-connected (by the condition in the theorem). So $Y \subseteq J_1$.

Let M be a perfect matching in $G - u' - v_1$. So M intersects $\delta(J_1)$ in exactly two edges (since $|I(B_1) \setminus \{v_1\}| = |B_1| - 2$ and $u' \in K(B_1)$, as $u' \not\in J_1$). If $G[J_1 \setminus \{v\}]$ is connected, then $Y = J_1 \setminus \{v\}$. Then M contains an edge leaving J_1 and not incident with v. This contradicts the fact that $N_{G-e}(Y) \subseteq \{u', v\}$ and that M does not cover u'.

So v is a cut vertex of $G[J_1]$, and hence by Theorem 38.7(vi), v belongs to $I(B_1)$. Now $v \neq v_1$, since $v_1 \in Y$. By Theorem 38.7(vii), $G[J_1 \setminus \{v\}]$ has two components, one containing v_1 and one consisting only of some neighbour, w say, of v. So $Z \cap J_1 = \{w\}$ and $|(Y \setminus \{v_1\}) \cup \{v\}|$ is even. Then M contains a matching with union $(Y \setminus \{v_1\}) \cup \{v\}$. Hence at most one edge in M leaves J_1, a contradiction. This shows (38.21). ∎

38.6. The perfect matching lattice of a brick

We now prove the theorem of Lovász [1987]:

Theorem 38.11. *Let $G = (V, E)$ be a brick different from the Petersen graph. Then the perfect matching lattice of G is equal to the set of integer vectors in the perfect matching space of G.*

Proof. We choose a counterexample with $|V| + |E|$ minimal. Let x be an integer vector in the perfect matching space of G that is not in the perfect matching lattice of G. We can assume that $x(\delta(v)) = 0$ for each vertex v (this can be achieved by adding an appropriate integer multiple of χ^M to x, for some perfect matching M in G).

Claim 1. Let $\delta(U)$ be an odd cut in G such that both G/U and $G/\overline{U}$ are matching-covered and have exactly one brick in their brick decompositions. Then there exist no perfect matchings M and N of G with $|M \cap \delta(U)| - |N \cap \delta(U)| = 2$.

Proof of Claim 1. Suppose to the contrary that such perfect matchings M, N exist. In particular, $|U|, |\overline{U}| \geq 3$. As $x(\delta(U))$ is even (since $x(\delta(v))$ is even for each vertex v), by adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x(\delta(U)) = 0$.

Let x' and x'' be the projections of x to the edges of $G/\overline{U}$ and G/U, respectively. Let $H := G/\overline{U}$.

Consider any minimal subset W of U, such that $|W| \geq 3$ and such that $\delta(W)$ is a tight cut of H. (Such a set exists, since $\delta(U)$ is tight in H.) Since H has exactly one brick in its brick decomposition, we know that $H/\overline{W}$ or H/W is bipartite and matching-covered. If $H/\overline{W}$ is bipartite and matching-covered, the colour class of $H/\overline{W}$ not containing vertex $\overline{W}$ would be a nontrivial barrier in G. This contradicts the fact that G is a brick.

So H/W is bipartite and matching-covered. Hence the projection of x' to the edges of H/W belongs to the perfect matching lattice of H/W. So (by Theorem 38.6) the projection y of x' to the edges of $I := H/\overline{W}$ is not in the perfect matching lattice of I.

By the minimality of W, I is a brick. Since y is not in the perfect matching lattice of I, by the minimality of $|V| + |E|$, I is the Petersen graph and has a 5-circuit disjoint from vertex $\overline{W}$ of I with $y(EC)$ odd.

As $\delta(W)$ is not tight in G (since G is a brick), G has a perfect matching L satisfying $|L \cap \delta(W)| \geq 3$, and hence $|L \cap \delta(W)| = 3$ (since I is the Petersen graph). Then by Theorem 38.2 (defining $b(\overline{W}) := 3$ and $b(v) := 1$ for each vertex $v \neq \overline{W}$ of I), we can modify L on the edges of I not incident with $\overline{W}$ to obtain a perfect matching L' of G such that the intersections of L and L' with EC have different parities. Resetting $x := x + \chi^L - \chi^{L'}$ we achieve that $x(EC)$, and hence $x'(EC)$, is even.

Hence the projection of the new x on the edges of $G/\overline{U}$ is in the perfect matching lattice of $G/\overline{U}$. We can perform similar resettings to achieve that the projection of the new x on the edges of G/U is in the perfect matching lattice of G/U. Then the new x, and hence also the original x, belongs to the perfect matching lattice of G, by Theorem 38.6. This contradicts our assumption. *End of Proof of Claim 1*

There exists an edge e with $G - e$ matching-covered

To see this, we first show:

Claim 2. There are no edges e and f such that $G - e - f$ is matching-covered and bipartite.

Proof of Claim 2. Suppose that such e and f exist. As $G - e - f$ is matching-covered, the colour classes of $G - e - f$ have the same size, and as G is matching-covered and nonbipartite, e is spanned by one of the colour classes, and f by the other.

Let M be a perfect matching in G containing e and f and let N be a perfect matching in G not containing e and f. By adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x_e = 0$. Since x is in the perfect matching space of G, this implies that $x_f = 0$. By Corollary 20.12a, the restriction of x to $G - e - f$ is in the perfect matching lattice of $G - e - f$. Hence x belongs to the perfect matching lattice of G, contradicting our assumption. *End of Proof of Claim 2*

This gives:

Claim 3. There is an edge e such that $G - e$ is matching-covered.

Proof of Claim 3. For each edge e, let $\mathcal{M}_e$ denote the collection of perfect matchings of G containing e. Choose any edge e with $\mathcal{M}_e$ inclusionwise minimal. We prove that $G-e$ is matching-covered.

Suppose that $G-e$ is not matching-covered. Hence there is an edge $f \neq e$ such that each perfect matching of G containing f, also contains e; that is, $\mathcal{M}_f \subseteq \mathcal{M}_e$. By the minimality of $\mathcal{M}_e$, $\mathcal{M}_f = \mathcal{M}_e$. Hence there is no perfect matching containing exactly one of e, f. We show that

(38.22) $\quad G - e - f$ is bipartite.

As there is no perfect matching containing e but not containing f, by Tutte's 1-factor theorem, there exists a subset B of V spanning e such that $G-f-B$ has more than $|B|-2$ odd components; hence, by parity, at least $|B|$ odd components. As $|B| \geq 2$ and as G is bicritical, f connects two distinct odd components, K_1 and K_2 say, of $G-f-B$. Moreover, as G is bicritical, each component of $G-f-B$ is odd.

We show that $G-e-f$ is bipartite with colour classes B and $W := V \setminus B$. That is, e is the only edge contained in B, and each component of $G-f-B$ is a singleton.

To see this, first assume that some component K of $G-f-B$ is not a singleton. Then $\delta(K)$ is a nontrivial cut, and hence it is not tight. So there exists a perfect matching M with $|M \cap \delta(K)| \geq 3$. If $f \notin M$, then (adding up over all components of $G-f-B$), $|M \cap \delta(B)| \geq |B|+2$, a contradiction. If $f \in M$, then similarly $|M \cap \delta(B)| \geq |B|$, again a contradiction (since $e \in M$).

Second assume that B spans some edge e' different from e. Let M be a perfect matching containing e'. If $f \notin M$, then $|M \cap \delta(B)| \geq |B|$, contradicting the fact that $e' \in M$. If $f \in M$, then $|M \cap \delta(B)| \geq |B|-2$, contradicting the fact that both e and e' belong to M. This shows (38.22).

In particular, any odd circuit in G contains exactly one of e and f. By Claim 2, $G-e-f$ is not matching-covered. Hence there is an edge g such that each perfect matching containing g contains e or f. Hence $\mathcal{M}_g = \mathcal{M}_e = \mathcal{M}_f$. So, as before, each of $G-e-f$, $G-e-g$, $G-f-g$ is bipartite. Hence each odd circuit in G contains exactly one edge from each pair taken from e, f, g, a contradiction. *End of Proof of Claim 3*

Each maximal barrier of $G - e$ is simple

We fix an edge e with $G-e$ matching-covered. Let e connect vertices v_1 and v_2.

Claim 4. Let B be a maximal barrier of $G-e$. Then B is simple and $G/\overline{K(B)}$ is a brick.

Proof of Claim 4. As the claim is trivial if $|B| = 1$, we can assume $|B| \geq 2$; that is, B is nontrivial. Since G has no nontrivial barrier, B is not a barrier of G, and hence e connects two different components of $G-e-B$.

By Theorem 38.4, each component K of $G - e - B$ is factor-critical. So it suffices to show (by Theorem 38.3) that $G[K(B)]$ is 2-connected. In other words, $G - e - B$ has precisely one block[22].

Let $\mathcal{K}$ denote the collection of components of $G-e-B$, and let $\mathcal{L}$ denote the collection of blocks of $G - e - B$. For $K \in \mathcal{K}$, let $\mathcal{L}_K$ denote the set of blocks of $G[K]$.

It is useful to state the following formulas (38.23) and (38.25). For any perfect matching M of G and any $K \in \mathcal{K}$ one has

$$\text{(38.23)} \qquad \sum_{L\in\mathcal{L}_K} (|M \cap \delta(L)| - 1) = |M \cap \delta(K)| - 1.$$

This can be shown inductively as follows. Consider any subsets U' and U'' of a set U of vertices with $U' \cup U'' = U$, $|U' \cap U''| = 1$, and no edge connecting $U' \setminus U''$ and $U'' \setminus U'$. Then $|M\cap\delta(U)|-1 = (|M\cap\delta(U')|-1)+(|M\cap\delta(U'')|-1)$, since

$$\text{(38.24)} \qquad \begin{aligned}&|M \cap \delta(U')| + |M \cap \delta(U'')| = |M \cap \delta(U' \cup U'')| + |M \cap \delta(U' \cap U'')| \\ &= |M \cap \delta(U)| + 1.\end{aligned}$$

One also has

$$\text{(38.25)} \qquad \sum_{K\in\mathcal{K}} (|M \cap \delta(K)| - 1) = 2|M \cap \{e\}|,$$

since

$$\text{(38.26)} \qquad \begin{aligned}&\sum_{K\in\mathcal{K}} |M \cap \delta(K)| = |M \cap \delta(B)| + 2|M \cap \{e\}| = |B| + 2|M \cap \{e\}| \\ &= |\mathcal{K}| + 2|M \cap \{e\}|.\end{aligned}$$

Suppose now that the claim is not true — that is, $|\mathcal{L}| \geq 2$. We derive:

(38.27) for each $L \in \mathcal{L}$ and for each edge $f \in \delta(L)$, G has a perfect matching M with $M \cap \delta(L) = \{f\}$.

Indeed, if $f \neq e$, let M be a perfect matching of $G-e$ containing f. By (38.23) and (38.25), M intersects $\delta(L)$ in exactly one edge. So $M \cap \delta(L) = \{f\}$.

Suppose next that $f = e$. As $|\mathcal{L}| \geq 2$ by assumption, there exists a block $L' \neq L$. As G has no tight nontrivial cuts, G has a perfect matching M with $|M \cap \delta(L')| \geq 3$, and hence by (38.23) and (38.25), $|M \cap \delta(L)| = 1$, that is, $M \cap \delta(L) = \{e\}$. This proves (38.27).

Now for each $L \in \mathcal{L}$ there exists a perfect matching M with $|M\cap\delta(L)| \geq 3$, and hence, by (38.23) and (38.25), $|M \cap \delta(L)| = 3$ and $|M \cap \delta(L')| = 1$ for all other $L' \in \mathcal{L}$. Moreover, let N be a perfect matching not containing e. Then adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x(\delta(L)) = 0$, while $x(\delta(L'))$ does not change for any other $L' \in \mathcal{L}$.

As we can do this for all $L \in \mathcal{L}$, we can assume that

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[22] A *block* of a graph H is an inclusionwise maximal set L of vertices with $|L| \geq 2$ and with $G[L]$ 2-connected.

(38.28) $x(\delta(L)) = 0$ for all $L \in \mathcal{L}$.

Since x is in the perfect matching space, with (38.23) this gives that

(38.29) $x(\delta(K)) = 0$ for all $K \in \mathcal{K}$.

Moreover, $x_e = 0$, since

(38.30) $$2x_e = \sum_{K\in\mathcal{K}} x(\delta(K)) - x(\delta(B)) = -x(\delta(B)) = -\sum_{v\in B} x(\delta(v)) = 0.$$

Let H be the matching-covered bipartite graph obtained from $G - e$ by contracting each $K \in \mathcal{K}$ to a vertex. Since $x(\delta(K)) = 0$ for each $K \in \mathcal{K}$ and $x(\delta(v)) = 0$ for each $v \in B$, and since $x_e = 0$, we know from Corollary 20.12a that $x|EH$ is in the perfect matching lattice of H. Now for each $K \in \mathcal{K}$ and for each $f \in \delta(K)$ with $f \neq e$, there exists a matching M in $G-e$ containing f, and hence there is a matching with union $K \setminus \{v\}$, where v is the vertex in K incident with f. We therefore can extend each perfect matching of H to a perfect matching of $G-e$ intersecting each $\delta(K)$ in one edge. This implies that we may assume that $x_f = 0$ for each $f \in \delta(B)$.

Hence each edge f with $x_f \neq 0$ is spanned by some $L \in \mathcal{L}$. Let $\mathcal{L}'$ be the collection of those blocks $L \in \mathcal{L}$ spanning at least one edge f with $x_f \neq 0$. We choose x satisfying all previous assumptions and such that $|\mathcal{L}'|$ is as small as possible.

As each $K \in \mathcal{K}$ is factor-critical, each $L \in \mathcal{L}$ is factor-critical. Hence, by Theorem 38.3,

(38.31) $G/\overline{L}$ is a brick for each $L \in \mathcal{L}$.

Moreover,

(38.32) we can assume that, for each $L \in \mathcal{L}$ with $G/\overline{L}$ the Petersen graph, there is a 5-circuit C in $G[L]$ with $x(EC)$ even.

Indeed, choose any 5-circuit C in $G[L]$, and suppose that $x(EC)$ is odd. Let M be a perfect matching in G with $|M \cap \delta(L)| = 3$. By Theorem 38.2, we can modify M on the edges spanned by L so as to obtain a perfect matching N with $|N \cap EC|$ having parity different from $|M \cap EC|$, and such that M and N coincide for all edges not spanned by L. Now adding $\chi^M - \chi^N$ to x makes $x(EC)$ even, and does not invalidate our previous assumptions. This shows (38.32).

We show next:

(38.33) for each $\mathcal{L}_0 \subseteq \mathcal{L}$ with $x_f = 0$ for each $f \in \delta(\bigcup \mathcal{L}_0)$, one has $\mathcal{L}_0 \subseteq \mathcal{L}'$.

We show this by induction on $|\mathcal{L}_0|$. If $\mathcal{L}_0 = \emptyset$, this is trivial. If $\mathcal{L}_0 \neq \emptyset$, we can choose an $L \in \mathcal{L}_0$ such that L has a vertex v such that each $L' \in \mathcal{L}_0$ with $L' \neq L$ is disjoint from $L \setminus \{v\}$. Hence each $f \in \delta(L)$ with $x_f \neq 0$ is incident with v. By (38.31) and (38.32), $x|E(G/\overline{L})$ is in the perfect matching lattice of $G/\overline{L}$. So

$$(38.34)\qquad x|E(G/\overline{L})=\sum_M \lambda_M\chi^M,$$

where M ranges over perfect matchings of $G/\overline{L}$ and where $\lambda_M\in\mathbb{Z}$. Let $\mathcal{M}$ denote the collection of perfect matchings of $G/\overline{L}$ not containing an edge leaving L at v. So if $f\in M\in\mathcal{M}$ and f is incident with $\overline{L}$, then $x_f=0$. By (38.27), for each $f\in\delta(L)$ we can choose a perfect matching N_f of G/L containing f. Then for each perfect matching M of $G/\overline{L}$, let $\widetilde{M}:=M\cup N_f$ where f is the edge of M leaving L. Then, by replacing x by

$$(38.35)\qquad x-\sum_{M\in\mathcal{M}}\lambda_M\chi^{\widetilde{M}},$$

x changes only on edges spanned by L, and we achieve that $x_f=0$ for each edge $f\in\delta(v)$ spanned by L. Hence for $\mathcal{L}_0':=\mathcal{L}_0\setminus\{L\}$ we have $x_f=0$ for each $f\in\delta(\bigcup\mathcal{L}_0')$. Therefore, by the induction hypothesis, $\mathcal{L}_0'\subseteq\mathcal{L}'$. So $x_f=0$ for each $f\in\delta(L)$. Hence, taking the λ_M as above, by replacing x by

$$(38.36)\qquad x-\sum_M\lambda_M\chi^{\widetilde{M}},$$

where M ranges over all perfect matchings of $G/\overline{L}$, we achieve that $x|E(G/\overline{L})=\mathbf{0}$. This proves (38.33).

Applying (38.33) to $\mathcal{L}_0:=\mathcal{L}$, we derive that $x=\mathbf{0}$, a contradiction.

End of Proof of Claim 4

We remind that for each maximal nontrivial barrier B of $G-e$ one has $e\in\delta(K(B))$ and:

(38.37) for each perfect matching M of G: $e\in M \iff |M\cap\delta(K(B))|=3$.

Pairs of simple barriers of $G-e$

Claim 5. Let B_1 and B_2 be simple barriers of $G-e$ and let $J_i:=B_i\cup I(B_i)$ (for $i=1,2$), with $J_1\cap J_2=\emptyset$ and $v_i\in J_i$ (for $i=1,2$). Then $H:=G-e/J_1/J_2$ is not a brick.

Proof of Claim 5. Suppose that H is a brick. By adding an appropriate integer multiple of $\chi^M-\chi^N$ to x, where M and N are perfect matchings in G containing e and not containing e, respectively, we can achieve $x_e=0$. Then, since $x(\delta(v))=0$ for each vertex v, we have that $x(\delta(J_1))=x(\delta(J_2))=0$. As $G-e/\overline{J_1}$ and $G-e/\overline{J_2}$ are bipartite and as H is a brick, it follows that the respective projections of x belong to the perfect matching space of $G-e/\overline{J_1}$, $G-e/\overline{J_2}$, and H.

As x is not in the perfect matching lattice of G, by Theorem 38.6 at least one of these projections is not in the corresponding perfect matching lattice. As $G-e/\overline{J_1}$ and $G-e/\overline{J_2}$ are bipartite, it follows (as G is a minimal counterexample to Theorem 38.11) that H is the Petersen graph and that $x(EC)$ is odd for some 5-circuit C in H disjoint from vertices J_1 and J_2 of H. Then it suffices to show:

(38.38) G has perfect matchings M and N, each containing e, such that M and N intersect EC in different parities,

since then adding $\chi^M - \chi^N$ to x turns the parity of $x(EC)$.

To prove (38.38), let

(38.39) $X := VG \setminus (J_1 \cup J_2) = K(B_1) \cap K(B_2)$.

So $VH = X \cup \{J_1, J_2\}$. We first show that for $i = 1, 2$:

(38.40) if $|J_i| \geq 3$, and a and b are distinct neighbours of vertex J_i of H with $a, b \in X$, then $\{aJ_i, bJ_i\}$ is the image of a matching in G.

To see this, we can assume that $i = 1$.

If J_1 and J_2 are adjacent vertices of H, then a and b are the only neighbours of J_1 in X. Choose $z \in B_2$. As G is bicritical, $G - v_1 - z$ has a perfect matching M. Then M matches up all vertices in $J_2 \setminus \{z\}$. Moreover, all but two vertices in B_1 are matched with vertices in $I(B_1) \setminus \{v_1\}$. Hence two edges of M connect B_1 and K. So M contains edges connecting a and b with B_1.

If J_1 and J_2 are nonadjacent vertices of H, let z be the vertex distinct from a, b adjacent in H to J_1. Since G is bicritical, $G - v_1 - z$ has a perfect matching M. All but two vertices in B_1 are matched with vertices in $I(B_1) \setminus \{v_1\}$. Since M misses z, M contains edges connecting a and b with B_1. This shows (38.40).

Moreover, we have:

(38.41) if J_1 and J_2 are adjacent vertices in H, and $|J_1| \geq 3$ and $|J_2| \geq 3$, then J_1 has a neighbour a_1 in X, and J_2 has a neighbour a_2 in X, such that $\{a_1J_1, J_1J_2, J_2a_2\}$ is the image of a matching in G.

Let f be an edge of $G-e$ connecting J_1 and J_2. By (38.40), J_1 has a neighbour a_1 in X such that there exists an edge connecting a_1 and J_1 disjoint from f. Similarly, J_2 has a neighbour a_2 in X such that there exists an edge connecting a_2 and J_2 disjoint from f. This gives the a_1 and a_2 required in (38.41).

By Theorem 38.2, we can find subsets F_1 and F_2 of the edge set of H such that for each $j = 1, 2$,

(38.42)
(i) each vertex in X is incident with exactly one edge in F_j,
(ii) for each $i = 1, 2$, if $|J_i| = 1$, then J_i is incident with none of the edges in F_j, and, if $|J_i| \geq 3$, then J_i is incident with exactly two edges in F_j,
(iii) $|F_1 \cap EC|$ and $|F_2 \cap EC|$ have different parities.

(Note that if $|J_1| = |J_2| = 1$, then J_1 and J_2 are not adjacent, as then $J_1 = \{v_1\}$ and $J_2 = \{v_2\}$, $e = v_1v_2$, and $H = G - e$.)

If J_1 and J_2 are adjacent vertices of H and $|J_1| \geq 3$, $|J_2| \geq 3$, we can choose the F_j such that moreover

(38.43) a_1J_1, a_2J_2 belong to both F_1 and F_2,

where a_1 and a_2 are as in (38.41). To see this, note that a_1 and a_2 are nonadjacent (as the Petersen graph has no 4-circuit). Then there exist by Theorem 38.2 subsets F_1' and F_2' of the edge set of H such that for each $j = 1, 2$, each vertex of H different from a_1 and a_2 is incident with exactly one edge in F_j', while a_1 and a_2 are not covered by F_j', and such that $|F_1' \cap EC|$ and $|F_2' \cap EC|$ have different parities. Extending the F_j' with the edges a_1J_1 and a_2J_2 gives F_j as required.

By Theorem 38.7(iii), (38.40) and (38.41), F_1 and F_2 are projections of perfect matchings M and N of G containing e, as required in (38.38).

End of Proof of Claim 5

This claim can be sharpened as follows:

Claim 6. Let B_1 and B_2 be simple barriers of $G-e$ and let $J_i := B_i \cup I(B_i)$ (for $i = 1, 2$), with $J_1 \cap J_2 = \emptyset$ and $v_i \in J_i$ (for $i = 1, 2$). Define $X := V \setminus (J_1 \cup J_2)$. If $G-e-u$ is factor-critical for each $u \in X$ and $H := G-e/J_1/J_2$ is bicritical, then $G/J_1/J_2$ has a 2-vertex-cut intersecting $\{J_1, J_2\}$.

Proof of Claim 6. If $G-u/J_1$ is not 2-connected for some $u \in X$, then $\{u, J_1\}$ is a 2-vertex-cut in G/J_1 (since G is 3-connected), hence in $G/J_1/J_2$, as required. So we may assume that $G-u/J_1$ and $G-u/J_2$ are 2-connected for each $u \in X$.

Let H be bicritical. By Claim 5, H is not a brick. Hence H is not 3-connected. Let $\{u, u'\}$ be a 2-vertex-cut of H. If $\{u, u'\}$ intersects $\{J_1, J_2\}$ we are done. So suppose that $\{u, u'\}$ is disjoint from $\{J_1, J_2\}$. Since G is 3-connected and e connects J_1 and J_2, we know that $\{u, u'\}$ separates J_1 and J_2. Hence, by Theorem 38.10, we may assume that the components K and L of $G-e-u-u'$ are such that $G/(K \cup \{u\})$ and $G/\overline{K \cup \{u\}}$ are bricks.

Define $U := K \cup \{u\}$. Then G has a perfect matching M with $|M \cap \delta(U)| \geq 3$, since G has no nontrivial tight cuts. As each edge in $\delta(U) \setminus \{e\}$ is incident with u or u', we know $|M \cap \delta(U)| = 3$. Let $f \in \delta(U) \setminus \{e\}$ and let N be a perfect matching in $G-e$ containing f. Then $|N \cap \delta(U)| = 1$, contradicting Claim 1.

End of Proof of Claim 6

$G-e$ has exactly two maximal nontrivial barriers

By Corollary 24.11a, we know:

(38.44) any two distinct maximal barriers of $G-e$ are disjoint.

Since each maximal nontrivial barrier B contains $N(v_1)\setminus\{v_2\}$ or $N(v_2)\setminus\{v_1\}$ (as e connects $I(B)$ and $K(B)$), we know that $G-e$ has at most two maximal nontrivial barriers. In fact:

Claim 7. $G-e$ has exactly two maximal nontrivial barriers B_1 and B_2.

Proof of Claim 7. First assume that $G-e$ has no nontrivial barriers; that is, $G-e$ is bicritical. This contradicts Claim 6 for $B_1 := \{v_1\}$ and $B_2 := \{v_2\}$. ($G-e-u$ is factor-critical for each $u \in V$ by (38.3).) So $G-e$ has at least one maximal nontrivial barrier, B_1 say. Let $J_1 := B_1 \cup I(B_1)$, and assume without loss of generality that $v_1 \in I(B_1)$.

Assume that there is exactly one maximal nontrivial barrier. Then $G-e/J_1$ has no nontrivial barrier; that is, it is bicritical. By Claim 4, G/J_1 is a brick, and hence is 3-connected. This contradicts Claim 6, taking $B_2 := \{v_2\}$. ($G-e-u$ is factor-critical for each $u \in V \setminus J_1$ by (38.3).)

End of Proof of Claim 7

Decomposition of G

Having the two maximal nontrivial barriers B_1 and B_2, assuming $v_1 \in I(B_1)$ and $v_2 \in I(B_2)$, we define

(38.45) $\quad J_1 := B_1 \cup I(B_1)$ and $J_2 := B_2 \cup I(B_2)$.

Note that J_1 and J_2 might intersect. Define $J_1' := J_1 \setminus J_2$, $J_2' := J_2 \setminus J_1$, $B_1' := B_1 \setminus I(B_2)$, and $B_2' := B_2' \setminus I(B_1)$. By Theorem 38.9, B_1' and B_2' are simple barriers again, with $I(B_1') = I(B_1) \setminus B_2$ and $I(B_2') = I(B_2) \setminus B_1$.

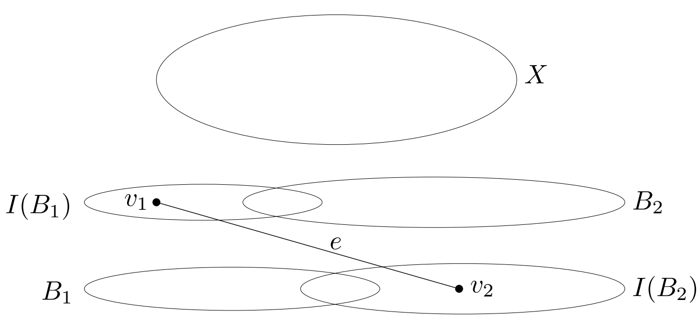

Figure 38.3

Thus we obtain a decomposition of V into

(38.46) $\quad B_1'$, B_2', $I(B_1')$, $I(B_2')$, $B_1 \cap I(B_2)$, $B_2 \cap I(B_1)$,
$\quad X := K(B_1) \cap K(B_2)$,

where e connects $I(B_1')$ and $I(B_2')$.

By Theorem 38.9, $G-X$ is bipartite, with colour classes $B_1 \cup I(B_2)$ and $B_2 \cup I(B_1)$.

$G[X]$ has exactly two components

Claim 8. $G[X]$ is disconnected.

Proof of Claim 8. Consider $H := G - e/J_1/J_2'$. Note that H is isomorphic to $G - e/J_1'/J_2$, since, if $J_1 \cap J_2 \neq \emptyset$, then $J_1 \cap J_2$ has neighbours both in J_1' and J_2', and nowhere else (by Theorems 38.5 and 38.9).

By Claim 5, H is not a brick. However,

(38.47) H is bicritical.

To see this, choose two distinct vertices v, v' of H. We can assume that $v \neq J_2'$ and $v' \neq J_1$. (If $v = J_2'$ or $v' = J_1$ then exchange v and v'.) Let w be equal to v if $v \neq J_1$ and let w be any vertex in B_1 if $v = J_1$. Similarly, let w' be equal to v' if $v' \neq J_2'$ and let w' be any vertex in B_2' if $v' = J_2'$. Then $G - e - w - w'$ has a perfect matching, since $\{w, w'\}$ is neither contained in B_1 nor in B_2. As B_1 is a simple barrier in $G-e$, each vertex in $I(B_1)$ is matched to a vertex in B_1. Similarly, each vertex in $I(B_2')$ is matched to a vertex in B_2'. Hence this perfect matching gives a perfect matching of $H - v - v'$. This proves (38.47).

By Claim 6, $G/J_1/J_2'$ has a 2-vertex-cut $\{u, u'\}$ intersecting $\{J_1, J_2'\}$. ($G - e - u$ is factor-critical for each $u \in X$ by (38.3).) If $\{u, u'\} = \{J_1, J_2'\}$ we are done. So we can assume that $u' \notin \{J_1, J_2'\}$. If $u = J_1$, then u' is a cut vertex of $G - J_1$, contradicting Claim 4. If $u = J_2'$, observe that $G/J_1/J_2'$ is isomorphic to $G/J_1'/J_2$, where the isomorphism brings vertex J_1 to vertex J_1', and vertex J_2' to vertex J_2. So u' is a cut vertex of $G - J_2$, again contradicting Claim 4.

End of Proof of Claim 8

We have that

(38.48) each component of $G[X]$ is even,

as for any $u \in B_1'$, $G[K(B_2)] - u$ has a perfect matching M. Then trivially no edge in M connects $K(B_2)$ and J_2. Moreover, no edge in M connects $K(B_1)$ and J_1, since $e \notin M$ (as e is not contained in $K(B_2)$) and since each vertex in $I(B_1')$ is matched to a vertex in $B_1' \setminus \{u\}$ (note that $J_1' \subseteq K(B_2)$).

For any subset L of X, any perfect matching M of G, and any $i \in \{1, 2\}$, define

(38.49) $\lambda_i(M, L) :=$ the number of edges in M connecting L and B_i.

Claim 9. For any component L of $G[X]$ and any perfect matching M of G containing e one has $\{\lambda_1(M, L), \lambda_2(M, L)\} = \{0, 2\}$.

Proof of Claim 9. Since $\lambda_1(M,L)+\lambda_2(M,L)=|M\cap\delta(L)|$ is even (as $|L|$ is even by (38.48)) and since $\lambda_i(M,L)\leq 2$ for $i=1,2$ (since M has two edges connecting $K(B_i)$ and B_i), it suffices to show that $\lambda_1(M,L)\neq\lambda_2(M,L)$.

Suppose that $\lambda_1(M,L)=\lambda_2(M,L)$. Since $e\in M$, $|M\cap\delta(J_1')|=3$. As no edge connects L and $I(B_i)$ (since e is the only edge connecting $K(B_i)$ and $I(B_i)$, but $v_1,v_2\not\in L$), we have that M has $\lambda_1(M,L)$ edges connecting L and J_1'. Hence for $U:=J_1'\cup L$,

(38.50) $$\begin{aligned}|M\cap\delta(U)| &= |M\cap\delta(J_1')|+|M\cap\delta(L)|-2\lambda_1(M,L)\\ &= 3+\lambda_1(M,L)+\lambda_2(M,L)-2\lambda_1(M,L)=3.\end{aligned}$$

Moreover, any perfect matching N of $G-e$ satisfies $|N\cap\delta(U)|=1$. Indeed, $|N\cap\delta(J_1')|=1$ and $|N\cap\delta(J_2)|=1$. So $|N\cap\delta(X)|\leq 2$. Hence if $|N\cap\delta(U)|\geq 3$, then $N\cap\delta(U)$ contains an edge leaving neither J_1' nor J_2. Hence N has an edge connecting L and $X\setminus L$, a contradiction. So $|N\cap\delta(U)|=1$.

We show that both $G/\overline{U}$ and G/U are matching-covered, AND THat each has a unique brick in its brick decomposition, contradicting Claim 1.

Consider $G':=G/J_2$. Then G' is a brick by Claim 4, and L is a nonempty union of components of $G'-J_1'-\{J_2\}$. Moreover, $G'-e$ is matching-covered (since each perfect matching of $G-e$ has exactly one edge in $\delta(J_2)$) and B_1' is a simple barrier of $G'-e$. So by Theorem 38.8 (taking $Z:=X\setminus L$ and $v_2=J_2$), $G'/\overline{U}=G/\overline{U}$ is matching-covered and has a unique brick in its brick decomposition.

Let $U':=J_2'\cup(X\setminus L)$. Similarly, $G/\overline{U'}$ is matching-covered and has a unique brick in its brick decomposition. Since $\overline{U'}=U\cup(J_1\cap J_2)$, we have $\overline{U\cup U'}=J_1\cap J_2$. So $G/U/U'$ is matching-covered and bipartite. As U' gives a tight cut in G/U, also G/U is matching-covered and has a unique brick in its brick decomposition. *End of Proof of Claim 9*

Claim 10. $G[X]$ has exactly two components.

Proof of Claim 10. Let M be any perfect matching of G containing e. Then $\lambda_i(M,X)\leq 2$ for $i=1,2$, and hence by Claim 9, $G[X]$ has exactly two components. *End of Proof of Claim 10*

Conclusion

Let L_1 and L_2 be the components of $G[X]$. For $j=1,2$, let Z_j be the set of pairs $\{b,b'\}$ with $b\in B_1$, $b'\in B_2'$ such that $L_j\cup\{b,b'\}$ is matchable. In particular, if $b\in B_1$ and $b'\in B_2'$ are adjacent, then $\{b,b'\}\in Z_1\cap Z_2$. Then

Claim 11. For each $j=1,2$, any $b\in N(L_j)$ belongs to some pair in Z_j.

Proof of Claim 11. As $b\in N(L_j)$, there is an edge f joining b and L_j. Let M be a perfect matching of $G-e$ containing f. Then $\lambda_1(M,L_j)=\lambda_2(M,L_j)=1$, and hence $\{b,b'\}\in Z_j$ for some b'. *End of Proof of Claim 11*

Note that if $b \in N(L_j)$ for some j, then $b \in B_1' \cup B_2'$ (since X has no neighbour in $I(B_1) \cup I(B_2)$).

Claim 12. Each pair in Z_1 intersects each pair in Z_2.

Proof of Claim 12. Suppose to the contrary that there exist disjoint pairs $\{b, b'\} \in Z_1$ and $\{c, c'\} \in Z_2$, taking $b, c \in B_1$ and $b', c' \in B_2'$. By definition of Z_j, $L_1 \cup \{b, b'\}$ and $L_2 \cup \{c, c'\}$ are matchable. Moreover, by Theorem 38.7, also $J_1 \setminus \{b, c, v_1\}$ and $J_2' \setminus \{b', c', v_2\}$ are matchable. Together with e, this gives a perfect matching M of G containing e with $\lambda_1(M, L_1) \leq 1$ and $\lambda_2(M, L_1) \leq 1$. This contradicts Claim 9. *End of Proof of Claim 12*

Claim 13. $Z_1 \cap Z_2 = \emptyset$, $|B_1| = |B_2| = 2$, $I(B_1) \cap B_2 = I(B_2) \cap B_1 = \emptyset$, $B_1 \cup B_2$ is a stable set, and Z_1 and Z_2 are perfect matchings on $B_1 \cup B_2$.

Proof of Claim 13. We have $|N(L_j) \cap B_i| \geq 2$ for $j = 1, 2$ and $i = 1, 2$, since (for $j = 1$, $i = 1$, say) L_1 has at least two neighbours in $K(B_2)$ (as $G[K(B_2)]$ is 2-connected), which must belong to B_1.

Assume that $Z_1 \cap Z_2 \neq \emptyset$. Let $\{c, c'\} \in Z_1 \cap Z_2$ with $c \in B_1$ and $c' \in B_2'$. We can choose $b \in N(L_1) \cap B_1$ with $b \neq c$. Then $\{b, c'\} \in Z_1$ (by Claims 11 and 12). We can choose $b' \in N(L_2) \cap B_2'$ with $b' \neq c'$. Again, $\{b', c\} \in Z_2$. As $\{b, c'\}$ and $\{b', c\}$ are disjoint, this contradicts Claim 12. So $Z_1 \cap Z_2 = \emptyset$.

Then $B_1 \cup B_2'$ is a stable set, since if there is an edge connecting $b \in B_1$ and $b' \in B_2'$, then $L_1 \cup \{b, b'\}$ and $L_2 \cup \{b, b'\}$ are matchable, and hence $\{b, b'\} \in Z_1 \cap Z_2$, a contradiction.

This implies $B_1 \cap I(B_2) = \emptyset$, since otherwise there is an edge connecting $b \in B_1 \cap I(B_2)$ and $b' \in B_2' = B_2 \setminus I(B_1)$ (since $B_1 \cap I(B_2)$ has more than $|B_1 \cap I(B_2)| = |B_2 \cap I(B_1)|$ neighbours in B_2, by Theorem 38.5). Hence, by (38.17)(iii), $B_2 \cap I(B_1) = \emptyset$. So $B_2' = B_2$.

Next, for each $j = 1, 2$, no two pairs in Z_j intersect. For assume that $\{b, b'\}$, $\{b, c'\}$ belong to Z_1 with b', c' different vertices in B_2. As $|N(L_2) \cap B_1| \geq 2$, we can choose (by Claim 11) $\{d, d'\} \in Z_2$, with $d \in B_1$ and $d \neq b$. However, then $d' = b'$ and $d' = c'$ by Claim 12, a contradiction, as $b' \neq c'$.

So Z_j consists of disjoint pairs. As each pair in Z_1 intersects each pair in Z_2, we have that each Z_j consists of two disjoint pairs, that Z_1 and Z_2 cover the same set of vertices, and that $Z_1 \cap Z_2 = \emptyset$. In particular,

(38.51) $$|N(X) \cap B_1| = |N(X) \cap B_2| = 2.$$

Finally we show that $|B_i| = 2$ for $i = 1, 2$. Suppose that (say) $|B_1| \geq 3$. Then $|I(B_1)| \geq 2$. Choose $v \in I(B_1) \setminus \{v_1\}$. As G is bicritical, $G - v - v_1$ has a perfect matching M. Necessarily, at least three edges of M connect B_1 and $K(B_1)$, hence (as $B_1 \cup B_2$ is stable) M has at least three edges connecting X and B_1. So $|N(X) \cap B_1| \geq 3$, contradicting (38.51).

End of Proof of Claim 13

This claim in particular implies that

(38.52) $\quad v_1$ and v_2 have degree 3

(since all neighbours of v_1 belong to $B_1 \cup \{v_2\}$). We can set

(38.53) $$B_1 = \{b_1, b'_1\},\ B_2 = \{b_2, b'_2\},$$
$$Z_1 = \{\{b_1, b'_2\}, \{b'_1, b_2\}\},\ Z_2 = \{\{b_1, b_2\}, \{b'_1, b'_2\}\}.$$

Claim 14. $L_j \cup B_i$ is matchable for all $i, j \in \{1, 2\}$.

Proof of Claim 14. We may assume $i = 2$, $j = 1$. Let M and N be matchings spanning $L_1 \cup \{b'_1, b_2\}$ and $L_1 \cup \{b_1, b'_2\}$, respectively. The path P in $M \cup N$ starting at b'_1 ends at b'_2, as if P would end at b_1, then $L_1 \cup \{b_1, b_2\}$ is matchable (while $\{b_1, b_2\} \notin Z_1$), and if it would end at b_2, then $L_1 \cup \{b_1, b'_1, b_2, b'_2\}$ is matchable, implying that G has a perfect matching M' containing e with $\lambda_1(M', L_1) = \lambda_2(M', L_1) = 2$, contradicting Claim 9. So $M \triangle EP$ is a perfect matching on $L_1 \cup \{b_2, b'_2\}$. *End of Proof of Claim 14*

Claim 15. $G - e'$ is matching-covered for each edge e' of G.

Proof of Claim 15. Since G is connected and e is chosen arbitrarily under the condition that $G - e$ is matching-covered, we can assume that e' is incident with e. In particular, we can assume that e' connects v_1 and b_1. Suppose that $G - e'$ is not matching-covered. Then there exists an edge $f \neq e'$ such that each perfect matching of G containing f also contains e'. So f is disjoint from e'.

First assume that f is incident with v_2. We may assume that f connects v_2 with vertex b_2. By definition of Z_1, $L_1 \cup \{b_1, b'_2\}$ is matchable. Since also L_2 is matchable, we can find a perfect matching of G containing f but not e', contradicting our assumption.

So we may assume that f is incident with L_1. Let M' be a perfect matching of G containing f. If M' does not intersect $\delta(L_1)$, we can extend $M'[L_1] \cup \{v_1 b'_1, v_2 b'_2\}$ by a matching spanning $L_2 \cup \{b_1, b_2\}$ to obtain a perfect matching containing f but not e', a contradiction. So M' intersects $\delta(L_1)$. Hence, necessarily, it contains an edge joining L_1 with b'_1 (as $e' \in M'$). So also it contains an edge joining L_1 and b_2. Therefore, M' contains a matching M spanning $L_1 \cup \{b'_1, b_2\}$. Let N be a matching spanning $L_1 \cup \{b_1, b'_2\}$.

Like in Claim 14, the path P in $M \cup N$ starting at b'_1 ends at b'_2. Similarly, the path Q in $M \cup N$ starting at b_2 ends at b_1. At least one of $M \triangle EP$ and $M \triangle EQ$ contains f (since f is in M and on at most one of P, Q). As $L_2 \cup \{b_1, b'_1\}$ and $L_2 \cup \{b_2, b'_2\}$ are matchable (by Claim 14), there is a perfect matching containing f and not e', a contradiction. *End of Proof of Claim 15*

This gives with (38.52) that

(38.54) $\quad G$ is 3-regular,

since by Claim 15 we can take for e any edge of G.

Claim 16. $|L_1| = |L_2| = 2$.

Proof of Claim 16. Since G is 3-regular, each $b \in B_1 \cup B_2$ has a unique neighbour in L_j, for each $j = 1, 2$. In fact, for any $j = 1, 2$,

(38.55) if $b \in B_1$, $b' \in B_2$, and $\{b, b'\} \notin Z_j$, then the neighbours of b and b' in L_j coincide.

For assume that the neighbour c of b in L_j differs from the neighbour c' of b' in L_j. As G is bicritical, $G - c - c'$ has a perfect matching M. Let M' be the set of edges in M intersecting L_j. As $|L_j|$ is even, M' spans either $L_j - c - c'$ or $(L_j - c - c') \cup (B_1 - b) \cup (B_2 - b')$. Extending M' with the edges bc and $b'c'$, we obtain a matching spanning $L_j \cup \{b, b'\}$, contradicting $\{b, b'\} \notin Z_j$, or spanning $L_j \cup B_1 \cup B_2$, contradicting Claim 9. This shows (38.55).

Now (38.55) implies that $N(B_1) \cap L_1 = N(B_2) \cap L_1$. As this set is not a 2-vertex-cut of G, we have $|L_1| = 2$. Similarly, $|L_2| = 2$.

End of Proof of Claim 16

So both L_1 and L_2 consist of a single edge. Therefore, G is the Petersen graph, contradicting our assumption. ∎

38.7. Synthesis and further consequences of the previous results

The previous results imply a characterization of the matching lattice for matching-covered graphs (Lovász [1987]):

Corollary 38.11a. *Let $G = (V, E)$ be a matching-covered graph and let $x \in \mathbb{Z}^E$. Then x belongs to the perfect matching lattice of G if and only if for some maximal cross-free collection $\mathcal{F}$ of nontrivial tight cuts:*

(38.56) (i) *$x(D) = x(\delta(v))$ for each $D \in \mathcal{F}$ and each $v \in V$;*
(ii) *for every Petersen brick resulting from the given tight cut decomposition, and for some 5-circuit C in that brick, the sum of the x_e over edges e mapping to EC, is even.*

Proof. Directly from Theorems 38.6, 38.1, and 38.11. ∎

Corollary 38.11a implies the following (conjectured by Lovász [1985]):

Corollary 38.11b. *Let $G = (V, E)$ be a matching-covered graph and let $x \in 2\mathbb{Z}^E$ be such that $x(C) = x(C')$ for any two tight cuts C and C'. Then x belongs to the perfect matching lattice of G.*

Proof. Directly from Corollary 38.11a. ∎

Moreover, there is the following corollary for regular graphs (recall that a *k-graph* is a k-regular graph with $|C| \geq k$ for each odd cut):

Corollary 38.11c. *Let $G = (V, E)$ be a k-graph. Then the all-2 vector $\mathbf{2}$ belongs to the perfect matching lattice of G. If G has no subgraph homeomorphic to the Petersen graph, then the all-1 vector belongs to the perfect matching lattice of G.*

Proof. Directly from Corollary 38.11a. ∎

A special case is the following result of Seymour [1979a], which also follows from the conjecture of Tutte [1966], proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000], that each bridgeless cubic graph without Petersen graph minor, is 3-edge-colourable.

Corollary 38.11d. *Let $G = (V, E)$ be a bridgeless cubic graph without Petersen graph minor. Then the all-1 vector $\mathbf{1}$ belongs to the perfect matching lattice of G.*

Proof. This is a special case of Corollary 38.11c. ∎

Similarly, the following consequence, a theorem of Seymour [1979a], supports a positive answer to the question of Fulkerson [1971a] whether each cubic graph G satisfies $\chi'(G_2) = 6$:

Corollary 38.11e. *Let $G = (V, E)$ be a bridgeless cubic graph. Then the all-2 vector $\mathbf{2}$ in $\mathbb{R}^E$ belongs to the perfect matching lattice of G.*

Proof. Again, this is a special case of Corollary 38.11c. ∎

38.8. What further might (not) be true

The conjecture that the perfect matchings in any graph would constitute a Hilbert base, is too bold: Let G be the graph obtained from the Petersen graph by adding one additional edge (connecting nonadjacent vertices of the Petersen graph). Let $x_e := 1$ if e is an edge of the Petersen graph, and $x_e := 0$ if e is the new edge. Then x belongs to the perfect matching cone[23] and to the perfect matching lattice (since G is a brick). However, x is not a nonnegative integer combination of perfect matchings, since the Petersen graph is not 3-edge-colourable. (This example was given by Goddyn [1993].)

Two weaker conjectures might yet hold true. The first one is due to L. Lovász (cf. Goddyn [1993]):

[23] The *perfect matching cone* is the cone generated by the incidence vectors of the perfect matchings.

(38.57) (?) for any graph without Petersen graph minor, the incidence vectors of the perfect matchings form a Hilbert base. (?)

The second one was given in Section 28.6 above ((28.28)), and is due to Seymour [1979a] (the *generalized Fulkerson conjecture*):

(38.58) (?) each k-graph contains $2k$ perfect matchings, covering each edge exactly twice. (?)

(A *k-graph* is a k-regular graph $G = (V, E)$ with $d_G(U) \geq k$ for each odd $U \subseteq V$.) For $k = 3$, (38.58) was asked by Fulkerson [1971a]:

(38.59) (?) each bridgeless cubic graph has 6 perfect matchings covering each edge precisely twice. (?)

What has been proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000] is:

(38.60) each bridgeless cubic graph without Petersen graph minor is 3-edge-colourable.

This is a special case of conjecture (38.57), and of the 4-*flow conjecture* of Tutte [1966]:

(38.61) (?) each bridgeless graph without Petersen graph minor has three cycles covering each edge precisely twice. (?)

(A *cycle* is an edge-disjoint union of circuits.) Related is the following theorem of Alspach, Goddyn, and Zhang [1994]:

(38.62) the circuits of a graph G form a Hilbert base $\iff$ G has no Petersen graph minor.

It implies that the *circuit double cover conjecture* (asked by Szekeres [1973], conjectured by Seymour [1979b]):

(38.63) (?) each bridgeless graph has a family of circuits covering each edge precisely twice, (?)

is true for graphs without Petersen graph minor:

(38.64) each bridgeless graph without Petersen graph minor has a family of circuits covering each edge precisely twice.

(For cubic graphs this was shown by Alspach and Zhang [1993].) This is also a special case of the 4-flow conjecture (38.61).

Seymour [1979b] conjectures that

(38.65) (?) each *even* integer vector x in the circuit cone is a nonnegative integer combination of incidence vectors of circuits. (?)

This is more general than the circuit double cover conjecture.

Bermond, Jackson, and Jaeger [1983] have proved that

(38.66) each bridgeless graph has a family of circuits covering each edge precisely four times.

Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

(38.67) (?) in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice. (?)

Finally, the 5-*flow conjecture* of Tutte [1954a]:

(38.68) (?) each bridgeless graph has a nowhere-zero 5-flow, (?)

can be formulated in terms of circuits as follows (by Theorem 28.4):

(38.69) (?) each bridgeless graph can be oriented such that there exist directed circuits, covering each edge at least once and at most four times. (?)

Seymour [1981b] showed that each bridgeless graph has a nowhere-zero 6-flow; equivalently:

(38.70) each bridgeless graph can be oriented such that there exist directed circuits, covering each edge at least once and at most five times.

It improves an earlier result of Jaeger [1976,1979] that each bridgeless graph has a nowhere-zero 8-flow. This is equivalent to: each bridgeless graph contains three cycles covering all edges.

Notes. More on nowhere-zero flows and circuit covers can be found in Itai, Lipton, Papadimitriou, and Rodeh [1981], Bermond, Jackson, and Jaeger [1983], Bouchet [1983], Steinberg [1984], Alon and Tarsi [1985], Fraisse [1985], Jaeger, Khelladi, and Mollard [1985], Tarsi [1986], Khelladi [1987], Möller, Carstens, and Brinkmann [1988], Catlin [1989], Goddyn [1989], Jamshy and Tarsi [1989,1992], Fan [1990,1993, 1995,1998], Jackson [1990], Zhang [1990,1993c], Raspaud [1991], Alspach and Zhang [1993], Fan and Raspaud [1994], Huck and Kochol [1995], Lai [1995], Steffen [1996], and Galluccio and Goddyn [2002]. Surveys were given by Jaeger [1979,1985,1988], Zhang [1993a,1993b], and Seymour [1995a], and a book was devoted to it by Zhang [1997b]. The extension to matroids is discussed in Section 81.10.

38.9. Further results and notes

38.9a. The perfect 2-matching space and lattice

Let $G = (V, E)$ be a graph. The *perfect* 2-*matching space* of G is the linear hull of the perfect 2-matchings in G. This space is easily characterized with the help of Corollary 30.2b:

Theorem 38.12. *The perfect 2-matching space of G consists of all vectors $x \in \mathbb{R}^E$ such that $x_e = 0$ if e is not in the support of any perfect 2-matching and such that $x(\delta(v)) = x(\delta(u))$ for all $u, v \in V$.*

Proof. Clearly each vector x in the perfect 2-matching space satisfies the condition. To see the reverse, let x satisfy the condition. By adding appropriate multiples of perfect 2-matchings, we can assume that $x \geq \mathbf{0}$. If $x = \mathbf{0}$ we are done, so we can assume $x \neq \mathbf{0}$. Then, by scaling, we can assume that $x(\delta(v)) = 2$ for each vertex v. Hence, by Corollary 30.2b, x belongs to the perfect 2-matching polytope of F, and therefore to the perfect 2-matching space. ∎

The *perfect 2-matching lattice* of G is the lattice generated by the perfect 2-matchings in G. Jungnickel and Leclerc [1989] showed that a characterization of the perfect 2-matching lattice can be easily derived from the theorem of Petersen that the edges of any $2k$-regular graph can be decomposed into k 2-factors (Corollary 30.7b):

Theorem 38.13. *The perfect 2-matching lattice of G consists of all integer vectors x in the perfect 2-matching space of G with $x(\delta(v))$ even for one (hence for each) vertex v.*

Proof. Trivially, each vector x in the perfect 2-matching lattice satisfies the condition. To see the reverse, let x satisfy the condition. By adding integer multiples of perfect 2-matchings, we can assume that $x \geq \mathbf{0}$. Replace each edge e by x_e parallel edges, yielding graph G', of degree $2k$ for some integer $k > 0$. Now by Corollary 30.7b, the edges of G' can be partitioned into k 2-factors. This gives a decomposition of x as a sum of k perfect 2-matchings in G. ∎

38.9b. Further notes

De Carvalho, Lucchesi, and Murty [2002a,2002b] showed that each brick G different from K_4, the prism $\overline{C_6}$, and the Petersen graph, has an edge e such that $G - e$ is a matching-covered graph with precisely one brick in its brick decomposition (conjectured by L. Lovász in 1987). Having this, the proof of Theorem 38.11 can be shortened considerably (de Carvalho, Lucchesi, and Murty [2002c]). (Earlier related work was done by de Carvalho and Lucchesi [1996].)

Naddef and Pulleyblank [1982] study the relation between ear-decompositions and the GF(2)-rank of the incidence vectors of the perfect matchings.

Kilakos [1996] characterized the lattice generated by the matchings M that have a positive coefficient in at least one fractional $\chi'^*(G)$-edge-colouring (these matchings form a face of the matching polytope of G).

Part IV

Matroids and Submodular Functions

Part IV: Matroids and Submodular Functions

Matroids form an important tool in combinatorial optimization. Among other, they apply to shortest and disjoint trees in undirected graphs, to bipartite matching, and to directed cut covering.
Matroids were introduced by Whitney in 1935, and equivalent axiom systems were considered in the 1930s by Nakasawa, Birkhoff, and van der Waerden. They were motivated by questions from algebra, geometry, and graph theory. The importance of matroids for combinatorial optimization was revealed by J. Edmonds in the 1960s, who found efficient algorithms and min-max relations for optimization problems involving matroids.
Matroids are exactly those structures where the greedy algorithm yields an optimum solution. Edmonds discovered that matroids have an even stronger algorithmic property: also optimization over intersections of two different matroids can be done efficiently. It is closely related to matroid union. Among the consequences of matroid intersection and union methods and results are min-max relations, polyhedral characterizations, and algorithms for bipartite matching, common transversals, and tree packing and covering. (In fact, tree packing and covering are best investigated within the structures offered by matroids. This insight was obtained already in the original paper of Nash-Williams on tree packing. That is why we discuss matroids before Part V on trees and forests.)
While bipartite matching is generalized by matroid intersection, nonbipartite matching is generalized by *matroid matching*. We prove in Chapter 43 Lovász's matroid matching theorem for linear matroids. For general matroids the problem is intractable.
The rank function of a matroid is a special case of a submodular function. Submodular functions give rise to a polyhedral generalization of matroids, the *polymatroids*. Most of matroid theory can be lifted to the level of submodular functions and polymatroids. Next to having applications by its own, it will also be used in Part V where we consider submodular functions defined on digraphs (Chapter 60). This applies to directed variants of tree and cut packing and covering, and to graph orientation and connectivity augmentation.

Chapters:

Chapter 39

Matroids

This chapter gives the basic definitions, examples, and properties of matroids. We use the shorthand notation

$$X + y := X \cup \{y\} \text{ and } X - y := X \setminus \{y\}.$$

39.1. Matroids

A pair $(S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of S satisfying:

(39.1) (i) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,
(ii) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + z \in \mathcal{I}$ for some $z \in J \setminus I$.

(These axioms are given by Whitney [1935].)

Given a matroid $M = (S, \mathcal{I})$, a subset I of S is called *independent* if I belongs to $\mathcal{I}$, and *dependent* otherwise. For $U \subseteq S$, a subset B of U is called a *base* of U if B is an inclusionwise maximal independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

It is not difficult to see that, under condition (39.1)(i), condition (39.1)(ii) is equivalent to:

(39.2) for any subset U of S, any two bases of U have the same size.

The common size of the bases of a subset U of S is called the *rank* of U, denoted by $r_M(U)$. If the matroid is clear from the context, we write $r(U)$ for $r_M(U)$.

A set is called simply a *base* if it is a base of S. The common size of all bases is called the *rank* of the matroid. A subset of S is called *spanning* if it contains a base as a subset. So bases are just the inclusionwise minimal spanning sets, and also just the independent spanning sets. A *circuit* of a matroid is an inclusionwise minimal dependent set. A *loop* is an element s such that $\{s\}$ is a circuit. Two elements s, t of S are called *parallel* if $\{s, t\}$ is a circuit.

Nakasawa [1935] showed the equivalence of axiom system (39.1) with an ostensibly weaker system, which will be useful in proofs:

Theorem 39.1. *Let S be a finite set and let $\mathcal{I}$ be a nonempty collection of subsets satisfying* (39.1)(i)*. Then* (39.1)(ii) *is equivalent to:*

(39.3) *if $I, J \in \mathcal{I}$ and $|I \setminus J| = 1$, $|J \setminus I| = 2$, then $I + z \in \mathcal{I}$ for some $z \in J \setminus I$.*

Proof. Obviously, (39.1)(ii) implies (39.3). Conversely, (39.1)(ii) follows from (39.3) by induction on $|I \setminus J|$, the case $|I \setminus J| = 0$ being trivial. If $|I \setminus J| \geq 1$, choose $i \in I \setminus J$. We apply the induction hypothesis twice: first to $I - i$ and J to find $j \in J \setminus I$ with $I - i + j \in \mathcal{I}$, and then to $I - i + j$ and J to find $j' \in J \setminus (I + j)$ with $I - i + j + j' \in \mathcal{I}$. Then by (39.3) applied to I and $I - i + j + j'$, we have that $I + j \in \mathcal{I}$ or $I + j' \in \mathcal{I}$. ∎

39.2. The dual matroid

With each matroid M, a dual matroid M^* can be associated, in such a way that $(M^*)^* = M$. Let $M = (S, \mathcal{I})$ be a matroid, and define

(39.4) $\mathcal{I}^* := \{I \subseteq S \mid S \setminus I \text{ is a spanning set of } M\}$.

Then (Whitney [1935]):

Theorem 39.2. *$M^* = (S, \mathcal{I}^*)$ is a matroid.*

Proof. Condition (39.1)(i) trivially holds for $\mathcal{I}^*$. To see (39.1)(ii), consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. By definition of $\mathcal{I}^*$, $S \setminus J$ contains some base B of M. As also $S \setminus I$ contains some base of M, and as $B \setminus I \subseteq S \setminus I$, there exists a base B' of M with $B \setminus I \subseteq B' \subseteq S \setminus I$. Then $J \setminus I \not\subseteq B'$, since otherwise (as $B \cap I \subseteq I \setminus J$, and as $B \setminus I$ and $J \setminus I$ are disjoint, since $B \cap J = \emptyset$)

(39.5) $|B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| < |J \setminus I| + |B \setminus I| \leq |B'|$,

which is a contradiction. As $J \setminus I \not\subseteq B'$, there is a $z \in J \setminus I$ with $z \notin B'$. So B' is disjoint from $I + z$. Hence $I + z \in \mathcal{I}^*$. ∎

The matroid M^* is called the *dual matroid* of M. The bases of M^* are precisely the complements of the bases of M. This implies $(M^*)^* = M$, which justifies the name dual.

Theorem 39.3. *The rank function r_{M^*} of the dual matroid M^* satisfies, for $U \subseteq S$:*

(39.6) $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$.

Proof. Let $\mathcal{B}$ and $\mathcal{B}^*$ denote the collections of bases of M and of M^*, respectively. Then

$$(39.7) \qquad \begin{aligned} r_{M^*}(U) &= \max\{|U \cap A| \bigm| A \in \mathcal{B}^*\} = \max\{|U \setminus B| \bigm| B \in \mathcal{B}\} \\ &= |U| - \min\{|B \cap U| \bigm| B \in \mathcal{B}\} \\ &= |U| - r_M(S) + \max\{|B \setminus U| \bigm| B \in \mathcal{B}\} \\ &= |U| - r_M(S) + r_M(S \setminus U). \end{aligned}$$

∎

The circuits of M^* are called the *cocircuits* of M. They are the inclusionwise minimal sets intersecting each base of M (as they are the inclusionwise minimal sets contained in no base of M^*, that is, not contained in the complement of any base of M). The loops of M^* are the *coloops* or *bridges* of M, and parallel elements of M^* are called *coparallel* or *in series* in M.

Let $M = (S, \mathcal{I})$ be a matroid, and suppose that we can test in polynomial time if any subset of S is independent in M (or we have an oracle for that). Then we can calculate, for any subset U of S, the rank $r_M(U)$ of U in polynomial time (by growing an independent set (starting from $\emptyset$) to an inclusionwise maximal independent subset of U). It follows that we can test in polynomial time if any subset U of S in independent in M^*, just by testing if $r_M(S \setminus U) = r_M(S)$.

A matroid $M = (S, \mathcal{I})$ is called *connected* if $r_M(U) + r_M(S \setminus U) > r_M(S)$ for each nonempty proper subset U of S. This is equivalent to: for any two elements $s, t \in S$ there exists a circuit containing both s and t. One may derive from (39.6) that a matroid M is connected if and only if M^* is connected.

39.3. Deletion, contraction, and truncation

We can derive matroids from matroids by 'deletion' and 'contraction'. Let $M = (S, \mathcal{I})$ be a matroid and let $Y \subseteq S$. Define

$$(39.8) \qquad \mathcal{I}' := \{Z \mid Z \subseteq Y, Z \in \mathcal{I}\}.$$

Then $M' = (Y, \mathcal{I}')$ is a matroid again, as directly follows from the matroid axioms (39.1). M' is called the *restriction* of M to Y, denoted by $M|Y$. If $Y = S \setminus Z$ with $Z \subseteq S$, we say that M' arises by *deleting* Z, and denote M' by $M \setminus Z$. Clearly, the rank function of $M|Y$ is the restriction of the rank function of M to subsets of Y.

Contraction is the operation dual to deletion. *Contracting* Z means replacing M by $(M^* \setminus Z)^*$. This matroid is denoted by M/Z. If $Y = S \setminus Z$, then we denote $M \cdot Y := M/Z$. Theorem 39.3 implies that the rank function r' of M/Z satisfies

$$(39.9) \qquad r_{M/Z}(X) = r(X \cup Z) - r(Z)$$

for $X \subseteq S \setminus Z$.

We can describe contraction as follows. Let $Z \subseteq S$ and let X be a base of Z. Then

(39.10) a subset I of $S \setminus Z$ is independent in M/Z if and only if $I \cup X$ is independent in M.

Note that for disjoint subsets Y, Z of S one has $(M \setminus Y) \setminus Z = M \setminus (Y \cup Z)$ and hence $(M/Y)/Z = M/(Y \cup Z)$. Moreover, deletion and contraction commute, as for any two distinct $x, y \in S$ and any $Z \subseteq S \setminus \{x, y\}$ one has (using (39.9)):

$$\begin{aligned}(39.11) \qquad r_{M\setminus x/y}(Z) &= r_{M\setminus x}(Z\cup\{y\}) - r_{M\setminus x}(\{y\}) = r_M(Z\cup\{y\}) - r_M(\{y\}) \\ &= r_{M/y}(Z) = r_{M/y\setminus x}(Z).\end{aligned}$$

If matroid M' arises from M by a series of deletions and contractions, M' is called a *minor* of M.

The circuits of $M|Y$ are exactly the circuits of M contained in Y, and the circuits of $M \cdot Y$ are exactly the minimal nonempty sets $C \cap Y$, where C is a circuit of M.

Another operation is that of 'truncation'. Let $M = (S, \mathcal{I})$ be a matroid and let k be a natural number. Define $\mathcal{I}' := \{I \in \mathcal{I} \mid |I| \leq k\}$. Then $(S, \mathcal{I}')$ is again a matroid, called the *k-truncation* of M.

39.4. Examples of matroids

We describe some basic classes of matroids.

Uniform matroids. An easy class of matroids is given by the *uniform matroids*. They are determined by a set S and a number k: the independent sets are the subsets I of S with $|I| \leq k$. This trivially gives a matroid, called a *k-uniform matroid* and denoted by U_n^k, where $n := |S|$.

Linear matroids (Grassmann [1862], Steinitz [1913]). Let A be an $m \times n$ matrix. Let $S := \{1, \ldots, n\}$ and let $\mathcal{I}$ be the collection of all those subsets I of S such that the columns of A with index in I are linearly independent. That is, such that the submatrix of A consisting of the columns with index in I has rank $|I|$.

Then $(S, \mathcal{I})$ is a matroid (property (39.1)(ii) was proved by Grassmann [1862] and by Steinitz [1913], and is called *Steinitz' exchange property*). Condition (39.1)(i) is trivial. To see condition (39.1)(ii), let $I, J \in \mathcal{I}$ with $|I| < |J|$. Then I spans an $|I|$-dimensional space $\overline{I}$. So $J \not\subseteq \overline{I}$. Take $j \in J \setminus \overline{I}$. Then $I + j \in \mathcal{I}$ and $j \in J \setminus I$.

Any matroid obtained in this way, or isomorphic to such a matroid, is called a *linear matroid*. If A has entries in a field $\mathbb{F}$, then M is called *representable over* $\mathbb{F}$. We will also say that M is *represented by* (the columns of) A, and A is called a *representation* of M.

Note that the rank $r_M(U)$ of any subset U of S is equal to the rank of the matrix formed by the columns indexed by U.

The dual matroid of a matroid representable over a field $\mathbb{F}$ is again representable over $\mathbb{F}$. Indeed, we can assume that the matrix A is of the form $[I_m \;\; B]$, where I_m is the $m \times m$ identity matrix, and B is an $m \times (n - m)$

matrix. Then the dual matroid can be represented by the matrix $[B^\mathsf{T} \;\; I_{n-m}]$, as follows directly from elementary linear algebra. This implies that the class of matroids representable over $\mathbb{F}$ is closed under taking minors.

MacLane [1936] (and also Lazarson [1958]) showed that nonlinear matroids exist.

Binary matroids. A matroid representable over GF(2) — the field with two elements — is called a *binary matroid*. For later purposes, we give some characterizations of binary matroids. The following is direct (Whitney [1935]):

(39.12) a matroid M is binary if and only if for each choice of circuits $C_1, \ldots, C_t$, the set $C_1 \triangle \cdots \triangle C_t$ can be partitioned into circuits.

In a binary matroid M, disjoint unions of circuits are called the *cycles* of M. Of special interest is the *Fano matroid* F_7, represented by the nonzero vectors in $\mathrm{GF}(2)^3$.

Tutte [1958a,1958b] showed that the unique minor-minimal nonbinary matroid is U_4^2, the 2-uniform matroid on 4 elements. (We follow the proof suggested by A.M.H. Gerards.)

Theorem 39.4. *A matroid is binary if and only if it has no U_4^2 minor.*

Proof. Necessity follows from the facts that the class of binary matroids is closed under taking minors and that U_4^2 is not binary.

To see sufficiency, we first show the following. Let M and N be matroids on the same set S. Call a set *wrong* if it is a base of precisely one of M and N. A *far base* is a common base B of M and N such that there is no wrong set X with $|B \triangle X| = 2$. We first show:

(39.13) if M and N are different and have a far base, then M or N has a U_4^2 minor.

Let M, N form a counterexample with S as small as possible. Let B be a far base and X be a wrong set with $|B \triangle X|$ minimal. Then $B \cup X = S$, since we can delete $S \setminus (B \cup X)$. Similarly (by considering M^* and N^*), $B \cap X = \emptyset$. Then, by the minimality of $|B \triangle X|$, X is the only wrong set. By symmetry, we may assume that X is a base of M. Then M has a base B' with $|B \triangle B'| = 2$. By the uniqueness of X, B' is also a base of N. By the minimality of $|B \triangle X|$, B' is not far. Hence, by the uniqueness of X, $|B' \triangle X| = 2$. So $|S| = 4$.

Let $S = \{a, b, c, d\}$, $B = \{a, b\}$, $X = \{c, d\}$. Since $M \neq U_4^2$ by assumption, we may assume that $\{a, c\}$ is not a base of M. Hence, since $\{a\}$ and $\{c, d\}$ are independent in M, $\{a, d\}$ is a base of M. Similarly, since $\{c\}$ and $\{a, b\}$ are independent in M, $\{b, c\}$ is a base of M.

Since B is far, $\{a, d\}$ and $\{b, c\}$ are bases also of N, and $\{a, c\}$ is not a base of N. So $\{c\}$ is independent in N, implying that $\{c, a\}$ or $\{c, d\}$ is a base of N, a contradiction. This proves (39.13).

Now let M be a nonbinary matroid on a set S. Choose a base B of M. Let $\{x_b \mid b \in B\}$ be a collection of linearly independent vectors over GF(2). For each $s \in S \setminus B$, let C_s be the circuit contained in $B \cup \{s\}$, and define

$$(39.14) \qquad x_s := \sum_{b \in C_s \setminus \{s\}} x_b.$$

Let N be the binary matroid represented by $\{x_s \mid s \in S\}$. Now for each $b \in B$ and each $s \in S \setminus B$ one has that $(B \setminus \{b\}) \cup \{s\}$ is a base of M if and only if it is a base of N. So B is a far base. Since N is binary, we know that $N \neq M$ and that N has no U_4^2 minor. Hence, by (39.13), M has a U_4^2 minor. ∎

Regular matroids. A matroid is called *regular* if it is representable over each field. It is equivalent to requiring that it can be represented over $\mathbb{R}$ by the columns of a totally unimodular matrix.

Regular matroids are characterized by Tutte [1958a,1958b] as those binary matroids not having an F_7 or F_7^* minor. (Gerards [1989b] gave a short proof.)

A basic decomposition theorem of Seymour [1980a] states that each regular matroid can be obtained by taking 1-, 2-, and 3-sums from graphic and cographic matroids and from copies of a 10-element matroid called R_{10}. (We do not use this theorem in this book. Background can be found in the book of Truemper [1992].)

Algebraic matroids (Steinitz [1910]). Let L be a field extension of a field K and let S be a finite subset of L. Let $\mathcal{I}$ be the collection of all subsets $\{s_1, \ldots, s_n\}$ of S that consist of algebraically independent elements over K. That is, there is no nonzero polynomial $p(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ with $p(s_1, \ldots, s_n) = 0$. Then $(S, \mathcal{I})$ is a matroid, and matroids arising in this way are called *algebraic* (*over* K). (Steinitz [1910] showed that $(S, \mathcal{I})$ satisfies the matroid axioms, although the term matroid was not yet introduced.)

To see that $(S, \mathcal{I})$ is a matroid, we check (39.3). It suffices to show that for all $s_1, \ldots, s_n \in S$ one has:

(39.15) if $\{s_1, s_2, s_3, \ldots, s_{n-1}\} \in \mathcal{I}$ and $\{s_3, \ldots, s_{n-1}, s_n\} \in \mathcal{I}$, then $\{s_1, s_3, \ldots, s_n\} \in \mathcal{I}$ or $\{s_2, s_3, \ldots, s_n\} \in \mathcal{I}$.

Suppose not. Then there exist nonzero polynomials $p(x_1, x_3, \ldots, x_n)$ and $q(x_2, x_3, \ldots, x_n)$ over K with $p(s_1, s_3, \ldots, s_n) = 0$ and $q(s_2, s_3, \ldots, s_n) = 0$. We may assume that p and q are irreducible. Moreover, since $\{s_3, \ldots, s_n\} \in \mathcal{I}$, p and q are relatively prime. Define $F := K(x_1, x_2, \ldots, x_{n-1})$. So p and q belong to the Euclidean ring $F[x_n]$. Let r be the g.c.d. of p and q in $F[x_n]$. As p and q are relatively prime, we know $r \in F$, and hence we may assume $r \in K[x_1, \ldots, x_{n-1}]$. Now $r = \alpha p + \beta q$ for some $\alpha, \beta \in F[x_n]$. So $r(s_1, \ldots, s_{n-1}) = 0$, contradicting the fact that $\{s_1, \ldots, s_{n-1}\} \in \mathcal{I}$. This proves (39.15).

Each linear matroid is algebraic (as we can consider the linear relations between the elements as polynomials of rank 1), while Ingleton [1971] gave an example of a nonlinear algebraic matroid. Examples of nonalgebraic matroids were given by Ingleton and Main [1975] and Lindström [1984,1986]. The class of algebraic matroids can be easily seen to be closed under taking minors (deletion is direct, while contraction of an element t corresponds to replacing K by $K(t)$), but it is unknown if it is closed under duality.

In fact, for any field K, the class of matroids that are algebraic over K is closed under taking minors, since Lindström [1989] showed that any matroid algebraic over $K(t)$ (for any t), is also algebraic over K.

For an in-depth survey on algebraic matroids, see Oxley [1992].

Graphic matroids (Birkhoff [1935c], Whitney [1935]). Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the collection of all subsets of E that form a forest. Then $M = (E, \mathcal{I})$ is a matroid. Condition (39.1)(i) is trivial. To see that condition (39.2) holds, let $F \subseteq E$. Then, by definition, each base U of F is an inclusionwise maximal forest contained in F. Hence U forms a spanning tree in each component of the graph (V, F). So U has $|V| - k$ elements, where k is the number of components of (V, F). So each base of F has $|V| - k$ elements, proving (39.2).

The matroid M is called the *cycle matroid* of G, denoted by $M(G)$. Any matroid obtained in this way, or isomorphic to such a matroid, is called a *graphic matroid*.

Trivially, the circuits of $M(G)$, in the matroid sense, are exactly the circuits of G, in the graph sense. The bases of $M(G)$ are exactly the inclusionwise maximal forests F of G. So if G is connected, the bases are the spanning trees.

The rank function of $M(G)$ can be described as follows. For each subset F of E, let $\kappa(V, F)$ denote the number of components of the graph (V, F). Then for each $F \subseteq E$:

(39.16) $$r_{M(G)}(F) = |V| - \kappa(V, F).$$

Note that deletion and contraction in the matroid correspond to deletion and contraction of edges in the graph.

Graphic matroids are regular, that is, representable over any field: orient the edges of G arbitrarily, and consider the $V \times E$ matrix L given by: $L_{v,e} = +1$ if v is the head of e, $L_{v,e} := -1$ if v is the tail of e, and $L_{v,e} := 0$ otherwise (for $v \in V$, $e \in E$). Then a subset F of E is a forest if and only if the set of columns with index in F is linearly independent.

By a theorem of Tutte [1959], the graphic matroids are precisely those regular matroids containing no $M(K_5)^*$ and $M(K_{3,3})^*$ minor. (Alternative proofs were given by Ghouila-Houri [1964] (Chapitre III), Seymour [1980d], Truemper [1985], Wagner [1985], and Gerards [1995b].)

Cographic matroids (Whitney [1935]). The dual of the cycle matroid $M(G)$ of a graph $G = (V, E)$ is called the *cocycle matroid* of G, and denoted by $M^*(G)$. Any matroid obtained in this way, or isomorphic to such a matroid, is called a *cographic matroid*.

So the bases of $M^*(G)$ are the complements of maximal forests of G. (So if G is connected, these are exactly the complements of the spanning trees in G.)

Hence the independent sets are those edge sets F for which $E \setminus F$ contains a maximal forest of G; that is, $(V, E \setminus F)$ has the same number of components as G.

A subset C of E is a circuit of $M^*(G)$ if and only if C is an inclusionwise minimal set with the property that $(V, E \setminus C)$ has more components than G. Hence C is a circuit of $M^*(G)$ if and only if C is an inclusionwise minimal nonempty cut in G.

The rank function of $M^*(G)$ can be described as follows. Again, for each subset F of E, let $\kappa(V, F)$ denote the number of components of the graph (V, F). Then (39.6) and (39.16) give that for each $F \subseteq E$:

(39.17) $$r_{M^*(G)}(F) = |F| - \kappa(V, E \setminus F) + \kappa(V, E).$$

Let G be an (embedded) planar graph, and let G^* be the dual planar graph of G. Then the cycle matroid $M(G^*)$ of G^* is isomorphic to the cocycle matroid $M^*(G)$ of G.

A theorem of Whitney [1933] implies that a matroid is both graphic and cographic if and only if it is isomorphic to the cycle matroid of a planar graph.

Transversal matroids (Edmonds and Fulkerson [1965], Mirsky and Perfect [1967]). Let $\mathcal{X} = (X_1, \ldots, X_n)$ be a family of subsets of a finite set S and let $\mathcal{I}$ be the collection of all partial transversals of $\mathcal{X}$. Then $M = (S, \mathcal{I})$ is a matroid, as follows directly from Corollary 22.4a. Any matroid obtained in this way, or isomorphic to such a matroid, is called a *transversal matroid* (*induced by* $\mathcal{X}$).

The bases of this matroid are the inclusionwise maximal partial transversals. If $\mathcal{X}$ has a transversal, the bases of M are the transversals of $\mathcal{X}$. In fact, Theorem 22.5 implies that we can assume the latter situation:

(39.18) Let M be the transversal matroid induced by the family $\mathcal{X}$. Then $\mathcal{X}$ has a subfamily $\mathcal{Y}$ such that M is equal to the transversal matroid induced by $\mathcal{Y}$ and such that $\mathcal{Y}$ has a transversal.

So we can assume that any transversal matroid has the transversals of a family of sets as bases.

It follows from Kőnig's matching theorem that the rank function r of the transversal matroid induced by $\mathcal{X}$ is given by

(39.19) $$\begin{aligned} r(U) &= \min_{T \subseteq U}(|U \setminus T| + |\{i \mid X_i \cap T \neq \emptyset\}|) \\ &= \min_{I \subseteq \{1,\ldots,n\}}(n - |I| + |\bigcup_{i \in I}(X_i \cap U)|) \end{aligned}$$

for $U \subseteq S$. This follows directly from Theorem 22.2 and Corollary 22.2a, applied to the family $(X_1 \cap U, \ldots, X_n \cap U)$.

Piff and Welsh [1970] (cf. Atkin [1972]) showed that

(39.20) any transversal matroid is representable over all fields, except for finitely many finite fields.

If the sets $X_1, \ldots, X_m$ form a partition of S, one speaks of a *partition matroid*. Trivially, each partition matroid is graphic and cographic (by considering a graph consisting of vertex-disjoint parallel classes of edges). Also uniform matroids are special cases of transversal matroids.

Gammoids (Perfect [1968]). An extension of transversal matroids is obtained by taking a directed graph $D = (V, A)$ and subsets U and S of V. For $X, Y \subseteq V$, call X *linked to* Y if $|X| = |Y|$ and D has $|X|$ vertex-disjoint $X - Y$ paths. (So X is the set of starting vertices of these paths, and Y the set of end vertices.)

Let $\mathcal{I}$ be the collection of subsets I of S such that some subset of U is linked to I. Then $M = (S, \mathcal{I})$ is a matroid. This follows from Theorem 9.11: let $I, J \in \mathcal{I}$ with $|I| < |J|$. Let $T := I \cup J$. Let k be the maximum number of disjoint $U - T$ paths. So $k \geq |J| > |I|$. By Theorem 9.11, there exist k disjoint $U - T$ paths covering I. Hence $I + j \in \mathcal{I}$ for some $j \in J \setminus I$. So M is a matroid.

Matroids obtained in this way are called *gammoids*. If $S = V$, the gammoid is called a *strict gammoid* (*induced by* D, U). Hence:

(39.21) gammoids are exactly the restrictions of strict gammoids.

The bases of the strict gammoid induced by D, U are the subsets B of V such that U is linked to B. In particular, U is a base.

From Menger's theorem (Corollary 9.1a) one easily derives the following formula for the rank function r_M of M:

(39.22) $$r_M(X) = \min\{|Y| \mid Y \text{ intersects each } U - X \text{ path}\}$$

for $X \subseteq S$. (One may prove easily that the right-hand side of (39.22) satisfies Theorem 39.8 below, thus proving again that M is a matroid.)

39.4a. Relations between transversal matroids and gammoids

Ingleton and Piff [1973] showed the following theorem (based on a duality of bipartite graphs and directed graphs similar to that described in Section 16.7c). The proof provides an alternative proof that gammoids are indeed matroids.

Theorem 39.5. *Strict gammoids are exactly the duals of the transversal matroids.*

Proof. Let M be the strict gammoid induced by the directed graph $D = (V, A)$ and $U \subseteq V$. We can assume that $(v, v) \in A$ for each $v \in V$. For each $v \in V$, let

(39.23) $$X_v := \{u \in V \mid (u, v) \in A\}.$$

Let L be the transversal matroid induced by the family $\mathcal{X} := (X_v \mid v \in V \setminus U)$. We show that $L = M^*$.

As $v \in X_v$ for each $v \in V \setminus U$, the set $V \setminus U$ is a transversal of $\mathcal{X}$. Hence the bases of L are the transversals of $\mathcal{X}$. As U is a base of the strict gammoid induced by D, U, it suffices to show, for each $B \subseteq V$:

(39.24) U is linked to B in D if and only if $V \setminus B$ is a transversal of $\mathcal{X}$.

To see necessity in (39.24), let U be linked to B in D and let $\mathcal{P}$ be a set of $|U|$ disjoint $U - B$ paths. Then for each $v \in V \setminus U$, let $x_v := u$ if v is entered by an arc (u, v) in a path P in $\mathcal{P}$ and let $x_v := v$ otherwise. Then:

(39.25) (i) $x_v \in X_v$, (ii) $x_v \neq x_{v'}$ for $v \neq v' \in V \setminus U$, and (iii) $\{x_v \mid v \in V \setminus U\} = V \setminus B$.

So $V \setminus B$ is a transversal of $\mathcal{X}$.

To see sufficiency in (39.24), let $V \setminus B$ be a transversal of $\mathcal{X}$. Hence there exist x_v for $v \in V \setminus U$ satisfying (39.25). Let A' be the set of arcs (x_v, v) of D with $v \in V \setminus U$. Then $V \setminus U$ is the set of vertices entered by an arc in A', and $V \setminus B$ is the set of vertices left by an arc in A'. Hence U is linked to B in D.

This shows (39.24), and hence that $M^* = L$. So the dual of a strict gammoid is a transversal matroid.

To see that each transversal matroid is the dual of a strict gammoid, we show that the construction described above can be reversed. Let L be the transversal matroid induced by the family $\mathcal{X} = (X_i \mid i = 1, \ldots, m)$ of sets. By (39.18) we can assume that $\mathcal{X}$ has a transversal. Hence we can assume that $i \in X_i$ for $i = 1, \ldots, m$ (by renaming). Let $V := X_1 \cup \cdots \cup X_m$ and let

(39.26) $A := \{(u, v) \mid v \in \{1, \ldots, m\}, u \in X_v\}$.

Let $D = (V, A)$ and define $U := V \setminus \{1, \ldots, m\}$. Since D, U and $\mathcal{X}$ are related as in (39.23), we again have (39.25). So L is equal to the dual of the strict gammoid induced by D, U. ∎

This theorem has a number of implications for the interrelations of the classes of transversal matroids and of gammoids. Consider the following class of matroids, introduced by Ingleton and Piff [1973]. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W. Let $M = (V, \mathcal{I})$ be the transversal matroid induced by the family $(\{v\} \cup N(v) \mid v \in U)$ (where $N(v)$ is the set of neighbours of v). So $B \subseteq V$ is a base of M if and only if $(U \setminus B) \cup (W \cap B)$ is matchable in G (that is, it induces a subgraph of G having a perfect matching).

Any such matroid M is called a *deltoid* (*induced by* G, U, W). Then M^* is the deltoid induced by G, W, U. So

(39.27) the dual of a deltoid is a deltoid again.

Now

(39.28) transversal matroids are exactly those matroids that are the restriction of a deltoid.

Indeed, each deltoid is a transversal matroid, and hence the restriction of any deltoid is a transversal matroid (as the class of transversal matroids is closed under taking restrictions). Conversely, any transversal matroid, induced by (say) $X_1, \ldots, X_m$ is the restriction to W of the deltoid induced by the bipartite graph G with colour classes $U := \{1,\ldots,m\}$ and $W := X_1 \cup \cdots \cup X_m$, with $i \in U$ and $x \in W$ adjacent if and only if $x \in X_i$. (Assuming without loss of generality that $U \cap W = \emptyset$.) This shows (39.28).

Then (39.27) and (39.28) give with Theorem 39.5:

(39.29) the strict gammoids are exactly the contractions of the deltoids.

Indeed, the strict gammoids are the duals of transversal matroids, hence the duals of restrictions of deltoids, and therefore the contractions of (the duals of) deltoids.

This gives:

Corollary 39.5a. *The gammoids are exactly the contractions of the transversal matroids.*

Proof. Gammoids are the restrictions of strict gammoids, hence the restrictions of contractions of deltoids, hence the contractions of restrictions of deltoids, therefore the contractions of transversal matroids. ∎

Similarly:

(39.30) the gammoids are exactly the minors of deltoids,

which implies (with (39.27)) a result of Mason [1972]:

(39.31) the class of gammoids is closed under taking minors and duals.

Theorem 39.5 also implies, with (39.20), that gammoids are representable over all fields, except for a finite number of finite fields (Mason [1972]). In fact, Lindström [1973] showed that any gammoid $(S,\mathcal{I})$ is representable over each field with at least $2^{|S|}$ elements.

Edmonds and Fulkerson [1965] showed that one gets a transversal matroid as follows. Let $G = (V,E)$ be an undirected graph and let $S \subseteq V$. Let $\mathcal{I}$ be the collection of subsets of S which are covered by some matching in G. Then $M = (S,\mathcal{I})$ is a matroid (which is easy to show), called the *matching matroid* of G. In fact, any matching matroid is a transversal matroid. To prove this, we may assume $S = V$. Let $D(G)$, $A(G)$, $C(G)$ form the Edmonds-Gallai decomposition of G (Section 24.4b). Let $\mathcal{K}$ be the collection of components of $G[D(G)]$. Let $\mathcal{X}$ be the family of sets

(39.32)	$\{v\}$	for each $v \in A(G) \cup C(G)$,
	$N(v) \cap D(G)$	for each $v \in A(G)$,
	K, repeated $\lvert K\rvert - 1$ times,	for each $K \in \mathcal{K}$.

Then M is equal to the transversal matroid induced by $\mathcal{X}$, as is easy to derive from the properties of the Edmonds-Gallai decomposition. A min-max relation for the rank function is given by Theorem 24.6.

It is straightforward to see that, conversely, each transversal matroid is a matching matroid, by taking G bipartite.

39.5. Characterizing matroids by bases

In Section 39.1, the notion of matroid is defined by 'axioms' in terms of the independent sets. There are several other axiom systems that characterize matroids. In this and the next sections we give a number of them.

Clearly, a matroid is determined by the collection of its bases, since a set is independent if and only if it is contained in a base. Conditions characterizing a collection of bases of a matroid are given in the following theorem (Whitney [1935]).

Theorem 39.6. *Let S be a set and let $\mathcal{B}$ be a nonempty collection of subsets of S. Then the following are equivalent:*

(39.33)
(i) *$\mathcal{B}$ is the collection of bases of a matroid;*
(ii) *if $B, B' \in \mathcal{B}$ and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$;*
(iii) *if $B, B' \in \mathcal{B}$ and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Proof. (i)⇒(ii): Let $\mathcal{B}$ be the collection of bases of a matroid $(S,\mathcal{I})$. Then all sets in $\mathcal{B}$ have the same size. Now let $B, B' \in \mathcal{B}$ and $x \in B' \setminus B$. Since $B' - x \in \mathcal{I}$, there exists a $y \in B \setminus B'$ with $B'' := B' - x + y \in \mathcal{I}$. Since $|B''| = |B'|$, we know $B'' \in \mathcal{B}$.

(iii)⇒(i): (iii) directly implies that no set in $\mathcal{B}$ is contained in another. Let $\mathcal{I}$ be the collection of sets I with $I \subseteq B$ for some $B \in \mathcal{B}$. We check (39.3). Let $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. Let $I \setminus J = \{x\}$.

Consider sets $B, B' \in \mathcal{B}$ with $I \subseteq B$, $J \subseteq B'$. If $x \in B'$, we are done. So assume $x \notin B'$. Then by (iii), $B' - y + x \in \mathcal{B}$ for some $y \in B' \setminus B$. As $|J \setminus I| = 2$, there is a $z \in J \setminus I$ with $z \neq y$. Then $I + z \subseteq B' - y + x$, and so $I + z \in \mathcal{I}$.

(ii)⇒(iii): By the foregoing we know that (iii) implies (ii). Now axioms (ii) and (iii) interchange if we replace $\mathcal{B}$ by the collection of complements of sets in $\mathcal{B}$. Hence also the implication (ii)⇒(iii) holds. ∎

The equivalence of (ii) and (iii) also follows from the fact that the collection of complements of bases of a matroid is the collection of bases of the dual matroid. Conversely, Theorem 39.6 implies that the dual indeed is a matroid.

39.6. Characterizing matroids by circuits

A matroid is determined by the collection of its circuits, since a set is independent if and only if it contains no circuit. Conditions characterizing a collection of circuits of a matroid are given in the following theorem (Whitney

[1935] proved (i)⇔(iii), and Robertson and Weston [1958] (and also Lehman [1964] and Asche [1966]) proved (i)⇔(ii)).

Theorem 39.7. *Let S be a set and let $\mathcal{C}$ be a collection of nonempty subsets of S, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent:*

(39.34) (i) *$\mathcal{C}$ is the collection of circuits of a matroid;*
(ii) *if $C, C' \in \mathcal{C}$ with $C \neq C'$ and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
(iii) *if $C, C' \in \mathcal{C}$, $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y.*

Proof. (i)⇒(iii): Let $\mathcal{C}$ be the collection of circuits of a matroid $(S, \mathcal{I})$ and let $\mathcal{B}$ be its collection of bases. Let $C, C' \in \mathcal{C}$, $x \in C \cap C'$, and $y \in C \setminus C'$. We can assume that $S = C \cup C'$. Let $B, B' \in \mathcal{B}$ with $B \supseteq C - y$ and $B' \supseteq C' - x$. Then $y \notin B$ and $x \notin B'$ (since $C \not\subseteq B$ and $C' \not\subseteq B'$).

We can assume that $y \notin B'$. Otherwise, $y \in B' \setminus B$, and hence by (ii) of Theorem 39.6, there exists a $z \in B \setminus B'$ with $B'' := B' - y + z \in \mathcal{B}$. Then $z \neq x$, since otherwise $C' \subseteq B''$. Hence, replacing B' by B'' gives $y \notin B'$.

As $y \notin B'$, we know $B' \cup \{y\} \notin \mathcal{I}$, and hence there exists a $C'' \in \mathcal{C}$ contained in $B' \cup \{y\}$. As $C'' \not\subseteq B'$, we know $y \in C''$. Moreover, as $x \notin B'$ we know $x \notin C''$.

(iii)⇒(ii): is trivial.

(ii)⇒(i): Let $\mathcal{I}$ be the collection of sets containing no set in $\mathcal{C}$ as a subset. We check (39.3). Let $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. Assume that $I + z \notin \mathcal{I}$ for each $z \in J \setminus I$. Let y be the element of $I \setminus J$. If $J + y \in \mathcal{I}$, then $I \cup J \in \mathcal{I}$, contradicting our assumption. So $J + y$ contains a set $C \in \mathcal{C}$. Then C is the unique set in $\mathcal{C}$ contained in $J + y$. For suppose that there is another, C' say. Again, $y \in C'$, and hence by (39.34)(ii) there exists a $C'' \in \mathcal{C}$ contained in $(C \cup C') \setminus \{y\}$. But then $C'' \subseteq J$, a contradiction.

As $C \not\subseteq I$, C intersects $J \setminus I$. Choose $x \in C \cap (J \setminus I)$. Then $X := J + y - x$ contains no set in $\mathcal{C}$ (as C is the only set in $\mathcal{C}$ contained in $J + y$). So $X \in \mathcal{I}$, implying that $I + z \in \mathcal{I}$ for the $z \in J \setminus I$ with $z \neq x$. ∎

This theorem implies the following important property for a matroid $M = (S, \mathcal{I})$:

(39.35) for any independent set I and any $s \in S \setminus I$ there is at most one circuit contained in $I \cup \{s\}$.

39.6a. A characterization of Lehman

Lehman [1964] showed that the cocircuits of a matroid M are exactly the inclusionwise minimal nonempty subsets D of S with $|D \cap C| \neq 1$ for each circuit C of M.

To show this, it suffices to show that

(39.36) (i) $|D \cap C| \neq 1$ for each cocircuit D and circuit C,
(ii) for each nonempty $D \subseteq S$, if $|D \cap C| \neq 1$ for each circuit C, then D contains a cocircuit; that is, then D is dependent in M^*.

To see (i), suppose that $D \cap C = \{s\}$ for some circuit C and cocircuit D. As $D - s$ is independent in M^*, M has a base B disjoint from $D - s$. Since $C - s$ is disjoint from $D - s$ and since $C - s \in \mathcal{I}$, we can assume that $C - s \subseteq B$. Then $s \notin B$, and so B is disjoint from D. This implies that D is independent in M^*, contradicting the fact that D is a circuit in M^*. This shows (i).

To see (ii), let $\emptyset \neq D \subseteq S$ with $|D \cap C| \neq 1$ for each circuit C. We show that D is dependent in M^*. Suppose not. Then M has a base B disjoint from D. Choose $s \in D$. Then $B + s$ contains a circuit C with $s \in C$. Hence $D \cap C = \{s\}$, contradicting our assumption, thus showing (ii).

39.7. Characterizing matroids by rank functions

The *rank function* of a matroid $M = (S, \mathcal{I})$ is the function $r_M : \mathcal{P}(S) \to \mathbb{Z}_+$ given by:

(39.37) $$r_M(U) := \max\{|Z| \mid Z \in \mathcal{I}, Z \subseteq U\}$$

for $U \subseteq S$. Again, a matroid is determined by its rank function, as a set U is independent if and only if $r(U) = |U|$. Conditions characterizing a rank function are given by the following theorem (Whitney [1935]; necessity was also shown (in a different terminology) by Bergmann [1929] and Nakasawa [1935]):

Theorem 39.8. *Let S be a set and let $r : \mathcal{P}(S) \to \mathbb{Z}_+$. Then r is the rank function of a matroid if and only if for all $T, U \subseteq S$:*

(39.38) (i) $r(T) \leq r(U) \leq |U|$ *if* $T \subseteq U$,
(ii) $r(T \cap U) + r(T \cup U) \leq r(T) + r(U)$.

Proof. *Necessity.* Let r be the rank function of a matroid $(S, \mathcal{I})$. Choose $T, U \subseteq S$. Clearly (39.38)(i) holds. To see (ii), let I be an inclusionwise maximal set in $\mathcal{I}$ with $I \subseteq T \cap U$ and let J be an inclusionwise maximal set in $\mathcal{I}$ with $I \subseteq J \subseteq T \cup U$. Since $(S, \mathcal{I})$ is a matroid, we know that $r(T \cap U) = |I|$ and $r(T \cup U) = |J|$. Then

(39.39) $$\begin{aligned} r(T) + r(U) &\geq |J \cap T| + |J \cap U| = |J \cap (T \cap U)| + |J \cap (T \cup U)| \\ &\geq |I| + |J| = r(T \cap U) + r(T \cup U); \end{aligned}$$

that is, we have (39.38)(ii).

Sufficiency. Let $\mathcal{I}$ be the collection of subsets I of S with $r(I) = |I|$. We show that $(S, \mathcal{I})$ is a matroid, with rank function r.

Trivially, $\emptyset \in \mathcal{I}$. Moreover, if $I \in \mathcal{I}$ and $J \subseteq I$, then

(39.40) $$r(J) \geq r(I) - r(I \setminus J) \geq |I| - |I \setminus J| = |J|.$$

So $J \in \mathcal{I}$.

In order to check (39.3), let $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. Let $J \setminus I = \{z_1, z_2\}$. If $I + z_1, I + z_2 \notin \mathcal{I}$, we have $r(I + z_1) = r(I + z_2) = |I|$. Then by (39.38)(ii),

(39.41) $$r(J) \leq r(I + z_1 + z_2) \leq r(I + z_1) + r(I + z_2) - r(I) = |I| < |J|,$$

contradicting the fact that $J \in \mathcal{I}$.

So $(S, \mathcal{I})$ is a matroid. Its rank function is r, since $r(U) = \max\{|I| \mid I \subseteq U, I \in \mathcal{I}\}$ for each $U \subseteq S$. Here $\geq$ follows from (39.38)(i), since if $I \subseteq U$ and $I \in \mathcal{I}$, then $r(U) \geq r(I) = |I|$. Equality can be shown by induction on $|U|$, the case $U = \emptyset$ being trivial. If $U \neq \emptyset$, choose $y \in U$. By induction, there is an $I \subseteq U - y$ with $I \in \mathcal{I}$ and $|I| = r(U - y)$. If $r(U) = r(U - y)$ we are done, so assume $r(U) > r(U - y)$. Then $I + y \in \mathcal{I}$, since $r(I + y) \geq r(I) + r(U) - r(U - y) \geq |I| + 1$. Moreover, $r(U) \leq r(U - y) + r(\{y\}) \leq |I| + 1$. This proves equality for U. ∎

Set functions satisfying condition (39.38)(ii) are called *submodular*, and will be studied in Chapter 44.

Whitney [1935] also showed that (39.38) is equivalent to:

(39.42)
(i) $r(\emptyset) = 0$,
(ii) $r(U) \leq r(U + s) \leq r(U) + 1$ for $U \subseteq S$, $s \in S \setminus U$,
(iii) for all $U \subseteq S$, $s, t \in S \setminus U$, if $r(U + s) = r(U + t) = r(U)$, then $r(U + s + t) = r(U)$.

The proof above in fact uses only these properties of r.

The following equivalent form of Theorem 39.8 will be useful.

Corollary 39.8a. *Let S be a finite set and let $\mathcal{I}$ be a nonempty collection of subsets of S, closed under taking subsets. For $U \subseteq S$, let $r(U)$ be the maximum size of a subset of U that belongs to $\mathcal{I}$. Then $(S, \mathcal{I})$ is a matroid if and only if r satisfies* (39.38)(ii) *for all $T, U \subseteq S$.*

Proof. Necessity follows directly from Theorem 39.8. To see sufficiency, it is easy to see that r satisfies (39.38)(i). So by Theorem 39.8, r is the rank function of some matroid $M = (S, \mathcal{J})$. Now: $I \in \mathcal{J} \iff r(I) = |I| \iff I \in \mathcal{I}$. Hence $\mathcal{I} = \mathcal{J}$, and so $(S, \mathcal{I})$ is a matroid. ∎

Note that if we can test in polynomial time if a given set is independent, we can also test in polynomial time if a given set is a base, or a circuit, and we can determine the rank of a given set in polynomial time.

39.8. The span function and flats

With any matroid $M = (S, \mathcal{I})$ we can define the *span function* $\mathrm{span}_M : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows:

(39.43) $$\mathrm{span}_M(T) := \{s \in S \mid r_M(T \cup \{s\}) = r_M(T)\}$$

for $T \subseteq S$. If the matroid M is clear from the context, we write $\mathrm{span}(T)$ for $\mathrm{span}_M(T)$. Note that $T \subseteq \mathrm{span}_M(T)$ and that

(39.44) $$r_M(\mathrm{span}_M(T)) = r_M(T).$$

This follows directly from the fact that if $r_M(Y) > r_M(T)$, then $r_M(T \cup \{s\}) > r_M(T)$ for some $s \in Y$.

Note also that

(39.45) $$T \text{ is spanning } \iff \mathrm{span}_M(T) = S$$

for any $T \subseteq S$. To see $\Longrightarrow$, let T be spanning. Then for each $s \in T$: $r_M(T + s) \leq r_M(S) = r_M(T)$. To see $\Longleftarrow$, suppose $\mathrm{span}_M(T) = S$. Then $r_M(T) = r_M(\mathrm{span}_M(T)) = r_M(S)$.

A *flat* in a matroid $M = (S, \mathcal{I})$ is a subset F of S with $\mathrm{span}_M(F) = F$. A matroid is determined by its collection of flats, as is shown by:

(39.46) a subset I of S is independent if and only if for each $y \in I$ there is a flat F with $I - y \subseteq F$ and $y \notin F$.

Indeed, if I is independent and $y \in I$, let $F := \mathrm{span}_M(I - y)$. Then F is a flat containing $I - y$, but not y, since $r_M(F + y) \geq r_M(I) > r_M(I - y) = r_M(F)$. Conversely, if I is not independent, then $y \in \mathrm{span}_M(I - y)$ for some $y \in I$, and hence each flat containing $I - y$ also contains y.

39.8a. Characterizing matroids by span functions

It was observed by Mac Lane [1938] that the following characterizes span functions of matroids (sufficiency was shown by van der Waerden [1937]).

Theorem 39.9. *Let S be a finite set. A function* $\mathrm{span} : \mathcal{P}(S) \to \mathcal{P}(S)$ *is the span function of a matroid if and only if:*

(39.47)
(i) *if $T \subseteq S$, then $T \subseteq \mathrm{span}(T)$;*
(ii) *if $T, U \subseteq S$ and $U \subseteq \mathrm{span}(T)$, then $\mathrm{span}(U) \subseteq \mathrm{span}(T)$;*
(iii) *if $T \subseteq S$, $t \in S \setminus T$, and $s \in \mathrm{span}(T + t) \setminus \mathrm{span}(T)$, then $t \in \mathrm{span}(T + s)$.*

Proof. *Necessity.* Let span be the span function of a matroid $M = (S, \mathcal{I})$ with rank function r. Clearly, (39.47)(i) is satisfied. To see (39.47)(ii), let $U \subseteq \mathrm{span}(T)$ and $s \in \mathrm{span}(U)$. We show $s \in \mathrm{span}(T)$. We can assume $s \notin T$. Then, by the submodularity of r,

(39.48) $$\begin{aligned} r(T \cup \{s\}) &\leq r(T \cup U \cup \{s\}) \leq r(T \cup U) + r(U \cup \{s\}) - r(U) \\ &= r(T \cup U) = r(T). \end{aligned}$$

(The last equality follows from (39.44).) This shows that $s \in \mathrm{span}(T)$.

To see (39.47)(iii), note that $s \in \mathrm{span}(T+t) \setminus \mathrm{span}(T)$ is equivalent to: $r(T+t+s) = r(T+t)$ and $r(T+s) > r(T)$. Hence

(39.49) $$r(T+t+s) = r(T+t) \leq r(T) + 1 \leq r(T+s),$$

that is, $t \in \mathrm{span}(T+s)$. This shows necessity of the conditions (39.47).

Sufficiency. Let a function span satisfy (39.47), and define

(39.50) $$\mathcal{I} := \{I \subseteq S \mid s \notin \mathrm{span}(I-s) \text{ for each } s \in I\}.$$

We first show the following:

(39.51) $$\text{if } I \in \mathcal{I}, \text{ then } \mathrm{span}(I) = I \cup \{t \mid I+t \notin \mathcal{I}\}.$$

Indeed, if $t \in \mathrm{span}(I)\setminus I$, then $I+t \notin \mathcal{I}$, by definition of $\mathcal{I}$. Conversely, $I \subseteq \mathrm{span}(I)$ by (39.47)(i). Moreover, if $I+t \notin \mathcal{I}$, then by definition of $\mathcal{I}$, $s \in \mathrm{span}(I+t-s)$ for some $s \in I+t$. If $s = t$, then $t \in \mathrm{span}(I)$ and we are done. So assume $s \neq t$; that is, $s \in I$. As $I \in \mathcal{I}$, we know that $s \notin \mathrm{span}(I-s)$. So by (39.47)(iii) (for $T := I - s$), $t \in \mathrm{span}(I)$, proving (39.51).

We now show that $M = (S, \mathcal{I})$ is a matroid. Trivially, $\emptyset \in \mathcal{I}$. To see that $\mathcal{I}$ is closed under taking subsets, let $I \in \mathcal{I}$ and $J \subseteq I$. We show that $J \in \mathcal{I}$. Suppose to the contrary that $s \in \mathrm{span}(J-s)$ for some $s \in J$. By (39.47)(ii), $\mathrm{span}(J-s) \subseteq \mathrm{span}(I-s)$. Hence $s \in \mathrm{span}(I-s)$, contradicting the fact that $I \in \mathcal{I}$.

In order to check (39.3), let $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. Let $I \setminus J = \{i\}$ and $J \setminus I = \{j_1, j_2\}$. Assume that $I + j_1 \notin \mathcal{I}$. That is, $J + i - j_2 \notin \mathcal{I}$, and so, by (39.51) applied to $J - j_2$, $i \in \mathrm{span}(J-j_2)$. Therefore, $I \subseteq \mathrm{span}(J-j_2)$, and so $\mathrm{span}(I) \subseteq \mathrm{span}(J-j_2)$. So $j_2 \notin \mathrm{span}(I)$ (as $J \in \mathcal{I}$), and therefore, by (39.51) applied to I, $I + j_2 \in \mathcal{I}$.

So M is a matroid. We finally show that $\mathrm{span} = \mathrm{span}_M$. Choose $T \subseteq S$. To see that $\mathrm{span}(T) = \mathrm{span}_M(T)$, let I be a base of T (in M). Then (using (39.51)),

(39.52) $$\mathrm{span}_M(T) = I \cup \{x \mid I+x \notin \mathcal{I}\} = \mathrm{span}(I) \subseteq \mathrm{span}(T).$$

So we are done by showing $\mathrm{span}(T) \subseteq \mathrm{span}(I)$; that is, by (39.47)(ii), $T \subseteq \mathrm{span}(I)$. Choose $t \in T \setminus I$. By the maximality of I, we know $I + t \notin \mathcal{I}$, and hence, by (39.51), $t \in \mathrm{span}(I)$. ∎

39.8b. Characterizing matroids by flats

Conditions characterizing collections of flats of a matroid are given in the following theorem (Bergmann [1929]):

Theorem 39.10. *Let S be a set and let $\mathcal{F}$ be a collection of subsets of S. Then $\mathcal{F}$ is the collection of flats of a matroid if and only if:*

(39.53)
(i) *$S \in \mathcal{F}$;*
(ii) *if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;*
(iii) *if $F \in \mathcal{F}$ and $t \in S \setminus F$, and F' is the smallest flat containing $F + t$, then there is no flat F'' with $F \subset F'' \subset F'$.*

Proof. *Necessity.* Let $\mathcal{F}$ be the collection of flats of a matroid $M = (S, \mathcal{I})$. Condition (39.53)(i) is trivial, and condition (39.53)(ii) follows from $\mathrm{span}_M(F_1 \cap F_2) \subseteq \mathrm{span}_M(F_1) \cap \mathrm{span}_M(F_2) = F_1 \cap F_2$. To see (39.53)(iii), suppose that such an F'' exists. Choose $s \in F'' \setminus F$. So $s \notin \mathrm{span}_M(F)$. As $F' \not\subseteq F''$, we have $t \notin \mathrm{span}_M(F+s)$. Therefore, by (39.47)(iii) for $T := F$, $s \notin \mathrm{span}_M(F) = F'$, a contradiction.

Sufficiency. Let $\mathcal{F}$ satisfy (39.53). For $Y \subseteq S$, let $\mathrm{span}(Y)$ be the smallest set in $\mathcal{F}$ containing Y. Since $F \in \mathcal{F} \iff \mathrm{span}(F) = F$, it suffices to show that span satisfies the conditions (39.47). Here (39.47)(i) and (ii) are trivial. To see (39.47)(iii), let $T \subseteq S$, $t \in S \setminus T$, and $s \in \mathrm{span}(T+t) \setminus \mathrm{span}(T)$. Then $\mathrm{span}(T) \subset \mathrm{span}(T+s) \subseteq \mathrm{span}(T+t)$. Hence, by (39.53)(iii), $\mathrm{span}(T+s) = \mathrm{span}(T+t)$, and hence $t \in \mathrm{span}(T+s)$. ∎

39.8c. Characterizing matroids in terms of lattices

Bergmann [1929] and Birkhoff [1935a] characterized matroids in terms of lattices. A partially ordered set $(L, \leq)$ is called a *lattice* if

(39.54)
(i) for all $A, B \in L$ there is a unique element, called $A \wedge B$, satisfying $A \wedge B \leq A, B$ and $C \leq A \wedge B$ for all $C \leq A, B$;
(ii) for all $A, B \in L$ there is a unique element, called $A \vee B$, satisfying $A \vee B \geq A, B$ and $C \geq A \vee B$ for all $C \geq A, B$.

$A \wedge B$ and $A \vee B$ are called the *meet* and *join* respectively of A and B. Here we assume lattices to be finite. Then a lattice has a unique minimal element, denoted by 0. The *rank* of an element A is the maximum number n of elements $x_1, \ldots, x_n$ with $0 < x_1 < \cdots < x_n = A$. An element of rank 1 is called a *point* or *atom*.

Call a lattice a *point lattice* if each element is a join of points, and a *matroid lattice* (or a *geometric lattice*) if it is isomorphic to the lattice of flats of a matroid. Trivially, each matroid lattice is a point lattice. Moreover, a matroid without loops and parallel elements is completely determined by the lattice of flats.

In the following theorem, the equivalence of (i) and (ii), and the implication (ii)⇒(iv) are due (in a different terminology) to Bergmann [1929]; the equivalence of (iii) and (iv) was shown by Birkhoff [1933], and the implication (iii)⇒(i) was shown by Birkhoff [1935a].

In a partially ordered set $(L, \leq)$ an element y is said to *cover* an element x if $x < y$ and there is no z with $x < z < y$.

Theorem 39.11. *For any finite point lattice $(L, \leq)$, with rank function r, the following are equivalent:*

(39.55)
(i) *L is a matroid lattice;*
(ii) *for each $a \in L$ and each point p, if $p \not\leq a$, then $a \vee p$ covers a;*
(iii) *for each $a, b \in L$, if a and b cover $a \wedge b$, then $a \vee b$ covers a and b;*
(iv) *$r(a) + r(b) \geq r(a \vee b) + r(a \wedge b)$ for all $a, b \in L$.*

Proof. (i)⇒(iv): Let L be the lattice of flats of a matroid $M = (S, \mathcal{I})$, with rank function r_M. We can assume that M has no loops and no parallel elements. Then for any flat F we have $r(F) = r_M(F)$, since $r_M(F)$ is equal to the maximum number k of nonempty flats $F_1 \subset \cdots \subset F_k$ with $F_k = F$. So (iv) follows from Theorem 39.8.

(iv)⇒(iii): We first show that (iv) implies that if b covers a, then $r(b) = r(a)+1$. As b is a join of points, and as b covers a, we know that $b = a \vee p$ for some point p with $p \not\leq a$. Hence $r(b) = r(a \vee p) \leq r(a)+r(p)-r(a \wedge p) = r(a)+r(p)-r(0) = r(a)+1$. As $r(b) > r(a)$, we have $r(b) = r(a) + 1$.

To derive (iii) from (iv), let a and b cover $a \wedge b$. Then $r(a) = r(b) = r(a \wedge b)+1$. Hence $r(a \vee b) \leq r(a) + r(b) - r(a \wedge b) = r(a) + 1$. Hence $a \vee b$ covers a. Similarly, $a \vee b$ covers b.

(iii)⇒(ii): We derive (ii) from (iii) by induction on $r(a)$. If $a = 0$, the statement is trivial. If $a > 0$, let a' be an element covered by a. Then, by induction, $a' \vee p$ covers a'. So $a' = a \wedge (a' \vee p)$. Hence by (iii), $a \vee (a' \vee p) = a \vee p$ covers a.

(ii)⇒(i): Let S be the set of points of L, and for $f \in L$ define $F_f := \{s \in S \mid s \leq f\}$. Let $\mathcal{F} := \{F_f \mid f \in L\}$. Then for all $f_1, f_2 \in L$ we have:

(39.56) $$f_1 \leq f_2 \iff F_{f_1} \subseteq F_{f_2}.$$

Here $\Longrightarrow$ is trivial, while $\Longleftarrow$ follows from the fact that for each $f \in L$ we have $f = \bigvee F_f$, as L is a point lattice.

By (39.56), $(L, \leq)$ is isomorphic to $(\mathcal{F}, \subseteq)$. Moreover, by (39.54)(i), $F_{f_1 \wedge f_2} = F_{f_1} \cap F_{f_2}$. So $\mathcal{F}$ is closed under intersections, implying (39.53)(ii), while (39.53)(i) is trivial. Finally, (39.53)(iii) follows from (39.55)(ii). ∎

Lattices satisfying (39.55)(iii) are called *upper semimodular*.

39.9. Further exchange properties

In this section we prove a number of exchange properties of bases, as a preparation to the forthcoming sections on matroid intersection algorithms.

An exchange property of bases, stronger than given in Theorem 39.6, is (Brualdi [1969c]):

Theorem 39.12. *Let $M = (S, \mathcal{I})$ be a matroid. Let B_1 and B_2 be bases and let $x \in B_1 \setminus B_2$. Then there exists a $y \in B_2 \setminus B_1$ such that both $B_1 - x + y$ and $B_2 - y + x$ are bases.*

Proof. Let C be the unique circuit in $B_2 + x$ (cf. (39.35)). Then $(B_1 \cup C) - x$ is spanning, since $x \in \mathrm{span}_M(C - x) \subseteq \mathrm{span}_M((B_1 \cup C) - x)$, implying $\mathrm{span}((B_1 \cup C) - x) = \mathrm{span}(B_1 \cup C) = S$.

Hence there is a base B_3 with $B_1 - x \subseteq B_3 \subseteq (B_1 \cup C) - x$. So $B_3 = B_1 - x + y$ for some y in $C - x$. Therefore, $B_2 - y + x$ is a base, as it contains no circuit (since C is the only circuit in $B_2 + x$). ∎

Let $M = (S, \mathcal{I})$ be a matroid. For any $I \in \mathcal{I}$ define the (bipartite) directed graph $D_M(I) = (S, A_M(I))$, or briefly $(S, A(I))$, by:

(39.57) $$A(I) := \{(y, z) \mid y \in I, z \in S \setminus I, I - y + z \in \mathcal{I}\}.$$

Repeated application of the exchange property described in Theorem 39.12 gives (Brualdi [1969c]):

Corollary 39.12a. *Let $M = (S, \mathcal{I})$ be a matroid and let $I, J \in \mathcal{I}$ with $|I| = |J|$. Then $A(I)$ contains a perfect matching on $I \triangle J$.*[1]

Proof. By truncating M, we can assume that I and J are bases of M. We prove the lemma by induction on $|I \setminus J|$. We can assume $|I \setminus J| \geq 1$. Choose $y \in I \setminus J$. By Theorem 39.12, $I - y + z \in \mathcal{I}$ and $J - z + y \in \mathcal{I}$ for some $z \in J \setminus I$. By induction, applied to I and $J' := J - z + y$, $A(I)$ has a perfect matching N on $I \triangle J'$. Then $N \cup \{(y, z)\}$ is a perfect matching on $I \triangle J$. ∎

Corollary 39.12a implies the following characterization of maximum-weight bases:

Corollary 39.12b. *Let $M = (S, \mathcal{I})$ be a matroid, let B be a base of M, and let $w : S \to \mathbb{R}$ be a weight function. Then B is a base of maximum weight $\iff$ $w(B') \leq w(B)$ for every base B' with $|B' \setminus B| = 1$.*

Proof. Necessity being trivial, we show sufficiency. Suppose to the contrary that there is a base B' with $w(B') > w(B)$. Let N be a perfect matching in $A(B)$ covering $B \triangle B'$. As $w(B') > w(B)$, there is an edge (y, z) in N with $w(z) > w(y)$, where $y \in B \setminus B'$ and $z \in B' \setminus B$. Hence $w(B - y + z) > w(B)$, contradicting the condition. ∎

The following forms a counterpart to Corollary 39.12a (Krogdahl [1974, 1976,1977]):

Theorem 39.13. *Let $M = (S, \mathcal{I})$ be a matroid and let $I \in \mathcal{I}$. Let $J \subseteq S$ be such that $|I| = |J|$ and such that $A(I)$ contains a unique perfect matching N on $I \triangle J$. Then J belongs to $\mathcal{I}$.*

Proof. Since N is unique, we can order N as $(y_1, z_1), \ldots, (y_t, z_t)$ such that $(y_i, z_j) \notin A(I)$ if $1 \leq i < j \leq t$. Suppose that $J \notin \mathcal{I}$, and let C be a circuit contained in J. Choose the smallest i with $z_i \in C$. Then $(y_i, z) \notin A(I)$ for all $z \in C - z_i$ (since $z = z_j$ for some $j > i$). Therefore, $z \in \operatorname{span}(I - y_i)$ for all $z \in C - z_i$. So $C - z_i \subseteq \operatorname{span}(I - y_i)$, and therefore $z_i \in C \subseteq \operatorname{span}(C - z_i) \subseteq \operatorname{span}(I - y_i)$, contradicting the fact that $I - y_i + z_i$ is independent. ∎

This implies:

Corollary 39.13a. *Let $M = (S, \mathcal{I})$ be a matroid and let $I \in \mathcal{I}$. Let $J \subseteq S$ be such that $|I| = |J|$ and $r_M(I \cup J) = |I|$, and such that $A(I)$ contains a unique perfect matching N on $I \triangle J$. Let $s \notin I \cup J$ with $I + s \in \mathcal{I}$. Then $J + s \in \mathcal{I}$.*

Proof. Let t be a new element and let $M' = (S \cup \{t\}, \mathcal{I}')$ be the matroid with $F \in \mathcal{I}'$ if and only if $F \setminus \{t\} \in \mathcal{I}$. Then $N' := N \cup \{(t, s)\}$ forms a

[1] A *perfect matching on* a vertex set U in a digraph is a set of vertex-disjoint arcs such that U is the set of tails and heads of these arcs.

unique perfect matching on $(I \triangle J) \cup \{s,t\}$ in $D_{M'}(I \cup \{t\})$ (since there is no arc from t to $J \setminus I$, as $I + j \notin \mathcal{I}$ for all $j \in J \setminus I$, since $r_M(I \cup J) = |I|$). So by Theorem 39.13, $J \cup \{s\}$ is independent in M', and hence in M. ∎

39.9a. Further properties of bases

Bases satisfy the following exchange property, stronger than that described in Theorem 39.12 (conjectured by G.-C. Rota, and proved by Brylawski [1973], Greene [1973], Woodall [1974a]):

(39.58) if B_1 and B_2 are bases and B_1 is partitioned into X_1 and Y_1, then B_2 can be partitioned into X_2 and Y_2 such that $X_1 \cup Y_2$ and $Y_1 \cup X_2$ are bases.

This will be proved in Section 42.1a (using the matroid union theorem).

Other exchange properties of bases were given by Greene [1974a] and Kung [1978a]. Decomposing exchanges was studied by Gabow [1976b].

In Schrijver [1979c] it was shown that the exchange property described in Corollary 16.8b for bipartite graphs and, more generally, in Theorem 9.12 for directed graphs, in fact characterizes systems that correspond to matroids.

To this end, let U and W be disjoint sets and let Λ be a collection of pairs (X,Y) with $X \subseteq U$ and $Y \subseteq W$. Call (U,W,Λ) a *bimatroid* (or *linking system*) if:

(39.59)
(i) $(\emptyset,\emptyset) \in \Lambda$;
(ii) if $(X,Y) \in \Lambda$ and $x \in X$, then $(X-x, Y-y) \in \Lambda$ for some $y \in Y$;
(iii) if $(X,Y) \in \Lambda$ and $y \in Y$, then $(X-x, Y-y) \in \Lambda$ for some $x \in X$;
(iv) if $(X_1,Y_1),(X_2,Y_2) \in \Lambda$, then there is an $(X,Y) \in \Lambda$ with $X_1 \subseteq X \subseteq X_1 \cup X_2$ and $Y_2 \subseteq Y \subseteq Y_1 \cup Y_2$.

Note that (ii) and (iii) imply that $|X| = |Y|$ for each $(X,Y) \in \Lambda$.

To describe the relation with matroids, define:

(39.60) $\mathcal{B} := \{(U \setminus X) \cup Y \mid (X,Y) \in \Lambda\}$.

So $\mathcal{B}$ determines Λ. Then (Schrijver [1979c]):

(39.61) (U,W,Λ) is a bimatroid if and only if $\mathcal{B}$ is the collection of bases of a matroid on $U \cup W$, with $U \in \mathcal{B}$.

So bimatroids are in one-to-one correspondence with pairs (M,B) of a matroid M and a base B of M, and the conditions (39.59) yield a characterization of matroids. An equivalent axiom system characterizing matroids was given by Kung [1978b].

(Bapat [1994] gave an extension of Kőnig's matching theorem to bimatroids.)

39.10. Further results and notes

39.10a. Further notes

Dilworth [1944] showed that if $r : \mathcal{P}(S) \to \mathbb{Z}$ satisfies (39.38) and $r(U) \geq 0$ if $U \neq \emptyset$, then

(39.62) $\mathcal{I} := \{I \subseteq S \mid \forall \text{ nonempty } U \subseteq I : |U| \leq r(U)\}$

is the collection of independent sets of a matroid M. Its rank function satisfies:

(39.63) $r_M(U) = \min(r(U_1) + \cdots + r(U_t))$,

where the minimum ranges over partitions of U into nonempty subsets $U_1, \ldots, U_t$ ($t \geq 0$). If $G = (V, E)$ is a graph, and we define $r(F) := \left|\bigcup F\right| - 1$ for $F \subseteq E$, we obtain the cycle matroid of G (this also was shown by Dilworth [1944]).[2]

Conforti and Laurent [1988] showed the following sharpening of Corollary 39.8a. Let $\mathcal{C}$ be a collection of subsets of a set S and let $f : \mathcal{C} \to \mathbb{Z}_+$. Let $\mathcal{I}$ be the collection of subsets T of S with $|T \cap U| \leq f(U)$ for each $U \in \mathcal{C}$. For $T \subseteq S$, let $r(T)$ be the maximum size of a subset of T that belongs to $\mathcal{I}$. Then $(S, \mathcal{I})$ is a matroid if and only if r satisfies the submodular inequality (39.38)(ii) for all $Y, Z \in \mathcal{C}$ with $Y \cap Z \neq \emptyset$. In fact, in the right-hand side of this inequality, r may be replaced by f.

Jensen and Korte [1982] showed that there is no polynomial-time algorithm to find the minimum size of a circuit of a matroid, if the matroid is given by an oracle for testing independence. For binary matroids (represented by binary vectors), the problem of finding a minimum-size circuit was shown by Vardy [1997] to be NP-complete (solving a problem of Berlekamp, McEliece, and van Tilborg [1978], who showed the NP-completeness of finding the minimum size of a circuit containing a given element of the matroid, and of finding a circuit of given size). If we know that a matroid is binary, a vector representation can be derived by a polynomially bounded number of calls from an independence testing oracle.

For further studies of the complexity of matroid properties, see Hausmann and Korte [1978], Robinson and Welsh [1980], and Jensen and Korte [1982].

Extensions of matroid theory to infinite structures were considered by Rado [1949a], Bleicher and Preston [1961], Johnson [1961], and Dlab [1962,1965].

Standard references on matroid theory are Welsh [1976] and Oxley [1992]. The book by Truemper [1992] focuses on decomposition of matroids. Earlier texts were given by Tutte [1965a,1971]. Elementary introductions to matroids were given by Wilson [1972b,1973], and a survey with applications to electrical networks and statics by Recski [1989]. Bixby [1982], Faigle [1987], Lee and Ryan [1992], and Bixby and Cunningham [1995] survey matroid optimization and algorithms. White [1986, 1987,1992] offers a collection of surveys on matroids, and Kung [1986] is a source book on matroids. Stern [1999] focuses on semimodular lattices. Books discussing matroid optimization include Lawler [1976b], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Nemhauser and Wolsey [1988], Parker and Rardin [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], and Korte and Vygen [2000].

39.10b. Historical notes on matroids

The idea of a matroid, that is, of abstract dependence, seems to have been developed historically along a number of independent lines during the period 1900-1935. Independently, different axiom systems were given, each of which is equivalent to

[2] $\bigcup F$ denotes the union of the edges (as sets) in F.

that of a matroid. It indicates the naturalness of the concept. Only at the end of the 1930s a synthesis of the different streams was obtained.

There is a line, starting with the *Dualgruppen* (dual groups = lattices) of Dedekind [1897,1900], introduced in order to study modules (= additive subgroups) of numbers. They give rise to lattices satisfying what Dedekind called the *Modulgesetz* (module law). Later, independently, Birkhoff [1933] studied such lattices, calling them initially B-lattices, and later (after he had learned about Dedekind's earlier work), *modular lattices*. Both Dedekind and Birkhoff considered, in their studies of modular lattices, an auxiliary property that characterizes so-called *semimodular lattices*. If the lattice is a point lattice (that is, each element of the lattice is a join of atoms (points)), then such semimodular lattices are exactly the lattices of flats of a matroid. This connection was pointed out by Birkhoff [1935a] directly after Whitney's introduction of matroids.

A second line concerns exchange properties of bases. It starts with the new edition of the *Ausdehnungslehre* of Grassmann [1862], where he showed that each linearly independent set can be extended to a bases, using elements from a given base. Next Steinitz [1910], in his fundamental paper *Algebraische Theorie der Körper* (Algebraic Theory of Fields), showed that algebraic dependence has a number of basic properties, which makes it into a matroid (like the equicardinality of bases), and he derived some other properties from these basic properties (thus deriving essentially properties of matroids). In a subsequent paper, Steinitz [1913] gave, as an auxiliary result, the property that is now called *Steinitz' exchange property* for linearly independent sets of vectors. Steinitz did not mention the similarities to his earlier results on algebraic dependence. These similarities were observed by Haupt [1929a] and van der Waerden [1930] in their books on 'modern' algebra. They formulated properties shared by linear and algebraic dependence that are equivalent to matroids. In the second edition of his book, van der Waerden [1937] condensed these properties to three properties, and gave a unified treatment of linear and algebraic dependence. Mac Lane [1938] observed the relation of this work to the work on lattices and matroids.

A third line pursued the axiomatization of geometry, which clearly can be rooted back to as early as Euclid. At the beginning of the 20th century this was considered by, among others, Hilbert and Veblen. Bergmann [1929] aimed at giving a lattice-theoretical basis for affine geometry, and from lattice-theoretical conditions equivalent to matroids (cf. Theorem 39.11 above) he derived a number of properties, like the equicardinality of bases and the submodularity of the rank function. In their book *Grundlagen der Mathematik I* (Foundations of Mathematics I), Hilbert and Bernays [1934] gave axioms for the collinearity of triples of points, amounting to the fact that any two distinct points belong to exactly one line. A direct extension of these axioms to general dimensions gives the axioms described by Nakasawa [1935], that are again equivalent to the matroid axioms. He introduced the concept of a $\mathcal{B}_1$-space, equivalent to a matroid. In fact, the only reference in Nakasawa [1935] is to the book *Grundlagen der Elementargeometrie* (Foundations of Elementary Geometry) of Thomsen [1933], in which a different axiom system, the *Zyklenkalkül* (cycle calculus), was given (not equivalent to matroids). Nakasawa only gave subsets of linear spaces as an example. In a sequel to his paper, Nakasawa [1936b] observed that his axioms are equivalent to those of Whitney. The same axiom system as Nakasawa's, added with a continuity axiom, was given by Pauc [1937]. In Haupt, Nöbeling, and Pauc [1940] the concept of an *Abhängigkeitsraum* (dependence space) based on these axioms was investigated.

The fourth 'line' was that of Whitney [1935], who introduced the notion of a matroid as a concept by itself. He was motivated by generalizing certain separability and duality phenomena in graphs, studied by him before. This led him to show that each matroid has a dual. While Whitney showed the equivalence of several axiom systems for matroids, he did not consider an axiom system based on a closure operation or on flats. Whitney gave linear dependence as an example, but not algebraic dependence. In a paper in the same year and journal, Birkhoff [1935a] showed the relation of Whitney's work with lattices.

We now discuss some historical papers more extensively, in a more or less chronological order.

1894-1900: Dedekind: lattices

In the supplements to the fourth edition of *Vorlesungen über Zahlentheorie* (Lectures on Number Theory) by Lejeune Dirichlet [1894], R. Dedekind introduced the notion of a *module* as any nonempty set of (real or complex) numbers closed under addition and subtraction, and he studied the lattice of all modules ordered by inclusion. He called A *divisible by* B if $A \subseteq B$. Trivially, the lattice operations are given by $A \wedge B = A \cap B$ and $A \vee B = A + B$. In fact, Dedekind denoted $A \cap B$ by $A - B$.

He gave the following 'charakteristischen Satz' (characteristic theorem):

> *Ist m theilbar durch d, und a ein beliebiger Modul, so ist*
>
> $$m + (a - d) = (m + a) - d.$$ [3]

In modern notation, for all a, b, c:

(39.64) if $a \leq c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$,

which is now known as the *modular law*, and lattices obeying it are called *modular lattices*.

Next, Dedekind [1897] introduced the notion of a lattice under the name *Dualgruppe* (dual group), motivated by similarities observed by him between operations on modules and those for logical statements as given in the book *Algebra der Logik* (Algebra of Logic) by Schröder [1890]. Dedekind mentioned, as examples, subsets of a set, modules, ideals in a finite field, subgroups of a group, and all fields, and he introduced the name *module law* for property (39.64):

> ich will es daher das Modulgesetz nennen, und jede Dualgruppe, in welcher es herrscht, mag eine Dualgruppe vom Modultypus heißen.[4]

---

[3] *If m is divisible by d, and a is an arbitrary module, then*

$$m + (a - d) = (m + a) - d.$$

[4] I will therefore call it the module law, and every dual group in which it holds, may be called a dual group of module type.

Dedekind [1900] continued the study of modular lattices, and showed that each modular lattice allows a rank function $r : M \to \mathbb{Z}_+$ with the property that for all a, b:

(39.65) (i) $r(0) = 0$;
(ii) $r(b) = r(a) + 1$ if b covers a;
(iii) $r(a \wedge b) + r(a \vee b) = r(a) + r(b)$.

In fact, this characterizes modular lattices.

In proving (39.65), Dedekind showed that each modular lattice satisfies

(39.66) if a and b cover c, and $a \neq b$, then $a \vee b$ covers a and b,

which is the property characterizing *upper semimodular lattices*, a structure equivalent to matroids.

1862-1913: Grassmann, Steinitz: linear and algebraic dependence

The basic exchange property of linear independence was formulated by Grassmann [1862], in his book *Die Ausdehnungslehre*, as follows (in his terminology, vectors are quantities):

> **20.** *Wenn m Grössen $a_1, \ldots a_m$, die in keiner Zahlbeziehung zu einander stehen, aus n Grössen $b_1, \ldots b_n$ numerisch ableitbar sind, so kann man stets zu den m Grössen $a_1, \ldots a_m$ noch $(n-m)$ Grössen $a_{m+1}, \ldots a_n$ von der Art hinzufügen, dass sich die Grössen $b_1, \ldots b_n$ auch aus $a_1, \ldots a_n$ numerisch ableiten lassen, und also das Gebiet der Grössen $a_1, \ldots a_n$ identisch ist dem Gebiete der Grössen $b_1, \ldots b_n$; auch kann man jene $(n-m)$ Grössen aus den Grössen $b_1, \ldots b_n$ selbst entnehmen.*[5]

This property was also given by Steinitz [1913] (see below), but before that, Steinitz proved it for algebraic independence. In his fundamental paper *Algebraische Theorie der Körper* (Algebraic Theory of Fields), Steinitz [1910] studied, in § 22, algebraic dependence in field extensions. The statements proved are as follows, where L is a field extension of field K. Throughout, a is *algebraically dependent* on S if a is algebraic with respect to the field extension $K(S)$; in other words, if there is a nonzero polynomial $p(x) \in K(S)[x]$ with $p(a) = 0$.

Calling a set a *system*, he first observed:

> 1. *Hängt das Element a vom System S algebraisch ab, so gibt es ein endliches Teilsystem S' von S, von welchem a algebraisch abhängt.*[6]

and next he showed:

> 2. *Hängt S_3 von S_2, S_2 von S_1 algebraisch ab, so ist S_3 algebraisch abhängig von S_1.*[7]

---

[5] **20.** *If m quantities $a_1, \ldots a_m$, that stand in no number relation to each other, are numerically derivable from n quantities $b_1, \ldots b_n$, then one can always add to the m quantities $a_1, \ldots a_m$ another $(n-m)$ quantities $a_{m+1}, \ldots a_n$ such that the quantities $b_1, \ldots b_n$ can also be derived numerically from $a_1, \ldots a_n$, and that hence the domain of the quantities $a_1, \ldots a_n$ is identical to the domain of the quantities $b_1, \ldots b_n$; one also can take those $(n-m)$ quantities from the quantities $b_1, \ldots b_n$ themselves.*

[6] 1. *If element a depends algebraically on the system S, then there is a finite subsystem S' of S on which a depends algebraically.*

[7] 2. *If S_3 depends algebraically on S_2, and S_2 on S_1, then S_3 is algebraically dependent on S_1.*

He called two sets S_1 and S_2 *equivalent* if S_1 depends algebraically on S_2, and conversely. A set is *reducible* if it has a proper subset equivalent to it. He showed:

> 3. *Jedes Teilsystem eines irreduziblen Systems ist irreduzibel.*
> 4. *Jedes reduzible System enthält ein endliches reduzibles Teilsystem.*[8]

and (after statement 5, saying that any two field extensions by equicardinal irreducible systems are isomorphic):

> 6. *Wird ein irreduzibles System S durch Hinzufügung eines Elementes a reduzibel, so ist a von S algebraisch abhängig.*[9]

From these properties, Steinitz derived:

> 7. *Ist S ein (in bezug auf K) irreduzibles System, das Element a in bezug auf K transzendent, aber von S algebraisch abhängig, so enthält S ein bestimmtes endliches Teilsystem T von folgender Beschaffenheit: a ist von T algebraisch abhängig; jedes Teilsystem von S, von welchem a algebraisch abhängt, enthält das System T; wird irgendein Element aus T durch a ersetzt, so geht S in ein äquivalentes irreduzibles System über; keinem der übrigen Elemente von S kommt diese Eigenschaft zu.*[10]

Steinitz proved this using only the properties given above (together with the fact that any $s \in S$ is algebraically dependent on S). Moreover, he derived from 7, (what is now called) *Steinitz' exchange property* for algebraic dependence:

> 8. *Es seien U und B endliche irreduzible Systeme von m bzw. n Elementen; es sei $n \leq m$ und B algebraisch abhängig von U. Dann sind im Falle $m = n$ die Systeme U und B äquivalent, im Falle $n < m$ aber ist U einem irreduziblen System äquivalent, welches aus B und $m - n$ Elementen aus U besteht.*[11]

This in particular implies that any two equivalent irreducible systems have the same size, and that the properties are equivalent to that determining a matroid.

In a subsequent paper, Steinitz [1913] proved a number of auxiliary statements on linear equations. Among other things, he showed (in his terminology, vectors are numbers, and a vector space is a module):

> *Besitzt der Modul M eine Basis von p Zahlen, und enthält er r linear unabhängige Zahlen $\beta_1, \ldots, \beta_r$, so besitzt er auch eine Basis von p Zahlen, unter denen die Zahlen $\beta_1, \ldots, \beta_r$ sämtlich vorkommen.*[12]

---

[8] 3. *Every subsystem of an irreducible system is irreducible.*
4. *Every reducible system contains a finite reducible subsystem.*

[9] 6. *If an irreducible system S becomes reducible by adding an element a, then a is algebraically dependent on S.*

[10] 7. *If S is an irreducible system (with respect to K), [and] the element a transcendent with respect to K, but algebraically dependent on S, then S contains a certain finite subsystem T with the following quality: a is algebraically dependent on T; every subsystem of S on which a depends algebraically, contains the system T; if any element from T is replaced by a, then S passes into an equivalent irreducible system; this property belongs to none of the other elements of S .*

[11] 8. *Let U and B be finite irreducible systems of m and n elements respectively; let $n \leq m$ and let B be algebraically dependent on U. Then, in case $m = n$, the systems U and B are equivalent, but in case $n < m$, U is equivalent to an irreducible system which consists of B and $m - n$ elements from U.*

[12] *If a module M possesses a base of p numbers, and it contains r linearly independent numbers $\beta_1, \ldots, \beta_r$, then it possesses also a base of p numbers, among which the numbers $\beta_1, \ldots, \beta_r$ all occur.*

Steinitz' proof of this in fact gives a stronger result, known as *Steinitz' exchange property*: the new base is obtained by extending $\beta_1, \ldots, \beta_r$ with vectors from the given base. So Steinitz came to the same result as Grassmann [1862] quoted above. In his paper, Steinitz [1913] did not make a link with similar earlier results in Steinitz [1910] on algebraic dependence.

1929: Bergmann

Inspired by Menger [1928a], who aimed at giving an axiomatic foundation for projective geometry on a lattice-theoretical basis, Bergmann [1929] gave an axiomatic foundation of affine geometry, again on the basis of lattices. Bergmann's article contains a number of proofs that in fact concern matroids, while he assumed, but not used, a complementation axiom (since he aimed at characterizing full affine spaces, not subsets of it): for each pair of elements $A \leq B$ there exist C_1 and C_2 with $A \vee C_1 = B$, $A \wedge C_1 = 0$, $B \wedge C_2 = A$, and $B \vee C_2 = 1$. This obviously implies (in the finite case) that

(39.67) each element of the lattice is a join of points.

(A *point* is a minimal nonzero element.) It is property (39.67) that Bergmann uses in a number of subsequent arguments (and not the complementation axiom). His further axiom is:

(39.68) for any element A and any point P of the lattice, there is no element B with $A < B < A \vee P$.

He called an ordered sequence $(P_1, \ldots, P_n)$ of points a *chain* (Kette) (*of* an element A), if $P_i \not\leq P_1 \vee \cdots \vee P_{i-1}$ for $i = 1, \ldots, n$ (and $A = P_1 \vee \cdots \vee P_n$). He derived from (39.67) and (39.68) that being a chain is independent of the order of the elements in the chain, and that any two chains of an element A have the same length:

Satz: Alle Ketten eines Elementes A haben dieselbe Gliederzahl.[13]

He remarked that under condition (39.67), this in turn implies (39.68).

Denoting the length of any chain of A by $|A|$, Bergmann showed that it is equal to the rank of A in the lattice, and he derived the submodular inequality:

$$|A| + |B| \geq |A + B| + |A \cdot B|.$$

(Bergmann denoted $\vee$ and $\wedge$ by $+$ and $\cdot$.) Thus he proved the submodularity of the rank function of a matroid. These results were also given by Alt [1936] in Menger's *mathematischen Kolloquium* in Vienna on 1 March 1935 (cf. Menger [1936a,1936b]).

1929-1937: Haupt, van der Waerden

Inspired by the work of Steinitz, in the books *Einführung in die Algebra* (Introduction to Algebra) by Haupt [1929a,1929b] and *Moderne Algebra* (Modern Algebra) by van der Waerden [1930], the analogies between proof methods for linear and algebraic dependence were observed.

Haupt mentioned in his preface (after saying that his book will contain the modern developments of algebra):

[13] *Theorem: All chains of an element A have the same number of members.*

> Demgemäß ist das vorliegende Buch durchweg beeinflußt von der bahnbrechenden ,,Algebraischen Theorie der Körper" von Herrn E. Steinitz, was hier ein für allemal hervorgehoben sei. Ferner stützt sich die Behandlung der linearen Gleichungen (vgl. *9,1* bis *9,4*), einer Anregung von Frl. E. Noether folgend, auf die von Herrn E. Steinitz gegebene Darstellung (vgl. das Zitat in *9,0*).[14]

(The quotation in Haupt's '*9,0*' is to Steinitz [1910,1913].)

A number of theorems on algebraic dependence were proved in Chapter 23 of Haupt [1929b] by referring to the proofs of the corresponding results on linear dependence in Chapter 9 of Haupt [1929a]. In the introduction of his Chapter 9, Haupt wrote:

> Die Behandlung der linearen Gleichungen ist (soweit es geht) so angelegt, daß sich ein Teil der dabei gewonnenen Sätze auf *Systeme von algebraisch abhängigen Elementen überträgt*, was später (*23,6*) dargelegt wird.[15]

In the first edition of his book, van der Waerden [1930] listed the properties of algebraic dependence:

> Die Relation der algebraischen Abhängigkeit hat demnach die folgenden Eigenschaften:
> 1. a ist abhängig von sich selbst, d.h. von der Menge $\{a\}$.
> 2. Ist a abhängig von M, so hängt es auch von jeder Obermenge von M ab.
> 3. Ist a abhängig von M, so ist a schon von einer endlichen Untermenge $\{m_1, \ldots, m_n\}$ von M (die auch leer sein kann) abhängig.
> 4. Wählt man diese Untermenge minimal, so ist jedes m_i von a und den übrigen m_j abhängig.
> Weiter gilt:
> 5. Ist a abhängig von M und jedes Element von M abhängig von N, so ist a abhängig von N.[16]

Following Steinitz, van der Waerden called two sets *equivalent* if each element of the one set depends algebraically on the other set, and vice versa, while a set is *irreducible* if no element of it depends algebraically on the remaining.

Using only the properties 1-5, van der Waerden derived that each set contains an irreducible set equivalent to it, and that if $M \subseteq N$, then each irreducible subset of M equivalent to M can be extended to an irreducible subset of N equivalent to N — in other words, inclusionwise minimal subsets of M equivalent to M are

[14] Accordingly, the present book is invariably influenced by the pioneering 'Algebraic Theory of Fields' by Mr E. Steinitz, which be emphasized here once and for all. Further, following a suggestion by Miss E. Noether, the treatment of linear equations (cf. *9,1* to *9,4*) leans on the presentation by Mr E. Steinitz (cf. the quotation in *9,0*).

[15] The treatment of linear equations is (as far as it goes) made such that a part of the theorems obtained therewith *transfers to systems of algebraically dependent elements*, which will be discussed later (*23,6*).

[16] The relation of algebraic dependence has therefore the following properties:
1. a is dependent on itself, that is, on the set $\{a\}$.
2. If a is dependent on M, then it also depends on every superset of M.
3. If a is dependent on M, then a is dependent already on a finite subset $\{m_1, \ldots, m_n\}$ of M (that can also be empty).
4. If one chooses this subset minimal, then every m_i is dependent on a and the remaining m_j.
Further it holds:
5. If a is dependent on M and every element of M is dependent on N, then a is dependent on N.

independent, and inclusionwise maximal independent subsets of M are equivalent to M.

Van der Waerden [1930] also showed that two equivalent irreducible systems have the same size, but in the proof he uses polynomials. This is not necessary, since the properties 1-5 determine a matroid.

Van der Waerden noticed the analogy with linear dependence, treated in his § 28, where he uses specific facts on linear equations:

> Tatsächlich gelten für die dort betrachtete lineare Abhängigkeit dieselben Regeln 1 bis 5, die für die algebraische Abhängigkeit in § 61 aufgestellt wurden; man kann also alle Beweise wörtlich übertragen.[17]

In the second edition of his book, van der Waerden [1937] gave a unified treatment of linear and algebraic dependence, slightly different from the first edition. As for linear dependence he stated in § 33:

> Drei Grundsätze genügen. Der erste ist ganz selbstverständlich.
> Grundsatz 1. *Jedes* u_i $(i = 1, \ldots, n)$ *ist von* $u_1, \ldots, u_n$ *linear abhängig.*
> Grundsatz 2. *Ist* v *linear abhängig von* $u_1, \ldots, u_n$*, aber nicht von* $u_1, \ldots, u_{n-1}$*, so ist* u_n *linear abhängig von* $u_1, \ldots, u_{n-1}, v$.
> [· · ·]
> Grundsatz 3. *Ist* w *linear abhängig von* $v_1, \ldots, v_s$ *und ist jedes* v_j $(j = 1, \ldots, s)$ *linear abhängig von* $u_1, \ldots, u_n$*, so ist* w *linear abhängig von* $u_1, \ldots, u_n$.[18]

The same axioms are given in § 64 of van der Waerden [1937], with 'linear' replaced by 'algebraisch'.

Next, van der Waerden called elements $u_1, \ldots, u_n$ (linearly or algebraically) independent if none of them depend on the rest of them. Among the consequences of these principles, he mentioned that if $u_1, \ldots, u_{n-1}$ are independent but $u_1, \ldots, u_{n-1}, u_n$ are not, then u_n is dependent on $u_1, \ldots, u_{n-1}$, and that each finite system of elements $u_1, \ldots, u_n$ contains a (possibly empty) independent subsystem on which each u_i is dependent. He called two systems $u_1, \ldots, u_n$ and $v_1, \ldots, v_s$ *equivalent* if each v_k depends on $u_1, \ldots, u_n$ and each u_i depends on $v_1, \ldots, v_s$, and he now derived from the three principles that two equivalent independent systems have the same size.

Mac Lane [1938] observed that the axioms introduced by Whitney [1935] and those by van der Waerden [1937] determine equivalent structures.

1934: Hilbert, Bernays: collinearity axioms

Axiom systems for points and lines in a plane were given by Hilbert [1899] in his book *Grundlagen der Geometrie* (Foundations of Geometry), and by Veblen [1904].

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[17] In fact, the same rules 1 to 5, that were formulated for algebraic dependence in § 61, hold for the linear dependence considered there; one can transfer therefore all proofs word for word.

[18] Three principles suffice. The first one is fully self-evident.
Principle 1. *Every* u_i $(i = 1, \ldots, n)$ *is linearly dependent on* $u_1, \ldots, u_n$.
Principle 2. *If* v *is linearly dependent on* $u_1, \ldots, u_n$*, but not on* $u_1, \ldots, u_{n-1}$*, then* u_n *is linearly dependent on* $u_1, \ldots, u_{n-1}, v$.
[· · ·]
Principle 3. *If* w *is linearly dependent on* $v_1, \ldots, v_s$ *and every* v_j $(j = 1, \ldots, s)$ *is linearly dependent on* $u_1, \ldots, u_n$*, then* w *is linearly dependent on* $u_1, \ldots, u_n$.

Basis is the axiom that any two distinct points are in exactly one line. Note that this axiom determines precisely all matroids of rank at most 3 with no parallel elements (by taking the lines as maximal flats).

One of the axioms of Veblen is:

> Axiom VI. If points C and D ($C \neq D$) lie on the line AB, then A lies on the line CD.

This axiom corresponds to axiom 3) in the book *Grundlagen der Mathematik* (Foundations of Mathematics) of Hilbert and Bernays [1934], who aim to make an axiom system based on points only:

> Dabei empfiehlt es sich für unseren Zweck, von dem HILBERTschen Axiomensystem darin abzuweichen, daß wir nicht die Punkte und die Geraden als zwei Systeme von Dingen zugrunde legen, sondern *nur die Punkte als Individuen nehmen.*[19]

The axiom system of Hilbert and Bernays is in terms of a relation Gr to describe collinearity of triples of points (where (x) stands for $\forall x$, (Ex) for $\exists x$, and $\overline{P}$ for the negation of P):

> I. Axiome der Verknüpfung.
>
> 1) $(x)(y)Gr(x, x, y)$
> ,,x, x, y liegen stets auf einer Geraden.“
> 2) $(x)(y)(z)(Gr(x, y, z) \to Gr(y, x, z)\&Gr(x, z, y))$.
> ,,Wenn x, y, z auf einer Geraden liegen, so liegen stets auch y, x, z sowie auch x, z, y auf einer Geraden.“
> 3) $(x)(y)(z)(u)(Gr(x, y, z)\&Gr(x, y, u)\&x \neq y \to Gr(x, z, u))$.
> ,,Wenn x, y, verschiedene Punkte sind und wenn x, y, z sowie x, y, u auf einer Geraden liegen, so liegen stets auch x, z, u auf einer Geraden.“
> 4) $(Ex)(Ey)(Ez)\overline{Gr(x, y, z)}$.
> ,,Es gibt Punkte x, y, z, die nicht auf einer Geraden liegen.“[20]

The axioms 1) and 2) in fact tell that the relation Gr is determined by unordered triples of distinct points. The exchange axiom 3) is a special case of the matroid axiom for circuits in a matroid.

Hilbert and Bernays extended the system by axioms for a betweenness relation Zw for ordered triples of points, and a parallelism relation Par for ordered quadruples of points.

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[19] At that it is advisable for our purpose to deviate from HILBERT's axiom system in that we do not lay the points and the lines as two systems of things as base, but *take only the points as individuals.*

[20] I. Axioms of connection.

1) $(x)(y)Gr(x, x, y)$
‘x, x, y always lie on a line.’
2) $(x)(y)(z)(Gr(x, y, z) \to Gr(y, x, z)\&Gr(x, z, y))$.
‘If x, y, z lie on a line, then also y, x, z as well as x, z, y always lie on a line.’
3) $(x)(y)(z)(u)(Gr(x, y, z)\&Gr(x, y, u)\&x \neq y \to Gr(x, z, u))$.
‘If x, y, are different points and if x, y, z as well as x, y, u lie on a line, then also x, z, u always lie on a line.’
4) $(Ex)(Ey)(Ez)\overline{Gr(x, y, z)}$.
‘There are points x, y, z, that do not lie on a line.’

1933–1935: Birkhoff: Lattices

In his paper 'On the combination of subalgebras', Birkhoff [1933] ('Received 15 May 1933') wrote:

> The purpose of this paper is to provide a point of vantage from which to attack combinatorial problems in what may be termed modern, synthetic, or abstract algebra. In this spirit, a research has been made into the consequences and applications of seven or eight axioms, only one [V] of which itself is new.

The axioms are those for a lattice, added with axiom V, that amounts to (39.64) above. Any lattice satisfying this condition is called by Birkhoff in this paper a 'B-lattice'. In an addendum, Birkhoff [1934b] mentioned that O. Ore had informed him that part of his results had been obtained before by Dedekind [1900]. Therefore, Birkhoff [1935b] renamed it to *modular lattice*.

Birkhoff [1933] mentioned, as examples, the classes of normal subgroups and of characteristic subgroups of a group. Other examples mentioned are the ideals of a ring, and the linear subspaces of Euclidean space. (Both examples actually give sublattices of the lattice of all normal subgroups of the corresponding groups.)

Like Dedekind, Birkhoff [1933] showed that (39.64) implies (39.66). Lattices satisfying (39.66) are called (upper) *semimodular*. Birkhoff showed that any upper semimodular lattice has a rank function satisfying (39.65)(i) and (ii) and satisfying the submodular law:

(39.69) $$r(a \cap b) + r(a \cup b) \leq r(a) + r(b).$$

This characterizes upper semimodular lattices.

Birkhoff noticed that this implies that the modular lattices are exactly those lattices satisfying both (39.66) and its symmetric form:

(39.70) if c covers a and b and $a \neq b$, then a and b cover $a \wedge b$.

Birkhoff [1935c] showed that the partition lattice is upper semimodular, that is, satisfies (39.66), and hence has a rank function satisfying the submodular inequality[21]. Thus the complete graph, and hence any graph, gives a geometric lattice (and hence a matroid — however, Whitney's work seems not to have been known yet to Birkhoff at the time of writing this paper).

In a number of other papers, Birkhoff [1934a,1934c,1935b] made a further study of modular lattices, and gave relations to projective geometries (in which the collection of all flats gives a modular lattice). Klein-Barmen [1937] further investigated semimodular lattices (called by him *Birkhoffsche Verbände* (Birkhoff lattices)), of which he found several lattice-theoretical characterizations.

1935: Whitney: Matroids

Whitney [1935] (presented to the American Mathematical Society, September 1934) introduces the notion of matroid as follows:

[21] In fact, Birkhoff [1935c] claimed the modular equality for the rank function of a partition lattice (page 448), but this must be a typo, witness the formulation of, and the reference in, the first footnote on that page.

> Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix M. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:
>
> (a) Any subset of an independent set is independent.
>
> (b) If N_p and N_{p+1} are independent sets of p and $p+1$ columns respectively, then N_p together with some column of N_{p+1} forms an independent set of $p+1$ columns.
>
> There are other theorems not deducible from this; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

In the paper, Whitney observed that forests in a graph form the independent sets of a matroid, for which reason he carried over various terms from graphs to matroids.

Whitney described several equivalent axiom systems for the notion of matroid. First, he showed that the rank function is characterized by (39.42), and he derived that it is submodular. Next, he showed that the collection of bases is characterized by (39.33)(ii), and the collection of circuits by (39.34)(iii). Moreover, he showed that complementing all bases gives again a matroid, the dual matroid, and that the dual of a linear matroid is again a linear matroid. In the paper, he also studied separability and representability of matroids. The example given in Whitney's § 16 (mentioned in the above quotation), is in fact the well-known Fano matroid — he apparently did not consider matrices over GF(2). However, in an appendix of the paper, he characterized the matroids representable by a matrix 'of integers mod 2': a matroid is representable over GF(2) if and only if any sum (mod 2) of circuits can be partitioned into circuits.

In a subsequent paper 'Abstract linear independence and lattices', Birkhoff [1935a] pointed out the relations of Whitney's work with Birkhoff's earlier work on semimodular lattices. He stated:

> In a preceding paper, Hassler Whitney has shown that it is difficult to distinguish theoretically between the properties of linear dependence of ordinary vectors, and those of elements of a considerably wider class of systems, which he has called "matroids."
>
> Now it is obviously impossible to incorporate all of the heterogeneous abstract systems which are constantly being invented, into a body of systematic theory, until they have been classified into two or three main species. The purpose of this note is to correlate matroids with abstract systems of a very common type, which I have called "lattices."

Birkhoff showed that a lattice is isomorphic to the lattice of flats of a matroid if and only if the lattice is semimodular, that is, satisfies (39.66), and each element is a join of atoms.

In the paper 'Some interpretations of abstract linear dependence in terms of projective geometry', MacLane [1936] gave a geometric interpretation of matroids. He introduced the notion of a 'schematic n-dimensional figure', consisting of 'k-dimensional planes' for $k = 1, 2, \ldots$. Each such plane is a subset of an (abstract) set of 'points', with the following axioms (for any appropriate k):

(39.71) (i) any k points belonging to no $k-1$-dimensional plane, belong to a unique k-dimensional plane; moreover, this plane is contained in any plane containing these k points;
(ii) every k-dimensional plane contains k points that belong to no $k-1$-dimensional plane.

MacLane mentioned that there is a 1-1 correspondence between schematic figures and the collections of flats of matroids. As a consequence he mentioned that a schematic n-dimensional figure is completely determined by its collection of $n-1$-dimensional planes (as a matroid is determined by its hyperplanes = complements of cocircuits).

1935: Nakasawa: Abhängigkeitsräume

In the paper *Zur Axiomatik der linearen Abhängigkeit. I* (On the axiomatics of linear dependence. I) in *Science Reports of the Tokyo Bunrika Daigaku* (Tokyo University of Literature and Science), Nakasawa [1935] introduced an axiom system for dependence, that he proved to be equivalent to matroids (in a different terminology).

He was motivated by an axiom system described by Thomsen [1933] in his book *Grundlagen der Elementargeometrie* (Foundations of Elementary Geometry). Thomsen's 'cycle calculus' is an attempt to axiomatize relations (like coincidence, orthogonality, parallelism) between geometric objects (points, lines, etc.). Thomsen emphasized that existence questions often are inessential in elementary geometry:

> In der Tat erscheinen uns ja auch die Existenzaussagen als ein verhältnismäßig unwesentliches Beiwerk der Elementargeometrie. Ohne Zweifel empfinden wir als die eigentlich inhaltsvollsten und die wichtigsten Einzelaussagen der Elementargeometrie die von der folgenden reinen Form: ,,Wenn eine Reihe von geometrischen Gebilden, d.h. eine Anzahl von Punkten, Geraden, usw., gegeben vorliegt, und zwar derart, daß zwischen den gegebenen Punkten, Geraden usw. die und die geometrischen Lagebeziehungen bestehen (Koinzidenz, Senkrechtstehen, Parallellaufen, ,,Mittelpunkt sein" und anderes mehr), dann ist eine notwendige Folge dieser Annahme, daß auch noch diese bestimmte weitere geometrische Lagebeziehung gleichzeitig besteht." In Sätzen dieser Form kommt nichts von Existenzaussagen vor. Was das Wichtigste ist, nicht in den Folgerungen. Dann aber auch nicht in den Annahmen. Wir nehmen an: *Wenn* die und die Dinge in den und den Beziehungen gegeben vorliegen..., usw. Wir machen aber keinerlei Voraussetzungen darüber, ob eine solche Konfiguration in unserer Geometrie existieren kann. Der Schluß ist nur: Wenn sie existieren, dann .... Falls die Konfiguration gar nicht existiert, der Satz also gegenstandslos wird, betrachten wir ihn nach der üblichen Konvention ,,gegenstandslos, also richtig" als richtig.[22]

[22] Indeed, also the existence statements seem to us a relatively inessential side issue of elementary geometry. Undoubtedly, we find as the really most substantial and most important special statements of elementary geometry those of the following pure form: 'If a sequence of geometric creations, that is, a number of points, lines etc., are given to us, and that in such a way, that those and those geometric position relations exist between the given points, lines etc. (coincidence, orthogonality, parallelism, "being a centre", and other), then a necessary consequence of this assumption is that also this certain further geometric position relation exists at the same time.' In theorems of this form, no existence statements occur. What is most important: not in the consequences. But then neither in the assumptions. We assume: *If* those and those things are given

Thomsen aimed at founding axiomatically 'the partial geometry of all elementary geometric theorems without existence statements'. To that end, he introduced the concept of a *cycle*, which is an ordered finite sequence of abstract objects, which can be thought of as points, lines, etc. Certain cycles are 'correct' and the other 'incorrect' (essentially they represent a system of relations defining any binary group):

> A) *Axiom der Grundzyklen:* Der Zyklus $\alpha\alpha$ ist für jedes α richtig, der Zyklus α für kein α.
> B) *Axiom des Löschens:* $\beta_1\beta_2\ldots\beta_n\alpha\alpha \to \beta_1\beta_2\ldots\beta_n$; in Worten: Aus der Richtigkeit des Zyklus $\beta_1\beta_2\ldots\beta_n\alpha\alpha$ folgt auch die des Zyklus $\beta_1\beta_2\ldots\beta_n$.
> C) *Axiom des Umstellens:* $\beta_1\beta_2\ldots\beta_n \to \beta_2\beta_3\ldots\beta_n\beta_1$.
> D) *Axiom des Umkehrens:* $\beta_1\beta_2\ldots\beta_{n-1}\beta_n \to \beta_n\beta_{n-1}\ldots\beta_2\beta_1$.
> E) *Axiom des Anfügens:* $\beta_1\beta_2\ldots\beta_n$ und $\gamma_1\gamma_2\ldots\gamma_r \to \beta_1\beta_2\ldots\beta_n\gamma_1\gamma_2\ldots\gamma_r$.[23]

Axiom B) can be considered as a variant of Steinitz' exchange property. With the other axioms it implies that if $\beta_1\cdots\beta_n\alpha$ and $\gamma_1\cdots\gamma_r\alpha$ are cycles, then $\beta_1\cdots\beta_n\gamma_1\cdots\gamma_r$ is a cycle. Therefore, the set of all inclusionwise minimal nonempty sets containing a cycle form the circuits of a matroid.

The purpose of Nakasawa [1935] is to generalize Thomsen's axiom system:

> In der vorliegenden Untersuchung soll ein Axiomensystem für eine neue Formulierung der linearen Abhängigkeit des n-dimensionalen projektiven Raumes angegeben werden, indem wir hauptsächlich den *Zyklenkalkül*, den Herr G. Thomsen bei seiner Grundlegung der elementaren Geometrie hergestellt hat, hier in einem noch abstrakteren Sinne verwenden.[24]

While Thomsen's cycles relate to unions of circuits in a matroid, those of Nakasawa form the dependent sets of a matroid. His axiom system can be considered as a direct extension to higher dimensions of the collinearity axioms of Hilbert and Bernays given above.

He called the structure *der erste Verknüpfungsraum* (the first connection space), or a $\mathcal{B}_1$-*Raum* ($\mathcal{B}_1$-space), writing $a_1\cdots a_s$ for $a_1\cdots a_s = 0$:

> **Grundannahme:** Wir denken uns eine gewisse Menge der Elementen; $\mathcal{B}_1 \ni a_1, a_2, \cdots, a_s, \cdots$. Für gewisse Reihen der Elementen, die wir *Zyklen* nennen wollen, denken wir dazu die Relationen "*gelten*" oder "*gültig sein*", in Zeichen $a_1\cdots a_s = 0$, bzw. "*nicht gelten*" oder "*nicht gültig sein*", in Zeichen $a_1\cdots a_s \neq 0$. Diese Relationen sollen nun folgenden Axiomen genügen;

to us in those and those relations..., etc. We do not make any assumption on the fact if such a configuration can exist in our geometry. The conclusion is only: If they exist, then .... In case the configuration does not exist at all, and the theorem thus becomes meaningless, we consider it by the usual convention 'meaningless, hence correct' as correct.

23

A) *Axiom of ground cycles:* The cycle $\alpha\alpha$ is correct for each α, the cycle α for no α.
B) *Axiom of solving:* $\beta_1\beta_2\ldots\beta_n\alpha\alpha \to \beta_1\beta_2\ldots\beta_n$; in words: From the correctness of the cycle $\beta_1\beta_2\ldots\beta_n\alpha\alpha$ follows that of the cycle $\beta_1\beta_2\ldots\beta_n$.
C) *Axiom of transposition:* $\beta_1\beta_2\ldots\beta_n \to \beta_2\beta_3\ldots\beta_n\beta_1$.
D) *Axiom of inversion:* $\beta_1\beta_2\ldots\beta_{n-1}\beta_n \to \beta_n\beta_{n-1}\ldots\beta_2\beta_1$.
E) *Axiom of addition:* $\beta_1\beta_2\ldots\beta_n$ and $\gamma_1\gamma_2\ldots\gamma_r \to \beta_1\beta_2\ldots\beta_n\gamma_1\gamma_2\ldots\gamma_r$.

24 In the present research, an axiom system for a new formulation of linear dependence of the n-dimensional projective space should be indicated, while we use here mainly the *cycle calculus*, which Mr G. Thomsen has constructed in his foundation of elementary geometry, in a still more abstract sense.

> **Axiom 1.** (Reflexivität) : aa.
> **Axiom 2.** (Folgerung) : $a_1 \cdots a_s \to a_1 \cdots a_s x$, $(s = 1, 2, \cdots)$.
> **Axiom 3.** (Vertauschung) : $a_1 \cdots a_i \cdots a_s \to a_i \cdots a_1 \cdots a_s$, $(s = 2, 3, \cdots; i = 2, \cdots, s)$.
> **Axiom 4.** (Transitivität) : $a_1 \cdots a_s \neq 0$, $xa_1 \cdots a_s$, $a_1 \cdots a_s y \to xa_1 \cdots a_{s-1} y$, $(s = 1, 2, \cdots)$.
>
> **Definition I.** Eine solche Menge $\mathcal{B}_1$ heisst der erste Verknüpfungsraum, in kurzen Worten, $\mathcal{B}_1$-**Raum**.[25]

Axiom 3 corresponds to condition (39.3).

Nakasawa introduced the concept of span, and he derived that any two independent sets having the same span, have the same size. It implies that $\mathcal{B}_1$-spaces are the same structures as matroids. Moreover, he gave a submodular law for a rank concept.

In a second paper, Nakasawa [1936a] added a further axiom on intersections of subspaces, yielding a '$\mathcal{B}_2$-space', which corresponds to a projective space (in which the rank is modular), and in a third paper, Nakasawa [1936b] observed that his $\mathcal{B}_1$-spaces form the same structure as the matroids of Whitney.

1937-1940: Pauc, Haupt, Nöbeling

The axioms presented by Nakasawa were also given by Pauc [1937], added with an axiom describing the limit behaviour of dependence, if the underlying set is endowed with a topology:

> Introduction axiomatique d'une notion de dépendance sur une classe limite. — Soit D un prédicat relatif aux systèmes finis non ordonnés de points d'une classe limite $\mathcal{L}$, assujetti aux axiomes (notation d'Hilbert-Bernays)
>
> (A$_1$) $(x_1)(x_2)(D[x_1, x_2] \sim (x_1 = x_2))$,
>
> (A$_2$) $(x_1)(x_2)\ldots(x_p)(y)(D[x_1, x_2, \ldots, x_p] \to D[x_1, x_2, \ldots, x_p, y])$,
>
> (A$_3$) $(x_1)(x_2)\ldots(x_p)(y)(z)(\overline{D[x_1, \ldots, x_p]} \& D[x_1, \ldots, x_p, y] \& D[x_1, \ldots, x_p, z] \to D[x_2, \ldots, x_p, y, z])$,
>
> (A$_4$) Quels que soient les points $x_1, x_2, \ldots, x_p$ et la suite $y_1, y_2, \ldots, y_q, \ldots$ de $\mathcal{L}$ $(\lim_{q\to\infty} y_q = y) \& (q) D[x_1, x_2, \ldots, x_p, y_q] \to D[x_1, x_2, \ldots, x_p, y]$.[26]

In a subsequent paper, Haupt, Nöbeling, and Pauc [1940] studied systems, called *A-Mannigfaltigkeit*, (*A*-manifolds) that satisfy the axioms A$_1$-A$_3$. They mentioned

---

[25] **Basic assumption:** We imagine ourselves a certain set of elements; $\mathcal{B}_1 \ni a_1, a_2, \cdots, a_s, \cdots$. For certain sequences of the elements, which we want to call *cycles*, we think the relations on them '*to hold*' or '*to be valid*', in notation $a_1 \cdots a_s = 0$, and '*not to hold*' or '*not to be valid*', in notation $a_1 \cdots a_s \neq 0$, respectively. These relations now should satisfy the following axioms;

Axiom 1. (reflexivity) : aa.
Axiom 2. (deduction) : $a_1 \cdots a_s \to a_1 \cdots a_s x$, $(s = 1, 2, \cdots)$.
Axiom 3. (exchange) : $a_1 \cdots a_i \cdots a_s \to a_i \cdots a_1 \cdots a_s$, $(s = 2, 3, \cdots; i = 2, \cdots, s)$.
Axiom 4. (transitivity) : $a_1 \cdots a_s \neq 0$, $xa_1 \cdots a_s$, $a_1 \cdots a_s y \to xa_1 \cdots a_{s-1} y$, $(s = 1, 2, \cdots)$.

Definition I. Such a set $\mathcal{B}_1$ is called the first connection space, in short, $\mathcal{B}_1$-**space**.

[26] Axiomatic introduction of a notion of dependence on a limit class. — Let D be a predicate relative to the finite unordered systems of points from a limit class $\mathcal{L}$, subject to the axioms (notation of Hilbert-Bernays)

that this axiom system was indeed inspired by those for collinearity of Hilbert-Bernays quoted above. They commented that its relation with Birkhoff's lattices, is analogous to the relation of the Hilbert-Bernays collinearity axioms with those of Hilbert for points and lines.

Haupt, Nöbeling, and Pauc [1940] gave, as examples, linear and algebraic dependence, and derived several basic facts (all bases have the same size, each independent set is contained in a base, for each pair of bases B, B' and $x \in B \setminus B'$ there is a $y \in B' \setminus B$ such that $B - x + y$ is a base, and the rank is submodular).

The authors mentioned that they were informed by G. Köthe about the relations of their work with the lattice formulation of algebraic dependence of Mac Lane [1938], but no connection is made with Whitney's matroid.

Among the further papers related to matroids are Menger [1936b], giving axioms for (full) affine spaces, and Wilcox [1939,1941,1942,1944] and Dilworth [1941a, 1941b,1944] on matroid lattices. The notion of M-symmetric lattice introduced by Wilcox [1942] was shown in Wilcox [1944] to be equivalent to upper semimodular lattice.

Rado

Rado was one of the first to take the independence structure as a source for further theorems, and to connect it with matching type theorems and combinatorial optimization. He had been interested in Kőnig-Hall type theorems (Rado [1933,1938]), and in his paper Rado [1942], he extended Hall's marriage theorem to transversals that are independent in a given matroid — a precursor of matroid intersection. In fact, with an elementary construction, Rado's theorem implies the matroid union theorem, and hence also the matroid intersection theorem (to be discussed in Chapters 41 and 42).

Rado [1942] did not refer to any earlier literature when introducing the concept of an *independence relation*, but the axioms are similar to those of Whitney for the independent sets in a matroid. Rado mentioned only linear independence as a special case.

He proved that a family of subsets of a matroid has an independent transversal if and only if the union of any k of the subsets contains an independent set of size k, for all k. Rado also showed that this theorem characterizes matroids.

Rado [1949a] extended the concept of matroid to infinite matroids, where he says that he extends the axioms of Whitney [1935].

Rado [1957] showed that if the elements of a matroid are linearly ordered by $\leq$, there is a unique minimal base $\{b_1, \ldots, b_r\}$ with $b_1 < b_2 < \cdots < b_r$ such that for each $i = 1, \ldots, r$ all elements $s < b_i$ belong to $\text{span}(\{b_1, \ldots, b_{i-1}\})$. Rado derived that for any independent set $\{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ one has $b_i \leq a_i$ for $i = 1, \ldots, k$. Therefore, the greedy method gives an optimum solution when

---

(A$_1$) $$(x_1)(x_2)(D[x_1, x_2] \sim (x_1 = x_2)),$$

(A$_2$) $$(x_1)(x_2)\ldots(x_p)(y)(D[x_1, x_2, \ldots, x_p] \to D[x_1, x_2, \ldots, x_p, y]),$$

(A$_3$) $$(x_1)(x_2)\ldots(x_p)(y)(z)(\overline{D[x_1, \ldots, x_p]}\& D[x_1, \ldots, x_p, y]\& D[x_1, \ldots, x_p, z] \to D[x_2, \ldots, x_p, y, z]),$$

(A$_4$) $$\begin{cases} \text{Whatever are the points } x_1, x_2, \ldots, x_p \text{ and the sequence } y_1, y_2, \ldots, y_q, \\ \ldots \text{ from } \mathcal{L} \\ (\lim_{q\to\infty} y_q = y)\&(q)D[x_1, x_2, \ldots, x_p, y_q] \to D[x_1, x_2, \ldots, x_p, y]. \end{cases}$$

applied to find a minimum-weight base. Rado mentioned that it extends the work of Borůvka and Kruskal on finding a shortest spanning tree in a graph.

For notes on the history of matroid union, see Section 42.6f. For an excellent survey of early literature on matroids, with reprints of basic articles, see Kung [1986].

Chapter 40

The greedy algorithm and the independent set polytope

We now pass to algorithmic and polyhedral aspects of matroids. We show that the greedy algorithm characterizes matroids and that it implies a characterization of the independent set polytope (the convex hull of the incidence vectors of the independent sets).
Algorithmic and polyhedral aspects of the *intersection* of two matroids will be studied in Chapter 41.

40.1. The greedy algorithm

Let $\mathcal{I}$ be a nonempty collection of subsets of a finite set S closed under taking subsets. For any weight function $w : S \to \mathbb{R}$ we want to find a set I in $\mathcal{I}$ maximizing $w(I)$. The *greedy algorithm* consists of setting $I := \emptyset$, and next repeatedly choosing $y \in S \setminus I$ with $I \cup \{y\} \in \mathcal{I}$ and with $w(y)$ as large as possible. We stop if no such y exists.

For general collections $\mathcal{I}$ of this kind this need not lead to an optimum solution. Indeed, matroids are precisely the structures where it always works, as the following theorem shows (Rado [1957] (necessity) and Gale [1968] and Edmonds [1971] (sufficiency)):

Theorem 40.1. *Let $\mathcal{I}$ be a nonempty collection of subsets of a set S, closed under taking subsets. Then the pair $(S, \mathcal{I})$ is a matroid if and only if for each weight function $w : S \to \mathbb{R}_+$, the greedy algorithm leads to a set I in $\mathcal{I}$ of maximum weight $w(I)$.*

Proof. *Necessity.* Let $(S, \mathcal{I})$ be a matroid and let $w : S \to \mathbb{R}_+$ be any weight function on S. Call an independent set I *good* if it is contained in a maximum-weight base. It suffices to show that if I is good, and y is an element in $S \setminus I$ with $I + y \in \mathcal{I}$ and with $w(y)$ as large as possible, then $I + y$ is good.

As I is good, there exists a maximum-weight base $B \supseteq I$. If $y \in B$, then $I + y$ is good again. If $y \notin B$, then there exists a base B' containing $I + y$ and contained in $B + y$. So $B' = B - z + y$ for some $z \in B \setminus I$. As $w(y)$ is chosen maximum and as $I + z \in \mathcal{I}$ since $I + z \subseteq B$, we know $w(y) \geq w(z)$.

Hence $w(B') \geq w(B)$, and therefore B' is a maximum-weight base. So $I + y$ is good.

Sufficiency. Suppose that the greedy algorithm leads to an independent set of maximum weight for each weight function $w : S \to \mathbb{R}_+$. We show that $(S, \mathcal{I})$ is a matroid.

Condition (39.1)(i) is satisfied by assumption. To see condition (39.1)(ii), let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose that $I + z \notin \mathcal{I}$ for each $z \in J \setminus I$.

Let $k := |I|$. Consider the following weight function w on S:

$$(40.1) \qquad w(s) := \begin{cases} k+2 & \text{if } s \in I, \\ k+1 & \text{if } s \in J \setminus I, \\ 0 & \text{if } s \in S \setminus (I \cup J). \end{cases}$$

Now in the first k iterations of the greedy algorithm we find the k elements in I. By assumption, at any further iteration, we cannot choose any element in $J \setminus I$. Hence any further element chosen, has weight 0. So the greedy algorithm yields an independent set of weight $k(k+2)$.

However, J has weight at least $|J|(k+1) \geq (k+1)(k+1) > k(k+2)$. Hence the greedy algorithm does not give a maximum-weight independent set, contradicting our assumption. ∎

The theorem restricts w to nonnegative weight functions. However, it is shown similarly that for matroids $M = (S, \mathcal{I})$ and arbitrary weight functions $w : S \to \mathbb{R}$, the greedy algorithm finds a maximum-weight base. By replacing 'as large as possible' in the greedy algorithm by 'as small as possible', one obtains an algorithm finding a *minimum*-weight base in a matroid. Moreover, by deleting elements of negative weight, the algorithm can be adapted to yield an independent set of maximum weight, for any weight function $w : S \to \mathbb{R}$.

Throughout we assume that the matroid $M = (S, \mathcal{I})$ is given by an algorithm testing if a given subset of S belongs to $\mathcal{I}$. We call this an *independence testing oracle*. So the full list of all independent sets is not given explicitly (such a list would increase the size of the input exponentially, making most complexity issues meaningless).

In explicit applications, the matroid usually can be described by such a polynomial-time algorithm (polynomial in $|S|$). For instance, we can test if a given set of edges of a graph $G = (V, E)$ is a forest in time polynomially bounded by $|V| + |E|$. So the matroid $(E, \mathcal{F})$ can be described by such an algorithm.

Under these assumptions we have:

Corollary 40.1a. *A maximum-weight independent set in a matroid can be found in strongly polynomial time.*

Proof. See above. ∎

Similarly, for minimum-weight bases:

Corollary 40.1b. *A minimum-weight base in a matroid can be found in strongly polynomial time.*

Proof. See above. ∎

40.2. The independent set polytope

The algorithmic results obtained in the previous section have interesting consequences for polyhedra associated with matroids, as was shown by Edmonds [1970b,1971,1979].

The *independent set polytope* $P_{\text{independent set}}(M)$ of a matroid $M = (S, \mathcal{I})$ is, by definition, the convex hull of the incidence vectors of the independent sets of M. So $P_{\text{independent set}}(M)$ is a polytope in $\mathbb{R}^S$.

Each vector x in $P_{\text{independent set}}(M)$ satisfies the following linear inequalities:

(40.2) $$\begin{aligned} &x_s \geq 0 && \text{for } s \in S, \\ &x(U) \leq r_M(U) && \text{for } U \subseteq S, \end{aligned}$$

because the incidence vector χ^I of any independent set I of M satisfies (40.2). Note that x is an integer vector satisfying (40.2) if and only if x is the incidence vector of some independent set of M.

Edmonds showed that system (40.2) fully determines the independent set polytope, by deriving it from the following formula (yielding a good characterization):

Theorem 40.2. *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r. Then for any weight function $w : S \to \mathbb{R}_+$:*

(40.3) $$\max\{w(I) \mid I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i),$$

where $U_1 \subset \cdots \subset U_n \subseteq S$ and where $\lambda_i \geq 0$ satisfy

(40.4) $$w = \sum_{i=1}^{n} \lambda_i \chi^{U_i}.$$

Proof. Order the elements of S as $s_1, \ldots, s_n$ such that $w(s_1) \geq w(s_2) \geq \cdots \geq w(s_n)$. Define

(40.5) $$U_i := \{s_1, \ldots, s_i\}$$

for $i = 0, \ldots, n$, and

(40.6) $$I := \{s_i \mid r(U_i) > r(U_{i-1})\}.$$

So I is the output of the greedy algorithm. Hence I is a maximum-weight independent set.

Next let:

(40.7)
$$\begin{aligned}\lambda_i &:= w(s_i) - w(s_{i+1}) \text{ for } i = 1, \ldots, n-1,\\ \lambda_n &:= w(s_n).\end{aligned}$$

This implies (40.3):

(40.8)
$$\begin{aligned} w(I) &= \sum_{s \in I} w(s) = \sum_{i=1}^{n} w(s_i)(r(U_i) - r(U_{i-1}))\\ &= w(s_n)r(U_n) + \sum_{i=1}^{n-1}(w(s_i) - w(s_{i+1}))r(U_i) = \sum_{i=1}^{n} \lambda_i r(U_i).\end{aligned}$$

By taking any ordering of S for which w is nonincreasing, (40.5) gives any chain of subsets U_i satisfying (40.4) for some $\lambda_i \geq 0$. Hence we have the theorem. ∎

This can be interpreted in terms of LP-duality. For any weight function $w : S \to \mathbb{R}$, consider the linear programming problem

(40.9)
$$\begin{array}{lll} \text{maximize} & w^{\mathsf{T}}x, & \\ \text{subject to} & x_s \geq 0 & (s \in S),\\ & x(U) \leq r_M(U) & (U \subseteq S),\end{array}$$

and its dual:

(40.10)
$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{U \subseteq S} y_U r_M(U), & \\ \text{subject to} & y_U \geq 0 & (U \subseteq S),\\ & \displaystyle\sum_{U \subseteq S} y_U \chi^U \geq w. & \end{array}$$

Corollary 40.2a. *If $w : S \to \mathbb{Z}$, then* (40.9) *and* (40.10) *have integer optimum solutions.*

Proof. We can assume that $w(s) \geq 0$ for each $s \in S$ (as neither the maximum nor the minimum changes by resetting $w(s)$ to 0 if negative). Then (40.4) implies that the λ_i are integer. This gives integer optimum solutions of (40.9) and (40.10). ∎

In polyhedral terms, Theorem 40.2 implies:

Corollary 40.2b. *The independent set polytope is determined by* (40.2).

Proof. Immediately from Theorem 40.2 (with (40.10)). ∎

Moreover, in TDI terms:

Corollary 40.2c. *System* (40.2) *is totally dual integral.*

Proof. Immediately from Corollary 40.2a. ∎

Similar results hold for the base polytope. For any matroid M, let $P_{\text{base}}(M)$ be the *base polytope* of M, defined as the convex hull of the incidence vectors of bases of M. Then:

Corollary 40.2d. *The base polytope of a matroid $M = (S, \mathcal{I})$ is determined by*

$$
(40.11) \qquad \begin{array}{ll} x_s \geq 0 & \text{for } s \in S, \\ x(U) \leq r_M(U) & \text{for } U \subseteq S, \\ x(S) = r_M(S). & \end{array}
$$

Proof. This follows directly from Corollary 40.2b, since the base polytope is the intersection of the independent set polytope with the hyperplane $\{x \mid x(S) = r_M(S)\}$, as an independent set I is a base if and only if $|I| \geq r_M(S)$. ∎

The corresponding TDI result reads:

Corollary 40.2e. *System* (40.11) *is totally dual integral.*

Proof. By Theorem 5.25 from Corollary 40.2c. ∎

One can similarly describe the *spanning set polytope* $P_{\text{spanning set}}(M)$ of M, which is, by definition, the convex hull of the incidence vectors of the spanning sets of M. It is determined by the system:

$$
(40.12) \qquad \begin{array}{ll} 0 \leq x_s \leq 1 & \text{for } s \in S, \\ x(U) \geq r_M(S) - r_M(S \setminus U) & \text{for } U \subseteq S. \end{array}
$$

Corollary 40.2f. *The spanning set polytope is determined by* (40.12).

Proof. A subset U of S is spanning in M if and only if $S \setminus U$ is independent in M^*. Hence for any $x \in \mathbb{R}^S$ we have:

$$
(40.13) \qquad x \in P_{\text{spanning set}}(M) \iff \mathbf{1} - x \in P_{\text{independent set}}(M^*).
$$

By Corollary 40.2b, $\mathbf{1} - x$ belongs to $P_{\text{independent set}}(M^*)$ if and only if x satisfies:

$$
(40.14) \qquad \begin{array}{ll} 1 - x_s \geq 0 & \text{for } s \in S, \\ |U| - x(U) \leq r_{M^*}(U) & \text{for } U \subseteq S. \end{array}
$$

Since $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$, the present corollary follows. ∎

Corollary 40.2c gives similarly the TDI result:

Corollary 40.2g. *System* (40.12) *is totally dual integral.*

Proof. By reduction to Corollary 40.2c, by a similar reduction as in the proof of the previous corollary. ∎

Note that

(40.15) $$P_{\text{base}}(M) = P_{\text{independent set}}(M) \cap P_{\text{spanning set}}(M),$$
$$P_{\text{independent set}}(M) = P^{\downarrow}_{\text{base}}(M) \cap [0,1]^S,$$
$$P_{\text{spanning set}}(M) = P^{\uparrow}_{\text{base}}(M) \cap [0,1]^S.$$

The following consequence on the intersection of the base polytope with a box was observed by Hell and Speer [1984]:

Corollary 40.2h. *Let $M = (S, \mathcal{I})$ be a matroid and let $l, u \in \mathbb{R}^S$ with $l \leq u$. Then there is an $x \in P_{\text{base}}(M)$ with $l \leq x \leq u$ if and only if $l \in P^{\downarrow}_{\text{base}}(M)$ and $u \in P^{\uparrow}_{\text{base}}(M)$.*

Proof. Necessity being trivial, we show sufficiency. We may assume that $l, u \in [0,1]^S$. So $l \in P_{\text{independent set}}(M)$ and $u \in P_{\text{spanning set}}(M)$. Choose l', u' such that $l \leq l' \leq u' \leq u$, $l' \in P_{\text{independent set}}(M)$, $u' \in P_{\text{spanning set}}(M)$, and $\|u' - l'\|_1$ minimal.

If $l' = u'$ we are done, so assume that there is an $s \in S$ with $l'(s) < u'(s)$. As we cannot increase $l'(s)$, there is a $T \subseteq S$ with $s \in T$ and $l'(T) = r(T)$. Similarly, as we cannot decrease $u'(s)$, there is a $U \subseteq S$ with $s \notin U$ and $u'(S \setminus U) = r(S) - r(U)$. Then we have the contradiction

(40.16) $$l'(T \cap U) + u'(T \cup U) \leq r(T \cap U) + u'(S) + r(T \cup U) - r(S)$$
$$\leq r(T) + r(U) + u'(S) - r(S) = l'(T) + u'(U)$$
$$< l'(T \cap U) + u'(T \cup U).$$

The last inequality follows from

(40.17) $$u'(T \cup U) - u'(U) = u'(T \setminus U) > l'(T \setminus U) = l'(T) - l'(T \cap U),$$

since $s \in T \setminus U$ and $u'(s) > l'(s)$. ∎

40.3. The most violated inequality

We now consider the problem to find, for any matroid $M = (S, \mathcal{I})$ and any $x \in \mathbb{R}^S_+$ not in the independent set polytope of M, an inequality among (40.2) most violated by x. That is, to find $U \subseteq S$ maximizing $x(U) - r_M(U)$.

The following theorem implies a min-max relation for this (Edmonds [1970b]):

Theorem 40.3. *Let $M = (S, \mathcal{I})$ be a matroid and let $x \in \mathbb{R}^S_+$. Then*

(40.18) $$\max\{z(S) \mid z \in P_{\text{independent set}}(M), z \leq x\} \\ = \min\{r_M(U) + x(S \setminus U) \mid U \subseteq S\}.$$

Proof. The inequality $\leq$ in (40.18) follows from

(40.19) $$z(S) = z(U) + z(S \setminus U) \leq r_M(U) + x(S \setminus U).$$

To see equality, let z attain the maximum. Then for each $s \in S$ with $z_s < x_s$ there exists a $U \subseteq S$ with $s \in U$ and $z(U) = r_M(U)$ (otherwise we can increase z_s). Now the collection of sets $U \subseteq S$ satisfying $z(U) = r_M(U)$ is closed under taking unions (and intersections), since if $z(T) = r_M(T)$ and $z(U) = r_M(U)$, then

(40.20) $$z(T \cup U) = z(T) + z(U) - z(T \cap U) \geq r_M(T) + r_M(U) - r_M(T \cap U) \\ \geq r_M(T \cup U).$$

Hence there exists a $U \subseteq S$ such that $z(U) = r_M(U)$ and such that U contains each $s \in S$ with $z_s < x_s$. Hence:

(40.21) $$z(S) = z(U) + z(S \setminus U) = r_M(U) + x(S \setminus U),$$

giving (40.18). ∎

Cunningham [1984] showed that from an independence testing oracle for a matroid one can derive a strongly polynomial time algorithm to find for any given vector x, a maximum violated inequality for the independent set polytope.

More strongly, Cunningham showed that one can solve the following problem in strongly polynomial time:

(40.22) given: a matroid $M = (S, \mathcal{I})$, by an independence testing oracle, and an $x \in \mathbb{Q}_+^S$;
find: a $z \in P_{\text{independent set}}(M)$ with $z \leq x$ maximizing $z(S)$, with a decomposition of z as convex combination of incidence vectors of independent sets, and a subset U of S satisfying $z(S) = r_M(U) + x(S \setminus U)$.

By (40.18), the set U certifies that z maximizes $z(S)$. In the algorithm for (40.22), Cunningham utilized the 'consistent breadth-first search' based on lexicographic order, given by Schönsleben [1980] and Lawler and Martel [1982a].

To prove Cunningham's result, we first show two lemmas. The first lemma is used only to prove the second lemma. As in Section 39.9, we define for any independent set I of a matroid $M = (S, \mathcal{I})$:

(40.23) $$A(I) := \{(y, z) \mid y \in I, z \in S \setminus I, I - y + z \in \mathcal{I}\}.$$

Lemma 40.4α. *Let $M = (S, \mathcal{I})$ be a matroid and let $I \in \mathcal{I}$. Let $(s, t) \in A(I)$, define $I' := I - s + t$, and let $(u, v) \in A(I') \setminus A(I)$. Then $t = u$ or $(u, t) \in A(I)$, and $s = v$ or $(s, v) \in A(I)$.*

Proof. By symmetry, it suffices to show that $t = u$ or $(u,t) \in A(I)$ (as we may assume that I is a base, and hence the second part follows by duality). We can assume that $t \neq u$. Then $t \neq v$, since $v \notin I' = I - s + t$, as $(u,v) \in A(I')$.

If $v = s$, then $I - u + t = I - u - s + t + v = I' - u + v \in \mathcal{I}$ and hence $(u,t) \in A(I)$. If $v \neq s$, then $I - u \in \mathcal{I}$ and $I - u - s + t + v \in \mathcal{I}$, and therefore $I - u + t \in \mathcal{I}$ or $I - u + v \in \mathcal{I}$; that is, $(u,v) \in A(I)$ or $(u,t) \in A(I)$. ∎

Lemma 40.4β. *Let $M = (S,\mathcal{I})$ be a matroid and let q be a new element. For any $I \in \mathcal{I}$, define*

$$\widetilde{A}(I) := \{(u,v) \mid u \in I + q, v \in S \setminus I, I - u + v \in \mathcal{I}\}. \tag{40.24}$$

Let $(s,t) \in A(I)$, define $I' := I - s + t$, and let $(u,v) \in \widetilde{A}(I') \setminus \widetilde{A}(I)$. Then $t = u$ or $(u,t) \in \widetilde{A}(I)$, and $s = v$ or $(s,v) \in \widetilde{A}(I)$.

Proof. Let $\widetilde{\mathcal{I}} := \{J \subseteq S + q \mid J - q \in \mathcal{I}\}$. Then the present lemma follows from Lemma 40.4α applied to the matroid $(S + q, \widetilde{\mathcal{I}})$. ∎

Now we can derive Cunningham's result:

Theorem 40.4. *Problem* (40.22) *is solvable in strongly polynomial time.*

Proof. We keep a vector $z \leq x$ in the independent set polytope of M and a decomposition

$$z = \sum_{i=1}^{k} \lambda_i \chi^{I_i}, \tag{40.25}$$

with $I_1, \ldots, I_k \in \mathcal{I}$, $\lambda_1, \ldots, \lambda_k > 0$, and $\sum_i \lambda_i = 1$. Initially $z := \mathbf{0}$, $k := 1$, $I_1 := \emptyset$, $\lambda_1 := 1$.

Let

$$T := \{s \in S \mid z_s < x_s\}. \tag{40.26}$$

Let q be a new element. For each i, define $\widetilde{A}(I_i)$ as in (40.24), and let $D = (S + q, A)$ be the directed graph with

$$A := \widetilde{A}(I_1) \cup \cdots \cup \widetilde{A}(I_k). \tag{40.27}$$

Fix an arbitrary linear order of the elements of $S + q$, by setting $S + q = \{1, \ldots, n\}$.

Case 1: D has no $q - T$ path. Let U be the set of $s \in S$ for which D has an $s - T$ path. As $T \subseteq U$, we know $z(S \setminus U) = x(S \setminus U)$. Also, as no arc of D enters U, we have $|U \cap I_i| = r_M(U)$ for all i, implying

$$z(U) = \sum_{i=1}^{k} \lambda_i |U \cap I_i| = \sum_{i=1}^{k} \lambda_i r_M(U) = r_M(U). \tag{40.28}$$

Hence $z(S) = r_M(U) + x(S \setminus U)$ as required.

Case 2: D has a $q-T$ path. For each $v \in S+q$, let $d(v)$ denote the distance in D from q to v (set to ∞ if no $q-v$ path exists). Choose a $t \in T$ with $d(t)$ finite and maximal, and among these t we choose the largest t. Let $(s,t) \in A$, with $d(s) = d(t) - 1$, and s largest. We can assume that $(s,t) \in \widetilde{A}(I_1)$. Let

(40.29) $\qquad \alpha := \min\{x_t - z_t, \lambda_1\}$

and define z' by

(40.30) $\qquad z' := z + \alpha(\chi^t - \chi^s)$ if $s \neq q$, and $z' := z + \alpha\chi^t$ if $s = q$.

Let $I_1' := I_1 - s + t$ (so $I_1' = I_1 + t$ if $s = q$).

Then

(40.31) $$z' = \alpha\chi^{I_1'} + (\lambda_1 - \alpha)\chi^{I_1} + \sum_{i=2}^{k} \lambda_i \chi^{I_i}.$$

If $\alpha = \lambda_1$, we delete the second term. We obtain a decomposition of z' as a convex combination of at most $k+1$ independent sets, and we can iterate.

Running time. We show that the number of iterations is at most $|S|^9$. Consider any iteration. Let d' and A' be the objects d and A of the next iteration. We first show:

(40.32) $\qquad$ for each $v \in S + q$: $d'(v) \geq d(v)$.

To show this, we can assume that $d'(v) < \infty$. We show (40.32) by induction on $d'(v)$, the case $d'(v) = 0$ being trivial (as it means $v = q$). Assume $d'(v) > 0$. Let u be such that $(u,v) \in A'$ and $d'(u) = d'(v) - 1$. By induction we know $d'(u) \geq d(u)$.

If $(u,v) \in A$, then $d(v) \leq d(u) + 1 \leq d'(u) + 1 = d'(v)$, as required. If $(u,v) \notin A$, then $(u,v) \in \widetilde{A}(I_1')$ and $(u,v) \notin \widetilde{A}(I_1)$. By Lemma 40.4$\beta$, $t = u$ or $(u,t) \in \widetilde{A}(I_1)$, and $s = v$ or $(s,v) \in \widetilde{A}(I_1)$. Hence

(40.33) $\qquad d(v) \leq d(s) + 1 = d(t) \leq d(u) + 1 \leq d'(u) + 1 = d'(v).$

So $d(v) \leq d'(v)$. This shows (40.32).

Let β be the number of $j = 1, \ldots, k$ with $(s,t) \in \widetilde{A}(I_j)$. Let T', t', s', and β' be the objects T, t, s, β in the next iteration. We show:

(40.34) $\qquad$ if $d'(v) = d(v)$ for each $v \in S + q$, then $(d'(t'), t', s', \beta')$ is lexicographically less than $(d(t), t, s, \beta)$.

Indeed, if $\alpha = x_t - z_t$, then $T' = T - t + s$ or $T' = T - t$. So $d'(t') < d(t)$, or $d'(t') = d(t)$ and $t' < t$. If $\alpha < x_t - z_t$, then $T' = T + s$ or $T' = T$. Moreover, $\alpha = \lambda_1$, so I_1 has been omitted from the convex combination. So, as $t \in T'$ and $d(s) < d(t)$, we know that $t' = t$ and $d'(t') = d(t)$. As $t \in I_1'$, we know $(s', t) \notin \widetilde{A}(I_1')$. Hence, as $(s', t) \in A'$, we have $(s', t) \in \widetilde{A}(I_j)$ for some $j = 2, \ldots, k$. Hence $(s', t) \in A$. By the choice of s, we know $s' \leq s$. If $s' < s$, we have (40.34), so assume $s' = s$. Then $\beta' = \beta - 1$, as $(s,t) \notin \widetilde{A}(I_1')$. This proves (40.34).

The number k of independent sets in the decomposition grows by 1 if $\alpha = x_t - z_t < \lambda_1$. In that case, $d'(v) = d(v)$ for each $v \in S + q$ (by (40.32), as $A' \supseteq A$). Moreover, $d'(t') < d(t)$ or $t' < t$ (since $T' \subseteq T - t + s$). So k does not exceed $|S|^4$, and hence β is at most $|S|^4$. Concluding, the number of iterations is at most $|S|^9$. ∎

With Gaussian elimination, we can reduce the number k in each iteration to at most $|S|$ (by Carathéodory's theorem). Incorporating this reduces the number of iterations to $|S|^6$.

Theorem 40.4 immediately implies that one can test if a given vector belongs to the independent set polytope of a matroid:

Corollary 40.4a. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and an $x \in \mathbb{Q}^S$, one can test in strongly polynomial time if x belongs to $P_{\text{independent set}}(M)$, and if so, decompose x as a convex combination of incidence vectors of independent sets.*

Proof. Directly from Theorem 40.4. ∎

One can derive a similar result for the spanning set polytope:

Corollary 40.4b. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and an $x \in \mathbb{Q}^S$, one can test in strongly polynomial time if x belongs to $P_{\text{spanning set}}(M)$, and if so, decompose x as a convex combination of incidence vectors of spanning sets.*

Proof. x belongs to the spanning set polytope of M if and only if $\mathbf{1} - x$ belongs to the independent set polytope of the dual matroid M^*. Also convex combinations of spanning sets of M and independent sets of M^* transfer to each other by this operation. Since $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$ for each $U \subseteq S$, also an independence testing oracle for M^* is easily obtained from one for M. ∎

The theorem also implies that the following *most violated inequality problem* can be solved in strongly polynomial time:

(40.35) given: a matroid $M = (S, \mathcal{I})$ by an independence testing oracle, and a vector $x \in \mathbb{Q}^S$;
find: a subset U of S minimizing $r_M(U) - x(U)$.

Corollary 40.4c. *The most violated inequality problem can be solved in strongly polynomial time.*

Proof. Any negative component of x can be reset to 0, as this does not change the problem. So we can assume that $x \geq \mathbf{0}$. Then by Theorem 40.4 we can find a $U \subseteq S$ minimizing $r_M(U) + x(S \setminus U)$ in strongly polynomial time. This U is as required. ∎

40.3a. Facets and adjacency on the independent set polytope

Let $M = (S, \mathcal{I})$ be a matroid, with rank function r. Trivially, the independent set polytope P of M is full-dimensional if and only if M has no loops. If P is full-dimensional there is a unique minimal collection of linear inequalities defining P (up to scalar multiplication), which corresponds to the facets of P. Edmonds [1970b] found that this collection is given by the following theorem. Recall that a subset F of S is called a *flat* if for all s in $S \setminus F$ one has $r(F + s) > r(F)$. A subset F is called *inseparable* if there is no partition of F into nonempty sets F_1 and F_2 with $r(F) = r(F_1) + r(F_2)$. Then:

Theorem 40.5. *If M is loopless, the following is a minimal system for the independent set polytope of M:*

(40.36) (i) $x_s \geq 0$ $(s \in S)$,
(ii) $x(F) \leq r(F)$ (F is a nonempty inseparable flat).

Proof. As M is loopless, the independent set polytope of M is full-dimensional. It is easy to see that (40.36) determines the independent set polytope, as any other inequality $x(U) \leq r(U)$ is implied by the inequalities $x(F_i) \leq r(F_i)$, where $F_1, \ldots, F_t$ is a maximal partition of $F := \mathrm{span}_M(U)$ such that $r(F_1) + \cdots + r(F_t) = r(F)$.

The irredundancy of collection (40.36) can be seen as follows. Each inequality $x_s \geq 0$ is irredundant, since the vector $-\chi^s$ satisfies all other inequalities.

We show that also the inequalities (40.36)(ii) are irredundant, by showing that for any two nonempty nonseparable flats T, U there exists a base I of T with $|I \cap U| < r(U)$ (implying that the face determined by T is contained in no (other) facet).

To show this, let I be a base of T with $|I \cap (T \setminus U)| = r(T \setminus U)$. Suppose $|I \cap U| = r(U)$. Then

(40.37) $$r(U) \geq r(T \cap U) \geq r(T) - r(T \setminus U) = |I \cap U| = r(U).$$

Hence we have equality throughout. This implies (as T is inseparable) that $T \setminus U = \emptyset$ or $T \cap U = \emptyset$, and that $r(U) = r(T \cap U)$. If $T \setminus U = \emptyset$, then $T \subset U$, and hence (as T is a flat) $r(U) > r(T) \geq r(T \cap U)$, a contradiction. If $T \cap U = \emptyset$, then $r(U) = r(T \cap U) = 0$, implying that $U = \emptyset$ (as M has no loops), again a contradiction. ∎

It follows that the base polytope, which is the face $\{x \in P \mid x(S) = r(S)\}$ of P, has dimension $|S| - 1$ if and only if S is inseparable (that is, the matroid is *connected*).

As for adjacency of vertices of the independent set polytope, we have:

Theorem 40.6. *Let $M = (S, \mathcal{I})$ be a loopless matroid and let I and J be distinct independent sets. Then χ^I and χ^J are adjacent vertices of the independent set* *polytope of M if and only if $|I \triangle J| = 1$, or $|I \setminus J| = |J \setminus I| = 1$ and $r_M(I \cup J) = |I| = |J|$.*

Proof. To see sufficiency, note that the condition implies that I and J are the only two independent sets with incidence vector x satisfying $x(I \cap J) = r_M(I \cap J)$, $x_s = 0$ for $s \notin I \cup J$, and (if $|I \triangle J| = 2$) $x(I \cup J) = r_M(I \cup J)$. Hence I and J are adjacent.

To see necessity, assume that χ^I and χ^J are adjacent. If I is not a base of $I \cup J$, then $I + j$ is independent for some $j \in J \setminus I$. Hence

$$\text{(40.38)} \qquad \tfrac{1}{2}(\chi^I + \chi^J) = \tfrac{1}{2}(\chi^{I+j} + \chi^{J-j}),$$

implying (as χ^I and χ^J are adjacent) that $I+j = J$ and $J-j = I$, that is $|I \triangle J| = 1$.

So we can assume that I and J are bases of $I \cup J$. Choose $i \in I \setminus J$. By Theorem 39.12, there is a $j \in J \setminus I$ such that $I - i + j$ and $J - j + i$ are bases of $I \cup J$. Then

$$\text{(40.39)} \qquad \tfrac{1}{2}(\chi^I + \chi^J) = \tfrac{1}{2}(\chi^{I-i+j} + \chi^{J-j+i}),$$

implying (as χ^I and χ^J are adjacent) that $I - i + j = J$ and $J - j + i = I$, that is we have the second alternative in the condition. ∎

More on the combinatorial structure of the independent set polytope can be found in Naddef and Pulleyblank [1981a].

40.3b. Further notes

Prodon [1984] showed that the separation problem for the independent set polytope of a matching matroid can be solved by finding a minimum-capacity cut in an auxiliary directed graph.

Frederickson and Solis-Oba [1997,1998] gave strongly polynomial-time algorithm for measuring the sensitivity of the minimum weight of a base under perturbing the weight. (Related analysis was given by Libura [1991].)

Narayanan [1995] described a rounding technique for the independent set polytope membership problem, leading to an $O(n^3r^2)$-time algorithm, where n is the size of the underlying set of the matroid and r is the rank of the matroid.

A strongly polynomial-time algorithm maximizing certain convex objective functions over the bases was given by Hassin and Tamir [1989].

For studies of structures where the greedy algorithm applies if condition (39.1)(i) is deleted, see Faigle [1979,1984b], Hausmann, Korte, and Jenkyns [1980], Korte and Lovász [1983,1984a,1984b,1984c,1985a,1985b,1989], Bouchet [1987a], Goecke [1988], Dress and Wenzel [1990], Korte, Lovász, and Schrader [1991], Helman, Moret, and Shapiro [1993], and Faigle and Kern [1996].

Chapter 41

Matroid intersection

Edmonds discovered that matroids have even more algorithmic power than just that of the greedy method. He showed that there exist efficient algorithms also for *intersections* of matroids. That is, a maximum-weight common independent set in *two* matroids can be found in strongly polynomial time. Edmonds also found good min-max characterizations for matroid intersection.

Matroid intersection yields a motivation for studying matroids: we may apply it to two matroids from different classes of examples of matroids, and thus we obtain methods that exceed the bounds of any particular class.

We should note here that if $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, then $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ need not be a matroid. (An example with $|S| = 3$ is easy to construct.)

Moreover, the problem of finding a maximum-size common independent set in *three* matroids is NP-complete (as finding a Hamiltonian circuit in a directed graph is a special case; also, finding a common transversal of three partitions is a special case).

41.1. Matroid intersection theorem

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids, on the same set S. Consider the collection $\mathcal{I}_1 \cap \mathcal{I}_2$ of *common independent sets*. The pair $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is generally *not* a matroid again.

Edmonds [1970b] showed the following formula, for which he gave two proofs — one based on linear programming duality and total unimodularity (see the proof of Theorem 41.12 below), and one reducing it to the matroid union theorem (see Corollary 42.1a and the remark thereafter). We give the direct proof implicit in Brualdi [1971e].

Theorem 41.1 (matroid intersection theorem). *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is equal to*

$$(41.1) \qquad \min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)).$$

Proof. Let k be equal to (41.1). It is easy to see that the maximum is not more than k, since for any common independent set I and any $U \subseteq S$:

(41.2) $$|I| = |I \cap U| + |I \setminus U| \leq r_1(U) + r_2(S \setminus U).$$

We prove equality by induction on $|S|$, the case $|S| \leq 1$ being trivial. So assume that $|S| \geq 2$.

If minimum (41.1) is attained only by $U = S$ or $U = \emptyset$, choose $s \in S$. Then $r_1(U) + r_2(S \setminus (U \cup \{s\})) \geq k$ for each $U \subseteq S \setminus \{s\}$, since otherwise both U and $U \cup \{s\}$ would attain (41.1), whence $\{U, U \cup \{s\}\} = \{\emptyset, S\}$, contradicting the fact that $|S| \geq 2$. Hence, by induction, $M_1 \setminus s$ and $M_2 \setminus s$ have a common independent set of size k, implying the theorem.

So we can assume that (41.1) is attained by some U with $\emptyset \neq U \neq S$. Then $M_1|U$ and $M_2 \cdot U$ have a common independent set I of size $r_1(U)$. Otherwise, by induction, there exists a subset T of U with

(41.3) $$r_1(U) > r_{M_1|U}(T) + r_{M_2 \cdot U}(U \setminus T) = r_1(T) + r_2(S \setminus T) - r_2(S \setminus U),$$

contradicting the fact that U attains (41.1). Similarly, $M_1 \cdot (S \setminus U)$ and $M_2|(S \setminus U)$ have a common independent set J of size $r_2(S \setminus U)$.

Now $I \cup J$ is a common independent set of M_1 and M_2. Indeed, $I \cup J$ is independent in M_1, as I is independent in $M_1|U$ and J is independent in $M_1 \cdot (S \setminus U) = M_1/U$ (cf. (39.10)). Similarly, $I \cup J$ is independent in M_2. As $|I \cup J| = r_1(U) + r_2(S \setminus U)$, this proves the theorem. ∎

This implies a characterization of the existence of a common base in two matroids:

Corollary 41.1a. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively, such that $r_1(S) = r_2(S)$. Then M_1 and M_2 have a common base if and only if $r_1(U) + r_2(S \setminus U) \geq r_1(S)$ for each $U \subseteq S$.*

Proof. Directly from Theorem 41.1. ∎

It is easy to derive from the matroid intersection theorem a similar min-max relation for the minimum size of a common spanning set:

Corollary 41.1b. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the minimum size of a common spanning set of M_1 and M_2 is equal to*

(41.4) $$\max_{U \subseteq S}(r_1(S) - r_1(U) + r_2(S) - r_2(S \setminus U)).$$

Proof. The minimum is equal to the minimum of $|B_1 \cup B_2|$ where B_1 and B_2 are bases of M_1 and M_2 respectively. Hence the minimum is equal to $r_1(S) + r_2(S)$ minus the maximum of $|B_1 \cap B_2|$ over such B_1, B_2. This last maximum is characterized in the matroid intersection theorem, yielding the present corollary. ∎

The following result of Rado [1942] (a generalization of Hall's marriage theorem (Theorem 22.1), and therefore sometimes called the Rado-Hall theorem) may be derived from the matroid intersection theorem, applied to M and the transversal matroid M_2 induced by $\mathcal{X}$.

Corollary 41.1c (Rado's theorem). *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $\mathcal{X} = (X_1, \ldots, X_n)$ be a family of subsets of S. Then $\mathcal{X}$ has a transversal which is independent in M if and only if*

$$(41.5)\qquad r(\bigcup_{i\in I} X_i) \geq |I|$$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Let r_2 be the rank function of the transversal matroid M_2 induced by $\mathcal{X}$. By the matroid intersection theorem, M and M_2 have a common independent set of size n if and only if

$$(41.6)\qquad r(U) + r_2(S \setminus U) \geq n \text{ for each } U \subseteq S.$$

Now for each $T \subseteq S$ one has (by Kőnig's matching theorem (cf. Corollary 22.2a)):

$$(41.7)\qquad r_2(T) = \min_{I\subseteq\{1,\ldots,n\}} (\big|\bigcup_{i\in I} X_i \cap T\big| + n - |I|).$$

So (41.6) is equivalent to:

$$(41.8)\qquad r(U) + \big|\bigcup_{i\in I} X_i \setminus U\big| + n - |I| \geq n$$

for all $U \subseteq S$ and $I \subseteq \{1, \ldots, n\}$. We can assume that $U = \bigcup_{i\in I} X_i$, since replacing U by $\bigcup_{i\in I} X_i$ does not increase the left-hand side in (41.8). So the condition is equivalent to (41.5), proving the corollary. ∎

Notes. Mirsky [1971a] gave an alternative proof of Rado's theorem. Welsh [1970] showed that, in turn, Rado's theorem implies the matroid intersection theorem. Las Vergnas [1970] gave an extension of Rado's theorem. Rado [1942] (and also Welsh [1971]) showed that Rado's theorem in fact characterizes matroids. Perfect [1969a] generalized Rado's theorem to characterizing the maximum size of an independent partial transversal. Related results are in Perfect [1971].

41.1a. Applications of the matroid intersection theorem

In this section we mention a number of applications of the matroid intersection theorem. Further applications will be given in the next chapter on matroid union.

Kőnig's theorems. Let $G = (V, E)$ be a bipartite graph, with colour classes U_1 and U_2. For $i = 1, 2$, let $M_i = (E, \mathcal{I}_i)$ be the matroid with $F \subseteq E$ independent if and only if each vertex in U_i is covered by at most one edge in F.

So M_1 and M_2 are partition matroids. The common independent sets in M_1 and M_2 are the matchings in G, and the common spanning sets are the edge covers in G. For $i = 1, 2$ and $F \subseteq E$, the rank $r_i(F)$ of F in M_i is equal to the number of vertices in U_i covered by F.

By the matroid intersection theorem, the maximum size of a matching in G is equal to the minimum of $r_1(F) + r_2(E \setminus F)$ taken over $F \subseteq E$. This last is equal to the minimum size of a vertex cover in G. So we have Kőnig's matching theorem (Theorem 16.2).

Similarly, by Corollary 41.1b, the minimum size of an edge cover in G (assuming G has no isolated vertices), is equal to the maximum of $|V| - r_1(F) - r_2(E \setminus F)$ taken over $F \subseteq E$. This last is equal to the maximum size of a stable set in G. So we have the Kőnig-Rado edge cover theorem (Theorem 19.4).

Common transversals. Let $\mathcal{X} = (X_1, \ldots, X_m)$ and $\mathcal{Y} = (Y_1, \ldots, Y_m)$ be families of subsets of a finite set S. Then the matroid intersection theorem implies Theorem 23.1 of Ford and Fulkerson [1958c]: $\mathcal{X}$ and $\mathcal{Y}$ have a common transversal if and only if

$$|X_I \cap Y_J| \geq |I| + |J| - m \tag{41.9}$$

for all subsets I and J of $\{1, \ldots, m\}$, where $X_I := \bigcup_{i \in I} X_i$ and $Y_J := \bigcup_{j \in J} Y_j$.

To see this, let M_1 and M_2 be the transversal matroids induced by $\mathcal{X}$ and $\mathcal{Y}$ respectively, with rank functions r_1 and r_2 say. So $\mathcal{X}$ and $\mathcal{Y}$ have a common transversal if and only if M_1 and M_2 have a common independent set of size m. By Theorem 41.1, this last holds if and only if $r_1(Z) + r_2(S \setminus Z) \geq m$ for each $Z \subseteq S$. Using Kőnig's matching theorem, this is equivalent to:

$$\min_{I \subseteq \{1,\ldots,m\}} (m - |I| + |X_I \cap Z|) + \min_{J \subseteq \{1,\ldots,m\}} (m - |J| + |Y_J \setminus Z|) \geq m \tag{41.10}$$

for each $Z \subseteq S$. Equivalently, for all $I, J \subseteq \{1, \ldots, m\}$:

$$\min_{Z \subseteq S} (m - |I| + |X_I \cap Z| + m - |J| + |Y_J \setminus Z|) \geq m. \tag{41.11}$$

As this minimum is attained by $Z := Y_J$, this is equivalent to (41.9).

Coloured trees. Let $G = (V, E)$ be a graph and let the edges of G be coloured with k colours. That is, we have partitioned E into sets $E_1, \ldots, E_k$, called *colours*. Then there exists a spanning tree with all edges coloured differently if and only if $G - F$ has at most $t + 1$ components, for any union F of t colours, for any $t \geq 0$. This follows from the matroid intersection theorem applied to the cycle matroid $M(G)$ of G and the partition matroid N induced by $E_1, \ldots, E_k$.

Indeed, $M(G)$ and N have a common independent set of size $|V| - 1$ if and only if $r_{M(G)}(E \setminus F) + r_N(F) \geq |V| - 1$ for each $F \subseteq E$. Now $r_N(F)$ is equal to the number of E_i intersecting F. So we can assume that F is equal to the union of t of the E_i, with $t := r_N(F)$. Moreover, $r_{M(G)}(E \setminus F)$ is equal to $|V| - \kappa(G - F)$, where $\kappa(G - F)$ is the number of components of $G - F$. So the requirement is that $|V| - \kappa(G - F) + t \geq |V| - 1$. In other words, $\kappa(G - F) \leq t + 1$.

Detachments. The following is a special case of a theorem of Nash-Williams [1985], which he derived from the matroid intersection theorem — in fact it is a consequence of the result on coloured trees given above.

Let $G = (V, E)$ be a graph and let $b : V \longrightarrow \mathbb{Z}_+$. Call a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ a *b-detachment* of G if there is a function $\phi : \widetilde{V} \longrightarrow V$ such that $|\phi^{-1}(v)| = b(v)$ for each $v \in V$, and such that there is a one-to-one function $\psi : \widetilde{E} \longrightarrow E$ with $\psi(e) = \{\phi(u), \phi(v)\}$ for each edge $e = uv$ of $\widetilde{G}$.

Then there exists a connected b-detachment if and only if

(41.12) $\qquad b(U) + \kappa(G - U) \leq |E_U| + 1$ for each $U \subseteq V$,

where $\kappa(G')$ denotes the number of components of graph G' and where E_U denotes the set of edges intersecting U.

To see this, let $H = (\widetilde{V}, E')$ be the graph obtained from G by replacing each vertex v by $b(v)$ new vertices, and by connecting for each edge $e = uv$ of G, the $b(u)$ new vertices associated with u with the $b(v)$ new vertices associated with v. We assign to these $b(u)b(v)$ edges the 'colour' e.

Then there exists a connected b-detachment if and only if H has a spanning tree in which all edges have a different colour. By the previous example, such a spanning tree exists if and only if for each $F \subseteq E$, deleting from H the edges with colour in F gives a graph H' with at most $|F| + 1$ components.

Now the number of components of H' is equal to the $\kappa(G - F) + b(I_F) - |I_F|$, where I_F denotes the set of isolated (hence loopless) vertices of $G - F$. So the condition is equivalent to: $\kappa(G - F) - |F| + b(I_F) - |I_F| \leq 1$. As $\kappa(G - F) - |F|$ does not decrease by removing edges from F, we can assume that F is equal to the set of edges incident with I_F. So F is determined by $U := I_F$, namely $F = E_U$. Then $\kappa(G - F) - |I_F| = \kappa(G - U)$. So the condition is equivalent to (41.12).

41.1b. Woodall's proof of the matroid intersection theorem

P.D. Seymour attributed the following proof of the matroid intersection theorem to D.R. Woodall (cf. Seymour [1976a]):

Let k be the value of (41.1). Let $x \in S$ be such that $r_1(\{x\}) = r_2(\{x\}) = 1$. (If no such x exists the theorem is trivial, as in that case the minimum is 0.) Let $Y := S \setminus \{x\}$. Now we may assume that the restrictions $M_1 \setminus x$ and $M_2 \setminus x$ have no common independent set of size k. So, by induction,

(41.13) $\qquad r_1(A_1) + r_2(A_2) \leq k - 1,$

for some partition A_1, A_2 of Y. Moreover, the contractions M_1/x and M_2/x have no common independent set of size $k-1$ (otherwise we can add x to obtain a common independent set of size k for M_1 and M_2). So, by induction,

(41.14) $\qquad r_1(B_1 \cup \{x\}) - 1 + r_2(B_2 \cup \{x\}) - 1 \leq k - 2$

(cf. (39.9) above), for some partition B_1, B_2 of Y. However,

(41.15)
$$\begin{aligned} r_1(A_1 \cap B_1) + r_1(A_1 \cup B_1 \cup \{x\}) &\leq r_1(A_1) + r_1(B_1 \cup \{x\}),\\ r_2(A_2 \cap B_2) + r_2(A_2 \cup B_2 \cup \{x\}) &\leq r_2(A_2) + r_2(B_2 \cup \{x\}), \end{aligned}$$

by the submodularity (cf. (39.38)(ii)) of the rank functions. Moreover, by the definition of k,

(41.16)
$$\begin{aligned} k &\leq r_1(A_1 \cap B_1) + r_2(A_2 \cup B_2 \cup \{x\}),\\ k &\leq r_1(A_1 \cup B_1 \cup \{x\}) + r_2(A_2 \cap B_2), \end{aligned}$$

as $A_1 \cap B_1, A_2 \cup B_2 \cup \{x\}$ and $A_1 \cup B_1 \cup \{x\}, A_2 \cap B_2$ form partitions of S. Adding the inequalities in (41.13), (41.14), (41.15), and (41.16) gives a contradiction.

41.2. Cardinality matroid intersection algorithm

A maximum-size common independent set can be found in polynomial time. This result follows from the matroid union algorithm of Edmonds [1968], since (as Edmonds [1970b] and Lawler [1970] observed) cardinality matroid intersection can be reduced to matroid union.

We describe below the direct algorithm given by Aigner and Dowling [1971] and Lawler [1975], based on finding paths in auxiliary graphs. A different algorithm was given by Edmonds [1979].

Note that the examples given in Section 41.1a provide applications for the matroid intersection algorithm. We should note that in the algorithm we require that in any matroid $M = (S, \mathcal{I})$, we can test in polynomial time if any subset of S belongs to $\mathcal{I}$ — no explicit list of all sets in $\mathcal{I}$ is required. Thus complexity results are all relative to the complexity of testing independence. As such a membership testing algorithm exists in each example mentioned, we obtain polynomial-time algorithms for these special cases.

For any two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we define a directed graph $D_{M_1,M_2}(I)$, with vertex set S, as follows. For any $y \in I, x \in S \setminus I$,

(41.17) (y, x) is an arc of $D_{M_1,M_2}(I)$ if and only if $I - y + x \in \mathcal{I}_1$,
(x, y) is an arc of $D_{M_1,M_2}(I)$ if and only if $I - y + x \in \mathcal{I}_2$.

These are all arcs of $D_{M_1,M_2}(I)$. So this graph is the union of the graphs $D_{M_1}(I)$ and the *reverse* of $D_{M_2}(I)$ defined in Section 39.9.

The following is the base for finding a maximum-size common independent set in two matroids.

Cardinality common independent set augmenting algorithm

input: matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$;
output: a set $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|I'| > |I|$ (if any).
description of the algorithm: Consider the sets

(41.18) $X_1 := \{x \in S \setminus I \mid I \cup \{x\} \in \mathcal{I}_1\}$,
$X_2 := \{x \in S \setminus I \mid I \cup \{x\} \in \mathcal{I}_2\}$.

Moreover, consider the directed graph $D_{M_1,M_2}(I)$ defined above. There are two cases.

Case 1: $D_{M_1,M_2}(I)$ has an $X_1 - X_2$ path P. (Possibly of length 0 if $X_1 \cap X_2 \neq \emptyset$.) We take a shortest such path P (that is, with a minimum number of arcs). Now output $I' := I \triangle VP$.

Case 2: $D_{M_1,M_2}(I)$ has no $X_1 - X_2$ path. Then I is a maximum-size common independent set. ∎

This finishes the description of the algorithm. The correctness of the algorithm is given by the following two theorems.

Theorem 41.2. *If Case 1 applies, then $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Assume that Case 1 applies. By symmetry it suffices to show that I' belongs to $\mathcal{I}_1$.

Let P start at $z_0 \in X_1$. The arcs in P leaving I form the only matching in $D_{M_1}(I)$ with union equal to $VP - z_0$, since otherwise P would have a shortcut. Moreover, for each $z \in VP \setminus I$ with $z \neq z_0$, one has $I + z \notin \mathcal{I}_1$, since otherwise $z \in X_1$, and hence P would have a shortcut. So by Corollary 39.13a, I' belongs to $\mathcal{I}_1$. ∎

Theorem 41.3. *If Case 2 applies, then I is a maximum-size common independent set.*

Proof. As Case 2 applies, there is no $X_1 - X_2$ path in $D_{M_1,M_2}(I)$. Hence there is a subset U of S with $X_1 \cap U = \emptyset$ and $X_2 \subseteq U$, and such that no arc enters U. We show

(41.19) $$r_{M_1}(U) + r_{M_2}(S \setminus U) \leq |I|.$$

To this end, we first show

(41.20) $$r_{M_1}(U) \leq |I \cap U|.$$

Suppose that $r_{M_1}(U) > |I \cap U|$. Then there exists an x in $U \setminus I$ such that $(I \cap U) \cup \{x\} \in \mathcal{I}_1$. Since $I \cup \{x\} \notin \mathcal{I}_1$ (as $x \notin X_1$), there is a $y \in I \setminus U$ with $I - y + x \in \mathcal{I}_1$. But then $D_{M_1}(I)$ has an arc from y to x, contradicting the facts that $x \in U$ and $y \notin U$ and that no arc enters U.

This shows (41.20). Similarly, $r_{M_2}(S \setminus U) \leq |I \setminus U|$. Hence we have (41.19). So by the matroid intersection theorem, I is a maximum-size common independent set. ∎

Clearly, the running time of the algorithm is polynomially bounded, since we can construct the auxiliary directed graph $D_{M_1,M_2}(I)$ and find the path P (if any), in polynomial time. Therefore:

Theorem 41.4. *A maximum-size common independent set in two matroids can be found in polynomial time.*

Proof. Directly from the above, as we can find a maximum-size common independent set after applying at most $|S|$ times the common independent set augmenting algorithm. ∎

The algorithm also yields a proof of the matroid intersection theorem (Theorem 41.1 above): if the algorithm stops with set I, we obtain a set U for which (41.19) holds.

Notes. The above algorithm can be shown to take $O(n^2m(n+Q))$ time, where n is the maximum size of a common independent set, m is the size of the underlying set, and Q is the time needed to test if a given set is independent (in either matroid). Cunningham [1986] showed that if one chooses a shortest path as augmenting path, the sum of the lengths of all augmenting paths chosen is $O(n \log n)$, which gives an $O(n^{3/2}mQ)$-time algorithm. This algorithm extends several of the ideas behind the $O(n^{1/2}m)$ algorithm of Hopcroft, Karp, and Karzanov for cardinality bipartite matching (see Section 16.4). For more efficient algorithms, see Gabow and Tarjan [1984], Gusfield [1984], Gabow and Stallmann [1985], Frederickson and Srinivas [1989], Gabow and Xu [1989,1996], and Fujishige and Zhang [1995].

The problem of finding a maximum-size common independent set in *three* matroids is NP-complete, as finding a Hamiltonian circuit in a directed graph is a special case (as was observed by Held and Karp [1970]). Another special case is finding a common transversal of three collections of sets, which is also NP-complete (Theorem 23.16). In particular, the k-intersection problem can be reduced to the 3-intersection problem (cf. Lawler [1976b]).

Barvinok [1995] gave an algorithm for finding a maximum-size common independent set in k linear matroids, represented by given vectors over the rationals. The running time is linear in the cardinality of the underlying set and singly polynomial in the maximum rank of the matroids.

41.3. Weighted matroid intersection algorithm

Also a maximum-*weight* common independent set can be found in strongly polynomial time. This result was announced by Edmonds [1970b], who published an algorithm in Edmonds [1979]. An alternative algorithm (which we describe below) was announced by Lawler [1970] and described in Lawler [1975,1976b] — the correctness of this algorithm was proved by Krogdahl [1974,1976], using the results described in Section 39.9. A similar algorithm was described by Iri and Tomizawa [1976].

This algorithm is an extension of the cardinality matroid intersection algorithm given in Section 41.2. In each iteration, instead of finding a path P with a minimum number of arcs in $D_{M_1,M_2}(I)$, we will now require P to have minimum length with respect to some length function defined on $D_{M_1,M_2}(I)$.

To describe the algorithm, if matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and a weight function $w : S \to \mathbb{R}$ are given, call a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ *extreme* if $w(J) \leq w(I)$ for each $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ satisfying $|J| = |I|$.

Weighted common independent set augmenting algorithm

input: matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, a weight function $w : S \to \mathbb{Q}$, and an extreme common independent set I;
output: an extreme common independent set I' with $|I'| = |I| + 1$ (if any).
description of the algorithm: Consider again the sets X_1 and X_2 and the directed graph $D_{M_1,M_2}(I)$ on S, as in the cardinality case.

For any $x \in S$ define the 'length' $l(x)$ of x by:

$$(41.21) \qquad l(x) := \begin{cases} w(x) & \text{if } x \in I, \\ -w(x) & \text{if } x \notin I. \end{cases}$$

The *length* of a path P, denoted by $l(P)$, is equal to the sum of the lengths of the vertices traversed by P.

Case 1: $D_{M_1,M_2}(I)$ has an $X_1 - X_2$ path P. We choose P such that $l(P)$ is minimal and such that (secondly) P has a minimum number of arcs among all minimum-length $X_1 - X_2$ paths. Set $I' := I \triangle VP$.

Case 2: $D_{M_1,M_2}(I)$ has no $X_1 - X_2$ path. Then there is no common independent set larger than I. ∎

This finishes the description of the algorithm. The correctness of the algorithm if Case 2 applies follows directly from Theorem 41.3. In order to show the correctness if Case 1 applies, we first prove the following basic property of the length function l.

Lemma 41.5α. *Let C be a directed circuit in $D_{M_1,M_2}(I)$ and let $t \in VC$. Define $J := I \triangle VC$. If $J \notin \mathcal{I}_1 \cap \mathcal{I}_2$, then there exists a directed circuit C' with $VC' \subset VC$ such that $l(VC') < 0$, or $l(VC') \leq l(VC)$ and $t \in VC'$.*

Proof. By symmetry we can assume that $J \notin \mathcal{I}_1$. Let N_1 and N_2 be the sets of arcs in C belonging to $D_{M_1}(I)$ and $D_{M_2}(I)$ respectively. As $J \notin \mathcal{I}_1$, there exists, by Theorem 39.13, a matching N_1' in $D_{M_1}(I)$ with union VC and with $N_1' \neq N_1$. Consider the directed graph $D = (VC, A)$ formed by the arcs in N_1, N_1' (taking arcs in $N_1 \cap N_1'$ parallel), and by the arcs in N_2 taking each of them twice (parallel). Then each vertex in VC is entered and left by exactly two arcs of D. Moreover, since $N_1' \neq N_1$, D contains a directed circuit C_1 with $VC_1 \subset VC$ (as N_1' contains a chord of C). As D is Eulerian, we can extend this to a decomposition of A into directed circuits $C_1, \ldots, C_k$. Then

$$(41.22) \qquad \chi^{VC_1} + \cdots + \chi^{VC_k} = 2 \cdot \chi^{VC}.$$

Since $VC_1 \neq VC$ we know that $VC_j = VC$ for at most one j. If, say $VC_k = VC$, then (41.22) implies that either $l(VC_j) < 0$ for some $j < k$ or $l(VC_j) \leq l(VC)$ for all $j < k$, implying the proposition.

Suppose next that $VC_j \neq VC$ for all j. If $l(VC_j) < 0$ for some $j \leq k$ we are done. So assume $l(VC_j) \geq 0$ for each $j \leq k$. We can assume that C_1 and C_2 traverse t. Then

$$(41.23) \qquad l(VC_1) + l(VC_2) \leq l(VC_1) + \cdots + l(VC_k) = 2l(VC).$$

Hence $l(VC_1) \leq l(VC)$ or $l(VC_2) \leq l(VC)$, and again we are done. ∎

This implies (Krogdahl [1976], Fujishige [1977a]):

Theorem 41.5. *Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. Then I is extreme if and only if $D_{M_1,M_2}(I)$ has no directed circuit of negative length.*

Proof. To see necessity, suppose that $D_{M_1,M_2}(I)$ has a directed circuit C of negative length. Choose C with $|VC|$ minimal. Consider $J := I \triangle VC$. Since $w(J) = w(I) - l(C) > w(I)$, while $|J| = |I|$, we know that $J \notin \mathcal{I}_1 \cap \mathcal{I}_2$. Hence by Lemma 41.5$\alpha$, $D_{M_1,M_2}(I)$ has a negative-length directed circuit covering fewer than $|VC|$ vertices, contradicting our assumption.

To see sufficiency, consider a $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|J| = |I|$. By Corollary 39.12a, both $D_{M_1}(I)$ and $D_{M_2}(I)$ have a perfect matching on $I \triangle J$. These two matchings together form a vertex-disjoint union of a number of directed circuits $C_1, \ldots, C_t$. Then

$$(41.24) \qquad w(I) - w(J) = \sum_{j=1}^{t} l(VC_j) \geq 0,$$

implying $w(J) \leq w(I)$. So I is extreme. ∎

This theorem implies that we can find a shortest path P, in Case 1 of the algorithm, in strongly polynomial time (with the Bellman-Ford method). It also gives:

Theorem 41.6. *If Case 1 applies, I' is an extreme common independent set.*

Proof. We first show that $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$. To this end, let t be a new element, and extend (for each $i = 1, 2$), M_i to a matroid $M'_i = (S + t, \mathcal{I}'_i)$, where for each $T \subseteq S + t$:

$$(41.25) \qquad T \in \mathcal{I}'_i \text{ if and only if } T - t \in \mathcal{I}_i.$$

Note that $D_{M'_1,M'_2}(I + t)$ arises from $D_{M_1,M_2}(I)$ by extending it with a new vertex t and adding arcs from t to each vertex in X_1, and from each vertex in X_2 to t.

Let P be the path found in the algorithm. Define

$$(41.26) \qquad w(t) := l(t) := -l(P).$$

As P is a shortest $X_1 - X_2$ path, this makes that $D_{M'_1,M'_2}(I + t)$ has no negative-length directed circuit. Hence, by Theorem 41.5, $I + t$ is an extreme common independent set of M'_1 and M'_2.

Let P run from $z_1 \in X_1$ to $z_2 \in X_2$. Extend P by the arcs (t, z_1) and (z_2, t) to a directed circuit C. So $J = (I + t) \triangle VC$. As P has a minimum number of arcs among all shortest $X_1 - X_2$ paths, and as $D_{M'_1,M'_2}(I + t)$ has no negative-length directed circuits, by Lemma 41.5α we know that $J \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Moreover, J is extreme, since $I + t$ is extreme and $w(J) = w(I + t)$. ∎

So the weighted common independent set augmenting algorithm is correct. It obviously has strongly polynomially bounded running time. Therefore:

Theorem 41.7. *A maximum-weight common independent set in two matroids can be found in strongly polynomial time.*

Proof. Starting with the extreme common independent set $I_0 := \emptyset$ we can find iteratively extreme common independent sets $I_0, I_1, \ldots, I_k$, where $|I_i| = i$ for $i = 0, \ldots, k$ and where I_k is a maximum-size common independent set. Taking one among $I_0, \ldots, I_k$ of maximum weight, we have a maximum-weight common independent set. ∎

The above algorithm gives a maximum-weight common independent set of size k, for each k. In particular, a maximum-weight common base can be found with the algorithm. Similarly for minimum-weight:

Theorem 41.8. *A minimum-weight common base in two matroids can be found in strongly polynomial time.*

Proof. The last extreme common independent set in the above algorithm is a maximum-weight common base. By flipping the signs of the weights, this can be turned into a minimum-weight common base algorithm. ∎

Notes. Frank [1981a] gave an $O(\tau n^3)$-time implementation of this algorithm, where τ is the time needed to test for any $I \in \mathcal{I}_i$ and any $s \in S$ whether or not $I \cup \{s\} \in \mathcal{I}_i$, and if not, to find a circuit of M_i contained in $I \cup \{s\}$.

Clearly, a maximum-weight common independent set need not be a common base, even if common bases exist and all weights are positive: Let $S = \{1, 2, 3\}$ and let M_i be the matroid on S with unique circuit $S \setminus \{i\}$ (for $i = 1, 2$). Define $w(1) := w(2) := 1$ and $w(3) := 3$. Then $\{3\}$ is the unique maximum-weight common independent set, while $\{1, 2\}$ is the unique common base.

41.3a. Speeding up the weighted matroid intersection algorithm

The algorithm described in Section 41.3 is strongly polynomial-time, since we can find a shortest path P in strongly polynomial time, as in each iteration the graph $D_{M_1,M_2}(I)$ has no negative-length directed circuit. Hence we can apply the Bellman-Ford method. To bound the running time, suppose that we can construct, for any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ the graph $D_{M_1,M_2}(I)$ in time T. Then any iteration can be done in time $O(T + n^3)$, where $n := |S|$.

We can improve this to $O(T + n \log n)$ as follows (Frank [1981a], Brezovec, Cornuéjols, and Glover [1986]). The idea is that, in each iteration, with the extreme common independent set I, we give a 'certificate' of extremity, by specifying a potential for the length function; that is, a function $p \in \mathbb{Q}^S$ satisfying

(41.27) $\qquad l(v) \geq p(v) - p(u)$

for each arc (u, v) of $D_{M_1,M_2}(I)$. By Theorem 41.5, such a potential certifies extremity of I. We call such a p a potential *for* I.

Having the potential, we can apply Dijkstra's method instead of the Bellman-Ford method, as with the potential we can transform the length function (if defined on arcs) to a nonnegative length function.

It is convenient to associate the following functions $w_1, w_2 : S \to \mathbb{R}$ to $p, w : S \to \mathbb{R}$:

(41.28) $w_1(v) = p(v)$ and $w_2(v) = w(v) - p(v)$ if $v \in I$,
$w_1(v) = w(v) + p(v)$ and $w_2(v) = -p(v)$ if $v \in S \setminus I$.

So $w = w_1 + w_2$. Then:

Theorem 41.9. *Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and let $p, w, w_1, w_2 : S \to \mathbb{R}$ satisfy* (41.28). *Then p is a potential for $D_{M_1,M_2}(I)$ if and only if for $i = 1, 2$ one has*

(41.29) *I maximizes $w_i(X)$ over all $J \in \mathcal{I}_i$ satisfying $|J| = |I|$.*

Proof. The theorem follows easily with Corollary 39.12b. Indeed, there is an arc (u, v) leaving I if and only if $I - u + v \in \mathcal{I}_1$. Then

(41.30) $w_1(v) \leq w_1(u) \iff l(v) \geq p(v) - p(u)$,

since $l(v) = -w(v) = -w_2(v) - w_1(v)$ and $-w_2(v) - w_1(u) = p(v) - p(u)$.

Similarly, there is an arc (u, v) entering I if and only if $I - v + u \in \mathcal{I}_2$. Then

(41.31) $w_2(v) \geq w_2(u) \iff l(v) \geq p(v) - p(u)$,

since $l(v) = w(v) = w_2(v) + w_1(v)$ and $w_2(u) + w_1(v) = p(v) - p(u)$. ∎

We trivially have a potential for $I := \emptyset$. Consider next an arbitrary iteration, with as input a common independent set I and a potential p for I. Construct $D_{M_1,M_2}(I)$ and l as before. Let P be an $X_1 - X_2$ path with $l(P)$ minimum, and, under this condition, with $|VP|$ minimum. (Using the potential described above, we can find P with Dijkstra's algorithm.) Let $I' := I \triangle VP$.

We now reset the potential p such that for any $v \in S$ with v reachable from X_1, $p(v)$ is equal to the distance from X_1 to v (= the minimum of $l(VQ)$ over all $X_1 - v$ paths Q in $D_{M_1,M_2}(I)$).

Let w_1 and w_2 satisfy (41.28) with respect to I, (the new) p, and w. Then:

Theorem 41.10. *w_1, w_2 satisfy* (41.29) *with respect to I'.*

Proof. Extend M_1 and M_2 to matroids $M_1' = (S + t, \mathcal{I}_1')$ and $M_2' = (S + t, \mathcal{I}_2')$ as in (41.25). Let P run from $z_1 \in X_1$ to $z_2 \in X_2$. Define $w(t) := l(t) := -l(P)$, $p(t) := 0$, $w_1(t) := 0$, and $w_2(t) := w(t)$. Now it suffices to show:

(41.32) (i) $w_i(I + t) = w_i(I')$ for $i = 1, 2$;
(ii) w_1, w_2 satisfy (41.29) with respect to M_1', M_2', and $I + t$.

Let C be the directed circuit obtained by extending P by the arcs (t, z_1) and (z_2, t). Now, since $I' = (I + t) \triangle VC$, to show (41.32), it suffices to show, for each arc (u, v):

(41.33) if (u, v) leaves $I + t$, then $w_1(v) \leq w_1(u)$, with equality if (u, v) is on C;
if (u, v) enters $I + t$, then $w_2(u) \leq w_2(v)$, with equality if (u, v) is on C.

Note that for each arc (u, v) of $D_{M_1',M_2'}(I + t)$ one has $p(v) \leq p(u) + l(v)$, with equality if (u, v) is on C. Hence, if (u, v) leaves $I + t$, then:

(41.34) $w_1(v) = p(v) + w(v) = p(v) - l(v) \leq p(u) = w_1(u),$

with equality if (u, v) is on C.

Similarly, if (u, v) enters $I + t$, then:

(41.35) $w_2(v) = w(v) - p(v) = l(v) - p(v) \geq -p(u) = w_2(u),$

with equality if (u, v) is on C. This proves (41.33). ∎

Using (41.28) and Theorem 41.9, we can obtain from w_1, w_2 a potential for I'. This implies:

Corollary 41.10a. *A maximum-weight common independent set can be found in time $O(k(T+n\log n))$, where $n := |S|$, k is the maximum size of a common independent set, and T is the time needed to find $D_{M_1,M_2}(I)$ for any common independent set I.*

Proof. Each iteration can be done in time $O(T + n \log n)$, since constructing the graph $D_{M_1,M_2}(I)$ takes T time, implying that there are $O(T)$ arcs. Hence, by Corollary 7.7a, a shortest $X_1 - X_2$ path P can be found in $O(T + n \log n)$ time. Hence I', and a potential for I' can be found in time $O(T + n \log n)$.

Since there are k iterations, we have the time bound given. ∎

In applications where the matroids are specifically given, one can often derive a better time bound, by obtaining $D_{M_1,M_2}(I')$ not from scratch, but by adapting $D_{M_1,M_2}(I)$. See also Brezovec, Cornuéjols, and Glover [1986] and Gabow and Xu [1989,1996].

41.4. Intersection of the independent set polytopes

It turns out that the intersection of the independent set polytopes of two matroids gives exactly the convex hull of the common independent sets, as was shown by Edmonds [1970b][27].

We first prove a very useful theorem, due to Edmonds [1970b], which we often will apply in this part. (A more general statement and interpretation in terms of network matrices will be given in Section 13.4.)

A family $\mathcal{C}$ of sets is called *laminar* if

(41.36) $Y \subseteq Z$ or $Z \subseteq Y$ or $Y \cap Z = \emptyset$

for all $Y, Z \in \mathcal{C}$.

Theorem 41.11. *Let $\mathcal{C}$ be the union of two laminar families of subsets of a set X. Let A be the $\mathcal{C} \times X$ incidence matrix of $\mathcal{C}$. Then A is totally unimodular.*

[27] Lawler [1976b] wrote that this result was announced by Edmonds 'at least as long ago as 1964'.

Proof. Let A be a counterexample with $|\mathcal{C}| + |X|$ minimal, and (secondly) with a minimal number of 1's. Then A is nonsingular and has determinant $\neq \pm 1$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be laminar families, with union $\mathcal{C}$.

If each $\mathcal{C}_i$ consists of pairwise disjoint sets, then A is the incidence matrix of a bipartite graph, added with some unit base vectors. Hence A is totally unimodular, a contradiction.

If say $\mathcal{C}_1$ does not consist of pairwise disjoint sets, $\mathcal{C}_1$ contains a smallest nonempty set Y that is contained in some other set Z in $\mathcal{C}_1$. Choose Z smallest. Replacing Z by $Z \setminus Y$, maintains laminarity of $\mathcal{C}_1$. As this does not change the determinant of the corresponding matrix (as it amounts to subtracting row indexed Y from row indexed Z), we would have a counterexample with a smaller number of 1's, a contradiction. ∎

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2. By Corollary 40.2a, the intersection $P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2)$ of the independent set polytopes associated with the matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ is determined by:

(41.37)
$$\begin{array}{lll} \text{(i)} & x_s \geq 0 & \text{for } s \in S, \\ \text{(ii)} & x(U) \leq r_i(U) & \text{for } i = 1, 2 \text{ and } U \subseteq S. \end{array}$$

Trivially, this intersection contains the convex hull of the incidence vectors of common independent sets of M_1 and M_2. We shall see that these two polytopes are equal.

Basis is the following result of Edmonds [1970b], whose proof we follow (it constitutes the base of a fundamental technique developed further in several other results).

Theorem 41.12. *System* (41.37) *is box-totally dual integral.*

Proof. Choose $w \in \mathbb{Z}^S$. Consider the linear programming problem dual to maximizing $w^{\mathsf{T}} x$ over the constraints (41.37)(ii):

(41.38)
$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{U \subseteq S} (y_1(U) r_1(U) + y_2(U) r_2(U)) \\ \text{where} & y_1, y_2 \in \mathbb{R}_+^{\mathcal{P}(S)}, \\ & \displaystyle\sum_{U \subseteq S} (y_1(U) + y_2(U)) \chi^U = w. \end{array}$$

Let y_1, y_2 attain this minimum, such that

(41.39)
$$\sum_{U \subseteq S} (y_1(U) + y_2(U)) |U| |S \setminus U|$$

is minimized. Define

(41.40)
$$\mathcal{F}_i := \{U \subseteq S \mid y_i(U) > 0\},$$

for $i = 1, 2$. We show that for $i = 1, 2$, the collection $\mathcal{F}_i$ is a chain; that is, (41.41) if $T, U \in \mathcal{F}_i$, then $T \subseteq U$ or $U \subseteq T$.

Suppose not. Choose $\alpha := \min\{y_i(T), y_i(U)\}$, and decrease $y_i(T)$ and $y_i(U)$ by α, and increase $y_i(T \cap U)$ and $y_i(T \cup U)$ by α. Since

(41.42) $$\chi^T + \chi^U = \chi^{T\cap U} + \chi^{T\cup U},$$

y_1, y_2 remains a feasible solution of (41.38); and since

(41.43) $$r_i(T) + r_i(U) \geq r_i(T \cap U) + r_i(T \cup U),$$

it remains optimum. However, sum (41.39) decreases (by Theorem 2.1), contradicting the minimality assumption. So $\mathcal{F}_1$ and $\mathcal{F}_2$ are chains.

As the constraints in (41.37)(ii) corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$ form a totally unimodular matrix (by Theorem 41.11), by Theorem 5.35 system (41.37)(ii) is box-TDI, and hence (41.37) is box-TDI. ∎

(The fact that the $\mathcal{F}_i$ can be taken to be chains also follows directly from the proof method of Theorem 40.2.)

This implies a characterization of the *common independent set polytope*

(41.44) $$P_{\text{common independent set}}(M_1, M_2)$$

of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, being the convex hull of the incidence vectors of the common independent sets of M_1 and M_2:

Corollary 41.12a. $P_{\text{common independent set}}(M_1, M_2)$ *is determined by* (41.37).

Proof. Directly from Theorem 41.12, since it implies that the vertices of the polytope determined by (41.37) are integer, and hence are the incidence vectors of common independent sets. ∎

Another way of stating this is:

Corollary 41.12b.

(41.45) $$P_{\text{common independent set}}(M_1, M_2) = P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2).$$

Proof. From Corollary 41.12a, using the fact that (41.37) is the union of the constraints for the independent set polytopes of M_1 and M_2, by Corollary 40.2b. ∎

The total dual integrality of (41.37) gives the following extension of the matroid intersection theorem:

Corollary 41.12c. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively, and let $w \in \mathbb{Z}_+^S$. Then the maximum value of $w(I)$ over $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is equal to the minimum value of*

(41.46) $\qquad r_1(U_1) + \cdots + r_1(U_k) + r_2(T_1) + \cdots + r_2(T_l)$,

where $U_1 \subseteq \cdots \subseteq U_k \subseteq S$ *and* $T_1 \subseteq \cdots \subseteq T_l \subseteq S$ *such that each element* s *of* S *occurs in precisely* $w(s)$ *sets among* $U_1, \ldots, U_k, T_1, \ldots, T_l$.

Proof. Directly from Theorem 41.12 and its proof. ∎

(Edmonds [1979] gave an algorithmic proof of this result.)

These corollaries cannot be extended to the intersection of the independent set polytopes of three matroids. Let $S = \{1, 2, 3\}$, and for $i = 1, 2, 3$, let M_i be the matroid on S with $S \setminus \{i\}$ as unique circuit. Then $P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2) \cap P_{\text{independent set}}(M_3)$ contains the all-$\frac{1}{2}$ vector, while each integer vector in this intersection contains at most one 1. So the intersection is *not* the convex hull of the common independent sets.

Similar results hold for the common base polytope. For matroids M_1 and M_2, let the *common base polytope* $P_{\text{common base}}(M_1, M_2)$ be the convex hull of the incidence vectors of common bases of M_1 and M_2. Then:

Corollary 41.12d. $P_{\text{common base}}(M_1, M_2) = P_{\text{base}}(M_1) \cap P_{\text{base}}(M_2)$.

Proof. Directly from the foregoing. ∎

So the common base polytope is determined by:

$$
\text{(41.47)} \qquad
\begin{array}{ll}
x_s \geq 0 & \text{for } s \in S, \\
x(U) \leq r_i(U) & \text{for } i = 1, 2 \text{ and } U \subseteq S, \\
x(S) = r_i(S) & \text{for } i = 1, 2.
\end{array}
$$

Corollary 41.12e. *System* (41.47) *is box-TDI.*

Proof. From Theorem 41.12, with Theorem 5.25. ∎

Moreover, similar results hold for the common spanning set polytope. For matroids M_1 and M_2, let the *common spanning set polytope*, in notation $P_{\text{common spanning set}}(M_1, M_2)$, be the convex hull of the incidence vectors of common spanning sets of M_1 and M_2. Then:

Corollary 41.12f.

$$
\text{(41.48)} \qquad
\begin{array}{l}
P_{\text{common spanning set}}(M_1, M_2) \\
= P_{\text{spanning set}}(M_1) \cap P_{\text{spanning set}}(M_2).
\end{array}
$$

Proof. This can be reduced to Corollary 41.12b on the common independent set polytope, by duality: x belongs to the spanning set polytope of M_i if and only if $\mathbf{1} - x$ belongs to the independent set polytope of M_i^*.

Similarly, x belongs to the common spanning set polytope of M_1 and M_2 if and only if $\mathbf{1} - x$ belongs to the common independent set polytope of M_1^* and M_2^*. ∎

So the common spanning set polytope is determined by:

(41.49) $$\begin{array}{ll} 0 \leq x_s \leq 1 & \text{for } s \in S, \\ x(U) \leq r_i(S) - r_i(S \setminus U) & \text{for } i = 1,2 \text{ and } U \subseteq S. \end{array}$$

Corollary 41.12g. *System* (41.49) *is box-TDI.*

Proof. Again, this can be derived from Theorem 41.12, by replacing x by $\mathbf{1} - x$. ∎

Another consequence of Theorem 41.12 is:

Corollary 41.12h. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids and let $x \in \mathbb{R}_+^S$. Then*

(41.50) $$\begin{array}{l} \max\{z(S) \mid z \leq x, z \in P_{\text{common independent set}}(M_1, M_2)\} \\ = \min\{r(U) + x(S \setminus U) \mid U \subseteq S\}, \end{array}$$

where $r(U)$ denotes the maximum size of a common independent set contained in U.

Proof. This follows from the box-total dual integrality of (41.37), using the fact that $r(U_1 \cup U_2) \leq r_1(U_1) + r_2(U_2)$ for disjoint U_1, U_2. ∎

Cunningham [1984] showed that, if matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are given by independence testing oracles, one can find in strongly polynomial time for any $x \in \mathbb{Q}^S$, optimum solutions of (41.50). This will follow from the results in Section 47.4.

The result of Cunningham [1984] also implies:

Theorem 41.13. *Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ by independence testing oracles, and given $x \in \mathbb{Q}^S$, one can test in strongly polynomial time if x belongs to the common independent set polytope, and if so, decompose x as a convex combination of incidence vectors of common independent sets.*

Proof. Let r_i be the rank function of M_i ($i = 1, 2$) and let $r(U) := \min\{r_1(U), r_2(U)\}$ for $i = 1, 2$. Let P be the common independent set polytope. Corollaries 40.4a and 41.12b imply that one can test in strongly polynomial time if x belongs to P.

So we can assume that x belongs to P. We decompose x as a convex combination of incidence vectors of common independent sets. Iteratively resetting x, we keep a collection $\mathcal{U}$ of subsets of S with $x(U) = r(U)$ for each $U \in \mathcal{U}$. Initially, $\mathcal{U} := \emptyset$. We describe the iteration.

Define

(41.51) $$F := \{y \in P \mid \forall s \in S : x_s = 0 \Rightarrow y_s = 0; \forall U \in \mathcal{U} : y(U) = r(U)\}.$$

So F is a face of P containing x.

Find a common independent set I with $\chi^I \in F$. This can be done by finding a common independent set $I \subseteq \text{supp}(x)$ maximizing $w^{\mathsf{T}}x$, where $w := \sum_{U \in \mathcal{U}} \chi^U$. (Here supp$(x)$ is the support of x; so supp$(x) = \{s \in S \mid x_s > 0\}$.)

If $x = \chi^I$ we stop. Otherwise, define $u := x - \chi^I$. Let λ be the largest rational such that

(41.52) $$\chi^I + \lambda u$$

belongs to P.

We describe an inner iteration to find λ. We consider vectors z along the halfline $L = \{\chi^I + \lambda u \mid \lambda \geq 0\}$. First we let λ be the largest rational with $\chi^I + \lambda u \geq \mathbf{0}$, and set $z := \chi^I + \lambda u$.

We iteratively reset z. We check if z belongs to the common independent set polytope, and if not, we find a $U \subseteq S$ minimizing $r(U) - z(U)$ (with Corollary 40.4c). Let z' be the (unique) vector on L achieving $x(U) \leq r(U)$ with equality; that is, satisfying $z'(U) = r(U)$.

Consider any inequality $x(U') \leq r(U')$ violated by z'. Then

(41.53) $$r(U') - |U' \cap I| < r(U) - |U \cap I|.$$

This can be seen by considering the function

(41.54) $$d(y) := (r(U) - y(U)) - (r(U') - y(U')).$$

We have $d(z) \leq 0$ (since U minimizes $r(U) - z(U)$) and $d(z') > 0$ (since $z'(U) = r(U)$ and $z'(U') > r(U')$). Hence, as d is linear, $d(\chi^I) > 0$; that is, we have (41.53). This implies that resetting $z := z'$, there are at most $r(S)$ inner iterations.

Let x' be the final z found. If we apply no inner iteration, then $x'_s = 0$ for some $s \in I \subseteq \text{supp}(x)$ (since we chose λ largest with $\chi^I + \lambda u \geq \mathbf{0}$). If we do at least one inner iteration, we find a U such that x' satisfies $x'(U) = r(U)$ while $|U \cap I| < r(U)$ (since x' is the unique vector on L satisfying $x'(U) = r(U)$ and since $x' \neq \chi^I$).

In the latter case, set $\mathcal{U}' := \mathcal{U} \cup \{U\}$; otherwise set $\mathcal{U}' := \mathcal{U}$. Then resetting x to x' and $\mathcal{U}$ to $\mathcal{U}'$, the dimension of F decreases (as χ^I does not belong to the new F). So the number of iterations is at most $|S|$. This shows that the method is strongly polynomial-time. ∎

41.4a. Facets of the common independent set polytope

Since the common independent set polytope of two matroids is the intersection of their independent set polytopes, each facet-inducing inequality for the intersection is facet-inducing for (at least) one of the independent set polytopes, but not necessarily conversely. Giles [1975] characterized which inequalities are facet-inducing for the common independent set polytope. If this polytope is full-dimensional, then each inequality $x_s \geq 0$ is facet-inducing. As for the other inequalities, Giles proved:

Theorem 41.14. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be loopless matroids, with rank functions r_1 and r_2. For $U \subseteq S$, define $r(U) := \min\{r_1(U), r_2(U)\}$. Then, for $U \subseteq S$, the inequality*

(41.55) $\qquad x(U) \leq r(U)$

is facet-inducing for $P_{\text{common independent set}}(M_1, M_2)$ if and only if there is no partition of U into nonempty proper subsets U_1, U_2 with

(41.56) $\qquad r(U) \geq r(U_1) + r(U_2)$

and there is no proper superset U' of U with $r(U') \leq r(U)$.

Proof. By symmetry, we can assume that $r(U) = r_1(U)$.

Necessity is easy: Assume that $x(U) \leq r_1(U)$ is facet-inducing. If (41.56) would hold, then each common independent set I with $|I \cap U| = r_1(U)$ satisfies $|I \cap U_1| = r(U_1)$ (since $|I \cap U_1| = |I \cap U| - |I \cap U_2| \geq r(U) - r(U_2) \geq r(U_1)$). Hence each x in the facet determined by $x(U) \leq r_1(U)$ satisfies $x(U_1) = r(U_1)$, a contradiction. Similarly, if $r(U') \leq r_1(U)$ for some proper superset U' of U, then each common independent set I with $|I \cap U| = r_1(U)$ satisfies $|I \cap U'| = r(U')$, implying that each x in the facet determined by $x(U) \leq r_1(U)$ satisfies $x(U') = r(U')$, again a contradiction.

To see sufficiency, suppose that (41.55) satisfies the conditions, but is not facet-inducing for the common independent set polytope. This implies that the inequality $x(U) \leq r_1(U)$ is implied by other inequalities in (41.37). So there exist $\lambda_i : \mathcal{P}(S) \to \mathbb{Q}_+$ (for $i = 1, 2$) such that

(41.57)
$$\sum_{T \in \mathcal{P}(S)} (\lambda_1(T) + \lambda_2(T))\chi^T \geq \chi^U \text{ and}$$
$$\sum_{T \in \mathcal{P}(S)} (\lambda_1(T) r_1(T) + \lambda_2(T) r_2(T)) \leq r_1(U),$$

and such that $\lambda_i(U) = 0$ for $i = 1, 2$. Let D be the least common denominator of the values of the λ_i. Choose the λ_i such that D is as small as possible and (secondly) such that

(41.58)
$$D \cdot \sum_{T \subseteq S} (\lambda_1(T) + \lambda_2(T))|T|(|S \setminus T| + 1)$$

is as small as possible. For $i = 1, 2$, define

(41.59) $\qquad \mathcal{F}_i := \{T \subseteq S \mid \lambda_i(T) > 0\}.$

We claim that for $i = 1, 2$:

(41.60) $\qquad \mathcal{F}_i$ is a chain.

Suppose to the contrary that $T_1, T_2 \in \mathcal{F}_i$ satisfy $T_1 \not\subseteq T_2 \not\subseteq T_1$. Then decreasing $\lambda_i(T_1)$ and $\lambda_i(T_2)$ by $1/D$ and increasing $\lambda_i(T_1 \cap T_2)$ and $\lambda_i(T_1 \cup T_2)$ by $1/D$ maintains (41.57) but decreases (41.58). This would be a contradiction, except if $T_1 \cap T_2$ or $T_1 \cup T_2$ equals U. If one of these sets equals U and $D \geq 2$, we can reset $\lambda_i(U) := 0$, and multiply all values of λ_1 and λ_2 by $D/(D-1)$. This again maintains (41.57) but decreases the least common divisor of the denominators. So the contradiction would remain, except if $D = 1$. Then (41.57) implies $r_i(T_1) + r_i(T_2) \leq r_1(U)$. Now if $T_1 \cap T_2 = U$, then $U \subset T_1$ and

(41.61) $$r(T_1) \leq r_i(T_1) \leq r_i(T_1) + r_i(T_2) \leq r_1(U),$$

contradicting the condition. If $T_1 \cup T_2 = U$, then

(41.62) $$r(T_1) + r(U \setminus T_1) \leq r_i(T_1) + r_i(U \setminus T_1) \leq r_i(T_1) + r_i(T_2) \leq r_1(U),$$

again contradicting the condition.

This proves (41.60). As each $\mathcal{F}_i$ is a chain, the incidence matrix of $\mathcal{F}_1 \cup \mathcal{F}_2$ is totally unimodular (by Theorem 41.11). Therefore, there are integer-valued λ_i satisfying (41.57), with $\lambda_i(T) = 0$ for $T \notin \mathcal{F}_i$. Then we can assume that $|\mathcal{F}_i| \leq 1$ for $i = 1, 2$, since if $T, T' \in \mathcal{F}_i$ and $T \subset T'$, we can decrease $\lambda_i(T)$ by 1 without violating (41.57). If $U' \in \mathcal{F}_i$ with $U' \supset U$, then $r(U') \leq r_i(U') \leq r(U)$, contradicting the condition. So each $\mathcal{F}_i$ contains a set $U_i \not\supseteq U$, implying $r(U_1) + r(U \setminus U_1) \leq r(U_1) + r(U_2) \leq r_1(U_1) + r_2(U_2) \leq r(U)$, again contradicting the condition. ∎

This theorem can be seen to imply a variant of it, in which, instead of $r(U) := \min\{r_1(U), r_2(U)\}$, we define

(41.63) $$r(U) := \max\{|I| \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{T \subseteq U}(r_1(T) + r_2(U \setminus T)).$$

Fonlupt and Zemirline [1983] characterized the dimension of the common base polytope of two matroids.

41.4b. Up and down hull of the common base polytope

We saw in Corollary 41.12d a characterization of the common base polytope $P_{\text{common base}}(M_1, M_2)$ of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$. The up hull of this polytope:

(41.64) $$P^{\uparrow}_{\text{common base}}(M_1, M_2) := P_{\text{common base}}(M_1, M_2) + \mathbb{R}^S_+$$

was characterized by Cunningham [1977] and McDiarmid [1978] as follows (proving a conjecture of Fulkerson [1971a]).

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids having a common base. Then $P^{\uparrow}_{\text{common base}}(M_1, M_2)$ is determined by:

(41.65) $$x(U) \geq r(S) - r(S \setminus U) \text{ for } U \subseteq S,$$

where $r(Z) :=$ the maximum size of a common independent set contained in Z. (A weaker version of this was proved by Edmonds and Giles [1977].)

For a proof we refer to Section 46.7a, where it is also shown that (41.65) is TDI (Gröflin and Hoffman [1981]). (Frank and Tardos [1984a] derived this, with a direct algorithmic construction, from the total dual integrality of (41.47).)

Note that by the matroid intersection theorem, the inequalities (41.65) are equivalent to:

(41.66) $$x(U) \geq k - r_1(A) - r_2(B) \text{ for each partition } U, A, B \text{ of } S,$$

where r_1 and r_2 are the rank functions of M_1 and M_2 respectively, and where k is the size of a common base. This implies that if we add $x \leq \mathbf{1}$ to (41.66) we obtain the convex hull of the subsets of S that contain a common base.

Similarly, the down hull of the common base polytope:

(41.67) $$P^{\downarrow}_{\text{common base}}(M_1, M_2) := P_{\text{common base}}(M_1, M_2) - \mathbb{R}^S_+,$$

is determined by

(41.68) $$x(U) \leq r_1(S \setminus A) + r_2(S \setminus B) - k \text{ for each partition } U, A, B \text{ of } S.$$

This can be derived from the description of the up hull of the common base polytope, since

(41.69) $$P^{\downarrow}_{\text{common base}}(M_1, M_2) = \mathbf{1} - P^{\uparrow}_{\text{common base}}(M_1^*, M_2^*)$$

(where $\mathbf{1}$ stands for the all-one vector in $\mathbb{R}^S$).

This implies that the convex hull of the incidence vectors of the subsets of common bases is determined by $x \geq \mathbf{0}$ and (41.68).

Cunningham [1984] gave a strongly polynomial-time algorithm to test if a vector belongs to $P^{\uparrow}_{\text{common base}}(M_1, M_2)$, or to $P^{\downarrow}_{\text{common base}}(M_1, M_2)$, using only independence testing oracles for M_1 and M_2.

41.5. Further results and notes

41.5a. Menger's theorem for matroids

Tutte [1965b] showed a special case of the matroid intersection theorem, namely when both M_1 and M_2 are minors of one matroid. Specialized to graphic matroids, it gives the vertex-disjoint, undirected version of Menger's theorem.

Let $M = (E, \mathcal{I})$ be a matroid, with rank function r, and let U and W be disjoint subsets of E. Then the maximum size of a common independent set in $M/U \setminus W$ and $M/W \setminus U$ is equal to the minimum value of

(41.70) $$r(X) - r(U) + r(E \setminus X) - r(W)$$

taken over sets X with $U \subseteq X \subseteq E \setminus W$. This is the special case of the matroid intersection theorem for the matroids $M/U \setminus W$ and $M/W \setminus U$, since for $Y \subseteq E \setminus (U \cup W)$ one has

(41.71) $$r_{M/U\setminus W}(Y) = r(Y \cup U) - r(U),$$

and similarly for $M/W \setminus U$.

To see that this implies the vertex-disjoint, undirected version of Menger's theorem, let $G = (V, E)$ be a graph and let S and T be disjoint nonempty subsets of V. We show that the above theorem implies that the maximum number of disjoint $S - T$ paths in G is equal to the minimum number of vertices intersecting each $S - T$ path.

To this end, we can assume that G is connected, and that E contains subsets U and W such that (S, U) and (T, W) are trees. (Adding appropriate edges does not modify the result to be proved.)

Let $M := M(G)$ be the cycle matroid of G. Define $R := V \setminus (S \cup T)$. Then

(41.72) the maximum number of disjoint $S-T$ paths is at least the maximum size of a common independent set I of $M/U\setminus W$ and $M/W\setminus U$, minus $|R|$.

(In fact, there is equality.)

To prove (41.72), let I be a maximum-size common independent in $M/U\setminus W$ and $M/W\setminus U$. So I is a forest. Consider any component K of I. Since I is independent in M/U, K intersects S in at most one vertex. Similarly, K intersects T in at most one vertex. Let p be the number of components K intersecting both S and T. By deleting p edges we obtain a forest I' such that no component of I' intersects both S and T. So $|I'|\leq|R|$ (since I' remains a forest after contracting (in the graphical sense) $S\cup T$ to one vertex). Hence $p=|I|-|I'|\geq|I|-|R|$. So we have (41.72).

On the other hand,

(41.73) the minimum size of a set of vertices intersecting each $S-T$ path is at most the minimum value of (41.70), minus $|R|$.

(Again, we have in fact equality.)

To prove (41.73), let X attain the minimum value of (41.70). So $U\subseteq X\subseteq E\setminus W$. Let K be the component of (V,X) containing S and let L be the component of $(V,E\setminus X)$ containing T. We choose X with $|K\cup L|$ maximized.

Then $K\cup L=V$. For suppose not. Then, as G is connected, there is an edge e of G leaving $K\cup L$. By symmetry, we can assume that $e\in X$. Let K' be the component of (V,X) containing e. So $K'\neq K$ and $E[K']\cap U=\emptyset$. Resetting X by $X\setminus E[K']$, $r(X)$ decreases by $|K'|-1$, while $r(E\setminus X)$ increases by at most $|K'|-1$. So the new X again attains the minimum in (41.70), while $K\cup L$ increases. This contradicts our maximality assumption.

So $K\cup L=V$. Hence $K\cap L$ intersects each $S-T$ path (since $S\subseteq K$ and $T\subseteq L$, and there is no edge connecting $K\setminus L$ and $L\setminus K$). Moreover

(41.74)
$$\begin{aligned}|K\cap L|&=|K|+|L|-|V|\leq(r(X)+1)+(r(E\setminus X)+1)-|V|\\&=r(X)+r(E\setminus X)-|V|+2=r(X)+r(E\setminus X)-r(U)-r(W)-|R|.\end{aligned}$$

So we have (41.73).

Since the maximum number of disjoint $S-T$ paths is trivially not more than the minimum number of vertices intersecting all $S-T$ paths, we thus obtain Menger's theorem (and also equality in (41.72) and (41.73)).

(Tomizawa [1976a] gave an algorithm for Menger's theorem for matroids.)

41.5b. Exchange properties

Kundu and Lawler [1973] showed the following extension of the exchange property of bipartite graphs given in Theorem 16.8. Let $M_1=(S,\mathcal{I}_1)$ and $M_2=(S,\mathcal{I}_2)$ be matroids, with span functions span_1 and span_2. Then

(41.75) For any $I_1,I_2\in\mathcal{I}_1\cap\mathcal{I}_2$ there exists an $I\in\mathcal{I}_1\cap\mathcal{I}_2$ with $I_1\subseteq\mathrm{span}_1(I)$ and $I_2\subseteq\mathrm{span}_2(I)$.

(Theorem 16.8 is equivalent to the case where the M_i are partition matroids.)

To prove (41.75), choose $I\in\mathcal{I}_1\cap\mathcal{I}_2$ with $I_1\subseteq\mathrm{span}_1(I)$ and $|I\cap I_2|$ maximized. Suppose that $I_2\not\subseteq\mathrm{span}_2(I)$. Choose $s\in I_2\setminus\mathrm{span}_2(I)$ with $I\cup\{s\}\in\mathcal{I}_2$. By the maximality of $|I\cap I_2|$ we know that $I\cup\{s\}\notin\mathcal{I}_1$. So M_1 has a circuit C

contained in $I \cup \{s\}$. Since $I_2 \in \mathcal{I}_1$ we know that $C \not\subseteq I_2$. Choose $t \in C \setminus I_2$. Then for $I' := I - t + s$ we have $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$, while $\mathrm{span}_1(I') = \mathrm{span}_1(I)$. Since $|I' \cap I_2| > |I \cap I_2|$ this contradicts the maximality assumption.

A second exchange property was shown by Davies [1976]:

(41.76) Two matroids M_1 and M_2 have bases B_1 and B_2 (respectively) with $|B_1 \cap B_2| = k$ if and only if M_1 has bases X_1 and Y_1 and M_2 has bases X_2 and Y_2 with $|X_1 \cap X_2| \leq k$ and $|Y_1 \cap Y_2| \geq k$.

To see this, we may assume that $X_2 = Y_2$, since if $|X_1 \cap Y_2| \leq k$ we can reset $X_2 := Y_2$, and if $|X_1 \cap Y_2| > k$ we can reset $Y_1 := X_1$ and exchange indices.

By (39.33)(ii), there exists a series of bases $Z_0, \ldots, Z_t$ of M_1 such that $Z_0 = X_1$, $Z_t = Y_1$, and $|Z_{i-1} \triangle Z_i| = 2$ for $i = 1, \ldots, t$. Hence

(41.77) $\big||Z_{i-1} \cap X_2| - |Z_i \cap X_2|\big| \leq 1$

for $i = 1, \ldots, t$. Since $|Z_0 \cap X_2| \leq k$ and $|Z_t \cap X_2| \geq k$, we know $|Z_i \cap X_2| = k$ for some i. This proves (41.76).

41.5c. Jump systems

A framework that includes both matroid intersection and maximum-size matching was introduced by Bouchet and Cunningham [1995]. For $x, y \in \mathbb{Z}^n$, let $[x, y]$ be the set of vectors $z \in \mathbb{Z}^n$ with $\|x - y\|_1 = \|x - z\|_1 + \|z - y\|_1$. So $[x, y]$ consists of all integer vectors z in the box $x \wedge y \leq z \leq x \vee y$.

Call a vector z a *step from x to y* if $z \in [x, y]$ and $\|z - x\|_1 = 1$. A *jump system* is a finite subset J of $\mathbb{Z}^n$ satisfying the following axiom:

(41.78) if $x, y \in J$ and z is a step from x to y, then $z \in J$ or J contains a step from z to y.

Trivially, for any jump system J and any $x, y \in \mathbb{Z}^n$, the intersection $J \cap [x, y]$ is again a jump system. Moreover, being a jump system is maintained under translations by an integer vector and by reflections in a coordinate hyperplane. Bouchet and Cunningham [1995] showed that the sum of jump systems is again a jump system (attributing the proof below to A. Sebő):

Theorem 41.15. *If J_1 and J_2 are jump systems in $\mathbb{Z}^n$, then $J_1 + J_2$ is a jump system.*

Proof. For $x, y \in J_1 + J_2$ we prove (41.78) by induction on the minimum value of

(41.79) $\|y' - x'\|_1 + \|y'' - x''\|_1$,

where $x', y' \in J_1$, $x'', y'' \in J_2$, $x' + x'' = x$, and $y' + y'' = y$.

Let z be a step from x to y. By reflection and permutation of coordinates, we can assume that $z = x + \chi^1$. So $x_1 < y_1$. Hence, by symmetry of J_1 and J_2, we can assume that $x'_1 < y'_1$. Next, by reflection, we can assume that $x' \leq y'$.

Now $x' + \chi^1$ is a step from x' to y'. If $x' + \chi^1 \in J_1$, then $z = x' + \chi^1 + x'' \in J_1 + J_2$, and we have (41.78). So we can assume that $x' + \chi^1 \notin J_1$. Hence, by (41.78) applied to J_1, there is an $i \in \{1, \ldots, n\}$ with $\tilde{x}' := x' + \chi^1 + \chi^i \in J_1$ and $\tilde{x}' \leq y'$.

So $z + \chi^i = \tilde{x}' + x'' \in J_1 + J_2$. If $z + \chi^i \in [x, y]$, we have (41.78). If $z + \chi^i \notin [x, y]$, then as $z \in [x, y]$, we have $z_i = y_i$. So z is a step from $z + \chi^i$ to y. Also,

$\|y' - \tilde{x}'\|_1 = \|y' - x'\|_1 - 2$. Hence, by our induction hypothesis applied to $z + \chi^i$ and y, we have (41.78). ∎

As Bouchet and Cunningham [1995] observed, this theorem implies that the following two constructions give jump systems $J \subseteq \mathbb{Z}^V$.

For any matroid $M = (S, \mathcal{I})$, the set $\{\chi^B \mid B \text{ base of } M\}$ is a jump system in $\mathbb{Z}^S$, as follows directly from the axioms (39.33). With Theorem 41.15, this implies that for matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, the set

(41.80) $$J := \{\chi^{B_1} - \chi^{B_2} \mid B_i \text{ base of } M_i \ (i = 1, 2)\}$$

is a jump system.

Let $G = (V, E)$ be an undirected graph and let

(41.81) $$J := \{\deg_F \mid F \subseteq E\} \subseteq \mathbb{Z}^V;$$

that is, J is the collection of degree sequences of spanning subgraphs of G. Again, J is a jump system. This follows from Theorem 41.15, since for each edge $e = uv$ the set $\{\mathbf{0}, \chi^{\{u,v\}}\}$ is trivially a jump system in $\mathbb{Z}^V$ and since J is the sum of these jump systems.

Bouchet and Cunningham [1995] showed that the following greedy approach finds, for any $w \in \mathbb{R}^n$, a vector $x \in J$ maximizing $w^\mathsf{T} x$. By reflecting, we can assume that $w \geq \mathbf{0}$. We can also assume that $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $J_0 := J$, and for $i = 1, \ldots, n$, let J_i be the set of vectors x in J_{i-1} maximizing x_i over J_{i-1}. Trivially, J_n consists of one vector, y say. Then:

Theorem 41.16. *y maximizes $w^\mathsf{T} x$ over J.*

Proof. It suffices to show that the maximum value of $w^\mathsf{T} x$ over J_1 is the same as over J (since applying this to the jump systems $J_1, \ldots, J_n$ gives the theorem). Let the maximum over J be attained by x and over J_1 by y. Suppose $w^\mathsf{T} y < w^\mathsf{T} x$. So $x \notin J_1$, and hence $x_1 < y_1$. We choose x, y such that $y_1 - x_1$ is minimal. Let $z := x + \chi^1$. So z is a step from x to y.

Then $w^\mathsf{T} z = w^\mathsf{T} x + w_1 \geq w^\mathsf{T} x$. Hence $z \notin J$, since otherwise we can replace x by z, contradicting the minimality of $y_1 - x_1$. So, by (41.78), J contains a step u from z to y. So $u = z \pm \chi^i$ for some $i \in \{1, \ldots, n\}$. Then

(41.82) $$w^\mathsf{T} u = w^\mathsf{T} z \pm w_i \geq w^\mathsf{T} z - w_i = w^\mathsf{T} x + w_1 - w_i \geq w^\mathsf{T} x.$$

So we can replace x by u, again contradicting the minimality of $y_1 - x_1$ (as $u_1 > x_1$). ∎

Lovász [1997] gave a min-max relation for the minimum l_1-distance of an integer vector to a jump system of special type. It can be considered as a common generalization of the matroid intersection theorem (Theorem 41.1) and the Tutte-Berge formula (Theorem 24.1).

For a survey, see Cunningham [2002].

41.5d. Further notes

A special case of the weighted matroid intersection algorithm (where one matroid is a partition matroid) was studied by Brezovec, Cornuéjols, and Glover [1988].

Data structures for on-line updating of matroid intersection solutions were given by Frederickson and Srinivas [1984,1987], and a randomized parallel algorithm for linear matroid intersection by Narayanan, Saran, and Vazirani [1992,1994].

An extension of matroid intersection to 'supermatroid' intersection was given by Tardos [1990]. Fujishige [1977a] gave a primal approach to weighted matroid intersection, and Shigeno and Iwata [1995] a dual approximation approach. Camerini and Maffioli [1975,1978] studied 3-matroid intersection problems.

Chapter 42

Matroid union

Matroid union is closely related to matroid intersection, and most of the basic matroid union results follow from basic matroid intersection results, and vice versa. But matroid union also gives a shift in focus and offers a number of specific algorithmic questions.

42.1. Matroid union theorem

The matroid union theorem will be derived from the following basic result given by Nash-Williams [1967], suggested by earlier unpublished work of J. Edmonds[28]:

Theorem 42.1. *Let $M' = (S', \mathcal{I}')$ be a matroid, with rank function r', and let $f : S' \to S$. Define*

(42.1) $$\mathcal{I} := \{f(I') \mid I' \in \mathcal{I}'\}$$

(where $f(I') := \{f(s) \mid s \in I'\}$). Then $M = (S, \mathcal{I})$ is a matroid, with rank function r given by

(42.2) $$r(U) = \min_{T \subseteq U}(|U \setminus T| + r'(f^{-1}(T)))$$

for $U \subseteq S$.

Proof. Trivially, $\mathcal{I}$ is nonempty and closed under taking subsets. To see condition (39.1)(ii), let $I, J \in \mathcal{I}$ with $|I| < |J|$. Choose $I', J' \in \mathcal{I}'$ with $f(I') = I$, $f(J') = J$, $|I'| = |I|$, $|J'| = |J|$, and $|I' \cap J'|$ as large as possible. As M' is a matroid, $I' + j \in \mathcal{I}'$ for some $j \in J' \setminus I'$. If $f(j) \in f(I')$, say $f(j) = f(i)$ for $i \in I'$, replacing I' by $I' - i + j$ would increase $|I' \cap J'|$, contradicting our assumption. So $f(j) \in J \setminus I$ and $f(I') + f(j) = f(I' + j) \in \mathcal{I}$. This proves (39.1)(ii), and hence M is a matroid.

The rank $r(U)$ of a subset U of S is equal to the maximum size of a common independent set in M' and the partition matroid $N = (S', \mathcal{J})$ induced by the family $(f^{-1}(s) \mid s \in U)$. By the matroid intersection theorem (Theorem 41.1), this is equal to the right-hand side of (42.2). ∎

[28] as mentioned in the footnote on page 20 of Pym and Perfect [1970] (quoted in Section 42.6f below).

(In his paper, Nash-Williams suggested a direct proof, by decomposing f as a product of 'elementary' functions in which only two elements are merged. Welsh [1970] observed that the rank formula (42.2) also follows directly from Rado's theorem (Corollary 41.1c) of Rado [1942].)

Theorem 42.1 implies the following result, formulated explicitly by Edmonds [1968] (and for all M_i equal by Nash-Williams [1967]).

Let $M_1 = (S_1, \mathcal{I}_1), \ldots, M_k = (S_k, \mathcal{I}_k)$ be matroids. Define the *union* of these matroids as $M_1 \vee \cdots \vee M_k = (S_1 \cup \cdots \cup S_k, \mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k)$, where

(42.3) $$\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k := \{I_1 \cup \ldots \cup I_k \mid I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\}.$$

Corollary 42.1a (matroid union theorem). *Let $M_1 = (S_1, \mathcal{I}_1), \ldots, M_k = (S_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$, respectively. Then $M_1 \vee \cdots \vee M_k$ is a matroid again, with rank function r given by:*

(42.4) $$r(U) = \min_{T \subseteq U} (|U \setminus T| + r_1(T \cap S_1) + \cdots + r_k(T \cap S_k)).$$

for $U \subseteq S_1 \cup \cdots \cup S_k$.

Proof. To see that $M_1 \vee \cdots \vee M_k$ is a matroid, let for each i, $M'_i = (S'_i, \mathcal{I}'_i)$ be a copy of M_i with $S'_1, \ldots, S'_k$ disjoint. Then trivially $M'_1 \vee \cdots \vee M'_k$ is a matroid. Now define $f : S'_1 \cup \cdots \cup S'_k \to S_1 \cup \cdots \cup S_k$ by, for $i = 1, \ldots, k$ and $s \in S'_i$: $f(s)$ is the original of s in S_i. Then the matroid obtained in Theorem 42.1 is equal to $M_1 \vee \cdots \vee M_k$, proving that the latter indeed is a matroid, and (42.4) follows from (42.2). ∎

Conversely, the matroid intersection theorem may be derived from the matroid union theorem (as was shown by Edmonds [1970b]): the maximum size of a common independent set in two matroids M_1 and M_2, is equal to the maximum size of an independent set in the union $M_1 \vee M_2^*$, minus the rank of M_2^*.

Application of the matroid union theorem to a number of copies of the same matroid gives the following results. First:

Corollary 42.1b. *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k independent sets is equal to*

(42.5) $$\min_{U \subseteq S} (|S \setminus U| + k \cdot r(U)).$$

Proof. This follows by applying Corollary 42.1a to $M_1 = \cdots = M_k = M$. ∎

This implies that the minimum number of independent sets (or bases) needed to cover the underlying set is described by the following result of Edmonds [1965c][29]:

[29] This result was also given, without proof, by Rado [1966], saying that the argument of Horn [1955] for linear matroids can be extended to arbitrary matroids. The result con-

Corollary 42.1c (matroid base covering theorem)**.** *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $k \in \mathbb{Z}_+$. Then S can be covered by k independent sets if and only if*

$$(42.6) \qquad k \cdot r(U) \geq |U|$$

for each $U \subseteq S$.

Proof. M can be covered by k independent sets if and only if there is a union of k independent sets of size $|S|$. By Corollary 42.1b, this is the case if and only if

$$(42.7) \qquad \min_{U \subseteq S}(|S \setminus U| + k \cdot r(U)) \geq |S|,$$

that is, if and only if $k \cdot r(U) \geq |U|$ for each subset U of S. ∎

One similarly has for the maximum number of disjoint bases in a matroid (Edmonds [1965a]):

Corollary 42.1d (matroid base packing theorem)**.** *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $k \in \mathbb{Z}_+$. Then there exist k disjoint bases if and only if*

$$(42.8) \qquad k \cdot (r(S) - r(U)) \leq |S \setminus U|$$

for each $U \subseteq S$.

Proof. M has k disjoint bases if and only if the maximum size of the union of k independent sets is equal to $k \cdot r(S)$. By Corollary 42.1b, this is the case if and only if

$$(42.9) \qquad \min_{U \subseteq S}(|S \setminus U| + k \cdot r(U)) \geq k \cdot r(S),$$

that is, if and only if $|S \setminus U| \geq k \cdot (r(S) - r(U))$ for each subset U of S. ∎

The more general forms of Corollaries 42.1c and 42.1d, with different matroids, were shown by Edmonds and Fulkerson [1965].

42.1a. Applications of the matroid union theorem

We describe a number of applications of the matroid union theorem. Further applications will follow in Chapter 51 on packing and covering of trees and forests.

Transversal matroids. Let $\mathcal{X} = (X_1, \ldots, X_n)$ be a family of subsets of a finite set S, and define for each $i = 1, \ldots, n$ a matroid M on S by: Y is independent in M_i if and only if $Y \subseteq X_i$ and $|Y| \leq 1$. Now the union $M_1 \vee \cdots \vee M_n$ is the same

firms a question of Rado [1962a,1962b] (in fact, the result also follows by an elementary construction from Rado's theorem (Corollary 41.1c) given in Rado [1942]).

as the transversal matroid induced by $\mathcal{X}$, so in this way one can prove again that transversal matroids indeed are matroids.

Disjoint transversals. Let $\mathcal{X} = (X_1, \ldots, X_n)$ be a family of subsets of a finite set S. Then $\mathcal{X}$ has k disjoint transversals if and only if

(42.10) $$|\bigcup_{i \in I} X_i| \geq k \cdot |I|$$

for each $I \subseteq \{1, \ldots, n\}$. This easy consequence of Hall's marriage theorem (cf. Theorem 22.10) can also be derived by applying the matroid base packing theorem to the transversal matroid induced by $\mathcal{X}$, using (39.19).

Similarly, it can be derived from the matroid base covering theorem that S can be partitioned into k partial transversals of $\mathcal{X}$ if and only if

(42.11) $$k(n - |I|) \geq |S \setminus \bigcup_{i \in I} X_i|$$

for each $I \subseteq \{1, \ldots, n\}$ (cf. Theorem 22.12).

Vector spaces. A finite subset S of a vector space can be covered by k linearly independent sets if and only if

(42.12) $|U| \leq k \cdot \text{rank}(U)$ for each $U \subseteq S$.

This conjecture of K.F. Roth and R. Rado was shown by Horn [1955][30]. It is the special case of the matroid base covering theorem for linear matroids (see also Section 42.1b below).

As a similar consequence of the matroid base packing theorem one has that the n-dimensional vector space S over the field $GF(q)$ contains $k := \lfloor (q^n - 1)/n \rfloor$ disjoint bases. Indeed, for each $U \subseteq S$ one has $k(n - r(U)) \leq q^n - |U|$, as $|U| \leq q^{r(U)}$.

An exchange property of bases. The matroid union theorem also implies the following stronger exchange property of bases of a matroid (stronger than given in the 'axioms' in Theorem 39.6). In any matroid $M = (S, \mathcal{I})$,

(42.13) for any two bases B_1 and B_2 and for any partition of B_1 into X_1 and Y_1, there is a partition of B_2 into X_2 and Y_2 such that both $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are bases.

This property was conjectured by G.-C. Rota, and proved by Brylawski [1973], Greene [1973], and Woodall [1974a] — we follow the proof of McDiarmid [1975a].

Consider the matroids $M_1 := M/Y_1$ and $M_2 := M/X_1$. Note that M_1 has rank $|X_1|$ and that M_2 has rank $|Y_1|$. We must show that B_2 is the union of an independent set X_2 of M_1 and an independent set Y_2 of M_2. By the submodularity of the rank functions ((39.38)(ii)) we have for each $T \subseteq B_2$:

30 Horn [1955] thanked Rado 'for improvements in the setting out of the argument'. The result was also published, in the same journal, by Rado [1962a]. This paper does not mention Horn's paper. The proof by Rado [1962a] is the same as that of Horn [1955] and uses the same notation. But Rado [1966] said that the theorem was first proved by Horn [1955].

(42.14) $$\begin{aligned}&|B_2 \setminus T| + r_{M_1}(T \setminus Y_1) + r_{M_2}(T \setminus X_1)\\ &= |B_2 \setminus T| + r(T \cup Y_1) - |Y_1| + r(T \cup X_1) - |X_1|\\ &\geq r(T) + r(T \cup Y_1 \cup X_1) - |T| = |B_2|.\end{aligned}$$

Hence, by the matroid union theorem (Corollary 42.1a), we have the required result.

Repeated application of this exchange phenomenon implies the following stronger property, given by Greene and Magnanti [1975]:

(42.15) for any two bases B_1 and B_2 and any partition of B_1 into $X_1, \ldots, X_k$, there is a partition of B_2 into $Y_1, \ldots, Y_k$ such that $(B_1 \setminus X_i) \cup Y_i$ is a base, for each $i = 1, \ldots, k$.

This extends Corollary 39.12a, which is the special case where each X_i is a singleton.

42.1b. Horn's proof

The proof of Horn [1955] of the matroid base covering theorem for linear matroids directly extends to general matroids (as was observed by Rado [1966]):

Consider a counterexample to the matroid base covering theorem (Corollary 42.1c) with smallest $|S|$. For subsets $S_1, \ldots, S_n$ of S, define inductively:

(42.16) $$[S_1, \ldots, S_n] := \begin{cases} S & \text{if } n = 0,\\ \text{span}([S_1, \ldots, S_{n-1}] \cap S_n) & \text{if } n \geq 1.\end{cases}$$

By the minimality of $|S|$, we know that for each $s \in S$, $S \setminus \{s\}$ can be partitioned into k independent sets $I_1, \ldots, I_k$. We first show:

(42.17) for each $s \in S$ and $I_1, \ldots, I_k$ partitioning $S \setminus \{s\}$, there exist $j_1, \ldots, j_n \in \{1, \ldots, k\}$ with $s \notin [I_{j_1}, \ldots, I_{j_n}]$.

Indeed, choose $j_1, \ldots, j_n \in \{1, \ldots, k\}$ with the rank of $[I_{j_1}, \ldots, I_{j_n}]$ as small as possible. Define $A := [I_{j_1}, \ldots, I_{j_n}]$. By the minimality of the rank of A, we have $r(A \cap I_j) = r(A)$ for each $j = 1, \ldots, k$. Hence, by (42.6),

(42.18) $$|A| \leq k \cdot r(A) = \sum_{j=1}^{k} r(A \cap I_j) \leq \sum_{j=1}^{k} |A \cap I_j| = |A \setminus \{s\}|.$$

So $s \notin A$, proving (42.17).

Now choose s, $I_1, \ldots, I_k$, and $j_1, \ldots, j_n$ as in (42.17) with n as small as possible. For $t = 0, \ldots, n$, define

(42.19) $$B_t := [I_{j_1}, \ldots, I_{j_t}].$$

As we have a counterexample, we know that $s \in \text{span}(I_{j_n})$ (otherwise we can add s to I_{j_n}). Let C be the circuit in $I_{j_n} \cup \{s\}$. As $s \notin B_n = \text{span}(B_{n-1} \cap I_{j_n})$, we know that $C \setminus \{s\}$ is not contained in B_{n-1} (otherwise $C \setminus \{s\} \subseteq B_{n-1} \cap I_{j_n}$, and hence $s \in \text{span}(B_{n-1} \cap I_{j_n})$). So we can choose $z \in C \setminus \{s\}$ with $z \notin B_{n-1}$.

Define $I'_{j_n} := I_{j_n} - z + s$ and $I'_j := I_j$ for $j \neq j_n$. Then $I'_1, \ldots, I'_k$ are independent sets partitioning $S \setminus \{z\}$. Define, for $t = 0, \ldots, n$:

(42.20) $$B'_t := [I'_{j_1}, \ldots, I'_{j_t}].$$

By the minimality of n we know that $z \in B'_{n-1}$. Since $z \notin B_{n-1}$, we have $B'_{n-1} \not\subseteq B_{n-1}$. Choose the smallest $q \leq n-1$ with $B'_q \not\subseteq B_q$. Then $q \geq 1$ and $B'_{q-1} \subseteq B_{q-1}$. By the minimality of n we know that $s \in B_q$ (as $q < n$). So

(42.21) $$B'_q = \text{span}(B'_{q-1} \cap I'_{j_q}) \subseteq \text{span}((B_{q-1} \cap I_{j_q}) \cup \{s\}) \subseteq \text{span}(B_q) = B_q,$$

a contradiction.

42.2. Polyhedral applications

The matroid base packing and covering theorems imply (in fact, are equivalent to) the following polyhedral result:

Corollary 42.1e. *For any matroid, the independent set polytope, the base polytope, and the spanning set polytope have the integer decomposition property.*

Proof. Let $M = (S, \mathcal{I})$ be a matroid. Choose $k \in \mathbb{Z}_+$ and an integer vector $x \in k \cdot P_{\text{independent set}}(M)$. Replace each element s of S by x_s parallel elements, thus obtaining the matroid $N = (T, \mathcal{J})$ say. Now for each $U \subseteq T$, one has $k \cdot r_N(U) \geq |U|$, since if W denotes the set of elements s in S such that U intersects the parallel class of s, then

$$r_N(U) = r_M(W) \geq x(W)/k \geq |U|/k, \tag{42.22}$$

since x/k belongs to $P_{\text{independent set}}(M)$. So by the matroid base covering theorem (Corollary 42.1c), T can be partitioned into k independent sets of N. Hence x is the sum of k incidence vectors of independent sets of M.

To see that the base polytope has the integer decomposition property, let $x \in k \cdot P_{\text{base}}(M)$. By the above, x is the sum of the incidence vectors of k independent sets. As $x(S) = k \cdot r(S)$, each of these independent sets is a base.

One similarly derives from the matroid base packing theorem (Corollary 42.1d) that the spanning set polytope has the integer decomposition property. ∎

The matroid base packing and covering theorems imply generalizations to the capacitated case, by splitting elements into parallel elements. For the matroid base covering theorem this gives:

Theorem 42.2. *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $c : S \to \mathbb{Z}_+$. Then the minimum value of $\sum_{I \in \mathcal{I}} \lambda_I$, where $\lambda : \mathcal{I} \to \mathbb{Z}_+$ satisfies*

$$\sum_{I \in \mathcal{I}} \lambda_I \chi^I = c, \tag{42.23}$$

is equal to the maximum value of

$$\left\lceil \frac{c(U)}{r(U)} \right\rceil \tag{42.24}$$

taken over $U \subseteq S$ with $r(U) \geq 1$.

Proof. Directly from the matroid base covering theorem (Corollary 42.1c), by splitting each $s \in S$ into $c(s)$ parallel elements. ∎

In other words, the system defining the antiblocking polyhedron of the independent set polytope:

$$\text{(42.25)} \qquad \begin{array}{ll} x_s \geq 0 & \text{for } s \in S, \\ x(I) \leq 1 & \text{for } I \in \mathcal{I}, \end{array}$$

has the integer rounding property (the optimum integer solution to the dual of maximizing $c^{\mathsf{T}}x$ over (42.25) has value equal to the upper integer part of the value of the optimum (fractional) solution, for any integer objective function c).

Similarly, the matroid base packing theorem gives:

Theorem 42.3. *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $c : S \to \mathbb{Z}_+$. Let $\mathcal{B}$ be the collection of bases of M. Then the maximum value of $\sum_{B\in\mathcal{B}} \lambda_B$, where $\lambda : \mathcal{B} \to \mathbb{Z}_+$ satisfies*

$$\text{(42.26)} \qquad \sum_{B\in\mathcal{B}} \lambda_B \chi^B \leq c,$$

is equal to the minimum value of

$$\text{(42.27)} \qquad \left\lfloor \frac{c(S \setminus U)}{r(S) - r(U)} \right\rfloor$$

taken over $U \subseteq S$ with $r(S) - r(U) \geq 1$.

Proof. Directly from the matroid base packing theorem (Corollary 42.1d), by splitting each $s \in S$ into $c(s)$ parallel elements. ∎

In other words, the system defining the blocking polyhedron of the base polytope:

$$\text{(42.28)} \qquad \begin{array}{ll} x_s \geq 0 & \text{for } s \in S, \\ x(B) \geq 1 & \text{for } B \in \mathcal{B}, \end{array}$$

has the integer rounding property.

De Pina and Soares [2000] showed that, in Theorem 42.3, the number of bases B with $\lambda_B > 0$ can be restricted to at most $|S| + r$, where r is the rank of M. This strengthens a result of Cook, Fonlupt, and Schrijver [1986].

42.3. Matroid union algorithm

A polynomial-time algorithm for partitioning a matroid in as few independent sets as possible may be derived from the matroid intersection algorithm, with the construction given in the proof of Theorem 42.1. A direct algorithm was given by Edmonds [1968]. We give the algorithm described by Knuth [1973] and Greene and Magnanti [1975], which is similar to the algorithm described in Section 41.2 for cardinality matroid intersection.

Let $M_1 = (S, \mathcal{I}_1), \ldots, (S, \mathcal{I}_k)$ be matroids. Let $I_i \in \mathcal{I}_i$, for $i = 1, \ldots, k$, with $I_i \cap I_j = \emptyset$ if $i \neq j$. Let D be the union of the graphs $D_{M_i}(I_i)$ as defined in Section 39.9.

For each i, let F_i be the set of elements $s \notin I_i$ with $I_i \cup \{s\} \in \mathcal{I}_i$. Define $I := I_1 \cup \cdots \cup I_k$, $F := F_1 \cup \cdots \cup F_k$, and $\mathcal{I} := \mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$.

Theorem 42.4. *For any $s \in S \setminus I$ one has: $I \cup \{s\} \in \mathcal{I} \iff D$ has an $F - s$ path.*

Proof. To see necessity, suppose that D has no $F - s$ path. Let T be the set of elements of S that can reach s in D. So $s \in T$, $T \cap F = \emptyset$, and no arc of D enters T. Then $r_i(T) = |I_i \cap T|$ for each $i = 1, \ldots, k$. Otherwise, there exists a $t \in T \setminus I_i$ with $(I_i \cap T) \cup \{t\} \in \mathcal{I}_i$. Since $t \notin F$, $I_i \cup \{t\} \notin \mathcal{I}_i$. So there is a $u \in I_i \setminus T$ with $I_i - u + t \in \mathcal{I}_i$. But then (u, t) is an arc of D entering T, a contradiction.

So $r_i(T) = |I_i \cap T|$ for each i. Hence $r_1(T) + \cdots + r_k(T) = |I \cap T|$. As $s \in T \setminus I$, this implies $(I \cap T) \cup \{s\} \notin \mathcal{I}$, and so $I \cup \{s\} \notin \mathcal{I}$.

To see sufficiency, let $P = (s_0, s_1, \ldots, s_p)$ be a shortest $F - s$ path in D. We can assume by symmetry that $s_0 \in F_1$; so $s_0 \notin I_1$ and $I_1 \cup \{s_0\} \in \mathcal{I}_1$. Since P is a shortest path, for each $i = 1, \ldots, k$, the set N_i of edges (s_{j-1}, s_j) with $j = 1, \ldots, p$ and $s_{j-1} \in I_i$, forms a unique perfect matching in $D_{M_i}(I_i)$ on the set S_i covered by N_i. So by Theorem 39.13, $I_i \triangle S_i$ belongs to $\mathcal{I}_i$ for each i. Moreover, by Corollary 39.13a, $(I_1 \triangle S_1) \cup \{s_0\} \in \mathcal{I}_1$. So $I \cup \{s\} \in \mathcal{I}$. ∎

This implies that a maximum-size set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$ can be found in polynomial time (by greedily growing an independent set in $M_1 \vee \cdots \vee M_k$). Similarly, we can find with the greedy algorithm a maximum-weight set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$.

In particular, we can test if a given set is independent in $M_1 \vee \cdots \vee M_k$. Cunningham [1986] gave an $O((n^{3/2} + k)mQ + n^{1/2}km)$ algorithm to find a maximum-size set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$, where n is the maximum size of a set in $\mathcal{I}_1 \vee \cdots \vee \mathcal{I}_k$, m is the size of the underlying set, and Q is the time needed to test if a given set belongs to $\mathcal{I}_j$ for any given j.

These methods (including the reduction to matroid intersection) also imply:

Theorem 42.5. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle, we can find a maximum number of disjoint bases, and a minimum number of independent sets covering S, in polynomial time.*

Proof. See above. ∎

42.4. The capacitated case: fractional packing and covering of bases

The complexity of the capacitated and fractional cases of the above packing and covering problems can be studied with the help of the strong polynomial-time solvability of the *most violated inequality problem* for a matroid $M = (S, \mathcal{I})$, with rank function r:

(42.29) given: a vector $x \in \mathbb{Q}_+^S$;
find: a subset U of S minimizing $r(U) - x(U)$.

The strong polynomial-time solvability of this problem was shown in Corollary 40.4c, and is a result of Cunningham [1984].

If x belongs to $P_{\text{independent set}}(M)$, we can decompose x as a convex combination of incidence vectors of independent sets. This decomposition can be found in strongly polynomial time, by Corollary 40.4a.

We now consider the problem of finding a maximum fractional packing of bases subject to a given capacity function, and its dual, finding a minimum fractional covering by independent sets of a demand function.

With a method given by Picard and Queyranne [1982a] and Padberg and Wolsey [1984] one finds:

Theorem 42.6. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Q}_+^S$, we can find the minimum value of λ such that $y \in \lambda \cdot P_{\text{independent set}}(M)$ in strongly polynomial time.*

Proof. Let r be the rank function of M. We can assume that y does not belong to the independent set polytope. Let L be the line through 0 and y. We iteratively reset y as follows. By Corollary 40.4c, we can find a subset U of S minimizing $r(U) - y(U)$. Let y' be the vector on L with $y'(U) = r(U)$.

Now, for any $U' \subseteq S$, if y' violates $x(U') \leq r(U')$, then $r(U') < r(U)$, since the function $d(x) := (r(U) - x(U)) - (r(U') - x(U'))$ is nonpositive at y and positive at y', implying that it is positive at 0 (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{independent set}}(M)$. So after at most $r(S)$ iterations the process terminates, with a y on the boundary of $P_{\text{independent set}}(M)$. Comparing the final y with the original y gives the required λ. ∎

Theorem 42.6 implies an algorithm for capacitated fractional covering by independent sets:

Corollary 42.6a. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Q}_+^S$, we can find independent sets $I_1, \ldots, I_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

(42.30) $$y = \lambda_1 \chi^{I_1} + \cdots + \lambda_k \chi^{I_k}$$

with $\lambda_1 + \cdots + \lambda_k$ minimal, in strongly polynomial time.

Proof. Without loss of generality, $y \neq \mathbf{0}$. By Theorem 51.7, we can find the minimum value of λ such that y belongs to $\lambda \cdot P_{\text{independent set}}(M)$. By Corollary 40.4a, we can decompose $\frac{1}{\lambda} \cdot y$ as a convex combination of incidence vectors of independent sets. This gives the required decomposition of y. ∎

One similarly shows for the spanning set polytope:

Theorem 42.7. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Q}_+^S$, we can find the maximum value of λ such that $y \in \lambda \cdot P_{\text{spanning set}}(M)$, in strongly polynomial time.*

Proof. Let r be the rank function of M. By Corollary 40.2f, the spanning set polytope of M is determined by the constraints $\mathbf{0} \leq x \leq \mathbf{1}$ and

(42.31) $\quad r(U) - x(U) \geq r(S) - x(S)$ for $U \subseteq S$.

We can assume that $y \notin P_{\text{spanning set}}(M)$ and that the support of y is a spanning set. Let L be the line through 0 and y. We iteratively reset y as follows.

Find a $U \subseteq S$ minimizing $r(U) - y(U)$ (this can be done in strongly polynomial time, by Corollary 40.4c). If y does not belong to the spanning set polytope, we know that y violates the constraint $r(U) - x(U) \geq r(S) - x(S)$. Let y' be the vector on L satisfying $r(U) - y'(U) = r(S) - y'(S)$.

Now for any $U' \subseteq S$, if y' violates $r(U') - x(U') \geq r(S) - x(S)$, then $r(U') > r(U)$, since the function $d(x) := (r(U) - x(U)) - (r(U') - x(U'))$ is nonpositive at y and positive at y', implying that it is negative at 0 (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{spanning set}}(M)$. So after at most $r(S)$ iterations the process terminates, in which case y is on the boundary of $P_{\text{spanning set}}(M)$. Comparing the final y with the original y gives the required λ. ∎

In turn, this gives an algorithm for capacitated fractional base packing:

Corollary 42.7a. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Q}_+^S$, we can find bases $B_1, \ldots, B_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

(42.32) $\quad y \geq \lambda_1 \chi^{B_1} + \cdots + \lambda_k \chi^{B_k}$

with $\lambda_1 + \cdots + \lambda_k$ maximal, in strongly polynomial time.

Proof. By Theorem 42.7, we can find the maximum value of λ such that y belongs to $\lambda \cdot P_{\text{spanning set}}(M)$. If $\lambda = 0$, we take $k = 0$. If $\lambda > 0$, by Corollary 40.4b we can decompose $\frac{1}{\lambda} \cdot y$ as a convex combination of incidence vectors of spanning sets. This gives the required decomposition of y. ∎

42.5. The capacitated case: integer packing and covering of bases

It is not difficult to derive integer versions of the above algorithms, but they are not strongly polynomial-time, as we round numbers in it. In fact, an integer packing or covering cannot be found in strongly polynomial time, as it would imply a strongly polynomial-time algorithm for testing if an integer k is even (which algorithm does not exist[31]): Let M be the 2-uniform matroid on 3 elements and let $k \in \mathbb{Z}_+$. Then k is even if and only if M has $\frac{3}{2}k$ bases containing each element of M at most k times.

Polynomial-time algorithms follow directly from the fractional versions with the help of the matroid base packing and covering theorems.

Theorem 42.8. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Z}_+^S$, we can find independent sets $I_1, \ldots, I_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

(42.33) $$y = \lambda_1 \chi^{I_1} + \cdots + \lambda_t \chi^{I_t}$$

with $\lambda_1 + \cdots + \lambda_t$ minimal, in polynomial time.

Proof. First find $I_1, \ldots, I_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 42.6a. We can assume that $k \leq |S|$ (by Carathéodory's theorem, applying Gaussian elimination). Let

(42.34) $$y' := \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{I_i} = y - \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \chi^{I_i}.$$

So y' is integer.

Replace each $s \in S$ by $y'(s)$ parallel elements, making matroid $M' = (S', \mathcal{I}')$. By Theorem 42.5, we can find a minimum number of independent sets partitioning S', in polynomial time (as $y'(s) \leq |S|$ for each $s \in S$). This gives independent sets $I_{k+1}, \ldots, I_t$ of M.

Setting $\lambda_i := 1$ for $i = k+1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (42.33) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). By the matroid base covering theorem applied to M' (as (42.34) gives a fractional decomposition of S' into independent sets),

(42.35) $$t - k \leq \lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \rceil.$$

Therefore,

(42.36) $$\sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t - k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \leq \lceil \sum_{i=1}^{k} \lambda_i \rceil,$$

[31] For any strongly polynomial-time algorithm with one integer k as input, there is a number L and a rational function $q : \mathbb{Z} \to \mathbb{Q}$ such that if $k > L$, then the output equals $q(k)$. (This can be proved by induction on the number of steps of the algorithm, which is a fixed number as the input consists of only one number.) However, there do not exist a rational function q and number L such that for $k > L$, $q(k) = 0$ if k is even, and $q(k) = 1$ if k is odd.

proving that the decomposition is optimum (cf. Theorem 42.2). ∎

One similarly shows for packing bases:

Theorem 42.9. *Given a matroid $M = (S, \mathcal{I})$ by an independence testing oracle and given $y \in \mathbb{Z}_+^S$, we can find bases $B_1, \ldots, B_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

$$(42.37) \qquad y \geq \lambda_1 \chi^{B_1} + \cdots + \lambda_t \chi^{B_t}$$

with $\lambda_1 + \cdots + \lambda_t$ maximal, in polynomial time.

Proof. First find bases $B_1, \ldots, B_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 42.7a. Again we can assume that $k \leq |S|$. Let

$$(42.38) \qquad y' := \lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{B_i} \rceil.$$

Replace each $s \in S$ by $y'(s)$ parallel elements, making matroid M'. By Theorem 42.5, we can find a maximum number of disjoint bases in M' in polynomial time (as $y'(s) \leq |S|$ for each $s \in S$). This gives bases $B_{k+1}, \ldots, B_t$ in M.

Setting $\lambda_i := 1$ for $i = k+1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (42.37) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). Again, now by the matroid base packing theorem applied to M', using (42.38),

$$(42.39) \qquad t - k \geq \lfloor \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \rfloor.$$

Therefore,

$$(42.40) \qquad \sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t-k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \geq \lfloor \sum_{i=1}^{k} \lambda_i \rfloor,$$

proving that the decomposition is optimum (cf. Theorem 42.3). ∎

De Pina and Soares [2000] showed that, in this theorem we can make the additional condition that $t \leq |S| + r$, where r is the rank of M.

42.6. Further results and notes

42.6a. Induction of matroids

An application of matroid intersection and union is the following 'induction of a matroid through a directed graph', discovered by Perfect [1969b] (for bipartite graphs) and Brualdi [1971c]. In fact, it forms a generalization of the basic Theorem 42.1.

Let $D = (V, A)$ be a directed graph, let $U, W \subseteq V$, and let $M = (U, \mathcal{I})$ be a matroid. Let $\mathcal{J}$ be the collection of subsets Y of W such that there exists an $X \in \mathcal{I}$ with X linked to Y. (Set X is *linked to* Y if $|X| = |Y|$ and D has $|X|$ disjoint $X - Y$ paths.)

Then:

(42.41) $\quad N = (W, \mathcal{J})$ is a matroid.

To show that N is a matroid, we can assume that U and W are disjoint. (Otherwise, add a new vertex w' and new arc (w, w') for each $w \in W$.) Let L be the gammoid induced by $D, U, U \cup W$. Then $N = (M \vee L)/U$. Indeed, since U is independent in L and hence in $M \vee L$, a subset Y of W is independent in $(M \vee L)/U$ if and only if $Y \cup U$ is independent in $M \vee L$. This is easily seen to be equivalent to: $Y \in \mathcal{J}$. So N is a matroid.

The rank function r_N of N can be described by (for $Y \subseteq W$):

(42.42) $\quad r_N(Y) = \min\{r_M(X) + |Z| \mid X \subseteq U, Z \subseteq V, Z$ intersects each $U \setminus X - Y$ path$\}$.

This can be derived from the matroid union theorem, but also (and simpler) from the matroid intersection theorem, as follows. Let K be the gammoid induced by D^{-1}, Y, U, where D^{-1} arises from D by reversing the orientations of all arcs. Then $r_N(Y)$ is equal to the maximum size of a common independent set in M and K. So, by the matroid intersection theorem (Theorem 41.1),

(42.43) $$r_N(Y) = \min_{X \subseteq U} (r_M(X) + r_K(U \setminus X)),$$

which by Menger's theorem is equal to the right-hand side of (42.42).

Applying the matroid intersection theorem again gives the following result of Brualdi [1971e] (generalizing Brualdi [1970a]).

Let $D = (V, A)$ be a directed graph, let $U, W \subseteq V$, and let $M = (U, \mathcal{I})$ and $M' = (W, \mathcal{I}')$ be matroids. Then the maximum size of an independent set in M that is linked to an independent set in M' is equal to the minimum value of

(42.44) $\quad r_M(X) + |Z| + r_{M'}(Y)$,

where $X \subseteq U$, $Y \subseteq W$, and $Z \subseteq V$, such that Z intersects each $U \setminus X - W \setminus Y$ path. (This follows directly by considering the maximum size of a common independent set in M' and N as defined above.)

Related results are given by McDiarmid [1975b] and Woodall [1975]. These results are generalized in Schrijver [1979c]. For an algorithm, see Fujishige [1977b].

42.6b. List-colouring

Seymour [1998] showed the following matroid list-colouring theorem (cf. Section 20.9c):

Theorem 42.10. *Let $M = (S, \mathcal{I})$ be a matroid such that S can be partitioned into k independent sets, and let $m \in \mathbb{Z}_+$. For each $s \in S$, let $L_s \subseteq \{1, \ldots, m\}$ be a set of size k. Then S can be partitioned into independent sets $I_1, \ldots, I_m$ such that for each $j = 1, \ldots, m$: if $s \in I_j$, then $j \in L_s$.*

Proof. For each $j = 1, \ldots, m$, let $U_j := \{s \in S \mid j \in L_s\}$. We need to prove that for all j, there exists an independent set $I_j \subseteq U_j$ such that $S = I_1 \cup \cdots \cup I_n$.

Since S can be partitioned into k independent sets, we know that $|X| \leq k{\cdot}r_M(X)$ for each $X \subseteq S$. Hence, for each $T \subseteq S$,

$$\text{(42.45)} \qquad \sum_{j=1}^{m} r_M(U_j \cap T) \geq \sum_{j=1}^{m} \frac{1}{k}|U_j \cap T| = |T|,$$

since each $s \in T$ belongs to k of the U_j. So by the matroid union theorem (Corollary 42.1a), applied to the matroids $M|U_j$, the independent sets I_j as required exist. ∎

42.6c. Strongly base orderable matroids

In general it is not true that given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ such that S can be partitioned into k independent sets of M_1, and also into k independent sets of M_2, then S can be partitioned into k common independent sets of M_1 and M_2. This could yield a 'matroid union intersection theorem'. However, taking for M_1 is the cycle matroid of K_4 and for M_2 the matroid with independent sets all sets of pairwise intersecting edges of K_4 (which is a partition matroid), shows that the statement is false for $k = 2$.

But the assertion is true if both M_1 and M_2 belong to the class of so-called strongly base orderable matroids, introduced by Brualdi [1970b]. A matroid $M = (S, \mathcal{I})$ is called *strongly base orderable* if for each two bases B_1, B_2 of M there exists a bijection $\pi : B_1 \to B_2$ such that for each subset X of B_1 the set $\pi(X) \cup (B_1 \setminus X)$ is a base again.

One easily checks that for such π, the function $\pi|B_1 \cap B_2$ is the identity map. It is also straightforward to check that if M is strongly base orderable, then also the dual of M and any contraction of M is strongly base orderable, and hence also any restriction, and therefore any minor is strongly base orderable. Moreover, Brualdi [1970b] showed:

Theorem 42.11. *Any truncation of a strongly base orderable matroid is strongly base orderable again.*

Proof. Let $M = (S, \mathcal{I})$ be a strongly base orderable matroid, with rank function r, and let $k := r(S) - 1$. It suffices to show that the k-truncation of M is strongly base orderable. Let I and J be independent sets of size k, and restrict M to $I \cup J$. If $r(I \cup J) = k$, we are done, since then I and J are bases of the strongly base orderable matroid $M|I \cup J$. So suppose $r(I \cup J) = r(S) = k + 1$, and let $i \in I \setminus J$ and $j \in J \setminus I$ be such that $I \cup \{j\}$ and $J \cup \{i\}$ are bases of M. As M is strongly base orderable, there exists a bijection $\pi : I \cup \{j\} \to J \cup \{i\}$ with the prescribed exchange property. So $\pi(j) = j$ and $\pi(i) = i$. Define $\pi' : I \to J$ by $\pi'(s) := \pi(s)$ if $s \neq i$, and $\pi'(i) = j$. We show that this bijection is as required. To prove this, choose $X \subseteq I$. We must show that $\pi'(X) \cup (I \setminus X)$ is independent.

If $i \notin X$, then $\pi'(X) = \pi(X)$, hence $\pi'(X) \cup (I \setminus X)$ is independent, since

$$\text{(42.46)} \qquad \pi'(X) \cup (I \setminus X) = \pi(X) \cup (I \setminus X) \subseteq \pi(X) \cup ((I \cup \{j\}) \setminus X)$$

and the last set is independent.

If $i \in X$, then $\pi'(X) = \pi(X \setminus \{i\}) \cup \{j\}$, hence $\pi'(X) \cup (I \setminus X)$ is independent, since

(42.47) $$\pi'(X)\cup(I\setminus X)=\pi(X\setminus\{i\})\cup\{j\}\cup(I\setminus X)=\pi(X\setminus\{i\})\cup((I\cup\{j\})\setminus X)$$
$$\subseteq \pi(X\setminus\{i\})\cup((I\cup\{j\})\setminus(X\setminus\{i\}))$$

and the last set is independent. ∎

One also easily checks that strong base orderability is closed under making parallel extensions. (Given a matroid $M=(S,\mathcal{I})$ a *parallel extension* in $s\in S$ is obtained by extending S with some new element s', and $\mathcal{I}$ with $\{(I\setminus\{s\})\cup\{s'\}\mid s\in I\in\mathcal{I}\}$.)

Since transversal matroids are strongly base orderable, also gammoids are strongly base orderable (Brualdi [1971c]):

Theorem 42.12. *Each gammoid is strongly base orderable.*

Proof. Since strong base orderability is closed under taking contractions and since each gammoid is a contraction of a transversal matroid (by Corollary 39.5a), it suffices to show that any transversal matroid is strongly base orderable.

Let M be the transversal matroid induced by a family $\mathcal{X}=(X_1,\dots,X_m)$ of subsets of a set S. We may assume that $\mathcal{X}$ has a transversal (cf. (39.18)). Consider two transversals $T_1=\{x_1,\dots,x_m\}$ and $T_2=\{y_1,\dots,y_m\}$ of $\mathcal{X}$, where $x_i,y_i\in X_i$ for $i=1,\dots,m$.

Consider the bipartite graph on $\{1,\dots,m\}\cup S$ with edges all pairs $\{i,s\}$ with $i\in\{1,\dots,m\}$ and $s\in X_i$ (assuming without loss of generality that $\{1,\dots,m\}\cap S=\emptyset$). Then $M_1:=\{\{i,x_i\}\mid i=1,\dots,m\}$ and $M_2:=\{\{i,y_i\}\mid i=1,\dots,m\}$ are matchings in G. Define $\pi:T_1\to T_2$ as follows. If $s\in T_1\cap T_2$, define $\pi(s):=s$. If $s\in T_1\setminus T_2$, let $\pi(s)$ be the (other) end of the path in $M_1\cup M_2$ starting at s. This defines a bijection as required. ∎

Brualdi [1971c] showed more generally that strong base orderability is maintained under induction of matroids through a directed graph, as described in Section 42.6a. However, not every strongly base orderable matroid is a gammoid (cf. Oxley [1992] p. 411).

Davies and McDiarmid [1976] (cf. McDiarmid [1976]) showed the following.

Theorem 42.13. *Let $M_1=(S,\mathcal{I}_1)$ and $M_2=(S,\mathcal{I}_2)$ be strongly base orderable matroids, let $k\in\mathbb{Z}_+$, and suppose that S can be split into k independent sets of M_1, and also into k independent sets of M_2. Then S can be split into k common independent sets of M_1 and M_2.*

Proof. In order to prove this, let $\mathcal{X}=(X_1,\dots,X_k)$ and $\mathcal{Y}=(Y_1,\dots,Y_k)$ be partitions of S into independent sets of M_1 and M_2, respectively, with

(42.48) $$\sum_{i=1}^{k}|X_i\cap Y_i|$$

as large as possible. If this sum is equal to $|S|$ we are done, so suppose that this sum is less than $|S|$. Hence there are i and j with $X_i\cap Y_j\neq\emptyset$ and $i\neq j$. Extend X_i and X_j to bases C_i and C_j of M_1. Similarly, extend Y_i and Y_j to bases D_i and D_j of M_2. Since M_1 and M_2 are strongly base orderable, there exist bijections

$\pi_1 : C_i \to C_j$ and $\pi_2 : D_i \to D_j$ with the exchange property. So $p_1(s) = s$ for each $s \in C_i \cap C_j$ and $p_2(s) = s$ for each $s \in D_i \cap D_j$.

Let G be the bipartite graph with vertex set $C_i \cup C_j \cup D_i \cup D_j$, and edges the pairs $\{s, \pi_1(s)\}$ with s in $C_i \setminus C_j$ and the pairs $\{s, \pi_2(s)\}$ with s in $D_i \setminus D_j$. Split the vertex set into colour classes S and T, say. Define

(42.49) $$X'_i := S \cap (X_i \cup X_j),\ X'_j := T \cap (X_i \cup X_j),$$ $$Y'_i := S \cap (Y_i \cup Y_j),\ Y'_j := T \cap (Y_i \cup Y_j).$$

So $X'_i \cap Y'_j = \emptyset$ and $X'_j \cap Y'_i = \emptyset$. Moreover, X'_i and X'_j are independent in M_1, since, by the exchange property of π, $S \cap (C_i \cup C_j)$ and $T \cup (C_i \cup C_j)$ are independent in M_1. Similarly, Y'_i and Y'_j are independent in M_2.

So replacing the classes X_i and X_j of $\mathcal{X}$ by X'_i and X'_j, and the classes Y_i and Y_j of $\mathcal{Y}$ by Y'_i and Y'_j yields partitions as required. However, since $X'_i \cap Y'_j = \emptyset$ and $X'_j \cap Y'_i = \emptyset$, we have

(42.50) $$|X'_i \cap Y'_i| + |X'_j \cap Y'_j| > |X_i \cap Y_i| + |X_j \cap Y_j|,$$

contradicting the maximality of (42.48). ∎

The proof also shows that the required partition can be found in polynomial time, provided that there is a polynomial-time algorithm to find the exchange bijection π. (This is the case for transversal matroids induced by a given family of sets.)

By the matroid base covering theorem (Corollary 42.1c), Theorem 42.13 is equivalent to:

Corollary 42.13a. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be loopless, strongly base orderable matroids, with rank functions r_1 and r_2. Then the minimum number of common independent sets needed to cover S, is equal to*

(42.51) $$\max\{\lceil \frac{|U|}{r_i(U)} \rceil \mid \emptyset \neq U \subseteq S, i = 1, 2\}.$$

Proof. Directly from Theorem 42.13 with the matroid base covering theorem. ∎

Applying Corollary 42.13a to transversal matroids gives Corollary 23.9a. Similarly, it follows from Theorem 42.13 that:

Corollary 42.13b. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be strongly base orderable matroids, with rank functions r_1 and r_2, satisfying $r_1(S) = r_2(S)$. Then M_1 and M_2 have k disjoint common bases if and only if*

(42.52) $$|S \setminus (T \cup U)| \geq k(r_1(S) - r_1(T) - r_2(U))$$

for all $T, U \subseteq S$.

Proof. Indeed, from Theorem 42.13 we have that M_1 and M_2 have k disjoint common bases if and only if the matroids $M_1 \vee \cdots \vee M_1$ and $M_2 \vee \cdots \vee M_2$ (k-fold unions) have a common independent set of size $k \cdot r_1(S)$. By the matroid union and intersection theorems, this last is equivalent to the condition stated in the present corollary. ∎

By truncating M_1 and M_2 one has similar results if we replace 'common bases' by 'common independent sets of size t'. Application to transversal matroids yields Corollary 23.9d.

Another consequence of Theorem 42.13 is:

Corollary 42.13c. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be strongly base orderable matroids. Then M_1 and M_2 have k disjoint common spanning sets if and only if both M_1 and M_2 have k disjoint bases.*

Proof. This can be deduced as follows. Let N_i arise from the dual matroid of M_i by replacing each element s of S by $k-1$ parallel elements (for $i = 1, 2$). So N_1 and N_2 are strongly base orderable again, with an underlying ground set of size $(k-1)|S|$. Now M_1 and M_2 have k disjoint (common) spanning sets, if and only if N_1 and N_2 have k (common) independent sets covering the underlying set. This directly implies the present corollary. ∎

Applying Corollary 42.13c to transversal matroids gives Theorem 23.11.

Corollary 42.13d. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be strongly base orderable matroids, with rank functions r_1 and r_2, satisfying $r_1(S) = r_2(S)$. Then S can be covered by k common bases of M_1 and M_2 if and only if*

(42.53) $$k(r_1(T) + r_2(U) - r_1(S)) \geq |T \cap U|$$

for all $T, U \subseteq S$.

Proof. Condition (42.53) is equivalent to:

(42.54) $$(k-1)|S \setminus (T \cup U)| \geq k(r_1^*(S) - r_1^*(T) - r_2^*(U))$$

for all $T, U \subseteq S$. Let N_1 and N_2 be the matroids defined in the proof of Corollary 42.13c. By Corollary 42.13b, condition (42.54) implies that N_1 and N_2 contain k disjoint common bases. So M_1^* and M_2^* have k common bases covering each element at most $k-1$ times. Hence M_1 and M_2 have k common bases covering S. ∎

Applying Corollary 42.13d to transversal matroids gives Theorem 23.12.

42.6d. Blocking and antiblocking polyhedra

We next investigate the blocking and antiblocking polyhedra corresponding to intersections of independent set polytopes of two matroids. Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be loopless matroids, with rank functions r_1 and r_2 respectively, and independent set polytopes P_1 and P_2 respectively. So $P_1 \cap P_2$ is the convex hull of the incidence vectors of common independent sets. Hence its antiblocking polyhedron $A(P_1 \cap P_2)$ is determined by the linear inequalities

(42.55) $$\begin{aligned} x_s &\geq 0 \quad && (s \in S), \\ x(I) &\leq 1 && (I \in \mathcal{I}_1 \cap \mathcal{I}_2). \end{aligned}$$

Since $P_1 \cap P_2$ is determined by the linear inequalities (41.37), $A(P_1 \cap P_2)$ consists of all vectors $x \geq \mathbf{0}$ for which there exists a $y \geq x$ which is a convex combination of vectors

(42.56) $$\frac{1}{r_i(U)}\chi^U$$

where U is a nonempty subset of S and $i = 1, 2$. Then $A(P_1 \cap P_2)$ gives rise to the following linear programming duality equation, for $c : S \to \mathbb{R}_+$:

(42.57) $$\max\{c^{\mathsf{T}}x \mid x \in A(P_1 \cap P_2)\} = \max\{\frac{c(U)}{r_i(U)} \mid \emptyset \neq U \subseteq S; i = 1, 2\}$$
$$= \min\{\sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} y(I) \mid y \in \mathbb{R}_+^{\mathcal{I}_1\cap\mathcal{I}_2}, \sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} y(I)\chi^I \geq c\}.$$

For integer c, an integer optimum solution y need not exist (for instance, if $|S| = 3$, $r_i(U) := \min\{|U|, 2\}$, and $c = \mathbf{1}$). That is, system (42.55) need not be totally dual integral. In fact, it generally does not have the integer rounding property. That is, it is not true, for each pair of matroids, that the minimum in (42.57) with y restricted to be integer:

(42.58) $$\min\{\sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} y(I) \mid y \in \mathbb{Z}_+^{\mathcal{I}_1\cap\mathcal{I}_2}, \sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} y(I)\chi^I \geq c\},$$

is equal to the upper integer part of the common value of (42.57). For instance, take for M_1 the cycle matroid of K_4, and for M_2 the matroid with independent sets all sets of pairwise intersecting edges in K_4, and let $c = \mathbf{1}$; then the common value in (42.57) is 2, while (42.58) is equal to 3. However, Corollary 42.13a implies that if M_1 and M_2 are strongly base orderable matroids, then (42.58) is equal to the upper integer part of (42.57). That is, for strongly base orderable matroids, system (42.57) has the integer rounding property.

Similar results hold if we consider the blocker $B(Q_1 \cap Q_2)$ of the intersection of the spanning set polytopes Q_1 and Q_2 of M_1 and M_2. In particular, Corollary 42.13c implies that the system

(42.59) $$\begin{aligned} &x_s \geq 0 && (s \in S),\\ &x(U) \geq 1 && (U \text{ common spanning set of } M_1 \text{ and } M_2)\end{aligned}$$

has the integer rounding property, if M_1 and M_2 are strongly base orderable.

Moreover, Corollaries 42.13b and 42.13d imply that the systems

(42.60) $$\begin{aligned} &x_s \geq 0 && (s \in S),\\ &x(B) \geq 1 && (B \text{ common base of } M_1 \text{ and } M_2)\end{aligned}$$

and

(42.61) $$\begin{aligned} &x_s \geq 0 && (s \in S),\\ &x(B) \leq 1 && (B \text{ common base of } M_1 \text{ and } M_2)\end{aligned}$$

have the integer rounding property, if M_1 and M_2 are strongly base orderable. Here the results of Section 41.4b are used: to prove that (42.60) has the integer rounding property, let $w \in \mathbb{Z}_+^S$. Let Q be the polytope determined by (42.60), let $r(U)$ be the maximum size of a common independent set contained in U, and let $\mathcal{B}$ denote the collection of common bases. Then

(42.62) $$\lceil\min\{w^{\mathsf{T}}x \mid x \in Q\}\rceil$$
$$= \min\{\lceil\frac{w(U)}{r(S) - r(S \setminus U)}\rceil \mid U \subseteq S, r(S) > r(S \setminus U)\}$$
$$= \max\{\sum_{B\in\mathcal{B}} y_B \mid y \in \mathbb{Z}_+^{\mathcal{B}}, \sum_{B\in\mathcal{B}} y_B\chi^B \leq w\}.$$

The first equality holds as the vertices of Q are given by the vectors

(42.63) $$\frac{1}{r(S) - r(S \setminus U)} \chi^U,$$

since Q is the blocking polyhedron of the common base polytope (cf. Section 41.4b). The second equality follows from Corollary 42.13b, using the fact that strong base orderability is maintained under adding parallel elements.

Related results on integer decomposition of the intersection of the independent set polytopes of two strongly base orderable matroids can be found in McDiarmid [1983].

42.6e. Further notes

Krogdahl [1976] observed that the following, general problem is solvable in polynomial time, by reduction to matroid intersection: given matroids $(S, \mathcal{I}_1), \ldots, (S, \mathcal{I}_k)$, weight functions $w_1, \ldots, w_k \in \mathbb{R}^S$, and $l \leq k$, find the maximum value of $w_1(I_1) + \cdots + w_k(I_k)$, where $I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k$, with $I_1, \ldots, I_l$ disjoint and $I_{l+1}, \ldots, I_k$ disjoint, and with $I_1 \cup \ldots \cup I_l = I_{l+1} \cup \ldots \cup I_k$.

With matroid union, several new classes of matroids can be constructed. One of them is formed by the *bicircular matroids*, which are the union of the cycle matroid $M(G)$ of a graph $G = (V, E)$ and the matroid on E in which $F \subseteq E$ is independent if and only if $|F| \leq 1$. The independent sets of this matroid are the edge sets containing at most one circuit.

A randomized parallel algorithm for linear matroid union was given by Narayanan, Saran, and Vazirani [1992,1994]. For matroid base packing algorithms, see Knuth [1973] and Karger [1993,1998].

42.6f. Historical notes on matroid union

As the matroid base covering theorem can be derived by an elementary construction from Rado's theorem (proved by Rado [1942]), it is surprising that, for a long time, it had remained an open question, posed by Rado himself.

In fact, it was Horn [1955] who showed that a set X of vectors is the union of k linearly independent sets of vectors if and only if each finite subset Y of X has rank at least $|Y|/k$. He mentioned that this was conjectured by K.F. Roth and R. Rado, and he did not refer to matroids. Horn also acknowledged the help of Rado.

Surprisingly, the same theorem was also published by Rado [1962a] (in the same journal). The proof method (including notation) is the same as that of Horn, but no reference to Horn's paper is given. Rado wondered if the theorem can be generalized to matroids:

> It can be seen that some steps of the argument can be adapted to the more general situation of abstract independence functions but there does not appear to be an obvious way of making the whole argument apply to the more general case.

Rado [1962b] presented the vector theorem at the International Congress of Mathematicians in Stockholm in 1962, where he mentioned again that its proof has not yet been extended to 'abstract independence relations' (matroids). He wondered if the property in fact would *characterize* linear matroids.

Finally, two years later, at the Conference on General Algebra in Warsaw, 7–11 September 1964, Rado announced the base covering theorem. Simultaneously, there was the Seminar on Matroids at the National Bureau of Standards in Washington, D.C., 31 August–11 September 1964, where Edmonds [1965c] presented the base covering theorem.

In the paper based on his lecture in Warsaw, Rado [1966] did not give a proof of the matroid base cover theorem, but just said that the argument of Horn [1955] can be adapted so as to yield the more general version (as we did in Section 42.1b).

The matroid base covering theorem generalizes also the min-max relation of Nash-Williams [1964] for the minimum number of forests needed to cover the edges of a graph. (As each graphic matroid is linear, this follows also from the result of Horn [1955] described above.)

The basic unifying result (Theorem 42.1) on matroid union was given in Nash-Williams [1967], which has as special case the matroid union theorem given by Edmonds [1968]. In a footnote on page 20 of Pym and Perfect [1970], it is remarked that:

> Professor Nash-Williams has written to inform us that these results were suggested by earlier unpublished work of Professor J. Edmonds on the relation between independence structures and submodular functions.

It seems in fact much easier to prove the matroid union theorem in general, than just its special case for graphic matroids (for instance, the covering forests theorem). It also generalizes theorems of Higgins [1959] on disjoint transversals (Theorem 22.11), and of Tutte [1961a] and Nash-Williams [1961b] on disjoint spanning trees in a graph (Corollary 51.1a). (These papers mention no possible generalization to matroids.)

Welsh [1976] mentioned on these results:

> They illustrate perfectly the principle that mathematical generalization often lays bare the important bits of information about the problem at hand.

Chapter 43

Matroid matching

We saw two generalizations of Kőnig's matching theorem for bipartite graphs: the Tutte-Berge formula on matchings in arbitrary graphs and the matroid intersection theorem. This raises the demand for a common generalization of these last two theorems. A solution to the following *matroid matching problem*, posed by Lawler [1971b,1976b], could yield such a generalization: given an undirected graph $G = (S, E)$ and a matroid $M = (S, \mathcal{I})$, what is the maximum number of disjoint edges of G whose union is independent in M?

By taking M trivial, the matroid matching problem reduces to the matching problem, and by taking G regular of degree one, and M to be the disjoint sum of two matroids defined on the two colour classes of the bipartite graph G, we obtain the matroid intersection problem.

However, the general matroid matching problem has been shown to be NP-complete in the regular NP-framework, and unsolvable in polynomial time in an oracle framework.

On the other hand, Lovász [1980b] gave a strongly polynomial-time algorithm in case the matroid M is linear. Moreover, Lovász [1980a] gave a min-max relation, which was extended by Dress and Lovász [1987] to algebraic matroids.

No extension to the weighted case has been discovered, even not for the linear case: no polyhedral characterization or polynomial-time algorithm for finding a maximum-weight matroid matching has been found.

43.1. Infinite matroids

In this chapter, we need an extension of the notion of matroids to infinite matroids. An *infinite matroid* is defined as a pair $M = (S, \mathcal{I})$, where S is an infinite set and $\mathcal{I}$ is a nonempty collection of subsets of S satisfying:

(43.1)
- (i) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,
- (ii) if $I \subseteq S$ and each finite subset of I belongs to $\mathcal{I}$, then I belongs to $\mathcal{I}$;
- (iii) if I, J are finite sets in $\mathcal{I}$ and $|I| < |J|$, then $I \cup \{j\} \in \mathcal{I}$ for some $j \in J \setminus I$.

Standard matroid terminology transfers to infinite matroids. The sets in $\mathcal{I}$ are called *independent* and those subsets of S not in $\mathcal{I}$ *dependent.* An inclusionwise minimal dependent set is a *circuit*. By (43.1)(ii), each *circuit* of M is finite. We will restrict ourselves to infinite matroids of *finite rank*. That is, there is a finite upper bound on the size of the sets in $\mathcal{I}$.

Examples of infinite matroids are linear spaces, where $\mathcal{I}$ is the collection of linearly independent subsets, and field extensions L of a field K, where $\mathcal{I}$ is the collection of subsets of L that are algebraically independent over K. In fact, these are the only two classes of infinite matroids that we will consider.

We call a matroid $M = (S, \mathcal{I})$ with S finite also a *finite matroid*.

43.2. Matroid matchings

Let $(S, \mathcal{I})$ be a (finite or infinite) matroid, with rank function r and span function span. Let E be a finite collection of unordered pairs from S, such that each pair is an independent set of $(S, \mathcal{I})$. For $F \subseteq E$ define

(43.2) $$\mathrm{span}(F) := \mathrm{span}(\bigcup F)$$

(where $\bigcup F$ denotes the union of the pairs in F), and

(43.3) $$r(F) := r(\mathrm{span}(F)).$$

Then for $X, Y \subseteq E$ one has

(43.4) $$r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y),$$

since

(43.5) $$\begin{aligned} r(X) + r(Y) &= r(\mathrm{span}(X)) + r(\mathrm{span}(Y)) \\ &\geq r(\mathrm{span}(X) \cap \mathrm{span}(Y)) + r(\mathrm{span}(X) \cup \mathrm{span}(Y)) \\ &\geq r(\mathrm{span}(X \cap Y)) + r(\mathrm{span}(X \cup Y)) = r(X \cap Y) + r(X \cup Y). \end{aligned}$$

Call a subset M of E a *matroid matching*, or just a *matching*, if

(43.6) $$r(M) = 2|M|.$$

So M is a matroid matching if and only if M consists of disjoint pairs and the union of the pairs in M belongs to $\mathcal{I}$. Hence each subset of a matching is a matching again. The maximum size of a matching in E is denoted by $\nu(E)$, or just by ν. A matching of size $\nu(E)$ is called a *base* of E. (We should be aware of the difference between a matching in a graph and a matroid matching, and between a base of a matroid and a base of a collection of pairs in a matroid. Below we will see moreover the notion of a circuit in a set of pairs in a matroid. We will be careful to avoid confusion.[32])

Consider the function s defined on subsets F of E by

(43.7) $$s(F) := 2|F| - r(F).$$

[32] As we denote a matching by M, we denote a matroid, for the time being, just by $(S, \mathcal{I})$.

So a subset M of E is a matching if and only if $s(M) = 0$.

Then for all collections X and Y:

(43.8) (i) $s(X) \leq s(Y)$ if $X \subseteq Y$,
(ii) $s(X) + s(Y) \leq s(X \cap Y) + s(X \cup Y)$.

Here (i) follows from

(43.9) $r(Y) \leq r(X) + r(Y \setminus X) \leq r(X) + 2|Y| - 2|X|$.

(43.8)(ii) follows from (43.4).

(43.8) implies:

(43.10) each $F \subseteq E$ contains a unique inclusionwise minimal subset X with $s(X) = s(F)$.

For let F contain subsets X and Y with $s(X) = s(Y) = s(F)$. Then by (43.8)(i), $s(X \cap Y) \leq s(F)$ and $s(X \cup Y) = s(F)$, and by (43.8)(ii), $s(X \cap Y) \geq s(X) + s(Y) - s(X \cup Y) = s(F)$. So $s(X \cap Y) = s(F)$.

43.3. Circuits

A subset C of E is called a *circuit* if it is an inclusionwise minimal set satisfying $r(C) = 2|C| - 1$. By (43.10):

(43.11) each $F \subseteq E$ with $r(F) = 2|F| - 1$ contains a unique circuit.

It implies that for each matching M and each $e \in E$ with $r(M + e) = r(M) + 1$, there is a unique circuit contained in $M + e$. This circuit is denoted by $C(M, e)$, and is called a *fundamental circuit* (*of* M). (Here and below, $M + e := M \cup \{e\}$ and $M - e := M \setminus \{e\}$.)

Such circuits have a useful exchange property:

(43.12) for each $f \in C(M, e)$, $M + e - f$ is a matching again.

Indeed, if $M + e - f$ is not a matching, then $s(M + e - f) \geq 1$. In fact, $s(M + e - f) = 1$, since $s(M + e - f) \leq s(M + e) = 1$. So $M + e - f$ contains a circuit C. As $f \notin C$, we know $C \neq C(M, e)$, contradicting (43.11).

43.4. A special class of matroids

The min-max equality for matroid matching to be proved, holds for (finite or infinite) matroids $(S, \mathcal{I})$ satisfying the following condition:

(43.13) for each pair of circuits C_1, C_2 of $(S, \mathcal{I})$ with $C_1 \cap C_2 \neq \emptyset$ and $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$, the intersection of $\text{span}(C)$ taken over all circuits $C \subseteq C_1 \cup C_2$ has positive rank.

Examples of such matroids will be seen in Section 43.6.

In (43.13), 'circuits' are meant in the original meaning: as subsets of S. But the property transfers to subsets of E, as follows:

Lemma 43.1α. *Let $(S,\mathcal{I})$ be a matroid satisfying (43.13) and let E be a collection of pairs from S. Then for each pair of circuits $C_1, C_2 \subseteq E$ with $C_1 \cap C_2 \neq \emptyset$ and $s(C_1 \cup C_2) = 2$, the intersection of* span(C) *taken over all circuits $C \subseteq C_1 \cup C_2$ has positive rank.*

Proof. Let $F := C_1 \cup C_2$. By assumption, $s(F) = 2$. Each proper subcollection F' of F satisfies $s(F') \leq 1$, since if $e \in C_i$, then $s(F - e) \leq s(F) + s(C_i - e) - s(C_i) = 2 + 0 - 1 = 1$.

Let $C_1, \ldots, C_k$ be the circuits contained in F. We can assume that $k \geq 3$ (otherwise the lemma trivially holds, since $C_1 \cap C_2 \neq \emptyset$ by assumption).

Then

(43.14) $\quad C_i \cup C_j = F$ for all distinct $i, j = 1, \ldots, k$,

since for any $e \in F \setminus (C_i \cup C_j)$ we would have that $s(F - e) = 1$ and that $F - e$ contains two distinct circuits, which contradicts (43.11).

An equivalent way of stating (43.14) is:

(43.15) $\quad F \setminus C_1, \ldots, F \setminus C_k$ are pairwise disjoint.

Now first assume that there exist distinct $e, f \in F$ with $e \cap f \neq \emptyset$. Then $|e \cup f| = 3$, so $\{e, f\}$ is a circuit, and therefore by (43.15), each C_i intersects $\{e, f\}$ (as $k \geq 3$). So each span(C_i) contains $e \cap f$, and therefore the intersection of the span(C_i) is nonempty, as required.

So we can assume that the pairs in F are disjoint. Consider any i. Then $\bigcup C_i$ is a subset of S, containing a unique circuit C_i' (as subset of S). This follows from:

(43.16) $\quad r(\bigcup C_i) = \left|\bigcup C_i\right| - 1$

(as C_i is a circuit in E), because (43.16) implies that $\bigcup C_i$ contains an independent set of size $\left|\bigcup C_i\right| - 1$.

Then

(43.17) $\quad C_i' \neq C_j'$ if $i \neq j$.

Indeed, C_i' intersects each pair in C_i, since for each $e \in C_i$ the union of the $f \in C_i - e$ has rank $2|C_i - e|$, hence is independent. As the pairs in F are disjoint, this shows (43.17).

Moreover, if $i \neq j$ and $h \in \{1, \ldots, k\}$, then

(43.18) $\quad C_h' \subseteq C_i' \cup C_j'$.

Otherwise, choose $x \in C_i'$, $y \in C_j' \setminus C_i'$, and $z \in C_h' \setminus (C_i' \cup C_j')$. So $x, y, z \in$ span$((C_i' \cup C_j' \cup C_h') \setminus \{x, y, z\})$. Hence $r(C_i' \cup C_j' \cup C_h') \leq |C_i' \cup C_j' \cup C_h'| - 3$, and so

(43.19) $\quad r(F) \leq r(C'_i \cup C'_j \cup C'_h) + \left|\bigcup F\right| - |C'_i \cup C'_j \cup C'_h| \leq \left|\bigcup F\right| - 3,$

a contradiction, since $s(F) = 2$.

This proves (43.18), which implies that $C'_1 \cap C'_2 \neq \emptyset$ (since $C'_1 \subseteq C'_2 \cup C'_3$ and $C'_1 \not\subseteq C'_3$). Then by (43.13), the intersection of $\mathrm{span}(C'_i)$ over all i has positive rank. Hence the intersection of $\mathrm{span}(C_i)$ over all i has positive rank. ∎

For any collection E of pairs from S, let H_E be the hypergraph with vertex set E and edges all fundamental circuits. The following theorem will be used in deriving a general min-max relation.

Theorem 43.1. *Let $(S, \mathcal{I})$ be a matroid satisfying (43.13) and let E be a collection of pairs from S such that the intersection of $\mathrm{span}(B)$ over all bases B of E has rank 0. Then*

(43.20) $\quad |B \cap F| = \lfloor \frac{1}{2} r(F) \rfloor$

for each base B and each component F of H_E.

Proof. I. Call two fundamental circuits C, D *far* if there exist a base B and $e, g \in E$ with $r(B + e + g) = 2\nu + 2$ and with $C = C(B, e)$ and $D = C(B, g)$. We first show:

(43.21) $\quad$ far fundamental circuits are disjoint.

Suppose to the contrary that there exist a base B and $e, g \in E$ with $r(B + e + g) = 2\nu + 2$ and $C(B, e) \cap C(B, g) \neq \emptyset$. Let $D := C(B, e) \cup C(B, g)$. Then

(43.22) $\quad s(D) \geq s(C(B, e)) + s(C(B, g)) - s(C(B, e) \cap C(B, g)) = 2$

and

(43.23) $\quad s(D) \leq s(B + e + g) = 2.$

So $s(D) = 2$. If C is any circuit contained in $B + e + g$, then $C \subseteq D$, since otherwise $s(C \cap D) = 0$, and hence

(43.24) $\quad 2 = 0 + s(B + e + g) \geq s(C \cap D) + s(C \cup D) \geq s(C) + s(D) = 3,$

a contradiction.

By Lemma 43.1α, there is a nonloop p that is contained in $\mathrm{span}(C)$ for each circuit $C \subseteq D$. By assumption, there is a base B' with $p \notin \mathrm{span}(B')$. Choose B' with $|B' \cap (B+e+g)|$ maximal. Then $r(B'+p) = 2\nu+1 < r(B+e+g)$, and hence $f \not\subseteq \mathrm{span}(B'+p)$ for some $f \in B+e+g$. Then $p \notin \mathrm{span}(B'+f)$ (since $r(B' + f) \leq 2\nu + 1$), and therefore $p \notin \mathrm{span}(C(B', f))$. So $C(B', f)$ is not one of the circuits contained in $B+e+g$. Choose $h \in C(B', f) \setminus (B+e+g)$. Hence, resetting B' to $B'-h+f$ would give a larger intersection with $B+e+g$, a contradiction. This shows (43.21).

II. We next show the theorem assuming that H_E is connected. Suppose to the contrary that $r(E) \geq 2\nu(E) + 2$. Then far fundamental circuits exist, since for any base B, there exist $e, g \in E$ with $r(B+e+g) = 2\nu+2$, since $r(E) \geq r(B)+2$. Then (43.21) implies, as H_E is connected, that there exist fundamental circuits C, C', D with C and D far, $C \cap C' \neq \emptyset$, and C' and D not far.

Choose $e \in C \cap C'$ and $f \in D$. As C and D are far fundamental circuits, there is a base B with $r(B+e+f) = 2\nu+2$ and $C = C(B,e)$, $D = C(B,f)$. Also, as C' is a fundamental circuit, there is a base B' with $r(B'+e) = 2\nu+1$ and $C' = C(B',e)$. Choose such a B' with $|B' \cap (B+f)|$ maximal.

As $r(B+e+f) > r(B'+e)$, there exists a $g \in B+f$ with $r(B'+e+g) = 2\nu+2$. As C' and D are not far, $C(B',g) \neq D = C(B,f)$. So $C(B',g) \not\subseteq B+f$, and hence there exists an $h \in C(B',g) \setminus (B+f)$. Set $B'' := B'-h+g$. Then $r(B''+h+e) = r(B'+g+e) = 2\nu+2$, and hence $r(B''+e) = 2\nu+1$. As, by (43.21), $C(B',g)$ and $C(B',e)$ are disjoint, we know $h \notin C(B',e)$, so $C(B',e) \subseteq B''+e$, and hence $C(B'',e) = C(B',e) = C'$. As $|B'' \cap (B+f)| > |B' \cap (B+f)|$ this contradicts the maximality of $|B' \cap (B+f)|$.

III. We finally prove the theorem in general. Let F be a component of H_E. Suppose that there is a base B of E with $|B \cap F| < \lfloor \frac{1}{2} r(F) \rfloor$. Then

(43.25) there is a base B of E and a base M of F with $|M| > |B \cap F|$.

Otherwise, for each base B of E, $B \cap F$ is a base of F. Then H_F consists of one component (as each fundamental circuit of E contained in F is a fundamental circuit of F). Hence, by part II of this proof, $|B \cap F| = \nu(F) = \lfloor \frac{1}{2} r(F) \rfloor$, contradicting our assumption.

So (43.25) holds. Choose B and M as in (43.25) with $|M \cap B|$ maximal. Then

(43.26) $\text{span}(M) \subseteq \text{span}(B)$,

since otherwise there is an $e \in M$ with $e \not\subseteq \text{span}(B)$, and we can choose $f \in C(B,e) \setminus M$ and replace B by $B-f+e$, thereby increasing $|M \cap B|$, contradicting the maximality of $|M \cap B|$.

Moreover,

(43.27) for each $e \in F$ with $e \not\subseteq \text{span}(B)$, we have $C(M,e) = C(B,e)$.

Otherwise, choose $f \in C(B,e) \setminus (M+e)$ and $g \in C(M,e) \setminus (B+e)$. Replacing B and M by $B-f+e$ and $M-g+e$ respectively, increases $|M \cap B|$, a contradiction.

As $M \setminus B \neq \emptyset$, there is an $h \in M \setminus B$. Then there is a base B' of E with $h \not\subseteq \text{span}(B')$ (as by the condition in the theorem, there is no nonloop that is contained in the span of each base). We assume that we have chosen M, B, and B' with $|B \cap B'|$ maximal (under the primary condition that $|M \cap B|$ is maximum).

Since $h \not\subseteq \text{span}(B')$, we know by (43.26) that $\text{span}(B) \neq \text{span}(B')$. Hence there exists an $e \in B'$ with $e \not\subseteq \text{span}(B)$.

If $e \notin F$, then $C(B,e)$ is disjoint from F (as F is a component of H_E). Choose $f \in C(B,e) \setminus B'$. Then replacing B by $B-f+e$ maintains M, $B \cap F$, and $M \cap B$, but increases $|B \cap B'|$, contradicting our assumption.

So $e \in F$. By (43.27), $C(B,e) = C(M,e)$. Choose $f \in C(B,e) \setminus B'$. Then replacing M and B by $M-f+e$ and $B-f+e$ respectively, maintains $|M|$, $|B \cap F|$, and $|M \cap B|$, but increases $|B \cap B'|$, contradicting our assumption. ∎

43.5. A min-max formula for maximum-size matroid matching

We can now derive a min-max formula for the maximum size of a matching in matroids satisfying (43.13) in an hereditary way, due to Lovász [1980a]:

Theorem 43.2 (matroid matching theorem). *Let $M = (S, \mathcal{I})$ be a (finite or infinite) matroid (with rank function r) such that each contraction of M satisfies* (43.13). *Let E be a finite set of pairs from S. Then the maximum size $\nu(E)$ of a matching in E satisfies*

$$\nu(E) = \min(r(F) + \sum_{i=1}^{k} \lfloor \tfrac{1}{2}(r(F_i) - r(F)) \rfloor), \tag{43.28}$$

where $F, F_1, \ldots, F_k$ are flats such that $F \subseteq F_i$ for $i = 1, \ldots, k$, and such that each $e \in E$ is contained in some F_i.

Proof. We first show that $\leq$ holds in (43.28). Let B be a base of E, and partition B into $B_1, \ldots, B_k$ such that $\mathrm{span}(B_i) \subseteq F_i$ for $i = 1, \ldots, k$. Define $F_i' := \mathrm{span}(B_i)$.

By induction on l we show that for each $l = 0, \ldots, k$:

$$r(F \cup F_1' \cup \cdots \cup F_l') \leq r(F) + \sum_{i=1}^{l} (|B_i| + \lfloor \tfrac{1}{2}(r(F \cup F_i') - r(F)) \rfloor). \tag{43.29}$$

For $l = 0$ this is trivial. For $l \geq 1$ we have (by induction and submodularity):

$$\begin{aligned} r(F \cup F_1' \cup \cdots \cup F_l') &\leq r(F \cup F_1' \cup \cdots \cup F_{l-1}') + r(F \cup F_l') - r(F) \\ &\leq r(F \cup F_l') + \sum_{i=1}^{l-1} (|B_i| + \lfloor \tfrac{1}{2}(r(F \cup F_i') - r(F)) \rfloor) \\ &\leq r(F) + \sum_{i=1}^{l} (|B_i| + \lfloor \tfrac{1}{2}(r(F \cup F_i') - r(F)) \rfloor), \end{aligned} \tag{43.30}$$

since

$$r(F \cup F_l') \leq r(F) + |B_l| + \tfrac{1}{2}(r(F \cup F_l') - r(F)), \tag{43.31}$$

as $|B_l| = \frac{1}{2}r(F_l')$. This shows (43.29), which for $l = k$ implies that $\nu(E)$ is at most (43.28), since

$$
\begin{aligned}
(43.32) \qquad 2\nu(E) &\leq r(F \cup F_1' \cup \cdots \cup F_k') \\
&\leq r(F) + \sum_{i=1}^{k} \left(|B_i| + \left\lfloor \tfrac{1}{2}(r(F \cup F_i') - r(F)) \right\rfloor\right) \\
&= \nu(E) + r(F) + \sum_{i=1}^{l} \left\lfloor \tfrac{1}{2}(r(F \cup F_i) - r(F)) \right\rfloor.
\end{aligned}
$$

Equality is shown by induction on $r(M)$. First assume that there is a nonloop p that is contained in $\mathrm{span}(B)$ for each base B of E. Let M' be the matroid M/p obtained by contracting p. Let E' be the set of pairs $\{s,t\}$ in E such that $s,t \neq p$ and such that s and t are not parallel in M'. Let ν' be the maximum size of a base $B' \subseteq E'$ with respect to M'.

Then $\nu' < \nu(E)$. For suppose that $\nu' \geq \nu(E)$. Let B' be a base of E' with respect to M'. As $|B'| \geq \nu(E)$, B' is also a base of E with respect to M. As $r_{M'}(B') = 2|B'| = r_M(B')$, we have $p \notin \mathrm{span}_M(B)$. This contradicts our assumption.

So $\nu' < \nu(E)$. By induction, M' has flats $F', F_1', \ldots, F_{k'}'$ with $F' \subseteq F_i'$ for $i = 1, \ldots, k'$, such that each $e \in E'$ is contained in some F_i' and such that

$$
(43.33) \qquad \nu' = r_{M'}(F') + \sum_{i=1}^{k'} \left\lfloor \tfrac{1}{2}(r_{M'}(F_i') - r_{M'}(F')) \right\rfloor.
$$

Define $F := \mathrm{span}_M(F' + p)$ and $F_i := \mathrm{span}_M(F_i' + p)$ for $i = 1, \ldots, k'$. Moreover, for each $e \in E$ not occurring in E', introduce a new F_i with $F_i := \mathrm{span}_M(F + e)$. As $p \in F$, we have $r_M(F_i) \leq r_M(F) + 1$ for each of these F_i.

This gives $F, F_1, \ldots, F_k$ such that $F \subseteq F_i$ for $i = 1, \ldots, k$, such that each $e \in E$ is contained in some F_i and such that

$$
\begin{aligned}
(43.34) \qquad \nu(E) \geq \nu' + 1 &= r_{M'}(F') + 1 + \sum_{i=1}^{k'} \left\lfloor \tfrac{1}{2}(r_{M'}(F_i') - r_{M'}(F')) \right\rfloor \\
&= r(F) + \sum_{i=1}^{k} \left\lfloor \tfrac{1}{2}(r(F_i) - r(F)) \right\rfloor.
\end{aligned}
$$

So we can assume that there is no nonloop p contained in $\mathrm{span}(B)$ for all bases B of E. Let $E_1, \ldots, E_k$ be the components of H_E and let $F_i := \mathrm{span}(E_i)$ for $i = 1, \ldots, k$. Let B be a base of E. Then by (43.20),

$$
(43.35) \qquad \nu(E) = |B| = \sum_{i=1}^{k} |B \cap E_i| = \sum_{i=1}^{k} \left\lfloor \tfrac{1}{2} r(F_i) \right\rfloor.
$$

So taking $F := \emptyset$ gives (43.28). ∎

43.6. Applications of the matroid matching theorem

We now consider specific classes of matroids satisfying (43.13), such that we know that the min-max equality holds. First, the linear matroids (Lovász [1980b]):

Corollary 43.2a. *If E is a finite set of pairs from a linear space S, then* (43.28) *holds, where flats are linear subspaces of S.*

Proof. Let $\mathcal{I}$ be the collection of sets of linearly independent vectors in S. We must show that each contraction of the infinite matroid $M = (S, \mathcal{I})$ satisfies (43.13). It suffices to show that M satisfies (43.13), since each contraction of M is again coming from a linear space, up to loops and parallel elements.

Let C_1 and C_2 be intersecting circuits in M with $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$. As C_1 is a circuit, there is a nonzero vector p in $\text{span}(C_1 \setminus C_2) \cap \text{span}(C_1 \cap C_2)$, since $r(C_1 \setminus C_2) + r(C_1 \cap C_2) > r(C_1)$. Consider any circuit C contained in $C_1 \cup C_2$.

Suppose $p \notin \text{span}(C)$. As $p \in \text{span}(C_1 \setminus C_2) \cap \text{span}(C_1 \cap C_2)$, C misses an element $s \in C_1 \setminus C_2$ and an element $t \in C_1 \cap C_2$. Now $t \in \text{span}(C_2 - t)$ and $s \in \text{span}(C_1 - s)$. Hence $(C_1 \cup C_2) - s - t$ spans $C_1 \cup C_2$, and hence, as $C_1 \cup C_2$ has rank $|C_1 \cup C_2| - 2$, we have that $(C_1 \cup C_2) - s - t$ is independent. This contradicts the fact that C is contained in $(C_1 \cup C_2) - s - t$. ∎

Dress and Lovász [1987] proved that a similar result holds for algebraic dependence in field extensions (where $\text{tr}_K(E)$ denotes the transcendence degree of $\bigcup E$ over K):

Corollary 43.2b. *Let E be a finite set of pairs from a field extension L of a field K. Then the maximum number of disjoint pairs from E such that the union is algebraically independent over K is equal to the minimum value of*

$$\text{(43.36)} \qquad \text{tr}_K(F) + \sum_{i=1}^{k} \lfloor \tfrac{1}{2} \text{tr}_F(E_i) \rfloor,$$

where F ranges over all field extensions of K in L and where $E_1, \ldots, E_k$ ranges over all partitions of E.

Proof. Let $M = (L, \mathcal{I})$ be the infinite matroid with $\mathcal{I}$ consisting of all subsets of L that are algebraically independent over K.

Similarly as for the previous corollary, it suffices to show that for any two intersecting circuits C_1 and C_2 of M with $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$ there is an $\alpha \in L \setminus \text{span}(K)$ such that α belongs to $\text{span}(C)$ for each circuit C contained in $C_1 \cup C_2$.

Let $I := C_1 \setminus C_2$. Then

(43.37) $\qquad I$ is a circuit in M/C_2.

To see this, trivially I is dependent in M/C_2. Consider any circuit $C \subseteq C_1 \cup C_2$ intersecting I. We must show that $I \subseteq C$. Suppose that there is an $s \in I \setminus C$. As C intersects I, C misses at least one element of C_2, say t. So $C \subseteq (C_1 \cup C_2) - s - t$. Now $(C_1 \cup C_2) - s - t$ spans $C_1 \cup C_2$ (since $t \in \mathrm{span}(C-t)$ and $s \in \mathrm{span}(C_1 \cup C_2 - s)$). This implies that $(C_1 \cup C_2) - s - t$ is independent (as $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$), contradicting the fact that it contains a circuit. This proves (43.37).

Let $I = \{s_1, \ldots, s_n\}$. Since I is a circuit in M/C_2, there exists an irreducible polynomial p in $\mathrm{span}(C_2)[x_1, \ldots, x_n]$ with $p(s_1, \ldots, s_n) = 0$. We can choose p such that at least one coefficient of p equals 1. Note that p has at least one coefficient, α say, that is not in $\mathrm{span}(K)$, since I is independent over K. It therefore is enough to show that all coefficients of p belong to $\mathrm{span}(C)$ for each circuit C contained in $C_1 \cup C_2$, since then α belongs to each $\mathrm{span}(C)$.

Choose a circuit $C \neq C_2$ with $C \subseteq C_1 \cup C_2$. As I is a circuit in M/C_2, we have $C \setminus C_2 = I$. So I is a circuit in $M/(C \cap C_2)$. Hence there exists an irreducible polynomial q in $\mathrm{span}(C \cap C_2)[x_1, \ldots, x_n]$ with $q(s_1, \ldots, s_n) = 0$. As $\mathrm{span}(C \cap C_2)$ is algebraically closed in $\mathrm{span}(C_2)$, q is also irreducible in $\mathrm{span}(C_2)[x_1, \ldots, x_n]$[33]. Then p and q are also irreducible in $\mathrm{span}(C_2)(x_1, \ldots, x_{n-1})[x_n]$ (cf., for instance, Section IV:6 of Jacobson [1951]). Therefore, p is a multiple of q in $\mathrm{span}(C_2)(x_1, \ldots, x_{n-1})$; that is, there are nonzero $r, s \in \mathrm{span}(C_2)[x_1, \ldots, x_{n-1}]$ with $rp = sq$. Hence by the unique factorization theorem (cf., for instance, Section IV:6 of Jacobson [1951]), $p = \lambda q$ for some $\lambda \in \mathrm{span}(C_2)$. As some coefficient of p equals 1, $\lambda \in \mathrm{span}(C \cap C_2)$. Hence $p \in \mathrm{span}(C \cap C_2)[x_1, \ldots, x_n]$. ∎

(The property of algebraic matroids shown in this proof generalizes a property shown by Ingleton and Main [1975].)

We also formulate the special case of graphic matroids:

Corollary 43.2c. *Let $G = (V, E)$ be a graph and let $\mathcal{P}$ be a partition of E into pairs. Then the maximum size of a forest $F \subseteq E$ that is the union of classes of $\mathcal{P}$ is equal to the minimum value of*

$$(43.38) \qquad 2|V| - 2|\mathcal{Q}| + 2\sum_{i=1}^{k} \lfloor \tfrac{1}{2}\delta_{\mathcal{Q}}(E_i) \rfloor,$$

---

[33] This can be seen as follows. Let L be a field extension of field K, such that K is algebraically closed in L. Then if p is an irreducible polynomial in $K[x_1, \ldots, x_n]$, then p is irreducible also in $L[x_1, \ldots, x_n]$. For suppose to the contrary that $p = p_1 p_2$ for nonconstant polynomials p_1, p_2 in $L[x_1, \ldots, x_n]$. We can assume that p_1 has at least one coefficient in K. Hence, as p is irreducible in $K[x_1, \ldots, x_n]$, p_1 has at least one coefficient not in K. Choose a large enough natural number k such that substituting x_i by x^{k^i} for $i = 1, \ldots, n$, transforms p_1 to a polynomial $\tilde{p}_1$ in $L[x] \setminus K[x]$. Let $\tilde{p} \in K[x]$ be obtained similarly from p. Now the algebraic closure of K contains all roots of $\tilde{p}$, hence all roots of $\tilde{p}_1$, and hence all coefficients of $\tilde{p}_1$. As each element in $L \setminus K$ is transcendental over K, we have a contradiction.

where $\mathcal{Q}$ ranges over partitions of V into nonempty classes and where $E_1, \ldots, E_k$ ranges over partitions of E such that each E_i is a union of pairs in $\mathcal{P}$. In (43.38)*, $\delta_{\mathcal{Q}}(E_i)$ denotes the size of a largest forest in the graph obtained from (V, E_i) by contracting each class in $\mathcal{Q}$ to one vertex.*

Proof. We apply Theorem 43.2 to the cycle matroid M of the graph H obtained from the complete graph on V by adding a parallel edge for each edge in E. Then (43.13) is satisfied for each contraction of M.

Now for each flat F of M there is a partition $\mathcal{Q}$ of V such that F is the set of edges of H contained in a class of $\mathcal{Q}$. The rank $r(F)$ of F (in M) is equal to $|V| - |\mathcal{Q}|$. For any $E' \subseteq E$, the smallest flat F' containing $F \cup E'$ has rank $r(F') = \delta_{\mathcal{Q}}(E') + r(F)$. Hence the corollary follows from Theorem 43.2. ∎

This corollary implies the following result on 3-uniform hypergraphs. A *hypergraph* is a pair $H = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a family of subsets of V. The hypergraph is called *k-uniform* if $|U| = k$ for each $U \in \mathcal{E}$.

A subfamily $\mathcal{F}$ of $\mathcal{E}$ is called a *forest* if there do not exist distinct $v_1, \ldots, v_t \in V$ and distinct $U_1, \ldots, U_t \in \mathcal{F}$ such that $t \geq 2$ and $v_{i-1}, v_i \in U_i$ for $i = 1, \ldots, t$, setting $v_0 := v_t$.

Corollary 43.2c implies a min-max relation for the maximum size of a forest in a given 3-uniform hypergraph (Lovász [1980a]):

Corollary 43.2d. *Let $H = (V, \mathcal{E})$ be a 3-uniform hypergraph. Then the maximum size of a forest $\mathcal{F} \subseteq \mathcal{E}$ is equal to the minimum value of*

$$(43.39) \qquad |V| - |\mathcal{Q}| + \sum_{\mathcal{S} \in \Sigma} \lfloor \tfrac{1}{2}(\phi_{\mathcal{Q}}(\mathcal{S}) - 1) \rfloor,$$

where $\mathcal{Q}$ and Σ range over partitions of V and $\mathcal{E}$, respectively. Here $\phi_{\mathcal{Q}}(\mathcal{S})$ denotes the number of classes of $\mathcal{Q}$ intersected by $\bigcup \mathcal{S}$.

Proof. For each $U \in \mathcal{E}$, choose two different pairs $e_U, f_U \subseteq U$, and let $G = (V, E)$ be the graph with edges all e_U and f_U. Let $\mathcal{P}$ be the partition of E into the pairs e_U, f_U. Then the maximum size of a forest $\mathcal{F} \subseteq \mathcal{E}$ is equal to half of the maximum size of a forest in E that is the union of pairs in $\mathcal{P}$. So to see that Corollary 43.2c implies the present corollary, it suffices to show that minimum (43.39) is equal to half of minimum (43.38).

First, let $\mathcal{Q}$ and Σ attain minimum (43.39). The partition Σ of $\mathcal{E}$ induces a partition of E into classes $\{e_U, f_U \mid U \in \mathcal{S}\}$ for $\mathcal{S} \in \Sigma$. One easily checks that for each $\mathcal{S} \in \Sigma$:

$$(43.40) \qquad \delta_{\mathcal{Q}}(\{e_U, f_U \mid U \in \mathcal{S}\}) \leq \phi_{\mathcal{Q}}(\mathcal{S}) - 1,$$

which implies that the minimum (43.39) is not less than half of minimum (43.38).

Second, to see the reverse inequality, let $\mathcal{Q}$, $E_1, \ldots, E_k$ attain minimum (43.38). Consider any $i = 1, \ldots, k$. Let $\mathcal{Q}'$ be the set of those classes in $\mathcal{Q}$

intersected by E_i and let t be the number of components of the hypergraph $(V, \mathcal{Q}' \cup E_i)$. Then $\delta_{\mathcal{Q}}(E_i) = |\mathcal{Q}'| - t$. The components partition E_i into $E_{i,1}, \ldots, E_{i,t}$. Then

$$(43.41) \qquad \delta_{\mathcal{Q}}(E_i) = |\mathcal{Q}'| - t = \sum_{j=1}^{t} (\phi_{\mathcal{Q}}(E_{i,j}) - 1).$$

So letting Σ to be the partition of $\mathcal{E}$ into classes $\mathcal{S}_{i,j} := \{U \mid e_U, f_U \in E_{i,j}\}$ (for all i, j), we have that minimum (43.38) is not less than twice minimum (43.39). ∎

(Szigeti [1998a] gave a direct proof of this theorem for the case where the hypergraph consists of all triangles of a given graph.)

Other applications of matroid matching are a derivation of Mader's theorem on maximum packings of T-paths (cf. Chapter 73), to rigidity (see Lovász [1980a]), and to matching forests (an easy application, see Section 59.6b).

43.7. A Gallai theorem for matroid matching and covering

We prove a Gallai-type theorem that relates the maximum size of a matroid matching to the minimum number of pairs spanning the matroid.

Let E be a collection of pairs of elements from a matroid $(S, \mathcal{I})$ such that each pair is an independent set and such that $\text{span}(E) = S$. Call $F \subseteq E$ a *matroid cover* if $\text{span}(F) = S$. Let $\rho(E)$ be the minimum size of a matroid cover. The following relation between $\nu(E)$ and $\rho(E)$ was observed by Lovász and extends Gallai's theorem (Theorem 19.1):

Theorem 43.3. *Let $(S, \mathcal{I})$ be a matroid, with rank function r, and let E be a collection of pairs from S spanning S. Then $\nu(E) + \rho(E) = r(S)$.*

Proof. To see $\leq$, let M be matching of size $\nu(E)$. Then by adding at most $r(S) - r(M)$ pairs from E to M we obtain a matroid cover F. So $\rho(E) \leq |F| \leq |M| + (r(S) - r(M)) = \nu(E) + r(S) - 2\nu(E) = r(S) - \nu(E)$.

To see $\geq$, let F be a matroid cover of size $\rho(E)$. Let $M := F$. As long as M contains an element e with $r(M - e) \geq r(M) - 1$, delete e from M. We end up with a matching M. For suppose not. Let M' be a maximum-size matching contained in M, and choose $e \in M \setminus M'$. Then $r(M - e) \leq r(M) - 2$ (otherwise we would delete e from M). Hence:

$$(43.42) \qquad r(M' + e) \geq r(M') + r(M) - r(M - e) \geq r(M') + 2 = 2|M'| + 2.$$

So $M' + e$ is a matching, contradicting the maximality of M'.

So M is a matching. Each time we have deleted an edge from M, its rank drops by at most 1. Hence $r(M) \geq r(S) - (|F| - |M|)$. Therefore $\nu(E) \geq |M| = r(M) - |M| \geq r(S) - |F| = r(S) - \rho(E)$. ∎

This theorem implies that formula (43.28) for the maximum size of a matching yields a formula for the minimum number of lines spanning all space.

43.8. Linear matroid matching algorithm

Jensen and Korte [1982] and Lovász [1981] showed that no polynomial-time algorithm exists for the matroid matching problem in general (see Section 43.9). On the other hand, Lovász [1981] gave a strongly polynomial-time algorithm for the matroid matching problem for linear matroids (an explicit representation over a field is required). This extends, e.g., Edmonds' polynomial-time algorithm finding a maximum matching in an undirected graph (cf. Section 24.2). It does not extend Edmonds' algorithm for a maximum-size common independent set in two matroids, as this algorithm also works for nonlinear matroids.

Theorem 43.4. *Given a set E of pairs of vectors in a linear space L, a maximum-size matching can be found in strongly polynomial time.*

Proof. The algorithm is a 'brute-force' polynomial-time algorithm, based on collecting many matchings and utilizing standard linear-algebraic operations, which can be performed in strongly polynomial time. Since we deal with subsets of a vector space, we can use $X + Y := \{x + y \mid x \in X, y \in Y\}$. For each $X \subseteq L$, $\mathrm{span}(X)$ is a subspace of L.

Throughout this proof, $\mathcal{B}$ will be a collection of matchings, all of the same size ν (say). Define:

(43.43) $K_{\mathcal{B}} := \bigcap\{\mathrm{span}(B) \mid B \in \mathcal{B}\}$ and $H_{\mathcal{B}} :=$ the hypergraph with vertex set E and edges all fundamental circuits of all $B \in \mathcal{B}$.

We say that we *improve* $\mathcal{B}$ if we find, in strongly polynomial time, either a matching B of size $\nu + 1$, or of size ν such that $K_{\mathcal{B}} \not\subseteq \mathrm{span}(B)$, or of size ν such that $H_{\mathcal{B}\cup\{B\}}$ has fewer components than $H_{\mathcal{B}}$. So replacing $\mathcal{B}$ by $\{B\}$ if $|B| = \nu + 1$, and by $\mathcal{B} \cup \{B\}$ if $|B| = \nu$, we can have at most $2|E|$ improvements.

I. We first show (where a component is called *nontrivial* if it has more than one element):

(43.44) We can improve $\mathcal{B}$ if we have a union F of nontrivial components of $H_{\mathcal{B}}$, a matching $M \subseteq F$, and a $B \in \mathcal{B}$ such that $r(M \cup A) > |B \cap F| + |M|$, where $A := \mathrm{span}(B \cap F) \cap K_{\mathcal{B}}$.

Here and below, $r(X \cup Y) := r(\bigcup X \cup Y)$ for $X \subseteq E$ and $Y \subseteq S$.

To see (43.44), apply the first applicable case of the following five cases, and then iterate. If Case 1 applies, we improve $\mathcal{B}$. In any of the other cases, we reset B or M or both, add the reset B to $\mathcal{B}$, and iterate with the reset B and M. The input condition given in (43.44) is maintained, as will be shown after describing the five cases.

Case 1: There is a $B' \in \mathcal{B}$ and an $e \in E$ such that $B' + e$ is a matching, or such that $C(B', e)$ intersects both F and $E \setminus F$, or such that $K_{\mathcal{B}} \not\subseteq \mathrm{span}(B' - f + e)$ for some $f \in C(B', e)$. Output $B' + e$, B', or $B' - f + e$ (thus we improve $\mathcal{B}$).

Note that if Case 1 does not apply, then

(43.45) $\quad f \not\subseteq K_{\mathcal{B}}$ for each $f \in F$.

Indeed, as f is in a nontrivial component of $H_{\mathcal{B}}$, f is contained in some fundamental circuit $C(B', e)$ for some $B' \in \mathcal{B}$. As Case 1 does not apply, we know $K_{\mathcal{B}} \subseteq \mathrm{span}(B' - f + e)$. Hence, if $f \subseteq K_{\mathcal{B}}$, then $f \subseteq \mathrm{span}(B' - f + e)$, hence $2\nu + 1 = r(B' + e) = r(B' - f + e) = 2\nu$, a contradiction.

Case 2: There is an $e \in F$ such that $M + e$ is a matching and $r((M \cup A) + e) \geq r(M \cup A) + 1$. Reset $M := M + e$.

Case 3: $\mathrm{span}(M) \not\subseteq \mathrm{span}(B)$. Choose $e \in M$ with $e \not\subseteq \mathrm{span}(B)$, choose $f \in C(B, e) \setminus M$, and reset $B := B - f + e$.

Case 4: There is an $e \in F$ such that $e \not\subseteq \mathrm{span}(B)$ and $C(B, e) \neq C(M, e)$. (Note: $e \not\subseteq \mathrm{span}(M) + A$, since $\mathrm{span}(M) + A \subseteq \mathrm{span}(B)$ (as Case 3 does not apply). So, as Case 2 does not apply, $M + e$ is not a matching. Hence $C(M, E)$ is defined.)

Choose $f \in C(B, e) \setminus (M + e)$ and $g \in C(M, e) \setminus (B + e)$, and reset $B := B - f + e$ and $M := M - g + e$.

Case 5. Choose $B' \in \mathcal{B}$ with $\mathrm{span}(M \triangle (B \cap F)) \not\subseteq \mathrm{span}(B')$ and with $|B \cap B'|$ maximal. (This is possible, since $M \neq B \cap F$, since $r((B \cap F) \cup A) = r(B \cap F) = 2|B \cap F|$ and $r(M \cup A) > |B \cap F| + |M|$ by assumption. As $M \triangle (B \cap F) \subseteq F$, such a B' exists, by (43.45).)

Choose $e \in B'$ with $e \not\subseteq \mathrm{span}(B)$. (This is possible since $\mathrm{span}(M \triangle (B \cap F)) \subseteq \mathrm{span}(B)$, so $\mathrm{span}(B) \neq \mathrm{span}(B')$.)

Choose $f \in C(B, e) \setminus B'$. If $e \notin F$, reset $B := B - f + e$. If $e \in F$, reset $B := B - f + e$ and $M := M - f + e$. (Note that if $e \in F$, then $C(B, e) = C(M, e)$ as Case 4 does not apply.)

Running time. The number of iterations is polynomially bounded, since in each iteration (except the last, where Case 1 applies), the vector $(|M|, |M \cap B|, |B \cap B'|)$ increases lexicographically. Here it is important to note that Case 5 does not modify the set $M \triangle (B \cap F)$, and increases the intersection of this set with B'.

We finally prove that the resettings in Cases 2-5 indeed maintain the condition given in (43.44). Let $\widetilde{B}$, $\widetilde{M}$, and $\widetilde{A}$ denote B, M, and A after resetting (taking $\widetilde{B}$ or $\widetilde{M}$ equal to B or M if they are not reset). We must show

(43.46) $\quad r(\widetilde{M} \cup \widetilde{A}) > |\widetilde{B} \cap F| + |\widetilde{M}|.$

Note that, as Case 1 does not apply, $|\widetilde{B} \cap F| = |B \cap F|$.

We first show:

(43.47) $\quad A \subseteq \widetilde{A}$.

This is equivalent to (since $K_{\mathcal{B}}$ does not change, as Case 1 does not apply):

(43.48) $\quad A \subseteq \text{span}(\widetilde{B} \cap F)$.

This is trivial if $\widetilde{B} \cap F = B \cap F$. So we can assume that $\widetilde{B} \cap F \neq B \cap F$. Hence $\widetilde{B} = B - f + e$ for some $e, f \in F$. Then (43.48) follows from

(43.49)
$$\begin{aligned} r((\widetilde{B} \cap F) \cup A) &\leq r((\widetilde{B} \cap F) \cup A + f) - 1 = r((B \cap F) \cup A + e) - 1 \\ &= r((B \cap F) + e) - 1 \leq r(B \cap F) = 2|B \cap F| = 2|\widetilde{B} \cap F| \\ &= r(\widetilde{B} \cap F). \end{aligned}$$

Here the first inequality holds as $f \not\subseteq \text{span}(\widetilde{B} \cap F) + A$, since $f \not\subseteq \text{span}(\widetilde{B})$ and $\text{span}(\widetilde{B} \cap F) + A \subseteq \text{span}(\widetilde{B})$. (We use that $A \subseteq K_{\mathcal{B}} \subseteq \text{span}(\widetilde{B})$, as Case 1 does not apply.) The last inequality holds as $(B \cap F) + e$ is not a matching, since it contains $C(B, e)$ (as Case 1 does not apply). This shows (43.48), and hence (43.47).

We finally show (43.46). In Case 2, we have $\widetilde{B} = B$, $\widetilde{M} = M + e$, and $\widetilde{A} = A$, and hence

(43.50)
$$\begin{aligned} r(\widetilde{M} \cup \widetilde{A}) &= r((M \cup A) + e) \geq r(M \cup A) + 1 > |B \cap F| + |M| + 1 \\ &= |\widetilde{B} \cap F| + |\widetilde{M}|, \end{aligned}$$

as required.

In Case 3, (43.47) implies (as $\widetilde{M} = M$) that $r(\widetilde{M} \cup \widetilde{A}) \geq r(M \cup A) > |B \cap F| + |M| = |\widetilde{B} \cap F| + |M|$.

In Cases 4 and 5 we have $\widetilde{M} = M - g + e$ (possible $g = f$). Then

(43.51)
$$\begin{aligned} r(\widetilde{M} \cup \widetilde{A}) &\geq r(\widetilde{M} \cup A) \geq r((\widetilde{M} \cup A) + g) - 1 = r((M \cup A) + e) - 1 \\ &\geq r(M \cup A) > |B \cap F| + |M| = |\widetilde{B} \cap F| + |\widetilde{M}|. \end{aligned}$$

The first inequality follows from (43.47). Next, $e \not\subseteq \text{span}(M \cup A)$ (as $e \not\subseteq \text{span}(B)$ and as $A \subseteq \text{span}(B)$ and $\text{span}(M) \subseteq \text{span}(B)$, since Case 3 does not apply). This gives the third inequality. To see the second inequality, suppose it does not hold. Then $\widetilde{M} + g$ is a matching, hence $M + e$ is a matching. Therefore, as Case 2 does not apply, $r(M \cup A + e) = r(M \cup A)$, contradicting the fact that $e \not\subseteq \text{span}(M \cup A)$.

II. Secondly,

(43.52) $\quad$ we can improve $\mathcal{B}$ if $K_{\mathcal{B}} = \{\mathbf{0}\}$, $H_{\mathcal{B}}$ is connected, and $\nu < \lfloor \frac{1}{2} r(E) \rfloor$.

(In this case, $\mathcal{B}$ can only be improved by finding a matching larger than B.)

The algorithm follows the framework of parts I and II in the proof of Theorem 43.1. Again, the algorithm iteratively applies the first applicable case. Call two circuits C_1, C_2 *far* if there exist $B \in \mathcal{B}$ and $e, g \in E$ with $r(B+e+g) = 2\nu+2$ and $C_1 = C(B,e)$ and $C_2 = C(B,g)$.

Case 1: There exists a $B \in \mathcal{B}$ and $e \in E$ such that $B+e$ is a matching of size $\nu+1$. Output $B+e$.

Case 2: There exist far circuits C_1 and C_2 with $C_1 \cap C_2 \neq \emptyset$. We will create a matching of size $\nu+1$.

Let $C_1 = C(B,e)$ and $C_2 = C(B,g)$ for some $B \in \mathcal{B}$ with $e, g \in E$ and $r(B+e+g) = 2\nu+2$. Define $D := C_1 \cup C_2$. As is shown in the proof of Corollary 43.2a, there is a $p \neq \mathbf{0}$ contained in $\text{span}(C)$ for each circuit $C \subseteq D$. Since $K_{\mathcal{B}} = \{\mathbf{0}\}$, there is a $B' \in \mathcal{B}$ with $p \notin \text{span}(B')$. Now $r(B'+p) = 2\nu+1 < r(B+e+g)$, and hence $f \not\subseteq \text{span}(B'+p)$ for some $f \in B+e+g$. Then (as $B'+f$ is not a matching, since Case 1 does not apply) $p \notin \text{span}(B'+f)$, and therefore $p \notin \text{span}(C(B',f))$. So $C(B',f)$ is not contained in $B+e+g$. Choose $h \in C(B',f) \setminus (B+e+g)$. Hence, resetting B' to $B'-h+f$ increases $|B' \cap (B+e+g)|$. So iterating this, we finally obtain a matching larger than ν.

Case 3. We show that we can create a matching of size $\nu+1$, or make that Case 1 or 2 applies.

Far circuits exist, since for any base B, there exist $e, g \in E$ with $r(B+e+g) = 2\nu+2$, since $r(E) \geq r(B)+2$. Choose far circuits C, D that are closest[34] together in the hypergraph $H_{\mathcal{B}}$. Assuming that Case 2 does not apply, we know $C \cap D = \emptyset$. Hence there is an intermediate set C' on a shortest path from C to D. Let $C = C(B,e)$, $D = C(B,g)$, and $C' = C(B',f)$ for $B, B' \in \mathcal{B}$ and $e, f, g \in E$ with $r(B+e+g) = 2\nu+2$. We choose B' such that $|B' \cap (B+e+g)|$ is maximal. Choose $h \in B+e+g$ with $h \not\subseteq \text{span}(B'+f)$.

$C(B',h)$ and $C(B',f)$ are disjoint, since otherwise we can apply Case 2. Moreover,

(43.53) $\qquad C(B',h) \not\subseteq B+e+g,$

Otherwise, $C(B',h) = C(B,e)$ or $C(B',h) = C(B,g)$. Hence C' and C or D are far, contradicting the minimality of the distance of C and D.

Hence we have (43.53). Choose $i \in C(B',h) \setminus (B+e+g)$ and add $B'' := B'-i+h$ to $\mathcal{B}$. Iterate Case 3 with B' replaced by B'' (note that $C' = C(B'',f)$). As $|B'' \cap B| > |B' \cap B|$, the number of iterations of Case 3 is at most ν.

III. Combination of the previous two algorithms implies:

(43.54) $\qquad$ we can improve $\mathcal{B}$ if $K_{\mathcal{B}} = \{\mathbf{0}\}$ and $\nu < \nu(E)$.

As $\nu < \nu(E)$, there is a component F of $H_{\mathcal{B}}$ with $|B \cap F| < \nu(F) \leq \lfloor \frac{1}{2} r(F) \rfloor$ for at least one $B \in \mathcal{B}$. If there exist $B, B' \in \mathcal{B}$ with $|B \cap F| < |B' \cap F|$, set

[34] Here the distance of fundamental circuits C, D is the minimum length of a path connecting C and D. A *path* connecting C and D is a sequence $C = C_0, \ldots, C_k = D$ of fundamental circuits such that $C_{i-1} \cap C_i \neq \emptyset$ for $i = 1, \ldots, k$. Its length is k.

$M := B' \cap F$. Otherwise (that is, if $|B \cap F| = |B' \cap F|$ for all $B' \in \mathcal{B}$), apply (43.52) to $\mathcal{B}' := \{B \cap F \mid B \in \mathcal{B}\}$ and $B \cap F$ for any $B \in \mathcal{B}$, to obtain a matching $M \subseteq F$ with $|M| = |B \cap F| + 1$.

Now applying (43.44) to $\mathcal{B}$, F, B, and M improves $\mathcal{B}$. (Since $A \subseteq K_{\mathcal{B}}$, we have $A = \{\mathbf{0}\}$, and hence $r(M \cup A) = r(M) = 2|M| > |M| + |B \cap F|$.)

IV. Finally:

(43.55) We can improve $\mathcal{B}$ if $\mathcal{B} \neq \emptyset$ and $\nu < \nu(E)$.

Define F to be the union of all fundamental circuits of the $B \in \mathcal{B}$. This implies

(43.56) $\operatorname{span}(E \setminus F) \subseteq K_{\mathcal{B}}$.

If there exist $B, B' \in \mathcal{B}$ with $|B \cap F| < |B' \cap F|$, then applying (43.44) to B and $M := B' \cap F$ improves $\mathcal{B}$. So we can assume that $|B \cap F| = \beta$ for all $B \in \mathcal{B}$. Choose $B_0 \in \mathcal{B}$ with $r(\operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}})$ maximal. Define

(43.57) $A := \operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}}$, $E' := F/A$, and $\nu' := \beta - r(A)$.

For each $B \in \mathcal{B}$ there is a matching M_B in $(B \cap F)/A$ of size ν', since

(43.58)
$$\begin{aligned}&\beta - r(\operatorname{span}(B \cap F) \cap A) \geq \beta - r(\operatorname{span}(B \cap F) \cap K_{\mathcal{B}})\\ &\geq \beta - r(\operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}}) = \beta - r(A) = \nu'.\end{aligned}$$

Let $\mathcal{B}' := \{M_B \mid B \in \mathcal{B}\}$. Then $K_{\mathcal{B}'} = \{\mathbf{0}\}$, since

(43.59)
$$\begin{aligned}&\bigcap_{B \in \mathcal{B}} \operatorname{span}(B \cap F) \subseteq \operatorname{span}(B_0 \cap F) \cap \bigcap_{B \in \mathcal{B}} \operatorname{span}(B)\\ &= \operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}} = A.\end{aligned}$$

Since $\nu(E) > \nu$ we have $\nu(E') > \nu'$. Indeed, let B' be a matching in E of size $\nu + 1$. Then B'/A contains a matching of size $|B' \cap F| - r(\operatorname{span}(B' \cap F) \cap A)$. Hence

(43.60)
$$\begin{aligned}&\nu(E') \geq |B' \cap F| - r(\operatorname{span}(B' \cap F) \cap A)\\ &= |B'| - |B' \setminus F| - r(\operatorname{span}(B' \cap F) \cap A)\\ &= |B'| - \tfrac{1}{2}(r(B' \setminus F) + r(\operatorname{span}(B' \cap F) \cap A)) - \tfrac{1}{2}r(\operatorname{span}(B' \cap F) \cap A)\\ &\geq |B'| - \tfrac{1}{2}r(K_{\mathcal{B}}) - \tfrac{1}{2}r(A) > |B_0| - \tfrac{1}{2}r(K_{\mathcal{B}}) - \tfrac{1}{2}r(A)\\ &\geq |B_0| - \tfrac{1}{2}(r(B_0 \setminus F) + r(A)) - \tfrac{1}{2}r(A) = |B_0 \cap F| - r(A) = \nu'.\end{aligned}$$

The second inequality holds as $\operatorname{span}(B' \setminus F)$ and $\operatorname{span}(B' \cap F) \cap A$ are subspaces of $K_{\mathcal{B}}$ having intersection $\{\mathbf{0}\}$ (since B is a matching and as (43.56) holds). The last inequality follows from

(43.61)
$$\begin{aligned}&r(K_{\mathcal{B}}) = r(\operatorname{span}(B_0) \cap K_{\mathcal{B}})\\ &= r((\operatorname{span}(B_0 \setminus F) + \operatorname{span}(B_0 \cap F)) \cap K_{\mathcal{B}})\\ &= r(\operatorname{span}(B_0 \setminus F) + (\operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}}))\\ &= r(\operatorname{span}(B_0 \setminus F) + r(\operatorname{span}(B_0 \cap F) \cap K_{\mathcal{B}})).\end{aligned}$$

Here we use that

(43.62) $$\begin{aligned}&(\text{span}(B_0 \setminus F) + \text{span}(B_0 \cap F)) \cap K_{\mathcal{B}}\\ &= \text{span}(B_0 \setminus F) + (\text{span}(B_0 \cap F) \cap K_{\mathcal{B}}),\end{aligned}$$

which holds since if $x \in \text{span}(B_0 \setminus F)$ and $y \in \text{span}(B_0 \cap F)$ with $x + y \in K_{\mathcal{B}}$, then $y \in K_{\mathcal{B}}$ (since $x \in \text{span}(B_0 \setminus F) \subseteq K_{\mathcal{B}}$ by (43.56)).

Now applying (43.54) repeatedly to $\mathcal{B}'$, we finally find a matching M' in E' with $|M'| = \nu' + 1$. It corresponds to a matching M in F with

(43.63) $$r(M \cup A) = 2|M| + r(A) = |M| + \nu' + 1 + r(A) = |B_0 \cap F| + |M| + 1.$$

Then applying (43.44) improves $\mathcal{B}$. ∎

The proof also yields an alternative proof of Theorem 43.2.

While most of the matroids we meet in daily life are linear, it might yet be interesting to extend the algorithm to the class of algebraic matroids. As Dress and Lovász [1987] remark, this requires the development of algorithmic techniques for algebraic matroids, for instance, for testing algebraic independence, and for finding a point p in the intersection of certain flats. If such techniques are available, pursuing the layout of the above algorithm for linear matroids might yield a polynomial-time algorithm for algebraic matroids.

An augmenting path algorithm for linear matroid matching, of complexity $O(n^3 m)$ (where $n :=$ rank, $m := |S|$) was given by Stallmann and Gabow [1984] and Gabow and Stallmann [1986] and an $O(n^4 m)$-time algorithm (by solving a sequence of matroid intersection algorithms) by Orlin and Vande Vate [1990] (these bounds can be improved to $O(n^{2.376} m)$ and $O(n^{3.376} m)$, respectively, with fast matrix multiplication).

43.9. Matroid matching is not polynomial-time solvable in general

Theorem 43.2 characterizes the matroid matching problem for algebraic matroids, and one is challenged to extend this to general matroids. A main objection to do this in a direct way is that in Theorem 43.2 a line of E may intersect the flat F in a point not contained in the original matroid. So we need to extend the matroid in some way, which is quite natural for linear matroids, but, as Lovász remarks, 'in general, there seems to be no hope to extend the original matroid so as to achieve the validity of [Theorem 43.2]. The possibility of "simulating" the flat F inside the matroid seems to be a difficult, and probably not only technical, question.'

Jensen and Korte [1982] and Lovász [1981] showed that, for matroids in general, the matroid matching problem is not solvable in polynomial time, if the matroid is given by an independence testing oracle (an oracle telling if a given set is independent or not). The construction in both papers is as follows.

Let $\nu \in \mathbb{Z}$, let S be a set, and let E be a partition of S into pairs. Let M be the matroid on S of rank 2ν, where $T \subseteq S$ is independent if and only if $|T| \leq 2\nu - 1$, or $|T| = 2\nu$ and T is not the union of ν pairs in E.

For each subset F of E of size ν, let M_F be the matroid on S obtained from M by adding $\bigcup F$ as independent subset.

It is easy to check that M and each of the M_F are matroids, and that E has no matroid matching of size ν with respect to M, while F is the unique matroid matching of size ν in M_F.

Suppose now that we want to find the maximum size of a matroid matching in a matroid, and that we know that the matroid is equal to M or to M_F for some ν-element $F \subseteq E$. Then we must ask the oracle for the independence of $\bigcup F$ for each ν-element subset F of E, in order to know if there exists a matroid matching of size ν. This takes exponential time.

This example shows that the matroid matching problem even does not belong to (oracle) co-NP, since any certificate that the matching number is at most $\nu - 1$, needs the oracle output that $\bigcup F$ is dependent, for all ν-element subsets F of E.

The example can be easily adapted to remove the oracle, and to obtain a proper problem in NP that is NP-complete. Let G be an undirected graph with vertex set V and let $\nu \in \mathbb{Z}_+$. For each vertex v of G, let p_v be a pair of elements, such that $p_u \cap p_v = \emptyset$ if $u \neq v$. Let $S := \bigcup_{v \in V} p_v$ and $E := \{p_v | v \in V\}$. So E is a partition of S into pairs. Define a matroid on S by extending the matroid M above by an independent set

(43.64) $$I := \bigcup_{v \in C} p_v$$

for each clique C of G with $|C| = \nu$. Then E contains a matroid matching of size ν if and only if G has a clique of size ν. As the maximum-size clique problem is NP-complete, also the matroid matching problem for such matroids is NP-complete.

43.10. Further results and notes

43.10a. Optimal path-matching

Cunningham and Geelen [1996,1997] gave the following generalization of nonbipartite matching and matroid intersection.

Let $G = (V, E)$ be an undirected graph, let S_1 and S_2 be two disjoint stable subsets of V, and let $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, such that $r_1(S_2) = r_2(S_2) =: \rho$. Define $R := V \setminus (S_1 \cup S_2)$. A *basic path-matching* is a collection of ρ vertex-disjoint $B_1 - B_2$ paths, each having all internal vertices in R, where B_1 and B_2 are bases of M_1 and M_2 respectively, together with a perfect matching on the vertices of R not covered by these paths.

If $R = V$, a basic path-matching is just a perfect matching. If $R = \emptyset$ and E consists of disjoint edges linking S_1 and S_2, then a basic path-matching corresponds to a common base.

Geelen and Cunningham showed that a basic path-matching exists if and only if for each $U_1 \subseteq S_1 \cup R$ and $U_2 \subseteq S_2 \cup R$ such that there is no edge connecting two sets among $U_1 \cap U_2$, $U_1 \setminus U_2$, $U_2 \setminus U_1$, one has

(43.65) $$r_1(S_1 \setminus U_1) + r_2(S_2 \setminus U_2) + |R \setminus (U_1 \cup U_2)| \geq \rho + o(G[U_1 \cap U_2]),$$

where $o(H)$ is the number of odd components of a graph H. Moreover, they gave a polynomial-time algorithm to decide whether there exists a basic path-matching.

More generally, they introduced the concept of an *independent path-matching*, which is a set F of edges such that each nonsingleton component of the graph (V, F) is an $S_1 \cup R - S_2 \cup R$ path all of whose internal vertices are in R, and such that the vertices in S_i covered by the paths is independent in M_i $(i = 1, 2)$. The corresponding *independent path-matching vector* is the vector $x \in \mathbb{Z}_+^E$ with $x(e) = 0$ if $e \notin F$, $x(e) = 2$ if $e \in F$ forms a component of (V, F) with both ends of e in R, and $x(e) = 1$ otherwise.

Geelen and Cunningham showed that the convex hull of the independent path-matching vectors is determined by:

(43.66) $$\begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(\delta(v)) \leq 2 & \text{for } v \in R, \\ x(E[U]) \leq |U \cap R| & \text{for } U \subseteq V \text{ with } U \cap S_1 = \emptyset \text{ or } U \cap S_2 = \emptyset, \\ x(E[U]) \leq |U| - 1 & \text{for } U \subseteq R, \\ x(\delta(U)) \leq r_i(U) & \text{for } U \subseteq S_i \text{ and } i = 1, 2, \end{array}$$

and that this system is TDI. It implies that the maximum of $\mathbf{1}^{\mathsf{T}}x$ over independent path-matching vectors is equal to the minimum of

(43.67) $$r_1(S_1 \setminus U_1) + r_2(S_2 \setminus U_2) + |R \setminus (U_1 \cup U_2)| + |R| - o(G[U_1 \cap U_2])$$

over all $U_i \subseteq S_i \cup R$ $(i = 1, 2)$ such that there is no edge connecting two sets among $U_1 \cap U_2$, $U_1 \setminus U_2$, $U_2 \setminus U_1$. (A simplified proof of this was given by Frank and Szegő [2002].)

Cunningham and Geelen argue that the set of inequalities (43.66) can be checked in polynomial time, implying (with the ellipsoid method) that, for any weight function w, an independent path-matching vector x maximizing $w^{\mathsf{T}}x$ can be found in strongly polynomial time. A combinatorial algorithm for the unweighted version was given by Spille and Weismantel [2002a,2002b].

For a survey, see Cunningham [2002].

43.10b. Further notes

Hochstättler and Kern [1989] showed that condition (43.13) is implied by the following:

(43.68) for any three flats A, B, C with

$$r(A \cup C) - r(A) = r(B \cup C) - r(B) = r(A \cup B \cup C) - r(A \cup B),$$

one has

$$r(\mathrm{span}(A \cup C) \cap \mathrm{span}(B \cup C)) - r(A \cap B) = r(A \cup C) - r(A).$$

Matroids with this property are called *pseudomodular* by Björner and Lovász [1987], who proved that full linear matroids (infinite matroids determined by linear independence of a linear space), full algebraic matroids (infinite matroids determined by algebraic independence of a field extension of a field), and full graphic matroids (cycle matroids of a complete graph) are pseudomodular. See also Lindström [1988], Dress, Hochstättler, and Kern [1994], and Tan [1997].

A randomized parallel algorithm for linear matroid matching was given by Narayanan, Saran, and Vazirani [1992,1994]. Stallmann and Gabow [1984] gave an algorithm for graphic matroid matching with running time $O(n^2m)$, which was improved by Gabow and Stallmann [1985] to $O(nm\log^6 n)$. Tong, Lawler, and Vazirani [1984] found a polynomial-time algorithm for *weighted* matroid matching for gammoids (by reduction to weighted matching). Structural properties of matroid matching, including an Edmonds-Gallai type decomposition, were given by Vande Vate [1992], which paper also studied the matroid matching polytope and a fractional relaxation of it.

The matroid matching problem generalizes the *matchoid problem* of J. Edmonds (cf. Jenkyns [1974]): given a graph $G = (V, E)$ and a matroid $M_v = (\delta(v), \mathcal{I}_v)$ for each v in V, what is the maximum number of edges such that the restriction to $\delta(v)$ forms an independent set in M_v, for each v in V?

Chapter 44

Submodular functions and polymatroids

In this chapter we describe some of the basic properties of a second main object of the present part, the submodular function. Each submodular function gives a polymatroid, which is a generalization of the independent set polytope of a matroid. We prove as a main result the theorem of Edmonds [1970b] that the vertices of a polymatroid are integer if and only if the associated submodular function is integer.

44.1. Submodular functions and polymatroids

Let f be a *set function* on a set S, that is, a function defined on the collection $\mathcal{P}(S)$ of all subsets of S. The function f is called *submodular* if

(44.1) $$f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$$

for all subsets T, U of S. Similarly, f is called *supermodular* if $-f$ is submodular, i.e., if f satisfies (44.1) with the opposite inequality sign. f is *modular* if f is both submodular and supermodular, i.e., if f satisfies (44.1) with equality.

A set function f on S is called *nondecreasing* if $f(T) \leq f(U)$ whenever $T \subseteq U \subseteq S$, and *nonincreasing* if $f(T) \geq f(U)$ whenever $T \subseteq U \subseteq S$.

As usual, denote for each function $w : S \to \mathbb{R}$ and for each subset U of S,

(44.2) $$w(U) := \sum_{s \in U} w(s).$$

So w may be considered also as a set function on S, and one easily sees that w is modular, and that each modular set function f on S with $f(\emptyset) = 0$ may be obtained in this way. (More generally, each modular set function f on S satisfies $f(U) = w(U) + \gamma$ (for $U \subseteq S$), for some unique function $w : S \to \mathbb{R}$ and some unique real number γ.)

In a sense, submodularity is the discrete analogue of convexity. If we define, for any $f : \mathcal{P}(S) \to \mathbb{R}$ and any $x \in S$, a function $\delta f_x : \mathcal{P}(S) \to \mathbb{R}$ by: $\delta f_x(T) := f(T \cup \{x\}) - f(T)$, then f is submodular if and only if δf_x is nonincreasing for each $x \in S$.

In other words:

Theorem 44.1. *A set function f on S is submodular if and only if*

(44.3) $$f(U \cup \{s\}) + f(U \cup \{t\}) \geq f(U) + f(U \cup \{s,t\})$$

for each $U \subseteq S$ and distinct $s,t \in S \setminus U$.

Proof. Necessity being trivial, we show sufficiency. We prove (44.1) by induction on $|T \triangle U|$, the case $|T \triangle U| \leq 2$ being trivial (if $T \subseteq U$ or $U \subseteq T$) or being implied by (44.3). If $|T \triangle U| \geq 3$, we may assume by symmetry that $|T \setminus U| \geq 2$. Choose $t \in T \setminus U$. Then, by induction,

(44.4) $$f(T \cup U) - f(T) \leq f((T \setminus \{t\}) \cup U) - f(T \setminus \{t\}) \leq f(U) - f(T \cap U),$$

(as $|T \triangle ((T \setminus \{t\}) \cup U)| < |T \triangle U|$ and $|(T \setminus \{t\}) \triangle U| < |T \triangle U|$). This shows (44.1). ∎

Define two polyhedra associated with a set function f on S:

(44.5) $$\begin{aligned} P_f &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \leq f(U) \text{ for each } U \subseteq S\}, \\ EP_f &:= \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for each } U \subseteq S\}. \end{aligned}$$

Note that P_f is nonempty if and only if $f \geq \mathbf{0}$, and that EP_f is nonempty if and only if $f(\emptyset) \geq 0$.

If f is a submodular function, then P_f is called the *polymatroid associated with* f, and EP_f the *extended polymatroid associated with* f. A polyhedron is called an (extended) polymatroid if it is the (extended) polymatroid associated with some submodular function. A polymatroid is bounded (since $0 \leq x_s \leq f(\{s\})$ for each $s \in S$), and hence is a polytope.

The following observation presents a basic technique in proofs for submodular functions, which we often use without further reference:

Theorem 44.2. *Let f be a submodular set function on S and let $x \in EP_f$. Then the collection of sets $U \subseteq S$ satisfying $x(U) = f(U)$ is closed under taking unions and intersections.*

Proof. Suppose $x(T) = f(T)$ and $x(U) = f(U)$. Then

(44.6) $$\begin{aligned} f(T) + f(U) &\geq f(T \cap U) + f(T \cup U) \geq x(T \cap U) + x(T \cup U) \\ &= x(T) + x(U) = f(T) + f(U), \end{aligned}$$

implying that equality holds throughout. So $x(T \cap U) = f(T \cap U)$ and $x(T \cup U) = f(T \cup U)$. ∎

A vector x in EP_f (or in P_f) is called a *base vector* of EP_f (or of P_f) if $x(S) = f(S)$. A *base vector* of f is a base vector of EP_f. The set of all base vectors of f is called the *base polytope* of EP_f or of f. It is a face of EP_f, and denoted by B_f. So

(44.7) $$B_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for all } U \subseteq S, \, x(S) = f(S)\}.$$

(It is a polytope, since $x_s = x(S) - x(S \setminus \{s\}) \geq f(S) - f(S \setminus \{s\})$ for each $s \in S$.)

Let f be a submodular set function on S and let $a \in \mathbb{R}^S$. Define the set function $f|a$ on S by

(44.8) $$(f|a)(U) := \min_{T \subseteq U}(f(T) + a(U \setminus T))$$

for $U \subseteq S$. It is easy to check that $f|a$ again is submodular and that

(44.9) $$EP_{f|a} = \{x \in EP_f \mid x \leq a\} \text{ and } P_{f|a} = \{x \in P_f \mid x \leq a\}.$$

It follows that if P is an (extended) polymatroid, then also the set $P \cap \{x \mid x \leq a\}$ is an (extended) polymatroid, for any vector a. In fact, as Lovász [1983c] observed, if $f(\emptyset) = 0$, then $f|a$ is the unique largest submodular function f' satisfying $f'(\emptyset) = 0$, $f' \leq f$, and $f'(U) \leq a(U)$ for each $U \subseteq V$.

44.1a. Examples

Matroids. Let $M = (S, \mathcal{I})$ be a matroid. Then the rank function r of M is submodular and nondecreasing. In Theorem 39.8 we saw that a set function r on S is the rank function of a matroid if and only if r is nonnegative, integer, nondecreasing and submodular with $r(U) \leq |U|$ for all $U \subseteq S$. (This last condition may be replaced by: $r(\emptyset) = 0$ and $r(\{s\}) \leq 1$ for each s in S.) Then the polymatroid P_r associated with r is equal to the independent set polytope of M (by Corollary 40.2b).

A generalization is obtained by partitioning S into sets $S_1, \ldots, S_k$, and defining

(44.10) $$f(J) := r(\bigcup_{i \in J} S_i)$$

for $J \subseteq \{1, \ldots, k\}$. It is not difficult to show that each integer nondecreasing submodular function f with $f(\emptyset) = 0$ can be constructed in this way (see Section 44.6b).

As another generalization, if $w : S \to \mathbb{R}_+$, define $f(U)$ to be the maximum of $w(I)$ over $I \in \mathcal{I}$ with $I \subseteq U$. Then f is submodular. (To see this, write $w = \lambda_1 \chi^{T_1} + \cdots + \lambda_n \chi^{T_n}$, with $\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_n \subseteq S$. Then by (40.3), $f(U) = \sum_{i=1}^n \lambda_i r(U \cap T_i)$, implying that f is submodular.)

For more on the relation between submodular functions and matroids, see Sections 44.6a and 44.6b.

Matroid intersection. Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2 respectively. Then the function f given by

(44.11) $$f(U) := r_1(U) + r_2(S \setminus U)$$

for $U \subseteq S$, is submodular. By the matroid intersection theorem (Theorem 41.1), the minimum value of f is equal to the maximum size of a common independent set.

Set unions. Let $T_1, \ldots, T_n$ be subsets of a finite set T and let $S = \{1, \ldots, n\}$. Define

(44.12) $$f(U) := |\bigcup_{i \in U} T_i|$$

for $U \subseteq S$. Then f is nondecreasing and submodular. More generally, for $w : T \to \mathbb{R}_+$, the function f defined by

(44.13) $$f(U) := w(\bigcup_{i \in U} T_i)$$

for $U \subseteq S$, is nondecreasing and submodular.

More generally, for any nondecreasing submodular set function g on T, the function f defined by

(44.14) $$f(U) := g(\bigcup_{i \in U} T_i)$$

for $U \subseteq S$, again is nondecreasing and submodular.

Let $G = (V, E)$ be the bipartite graph corresponding to $T_1, \ldots, T_n$. That is, G has colour classes S and T, and $s \in S$ and $t \in T$ are adjacent if and only if $t \in T_s$. Then we have: $x \in P_f$ if and only if there exist $z \in P_g$ and $y : E \to \mathbb{Z}_+$ such that

(44.15) $$\begin{aligned} y(\delta(v)) &= x(v) \quad \text{for all } v \in S, \\ y(\delta(v)) &= z(v) \quad \text{for all } v \in T. \end{aligned}$$

So y may be considered as an 'assignment' of a 'supply' z to a 'demand' x. If g and x are integer we can take also y and z integer.

Directed graph cut functions. Let $D = (V, A)$ be a directed graph and let $c : A \to \mathbb{R}_+$ be a 'capacity' function on A. Define

(44.16) $$f(U) := c(\delta^{\text{out}}(U))$$

for $U \subseteq V$ (where $\delta^{\text{out}}(U)$ denotes the set of arcs leaving U). Then f is submodular (but in general not nondecreasing). A function f arising in this way is called a *cut function.*

Hypergraph cut functions. Let $(V, \mathcal{E})$ be a hypergraph. For $U \subseteq V$, let $f(U)$ be the number of edges $E \in \mathcal{E}$ split by U (that is, with both $E \cap U$ and $E \setminus U$ nonempty). Then f is submodular.

Directed hypergraph cut functions. Let V be a finite set and let $(E_1, F_1), \ldots, (E_m, F_m)$ be pairs of subsets of V. For $U \subseteq V$, let $f(U)$ be the number of indices i with $U \cap E_i \neq \emptyset$ and $F_i \not\subseteq U$. Then f is submodular. (In proving this, we can assume $m = 1$, since any sum of submodular functions is submodular again.)

More generally, we can choose $c_1, \ldots, c_m \in \mathbb{R}_+$ and define

(44.17) $$f(U) = \sum(c_i \mid U \cap E_i \neq \emptyset, F_i \not\subseteq U)$$

for $U \subseteq V$. Again, f is submodular. This generalizes the previous two examples (where $E_i = F_i$ for each i or $|E_i| = |F_i| = 1$ for each i).

Maximal element. Let V be a finite set and let $h : V \to \mathbb{R}$. For nonempty $U \subseteq V$, define

(44.18) $$f(U) := \max\{h(u) \mid u \in U\},$$

and define $f(\emptyset)$ to be the minimum of $h(v)$ over $v \in V$. Then f is submodular.

Subtree diameter. Let $G = (V, E)$ be a forest (a graph without circuits), and for each $X \subseteq E$ define

(44.19) $$f(X) := \sum_K \text{diameter}(K),$$

where K ranges over the components of the graph (V, X). Here diameter(K) is the length of a longest path in K. Then f is submodular (Tamir [1993]); that is:

(44.20) $$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

for $X, Y \subseteq E$.

To see this, denote, for any $X \subseteq E$, the set of vertices covered by X by VX. We first show (44.20) for $X, Y \subseteq E$ with (VX, X) and (VY, Y) connected and $VX \cap VY \neq \emptyset$. Note that in this case $X \cap Y$ and $X \cup Y$ give connected subgraphs again.

The proof of (44.20) is based on the fact that for all $s, t, u, v \in V$ one has:

(44.21) $$\begin{aligned} &\text{dist}(s, u) + \text{dist}(t, v) \geq \text{dist}(s, t) + \text{dist}(u, v) \\ &\text{or } \text{dist}(t, u) + \text{dist}(s, v) \geq \text{dist}(s, t) + \text{dist}(u, v), \end{aligned}$$

where dist denotes the distance in G.

To prove (44.20), let P and Q be longest paths in $X \cap Y$ and $X \cup Y$ respectively. If EQ is contained in X or in Y, then (44.20) follows, since P is contained in X and in Y. So we can assume that EQ is contained neither in X nor in Y. Let Q have ends u, v, with $u \in VX$ and $v \in VY$. Let P have ends s, t. So $s, t, u \in VX$ and $s, t, v \in VY$. Hence (44.21) implies (44.20).

We now derive (44.20) for all $X, Y \subseteq E$. Let $\mathcal{X}$ and $\mathcal{Y}$ be the collections of edge sets of the components of (V, X) and of (V, Y) respectively. Let $\mathcal{F}$ be the family made by the union of $\mathcal{X}$ and $\mathcal{Y}$, taking the sets in $\mathcal{X} \cap \mathcal{Y}$ twice. Then

(44.22) $$f(X) + f(Y) \geq \sum_{Z \in \mathcal{F}} f(Z).$$

We now modify $\mathcal{F}$ iteratively as follows. If $Z, Z' \in \mathcal{F}$, $Z \not\subseteq Z' \not\subseteq Z$, and $VZ \cap VZ' \neq \emptyset$, we replace Z, Z' by $Z \cap Z'$ and $Z \cup Z'$. By (44.20), (44.22) is maintained. By Theorem 2.1, these iterations stop. We delete the empty sets in the final $\mathcal{F}$.

Then the inclusionwise maximal sets in $\mathcal{F}$ have union equal to $X \cup Y$ and form the nonempty edge sets of the components of $(V, X \cup Y)$. Similarly, the inclusionwise minimal sets in $\mathcal{F}$ form the nonempty edge sets of the components of $(V, X \cap Y)$. So

(44.23) $$\sum_{Z \in \mathcal{F}} f(Z) = f(X \cap Y) + f(X \cup Y),$$

and we have (44.20).

Further examples. Choquet [1951,1955] showed that the classical Newtonian capacity in $\mathbb{R}^3$ is submodular. Examples of submodular functions based on probability are given by Fujishige [1978b] and Han [1979], and other examples by Lovász [1983c].

44.2. Optimization over polymatroids by the greedy method

Edmonds [1970b] showed that one can optimize a linear function $w^{\mathsf{T}}x$ over an (extended) polymatroid by an extension of the greedy algorithm. The submodular set function f on S is given by a *value giving oracle*, that is, by an oracle that returns $f(U)$ for any $U \subseteq S$.

Let f be a submodular set function on S, and suppose that we want to maximize $w^{\mathsf{T}}x$ over EP_f, for some $w : S \to \mathbb{R}$. We can assume that $EP_f \neq \emptyset$, that is $f(\emptyset) \geq 0$, and hence that $f(\emptyset) = 0$ (since decreasing $f(\emptyset)$ maintains submodularity). We can also assume that $w \geq \mathbf{0}$, since if some component of w is negative, the maximum value is unbounded.

Now order the elements in S as $s_1, \ldots, s_n$ such that $w(s_1) \geq \cdots \geq w(s_n)$. Define

(44.24) $$U_i := \{s_1, \ldots, s_i\} \text{ for } i = 0, \ldots, n,$$

and define $x \in \mathbb{R}^S$ by

(44.25) $$x(s_i) := f(U_i) - f(U_{i-1}) \text{ for } i = 1, \ldots, n.$$

Then x maximizes $w^{\mathsf{T}}x$ over EP_f, as will be shown in the following theorem.

To prove it, consider the following linear programming duality equation:

(44.26) $$\begin{aligned}&\max\{w^{\mathsf{T}}x \mid x \in EP_f\}\\ &= \min\{\sum_{T \subseteq S} y(T)f(T) | y \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{T \in \mathcal{P}(S)} y(T)\chi^T = w\}.\end{aligned}$$

Define:

(44.27) $$\begin{aligned}y(U_i) &:= w(s_i) - w(s_{i+1}) \quad && (i = 1, \ldots, n-1),\\ y(S) &:= w(s_n), &&\\ y(T) &:= 0 && (T \neq U_i \text{ for each } i).\end{aligned}$$

Theorem 44.3. *Let f be a submodular set function on S with $f(\emptyset) = 0$ and let $w : S \to \mathbb{R}_+$. Then x and y given by* (44.25) *and* (44.27) *are optimum solutions of* (44.26).

Proof. We first show that x belongs to EP_f; that is, $x(T) \leq f(T)$ for each $T \subseteq S$. This is shown by induction on $|T|$, the case $T = \emptyset$ being trivial. Let $T \neq \emptyset$ and let k be the largest index with $s_k \in T$. Then by induction,

(44.28) $$x(T \setminus \{s_k\}) \leq f(T \setminus \{s_k\}).$$

Hence

(44.29) $$x(T) \leq f(T \setminus \{s_k\}) + x(s_k) = f(T \setminus \{s_k\}) + f(U_k) - f(U_{k-1}) \leq f(T)$$

(the last inequality follows from the submodularity of f). So $x \in EP_f$.

Also, y is feasible for (44.26). Trivially, $y \geq \mathbf{0}$. Moreover, for any i we have by (44.27):

(44.30) $$\sum_{T \ni s_i} y(T) = \sum_{j \geq i} y(U_j) = w(s_i).$$

So y is a feasible solution of (44.26).

Optimality of x and y follows from:

(44.31) $$\begin{aligned} w^{\mathsf{T}} x &= \sum_{s \in S} w(s) x_s = \sum_{i=1}^{n} w(s_i)(f(U_i) - f(U_{i-1})) \\ &= \sum_{i=1}^{n-1} f(U_i)(w(s_i) - w(s_{i+1})) + f(S) w(s_n) = \sum_{T \subseteq S} y(T) f(T). \end{aligned}$$

The third equality follows from a straightforward reordering of the terms, using that $f(\emptyset) = 0$. ∎

Note that if f is integer, then x is integer, and that if w is integer, then y is integer. Moreover, if f is nondecreasing, then x is nonnegative. Hence, in that case, x and y are optimum solutions of

(44.32) $$\begin{aligned} &\max\{w^{\mathsf{T}} x \mid x \in P_f\} \\ &= \min\{\sum_{T \subseteq S} y(T) f(T) \mid y \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{T \in \mathcal{P}(S)} y(T) \chi^T \geq w\}. \end{aligned}$$

Therefore:

Corollary 44.3a. *Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$ and let $w : S \to \mathbb{R}_+$. Then x and y given by* (44.25) *and* (44.27) *are optimum solutions for* (44.32).

Proof. Directly from Theorem 44.3, using the fact that $x \geq \mathbf{0}$ if f is nondecreasing. ∎

As for complexity we have:

Corollary 44.3b. *Given a submodular set function f on a set S (by a value giving oracle) and a function $w \in \mathbb{Q}^S$, we can find an $x \in EP_f$ maximizing $w^{\mathsf{T}} x$ in strongly polynomial time. If f is moreover nondecreasing, then $x \in P_f$ (and hence x maximizes $w^{\mathsf{T}} x$ over P_f).*

Proof. By the extension of the greedy method given above. ∎

The greedy algorithm can be interpreted geometrically as follows. Let w be some linear objective function on S, with $w(s_1) \geq \ldots \geq w(s_n)$. Travel via the vertices of P_f along the edges of P_f, by starting at the origin, as follows: first go from the origin as far as possible (in P_f) in the positive s_1-direction, say to vertex x_1; next go from x_1 as far as possible in the positive s_2-direction, say to x_2, and so on. After n steps one reaches a vertex x_n maximizing $w^{\mathsf{T}} x$

over P_f. In fact, the effectiveness of this algorithm characterizes polymatroids (Dunstan and Welsh [1973]).

44.3. Total dual integrality

Theorem 44.3 implies the box-total dual integrality of the following system:

(44.33) $\quad x(U) \leq f(U)$ for $U \subseteq S$.

Corollary 44.3c. *If f is submodular, then* (44.33) *is box-totally dual integral.*

Proof. Consider the dual of maximizing $w^{\mathsf{T}}x$ over (44.33), for some $w \in \mathbb{Z}_+^S$. By Theorem 44.3, it has an optimum solution $y : \mathcal{P}(S) \to \mathbb{R}_+$ with the sets $U \subseteq S$ having $y(U) > 0$ forming a chain. So these constraints give a totally unimodular submatrix of the constraint matrix (by Theorem 41.11). Therefore, by Theorem 5.35, (44.33) is box-TDI. ∎

This gives the integrality of polyhedra:

Corollary 44.3d. *For any integer submodular set function f, the polymatroid P_f and the extended polymatroid EP_f are integer.*

Proof. Directly from Corollary 44.3c. (In fact, integer optimum solutions are explicitly given by Theorem 44.3 and Corollary 44.3a.) ∎

44.4. f is determined by EP_f

Theorem 44.3 implies that for any extended polymatroid P there is a unique submodular function f satisfying $f(\emptyset) = 0$ and $EP_f = P$, since:

Corollary 44.3e. *Let f be a submodular set function on S with $f(\emptyset) = 0$. Then*

(44.34) $\quad f(U) = \max\{x(U) \mid x \in EP_f\}$

for each $U \subseteq S$.

Proof. Directly from Theorem 44.3 by taking $w := \chi^U$. ∎

So there is a one-to-one correspondence between nonempty extended polymatroids and submodular set functions f with $f(\emptyset) = 0$. The correspondence relates integer extended polymatroids with integer submodular functions.

There is a similar correspondence between nonempty polymatroids and *nondecreasing* submodular set functions f with $f(\emptyset) = 0$. For any (not necessarily nondecreasing) nonnegative submodular set function f, define $\bar{f}$ by:

(44.35) $$\bar{f}(\emptyset) = 0,$$
$$\bar{f}(U) = \min_{T \supseteq U} f(T) \quad \text{for nonempty } U \subseteq S.$$

It is easy to see that $\bar{f}$ is nondecreasing and submodular and that $P_{\bar{f}} = P_f$ (Dunstan [1973]). In fact, $\bar{f}$ is the unique nondecreasing submodular set function associated with P_f, with $\bar{f}(\emptyset) = 0$, as (Kelley [1959]):

Corollary 44.3f. *If f is a nondecreasing submodular function with $f(\emptyset) = 0$, then*

(44.36) $$f(U) = \max\{x(U) \mid x \in P_f\}$$

for each $U \subseteq S$.

Proof. This follows from Corollary 44.3a by taking $w := \chi^T$. ∎

This one-to-one correspondence between polymatroids and nondecreasing submodular set functions f with $f(\emptyset) = 0$ relates integer polymatroids to integer such functions:

Corollary 44.3g. *For each integer polymatroid P there exists a unique integer nondecreasing submodular function f with $f(\emptyset) = 0$ and $P = P_f$.*

Proof. By Corollary 44.3d and (44.36). ∎

By (44.36) we have for any nonnegative submodular set function f that $\bar{f}(U) = \max\{x(U) \mid x \in P_f\}$. Since we can optimize over EP_f in polynomial time (with the greedy algorithm described above), with the ellipsoid method we can optimize over $P_f = EP_f \cap \mathbb{R}^S_+$ in polynomial time. Hence we can calculate $\bar{f}(U)$ in polynomial time. Alternatively, calculating $\bar{f}(U)$ amounts to minimizing the submodular function $f'(T) := f(T \cup U)$.

In fact $\bar{f}$ is the largest among all nondecreasing submodular set functions g on S with $g(\emptyset) = 0$ and $g \leq f$, as can be checked straightforwardly.

44.5. Supermodular functions and contrapolymatroids

Similar results hold for supermodular functions and the associated contrapolymatroids. Associate the following polyhedra with a set function g on S:

(44.37) $$Q_g := \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \geq g(U) \text{ for each } U \subseteq S\},$$
$$EQ_g := \{x \in \mathbb{R}^S \mid x(U) \geq g(U) \text{ for each } U \subseteq S\}.$$

If g is supermodular, then Q_g and EQ_g are called the *contrapolymatroid* and the *extended contrapolymatroid associated with* g, respectively. A vector $x \in EQ_g$ (or Q_g) is called a *base vector* of EQ_g (or Q_g) if $x(S) = g(S)$. A *base vector* of g is a base vector of EQ_g.

Since $EQ_g = -EP_{-g}$, we can reduce most problems on (extended) contrapolymatroids to (extended) polymatroids. Again we can minimize a linear function $w^\mathsf{T} x$ over EQ_g with the greedy algorithm, as described in Section 44.2. (In fact, we can apply the same formulas (44.25) and (44.27) for g instead of f.) If g is nondecreasing, it yields a nonnegative optimum solution, and hence a vector x minimizing $w^\mathsf{T} x$ over Q_g.

Similarly, the system

(44.38) $\quad x(U) \geq g(U)$ for $U \subseteq S$

is box-TDI, as follows directly from the box-total dual integrality of

(44.39) $\quad x(U) \leq -g(U)$ for $U \subseteq S$.

Let EP_f be the extended polymatroid associated with the submodular function f with $f(\emptyset) = 0$. Let B_f be the face of base vectors of EP_f, i.e.,

(44.40) $\quad B_f = \{x \in EP_f \mid x(S) = f(S)\}$.

A vector $y \in \mathbb{R}^S$ is called *spanning* if there exists an x in B_f with $x \leq y$. Let Q be the set of spanning vectors.

A vector y belongs to Q if and only if $(f|y)(S) = f(S)$, that is (by (44.8) and (44.9)) if and only if

(44.41) $\quad y(U) \geq f(S) - f(S \setminus U)$

for each $U \subseteq S$. So Q is equal to the contrapolymatroid EQ_g associated with the submodular function g defined by $g(U) := f(S) - f(S \setminus U)$ for $U \subseteq S$. Then B_f is equal to the face of minimal elements of EQ_g.

There is a one-to-one correspondence between submodular set functions f on S with $f(\emptyset) = 0$ and supermodular set functions g on S with $g(\emptyset) = 0$, given by the relations

(44.42) $\quad g(U) = f(S) - f(S \setminus U)$ and $f(U) = g(S) - g(S \setminus U)$

for $U \subseteq S$.

Then the pair $(-g, -Q)$ is related to the pair (f, P) by a relation similar to the duality relation of matroids (cf. Section 44.6f).

44.6. Further results and notes

44.6a. Submodular functions and matroids

Let P be the polymatroid associated with the nondecreasing integer submodular set function f on S, with $f(\emptyset) = 0$. Then the collection

(44.43) $\quad \mathcal{I} := \{I \subseteq S \mid \chi^I \in P\}$

forms the collection of independent sets of a matroid $M = (S, \mathcal{I})$ (this result was announced by Edmonds and Rota [1966] and proved by Pym and Perfect [1970]). By Corollary 40.2b, the subpolymatroid (cf. Section 44.6c)

(44.44) $$P|\mathbf{1} = \{x \in P \mid x \leq \mathbf{1}\}$$

is the convex hull of the incidence vectors of the independent sets of M. By (44.8), the rank function r of M satisfies

(44.45) $$r(U) = \min_{T \subseteq U}(|U \setminus T| + f(T))$$

for $U \subseteq S$.

As an example, if f is the submodular function given in the set union example in Section 44.1a, we obtain the transversal matroid on $\{1, \ldots, n\}$ with $I \subseteq \{1, \ldots, n\}$ independent if and only if the family $(T_i \mid i \in I)$ has a transversal (Edmonds [1970b]).

44.6b. Reducing integer polymatroids to matroids

In fact, each integer polymatroid can be derived from a matroid as follows (Helgason [1974]). Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$. Choose for each s in S, a set X_s of size $f(\{s\})$, such that the sets X_s $(s \in S)$ are disjoint. Let $X := \bigcup_{s \in S} X_s$, and define a set function r on X by

(44.46) $$r(U) := \min_{T \subseteq S}(|U \setminus \bigcup_{s \in T} X_s| + f(T))$$

for $U \subseteq X$. One easily checks that r is the rank function of a matroid M (by checking the axioms (39.38)), and that for each subset T of S

(44.47) $$f(T) = r(\bigcup_{s \in T} X_s).$$

Therefore, f arises from the rank function of M, as in the Matroids example in Section 44.1a. The polymatroid P_f associated with f is just the convex hull of all vectors x for which there exists an independent set I in M with $x_s = |I \cap X_s|$ for all s in S.

Given a nondecreasing submodular set function f on S with $f(\emptyset) = 0$, Lovász [1980a] called a subset $U \subseteq S$ a *matching* if

(44.48) $$f(U) = \sum_{s \in U} f(\{s\}).$$

If $f(\{s\}) = 1$ for each s in S, f is the rank function of a matroid, and U is a matching if and only if U is independent in this matroid. If $f(\{s\}) = 2$ for each s in S, the elements of S correspond to certain flats of rank 2 in a matroid. Now determining the maximum size of a matching is just the matroid matching problem (cf. Chapter 43).

44.6c. The structure of polymatroids

Vertices of polymatroids (Edmonds [1970b], Shapley [1965,1971]). Let f be a submodular set function on a set $S = \{s_1, \ldots, s_n\}$ with $f(\emptyset) = 0$. Let P_f be the polymatroid associated with f. It follows immediately from the greedy algorithm, as in the proof of Corollary 44.3a, that the vertices of P_f are given by (for $i = 1, \ldots, n$):

(44.49) $$x(s_{\pi(i)}) = \begin{cases} f(\{s_{\pi(1)}, \ldots, s_{\pi(i)}\}) - f(\{s_{\pi(1)}, \ldots, s_{\pi(i-1)}\}) & \text{if } i \leq k, \\ 0 & \text{if } i > k, \end{cases}$$

where π ranges over all permutations of $\{1, \ldots, n\}$ and where k ranges over $0, \ldots, n$.

Similarly, for any submodular set function f on S with $f(\emptyset) = 0$, the vertices of the extended polymatroid EP_f are given by

(44.50) $$x(s_{\pi(i)}) = f(\{s_{\pi(1)}, \ldots, s_{\pi(i)}\}) - f(\{s_{\pi(1)}, \ldots, s_{\pi(i-1)}\})$$

for $i = 1, \ldots, n$, where π ranges over all permutations of $\{1, \ldots, n\}$.

Topkis [1984] characterized adjacency of the vertices of a polymatroid, while Bixby, Cunningham, and Topkis [1985] and Topkis [1992] gave further results on vertices of and paths on a polymatroid and on related partial orders of S.

Facets of polymatroids. Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$. One easily checks that P_f is full-dimensional if and only if $f(\{s\}) > 0$ for all s in S. If P_f is full-dimensional there is a unique minimal collection of linear inequalities defining P_f (clearly, up to scalar multiplication). They correspond to the facets of P_f. Edmonds [1970b] found that this collection is given by the following theorem. A subset $U \subseteq S$ is called an *f-flat* if $f(U \cup \{s\}) > f(U)$ for all $s \in S \setminus U$, and U is called *f-inseparable* if there is no partition of U into nonempty sets U_1 and U_2 with $f(U) = f(U_1) + f(U_2)$. Then:

Theorem 44.4. *Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$ and $f(\{s\}) > 0$ for each $s \in S$. The following is a minimal system determining the polymatroid P_f:*

(44.51) $$\begin{array}{ll} x_s \geq 0 & (s \in S), \\ x(U) \leq f(U) & (U \text{ is a nonempty } f\text{-inseparable } f\text{-flat}). \end{array}$$

Proof. It is easy to see that (44.51) determines P_f, as any other inequality $x(U) \leq f(U)$ follows from (44.51). The irredundancy of collection (44.51) can be seen as follows.

Clearly, each inequality $x_s \geq 0$ determines a facet. Next consider a nonempty f-inseparable f-flat U. Suppose that the face determined by U is not a facet. Then it is contained in another face, say determined by T. Let x be a vertex of P_f with $x(U \setminus T) = f(U \setminus T)$, $x(U) = f(U)$, and $x(S \setminus U) = 0$. Such a vertex exists by the greedy algorithm (cf. (44.49)).

Since x is on the face determined by U, it is also on the face determined by T. So $x(T) = f(T)$. Hence $f(T) = x(T) = x(T \cap U) = f(U) - f(U \setminus T)$. So we have equality throughout in:

(44.52) $$f(U \setminus T) + f(T) \geq f(U \setminus T) + f(T \cap U) \geq f(U).$$

This implies that $U \setminus T = \emptyset$ or $T \cap U = \emptyset$ (as U is f-inseparable), and that $f(T) = f(T \cap U)$. If $U \setminus T = \emptyset$, then $U \subset T$, and hence (as U is an f-flat) $f(T) > f(U) \geq f(T \cap U)$, a contradiction. If $T \cap U = \emptyset$, then $f(T) = f(T \cap U) = 0$, implying that $T = \emptyset$, again a contradiction. ∎

It follows that the face $\{x \in P_f \mid x(S) = f(S)\}$ of maximal vectors in P_f is a facet if and only if $f(U) + f(S \setminus U) > f(S)$ for each proper nonempty subset U of S. More generally, its codimension is equal to the number of inclusionwise minimal nonempty sets U with $f(U) + f(S \setminus U) = f(S)$ (cf. Fujishige [1984a]).

Faces of polymatroids (Giles [1975]). We now extend the characterizations of vertices and facets of polymatroids given above to arbitrary faces. Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$. Suppose that P is full-dimensional. If $\emptyset \neq S_1 \subset \cdots \subset S_k \subseteq T \subseteq S$, then

(44.53) $$F = \{x \in P \mid x(S_1) = f(S_1), \ldots, x(S_k) = f(S_k), x(S \setminus T) = 0\}$$

is a face of P of dimension at most $|T| - k$. (Indeed, F is nonempty by the characterization (44.49) of vertices, while $\dim(F) \leq |T| - k$, as the incidence vectors of $S_1, \ldots, S_k$ are linearly independent.)

In fact, each face has a representation (44.53). Indeed, let F be a face of P. Define $T = \{s \in S \mid x_s > 0$ for some x in $F\}$, and let $S_1 \subset \cdots \subset S_k$ be any maximal chain of nonempty subsets of T with the property that

(44.54) $$F \subseteq \{x \in P \mid x(S_1) = f(S_1), \ldots, x(S_k) = f(S_k), x(S \setminus T) = 0\}.$$

Then we have equality in (44.54), and $\dim(F) = |T| - k$. (Here a maximal chain is a chain which is contained in no larger chain satisfying (44.54) — since the empty chain satisfies (44.54), there exist maximal chains.)

In order to prove this assertion, suppose that F has dimension d. As the right-hand side of (44.54) is a face of P of dimension at most $|T| - k$, it suffices to show that $d = |T| - k$. Therefore, suppose $d < |T| - k$. Then there exists a subset U of S such that $x(U) = f(U)$ for all x in F, and such that the incidence vector of $U \cap T$ is linearly independent of the incidence vectors of $S_1, \ldots, S_k$. That is, $U \cap T$ is not the union of some of the sets $S_i \setminus S_{i-1}$ ($i = 1, \ldots, k$). Since $x(U \cap T) = x(U) = f(U) \geq f(U \cap T)$ for all x in F, we may assume that $U \subseteq T$. Since the collection of subsets U of S with $x(U) = f(U)$ is closed under taking unions and intersections, we may assume moreover that U is comparable with each of the sets in the chain $S_1 \subset \cdots \subset S_k$. Hence U could be added to the chain to obtain a larger chain, contradicting our assumption. So $d = |T| - k$.

Note that a chain $S_1 \subset \cdots \subset S_k$ of nonempty subsets of T is a maximal chain satisfying (44.54) if and only if there is equality in (44.54) and (setting $S_0 := \emptyset$):

(44.55) $f(S_k \cup \{s\}) > f(S_k)$ for all s in $T \setminus S_k$, and each of the sets $S_i \setminus S_{i-1}$ is f_i-inseparable, where f_i is the submodular set function on $S_i \setminus S_{i-1}$ given by $f_i(U) := f(U \cup S_{i-1}) - f(S_{i-1})$ for $U \subseteq S_i \setminus S_{i-1}$.

This may be derived straightforwardly from the existence, by (44.49), of appropriate vertices of F.

It is not difficult to show that if F has a representation (44.53), then F is the direct sum of $F_1, \ldots, F_k$ and Q, where F_i is the face of maximal vectors in the polymatroid associated with f_i ($i = 1, \ldots, k$), and Q is the polymatroid associated with the submodular set function g on $T \setminus S_k$ given by $g(U) := f(U \cup S_k) - f(S_k)$ for $U \subseteq T \setminus S_k$. Since $\dim(F_i) \leq |S_i \setminus S_{i-1}| - 1$ and $\dim(Q) \leq |T \setminus S_k|$, this yields that $\dim(F) = |T| - k$ if and only if $\dim(F_i) = |S_i \setminus S_{i-1}| - 1$ ($i = 1, \ldots, k$) and $\dim(Q) = |T \setminus S_k|$. From this, characterization (44.55) can be derived again. It also yields that if F, represented by (44.53), has dimension $|T| - k$, then the unordered partition $\{S_1, S_2 \setminus S_1, \ldots, S_k \setminus S_{k-1}, T \setminus S_k\}$ is the same for all maximal chains $S_1 \subset \cdots \subset S_k$.

For a characterization of the faces of a polymatroid, see Fujishige [1984a].

44.6d. Characterization of polymatroids

Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$. The following three observations are easily derived from the representation (44.49) of vertices of P. (a) If x_0 is a vertex of P, there exists a vertex x_1 of P such that $x_1 \geq x_0$ and x_1 has the form (44.49) with $k = n$. (b) A vertex x_1 of P can be represented as (44.49) with $k = n$ if and only if $x_1(S) = f(S)$. (c) The convex hull of the vertices x_1 of P with $x_1(S) = f(S)$ is the face $\{x \in P \mid x(S) = f(S)\}$ of P. It follows directly from (a), (b) and (c) that $x \in P$ is a maximal element of P (with respect to $\leq$) if and only if $x(S) = f(S)$. So for each vector y in P there is a vector x in P with $y \leq x$ and $x(S) = f(S)$.

Applying this to the subpolymatroids $P|a = P \cap \{x \mid x \leq a\}$ (cf. Section 44.1), one finds the following property of polymatroids:

(44.56) for each $a \in \mathbb{R}^S_+$ there exists a number $r(a)$ such that each maximal vector x of $P \cap \{x \mid x \leq a\}$ satisfies $x(S) = r(a)$.

Here *maximal* is maximal in the partial order $\leq$ on vectors. The number $r(a)$ is called the *rank* of a, and any x with the properties mentioned in (44.56) is called a *base* of a.

Edmonds [1970b] (cf. Dunstan [1973], Woodall [1974b]) noticed the following (we follow the proof of Welsh [1976]):

Theorem 44.5. *Let $P \subseteq \mathbb{R}^S_+$. Then P is a polymatroid if and only if P is compact, and satisfies* (44.56) *and*

(44.57) *if $\mathbf{0} \leq y \leq x \in P$, then $y \in P$.*

Proof. Necessity was observed above. To see sufficiency, let f be the set function on S defined by

(44.58) $$f(U) := \max\{x(U) \mid x \in P\}$$

for $U \subseteq S$. Then f is nonnegative and nondecreasing. Moreover, f is submodular. To see this, consider $T, U \subseteq S$. Let x be a maximal vector in P satisfying $x_s = 0$ if $s \notin T \cup U$, and let y be a maximal vector in P satisfying $y(s) = 0$ if $s \notin T \cap U$ and $x \leq y$. Note that (44.56) and (44.58) imply that $x(T \cap U) = f(T \cap U)$ and $y(T \cup U) = f(T \cup U)$. Hence

(44.59) $$f(T)+f(U) \geq y(T)+y(U) = y(T\cap U)+y(T\cup U) \geq x(T\cap U)+y(T\cup U) = f(T \cap U) + f(T \cup U),$$

that is, f is submodular.

We finally show that P is equal to the polymatroid P_f associated to f. Clearly, $P \subseteq P_f$, since if $x \in P$ then $x(U) \leq f(U)$ for each $U \subseteq S$, by definition (44.58) of f.

To see that $P_f = P$, suppose $v \in P_f \setminus P$. Let u be a base of v (that is, a maximal vector $u \in P$ satisfying $u \leq v$). Choose u such that the set

(44.60) $$U := \{s \in S \mid u_s < v_s\}$$

is as large as possible. Since $v \notin P$, we have $u \neq v$, and hence $U \neq \emptyset$. As $v \in P_f$, we know

(44.61) $u(U) < v(U) \leq f(U)$.

Define

(44.62) $w := \frac{1}{2}(u + v)$.

So $u \leq w \leq v$. Hence u is a base of w, and each base of w is a base of v.

For any $z \in \mathbb{R}^S$, define z' as the projection of z on the subspace $L := \{x \in \mathbb{R}^S \mid x_s = 0 \text{ if } s \in S \setminus U\}$. That is:

(44.63) $z'(s) := z(s)$ if $s \in U$, and $z'(s) := 0$ if $s \in S \setminus U$.

By definition of f, there is an $x \in P$ with $x(U) = f(U)$. We may assume that $x \in L$. Choose $y \in L$ with $x \leq y$ and $u' \leq y$. Then

(44.64) $x(S) = x(U) = f(U) > u(U) = u'(U) = u'(S)$.

So $r(y) > u'(S)$. Hence, by (44.56), there exists a base z of y with $u' \leq z$ and $z(S) > u'(S)$. So $u'_s < z_s$ for at least one $s \in U$. This implies, since $u'_s < w'_s$ for each $s \in S$, that there is an $a \in P$ with $u' \leq a \leq w'$ and $a \neq u'$, hence $a(U) > u'(U)$.

Since $a \leq w' \leq w$, there is a base b of w with $a \leq b$. Then $b(S) = u(S)$ (since also u is a base of w) and $b(U) \geq a(U) > u'(U) = u(U)$. Hence $b_s < u_s = v_s$ for some $s \in S \setminus U$. Moreover, $b_s \leq w_s < v_s$ for each $s \in U$. So U is properly contained in $\{s \in S \mid b_s < v_s\}$, contradicting the maximality of U. ∎

(For an alternative characterization, see Welsh [1976].)

By (44.8) and (44.9) the rank of a is given by

(44.65) $r(a) = \min_{U \subseteq S}(a(S \setminus U) + f(U))$

(from this one may derive a 'submodular law' for r: $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$, where $\wedge$ and $\vee$ are the meet and join in the lattice $(\mathbb{R}^S, \leq)$ (Edmonds [1970b])).

Since if P has integer vertices and a is integer, the intersection $P|a = \{x \in P \mid x \leq a\}$ is integer again, we know that for integer polymatroids (44.56) also holds if we restrict a and x to integer vectors. So if a is integer, then there exists an integer vector $x \leq a$ in P with $x(S) = r(a)$.

Theorem 44.5 yields an analogous characterization of extended polymatroids. Let f be a submodular set function on S with $f(\emptyset) = 0$. Choose $c \in \mathbb{R}^S_+$ such that

(44.66) $g(U) := f(U) + c(U)$

is nonnegative for all $U \subseteq S$. Clearly, g again is submodular, and $g(\emptyset) = 0$. Then the extended polymatroid EP_f associated with f and the polymatroid P_g associated with g are related by:

(44.67) $P_g = \{x \mid x \geq \mathbf{0}, x - c \in EP_f\} = (c + EP_f) \cap \mathbb{R}^S_+$.

Since P_g is a polymatroid, by (44.56) we know that EP_f satisfies:

(44.68) for each a in $\mathbb{R}^S$ there exists a number $r(a)$ such that each maximal vector x in $EP_f \cap \{x \in \mathbb{R}^S \mid x \leq a\}$ satisfies $x(S) = r(a)$.

One easily derives from Theorem 44.5 that (44.68) together with

(44.69) if $y \leq x \in EP_f$, then $y \in EP_f$,

characterizes the class of all extended polymatroids among the closed subsets of $\mathbb{R}^S$.

44.6e. Operations on submodular functions and polymatroids

The class of submodular set functions on a given set is closed under certain operations. Obviously, the sum of two submodular functions is submodular again. In particular, adding a constant t to all values of a submodular function maintains submodularity. Also the multiplication of a submodular function by a nonnegative scalar maintains submodularity. Moreover, if f is a nondecreasing submodular set function on S, and q is a real number, then the function f' given by $f'(U) := \min\{q, f(U)\}$ for $U \subseteq S$, is submodular again. (Monotonicity cannot be deleted, as is shown by taking $S := \{a, b\}$, $f(\emptyset) = f(S) = 1$, $f(\{a\}) = 0$, $f(\{b\}) = 2$, and $q = 1$.)

It follows that the class of all submodular set functions on S forms a convex cone C in $\mathbb{R}^{\mathcal{P}(S)}$. This cone is polyhedral as the constraints (44.1) form a finite set of linear inequalities defining C. Edmonds [1970b] raised the problem of determining the extreme rays of the cone of all nonnegative nondecreasing submodular set functions f on S with $f(\emptyset) = 0$. It is not difficult to show that the rank function r of a matroid M determines an extreme ray of this cone if and only if r is not the sum of the rank functions of two other matroids, i.e., if and only if M is the sum of a connected matroid and a number of loops. But these do not represent all extreme rays: if $S = \{1, \ldots, 5\}$ and $w(1) = 2, w(s) = 1$ for $s \in S \setminus \{1\}$, let $f(U) := \min\{3, w(U)\}$ for $U \subseteq S$; then f is on an extreme ray, but cannot be decomposed as the sum of rank functions of matroids (L. Lovász's example; cf. also Murty and Simon [1978] and Nguyen [1978]).

Lovász [1983c] observed that if f_1 and f_2 are submodular and $f_1 - f_2$ is nondecreasing, then $\min\{f_1, f_2\}$ is submodular.

Let f be a nonnegative submodular set function on S. Clearly, for any $\lambda \geq 0$ we have $P_{\lambda f} = \lambda P_f$ (where $\lambda P_f = \{\lambda x \mid x \in P_f\}$). If $q \geq 0$, and f' is given by $f'(U) = \min\{q, f(U)\}$ for $U \subseteq S$, then f' is submodular and

(44.70) $$P_{f'} = \{x \in P_f \mid x(S) \leq q\},$$

as can be checked easily. So the class of polymatroids is closed under intersections with affine halfspaces of the form $\{x \in \mathbb{R}^S \mid x(S) \leq q\}$, for $q \geq 0$.

Let f_1 and f_2 be nondecreasing submodular set functions on S, with $f_1(\emptyset) = f_2(\emptyset) = 0$, and associated polymatroids P_1 and P_2 respectively. Let P be the polymatroid associated with $f := f_1 + f_2$. Then (McDiarmid [1975c]):

Theorem 44.6. $P_{f_1+f_2} = P_{f_1} + P_{f_2}$.

Proof. It is easy to see that $P_{f_1+f_2} \supseteq P_{f_1} + P_{f_2}$. To prove the reverse inclusion, let x be a vertex of $P_{f_1+f_2}$. Then x has the form (44.49). Hence, by taking the same permutation π and the same $k, x = x_1 + x_2$ for certain vertices x_1 of P_{f_1} and x_2 of P_{f_2}. Since $P_{f_1} + P_{f_2}$ is convex it follows that $P_{f_1+f_2} = P_{f_1} + P_{f_2}$. ∎

In fact, if f_1 and f_2 are integer, each *integer* vector in $P_{f_1} + P_{f_2}$ is the sum of *integer* vectors in P_{f_1} and P_{f_2} — see Corollary 46.2c. Similarly, if f_1 and f_2 are integer, each integer vector in $EP_{f_1} + EP_{f_2}$ is the sum of integer vectors in EP_{f_1} and EP_{f_2}.

Faigle [1984a] derived from Theorem 44.6 that, for any submodular function f, if $x, y \in P_f$ and $x = x_1 + x_2$ with $x_1, x_2 \in P_f$, then there exist $y_1, y_2 \in P_f$ with

$y = y_1 + y_2$ and $x_1 + y_1, x_2 + y_2 \in P_f$. (Proof: $y \in P_f \subseteq P_{2f-x} = P_{f-x_1} + P_{f-x_2}$.) An integer version of this can be derived from Corollary 46.2c and generalizes (42.13).

If $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, with rank functions r_1 and r_2 and corresponding independent set polytopes P_1 and P_2, respectively, then by Section 44.6c above, $P_1 + P_2$ is the convex hull of sums of incidence vectors of independent sets in M_1 and M_2. Hence the 0,1 vectors in $P_1 + P_2$ are just the incidence vectors of the sets $I_1 \cup I_2$, for $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Therefore, the polyhedron

(44.71) $$(P_1 + P_2)|\mathbf{1} = \{x \in P_1 + P_2 \mid x \leq \mathbf{1}\}$$

is the convex hull of the independent sets of $M_1 \vee M_2$. By Theorem 44.6 and (44.45), it follows that the rank function r of $M_1 \vee M_2$ satisfies

(44.72) $$r(U) = \min_{T \subseteq U} (|U \setminus T| + r_1(T) + r_2(T))$$

for $U \subseteq S$. Thus we have derived the matroid union theorem (Corollary 42.1a).

44.6f. Duals of polymatroids

McDiarmid [1975c] described the following duality of polymatroids. Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$ and let a be a vector in $\mathbb{R}^S$ with $a \geq x$ for all x in P (i.e., $a(s) \geq f(\{s\})$ for all s in S). Define

(44.73) $$f^*(U) := a(U) + f(S \setminus U) - f(S)$$

for $U \subseteq S$. One easily checks that f^* again is nondecreasing and submodular, and that $f^*(\emptyset) = 0$. We call f^* the *dual* of f (with respect to a). Then $f^{**} = f$ taking the second dual with respect to the same a, as follows immediately from (44.73).

Let P^* be the polymatroid associated with f^*, and call P^* the *dual* polymatroid of P (with respect to a). Now the maximal vertices of P and P^* are given by (44.49) by choosing $k = n$. It follows that x is a maximal vertex of P if and only if $a - x$ is a maximal vertex of P^*. Since the maximal vectors of a polymatroid form just the convex hull of the maximal vertices, we may replace in the previous sentence the word 'vertex' by 'vector'. So the set of maximal vectors of P^* arises from the set of maximal vectors of P by reflection in the point $\frac{1}{2}a$.

Clearly, duals of matroids correspond in the obvious way to duals of the related polymatroids (with respect to the vector $\mathbf{1}$).

44.6g. Induction of polymatroids

Let $G = (V, E)$ be a bipartite graph, with colour classes S and T. Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$, and define

(44.74) $$g(U) := f(N(U))$$

for $U \subseteq T$ (cf. Section 44.1a). (As usual, $N(U)$ denotes the set of vertices not in U adjacent to at least one vertex in U.)

The function g again is nondecreasing and submodular. Similarly to Rado's theorem (Corollary 41.1c), one may prove that a vector x belongs to P_g if and only if there exist $y \in \mathbb{R}^E_+$ and $z \in P_f$ such that

(44.75) $$y(\delta(t)) = x_t \quad (t \in T),$$
$$y(\delta(s)) = z_s \quad (s \in S).$$

Moreover, if f and g are integer, we can take y and z to be integer. This procedure gives an 'induction' of polymatroids through bipartite graphs, and yields 'Rado's theorem for polymatroids' (cf. McDiarmid [1975c]).

In case f is the rank function of a matroid on S, a 0,1 vector x belongs to P_g if and only if there exists a matching in G whose end vertices in S form an independent set of the matroid, and the end vertices in T have x as incidence vector. So these 0,1 vectors determine a matroid on T, with rank function r given by

(44.76) $$r(U) = \min_{W \subseteq U} (|U \setminus W| + f(N(W)))$$

for $U \subseteq T$ (cf. (44.45) and (44.74)).

Another extension is the following. Let $D = (V, A)$ be a directed graph and let V be partitioned into classes S and T. Let furthermore a 'capacity' function $c : A \to \mathbb{R}_+$ be given. Define the set function g on T by

(44.77) $$g(U) := c(\delta^{\text{out}}(U))$$

for $U \subseteq T$, where $\delta^{\text{out}}(U)$ denotes the set of arcs leaving U. Then g is nonnegative and submodular, and it may be derived straightforwardly from the max-flow min-cut theorem (Theorem 10.3) that a vector x in $\mathbb{R}^T_+$ belongs to P_g if and only if there exist $T - S$ paths $Q_1, \ldots, Q_k$ and nonnegative numbers $\lambda_1, \ldots, \lambda_k$ (for some k), such that

(44.78) $$\sum_{i=1}^{k} \lambda_i \chi^{AQ_i} \leq c \text{ and } \sum_{i=1}^{k} \lambda_i \chi^{b(Q_i)} = x,$$

where $b(Q_i)$ is the beginning vertex of Q_i. If the c and x are integer, we can take also the λ_i integer.

Here the function g in general is not nondecreasing, but the value

(44.79) $$\bar{g}(U) = \min\{g(W) \mid U \subseteq W \subseteq T\}$$

of the associated nondecreasing submodular function (cf. (44.35)) is equal to the minimum capacity of a cut separating U and S, which is equal to the maximum amount of flow from U to S, subject to the capacity function c (by the max-flow min-cut theorem).

In an analogous way, one can construct polymatroids by taking vertex-capacities instead of arc-capacities. Moreover, the notion of induction of polymatroids through bipartite graphs can be extended in a natural way to the induction of polymatroids through directed graphs (cf. McDiarmid [1975c], Schrijver [1978]).

44.6h. Lovász's generalization of Kőnig's matching theorem

Lovász [1970a] gave the following generalization of Kőnig's matching theorem (Theorem 16.2).

For a graph $G = (V, E)$, $U \subseteq V$, and $F \subseteq E$, let $N_F(U)$ denote the set of vertices not in U that are adjacent in (V, F) to at least one vertex in U. Kőnig's matching theorem follows by taking $g(X) := |X|$ in the following theorem.

Theorem 44.7. *Let $G = (V, E)$ be a simple bipartite graph, with colour classes S and T. Let g be a supermodular set function on S, such that $g(\{v\}) \geq 0$ for each $v \in S$ and such that*

(44.80) $$g(U \cup \{v\}) \leq g(U) + g(\{v\}) \text{ for nonempty } U \subseteq S \text{ and } v \in S \setminus U.$$

Then E has a subset F with $\deg_F(v) = g(\{v\})$ for each $v \in V$ and $|N_F(U)| \geq g(U)$ for each nonempty $U \subseteq S$ if and only if $|N_E(U)| \geq g(U)$ for each nonempty $U \subseteq S$.

Proof. Necessity being trivial, we show sufficiency. Choose $F \subseteq E$ such that

(44.81) $$|N_F(U)| \geq g(U)$$

for each nonempty $U \subseteq S$, with $|F|$ as small as possible. We show that F is as required.

Suppose to the contrary that $\deg_F(v) > g(\{v\})$ for some $v \in S$. By the minimality of F, for each edge $e = vw \in F$, there is a subset U_e of S with $v \in U_e$, $|N_F(U_e)| = g(U_e)$, and $w \notin N_F(U_e \setminus \{v\})$. Since the function $|N_F(U)|$ is submodular, the intersection U of the U_e over $e \in \delta(v)$ satisfies $|N_F(U)| = g(U)$ (using (44.81)). Then no neighbour w of v is adjacent to U. Hence $N_F(v)$ and $N_F(U \setminus \{v\})$ are disjoint. Moreover, $U \neq \{v\}$, since $N_F(U) = g(U)$ and $N_F(\{v\}) > g(v)$. This gives the contradiction

(44.82) $$g(U) \leq g(U \setminus \{v\}) + g(\{v\}) < |N_F(U \setminus \{v\})| + |N_F(v)| = |N_F(U)|. \quad \blacksquare$$

For a derivation of this theorem with the Edmonds-Giles method, see Frank and Tardos [1989].

44.6i. Further notes

Edmonds [1970b] and D.A. Higgs (as mentioned in Edmonds [1970b]) observed that if f is a set function on a set S, we can define recursively a submodular function $\bar{f}$ as follows:

(44.83) $$\bar{f}(T) := \min\{f(T), \min(\bar{f}(S_1) + \bar{f}(S_2) - \bar{f}(S_1 \cap S_2))\},$$

where the second minimum ranges over all pairs S_1, S_2 of proper subsets of T with $S_1 \cup S_2 = T$.

Lovász [1983c] gave the following characterization of submodularity in terms of convexity. Let f be a set function on S and define for each $c \in \mathbb{R}^S_+$

(44.84) $$\hat{f}(c) := \sum_{i=1}^{k} \lambda_i f(U_i),$$

where $\emptyset \neq U_1 \subset U_2 \subset \cdots \subset U_k \subseteq S$ and $\lambda_1, \ldots, \lambda_k > 0$ are such that $c = \sum_{i=1}^k \lambda_i \chi^{U_i}$. Then f is submodular if and only if $\hat{f}$ is convex. Similarly, f is supermodular if and only if $\hat{f}$ is concave. Related is the 'subdifferential' of a submodular function, investigated by Fujishige [1984d].

Korte and Lovász [1985c] and Nakamura [1988a] studied polyhedral structures where the greedy algorithm applies. Federgruen and Groenevelt [1986] extended the greedy method for polymatroids to 'weakly concave' objective functions (instead of linear functions). (Related work was reported by Bhattacharya, Georgiadis, and

Tsoucas [1992].) Nakamura [1993] extended polymatroids and submodular functions to $\varDelta$-polymatroids and $\varDelta$-submodular functions.

Gröflin and Liebling [1981] studied the following example of 'transversal polymatroids'. Let $G = (V, E)$ be an undirected graph, and define the submodular set function f on E by $f(F) := \left|\bigcup F\right|$ for $F \subseteq E$. Then the vertices of the associated polymatroid are all $\{0, 1, 2\}$ vectors x in $\mathbb{R}^E$ with the property that the set $F := \{e \in E \mid x_e \geq 1\}$ forms a forest each component of which contains at most one edge e with $x_e = 2$. If x is a maximal vertex, then each component contains exactly one edge e with $x_e = 2$.

Narayanan [1991] studied, for a given submodular function f on S, the lattice of all partitions $\mathcal{P}$ of S into nonempty sets such that there exists a $\lambda \in \mathbb{R}$ for which $\mathcal{P}$ attains $\min \sum_{U \in \mathcal{P}} (f(U) - \lambda)$ (taken over all partitions $\mathcal{P}$). Fujishige [1980b] studied minimum values of submodular functions.

For results on the (NP-hard) problems of *maximizing* a submodular function and of submodular set cover, see Fisher, Nemhauser, and Wolsey [1978], Nemhauser and Wolsey [1978,1981], Nemhauser, Wolsey, and Fisher [1978], Wolsey [1982a,1982b], Conforti and Cornuéjols [1984], and Fujito [1999].

Cunningham [1983], Fujishige [1983], and Nakamura [1988c] presented decomposition theories for submodular functions. Benczúr and Frank [1999] considered covering symmetric supermodular functions by graphs.

For surveys and books on polymatroids and submodular functions, see McDiarmid [1975c], Welsh [1976], Lovász [1983c], Lawler [1985], Nemhauser and Wolsey [1988], Fujishige [1991], Narayanan [1997], and Murota [2002]. For a survey on applications of submodular functions, see Frank [1993a].

Historically, submodular functions arose in lattice theory (Bergmann [1929], Birkhoff [1933]), while submodularity of the rank function of a matroid was shown by Bergmann [1929] and Whitney [1935]. Choquet [1951,1955] and Kelley [1959] studied submodular functions in relation to the Newton capacity and to measures in Boolean algebras. The relevance of submodularity for optimization was revealed by Edmonds [1970b].

Several alternative names have been proposed for submodular functions, like sub-valuation (Choquet [1955]), β-function (Edmonds [1970b]), and ground set rank function (McDiarmid [1975c]). The set of integer vectors in an integer polymatroid was called a hypermatroid by Helgason [1974] and Lovász [1977c]. A generalization of polymatroids (called supermatroids) was studied by Dunstan, Ingleton, and Welsh [1972].

Chapter 45

Submodular function minimization

This chapter describes a strongly polynomial-time algorithm to find the minimum value of a submodular function. It suffices that the submodular function is given by a value giving oracle.
One application of submodular function minimization is optimizing over the intersection of two polymatroids. This will be discussed in Chapter 47.

45.1. Submodular function minimization

It was shown by Grötschel, Lovász, and Schrijver [1981] that the minimum value of a rational-valued submodular set function f on S can be found in polynomial time, if f is given by a value giving oracle and an upper bound B is given on the numerators and denominators of the values of f. The running time is bounded by a polynomial in $|S|$ and $\log B$. This algorithm is based on the ellipsoid method: we can assume that $f(\emptyset) = 0$ (by resetting $f(U) := f(U) - f(\emptyset)$ for all $U \subseteq S$); then with the greedy algorithm, we can optimize over EP_f in polynomial time (Corollary 44.3b), hence the separation problem for EP_f is solvable in polynomial time, hence also the separation problem for

(45.1) $\qquad P := EP_f \cap \{x \mid x \leq \mathbf{0}\},$

and therefore also the optimization problem for P. Now the maximum value of $x(S)$ over P is equal to the minimum value of f (by (44.8), (44.9), and (44.34)).

Having a polynomial-time method to find the minimum value of a submodular function, we can turn it into a polynomial-time method to find a subset T of S minimizing $f(T)$: For each $s \in S$, we can determine if the minimum value of f over all subsets of S is equal to the minimum value of f over subsets of $S \setminus \{s\}$. If so, we reset $S := S \setminus \{s\}$. Doing this for all elements of S, we are left with a set T minimizing f over all subsets of (the original) S.

Grötschel, Lovász, and Schrijver [1988] showed that this algorithm can be turned into a strongly polynomial-time method. Cunningham [1985b] gave a combinatorial, pseudo-polynomial-time algorithm for minimizing a submodular function f (polynomial in the size of the underlying set and the maximum absolute value of f (assuming f to be integer)). Inspired by Cunningham's method, combinatorial strongly polynomial-time algorithms were found by Iwata, Fleischer, and Fujishige [2000,2001] and Schrijver [2000a]. We will describe the latter algorithm.

45.2. Orders and base vectors

Let f be a submodular set function on a set S. In finding the minimum value of f, we can assume $f(\emptyset) = 0$, as resetting $f(U) := f(U) - f(\emptyset)$ for all $U \subseteq S$ does not change the problem. So throughout we assume that $f(\emptyset) = 0$.

Moreover, we assume that f is given by a *value giving oracle*, that is, an oracle that returns $f(U)$ for any given subset U of S. We also assume that the numbers returned by the oracle are rational (or belong to any ordered field in which we can perform the elementary arithmetic operations algorithmically).

Recall that the *base polytope* B_f of f is defined as the set of base vectors of f:

(45.2) $$B_f := \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for all } U \subseteq S, x(S) = f(S)\}.$$

Consider any total order $\prec$ on S.[35] For any $v \in S$, denote

(45.3) $$v_\prec := \{u \in S \mid u \prec v\}.$$

Define a vector $b^\prec$ in $\mathbb{R}^S$ by:

(45.4) $$b^\prec(v) := f(v_\prec \cup \{v\}) - f(v_\prec)$$

for $v \in S$. Theorem 44.3 implies that $b^\prec$ belongs to B_f.

Note that $b^\prec(U) = f(U)$ for each lower ideal U of $\prec$ (where a *lower ideal* of $\prec$ is a subset U of S such that if $v \in U$ and $u \prec v$, then $u \in U$).

45.3. A subroutine

In this section we describe a subroutine that is important in the algorithm. It replaces a total order $\prec$ by other total orders, thereby reducing some interval $(s,t]_\prec$, where

(45.5) $$(s,t]_\prec := \{v \mid s \prec v \preceq t\}$$

for $s, t \in S$.

Let $\prec$ be a total order on S. For any $s, u \in S$ with $s \prec u$, let $\prec^{s,u}$ be the total order on S obtained from $\prec$ by resetting $v \prec u$ to $u \prec v$ for each

[35] As usual, we use $\prec$ for strict inequality and $\preceq$ for nonstrict inequality. We refer to the order by the strict inequality sign $\prec$.

v satisfying $s \preceq v \prec u$. Thus in the ordering, we move u to the position just before s. Hence $(s,t]_{\prec^{s,u}} = (s,t]_\prec \setminus \{u\}$ if $u \in (s,t]_\prec$.

We show that there is a strongly polynomial-time subroutine that

(45.6) for any $s,t \in S$ with $s \prec t$, finds a $\delta \geq 0$ and describes $b^\prec + \delta(\chi^t - \chi^s)$ as a convex combination of the $b^{\prec^{s,u}}$ for $u \in (s,t]_\prec$.

To describe the subroutine, we can assume that $b^\prec = \mathbf{0}$, by replacing (temporarily) $f(U)$ by $f(U) - b^\prec(U)$ for each $U \subseteq S$.

We investigate the signs of the vector $b^{\prec^{s,u}}$. We show that for each $v \in S$:

(45.7)
$$\begin{aligned} &b^{\prec^{s,u}}(v) \leq 0 \text{ if } s \preceq v \prec u,\\ &b^{\prec^{s,u}}(v) \geq 0 \text{ if } v = u,\\ &b^{\prec^{s,u}}(v) = 0 \text{ otherwise.} \end{aligned}$$

To prove this, observe that if $T \subseteq U \subseteq S$, then for any $v \in S \setminus U$ we have by the submodularity of f:

(45.8)
$$f(U \cup \{v\}) - f(U) \leq f(T \cup \{v\}) - f(T).$$

To see (45.7), if $s \preceq v \prec u$, then by (45.8),

(45.9)
$$\begin{aligned} b^{\prec^{s,u}}(v) &= f(v_{\prec^{s,u}} \cup \{v\}) - f(v_{\prec^{s,u}}) \leq f(v_\prec \cup \{v\}) - f(v_\prec)\\ &= b^\prec(v) = 0, \end{aligned}$$

since $v_{\prec^{s,u}} = v_\prec \cup \{u\} \supset v_\prec$.

Similarly,

(45.10)
$$\begin{aligned} b^{\prec^{s,u}}(u) &= f(u_{\prec^{s,u}} \cup \{u\}) - f(u_{\prec^{s,u}}) \geq f(u_\prec \cup \{u\}) - f(u_\prec)\\ &= b^\prec(u) = 0, \end{aligned}$$

since $u_{\prec^{s,u}} = s_\prec \subset u_\prec$.

Finally, if $v \prec s$ or $u \prec v$, then $v_{\prec^{s,u}} = v_\prec$, and hence $b^{\prec^{s,u}}(v) = b^\prec(v) = 0$. This shows (45.7).

By (45.7), the matrix $M = (b^{\prec^{s,u}}(v))_{u,v}$ with rows indexed by $u \in (s,t]_\prec$ and columns indexed by $v \in S$, in the order given by $\prec$, has the following, partially triangular, shape, where $+$ means that the entry is ≥ 0, and $-$ that the entry is ≤ 0:

$$\begin{array}{c|cccccccccccccc}
 & & & & s & & & & & & & t & & & \\
\hline
 & 0 & \cdots & 0 & - & + & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0\\
 & \vdots & & \vdots & - & - & + & \ddots & & & \vdots & \vdots & \vdots & & \vdots\\
 & \vdots & & \vdots & - & - & - & \ddots & \ddots & & \vdots & \vdots & \vdots & & \vdots\\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots\\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0 & 0 & \vdots & & \vdots\\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \ddots & + & 0 & \vdots & & \vdots\\
t & 0 & \cdots & 0 & - & - & - & \cdots & \cdots & \cdots & - & + & 0 & \cdots & 0
\end{array}$$

As each row of M represents a vector $b^{\prec^{s,u}}$, to obtain (45.6) we must describe $\delta(\chi^t - \chi^s)$ as a convex combination of the rows of M, for some $\delta \geq 0$.

We call the $+$ entries in the matrix the 'diagonal' elements. Now for each row of M, the sum of its entries is 0, as $b^{\prec^{s,u}}(S) = f(S) = b^{\prec}(S) = 0$. Hence, if a 'diagonal' element $b^{\prec^{s,u}}(u)$ is equal to 0 for some $u \in (s,t]_{\prec}$, then the corresponding row of M is all-zero. So in this case we can take $\delta = 0$ in (45.6).

If $b^{\prec^{s,u}}(u) > 0$ for each $u \in (s,t]_{\prec}$ (that is, if each 'diagonal' element is strictly positive), then $\chi^t - \chi^s$ can be described as a nonnegative combination of the rows of M (by the sign pattern of M and since the entries in each row of M add up to 0). Hence $\delta(\chi^t - \chi^s)$ is a convex combination of the rows of M for some $\delta > 0$, yielding again (45.6).

45.4. Minimizing a submodular function

We now describe the algorithm to find the minimum value of a submodular set function f on S. We assume $f(\emptyset) = 0$ and $S = \{1, \ldots, n\}$.

We iteratively update a vector $x \in B_f$, given as a convex combination

(45.11) $\qquad x = \lambda_1 b^{\prec_1} + \cdots + \lambda_k b^{\prec_k},$

where the $\prec_i$ are total orders of S, and where the λ_i are positive and sum to 1. Initially, we choose an arbitrary total order $\prec$ and set $x := b^{\prec}$ (so $k = 1$ and $\prec_1 = \prec$).

We describe the iteration. Consider the directed graph $D = (S, A)$, with

(45.12) $\qquad A := \{(u, v) \mid \exists i = 1, \ldots, k : u \prec_i v\}.$

Define

(45.13) $\qquad P := \{v \in S \mid x(v) > 0\}$ and $N := \{v \in S \mid x(v) < 0\}.$

Case 1: D has no path from P to N. Then let U be the set of vertices of D that can reach N by a directed path. So $N \subseteq U$ and $U \cap P = \emptyset$; that is, U contains all negative components of x and no positive components. Hence $x(W) \geq x(U)$ for each $W \subseteq S$. As no arcs of D enter U, U is a lower ideal of $\prec_i$, and hence $b^{\prec_i}(U) = f(U)$, for each $i = 1, \ldots, k$. Therefore, for each $W \subseteq S$:

(45.14) $$f(U) = \sum_{i=1}^{k} \lambda_i b^{\prec_i}(U) = x(U) \leq x(W) \leq f(W).$$

So U minimizes f.

Case 2: D has a path from P to N. Let $d(v)$ denote the distance in D from P to v (= minimum number of arcs in a directed path from P to v). Set $d(v) := \infty$ if v is not reachable from P. Choose $s, t \in S$ as follows.

Let t be the element in N reachable from P with $d(t)$ maximum, such that t is largest. Let s be the element with $(s,t) \in A$, $d(s) = d(t) - 1$, and s largest. Let α be the maximum of $|(s,t]_{\prec_i}|$ over $i = 1, \ldots, k$. Reorder indices such that $|(s,t]_{\prec_1}| = \alpha$.

By (45.6), we can find $\delta \geq 0$ and describe

(45.15) $\qquad b^{\prec_1} + \delta(\chi^t - \chi^s)$

as a convex combination of the $b^{\prec_1^{s,u}}$ for $u \in (s,t]_{\prec_1}$. Then with (45.11) we obtain

(45.16) $\qquad y := x + \lambda_1 \delta(\chi^t - \chi^s)$

as a convex combination of $b^{\prec_i}$ $(i = 2, \ldots, k)$ and $b^{\prec_1^{s,u}}$ $(u \in (s,t]_{\prec_1})$.

Let x' be the point on the line segment $\overline{xy}$ closest to y satisfying $x'(t) \leq 0$. (So $x'(t) = 0$ or $x' = y$.) We can describe x' as a convex combination of $b^{\prec_i}$ $(i = 1, \ldots, k)$ and $b^{\prec_1^{s,u}}$ $(u \in (s,t]_{\prec_1})$. Moreover, if $x'(t) < 0$, then we can do without $b^{\prec_1}$.

We reduce the number of terms in the convex decomposition of x' to at most $|S|$ by linear algebra: any affine dependence of the vectors in the decomposition yields a reduction of the number of terms in the decomposition, as in the standard proof of Carathéodory's theorem (subtract an appropriate multiple of the linear expression giving the affine dependence, from the linear expression giving the convex combination, such that all coefficients remain nonnegative, and at least one becomes 0). As all $b^{\prec}$ belong to a hyperplane, this reduces the number of terms to at most $|S|$.

Then reset $x := x'$ and iterate. This finishes the description of the algorithm.

45.5. Running time of the algorithm

We show that the number of iterations is at most $|S|^6$. Consider any iteration. Let

(45.17) $\qquad \beta :=$ number of $i \in \{1, \ldots, k\}$ with $|(s,t]_{\prec_i}| = \alpha$.

Let x', d', A', P', N', t', s', α', β' be the objects x, d, A, P, N, t, s, α, β in the next iteration (if any). Then

(45.18) $\qquad$ for all $v \in S$, $d'(v) \geq d(v)$,

and

(45.19) $\qquad$ if $d'(v) = d(v)$ for all $v \in S$, then $(d'(t'), t', s', \alpha', \beta')$ is lexicographically less than $(d(t), t, s, \alpha, \beta)$.

Since each of $d(t), t, s, \alpha, \beta$ is at most $|S|$, and since (if $d(v)$ is unchanged for all v) there are at most $|S|$ pairs $(d(t), t)$, (45.19) implies that in at most $|S|^4$

iterations $d(v)$ increases for at least one v. Any fixed v can have at most $|S|$ such increases, and hence the number of iterations is at most $|S|^6$.

In order to prove (45.18) and (45.19), notice that

(45.20) for each arc $(v,w) \in A' \setminus A$ we have $s \preceq_1 w \prec_1 v \preceq_1 t$.

Indeed, as $(v,w) \notin A$ we have $w \prec_1 v$. As $(v,w) \in A'$, we have $v \prec_1^{s,u} w$ for some $u \in (s,t]_{\prec_1}$. Hence the definition of $\prec_1^{s,u}$ gives $v = u$ and $s \preceq_1 w \prec_1 u$. This shows (45.20).

If (45.18) does not hold, then $A' \setminus A$ contains an arc (v,w) with $d(w) \geq d(v) + 2$ (using that $P' \subseteq P$). By (45.20), $s \preceq_1 w \prec_1 v \preceq_1 t$, and so $d(w) \leq d(s) + 1 = d(t) \leq d(v) + 1$, a contradiction. This shows (45.18).

To prove (45.19), assume that $d'(v) = d(v)$ for all $v \in S$. As $x'(t') < 0$, we have $x(t') < 0$ or $t' = s$. So by our criterion for choosing t (maximizing $(d(t), t)$ lexicographically), and since $d(s) < d(t)$, we know that $d(t') \leq d(t)$, and that if $d(t') = d(t)$, then $t' \leq t$.

Next assume that moreover $d(t') = d(t)$ and $t' = t$. As $(s', t) \in A'$, and as (by (45.20)) $A' \setminus A$ contains no arc entering t, we have $(s', t) \in A$, and so $s' \leq s$, by the maximality of s.

Finally assume that moreover $s' = s$. As $(s,t]_{\prec_1^{s,u}}$ is a proper subset of $(s,t]_{\prec_1}$ for each $u \in (s,t]_{\prec_1}$, we know that $\alpha' \leq \alpha$. Moreover, if $\alpha' = \alpha$, then $\beta' < \beta$, since $\prec_1$ does not occur anymore among the linear orders making the convex combination, as $x'(t) < 0$. This proves (45.19).

We therefore have proved:

Theorem 45.1. *Given a submodular function f by a value giving oracle, a set U minimizing $f(U)$ can be found in strongly polynomial time.*

Proof. See above. ∎

This algorithm performs the elementary arithmetic operations on function values, including multiplication and division (in order to solve certain systems of linear equations). One would wish to have a 'fully combinatorial' algorithm, in which the function values are only compared, added, and subtracted. The existence of such an algorithm was shown by Iwata [2002a, 2002c], by extending the algorithm of Iwata, Fleischer, and Fujishige [2000, 2001].

Notes. In the algorithm, we have chosen t and s largest possible, in some fixed order of S. To obtain the above running time bound it only suffices to choose t and s in a consistent way. That is, if the set of choices for t is the same as in the previous iteration, then we should choose the same t — and similarly for s. This roots in the idea of 'consistent breadth-first search' of Schönsleben [1980].

The observation that the number of iterations in the algorithm of Section 45.4 is $O(|S|^6)$ instead of $O(|S|^7)$ is due to L.K. Fleischer. Vygen [2002] showed that the number of iterations can in fact be bounded by $O(|S|^5)$.

The algorithm described above has been speeded up by Fleischer and Iwata [2000,2002], by incorporating a push-relabel type of iteration based on approximate distances instead of precise distances (like Goldberg's method for maximum flow, given in Section 10.7). Iwata [2002b] combined the approaches of Iwata, Fleischer, and Fujishige [2000,2001] and Schrijver [2000a] to obtain a faster algorithm. A descent method for submodular function minimization based on an oracle for membership of the base polytope was given by Fujishige and Iwata [2002].

Surveys and background on submodular function minimization are given by Fleischer [2000b] and McCormick [2001].

45.6. Minimizing a symmetric submodular function

A set function f on S is called *symmetric* if $f(U) = f(S \setminus U)$ for each $U \subseteq S$. The minimum of a symmetric submodular function f is attained by $\emptyset$, since for each $U \subseteq S$ one has

(45.21) $$2f(U) = f(U) + f(S \setminus U) \geq f(\emptyset) + f(S) = 2f(\emptyset).$$

By extending a method of Nagamochi and Ibaraki [1992b] for finding the minimum nonempty cut in an undirected graph, Queyranne [1995,1998] gave an easy combinatorial algorithm to find a nonempty proper subset U of S minimizing $f(U)$, where f is given by a value giving oracle. We may assume that $f(\emptyset) = f(S) = 0$, by resetting $f(U) := f(U) - f(\emptyset)$ for all $U \subseteq S$.

Call an ordering $s_1, \ldots, s_n$ of the elements of S a *legal order* of S for f, if, for each $i = 1, \ldots, n$,

(45.22) $$f(\{s_1, \ldots, s_{i-1}, x\}) - f(\{x\})$$

is minimized over $x \in S \setminus \{s_1, \ldots, s_{i-1}\}$ by $x = s_i$. One easily finds a legal order, by $O(|S|^2)$ oracle calls (for the value of f).

Now the algorithm is (where a set U *splits* a set X if both $X \cap U$ and $X \setminus U$ are nonempty):

(45.23) Find a legal order $(s_1, \ldots, s_n)$ of S for f.
Determine (recursively) a nonempty proper subset T of S not splitting $\{s_{n-1}, s_n\}$, minimizing $f(T)$. (This can be done by identifying s_{n-1} and s_n.)
Then the minimum value of $f(U)$ over nonempty proper subsets U of S is equal to $\min\{f(T), f(\{s_n\})\}$.

The correctness of the algorithm follows from, for $n \geq 2$:

(45.24) $f(U) \geq f(\{s_n\})$ for each $U \subseteq S$ splitting $\{s_{n-1}, s_n\}$.

This can be proved as follows. Define $t_0 := s_1$. For $i = 1, \ldots, n-1$, define $t_i := s_j$, where j is the smallest index such that $j > i$ and such that U splits $\{s_i, s_j\}$. For $i = 0, \ldots, n$, let $U_i := \{s_1, \ldots, s_i\}$. Note that for each $i = 1, \ldots, n-1$ one has

(45.25) $$f(U_{i-1} \cup \{t_i\}) - f(\{t_i\}) \geq f(U_{i-1} \cup \{t_{i-1}\}) - f(\{t_{i-1}\}),$$

since if $t_{i-1} = t_i$ this is trivial, and if $t_{i-1} \neq t_i$, then $t_{i-1} = s_i$, in which case (45.25) follows from the legality of the order.

Moreover, for each $i = 1, \ldots, n-1$ (setting $\overline{U} := S \setminus U$):

(45.26) $$\begin{aligned} & f(U_i \cup U) - f(U_{i-1} \cup U) + f(U_i \cup \overline{U}) - f(U_{i-1} \cup \overline{U}) \\ & \leq f(U_i \cup \{t_i\}) - f(U_{i-1} \cup \{t_i\}). \end{aligned}$$

In proving this, we may assume (by symmetry of U and $\overline{U}$) that $s_i \in \overline{U}$. Then $U_i \cup \overline{U} = U_{i-1} \cup \overline{U}$ and $t_i \in U$. So $f(U_i \cup \{t_i\}) + f(U_{i-1} \cup U) \geq f(U_{i-1} \cup \{t_i\}) + f(U_i \cup U)$, by submodularity. This gives (45.26).

Then we have:

(45.27) $$\begin{aligned} & f(s_n) - 2f(U) \\ & = f(U_{n-1} \cup U) + f(U_{n-1} \cup \overline{U}) - f(U_0 \cup U) - f(U_0 \cup \overline{U}) \\ & = \sum_{i=1}^{n-1} (f(U_i \cup U) - f(U_{i-1} \cup U) + f(U_i \cup \overline{U}) - f(U_{i-1} \cup \overline{U})) \\ & \leq \sum_{i=1}^{n-1} (f(U_i \cup \{t_i\}) - f(U_{i-1} \cup \{t_i\})) \\ & \leq \sum_{i=1}^{n-1} (f(U_i \cup \{t_i\}) - f(U_{i-1} \cup \{t_{i-1}\}) + f(\{t_{i-1}\}) - f(\{t_i\})) \\ & = f(U_{n-1} \cup \{t_{n-1}\}) - f(\{t_{n-1}\}) - f(\{t_0\}) + f(\{t_0\}) = -f(s_n) \end{aligned}$$

(where the first inequality follows from (45.26), and the second inequality from (45.25)). This shows (45.24).

Notes. Fujishige [1998] gave an alternative correctness proof. Nagamochi and Ibaraki [1998] extended the algorithm to minimizing submodular functions f satisfying

(45.28) $$f(T) + f(U) \geq f(T \setminus U) + f(U \setminus T)$$

for all $T, U \subseteq S$. Rizzi [2000b] gave an extension.

45.7. Minimizing a submodular function over the odd sets

From the strong polynomial-time solvability of submodular function minimization, one can derive that also a set of odd cardinality minimizing f (over the odd sets) is solvable in strongly polynomial time (Grötschel, Lovász, and Schrijver [1981,1984a,1988] (the second paper corrects a wrong argument given in the first paper)).

Theorem 45.2. *Given a submodular set function f on S (by a value giving oracle) and a nonempty subset T of S, one can find in strongly polynomial time a set $W \subseteq S$ minimizing $f(W)$ over W with $|W \cap T|$ odd.*

Proof. The case T odd can be reduced to the case T even as follows. Find for each $t \in T$ a subset W_t of $S - t$ with $W_t \cap (T - t)$ odd, and minimizing $f(W_t)$. Moreover, find a subset U of S minimizing $f(U)$ over $U \supseteq T$. Then a set that attains the minimum among $f(U)$ and the $f(W_t)$, is an output as required.

So we can assume that T is even. We describe a recursive algorithm. Say that a set U *splits* T if both $T \cap U$ and $T \setminus U$ are nonempty. First find a set U minimizing $f(U)$ over all subsets U of S splitting T. This can be done by finding for all $s, t \in T$ a set $U_{s,t}$ minimizing $f(U_{s,t})$ over all subsets of S containing s and not containing t (this amounts to submodular function minimization), and taking for U a set that minimizes $f(U_{s,t})$ over all such s, t.

If $U \cap T$ is odd, we output $W := U$. If $U \cap T$ is even, then recursively we find a set X minimizing $f(X)$ over all X with $X \cap (T \cap U)$ odd, and not splitting $T \setminus U$. This can be done by shrinking $T \setminus U$ to one element. Also, recursively we can find a set Y minimizing $f(Y)$ over all Y with $Y \cap (T \setminus U)$ odd, and not splitting $T \cap U$. Output an X or Y attaining the minimum of $f(X)$ and $f(Y)$.

This gives a strongly polynomial-time algorithm as the total number of recursive calls is at most $|T| - 2$ (since $2 + (|T \cap U| - 2) + (|T \setminus U| - 2) = |T| - 2$).

To see the correctness, let W minimize $f(W)$ over those W with $|W \cap T|$ odd. Suppose that $f(W) < f(X)$ and $f(W) < f(Y)$. As $f(W) < f(X)$, W splits $T \setminus U$, and hence $W \cup U$ splits T. Similarly, $f(W) < f(Y)$ implies that $W \cap U$ splits T.

Since $W \cap T$ is odd and $U \cap T$ is even, either $(W \cap U) \cap T$ or $(W \cup U) \cap T$ is odd.

If $(W \cap U) \cap T$ is odd, then $f(W \cap U) \geq f(W)$ (as W minimizes $f(W)$ over W with $W \cap T$ odd) and $f(W \cup U) \geq f(U)$ (as $W \cup U$ splits T and as U minimizes $f(U)$ over U splitting T). Hence, by the submodularity of f, $f(W \cap U) = f(W)$. Since $(W \cap U) \cap (T \cap U) = (W \cap U) \cap T$ is odd and since $W \cap U$ does not split $T \setminus U$, we have $f(W) = f(W \cap U) \geq f(X)$, contradicting our assumption.

If $(W \cup U) \cap T$ is odd, a symmetric argument gives a contradiction. ∎

This generalizes the strong polynomial-time solvability of finding a minimum-capacity odd cut in a graph, proved by Padberg and Rao [1982] (Corollary 25.6a). For a further generalization, see Section 49.11a.

Chapter 46

Polymatroid intersection

The intersection of two polymatroids behaves as nice as the intersection of two matroids, as was shown by Edmonds again. We study in this chapter min-max relations, polyhedral characterizations, and total dual integrality results. In the next chapter we go over to the algorithmic questions.

46.1. Box-total dual integrality of polymatroid intersection

We saw in Section 44.2 that the greedy algorithm yields a proof that an integer-valued submodular function gives an integer polymatroid. The interest of polymatroids for combinatorial optimization is enlarged by the fundamental result of Edmonds [1970b] that also the intersection of two integer polymatroids is integer, thus generalizing the matroid intersection theorem. In order to obtain this result, we first show a more general theorem (also due to Edmonds [1970b]).

Consider, for submodular set functions f_1, f_2 on S, the system:

(46.1) $$\begin{aligned} x(U) &\leq f_1(U) \quad \text{for } U \subseteq S, \\ x(U) &\leq f_2(U) \quad \text{for } U \subseteq S. \end{aligned}$$

Then:

Theorem 46.1. *If f_1 and f_2 are submodular, then* (46.1) *is box-TDI.*

Proof. Choose $w : S \to \mathbb{R}$. Let y_1, y_2 attain

(46.2) $$\min\{\sum_{U \subseteq S} (y_1(U)f_1(U) + y_2(U)f_2(U)) \mid y_1, y_2 \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{U \subseteq S} (y_1(U) + y_2(U))\chi^U = w\}.$$

For $i = 1, 2$, define $w_i : S \to \mathbb{R}$ by

(46.3) $$w_i := \sum_{U \subseteq S} y_i(U)\chi^U.$$

Then y_i attains

(46.4) $$\min\{\sum_{U\subseteq S} y_i(U)f_i(U) \mid y_i \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{U\subseteq S} y_i(U)\chi^U = w_i\}.$$

So by Theorem 44.3, we can assume that the collections

(46.5) $$\mathcal{F}_i := \{U \subseteq S \mid y_i(U) > 0\}$$

are chains. Hence, by Theorem 41.11, (46.2) has an optimum solution such that the inequalities with positive coefficients have a totally unimodular constraint matrix. Therefore, by Theorem 5.35, (46.1) is box-TDI. ∎

(This proof method is due to Edmonds [1970b].)

46.2. Consequences

Theorem 46.1 has the following consequences. First, the integrality of the intersection of two polymatroids:

Corollary 46.1a (polymatroid intersection theorem)**.** *The intersection of two integer (extended) polymatroids is box-integer.*

Proof. If P_{f_1} and P_{f_2} are integer polymatroids, f_1 and f_2 can be taken to be integer-valued, by Corollary 44.3g. Hence (46.1) determines a box-integer polyhedron. ∎

Next, a min-max relation:

Corollary 46.1b. *Let f_1 and f_2 be submodular set functions on S with $f_1(\emptyset) = f_2(\emptyset) = 0$. Then*

(46.6) $$\max\{x(U) \mid x \in EP_{f_1} \cap EP_{f_2}\} = \min_{T\subseteq U}(f_1(T) + f_2(U \setminus T))$$

for each $U \subseteq S$.

Proof. This follows by maximizing $w^{\mathsf{T}}x$ over (46.1) for $w := \chi^U$, and applying Theorem 46.1. ∎

Similarly, for (nonextended) polymatroids:

Corollary 46.1c. *Let f_1 and f_2 be nondecreasing submodular set functions on S with $f_1(\emptyset) = f_2(\emptyset) = 0$. Then*

(46.7) $$\max\{x(U) \mid x \in P_{f_1} \cap P_{f_2}\} = \min_{T\subseteq U}(f_1(T) + f_2(U \setminus T))$$

for each $U \subseteq S$.

Proof. As the previous corollary. ∎

Let f_1 and f_2 be submodular set functions on S with $f_1(\emptyset) = f_2(\emptyset) = 0$. Define

(46.8) $$f(U) := \min_{T \subseteq U}(f_1(T) + f_2(U \setminus T))$$

for $U \subseteq S$. It is easy to see that a vector x belongs to $P_{f_1} \cap P_{f_2}$ if and only if

(46.9) $$\begin{array}{ll} x_s \geq 0 & (s \in S), \\ x(U) \leq f(U) & (U \subseteq S). \end{array}$$

Moreover, system (46.9) is box-totally dual integral, since $f(U) \leq f_i(U)$ for each $U \subseteq S$ and $i = 1, 2$.

A consequence is that $P_{f_1} \cap P_{f_2}$ is integer if and only if f is integer. It may occur that P_{f_1} and P_{f_2} are not integer (i.e., f_1 and f_2 are not integer), while $P_{f_1} \cap P_{f_2}$ is integer (i.e., f is integer). For instance, take $P_{f_1} = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq \frac{3}{2}\}$ and $P_{f_2} = \{(x_1, x_2) \mid (x_2, x_1) \in P_{f_1}\}$.

Many other results on polymatroid intersection may be deduced from Theorem 46.1, by considering derived polymatroids (cf. McDiarmid [1978]). For instance, if P_{f_1} and P_{f_2} are integer polymatroids in $\mathbb{R}^S$, v and w are integer vectors, and k and ℓ are integers, then the polytope

(46.10) $$\{x \in P_{f_1} \cap P_{f_2} \mid v \leq x \leq w, k \leq x(S) \leq \ell\}$$

is integer again. To see this, it suffices to show that the polytope $P_{f_1} \cap P_{f_2} \cap \{x \mid x(S) = k\}$ is integer for any integer k. We can reset $f_1(S) := \min\{f_1(S), k\}$. Then the polytope is a face of $P_{f_1} \cap P_{f_2}$, and hence is integer. In fact, the system determining (46.10) is box-TDI — see Corollary 49.12d.

The intersection of *three* integer polymatroids can have noninteger vertices, as the following example shows. Let $S = \{1, 2, 3, 4\}$ and let P_1, P_2 and P_3 be the following polymatroids:

(46.11) $$\begin{aligned} P_1 &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(\{1,2\}) \leq 1, x(\{3,4\}) \leq 1\}, \\ P_2 &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(\{1,3\}) \leq 1, x(\{2,4\}) \leq 1\}, \\ P_3 &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(\{1,4\}) \leq 1, x(\{2,3\}) \leq 1\}. \end{aligned}$$

(So each P_i is the independent set polytope of a partition matroid.) Now the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in $P_1 \cap P_2 \cap P_3$, but the only integer vectors in $P_1 \cap P_2 \cap P_3$ are the 0,1 vectors with at most one 1.

46.3. Contrapolymatroid intersection

Similar results as in the previous sections can be shown for the intersection of two contrapolymatroids. Such results can be proved similarly, or can be derived from the corresponding results for polymatroids.

Consider the system, for set functions g_1, g_2 on S:

(46.12) $$\begin{array}{ll} x(U) \geq g_1(U) & \text{for } U \subseteq S, \\ x(U) \geq g_2(U) & \text{for } U \subseteq S. \end{array}$$

Then Theorem 46.1 gives:

Corollary 46.1d. *If g_1 and g_2 are supermodular, then* (46.12) *is box-TDI.*

Proof. This follows from the box-total dual integrality of (46.1) taking $f_i := -g_i$ for $i = 1, 2$. ∎

46.4. Intersecting a polymatroid and a contrapolymatroid

Let S be a finite set. For set functions f and g on S consider the system

$$\begin{aligned} &x(U) \leq f(U) && \text{for } U \subseteq S, \\ &x(U) \geq g(U) && \text{for } U \subseteq S. \end{aligned} \tag{46.13}$$

Theorem 46.2. *If f is submodular and g is supermodular, then system* (46.13) *is box-TDI.*

Proof. We can assume that $f(\emptyset) \geq 0$ and $g(\emptyset) \leq 0$. Choose $w \in \mathbb{R}^S$, and consider the dual problem of maximizing $w^{\mathsf{T}}x$ over (46.13):

$$\begin{aligned} \min\{ &\sum_{U \subseteq S} y(U)f(U) - \sum_{U \subseteq S} z(U)g(U) \mid \\ &y, z \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{U \subseteq S} y(U)\chi^U - \sum_{U \subseteq S} z(U)\chi^U = w\}. \end{aligned} \tag{46.14}$$

Let y, z attain this minimum. Define

$$u := \sum_{U \subseteq S} y(U)\chi^U \text{ and } v := \sum_{U \subseteq S} z(U)\chi^U. \tag{46.15}$$

So y attains

$$\min\{\sum_{U \subseteq S} y(U)f(U) \mid y \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{U \subseteq S} y(U)\chi^U = u\} \tag{46.16}$$

and z attains

$$\max\{\sum_{U \subseteq S} z(U)g(U) \mid z \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{U \subseteq S} z(U)\chi^U = v\}. \tag{46.17}$$

By Theorem 44.3, (46.16) has an optimum solution y with $\mathcal{F} := \{U \mid y(U) > 0\}$ is a chain. Similarly, (46.17) has an optimum solution z with $\mathcal{G} := \{U \mid z(U) > 0\}$ is a chain. Hence by Theorem 41.11, minimum (46.14) has an optimum solution such that the inequalities corresponding to positive coefficients have a totally unimodular constraint matrix. Hence by Theorem 5.35, (46.13) is box-TDI. ∎

So for the intersection of a polymatroid and a contrapolymatroid one gets:

Corollary 46.2a. *The intersection of an integer extended polymatroid and an integer extended contrapolymatroid is integer.*

Proof. Directly from the fact that an integer extended polymatroid is the solution set of $x(U) \leq f(U)$ ($U \subseteq S$) for some integer submodular set function on S, and an integer extended contrapolymatroid is the solution set of $x(U) \geq g(U)$ ($U \subseteq S$) for some integer submodular set function on S. Hence, by Theorem 46.2, the intersection is determined by a TDI system with integer right-hand sides. So the intersection is integer. ∎

46.5. Frank's discrete sandwich theorem

Frank [1982b] showed the following 'discrete sandwich theorem' (analogous to the 'continuous sandwich theorem', stating the existence of a linear function between a convex and a concave function):

Corollary 46.2b (Frank's discrete sandwich theorem)**.** *Let f be a submodular and g a supermodular set function on S, with $g \leq f$. Then there exists a modular set function h on S with $g \leq h \leq f$. If f and g are integer, h can be taken integer.*

Proof. We can assume that $g(\emptyset) = 0 = f(\emptyset)$, by resetting $f(U) := f(U) - f(\emptyset)$ and $g(U) := g(U) - f(\emptyset)$, for each $U \subseteq S$, and $g(\emptyset) := 0$.

Define $f'(U) := f(S) - g(S \setminus U)$ for each $U \subseteq S$. Then f' is submodular. Now by Corollary 46.1b:

(46.18) $$\max\{x(S) \mid x(U) \leq f(U), x(U) \leq f'(U) \text{ for each } U \subseteq S\} \\ = \min\{f(T) + f'(S \setminus T) \mid T \subseteq S\}.$$

The minimum is at least $f(S)$, since $f(T) + f'(S \setminus T) = f(T) + f(S) - g(T) \geq f(S)$. Hence there exists an $x \in \mathbb{R}^S$ with $x(U) \leq f(U)$ and $x(U) \leq f'(U)$ for each $U \subseteq S$ and with $x(S) = f(S)$. Defining $h(U) := x(U)$, gives the modular function as required, since for each $U \subseteq S$:

(46.19) $$g(U) = f(S) - f'(S \setminus U) \leq x(S) - x(S \setminus U) = x(U) \leq f(U).$$

If f and g are integer, we can choose x integer, implying that h is integer. ∎

As Lovász [1983c] observed, the first part of this result can be derived from the continuous sandwich theorem: define $\tilde{f} : \mathbb{R}^S_+ \to \mathbb{R}$ by

(46.20) $$\tilde{f}(x) := \sum_{i=1}^{k} \lambda_i f(U_i),$$

where $\emptyset \neq U_1 \subset U_2 \subset \cdots \subset U_n \subseteq S$ and $\lambda_1, \ldots, \lambda_k > 0$ are such that $x = \sum_{i=1}^{k} \lambda_i \chi^{U_i}$. Define $\tilde{g}$ similarly. Then $\tilde{f}$ is convex and $\tilde{g}$ is concave, and $\tilde{g} \leq \tilde{f}$. Hence there is a linear function $\tilde{h}$ satisfying $\tilde{g} \leq \tilde{h} \leq \tilde{f}$. This gives the function h as required.

46.6. Integer decomposition

Integer polymatroids have the integer decomposition property. More generally:

Corollary 46.2c. *Let $P_1, \ldots, P_k$ be integer polymatroids. Then each integer vector in $P_1 + \cdots + P_k$ is a sum $x_1 + \cdots + x_k$ of integer vectors $x_1 \in P_1, \ldots, x_k \in P_k$.*

Proof. It suffices to show this for $k = 2$; the general case follows by induction (as the sum of integer polymatroids is again an integer polymatroid, by Theorem 44.6). Choose an integer vector $x \in P_1 + P_2$. Let Q be the contrapolymatroid given by

(46.21) $\qquad Q := x - P_2.$

Then $P_1 \cap Q \neq \emptyset$, since $x = x_1 + x_2$ for some $x_1 \in P_1$ and $x_2 \in P_2$, implying $x_1 \in P_1 \cap Q$. Now Q is integer as x and P_2 are integer. Hence by Corollary 46.2a, $P_1 \cap Q$ contains an integer vector y. Then $x - y \in P_2$, and so x is the sum of $y \in P_1$ and $x - y \in P_2$. ∎

This implies the integer decomposition property for integer polymatroids, proved by Giles [1975] (also by Baum and Trotter [1981]):

Corollary 46.2d. *An integer polymatroid has the integer decomposition property.*

Proof. Directly from Corollary 46.2c, by taking all P_i equal. ∎

This gives the following integer rounding properties (Baum and Trotter [1981]). Let P_f be the integer polymatroid determined by some integer submodular set function f on S. Let $\mathcal{B}$ be the collection of integer base vectors of P_f. Let B be the $\mathcal{B} \times S$ incidence matrix. Then for each $c \in \mathbb{Z}_+^S$, one has

(46.22) $$\begin{aligned}&\min\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{Z}_+^{\mathcal{B}}, y^{\mathsf{T}}B \geq c^{\mathsf{T}}\}\\ &= \lceil \min\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{B}}, y^{\mathsf{T}}B \geq c^{\mathsf{T}}\}\rceil.\end{aligned}$$

Indeed, $\geq$ is trivial. To see equality, let k be equal to the right-hand side. Then $c \in k \cdot P_f$, and hence, by Corollary 46.2d, $c \leq b_1 + \cdots + b_k$ for rows $b_1, \ldots, b_k$ of B. This shows equality.

Note that the right-hand side in (46.22) is equal to $\lceil \max\{c^{\mathsf{T}}x \mid x \in A(P_f)\}\rceil$, where $A(P_f)$ is the antiblocking polyhedron of P_f.

Similarly, one has:

(46.23) $$\begin{aligned}&\max\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{Z}_+^{\mathcal{B}}, y^{\mathsf{T}}B \leq c^{\mathsf{T}}\}\\ &= \lfloor \max\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{B}}, y^{\mathsf{T}}B \leq c^{\mathsf{T}}\}\rfloor.\end{aligned}$$

Now the right-hand side is equal to $\lfloor \min\{c^{\mathsf{T}}x \mid x \in B(Q)\}\rfloor$, where $B(Q)$ is the blocking polyhedron of $Q := \{x \in \mathbb{R}^S \mid x(U) \geq f(S) - f(S \setminus U)$ for $U \subseteq S\}$.

If f is the rank function of a matroid, then (46.22) describes the minimum number of bases covering S, while (46.23) describes the maximum number of disjoint bases.

46.7. Further results and notes

46.7a. Up and down hull of the common base vectors

Let f_1 and f_2 be nondecreasing submodular set functions on S, with $f_1(\emptyset) = f_2(\emptyset) = 0$ and $f_1(S) = f_2(S)$, and let P_1 and P_2 be the associated polymatroids. Let F_1 and F_2 be the faces of base vectors of P_1 and of P_2, respectively. Suppose that $F_1 \cap F_2 \neq \emptyset$, equivalently that

(46.24) $$f_1(S) = f_2(S) = \max\{x(S) \mid x \in P_1 \cap P_2\} = \min_{U \subseteq S} f_1(U) + f_2(S \setminus U).$$

Consider the polyhedra P and Q defined by

(46.25) $$\begin{aligned} P &:= \{x \in \mathbb{R}^S_+ \mid x \leq y \text{ for some } y \text{ in } F_1 \cap F_2\}, \\ Q &:= \{x \in \mathbb{R}^S_+ \mid x \geq y \text{ for some } y \text{ in } F_1 \cap F_2\}. \end{aligned}$$

So if f_1 and f_2 are the rank functions of matroids on S, then P is just the convex hull of incidence vectors of subsets of S which are contained in a common base.

Note that F_1 and F_2 are the faces of minimal vectors in the contrapolymatroids Q_1 and Q_2 associated with the supermodular set functions g_1 and g_2 on S given by

(46.26) $$g_i(U) := f_i(S) - f_i(S \setminus U)$$

for $U \subseteq S$ and $i = 1, 2$ (cf. Section 44.5). So $P \subseteq P_1 \cap P_2$ and $Q \subseteq Q_1 \cap Q_2$.

Let the set functions f and g on S be defined by

(46.27) $$\begin{aligned} f(U) &:= \max\{x(U) \mid x \in P_1 \cap P_2\} = \min_{T \subseteq U}(f_1(T) + f_2(U \setminus T)), \\ g(U) &:= \min\{x(U) \mid x \in Q_1 \cap Q_2\} = \max_{T \subseteq U}(g_1(T) + g_2(U \setminus T)), \end{aligned}$$

for $U \subseteq S$ (cf. Corollary 46.1c). Then $f(S) = g(S) = f_1(S) = f_2(S) = g_1(S) = g_2(S)$.

It is easy to see that if x belongs to Q, then

(46.28) $$x(U) \geq f(S) - f(S \setminus U) \text{ for each } U \subseteq S$$

(note that $x \geq \mathbf{0}$ follows from (46.28) by taking $U = \{s\}$). Indeed, if $x \geq z$ with $z \in F_1 \cap F_2$, then $x(U) \geq z(U) = f(S) - z(S \setminus U) \geq f(S) - f(S \setminus U)$, as $z \in P_1 \cap P_2$.

Similarly, if x belongs to P, then

(46.29) $$\begin{aligned} &x_s \geq 0 && (s \in S), \\ &x(U) \leq g(S) - g(S \setminus U) && (U \subseteq S). \end{aligned}$$

In fact, the systems (46.28) and (46.29) determine Q and P respectively. This was shown by Cunningham [1977] and McDiarmid [1978], thus proving a conjecture of Fulkerson [1971a] (cf. Weinberger [1976]).

Theorem 46.3. *Polyhedron Q is determined by* (46.28). *Polyhedron P is determined by* (46.29).

Proof. Consider $x \in \mathbb{R}_+^S$ and let P_i' be the polymatroid $P_i' := \{y \in P_i \mid y \leq x\}$ for $i = 1, 2$ (cf. Section 44.1). By (44.8), the submodular function f_i' associated with P_i' is given by

(46.30) $$f_i'(U) = \min_{T \subseteq U}(f_i(T) + x(U \setminus T))$$

for $U \subseteq S$ and $i = 1, 2$. Now x is in Q if and only if there is a vector z in $P_1 \cap P_2$ with $z \leq x$ and $z(S) = f(S)$, i.e., if and only if there is a vector z in $P_1' \cap P_2'$ with $z(S) = f(S)$. By (46.7) such a vector exists if and only if

(46.31) $$\min_{U \subseteq S}(f_1'(U) + f_2'(S \setminus U)) \geq f(S).$$

Substituting (46.30) one finds that (46.31) is equivalent to (46.28).

The second statement of Theorem 46.3 is proved similarly. ∎

This theorem has a self-refining character. If k is a rational number with $k \leq f(S)$ and if $w \in Q$, then

(46.32) $$\begin{aligned}&\{x \in \mathbb{R}_+^S \mid x \geq z \text{ for some } z \text{ in } P_1 \cap P_2 \text{ with } z(S) = k\}\\ &= \{x \in \mathbb{R}_+^S \mid x(U) \geq k - f(S \setminus U) \text{ for all } U \subseteq S\}\end{aligned}$$

and

(46.33) $$\begin{aligned}&\{x \in \mathbb{R}_+^S \mid x \geq z \text{ for some } z \text{ in } F_1 \cap F_2 \text{ with } z \leq w\}\\ &= \{x \in \mathbb{R}_+^S \mid x(S \setminus (T \cup U)) \geq f(S) - w(U) - f(T) \text{ for disjoint}\\ &\qquad T, U \subseteq S\},\end{aligned}$$

as can be seen by taking appropriate subpolymatroids of P_1 and P_2 (cf. also McDiarmid [1976,1978]).

This has the following applications. Let $G = (V, E)$ be a bipartite graph, let $x \in \mathbb{R}_+^E$, and let k be a natural number. Then there exists a vector $z \leq x$ such that z is a convex combination of incidence vectors of matchings in G of size k if and only if

(46.34) $$x(E[U]) \geq k - |V| + |U|$$

for all $U \subseteq V$ (where $E[U]$ denotes the set of edges spanned by U). This can be derived as follows. Let V_1 and V_2 be the colour classes of G. For $F \subseteq E$, let $f_i(F)$ be the number of vertices in V_i covered by F (for $i = 1, 2$). Then $f(F)$ equals the maximum size of a matching in F, which is equal to the minimum number of vertices covering F. Hence the inequalities $x(F) \geq k - f(E \setminus F)$ (for $F \subseteq E$) follow from $x(E[U]) \geq k - |V \setminus U|$ (for $U \subseteq V$).

Another application is Corollary 52.3a on the up hull of the r-arborescence polytope (cf. Section 52.1a).

Gröflin and Hoffman [1981] gave a method to show the following:

Theorem 46.4. (46.28) *and* (46.29) *are box-TDI.*

Proof. We prove that (46.28) is box-TDI. The box-total dual integrality of (46.29) is proved similarly.

Let $\mathcal{R}$ be the collection of all pairs (T, U) of subsets of S with $T \cap U = \emptyset$. Then the system

(46.35) $$x(S \setminus (T \cup U)) \geq f(S) - f_1(T) - f_2(U) \quad ((T, U) \in \mathcal{R})$$

is equivalent to (46.28), in the following sense: by (46.27), (46.35) determines Q, and (46.35) contains all inequalities occurring in (46.28); moreover, all inequalities in (46.35) satisfied with equality by some $x \in Q$, also occur in (46.28). Hence, if (46.35) is box-totally dual integral, also (46.28) is box-totally dual integral. So it suffices to show the box-total dual integrality of (46.35). To this end, let $w \in \mathbb{Z}_+^S$, and consider the dual of minimizing $w^{\mathsf{T}} x$ over (46.35):

(46.36) $$\max\{\sum_{(T,U)\in\mathcal{R}} y(T, U)(f(S) - f_1(T) - f_2(U)) \mid y \in \mathbb{R}_+^{\mathcal{R}}, \sum_{(T,U)\in\mathcal{R}} y(T, U)\chi^{S\setminus(T\cup U)} = w\}.$$

We show that it is attained by an integer vector y if w is integer.

To this end, let y attain the maximum (46.36) such that

(46.37) $$\sum_{(T,U)\in\mathcal{R}} y(T, U)(|T| + |S \setminus U|)(|U| + |S \setminus T|)$$

is as small as possible. Then:

(46.38) if $y(A, B) > 0$ and $y(C, D) > 0$, then either $A \subseteq C$ and $B \supseteq D$, or $A \supseteq C$ and $B \subseteq D$.

Suppose not. Define $\alpha := \min\{y(A, B), y(C, D)\}$. Define $y' : \mathcal{R} \to \mathbb{R}_+$ by

(46.39) $$\begin{aligned} y'(A, B) &:= y(A, B) - \alpha, \\ y'(C, D) &:= y(C, D) - \alpha, \\ y'(A \cap C, B \cup D) &:= y(A \cap C, B \cup D) + \alpha, \\ y'(A \cup C, B \cap D) &:= y(A \cup C, B \cap D) + \alpha, \end{aligned}$$

and let y' coincide with y in the other components. One easily checks that

(46.40) $$\sum_{(T,U)\in\mathcal{R}} y'(T, U)\chi^{S\setminus(T\cup U)} = \sum_{(T,U)\in\mathcal{R}} y(T, U)\chi^{S\setminus(T\cup U)} \text{ and}$$
$$\sum_{(T,U)\in\mathcal{R}} y'(T, U)(f(S) - f_1(T) - f_2(U)) \geq \sum_{(T,U)\in\mathcal{R}} y(T, U)(f(S) - f_1(T) - f_2(U)),$$

by the submodularity of f_1 and f_2. So y' also attains the maximum (46.36). Moreover, one straightforwardly checks that replacing y by y' decreases (46.37).[36] This contradicts our assumption, and therefore proves (46.38).

[36] This can be seen with Theorem 2.1: Make a copy $\widetilde{S}$ of S, and, for any $U \subseteq S$, let $\widetilde{U}$ be the set of copies of elements of U. Define $X_{T,U} := T \cup (\widetilde{S} \setminus \widetilde{U})$. Then $|T| + |S \setminus U| = |X_{T,U}|$ and $|U| + |S \setminus T| = |(S \cup \widetilde{S}) \setminus X_{T,U}|$. Moreover, for (A, B) and (C, D) in $\mathcal{R}$ we have $X_{A\cap C, B\cup D} = X_{A,B} \cap X_{C,D}$ and $X_{A\cup C, B\cap D} = X_{A,B} \cup X_{C,D}$. So the replacements decrease (46.37) by Theorem 2.1, since $X_{A,B} \not\subseteq X_{C,D} \not\subseteq X_{A,B}$.

Let $\mathcal{R}_0 := \{(T,U) \in \mathcal{R} \mid y(T,U) > 0\} = \{(T_1,U_1),\ldots,(T_n,U_n)\}$ with $T_1 \subseteq \cdots \subseteq T_n$ and $U_n \subseteq \cdots \subseteq U_1$ (this is possible by (46.38)). Let M be the $\{0,1\}$ matrix with rows indexed by $\mathcal{R}_0$ and columns indexed by S, such that $M_{(T,U),s} = 1$ if and only if $s \notin T \cup U$. Then for each s in S, the indices i for which $M_{(T_i,U_i),s} = 1$ form a contiguous interval of $\{1,\ldots,n\}$. Hence M is totally unimodular (as it is a network matrix with directed tree being a directed path). So we have the box-total dual integrality of (46.35) by Theorem 5.35. ∎

Frank and Tardos [1984a] indicated a direct derivation of this theorem from the total dual integrality of (46.1).

There are a number of straightforward corollaries. As for the integrality of polyhedra:

Corollary 46.4a. *If f (or, equivalently, g) is integer, then the polyhedra P, Q, and $F_1 \cap F_2$ are integer.*

Proof. This follows directly from Theorem 46.4. Note that $F_1 \cap F_2$ is integer if and only if P is integer. ∎

Also a min-max relation follows:

Corollary 46.4b. *Let $M_1 = (S,\mathcal{I}_1)$ and $M_2 = (S,\mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2 and let k be the maximum size of a common independent set. Then for any subset U of S,*

(46.41) $$\min_I |U \cap I| = \max_{S_1,\ldots,S_t} \sum_{i=1}^{t} (k - r(S \setminus S_i)),$$

where the minimum ranges over all common independent sets I with $|I| = k$, and where the maximum ranges over all partitions of U into sets $S_1,\ldots,S_t$ ($t \geq 0$), and where $r(T)$ denotes the maximum size of a common independent set contained in T.

Proof. Apply Theorem 46.4, taking $c := \chi^U$, $f_i := r_i$, and $f := r$. ∎

It is not necessarily true that if $F_1 \cap F_2$ is integer, then also $P_1 \cap P_2$ (or $Q_1 \cap Q_2$) is integer — i.e., that the converse implication of Corollary 46.4a holds. For instance, if

(46.42) $$\begin{aligned} P_1 &:= \{(x,y,z)^{\mathsf{T}} \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq \tfrac{3}{2}, x+z \leq 2\},\\ P_2 &:= \{(x,y,z)^{\mathsf{T}} \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq \tfrac{3}{2}, y+z \leq 2\}, \end{aligned}$$

then $F_1 \cap F_2 = \{(1,1,1)^{\mathsf{T}}\}$, but $(\frac{1}{2},\frac{1}{2},\frac{3}{2})^{\mathsf{T}}$ is a vertex of $P_1 \cap P_2$.

Related results on integer decomposition of integer polymatroids in McDiarmid [1983].

46.7b. Further notes

Giles [1975] characterized the facets of the intersection of two polymatroids. Ageev [1988] studied the problem of maximizing a concave function over the intersection of polymatroids.

Chapter 47

Polymatroid intersection algorithmically

In this chapter we consider the problem of finding a vector of maximum weight in the intersection of two (extended) polymatroids algorithmically. We describe a strongly polynomial-time algorithm for this problem in four stages (where f_1 and f_2 are submodular set functions on S):

- a strongly polynomial-time algorithm finding a maximum-size vector in $EP_{f_1} \cap EP_{f_2}$ (Section 47.1),
- a strongly polynomial-time algorithm finding a common base vector of f_1 and f_2 maximizing $x(s)$ for some prescribed $s \in S$ (Section 47.2),
- a polynomial-time algorithm finding a maximum-weight common base vector of f_1 and f_2 (Section 47.3),
- a strongly polynomial-time algorithm finding a maximum-weight common base vector of f_1 and f_2 (Section 47.4).

At the base of the algorithms is submodular function minimization, which leads back to the 'consistent breadth-first search' technique proposed in a pioneering paper of Schönsleben [1980] on polymatroid intersection.

47.1. A maximum-size common vector in two polymatroids

We first consider the problem:

(47.1) given: submodular set functions f_1 and f_2 on a set S (by value giving oracles),
find: an $x \in EP_{f_1} \cap EP_{f_2}$ maximizing $x(S)$, and a $T \subseteq S$ with $x(S) = f_1(T) + f_2(S \setminus T)$.

So T certifies that x indeed maximizes $x(S)$ over $EP_{f_1} \cap EP_{f_2}$, since for any $x' \in EP_{f_1} \cap EP_{f_2}$ we have:

(47.2) $$x'(S) = x'(T) + x'(S \setminus T) \leq f_1(T) + f_2(S \setminus T) = x(S).$$

On the other hand, x certifies that T minimizes $f_1(T) + f_2(S \setminus T)$.

Then (Lawler and Martel [1982a], extending a weakly polynomial bound of Schönsleben [1980]):

Theorem 47.1. *Problem* (47.1) *is solvable in strongly polynomial-time.*

Proof. We can assume that $f_1(\emptyset) = 0$ and $f_2(\emptyset) = 0$. Define the submodular set function f on S by

$$f(U) := f_1(U) + f_2(S \setminus U) - f_2(S) \tag{47.3}$$

for $U \subseteq S$. With the submodular function minimization algorithm described in Section 45.4 we find a subset T of S minimizing f. So, by Corollary 46.1b, $f(T) + f_2(S)$ is equal to the maximum of $x(S)$ over $EP_{f_1} \cap EP_{f_2}$.

The submodular function minimization algorithm of Section 45.4 also gives vertices $b_1, \ldots, b_k$ of EP_f and $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$ such that for

$$y := \lambda_1 b_1 + \cdots + \lambda_k b_k \tag{47.4}$$

we have $y(T) = f(T)$, $\mathrm{supp}^-(y) \subseteq T$, and $\mathrm{supp}^+(y) \subseteq S \setminus T$. (Here, as usual, $\mathrm{supp}^+(x) := \{s \in S \mid x(s) > 0\}$ and $\mathrm{supp}^-(x) := \{s \in S \mid x(s) < 0\}$.)

Now for each $i = 1, \ldots, k$, we can find $b'_i \in EP_{f_1}$ and $b''_i \in EP_{f_2}$ with $b_i = b'_i - b''_i$. Indeed, let $u_1, \ldots, u_n$ be a total order of S generating b_i. (That is, $b_i(u_j) = f(\{u_1, \ldots, u_j\}) - f(\{u_1, \ldots, u_{j-1}\})$ for $j = 1, \ldots, n$. These orderings are also implied by the submodular function minimization algorithm.) Let b'_i be the vertex of EP_{f_1} generated by the order $u_1, \ldots, u_n$ (that is, $b'_i(u_j) = f_1(\{u_1, \ldots, u_j\}) - f_1(\{u_1, \ldots, u_{j-1}\})$ for $j = 1, \ldots, n$). Let b''_i be the vertex of EP_{f_2} generated by the order $u_n, u_{n-1}, \ldots, u_1$ (that is, $b''_i(u_j) = f_2(\{u_j, \ldots, u_n\}) - f_2(\{u_{j+1}, \ldots, u_n\})$ for $j = 1, \ldots, n$). Then by definition of f we have $b_i = b'_i - b''_i$, since for each j:

$$\begin{aligned} b_i(u_j) &= f(\{u_1, \ldots, u_j\}) - f(\{u_1, \ldots, u_{j-1}\}) \\ &= f_1(\{u_1, \ldots, u_j\}) + f_2(\{u_{j+1}, \ldots, u_n\}) - f_1(\{u_1, \ldots, u_{j-1}\}) \\ &\quad - f_2(\{u_j, \ldots, u_n\}) = b'_i(u_j) - b''_i(u_j). \end{aligned} \tag{47.5}$$

Define

$$x' := \sum_{i=1}^{k} \lambda_i b'_i,\ x'' := \sum_{i=1}^{k} \lambda_i b''_i, \text{ and } x := x' \wedge x'', \tag{47.6}$$

where $\wedge$ stands for taking coordinatewise minimum. As $x' \in EP_{f_1}$ and $x'' \in EP_{f_2}$, we know $x \in EP_{f_1} \cap EP_{f_2}$. Also, as $y = x' - x''$, we know that if $u \in T$, then $y(u) \leq 0$, hence $x''(u) \geq x'(u)$, and therefore $x(u) = x'(u)$. Similarly, if $u \in S \setminus T$, then $x(u) = x''(u)$. Hence

$$\begin{aligned} x(S) &= x(T) + x(S \setminus T) = x'(T) + x''(S \setminus T) = (x' - x'')(T) + x''(S) \\ &= y(T) + x''(S) = f(T) + f_2(S) = f_1(T) + f_2(S \setminus T), \end{aligned} \tag{47.7}$$

as required. ∎

47.2. Maximizing a coordinate of a common base vector

Theorem 47.1 implies the strong polynomial-time solvability of:

(47.8) given: submodular set functions f_1 and f_2 on a set S (by value giving oracles) and an element $s \in S$,
find: a common base vector x of f_1 and f_2 maximizing $x(s)$, and subsets S_1 and S_2 of S with $S_1 \cap S_2 = \{s\}$, $S_1 \cup S_2 = S$, and $x(S_i) = f_i(S_i)$ for $i = 1, 2$.

This is a result of Frank [1984a]:

Theorem 47.2. *Problem* (47.8) *is solvable in strongly polynomial time.*

Proof. We can assume that $f_1(S) = f_2(S)$ and that f_1 and f_2 have a common base vector (this can be checked by Theorem 47.1). Hence

(47.9) $f_1(S) \leq f_1(U) + f_2(S \setminus U)$

for each $U \subseteq S$. Define $S' := S \setminus \{s\}$.

First determine S_1, S_2 with $S_1 \cap S_2 = \{s\}$ and $S_1 \cup S_2 = S$ and minimizing $f_1(S_1) + f_2(S_2)$. This can be done by minimizing the submodular function $f_1(U + s) + f_2(S \setminus U)$ over $U \subseteq S'$.

Define

(47.10) $\alpha := f_1(S_1) + f_2(S_2) - f_1(S)$.

For $i = 1, 2$ and $U \subseteq S'$, define

(47.11) $g_i(U) := \min\{f_i(U), f_i(U + s) - \alpha\}$.

Then g_1 and g_2 are submodular set functions on S', as is easy to check. Moreover,

(47.12) $g_i(S') = f_i(S) - \alpha$.

To show this, we may assume that $i = 1$. Then we must show:

(47.13) $f_1(S') \geq f_1(S) - \alpha = 2f_1(S) - f_1(S_1) - f_2(S_2)$.

Now $f_1(S_1 \setminus \{s\}) + f_2(S_2) \geq f_1(S)$ (since f_1 and f_2 have a common base vector) and $f_1(S_1) - f_1(S_1 \setminus \{s\}) \geq f_1(S) - f_1(S')$ (by the submodularity of f_1). These two inequalities imply (47.13).

Then

(47.14) g_1 and g_2 have a common base vector.

Otherwise, S' can be partitioned into sets R_1 and R_2 with

(47.15) $g_1(R_1) + g_2(R_2) < g_1(S')$.

If $g_1(R_1) = f_1(R_1)$ and $g_2(R_2) = f_2(R_2)$, then (47.15) implies

(47.16) $$2f_1(S) > f_1(S_1) + f_2(S_2) + f_1(R_1) + f_2(R_2) \\ \geq f_1(S_1 \cap R_1) + f_1(S_1 \cup R_1) + f_2(S_2 \cap R_2) + f_2(S_2 \cup R_2).$$

By symmetry, we can assume that $f_1(S) > f_1(S_1 \cap R_1) + f_2(S_2 \cup R_2)$. However, $S_1 \cap R_1$ and $S_2 \cup R_2$ partition S, contradicting (47.9).

If $g_1(R_1) = f_1(R_1)$ and $g_2(R_2) = f_2(R_2 + s) - \alpha$, then (47.15) implies

(47.17) $$f_1(S) - \alpha > f_1(R_1) + f_2(R_2 + s) - \alpha,$$

and hence $f_1(S) > f_1(R_1) + f_2(R_2 + s)$, contradicting (47.9).

If $g_1(R_1) = f_1(R_1 + s) - \alpha$ and $g_2(R_2) = f_2(R_2 + s) - \alpha$, then (47.15) implies

(47.18) $$f_1(S) - \alpha > f_1(R_1 + s) + f_2(R_2 + s) - 2\alpha,$$

implying $f_1(S_1) + f_2(S_2) > f_1(R_1 + s) + f_2(R_2 + s)$, contradicting the minimality of $f_1(S_1) + f_2(S_2)$. This proves (47.14).

By Theorem 47.1, we can find in strongly polynomial time a common base vector x of g_1 and g_2. So $x(S') = g_1(S')$. Extend x to S by defining $x(s) := \alpha$. Then

(47.19) x is a common base vector of f_1 and f_2.

By symmetry, it suffices to show that x is a base vector of f_1. First, x belongs to EP_{f_1}, since for each $U \subseteq S'$ we have

(47.20) $$x(U) \leq g_1(U) \leq f_1(U) \text{ and} \\ x(U + s) = x(U) + \alpha \leq g_1(U) + \alpha \leq f_1(U + s).$$

Next, x is a base vector of f_1, since

(47.21) $$x(S) = x(S') + \alpha = g_1(S') + \alpha = f_1(S),$$

by (47.12). This proves (47.19).

Moreover,

(47.22) $$x(S_i) = f_i(S_i)$$

for $i = 1, 2$. Indeed (for $i = 1$),

(47.23) $$x(S_1) = x(S) - x(S_2 \setminus \{s\}) \geq f_1(S) - g_2(S_2 \setminus \{s\}) \\ \geq f_1(S) - f_2(S_2) + \alpha = f_1(S_1).$$

This proves (47.22), which implies that x is a common base vector of f_1 and f_2 maximizing $x(s)$, as for any common base vector z of f_1 and f_2 we have

(47.24) $$z(s) = z(S_1) + z(S_2) - z(S) \leq f_1(S_1) + f_2(S_2) - f_1(S) \\ = x(S_1) + x(S_2) - x(S) = x(s).$$

So x, S_1, and S_2 have the required properties. ∎

47.3. Weighted polymatroid intersection in polynomial time

It may be shown with the ellipsoid method that the following problem is solvable in polynomial time:

(47.25) given: submodular functions f_1, f_2 on a set S (by value giving oracles) and a function $w : S \to \mathbb{Z}$,
find: a common base vector x of f_1 and f_2 maximizing $w^{\mathsf{T}}x$, and $w_1, w_2 : S \to \mathbb{Z}$ with $w = w_1 + w_2$ such that, for each $i = 1, 2$, x maximizes $w_i^{\mathsf{T}}x$ over all base vectors of f_i.

Cunningham and Frank [1985] gave, with the help of Theorem 47.2, a combinatorial polynomial-time algorithm (using submodular function minimization).

In order to describe this, we first give an auxiliary result concerning polymatroids. Let f be a submodular set function on S and let F be a face of EP_f. Define

(47.26) $$F^{\downarrow} := F - \mathbb{R}^S_+.$$

Then $F^{\downarrow}$ is an extended polymatroid again. Moreover, algorithmic properties for $F^{\downarrow}$ can be deduced from those for EP_f:

Lemma 47.3α. *Let f be a submodular set function on S, let $w : S \to \mathbb{Z}_+$, and let F be the set of vectors x in EP_f maximizing $w^{\mathsf{T}}x$. Then there is a submodular set function f' on S with $F^{\downarrow} = EP_{f'}$. Moreover, if f is given by a value giving oracle, $f'(U)$ can be computed in strongly polynomial time, for any $U \subseteq S$.*

Proof. We can assume that $f(\emptyset) = 0$. Let $\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_{k-1} \subset T_k = S$ be the (unique) sets satisfying

(47.27) $$w = \lambda_1 \chi^{T_1} + \cdots + \lambda_k \chi^{T_k},$$

for some $\lambda_1, \ldots, \lambda_{k-1} > 0$ and $\lambda_k \geq 0$. Set $T_0 := \emptyset$, and define f' by:

(47.28) $$f'(U) := \sum_{i=1}^{k} (f((U \cap T_i) \cup T_{i-1}) - f(T_{i-1})),$$

for $U \subseteq S$. Then f' is submodular, as it is the sum of k submodular functions. Also,

(47.29) $$f' \leq f,$$

since for each U we have, by the submodularity of f:

(47.30) $$\begin{aligned} f'(U) &= \sum_{i=1}^{k} (f((U \cap T_i) \cup T_{i-1}) - f(T_{i-1})) \\ &\leq \sum_{i=1}^{k} (f(U \cap T_i) - f(U \cap T_{i-1})) = f(U). \end{aligned}$$

We show:

(47.31) $F^{\downarrow} = EP_{f'}$.

To see $\subseteq$, it suffices to show that $F \subseteq EP_{f'}$. Let $x \in F$. So $x(T_i) = f(T_i)$ for $i = 0, \ldots, k-1$. Hence

$$
\begin{aligned}
(47.32) \qquad x(U) &= \sum_{i=1}^{k} x(U \cap (T_i \setminus T_{i-1})) = \sum_{i=1}^{k} (x((U \cap T_i) \cup T_{i-1}) - x(T_{i-1})) \\
&\leq \sum_{i=1}^{k} (f((U \cap T_i) \cup T_{i-1}) - f(T_{i-1})) = f'(U)
\end{aligned}
$$

for each $U \subseteq S$. This proves that $x \in EP_{f'}$.

To see $\supseteq$ in (47.31), it suffices to show that any $x \in EP_{f'}$ with $x(S) = f'(S)$ belongs to F. As $f' \leq f$ we know that $x \in EP_f$. So it suffices to show that $x(T_j) = f(T_j)$ for $j = 1, \ldots, k$ (as this implies that x maximizes $w^{\mathsf{T}}x$ over EP_f, by the greedy algorithm). Now, as $f'(S) = f(S)$:

$$
\begin{aligned}
(47.33) \qquad x(T_j) &= x(S) - x(S \setminus T_j) \geq f'(S) - f'(S \setminus T_j) \\
&= f(S) - \sum_{i=1}^{k} (f((T_i \setminus T_j) \cup T_{i-1}) - f(T_{i-1})) \\
&= f(S) - \sum_{i=j+1}^{k} (f(T_i) - f(T_{i-1})) = f(T_j).
\end{aligned}
$$

This proves (47.31). ∎

We also will use the following lemma:

Lemma 47.3β. *Let f be a submodular set function on S, let $w : S \to \mathbb{Z}$, and let F be the set of base vectors x of f maximizing $w^{\mathsf{T}}x$. Let $U \subseteq S$ and let x maximize $x(U)$ over F. Then x maximizes $(w + \chi^U)^{\mathsf{T}}x$ over all base vectors of f.*

Proof. Let $w' := w + \chi^U$. As x maximizes $x(U)$ over F, we know that x maximizes ${w'}^{\mathsf{T}}x$ over F. Also, some $z \in F$ maximizes ${w'}^{\mathsf{T}}z$ over EP_f, by the greedy method, as there is an ordering of S in which both w and w' are monotonically nondecreasing, and so EP_f has a vertex z maximizing both $w^{\mathsf{T}}z$ and ${w'}^{\mathsf{T}}z$.

As ${w'}^{\mathsf{T}}x \geq {w'}^{\mathsf{T}}z$, x maximizes ${w'}^{\mathsf{T}}x$ over EP_f. ∎

Now we can derive:

Theorem 47.3. *Problem* (47.25) *is solvable in polynomial time.*

Proof. We give a polynomial-time algorithm to transform a solution of (47.25) for some w to a solution of (47.25) for $w := w + \chi^s$, for any $s \in S$. This implies a polynomial-time algorithm for (47.25), since we can assume that $w \geq \mathbf{0}$, and since any $w \geq \mathbf{0}$ can be obtained from $w = \mathbf{0}$ by a polynomially bounded number of resettings $w := 2w$ and $w := w + \chi^s$ for $s \in S$. Note that for $w = \mathbf{0}$, (47.25) is trivial, and that a solution x, w_1, w_2 for w yields a solution $x, 2w_1, 2w_2$ for $2w$.

Let $s \in S$. Let x, w_1, w_2 be a solution of (47.25) for some w. For $i = 1, 2$, let F_i be the set of all vectors $x \in EP_{f_i}$ maximizing $w_i^{\mathsf{T}} x$ and let f_i' be a submodular function satisfying $F_i^{\downarrow} = EP_{f_i'}$ (Lemma 47.3α). Applying Theorem 47.2 to f_1', f_2', we find a common base vector x' of f_1' and f_2' maximizing $x'(s)$, and subsets S_1, S_2 of S with $S_1 \cap S_2 = \{s\}$, $S_1 \cup S_2 = S$, and $x'(S_1) = f_1'(S_1)$, $x'(S_2) = f_2'(S_2)$. Then x' maximizes $x'(S_1)$ over $EP_{f_1'}$, and x' maximizes $x'(S_2)$ over $EP_{f_2'}$. Hence, by Lemma 47.3β, x' is a base vector of f_1' maximizing $(w_1 + \chi^{S_1})^{\mathsf{T}} x'$, and also, x' is a base vector of f_2' maximizing $(w_2 + \chi^{S_2})^{\mathsf{T}} x'$. So

(47.34) $$x', w_1' := w_1 + \chi^{S_1} - \chi^S, w_2' := w_2 + \chi^{S_2},$$

gives a solution of (47.25) for $w + \chi^s$. ∎

47.4. Weighted polymatroid intersection in strongly polynomial time

A general simultaneous diophantine approximation method of Frank and Tardos [1985,1987] implies that (47.25) is *strongly* polynomial-time solvable. Fujishige, Röck, and Zimmermann [1989] showed that from Theorem 47.3 a combinatorial strongly polynomial-time algorithm can be derived, by extending the method of Tardos [1985a] for the minimum-cost circulation problem.

To prove this, we first show a sensitivity result. Let f_1, f_2 be submodular set functions on S. Call a pair $w_1, w_2 : S \to \mathbb{R}$ *good* if there exists an x that maximizes $w_i^{\mathsf{T}} x$ over EP_{f_i}, for both $i = 1$ and $i = 2$.

Lemma 47.4α. *Let $w : S \to \mathbb{Q}$ and let w_1, w_2 be a good pair with $w = w_1 + w_2$. Then for any $\tilde{w} : S \to \mathbb{Q}$ with $\tilde{w} \geq w$ there exists a good pair $\tilde{w}_1, \tilde{w}_2$ with $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$ and $\|\tilde{w}_i - w_i\|_\infty \leq \|\tilde{w} - w\|_1$ for $i = 1, 2$.*

Proof. We can assume that w and $\tilde{w}$ are integer, and that $\|\tilde{w} - w\|_1 = 1$ (as the general case then follows inductively).

Let F_i be the set of all x maximizing $w_i^{\mathsf{T}} x$ over EP_{f_i}. Let f_i' be a submodular function satisfying $F_i^{\downarrow} = EP_{f_i'}$. Let s be such that $\tilde{w}(s) = w(s) + 1$. By the solvability of problem (47.8), there is a common base vector x of f_1' and f_2' maximizing x_s, and there exist S_1, S_2 with $S_1 \cap S_2 = \{s\}$ and $S_1 \cup S_2 = S$ such that $x(S_1) = f_1'(S_1)$ and $x(S_2) = f_2'(S_2)$. Define

(47.35) $$\tilde{w}_1 := w_1 + \chi^{S_1} - \chi^S, \tilde{w}_2 := w_2 + \chi^{S_2}.$$

So $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$. By Lemma 47.3$\beta$, x maximizes $\tilde{w}_i^{\mathsf{T}}x$ over EP_{f_i} for $i = 1, 2$. Therefore, the pair $\tilde{w}_1, \tilde{w}_2$ is good. As $\|\tilde{w}_i - w_i\|_\infty \leq 1$ for $i = 1, 2$, this proves the lemma. ∎

Theorem 47.4. *Given submodular functions f_1, f_2 on a set S and $w \in \mathbb{Q}^S$, one can find a common base vector x of f_1 and f_2 maximizing $w^{\mathsf{T}}x$, in strongly polynomial time.*

Proof. Let be given submodular functions f_1, f_2 on a set S and a function $w : S \to \mathbb{Q}$. We may assume that f_1 and f_2 have a common base vector. (This can be checked by Theorem 47.1.)

We keep chains $\mathcal{C}_1, \mathcal{C}_2$ of subsets of S such that for $i = 1, 2$ and each $U \in \mathcal{C}_i$:

(47.36) $x(U) = f_i(U)$ for each common base vector x of f_1 and f_2 maximizing $w^{\mathsf{T}}x$,

and such that $S \in \mathcal{C}_1$ and $S \in \mathcal{C}_2$. Initially we set $\mathcal{C}_i := \{S\}$ for $i = 1, 2$. We describe an iteration that either extends $\mathcal{C}_1$ or $\mathcal{C}_2$, or gives a solution x.

We can assume that, for $i = 1, 2$,

(47.37) each base vector x of f_i satisfies $x(U) = f_i(U)$ for each $U \in \mathcal{C}_i$.

Indeed, let F_i be the set of vectors x in EP_{f_i} with $x(U) = f_i(U)$ for each $U \in \mathcal{C}_i$. So F_i is equal to the set of $x \in EP_{f_i}$ maximizing $c_i^{\mathsf{T}}x$, where $c_i := \sum_{U \in \mathcal{C}_i} \chi^U$. By Lemma 47.3$\alpha$, we can find f_i' with $F_i^{\downarrow} = EP_{f_i'}$. By (47.36), replacing the f_i by f_i' does not change the set of optimum solutions x of our problem.

Let

(47.38) $L :=$ linear hull of $\{\chi^U \mid U \in \mathcal{C}_1 \cup \mathcal{C}_2\}$.

Determine $y \in L$ minimizing

(47.39) $\|w - y\|_\infty$.

This can be done in strongly polynomial time as follows. For $i = 1, 2$, let $\mathcal{P}_i$ be the partition of S into nonempty classes such that u and v belong to the same class if and only if there is no set in $\mathcal{C}_i$ containing exactly one of u, v. Let D be the directed graph with vertex set $\mathcal{P}_1 \cup \mathcal{P}_2$ such that for each $v \in S$ there is an arc of length $w(v)$ from $U \in \mathcal{P}_1$ to $W \in \mathcal{P}_2$ with $v \in U \cap W$, and an arc of length $-w(v)$ in the reverse direction. Determine the minimum mean-length α of a directed circuit in D (cf. Section 8.5). It is the minimum α for which there exist $p_i : \mathcal{P}_i \to \mathbb{Q}$ such that

(47.40) $-\alpha \leq w(v) + p_1(U) - p_2(W) \leq \alpha$

for each arc as described. Then

(47.41) $$y := -\sum_{U \in \mathcal{P}_1} p_1(U)\chi^U + \sum_{W \in \mathcal{P}_2} p_2(W)\chi^W$$

minimizes (47.39).

Let α be the value of (47.39). If $\alpha = 0$, then $w \in L$, and so

(47.42) $$w = \sum_{i=1}^{2} \sum_{U \in \mathcal{C}_i} \lambda_i(U)\chi^U$$

for functions $\lambda_i : \mathcal{C}_i \to \mathbb{Q}$. Then for any common base vector x of f_1 and f_2 we have

(47.43) $$w^{\mathsf{T}}x = \sum_{i=1}^{2} \sum_{U \in \mathcal{C}_i} \lambda_i(U)x(U) = \sum_{i=1}^{2} \sum_{U \in \mathcal{C}_i} \lambda_i(U)f_i(U).$$

So each common base vector is optimum. As we can find any common base vector in strongly polynomial time (by Theorem 47.1), we have solved the problem.

So we can assume that $\alpha > 0$. Define $w' : S \to \mathbb{Z}$ by

(47.44) $$w' := \lfloor \frac{5n^2}{\alpha}(w - y) \rfloor,$$

where $n := |S|$. By definition of α, $\|w'\|_\infty = 5n^2$. Hence by Theorem 47.3, we can find in strongly polynomial time a common base vector x' of f_1 and f_2 and $w'_1, w'_2 : S \to \mathbb{Z}$ with $w' = w'_1 + w'_2$ such that x' is a base vector of f_i maximizing ${w'}_i^{\mathsf{T}} x'$, for $i = 1, 2$.

For $i = 1, 2$, we can determine a chain $\mathcal{D}_i$ of subsets of S (with $S \in \mathcal{D}_i$) and a function $\lambda_i : \mathcal{D}_i \to \mathbb{Z}$ such that

(47.45) $$w'_i = \sum_{W \in \mathcal{D}_i} \lambda_i(W)\chi^W$$

and such that $\lambda_i(W) > 0$ if $W \neq S$. We show that

(47.46) there exist $i \in \{1, 2\}$ and $W \in \mathcal{D}_i$ with $\lambda_i(W) > 2n$ and $\chi^W \notin L$.

Suppose not. Let $\mathcal{D}'_i := \{W \in \mathcal{D}_i \mid \chi^W \notin L\}$, and $\mathcal{D}''_i := \mathcal{D}_i \setminus \mathcal{D}'_i$, for $i = 1, 2$. So if $W \in \mathcal{D}'_i$, then $\lambda_i(W) \leq 2n$. This gives the contradiction:

(47.47) $$\begin{aligned} 4n^2 &\geq \|\sum_{i=1}^{2} \sum_{W \in \mathcal{D}'_i} \lambda_i(W)\chi^W\|_\infty = \|w' - \sum_{i=1}^{2} \sum_{W \in \mathcal{D}''_i} \lambda_i(W)\chi^W\|_\infty \\ &> \|\frac{5n^2}{\alpha}(w - y) - \sum_{i=1}^{2} \sum_{W \in \mathcal{D}''_i} \lambda_i(W)\chi^W\|_\infty - 1 \geq 5n^2 - 1. \end{aligned}$$

The last inequality holds as y minimizes $\|w - y\|_\infty$ over $y \in L$.

This shows (47.46). We can assume that $W \in \mathcal{D}'_1$ is such that $\lambda_1(W) > 2n$. Then:

(47.48) each optimum common base vector x of f_1 and f_2 satisfies $x(W) = f_1(W)$.

To see this, let

$$(47.49)\qquad \tilde{w} := \frac{5n^2}{\alpha}(w - y).$$

Replacing w by $\tilde{w}$ does not change the set of optimum common base vectors, since y belongs to L (implying (by our assumption (47.37)) that $y^{\mathsf{T}}x$ is the same for all common base vectors x of f_1 and f_2).

By Lemma 47.4α, there exists a good pair $\tilde{w}_1, \tilde{w}_2$ with $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$ and

$$(47.50)\qquad \|\tilde{w}_i - w'_i\|_\infty \leq \|\tilde{w} - w'\|_1 < n$$

for $i = 1, 2$. Now for any $v \in W$ and $u \in S \setminus W$ we have $w'_1(v) > w'_1(u) + 2n$, as $\lambda_1(W) > 2n$, and as $\lambda_1(W') \geq 0$ for each $W' \in \mathcal{D}_1 \setminus \{S\}$. Hence, by (47.50), $\tilde{w}_1(v) > \tilde{w}_1(u)$. So (by the greedy method) any base vector x of f_1 maximizing $\tilde{w}_1^{\mathsf{T}}x$ satisfies $x(W) = f_1(W)$. This shows (47.48).

Let $\mathcal{C}_1 = \{U_1 \subset U_2 \subset \cdots \subset U_t = S\}$. For $j = 1, \ldots, t$, let $W_j := (W \cap U_j) \cup U_{j-1}$, where $U_0 := \emptyset$. Then $x(W_j) = f(W_j)$ for each optimum common base vector x (since W_j arises by taking unions and intersections from W, U_j, and U_{j-1}).

Moreover, $W_j \notin \mathcal{C}_1$ for at least one $j = 1, \ldots, t$, since

$$(47.51)\qquad \chi^W = \sum_{j=1}^{t}(\chi^{W_j} - \chi^{U_{j-1}})$$

while χ^W does not belong to L, implying that not all χ^{W_j} belong to L, and so some W_j does not belong to $\mathcal{C}_1$. So W_j can be added to $\mathcal{C}_1$, and we can iterate. ∎

From an optimum common base vector x, an optimum 'dual solution' w_1, w_2 can be derived, with a method of Cunningham and Frank [1985]. This gives:

Corollary 47.4a. *Problem* (47.25) *is solvable in strongly polynomial time.*

Proof. By Theorem 47.4, we can find a common base vector x of f_1 and f_2 maximizing $w^{\mathsf{T}}x$, in strongly polynomial time. Define a directed graph $D = (S, A)$ as follows.

For $i = 1, 2$, let A_i consist of all pairs (u, v) with $u, v \in S$ such that for each $U \subseteq V$:

$$(47.52)\qquad \text{if } x(U) = f_i(U) \text{ and } u \in U \text{ then } v \in U.$$

We can find A_i in strongly polynomial time, by finding the minimum of $f_i(U) - x(U)$ over subsets U of S with $u \in U$ and $v \notin U$ (with any strongly polynomial-time submodular function minimization algorithm).

Let D have arc set $A := A_1 \cup A_2^{-1}$ (taking two parallel arcs from u to v in case $(u, v) \in A_1$ and $(v, u) \in A_2$). Define a length function l on A by, for $(u, v) \in A$:

(47.53) $$l(u,v) := \begin{cases} w(v) - w(u) & \text{for } (u,v) \in A_1, \\ 0 & \text{for } (v,u) \in A_2. \end{cases}$$

We claim:

(47.54) D has no negative-length directed circuits.

For suppose that C is a negative-length directed circuit. We take such a C with $|AC|$ smallest. Then two consecutive arcs in C neither both belong to A_1 nor both belong to A_2^{-1}. For suppose that $a = (t,u)$ and $a' = (u,v)$ are in C and that they both belong to A_1. Then $(t,v) \in A_1$ and $l(a)+l(a') = l(t,v)$, contradicting the minimality of $|AC|$. This similarly gives a contradiction if $a, a' \in A_2^{-1}$.

So we can assume that C traverses the vertices $u_0, u_1, \ldots, u_k$ in this order, with $u_0 = u_k$, such that (u_{i-1}, u_i) belongs to A_1 if i is odd, and to A_2^{-1} if i is even. Let $X := \{u_1, u_3, \ldots, u_{k-1}\}$ and $Y := \{u_2, u_4, \ldots, u_k\}$. As C has negative length, we know $l(AC) < 0$, and hence $w(X) < w(Y)$.

By (47.52), for each $i = 1,2$ and for each $U \subseteq V$ with $x(U) = f_i(U)$ we have $|U \cap Y| \geq |U \cap X|$. Hence there exists an $\varepsilon > 0$ such that the vector

(47.55) $$x' := x + \varepsilon(\chi^X - \chi^Y)$$

belongs to EP_{f_1} and to EP_{f_2}. So, since $x'(S) = x(S)$, x' is again a common base vector of f_1 and f_2. However, $w^{\mathsf{T}}x' = w^{\mathsf{T}}x + w(X) - w(Y) > w^{\mathsf{T}}x$, contradicting the fact that x maximizes $w^{\mathsf{T}}x$. This proves (47.54).

By Theorem 8.7, we can find a potential $p : S \to \mathbb{Z}$ for D with respect to l, in strongly polynomial time. Then p satisfies

(47.56) $$\begin{array}{ll} p(v) - p(u) \leq w(v) - w(u) & \text{for each } (u,v) \in A_1, \\ p(v) - p(u) \geq 0 & \text{for each } (u,v) \in A_2. \end{array}$$

Define $w_1 := w - p$ and $w_2 := p$. We show that w_1 and w_2 are as required in (47.25).

(47.56) implies that, for each $i = 1,2$,

(47.57) if $(u,v) \in A_i$ then $w_i(v) \geq w_i(u)$.

We show that (47.57) implies that, for each $i = 1,2$, x is a base vector of f_i maximizing $w_i^{\mathsf{T}}x$, as required. We may assume $i = 1$.

Let μ and ν be the minimum and maximum value (respectively) of the entries in w_1. For $j \in \mathbb{Z}$, let $U_j := \{v \in S \mid w_1(v) \geq \mu + j\}$. Then, taking $k := \nu - \mu$,

(47.58) $$w_1 = \mu \cdot \chi^S + \sum_{j=1}^{k} \chi^{U_j}.$$

Moreover,

(47.59) $x(U_j) = f_1(U_j)$ for each $j = 1, \ldots, k$.

Indeed, for all $s \in U_j$ and $t \in S \setminus U_j$ we have $(s,t) \notin A_1$ (by (47.57), since $w_1(t) < \mu + j \leq w_1(s)$). Hence, by definition of A_1, there is a set $T_{s,t}$ with $s \in T_{s,t}$, $t \notin T_{s,t}$, and $x(T_{s,t}) = f_1(T_{s,t})$. As the collection of sets U with $x(U) = f_1(U)$ is closed under taking unions and intersections, (47.59) follows.

Then for any base vector x' of f_1 we have

$$(47.60) \qquad w_1^{\mathsf{T}} x' = \mu x'(S) + \sum_{j=1}^{k} x'(U_j) \leq \mu f_1(S) + \sum_{j=1}^{k} f_1(U_j).$$

By (47.59), we here have equality throughout for $x' := x$, which proves that x maximizes $w_1^{\mathsf{T}} x$ over all base vectors of f_1. ∎

Theorem 47.4 implies for (nonextended) polymatroids (extending a result of Schönsleben [1980] for integer f_1 and f_2 for which there is a fixed K with $P_{f_1} \cap P_{f_2} \subseteq [0,K]^S$):

Corollary 47.4b. *Given submodular set functions f_1, f_2 on S (by value giving oracles) and a weight function $w \in \mathbb{Q}^S$, we can find a maximum-weight vector $x \in P_{f_1} \cap P_{f_2}$ in strongly polynomial time.*

Proof. We can assume that $f_1(\emptyset) = f_2(\emptyset) = 0$ and that f_1 and f_2 are nondecreasing (as we can replace $f_i(U)$ by $\min_{T \supseteq U} f_i(T)$). Extend S with a new element t to a set $S' := S + t$. Define set functions f_1' and f_2' on S' by:

$$(47.61) \qquad f_i'(U) := f_i(U) \text{ and } f_i'(U+t) := 0$$

for $U \subseteq S$ and $i = 1,2$. Then f_1' and f_2' are submodular (using the nondecreasingness of f_1 and f_2). Moreover, consider any $x' \in \mathbb{R}^{S'}$ with $x'(S') = 0$. Let x be the restriction of x' to S. Then:

$$(47.62) \qquad x' \in EP_{f_i'} \text{ if and only if } x \in P_{f_i}.$$

Indeed, if $x' \in EP_{f_i'}$, then $x'(s) \geq 0$ for each $s \in S$, since $x'(S'-s) \leq f'(S'-s) = 0$ and $x'(S') = 0$, implying that $x(s) = x'(s) \geq 0$. Moreover, for each $U \subseteq S$ one has $x(U) = x'(U) \leq f_i'(U) = f_i(U)$. So $x \in P_{f_i}$.

Conversely, if $x \in P_f$, then for each $U \subseteq S$ one has $x'(U) = x(U) \leq f_i(U) = f_i'(U)$ and $x'(U+t) = x(U) - x(S) = -x(S \setminus U) \leq 0 = f_i'(U+t)$. So $x' \in EP_{f_i'}$. This proves (47.62).

Define $w' \in \mathbb{Q}^{S'}$ by $w'(v) := w(v)$ if $v \in S$, and $w'(t) := 0$. By Theorem 47.4, we can find a common base vector x' of f_1' and f_2' maximizing $w'^{\mathsf{T}} x'$ in strongly polynomial time. Let x be the restriction of x' to S. By (47.62), x maximizes $w^{\mathsf{T}} x$ over $P_{f_1} \cap P_{f_2}$. ∎

Similarly for maximum-weight common base vectors in (nonextended) polymatroids:

Corollary 47.4c. *Given submodular set functions f_1, f_2 on S (by value giving oracles) and a weight function $w \in \mathbb{Q}^S$, we can find a maximum-weight common base vector x of P_{f_1} and P_{f_2} in strongly polynomial time.*

Proof. Again, we can assume that $f_1(\emptyset) = f_2(\emptyset) = 0$ and that f_1 and f_2 are nondecreasing. Then the present corollary follows directly from Theorem 47.4, since

$$(47.63) \qquad P_{f_1} \cap P_{f_2} \cap \{x \mid x(S) = f_1(S)\} = EP_{f_1} \cap EP_{f_2} \cap \{x \mid x(S) = f_1(S)\}.$$

Indeed, if $x \in EP_{f_i}$ and $x(S) = f_i(S)$, then $x \geq \mathbf{0}$, since for any $s \in S$ one has $x_s = x(S) - x(S - s) \geq f_i(S) - f_i(S - s) \geq 0$. ∎

Back to extended polymatroids, Corollary 47.4b yields that we can optimize over the intersection of two extended polymatroids in strongly polynomial time:

Corollary 47.4d. *Given submodular set functions f_1, f_2 on S (by value giving oracles) and a weight function $w \in \mathbb{Q}_+^S$, we can find a maximum-weight vector $x \in EP_{f_1} \cap EP_{f_2}$ in strongly polynomial time.*

Proof. We may assume that $f_1(\emptyset) = f_2(\emptyset) = 0$. Let

$$(47.64) \qquad L := \max_{i=1,2}(|f_i(S)| + \sum_{s \in S} |f_i(\{s\})|).$$

Then $|f_i(U)| \leq L$ for each $i = 1, 2$ and $U \subseteq S$, since

$$(47.65) \qquad f_i(U) \leq \sum_{s \in U} f_i(\{s\}) \leq L$$

and

$$(47.66) \qquad f_i(U) \geq f_i(S) - f_i(S \setminus U) \geq f_i(S) - \sum_{s \in S \setminus U} f_i(\{s\}) \geq -L.$$

Let $K := |S| \cdot L + 1$. Then for any vertex x of $EP_{f_1} \cap EP_{f_2}$ and any $s \in S$:

$$(47.67) \qquad x(s) > -K,$$

since $x = A^{-1}b$ for some totally unimodular matrix A and some vector b whose entries are values of f_1 and f_2 (as in the proof of Theorem 46.1; observe that the entries of A^{-1} belong to $\{0, \pm 1\}$).

Define $f_i'(U) := f_i(U) + K \cdot |U|$. Then

$$(47.68) \qquad EP_{f_i'} = K \cdot \mathbf{1} + EP_{f_i}$$

for $i = 1, 2$. Hence, by (47.67), all vertices of $EP_{f_1'} \cap EP_{f_2'}$ are nonnegative. So any vector x maximizing $w^\mathsf{T} x$ over $P_{f_1'} \cap P_{f_2'}$ also maximizes $w^\mathsf{T} x$ over $EP_{f_1'} \cap EP_{f_2'}$. By Corollary 47.4b, x can be found in strongly polynomial time. ∎

47.5. Contrapolymatroids

Similar results hold for intersections of contrapolymatroids, by reduction to polymatroids. Given supermodular set functions g_1 and g_2 on S (by value giving oracles) and a weight function $w \in \mathbb{Q}^S$, we can find in strongly polynomial time:

(47.69) (i) a minimum-weight vector in $EQ_{g_1} \cap EQ_{g_2}$,
(ii) a minimum-weight common base vector of EQ_{g_1} and EQ_{g_2},
(iii) a minimum-weight vector in $Q_{g_1} \cap Q_{g_2}$, and
(iv) a minimum-weight common base vector of Q_{g_1} and Q_{g_2}.

Here (i) and (ii) follow from Corollary 47.4d and Theorem 47.4 applied to the submodular functions $-g_1$ and $-g_2$. Moreover, (iii) and (iv) follow by application of (i) and (ii) to the supermodular functions $\bar{g}_i$ given by $\bar{g}_i(U) = \max_{T \subseteq U} g_i(T)$ for $U \subseteq S$ and $i = 1, 2$ (assuming without loss of generality $g_1(\emptyset) = g_2(\emptyset) = 0$).

47.6. Intersecting a polymatroid and a contrapolymatroid

Let f be a submodular, and g a supermodular, set function on S. The results on polymatroid intersection also imply that

(47.70) a maximum-weight vector in the intersection of the extended polymatroid EP_f and the extended contrapolymatroid EQ_g can be found in strongly polynomial time,

assuming that we have value giving oracles for f and g.

To see this, we can assume that $f(\emptyset) = g(\emptyset) = 0$ and $g(S) \leq f(S)$. Let t be a new element. Define submodular set functions f_1 and f_2 on $S + t$ by:

(47.71) $f_1(U) := f(U)$, $f_1(U+t) := f(U) - g(S)$, $f_2(U) := f(S) - g(S \setminus U)$, $f_2(U + t) := -g(S \setminus U)$,

for $U \subseteq S$. Reset $f_1(S + t) := 0$. Then for each $x \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$:

(47.72) (x, λ) is a common base vector of EP_{f_1} and EP_{f_2}
$\iff \lambda = -x(S)$ and $x \in EP_f \cap EQ_g$.

To see necessity, let (x, λ) be a common base vector of EP_{f_1} and EP_{f_2}. As $f_1(S + t) = 0$, we have $\lambda = -x(S)$. Moreover, for any $U \subseteq S$, we have

(47.73) $x(U) \leq f_1(U) = f(U)$ and
$x(U) = x(S) - x(S \setminus U) = -\lambda - x(S \setminus U) \geq -f_2((S \setminus U) + t) = g(U)$.

So $x \in EP_f \cap EQ_g$.

To see sufficiency, assume $\lambda = -x(S)$ and $x \in EP_f \cap EQ_g$. Then for each $U \subseteq S$ we have:

(47.74)
$$\begin{aligned}
&x(U) \leq f(U) = f_1(U),\\
&x(U+t) = x(U) + \lambda = x(U) - x(S) \leq f(U) - g(S) = f_1(U+t),\\
&x(U) = x(S) - x(S \setminus U) \leq f(S) - g(S \setminus U) = f_2(U),\\
&x(U+t) = x(U) + \lambda = x(U) - x(S) = -x(S \setminus U) \leq -g(S \setminus U)\\
&= f_2(U+t).
\end{aligned}$$

So (x, λ) is a common base vector of EP_{f_1} and EP_{f_2}.

This shows (47.72), which implies that finding a minimum-weight vector in $EP_f \cap EQ_g$ amounts to finding a minimum-weight common base vector of EP_{f_1} and EP_{f_2}.

Similarly, we can find a modular function h satisfying $g \leq h \leq f$ in strongly polynomial time, if $g \leq f$ (Frank's discrete sandwich theorem (Corollary 46.2b)). To see this, let f_1 and f_2 be as above, and find an (x, λ) in $EP_{f_1} \cap EP_{f_2}$ maximizing $x(S) + \lambda$. If $x(S) + \lambda \geq 0$, then $x \in EP_f \cap EQ_g$, that is x gives a modular function h with $g \leq h \leq f$.

47.6a. Further notes

Polymatroid intersection is a special case of submodular flow, as discussed in Chapter 60. We therefore refer for further algorithmic work to the notes in Section 60.3e.

A preflow-push algorithm for finding a maximum common vector in the intersection of two polymatroids was presented by Fujishige and Zhang [1992].

Tardos, Tovey, and Trick [1986] gave an improved version of Cunningham and Frank's polynomial-time algorithm for weighted polymatroid intersection. Fujishige [1978a] gave a (non-polynomial-time) algorithm for weighted polymatroid intersection. Optimizing over the intersection of a base polytope and an affine space was considered by Hartvigsen [1996,1998a,2001a].

Frank [1984c] and Fujishige and Iwata [2000] gave surveys.

Chapter 48

Dilworth truncation

If a submodular function f has $f(\emptyset) < 0$, the associated extended polymatroid is empty, as the conditions $x(U) \leq f(U)$ for all U include $x(\emptyset) < f(\emptyset)$. However, by ignoring the condition for $U = \emptyset$, the obtained polyhedron is yet an extended polymatroid, for a different submodular function, denoted by $\hat{f}$. This function $\hat{f}$ is called the *Dilworth truncation* of f.

48.1. If $f(\emptyset) < 0$

Let f be a submodular set function on S. If $f(\emptyset) < 0$, the associated extended polymatroid EP_f is empty. However, by ignoring the empty set, we yet obtain an extended polymatroid. (The interest in this goes back to Dilworth [1944].)

Consider the system

(48.1) $\qquad x(U) \leq f(U)$ for $U \in \mathcal{P}(S) \setminus \{\emptyset\}$,

and the problem dual to maximizing $w^{\mathsf{T}}x$ over (48.1), for $w \in \mathbb{R}^S_+$:

(48.2)
$$\min\{\sum_{U \in \mathcal{P}(S)\setminus\{\emptyset\}} y(U)f(U) \mid y \in \mathbb{R}_+^{\mathcal{P}(S)\setminus\{\emptyset\}}, \sum_{U \in \mathcal{P}(S)\setminus\{\emptyset\}} y(U)\chi^U = w\}.$$

Recall that a collection $\mathcal{F}$ of sets is called *laminar* if

(48.3) $\qquad T \cap U = \emptyset$ or $T \subseteq U$ or $U \subseteq T$ for all $T, U \in \mathcal{F}$.

Then a basic result of Edmonds [1970b] is:

Theorem 48.1. *If f is a submodular set function on S, then* (48.2) *has an optimum solution y with $\mathcal{F} := \{U \in \mathcal{P}(S) \setminus \{\emptyset\} \mid y(U) > 0\}$ laminar.*

Proof. Let $y : \mathcal{P}(S) \setminus \{\emptyset\} \to \mathbb{R}_+$ achieve this minimum, with

(48.4)
$$\sum_{U \in \mathcal{P}(S)\setminus\{\emptyset\}} y(U)|U||S \setminus U|$$

as small as possible. Assume that $\mathcal{F}$ is not laminar, and choose $T, U \in \mathcal{F}$ violating (48.3). Let $\alpha := \min\{y(T), y(U)\}$. Decrease $y(T)$ and $y(U)$ by α, and increase $y(T \cap U)$ and $y(T \cup U)$ by α. Since

(48.5) $$\chi^{T\cap U} + \chi^{T\cup U} = \chi^{T} + \chi^{U},$$

y remains a feasible solution of (48.2). As moreover

(48.6) $$f(T \cap U) + f(T \cup U) \leq f(T) + f(U),$$

f remains optimum. However, by Theorem 2.1, sum (48.4) decreases, contradicting our assumption. ∎

This implies that system (48.1) is TDI. More generally, it implies the box-total dual integrality of (48.1):

Corollary 48.1a. *For any submodular set function f on S, system* (48.1) *is box-totally dual integral.*

Proof. Consider some $w : S \to \mathbb{Z}_+$, and problem (48.2) dual to maximizing $w^{\mathsf{T}}x$ over (48.1). By Theorem 48.1, this minimum is attained by a y with $\mathcal{F} := \{U \in \mathcal{P}(S) \setminus \{\emptyset\} \mid y(U) > 0\}$ laminar. Hence the constraints corresponding to positive entries in y form a totally unimodular matrix (by Theorem 41.11). Therefore, by Theorem 5.35, (48.1) is box-TDI. ∎

Let EP'_f denote the solution set of (48.1). So EP'_f is nonempty for each submodular function f. As for integrality we have:

Corollary 48.1b. *If f is submodular and integer, then EP'_f is integer.*

Proof. Directly from Corollary 48.1a. ∎

In fact, as we shall see in Section 48.2, EP'_f is again an extended polymatroid.

48.2. Dilworth truncation

For each submodular function f, there exists a unique largest submodular function $\hat{f}$ with the property that $\hat{f}(U) \leq f(U)$ for each nonempty $U \subseteq S$, and $\hat{f}(\emptyset) = 0$. This follows from a method of Dilworth [1944].

Let f be a submodular set function on S. The *Dilworth truncation* $\hat{f} : \mathcal{P}(S) \to \mathbb{R}$ of f is given by:

(48.7) $$\hat{f}(U) := \min\{\sum_{P\in\mathcal{P}} f(P) \mid \mathcal{P} \text{ is a partition of } U \text{ into nonempty sets}\}$$

for $U \subseteq S$. So $\hat{f}(\emptyset) = 0$ (as for $U = \emptyset$, only $\mathcal{P} = \emptyset$ qualifies in (48.7)). Dunstan [1976] showed:

Theorem 48.2. *$\hat{f}$ is submodular.*

Proof. Choose $T, U \subseteq S$, and let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of T and U (respectively) into nonempty sets with

$$\hat{f}(T) = \sum_{P \in \mathcal{P}} f(P) \text{ and } \hat{f}(U) = \sum_{Q \in \mathcal{Q}} f(Q). \tag{48.8}$$

Consider the family $\mathcal{F}$ made by $\mathcal{P}$ and $\mathcal{Q}$ (taking a set twice if it occurs in both partitions). We can transform $\mathcal{F}$ iteratively into a laminar family, by replacing any $X, Y \in \mathcal{F}$ with $X \cap Y \neq \emptyset$ and $X \not\subseteq Y \not\subseteq X$ by $X \cap Y, X \cup Y$. In each iteration, the sum

$$\sum_{Z \in \mathcal{F}} f(Z) \tag{48.9}$$

does not increase (as f is submodular). As at each iteration the sum

$$\sum_{Z \in \mathcal{F}} |Z||S \setminus Z| \tag{48.10}$$

decreases (by Theorem 2.1), this process terminates. We end up with a laminar family $\mathcal{F}$.

The inclusionwise maximal sets in $\mathcal{F}$ form a partition $\mathcal{R}$ of $T \cup U$, and the remaining sets form a partition $\mathcal{S}$ of $T \cap U$. Therefore,

$$\begin{aligned} \hat{f}(T \cup U) + \hat{f}(T \cap U) &\leq \sum_{X \in \mathcal{R}} f(X) + \sum_{Y \in \mathcal{S}} f(Y) \\ &\leq \sum_{P \in \mathcal{P}} f(P) + \sum_{Q \in \mathcal{Q}} f(Q) = \hat{f}(T) + \hat{f}(U), \end{aligned} \tag{48.11}$$

showing that $\hat{f}$ is submodular. ∎

Lovász [1983c] observed that $\hat{f}$ is the unique largest among all submodular set functions g on S with $g(\emptyset) = 0$ and $g(U) \leq f(U)$ for $U \neq \emptyset$. Indeed, each subset U of S can be partitioned into nonempty sets $U_1, \ldots, U_t$ such that

$$g(U) \leq \sum_{i=1}^{t} g(U_i) \leq \sum_{i=1}^{t} f(U_i) = \hat{f}(U) \tag{48.12}$$

(the first inequality follows from the submodularity of g, as $g(\emptyset) = 0$).

Trivially, $EP_{\hat{f}} = EP'_f$. In particular, EP'_f is an extended polymatroid. Moreover, by (44.34),

Theorem 48.3.

$$\hat{f}(U) = \max\{x(U) \mid x \in EP'_f\}. \tag{48.13}$$

Proof. By (44.34), since $EP'_f = EP_{\hat{f}}$. ∎

$\hat{f}(U)$ can be computed in strongly polynomial time:

Theorem 48.4. *If a submodular set function f on S is given by a value giving oracle, then for each given $U \subseteq S$, $\hat{f}(U)$ can be computed in strongly polynomial time.*

Proof. We can assume that $U = S$. Order $S = \{s_1, \ldots, s_n\}$ arbitrarily. For $i = 1, \ldots, n$, define $U_i := \{s_1, \ldots, s_i\}$. Set $x := \mathbf{0}$. Iteratively, for $i = 1, \ldots, n$, determine

(48.14) $$\mu := \min\{f(T) - x(T) \mid s_i \in T \subseteq U_i\}$$

(with a submodular function minimization algorithm), and reset $x(s_i) := x(s_i) + \mu$.

We end up with $x \in EP'_f$ and for each $u \in S$ a subset T_u of S with $u \in T_u$ and $x(T_u) = f(T_u)$. As the collection of subsets T of S with $x(T) = f(T)$ is closed under unions and intersections of intersecting sets (cf. Theorem 44.2), we can modify the T_u in such a way that they form a partition $U_1, \ldots, U_k$ of S. Then $\hat{f}(S) = f(U_1) + \cdots + f(U_k)$, as x attains the maximum in (48.13). ∎

As a consequence, given a submodular set function f on S (by a value giving oracle), we can optimize over EP'_f in strongly polynomial time (by Corollary 44.3b, as $EP'_f = EP_{\hat{f}}$ and as we can compute $\hat{f}$).

48.2a. Applications and interpretations

Graphic matroids (Dilworth [1944], also Edmonds [1970b], Dunstan [1976]). Let $G = (V, E)$ be an undirected graph and let for each $F \subseteq E$, $f(F)$ be given by

(48.15) $$f(F) := \left|\bigcup F\right| - 1.$$

It is easily checked that the function f is submodular, and that the function $\hat{f}$ as given by (48.7) satisfies

(48.16) $$\hat{f}(F) = |V| \text{ minus the number of components of the graph } (V, F),$$

i.e., $\hat{f}$ is the rank function of the cycle matroid of G.

Geometric interpretation. The operation of making $\hat{f}$ from f can be interpreted geometrically as follows (Lovász [1977c], Mason [1977,1981]).

Let $\mathcal{F}$ be a collection of flats (subspaces) in a projective space, and define for each subset $\mathcal{F}'$ of $\mathcal{F}$, the rank $r(\mathcal{F}')$ by

(48.17) $$r(\mathcal{F}') := \text{the (projective) dimension of } \bigcup \mathcal{F}'.$$

One easily checks that r is nondecreasing and submodular and that $r(\emptyset) = 0$. Now let

(48.18) $\quad f(\mathcal{F}') := r(\mathcal{F}') - 1$

for $\mathcal{F}' \subseteq \mathcal{F}$, and consider the function $\hat{f}$. Then $\hat{f}$ can be interpreted geometrically as follows. Let H be some hyperplane 'in general position' in the projective space. Then $\hat{f}(\mathcal{F}')$ is equal to the projective dimension of $H \cap \bigcup \mathcal{F}'$, i.e., $\hat{f}$ is as given by (48.17) if we replace $\mathcal{F}$ by $\{F \cap H \mid F \in \mathcal{F}\}$ (see Lovász [1977c] and Mason [1977, 1981]).

Rigidity. Let $M = (S, \mathcal{I})$ be a loopless matroid, with rank function r. Let d be a natural number. Define the set function f on S by

(48.19) $\quad f(U) := d \cdot r(U) - d + 1,$

for $U \subseteq S$. Again, f is submodular and nondecreasing. Moreover, the function $\hat{f}$ is the rank function of a loopless matroid, as $\hat{f}(\{s\}) = f(\{s\}) = 1$ for all s in S.

Let $M_d = (S, \mathcal{I}_d)$ be this matroid. Since $EP'_f = EP_{\hat{f}}$, a subset I of S is independent in M_d if and only if

(48.20) $\quad |U| \leq d \cdot r(U) - d + 1$

for all $U \subseteq I$.

In case M is the cycle matroid of a connected graph $G = (V, E)$, this relates to the following (cf. Crapo [1979] and Crapo and Whiteley [1978]). Let the vertices of G be placed 'in general position' in the d-dimensional Euclidean space. Make the edges 'rigid bars'. Suppose now that the whole graph G is rigid (which only depends on G and not on the embedding, since the vertices are 'in general position'). Then G is called *rigid (in d dimensions)*. It is not difficult to see that the minimal sets F of edges of G for which the subgraph (V, F) is rigid, form the bases of a matroid. For $d = 1$ this matroid is just the cycle matroid of G, as can be checked easily. Laman [1970] (cf. Asimow and Roth [1978,1979]) showed that for $d = 2$, a graph $G = (V, E)$ is a base (i.e., a minimal rigid graph), if and only if

(48.21) $\quad$ (i) $|E| = 2|V| - 3$,
$\quad$ (ii) $|E[U]| \leq 2|U| - 3$, for each $U \subseteq V$.

Now if M is the cycle matroid of a rigid graph G, with rank function r, then (48.21)(ii) is equivalent to

(48.22) $\quad |F| \leq 2r(F) - 1$, for each subset F of E,

that is, by (48.20), to: E is independent in the matroid M_2, as given above. Condition (48.21)(i) implies that M_2 has rank $2r(E) - 1$. Hence, if G is rigid in 2 dimensions, then the bases of M_2 are the minimally rigid subgraphs of G in 2 dimensions.

In general, the matroid of rigid subgraphs of a graph $G = (V, E)$ (in d dimensions) has rank $d|V| - \binom{d+1}{2}$. However, it is not necessarily true that G is minimally rigid in d dimensions if and only if G has $d|V| - \binom{d+1}{2}$ edges and each subgraph (U, F) of G has at most $d|U| - \binom{d+1}{2}$ edges. For instance, if G arises from glueing two copies of the complete graph K_5 together in two vertices, and deleting the edge connecting these two vertices, then G is not rigid in 3 dimensions, but it satisfies the conditions given above for $d = 3$. (These conditions are easily seen to be necessary.)

More on the relation between rigidity and matroid union can be found in Whiteley [1988].

48.3. Intersection

Corollaries 48.1a and 48.1b on submodular functions f not necessarily satisfying $f(\emptyset) \geq 0$, can be extended to pairs of functions. Let f_1 and f_2 be submodular set functions on S, and consider the system

(48.23) $\quad x(U) \leq f_i(U)$ for $U \in \mathcal{P}(S) \setminus \{\emptyset\}$ and $i = 1, 2$.

Then:

Theorem 48.5. *System* (48.23) *is box-totally dual integral.*

Proof. Choose $w \in \mathbb{Z}^S$, and consider the problem dual to maximizing $w^{\mathsf{T}} x$ over (48.23):

(48.24)
$$\min\{\sum_{U \in \mathcal{P}(S) \setminus \{\emptyset\}} (y_1(U) f_1(U) + y_2(U) f_2(U)) \mid y_1, y_2 \in \mathbb{R}_+^{\mathcal{P}(S) \setminus \{\emptyset\}}, \sum_{U \in \mathcal{P}(S) \setminus \{\emptyset\}} (y_1(U) + y_2(U)) \chi^U = w\}.$$

Let y_1, y_2 attain the minimum.

For $i \in \{1, 2\}$, define

(48.25)
$$w_i := \sum_{U \in \mathcal{P}(S) \setminus \{\emptyset\}} y_i(U) \chi^U.$$

By Theorem 48.1, for each $i = 1, 2$,

(48.26)
$$\min\{\sum_{U \in \mathcal{P}(S) \setminus \{\emptyset\}} y_i(U) f_i(U) \mid y_i \in \mathbb{R}_+^{\mathcal{P}(S) \setminus \{\emptyset\}}, \sum_{U \in \mathcal{P}(S) \setminus \{\emptyset\}} y_i(U) \chi^U = w_i\}$$

has an optimum solution y_i with $\mathcal{F}_i := \{U \mid y_i(U) > 0\}$ laminar.

These (modified) y_1, y_2 again are optimum in (48.24). As the constraints corresponding to positive components of y_1, y_2 give a totally unimodular matrix (by Theorem 41.11), Theorem 5.35 implies that system (48.23) is box-TDI. ∎

Theorem 48.5 implies primal integrality:

Corollary 48.5a. *If f_1 and f_2 are submodular and integer, then $EP'_{f_1} \cap EP'_{f_2}$ is box-integer.*

Proof. Directly from Theorem 48.5. ∎

Given submodular functions f_1 and f_2 (by value giving oracles), we can optimize over $EP'_{f_1} \cap EP'_{f_2}$ in strongly polynomial time (by Corollary 47.4d, as $EP'_{f_1} = EP_{\hat{f}_1}$ and $EP'_{f_2} = EP_{\hat{f}_2}$).

Chapter 49

Submodularity more generally

We now discuss a number of generalizations of submodular functions, namely those defined on a subcollection $\mathcal{C}$ of the collection of all subsets of a set S. The results are similar to those for submodular functions defined on all subsets on S. Often, the corresponding polyhedra form a polymatroid for some derived submodular function defined on all subsets of S.
We consider three kinds of collections, in order of increasing generality: lattice families, intersecting families, and crossing families.

49.1. Submodular functions on a lattice family

We first consider the generalization of submodular functions to those defined on a 'lattice family'.

Let S be a finite set. A family $\mathcal{C}$ of sets is called a *lattice family* if

(49.1) $\quad T \cap U, T \cup U \in \mathcal{C}$ for all $T, U \in \mathcal{C}$.

For a lattice family $\mathcal{C}$, a function $f : \mathcal{C} \to \mathbb{R}$ is called *submodular* if

(49.2) $\quad f(T \cap U) + f(T \cup U) \leq f(T) + f(U)$

for all $T, U \in \mathcal{C}$. Consider the system

(49.3) $\quad x(U) \leq f(U)$ for $U \in \mathcal{C}$,

and the problem dual to maximizing $w^{\mathsf{T}}x$ over (49.3), for $w \in \mathbb{R}^S$:

(49.4) $$\min\{\sum_{U \in \mathcal{C}} y(U)f(U) \mid y \in \mathbb{R}_+^{\mathcal{C}}, \sum_{U \in \mathcal{C}} y(U)\chi^U = w\}.$$

Theorem 49.1. *Let $\mathcal{C}$ be a lattice family, $f : \mathcal{C} \to \mathbb{R}$ a submodular function, and $w \in \mathbb{R}^S$. Then* (49.4) *has an optimum solution y with $\mathcal{F} := \{U \in \mathcal{C} \mid y(U) > 0\}$ a chain.*

Proof. Let $y : \mathcal{C} \to \mathbb{R}_+$ achieve this minimum, with

(49.5) $$\sum_{U \in \mathcal{C}} y(U)|U||S \setminus U|$$

as small as possible. Assume that $\mathcal{F}$ is not a chain, and choose $T, U \in \mathcal{F}$ with $T \not\subseteq U$ and $U \not\subseteq T$. Let $\alpha := \min\{y(T), y(U)\}$. Decrease $y(T)$ and $y(U)$ by α, and increase $y(T \cap U)$ and $y(T \cup U)$ by α. Since

$$\chi^{T\cap U} + \chi^{T\cup U} = \chi^T + \chi^U, \tag{49.6}$$

y remains a feasible solution of (49.4). As moreover

$$f(T \cap U) + f(T \cup U) \leq f(T) + f(U), \tag{49.7}$$

f remains optimum. However, by Theorem 2.1, sum (49.5) decreases, contradicting our assumption. ∎

This implies the box-total dual integrality of (49.3):

Corollary 49.1a. *If $\mathcal{C}$ is a lattice family and $f : \mathcal{C} \to \mathbb{R}$ is submodular, then system* (49.3) *is box-TDI.*

Proof. Consider some $w : \mathcal{C} \to \mathbb{Z}$, and problem (49.4) dual to maximizing $w^{\mathsf{T}}x$ over (49.3). By Theorem 49.1, this minimum is attained by a y with $\mathcal{F} := \{U \in \mathcal{C} \mid y(U) > 0\}$ a chain. So the constraints corresponding to positive components of y form a totally unimodular matrix (by Theorem 41.11). Hence by Theorem 5.35, (49.3) is box-TDI. ∎

For any $\mathcal{C} \subseteq \mathcal{P}(S)$ and $f : \mathcal{C} \to \mathbb{R}$, define:

$$\begin{aligned} P_f &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \leq f(U) \text{ for each } U \in \mathcal{C}\}, \\ EP_f &:= \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for each } U \in \mathcal{C}\}. \end{aligned} \tag{49.8}$$

Then Corollary 49.1a implies:

Corollary 49.1b. *If $\mathcal{C}$ is a lattice family and $f : \mathcal{C} \to \mathbb{R}$ is submodular and integer, then EP_f is box-integer.*

Proof. Directly from Corollary 49.1a. ∎

Another consequence of Theorem 49.1 is that a submodular function f on a lattice family is uniquely determined by EP_f (given the lattice family):

Corollary 49.1c. *If $\mathcal{C}$ is a lattice family and $f : \mathcal{C} \to \mathbb{R}$ is submodular, then*

$$f(U) = \max\{x(U) \mid x \in EP_f\} \tag{49.9}$$

for each $U \in \mathcal{C}$.

Proof. Let $w := \chi^U$ and let y attain minimum (49.4), with $\mathcal{F} := \{T \in \mathcal{C} \mid y(T) > 0\}$ a chain. Since

$$\chi^U = w = \sum_{T\in\mathcal{F}} y(T)\chi^T, \tag{49.10}$$

we know that $\mathcal{F} = \{U\}$ and $y(U) = 1$. So the maximum in (49.9) is equal to $\sum_{T\in\mathcal{C}} y(T)f(T) = y(U)f(U) = f(U)$. ∎

We note that for any lattice family $\mathcal{C} \subseteq \mathcal{P}(S)$ with $\bigcup\mathcal{C} = S$, and any submodular function $f : \mathcal{C} \to \mathbb{R}$, the polytope P_f is a polymatroid. Indeed, define

$$(49.11) \qquad f'(U) := \min\{f(T) \mid T \in \mathcal{C}, T \supseteq U\}.$$

for $U \subseteq S$. Then f' is submodular, and $P_{f'} = P_f$.

49.2. Intersection

Also the intersection of two of the polyhedra EP_f is tractable. Let S be a finite set. For $i = 1, 2$, let $\mathcal{C}_i$ be a lattice family on S and let $f_i : \mathcal{C}_i \to \mathbb{R}$ be submodular. Consider the system

$$(49.12) \qquad \begin{aligned} x(U) &\leq f_1(U) \text{ for } U \in \mathcal{C}_1,\\ x(U) &\leq f_2(U) \text{ for } U \in \mathcal{C}_2. \end{aligned}$$

Then:

Corollary 49.1d. *System* (49.12) *is box-TDI.*

Proof. Choose $w \in \mathbb{R}^S$, and consider the problem dual to maximizing $w^{\mathsf{T}}x$ over (49.12):

$$(49.13) \qquad \begin{aligned} \min\{&\sum_{U\in\mathcal{C}_1} y_1(U)f_1(U) + \sum_{U\in\mathcal{C}_2} y_2(U)f_2(U) \mid \\ &y_1 \in \mathbb{R}_+^{\mathcal{C}_1}, y_2 \in \mathbb{R}_+^{\mathcal{C}_2}, \sum_{U\in\mathcal{C}_1} y_1(U)\chi^U + \sum_{U\in\mathcal{C}_2} y_2(U)\chi^U = w\}. \end{aligned}$$

Let y_1, y_2 attain the minimum.

For $i \in \{1, 2\}$, define

$$(49.14) \qquad w_i := \sum_{U\in\mathcal{C}_i} y_i(U)\chi^U.$$

By Theorem 49.1, for each $i = 1, 2$,

$$(49.15) \qquad \min\{\sum_{U\in\mathcal{C}_i} y_i(U)f_i(U) \mid y_i \in \mathbb{R}_+^{\mathcal{C}_i}, \sum_{U\in\mathcal{C}_i} y_i(U)\chi^U = w_i\}$$

has an optimum solution y_i with $\mathcal{F}_i := \{U \in \mathcal{C}_i \mid y_i(U) > 0\}$ a chain.

These y_1, y_2 again are optimum in (49.13). So, by Theorem 41.11, the constraints corresponding to positive components of y form a totally unimodular matrix. Hence by Theorem 5.35, (49.12) is box-TDI. ∎

This implies primal integrality:

Corollary 49.1e. *If f_1 and f_2 are submodular and integer, then $EP_{f_1} \cap EP_{f_2}$ is box-integer.*

Proof. Directly from Corollary 49.1d. ∎

49.3. Complexity

To find the minimum of a submodular function f defined on a lattice family $\mathcal{C}$ in polynomial time, just an oracle telling if a set U belongs to $\mathcal{C}$, and if so, giving $f(U)$, is not sufficient: if $\mathcal{C} = \{\emptyset, T, S\}$ for some $T \subseteq S$, with $f(\emptyset) = f(S) = 0$ and $f(T) = -1$, we cannot find T by a polynomially bounded number of oracle calls. So we need to have more information on $\mathcal{C}$.

A lattice family $\mathcal{C}$ is fully characterized by the smallest set M and the largest set L in $\mathcal{C}$, together with the pre-order $\preceq$ on S defined by:

(49.16) $\quad u \preceq v \iff$ each $U \in \mathcal{C}$ containing v also contains u.

Then $\preceq$ is a pre-order (that is, it is reflexive and transitive). A subset U of S belongs to $\mathcal{C}$ if and only if $M \subseteq U \subseteq L$ and U is a lower ideal in $\preceq$ (that is, if $v \in U$ and $u \preceq v$, then $u \in U$).

Hence $\mathcal{C}$ has a description of size $O(|S|^2)$, such that for given $U \subseteq S$ one can test in polynomial time if U belongs to $\mathcal{C}$.

For $U \subseteq S$, define

(49.17) $\quad U^{\downarrow} := \{s \in S \mid \exists t \in U : s \preceq t\}$ and
$\quad U^{\uparrow} := \{s \in S \mid \exists t \in U : t \preceq s\}$.

Set

(49.18) $\quad v^{\uparrow} := \{v\}^{\uparrow}$, $v^{\downarrow} := \{v\}^{\downarrow}$, $\tilde{v} := v^{\uparrow} \cap v^{\downarrow}$.

For any $U \subseteq S$, let $\overline{U}$ be the (unique) smallest set in $\mathcal{C}$ containing $U \cap L$; that is,

(49.19) $\quad \overline{U} = (U \cap L)^{\downarrow} \cup M$.

So having L, M, and $\preceq$, the set $\overline{U}$ can be determined in polynomial time.

Determine a number $\alpha > 0$ such that

(49.20) $\quad \alpha \geq f(S \setminus v^{\uparrow}) - f((S \setminus v^{\uparrow}) \cup \tilde{v})$ and $\alpha \geq f(v^{\downarrow}) - f(v^{\downarrow} \setminus \tilde{v})$

for all $v \in L \setminus M$. Such an α can be found by at most $4|S|$ oracle calls.

Then α satisfies, for any $X, Y \in \mathcal{C}$ with $X \subseteq Y$:

(49.21) $\quad |f(Y) - f(X)| \leq \alpha |Y \setminus X|$.

To show this, we can assume that $Y \setminus X = \tilde{v}$ for some $v \in L \setminus M$. Then $f(Y) - f(X) \leq f(v^{\downarrow}) - f(v^{\downarrow} \setminus \tilde{v}) \leq \alpha$ and $f(Y) - f(X) \geq f((S \setminus v^{\uparrow}) \cup \tilde{v}) - f(S \setminus v^{\uparrow}) \geq -\alpha$, implying (49.21).

Now define a function $\bar{f} : \mathcal{P}(S) \to \mathbb{R}$ by:

(49.22) $\quad \bar{f}(U) := f(\overline{U}) + \alpha|\overline{U} \triangle U|$

for $U \subseteq S$.

Then:

Theorem 49.2. *For any $\alpha > 0$ satisfying (49.20) for all $v \in L \setminus M$, the function $\bar{f}$ is submodular.*

Proof. First consider $T, U \subseteq L$. Then $T \subseteq \overline{T}$ and $U \subseteq \overline{U}$, and hence:

(49.23)
$$\begin{aligned}\bar{f}(T) + \bar{f}(U) &= f(\overline{T}) + \alpha|\overline{T} \setminus T| + f(\overline{U}) + \alpha|\overline{U} \setminus U| \\ &\geq f(\overline{T} \cap \overline{U}) + \alpha|(\overline{T} \cap \overline{U}) \setminus (T \cap U)| + f(\overline{T} \cup \overline{U}) + \alpha|(\overline{T} \cup \overline{U}) \setminus (T \cup U)| \\ &\geq f(\overline{T \cap U}) + \alpha|\overline{T \cap U} \setminus (T \cap U)| + f(\overline{T \cup U}) + \alpha|\overline{T \cup U} \setminus (T \cup U)| \\ &= \bar{f}(T \cap U) + \bar{f}(T \cup U).\end{aligned}$$

(The last inequality uses (49.21), since $\overline{T} \cap \overline{U} \supseteq \overline{T \cap U}$ (while $\overline{T} \cup \overline{U} = \overline{T \cup U}$).) Hence, for $T, U \subseteq S$ one has:

(49.24)
$$\begin{aligned}\bar{f}(T) + \bar{f}(U) &= \bar{f}(T \cap L) + \alpha|T \setminus L| + \bar{f}(U \cap L) + \alpha|U \setminus L| \\ &\geq \bar{f}((T \cap L) \cap (U \cap L)) + \bar{f}((T \cap L) \cup (U \cap L)) + \alpha|T \setminus L| + \alpha|U \setminus L| \\ &= \bar{f}((T \cap U) \cap L) + \bar{f}((T \cup U) \cap L) + \alpha|(T \cap U) \setminus L| + \alpha|(T \cup U) \setminus L| \\ &= \bar{f}(T \cap U) + \bar{f}(T \cup U).\end{aligned}$$

So $\bar{f}$ is submodular. ∎

The function $\bar{f}$ enables us to reduce optimization problems on submodular functions defined on a lattice family, to those defined on all subsets.

Minimization. By Theorem 45.1, the minimum of $\bar{f}$ can be found in strongly polynomial time. Hence

(49.25) if $\mathcal{C}$ is given by L, M, and $\preceq$, and a submodular function $f : \mathcal{C} \to \mathbb{R}$ is given by a value giving oracle, we can find a $U \in \mathcal{C}$ minimizing $f(U)$ in strongly polynomial time.

Indeed, if $\bar{f}$ attains its minimum at U, then $U \in \mathcal{C}$, since otherwise $\overline{U} \neq U$ and hence $\bar{f}(U) > \bar{f}(\overline{U})$ (as $\alpha > 0$), contradicting the fact that $\bar{f}$ attains its minimum at U. This shows (49.25).

Maximization over EP_f. Given a lattice family $\mathcal{C}$ of subsets of a set S, a submodular function $f : \mathcal{C} \to \mathbb{R}$, and a weight function $w \in \mathbb{Q}^S$, we can maximize $w^{\mathsf{T}}x$ over EP_f, by adapting the greedy algorithm as follows.

Note that $\max\{w^{\mathsf{T}}x \mid x \in EP_f\}$ is finite if and only if $w \geq \mathbf{0}$, $w(s) = 0$ for each $s \in S \setminus L$, and

(49.26) $\quad u \preceq v$ implies $w(u) \geq w(v)$

for all $u, v \in S$. If (49.26) is not the case, the maximum value is infinite, since if $u \preceq v$, then for any $x \in EP_f$, the vector $x + \lambda(\chi^v - \chi^u)$ belongs to EP_f for all $\lambda \geq 0$. Now, if $w(v) > w(u)$, the weight increases to infinity along this line, and therefore the maximum value is ∞.

So we can check in strongly polynomial time if $\max\{w^{\mathsf{T}}x \mid x \in EP_f\}$ is finite, and therefore we can assume that it is finite. Moreover, we can assume that $L = S$, since $w(s) = 0$ for each $s \in S \setminus L$, and hence we can delete $S \setminus L$. Similarly, we can assume that $\emptyset \in \mathcal{C}$ and $f(\emptyset) = 0$. For if $\emptyset \in \mathcal{C}$ and $f(\emptyset) < 0$, then $EP_f = \emptyset$, and if $f(\emptyset) > 0$, we can reset $f(\emptyset) := 0$, without violating the submodularity and without modifying EP_f. If $\emptyset \notin \mathcal{C}$, then we can add $\emptyset$ to $\mathcal{C}$ and set $f(\emptyset) := 0$, again maintaining submodularity and EP_f. Finally, we can assume that $\preceq$ is a partial order, since if $u \preceq v \preceq u$, then by (49.26), $w(u) = w(v)$, and each set in $\mathcal{C}$ either contains both u and v, or neither of them. So we can merge u and v; and in fact we can merge any set $\tilde{v}$ to one element.

Now let $\leq$ be a linear order such that for any u, v, if $u \preceq v$ or $w(u) > w(v)$, then $u \leq v$. By (49.26), the latter defines a partial order. So $\leq$ is a linear extension of it, and hence can be found in strongly polynomial time.

Let $S = \{s_1, \ldots, s_n\}$ with $s_1 < s_2 < \cdots < s_n$. For $i = 0, \ldots, n$, define $U_i := \{s_1, \ldots, s_i\}$. As $\leq$ is a linear extension of $\preceq$, each U_i is a lower ideal of $\preceq$, and hence each U_i belongs to $\mathcal{C}$. Define $x(s_i) := f(U_i) - f(U_{i-1})$ for $i = 1, \ldots, n$. Then x maximizes $w^{\mathsf{T}}x$ over EP_f.

To see this, let $\bar{f}$ be defined as above. Then by Theorem 44.3, x belongs to $EP_{\bar{f}}$ (as f and $\bar{f}$ coincide on each U_i), and hence x belongs to EP_f. To see that x is optimum, we have for any $z \in EP_f$:

$$(49.27)\qquad \begin{aligned} w^{\mathsf{T}}z &= \sum_{i=1}^{n-1} z(U_i)(w(s_i) - w(s_{i+1})) + z(S)w(s_n) \\ &\leq \sum_{i=1}^{n-1} f(U_i)(w(s_i) - w(s_{i+1})) + f(S)w(s_n) \\ &= \sum_{i=1}^{n} w(s_i)(f(U_i) - f(U_{i-1})) = \sum_{i=1}^{n} w(s_i)x(s_i) = w^{\mathsf{T}}x. \end{aligned}$$

This also gives a dual solution to the corresponding LP-formulation of the problem.

Maximization over intersections. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be lattice families of subsets of S and let f_1 and f_2 be submodular functions on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Let $\mathcal{C}_i$ be specified by L_i, M_i, and $\preceq_i$.

Find a number $\alpha > 0$ satisfying (49.20) for both $f = f_1$ and $f = f_2$. So by (49.21), $\alpha|S| + \max_{i=1,2} |f_i(L_i)|$ is an upper bound on $|f_i(U)|$ for each $i \in \{1, 2\}$ and each $U \in \mathcal{C}_i$. Define

$$(49.28)\qquad K := |S|(\alpha|S| + \max_{i=1,2} |f_i(L_i)|).$$

Now for $i = 1, 2$ and $U \subseteq S$, let $\bar{f}_i(U) := f_i(\overline{U}) + K|\overline{U} \triangle U|$ (where $\overline{U}$ is taken with respect to $\mathcal{C}_i$). So $\bar{f}_1$ and $\bar{f}_2$ are submodular (by Theorem 49.2). Then:

$$(49.29)\qquad \max\{w^{\mathsf{T}}x \mid x \in EP_{f_1} \cap EP_{f_2}\} = \max\{w^{\mathsf{T}}x \mid x \in EP_{\bar{f}_1} \cap EP_{\bar{f}_2}\},$$

if the first maximum is finite. Clearly $\geq$ holds in (49.29), since $EP_{\bar{f}_i} \subseteq EP_{f_i}$ for $i = 1, 2$. To see equality, each face of $EP_{f_1} \cap EP_{f_2}$ is determined by equations $x(U) = f_i(U)$ for $i = 1, 2$ and $U \in \mathcal{D}_i$, where $\mathcal{D}_i$ is a chain of sets in $\mathcal{C}_i$. So it is determined by a system of linear equations with totally unimodular constraint set and right-hand sides determined by function values of f_1 and f_2. So each face contains a vector x with $|x_s| \leq K$ for all $s \in S$ (by (49.28), since the inverse of a nonsingular totally unimodular matrix has all its entries in $\{0, \pm 1\}$). But any such x belongs to $EP_{\bar{f}_1} \cap EP_{\bar{f}_2}$, since for $i = 1, 2$ and $U \subseteq S$, we have:

(49.30) $$x(U) \leq x(\overline{U}) + K|\overline{U} \triangle U| \leq f_i(\overline{U}) + K|\overline{U} \triangle U| = \bar{f}_i(U)$$

(where $\overline{U}$ is taken with respect to $\mathcal{C}_i$). So we have (49.29).

Therefore, by Corollary 47.4d, we can maximize $w^{\mathsf{T}} x$ over $EP_{f_1} \cap EP_{f_2}$ in strongly polynomial time. Note that for any $w \in \mathbb{Q}^S$, we can decide in strongly polynomial time if the first maximum in (49.29) is finite. For this, we should decide if there exist $w_1, w_2 \in \mathbb{Q}^S_+$ such that $w = w_1 + w_2$ and such that for $i = 1, 2$: $w_i(s) = 0$ for $s \in S \setminus L_i$ and $u \preceq_i v$ implies $w_i(u) \geq w_i(v)$ for all u, v. This can be reduced to checking if a certain digraph with lengths has no negative-length directed circuit.

49.4. Submodular functions on an intersecting family

We next consider functions defined on a broader class of collections, the intersecting families, where the function satisfies a restricted form of submodularity. It yields an extension of the *Dilworth truncation* studied in Chapter 48.

A family $\mathcal{C}$ of sets is called an *intersecting family* if for all $T, U \in \mathcal{C}$ one has:

(49.31) if $T \cap U \neq \emptyset$, then $T \cap U, T \cup U \in \mathcal{C}$.

Let $\mathcal{C}$ be an intersecting family. A function $f : \mathcal{C} \to \mathbb{R}$ is called *submodular on intersecting pairs*, or *intersecting submodular*, if

(49.32) $$f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$$

for all $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$.

Consider the system

(49.33) $x(U) \leq f(U)$ for $U \in \mathcal{C}$,

and the problem dual to maximizing $w^{\mathsf{T}} x$ over (49.33), for $w \in \mathbb{R}^S$:

(49.34) $$\min\{\sum_{U \in \mathcal{C}} y(U) f(U) \mid y \in \mathbb{R}^{\mathcal{C}}_+, \sum_{U \in \mathcal{C}} y(U) \chi^U = w\}.$$

Recall that a collection $\mathcal{F}$ of sets is called *laminar* if

(49.35) $T \cap U = \emptyset$ or $T \subseteq U$ or $U \subseteq T$, for all $T, U \in \mathcal{F}$.

A basic result (proved with a method due to Edmonds [1970b]) is:

Theorem 49.3. *Let $\mathcal{C}$ be an intersecting family of subsets of a set S, let $f : \mathcal{C} \to \mathbb{R}$ be intersecting submodular and let $w \in \mathbb{R}^S$. Then* (49.34) *has an optimum solution y with $\mathcal{F} := \{U \in \mathcal{C} \mid y(U) > 0\}$ laminar.*

Proof. Let $y : \mathcal{C} \to \mathbb{R}_+$ achieve this minimum, with

(49.36) $$\sum_{U \in \mathcal{C}} y(U)|U||S \setminus U|$$

as small as possible. Assume that $\mathcal{F}$ is not laminar, and choose $T, U \in \mathcal{F}$ violating (49.35). Let $\alpha := \min\{y(T), y(U)\}$. Decrease $y(T)$ and $y(U)$ by α, and increase $y(T \cap U)$ and $y(T \cup U)$ by α. Since

(49.37) $$\chi^{T \cap U} + \chi^{T \cup U} = \chi^T + \chi^U,$$

y remains a feasible solution of (49.34). As moreover

(49.38) $$f(T \cap U) + f(T \cup U) \leq f(T) + f(U),$$

f remains optimum. However, by Theorem 2.1, sum (49.36) decreases, contradicting our assumption. ∎

It gives the box-total dual integrality of (49.33):

Corollary 49.3a. *Let $\mathcal{C}$ be an intersecting family of subsets of a set S and let $f : \mathcal{C} \to \mathbb{R}$ be intersecting submodular. Then system* (49.33) *is box-TDI.*

Proof. Consider problem (49.34) dual to maximizing $w^{\mathsf{T}}x$ over (49.33). By Theorem 49.3, this minimum is attained by a y with $\mathcal{F} := \{U \in \mathcal{C} \mid y(U) > 0\}$ laminar. As the matrix of constraints corresponding to $\mathcal{F}$ is totally unimodular (Theorem 41.11), Theorem 5.35 gives the corollary. ∎

49.5. Intersection

Again, these results can be extended in a natural way to pairs of functions, by derivation from Theorem 49.3.

Corollary 49.3b. *For $i = 1, 2$, let $\mathcal{C}_i$ be an intersecting family of subsets of a set S and let $f_i : \mathcal{C}_i \to \mathbb{R}$ be intersecting submodular. Then the system*

(49.39) $$\begin{aligned} x(U) &\leq f_1(U) \text{ for } U \in \mathcal{C}_1, \\ x(U) &\leq f_2(U) \text{ for } U \in \mathcal{C}_2, \end{aligned}$$

is box-TDI.

Proof. Choose $w \in \mathbb{R}^S$, and consider the problem dual to maximizing $w^\mathsf{T} x$ over (49.39):

(49.40) $$\min\{\sum_{U\in\mathcal{C}_1} y_1(U)f_1(U) + \sum_{U\in\mathcal{C}_2} y_2(U)f_2(U) \mid y_1 \in \mathbb{R}_+^{\mathcal{C}_1}, y_2 \in \mathbb{R}_+^{\mathcal{C}_2}, \sum_{U\in\mathcal{C}_1} y_1(U)\chi^U + \sum_{U\in\mathcal{C}_2} y_2(U)\chi^U = w\}.$$

Let y_1, y_2 attain the minimum. For $i \in \{1,2\}$, define

(49.41) $$w_i := \sum_{U\in\mathcal{C}_i} y_i(U)\chi^U.$$

By Theorem 49.3,

(49.42) $$\min\{\sum_{U\in\mathcal{C}_i} y_i(U)f_i(U) \mid y_i \in \mathbb{R}_+^{\mathcal{C}_i}, \sum_{U\in\mathcal{C}_i} y_i(U)\chi^U = w_i\}$$

has an optimum solution y_i with $\mathcal{F}_i := \{U \in \mathcal{C}_i \mid y_i(U) > 0\}$ laminar.

As $\mathcal{F}_1 \cup \mathcal{F}_2$ determine a totally unimodular matrix (by Theorem 41.11), Theorem 5.35 implies that system (49.39) is box-TDI. ∎

This implies the integrality of polyhedra:

Corollary 49.3c. *If f_1 and f_2 are integer, then $EP_{f_1} \cap EP_{f_2}$ is box-integer.*

Proof. Directly from Corollary 49.3b. ∎

49.6. From an intersecting family to a lattice family

Let $\mathcal{C}$ be an intersecting family of subsets of a set S and let $f : \mathcal{C} \to \mathbb{R}$ be submodular on intersecting pairs. Let $\check{\mathcal{C}}$ be the collection of all unions of sets in $\mathcal{C}$. Since $\mathcal{C}$ is closed under unions of intersecting sets, $\check{\mathcal{C}}$ is equal to the collection of *disjoint* unions of nonempty sets in $\mathcal{C}$. It is not difficult to show that $\check{\mathcal{C}}$ is a lattice family and that $\emptyset \in \check{\mathcal{C}}$.

Call a partition *proper* if its classes are nonempty. Define $\check{f} : \check{\mathcal{C}} \to \mathbb{R}$ by:

(49.43) $$\check{f}(U) := \min\{\sum_{P\in\mathcal{P}} f(P) \mid \mathcal{P} \subseteq \mathcal{C} \text{ is a proper partition of } U\}.$$

for $U \in \check{\mathcal{C}}$. So $\check{f}(\emptyset) = 0$. Then (Dunstan [1976]):

Theorem 49.4. *$\check{f}$ is submodular.*

Proof. Choose $T, U \in \check{\mathcal{C}}$, and let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of T and U (respectively) into nonempty sets in $\mathcal{C}$ with

(49.44) $$\check{f}(T) = \sum_{P \in \mathcal{P}} f(P) \text{ and } \check{f}(U) = \sum_{Q \in \mathcal{Q}} f(Q).$$

Consider the family $\mathcal{F}$ made by the union of $\mathcal{P}$ and $\mathcal{Q}$ (taking a set twice if it occurs in both partitions). We can transform $\mathcal{F}$ iteratively into a laminar family, by replacing any $X, Y \in \mathcal{F}$ with $X \cap Y \neq \emptyset$ and $X \not\subseteq Y \not\subseteq X$ by $X \cap Y, X \cup Y$. In each iteration, the sum

(49.45) $$\sum_{Z \in \mathcal{F}} f(Z)$$

does not increase (as f is submodular on intersecting pairs). As at each iteration the sum

(49.46) $$\sum_{Z \in \mathcal{F}} |Z||S \setminus Z|$$

decreases (by Theorem 2.1), this process terminates. We end up with a laminar family $\mathcal{F}$.

The inclusionwise maximal sets in $\mathcal{F}$ form a partition $\mathcal{R}$ of $T \cup U$, and the remaining sets form a partition $\mathcal{S}$ of $T \cap U$. Therefore,

(49.47) $$\check{f}(T \cup U) + \check{f}(T \cap U) \leq \sum_{X \in \mathcal{R}} f(X) + \sum_{Y \in \mathcal{S}} f(Y)$$
$$\leq \sum_{P \in \mathcal{P}} f(P) + \sum_{Q \in \mathcal{Q}} f(Q) = \check{f}(T) + \check{f}(U),$$

showing that $\check{f}$ is submodular. ∎

Trivially, if $\emptyset \notin \mathcal{C}$ or if $f(\emptyset) \geq 0$, then $EP_{\check{f}} = EP_f$. Hence, by (49.9),

(49.48) $$\check{f}(U) = \max\{x(U) \mid U \in EP_f\}.$$

As we shall see in Section 49.7, this enables us to calculate $\check{f}$ from a value giving oracle, using the greedy algorithm.

49.7. Complexity

The results of the previous section enable us to reduce algorithmic problems on intersecting submodular functions, to those on submodular functions on lattice families.

If $\mathcal{C}$ is an intersecting family on S, then for each $s \in S$, the collection $\mathcal{C}_s := \{U \in \mathcal{C} \mid s \in U\}$ is a lattice family. So (like in Section 49.3) we can assume that $\mathcal{C}$ is given by a representation of $\mathcal{C}_s$ for each $s \in S$, in terms of the pre-order $\preceq_s$ given by: $u \preceq_s v$ if and only if each set in $\mathcal{C}$ containing s and v also contains u, and by $M_s := \bigcap \mathcal{C}_s$ and $L_s := \bigcup \mathcal{C}_s$; next to that we should know if $\emptyset$ belongs to $\mathcal{C}$.

We can derive the information on $\check{\mathcal{C}}$ as follows:

(49.49) $$\bigcap \check{\mathcal{C}} = \emptyset, \bigcup \check{\mathcal{C}} = \bigcup_{s\in S} L_s;\ u \preceq v \text{ if and only if } u \in M_v.$$

So we can decide in polynomial time if a set U belongs to $\check{\mathcal{C}}$.

For any intersecting submodular function f on $\mathcal{C}$, the restriction f_s of f to $\mathcal{C}_s$ is submodular. So by the results of Section 49.3, we can find a set minimizing f in strongly polynomial time.

For any $U \in \check{\mathcal{C}}$, we can calculate $\check{f}(U)$, as defined in (49.43), in strongly polynomial time, having a value giving oracle for f. To see this, we use (49.48). We can assume that $\emptyset \notin \mathcal{C}$.

Order the elements of U as $t_1, \ldots, t_k$ such that if $L_{t_j} \subset L_{t_i}$, then $j < i$. For $i = 0, \ldots, k$, let $U_i := L_{t_1} \cup \cdots \cup L_{t_i}$. So $U_k = U$.

Initially, set $x(t) := 0$ for each $t \in U$. Next, for $i = 1, \ldots, k$, calculate

(49.50) $$\mu := \min\{f(T) - x(T) \mid T \in \mathcal{C}, t_i \in T \subseteq U_i\},$$

and reset $x(t_i) := x(t_i)+\mu$. We prove, by induction on i, that for $i = 0, 1, \ldots, k$ we have, after processing $t_1, \ldots, t_i$:

(49.51)
(i) $x(T) \le f(T)$ for each $T \in \mathcal{C}$ with $T \subseteq U_i$,
(ii) for each $j = 1, \ldots, i$ there exists a $T \in \mathcal{C}$ with $t_j \in T \subseteq U_i$ and $x(T) = f(T)$.

For $i = 0$ this is trivial. Let $i \ge 1$. Consider any $T \in \mathcal{C}$ with $T \subseteq U_i$. If $t_i \in T$, then $x(T) \le f(T)$, as at processing t_i we have added μ to $x(t_i)$. If $t_i \notin T$, then $T \subseteq U_{i-1}$. For suppose that there exists a $t_j \in T$ with $t_j \notin U_{i-1}$. So $j > i$ and $t_j \in L_{t_i}$, and therefore $L_{t_j} \subseteq L_{t_i}$, implying $L_{t_j} = L_{t_i}$ (since if $L_{t_j} \subset L_{t_i}$, then $j < i$). But then $t_i \in L_{t_j} \subseteq T$, contradicting the fact that $t_i \notin T$. So $T \subseteq U_{i-1}$. As $t_i \notin T$, $x(T)$ did not change at processing t_i, and hence we know $x(T) \le f(T)$ by induction. This proves (49.51)(i).

To see (49.51)(ii), choose $j \le i$. If $j = i$, there exists after processing t_i a T as required, as we have added μ to $x(t_i)$. If $j < i$, by induction there exists a $T \in \mathcal{C}$ with $t_j \in T \subseteq U_{i-1}$ and $x(T) = f(T)$ before processing t_i. If $t_i \notin T$, $x(T) = f(T)$ is maintained at processing t_i. If $t_i \in T$, then $t_i \in U_{i-1}$, and so $t_i \in L_{t_j}$ for some $j < i$. Hence $L_{t_i} \subseteq L_{t_j}$. So, by the choice of the order of U, $L_{t_i} = L_{t_j}$. Hence before processing t_i we have $x(T') \le f(T')$ for each $T' \subseteq U_i$. So, as $x(T) = f(T)$ and $t_i \in T$, $x(t_i)$ is not modified at processing t_i. Therefore, $x(T) = f(T)$ holds also after processing t_i. This proves (49.51)(ii).

This shows (49.51), which gives, taking $i = k$, that $x(T) \le f(T)$ for each $T \in \mathcal{C}$ with $T \subseteq U$, and that for each $t \in U$, we have a T containing t with $x(T) = f(T)$. We can replace any two T and T' with $T \cap T' \ne \emptyset$ by $T \cup T'$. We end up with a partition $\mathcal{T}$ of U with $x(U) = \sum_{T\in\mathcal{T}} f(T)$. Hence we know

(49.52) $$\check{f}(U) \ge x(U) = \sum_{T\in\mathcal{T}} x(T) = \sum_{T\in\mathcal{T}} f(T) \ge \check{f}(U),$$

and therefore we have equality throughout.

Having this, we can reduce the problem of maximizing $w^{\mathsf{T}}x$ over EP_f, where f is intersecting submodular, to that of maximizing $w^{\mathsf{T}}x$ over $EP_{\check{f}}$, which can be done in strongly polynomial time by the results of Section 49.3. Similarly for intersections of two such polyhedra EP_{f_1} and EP_{f_2}.

49.8. Intersecting a polymatroid and a contrapolymatroid

For an intersecting family $\mathcal{C}$, a function $g : \mathcal{C} \to \mathbb{R}$ is called *supermodular on intersecting pairs*, or *intersecting supermodular*, if $-g$ is intersecting submodular.

Let S be a finite set. Let $\mathcal{C}$ and $\mathcal{D}$ be collections of subsets of S and let $f : \mathcal{C} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$. Consider the system

$$\text{(49.53)} \qquad \begin{aligned} x(U) &\leq f(U) \quad \text{for } U \in \mathcal{C}, \\ x(U) &\geq g(U) \quad \text{for } U \in \mathcal{D}. \end{aligned}$$

Theorem 49.5. *If $\mathcal{C}$ and $\mathcal{D}$ are intersecting families, $f : \mathcal{C} \to \mathbb{R}$ is intersecting submodular, and $g : \mathcal{D} \to \mathbb{R}$ is intersecting supermodular, then system* (49.53) *is box-TDI.*

Proof. Choose $w \in \mathbb{Z}^S$, and consider the dual problem of maximizing $w^\mathsf{T} x$ over (49.53):

$$\text{(49.54)} \qquad \begin{aligned} \min\{\sum_{U \in \mathcal{C}} y(U) f(U) - \sum_{U \in \mathcal{D}} z(U) g(U) \mid \\ y \in \mathbb{R}_+^{\mathcal{C}}, z \in \mathbb{R}_+^{\mathcal{D}}, \sum_{U \in \mathcal{C}} y(U)\chi^U - \sum_{U \in \mathcal{D}} z(U)\chi^U = w\}. \end{aligned}$$

Let y, z attain this minimum. Define

$$\text{(49.55)} \qquad u := \sum_{U \in \mathcal{C}} y(U)\chi^U \text{ and } v := \sum_{U \in \mathcal{D}} z(U)\chi^U.$$

So y attains

$$\text{(49.56)} \qquad \min\{\sum_{U \in \mathcal{C}} y(U) f(U) \mid y \in \mathbb{R}_+^{\mathcal{C}}, \sum_{U \in \mathcal{C}} y(U)\chi^U = u\}$$

and z attains

$$\text{(49.57)} \qquad \max\{\sum_{U \in \mathcal{D}} z(U) g(U) \mid z \in \mathbb{R}_+^{\mathcal{D}}, \sum_{U \in \mathcal{D}} z(U)\chi^U = v\}.$$

By Theorem 49.3, (49.56) has an optimum solution y with $\mathcal{F} := \{U \in \mathcal{C} \mid y(U) > 0\}$ laminar. Similarly, (49.57) has an optimum solution z with $\mathcal{G} := \{U \in \mathcal{D} \mid z(U) > 0\}$ laminar. Now $\mathcal{F}$ and $\mathcal{G}$ determine a totally unimodular submatrix (by Theorem 41.11), and hence by Theorem 5.35, (49.53) is box-TDI. ∎

49.9. Submodular functions on a crossing family

Finally, we consider submodular functions defined on a crossing family. The results discussed above for submodular functions on intersecting families do not all transfer to crossing families. But certain restricted versions still hold.

A family $\mathcal{C}$ of subsets of a set S is called a *crossing family* if for all $T, U \in \mathcal{C}$ one has:

(49.58) if $T \cap U \neq \emptyset$ and $T \cup U \neq S$, then $T \cap U, T \cup U \in \mathcal{C}$.

A function $f : \mathcal{C} \to \mathbb{R}$, defined on a crossing family $\mathcal{C}$, is called *submodular on crossing pairs*, or *crossing submodular*, if for all $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$ and $T \cup U \neq S$:

(49.59) $f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$.

In general, the system

(49.60) $x(U) \leq f(U)$ for $U \in \mathcal{C}$

is *not* TDI. For instance, if $S = \{1, 2, 3\}$, $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, from S, and $f(U) := 1$ for each $U \in \mathcal{C}$, then (49.60) not even determines an integer polyhedron (as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{\mathsf{T}}$ is a vertex of it).

However, for any $k \in \mathbb{R}$, the system

(49.61) $x(U) \leq f(U)$ for $U \in \mathcal{C}$,
$x(S) = k$

is box-TDI. This can be done by reduction to Corollary 49.3a. Similarly for pairs of such functions. This was shown by Frank [1982b,1984a] and Fujishige [1984e].

Let $\mathcal{C}$ be a crossing family of subsets of a set S. Let $\hat{\mathcal{C}}$ be the collection of all nonempty intersections of sets in $\mathcal{C}$ (we allow the intersection of 0 sets, so $S \in \hat{\mathcal{C}}$). Since $\mathcal{C}$ is a crossing family, we know

(49.62) $\hat{\mathcal{C}} = \{U \mid U \neq \emptyset; \exists \mathcal{P} \subseteq \mathcal{C} : \mathcal{P}$ is a copartition of $S \setminus U\}$,

where a *copartition* of U is a collection $\mathcal{P}$ of subsets of S such that the collection $\{S \setminus T \mid T \in \mathcal{P}\}$ is a partition of U. We call the copartition *proper* if $T \neq S$ for each $T \in \mathcal{P}$.

Note that, in (49.62), restricting $\mathcal{P}$ to proper copartitions of $S \setminus U$, does not modify $\hat{\mathcal{C}}$. We allow $\mathcal{P} = \emptyset$, so $S \in \hat{\mathcal{C}}$.

Define $\hat{f} : \hat{\mathcal{C}} \to \mathbb{R}$ by:

(49.63) $\hat{f}(U) := \min\{\sum_{P \in \mathcal{P}} f(P) \mid \mathcal{P} \subseteq \mathcal{C}$ is a proper copartition of $S \setminus U\}$

for $U \in \hat{\mathcal{C}}$. So $\hat{f}(S) = 0$. Then:

Theorem 49.6. *$\hat{\mathcal{C}}$ is an intersecting family and $\hat{f}$ is submodular on intersecting pairs.*

Proof. Define, for $s \in S$,

(49.64) $\qquad \hat{\mathcal{C}}_s := \{U \in \hat{\mathcal{C}} \mid s \in U\}$ and $\mathcal{D} := \{S \setminus U \mid s \in U \in \mathcal{C}\}$.

As $\mathcal{C}$ is crossing, $\mathcal{D}$ is intersecting. Hence $\check{\mathcal{D}}$ is a lattice family. As $\hat{\mathcal{C}}_s = \{S \setminus U \mid U \in \check{\mathcal{D}}\}$, also $\hat{\mathcal{C}}_s$ is a lattice family. As this is true for each $s \in S$, $\hat{\mathcal{C}}$ is intersecting.

To prove that $\hat{f}$ is intersecting submodular, it suffices to show that for each $s \in S$, the restriction $\hat{f}_s$ of $\hat{f}$ to $\hat{\mathcal{C}}_s$ is submodular. Define $g : \mathcal{D} \to \mathbb{R}$ by $g(U) := f(S \setminus U)$ for $U \in \mathcal{D}$. Then g is intersecting submodular (as f is crossing submodular). Hence, by Theorem 49.4, $\check{g}$ is submodular on $\check{\mathcal{D}}$. As $\hat{f}_s(U) = \check{g}(S \setminus U)$ for $U \in \hat{\mathcal{C}}_s$, $\hat{f}_s$ is submodular. ∎

Fujishige [1984e] showed that the following box-TDI result can be derived from Corollary 49.3a:

Theorem 49.7. *Let $\mathcal{C}$ be a crossing family of subsets of S, let $f : \mathcal{C} \to \mathbb{R}$ be crossing submodular, and let $k \in \mathbb{R}$. Then system* (49.61) *is box-TDI and determines the polyhedron of maximal vectors of $EP_{f'}$ for some submodular function f' defined on a lattice family.*

Proof. We can assume that $k = 0$, since, choosing any $t \in S$ and resetting $f(U) := f(U) - k$ for all $U \in \mathcal{C}$ with $t \in U$, does not change the box-total dual integrality of (49.61). We can also assume that $\emptyset \notin \mathcal{C}$.

The box-total dual integrality of (49.61) follows from that of

(49.65)
$$\begin{aligned} &x(U) \le \hat{f}(U) \quad \text{for } U \in \hat{\mathcal{C}},\\ &x(S) = 0, \end{aligned}$$

as (49.65) and (49.61) have the same solution set, and as each constraint in (49.65) is a nonnegative integer combination of constraints in (49.61). The box-total dual integrality of (49.65) follows from Corollary 49.3a (using Theorem 5.25). It also shows that the solution set of (49.61) is the set of maximal vectors of $EP_{f'}$ for some submodular function f' defined on a lattice family. ∎

Frank and Tardos [1984b] observed that this implies a relation with matroids:

Corollary 49.7a. *If $\mathcal{C}$ is a crossing family of subsets of a set S, $f : \mathcal{C} \to \mathbb{Z}$ is crossing submodular, and $k \in \mathbb{Z}$, then the collection*

(49.66) $\qquad \{B \subseteq S \bigm| |B| = k, |B \cap U| \le f(U) \text{ for each } U \in \mathcal{C}\}$

forms the collection of bases of a matroid (if nonempty).

Proof. Directly from Theorem 49.7, using the observations on (44.43) and (49.11). ∎

Similarly, the box-total dual integrality of pairs of such systems follows:

Theorem 49.8. *For $i = 1, 2$, let $\mathcal{C}_i$ be a crossing family of subsets of a set S, let $f_i : \mathcal{C}_i \to \mathbb{R}$ be crossing submodular, and let $k \in \mathbb{R}$. Then the system*

(49.67)
$$\begin{aligned} &x(U) \leq f_1(U) \text{ for } U \in \mathcal{C}_1,\\ &x(U) \leq f_2(U) \text{ for } U \in \mathcal{C}_2,\\ &x(S) = k, \end{aligned}$$

is box-TDI.

Proof. Similar to the proof of the previous theorem, by reduction to Corollary 49.3b. ∎

This implies the integrality of polyhedra:

Corollary 49.8a. *For $i = 1, 2$, let $\mathcal{C}_i$ be a crossing family of subsets of a set S, let $f_i : \mathcal{C}_i \to \mathbb{R}$ be crossing submodular, and let $k \in \mathbb{R}$. If f_1, f_2, and k are integer, system* (49.67) *determines a box-integer polyhedron.*

Proof. Directly from Theorem 49.8. ∎

49.10. Complexity

The reduction given in the proof of Theorem 49.6 also enables us to calculate $\hat{f}(U)$ from a value giving oracle for f, similar to the proof in Section 49.7. We assume that $\mathcal{C}$ is given by descriptions of the lattice families $\mathcal{C}_{s,t} := \{U \in \mathcal{C} \mid s \in U, t \notin U\}$ as in Section 49.3.

Note that

(49.68)
$$EP_{\hat{f}} \cap \{x \mid x(S) = 0\} = EP_f \cap \{x \mid x(S) = 0\}.$$

This follows from the fact that if $x \in EP_f$ and $x(S) = 0$, then for any $U \in \hat{\mathcal{C}}$ and any proper copartition $\mathcal{P} = \{U_1, \ldots, U_p\}$ of $S \setminus U$ with $f(U) = \sum_{P \in \mathcal{P}} f(P)$, one has:

(49.69)
$$\begin{aligned} x(U) &= -x(S \setminus U) = -\sum_{P \in \mathcal{P}} x(S \setminus P) = \sum_{P \in \mathcal{P}} x(P) \leq \sum_{P \in \mathcal{P}} f(P)\\ &= f(U). \end{aligned}$$

Having this, the problem of maximizing $w^{\mathsf{T}}x$ over $EP_f \cap \{x \mid x(S) = 0\}$, where f is crossing submodular, is reduced to the problem of maximizing $w^{\mathsf{T}}x$ over $EP_{\hat{f}} \cap \{x \mid x(S) = 0\}$. The latter problem can be solved in strongly polynomial time by the results of Section 49.7. Similar results hold for intersections of two such polyhedra:

Theorem 49.9. *For crossing families $\mathcal{C}_1, \mathcal{C}_2$ of subsets of a set S, crossing submodular functions $f_1 : \mathcal{C}_1 \to \mathbb{Q}$ and $f_2 : \mathcal{C}_2 \to \mathbb{Q}$, $k \in \mathbb{Q}$, and $w \in \mathbb{Q}^S$, one* *can find an* $x \in EP_{f_1} \cap EP_{f_2}$ *with* $x(S) = k$ *and maximizing* $w^{\mathsf{T}}x$ *in strongly polynomial time.*

Proof. From the above. ∎

If $\mathcal{C}$ is a crossing family and $f : \mathcal{C} \to \mathbb{Q}$ is crossing submodular, then we can find its minimum value in polynomial time, as, for each $s, t \in S$, we can minimize f over the lattice family $\{U \in \mathcal{C} \mid s \in U, t \notin U\}$, and take the minimum of all these minima, and of the values in $\emptyset$ and S (if in $\mathcal{C}$).

Hence we can decide in polynomial time if a given vector $x \in \mathbb{Q}^S$ belongs to EP_f, by testing if the minimum value of the crossing submodular function $g : \mathcal{C} \to \mathbb{Q}$ defined by

$$g(U) := f(U) - x(U) \tag{49.70}$$

for $U \in \mathcal{C}$, is nonnegative.

49.10a. Nonemptiness of the base polyhedron

Let $\mathcal{C}$ be a crossing family of subsets of a set S and let $f : \mathcal{C} \to \mathbb{R}$ be crossing submodular. We give a theorem of Fujishige [1984e] characterizing when EP_f contains a vector x with $x(S) = 0$. If $S \in \mathcal{C}$ and $f(S) = 0$, then the set $EP_f \cap \{x \mid x(S) = 0\}$ is called the *base polyhedron* of f.

To give the characterization, again call a collection $\mathcal{P} \subseteq \mathcal{C}$ a *copartition* of S if the collection $\{S \setminus U \mid U \in \mathcal{P}\}$ is a partition of S.

Theorem 49.10. *EP_f contains a vector x satisfying $x(S) = 0$ if and only if*

$$\sum_{U \in \mathcal{P}} f(U) \geq 0 \tag{49.71}$$

for each partition or copartition $\mathcal{P} \subseteq \mathcal{C}$ of S. If moreover f is integer, there exists an integer such vector x.

Proof. The condition is necessary, since if $x \in EP_f$ satisfies $x(S) = 0$ and $\mathcal{P} \subseteq \mathcal{C}$ is a partition of S, then

$$\sum_{U \in \mathcal{P}} f(U) \geq \sum_{U \in \mathcal{P}} x(U) = x(S) = 0. \tag{49.72}$$

Similarly, if $\mathcal{P} \subseteq \mathcal{C}$ is a copartition of S, then

$$\sum_{U \in \mathcal{P}} f(U) \geq \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (x(S) - x(S \setminus U)) = |\mathcal{P}|x(S) - x(S) = 0. \tag{49.73}$$

To see sufficiency, let $\mathcal{D} := \hat{\mathcal{C}}$ and $g := \hat{f}$ (cf. (49.62)). By Theorem 49.6, $\mathcal{D}$ is an intersecting family and g is intersecting submodular. Moreover, $S \in \mathcal{D}$. Let $\mathcal{E} := \check{\mathcal{D}}$ and $h := \check{g}$. By Theorem 49.4, $\mathcal{E}$ is a lattice family and h is submodular.

Now if EP_h contains a vector x with $x(S) = 0$, then $x \in EP_g$, and hence $x \in EP_f$ (using $x(S) = 0$). So it suffices to show that EP_h contains a vector x with $x(S) = 0$.

By Corollary 49.1c, in order to show this, it suffices to show that $h(S) \geq 0$. The solution can be taken integer if f (hence h) is integer.

Suppose $h(S) < 0$. Since $h = \check{g}$, $\mathcal{D}$ contains a proper partition $\mathcal{P}$ of S with

(49.74) $$h(S) = \sum_{U \in \mathcal{P}} g(U).$$

Since $g = \hat{f}$, for each $U \in \mathcal{P}$, $\mathcal{C}$ contains a proper copartition $\mathcal{Q}_U$ of $S \setminus U$ such that

(49.75) $$g(U) = \sum_{T \in \mathcal{Q}_U} f(T).$$

Let $\mathcal{F}$ be the family consisting of the union of the $\mathcal{Q}_U$ over $U \in \mathcal{P}$, taking multiplicities into account. Then

(49.76) (i) all elements of S are contained in the same number of sets in $\mathcal{F}$;
(ii) $\sum_{T \in \mathcal{F}} f(T) < 0$.

Now apply the following operation as often as possible to $\mathcal{F}$: if $T, W \in \mathcal{F}$ with $T \cap W \neq \emptyset$, $T \cup W \neq S$, and $T \not\subseteq W \not\subseteq T$, replace T and W by $T \cap W$ and $T \cup W$. This maintains (49.76) and decreases $\sum_{T \in \mathcal{F}} |T||S \setminus T|$ (by Theorem 2.1). So the process terminates, and we end up with a *cross-free family*: for all $T, W \in \mathcal{F}$ we have $T \subseteq W$ or $W \subseteq T$ or $T \cap W = \emptyset$ or $T \cup W = S$.

We show that $\mathcal{F}$ contains a partition or copartition $\mathcal{P}$ of S. By (49.71), $\mathcal{F} \setminus \mathcal{P}$ again satisfies (49.76), and hence we can repeat. We end up with $\mathcal{F}$ empty, a contradiction.

To show that $\mathcal{F}$ contains a partition or copartition of S, choose $U \in \mathcal{F}$. If $U = \emptyset$ or $U = S$ we are done (taking $\mathcal{P} := \{U\}$). So we can assume that $\emptyset \neq U \neq S$. Let $\mathcal{X}$ be the collection of inclusionwise maximal subsets of $S \setminus U$ that belong to $\mathcal{F}$. Let $\mathcal{Y}$ be the collection of inclusionwise minimal supersets of $S \setminus U$ that belong to $\mathcal{F}$. Since $\mathcal{F}$ is cross-free and $U \neq \emptyset$, the sets in $\mathcal{X}$ are pairwise disjoint. Similarly, the complements of the sets in $\mathcal{Y}$ are pairwise disjoint.

If $\bigcup \mathcal{X} = S \setminus U$, then $\mathcal{X} \cup \{U\}$ is a partition of S as required. If $\bigcap \mathcal{Y} = S \setminus U$, then $\mathcal{Y} \cup \{U\}$ is a copartition of S as required. So we can assume that there exist $s \in (S \setminus U) \setminus \bigcup \mathcal{X}$ and $t \in U \cap \bigcap \mathcal{Y}$. Since each element of S is contained in the same number of sets in $\mathcal{F}$, and since $s \notin U$, and $t \in U$, there exists a $T \in \mathcal{F}$ with $s \in T$ and $t \notin T$. So $T \not\subseteq U \not\subseteq T$.

Hence $T \cap U = \emptyset$ or $T \cup U = S$. However, if $T \cap U = \emptyset$, then T is contained in some set in $\mathcal{X}$, and hence $s \in T \subseteq \bigcup \mathcal{X}$, a contradiction. If $T \cup U = S$, then T contains some set in $\mathcal{Y}$, and hence $t \notin T \supseteq \bigcap \mathcal{Y}$, again a contradiction. ∎

This theorem will be used in proving Theorem 61.8.

Fujishige and Tomizawa [1983] characterized the vertices of the base polyhedron of a submodular function defined on a lattice family.

49.11. Further results and notes

49.11a. Minimizing a submodular function over a subcollection of a lattice family

In Section 45.7 we saw that the minimum of a submodular function over the odd subsets can be found in strongly polynomial time. A generalization of minimizing a submodular function over the odd subsets (cf. Section 45.7), was given by Grötschel, Lovász, and Schrijver [1981,1984a] (the latter paper corrects a serious flaw in the first paper found by A. Frank). This was extended by Goemans and Ramakrishnan [1995] to the following.

Let $\mathcal{C}$ be a lattice family and let $\mathcal{D}$ be a subcollection of $\mathcal{C}$ with the following property:

(49.77) for all $X, Y \in \mathcal{C} \setminus \mathcal{D}$: $X \cap Y \in \mathcal{D} \iff X \cup Y \in \mathcal{D}$.

Examples are: $\mathcal{D} := \{X \in \mathcal{C} \bigm| |X| \not\equiv q (\mathrm{mod}\ p)\}$ for some natural numbers p, q, and $\mathcal{D} := \mathcal{C} \setminus \mathcal{A}$ for some antichain or some sublattice $\mathcal{A} \subseteq \mathcal{C}$.

To prove that for a submodular function on $\mathcal{C}$, the minimum over $\mathcal{D}$ can be found in strongly polynomial time, Goemans and Ramakrishnan gave the following interesting lemma:

Lemma 49.11α. *Let $\mathcal{C}$ be a lattice family, let f be a submodular function on $\mathcal{C}$, let $\mathcal{D} \subseteq \mathcal{C}$ satisfy (49.77), and let U minimize $f(U)$ over $U \in \mathcal{D}$. If $U \neq \emptyset$, then there exists a $u \in U$ such that $f(W) \geq f(U)$ for each subset W of U with $W \in \mathcal{C}$ and $u \in W$.*

Proof. Suppose not. Then for each $u \in U$ there exists a $W_u \in \mathcal{C}$ satisfying $u \in W_u \subseteq U$ and $f(W_u) < f(U)$. Choose each W_u inclusionwise maximal with this property. Then

(49.78) $$f(\bigcap_{u \in T} W_u) < f(U)$$

for each nonempty $T \subseteq U$. To prove this, choose a counterexample T with $|T|$ minimal. Then $|T| > 1$, since $f(W_u) < f(U)$ for each $u \in U$. Choose $t \in T$. Since $\bigcap_{u \in T} W_u \neq \bigcap_{u \in T-t} W_u$ by the minimality of T, we know that $\bigcap_{u \in T-t} W_u \not\subseteq W_t$, and hence W_t is a proper subset of $(\bigcap_{u \in T-t} W_u) \cup W_t$. So by the maximality of W_t, $f((\bigcap_{u \in T-t} W_u) \cup W_t) \geq f(U)$. Hence

(49.79) $$\begin{aligned} f(U) &\leq f(\bigcap_{u \in T} W_u) = f((\bigcap_{u \in T-t} W_u) \cap W_t) \\ &\leq f(\bigcap_{u \in T-t} W_u) + f(W_t) - f((\bigcap_{u \in T-t} W_u) \cup W_t) \\ &< f(U) + f(U) - f(U) = f(U), \end{aligned}$$

a contradiction.

This shows (49.78), which implies

(49.80) $$\bigcap_{u \in T} W_u \notin \mathcal{D}$$

for each nonempty $T \subseteq U$.

This can be extended to:

(49.81) $$X := (\bigcap_{u \in T} W_u) \cap (\bigcup_{u \in V} W_u) \notin \mathcal{D}$$

for all disjoint $T, V \subseteq U$ with V nonempty. Suppose to the contrary that $X \in \mathcal{D}$. Choose such X with $|V|$ minimal. By (49.80), $|V| \geq 2$. Choose $v \in V$. The minimality of V gives

$$(49.82)\qquad (\bigcap_{u\in T} W_u)\cap W_v \notin \mathcal{D} \text{ and } (\bigcap_{u\in T} W_u)\cap (\bigcup_{u\in V\setminus\{v\}} W_u)\notin \mathcal{D}.$$

By assumption, the union of these sets belongs to $\mathcal{D}$. Hence, by (49.77), also their intersection belongs to $\mathcal{D}$; that is

$$(49.83)\qquad (\bigcap_{u\in T\cup\{v\}} W_u)\cap (\bigcup_{u\in V\setminus\{v\}} W_u)\in \mathcal{D}$$

This contradicts the minimality of $|V|$.

This proves (49.81), which gives for $T := \emptyset$ and $V := U$ a contradiction, since then $X = U \in \mathcal{D}$. ∎

This lemma is used in proving the following theorem, where we assume that $\mathcal{C}$ is given as in Section 49.3, f is given by a value giving oracle, and $\mathcal{D}$ is given by an oracle telling if any given set in $\mathcal{C}$ belongs to $\mathcal{D}$:

Theorem 49.11. *Given a submodular function f on a lattice family $\mathcal{C}$, and a subcollection $\mathcal{D}$ of $\mathcal{C}$ satisfying* (49.77)*, a set U minimizing $f(U)$ over $U \in \mathcal{D}$ can be found in strongly polynomial time.*

Proof. We describe the algorithm. For all distinct $s,t \in S$ define

$$(49.84)\qquad \mathcal{C}_{s,t} := \{U \in \mathcal{C} \mid s \in U, t \notin U\}.$$

Let $U_{s,t}$ be the inclusionwise minimal set minimizing f over $\mathcal{C}_{s,t}$. ($U_{s,t}$ can be found by minimizing a slight perturbation of f.) Choose in

$$(49.85)\qquad \{\emptyset, S\}\cup\{U_{s,t} \mid s,t\in S\}$$

a $U \in \mathcal{D}$ minimizing f. Then U minimizes f over $\mathcal{D}$.

To see this, we must show that set (49.85) contains a set minimizing f over $\mathcal{D}$. Let W be a set minimizing f over $\mathcal{D}$, with $|W|$ minimal. If $W \in \{\emptyset, S\}$ we are done. So we can assume that $W \notin \{\emptyset, S\}$. By Lemma 49.11$\alpha$ (applied to the function $\tilde{f}(X) := f(S \setminus X)$ for $X \subseteq S$), there exists an element $t \in S \setminus W$ such that each $T \supseteq W$ with $t \notin T$ satisfies $f(T) \geq f(W)$.

The lemma also gives the existence of an $s \in W$ such that each $T \subset W$ with $s \in T$ satisfies $f(T) > f(W)$. Indeed, for small enough $\varepsilon > 0$, W minimizes $f(X)+\varepsilon|X|$ over $X \in \mathcal{D}$. Hence, by Lemma 49.11α, there exists an $s \in W$ such that each $T \subseteq W$ with $s \in T$ satisfies $f(T)+\varepsilon|T| \geq f(W)+\varepsilon|W|$. This implies $f(T) > f(W)$ if $T \neq W$.

We show that $W = U_{s,t}$. Indeed, W minimizes f over $\mathcal{C}_{s,t}$, since

$$(49.86)\qquad f(U_{s,t}) \geq f(W\cap U_{s,t}) + f(W\cup U_{s,t}) - f(W) \geq f(W)+f(W)-f(W) = f(W).$$

Moreover, $W \subseteq U_{s,t}$, as otherwise $W \cap U_{s,t} \subset W$, implying that the second inequality in (49.86) would be strict.

So $f(W) = f(U_{s,t})$, and hence, by the minimality of $U_{s,t}$, we have $W = U_{s,t}$. ∎

It is interesting to note that this algorithm implies that the set $\{\emptyset, S\}\cup\{U_{s,t} \mid s,t \in S\}$ contains a set minimizing f over $\mathcal{D}$, for any nonempty subcollection $\mathcal{D}$ of $\mathcal{C}$ satisfying (49.77).

Goemans and Ramakrishnan showed that if $\mathcal{C}$ and $\mathcal{D}$ are *symmetric* (that is, $U \in \mathcal{C} \iff S \setminus U \in \mathcal{C}$, and similarly for $\mathcal{D}$) and $\emptyset \notin \mathcal{D}$, then (49.77) is equivalent to: if $X, Y \in \mathcal{C} \setminus \mathcal{D}$ are disjoint, then $X \cup Y \in \mathcal{C} \setminus \mathcal{D}$.

Related work was reported by Benczúr and Fülöp [2000].

49.11b. Generalized polymatroids

We now describe a generalization, given by Frank [1984b], that comprises sub- and supermodular functions, and (extended) polymatroids and contrapolymatroids. (Hassin [1978,1982] described the case $\mathcal{C} = \mathcal{D} = \mathcal{P}(S)$.)

Let $\mathcal{C}$ and $\mathcal{D}$ be intersecting families of subsets of a finite set S and let $f : \mathcal{C} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$. We say that the pair (f, g) is *paramodular* if

(49.87)
(i) f is submodular on intersecting pairs,
(ii) g is supermodular on intersecting pairs,
(iii) if $T \in \mathcal{C}$ and $U \in \mathcal{D}$ with $T \setminus U \neq \emptyset$ and $U \setminus T \neq \emptyset$, then $T \setminus U \in \mathcal{C}$ and $U \setminus T \in \mathcal{D}$, and

$$f(T \setminus U) - g(U \setminus T) \leq f(T) - g(U).$$

If (f, g) is paramodular, the solution set P of the system (for $x \in \mathbb{R}^S$):

(49.88)
$$\begin{aligned} x(U) &\leq f(U) \quad \text{for } U \in \mathcal{C},\\ x(U) &\geq g(U) \quad \text{for } U \in \mathcal{D}, \end{aligned}$$

is called a *generalized polymatroid* (*determined by* (f, g)).

Generalized polymatroids generalize polymatroids (where $g(U) = 0$ for each $U \subseteq S$), extended polymatroids (where $\mathcal{D} = \emptyset$), contrapolymatroids (where $\mathcal{C} = \emptyset$ and $g(\{s\}) \geq 0$ for each $s \in S$), and extended contrapolymatroids (where $\mathcal{C} = \emptyset$).

The intersection of a generalized polymatroid with a 'box' $\{x \mid d \leq x \leq c\}$ (for $d, c \in \mathbb{R}^S$) is again a generalized polymatroid: we can add $\{s\}$ to $\mathcal{C}$ and to $\mathcal{D}$ if necessary, and (re)define $f(\{s\}) := w(s)$ and $g(\{s\}) := d(s)$, if necessary. This transformation does not violate the paramodularity of (f, g).

Another transformation is as follows. Let $P \subseteq \mathbb{R}^S$ be a generalized polymatroid and let $\kappa, \lambda \in \mathbb{R}$. Let t be a new element and let $S' := S \cup \{t\}$. Let P' be the polyhedron in $\mathbb{R}^{S'}$ given by

(49.89) $P' := \{(x, \eta) \mid x \in P, \lambda \leq x(S) + \eta \leq \kappa\}$.

Then P' again is a generalized polymatroid, determined by the functions obtained by extending $\mathcal{C}$ and $\mathcal{D}$ with S' and extending f, g with the values $f(S') := \kappa$ and $g(S') := \lambda$.

The class of generalized polymatroids is closed under projections. That is, for any generalized polymatroid $P \subseteq \mathbb{R}^S$ and any $t \in S$, the set

(49.90) $P' := \{x \in \mathbb{R}^{S-t} \mid \exists \eta : (x, \eta) \in P\}$

is again a generalized polymatroid. This will be shown as Corollary 49.13c.

The following theorem will imply that system (49.88) is TDI. Hence, if f and g are integer, then P is integer.

Theorem 49.12. *System* (49.88) *is box-TDI.*

Proof. Let t be a new element. Define

$$(49.91) \qquad \mathcal{B} := \mathcal{C} \cup \{(S \setminus D) \cup \{t\} \mid D \in \mathcal{D}\}.$$

Then $\mathcal{B}$ is a crossing family of subsets of $S \cup \{t\}$. Define $e : \mathcal{B} \to \mathbb{R}$ by: $e(C) := f(C)$ for $C \in \mathcal{C}$ and $e((S \setminus D) \cup \{t\}) := -g(D)$ for $D \in \mathcal{D}$. Then e is crossing submodular. Hence, by Theorem 49.7, system

$$(49.92) \qquad \begin{aligned} &x(U) \leq e(U) \qquad \text{for } U \in \mathcal{B}, \\ &x(S \cup \{t\}) = 0, \end{aligned}$$

is box-TDI. Therefore, by Theorem 5.27, system (49.88) is box-TDI. ∎

This gives for the integrality of generalized polymatroids:

Corollary 49.12a. *If (f, g) is paramodular and f and g are integer, the generalized polymatroid is box-integer.*

Proof. Directly from Theorem 49.12. ∎

More generally one has the box-total dual integrality of the system

$$(49.93) \qquad \begin{aligned} x(U) &\leq f_1(U) \quad \text{for } U \in \mathcal{C}_1, \\ x(U) &\geq g_1(U) \quad \text{for } U \in \mathcal{D}_1, \\ x(U) &\leq f_2(U) \quad \text{for } U \in \mathcal{C}_2, \\ x(U) &\geq g_2(U) \quad \text{for } U \in \mathcal{D}_2, \end{aligned}$$

for pairs of paramodular pairs (f_i, g_i):

Corollary 49.12b. *For $i = 1, 2$, let $\mathcal{C}_i$ and $\mathcal{D}_i$ be intersecting families and let $f_i : \mathcal{C}_i \to \mathbb{R}$, $g_i : \mathcal{D}_i \to \mathbb{R}$ form a paramodular pair. Then system (49.93) is box-TDI.*

Proof. Similar to the proof of Theorem 49.12, by reduction to Theorem 49.8. ∎

This gives for primal integrality:

Corollary 49.12c. *If f_1, g_1, f_2 and g_2 are integer, the intersection of the associated generalized polymatroids is box-integer.*

Proof. Directly from Corollary 49.12b. ∎

Another consequence is the following box-TDI result of McDiarmid [1978]:

Corollary 49.12d. *Let f_1 and f_2 be submodular set functions on a set S and let $\lambda, \kappa \in \mathbb{R}$. Then the system*

$$(49.94) \qquad \begin{aligned} &x(U) \leq f_1(U) \quad \text{for } U \subseteq S, \\ &x(U) \leq f_2(U) \quad \text{for } U \subseteq S, \\ &\lambda \leq x(S) \leq \kappa, \end{aligned}$$

is box-TDI.

Proof. Redefine $f_1(S) := \min\{f_1(S), \kappa\}$, and define $g_1 : \{S\} \to \mathbb{R}$ by $g_1(S) := \lambda$, and $g_2 : \emptyset \to \mathbb{R}$. Then (f_1, g_1) and (f_2, g_2) are paramodular pairs, and the box-total dual integrality of (49.94) is equivalent to the box-total dual integrality of (49.93). ∎

From Corollary 49.12c one can derive that the intersection of two integer generalized polymatroids is integer again. To prove this, we show that for any integer generalized polymatroid P there exists a paramodular pair (f, g) determining P, with f and g integer.

Let P be a generalized polymatroid, determined by the paramodular pair (f, g) of functions $f : \mathcal{C} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$, where $\mathcal{C}$ and $\mathcal{D}$ are intersecting families. For any $U \subseteq S$, define

(49.95) $$\tilde{f}(U) := \max\{x(U) \mid x \in P\} \text{ and } \tilde{g}(U) := \min\{x(U) \mid x \in P\}.$$

So $\tilde{f}$ and $\tilde{g}$ are integer if P is integer.

Let

(49.96) $$\widetilde{\mathcal{C}} := \{U \in \mathcal{C} \mid \tilde{f}(U) < \infty\} \text{ and } \widetilde{\mathcal{D}} := \{U \in \mathcal{D} \mid \tilde{g}(U) > -\infty\}.$$

We restrict $\tilde{f}$ and $\tilde{g}$ to $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ respectively. We show that $(\tilde{f}, \tilde{g})$ is a paramodular pair determining P.

It is convenient to note that if $w \in \mathbb{R}^S$ with $w = w_1 + w_2$, then

(49.97) $$\max\{w^{\mathsf{T}}x \mid x \in P\} \leq \max\{w_1^{\mathsf{T}}x \mid x \in P\} + \max\{w_2^{\mathsf{T}}x \mid x \in P\}.$$

Theorem 49.13. *For any generalized polymatroid P, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ are intersecting families, and the pair $(\tilde{f}, \tilde{g})$ is paramodular and determines P.*

Proof. We first show the following. Let $w \in \mathbb{Z}^S$ and let $\lambda > 0$ be such that $w_s \leq \lambda$ for each $s \in S$. Let $U := \{s \in S \mid w(s) = \lambda\}$ and $w' := w - \chi^U$. Then

(49.98) $$\max\{w^{\mathsf{T}}x \mid x \in P\} = \max\{w'^{\mathsf{T}}x \mid x \in P\} + \tilde{f}(U).$$

Here $\leq$ follows from (49.97), by definition of $\tilde{f}$. Equality is proved by induction on $|U|$, the case $U = \emptyset$ being trivial; so let $U \neq \emptyset$.

Let y, z be an optimum solution to the dual of $\max\{w^{\mathsf{T}}x \mid x \in P\}$:

(49.99) $$\min\{\sum_{T\in\mathcal{C}} y_T f(T) - \sum_{T\in\mathcal{D}} z_T g(T) \mid y \in \mathbb{R}_+^{\mathcal{C}}, z \in \mathbb{R}_+^{\mathcal{D}}, \sum_{T\in\mathcal{C}} y_T \chi^T - \sum_{T\in\mathcal{D}} z_T \chi^T = w\}.$$

Define $\mathcal{F} := \{T \in \mathcal{C} \mid y_T > 0\}$ and $\mathcal{G} := \{T \in \mathcal{D} \mid z_T > 0\}$. Similarly to Theorem 49.3, we can assume that $\mathcal{F} \cup \mathcal{G}$ is laminar.

Choose $u \in U$, and let W be an inclusionwise minimal set in $\mathcal{F}$ containing u. (Such a set exists, as $w(s) = \lambda > 0$.) Let $\mathcal{H}$ be the collection of inclusionwise maximal sets in $\mathcal{G}$ contained in $W - u$. As $\mathcal{G}$ is laminar, the sets in $\mathcal{H}$ are disjoint. Moreover, each $t \in W \setminus U$ is contained in some set in $\mathcal{H}$: since $w(t) < w(u)$ and since every set in $\mathcal{F}$ containing u also contains t (as $t \in W$), there exists an $X \in \mathcal{G}$ with $t \in X$ and $u \notin X$; as $\mathcal{F} \cup \mathcal{G}$ is laminar, we know that $X \subseteq W - u$.

Now let $Y := W \setminus \bigcup \mathcal{H}$. So Y is a nonempty subset of U. Define $w'' := w - \chi^Y$, let y' be obtained from y by decreasing $y(W)$ by 1, and let z' be obtained from z by decreasing $y(H)$ by 1 for each $H \in \mathcal{H}$. So (since $\chi^Y = \chi^W - \sum_{H\in\mathcal{H}} \chi^H$)

(49.100) $$\sum_{T\in\mathcal{C}} y'(T)\chi^T - \sum_{T\in\mathcal{D}} z'(T)\chi^T = w''$$

and

(49.101) $$\tilde{f}(Y) \leq f(W) - \sum_{H\in\mathcal{H}} g(H).$$

Moreover, setting $U' := U \setminus Y$, we have (by (49.97))

(49.102) $$\tilde{f}(U) \leq \tilde{f}(Y) + \tilde{f}(U'),$$

and by our induction hypothesis, as $|U'| < |U|$,

(49.103) $$\max\{w''^{\mathsf{T}}x \mid x \in P\} = \max\{w'^{\mathsf{T}}x \mid x \in P\} + \tilde{f}(U').$$

Hence

(49.104)
$$\begin{aligned}\max\{w^{\mathsf{T}}x \mid x \in P\} &= \sum_{T\in\mathcal{C}} y(T)f(T) - \sum_{T\in\mathcal{D}} z(T)g(T)\\ &= \sum_{T\in\mathcal{C}} y'(T)f(T) - \sum_{T\in\mathcal{D}} z'(T)g(T) + f(W) - \sum_{H\in\mathcal{H}} g(H)\\ &\geq \max\{w''^{\mathsf{T}}x \mid x \in P\} + \tilde{f}(Y) = \max\{w'^{\mathsf{T}}x \mid x \in P\} + \tilde{f}(U') + \tilde{f}(Y)\\ &\geq \max\{w'^{\mathsf{T}}x \mid x \in P\} + \tilde{f}(U),\end{aligned}$$

thus proving (49.98).

We next derive that $\tilde{f}$ is submodular on intersecting pairs. Choose $X, Y \in \widetilde{\mathcal{C}}$ with $X \cap Y \neq \emptyset$. Define $w := \chi^X + \chi^Y$. Then by (49.98) and (49.97),

(49.105) $$\tilde{f}(X\cap Y) + \tilde{f}(X\cup Y) = \max\{w^{\mathsf{T}}x \mid x \in P\} \leq \tilde{f}(X) + \tilde{f}(Y).$$

So $\tilde{\mathcal{C}}$ is an intersecting family and $\tilde{f}$ is submodular on intersecting pairs. By symmetry, it follows that $\tilde{\mathcal{D}}$ is an intersecting family and $\tilde{g}$ is supermodular on intersecting pairs.

Finally, to see that (f,g) is paramodular, let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Define $w := \chi^X - \chi^Y$. Again, by (49.98) and (49.97),

(49.106) $$\tilde{f}(X\setminus Y) - \tilde{g}(Y\setminus X) = \max\{w^{\mathsf{T}}x \mid x \in P\} \leq \tilde{f}(X) - \tilde{g}(Y).$$

So $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ are intersecting families, and the pair $(\tilde{f},\tilde{g})$ is paramodular. It determines P, since P is determined by upper and lower bounds on $x(U)$ for subsets U of S. ∎

Corollary 49.12a implies:

Corollary 49.13a. *A generalized polymatroid P is integer if and only if there is a paramodular pair (f,g) defining P with f and g integer.*

Proof. Sufficiency follows from Corollary 49.12a. Necessity follows from Theorem 49.13, as P is determined by $(\tilde{f},\tilde{g})$, where $\tilde{f}$ and $\tilde{g}$ are integer if P is integer. ∎

A second consequence is:

Corollary 49.13b. *The intersection of two integer generalized polymatroids is integer.*

Proof. Directly by combining Corollaries 49.12c and 49.13a. ∎

We should note that the collections $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ found in the proof of Theorem 49.13 are lattice families and that $\tilde{f}$ and $\tilde{g}$ are sub- and supermodular respectively. Moreover, $\tilde{f}(T \setminus U) - \tilde{g}(U \setminus T) \leq \tilde{f}(T) - \tilde{g}(U)$ for each pair $T \in \tilde{\mathcal{C}}$, $U \in \tilde{\mathcal{D}}$.

This implies that projections of generalized polymatroids are again generalized polymatroids (Frank [1984b]):

Corollary 49.13c. *Let $P \subseteq \mathbb{R}^S$ be a generalized polymatroid and let $t \in S$. Define $S' := S - t$. Then the projection*

(49.107) $$P' := \{x \in \mathbb{R}^{S'} \mid \exists \eta : (x, \eta) \in P\}$$

is again a generalized polymatroid.

Proof. We can assume that P is nonempty, that $\mathcal{C}$ and $\mathcal{D}$ are lattice families, and that P is determined by a paramodular pair $(f, g) = (\tilde{f}, \tilde{g})$ as above. Let $\mathcal{C}'$ and $\mathcal{D}'$ be the collections of sets in $\mathcal{C}$ and $\mathcal{D}$ respectively not containing t. Let $f' := f|\mathcal{C}'$ and $g' := g|\mathcal{D}'$.

Trivially, (f', g') is a paramodular pair. We claim that P' is equal to the generalized polymatroid Q determined by (f', g'). Trivially, $P' \subseteq Q$. To see the reverse inclusion, let $x \in Q$. Let η' be the largest real such that $x(T - t) + \eta' \leq f(T)$ for each $T \in \mathcal{C} \setminus \mathcal{C}'$. Let η'' be the smallest real such that $x(U - t) + \eta'' \geq g(U)$ for each $U \in \mathcal{D} \setminus \mathcal{D}'$.

If $x \notin P'$, then $\eta' < \eta''$, and hence there exist $T \in \mathcal{C}$ and $U \in \mathcal{D}$ with $t \in T \cap U$ and $f(T) - x(T - t) < g(U) - x(U - t)$. Hence

(49.108) $$\begin{aligned} x(T \setminus U) - x(U \setminus T) &= x(T - t) - x(U - t) > f(T) - g(U) \\ &\geq f(T \setminus U) - g(U \setminus T). \end{aligned}$$

This contradicts the fact that $x(T \setminus U) \leq f(T \setminus U)$ and $x(U \setminus T) \geq g(U \setminus T)$, as $x \in Q$. ∎

For results on the dimension of generalized polymatroids, see Frank and Tardos [1988], which paper surveys generalized polymatroids and submodular flows. More results on generalized polymatroids are reported by Fujishige [1984b], Nakamura [1988b], Naitoh and Fujishige [1992], and Tamir [1995].

49.11c. Supermodular colourings

A colouring-type of result on supermodular functions was shown by Schrijver [1985]. We give the proof based on generalized polymatroids found by Tardos [1985b].

Theorem 49.14. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be intersecting families of subsets of a set S, let $g_1 : \mathcal{C}_1 \to \mathbb{Z}$ and $g_2 : \mathcal{C}_2 \to \mathbb{Z}$ be intersecting supermodular, and let $k \in \mathbb{Z}_+$ with $k \geq 1$. Then S can be partitioned into classes $L_1, \ldots, L_k$ such that*

(49.109) $$g_i(U) \leq |\{j \in \{1, \ldots, k\} \mid L_j \cap U \neq \emptyset\}|$$

for each $i = 1, 2$ and each $U \in \mathcal{C}_i$ if and only if

(49.110) $$g_i(U) \leq \min\{k, |U|\}$$

for each $i = 1, 2$ *and each* $U \in \mathcal{C}_i$.

Proof. Necessity is easy. Sufficiency is shown by induction on k, the case $k = 0$ being trivial. By induction, it suffices to find a subset L of S such that

(49.111) $\quad |U \setminus L| \geq g_i(U) - 1$ and, if $g_i(U) = k$, then $U \cap L \neq \emptyset$.

Indeed, in that case we can apply induction to the functions $g_i' : \mathcal{C}_i' \to \mathbb{Z}$ on $\mathcal{C}_i' := \{U \setminus L \mid U \in \mathcal{C}\}$, defined by

(49.112) $$g_i'(U \setminus L) := \begin{cases} g_i(U) - 1 & \text{if } U \cap L \neq \emptyset, \\ g_i(U) & \text{if } U \cap L = \emptyset, \end{cases}$$

for $U \in \mathcal{C}_i$.

For $i = 1, 2$, consider the system:

(49.113)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for } s \in S, \\ \text{(ii)} & x(U) \leq |U| - g_i(U) + 1 & \text{for } U \in \mathcal{C}_i, \\ \text{(iii)} & x(U) \geq 1 & \text{for } U \in \mathcal{C}_i \text{ with } g_i(U) = k. \end{array}$$

This system determines an integer generalized polymatroid. This can be seen as follows. Let $\mathcal{D}_i$ be the collection of inclusionwise minimal sets in $\{U \in \mathcal{C}_i \mid g_i(U) = k\}$. So $\mathcal{D}_i$ consists of disjoint sets (as $\mathcal{C}_i$ is intersecting and as $g_i(U) \leq k$ for each $U \in \mathcal{C}_i$). Let

(49.114) $\quad \mathcal{C}_i' := \{U \in \mathcal{C}_i \mid \forall T \in \mathcal{D}_i : U \subseteq T \text{ or } T \cap U = \emptyset\}$.

Then (49.113) has the same solution set as:

(49.115)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_s \leq 1 & \text{for } s \in S, \\ \text{(ii)} & x(U) \leq |U| - g_i(U) + 1 & \text{for } U \in \mathcal{C}_i', \\ \text{(iii)} & x(U) \geq 1 & \text{for } U \in \mathcal{D}_i. \end{array}$$

Indeed, (49.115)(iii) implies (49.113)(iii) (as $x \geq \mathbf{0}$). Moreover, for any $U \in \mathcal{C}_i$ with $T \cap U \neq \emptyset$ for some $T \in \mathcal{D}_i$, one has

(49.116) $\quad g_i(T \cap U) \geq g_i(T) + g_i(U) - g_i(T \cup U) \geq g_i(U)$

(as $g_i(T \cup U) \leq k = g_i(T)$). So with (49.115)(iii) we have:

(49.117)
$$\begin{aligned} & x(U) \leq x(T \cap U) + |U \setminus T| \leq |T \cap U| - g_i(T \cap U) + 1 + |U \setminus T| \\ & \leq |U| - g_i(U) + 1 \end{aligned}$$

(as $x_s \leq 1$ for all $x \in U \setminus T$). Hence (49.113) and (49.115) have the same solution set.

Now (49.115) is a system defining a generalized polymatroid, as one easily checks (condition (49.87)(iii) follows, since if $T \in \mathcal{C}_i'$ and $U \in \mathcal{D}_i$ with $T \setminus U \neq \emptyset$ and $U \setminus T \neq \emptyset$, then, by definition of $\mathcal{C}_i'$, T and U are disjoint, and then the inequality in (49.87)(iii) is trivial). It is integer, as the right-hand sides in (49.115) are integer.

Also, the intersection of these generalized polymatroids for $i = 1$ and $i = 2$ is nonempty, since the vector $x := k^{-1} \cdot \mathbf{1}$ belongs to it. (49.115)(i) and (iii) hold trivially. To see (ii), we have

(49.118)
$$\begin{aligned} & x(U) = \tfrac{1}{k}|U| = |U| - \tfrac{k-1}{k}|U| \leq |U| - \tfrac{k-1}{k} g_i(U) = |U| - g_i(U) + \tfrac{1}{k} g_i(U) \\ & \leq |U| - g_i(U) + 1. \end{aligned}$$

Therefore, the intersection contains an integer vector x, which is, by (49.115)(i), the incidence vector of some subset L of S satisfying (49.111), as required. ∎

This theorem generalizes edge-colouring theorems for bipartite graphs $G = (V, E)$. Let V_1 and V_2 be the colour classes of G. Let $\mathcal{C}_i := \{\delta(v) \mid v \in V_i\}$ for $i = 1, 2$. If we define $g_i(\delta(v)) := |\delta(v)|$ for $v \in V_i$ $(i = 1, 2)$, Theorem 49.14 reduces to Kőnig's edge-colouring theorem (Theorem 20.1). If $g_i(\delta(v))$ is set to the minimum degree of G, we obtain Theorem 20.5, and if it is set to the minimum of k and $|\delta(v)|$, we obtain Theorem 20.6.

Theorem 49.14 can also be used in proving the 'disjoint bibranchings theorem' (Theorem 54.11 — see Section 54.7a). Szigeti [1999] gave a generalization of Theorem 49.14.

49.11d. Further notes

Let S and T be disjoint sets. A function $f : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathbb{R}$ is called *bisubmodular* if

(49.119) $$f(X_1 \cap X_2, Y_1 \cup Y_2) + f(X_1 \cup X_2, Y_1 \cap Y_2) \leq f(X_1, Y_1) + f(X_2, Y_2)$$

for all $X_1, X_2 \subseteq S$ and $Y_1, Y_2 \subseteq T$.

Bisubmodular functions were studied by Kung [1978b] and Schrijver [1978, 1979c]. Most of the results can be obtained from those for submodular functions, by considering the submodular set function f' on $S \cup T$ defined by $f'(X \cup Y) := f((S \setminus X) \cup Y)$ for $X \subseteq S$ and $Y \subseteq T$. Similarly for *bisupermodular* functions, where the inequality sign in (49.119) is reversed.

For an interesting related result of Frank and Jordán [1995b] yielding Győri's theorem, see Section 60.3d.

Fujishige [1984c] gave a framework that includes Theorem 46.2 on the total dual integrality of the intersection of a polymatroid and a contrapolymatroid system, Corollary 46.2b on the existence of a modular function between a sub- and a supermodular function, and Theorem 49.13 on the total dual integrality of the generalized polymatroid constraints (but not the total dual integrality of the intersection of two polymatroids). Fujishige [1984b] described generalized polymatroids as projections of base polyhedra of submodular functions.

Chandrasekaran and Kabadi [1988] introduced the concept of a *generalized submodular function* as a function $f : \mathcal{R} \to \mathbb{R}$, where $\mathcal{R} := \{(T, U) \mid T, U \subseteq S, T \cap U = \emptyset\}$ for some set S, satisfying

(49.120) $$\begin{aligned} &f(A, B) + f(C, D) \\ &\geq f(A \cap C, B \cap D) + f((A \setminus D) \cup (C \setminus B), (B \setminus C) \cup (D \setminus A)) \end{aligned}$$

for all $(A, B), (C, D) \in \mathcal{R}$. They showed that the system

(49.121) $$x(T) - x(U) \leq f(T, U) \text{ for } (T, U) \in \mathcal{R}$$

is box-TDI, and that for any $w \in \mathbb{R}^S$, an x maximizing $w^{\mathsf{T}} x$ over (49.121) can be found by a variant of the greedy method. Unions of two such systems need not define an integer polyhedron if the functions are integer, as is shown by an example with $|S| = 2$. A similar framework was considered by Nakamura [1990]. More results can be found in Dress and Havel [1986], Bouchet [1987a,1995], Bouchet, Dress, and

Havel [1992], Ando and Fujishige [1996], Fujishige [1997], and Fujishige and Iwata [2001].

It is direct to represent a lattice family on a set S of size n in $O(n^2)$ space (just by giving all pairs (u, v) for which each set in the family containing u also contains v). Gabow [1993b,1995c] gave an $O(n^2)$ representation for intersecting and crossing families. Related results were found by Fleiner and Jordán [1999].

Tardos [1985b] also studied *generalized matroids*, which form the special case of generalized polymatroids with $0, 1$ vertices. An instance of it we saw in the proof of Theorem 49.14.

More results on submodularity are given by Fujishige [1980b,1984f,1984g,1988], Nakamura [1988b,1988c,1993], Kabadi and Chandrasekaran [1990], Iwata [1995], Iwata, Murota, and Shigeno [1997], and Murota [1998]. Generalizations were studied by Qi [1988b] and Kashiwabara, Nakamura, and Takabatake [1999].

Part V

Trees, Branchings, and Connectors

Part V: Trees, Branchings, and Connectors

This part focuses on structures that are defined by connecting several pairs of vertices simultaneously, with most basic structure that of a *spanning tree*. A spanning tree can be characterized as a minimal set of edges that connects each pair of vertices by at least one path — that is, a minimal *connector*. Alternatively, it can be characterized as a maximal set of edge that connects each pair of vertices by at most one path — that is, a maximal *forest*.

Finding a shortest spanning tree belongs to classical combinatorial optimization, with lots of applications in planning road, energy, and communication networks, in chip design, and in clustering data in areas like biology, taxonomy, archeology, and, more generally, in any large data base. Spanning trees are well under control polyhedrally and algorithmically, both as to *shortest* and as to *disjoint* spanning trees. They form a prime area of application of matroid theory.

There are several variations and generalizations of the notion of spanning tree that are also well under control, like arborescences, branchings, biconnectors, bibranchings, directed cut covers, and matching forests.

An illustrious variant that is worse under control is the Hamiltonian circuit — in other words, the traveling salesman tour — which (in the directed case) can be considered as a smallest strongly connected subgraph. The traveling salesman problem is NP-complete and no complete polyhedral characterization is known. It implies that more general optimization problems like finding a shortest strong connector or a cheapest connectivity augmentation also are NP-complete. In this part we will however come across some special cases that *are* well-solvable and well-characterized.

In this part we also discuss the powerful framework designed by Edmonds and Giles, based on defining the concept of a *submodular flow* in a directed graph with a submodular function on its vertex set. It unifies several of the results and techniques of the present part and of the previous part on matroids and submodular functions.

Chapters:

Chapter 50

Shortest spanning trees

In this chapter we consider shortest spanning trees in undirected graphs. We show that the greedy algorithm finds a shortest spanning tree in a graph, and moreover yields min-max relations and polyhedral characterizations. These are special cases of results on matroids discussed in Chapter 40, but deserve special consideration since the graph framework allows a number of additional viewpoints and opportunities.

We recall some terminology and elementary facts. A graph $G = (V, E)$ is called a *tree* if G is connected and contains no circuit. For any graph $G = (V, E)$, a subset F of E is called:

- a *spanning tree* if (V, F) is a tree,
- a *forest* if F contains no circuit,
- a *maximal forest* if F is an inclusionwise maximal forest,
- a *connector* if (V, F) is connected.

A graph G has a spanning tree if and only if G is connected. For any connected graph $G = (V, E)$, each of the following characterizes a subset F of E as a spanning tree:

- F is a maximal forest;
- F is an inclusionwise minimal connector;
- F is a forest with $|F| = |V| - 1$;
- F is a connector with $|F| = |V| - 1$.

In any graph $G = (V, E)$, a maximal forest has $|V| - k$ edges, where k is the number of components of G; it forms a spanning tree in each of the components of G. So each inclusionwise maximal forest is a maximum-size forest; that is, each forest is contained in a maximum-size forest. Similarly, each connector contains a minimum-size connector.

50.1. Shortest spanning trees

Let $G = (V, E)$ be a connected graph and let $l : E \to \mathbb{R}$ be a function, called the *length* function. For any subset F of E, the *length* $l(F)$ of F is, by definition:

(50.1) $$l(F) := \sum_{e \in F} l(e).$$

In this section we consider the problem of finding a shortest spanning tree in G — that is, one of minimum length.

While this is a special case of finding a minimum-weight base in a matroid, and hence can be solved with the greedy algorithm (Section 40.1), spanning trees allow some variation on the method, essentially because we can exploit the presence of the vertex set (graphic matroids are defined on the edge set only).

Also these variants of the greedy method will be called greedy. Such methods go back to Borůvka [1926a]. The correctness of each of the variants follows from the following basic phenomenon.

Call a forest F *good* if there exists a shortest spanning tree T of G that contains F. (So we are out for a good spanning tree.) Then:

Theorem 50.1. *Let F be a good forest and let e be an edge not in F. Then $F \cup \{e\}$ is a good forest if and only if*

(50.2) *there exists a cut C disjoint from F such that e is shortest among the edges in C.*

Proof. To see necessity, let T be a shortest spanning tree containing $F \cup \{e\}$. Let C be the unique cut disjoint from $T \setminus \{e\}$. Then e is shortest in C, since if $f \in C$, then $T' := (T \setminus \{e\}) \cup \{f\}$ is again a spanning tree. As $l(T') \geq l(T)$ we have $l(f) \geq l(e)$.

To see sufficiency, let T be a shortest spanning tree containing F. Let P be the path in T between the ends of e. Then P contains at least one edge f that belongs to C. Then $T' := (T \setminus \{f\}) \cup \{e\}$ is a spanning tree again. By assumption, $l(e) \leq l(f)$ and hence $l(T') \leq l(T)$. Hence T' is a shortest spanning tree again. As $F \cup \{e\}$ is contained in T', it is a good forest. ∎

(The idea of this proof is in Jarník [1930].)

This theorem offers us a framework for an algorithm: starting with $F := \emptyset$, iteratively extend F by an edge e satisfying (50.2). We end up with a shortest spanning tree.

Rule (50.2) was formulated by Tarjan [1983], and is the most liberal rule in obtaining greedily a shortest spanning tree. The variants of the greedy method are obtained by specifying how to choose edge e.

The first variant, the *tree-growing method*, was given by Jarník [1930] (and by Kruskal [1956], Prim [1957], Dijkstra [1959]). It is also called the *Jarník-Prim method* or *Prim's method* (Prim was the first giving an $O(n^2)$ implementation):

(50.3) Fix a vertex r. Set $F := \emptyset$. As long as F is not a spanning tree, let K be the component of F containing r, let e be a shortest edge leaving K, and reset $F := F \cup \{e\}$.

Corollary 50.1a. *Prim's method yields a shortest spanning tree.*

Proof. Directly from Theorem 50.1, by taking $C := \delta(K)$. ∎

A second variant, the *forest-merging method* or *Kruskal's method*, is due to Kruskal [1956] (and to Loberman and Weinberger [1957] and Prim [1957]):

(50.4) Set $F := \emptyset$. As long as F is not a spanning tree, choose a shortest edge e for which $F \cup \{e\}$ is a forest, and reset $F := F \cup \{e\}$.

(So this version is the true specialization of the greedy algorithm for matroids to graphs.)

Corollary 50.1b. *Kruskal's method yields a shortest spanning tree.*

Proof. Again directly from Theorem 50.1, as e is shortest in the cut $\delta(K)$ for each of the two components K of F incident with e. ∎

Prim [1957] and Loberman and Weinberger [1957] observed that the optimality of the greedy method implies that each length function which gives the same order of the edges (like the logarithm or square of the lengths), has the same collection of shortest spanning trees. Similarly, the shortest spanning tree minimizes the *product* of the lengths (if nonnegative).

In a similar way one finds a *longest* spanning tree. The maximum length of a forest and the minimum length of a connector can also be found with the greedy method.

Note that the greedy method is flexible: We can change our rule of choosing the new edge e at any time throughout the algorithm, as long as at any choice of e, (50.2) is satisfied.

As Prim [1957] and Dijkstra [1959] remark, the value of any variant of the greedy method depends on its implementation. One should have efficient ways to store and update information on the components of F, and on finding an edge satisfying (50.2). We now consider such implementations for Prim's and for Kruskal's method.

50.2. Implementing Prim's method

Prim [1957] and Dijkstra [1959] described implementations of Prim's method that run in time $O(n^2)$. (Here we assume without loss of generality that the graph is simple.)

To this end, we indicate at any vertex v, whether or not v belongs to the component K containing r of the current forest F, and in case $v \notin K$, we store at v a shortest edge e_v connecting v with K (void if there is no such edge). Then at each iteration, we scan all vertices, and select one, v say, for which $v \notin K$ and e_v is shortest. We add e_v to F, and v to K, and for each edge vu incident with v, we replace e_u by vu if $u \notin K$ and vu is shorter than e_u (or if e_u is void).

As each iteration takes $O(n)$ time and as there are $n-1$ iterations we have the result stated by Dijkstra [1959]:

Theorem 50.2. *A shortest spanning tree can be found in time $O(n^2)$.*

Proof. See above. ∎

In fact, by applying 2-heaps (Section 7.3) one can obtain a running time bound of $O(m \log n)$ (E.L. Johnson, cf. Kershenbaum and Van Slyke [1972]), and with Fibonacci heaps (Section 7.4) one obtains (Fredman and Tarjan [1984,1987]):

Theorem 50.3. *A shortest spanning tree can be found in time $O(m + n \log n)$.*

Proof. Directly by applying Fibonacci heaps as described in Section 7.4. ∎

50.3. Implementing Kruskal's method

Bottleneck in implementing Kruskal's method is the necessity to scan the edges sorted by length. As the best bound for sorting is $O(m \log n)$, we cannot hope for implementations of Kruskal's method faster than that.

However, the bound $O(m \log n)$ is easy to achieve. In fact, as was noticed by Kershenbaum and Van Slyke [1972] (using ideas of Van Slyke and Frank [1972]), it is easy to implement Kruskal's method such that the time *after sorting* is $O(m + n \log n)$. This can be done with elementary data-structures like lists; no heaps are needed.

Indeed, it is not hard to design a simple data structure that tests in constant time if the ends of any edge belong to different components of the current forest F, and that merges components in time linear in the size of the smaller component[1].

Then the iterations take $O(m + n \log n)$ time, since checking if the ends of an edge belong to different components takes $O(m)$ time overall, while merging takes $O(n \log n)$ time overall: any vertex v belongs at most $\log_2 n$ times to the smaller component when merging, as, at any such event, the component containing v at least doubles in size.

---

[1] Consider any forest F. Represent each component K by a (singly) linked list. For any vertex v, let $r(v)$ be the first vertex of the list L_v containing v.

Initially, for each vertex v, $r(v) = v$, as $L_v = \{v\}$. At any iteration, the edge $e = uv$ considered connects different components of F if and only if $r(u) \neq r(v)$. Checking this takes constant time.

If $r(u) \neq r(v)$, we can determine which of the lists L_u, L_v is smallest in time $O(\min\{|L_u|, |L_v|\})$ (by scanning them in parallel, starting at $r(u)$ and $r(v)$). Assume without loss of generality that $|L_u| \leq |L_v|$. Then we reset $r(u') := r(v)$ for all u' in L_u, and we insert L_u into L_v directly after v. This can be done in time $O(|L_u|)$.

Tarjan [1983] showed that if the edges are presorted, a minimum spanning tree can be found in time $O(m\alpha(m,n))$ (where $\alpha(m,n)$ is the 'inverse Ackermann function — see Section 50.6a).

50.3a. Parallel forest-merging

A variant that suggests parallel implementation was given by Borůvka [1926a,1926b] — the *parallel forest-merging method* or *Borůvka's method*. (This method was also given by Choquet [1938] (without proof) and Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a].) It assumes that all edge lengths are different:

(50.5) Set $F := \emptyset$. As long as F is not a spanning tree do the following: choose for each component K of F the shortest edge leaving K, and add all chosen edges to F.

Theorem 50.4. *Assuming that all edge lengths are different, the parallel forest-merging variant yields a shortest spanning tree.*

Proof. We show that F remains a good forest throughout the iterations. Consider some iteration, and let F be the good forest at the start of the iteration. Let $e_1, \ldots, e_k$ be the edges added in the iteration, indexed such that $l(e_1) < l(e_2) < \cdots < l(e_k)$. By the selection rule (50.5), for each $i = 1, \ldots, k$, e_i is the shortest edge leaving some component K of F. Then K is also a component of $F \cup \{e_1, \ldots, e_{i-1}\}$, as none of $e_1, \ldots, e_{i-1}$ leave K (since e_i is shortest leaving K). Hence for each $i = 1, \ldots, k$, $F \cup \{e_1, \ldots, e_i\}$ is a good forest (by induction on i). Concluding, the iteration yields a good forest. ∎

50.3b. A dual greedy algorithm

We can consider a dual approach by iteratively decreasing a connector, instead of iteratively growing a forest. The analogy can be exhibited as follows.

Let $G = (V, E)$ be a connected graph and let $l : E \to\in \mathbb{R}$ be a length function. Call a connector $K \subseteq E$ *good* if K contains a shortest spanning tree. Then we have:

Theorem 50.5. *Let K be a good connector and let $e \in K$. Then $K \setminus \{e\}$ is a good connector if and only if*

(50.6) *K contains a circuit C such that e is a longest edge in C.*

Proof. To see necessity, let T be a shortest spanning tree contained in $K \setminus \{e\}$. Let C be the unique circuit contained in $T \cup \{e\}$. Then e is longest in C, since if $f \in C$, then $T' := (T \setminus \{f\}) \cup \{e\}$ is again a spanning tree. As $l(T') \geq l(T)$ we have $l(e) \geq l(f)$.

To see sufficiency, let T be a shortest spanning tree contained in K. If $e \notin T$, then also $K \setminus \{e\}$ contains T, and hence $K \setminus \{e\}$ is a good connector. So we can assume that $e \in T$. Let D be the cut determined by $T - e$. Then the circuit C contains at least one edge $f \neq e$ that belongs to D. So $T' := (T \setminus \{e\}) \cup \{f\}$ is a spanning tree again. By assumption, $l(e) \geq l(f)$ and hence $l(T') \leq l(T)$. Hence T'

is a shortest spanning tree again. It is contained in $K \setminus \{e\}$, which therefore is a good connector. ∎

So we can formulate the *dual greedy algorithm*: starting with $K := E$, iteratively remove from K an edge e satisfying (50.6). We end up with a shortest spanning tree.

A special case is the following algorithm, proposed by Kruskal [1956]: iteratively delete a longest edge e that is not a bridge. We end up with a shortest spanning tree.

50.4. The longest forest and the forest polytope

The greedy algorithm can be easily adapted so as to give:

Theorem 50.6. *A longest forest can be found in strongly polynomial time.*

Proof. It suffices to find a longest spanning tree in any component. This can be done with the greedy method. ∎

As Edmonds [1971] noticed, it is easy to derive with the greedy method a min-max relation for the maximum length of a forest in a graph $G = (V, E)$. This is similar to the results of Section 40.2.

Theorem 50.7. *Let $G = (V, E)$ be a graph and let $l \in \mathbb{Z}_+^E$. Then the maximum length of a forest is equal to the minimum value of*

$$(50.7)\qquad \sum_{U \in \mathcal{P}(V)\setminus\{\emptyset\}} y_U(|U| - 1),$$

where $y \in \mathbb{Z}_+^{\mathcal{P}(V)\setminus\{\emptyset\}}$ satisfies

$$(50.8)\qquad \sum_{U \in \mathcal{P}(V)\setminus\{\emptyset\}} y_U \chi^{E[U]} \geq l.$$

Proof. The maximum cannot be larger than the minimum, since for any forest F and any $y \in \mathbb{Z}_+^{\mathcal{P}(V)\setminus\{\emptyset\}}$ satisfying (50.8) one has:

$$(50.9)\qquad l(F) \leq \sum_{U \in \mathcal{P}(V)\setminus\{\emptyset\}} y_U |E[U] \cap F| \leq \sum_{U \in \mathcal{P}(V)\setminus\{\emptyset\}} y_U(|U| - 1).$$

To see equality, let $k := \max\{l(e) \mid e \in E\}$, and let E_i be the set of edges e with $l(e) \geq i$, for $i = 0, 1, \ldots, k$. For each $U \in \mathcal{P}(V) \setminus \{\emptyset\}$, let y_U be the number of $i \in \{1, \ldots, k\}$ such that U is a component of the graph (V, E_i). Then it is easy to see that y satisfies (50.8).

We can find a sequence of forests $F_k \subseteq \cdots \subseteq F_1 \subseteq F_0$, where for $i = 0, 1, \ldots, k$, F_i is a maximal forest in (V, E_i) containing F_{i+1}, setting $F_{k+1} := \emptyset$.

Then for $F := F_0$ we have:

$$(50.10) \qquad l(F) = \sum_{i=0}^{k} i|F_i \setminus F_{i+1}| = \sum_{i=1}^{k} |F_i| = \sum_{i=1}^{k} (|V| - \kappa(V, E_i)) = \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U(|U| - 1),$$

where $\kappa(V, E_i)$ denotes the number of components of the graph (V, E_i). ∎

(The series of forests $F_k \subseteq F_{k-1} \subseteq \cdots \subseteq F_1 \subseteq F_0$, corresponds to the greedy method.)

Note that this theorem gives, if G is connected, a min-max relation for the maximum length of a spanning tree.

For any graph $G = (V, E)$, let the *forest polytope* of G, denoted by $P_{\text{forest}}(G)$, be the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the forests of G. The following characterization of the forest polytope is (in matroid terms) due to Edmonds [1971] (announced in Edmonds [1967a]):

Corollary 50.7a. *The forest polytope of a graph G is determined by*

$$(50.11) \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(E[U]) \leq |U| - 1 & \text{for nonempty } U \subseteq V. \end{array}$$

Proof. Trivially, the incidence vector of any forest satisfies (50.11), and hence the forest polytope is contained in the polytope determined by (50.11). Suppose now that the latter polytope is larger. Then (since both polytopes are rational and down-monotone in $\mathbb{R}^E_+$) there exists a vector $l \in \mathbb{Q}^E_+$ such that the maximum value of $l^{\mathsf{T}}x$ over (50.11) is larger than the maximum of $l(F)$ over forests F. We can assume that l is integer. However, by Theorem 50.7, the maximum of $l(F)$ is at least the minimum value of the problem dual to maximizing $l^{\mathsf{T}}x$ over (50.11), a contradiction. ∎

Theorem 50.7 can be stated equivalently in TDI terms as follows:

Corollary 50.7b. *System* (50.11) *is totally dual integral.*

Proof. This follows from Theorem 50.7, by the definition of total dual integrality. ∎

Having a description of the forest polytope, we can derive a description of the *spanning tree polytope* $P_{\text{spanning tree}}(G)$ of a graph $G = (V, E)$, which is the convex hull of the incidence vectors of the spanning trees in G.

Corollary 50.7c. *The spanning tree polytope of a graph $G = (V, E)$ is determined by*

(50.12)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(E[U]) \leq |U| - 1 & \text{for nonempty } U \subseteq V, \\ \text{(iii)} & x(E) = |V| - 1. & \end{array}$$

Proof. Directly from Corollary 50.7a, since the spanning trees are exactly the forests of size $|V|-1$, and since there exist no forests larger than that. ∎

One also directly has a TDI result:

Corollary 50.7d. *System* (50.12) *is totally dual integral.*

Proof. Directly from Corollary 50.7b, since (50.12) arises from (50.11) by setting an inequality to equality (cf. Theorem 5.25). ∎

Theorem 40.5 implies that (if G is loopless) an inequality (50.12)(ii) is facet-inducing if and only if $|U| \geq 2$ and U induces a 2-connected subgraph of G (cf. Grötschel [1977a]).

In Section 51.4 we consider the problem of testing membership of the forest polytope.

50.5. The shortest connector and the connector polytope

The greedy method also provides a min-max relation for the minimum length of a connector in a graph $G = (V, E)$. Let Π denote the collection of partitions of V into nonempty subsets. For any partition $\mathcal{P}$ of V, let $\delta(\mathcal{P})$ denote the set of edges connecting two different classes of $\mathcal{P}$. So any connector contains at least $|\mathcal{P}| - 1$ edges in $\delta(\mathcal{P})$.

Theorem 50.8. *Let $G = (V, E)$ be a connected graph and let $l \in \mathbb{Z}_+^E$. Then the minimum length of a spanning tree is equal to the maximum value of*

(50.13)
$$\sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1),$$

where $y \in \mathbb{Z}_+^{\Pi}$ such that

(50.14)
$$\sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} \chi^{\delta(\mathcal{P})} \leq l.$$

Proof. The minimum cannot be smaller than the maximum, since for any spanning tree T and any $y \in \mathbb{Z}_+^{\Pi}$ satisfying (50.14) one has:

(50.15)
$$l(T) \geq \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} \chi^{\delta(\mathcal{P})}(T) = \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} |\delta(\mathcal{P}) \cap T| \geq \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1).$$

To see equality, define $k := \max\{l(e) \mid e \in E\}$ and for $i = 0, 1, \ldots, k$, let E_i be the set of edges e with $l(e) \leq i$. For each $\mathcal{P} \in \Pi$, let $y_{\mathcal{P}}$ be the number of

$i \in \{1, \ldots, k\}$ such that $\mathcal{P}$ is the collection of components of (V, E_i). Then it is easy to see that y satisfies (50.14).

We can find a sequence of forests $F_0 \subseteq F_1 \subseteq \cdots F_{k-1} \subseteq F_k$, where F_0 is a maximal forest in (V, E_0), and where for $i = 0, \ldots, k$, F_i is a maximal forest in (V, E_i) containing F_{i-1}, setting $F_{-1} := \emptyset$.

Then for $T := F_k$ we have:

$$\text{(50.16)} \qquad \begin{aligned} l(T) &= \sum_{i=0}^{k} i|F_i \setminus F_{i-1}| = k|T| - \sum_{i=0}^{k-1} |F_i| = \sum_{i=0}^{k-1}(|V| - 1 - |F_i|) \\ &= \sum_{i=1}^{k-1}(\kappa(V, E_i) - 1) = \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1), \end{aligned}$$

where $\kappa(V, E_i)$ denotes the number of components of the graph (V, E_i). ∎

For any graph $G = (V, E)$, let the *connector polytope* of G, denoted by $P_{\text{connector}}(G)$, be the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the connectors of G. The following characterization can be derived from Edmonds [1970b], and was stated explicitly by Fulkerson [1970b]:

Corollary 50.8a. *The connector polytope of a graph G is determined by*

$$\text{(50.17)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(\mathcal{P})) \geq |\mathcal{P}| - 1 & \text{for } \mathcal{P} \in \Pi. \end{array}$$

Proof. Trivially, the incidence vector of any connector satisfies (50.17), and hence the connector polytope is contained in the polytope determined by (50.17). Suppose now that the latter polytope is larger. Then (since both polytopes are rational and up-monotone in $[0,1]^E$) there exists a vector $l \in \mathbb{Q}_+^E$ such that the minimum value of $l^{\mathsf{T}}x$ over (50.17) is smaller than the minimum of $l(C)$ over connectors C. We can assume that l is integer. However, by Theorem 50.8, the minimum of $l(C)$ is at most the maximum value of the problem dual to minimizing $l^{\mathsf{T}}x$ over (50.17), a contradiction. ∎

Theorem 50.8 can be stated equivalently in TDI terms as follows:

Corollary 50.8b. *System* (50.17) *is totally dual integral.*

Proof. This follows from Theorem 50.8, by the definition of total dual integrality. ∎

Chopra [1989] described the facets of the connector polytope. In Section 51.4 we consider the problem of testing membership of the connector polytope.

50.6. Further results and notes

50.6a. Complexity survey for shortest spanning tree

	$O(nm)$	Jarník [1930]
	$O(n^2)$	Prim [1957], Dijkstra [1959]
	$O(m \log n)$	Kershenbaum and Van Slyke [1972], E.L. Johnson (cf. Kershenbaum and Van Slyke [1972])
	$O(m \log_{m/n} n)$	D.B. Johnson [1975b]
	$O(m\sqrt{\log n})$	R.E. Tarjan (cf. Yao [1975])
	$O(m \log \log n)$	Yao [1975]
	$O(m \log \log_{m/n} n)$	Cheriton and Tarjan [1976], Tarjan [1983]
*	$O((m + n \log L) \log \log L)$	D.B. Johnson [1977b]
	$O(m + n \log n)$	Fredman and Tarjan [1984,1987]
	$O(m\beta(m, n))$	Fredman and Tarjan [1984,1987]
	$O(m \log \beta(m, n))$	Gabow, Galil, Spencer, and Tarjan [1986] (cf. Gabow, Galil, and Spencer [1984])
	$O(m(\log_n L + \alpha(m, n)))$	Gabow [1983b,1985b]
	$O(m\alpha(m, n) \log \alpha(m, n))$	Chazelle [1997]
*	$O(m\alpha(m, n))$	Chazelle [2000]

As before, $*$ indicates an asymptotically best bound in the table. Moreover, $\beta(m, n) := \min\{i \mid \log_2^{(i)} n \leq m/n\}$ and $L := \max\{l(e) \mid e \in E\}$ (assuming l nonnegative integer). The function $\alpha(m, n)$ is the *inverse Ackermann function*, defined as follows. For $i, j \geq 1$, the *Ackermann function* $A(i, j)$ is defined recursively by:

$$\begin{aligned} A(1, j) &= 2^j && \text{for } j \geq 1, \\ A(i, 1) &= A(i-1, 2) && \text{for } i \geq 2, \\ A(i, j) &= A(i-1, A(i, j-1)) && \text{for } i, j \geq 2. \end{aligned} \tag{50.18}$$

Next, for $m \geq n \geq 1$,

$$\alpha(m, n) := \min\{i \geq 1 \mid A(i, \lfloor m/n \rfloor) > \log_2 n\}. \tag{50.19}$$

The function $\alpha(m, n)$ is extremely slowly growing.

Fredman and Willard [1990,1994] gave a 'strongly trans-dichotomous' linear-time minimum spanning tree algorithm (where capabilities of random access machines, like addressing, can be used). Based on sampling, Karger [1993,1998] found a simple linear-time approximative spanning tree algorithm, and an $O(m+n \log n)$-time minimum spanning tree algorithm not using Fibonacci heaps.

Katoh, Ibaraki, and Mine [1981] gave an algorithm to find the Kth shortest spanning tree in time $O(Km + \min\{n^2, m \log \log n\})$ (improving slightly Gabow

[1977]). They also gave an algorithm to find the second shortest spanning tree in time $O(\min\{n^2, m\alpha(m,n)\})$.

Pettie and Ramachandran [2000,2002a] showed that a shortest spanning tree can be found in time $O(\mathcal{T}^*(m,n))$, where $\mathcal{T}^*(m,n)$ is the minimum number of edge length comparisons needed to determine the solution.

Frederickson [1983a,1985] gave an $O(\sqrt{m})$-time algorithm to update a shortest spanning tree (and the data-structure) if one edge changes length. Spira and Pan [1973,1975] and Chin and Houck [1978] gave fast algorithms to update a shortest spanning tree if vertices are added or removed. More on sensitivity and most vital edges can be found in Tarjan [1982], Hsu, Jan, Lee, Hung, and Chern [1991], Dixon, Rauch, and Tarjan [1992], Iwano and Katoh [1993], Lin and Chern [1993], and Frederickson and Solis-Oba [1996,1999].

Tarjan [1979] showed that the minimality of a given spanning tree can be checked in time $O(m\alpha(m,n))$ (cf. Dixon, Rauch, and Tarjan [1992]). Komlós [1984, 1985] showed that the minimality of a given spanning tree can be checked by $O(m)$ comparisons of edge lengths. King [1997] gave a linear-time implementation in the unit-cost RAM model. A randomized linear-time algorithm was given by Klein and Tarjan [1994], and Karger, Klein, and Tarjan [1995].

Gabow and Tarjan [1984] (cf. Gabow and Tarjan [1979]) showed that the problem of finding a shortest spanning tree with a prescribed number of edges incident with a (one) given vertex r, is linear-time equivalent to the (unconstrained) shortest spanning tree problem. They also showed that if the edges of a graph are coloured red and blue, a shortest spanning tree having exactly k red edges (for given k) can be found in time $O(m \log\log_{2+\frac{m}{n}} n + n\log n)$.

Brezovec, Cornuéjols, and Glover [1988] gave an efficient algorithm to find a shortest spanning tree in a coloured graph with, for each colour, an upper and a lower bound on the number of edges in the tree of that colour.

Camerini [1978] showed that a spanning tree minimizing $\max_{e\in T} l(e)$ can be found in $O(m)$ time.

Geometric spanning trees (on vertices in Euclidean space, with Euclidean distance as length function) were considered by Bentley, Weide, and Yao [1980], Yao [1982], Supowit [1983], Clarkson [1984,1989], and Agarwal, Edelsbrunner, Schwarzkopf, and Welzl [1991].

50.6b. Characterization of shortest spanning trees

The following theorem is implicit in Kalaba [1960]:

Theorem 50.9. *Let $G = (V,E)$ be a graph, let $l \in \mathbb{R}^E$ be a length function, and let T be a spanning tree in G. Then T is a shortest spanning tree if and only if $l(f) \geq l(e)$ for all $e \in T$ and $f \in E \setminus T$ with $T - e + f$ a spanning tree.*

Proof. Necessity being trivial, we show sufficiency. Let the condition be satisfied, and suppose that T is not a shortest spanning tree. Choose a shorter spanning tree T' with $|T' \setminus T|$ minimal. Let $f \in T' \setminus T$. Let e be an edge on the circuit in $T \cup \{f\}$ with $e \neq f$, such that e connects the two components of $T' \setminus \{f\}$. Then $(T\setminus\{e\})\cup\{f\}$ is a spanning tree, and hence $l(f) \geq l(e)$. Define $T'' := (T'\setminus\{f\})\cup\{e\}$. Then $l(T'') \leq l(T') < l(T)$ and $|T'' \setminus T| < |T' \setminus T|$, contradicting our minimality assumption. ∎

This theorem gives a good characterization of the minimum length of a spanning tree. (As Kalaba [1960] pointed out, it also gives an algorithm to find a shortest spanning tree (by iteratively exchanging one edge for another if it makes the tree shorter), but it is not polynomial-time.)

Recall that a forest is called *good* if it is contained in a shortest spanning tree.

Corollary 50.9a. *Let $G = (V, E)$ be a connected graph, let $l \in \mathbb{R}^E$ be a length function, and let F be a forest. Then F is good if and only if for each $e \in F$ there exists a cut C with $C \cap F = \{e\}$ and with e shortest in C.*

Proof. To see necessity, let F be good and let $e \in F$. So there exists a shortest spanning tree T containing F. By Theorem 50.9, e is a shortest edge connecting the two components of $T - e$. This gives the required cut C.

Sufficiency is shown by induction on $|F|$, the case $F = \emptyset$ being trivial. Choose $e \in F$. By induction, $F \setminus \{e\}$ is good (as the condition is maintained for $F \setminus \{e\}$). The condition implies that (50.2) is satisfied, and hence F is good by Theorem 50.1. ∎

50.6c. The maximum reliability problem

Often, in designing a network, one is not primarily interested in minimizing the total length, but rather in maximizing 'reliability' (for instance when designing energy or communication networks).

Let $G = (V, E)$ be a connected graph and let $r : E \to \mathbb{R}_+$ be a function. Let us call $r(e)$ the *reliability* of edge e. For any path P in G, the *reliability* of P is, by definition, the minimum reliability of the edges occurring in P. The *reliability* $r_G(s, t)$ of two vertices s and t is equal to the maximum reliability of P where P ranges over all $s - t$ paths. That is,

$$r_G(s,t) := \max_{P} \min_{e \in EP} r(e), \tag{50.20}$$

where the maximum ranges over all $s - t$ paths P. (The value of $r_G(s, t)$ can be found with the method described in Section 8.6e.)

The problem now is to find a minimal subgraph H of G having the same reliability as G; that is, with $r_H = r_G$. Hu [1961] observed that there is a spanning tree carrying the reliability of G. More precisely, Hu showed that any spanning tree T of maximum total reliability is such a tree (also shown by Kalaba [1964]):

Corollary 50.9b. *Let $G = (V, E)$ be a graph, let $r \in \mathbb{R}^E$, and let T be any spanning tree. Then $r_T(s, t) = r_G(s, t)$ for all s, t if and only if T is a spanning tree in G maximizing $r(T)$.*

Proof. To see sufficiency, let T maximize $r(T)$. Choose $s, t \in V$, and let P be a path in G attaining maximum (50.20). Let e be an edge on the $s - t$ path in T with minimum $r(e)$. Then P contains an edge f connecting the two components of $T - e$. As T maximizes $r(T)$ we have $r(f) \leq r(e)$. Hence

$$r_T(s,t) = r(e) \geq r(f) \geq r_G(s,t). \tag{50.21}$$

Since trivially $r_T(s,t) \leq r_G(s,t)$, this shows sufficiency.

To see necessity, we apply Theorem 50.9. Choose $e \in T$, and suppose that there is an edge f connecting the components of $T - e$, with $r(f) > r(e)$. Then for the ends s, t of f we have

(50.22) $$r_G(s,t) \geq r(f) > r(e) \geq r_T(s,t),$$

a contradiction. ∎

Corollary 50.9b implies:

Corollary 50.9c. *Let $G = (V, E)$ be a complete graph and let $l : E \to \mathbb{R}_+$ be a length function satisfying*

(50.23) $$l(uw) \geq \min\{l(uv), l(vw)\}$$

for all distinct $u, v, w \in V$. Let T be a longest spanning tree in G. Then for all $u, w \in V$, $l(uw)$ is equal to the minimum length of the edges in the $u - w$ path in T.

Proof. Note that (50.23) implies that $l(uw)$ is equal to the reliability $r_G(u,w)$ of u and w, taking $r := l$. So the corollary follows from Corollary 50.9b. ∎

This implies the following. Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{R}_+$ be a capacity function. Let K be the complete graph on V. For each edge st of K, let the length $l(st)$ be the minimum capacity of any $s - t$ cut in G. (An $s - t$ cut is any subset $\delta(W)$ with $s \in W, t \notin W$.)

Let T be a longest spanning tree in K. Then for all $s, t \in V$, $l(st)$ is equal to the minimum length of the edges of T in the $s - t$ path in T.

(This tree need not be a Gomory-Hu tree, as is shown by the complete graph on vertices $1, 2, 3$ and $c(12) = 1$ and $c(13) = c(23) = 2$. Then edges 12 and 13 form a tree as above, but it is not a Gomory-Hu tree.)

50.6d. Exchange properties of forests

The following fundamental property of forests in fact is the basis of most theorems in this chapter. It is the 'exchange property' that makes the collection of forests into a matroid.

Theorem 50.10. *Let $G = (V, E)$ be a graph and let F and F' be forests with $|F| < |F'|$. Then $F \cup \{e\}$ is a forest for some $e \in F' \setminus F$.*

Proof. We can assume that $E = F \cup F'$. If no such edge e exists, then F is a maximal forest in G. This however implies that $|F| \geq |F'|$, a contradiction. ∎

Call a forest F *extreme* if $l(F') \geq l(F)$ for each forest F' satisfying $|F'| = |F|$. The forests made iteratively in Kruskal's method all are extreme, since:

Corollary 50.10a. *Let F be an extreme forest and let e be a shortest edge with $e \notin F$ and $F \cup \{e\}$ a forest. Then $F \cup \{e\}$ is extreme again.*

Proof. Let F' be an extreme forest with $|F'| = |F| + 1$. By Theorem 50.10, there exists an $e' \in F' \setminus F$ such that $F \cup \{e'\}$ is a forest. As F is extreme we have $l(F' \setminus \{e'\}) \geq l(F)$. Hence $l(F \cup \{e'\}) \leq l(F')$. Also, by the choice of e, $l(e) \leq l(e')$. So $l(F \cup \{e\}) \leq l(F')$. Concluding, $F \cup \{e\}$ is extreme (as F' is extreme). ∎

The following corollary is due to Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a]. Recall that a forest is called *good* if it is contained in a shortest spanning tree.

Corollary 50.10b. *Each extreme forest is good.*

Proof. Directly from Corollary 50.10a, since it implies that each extreme forest is contained in an extreme maximal forest, and hence in a shortest maximal forest; so it is good. ∎

We also can derive a 'slice-integrality' result:

Corollary 50.10c. *Let $G = (V, E)$ be a graph and let $k, l \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of forests F with $k \leq |F| \leq l$ is equal to the intersection of the forest polytope of G with $\{x \in \mathbb{R}^E \mid k \leq x(E) \leq l\}$.*

Proof. Let x be in the forest polytope with $k \leq x(E) \leq l$. Let $x = \sum_F \lambda_F \chi^F$, where F ranges over all forests and where the λ_F are nonnegative reals with $\sum_F \lambda_F = 1$. Choose the λ_F with

$$(50.24) \qquad \sum_F \lambda_F |F|^2$$

minimal. Then

$$(50.25) \qquad |F'| \leq |F| + 1 \text{ for all } F, F' \text{ with } \lambda_F > 0 \text{ and } \lambda_{F'} > 0.$$

Otherwise we can choose $e \in F' \setminus F$ such that $F \cup \{e\}$ is a forest (by Theorem 50.10). Let $\alpha := \min\{\lambda_F, \lambda_{F'}\}$. Then decreasing λ_F and $\lambda_{F'}$ by α and increasing $\lambda_{F \cup \{e\}}$ and $\lambda_{F' \setminus \{e\}}$ by α, decreases sum (50.24). This contradicts our assumption, and proves (50.25).

It implies that $k \leq |F| \leq l$ for each F with $\lambda_F > 0$, and we have the corollary. ∎

50.6e. Uniqueness of shortest spanning tree

Kotzig [1961b] characterized when there is a unique shortest spanning tree:

Theorem 50.11. *Let $G = (V, E)$ be a graph, let $l \in \mathbb{R}^E$ be a length function, and let T be a spanning tree in G. Then T is a unique shortest spanning tree if and only if $l(f) > l(e)$ for all $e \in T$ and $f \in E \setminus T$ such that $T - e + f$ is a spanning tree.*

Proof. As the proof of Theorem 50.9. ∎

This implies a sufficient condition given by Borůvka [1926a]:

Corollary 50.11a. *Let $G = (V, E)$ be a graph and let $l \in \mathbb{R}^E$ be a length function with $l(e) \neq l(f)$ if $e \neq f$. Then there is a unique shortest spanning tree.*

Proof. Directly from Theorem 50.11. ∎

Let $G = (V, E)$ be a connected graph and let $l \in \mathbb{R}^E$ be a length function, with $l(e) \neq l(f)$ if $e \neq f$. Define

(50.26) $T := \{e \in E \mid \exists \text{ cut } C \text{ such that } e \text{ is the shortest edge of } C\}.$

Then

(50.27) $E \setminus T = \{e \in E \mid \exists \text{ circuit } D \text{ such that } e \text{ is the longest edge in } D\}.$

This is easy, since if some edge e is contained in some cut C and some circuit D, then there exists an edge $f \neq e$ in $C \cap D$. If $l(f) < l(e)$, then e is not shortest in C, and if $l(f) > l(e)$, then e is not longest in D. Moreover, for any $e \in E$, if no circuit D as in (50.27) exists, then each circuit D containing e contains an edge f with $l(f) > l(e)$. Hence the set of edges f with $l(f) \geq l(e)$ contains a cut C containing e. This C is as in (50.26).

Now (Dijkstra [1960], Rosenstiehl [1967]):

(50.28) T is the unique shortest spanning tree in G.

Indeed, T is a forest, since each circuit D intersects $E \setminus T$ (namely, in the longest edge of D). Moreover, T is a connector, since each cut C intersects T (namely, in the shortest edge of C). T is the unique shortest spanning tree. This follows from Theorem 50.11, since for each $e \in T$ and each $f \notin T$, if $(T \setminus \{e\}) \cup \{f\}$ is a spanning tree, then $l(e) < l(f)$ as e is the shortest edge in the cut determined by $T - e$.

50.6f. Forest covers

Let $G = (V, E)$ be an undirected graph. A subset F of E is called a *forest cover* if F is both a forest and an edge cover. Forest covers turn out to be interesting algorithmically and polyhedrally.

As Gamble and Pulleyblank [1989] point out, White [1971] showed:

Theorem 50.12. *Given a graph $G = (V, E)$ and a weight function $w \in \mathbb{Q}^E$, a minimum-weight forest cover can be found in strongly polynomial time.*

Proof. Let E_- be the set of edges of negative weight and let V_- be the set of vertices covered by E_-. Let $V_+ := V \setminus V_-$. First find a subset F' of $E[V_+] \cup \delta(V_+)$ covering V_+, of minimum weight. This can be done in strongly polynomial time, by a variation of the strongly polynomial-time algorithm for the minimum weight edge cover problem. (In fact, it is a special case of Theorem 34.4.)

Next find a forest F'' in $E[V_-]$ of minimum weight. Again, this can be done in strongly polynomial time, by Theorem 50.6.

We can assume that any proper subset of F' does not cover V_+. It implies that F' is a forest and that for any vertex $v \in V_+$ incident with some edge e in F' with $e \in \delta(V_+)$, e is the only edge in F' incident with v.

This implies that $F' \cup F''$ is a forest. Moreover, it is an edge cover, since F' covers V_+ and F'' covers V_-, since any vertex in V_- is incident with an edge of negative weight.

So $F' \cup F''$ is a forest cover. To see that it has minimum weight, let $B \subseteq E$ be any forest cover. Let $B'' := B \cap E[V_-]$ and $B' := B \setminus B''$. Then $w(B') \geq w(F')$, since B' covers V_+. Also, $w(B'') \geq w(F'')$, since B'' is a forest. So $w(B) \geq w(F)$. ∎

Gamble and Pulleyblank [1989] showed that White's method implies a characterization of the *forest cover polytope* $P_{\text{forest cover}}(G)$ of a graph G, which is the convex hull of the incidence vectors of forest covers in G. It turns out to be equal to the intersection of the forest polytope (characterized in Corollary 50.7a) and the edge cover polytope (characterized in Corollary 27.3a):

Theorem 50.13. *For any undirected graph $G = (V, E)$:*

(50.29) $$P_{\text{forest cover}}(G) = P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G).$$

Proof. The inclusion $\subseteq$ is trivial, as any forest cover is both a forest and an edge cover. Suppose that the reverse inclusion does not hold, and let x be a vertex of $P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G)$ which is not in $P_{\text{forest cover}}(G)$. Let $w \in \mathbb{Q}^E$ be a weight function such that x uniquely minimizes $w^{\mathsf{T}}x$ over $P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G)$. We can assume that $w(e) \neq 0$ for each edge e (as we can perturb w slightly).

Again let E_- be the set of edges of negative weight, V_- be the set of vertices covered by E_-, and $V_+ := V \setminus V_-$. Since x is in the edge cover polytope, there exists a subset F' of $E[V_+] \cup \delta(V_+)$ covering V_+ with

(50.30) $$w(F') \leq \sum_{e \in E[V_+] \cup \delta(V_+)} w(e) x_e.$$

Similarly, since x is in the forest polytope, there is a forest F'' in $E[V_-]$ with

(50.31) $$w(F'') \leq \sum_{e \in E[V_-]} w(e) x_e.$$

Now, as in the proof of Theorem 50.12, $F := F' \cup F''$ is a forest cover. Since $w(F) \leq w^{\mathsf{T}}x$, this contradicts our assumptions on x and w. ∎

White [1971] also considered the problem of finding a minimum weight forest cover of given size k. Gamble and Pulleyblank [1989] showed that the convex hull of the incidence vectors of forest covers of size k is equal to the intersection of the forest cover polytope with the hyperplane $\{x \in \mathbb{R}^E \mid x(E) = k\}$.

Cerdeira [1994] related forest covers to matroid intersection.

50.6g. Further notes

Let $G = (V, E)$ be a graph. Call a subset U of V *circuit-free* if U spans no circuit; that is, it induces a forest as subgraph of G. Ding and Zang [1999] characterized the graphs G for which the convex hull of the incidence vectors of circuit-free sets is determined by

(50.32) $$\begin{array}{ll} 0 \leq x_v \leq 1 & \text{for each vertex } v, \\ x(VC) \leq |VC| - 1 & \text{for each circuit } C. \end{array}$$

Their characterization implies that (50.32) is totally dual integral as soon as it determines an integer polytope.

Goemans [1992] studied the convex hull of the incidence vectors of (not necessarily spanning) subtrees of a graph.

Brennan [1982] reported on good experimental results with an implementation of Kruskal's method by only partially sorting the edges until the successive shortest edges to be added to the current forest can be identified.

Győri [1978] and Lovász [1977a] showed that if $G = (V, E)$ is k-connected and $v_1, \ldots, v_k$ are distinct vertices, and $n_1, \ldots, n_k$ are positive integers with $n_1 + \cdots + n_k = |V|$, then G contains a forest F such that the component containing v_i has size n_i ($i = 1, \ldots, k$). For $k = 2$, Győri's proof gives an $O(nm)$-time algorithm. A linear-time algorithm for $k = 2$ was given by Suzuki, Takahashi, and Nishizeki [1990]. More can be found in Győri [1981].

Khuller, Raghavachari, and Young [1993,1995b] considered spanning trees that belance between shortest spanning trees and shortest paths trees.

Books covering shortest spanning trees include Even [1973,1979], Christofides [1975], Lawler [1976b], Minieka [1978], Hu [1982], Papadimitriou and Steiglitz [1982], Smith [1982], Aho, Hopcroft, and Ullman [1983], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Nemhauser and Wolsey [1988], Chen [1990], Cormen, Leiserson, and Rivest [1990], Lengauer [1990]. Ahuja, Magnanti, and Orlin [1993], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], and Korte and Vygen [2000]. Pierce [1975] and Golden and Magnanti [1977] gave bibliographies on algorithms for shortest spanning tree.

50.6h. Historical notes on shortest spanning trees

We refer to Graham and Hell [1985] for an extensive historical survey of shortest tree algorithms, with several quotes (with translations) from old papers. Our notes below have profited from their investigations.

We recall some terminology for a shortest spanning tree algorithm. We call it *tree-growing* if we keep a tree on a subset of the vertices, and iteratively extend it by adding an edge joining the tree with a vertex outside of the tree. It is *forest-merging* if we keep a forest, and iteratively merge two components by joining them by an edge. It is called *parallel forest-merging* if forest-merging is performed in parallel, by connecting each component to its nearest neighbouring component (assuming all lengths are distinct).

Borůvka: parallel forest-merging

Borůvka [1926a] described the problem of finding a shortest spanning tree as follows (the paper is in Czech; we quote from its German summary; for quotes from Czech with translation, see Graham and Hell [1985]):

> In dieser Arbeit löse ich folgendes Problem:
> Es möge eine Matrix der bis auf die Bedingungen $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$ positiven und von einander verschiedenen Zahlen $r_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \ldots n; n \geq 2$) gegeben sein.

> Aus dieser ist eine Gruppe von einander und von Null verschiedener Zahlen auszuwählen, so dass
> 1° in ihr zu zwei willkürlich gewählten natürlichen Zahlen p_1, p_2 ($\leq n$) eine Teilgruppe von der Gestalt
>
> $$r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2}$$
>
> existiere,
> 2° die Summe ihrer Glieder kleiner sei als die Summe der Glieder irgendeiner anderen, der Bedingung 1° genügenden Gruppe von einander und von Null verschiedenen Zahlen.[2]

So Borůvka stated that the spanning tree found is the unique shortest. He assumed that all edge lengths are different.

Borůvka next described parallel forest-merging, in a somewhat complicated way. (He did not have the language of graph theory at hand.) The idea is to update a number of vertex-disjoint paths $P_1, \ldots, P_k$ (initially $k = 0$). Along any P_i, the edge lengths are decreasing. Let v be the last vertex of P_k and let e be the edge of shortest length incident with v. If the other end vertex of e is not yet covered by any P_i, we extend P_k with e, and iterate. Otherwise, if not all vertices are covered yet by the P_i, we choose such a vertex v, and start a new path P_{k+1} at v. If all vertices are covered by the P_i, we shrink each of the P_i to one vertex, and iterate. At the end, the edges chosen throughout the iterations form a shortest spanning tree. It is easy to see that this in fact is 'parallel forest-merging'.

The interest of Borůvka in this problem came from a question of the Electric Power Company of Western Moravia in Brno, at the beginning of the 1920s, asking for the most economical construction of an electric power network (see Borůvka [1977]).

In a follow-up paper, Borůvka [1926b] gave a simple explanation of the method by means of an example. We refer to Nešetřil, Milková, and Nešetřilova [2001] for translations of and comments on the two papers of Borůvka.

Jarník: tree-growing

In a reaction to Borůvka's work, Jarník wrote on 12 February 1929 a letter to Borůvka in which he described a 'new solution of a minimal problem discussed by Mr Borůvka'. This 'new solution' is the tree-growing method. An extract of the letter was published as Jarník [1930]. We quote from the German summary:

> a_1 ist eine beliebige unter den Zahlen $1, 2, \ldots, n$.
> a_2 ist durch

[2] In this work, I solve the following problem:
A matrix may be given of positive distinct numbers $r_{\alpha\beta}$ ($\alpha, \beta = 1, 2 \ldots n$; $n \geq 2$), up to the conditions $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$.
From this, a group of numbers, different from each other and from zero, should be selected such that
1° for arbitrarily chosen natural numbers p_1, p_2 ($\leq n$) a subgroup of it exists of the form

$$r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2},$$

2° the sum of its members be smaller than the sum of the members of any other group of numbers different from each other and from zero, satisfying condition 1°.

$$r_{a_1,a_2} = \min_{\left(\substack{l=1,2,\ldots,n \\ l \neq a_1}\right)} r_{a_1,l}$$

> definiert.
> Wenn $2 \leq k < n$ und wenn $[a_1, a_2], \ldots, [a_{2k-3}, a_{2k-2}]$ bereits bestimmt sind, so wird $[a_{2k-1}, a_{2k}]$ durch
> $$r_{a_{2k-1},a_{2k}} = \min r_{i,j},$$
> definiert, wo i alle Zahlen $a_1, a_2, \ldots, a_{2k-2}$, j aber alle übrigen von den Zahlen $1, 2, \ldots, n$ durchläuft.[3]

Again, Jarník assumed that all lengths are distinct and showed that then the shortest spanning tree is unique. For a detailed discussion and translation of the article of Jarník [1930] (and of Jarník and Kössler [1934] on the Steiner tree problem), see Korte and Nešetřil [2001].

Other discoveries of parallel forest-merging

Parallel forest-merging was described also by Choquet [1938] (without proof), who gave as motivation the construction of road systems:

> Étant donné n villes du plan, il s'agit de trouver un réseau de routes permettant d'aller d'une quelconque de ces villes à une autre et tel que:
> 1° la longueur globale du réseau soit minimum;
> 2° exception faite des villes, on ne peut partir d'aucun point dans plus de deux directions, afin d'assurer la sûreté de la circulation; ceci entraîne, par exemple, que lorsque deux routes semblent se croiser en un point qui n'est pas une ville, elles passent en fait l'une au-dessus de l'autre et ne communiquent pas entre elles en ce point, qu'on appellera faux-croisement.[4]

He was one of the first concerned on the complexity of the method:

> Le réseau cherché sera tracé après $2n$ opérations élémentaires au plus, en appelant opération élémentaire la recherche du continu le plus voisin d'un continu donné.[5]

---

[3] a_1 is an arbitrary one among the numbers $1, 2, \ldots, n$.
a_2 is defined by
$$r_{a_1,a_2} = \min_{\left(\substack{l=1,2,\ldots,n \\ l \neq a_1}\right)} r_{a_1,l}.$$
If $2 \leq k < n$ and if $[a_1, a_2], \ldots, [a_{2k-3}, a_{2k-2}]$ are determined already, then $[a_{2k-1}, a_{2k}]$ is defined by
$$r_{a_{2k-1},a_{2k}} = \min r_{i,j},$$
where i runs through all numbers $a_1, a_2, \ldots, a_{2k-2}$, j however through all remaining of the numbers $1, 2, \ldots, n$.

[4] Being given n cities of the plane, the point is to find a network of routes allowing to go from an arbitrary of these cities to another and such that:
1° the global length of the network be minimum;
2° except for the cities, one cannot depart from any point in more than two directions, in order to assure the certainty of the circulation; this entails, for instance, that when two routes seem to cross each other in a point which is not a city, they pass in fact one above the other and do not communicate among them in this point, which we shall call a false crossing.

[5] The network looked for will be traced after at most $2n$ elementary operations, calling the search for the continuum closest to a given continuum an elementary operation.

Also Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a,1951b] described parallel forest-merging. They were motivated by clustering in anthropology, taxonomy, etc. In the latter paper, they apply the method to:

> 1° the capitals of Poland's provinces, 2° two collections of excavated skulls, 3° 42 archeological finds, 4° the liverworts of Silesian Beskid mountains with forests as their background, and to the forests of Silesian Beskid mountains with the liverworts appearing in them as their background.

Kruskal

Kruskal [1956] was motivated by Borůvka's first paper and by the application to the traveling salesman problem, described as follows (where [1] refers to Borůvka [1926a]):

> Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the "matrix of lengths" of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j. Of course, it is assumed that $a_{ij} = a_{ji}$ and that $a_{ii} = 0$ for all i and j.) The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.
> PROBLEM 1. Give a practical method for constructing a spanning subtree of minimum length.
> PROBLEM 2. Give a practical method for constructing an unbranched spanning subtree of minimum length.
> The construction in [1] is unnecessarily elaborate. In the present paper I give several simpler constructions which solve Problem 1, and I show how one of these constructions may be used to prove the theorem of [1]. Probably it is true that any construction which solves Problem 1 may be used to prove this theorem.

Kruskal described three algorithms: Construction A: iteratively choose the shortest edge that can be added (forest-merging); Construction B: fix a nonempty set U of vertices, and choose iteratively the shortest edge leaving some component intersecting U (a generalization of tree-growing); Construction A′: iteratively remove the longest edge that can be removed without making the graph disconnected. He proved that Construction A implies the uniqueness of shortest spanning tree if all lengths are distinct.

In his reminiscences, Kruskal [1997] wrote about Borůvka's method:

> In one way, the method of construction was very elegant. In another way, however, it was unnecessarily complicated. A goal which has always been important to me is to find simpler ways to describe complicated ideas, and that is all I tried to do here. I simplified the construction down to its essence, but it seems to me that the idea of Professor Borůvka's method is still present in my version.

Prim

Prim [1957] gave the following motivation:

> A problem of inherent interest in the planning of large-scale communication, distribution and transportation networks also arises in connection with the current rate structure for Bell System leased-line services.

He described the following algorithm: choose a component of the current forest, and connect it to the nearest component. He observed that Kruskal's constructions A and B are special cases of this.

Prim noticed that in fact only the order of the lengths determines if a spanning tree is shortest:

> The *shortest spanning subtree* of a connected labelled graph also minimizes all increasing symmetric functions, and maximizes all decreasing symmetric functions, of the edge "lengths."

Prim preferred starting at a vertex and growing a tree for computational reasons:

> This computational procedure is easily programmed for an automatic computer so as to handle quite large-scale problems. One of its advantages is its avoidance of checks for closed cycles and connectedness. Another is that it never requires access to more than two rows of distance data at a time — no matter how large the problem.

The implementation described by Prim has $O(n^2)$ running time.

Loberman and Weinberger

Loberman and Weinberger [1957] gave minimizing wire connections as motivation:

> In the construction of a digital computer in which high-frequency circuitry is used, it is desirable and often necessary when making connections between terminals to minimize the total wire length in order to reduce the capacitance and delay-line effects of long wire leads.

They described two methods: tree-growing and forest-merging. Only after they had designed their algorithms, they discovered that their algorithms were given earlier by Kruskal [1956].

> However, it is felt that the more detailed implementation and general proofs of the procedures justify this paper.

They next described how to implement Kruskal's method, in particular, how to merge forests. They also observed that the minimality of a spanning tree depends only on the order of the lengths, and not on their specific values:

> After the initial sorting into a list where the branches are of monotonically increasing length, the actual value of the length of any branch no longer appears explicitly in the subsequent manipulations. As a result, some other parameter such as the square of the length could have been used. More generally, the same minimum tree will persist for all variations in branch lengths that do not disturb the original relative order.

Dijkstra

Dijkstra [1959] gave again the tree-growing method, which he preferred (for computational reasons) above the forest-merging method of Kruskal and Loberman and Weinberger (overlooking the fact that these authors also gave the tree-growing method):

> The solution given here is to be preferred to the solution given by J.B. Kruskal [*1*] and those given by H. Loberman and A. Weinberger [*2*]. In their solutions all the — possibly $\frac{1}{2}n(n-1)$ — branches are first of all sorted according to length. Even if the length of the branches is a computable function of the node coordinates, their methods demand that data for all branches are stored simultaneously. Our method requires the simultaneous storing of the data for at most n branches, ...

Dijkstra described an $O(n^2)$ implementation.

Dijkstra [1960] gave the following alternative shortest spanning tree method: order edges arbitrarily, find the first edge that forms a circuit with previous edges; delete the longest edge from this circuit, and continue. (This method was also found by Rosenstiehl [1967].) This generalizes both forest-merging and tree-growing, by choosing the order appropriately.

Further work

Kalaba [1960] proposed the method of first choosing a spanning tree arbitrarily, and next adding, iteratively, an edge and removing the longest edge in the circuit arising.

Kotzig [1961b] gave again Kruskal's Algorithm A' (a referee pointed Kruskal's work out to Kotzig). Kotzig moreover showed that there is a unique minimum spanning tree T if and only if for each edge e not in T, e is the unique longest edge in the circuit in $T \cup \{e\}$.

As mentioned, Graham and Hell [1985] give an extensive survey on the history of the minimum spanning tree (and minimum Steiner tree) problem. See also Nešetřil [1997] for additional notes on the history of the minimum spanning tree problem.

Chapter 51

Packing and covering of trees

The basic facts on packing and covering of trees follow directly from those on matroid union. In this chapter we check what these results amount to in terms of graphs, and we give some more direct algorithms.

51.1. Unions of forests

For any graph $G = (V, E)$ and any partition $\mathcal{P}$ of V, let $\delta(\mathcal{P})$ denote the set of edges connecting distinct classes of $\mathcal{P}$. From the following consequence of the matroid union theorem we will derive other results on tree packing and covering:

Theorem 51.1. *Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k forests is equal to the minimum value of*

(51.1) $$|\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|)$$

taken over all partitions $\mathcal{P}$ of V into nonempty classes.

Proof. This follows directly from the matroid union theorem (Corollary 42.1a) applied to the cycle matroid M of G. Indeed, by Corollary 42.1b, the maximum size of the union of k forests is equal to the minimum value of

(51.2) $$|E \setminus F| + kr_M(F),$$

where $r_M(F)$ is the maximum size of a forest contained in F. We can assume that each component of (V, F) is an induced subgraph of G. So taking $\mathcal{P}$ equal to the set of components of (V, F), we see that $r_M(F) = |V| - |\mathcal{P}|$, and hence that the minimum of (51.2) is equal to the minimum of (51.1). ∎

51.2. Disjoint spanning trees

Theorem 51.1 has a number of consequences. First we have the following tree packing result of Tutte [1961a] and Nash-Williams [1961b]:

Corollary 51.1a (Tutte-Nash-Williams disjoint trees theorem)**.** *A graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if*

(51.3) $$|\delta(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$$

for each partition $\mathcal{P}$ of V into nonempty classes.

Proof. To see necessity of (51.3), each spanning tree contains at least $|\mathcal{P}|-1$ edges in $\delta(\mathcal{P})$. To show sufficiency, it is equivalent to show that there exist $k(|V|-1)$ edges that can be covered by k forests. By Theorem 51.1, this is indeed possible, since

(51.4) $$|\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|) \geq k(|\mathcal{P}| - 1) + k(|V| - |\mathcal{P}|) = k(|V| - 1)$$

for each partition $\mathcal{P}$ of V into nonempty sets. ∎

Gusfield [1983] observed that the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a) implies that each $2k$-edge-connected undirected graph has k edge-disjoint spanning trees (since $|\delta(\mathcal{P})| \geq k|\mathcal{P}| \geq k(|\mathcal{P}|-1)$).

Similarly to the line pursued in Section 42.2, Corollary 51.1a can be formulated equivalently in polyhedral terms:

Corollary 51.1b. *The connector polytope of a graph has the integer decomposition property.*

Proof. Similar to the proof of Corollary 42.1e. ∎

For any connected graph $G = (V, E)$, define the *strength* of G by:

(51.5) $$\begin{aligned}\text{strength}(G) &:= \max\{\lambda \mid \mathbf{1} \in \lambda \cdot P_{\text{connector}}(G)\} \\ &= \max\{\sum_T \lambda_T \mid \lambda_T \geq 0, \sum_T \lambda_T \chi^T \leq \mathbf{1}\},\end{aligned}$$

where T ranges over the spanning trees of G, and where $\mathbf{1}$ denotes the all-1 vector in $\mathbb{R}^E$.

The Tutte-Nash-Williams disjoint trees theorem is equivalent to: the maximum number of disjoint spanning trees in a graph $G = (V, E)$ is equal to $\lfloor \text{strength}(G) \rfloor$. Similarly, the capacitated version of the Tutte-Nash-Williams theorem is equivalent to the integer rounding property of the system (cf. Section 42.2):

(51.6) $$\begin{aligned} x_e &\geq 0 && \text{for } e \in E, \\ x(T) &\geq 1 && \text{for each spanning tree } T. \end{aligned}$$

51.3. Covering by forests

Dual to Corollary 51.1a is the following forest covering theorem of Nash-Williams [1964], where $E[U]$ denotes the set of edges contained in U. (The theorem is also a consequence of a theorem of Horn [1955] on covering vector sets by linearly independent sets, since each graphic matroid is linear.)

Corollary 51.1c (Nash-Williams' covering forests theorem)**.** *The edge set of a graph $G = (V, E)$ can be covered by k forests if and only if*

$$(51.7) \qquad |E[U]| \leq k(|U| - 1)$$

for each nonempty subset U of V.

Proof. Since any forest has at most $|U| - 1$ edges contained in U, we have necessity of (51.7). To see sufficiency, notice that (51.7) implies

$$(51.8) \qquad |E| - |\delta(\mathcal{P})| = \sum_{U \in \mathcal{P}} |E[U]| \leq \sum_{U \in \mathcal{P}} k(|U| - 1) = k(|V| - |\mathcal{P}|)$$

for any partition $\mathcal{P}$ of V into nonempty sets. So $|\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|) \geq |E|$, and hence Theorem 51.1 implies that there exist k forests covering E. ∎

(Nash-Williams [1964] derived Corollary 51.1c from Corollary 51.1a.)

Again, this corollary can be formulated in terms of the integer decomposition property:

Corollary 51.1d. *For any graph G, the forest polytope has the integer decomposition property.*

Proof. Similar to the proof of Corollary 42.1e. ∎

These results are equivalent to: the minimum number of forests needed to cover the edges of a graph $G = (V, E)$ is equal to

$$(51.9) \qquad \lceil \min\{\lambda \mid \mathbf{1} \in \lambda \cdot P_{\text{forest}}(G)\} \rceil,$$

where $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^E$. A similar relation holds for the capacitated case, which is equivalent to the integer rounding property of the system:

$$(51.10) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(F) \leq 1 & \text{for each forest } F. \end{array}$$

The minimum number of forests needed to cover the edges of a graph G is called the *arboricity* of G.

51.4. Complexity

The complexity results on matroid union in Sections 40.3, 42.3 and 42.4 imply that these packing and covering problems for forests and trees are solvable in polynomial time:

Theorem 51.2. *For any graph $G = (V, E)$, a maximum number of edge-disjoint spanning trees and a minimum number of forests covering E can be found in polynomial time.*

Proof. See Section 42.3. ∎

Also weighted versions of it can be solved in strongly polynomial time, for instance, finding a maximum-weight union of k forests in a graph. We give in this section some direct proofs.

To study the complexity of the capacitated and fractional cases, we first observe the following auxiliary result, that is (when applied to undirected graphs) at the base of several algorithms on the forest and connector polytopes, and was observed by Rhys [1970], Picard and Ratliff [1975,1978], Picard [1976], Trubin [1978], Picard and Queyranne [1982a], Padberg and Wolsey [1983], and Cunningham [1985a]. (It also follows from the strong polynomial-time solvability of submodular function minimization, but there is an easier direct method.)

Theorem 51.3. *Given a digraph $D = (V, A)$, $x \in \mathbb{Q}_+^A$, $y \in \mathbb{Q}^V$, and disjoint subsets S and T, we can find a set U with $T \subseteq U \subseteq V \setminus S$ minimizing*

(51.11) $$x(\delta^{\mathrm{in}}(U)) + y(U)$$

in strongly polynomial time.

Proof. Extend D by two new vertices s and t, and arcs (s, v) for $v \in V$ with $y_v > 0$ and (v, t) for $v \in V$ with $y_v < 0$. This gives the digraph $D' = (V \cup \{s, t\}, A')$. Define a capacity function c on A' by:

(51.12) $$\begin{array}{ll} c(u,v) := x(u,v) & \text{for } (u,v) \in A, \\ c(s,v) := y_v & \text{if } (s,v) \in A', \\ c(v,t) := -y_v & \text{for } (v,t) \in A'. \end{array}$$

Let $\kappa := -c(\delta^{\mathrm{in}}_{A'}(t))$ (the sum of the negative y_v's). Then

(51.13) $$\begin{aligned} c(\delta^{\mathrm{in}}_{A'}(U \cup \{t\})) &= x(\delta^{\mathrm{in}}_A(U)) + \sum_{\substack{v \in U \\ y_v > 0}} y_v - \sum_{\substack{v \in V \setminus U \\ y_v > 0}} y_v \\ &= x(\delta^{\mathrm{in}}_A(U)) + \sum_{v \in U} y_v - \sum_{\substack{v \in V \\ y_v < 0}} y_v = x(\delta^{\mathrm{in}}_A(U)) + y(U) - \kappa \end{aligned}$$

for any $U \subseteq V$. Thus minimizing $x(\delta^{\mathrm{in}}_A(U)) + y(U)$ is reduced to finding a minimum-capacity $(S \cup \{s\}) - (T \cup \{t\})$ cut in D'. ∎

Testing membership and finding most violated inequalities

A first consequence of Theorem 51.3 is that we can test membership, and find a most violated inequality, for the forest polytope (Picard and Queyranne [1982b] (suggested by W.H. Cunningham) and Padberg and Wolsey [1983]).

Corollary 51.3a. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can decide if x belongs to $P_{\text{forest}}(G)$, and if not, find a most violated inequality among* (50.11)*, in strongly polynomial time.*

Proof. Define $y_v := 2 - x(\delta(v))$ for $v \in V$. Then

$$
\begin{aligned}
(51.14) \qquad 2(x(E[U]) - |U|) &= \sum_{v \in U} x(\delta(v)) - x(\delta(U)) - 2|U| \\
&= -x(\delta(U)) - y(U).
\end{aligned}
$$

So any nonempty $U \subseteq V$ minimizing $x(\delta(U))+y(U)$, maximizes $x(E[U])-|U|$. By Theorem 51.3, we can find such a U in strongly polynomial time. If $x(E[U]) \leq |U| - 1$, x belongs to $P_{\text{forest}}(G)$, and otherwise U gives a most violated inequality. ∎

A similar result holds for the up hull of the connector polytope, which we show with a method of Jünger and Pulleyblank [1995]:

Corollary 51.3b. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can find a partition $\mathcal{P}$ of V into nonempty sets minimizing*

$$(51.15) \qquad x(\delta(\mathcal{P})) - |\mathcal{P}|$$

in strongly polynomial time.

Proof. We first construct a vector $y \in \mathbb{Q}^V$, by updating a vector y. Throughout, y satisfies

$$(51.16) \qquad y(U) \leq x(\delta(U)) - 2 \text{ for each nonempty } U \subseteq V.$$

Start with $y_v := -2$ for all $v \in V$. Successively, for each $v \in V$, reset y_v to $y_v + \alpha$, where α is the minimum value of

$$(51.17) \qquad x(\delta(U)) - 2 - y(U)$$

taken over all $U \subseteq V$ containing v. Such a U can be found in strongly polynomial time by Theorem 51.3.

We end up with a y satisfying (51.16). Moreover, each $v \in V$ is contained in some set U with $y(U) = x(\delta(U)) - 2$.

Let $\mathcal{P}$ be the inclusionwise maximal sets U satisfying $y(U) = x(\delta(U)) - 2$. Then $\mathcal{P}$ is a partition of V, since if $T, U \in \mathcal{P}$ and $T \cap U \neq \emptyset$, then (by the submodularity of $x(\delta(Y))$) $y(T \cup U) = x(\delta(T \cup U)) - 2$, and hence $T = U = T \cup U$.

This $\mathcal{P}$ is as required, since for each partition $\mathcal{Q}$ of V into nonempty sets we have

$$(51.18) \qquad 2x(\delta(\mathcal{Q})) - 2|\mathcal{Q}| = \sum_{U \in \mathcal{Q}} (x(\delta(U)) - 2) \geq \sum_{U \in \mathcal{Q}} y(U) = y(V),$$

with equality if $\mathcal{Q} = \mathcal{P}$. ∎

(This method is analogous to calculating the Dilworth truncation as discussed in Theorem 48.4.)

Corollary 51.3b implies for finding the most violated inequality:

Corollary 51.3c. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can decide if x belongs to $P^{\uparrow}_{\text{connector}}(G)$, and if not, find a most violated inequality among* (50.17)(ii)*, in strongly polynomial time.*

Proof. By Corollary 51.3b, we can find a partition $\mathcal{P}$ of V into nonempty sets, minimizing $x(\delta(\mathcal{P})) - |\mathcal{P}|$. If this value is at least -1, then x belongs to the up hull of the connector polytope, while otherwise $\mathcal{P}$ gives a most violated inequality among (50.17)(ii). ∎

Barahona [1992] showed that membership in the connector polytope can be tested by solving $O(n)$ maximum flow computations (improving Cunningham [1985c]).

Fractional decomposition into trees

By definition, any vector in $P_{\text{forest}}(G)$ or $P_{\text{connector}}(G)$ can be decomposed as a convex combination of incidence vectors of forests or of connectors. These decompositions can be found in strongly polynomial time, a result due to Cunningham [1984] and Padberg and Wolsey [1984] (for the forest polytope).

In order to decompose a vector in the forest polytope as a convex combination of forests, by the following theorem it suffices to have a method to decompose a vector in the spanning tree polytope as a convex combination of spanning trees:

Theorem 51.4. *Given a connected graph $G = (V, E)$ and $x \in P_{\text{forest}}(G)$, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \leq z$ in strongly polynomial time.*

Proof. We reset x successively for each edge $e = uv$ of G as follows. Reset x_e to $x_e + \alpha$, where α is the largest value such that x remains to belong to $P_{\text{forest}}(G)$. That is, α equals the minimum value of

$$(51.19) \qquad |U| - 1 - x(E[U]) = |U| - 1 - \tfrac{1}{2}\sum_{v \in U} x(\delta(v)) + \tfrac{1}{2}x(\delta(U)),$$

taken over subsets U of V with $u, v \in U$. Such a U can be found in strongly polynomial time by Theorem 51.3.

As $P_{\text{forest}}(G) = P^{\downarrow}_{\text{spanning tree}}(G) \cap \mathbb{R}_+^E$, the final x is a z as required. ∎

Hence, to decompose a vector in the forest polytope, we can do with decomposing vectors in the spanning tree polytope:

Theorem 51.5. *Given a graph $G = (V, E)$ and $y \in P_{\text{spanning tree}}(G)$, we can find spanning trees $T_1, \ldots, T_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

$$(51.20) \qquad y = \lambda_1 \chi^{T_1} + \cdots + \lambda_k \chi^{T_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. Iteratively resetting y, we keep an integer weight function w such that y maximizes $w^{\mathsf{T}} y$ over the spanning tree polytope. Initially, $w := \mathbf{0}$. We describe the iteration.

Let T be a spanning tree in G with $T \subseteq \text{supp}(y)$, maximizing $w(T)$. Let $a := y - \chi^T$. If $a = \mathbf{0}$ we stop; then $y = \chi^T$. If $a \neq \mathbf{0}$, let λ be the largest rational such that

$$(51.21) \qquad \chi^T + \lambda \cdot a$$

belongs to $P^{\uparrow}_{\text{spanning tree}}(G)$.

We describe an inner iteration to find λ. We iteratively consider vectors y along the halfline $L := \{\chi^T + \lambda \cdot a \mid \lambda \geq 0\}$. Note that the function $w^{\mathsf{T}} x$ is constant on L. First we let λ be the largest rational such that $\chi^T + \lambda \cdot a$ is nonnegative, and set $z := \chi^T + \lambda \cdot a$.

We iteratively reset z. We check if z belongs to the spanning tree polytope, and if not, we find a constraint among (50.12) most violated by z. That is, we find a nonempty subset U of V minimizing $|U| - 1 - z(E[U])$. Let z' be the vector on L attaining $x(E[U]) \leq |U| - 1$ with equality.

Consider any inequality $x(E[U']) \leq |U'| - 1$ violated by z'. Then

$$(51.22) \qquad |U'| - 1 - |T \cap E[U']| < |U| - 1 - |T \cap E[U]|.$$

This can be seen by considering the function $d(x) := (|U| - 1 - x(E[U])) - (|U'| - 1 - x(E[U']))$. We have $d(z) \leq 0$ and $d(z') > 0$, and hence, as d is linear, $d(\chi^T) > 0$; that is, we have (51.22). So resetting $z := z'$, there are at most $|V|$ inner iterations.

Let y' be the final z found. Since $\lambda \geq 1$ (as $y \in P_{\text{spanning tree}}(G)$) and $y = \lambda^{-1} \cdot y' + (1 - \lambda^{-1}) \cdot \chi^T$, any convex decomposition of y' into incidence vectors of spanning trees, yields such a decomposition of y. We show that this recursion terminates.

If we apply no iteration, then $\text{supp}(y') \subset \text{supp}(y)$. So replacing y, w by y', w gives a reduction.

If we do at least one iteration, we find a U such that y' satisfies $y'(E[U]) = |U| - 1$ while $|T \cap E[U]| < |U| - 1$. In this case we replace y, w by $y', w' := 2w + \chi^{E[U]}$.

Then y' maximizes $w'^{\mathsf{T}} x$ over the spanning tree polytope. Indeed, for any x in the spanning tree polytope, we have

$$(51.23) \qquad \begin{aligned} w'^{\mathsf{T}} x &= 2w^{\mathsf{T}} x + x(E[U]) \leq 2w^{\mathsf{T}} y + |U| - 1 = 2w^{\mathsf{T}} y' + y'(E[U]) \\ &= w'^{\mathsf{T}} y'. \end{aligned}$$

Moreover, each tree T' maximizing $w'(T')$ also maximizes $w(T')$ (by the greedy method: for any ordering of V for which w' is nondecreasing, also w is nondecreasing). However, T does not maximize $w'(T)$, since $w'(T) = 2w(T) + |T \cap E[U]| < 2w(T) + |U| - 1 = 2w^{\mathsf{T}}y + |U| - 1 = w'^{\mathsf{T}}y'$. So the dimension of the face of vectors x maximizing $w'^{\mathsf{T}}x$ is less than the dimension of the face of vectors x maximizing $w^{\mathsf{T}}x$.

So the number of iterations is at most $|E|$. This shows that the method is strongly polynomial-time. ∎

Now we can derive from the previous two theorems, an algorithmic result for fractional forest decomposition:

Corollary 51.5a. *Given a graph $G = (V, E)$ and $y \in P_{\text{forest}}(G)$, we can find forests $F_1, \ldots, F_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

(51.24) $$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_k \chi^{F_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. We can assume that G is connected, as we can consider each component of G separately. By Theorem 51.4, we can find a $z \in P_{\text{spanning tree}}(G)$ with $y \leq z$ in strongly polynomial time. By Theorem 51.5, we can decompose z as a convex combination of incidence vectors of spanning trees in strongly polynomial time. By restricting the spanning trees to subforests if necessary, we obtain a convex decomposition of y into incidence vectors of forests. ∎

We can proceed similarly for decomposing a vector in the connector polytope. To this end, we show the analogue for connectors of Theorem 51.4:

Theorem 51.6. *Given a graph $G = (V, E)$ and $x \in P^{\uparrow}_{\text{connector}}(G)$, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \geq z$, in strongly polynomial time.*

Proof. The method described in the proof of Corollary 51.3b gives a vector $y \in \mathbb{Q}^V$ satisfying

(51.25) $$y(U) \leq x(\delta(U)) - 2 \text{ for each nonempty } U \subseteq V,$$

and a partition $\mathcal{P}$ of V into nonempty sets with $y(U) = x(\delta(U)) - 2$ for each $U \in \mathcal{P}$. Hence

(51.26) $$y(V) = \sum_{U \in \mathcal{P}} (x(\delta(U)) - 2) = 2x(\delta(\mathcal{P})) - 2|\mathcal{P}| \geq -2.$$

By decreasing components of y appropriately, we can achieve that $y(V) = -2$, while maintaining (51.25).

We are going to modify y and x, maintaining (51.25) and $y(V) = -2$. For each $u, v \in V$ with $e = uv \in E$, we do the following. Let α be the minimum value of $x(\delta(U)) - 2 - y(U)$ taken over subsets U of V with $u \in U$, $v \notin U$. So $\alpha \geq 0$. Let $\beta := \min\{x_e, \frac{1}{2}\alpha\}$ and reset

(51.27) $\qquad x_e := x_e - \beta,\ y_u := y_u + \beta,\ y_v := y_v - \beta.$

Then (51.25) is maintained, and the collection $\mathcal{C}$ of subsets U having equality in (51.25) is not reduced. Moreover, in the new situation, $x_e = 0$ or there is a $U \in \mathcal{C}$ with $u \in U$ and $v \notin U$. Also, the new x belongs to $P^{\uparrow}_{\text{connector}}(G)$, as for any partition $\mathcal{Q}$ of V into nonempty sets we have

(51.28) $$\sum_{U \in \mathcal{Q}} x(\delta(U)) \geq y(V) + 2|\mathcal{Q}| = 2|\mathcal{Q}| - 2.$$

Doing this for each edge e (in both directions), we end up with x, y satisfying (51.25) such that

(51.29) for all adjacent u, v, if $x_{uv} > 0$, then there is a $U \in \mathcal{C}$ with $u \in U$ and $v \notin U$.

This implies that

(51.30) $$y_u = x(\delta(u)) - 2$$

for each $u \in V$. Indeed, $\mathcal{C}$ is closed under unions and intersections of intersecting sets. Let U be the smallest set in $\mathcal{C}$ containing u. (This exists, since $V \in \mathcal{C}$.) To show (51.30), we must show $U = \{u\}$. Suppose therefore that $U \neq \{u\}$. By (51.29), there is no edge e connecting u and $U \setminus \{u\}$ with $x_e > 0$. Hence

(51.31) $$\begin{aligned} y(U) &= y_u + y(U \setminus \{u\}) \leq x(\delta(u)) - 2 + x(\delta(U \setminus \{u\})) - 2 \\ &= x(\delta(U)) - 4 < x(\delta(U)) - 2, \end{aligned}$$

contradicting the fact that $U \in \mathcal{C}$. This proves (51.30).

Hence

(51.32) $$2x(E) = \sum_{u \in V} x(\delta(u)) = y(V) + 2|V| = 2(|V| - 1),$$

and so $x(E) = |V| - 1$. This implies that x belongs to the spanning tree polytope. ∎

This implies for fractional connector decomposition:

Corollary 51.6a. *Given a graph $G = (V, E)$ and $x \in P_{\text{connector}}(G)$, we can find connectors $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

(51.33) $$x = \lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. By Theorem 51.6, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \geq z$ in strongly polynomial time. By Theorem 51.5, we can decompose z as a convex combination of incidence vectors of spanning trees in strongly polynomial time. This gives a decomposition as required. ∎

Fractionally packing and covering trees and forests

We now consider the problem of finding a maximum fractional packing of spanning trees subject to a given capacity function, and its dual, finding a minimum fractional covering by forests of a given demand function.

Since we have proved above that convex decompositions can be found in strongly polynomial time, we only need to give a method to find the optimum *values* of the fractional packing and covering.

The method is a variant of a 'fractional programming method' initiated by Isbell and Marlow [1956], and developed by Dinkelbach [1967], Schaible [1976], Picard and Queyranne [1982a], Padberg and Wolsey [1984], and Cunningham [1985c].

It implies the following result of Picard and Queyranne [1982a] and Padberg and Wolsey [1984]:

Theorem 51.7. *Given a graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find the minimum λ such that $y \in \lambda \cdot P_{\text{forest}}(G)$, in strongly polynomial time.*

Proof. We can assume that y does not belong to the forest polytope. (Otherwise multiply y by a sufficiently large scalar.) Let L be the line through $\mathbf{0}$ and y. We iteratively reset y as follows. Find a nonempty subset U of V minimizing $|U| - 1 - y(E[U])$. Let y' be the vector on L with $|U| - 1 - y'(E[U]) = 0$.

Now if y' violates $x(E[U']) \leq |U'| - 1$ for some U', then $|U'| < |U|$, since the function $d(x) := (|U| - 1 - x(E[U])) - (|U'| - 1 - x(E[U']))$ is nonpositive at y and positive at y', implying that it is positive at $\mathbf{0}$ (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{forest}}(G)$. So after at most $|V|$ iterations the process terminates, with a y on the boundary of $P_{\text{forest}}(G)$. Comparing the final y with the original y gives the required λ. ∎

Hence we have for fractional forest covering:

Corollary 51.7a. *Given a graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find forests $F_1, \ldots, F_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

$$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_k \chi^{F_k} \tag{51.34}$$

with $\lambda_1 + \ldots + \lambda_k$ minimal, in strongly polynomial time.

Proof. By Theorem 51.7, we can find the minimum value of λ such that y belongs to $\lambda \cdot P_{\text{forest}}(G)$. If $\lambda = 0$, then $y = \mathbf{0}$, and (51.34) is trivial. If $\lambda > 0$, then by Corollary 51.5a we can decompose $\lambda^{-1} \cdot y$ as a convex combination of incidence vectors of forests. This gives the required decomposition of y. ∎

Similar results holds for fractional tree packing (Cunningham [1984, 1985c]). First one has:

Theorem 51.8. *Given a connected graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find the maximum λ such that $y \in \lambda \cdot P_{\text{connector}}(G)$, in strongly polynomial time.*

Proof. If $\text{supp}(y)$ is not a connector, then $\lambda = 0$. So we may assume that $\text{supp}(y)$ is a connector. We can also assume that $y \notin P_{\text{connector}}(G)$. Let L be the line through $\mathbf{0}$ and y. We iteratively reset y as follows. Find a partition $\mathcal{P}$ of V into nonempty sets minimizing $y(\delta(\mathcal{P})) - (|\mathcal{P}| - 1)$ (by Corollary 51.3b). Let y' be the vector on L with $y'(\delta(\mathcal{P})) = |\mathcal{P}| - 1$.

Now if y' violates $x(\delta(\mathcal{P}')) \geq |\mathcal{P}'| - 1$ for some partition $\mathcal{P}'$ of V into nonempty sets, then $|\mathcal{P}'| < |\mathcal{P}|$, since the function $d(x) := (x(\delta(\mathcal{P})) - |\mathcal{P}| + 1) - (x(\delta(\mathcal{P}')) - |\mathcal{P}'| + 1)$ is nonpositive at y and positive at y', implying that it is negative at $\mathbf{0}$ (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{connector}}(G)$. So after at most $|V|$ iterations the process terminates, in which case we have a y on the boundary of $P_{\text{connector}}(G)$. Comparing the final y with the original y gives the required λ. ∎

This implies for fractional tree packing:

Corollary 51.8a. *Given a connected graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can find spanning trees $T_1, \ldots, T_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

(51.35) $$x \geq \lambda_1 \chi^{T_1} + \cdots + \lambda_k \chi^{T_k}$$

with $\lambda_1 + \ldots + \lambda_k$ maximal, in strongly polynomial time.

Proof. By Theorem 51.8, we can find the maximum value of λ such that x belongs to $\lambda \cdot P_{\text{connector}}(G)$. If $\lambda = 0$, we take $k = 0$. If $\lambda > 0$, by Corollary 51.6a we can decompose $\lambda^{-1} \cdot x$ as a convex combination of incidence vectors of connectors. This gives the required decomposition of x. ∎

Integer packing and covering of trees

It is not difficult to derive integer versions of the above algorithms, but they are not strongly polynomial-time, as we round numbers in it. In fact, an integer packing or covering cannot be found in strongly polynomial time, as it would imply a strongly polynomial-time algorithm for testing if an integer k is even (which algorithm does not exist[6]): k is even if and only if K_3 has $\frac{3}{2}k$ spanning trees containing each edge at most k times.

[6] For any strongly polynomial-time algorithm with one integer k as input, there is a number L and a rational function $q : \mathbb{Z} \to \mathbb{Q}$ such that if $k > L$, then the output equals $q(k)$. (This can be proved by induction on the number of steps of the algorithm.) However, there do not exist a rational function q and a number L such that for $k > L$, $q(k) = 0$ if k is even, and $q(k) = 1$ if k is odd.

Weakly polynomial-time algorithms follow directly from the fractional case with the help of the theorems of Nash-Williams and Tutte on disjoint trees and covering forests.

Theorem 51.9. *Given a graph $G = (V, E)$ and $y \in \mathbb{Z}_+^E$, we can find forests $F_1, \ldots, F_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

(51.36) $$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_t \chi^{F_t}$$

with $\lambda_1 + \ldots + \lambda_t$ minimal, in polynomial time.

Proof. First find $F_1, \ldots, F_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 51.7a. We can assume that $k \leq |E|$ (by Carathéodory's theorem). Let

(51.37) $$y' := \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{F_i} = y - \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \chi^{F_i}.$$

So y' is integer.

Replace each edge e by y'_e parallel edges, making G'. By Theorem 51.2, we can find a minimum number of forests partitioning the edges of G', in polynomial time (as $y'_e \leq |E|$ for each $e \in E$). This gives forests $F_{k+1}, \ldots, F_t$ in G.

Setting $\lambda_i := 1$ for $i = k + 1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (51.36) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). By Nash-Williams' covering forests theorem (Theorem 51.1c), using (51.37),

(51.38) $$t - k \leq \left\lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \right\rceil.$$

Therefore,

(51.39) $$\sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t - k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \leq \left\lceil \sum_{i=1}^{k} \lambda_i \right\rceil,$$

proving that the decomposition is optimum. ∎

One similarly shows for tree packing:

Theorem 51.10. *Given a connected graph $G = (V, E)$ and $y \in \mathbb{Z}_+^E$, we can find spanning trees $T_1, \ldots, T_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

(51.40) $$y \geq \lambda_1 \chi^{T_1} + \cdots + \lambda_t \chi^{T_t}$$

with $\lambda_1 + \ldots + \lambda_t$ maximal, in polynomial time.

Proof. First find $T_1, \ldots, T_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 51.8a. We can assume that $k \leq |E|$ (by Carathéodory's theorem). Let

(51.41) $$y' := \left\lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{T_i} \right\rceil.$$

Replace each edge e by y'_e parallel edges, making G'. By Theorem 51.2, we can find a maximum number of edge-disjoint spanning trees in G', in polynomial time (as $y'_e \leq |E|$ for each $e \in E$). This gives spanning trees $T_{k+1}, \ldots, T_t$ in G.

Setting $\lambda_i := 1$ for $i = k+1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (51.40) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). By the Tutte-Nash-Williams disjoint trees theorem using (51.41),

$$(51.42) \qquad t - k \geq \lfloor \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \rfloor.$$

Therefore,

$$(51.43) \qquad \sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t-k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \geq \lfloor \sum_{i=1}^{k} \lambda_i \rfloor,$$

proving that the decomposition is optimum. ∎

51.5. Further results and notes

51.5a. Complexity survey for tree packing and covering

Complexity survey for finding a maximum number of (or k) disjoint spanning trees ($*$ indicates an asymptotically best bound in the table):

	$O(m^2 \log n)$	Imai [1983a]
	$O(m^2)$	Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
$*$	$O(m\sqrt{\frac{m}{n}}(m + n \log n) \log \frac{m}{n})$	Gabow and Westermann [1988,1992]
$*$	$O(nm \log \frac{m}{n})$	Gabow and Westermann [1988,1992]
$*$	$O(kn\sqrt{m + kn \log n})$	Gabow [1991a] (announced)

Complexity survey for finding a minimum number of forests covering all edges of the graph:

$O(n^4)$	Picard and Queyranne [1982a] (finding the number) *for simple graphs*
$O(n^2 m \log^2 n)$	Picard and Queyranne [1982a] (finding the number) *for simple graphs*

≫

continued

	$O(m^2)$	Imai [1983a], Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
*	$O(nm \log n)$	Gabow and Westermann [1988,1992]
	$O(m(m(m+n\log n)\log m)^{1/3})$	Gabow and Westermann [1988,1992]
*	$O(m^{3/2}\log(n^2/m))$	Gabow [1995b,1998]

Liu and Wang [1988] gave an $O(k^2n^2m(m + kn^2))$-time algorithm to find a minimum-weight union F of k edge-disjoint spanning trees in a graph $G = (V, E)$, where E is partitioned into classes $E_1, \ldots, E_t$, such that $a_i \leq |F \cap E_i| \leq b_i$ for each i, given a partition $E_1, \ldots, E_t$ of E and numbers a_i and b_i for all i.

Complexity survey for finding a maximum-size union of k forests:

	$O(k^2n^2)$	Imai [1983a], Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
	$O(k^{3/2}\sqrt{nm(m+n\log n)})$	Gabow and Stallmann [1985]
*	$O(k^{3/2}n\sqrt{m+n\log n})$	Gabow and Westermann [1988,1992]
*	$O(k^{1/2}m\sqrt{m+n\log n})$	Gabow and Westermann [1988,1992]
*	$O(kn^2\log k)$	Gabow and Westermann [1988,1992]
*	$O(\frac{m^2}{k}\log k)$	Gabow and Westermann [1988,1992]

Algorithms for finding a maximum-size union of two forests were given by Kishi and Kajitani [1967,1968,1969] and Kameda and Toida [1973].

Complexity survey for finding a maximum-weight union of k forests:

	$O(k^2n^2 + m\log m)$	Roskind and Tarjan [1985] *for simple graphs*
*	$O(kn^2\log k + m\log m)$	Gabow and Westermann [1988,1992]
*	$O(\frac{m^2}{k}\log k + m\log m)$	Gabow and Westermann [1988,1992]

Roskind and Tarjan [1985] (cf. Clausen and Hansen [1980]) gave an $O(k^2n^2 + m\log m)$-time algorithm for finding a maximum-weight union of k disjoint spanning trees, in a simple graph.

As for **the capacitated case**, the methods given in Section 51.4 indicate that packing and covering problems on forests and trees can be solved by a series of minimum-capacity cut problem (as they reduce to Theorem 51.3). A parametric minimum cut method designed by Gallo, Grigoriadis, and Tarjan [1989] allows to combine several consecutive minimum cut computations, improving the efficiency of the corresponding tree packing and covering problem, as was done by Gusfield [1991].

The published algorithms for integer packing and coverings of trees all are based on rounding the fractional version, not increasing the complexity of the problem, except that rounding is included as an operation. This blocks these algorithms from being strongly polynomial-time: as we saw in Section 51.4, it can be proved that there exists no strongly polynomial-time algorithm for finding an optimum integer packing of spanning trees under a given capacity (similarly, for integer covering by forests).

The following table gives a complexity survey for finding a maximum fractional packing of spanning trees subject to a given integer capacity function c, or a minimum fractional covering by forests subject to a given demand function c. Here it seems that the optimum value can be found faster than an explicit fractional packing or covering. The problems of finding an optimum fractional packing of trees is close to that of finding an optimum fractional covering of forests (or trees), so we present their complexity in one survey.

For any graph $G = (V, E)$ and $c : E \to \mathbb{R}_+$, the *strength* is the maximum value of λ such that c belongs to $\lambda \cdot P_{\text{connector}}(G)$. It is equal to the maximum size of a fractional packing of spanning trees subject to c. The *fractional arboricity* is the minimum value of λ such that c belongs to $\lambda \cdot P_{\text{forest}}(G)$. This is equal to the minimum size of a fractional c-covering by forest.

	$O(nm^8)$	Cunningham [1984]: finding an optimum fractional packing of trees
	$O(nm \cdot \text{MF}(n, n^2))$	Cunningham [1985c]: computing strength
	$O(n^4 m^2 \log^2 C)$	Gabow [1991a] (announced): computing strength
*	$O(n^3 m)$	Gusfield [1991]: computing strength
	$O(nm^2 \log(n^2/m))$	Gusfield [1991]: computing strength
	$O(n^3 \cdot \text{MF}(n, m))$	Trubin [1991]: finding an optimum fractional packing of trees
	$O(n^2 \cdot \text{MF}(n, n^2))$	Barahona [1992]: computing strength
	$O(n^2 \cdot \text{MF}(n, n^2))$	Barahona [1995]: finding optimum fractional packing of trees
*	$O(n \cdot \text{MF}(n, m))$	Cheng and Cunningham [1994]: computing strength
*	$O(n \cdot \text{MF}(n, m))$	Gabow [1995b,1998]: computing strength and fractional arboricity
*	$O(n^3 m \log(n^2/m))$	Gabow and Manu [1995,1998]: finding an optimum fractional packing of trees and an optimum fractional covering by forests
*	$O(n^2 m \log C \log(n^2/m))$	Gabow and Manu [1995,1998]: finding an optimum fractional packing of trees and an optimum fractional covering by forests

Here $\text{MF}(n, m)$ is the complexity of finding a maximum-value $s - t$ flow subject to c in a digraph with n vertices and m arcs, and $C := \|c\|_{\max}$ (assuming c integer).

51.5b. Further notes

A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

(51.44) (?) Let $G = (V, E)$ be an undirected graph and let $s \in V$. Suppose that for each vertex $t \neq s$, there exist k internally vertex-disjoint $s-t$ paths. Then G has k spanning trees such that for each vertex $t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint. (?)

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (51.44) was proved by Itai and Rodeh [1984,1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

Peng, Chen, and Koh [1991] showed that for any undirected graph $G = (V, E)$ and any $p, k \in \mathbb{Z}_+$, there exist k disjoint forests each with p components if and only if

(51.45) $$|\delta(\mathcal{P})| \geq k(|\mathcal{P}| - p)$$

for each partition $\mathcal{P}$ of V into nonempty sets. This in fact is the matroid base packing theorem (Corollary 42.1d) applied to the $(|V| - p)$-truncation of the cycle matroid of G.

Theorem 42.10 of Seymour [1998] implies that if the edges of a graph $G = (V, E)$ can be partitioned into k forests and if for each $e \in E$ a subset L_e of $\{1, 2, \ldots\}$ with $|L_e| = k$ is given, then we can partition E into forests $F_1, F_2, \ldots$ such that $j \in L_e$ for each $j \geq 1$ and each $e \in F_j$.

Henneberg [1911] and Laman [1970] characterized those graphs which have, after adding any edge, two edge-disjoint spanning trees. This was extended to k edge-disjoint spanning trees by Frank and Szegő [2001].

Farber, Richter, and Shank [1985] showed the following. Let $G = (V, E)$ be an undirected graph. Let $\mathcal{V}$ be the collection of pairs (T_1, T_2) of edge-disjoint spanning trees T_1 and T_2 in G. Call two pairs (T_1, T_2) and (S_1, S_2) in $\mathcal{V}$ *adjacent* if $|(T_1 \cup T_2) - (S_1 \cup S_2)| = 2$. Then this determines a connected graph on $\mathcal{V}$.

Cunningham [1985c] gave a strongly polynomial-time algorithm ($O(nm \min\{n^2, m \log n\})$) to find a minimum-cost set of capacities to be added to a capacitated graph so as to create the existence of k edge-disjoint spanning trees; that is, given $G = (V, E)$ and $c, k \in \mathbb{Z}_+^E$, solving

(51.46) $$\sum_{e \in E} k(e) x_e$$

where $x \in \mathbb{Z}_+^E$ satisfies

(51.47) $$(c + x)(\delta(\mathcal{P})) \geq k(|\mathcal{P}| - 1)$$

for each partition $\mathcal{P}$ of V into nonempty sets. (It amounts to finding a minimum-cost integer vector in a contrapolymatroid.) Related work can be found in Baïou, Barahona, and Mahjoub [2000].

Chapter 52

Longest branchings and shortest arborescences

We next consider trees in directed graphs. We recall some terminology and facts. Let $D = (V, A)$ be a digraph. A *branching* is a subset B of A such that B contains no undirected circuit and such that for each vertex v there is at most one arc in B entering v. A *root* of B is a vertex not entered by any arc in B. For any branching B, each weak component of (V, B) contains a unique root.
A branching B is called an *arborescence* if the digraph (V, B) is weakly connected; equivalently, if (V, B) is a rooted tree. So each arborescence B has a unique root r. We say that B is *rooted at* r, and we call B an *r-arborescence*. An r-arborescence can be characterized as a directed spanning tree B such that each vertex is reachable in B from r. A digraph $D = (V, A)$ contains an r-arborescence if and only if each vertex of D is reachable from r.

52.1. Finding a shortest r-arborescence

Let be given a digraph $D = (V, A)$, a vertex r, and a length function $l : A \to \mathbb{Q}_+$. We consider the problem of finding a shortest (= minimum-length) r-arborescence.

We cannot apply here the greedy method of starting at the root r and iteratively extending an r-arborescence on a subset U of V, by the shortest arc leaving U. This is shown by the example of Figure 52.1.

The following algorithm was given by Chu and Liu [1965], Edmonds [1967a], and Bock [1971]:

Algorithm to find a shortest r-arborescence. Let $A_0 := \{a \in A \mid l(a) = 0\}$. If A_0 contains an r-arborescence B, then B is a shortest r-arborescence. If A_0 contains no r-arborescence, there is a strong component K of (V, A_0) with $r \notin K$ and with $l(a) > 0$ for each $a \in \delta^{\text{in}}(K)$. Let $\alpha := \min\{l(a) \mid a \in \delta^{\text{in}}(K)\}$. Set $l'(a) := l(a) - \alpha$ if $a \in \delta^{\text{in}}(K)$ and $l'(a) := l(a)$ otherwise.

Find (recursively) a shortest r-arborescence B with respect to l'. As K is a strong component of (V, A_0), we can choose B such that $|B \cap \delta^{\text{in}}(K)| = 1$

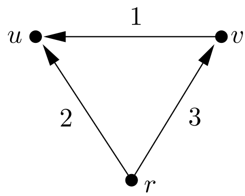

Figure 52.1
In a greedy method one would first choose the shortest arc leaving r, which is (r,u). This arc however is not contained in the shortest r-arborescence.

(since if $|B \cap \delta^{\text{in}}(K)| \geq 2$, then there exists an $a \in B \cap \delta^{\text{in}}(K)$ such that the set $(B \setminus \{a\}) \cup A_0$ contains an r-arborescence, say B', with $l'(B') \leq l'(B) - l'(a) \leq l'(B)$).

Then B is also a shortest r-arborescence with respect to l, since for any r-arborescence B':

$$(52.1) \qquad l(B') = l'(B') + \alpha|B' \cap \delta^{\text{in}}(K)| \geq l'(B') + \alpha \geq l'(B) + \alpha = l(B).$$

Since the number of iterations is at most m (as in each step A_0 increases), we have:

Theorem 52.1. *A shortest r-arborescence can be found in strongly polynomial time.*

Proof. See above. ∎

In fact, direct analysis gives the following result of Chu and Liu [1965], Edmonds [1967a], and Bock [1971]:

Theorem 52.2. *A shortest r-arborescence can be found in time $O(nm)$.*

Proof. First note that there are at most $2n$ iterations. This can be seen as follows. Let k be the number of strong components of (V, A_0), and let k_0 be the number of strong components K of (V, A_0) with $d^{\text{in}}_{A_0}(K) = 0$. Then at any iteration, the number $k + k_0$ decreases. Indeed, if the strong component K selected remains a strong component, then $d^{\text{in}}_{A_0}(K) \neq 0$ in the next iteration; so k_0 decreases. Otherwise, k decreases. Hence there are at most $2n$ iterations.

Next, each iteration can be performed in time $O(m)$. Indeed, in time $O(m)$ we can identify the set U of vertices not reachable in (V, A_0) from r. Next, by Theorem 6.6 one can identify the strong components of the subgraph of (V, A_0) induced by U, in time $O(m)$. Moreover, by Theorem 6.5 we can order the vertices in U pre-topologically. Then the first vertex in this order belongs to a strong component K such that each arc a entering K has $l(a) > 0$. ∎

Tarjan [1977] showed that this algorithm has an $O(\min\{n^2, m \log n\})$-time implementation.

52.1a. *r*-arborescences as common bases of two matroids

Let $D = (V, A)$ be a digraph and let $r \in V$. The r-arborescences can be considered as the common bases in two matroids on A: M_1 is the cycle matroid of the underlying undirected graph, and M_2 is the partition matroid on A induced by the sets $\delta^{\text{in}}(v)$ for $v \in V \setminus \{r\}$. We assume without loss of generality that no arc of D enters r.

Then the r-arborescences are exactly the common bases of M_1 and M_2. This gives us a reduction of polyhedral and algorithmic results to matroid intersection. In particular, Theorem 52.1 follows from the strong polynomial-time solvability of weighted matroid intersection.

52.2. Related problems

The complexity results of Section 52.1 immediately imply similar results for finding optimum branchings and arborescences without specifying a root. First we note:

Corollary 52.2a. *Given a digraph $D = (V, A)$, $r \in V$, and a length function $l : A \to \mathbb{Q}$, a longest r-arborescence can be found in $O(nm)$ time.*

Proof. Define $L := \max\{l(a) \mid a \in A\}$ and $l'(a) := L - l(a)$ for each $a \in A$. Then an r-arborescence B minimizing $l'(B)$ is an r-arborescence maximizing $l(B)$. ∎

Then we have for longest branching:

Corollary 52.2b. *Given a digraph $D = (V, A)$ and a length function $l \in \mathbb{Q}^A$, a longest branching can be found in time $O(nm)$.*

Proof. We can assume that l is nonnegative, by deleting all arcs of negative length. Extend D by a new vertex r and new arcs (r, v) for all $v \in V$, each of length 0. Let B be a longest r-arborescence in D' (this can be found in $O(nm)$-time by Corollary 52.2a). Then trivially $B \cap A$ is a longest branching in D. ∎

Similarly, for finding a shortest arborescence, without prescribing a root:

Corollary 52.2c. *Given a digraph $D = (V, A)$ and a length function $l \in \mathbb{Q}_+^A$, a shortest arborescence can be found in time $O(nm)$.*

Proof. Extend D by a new vertex r and arcs (r, v) for each $v \in V$, giving digraph D'. Let $l(r, v) := Ln$, where $L := \max\{l(a) \mid a \in A\}$. If D has an arborescence, then a shortest r-arborescence in D' has only one arc leaving r, and deleting this arc gives a shortest arborescence in D. ∎

52.3. A min-max relation for shortest r-arborescences

We now characterize the minimum length of an r-arborescence. Let $D = (V, A)$ be a digraph and let $r \in V$. Call a set C of arcs an *r-cut* if there exists a nonempty subset U of $V \setminus \{r\}$ with

(52.2) $\quad C = \delta^{\text{in}}(U).$

It is not difficult to show that

(52.3) the collection of inclusionwise minimal arc sets intersecting each r-arborescence is equal to the collection of inclusionwise minimal r-cuts,

and

(52.4) the collection of inclusionwise minimal arc sets intersecting each r-cut is equal to the collection of r-arborescences.

The following theorem follows directly from the method of Edmonds [1967a], and was stated explicitly by Bock [1971] (and also by Fulkerson [1974]):

Theorem 52.3 (optimum arborescence theorem). *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $l : A \to \mathbb{Z}_+$. Then the minimum length of an r-arborescence is equal to the maximum size of a family of r-cuts such that each arc a is in at most $l(a)$ of them.*

Proof. Clearly, the maximum is not more than the minimum, as each r-cut intersects each r-arborescence.

We prove the reverse inequality by induction on $\sum_{a \in A} l(a)$. Let $A_0 := \{a \in A \mid l(a) = 0\}$. If A_0 contains an r-arborescence, the minimum is 0, while the maximum is at least 0.

If A_0 contains no r-arborescence, there exists a strong component K of the digraph (V, A_0) with $r \notin K$ and with $l(a) > 0$ for each $a \in \delta^{\text{in}}(K)$. Define $l' := l - \chi^{\delta^{\text{in}}(K)}$. By induction there exist an r-arborescence B and r-cuts $C_1, \ldots, C_t$ such that each arc a is in at most $l'(a)$ of the C_i and such that $l'(B) = t$. We may assume that $|B \cap \delta^{\text{in}}(K)| = 1$, since if $|B \cap \delta^{\text{in}}(K)| \geq 2$, then for each $a \in B \cap \delta^{\text{in}}(K)$, $(B \setminus \{a\}) \cup A_0$ contains an r-arborescence, say B', with $l'(B') \leq l'(B) - l'(a) \leq l'(B)$.

It follows that $l(B) = t + 1$. Moreover, taking $C_{t+1} := \delta^{\text{in}}(K)$, each arc a is in at most $l(a)$ of the $C_1, \ldots, C_{t+1}$. ∎

Note that if B is a shortest r-arborescence, then $|B \cap C| = 1$ for any r-cut C in the maximum-size family. Moreover, for any $a \in B$, $l(a)$ is equal to the number of r-cuts C chosen with $a \in C$.

52.4. The r-arborescence polytope

Given a digraph $D = (V, A)$ and a vertex $r \in V$, the *r-arborescence polytope* is defined as the convex hull of the incidence vectors (in $\mathbb{R}^A$) of the r-arborescences; that is,

(52.5) $$P_{r\text{-arborescence}}(D) := \text{conv.hull}\{\chi^B \mid B \ r\text{-arborescence}\}.$$

Theorem 52.3 implies that the r-arborescence polytope of D is determined by:

(52.6)
$$\begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } r\text{-cut } C, \\ \text{(iii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for } v \in V \setminus \{r\}. \end{array}$$

To prove this, we first characterize the up hull of the r-arborescence polytope, where as usual the up hull of the r-arborescence polytope is defined as

(52.7) $$P^{\uparrow}_{r\text{-arborescence}}(D) := P_{r\text{-arborescence}}(D) + \mathbb{R}^A_+.$$

Corollary 52.3a. $P^{\uparrow}_{r\text{-arborescence}}(D)$ *is determined by*

(52.8)
$$\begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } r\text{-cut } C. \end{array}$$

Proof. The incidence vector of any r-arborescence trivially satisfies (52.8); hence $P^{\uparrow}_{r\text{-arborescence}}(D)$ is contained in the polyhedron Q determined by (52.8).

Suppose that the reverse inclusion does not hold. Then there exists a rational length function $l \in \mathbb{Q}^A_+$ such that the minimum value of $l^{\mathsf{T}}x$ over Q is less than the minimum length of an r-arborescence. We can assume that l is integer. However, the minimum value of $l^{\mathsf{T}}x$ over Q cannot be less than the maximum described in Theorem 52.3. So we have a contradiction. ∎

Since the r-arborescence polytope is a face of its up hull, this implies:

Corollary 52.3b. *The r-arborescence polytope is determined by* (52.6).

Proof. Directly from Corollary 52.3a. ∎

Corollary 52.3a also implies for the restriction to the unit cube:

Corollary 52.3c. *The convex hull of incidence vectors of arc sets containing an r-arborescence is determined by*

(52.9) (i) $0 \leq x_a \leq 1$ for $a \in A$,
(ii) $x(C) \geq 1$ for each r-cut C.

Proof. Directly from Corollary 52.3a with Theorem 5.19. ∎

Theorem 52.3 can be reformulated in TDI terms as:

Corollary 52.3d. *System* (52.8) *is TDI.*

Proof. Choose a length function $l \in \mathbb{Z}_+^A$, and consider the dual problem of minimizing $l^\mathsf{T} x$ over (52.8). For each r-cut C, let y_C be the number of times C is chosen in the maximum family in Theorem 52.3. Moreover, let B be a shortest r-arborescence. Then by Theorem 52.3, $x := \chi^B$ and the y_C form a dual pair of optimum solutions. As the y_C are integer, it follows that (52.8) is TDI. ∎

This in turn implies for the r-arborescence polytope:

Corollary 52.3e. *System* (52.6) *is TDI.*

Proof. Directly from Corollary 52.3d, with Theorem 5.25, since (52.6) arises from (52.8) by setting some of the inequalities to equality. ∎

For the intersection with the unit cube it gives:

Corollary 52.3f. *System* (52.9) *is TDI.*

Proof. Directly from Corollary 52.3d, with Theorem 5.23. ∎

In fact, (poly)matroid intersection theory gives the box-total dual integrality of (52.8):

Theorem 52.4. *System* (52.8) *is box-TDI.*

Proof. Let M_1 be the cycle matroid of the undirected graph underlying $D = (V, A)$, and let M_2 be the partition matroid induced by the sets $\delta^{\text{in}}(v)$ for $v \in V \setminus \{r\}$. By Corollary 46.1d, the system

(52.10) $x(B) \geq |V| - 1 - r_{M_i}(A \setminus B)$ for $i = 1, 2$ and $B \subseteq A$,

is box-TDI. Now any inequality in (52.10) is a nonnegative integer combination of inequalities (52.8).

Indeed, if $i = 1$, then $r_{M_1}(A \setminus B)$ is equal to $|V|$ minus the number of weak components of the digraph $(V, A \setminus B)$. So the inequality in (52.10) states that $x(B)$ is at least the number of weak components of $(V, A \setminus B)$ not containing r.

Hence it is a sum of the inequalities $x(\delta^{\text{in}}(K)) \geq 1$ for each weak component K of $(V, A \setminus B)$ not containing r, and of $x_a \geq 0$ for all $a \in B$ not entering any of these components.

If $i = 2$, then $r_{M_2}(A \setminus B)$ is equal to the number of $v \neq r$ entered by at least one arc in $A \setminus B$. So the inequality in (52.10) states that $x(B)$ is at least the number of $v \neq r$ with $\delta^{\text{in}}(v) \subseteq B$. It therefore is a sum of the inequalities $x(\delta^{\text{in}}(v)) \geq 1$ for these v, and $x_a \geq 0$ for all $a \in B$ not entering any of these vertices.

So Corollary 46.1d implies that (52.8) is box-TDI. ∎

52.4a. Uncrossing cuts

Edmonds and Giles [1977] and Frank [1979b] gave the following procedure of proving that system (52.8) is box-TDI (cf. Corollary 52.3b). The proof is longer than that given above, but it is a special case of a far more general approach (to be discussed in Chapter 60), and is therefore worth noting at this point.

System (52.8) is equivalent to:

$$\text{(52.11)} \qquad \begin{array}{lll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(U)) \geq 1 & \text{for } \emptyset \neq U \subseteq V \setminus \{r\} \end{array}$$

Consider any length function $l \in \mathbb{R}^A_+$. Let y_U form an optimum solution to the problem dual to minimizing $l^{\mathsf{T}}x$ over (52.11):

$$\text{(52.12)} \qquad \begin{array}{ll} \text{maximize} & \sum_U y_U \\ \text{subject to} & y_U \geq 0 \text{ for all } U, \\ & \sum_U y_U \chi^{\delta^{\text{in}}(U)} \leq l, \end{array}$$

where U ranges over the nonempty subsets of $V \setminus \{r\}$.

Choose the y_U in such a way that

$$\text{(52.13)} \qquad \sum_U y_U |U||V \setminus U|$$

is as small as possible. Then the collection

$$\text{(52.14)} \qquad \mathcal{F} := \{U \mid y_U > 0\}$$

is laminar; that is,

$$\text{(52.15)} \qquad U \cap W = \emptyset \text{ or } U \subseteq W \text{ or } W \subseteq U \text{ for all } U, W \in \mathcal{F}.$$

For suppose not. Let $\alpha := \min\{y_U, y_W\}$. Decrease y_U and y_W by α, and increase $y_{U\cap W}$ and $y_{U\cup W}$ by α. Then y remains a feasible dual solution, since

$$\text{(52.16)} \qquad \chi^{\delta^{\text{in}}(U\cap W)} + \chi^{\delta^{\text{in}}(U\cup W)} \leq \chi^{\delta^{\text{in}}(U)} + \chi^{\delta^{\text{in}}(W)}.$$

Moreover, y remains trivially optimum. However, sum (52.13) decreases (by Theorem 2.1), contradicting our assumption. So $\mathcal{F}$ is laminar.

Now the $\mathcal{F} \times A$ matrix M with

$$\text{(52.17)} \qquad M_{U,a} := \begin{cases} 1 & \text{if } a \in \delta^{\text{in}}(U), \\ 0 & \text{otherwise,} \end{cases}$$

is totally unimodular. In fact, it is a network matrix. For make a directed tree T as follows. The vertex set of T is the set $\mathcal{F}' := \mathcal{F} \cup \{V\}$, while for each $U \in \mathcal{F}$ there is an arc a_U from W to U where W is the smallest set in $\mathcal{F}'$ with $W \supset U$. This is in fact an arborescence with root V.

We also define a digraph $\widetilde{D} = (\mathcal{F}', \widetilde{A})$. For each arc $a = (u, v)$ of D, let $\tilde{a}$ be an arc from the smallest set in $\mathcal{F}'$ containing both u and v, to the smallest set in $\mathcal{F}'$ containing v. Let $\widetilde{A} := \{\tilde{a} \mid a \in A\}$.

Identifying any set U in $\mathcal{F}$ with the arc a_U of T, the network matrix generated by directed tree T and digraph $\widetilde{D}$ is an $\mathcal{F} \times \widetilde{A}$ matrix which is the same as M. So M is totally unimodular. Therefore, by Theorem 5.35, (52.11) is box-TDI.

52.5. A min-max relation for longest branchings

We now consider longest branchings. Characterizing the maximum size of a branching is easy:

Theorem 52.5. *Let $D = (V, A)$ be a digraph. Then the maximum size of a branching is equal to $|V|$ minus the number of strong components K of D with $d_A^{\text{in}}(K) = 0$.*

Proof. The theorem follows directly from: (i) each branching has at least one root in any strong component K of D with $d_A^{\text{in}}(K) = 0$, and (ii) if a set R intersects each such K, then there is a branching with root set R (since each vertex of D is reachable from R). ∎

From Theorem 52.3 one can derive a min-max relation for the maximum length of a branching in a digraph. The reduction is similar to the reduction of the algorithmic problem of finding a longest branching to that of finding a shortest r-arborescence.

However, a direct proof can be derived from matroid intersection. Consider the system:

$$\text{(52.18)} \quad \begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(v)) \leq 1 & \text{for } v \in V, \\ \text{(iii)} & x(A[U]) \leq |U| - 1 & \text{for } U \subseteq V,\ U \neq \emptyset. \end{array}$$

Theorem 52.6. *System* (52.18) *is TDI.*

Proof. Directly from Theorem 41.12, applied to the cycle matroid M_1 of the undirected graph underlying $D = (V, A)$, and the partition matroid M_2 induced by the sets $\delta^{\text{in}}(v)$ for $v \in V$. Then each inequality $x(B) \leq r_{M_1}(B)$ is the sum of the inequalities $x(A[U]) \leq |U| - 1$ for the weak components U of (V, B), and $-x_a \leq 0$ for those arcs $a \in A \setminus B$ contained in any weak component of (V, B). Each inequality $x(B) \leq r_{M_2}(B)$ is the sum of the inequalities $x(\delta^{\text{in}}(v)) \leq 1$ for those v entered by at least one arc in B, and

$-x_a \leq 0$ for those arcs $a \in A \setminus B$ that enter a vertex v entered by at least one arc in B. ∎

52.6. The branching polytope

The previous corollary immediately implies a description of the *branching polytope* $P_{\text{branching}}(D)$ of D, which is the convex hull of the incidence vectors of branchings in D (stated by Edmonds [1967a]):

Corollary 52.6a. *The branching polytope of $D = (V, A)$ is determined by (52.18).*

Proof. Directly from Theorem 52.6, since the integer solutions of (52.18) are the incidence vectors of the branchings. ∎

Also the following theorem of Edmonds [1967a] follows from matroid intersection theory:

Corollary 52.6b. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of branchings of size k is equal to the intersection of the branching polytope of D with the hyperplane $\{x \mid x(A) = k\}$.*

Proof. This is the common base polytope of the k-truncations of the matroids M_1 and M_2 defined in the proof of Theorem 52.6. ∎

In Corollary 53.3a we shall see that the convex hull of the incidence vectors of branchings of size k has the integer decomposition property (McDiarmid [1983]).

Giles and Hausmann [1979] characterized which pairs of branchings give adjacent vertices of the branching polytope, and Giles [1975,1978b] and Grötschel [1977a] characterized the facets of the branching polytope.

52.7. The arborescence polytope

The results on branchings in the previous section can be specialized to arborescences (without prescribed root). Given a digraph $D = (V, A)$, the *arborescence polytope* of D, denoted by $P_{\text{arborescence}}(D)$, is the convex hull of the incidence vectors of arborescences.

Corollary 52.6c. *The arborescence polytope is determined by*

$$
(52.19) \qquad \begin{array}{rll}
\text{(i)} & x_a \geq 0 & \text{for } a \in A, \\
\text{(ii)} & x(\delta^{\text{in}}(v)) \leq 1 & \text{for } v \in V, \\
\text{(iii)} & x(A[U]) \leq |U| - 1 & \text{for } U \subseteq V,\ U \neq \emptyset, \\
\text{(iv)} & x(A) = |V| - 1. &
\end{array}
$$

Proof. Directly from Corollary 52.6a, since $P_{\text{arborescence}}(D)$ is the face of $P_{\text{branching}}(D)$ determined by the hyperplane $x(A) = |V| - 1$. ∎

One similarly obtains from Theorem 52.6 the following, which yields a min-max relation for the minimum length of an arborescence:

Corollary 52.6d. *System* (52.19) *is TDI.*

Proof. From Theorem 52.6, with Theorem 5.25. ∎

52.8. Further results and notes

52.8a. Complexity survey for shortest r-arborescence

	$O(nm)$	Chu and Liu [1965], Edmonds [1967a], Bock [1971]
	$O(n^2)$	Tarjan [1977] (cf. Camerini, Fratta, and Maffioli [1979])
	$O(m \log n)$	Tarjan [1977] (cf. Camerini, Fratta, and Maffioli [1979])
	$O(n \log n + m \log\log\log_{m/n} n)$	Gabow, Galil, and Spencer [1984]
*	$O(m + n \log n)$	Gabow, Galil, Spencer, and Tarjan [1986]

As before, $*$ indicates an asymptotically best bound in the table.

X. Guozhi (see Guan [1979]), Gabow and Tarjan [1979,1984], and Gabow, Galil, Spencer, and Tarjan [1986] studied the problem of finding a shortest r-arborescence with exactly k arcs leaving r, yielding an $O(m+n \log n)$-time algorithm. Hou [1996] gave an $O(k^3 m^3)$-time algorithm to find the k shortest r-arborescences in a digraph.

Gabow and Tarjan [1988a] gave $O(m+n \log n)$- and $O(m \log^* n)$-time algorithms for the *bottleneck* r-arborescence problem (that is, minimizing the maximum arc cost), improving the $O(m \log n)$-time algorithm of Camerini [1978]. (Here $\log^* n$ is the minimum i with $\log_2^{(i)} n \leq 1$.)

52.8b. Concise LP-formulation for shortest r-arborescence

Wong [1984] and Maculan [1986] observed that the problem of finding a shortest r-arborescence can be formulated as a concise linear programming problem. In fact, the dominant $P^{\uparrow}_{r\text{-arborescence}}(D)$ of the r-arborescence polytope is the projection of a polyhedron in nm dimensions determined by at most $n(2m+n)$ constraints.

Theorem 52.7. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then $P^{\uparrow}_{r\text{-arborescence}}(D)$ is equal to the set Q of all vectors $x \in \mathbb{R}^A_+$ such that for each $u \in V \setminus \{r\}$ there exists an $r-u$ flow f_u of value 1 satisfying $f_u \leq x$.*

Proof. Since the incidence vector $x = \chi^B$ of any r-arborescence satisfies the constraints, we know that $P^{\uparrow}_{r\text{-arborescence}}(D)$ is contained in Q.

To see the reverse inclusion, let $x \in Q$. Then for each nonempty subset U of $V \setminus \{r\}$ one has

(52.20) $$x(\delta^{\text{in}}(U)) \geq f_u(\delta^{\text{in}}(U)) \geq 1,$$

where u is any vertex in U and where f_u is an $r-u$ flow of value 1 with $f_u \leq x$. So by Corollary 52.3a, x belongs to $P^{\uparrow}_{r\text{-arborescence}}(D)$. ∎

This implies that a shortest r-arborescence can be found by solving a linear programming problem of polynomial size:

Corollary 52.7a. *Let $D = (V, A)$ be a digraph and let $r \in V$ and $l \in \mathbb{R}^A_+$. Then the length of a shortest r-arborescence is equal to the minimum value of*

(52.21) $$\sum_{a \in A} l(a) x_a,$$

where $x \in \mathbb{R}^A$ is such that for each $u \in V \setminus \{r\}$ there exists an $r-u$ flow f_u of value 1 with $f_u \leq x$.

Proof. Directly from Theorem 52.7. ∎

52.8c. Further notes

Frank [1979b] showed the following. Let $D = (V, A)$ be a digraph and let $r \in V$. Then a subset A' of A is contained in an r-arborescence if and only if $|\mathcal{U}| \leq |V| - 1 - |A'|$ for each laminar collection $\mathcal{U}$ of nonempty subsets of $V \setminus \{r\}$ such that each arc of D enters at most one set in $\mathcal{U}$ and no arc in A' enters any set in $\mathcal{U}$.

Goemans [1992,1994] studied the convex hull of (not necessarily spanning) partial r-arborescences.

Karp [1972a] gave a shortening of the proof of Edmonds [1967a] of the correctness of the shortest r-arborescence algorithm.

Books covering shortest arborescences include Minieka [1978], Papadimitriou and Steiglitz [1982], and Gondran and Minoux [1984].

Chapter 53

Packing and covering of branchings and arborescences

Packing arborescences is a special case of packing common bases in two matroids. However, no general matroid theorem is known that covers this case. In Section 42.6c the maximum number of common bases in two strongly base orderable matroids was characterized, but this does not apply to packing arborescences, as graphic matroids are generally not strongly base orderable. Yet, min-max relations and polyhedral characterizations can be proved for packing arborescences, and similarly for covering by branchings.

53.1. Disjoint branchings

Edmonds [1973] gave the following characterization of the existence of disjoint branchings in a given directed graph $D = (V, A)$. We give the proof of Lovász [1976c]. The *root set* of a branching B is the set of roots of B, that is, the set of sources of the digraph (V, B).

Theorem 53.1 (Edmonds' disjoint branchings theorem)**.** *Let $D = (V, A)$ be a digraph and let $R_1, \ldots, R_k$ be subsets of V. Then there exist disjoint branchings $B_1, \ldots, B_k$ such that B_i has root set R_i (for $i = 1, \ldots, k$) if and only if*

(53.1) $\qquad d^{\text{in}}(U) \geq |\{i \mid R_i \cap U = \emptyset\}|,$

for each nonempty subset U of V.

Proof. Necessity being trivial, we show sufficiency, by induction on $|V \setminus R_1| + \cdots + |V \setminus R_k|$. If $R_1 = \cdots = R_k = V$, the theorem is trivial, so we can assume that $R_1 \neq V$. For each $U \subseteq V$, define

(53.2) $\qquad g(U) := |\{i \mid R_i \cap U = \emptyset\}|.$

Let W be an inclusionwise minimal set with the properties that $W \cap R_1 \neq \emptyset$, $W \setminus R_1 \neq \emptyset$, and $d^{\text{in}}(W) = g(W)$. Such a set exists, since $W = V$ would qualify.

Then

(53.3) $$d^{\text{in}}(W \setminus R_1) \geq g(W \setminus R_1) > g(W) = d^{\text{in}}(W),$$

and hence there exists an arc $a = (u, v)$ in A with $u \in W \cap R_1$ and $v \in W \setminus R_1$. It suffices to show that (53.1) is maintained after resetting $A := A \setminus \{a\}$ and $R_1 := R_1 \cup \{v\}$, since after resetting we can apply induction, and assign a to B_1.

To see that (53.1) is maintained, suppose that to the contrary there is a $U \subseteq V$ violating the condition after resetting. Then in resetting, $d^{\text{in}}(U)$ decreases by 1 while $g(U)$ is unchanged. So a enters U, and, before resetting we had $d^{\text{in}}(U) = g(U)$ and $U \cap R_1 \neq \emptyset$. This implies (before resetting):

(53.4) $$\begin{aligned} d^{\text{in}}(U \cap W) &\leq d^{\text{in}}(U) + d^{\text{in}}(W) - d^{\text{in}}(U \cup W) \\ &\leq g(U) + g(W) - g(U \cup W) \leq g(U \cap W). \end{aligned}$$

So we have equality throughout. Hence $d^{\text{in}}(U \cap W) = g(U \cap W)$ and $R_1 \cap (U \cap W) \neq \emptyset$ (as $R_1 \cap W \neq \emptyset$ and $R_1 \cap U \neq \emptyset$, and $g(U \cap W) = g(U) + g(W) - g(U \cup W)$). Also $(U \cap W) \setminus R_1 \neq \emptyset$ (since $v \in U \cap W$) and $U \cap W \subset W$ (as $u \notin U \cap W$). This contradicts the minimality of W. ∎

(Also the method of Tarjan [1974a] is based on the existence of an arc a as in this proof. Fulkerson and Harding [1976] gave another proof of the existence of such ar arc (more complicated than that of Lovász given above).)

53.2. Disjoint r-arborescences

The previous theorem implies a characterization of the existence of disjoint arborescences with prescribed roots:

Corollary 53.1a. *Let $D = (V, A)$ be a digraph and let $r_1, \ldots, r_k \in V$. Then there exist k disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i (for $i = 1, \ldots, k$) if and only if each nonempty subset U of V is entered by at least as many arcs as there exist i with $r_i \notin U$.*

Proof. Directly from Edmonds' disjoint branchings theorem (Theorem 53.1) by taking $R_i := \{r_i\}$ for all i. ∎

If all roots are equal, we obtain the following min-max relation, announced by Edmonds [1970b]. Recall that an *r-cut* is a cut $\delta^{\text{in}}(U)$ where U is a nonempty subset of $V \setminus \{r\}$.

Corollary 53.1b (Edmonds' disjoint arborescences theorem). *Let $D = (V, A)$ be a digraph and let $r \in V$. Then the maximum number of disjoint r-arborescences is equal to the minimum size of an r-cut.*

Proof. Directly from Corollary 53.1a by taking k equal to the minimum size of an r-cut and $r_i := r$ for $i = 1, \ldots, k$. ∎

Note that Edmonds' disjoint arborescences theorem implies Menger's theorem: for any digraph $D = (V, A)$ and $r, s \in V$, if k is the minimum size of an $r - s$ cut, we can extend D by k parallel arcs from s to v, for each vertex $v \neq s$; in the extended graph, the minimum size of an r-cut is k, and hence it contains k arc-disjoint r-arborescences. This gives k arc-disjoint $r - s$ paths in the original graph D.

One can reformulate Edmonds' disjoint arborescences theorem in a number of ways (Edmonds [1975]):

Corollary 53.1c. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then for each $k \in \mathbb{Z}_+$ the following are equivalent:*

(53.5)
(i) *there exist k disjoint r-arborescences;*
(ii) *for each nonempty $U \subseteq V \setminus \{r\}$, $d^{\text{in}}(U) \geq k$;*
(iii) *for each $s \neq r$ there exist k arc-disjoint $r - s$ paths in D;*
(iv) *there exist k edge-disjoint spanning trees in the underlying undirected graph such that for each $s \neq r$ there are exactly k arcs entering s covered by these trees.*

Proof. The equivalence of (i) and (ii) follows from Edmonds' disjoint arborescences theorem (Theorem 53.1b), and the equivalence of (ii) and (iii) is a direct consequence of Menger's theorem.

The implication (i) $\Rightarrow$ (iv) is trivial. To prove (iv) $\Rightarrow$ (ii), suppose that (iv) holds, and let U be a nonempty subset of $V \setminus \{r\}$. Each spanning tree has at most $|U| - 1$ arcs contained in U. So the spanning trees of (iv) together have at most $k(|U| - 1)$ arcs contained in U. Moreover, they have exactly $k|U|$ arcs with head in U. Hence, at least k arcs enter U. ∎

An interesting consequence of Edmonds' disjoint arborescences theorem was observed by Shiloach [1979a] and concerns the arc-connectivity of a directed graph:

Corollary 53.1d. *A digraph $D = (V, A)$ is k-arc-connected if and only if for all $s_1, t_1, \ldots, s_k, t_k \in V$ there exist arc-disjoint paths $P_1, \ldots, P_k$, where P_i runs from s_i to t_i $(i = 1, \ldots, k)$.*

Proof. Sufficiency follows by taking $s_1 = \cdots = s_k$ and $t_1 = \cdots = t_k$. To see necessity, extend D by a vertex r and arcs (r, s_i) for $i = 1, \ldots, k$. By Edmonds' disjoint arborescences theorem (Corollary 53.1b), the extended digraph has k disjoint r-arborescences, since each nonempty subset U of V is entered by at least k arcs of D'. Choosing the $s_i - t_i$ path in the r-arborescence containing (r, s_i), for $i = 1, \ldots, k$, we obtain paths as required. ∎

53.3. The capacitated case

The capacitated version of the min-max relation for disjoint r-arborescences reads:

Corollary 53.1e. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the minimum capacity of an r-cut is equal to the maximum value of $\sum_B \lambda_B$, where λ_B is a nonnegative integer for each r-arborescence B such that*

(53.6) $$\sum_B \lambda_B \chi^B \leq c.$$

Proof. Directly from Corollary 53.1b by replacing each arc a by $c(a)$ parallel arcs. ∎

One can equivalently formulate this in term of total dual integrality. To see this, consider the *r-cut polytope* $P_{r\text{-cut}}(D)$ of D, defined as the convex hull of the incidence vectors of the r-cuts in D. In particular, consider the up hull

(53.7) $$P^{\uparrow}_{r\text{-cut}}(D) := P_{r\text{-cut}}(D) + \mathbb{R}_+^A$$

of the r-cut polytope.

In Corollary 52.3a we saw that the up hull $P^{\uparrow}_{r\text{-arborescence}}(D)$ of the r-arborescence polytope of D is determined by:

(53.8)
(i) $x_a \geq 0$ for each arc a,
(ii) $x(C) \geq 1$ for each r-cut C.

By the theory of blocking polyhedra, this implies that $P^{\uparrow}_{r\text{-cut}}(D)$ is determined by:

(53.9)
(i) $x_a \geq 0$ for each arc a,
(ii) $x(B) \geq 1$ for each r-arborescence B.

In fact:

Corollary 53.1f. *System* (53.9) *determines $P^{\uparrow}_{r\text{-cut}}(D)$ and is TDI.*

Proof. The first part follows from the theory of blocking polyhedra applied to Corollary 52.3a, and the second part is equivalent to Corollary 53.1e. ∎

Another equivalent form is:

(53.10) For any digraph $D = (V, A)$ and $r \in V$, the r-arborescence polytope has the integer decomposition property.

By Theorem 5.30, the number of r-arborescences B with $\lambda_B \geq 1$ in Corollary 53.1e can be taken to be at most $2|A| - 1$. (This improves a result of Pevzner [1979a] giving an $O(nm)$ upper bound.) Gabow and Manu [1995, 1998] showed an upper bound of $|V| + |A| - 2$.

53.4. Disjoint arborescences

Frank [1979a,1981c] derived from Corollary 53.1a the following min-max relation for disjoint arborescences without a prescribed root. (A *subpartition* of V is a partition of a subset of V.)

Corollary 53.1g. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A contains k disjoint arborescences if and only if*

$$(53.11) \qquad \sum_{U \in \mathcal{P}} d^{\text{in}}(U) \geq k(|\mathcal{P}| - 1)$$

for each subpartition $\mathcal{P}$ of V with nonempty classes.

Proof. Necessity being easy, we show sufficiency. Choose $x \in \mathbb{Z}_+^V$ such that

$$(53.12) \qquad x(U) \geq k - d^{\text{in}}(U)$$

for each nonempty subset U of V, with $x(V)$ as small as possible. We show that $x(V) = k$. Since $x(V) \geq k$ by (53.12), it suffices to show $x(V) \leq k$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal nonempty sets having equality in (53.12). Then $\mathcal{P}$ is a subpartition, for suppose that $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$(53.13) \qquad \begin{aligned} x(U \cup W) &= x(U) + x(W) - x(U \cap W) \\ &\leq (k - d^{\text{in}}(U)) + (k - d^{\text{in}}(W)) - (k - d^{\text{in}}(U \cap W)) \\ &\leq (k - d^{\text{in}}(U \cup W)), \end{aligned}$$

and hence $U \cup W \in \mathcal{P}$. So $U = W$.

Now for each $v \in V$ with $x_v > 0$ there exists a set U in $\mathcal{P}$ containing v, since otherwise we could decrease x_v. Hence

$$(53.14) \qquad x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq k,$$

by (53.11).

So $x(V) = k$. Now let $r_1, \ldots, r_k$ be vertices such that any vertex v occurs x_v times among the r_i. Then by Corollary 53.1a there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i. This shows the corollary. ∎

53.5. Covering by branchings

Let $A[U]$ denote the set of arcs in A with both ends in U. Frank [1979a] observed that the following min-max relation for covering by branchings can be derived from Edmonds' disjoint arborescences theorem:

Corollary 53.1h. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k branchings if and only if*

(53.15) (i) $\deg^{\text{in}}(v) \leq k$ *for each* $v \in V$,
(ii) $|A[U]| \leq k(|U| - 1)$ *for each nonempty subset* U *of* V.

Proof. Necessity being trivial, we show sufficiency. Extend D by a new vertex r, and for each $v \in V$, $k - \deg^{\text{in}}(v)$ parallel arcs from r to v. Let D' be the digraph thus arising. So each vertex in V is entered by exactly k arcs of D', and D' has $k|V|$ arcs.

Now each nonempty subset U of V is entered by at least k arcs of D', since exactly $k|U|$ arcs have their head in U and at most $k(|U|-1)$ arcs have both ends in U. So by Edmonds' disjoint arborescences theorem (Theorem 53.1b), D' has k disjoint r-arborescences. Since D' has exactly $k|V|$ arcs, these arborescences partition the arc set of D'. Hence restricting them to the arcs of the original graph D, we obtain k branchings partitioning A. ∎

(This was also shown by Markosyan and Gasparyan [1986].)

Corollary 53.1h is equivalent to:

Corollary 53.1i. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k branchings if and only if* $\deg^{\text{in}}(v) \leq k$ *for each $v \in V$ and A can be covered by k forests of the underlying undirected graph.*

Proof. Directly from Corollary 53.1h with Corollary 51.1c. ∎

Corollary 53.1h implies a polyhedral result of Baum and Trotter [1981] (attributing the proof to R. Giles):

Corollary 53.1j. *The branching polytope of a digraph $D = (V, A)$ has the integer decomposition property.*

Proof. Let $k \in \mathbb{Z}_+$ and let x be an integer vector in $k \cdot P_{\text{branching}}(D)$. Let $D' = (V, A')$ be the digraph obtained from D by replacing any arc $a = (u, v)$ by x_a parallel arcs from u to v. Then by Corollary 53.1h, A' can be partitioned into k branchings. This gives a decomposition of x as a sum of the incidence vectors of k branchings in D. ∎

53.6. An exchange property of branchings

We derive an exchange property of branchings from Edmonds' disjoint branchings theorem (Theorem 53.1). It implies that the branchings in an optimum covering can be taken of almost equal size. It will also be used in Section 59.5 on the total dual integrality of the matching forest constraints.

We first show a lemma. For any branching B, let $R(B)$ denote the set of roots of B.

Lemma 53.2α. *Let B_1 and B_2 be branchings partitioning the arc set A of a digraph $D = (V, A)$. Let R_1 and R_2 be sets with $R_1 \cup R_2 = R(B_1) \cup R(B_2)$ and $R_1 \cap R_2 = R(B_1) \cap R(B_2)$. Then A can be split into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$ if and only if each strong component K of D with $d^{\text{in}}(K) = 0$ intersects both R_1 and R_2.*

Proof. Necessity is easy, since the root set of any branching intersects any strong component K with $d^{\text{in}}(K) = 0$.

To see sufficiency, by Edmonds' disjoint branchings theorem (Theorem 53.1), branchings B'_1 and B'_2 as required exist if and only if

(53.16) $\qquad d^{\text{in}}(U) \geq |\{i \in \{1, 2\} \mid U \cap R_i = \emptyset\}|$

for each nonempty $U \subseteq V$. (Actually, Edmonds' theorem gives the existence of disjoint branchings B'_1 and B'_2 satisfying $R(B'_i) = R_i$ for $i = 1, 2$. That $B'_1 \cup B'_2 = A$ follows from the fact that $|B'_1| + |B'_2| = |B_1| + |B_2|$, as $|R(B'_1)| + |R(B'_2)| = |R(B_1)| + |R(B_2)|$.)

Suppose that inequality (53.16) does not hold. Then the right-hand side is positive. If it is 2, then U is disjoint from both R_1 and R_2, and hence from both $R(B_1)$ and $R(B_2)$ (since $R_1 \cup R_2 = R(B_1) \cup R(B_2)$), implying that both B_1 and B_2 enter U, and so $d^{\text{in}}(U) \geq 2$.

So the right-hand side is 1, and hence the left-hand side is 0. We can assume that U is an inclusionwise minimal set with this property. It implies that U is a strong component of D. Then by the condition, U intersects both R_1 and R_2, contradicting the fact that the right-hand side in (53.16) is 1. ∎

First, this implies the following exchange property of branchings:

Theorem 53.2. *Let B_1 and B_2 be branchings in a digraph $D = (V, A)$. Let s be a root of B_2 and let r be the root of the arborescence in B_1 containing s. Then D contains branchings B'_1 and B'_2 satisfying*

(53.17) $\qquad B'_1 \cup B'_2 = B_1 \cup B_1,\ B'_1 \cap B'_2 = B_1 \cap B_2,$
$\qquad\qquad$ *and* $R(B'_1) = R(B_1) \cup \{s\}$ *or* $R(B'_1) = (R(B_1) \setminus \{r\}) \cup \{s\}$.

Proof. We may assume that B_1, B_2 partition A, since we can delete all arcs not occurring in $B_1 \cup B_2$, and add parallel arcs for those in $B_1 \cap B_2$. We may also assume that $s \neq r$ (since the theorem is trivial if $s = r$).

Let K be the strong component of D containing s. If no arc of D enters K, then $r \in K$ (as B_1 contains a directed path from r to s), and hence r is not a root of B_2 (as otherwise no arc enters r while K is strongly connected); define $R_1 := (R(B_1) \setminus \{r\}) \cup \{s\}$ and $R_2 := (R(B_2) \setminus \{s\}) \cup \{r\}$.

Alternatively, if some arc of D enters K, define $R_1 := R(B_1) \cup \{s\}$ and $R_2 := R(B_2) \setminus \{s\}$. Then Lemma 53.2$\alpha$ implies that A can be split into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$. ∎

The lemma also implies that a packing of branchings can be balanced in the following sense:

Theorem 53.3. *Let $D = (V, A)$ be a digraph. If A can be covered by k branchings, then A can be covered by k branchings each of size $\lfloor |A|/k \rfloor$ or $\lceil |A|/k \rceil$.*

Proof. Consider any two branchings B_1, B_2 in the covering which differ in size by at least 2. Consider the digraph $D' = (V, B_1 \cup B_2)$. We can find subsets R_1 and R_2 of V with $R_1 \cup R_2 = R(B_1) \cup R(B_2)$ and $R_1 \cap R_2 = R(B_1) \cap R(B_2)$, such that each strong component K of D' with $d^{\text{in}}_{D'}(K) = 0$ intersects both R_1 and R_2, and such that R_1 and R_2 differ by at most 1 in size. (We can first include, for any such component K, one element in $K \cap R(B_1)$ in R_1, and one element in $K \cap R(B_2)$ in R_2; next we distribute the remaining elements in $R(B_1)$ and $R(B_2)$ almost equally over R_1 and R_2).

Then, by Lemma 53.2α, $B_1 \cup B_2$ can be partitioned into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$. Then B'_1 and B'_2 differ by at most 1 in size. Replacing B_1 and B_2 in the covering by B'_1 and B'_2, and iterating this, we end up with a covering by k branchings, any two of which differ in size by at most 1. This is a covering as required. ∎

This theorem implies the integer decomposition property of the convex hull of branchings of size k (McDiarmid [1983]):

Corollary 53.3a. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of the branchings of size k has the integer decomposition property.*

Proof. Choose $p \in \mathbb{Z}_+$, and let x be an integer vector in $p \cdot \text{conv.hull}\{\chi^B \mid B$ branching, $|B| = k\}$. By Corollary 53.1j, x is a sum of the incidence vectors of p branchings. Let $D' = (V, A')$ be the digraph arising from D by replacing any arc a by x_a parallel arcs. Then A' can be partitioned into p branchings. Now $|A'|/p = x(A)/p = k$. So, by Theorem 53.3, we can take these branchings all of size k. Hence x is the sum of the incidence vectors of p branchings each of size k. ∎

53.7. Covering by r-arborescences

Vidyasankar [1978a] proved the following covering analogue of Edmonds' disjoint branchings theorem. (A weaker version was shown by Frank [1979a] (cf. Frank [1979b]).) For any digraph $D = (V, A)$ and $U \subseteq V$, let $H(U)$ denote the set of outneighbours of $V \setminus U$; that is, the set of the heads of the arcs entering U. So $H(U) \subseteq U$.

Theorem 53.4. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Then A can be covered by k r-arborescences if and only if*

(53.18) $\quad \deg^{\text{in}}(v) \leq k$ *for each* $v \in V$, *and* $\deg^{\text{in}}(r) = 0$,

and

(53.19)
$$\sum_{v \in H(U)} (k - \deg^{\text{in}}(v)) \geq k - d^{\text{in}}(U)$$

for each nonempty subset U of $V \setminus \{r\}$.

Proof. Necessity of (53.18) is trivial. To see necessity of (53.19), let U be a nonempty subset of $V \setminus \{r\}$. Then each r-arborescence B intersects the set

(53.20)
$$\bigcup_{v \in H(U)} \delta^{\text{in}}(v) \setminus \delta^{\text{in}}(U)$$

in at most $|H(U)| - 1$ arcs, since at least one arc of B should enter U. Hence if A can be covered by k r-arborescences, the size of set (53.20) is at most $k(|H(U)| - 1)$, implying (53.19).

To see sufficiency, we can assume that for any arc a of D, if we would add a parallel arc to a, then (53.18) or (53.19) is violated (since deleting parallel arcs does not increase the minimum number of r-arborescences needed to cover the arcs).

If $\deg^{\text{in}}(v) = k$ for each vertex $v \neq r$, then A can be decomposed into k r-arborescences by Edmonds' disjoint arborescences theorem (Corollary 53.1b), since then (53.19) implies that $d^{\text{in}}(U) \geq k$ for each nonempty subset U of $V \setminus \{r\}$.

So we can assume that there exists a vertex $u \neq r$ with $\deg^{\text{in}}(u) < k$. Consider the collection $\mathcal{C}$ of nonempty subsets U of $V \setminus \{r\}$ having equality in (53.19) and with $u \in H(U)$. Then $\mathcal{C}$ is closed under taking union and intersection. Indeed, let U and W be in $\mathcal{C}$. Then

(53.21)
$$\begin{aligned}
&\sum_{v \in H(U \cap W)} (k - \deg^{\text{in}}(v)) + \sum_{v \in H(U \cup W)} (k - \deg^{\text{in}}(v)) \\
&\leq \sum_{v \in H(U)} (k - \deg^{\text{in}}(v)) + \sum_{v \in H(W)} (k - \deg^{\text{in}}(v)) \\
&= (k - d^{\text{in}}(U)) + (k - d^{\text{in}}(W)) \\
&\leq (k - d^{\text{in}}(U \cap W)) + (k - d^{\text{in}}(U \cup W)).
\end{aligned}$$

The first inequality follows from

(53.22)
$$\begin{aligned}
&H(U \cap W) \cap H(U \cup W) \subseteq H(U) \cap H(W) \text{ and} \\
&H(U \cap W) \cup H(U \cup W) \subseteq H(U) \cup H(W),
\end{aligned}$$

as one easily checks.

By (53.19), (53.21) implies that we have equality throughout, As we have equality in the first inequality in (53.21), and as $k - \deg^{\text{in}}(u) > 0$, we know that $u \in H(U \cap W) \cap H(U \cup W)$. So $U \cap W$ and $U \cup W$ belong to $\mathcal{C}$.

Now for each arc a entering u, if we would add an arc parallel to a, (53.19) is violated for some U. This implies that for each arc a entering u there exists a $U \in \mathcal{C}$ such that the tail of a is in U. We can take for U the largest set in $\mathcal{C}$. Hence for each arc a entering u, the tail of a is in U. This contradicts the fact that $u \in H(U)$. ∎

Frank [1979b] showed the following consequence of this result:

Corollary 53.4a. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Then A can be covered by k r-arborescences if and only if*

(53.23) $\quad k \cdot s(A') \geq |A'|$

for each $A' \subseteq A$. Here $s(A')$ denotes the maximum of $|B \cap A'|$ over r-arborescences B.

Proof. As necessity is trivial, we show sufficiency, by showing that (53.23) implies (53.18) and (53.19). To see (53.18), apply (53.23) to $A' := \delta^{\text{in}}(v)$. To see (53.19), apply (53.23) to A' equal to the set (53.20). ∎

Note that for *acyclic* digraphs, the minimum number of r-arborescences needed to cover all arcs is easily characterized (Vidyasankar [1978a]):

Theorem 53.5. *Let $D = (V, A)$ be an acyclic digraph and let $r \in V$. Then A can be covered by k r-arborescences if and only if r is the only source of D and each indegree is at most k.*

Proof. Necessity being easy, we show sufficiency. Trivially, we can cover A by sets $B_1, \ldots, B_k$ such that each B_i enters each $v \neq r$ precisely once. As D is acyclic, each B_i is an r-arborescence. ∎

53.8. Minimum-length unions of k r-arborescences

Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Consider the following system in the variable $x \in \mathbb{R}^A$:

(53.24) $\quad$ (i) $x_a \geq 0$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$.

The following basic result of Frank [1979b] follows from Theorem 52.4.

Theorem 53.6. *System* (53.24) *is box-TDI.*

Proof. Directly from Theorem 52.4, since if a system $Ax \leq b$ is box-TDI, then for any $k \geq 0$, the system $Ax \leq k \cdot b$ is box-TDI. ∎

This theorem has several consequences. First consider the system

(53.25) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$.

The following (cf. Frank [1979b]) implies a min-max relation for the minimum length of the union of k disjoint r-arborescences in D:

Corollary 53.6a. *System* (53.25) *is TDI, and determines the convex hull of subsets of A containing k disjoint r-arborescences.*

Proof. Directly from Theorem 53.6 and Edmonds' disjoint arborescences theorem (Corollary 53.1b). ∎

Another consequence of Theorem 53.6 is as follows. Let $D = (V, A)$ and $D' = (V, A')$ be digraphs, let $r \in V$, and let $k \in \mathbb{Z}_+$. Consider the system in the variable $x \in \mathbb{R}^A$:

(53.26) (i) $x_a \geq 0$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}_A(U)) \geq k - d^{\text{in}}_{A'}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$.

Then:

Corollary 53.6b. *System* (53.26) *is box-TDI.*

Proof. Choose $d, c \in \mathbb{Z}^A_+$. We must show that the system

(53.27) (i) $d(a) \leq x_a \leq c(a)$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}_A(U)) \geq k - d^{\text{in}}_{A'}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$

is TDI. Let $D'' = (V, A'')$ be the digraph with $A'' := A \cup A'$ (taking arcs multiple if they occur both in A and A'). By Theorem 53.6, the following system in the variable $x \in \mathbb{R}^{A''}$ is TDI:

(53.28) (i) $d(a) \leq x_a \leq c(a)$ for each $a \in A$,
(ii) $1 \leq x_a \leq 1$ for each $a \in A'$,
(iii) $x(\delta^{\text{in}}_{A''}(U)) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$.

This implies the total dual integrality of (53.27) by Corollary 5.27a. ∎

Frank [1979a] derived the following 'rank' formula for coverings by k r-arborescences:

Corollary 53.6c. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $A' \subseteq A$. Then the maximum number of arcs in A' that can be covered by k r-arborescences is equal to the minimum value of*

(53.29) $$k(|V| - 1) + \sum_{i=1}^{t} (d^{\text{in}}_{A'}(V_i) - k),$$

where $V_1, \ldots, V_t$ is a laminar collection of nonempty subsets of $V \setminus \{r\}$ such that each arc in A enters at most one of these sets.

Proof. Let μ be the maximum number of arcs in A' that can be covered by k r-arborescences. Consider the system (in $x \in \mathbb{R}^A$)

(53.30) (i) $x_a \geq 0$ for $a \in A$,
(ii) $x(\delta^{\text{in}}_A(U)) \geq k - d^{\text{in}}_{A'}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$.

By Corollary 53.6b, this system is TDI. Let x be an integer vector attaining the minimum of $x(A)$ over (53.30). Then

(53.31) $\mu = k(|V| - 1) - x(A).$

Indeed, by (53.30) and by Edmonds' disjoint arborescences theorem, there exist k r-arborescences $B_1, \ldots, B_k$ with

(53.32) $x + \chi^{A'} \geq \chi^{B_1} + \cdots + \chi^{B_k}.$

Let A'' be the set of arcs in A' covered by no B_i. By the minimality of $x(A)$, we have that $x(A) + |A'| = k(|V| - 1) + |A''|$. As $\mu \geq |A'| - |A''|$ we have $\geq$ in (53.31). Since we can reverse this construction (starting from a set of k r-arborescences covering μ arcs in A', and making x), we have the equality in (53.31).

By the total dual integrality of (53.30), $x(A)$ is equal to the maximum value of

(53.33) $$\sum_{i=1}^{t}(k - d^{\text{in}}_{A'}(V_i)),$$

taken over nonempty subsets $V_1, \ldots, V_t$ of $V \setminus \{r\}$ such that each arc in A enters at most one of these sets. If, say, $V_1 \cap V_2 \neq \emptyset$ and $V_1 \not\subseteq V_2 \not\subseteq V_1$, we can replace V_1 and V_2 by $V_1 \cap V_2$ and $V_1 \cup V_2$ without violating these conditions. Such replacements terminate by Theorem 2.1. We end up with $V_1, \ldots, V_t$ laminar as required. Therefore, with (53.31) we have the corollary. ∎

Taking $A' = A$, we get (Frank [1979b]):

Corollary 53.6d. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then the maximum number of arcs that can be covered by k r-arborescences is equal to the minimum value of*

(53.34) $$k(|V| - 1) + \sum_{i=1}^{t}(d^{\text{in}}(V_i) - k),$$

where $V_1, \ldots, V_t$ form a laminar collection of nonempty subsets of $V \setminus \{r\}$ such that each arc enters at most one of these sets.

Proof. This is the case $A' = A$ in Corollary 53.6c. ∎

This directly implies a min-max characterization for the minimum number of r-arborescences needed to cover all arcs. However, Theorem 53.4 gives a stronger relation.

As for unions of k branchings, Frank [1979a] derived from Corollary 53.6c:

Corollary 53.6e. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. The maximum number of arcs of D that can be covered by k branchings is equal to the minimum value of*

$$(53.35) \qquad k(|V| - |\mathcal{P}|) + \sum_{U \in \mathcal{P}} d^{\text{in}}(U)$$

taken over all subpartitions $\mathcal{P}$ of V with nonempty classes.

Proof. Let D' be the digraph obtained from D by adding a new vertex r and arcs (r, v) for each $v \in V$. Then the maximum number of arcs of D that can be covered by k branchings in D is equal to the maximum number of arcs in A that can be covered by k r-arborescences in D'. So Corollary 53.6c gives a min-max relation for this.

The subsets V_i form a subpartition of V since if V_i and V_j would intersect in a vertex v say, then the arc (r, v) of D' enters two sets among the V_i, contradicting the condition. ∎

As for unions of k arborescences without prescribed root, Frank [1979a] derived:

Corollary 53.6f. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k arborescences if and only if*

$$(53.36) \qquad k(|V| - 1 + \lambda) \geq |A| + \sum_{i=1}^{t} (k - d^{\text{in}}(V_i))$$

for each laminar family $(V_1, \ldots, V_t)$ of nonempty sets such that no arc enters more than one of the V_i. Here λ denotes the maximum number of V_i's having nonempty intersection.

Proof. Necessity can be seen as follows. Let A be covered by arborescences $B_1, \ldots, B_k$. For each $v \in V$, let $r(v)$ be the number of B_i having v as root. So $r(V) = k$. For each $a \in A$, let $s(a)$ be the number of B_i containing a. So $s(a) \geq 1$ for each $a \in A$. Moreover, $s(\delta^{\text{in}}(V_i)) + r(V_i) \geq k$ for each i. Hence

$$\begin{aligned}(53.37) \qquad |A| + \sum_{i=1}^{t} (k - d^{\text{in}}(V_i)) &\leq |A| + \sum_{i=1}^{t} \big(r(V_i) + s(\delta^{\text{in}}(V_i)) - d^{\text{in}}(V_i)\big) \\ &\leq |A| + \sum_{a \in A} (s(a) - 1) + \sum_{i=1}^{t} r(V_i) = \sum_{a \in A} s(a) + \sum_{i=1}^{t} r(V_i) \\ &= k(|V| - 1) + \sum_{i=1}^{t} r(V_i) \leq k(|V| - 1) + k\lambda.\end{aligned}$$

The second inequality holds as each arc enters at most one of the V_i. For the last inequality, we use that $r(V) = k$ and that $V_1, \ldots, V_t$ can be partitioned into λ collections, each consisting of disjoint sets. (53.37) shows necessity.

To see sufficiency, extend D by a vertex r and by the arc set $A' := \{(r, v) \mid v \in V\}$, yielding the digraph $D' = (V \cup \{r\}, A \cup A')$. Consider the following constraints for $x \in \mathbb{R}^{A \cup A'}$:

$$\text{(53.38)} \qquad \begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for each } a \in A \cup A', \\ \text{(ii)} & x(\delta^{\text{in}}_{D'}(U)) \geq k - d^{\text{in}}_D(U) & \text{for each nonempty } U \subseteq V, \\ \text{(iii)} & x(\delta^{\text{in}}_{D'}(V)) = k. & \end{array}$$

Let x attain the minimum of $x(A)$ over (53.38). Since system (53.38) is TDI by Corollary 53.6b (with Theorem 5.25), we can assume that x is integer. We show

$$\text{(53.39)} \qquad x(A) = k(|V| - 1) - |A|.$$

First, $x(A) \geq k(|V| - 1) - |A|$, since

$$\text{(53.40)} \qquad \begin{aligned} & x(A) + |A| + k = x(A) + |A| + x(\delta^{\text{in}}_{A'}(V)) \\ & = \sum_{v \in V} \left(x(\delta^{\text{in}}_{A'}(v)) + x(\delta^{\text{in}}_{A}(v)) + d^{\text{in}}_A(v) \right) \geq k|V|. \end{aligned}$$

To see the reverse inequality, $x(A)$ is equal to the optimum value μ of the problem dual to the above minimization problem: maximize

$$\text{(53.41)} \qquad \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} z_U (k - d^{\text{in}}_A(U))$$

where $z \in \mathbb{R}^{\mathcal{P}(V) \setminus \{\emptyset\}}_+$ such that

$$\text{(53.42)} \qquad \sum_U z_U \chi^{\delta^{\text{in}}_{D'}(U)} \leq \chi^A.$$

So we should prove that $\mu \leq k(|V| - 1) - |A|$.

Now let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $z_U = 1$. We may assume that $\mathcal{U}$ is laminar. Let λ be the maximum number of $U \in \mathcal{U}$ containing any vertex. Then (53.42) implies that $\lambda \leq -z_V$ (since $\chi^A(a) = 0$ for each $a = (r, v)$). Hence

$$\text{(53.43)} \qquad \begin{aligned} & \mu = k \cdot z_V + \sum_{U \in \mathcal{U}} (k - d^{\text{in}}_A(U)) \leq -k\lambda + \sum_{U \in \mathcal{U}} (k - d^{\text{in}}_A(U)) \\ & \leq k(|V| - 1) - |A| \end{aligned}$$

by (53.36), and we have the required inequality. This proves (53.39).

Then the vector $y := x + \chi^A$ satisfies:

$$\text{(53.44)} \qquad \begin{aligned} & y(\delta^{\text{in}}_{D'}(U)) \geq k \text{ for each nonempty } U \subseteq V, \\ & y(\delta^{\text{in}}_{D'}(V)) = k, \\ & y(A \cup A') = x(A) + x(A') + |A| = k(|V| - 1) - |A| + k + |A| = k|V|. \end{aligned}$$

So by Edmonds' disjoint arborescences theorem (Corollary 53.1b), y is the sum of the incidence vectors of k r-arborescences, each with exactly one arc leaving r. Hence (by the definition of y) A can be covered by k arborescences. ∎

53.9. The complexity of finding disjoint arborescences

By Edmonds' disjoint arborescences theorem, the maximum number of disjoint r-arborescences can be calculated in polynomial time, just by determining the minimum size of an r-cut. This can be done by determining, for each $v \in V \setminus \{r\}$, the minimum size of an $r - v$ cut, and taking the minimum of these values.

Lovász [1976c] and Tarjan [1974a] showed that actually also a maximum collection of disjoint r-arborescences can be found in polynomial time.

The proof (due to Lovász [1976c]) of Theorem 53.1 described above gives such a polynomial-time algorithm. In fact, Lovász observed that it implies the following result (obtained also by Tarjan [1974a]). Call a subset B of the arc set A of a digraph $D = (V, A)$ a *partial r-arborescence* if B is an r-arborescence for the subgraph of D induced by the set $V(B)$ of vertices covered by B. We take $V(B) := \{r\}$ if B is empty.

Theorem 53.7. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, a maximum number k of disjoint r-arborescences can be found in time $O(k^2m^2)$.*

Proof. First, the number k can be determined in time $O(knm)$. Since k is equal to the minimum size of a cut $d^{\text{in}}(U)$ over nonempty subsets U of $V \setminus \{r\}$, we can determine for each $v \in V \setminus \{r\}$ a maximum set of arc-disjoint $r - v$ paths, by the augmenting path method described in Section 9.2. Actually, for $i = 1, \ldots, k$, we determine the ith augmenting paths for all $v \in V \setminus \{r\}$, before searching for the $(i + 1)$th augmenting paths. In this way we can stop if for some $v \in V \setminus \{r\}$ no augmenting path exists. So in total we do at most $(n - 1)(k + 1)$ augmenting path searches. Thus it takes $O(knm)$ time to determine k.

Next, we can find an r-arborescence B such that

(53.45) $\quad d^{\text{in}}_{A \setminus B}(U) \geq k - 1$ for each nonempty $U \subseteq V \setminus \{r\}$,

in time $O(km^2)$. This recursively implies the theorem.

To find B, as in the proof of Theorem 53.1, we can grow a partial r-arborescence B satisfying (53.45), starting with $B = \emptyset$. By the proof of Theorem 53.1, if $V(B) \neq V$, there exists an arc a leaving $V(B)$ such that resetting $B := B \cup \{a\}$ maintains (53.45). For any given arc a leaving $V(B)$ it amounts to testing if there exists a set $U \subseteq V \setminus \{r\}$ such that $a \in \delta^{\text{in}}(U)$ and $d^{\text{in}}_{A \setminus B}(U) = k - 1$. This can be done in $O(km)$ time with a minimum cut algorithm.

Now it is important to observe that for each arc a we need to do this test at most once: if the test result is negative, then in growing B we never have to consider arc a anymore; if the result is positive, a is added to B, and again we will not consider a again.

So to obtain an r-arborescence, we determine at most m minimum cuts, and so finding the r-arborescence B takes $O(km^2)$ time. ∎

Tong and Lawler [1983] observed that the following quite easily follows from Edmonds' disjoint arborescences theorem:

Theorem 53.8. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, we can find in time $O(knm)$ a set of arcs that is the union of a maximum number k of disjoint r-arborescences.*

Proof. As in the proof of Theorem 53.7 we can determine the number k in time $O(knm)$. Now consider any vertex $v \in V$. Find k arc-disjoint $r-v$ paths in D, and delete from D each arc entering v that is on none of these paths. After that we still have $d^{\text{in}}(U) \geq k$ for any nonempty $U \subseteq V \setminus \{r\}$, since if $v \notin U$, then no arc entering U has been deleted, and if $v \in U$, then k arcs entering U are maintained, as after deletion there are still k arc-disjoint $r-v$ paths in D.

Doing this successively for all vertices $v \in V$, we are left with a digraph D with $\deg^{\text{in}}(v) = k$ if $v \neq r$ and $\deg^{\text{in}}(r) = 0$, and with $d^{\text{in}}(U) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$. So the remaining arc set is the union of k disjoint r-arborescences. As k arc-disjoint $r-v$-paths can be found in time $O(km)$, we have the required result. ∎

This implies with Theorem 53.7 a sharpening of Theorem 53.7:

Corollary 53.8a. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, a maximum number k of disjoint r-arborescences can be found in time $O(knm + k^4n^2)$.*

Proof. By Theorem 53.8, we can find in time $O(knm)$ a set A' that is the union of k disjoint r-arborescences. So $m' := |A'| = k(n-1)$. Then by Theorem 53.7 we can find k disjoint r-arborescences in A', in time $O(k^2{m'}^2)$. Since $O(k^2{m'}^2) = O(k^4n^2)$, the corollary follows. ∎

Tong and Lawler [1983] in fact showed that the method of Lovász [1976c] has an $O(k^2nm)$-time implementation, yielding with Theorem 53.8 an $O(knm + k^3n^2)$-time algorithm for finding k disjoint r-arborescences.

Also the capacitated case can be solved in strongly polynomial time (Gabow [1991a,1995a]), as can be shown with the help of Edmonds' disjoint branchings theorem. (Pevzner [1979a] proved that it can be solved in *semi-strongly polynomial time*, that is, by taking rounding as one arithmetic step.)

Theorem 53.9. *Given a digraph $D = (V, A)$, $r \in V$, and a capacity function $c : A \to \mathbb{Z}_+$, we can find r-arborescences $B_1, \ldots, B_k$ and integers $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^{k} \lambda_i \chi^{B_i} \leq c$ and with $\sum_{i=1}^{k} \lambda_i$ maximized, in strongly polynomial time.*

Proof. We can find the maximum value in strongly polynomial time, as it is equal to the minimum capacity of an r-cut. To find the λ_i explicitly, we show more generally that the following problem is solvable in strongly polynomial time (where $R(B)$ denotes the set of roots of B):

(53.46) given: a digraph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_+$, a collection $\mathcal{R}$ of nonempty subsets of V, and a demand function $d : \mathcal{R} \to \mathbb{Z}_+$,
find: a collection $\mathcal{B}$ of branchings and a function $\lambda : \mathcal{B} \to \mathbb{Z}_+$, with $\sum_{B \in \mathcal{B}} \lambda_B \chi^B \leq c$ and $\sum(\lambda_B \mid B \in \mathcal{B}, R(B) = R) = d(R)$ for each $R \in \mathcal{R}$.

For any $U \subseteq V$, define

(53.47) $$g(U) := \sum(d(R) \mid R \in \mathcal{R}, R \cap U = \emptyset).$$

By replacing each arc a by $c(a)$ parallel arcs, it follows from Edmonds' disjoint branchings theorem (Theorem 53.1) that a necessary and sufficient condition for the existence of a solution of (53.46) is that

(53.48) $$c(\delta^{\text{in}}(U)) \geq g(U)$$

for each nonempty $U \subseteq V$.

We can assume that $c(a) > 0$ for each $a \in A$ and $d(R) > 0$ for each $R \in \mathcal{R}$, and that we have an $R_1 \in \mathcal{R}$ with $R_1 \neq V$.

We may also assume that (53.46) has a solution. This implies that there exists an arc $a = (u, v) \in A$ leaving R_1 and a $\mu \geq 1$ such that resetting $d(R_1) := d(R_1) - \mu$, $d(R_1 \cup \{v\}) := d(R_1 \cup \{v\}) + \mu$, $c(a) := c(a) - \mu$, maintains feasibility of (53.46). (If $R_1 \cup \{v\}$ did not belong to $\mathcal{R}$, we add it to $\mathcal{R}$.) We apply this for the maximum possible μ. This value of μ can be calculated in strongly polynomial time, as it satisfies

(53.49) $$\mu = \min\{c(a), \min\{c(\delta^{\text{in}}(W)) - g(W) \mid a \in \delta^{\text{in}}(W), W \cap R_1 \neq \emptyset\}\}$$

(for the original c and g).

To minimize $c(\delta^{\text{in}}(W)) - g(W)$ over W with $a \in \delta^{\text{in}}(W)$ and $W \cap R_1 \neq \emptyset$, add, for each $R \in \mathcal{R}$, a new vertex v_R and, for each $v \in R$, a new arc (v_R, v) of capacity $d(R)$. Moreover, add a new vertex r, and for each $R \in \mathcal{R}$, a new arc (r, v_R) of capacity $d(R)$. Let $D' = (V', A')$ be the extended digraph. With a minimum cut algorithm we can find a subset W' of $V' \setminus \{r\}$ with $a \in \delta^{\text{in}}(W)$

and $W \cap R_1 \neq \emptyset$, minimizing the capacity of $\delta^{\mathrm{in}}_{A'}(W)$. Then $W := W' \cap V$ is a set as required.

We next apply the algorithm recursively. This describes the algorithm.

Running time. In each iteration, the number of arcs a with $c(a) > 0$ decreases or the collection $\mathcal{C} := \{U \subseteq V \mid U \neq \emptyset, c(\delta^{\mathrm{in}}(U)) = g(U)\}$ increases. As $\mathcal{C}$ is an intersecting family, the number of times $\mathcal{C}$ increases is at most $|V|^3$ (since for each $v \in V$, the collection $\mathcal{C}_v := \{U \in \mathcal{C} \mid v \in U\}$ is a lattice family, and since each lattice family $\mathcal{L}$ is determined by the preorder $\preceq$ given by: $s \preceq t \iff$ each set in $\mathcal{L}$ containing t contains s; if $\mathcal{L}$ increases, then $\preceq$ decreases, which can happen at most $|V|^2$ times.)

So the number of iterations is at most $|A| + |V|^3$. ∎

With the reductions given earlier, this implies that the capacitated versions of packing arborescences and covering by branchings also can be solved in strongly polynomial time.

Edmonds [1975] observed that matroid intersection and union theory implies:

Theorem 53.10. *Given a digraph $D = (V, A)$, $r \in V$, $k \in \mathbb{Z}_+$, and a length function $l \in \mathbb{Q}^A$, we can find k disjoint r-arborescences $B_1, \ldots, B_k$ minimizing $l(B_1) + \cdots + l(B_k)$ in strongly polynomial time.*

Proof. This follows, with Corollary 53.1c and Theorem 53.7, from Theorem 41.8 applied to the intersection of two matroids: one being the union of k times the cycle matroid of the undirected graph underlying D; the other being the matroid in which a subset B of A is independent if and only if any $v \in V \setminus \{r\}$ is entered by at most k arcs in B. ∎

This implies:

Corollary 53.10a. *Given a digraph $D = (V, A)$, $r \in V$, $k \in \mathbb{Z}_+$, and a length function $l \in \mathbb{Q}^A$, we can find a minimum-length subset B of A with $\delta^{\mathrm{in}}_B(U) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$ in strongly polynomial time.*

Proof. Directly from Theorem 53.10, with Edmonds' disjoint arborescences theorem (Corollary 53.1b). ∎

53.10. Further results and notes

53.10a. Complexity survey for disjoint arborescences

Finding k disjoint r-arborescences in an uncapacitated digraph ($*$ indicates an asymptotically best bound in the table):

	$O(k^2m^2)$	Lovász [1976c], Tarjan [1974a]
	$O(knm + k^3n^2)$	Tong and Lawler [1983]
*	$O(k^2n^2 + m)$	Gabow [1991a,1995a]

(As noticed by Tong and Lawler [1983], the paper of Shiloach [1979a] claiming an $O(k^2nm)$ bound, contains an essential error (the set A constructed on page 25 of Shiloach [1979a] need not have the desired properties: it is maximal under the condition that $y \notin A$, while it should be maximal under the condition that $A \cup V(T) \neq V$).)

The $O(k^2m^2)$ bound for finding k pairwise disjoint r-arborescences implies the $O(n^2\Delta^4 \log \Delta)$ bound of Markosyan and Gasparyan [1986] for finding a minimum number of branchings covering all arcs (where Δ is the maximum indegree of the vertices), by the construction given in the proof of Corollary 53.1h (as we can take $m \leq n\Delta$ and $k \leq 2\Delta$).

Tarjan [1974c] gave an $O(m + n \log n)$-time algorithm to find two disjoint r-arborescences (actually, to find two r-arborescences with smallest intersection). This was improved to $O(m\alpha(m,n))$ by Tarjan [1976] (where $\alpha(m,n)$ is the inverse Ackermann function), and to $O(m)$ by Gabow and Tarjan [1985].

Clearly, each of the bounds in the table above implies a complexity bound for the capacitated case, by replacing arcs by multiple arcs. However, this can increase the number m of arcs dramatically, and does not lead to a polynomial-time algorithm. Better bounds are given in the following table:

	$O(n^3 \cdot \mathrm{MF}(n,m))$	Pevzner [1979a] *taking rounding as one arithmetic step*
*	$O(k^2n^2 + m)$	Gabow [1991a,1995a]
*	$O(n^3m \log \frac{n^2}{m})$	Gabow and Manu [1995,1998]
*	$O(n^2m \log C \log \frac{n^2}{m})$	Gabow and Manu [1995,1998]

In these bounds, m is the number of arcs in the original graph, $\mathrm{MF}(n,m)$ denotes the time needed to solve a maximum flow problem in a digraph with n vertices and m arcs, and C is the maximum capacity (for integer capacity function).

The bounds of Gabow [1991a,1995a] and Gabow and Manu [1995,1998] in these tables also apply to the problem considered in Edmonds' disjoint branching theorem (Theorem 53.1): finding k disjoint branchings $B_1, \ldots, B_k$ where B_i has a given root set R_i ($i = 1, \ldots, k$), finding minimum coverings by branchings, and related problems. Gabow and Manu [1995,1998] also gave an $O(n^3m \log \frac{n^2}{m})$ fractional packing algorithm of r-arborescences.

Gabow [1991a,1995a] announced $O(kn(m+n\log n)\log n)$- and $O(k\sqrt{n \log n}(m+kn\log n)\log(nK))$-time algorithms to find a minimum-cost union of k disjoint r-arborescences (where K is the maximum cost, with integer cost function).

53.10b. Arborescences with roots in given subsets

Let $D = (V, A)$ be a digraph. Call a vector $x \in \mathbb{Z}_+^V$ a *root vector* if there exist disjoint arborescences such that for each $v \in V$, exactly x_v of these arborescences have root v. By Corollary 53.1a, root vectors are the integer solutions of the following system:

$$\text{(53.50)} \qquad \begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(U) \leq d^{\text{out}}(U) & \text{for each } U \subset V. \end{array}$$

This system generally does not define an integer polytope P, as is shown by the digraph with vertices u, v, w and arcs (u, v), (v, w), and (w, u), where $\frac{1}{2} \cdot \mathbf{1}$ is in P, but each integer vector x in P satisfies $\mathbf{1}^{\mathsf{T}} x \leq 1$.

Moreover, sets R of vertices for which there exist $|R|$ disjoint arborescences, rooted at distinct vertices in R, do not form the independent sets of a matroid, as is shown by the graph of Figure 53.1.

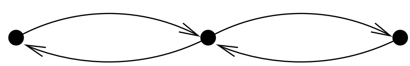

Figure 53.1

However, for any $k \in \mathbb{Z}_+$, the system

$$\text{(53.51)} \qquad x(U) \geq k - d^{\text{in}}(U) \text{ for each nonempty } U \subseteq V,$$

is box-TDI, since the right-hand side in (ii) is intersecting supermodular (cf. Sections 44.5 and 48.1).

Cai [1983] proved the following result, with a method (described below) of Frank [1981c] for proving a special case (Corollary 53.11a):

Theorem 53.11. *Let $D = (V, A)$ be a digraph such that D has k arc-disjoint arborescences. Let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then D has k arc-disjoint arborescences such that, for each $v \in V$, at least $l(v)$ and most $u(v)$ of these arborescences are rooted at v if and only if*

$$\text{(53.52)} \qquad u(U) + d^{\text{in}}(U) \geq k \text{ and } l(U) + \sum_{W \in \mathcal{P}} (k - d^{\text{in}}(W)) \leq k$$

for each nonempty subset U of V and each partition $\mathcal{P}$ of $V \setminus U$ into nonempty sets.

Proof. Necessity being easy, we show sufficiency. Choose $x \in \mathbb{Z}_+^V$ such that $l \leq x \leq u$ and such that (53.51) holds, with $x(V)$ as small as possible. (Such an x exists since $u(U) \geq k - d^{\text{in}}(U)$ for each nonempty subset U of V.)

We show that $x(V) = k$. Since $x(V) \geq k$ by (53.51), it suffices to show $x(V) \leq k$. Let $\mathcal{P}$ be the collection of inclusionwise maximal sets having equality in (53.51). Then $\mathcal{P}$ is a subpartition, for suppose that $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$\text{(53.53)} \qquad \begin{aligned} x(U \cup W) &= x(U) + x(W) - x(U \cap W) \\ &\leq (k - d^{\text{in}}(U)) + (k - d^{\text{in}}(W)) - (k - d^{\text{in}}(U \cap W)) \\ &\leq (k - d^{\text{in}}(U \cup W)), \end{aligned}$$

and hence $U \cup W \in \mathcal{P}$. So $U = W$.

Now for each $v \in V$ with $x_v > l(v)$ there exists a set W in $\mathcal{P}$ containing v, since otherwise we could decrease x_v. Hence

$$(53.54) \qquad x(V) - l(V) = \sum_{W \in \mathcal{P}} (x(W) - l(W)) = \sum_{W \in \mathcal{P}} (k - d^{\text{in}}(W) - l(W)) \\ \leq k - l(V),$$

by (53.52).

So $x(V) = k$. Now let $r_1, \ldots, r_k$ be vertices such that each vertex v occurs x_v times among the r_i. Then by Corollary 53.1a there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i. This shows the theorem. ∎

(In this proof we did not use the box-total dual integrality of (53.51), but we applied a similar argument.)

This has as special case the following result of Frank [1981c]:

Corollary 53.11a. *Let $D = (V, A)$ be a digraph such that D has k arc-disjoint arborescences. Let $u \in \mathbb{Z}_+^V$. Then D has k arc-disjoint arborescences such that, for each $v \in V$, at most $u(v)$ of these arborescences have their root in v if and only if*

$$(53.55) \qquad u(U) + d^{\text{in}}(U) \geq k$$

for each nonempty subset U of V.

Proof. Directly from Theorem 53.11. ∎

A related theorem is:

Theorem 53.12. *Let $D = (V, A)$ be a digraph and let $R_1, \ldots, R_k$ be subsets of V. Then there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has its root in R_i (for $i = 1, \ldots, k$) if and only if*

$$(53.56) \qquad \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq |\{i \mid R_i \cap \bigcup \mathcal{P} \neq \emptyset\}|$$

for each subpartition $\mathcal{P}$ of V with nonempty classes.

Proof. Necessity is easy, since if the B_i exist, with roots $r_i \in R_i$, then for each $U \in \mathcal{P}$ one has that $r_i \in U$ or B_i contains at least one arc entering U. That is,

$$(53.57) \qquad |\{r_i\} \cap U| + d^{\text{in}}_{B_i}(U) \geq 1.$$

Summing this inequality over $U \in \mathcal{P}$ and over $i = 1, \ldots, k$ we obtain (53.56), with R_i replaced by $\{r_i\}$. This implies (53.56) for the original R_i.

To see sufficiency, first observe that the condition implies that the R_i are nonempty (by taking $\mathcal{P} := \{V\}$). If the R_i are singletons, the theorem is equivalent to Corollary 53.1a. So we can assume that $|R_1| \geq 2$. Choose distinct vertices $u, w \in R_1$.

If the condition is maintained after replacing R_1 by $R_1 \setminus \{u\}$, the theorem follows by induction. So we can assume that this violates the condition. That is, there exists a subpartition $\mathcal{P}$ of V into nonempty classes such that (setting $X := \bigcup \mathcal{P}$):

(53.58) $$\sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) = |\{i \mid R_i \cap X \neq \emptyset\}|$$

and such that $X \cap R_1 = \{u\}$ (for the original R_1). Similarly we can assume that there exists a subpartition $\mathcal{Q}$ of V into nonempty classes such that (setting $Y := \bigcup \mathcal{Q}$):

(53.59) $$\sum_{U \in \mathcal{Q}} (k - d^{\text{in}}(U)) = |\{i \mid R_i \cap Y \neq \emptyset\}|$$

and such that $Y \cap R_1 = \{w\}$.

Let $\mathcal{F}$ be the union of $\mathcal{P}$ and $\mathcal{Q}$ (any set occurring both in $\mathcal{P}$ and in $\mathcal{Q}$ occurs twice in $\mathcal{F}$). Now iteratively replace any $T, U \in \mathcal{F}$ with $T \cap U \neq \emptyset$ and $T \not\subseteq U \not\subseteq T$ by $T \cap U$ and $T \cup U$. Then the final family $\mathcal{F}$ is laminar. Let $\mathcal{R}$ be the collection of inclusionwise minimal sets in $\mathcal{F}$ and let $\mathcal{S}$ be the collection of inclusionwise maximal sets in $\mathcal{F}$. Then $\mathcal{R}$ and $\mathcal{S}$ are subpartitions of V into nonempty classes, and $\bigcup \mathcal{R} = X \cap Y$ and $\bigcup \mathcal{S} = X \cup Y$. Moreover

(53.60) $$\begin{aligned} &\sum_{U \in \mathcal{R}} (k - d^{\text{in}}(U)) + \sum_{U \in \mathcal{S}} (k - d^{\text{in}}(U)) = \sum_{U \in \mathcal{F}} (k - d^{\text{in}}(U)) \\ &\geq \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) + \sum_{U \in \mathcal{Q}} (k - d^{\text{in}}(U)) \\ &= |\{i \mid R_i \cap X \neq \emptyset\}| + |\{i \mid R_i \cap Y \neq \emptyset\}| \\ &> |\{i \mid R_i \cap (X \cap Y) \neq \emptyset\}| + |\{i \mid R_i \cap (X \cup Y) \neq \emptyset\}|. \end{aligned}$$

The first inequality follows from the submodularity of $d^{\text{in}}(U)$. The last inequality holds as (i) if R_i intersects $X \cup Y$, then it intersects X or Y, (ii) if R_i intersects $X \cap Y$, then it intersects X and Y, and (iii) R_1 intersects X and Y but not $X \cap Y$, since $R_1 \cap X = \{u\}$ and $R_1 \cap Y = \{w\}$.

However, (53.60) contradicts (53.56). ∎

53.10c. Disclaimers

The equivalence of (i) and (iii) in Corollary 53.1c suggests the following question, raised by A. Frank (cf. Schrijver [1979b], Frank [1995]; it generalizes a similar question for the undirected case, described in Section 51.5b):

(53.61) (?) Let $D = (V, A)$ be a k-arc-connected digraph and let $r \in V$. Suppose that for each $s \in V$ there exist k internally vertex-disjoint $r - s$ paths in D. Then there exist k r-arborescences such that, for any vertex s, the k $r - s$ paths determined by the respective r-arborescences are internally vertex-disjoint. (?)

For $k = 2$ this was proved by Whitty [1987]. However, for $k = 3$, a counterexample was found by Huck [1995].

Two potential generalizations of Edmonds' disjoint arborescences theorem have been raised, neither of which holds however. For vertices s, t, let $\lambda(s, t)$ denote the maximum number of arc-disjoint $s - t$ paths. It is not true that for any digraph $D = (V, A)$, $r \in V$, and $T \subseteq V \setminus \{r\}$, there exist k disjoint subsets $A_1, \ldots, A_k$ of A such that each A_i contains an $r - t$ path for each $t \in T$ if and only if $\lambda(r, t) \geq k$ for each $t \in T$ (see Figure 53.2, for $k = 2$).

N. Robertson raised the question if it is true that in any digraph $D = (V, A)$ and any $r \in V$, there exist partial r-arborescences $B_1, B_2, \ldots$ such that each vertex $v \in V \setminus \{r\}$ is in exactly $\lambda(r, v)$ of them. Lovász [1973b] showed that Figure 53.3 is a counterexample. (Related work is reported by Bang-Jensen, Frank, and Jackson [1995] and Gabow [1996].)

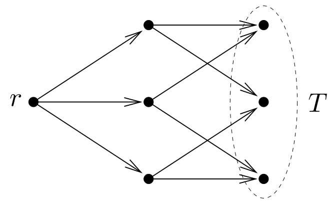

Figure 53.2

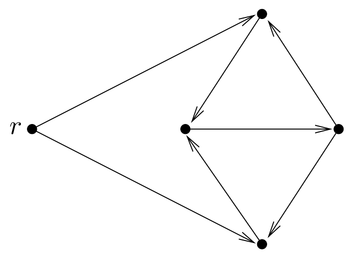

Figure 53.3

53.10d. Further notes

Frank [1981c] gave the following results for mixed graphs. Let $G = (V, E, A)$ be a mixed graph (that is, (V, E) is an undirected graph and (V, A) is a directed graph). A *mixed branching* is a subset B of $E \cup A$ such that the undirected edges in B can be oriented such that B becomes a branching. Then E can be covered by k mixed branchings if and only if

(53.62) (i) $d_A^{\text{in}}(U) + |E[U]| \leq k|U|$ for each $U \subseteq V$,
(ii) $|A[U]| + |E[U]| \leq k(|U| - 1)$ for each $\emptyset \neq U \subseteq V$.

Similarly, a *mixed r-arborescence* is a subset B of $E \cup A$ such that the undirected edges in B can be oriented such that B becomes an r-arborescence. Then for any $r \in V$, G has k disjoint mixed r-arborescences if and only if for each subpartition $\mathcal{P}$ of $V \setminus \{r\}$ with nonempty classes, the number of edges (directed or not) entering any class of $\mathcal{P}$, is at least $k|\mathcal{P}|$.

Cai [1989] characterized when, for given digraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$, $a, b \in \mathbb{Z}_+^V$, and $k \in \mathbb{Z}_+$, there exists an $r \in \mathbb{Z}_+^V$ with $a \leq r \leq b$ and there exist, for $i = 1, 2$, k disjoint arborescences in D_i such that for each $v \in V$, $r(v)$ of these arborescences have root v. This can be proved using polymatroid intersection theory, in particular the box-total dual integrality of

(53.63) $x(U) \geq k - d_{A_i}^{\text{in}}(U)$ for $i = 1, 2$ and nonempty $U \subseteq V$

(Theorem 48.5). (For a generalization, see Cai [1990a,1993].)

Cai [1990b] showed, for given digraph $D = (V, A)$, $r \in V$, $d, c \in \mathbb{Z}_+^A$, and $k \in \mathbb{Z}_+$: there exist k r-arborescences such that each arc is covered at least $d(a)$ times and at most $c(a)$ times, if and only if $d(\delta^{\text{in}}(v)) \leq k$, for each $v \in V \setminus \{r\}$, and

$$\sum_{v \in U} \min\{k - d(\delta^{\text{in}}(v) \setminus \delta^{\text{in}}(U)), c(\delta^{\text{in}}(v) \cap \delta^{\text{in}}(U))\} \geq k \tag{53.64}$$

for each nonempty subset U of $V \setminus \{r\}$.

Chapter 54

Biconnectors and bibranchings

The concept of biconnector is a generalization of that of a connector. Let $G = (V, E)$ be an undirected graph and let V be partitioned into classes R and S. An $R - S$ *biconnector* is a subset F of E such that each component of (V, F) intersects both R and S. So contracting R or S gives a connector. If R is a singleton, $R - S$ biconnectors are precisely the connectors.

For biconnectors, min-max relations, polyhedral characterizations, and complexity results similar to those for connectors hold.

In this chapter we also consider the forest analogue of biconnector, the biforest. An $R - S$ *biforest* is a forest F such that each component of (V, F) has at most one edge in the cut $\delta(R)$. So contracting R or S gives a forest. Also biforests show good polyhedral and algorithmical behaviour.

Similar results hold for the directed analogues of biconnectors and biforests, the bibranchings and the bifurcations. An $R - S$ *bibranching* is a set B of arcs such that for each $s \in S$, B contains an $R - s$ path and for each $r \in R$, B contains an $r - S$ path. Bibranchings form a generalization of arborescences, and give rise to similar min-max relations and polyhedral characterizations. An $R - S$ *bifurcation* is a set B of arcs containing no undirected circuit, such that each vertex in R is left by at most one arc in B, each vertex in S is entered by at most one arc in B, and B contains no arcs from S to R.

Theorem 54.11 on disjoint bibranchings will be the only result of this chapter that will be used later in this book, namely in Chapter 56 to obtain a dual form of the Lucchesi-Younger theorem, on packing directed cut covers in a source-sink connected digraph. The proof of Theorem 54.11 uses no other results from this chapter.

54.1. Shortest $R - S$ biconnectors

Let $G = (V, E)$ be a graph and let V be partitioned into two sets R and S. A subset F of E is called an $R - S$ *biconnector* if each component of the graph (V, F) intersects both R and S. So F is an $R - S$ biconnector if and only if each component of (V, F) has at least one edge in $\delta(R)$.

A min-max relation for the minimum size of an $R - S$ biconnector can be derived easily from the Kőnig-Rado edge cover theorem:

Theorem 54.1. *Let $G=(V,E)$ be a graph and let V be partitioned into sets R and S such that each component of G intersects both R and S. Then the minimum size of an $R-S$ biconnector is equal to the maximum size of a subset of V spanning no edge connecting R and S.*

Proof. To see that the minimum is not less than the maximum, let F be a minimum $R-S$ biconnector and let U attain the maximum. Then F is a forest. For each $r \in U \cap R$, let $\phi(r)$ be the first edge in any $r-S$ path in F; and for each $s \in U \cap S$ let $\phi(s)$ be the first edge in any $s-R$ path in F. Then ϕ is injective from U to F (as U spans no edge in $\delta(R)$). Hence $|U| \leq |F|$.

To see equality, let $H := (N(R) \cup N(S), \delta(R))$, where $N(R)$ and $N(S)$ are the sets of neighbours of R and of S respectively. (So $N(R) \subseteq S$ and $N(S) \subseteq R$, and H is bipartite.) Let U' be a maximum-size stable set in H. Let F' be a minimum-size edge cover in H. By the Kőnig-Rado edge cover theorem (Theorem 19.4) we know $|F'| = |U'|$. Let $U := U' \cup (V \setminus (N(R) \cup N(S)))$. Then U spans no edge connecting R and S. By adding $|V \setminus (N(R) \cup N(S))|$ edges to F' we obtain an $R-S$ biconnector F with

(54.1) $$|F| = |U'| + |V \setminus (N(R) \cup N(S))| = |U|.$$

This shows the required equality. ∎

To obtain a min-max relation for the minimum length of an $R-S$ biconnector (given a length function on the edges), consider the system

(54.2)
(i)	$0 \leq x_e \leq 1$	for each $e \in E$,
(ii)	$x(\delta(\mathcal{P})) \geq \lvert\mathcal{P}\rvert$	for each subpartition $\mathcal{P}$ of R or S with nonempty classes.

Here a *subpartition* of a set X is a partition of a subset of X (that is, a collection of disjoint subsets of X). $\delta(\mathcal{P})$ denotes the set of edges incident with but not spanned by any set in $\mathcal{P}$. Then system (54.2) determines the *$R-S$ biconnector polytope* — the convex hull of the incidence vectors of $R-S$ biconnectors:

Theorem 54.2. *System* (54.2) *is box-totally dual integral and determines the $R-S$ biconnector polytope.*

Proof. This follows from matroid intersection theory, applied to the matroids M_1 and M_2 on E, where M_1 is obtained from the cycle matroid $M(G)$ of G by contracting R to one vertex, making all edges spanned by R to a loop, and where M_2 is obtained similarly from $M(G)$ by contracting S to one vertex, making all edges spanned by S to a loop.

So the spanning sets of M_1 are the subsets F of E such that each component of (V,F) intersects R. Similarly, the spanning sets of M_2 are the subsets F of E such that each component of (V,F) intersects S. Hence the common spanning sets are precisely the $R-S$ biconnectors. Therefore, by Corollaries

41.12f and 50.8a, system (54.2) determines the convex hull of the incidence vectors of $R - S$ biconnectors. To see that the system is box-TDI, we use Corollary 41.12g and the fact that for each $F \subseteq E$, the inequality

(54.3) $$x(F) \geq r_{M_i}(E) - r_{M_i}(E \setminus F)$$

is a nonnegative integer combination of the inequalities (54.2). Indeed (for $i = 1$), if $\mathcal{P}$ denotes the collection of the components of $(V, E \setminus F)$ contained in S, then

(54.4) $$x(F) \geq x(\delta(\mathcal{P})) \geq |\mathcal{P}| \geq r_{M_1}(E) - r_{M_1}(E \setminus F),$$

as $r_{M_1}(E) \leq |S|$ and $r_{M_1}(E \setminus F) = |S| - |\mathcal{P}|$. ∎

This implies a min-max relation for the minimum length of an $R - S$ biconnector. The reduction to matroid intersection also immediately implies that one can find a shortest $R - S$ biconnector in strongly polynomial time.

54.2. Longest $R - S$ biforests

Again, let $G = (V, E)$ be a graph and let V be partitioned into two sets R and S. Call a subset F of E an $R - S$ *biforest* if F is a forest and each component of F contains at most one edge in $\delta(R)$.

A min-max relation for the maximum size of an $R - S$ biforest can be derived easily from Kőnig's matching theorem:

Theorem 54.3. *Let $G = (V, E)$ be a graph and let V be partitioned into sets R and S. Then the maximum size of an $R - S$ biforest is equal to the minimum value of $|V| - |\mathcal{U}|$, where $\mathcal{U}$ is a collection of components of $G - R$ and $G - S$ such that no edge connects any two sets in $\mathcal{U}$.*

Proof. We may assume that G has no loops. To see that the maximum is not more than the minimum, consider any $R - S$ biforest F and any collection $\mathcal{U}$ as in the theorem. Then F contains no path connecting two distinct sets in $\mathcal{U}$. Hence $|F| \leq |V| - |\mathcal{U}|$.

The reverse inequality is proved by induction on the number of edges not in $\delta(R)$.

If $E = \delta(R)$, then G is bipartite, and $R - S$ biforests coincide with matchings. Then the theorem is equivalent to Kőnig's matching theorem (Theorem 16.2).

If $E \neq \delta(R)$, choose an edge $e = uv$ in $E \setminus \delta(R)$. If we contract e, the minimum value in the theorem reduces by precisely 1. Moreover, the maximum reduces by at least 1, since any $R - S$ forest in the contracted graph gives with e an $R - S$ forest in the original graph. So we are done by induction. ∎

To obtain a min-max relation for the maximum length of an $R - S$ biforest (given a length function on the edges), consider the following system:

(54.5) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(E[U]) \leq |U| - 1$ for each nonempty subset U of R or S,
(iii) $x(E[U] \cup (\delta(U) \cap \delta(R))) \leq |U|$
for each subset U of R or S.

This determines the $R-S$ *biforest polytope* — the convex hull of the incidence vectors of $R-S$ biforests:

Theorem 54.4. *System* (54.5) *is box-totally dual integral and determines the* $R-S$ *biforest polytope.*

Proof. This can be reduced to matroid intersection theory, similar to the proof of Theorem 54.3. ∎

Again, this theorem implies a min-max relation for the maximum length of an $R-S$ biforest, and the reduction to matroid intersection also implies that a longest $R-S$ biforest can be found in strongly polynomial time.

54.3. Disjoint $R-S$ biconnectors

We give a min-max relation for the maximum number of disjoint $R-S$ biconnectors (Keijsper and Schrijver [1998]). It generalizes the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a) — which theorem however is used in the proof — and the disjoint edge covers theorem for bipartite graphs (Theorem 20.5).

We follow the (algorithmic) proof method of Keijsper [1998a], based on the following lemma:

Lemma 54.5α. *Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be edge-disjoint spanning trees and let $r \in V$. For each $e = rv \in \delta_{T_1}(r)$, let $\phi(e)$ be the first edge of the $v-r$ path in T_2 that leaves the component of $T_1 - e$ containing v. Let $B \subseteq \delta_{T_1}(r)$ be such that $\phi(B)$ contains at most one edge not in $\delta_{T_2}(r)$. Then $(E_1 \setminus B) \cup \phi(B)$ and $(E_2 \setminus \phi(B)) \cup B$ are spanning trees again.*

Proof. By induction on $|B|$, the case $|B| \leq 1$ being easy. Let $|B| \geq 2$. Then there exists an edge $f = rw \in B$ with $\phi(f) \in \delta_{T_2}(r)$ (by the condition given in the theorem). Define

(54.6) $T_1' := (T_1 \setminus \{f\}) \cup \{\phi(f)\}$ and $T_2' := (T_2 \setminus \{\phi(f)\}) \cup \{f\}$.

Let $B' := B \setminus \{f\}$. Then for each $e = rv \in B'$,

(54.7) $\phi(e)$ is equal to the first edge of the $v-r$ path in T_2' that leaves the component K of $T_1' - e$ containing v.

To see this, let L be the component of $T_1 - f$ containing w. Since $\phi(f)$ connects L and r, K is equal to the component of $T_1 - e$ containing v. Moreover, the

$v - r$ path P in T'_2 does not differ from the $v - r$ path in T_2 before entering L, and hence the first edge of P leaving K equals $\phi(e)$. This shows (54.7).

Now $(T_1 \setminus B) \cup \phi(B) = (T'_1 \setminus B') \cup \phi(B')$ and $(T_2 \setminus \phi(B)) \cup B = (T'_2 \setminus \phi(B')) \cup B'$, and by induction, they are spanning trees. ∎

Notice that the function $\phi : E_1 \cap \delta(r) \to E_2$ defined in the lemma is injective.

In the following lemma, we consider forests as edge sets. We recall that G/R denotes the graph obtained from G by contracting all vertices in R to one new vertex, denoted by R. The edges in the contracted graph are named after the edges in the original graph.

Lemma 54.5β. *Let $G = (V, E)$ be a graph and let V be partitioned into sets R and S. Let X_1 and X_2 be disjoint forests in G/R and let Y_1 and Y_2 be disjoint forests in G/S. Then there exist disjoint forests X'_1 and X'_2 in G/R and disjoint forests Y'_1 and Y'_2 in G/S with $X'_1 \cup X'_2 = X_1 \cup X_2$, $Y'_1 \cup Y'_2 = Y_1 \cup Y_2$, $X'_1 \cap Y'_2 = \emptyset$, and $X'_2 \cap Y'_1 = \emptyset$.*

Proof. By adding new edges spanned by S, we can assume that the X_i are spanning trees in G/R. Similarly, we can assume that the Y_i are spanning trees in G/S. (At the conclusion, we delete the new edges from the X'_i and Y'_i.)

If $X_1 \cap Y_2 = \emptyset$ and $X_2 \cap Y_1 = \emptyset$, we are done. So, by symmetry, we can assume that $X_1 \cap Y_2 \neq \emptyset$.

For each $e = rs \in X_1 \cap \delta(R)$, with $r \in R, s \in S$, let $\phi(e)$ be the first edge on the $s - R$ path in X_2 that leaves the component of $X_1 - e$ containing s. For each $e = rs \in Y_2 \cap \delta(R)$, with $r \in R, s \in S$, let $\psi(e)$ be the first edge on the $r - S$ path in Y_1 that leaves the component of $Y_2 - e$ containing r.

This gives injective functions

(54.8) $$\phi : X_1 \cap \delta(R) \to X_2 \text{ and } \psi : Y_2 \cap \delta(R) \to Y_1.$$

Observe that $X_1 \cap (Y_1 \cup Y_2) \subseteq \delta(R)$ and $Y_2 \cap (X_1 \cup X_2) \subseteq \delta(R)$. Consider the directed graph with vertex set E and arc set

(54.9) $$A := \{(e, \phi(e)) \mid e \in X_1 \cap \delta(R)\} \cup \{(e, \psi(e)) \mid e \in Y_2 \cap \delta(R)\}.$$

Choose $e_0 \in X_1 \cap Y_2$ and set $e_1 := \phi(e_0)$. Then D contains a unique directed path $e_0, e_1, \ldots, e_h$ such that $e_0, \ldots, e_{h-1} \in X_1 \cup Y_2$ and $e_h \notin X_1 \cup Y_2$. (This because each vertex in $X_1 \cap Y_2$ has outdegree 2 and indegree 0 in D, and each vertex in $(X_1 \cup Y_2) \setminus (X_1 \cap Y_2)$ has outdegree 1 and indegree at most 1.)

It follows that for each $j < h$ one has $e_{j+1} = \phi(e_j)$ if j is even and $e_{j+1} = \psi(e_j)$ if j is odd. Define

(54.10) $$B := \{e_j \mid 0 \leq j < h, j \text{ even}\} \text{ and } C := \{e_j \mid 1 \leq j < h, j \text{ odd}\}.$$

Then by Lemma 54.5α,

(54.11) $X_1' := (X_1 \setminus B) \cup \phi(B)$, $X_2' := (X_2 \setminus \phi(B)) \cup B$,
$Y_1' := (Y_1 \setminus \psi(C)) \cup C$, $Y_2' := (Y_2 \setminus C) \cup \psi(C)$,

are again spanning tree of G/R and G/S respectively. Note that $X_1' \cap X_2' = \emptyset$, $Y_1' \cap Y_2' = \emptyset$, $X_1' \cup X_2' = X_1 \cup X_2$ and $Y_1' \cup Y_2' = Y_1 \cup Y_2$.

Now $\phi(B) \cap \psi(C) = \emptyset$, $\phi(B) \cap (Y_2 \setminus C) = \emptyset$ (since $\phi(B) \cap Y_2 \subseteq C$, as $e_h \notin Y_2$), and $\psi(C) \cap (X_1 \setminus B) = \emptyset$ (since $\psi(C) \cap X_1 \subseteq B$, as $e_h \notin X_1$). So $X_1' \cap Y_2' \subseteq (X_1 \cap Y_2) \setminus \{e_0\}$ (since $e_0 \notin X_1'$).

Moreover, $B \cap C = \emptyset$, $B \cap (Y_1 \setminus \psi(C)) = \emptyset$ (since $B \cap Y_1 \subseteq \psi(C)$, as $e_0 \notin Y_1$), and $C \cap (X_2 \setminus \phi(B)) = \emptyset$ (since $C \cap X_2 \subseteq \phi(B)$, as $e_0 \notin X_2$). So $X_2' \cap Y_1' \subseteq X_2 \cap Y_1$.

Concluding, $|X_1' \cap Y_2'| + |X_2' \cap Y_1'| < |X_1 \cap Y_2| + |X_2 \cap Y_1|$. Therefore, iterating this, we obtain trees as required. ∎

Now a min-max relation for disjoint $R - S$ biconnectors can be deduced:

Theorem 54.5. *Let $G = (V, E)$ be a graph, let V be partitioned into sets R and S, and let $k \in \mathbb{Z}_+$. Then there exist k disjoint $R - S$ biconnectors if and only if $|\delta(\mathcal{P})| \geq k|\mathcal{P}|$ for each subpartition $\mathcal{P}$ of R or S with nonempty classes.*

Proof. Necessity being easy, we show sufficiency. By Corollary 51.1a, the graph G/R (obtained from G by contracting R) has k disjoint spanning trees $X_1, \ldots, X_k$. Similarly, the graph G/S has k disjoint spanning trees $Y_1, \ldots, Y_k$. Then $X_i \cap Y_j$ is a subset of $\delta(R)$, for all i, j. Choose the X_i and Y_i in such a way that

$$(54.12) \qquad \sum_{i=1}^{k} |X_i \cap Y_i|$$

is as large as possible.

Then $X_i \cap Y_j = \emptyset$ for all distinct i, j, for if, say, $X_1 \cap Y_2 \neq \emptyset$, we can replace X_1, X_2, Y_1, Y_2 by X_1', X_2', Y_1', Y_2' as in Lemma 54.5β. Then we have

$$(54.13) \qquad \begin{aligned} &|X_1' \cap Y_1'| + |X_2' \cap Y_2'| = |(X_1' \cup X_2') \cap (Y_1' \cup Y_2')| \\ &= |(X_1 \cup X_2) \cap (Y_1 \cup Y_2)| > |X_1 \cap Y_1| + |X_2 \cap Y_2|. \end{aligned}$$

This contradicts the maximality of sum (54.12).

Hence $X_1 \cup Y_1, \ldots, X_k \cup Y_k$ form k disjoint $R - S$ biconnectors as required. ∎

This proof gives a polynomial-time algorithm to find a maximum number of disjoint $R - S$ biconnectors. Keijsper [1998a] gave an $O(\mathrm{DT}(n, m) + nm)$-time algorithm for this problem, where $\mathrm{DT}(n, m)$ denotes the time needed to find a maximum number of disjoint spanning trees in an undirected graph with n vertices and m edges.

By replacing edges by parallel edges, one obtains a capacitated version of Theorem 54.5. The corresponding optimization problem can be solved in polynomial time, by a straightforward adaptation of the methods described in the proofs of Theorems 51.8 and 51.10 and Corollary 51.8a. However, the capacitated problem cannot be solved in strongly polynomial time if we do not allow rounding (cf. the argument given in Section 51.4).

A generalization of Theorem 54.5 is given by Keijsper [1998a].

54.4. Covering by $R-S$ biforests

With the foregoing two lemmas, one can also derive a min-max relation for the minimum number of $R-S$ biforests that cover all edges (Keijsper [1998b]). It generalizes the Nash-Williams' covering forests theorem (Corollary 51.1c) — which theorem however is used in the proof — and Kőnig's edge-colouring theorem for bipartite graphs (Theorem 20.1).

Theorem 54.6. *Let $G=(V,E)$ be a graph, let V be partitioned into sets R and S, and let $k \in \mathbb{Z}_+$. Then E can be covered by k $R-S$ biforests if and only if*

(54.14) $\quad |E[U]| \leq k(|U|-1)$ *and* $|E[U]| + |\delta(U) \cap \delta(R)| \leq k|U|$

for each nonempty subset U of R or S.

Proof. Necessity being easy, we show sufficiency. We can assume that G is connected, as otherwise we can consider any component of G separately.

By Corollary 51.1c, the edges of the graph G/R can be partitioned into k forests $X_1, \ldots, X_k$. Similarly, the edges of the graph G/S can be partitioned into k forests $Y_1, \ldots, Y_k$. So $X_i \cap Y_j \subseteq \delta(R)$, for all i, j. Choose the X_i and Y_i in such a way that sum (54.12) is as large as possible. Then, as in the proof of Theorem 54.5, $X_i \cap Y_j = \emptyset$ for distinct i, j. Hence each $e \in \delta(R)$ belongs to $X_i \cap Y_i$ for some $i = 1, \ldots, k$. Concluding, $X_1 \cup Y_1, \ldots, X_k \cup Y_k$ form $R-S$ biforests as required. ∎

This proof gives a polynomial-time algorithm for finding a minimum covering by $R-S$ biforests. The methods of Section 51.4 can be extended to imply the polynomial-time solvability of the corresponding capacitated version, while strong polynomial-time solvability is again impossible.

54.5. Minimum-size bibranchings

We now turn to the directed analogues of biconnectors and biforests. Let $D=(V,A)$ be a digraph and let V be partitioned into two sets R and S. Call a subset B of A an $R-S$ *bibranching* if in the graph (V,B), each vertex in S is reachable from R, and each vertex in R reaches S.

Similarly to minimum $R - S$ biconnectors, a min-max relation for the minimum size of an $R - S$ bibranching follows easily from the Kőnig-Rado edge cover theorem.

Theorem 54.7. *Let $D = (V, A)$ be a graph and let V be partitioned into sets R and S such that each vertex in R can reach S and such that each vertex in S is reachable from R. Then the minimum size of an $R - S$ bibranching is equal to the maximum size of a subset of V spanning no arc in $\delta^{\text{out}}(R)$.*

Proof. To see that the minimum is not less than the maximum, let B be a minimum-size $R - S$ bibranching and let U attain the maximum. For each $r \in U \cap R$, let $\phi(r)$ be any arc in B leaving r, and for each $s \in U \cap S$ let $\phi(s)$ be any arc in B entering s. Then ϕ is injective from U to B, and hence $|U| \leq |B|$.

To see equality, let U' be a maximum stable set in the bipartite graph H with colour classes $N^{\text{in}}(S) \subseteq R$ and $N^{\text{out}}(R) \subseteq S$, with $r \in N^{\text{in}}(S)$ and $s \in N^{\text{out}}(R)$ adjacent if and only if D has an arc from r to s. (Here $N^{\text{out}}(X)$ and $N^{\text{in}}(X)$ are the sets of outneighbours and of inneighbours of X, respectively.)

Let B' be a minimum-size edge cover in H. By the Kőnig-Rado edge cover theorem (Theorem 19.4) we know $|B'| = |U'|$. Now by adding $|V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S))|$ arcs to B' we obtain an $R - S$ bibranching B with

(54.15) $$|B| = |U'| + |V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S))| = |U|,$$

where $U := U' \cup (V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S)))$. This shows the required equality. ∎

If each arc of D belongs to $\delta^{\text{out}}(R)$, then Theorem 54.7 reduces to the Kőnig-Rado edge cover theorem (Theorem 19.4).

The proof gives a polynomial-time algorithm to find a minimum-size $R-S$ bibranching (as we can find a minimum-size edge cover in a bipartite graph in polynomial time (Corollary 19.3a)).

54.6. Shortest bibranchings

To obtain a min-max relation for the minimum length of an $R-S$ bibranching (given a length function on the arcs), define a set of arcs C to be an $R - S$ *bicut* if $C = \delta^{\text{in}}(U)$ for some nonempty proper subset U of V satisfying $U \subseteq S$ or $S \subseteq U$.

Consider the system:

(54.16)
(i) $x_a \geq 0$ for each $a \in A$,
(ii) $x(C) \geq 1$ for each $R - S$ bicut C.

Then the following implies a min-max relation for the minimum length of an $R-S$ bibranching.

Theorem 54.8. *System* (54.16) *is box-TDI.*

Proof. Let $w : A \to \mathbb{R}_+$. Let $\mathcal{U}$ be the collection of nonempty proper subsets U of V satisfying $U \subseteq S$ or $S \subseteq U$. Consider the maximum value of

(54.17) $$\sum_{U \in \mathcal{U}} y_U$$

where $y : \mathcal{U} \to \mathbb{R}_+$ satisfies

(54.18) $$\sum_{U \in \mathcal{U}} y_U \chi^{\delta^{\text{in}}(U)} \leq w.$$

Choose $y : \mathcal{U} \to \mathbb{R}_+$ attaining the maximum, such that

(54.19) $$\sum_{U \in \mathcal{U}} y_U |U||V \setminus U|$$

is minimized. We show that the collection $\mathcal{F} := \{U \in \mathcal{U} \mid y_U > 0\}$ is cross-free; that is, for all $T, U \in \mathcal{F}$ one has

(54.20) $$T \subseteq U \text{ or } U \subseteq T \text{ or } T \cap U = \emptyset \text{ or } T \cup U = V.$$

Suppose that this is not true. Let $\alpha := \min\{y_T, y_U\}$. Decrease y_T and y_U by α, and increase $y_{T\cap U}$ and $y_{T\cup U}$ by α. Now (54.18) is maintained, and (54.17) did not change. However, (54.19) decreases (by Theorem 2.1), contradicting our minimality assumption.

So $\mathcal{F}$ is cross-free. Now the $\mathcal{F} \times A$ matrix M with

(54.21) $$M_{U,a} := \begin{cases} 1 & \text{if } a \in \delta^{\text{in}}(U), \\ 0 & \text{otherwise,} \end{cases}$$

is totally unimodular. To see this, let $T = (W, B)$ and $\pi : V \to W$ form a tree-representation of $\mathcal{F}$ (see Section 13.4). That is, T is a directed tree and $\mathcal{F} = \{V_b \mid b \in B\}$, where

(54.22) $V_b := \{v \in V \mid \pi(v)$ belongs to the same component of $T - b$ as the head of $b\}$.

Then for any arc $a = (u, v)$ of D, the set of forward arcs in the undirected $\pi(u) - \pi(v)$ path in T is contiguous, that is, forms a directed path, say from u' to v'. This follows from the fact that there exist no arcs b, c, d in this order on the path with b and d forward and c backward.

Define $a' := (u', v')$, and let $D' = (W, A')$ be the digraph with $A' := \{a' \mid a \in A\}$. Then M is equal to the network matrix generated by T and D' (identifying $b \in B$ with the set V_b in $\mathcal{F}$ determined by b). Hence by Theorem 13.20, M is totally unimodular.

This implies with Theorem 5.35 that (54.16) is box-TDI. ∎

This implies that the $R-S$ *bibranching polytope* — the convex hull of the incidence vectors of $R-S$ bibranchings — can be described as follows:

Corollary 54.8a. *The $R-S$ bibranching polytope is determined by*

$$\text{(54.23)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_a \leq 1 & \text{for each } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } R-S \text{ bicut } C. \end{array}$$

Proof. By Theorem 54.8, (54.23) determines an integer polytope. Necessarily, each vertex of it is the incidence vector of an $R-S$ branching. ∎

The box-total dual integrality of (54.16) has as special case the total dual integrality of (54.23), which is equivalent to:

Corollary 54.8b (optimum bibranching theorem)**.** *Let $D = (V, A)$ be a digraph, let V be partitioned into sets R and S, and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of an $R-S$ bibranching is equal to the maximum size of a family of $R-S$ dicuts, such that each arc a is in at most $l(a)$ of them.*

Proof. This is a reformulation of the total dual integrality of (54.23), which follows from Theorem 54.8. ∎

We also note that Theorem 54.8 implies that for each $k \in \mathbb{Z}_+$ the system

$$\text{(54.24)} \qquad \begin{array}{lll} \text{(i)} & x_a \geq 0 & \text{for each } a \in A, \\ \text{(ii)} & x(C) \geq k & \text{for each } R-S \text{ bicut } C, \end{array}$$

is box-TDI (since if $Ax \leq b$ is box-TDI, then for each $k > 0$, $Ax \leq k \cdot b$ is box-TDI).

Keijsper and Pendavingh [1998] gave an $O(n'(m + n \log n))$ algorithm to find a shortest bibranching, where $n' := \min\{|R|, |S|\}$. The strong polynomial-time solvability follows also from the strong polynomial-time solvability of finding a minimum-length strong connector for a source-sink connected digraph, which by the method of Theorem 57.3 can be reduced to finding a minimum-length directed cut cover, which is a special case of weighted matroid intersection (Section 55.5).

54.6a. Longest bifurcations

Let $D = (V, A)$ be a digraph and let V be partitioned into two sets R and S. Call a subset B of A an $R-S$ *bifurcation* if B contains no undirected circuits, each vertex in R is left by at most one arc in B, each vertex in S is entered by at most one arc in B, and B contains no arcs from S to R. So B is an $R-S$ bifurcation if and only if contracting R gives a branching and contracting S gives a cobranching. (A *cobranching* is a set B of arcs whose reversal B^{-1} is a branching.)

Similarly to maximum $R-S$ biforests, a min-max relation for the maximum size of an $R-S$ bifurcation follows from Kőnig's matching theorem:

Theorem 54.9. *Let $D = (V, A)$ be a graph and let V be partitioned into sets R and S, with $\delta^{\text{in}}(R) = \emptyset$. Then the maximum size of an $R - S$ bifurcation is equal to the minimum size of $|V| - |\mathcal{L}|$, where $\mathcal{L}$ is a collection of strong components K of D with either $K \subseteq R$ and $\delta^{\text{out}}(K) \subseteq \delta^{\text{out}}(R)$, or $K \subseteq S$ and $\delta^{\text{in}}(K) \subseteq \delta^{\text{in}}(S)$, such that no arc connects two components in $\mathcal{L}$.*

Proof. To see that the minimum is not less than the maximum, let B be a maximum-size $R - S$ bifurcation and let $\mathcal{L}$ attain the minimum. Let U be the set of vertices v with $v \in R$ and $\delta^{\text{out}}_B(v) = \emptyset$, or $v \in S$ and $\delta^{\text{in}}_B(v) = \emptyset$. Then

(54.25) $$|B| = |V| - |U| - |B \cap \delta^{\text{out}}(R)| \leq |V| - |\mathcal{L}|,$$

since each $K \in \mathcal{L}$ contains a vertex in U or is entered or left by an arc in $B \cap \delta^{\text{out}}(R)$.

To see equality, consider the following bipartite graph H. H has vertex set the set $\mathcal{K}$ of strong components K of D with either $K \subseteq R$ and $\delta^{\text{out}}(K) \subseteq \delta^{\text{out}}(R)$, or $K \subseteq S$ and $\delta^{\text{in}}(K) \subseteq \delta^{\text{in}}(S)$. Two sets $K, L \in \mathcal{K}$ are adjacent if and only if there is an arc connecting K and L. (This implies that one of K, L is contained in R, the other in S.) Let $\mathcal{L}$ be a maximum-size stable set in H and let B' be a maximum-size matching in H. By Kőnig's matching theorem (Theorem 16.2), $|B'| + |\mathcal{L}| = |\mathcal{K}|$. Now by adding $|V| - |\mathcal{K}|$ arcs to the arc set in D corresponding to B', we can obtain an $R - S$ bifurcation of size $|B'| + |V| - |\mathcal{K}| = |V| - |\mathcal{L}|$. ∎

If each arc of D belongs to $\delta^{\text{out}}(R)$, then Theorem 54.9 reduces to Kőnig's matching theorem (Theorem 16.2).

We next give a min-max relation for the maximum length of an $R - S$ bifurcation, by reduction to Theorem 54.8 on minimum-length bibranching:

Theorem 54.10. *Let $D = (V, A)$ be a digraph and let V be partitioned into R and S such that there are no arcs from S to R. Let $l \in \mathbb{Z}^A_+$ be a length function. Then the maximum length of an $R - S$ bifurcation is equal to the minimum value of*

(54.26) $$\sum_{v \in V} y_v + \sum_{U \in \mathcal{U}} z_U(|U| - 1)$$

where $y \in \mathbb{Z}^V_+$ and $z \in \mathbb{Z}^{\mathcal{U}}_+$, with $\mathcal{U} := \{U \mid U \neq \emptyset, U \subseteq R \text{ or } U \subseteq S\}$, such that

(54.27) $$\sum_{v \in R} y_v \chi^{\delta^{\text{out}}(v)} + \sum_{v \in S} y_v \chi^{\delta^{\text{in}}(v)} + \sum_{U \in \mathcal{U}} z_U \chi^{A[U]} \geq l.$$

Proof. To see that the maximum is not more than the minimum, let B be any $R - S$ bifurcation and let y_v, z_U satisfy (54.27). Then

(54.28) $$\begin{aligned} l(B) &= \sum_{a \in B} l(a) \leq \sum_{a \in B} \Big(\sum_{\substack{v \in R \\ a \in \delta^{\text{out}}(v)}} y_v + \sum_{\substack{v \in S \\ a \in \delta^{\text{in}}(v)}} y_v + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U \Big) \\ &= \sum_{v \in R} y_v |B \cap \delta^{\text{out}}(v)| + \sum_{v \in S} y_v |B \cap \delta^{\text{in}}(v)| + \sum_{U \in \mathcal{U}} z_U |B \cap A[U]| \\ &\leq \sum_{v \in V} y_v + \sum_{U \in \mathcal{U}} z_U(|U| - 1). \end{aligned}$$

To see equality, extend D by two new vertices, r and s, and by arcs (r, v) for each $v \in S \cup \{s\}$ and (v, s) for each $v \in R$. This makes the digraph $D' = (V', A')$. Define $R' := R \cup \{r\}$ and $S' := S \cup \{s\}$. Let $L := \max\{l(a) \mid a \in A\} + 1$. Define $l' \in \mathbb{Z}^{A'}_+$ by:

$$l'(a) := \begin{cases} L - l(a) & \text{for each } a \in A[R] \cup A[S], \\ 2L - l(a) & \text{for each } a \in \delta^{\text{out}}(R), \\ L & \text{for each } a = (r,v) \text{ with } v \in S \text{ and } a = (v,s) \\ & \text{with } v \in R, \\ 0 & \text{for } a = (r,s). \end{cases} \tag{54.29}$$

Let $\mathcal{U}'$ be the collection of nonempty subsets U of R' or S'. By Theorem 54.8, applied to D', there exists an $R'-S'$ bibranching B' in D' and a $z : \mathcal{U}' \to \mathbb{Z}_+$ such that

$$l'(B') = \sum_{U \in \mathcal{U}'} z_U, \tag{54.30}$$

and

$$\sum_{\substack{U \in \mathcal{U}' \\ U \subseteq R'}} z_U \chi^{\delta^{\text{out}}(U)} + \sum_{\substack{U \in \mathcal{U}' \\ U \subseteq S'}} z_U \chi^{\delta^{\text{in}}(U)} \leq l'. \tag{54.31}$$

Since $l'(r,s) = 0$ we know that $z_U = 0$ if r or s belongs to U. That is, $z_U = 0$ if $U \in \mathcal{U}' \setminus \mathcal{U}$.

For each $v \in V$, define

$$y_v := L - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U. \tag{54.32}$$

Then $y_v \geq 0$ for each $v \in V$, as

$$y_v = L - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U \geq L - l'(r,v) = 0 \tag{54.33}$$

if $v \in S$, and similarly $y_v \geq 0$ if $v \in R$.

Also, y and z satisfy (54.27), since for any arc $a = (u,v)$ one has, if $u, v \in R$:

$$\begin{aligned} y_u + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U &= L - \sum_{\substack{U \in \mathcal{U} \\ u \in U}} z_U + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U = L - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{out}}(U)}} z_U \\ &\geq L - l'(a) = l(a). \end{aligned} \tag{54.34}$$

Similarly, if $u, v \in S$, then

$$y_v + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U \geq l(a). \tag{54.35}$$

Finally, if $u \in R$ and $v \in S$, then:

$$\begin{aligned} y_u + y_v &= 2L - \sum_{\substack{U \in \mathcal{U} \\ u \in U}} z_U - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U \\ &= 2L - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{out}}(U)}} z_U - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{in}}(U)}} z_U \geq 2L - l'(a) = l(a). \end{aligned} \tag{54.36}$$

So y and z satisfy (54.27).

Note that each $u \in R$ is left by a unique arc in B', since if there is more than one, all arcs leaving u should have their heads in S' (since if (u, v), $(u', v') \in B'$, $v \neq v'$, $v \notin S'$, then $B' \setminus \{(u, v)\}$ is again an $R' - S'$ bibranching). Then replacing one outgoing arc $(u, v) \in B'$ by the arc (r, v) keeps B' an $R - S$ bibranching, however of smaller length. This contradicts our assumption. So each vertex in R is left by exactly one arc in B', and similarly, each vertex in S is entered by exactly one arc in B'. This implies that $B := B' \cap A$ is an $R - S$ bifurcation.

We finally show that equality holds throughout in (54.28). Indeed, if $a \in B$, then $a \in B'$, and hence we have equality in (54.34), implying that the first inequality in (54.28) is satisfied with equality. Moreover, if $y_v > 0$ and $v \in S$, then we have strict inequality in (54.33), and hence $(r, v) \notin B'$. Therefore $|B \cap \delta^{\text{in}}(v)| = 1$. Similarly, $y_v > 0$ and $v \in R$ implies $|B \cap \delta^{\text{out}}(v)| = 1$. Finally, if $z_U > 0$ and (say) $U \subseteq R$, then $|B' \cap \delta^{\text{out}}(U)| = 1$, and hence $|B' \cap A[U]| = |U| - 1$ (since each $v \in R$ is left by precisely one arc in B'), implying $|B \cap A[U]| = |U| - 1$. This shows that also the second inequality in (54.28) is satisfied with equality. ∎

Theorem 54.10 is equivalent to the total dual integrality of the following system:

(54.37)

(i)	$x_a \geq 0$	for each $a \in A$,
(ii)	$x(\delta^{\text{out}}(v)) \leq 1$	for each $v \in R$,
(iii)	$x(\delta^{\text{in}}(v)) \leq 1$	for each $v \in S$,
(iv)	$x(A[U]) \leq \|U\| - 1$	for each nonempty U with $U \subseteq R$ or $U \subseteq S$.

It yields a description of the $R - S$ *bifurcation polytope* — the convex hull of the incidence vectors of the $R - S$ bifurcations in D.

Corollary 54.10a. *System* (54.37) *is TDI and determines the $R - S$ bifurcation polytope.*

Proof. This is equivalent to Theorem 54.10. ∎

As for the *complexity*, the reduction given in Theorem 54.10 also implies that a maximum-length $R - S$ bifurcation can be found in strongly polynomial time (since a minimum-length $R - S$ bibranching can be found in strongly polynomial time).

54.7. Disjoint bibranchings

Consider the system

(54.38)

(i)	$0 \leq x_a \leq 1$	for each $a \in A$,
(ii)	$x(B) \geq 1$	for each $R - S$ bibranching B.

By the theory of blocking polyhedra, Corollary 54.8a implies:

Corollary 54.10b. *System* (54.38) *determines the convex hull of the incidence vectors of arc sets containing an $R - S$ bicut.*

Proof. Directly from Corollary 54.8a with the theory of blocking polyhedra. ∎

System (54.38) in fact is TDI, which is equivalent to the following statement:

Theorem 54.11 (disjoint bibranchings theorem). *Let $D = (V, A)$ be a digraph and let V be partitioned into sets R and S. Then the maximum number of disjoint $R-S$ bibranchings is equal to the minimum size of an $R-S$ bicut.*

Proof. Let k be the minimum size of an $R - S$ bicut. Clearly, there are at most k disjoint $R - S$ bibranchings. We show equality. For any digraph $D = (V, A)$ and $r \in V$, call a subset B of A an *r-coarborescence* if the set B^{-1} of reverse arcs of B is an r-arborescence.

By Edmonds' disjoint arborescences theorem (Corollary 53.1b), the graph D/R (obtained from D by contracting R to one vertex) has k disjoint R-arborescences $B_1, \ldots, B_k$. Similarly, the graph D/S has k disjoint S-coarborescences $B'_1, \ldots, B'_k$. Choose the B_i and B'_i such that the sum

(54.39) $$\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} |B_i \cap B'_j|$$

is as small as possible. If the sum is 0, then

(54.40) $$B_1 \cup B'_1, \ldots, B_k \cup B'_k$$

are k disjoint $R - S$ bibranchings in D as required. So we can assume that the sum is positive. Without loss of generality, $B_1 \cap B'_2 \neq \emptyset$.

Define

(54.41) $$X := (B_1 \cup B_2) \cap A[S],\ X' := (B'_1 \cup B'_2) \cap A[R],$$
$$Y := (B_1 \cup B_2) \cap \delta^{\text{out}}(R),\ Y' := (B'_1 \cup B'_2) \cap \delta^{\text{out}}(R).$$

Let $\mathcal{K}$ be the collection of strong components K of the digraph (S, X) with $\delta^{\text{in}}_X(K) = \emptyset$. Similarly, let $\mathcal{K}'$ be the collection of strong components K of the digraph (R, X') with $\delta^{\text{out}}_{X'}(K) = \emptyset$.

Now $d^{\text{in}}_Y(K) = d^{\text{in}}_{B_1 \cup B_2}(K) \geq 2$ for each $K \in \mathcal{K}$, and similarly $d^{\text{out}}_{Y'}(K) \geq 2$ for each $K \in \mathcal{K}'$. Then we can split Y into Y_1 and Y_2 and Y' into Y'_1 and Y'_2 such that

(54.42) $$d^{\text{in}}_{Y_i}(K) \geq 1 \text{ for each } K \in \mathcal{K} \text{ and } i = 1, 2,$$
$$d^{\text{out}}_{Y'_i}(K) \geq 1 \text{ for each } K \in \mathcal{K}' \text{ and } i = 1, 2,$$
$$\text{and } Y_1 \cap Y'_2 = \emptyset \text{ and } Y_2 \cap Y'_1 = \emptyset.$$

This can be seen as follows. Select for each $U \in \mathcal{K}$ a pair e_U from $\delta^{\text{in}}_Y(U)$. Similarly, select for each $U \in \mathcal{K}'$ a pair e_U from $\delta^{\text{out}}_{Y'}(U)$. So the e_U for $U \in \mathcal{K}$ are disjoint, and the e_U for $U \in \mathcal{K}'$ are disjoint. Hence the e_U for $U \in \mathcal{K} \cup \mathcal{K}'$ form a bipartite graph on $Y \cup Y'$ (in fact, a set of vertex-disjoint paths and even circuits). The two colour classes of this bipartite graph give the partitions of Y and Y' as required.

Then by Lemma 53.2α, X can be split into two branchings X_1 and X_2 such that the set of roots of X_i is equal to the set of heads of Y_i ($i = 1, 2$). Similarly, X' can be split into two cobranchings X'_1 and X'_2 such that the set of coroots of X'_i is equal to the set of tails of Y'_i ($i = 1, 2$). (A *cobranching* is a set B of arcs whose reversal B^{-1} is a branching. A *coroot* of B is a root of B^{-1}.)

Define

(54.43) $\qquad \widetilde{B}_i := X_i \cup Y_i$ and $\widetilde{B}'_i := X'_i \cup Y'_i$

for $i = 1, 2$. Since $\widetilde{B}_1 \cap \widetilde{B}'_2 = \emptyset$ and $\widetilde{B}_2 \cap \widetilde{B}'_1 = \emptyset$, replacing B_1, B_2, B'_1, B'_2 by $\widetilde{B}_1, \widetilde{B}_2, \widetilde{B}'_1, \widetilde{B}'_2$ decreases sum (54.39), contradicting the minimality assumption. ∎

The capacitated case can be derived as a consequence:

Corollary 54.11a. *Let $D = (V, A)$ be a digraph, let V be partitioned into sets R and S, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the maximum number of $R - S$ bibranchings such that no arc a is in more than $c(a)$ of these bibranchings is equal to the minimum capacity of an $R - S$ bicut.*

Proof. This follows from Theorem 54.11 by replacing any arc a by $c(a)$ parallel arcs. ∎

Equivalently, in TDI terms:

Corollary 54.11b. *System* (54.38) *is totally dual integral.*

Proof. This is a reformulation of Corollary 54.11a. ∎

Another consequence is:

(54.44) For any digraph $D = (V, A)$ and any partition of V into R and S, the $R - S$ bibranching polytope has the integer decomposition property.

As for the *complexity*, the proof of Theorem 54.11 gives a polynomial-time algorithm for finding a maximum number of disjoint $R - S$ bibranchings. For the capacitated case there is a semi-strongly polynomial-time algorithm (that is, where rounding takes one arithmetic step): first find a fractional dual solution, then round (Grötschel, Lovász, and Schrijver [1988]). A combinatorial semi-strongly polynomial-time algorithm follows from the results in Section 57.5.

54.7a. Proof using supermodular colourings

We show how to derive Theorem 54.11 on disjoint bibranchings from Edmonds' disjoint branchings theorem (Theorem 53.1) and Theorem 49.14 on supermodular colourings.

Let $D = (V, A)$ be a digraph and let V be partitioned into R and S. Let $k \in \mathbb{Z}_+$. Define $H := \delta^{\text{out}}(R)$, and define the following collections of subsets of H:

(54.45) $$\mathcal{C}_1 := \{\delta^{\text{in}}_H(U) \mid \emptyset \neq U \subseteq S\} \text{ and } \mathcal{C}_2 := \{\delta^{\text{out}}_H(U) \mid \emptyset \neq U \subseteq R\}.$$

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are intersecting families on H. Define $g_j : \mathcal{C}_j \to \mathbb{Z}$ for $j = 1, 2$ by:

(54.46) $$g_1(B) := \max\{k - d^{\text{in}}_{A[S]}(U) \mid \emptyset \neq U \subseteq S, B = \delta^{\text{in}}_H(U)\} \text{ for } B \in \mathcal{C}_1,$$
$$g_2(B) := \max\{k - d^{\text{out}}_{A[R]}(U) \mid \emptyset \neq U \subseteq R, B = \delta^{\text{out}}_H(U)\} \text{ for } B \in \mathcal{C}_2.$$

Then g_1 and g_2 are intersecting supermodular. Moreover, if U attains the maximum in (54.46), then

(54.47) $$g_1(B) = k - d^{\text{in}}_{A[S]}(U) \leq d^{\text{in}}_A(U) - d^{\text{in}}_{A[S]}(U) = d^{\text{in}}_H(U) = |B| \text{ if } U \subseteq S$$
and
$$g_2(B) = k - d^{\text{out}}_{A[R]}(U) \leq d^{\text{out}}_A(U) - d^{\text{out}}_{A[R]}(U) = d^{\text{out}}_H(U) = |B| \text{ if } U \subseteq R.$$

Since $g_j(B) \leq k$ for $j = 1, 2$ and $B \in \mathcal{C}_j$, by Theorem 49.14 we can partition H into classes $H_1, \ldots, H_k$ such that:

(54.48)
(i) if $\emptyset \neq U \subseteq S$, then U is entered by at least $k - d^{\text{in}}_{A[S]}(U)$ of the classes H_i, and
(ii) if $\emptyset \neq U \subseteq R$, then U is left by at least $k - d^{\text{out}}_{A[R]}(U)$ of the classes H_i.

By Edmonds' disjoint branchings theorem, (i) implies that $A[S]$ contains disjoint branchings $B_1, \ldots, B_k$ such that, for each $i = 1, \ldots, k$, the root set of B_i is equal to the set of heads of the arcs in H_i; that is, each vertex in S is entered by at least one arc in $B_i \cup H_i$. Similarly, $A[R]$ contains disjoint cobranchings (= branchings if all orientations are reversed) $B'_1, \ldots, B'_k$ such that, for each $i = 1, \ldots, k$, each vertex in R is left by at least one arc in $B'_i \cup H_i$. Then the $B_i \cup H_i \cup B'_i$ form disjoint $R - S$ bibranchings.

54.7b. Covering by bifurcations

Theorem 54.11 also implies the following characterization of the minimum number of $R - S$ bifurcations needed to cover all arcs (Keijsper [1998b]):

Corollary 54.11c. *Let $D = (V, A)$ be a digraph and let V be partitioned into sets R and S, with no arc from S to R. Then A can be covered by k $R - S$ bifurcations if and only if*

(54.49)
(i) $\deg^{\text{out}}(v) \leq k$ *for each* $v \in R$;
(ii) $\deg^{\text{in}}(v) \leq k$ *for each* $v \in S$;
(iii) $|A[U]| \leq k(|U| - 1)$ *for each nonempty subset U of R or S.*

Proof. Necessity being easy, we show sufficiency. Extend D by two new vertices r and s, for each $v \in S$ by $k - \deg^{\text{in}}(v)$ parallel arcs from r to v, for each $v \in R$ by $k - \deg^{\text{out}}(v)$ parallel arcs from v to s, and by k parallel arcs from r to s. Let D' be the graph arising in this way. So in D', each $v \in R$ has outdegree k, and each $v \in S$ has indegree k. Define $R' := R \cup \{r\}$ and $S' := S \cup \{s\}$.

Then by Theorem 54.11, D' has k disjoint $R' - S'$ bibranchings. Indeed, any nonempty subset U of R' is left by $k|U| - |A[U]| \geq k$ arcs of D' if $r \notin U$ (since each vertex in R has outdegree k in D'), and by at least k arcs of D' if $r \in U$. Similarly, any nonempty subset of S' is entered by at least k arcs of D'.

Now each of these bibranchings leaves any $v \in R$ exactly once (as v has outdegree k in D'), and (similarly) enters any $v \in S$ exactly once. Moreover, these bibranchings cover A. Hence restricted to A we obtain a covering of A by k $R - S$ bifurcations. ∎

An equivalent way of saying this is (using Corollary 54.10a):

(54.50) For any digraph $D = (V, A)$ and any partition of V into R and S, the $R - S$ bifurcation polytope has the integer decomposition property.

As for the *complexity*, the reduction given in the proof of Corollary 54.11c implies a polynomial-time algorithm to find a minimum number of $R - S$ bifurcations covering the arc set (by reduction to finding a maximum number of disjoint bibranchings). The capacitated version can be solved in semi-strongly polynomial time, with the help of the ellipsoid method, by first finding a fractional packing, and next round (like in Section 51.4).

54.7c. Disjoint $R - S$ biconnectors and $R - S$ bibranchings

As in Keijsper and Schrijver [1998], one can derive Theorem 54.5 on disjoint $R{-}S$ biconnectors (in an undirected graph) from Theorem 54.11 on disjoint $R{-}S$ bibranchings (in a directed graph), with the help of the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a).

Indeed, the condition in Theorem 54.5 gives, with the Tutte-Nash-Williams disjoint trees theorem, that the graph G/R obtained from G by contracting R to one vertex, has k edge-disjoint spanning trees.

By orienting the edges in these trees appropriately, we see that G/R has an orientation such that any nonempty $U \subseteq S$ is entered by at least k arcs, and such that each edge incident with R is oriented away from R. Similarly, G/S has an orientation such that any nonempty $U \subseteq R$ is left by at least k arcs, and such that each edge incident with S is oriented towards S.

Combining the two orientations, we obtain an orientation $D = (V, A)$ of G such that each $R{-}S$ bicut has size at least k. Hence, by Theorem 54.11, D has k disjoint $R - S$ bibranchings, and hence, G has k disjoint $R - S$ biconnectors.

54.7d. Covering by $R - S$ biforests and by $R - S$ bifurcations

Similarly, one can derive Theorem 54.6 on covering $R - S$ biforests from Corollary 54.11c on covering $R - S$ bifurcations, with the help of Nash-Williams' covering forests theorem (Corollary 51.1c). Indeed, the condition in Theorem 54.6 gives, with Nash-Williams' covering forests theorem, that the edges of the graph G/R obtained from G by contracting R to one vertex, can be covered by k forests. Hence G/R has an orientation such that any vertex in S is entered by at most k arcs, and such that R is only left by arcs. Similarly, G/S has an orientation such that any vertex in R is left by at most k arcs, and such that S is only entered by arcs.

Combining the two orientations, we obtain an orientation $D = (V, A)$ of G satisfying the condition in Corollary 54.11c. Hence the arcs of D can be covered by k $R - S$ bifurcations, and hence the edges of G can be covered by k $R - S$ biforests.

Chapter 55

Minimum directed cut covers and packing directed cuts

A *directed cut* in a directed graph $D = (V, A)$ is a set of arcs $\delta^{\text{in}}(U)$ for some nonempty proper subset U of V with $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting each directed cut — equivalent, it is a set of arcs such that their contraction makes the graph strongly connected. For planar digraphs, a directed cut cover corresponds to a *feedback arc set* in the dual digraph — a set of arcs whose removal makes the digraph acyclic. Lucchesi and Younger showed that the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. This min-max relation is the basis for several other results on *shortest* directed cut covers, which we survey in this chapter. In the next chapter we consider the, less tractable, *disjoint* directed cut covers.

55.1. Minimum directed cut covers and packing directed cuts

Let $D = (V, A)$ be a digraph. A subset C of A is called a *directed cut* if there exists a nonempty proper subset U of V with $\delta^{\text{in}}(U) = C$ and $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting each directed cut.

It is easy to show that for any subset B of A the following are equivalent:

(55.1)
(i) B is a directed cut cover;
(ii) adding to D all arcs (u, v) with $(v, u) \in B$ makes the digraph strongly connected;
(iii) contracting all arcs in B makes the digraph strongly connected.

So a minimum directed cut cover gives a minimum number of arcs in D such that making them two-way we obtain a strongly connected digraph.

Moreover, A. Frank (cf. Lovász [1979a] p. 271) showed:

Theorem 55.1. *Let $D = (V, A)$ be a weakly connected digraph without cut arcs and let $B \subseteq A$. Then B is an inclusionwise minimal directed cut cover if and only if B is an inclusionwise minimal set such that if we invert the orientations of all arcs in B, the digraph becomes strongly connected.*

Proof. Define $\widetilde{A} := (A \setminus B) \cup B^{-1}$, where $B^{-1} := \{a^{-1} \mid a \in B\}$, and where a^{-1} is the arc arising from a by inverting its orientation.

Trivially, if $(V, \widetilde{A})$ is strongly connected, then B is a directed cut cover. Hence it suffices to show that if B is an inclusionwise minimal directed cut cover, then $\widetilde{D} = (V, \widetilde{A})$ is strongly connected.

Suppose that $\widetilde{D}$ is not strongly connected. Let K be a strong component of $\widetilde{D}$ with $\delta^{\text{in}}_{\widetilde{A}}(K) = \emptyset$. Let $\delta^{\text{in}}_{A}(K) = \{a_1, \ldots, a_t\}$. So $a_1, \ldots, a_t$ belong to B. Hence, as B is an inclusionwise minimal directed cut cover, for each $i = 1, \ldots, t$ there exists a subset U_i of V with $\delta^{\text{in}}_{A}(U_i) = \emptyset$ and $\delta^{\text{out}}_{B}(U_i) = \{a_i\}$.

Then for each i, $U_i \cap K = \emptyset$. For suppose that $U_i \cap K \neq \emptyset$. As the head of a_i does not belong to U_i, U_i splits K. Hence some arc $a \in \widetilde{A}$ enters U_i, with a spanned by K. As $\delta^{\text{in}}_{A}(U_i) = \emptyset$, we know $a \in B^{-1}$, and therefore $a^{-1} \in \delta^{\text{out}}_{B}(U_i)$ while $a \neq a_i$, a contradiction.

Also, $U_i \cap U_j = \emptyset$ for $i \neq j$, as $\delta^{\text{in}}_{A}(U_i \cap U_j) = \emptyset$ and

$$d^{\text{out}}_{B}(U_i \cap U_j) \leq d^{\text{out}}_{B}(U_i) + d^{\text{out}}_{B}(U_j) - d^{\text{out}}_{B}(U_i \cup U_j) = 0, \tag{55.2}$$

since both a_i and a_j leave $U_i \cup U_j$.

So $U_1, \ldots, U_t$ are disjoint subsets of $V \setminus K$. As D has no cut arcs, $d^{\text{out}}_{A}(U_i) \geq 2$ for each i. Hence, as no arc in A enters any U_i, and only one arc (namely a_i) leaves U_i to enter K, the set $W := V \setminus (K \cup U_1 \cup \cdots \cup U_t)$ is nonempty. Also, $\delta^{\text{out}}_{A}(W) = \emptyset$, and so $\delta^{\text{in}}_{B}(W) \neq \emptyset$, that is $\delta^{\text{out}}_{B}(K \cup U_1 \cup \cdots \cup U_t) \neq \emptyset$. However, $\delta^{\text{out}}_{B}(K) = \emptyset$ (since $\delta^{\text{in}}_{\widetilde{A}}(K) = \emptyset$) and $\delta^{\text{out}}_{B}(U_i) = \{a_i\}$, implying $\delta^{\text{out}}_{B}(K \cup U_i) = \emptyset$ for each i, a contradiction. ∎

55.2. The Lucchesi-Younger theorem

Lucchesi and Younger [1978] proved the following min-max relation for the minimum size of a directed cut cover, which was conjectured by N. Robertson and by Younger [1965,1969] (for planar graphs by Younger [1963a], inspired by a question suggested by J.P. Runyan to Seshu and Reed [1961]).

The proof below is a variant of the proof of Lovász [1976c] (cf. Lovász [1979b]).

Theorem 55.2 (Lucchesi-Younger theorem)**.** *Let $D = (V, A)$ be a weakly connected digraph. Then the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts.*

Proof. For any digraph D, let $\nu(D)$ be the maximum number of disjoint directed cuts in D and let $\tau(D)$ be the minimum size of a directed cut cover. Choose a counterexample $D = (V, A)$ with a minimum number of arcs.

For any $B \subseteq A$, let D_B be the graph obtained from D by replacing each arc (u, v) in B by a directed $u-v$ path of length 2 (the intermediate vertex being new). Choose an inclusionwise maximal subset B of A with $\nu(D_B) = \nu(D)$. Then $B \neq A$, as $\nu(D_A) \geq 2\nu(D) > \nu(D)$.

Choose $b \in A \setminus B$. So $\nu(D_{B\cup\{b\}}) > \nu(D)$. Moreover, as D is a smallest counterexample, the graph D' obtained from D by contracting b satisfies $\nu(D') = \tau(D') \geq \tau(D) - 1 \geq \nu(D)$. Combining a maximum-size packing of directed cuts in D' and one in $D_{B\cup\{b\}}$, we obtain a family $\mathcal{F}$ of nonempty proper subsets of the vertex set V' of D_B with the property that

(55.3) $|\mathcal{F}| = 2\nu(D) + 1$, and the $\delta^{\text{in}}(U)$ for $U \in \mathcal{F}$ are directed cuts in D_B covering any arc of D_B at most twice.

Now we choose $\mathcal{F}$ satisfying (55.3) such that

(55.4) $$\sum_{U\in\mathcal{F}} |U||V' \setminus U|$$

is minimized. Then $\mathcal{F}$ is a cross-free family. Indeed, if $X, Y \in \mathcal{F}$ with $X \not\subseteq Y \not\subseteq X$, $X \cap Y \neq \emptyset$ and $X \cup Y \neq V'$, we can replace X and Y by $X \cap Y$ and $X \cup Y$, while not violating (55.3) but decreasing sum (55.4) (by Theorem 2.1), contradicting its minimality.

So $\mathcal{F}$ is cross-free. For each $X \in \mathcal{F}$, define

(55.5) $\beta(X) := \{U \in \mathcal{F} \mid U \subseteq X \text{ or } U \cap X = \emptyset\}$.

Let $\mathcal{F}_2$ be the collection of sets occurring twice in $\mathcal{F}$ and let $\mathcal{F}_1$ be the collection of sets occurring precisely once in $\mathcal{F}$. Then

(55.6) if X and Y are distinct sets in $\mathcal{F}_1$ with $|\beta(X)| \equiv |\beta(Y)| \pmod 2$, then no arc of D enters both X and Y.

Suppose that to the contrary arc a enters both X and Y. As $\mathcal{F}$ is cross-free, we can assume that $X \subset Y$.

If $|\beta(Y)| \leq |\beta(X)|$, then (as $Y \in \beta(Y) \setminus \beta(X)$) there exists a Z in $\beta(X) \setminus \beta(Y)$. So $Z \not\subseteq Y$ and $Z \cap Y \neq \emptyset$. Hence $Z \not\subseteq X$, and so $Z \cap X = \emptyset$. So $Y \not\subseteq Z$, and hence (as $\mathcal{F}$ is cross-free) $Z \cup Y = V'$. So a leaves Z, a contradiction (since no arc leaves any set in $\mathcal{F}$).

If $|\beta(Y)| \geq |\beta(X)|+2$, then there exists a $Z \neq Y$ with $Z \in \beta(Y)\setminus\beta(X)$. So $Z \not\subseteq X$ and $Z \cap X \neq \emptyset$. Hence $Z \cap Y \neq \emptyset$, and so $Z \subseteq Y$. So $Z \cup X \neq V'$, and hence (as $\mathcal{F}$ is cross-free) $X \subset Z$. So a enters X, Y, and Z, a contradiction. This proves (55.6).

It follows that for some $j \in \{0, 1\}$, the collection

(55.7) $\mathcal{F}_2 \cup \{X \in \mathcal{F}_1 \bigm| |\beta(X)| \equiv j \pmod 2\}$

has size at least $\nu(D) + 1$. By (55.6), it gives $\nu(D) + 1$ disjoint directed cuts in D_B, contradicting our assumption. ∎

Equivalent to the Lucchesi-Younger theorem is the following weighted version of it:

Corollary 55.2a. *Let $D = (V, A)$ be a digraph and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a directed cut cover is equal to* *the maximum number of directed cuts such that each arc a is in at most $l(a)$ of them.*

Proof. Replace any arc a by a path of length $l(a)$ (contracting a if $l(a) = 0$). Then the Lucchesi-Younger theorem applied to the new graph gives the present corollary. ∎

This can be formulated in terms of the total dual integrality of the following system:

$$\text{(55.8)}\qquad \begin{array}{rll} \text{(i)} & 0 \leq x_a \leq 1 & \text{for each } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each directed cut } C. \end{array}$$

Define the *directed cut cover polytope* of D as the convex hull of the incidence vectors of directed cut covers. Then:

Corollary 55.2b. *System* (55.8) *is TDI and determines the directed cut cover polytope of D.*

Proof. The total dual integrality is a reformulation of Corollary 55.2a. The total dual integrality of (55.8) implies that it determines an integer polytope. Hence the second part of the corollary follows. ∎

55.3. Directed cut k-covers

In fact, system (55.8) is box-TDI, and more generally, the following system is box-TDI, as was shown by Edmonds and Giles [1977]:

$$\text{(55.9)}\qquad x(C) \geq 1 \quad \text{for each directed cut } C.$$

Edmonds and Giles' proof gives the following alternative way of proving the Lucchesi-Younger theorem.

Theorem 55.3. *System* (55.9) *is box-TDI.*

Proof. Let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $\delta^{\text{out}}(U) = \emptyset$. So $\{\delta^{\text{in}}(U) \mid U \in \mathcal{U}\}$ is the collection of all directed cuts.

Choose $w \in \mathbb{R}^A$, and let y achieve the maximum in the dual of minimizing $w^{\mathsf{T}}x$ over (55.9), that is, in:

$$\text{(55.10)}\qquad \max\{\sum_{U\in\mathcal{U}} y_U \mid y \in \mathbb{R}^{\mathcal{U}}_+, \sum_{U\in\mathcal{U}} y_U \chi^{\delta^{\text{in}}(U)} = w\},$$

such that

$$\text{(55.11)}\qquad \sum_{U\in\mathcal{U}} y_U |U||V \setminus U|$$

is as small as possible. Let $\mathcal{F} := \{U \in \mathcal{U} \mid y_U > 0\}$. Then $\mathcal{F}$ is cross-free. Suppose to the contrary that $T, U \in \mathcal{F}$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, $T \cup U \neq V$. Let $\alpha := \min\{y_T, y_U\} > 0$. Then decreasing y_T and y_U by α, and increasing $y_{T\cap U}$ and $y_{T\cup U}$ by α, maintains feasibility of y, while its value is not changed; so it remains an optimum solution. However, sum (55.11) decreases (by Theorem 2.1). This contradicts the minimality of (55.11).

So $\mathcal{F}$ is cross-free, and hence the constraints corresponding to $\mathcal{F}$ form a totally unimodular matrix (Corollary 13.21a). Hence, by Theorem 5.35, system (55.9) is box-TDI. ∎

This implies the box-total dual integrality of (for $k \geq 0$):

(55.12) $\quad x(C) \geq k \quad$ for each directed cut C.

Corollary 55.3a. *For each $k \in \mathbb{R}_+$, system* (55.12) *is box-TDI.*

Proof. Directly from Theorem 55.3, since if a system $Ax \leq b$ is box-TDI, then also $Ax \leq k \cdot b$ is box-TDI. ∎

This has the following consequences. Call a subset C of the arc set A of a digraph $D = (V, A)$ a *directed cut k-cover* if C intersects each directed cut in at least k arcs. Consider the system:

(55.13) $\quad 0 \leq x_a \leq 1 \quad$ for $a \in A$,
$\quad x(C) \geq k \quad$ for each directed cut C.

Then:

Corollary 55.3b. *System* (55.13) *is TDI and determines the convex hull of the incidence vectors of the directed cut k-covers.*

Proof. Directly from Corollary 55.3a. ∎

From this, a min-max relation for the minimum size of a directed cut k-cover can be derived:

Corollary 55.3c. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$, such that each directed cut has size at least k. Then the minimum size of a directed cut k-cover is equal to the maximum value of*

(55.14) $$\left|\bigcup \mathcal{C}\right| + k|\mathcal{C}| - \sum_{C \in \mathcal{C}} |C|$$

taken over all collections $\mathcal{C}$ of directed cuts.

Proof. By Corollary 55.3b, the minimum size of a directed cut k-cover is equal to the minimum value of $\mathbf{1}^{\mathsf{T}} x$ over (55.13). Hence, as (55.12) is TDI, the minimum size of a directed cut k-cover is equal to the maximum value of

(55.15) $$k \sum_{C} y_C - z(A),$$

where $y_C \in \mathbb{Z}_+$ for each directed cut C and where $z \in \mathbb{Z}_+^A$ such that

(55.16) $$\sum_{C} y_C \chi^C - z \leq \mathbf{1}.$$

Now we can assume that $y_C \in \{0, 1\}$ for each C, since if $y_C \geq 2$, then $z_a \geq 1$ for each $a \in C$ (by (55.16)). Hence decreasing y_C by 1 and decreasing z_a by 1 for each $a \in C$, maintains (55.16) while (55.15) is not decreased (as $|C| \geq k$ by assumption).

Let $\mathcal{C} := \{C \mid y_C = 1\}$. As $z(A)$ is minimized, we have

(55.17) $$z = \sum_{C \in \mathcal{C}} \chi^C - \chi^{\bigcup \mathcal{C}},$$

and hence that $z(A)$ is equal to $\sum_{C \in \mathcal{C}} |C| - \left|\bigcup \mathcal{C}\right|$. This proves the corollary. ∎

55.4. Feedback arc sets

The Lucchesi-Younger theorem implies a min-max relation for the minimum size of a feedback arc set in a planar digraph. A *feedback arc set* in a digraph $D = (V, A)$ is a set of arcs intersecting every directed circuit.

In fact, if D has no loops, then a set A' is an inclusionwise minimal feedback arc set if and only if A' is an inclusionwise minimal set of arcs such that inverting all arcs in A' makes the digraph acyclic (Grinberg and Dambit [1966], Gallai [1968a]).

E.L. Lawler and R.M. Karp (see Karp [1972b]) showed that finding a minimum-size feedback arc set in a digraph, is NP-complete. For planar digraphs one has however:

Theorem 55.4. *Let $D = (V, A)$ be a planar digraph. Then the minimum size of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits.*

Proof. Consider the dual digraph D^* of D. Then a set of arcs of D forms a directed circuit if and only if the set of dual arcs forms a directed cut in D^*. Hence the corollary follows immediately from the Lucchesi-Younger theorem (Theorem 55.2). ∎

Notes. Figure 55.1 (from Younger [1965]) shows that we cannot drop the planarity condition. This is a counterexample with a smallest number of vertices, since Barahona, Fonlupt, and Mahjoub [1994] showed that in a digraph with no $K_{3,3}$ minor, the minimum size of a feedback arc set is equal to the maximum number of disjoint

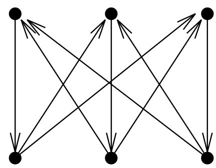

Figure 55.1
The minimum size of a feedback arc set equals 2, while there are no two disjoint directed cuts.

directed circuits. The proof is based on a theorem of Wagner [1937b] on decomposing graphs without $K_{3,3}$ minor into planar graphs and copies of K_5. (Nutov and Penn [1995] gave a similar proof. Related work is done is reported in Nutov and Penn [2000].)

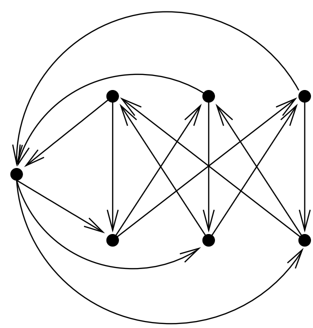

Figure 55.2
An Eulerian digraph where the minimum size of a feedback arc set equals 5, while there are no 5 disjoint directed cuts.

Moreover, Borobia, Nutov, and Penn [1996] showed that in an Eulerian digraph with at most 6 vertices, the minimum size of a feedback arc set is equal to the maximum value of a *fractional* packing of directed circuits. This is not the case if there are more than 6 vertices, as is shown by the graph in Figure 55.2.

Guenin and Thomas [2001] characterized the digraphs D that have the property that for every subhypergraph D' of D, the maximum number of disjoint circuits in D' is equal to the minimum size of a feedback arc set in D'.

More on the polytope determined by the feedback arc sets, equivalently on the *acyclic subgraph polytope* (the convex hull of the incidence vectors of arc sets containing no directed circuit) is presented in Young [1978], Grötschel, Jünger, and Reinelt [1984,1985a,1985b], Reinelt [1993], Leung and Lee [1994], Goemans and Hall [1996], and Bolotashvili, Kovalev, and Girlich [1999]. (Bowman [1972] wrongly claimed to give a system determining the acyclic subgraph polytope.)

The problem of finding a minimum-weight feedback arc set is equivalent to the *linear ordering problem*: given a matrix M, find a permutation matrix P such that the sum of the elements below the main diagonal of $P^{\mathsf{T}}MP$ is minimized. More on this can be found in Younger [1963b], Jünger [1985], Reinelt [1985], Berger and Shor [1990,1997], Arora, Frieze, and Kaplan [1996,2002], Fernandez de la Vega [1996], Frieze and Kannan [1996,1999], and Newman and Vempala [2001].

For feedback arc sets in linklessly embeddable graphs, see Section 55.6b. For feedback vertex sets, see Section 55.6c.

55.5. Complexity

It was shown by Lucchesi [1976], Karzanov [1979c,1981], and Frank [1981b] that a minimum-size directed cut cover and a maximum packing of directed cuts can be found in polynomial time. Lucchesi [1976] also gave a weakly polynomial-time algorithm for the weighted versions of these problems, and Frank [1981b] gave a strongly polynomial-time algorithm for these problems.

Frank and Tardos [1984b] showed that finding a minimum-length directed cut k-cover in fact can be reduced to a weighted matroid intersection problem. Thus all ingredients for a strongly polynomial-time algorithm are ready at hand.

We describe the reduction. Let $D = (V, A)$ be a digraph, let $l : A \to \mathbb{Q}_+$ be a length function, and let $k \in \mathbb{Z}_+$. We want to find a directed cut k-cover of minimum length.

Let $D^{-1} = (V, A^{-1})$ be the reverse digraph of D, where $A^{-1} := \{a^{-1} \mid a \in A\}$ and $a^{-1} = (v, u)$ if $a = (u, v)$. We will define matroids M_1 and M_2 on $A \cup A^{-1}$.

M_1 is easy: it is the partition matroid induced by the sets $\{a, a^{-1}\}$ for $a \in A$. To define M_2, let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $\delta^{\text{in}}_A(U) = \emptyset$. Define

(55.18) $P := \{x \in \mathbb{Z}^V_+ \mid x(V) = |A| \text{ and } x(U) \geq |A[U]| + k \text{ for each } U \in \mathcal{U}\}$.

Then:

(55.19) for $x, y \in P$ and $u \in V$ with $x_u < y_u$, there exists a $v \in V$ with $x_v > y_v$ and $x + \chi^u - \chi^v \in P$.

Indeed, let $\mathcal{K}$ be the collection of inclusionwise maximal subsets U of $V \setminus \{u\}$ with $U \in \mathcal{U}$ and $x(U) = |A[U]| + k$. As sets with this property are closed under unions of intersecting sets, $\mathcal{K}$ consists of disjoint sets, and no two of them are connected by an arc. Hence for $W := V \setminus \bigcup \mathcal{K}$, we have

(55.20) $$y(W) = y(V) - \sum_{U \in \mathcal{K}} y(U) \leq |A| - \sum_{U \in \mathcal{K}} (|A[U]| + k) = x(W).$$

As $x_u < y_u$ and $u \in W$, we know that $x_v > y_v$ for some $v \in W$. Also, $x + \chi^u - \chi^v \in P$, since there is no subset U of $V \setminus \{u\}$ with $\delta^{\text{in}}(U) = \emptyset$, $x(U) = |A[U]| + k$, and $v \in U$.

This shows (55.19), which implies that

(55.21) $\mathcal{B} := \{B \subseteq A \cup A^{-1} \mid \deg_B^{\text{in}} \in P\}$

forms the collection of bases of a matroid M_2 on $A \cup A^{-1}$, provided that $\mathcal{B}$ is nonempty; equivalently, provided that each directed cut in D has size at least k. (That M_2 is a matroid can also be derived from Corollary 49.7a.)

To test independence in M_2, it suffices to have one base of M_2 (which we have: A^{-1}), and to have a test of being a base. Equivalently, we should be able to test membership of P. Let $x \in \mathbb{Z}_+^V$ with $x(V) = |A|$. By Theorem 51.3, we can find a nonempty proper subset U of V minimizing

(55.22) $$x(U) - |A[U]| + (k + |A|)d^{\text{in}}(U)$$
$$= x(U) - \sum_{v \in U} \deg^{\text{out}}(v) + d^{\text{out}}(U) + (k + |A|)d^{\text{in}}(U),$$

in strongly polynomial time. If this minimum is at least k, then x belongs to P. If this minimum is less than k, then $d^{\text{in}}(U) = 0$, and hence $x(U) < |A[U]| + k$, implying that x does not belong to P.

Now a subset C of A is a directed cut k-cover if and only if $B := (A \setminus C) \cup C^{-1}$ is a common base of M_1 and M_2. Hence:

Theorem 55.5. *Given a digraph $D = (V, A)$, a length function $l : A \to \mathbb{Q}_+$, and $k \in \mathbb{Z}_+$, a minimum-length directed cut k-cover can be found in strongly polynomial time.*

Proof. Directly from the above, with Theorem 41.8. We apply the weighted matroid intersection algorithm to find a maximum-length common base B in the matroids M_1 and M_2 on $A \cup A^{-1}$, defining $l(a^{-1}) := 0$ for $a \in A$. Then $A \setminus B$ is a minimum-length directed cut cover. ∎

55.5a. Finding a dual solution

Also a maximum packing of directed cuts can be found in polynomial time. Let B be the maximum-length base found and let C be the directed cut k-cover with $B = (A \setminus C) \cup C^{-1}$.

The weighted matroid intersection algorithm also yields a dual solution. Indeed, if l is integer-valued, it gives length functions $l_1, l_2 : A \cup A^{-1} \to \mathbb{Z}$ such that $l = l_1 + l_2$ and such that B is an l_i-maximal base of M_i, for $i = 1, 2$ (cf. Section 41.3a).

Define

(55.23) $\mathcal{F} := \{U \subseteq V \mid d_A^{\text{in}}(U) = 0, d_C^{\text{out}}(U) = k\}$,

and define a pre-order $\preceq$ on V by:

(55.24) $u \preceq v \iff$ each $U \in \mathcal{F}$ containing u also contains v, for $u, v \in V$. It can be checked in polynomial time whether $u \preceq v$ holds, since it is equivalent to: $\deg_B^{\text{in}} - \chi^u + \chi^v \in P$. Indeed, $\deg_B^{\text{in}} - \chi^u + \chi^v$ belongs to P if and only if $v \in U$ for each $U \in \mathcal{U}$ satisfying $u \in U$ and $\sum_{s \in U} \deg_B^{\text{in}}(s) = |A[U]| + k$. Now $\sum_{s \in U} \deg_B^{\text{in}}(s) = |A[U]| + d_C^{\text{out}}(U)$. So it is equivalent to: $u \preceq v$.

Next define for each $u \in V$:

(55.25) $$p(u) := \max\{l_2(a) \mid \exists v \succeq u : a^{-1} \in \delta_B^{\text{out}}(v)\}.$$

Let $h_0 < \cdots < h_t$ be the elements of the set $\{p(u) \mid u \in V\}$. For $j = 1, \ldots, t$, define $V_j := \{u \mid p(u) \geq h_j\}$. Let $\mathcal{K}$ be the collection of all weak components of $D - V_j$, over all $j = 1, \ldots, t$, and for each $K \in \mathcal{K}$, let

(55.26) $$y_K := \sum(h_j - h_{j-1} \mid j = 1, \ldots, t;\ K \text{ is a weak component of } D - V_j).$$

So

(55.27) $$P = h_0 \chi^V + \sum_{K \in \mathcal{K}} y_K \chi^K.$$

Then:

Theorem 55.6. *Each $K \in \mathcal{K}$ belongs to $\mathcal{F}$. Moreover, for each $a = (u, v) \in A$:*

(55.28) $$\begin{array}{lll} \text{(i)} & \sum(y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) \leq l(a) & \text{if } a \in A \setminus C, \\ \text{(ii)} & \sum(y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) \geq l(a) & \text{if } a \in C. \end{array}$$

Proof. Consider any $j = 1, \ldots, t$. By definition of $p(u)$, we know that V_j is a lower ideal with respect to $\preceq$. That is, if $v \in V_j$ and $u \preceq v$, then $u \in V_j$. (Indeed, if $v \in V_j$, then $p(v) \geq h_j$, hence $l_2(a) \geq h_j$ for some a with $a^{-1} \in \delta_B^{\text{out}}(w)$ for some $w \succeq v$. Since $w \succeq u$ we have $p(u) \geq l_2(a) \geq h_j$.)

Hence, for each $v \in V_j$ and $u \notin V_j$ we have $u \not\preceq v$. Therefore, there is a $U \in \mathcal{F}$ with $u \in U$ and $v \notin U$. This implies, as $\mathcal{F}$ is a crossing family, that there is a partition of $V \setminus V_j$ into sets in $\mathcal{F}$. As $d_A^{\text{in}}(U) = 0$ and $d_C^{\text{out}}(U) = k$, it follows that this partition is equal to the collection of weak components of the digraph $D - V_j$. So each weak component K of $D - V_j$ satisfies $d_A^{\text{in}}(K) = 0$ and $d_C^{\text{out}}(K) = k$; that is, K belongs to $\mathcal{F}$.

Consider any arc $a = (v, u) \in B$. As B is an l_1-maximal base of M_1, we have $l_1(a^{-1}) \leq l_1(a)$. Let $p(u) = l_2(b)$ for some $b^{-1} \in \delta_B^{\text{out}}(w)$ and some $w \succeq u$. Since $u \preceq w$, we know that $\deg_B^{\text{in}} - \chi^u + \chi^w \in P$. So $(B \cup \{b\}) \setminus \{a\}$ is again a base of M_2. Hence we have (as B is an l_2-maximal base of M_2) $l_2(b) \leq l_2(a)$. So $l_2(a) \geq l_2(b) \geq p(u)$. Also $p(v) \geq l_2(a^{-1})$, by definition of $p(v)$. Hence

(55.29) $$\begin{aligned} l(a) - l(a^{-1}) &= l_1(a) + l_2(a) - l_1(a^{-1}) - l_2(a^{-1}) \geq l_2(a) - l_2(a^{-1}) \\ &\geq p(u) - p(v). \end{aligned}$$

If $a \in A \setminus C$, we have $l(a^{-1}) = 0$, and obtain (55.28)(i), since a enters no $K \in \mathcal{K}$ and so $p(u) \geq p(v)$. Hence

(55.30) $$l(a) \geq p(u) - p(v) = \sum(y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)).$$

If $a \in C^{-1}$, we have $l(a) = 0$ and obtain (55.28)(ii), since a^{-1} enters no $K \in \mathcal{K}$, and so $p(u) \leq p(v)$. Hence

(55.31) $$l(a^{-1}) \leq p(v) - p(u) = \sum(y_K \mid K \in \mathcal{K}, a^{-1} \in \delta^{\text{out}}(K)).$$ ∎

For each $a \in C$, let $s(a)$ be the difference of the two terms in (55.28)(ii), and for $a \in A \setminus C$ let $s(a) := 0$. Then

(55.32) $$\sum_{K \in \mathcal{K}} y_K \chi^{\delta_A^{\text{out}}(K)} - s \leq l$$

and

(55.33) $$\begin{aligned} k \sum_{K \in \mathcal{K}} y_K - s(A) &= \sum_{K \in \mathcal{K}} y_K |\delta_A^{\text{out}}(K) \cap C| - s(A) \\ &= \sum_{a \in C} \sum (y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) - s(A) = l(C). \end{aligned}$$

Thus we have an integer optimum dual solution to maximizing $l^{\mathsf{T}} x$ over the system $\mathbf{0} \leq x \leq \mathbf{1}$, $x(Y) \geq k$ (Y directed cut). If $k = 1$, we can do with $s = \mathbf{0}$, and obtain an integer optimum packing of directed cuts subject to l.

So we have:

Theorem 55.7. *Given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{Z}_+$, an optimum packing of directed cuts subject to l can be found in strongly polynomial time.*

Proof. See above. ∎

55.6. Further results and notes

55.6a. Complexity survey for minimum-size directed cut cover

	$O(n^5 \log n)$	Lucchesi [1976]
	$O(n^3 m)$	Frank [1981b]
	$O(n^2 M(n))$	Frank [1981b]
*	$O(n^2 m)$	Gabow [1993b,1995c]
*	$O(nM(n))$	Gabow [1993b,1995c]

As before, $*$ indicates an asymptotically best bound in the table. $M(n)$ denotes the time to multiply $n \times n$ matrices. Also Karzanov [1979c,1981] gave a polynomial-time algorithm to find a minimum-size directed cut cover. Lucchesi [1976] gave also a polynomial-time algorithm to find a minimum-weight directed cut cover, and Frank [1981b] and Gabow [1993a,1993b,1995c] gave strongly polynomial-time algorithms for this.

55.6b. Feedback arc sets in linklessly embeddable digraphs

An undirected graph is called *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ such that any two vertex-disjoint circuits C_1 and C_2 are unlinked (that is, there is a topological sphere S such that C_1 is in the interior of S and C_2 is in the exterior of S). A digraph is called linklessly embeddable if its underlying undirected graph is linklessly embeddable. (Linklessly embeddable graphs are characterized by Robertson, Seymour, and Thomas [1995] in terms of 'forbidden minors'.)

Seymour [1996] showed that in an Eulerian linklessly embeddable directed graph, the minimum-size of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits. We sketch the proof.

The basic combinatorial-topological part of the proof consists of showing:

Theorem 55.8. *Let D be an Eulerian linklessly embeddable digraph. Suppose that there exist $2k+1$ directed circuits such that any arc is in at most two of them. Then there exist $k+1$ arc-disjoint directed circuits.*

Sketch of proof. By a theorem of Robertson, Seymour, and Thomas [1995], D can be embedded in $\mathbb{R}^3$ such that for each undirected circuit C in D there exists an open disk B in $\mathbb{R}^3$ with boundary C and disjoint from D. (We identify D with its embedding.)

Let $C_1, \ldots, C_t$ be a maximum number of directed circuits in D such that any arc of D is in at most two of them. So $t \geq 2k+1$. Moreover, each arc of D is contained in *exactly* two of the C_i. Otherwise, the arcs not covered twice contain a directed circuit (as D is Eulerian). This contradicts the maximality of t.

For each $i = 1, \ldots, t$, let B_i by an open disk with boundary C_i and disjoint from D. We can assume that the B_i are pairwise disjoint, as can be seen as follows. First, we can assume that the B_i are tame and in general position. In particular, no point is in four of the B_i. Any point p in three of the B_i is the intersection point of three of the B_i, pairwise crossing at p. Any point p in two of the B_i is the intersection point of two of the B_i, crossing at p. Moreover, any two distinct B_i and B_j intersect each other in a finite number of closed and open curves, each representing crossings of B_i and B_j. Let $c(B_i, B_j)$ denote the number of such components.

We choose the C_i and B_i such that the sum of the $c(B_i, B_j)$ for $i \neq j$ is minimized.

Now it is elementary combinatorial topology to prove that there exist for any distinct i, j, with $B_i \cap B_j \neq \emptyset$, directed circuits C'_i and C'_j in D with

(55.34) $$\chi^{AC'_i} + \chi^{AC'_j} = \chi^{AC_i} + \chi^{AC_j}$$

and open disks B'_i and B'_j with boundaries C'_i and C'_j respectively, such that $B'_i \cap B'_j = \emptyset$ and $c(B'_i, B_h) + c(B'_j, B_h) \leq c(B_i, B_h) + c(B_j, B_h)$ for all $h \neq i, j$.

Hence, by the minimality of the sum of the $c(B_i, B_j)$, it follows that the B_i are disjoint. So D, together with the B_i forms the union of a number of compact surfaces, certain points of which are identified. As these surfaces are orientable (since they are embedded in $\mathbb{R}^3$), the B_i fall apart into two classes: those with boundary oriented clockwise, and those with boundary oriented counter-clockwise. Each of these classes have arc-disjoint boundaries, and at least one of these classes has size at least $k+1$. This proves the theorem. ∎

Seymour [1996] next continues by deriving (for linklessly embeddable graphs) the total dual integrality of the following system in $x \in \mathbb{R}^A$:

(55.35) $x(C) \geq 1$ for each directed circuit C in D.

Note that nonnegativity of x is not required here.

Corollary 55.8a. *For any linklessly embeddable digraph $D = (V, A)$, system* (55.35) *is totally dual integral.*

Proof. Let $w \in \mathbb{Z}^A$ be such that the minimum of $w^{\mathsf{T}} x$ over (55.35) is finite. Let $\mathcal{C}$ be the collection of directed circuits in D. By Theorem 5.29, it suffices to show that the maximum value μ of $\sum_{C\in\mathcal{C}} y(C)$ taken over $y : \mathcal{C} \to \frac{1}{2}\mathbb{Z}_+$ satisfying

$$(55.36) \qquad \sum_{C\in\mathcal{C}} y_C \chi^{AC} \leq w$$

is attained by an integer-valued y.

Now (as the minimum is finite) w belongs to the cone generated by the incidence vectors of directed circuits, and hence w is a nonnegative circulation. Replace any arc $a = (u, v)$ by $w(a)$ parallel arcs from u to v, giving the Eulerian digraph $D' = (V, A')$. Then μ is equal to half of the maximum number μ' of directed circuits in D' such that any arc of D' is in at most two of these circuits. By Theorem 55.8, D' contains at least $\lceil \frac{1}{2}\mu' \rceil$ arc-disjoint directed circuits. Since $\lceil \frac{1}{2}\mu' \rceil \geq \mu$, this gives in D an integer vector $y : \mathcal{C} \to \mathbb{Z}_+$ as required. ∎

This finally gives:

Theorem 55.9. *The minimum size of a feedback arc set in an Eulerian linklessly embeddable digraph $D = (V, A)$ is equal to the maximum number of arc-disjoint directed circuits.*

Proof. Consider the LP-duality relation for maximizing $x(U)$ over (55.35):

$$(55.37) \qquad \begin{aligned} &\min\{x(A) \mid x(C) \geq 1 \text{ for each directed circuit } C\} \\ &= \max\{\sum_C y_C \mid y_C \geq 0, \sum_C y_C \chi^{AC} = \mathbf{1}\}, \end{aligned}$$

where C ranges over all directed circuits. By Corollary 55.8a and the theory of total dual integrality (Theorem 5.22), both optima have an integer optimum solution. So the maximum is equal to the maximum number of arc-disjoint directed circuits. Let x attain the minimum. By Theorem 8.2, there exists a ('potential') $p : V \to \mathbb{Z}$ with $x_a \geq p(v) - p(u)$ for each arc $a = (u, v)$ of D. Define $x'(a) := x_a - p(v) + p(u)$ for each arc $a = (u, v)$. Then $x' \in \mathbb{Z}_+^A$, $x'(C) = x(C) \geq 1$ for each directed circuit C, and $x'(A) = x(A)$ (since D is Eulerian). Hence the set of arcs a with $x'(a) \geq 1$ forms a feedback arc set of size at most $x(A)$, proving the theorem. ∎

System (55.35) can be tested in polynomial time, for any digraph (with the Bellman-Ford method). It implies that in an Eulerian linklessly embeddable digraph, a minimum-size feedback arc set can be found in polynomial time (with the ellipsoid method).

55.6c. Feedback vertex sets

A *feedback vertex set* in a digraph $D = (V, A)$ is a subset U of V with $D - U$ acyclic — that is, U intersects every directed circuit. Reed, Robertson, Seymour, and Thomas [1996] proved:

(55.38) for each integer $k \geq 0$ there exists an integer $n_k \geq 0$ such that each digraph $D = (V, A)$ having no k vertex-disjoint directed circuits, has a feedback vertex set of size at most n_k.

(For $k = 2$ this answers a question of Gallai [1968b] and Younger [1973].)

Reed, Robertson, Seymour, and Thomas also showed that for each fixed integer k, there is a polynomial-time algorithm to find k vertex-disjoint directed circuits in a digraph if they exist.

Earlier, progress on (55.38) was made by McCuaig [1993], who proved it for $k = 2$, where $n_2 = 3$, by Seymour [1995b], who proved a fractional version of it (if there is no fractional packing of k directed circuits, then there is a feedback vertex set of size at most n_k), and by Reed and Shepherd [1996] for planar graphs.

According to Reed, Robertson, Seymour, and Thomas [1996], N. Alon proved that n_k is at least $Ck \log k$ for some constant C.

Cai, Deng, and Zang [1999,2002] characterized for which orientations $D = (V, A)$ of a complete bipartite graph the system

(55.39)
$$\begin{array}{ll} x_v \geq 0 & \text{for } v \in V, \\ x(VC) \geq 1 & \text{for each directed circuit } C, \end{array}$$

is totally dual integral. (Related results can be found in Cai, Deng, and Zang [1998].)

Guenin [2001b] gave a characterization of digraphs $D = (V, A)$ for which the linear system in $\mathbb{R}^{V \cup A}$ for feedback arc and vertex sets:

(55.40)
$$\begin{array}{ll} x_v \geq 0 & \text{for } v \in V, \\ x_a \geq 0 & \text{for } a \in A, \\ x(VC) + x(AC) \geq 1 & \text{for each directed circuit } C, \end{array}$$

is totally dual integral.

The undirected analogue of (55.38) was proved for $k = 2$ by Bollobás [1963], and for general k by Erdős and Pósa [1965]. Ding and Zang [1999] characterized the undirected graphs $G = (V, E)$ for which the system

(55.41)
$$\begin{array}{ll} x_v \geq 0 & \text{for } v \in V, \\ x(VC) \geq 1 & \text{for each circuit } C, \end{array}$$

is totally dual integral. Their characterization implies that (55.41) is totally dual integral if and only if it defines an integer polyhedron.

A polyhedral approach to the feedback vertex set problem was investigated by Funke and Reinelt [1996]. Approximation algorithms for feedback problems were given by Monien and Schulz [1982], Eades, Lin, and Smyth [1993], Bar-Yehuda, Geiger, Naor, and Roth [1994,1998], Becker and Geiger [1994,1996], Bafna, Berman, and Fujito [1995,1999], Even, Naor, Schieber, and Sudan [1995,1998], Even, Naor, and Zosin [1996,2000], Goemans and Williamson [1996,1998], Chudak, Goemans, Hochbaum, and Williamson [1998], Bar-Yehuda [2000], and Cai, Deng, and Zang [2001]. More on the feedback vertex set problem was presented by Smith and Walford [1975], Kevorkian [1980], Rosen [1982], Speckenmeyer [1988], Stamm [1991], Hackbusch [1997], and Pardalos, Qian, and Resende [1999].

55.6d. The bipartite case

McWhirter and Younger [1971] (cf. Younger [1970], Vidyasankar [1978b]) proved the Lucchesi-Younger theorem in case the arcs of D form a directed cut; that is, in case the underlying undirected graph is bipartite, while all arcs are oriented from one colour class to the other. It amounts to the following:

Theorem 55.10. *Let $G = (V, E)$ be a connected bipartite graph and let $\mathcal{F}$ be the collection of subsets $E[U]$ of E for which U is a vertex cover with $E[U]$ nonempty. Then the minimum size of a set of edges intersecting each set in $\mathcal{F}$ is equal to the maximum number of disjoint sets in $\mathcal{F}$.*

Proof. Let U and W be the colour classes of G and let digraph D be obtained from G by orienting each edge from U to W. Then a set of edges belongs to $\mathcal{F}$ if and only if it is a directed cut of D. Hence the theorem follows from the Lucchesi-Younger theorem. ∎

D.H. Younger (cf. Frank [1993b]) showed that the maximum number of disjoint nonempty cuts in a bipartite graph G is equal to the maximum number of disjoint directed cuts in the directed graph obtained from G by orienting all edges from one colour class to the other (cf. Corollary 29.13b). (Vidyasankar [1978b] showed that a set of edges J intersecting each set in $\mathcal{F}$ attains the minimum in Theorem 55.10 if and only if J intersects any circuit C of G in at most $\frac{1}{2}|EC|$ edges; that is, if and only if J is a join — cf. Section 29.11d.)

As noted by Younger [1979], the Lucchesi-Younger theorem, in the form of Corollary 55.2b, implies the Kőnig-Rado edge cover theorem (Theorem 19.4): the minimum size of an edge cover in a bipartite graph $G = (V, E)$ is equal to the maximum size of a stable set in G. To obtain this as a consequence, let U and W be the colour classes of G and let $D = (V, A)$ be the directed graph with vertex set V and arcs all pairs (u, v) with $u \in U$ and $v \in W$. Define a weight function $w : A \to \mathbb{Z}$ by $w(u, v) := 1$ if $uv \in E$, and ∞ otherwise. Then the minimum weight of a directed cut cover in D is equal to the minimum size of an edge cover in G. With this correspondence, Corollary 55.2b gives the Kőnig-Rado edge cover theorem.

55.6e. Further notes

Frank, Tardos, and Sebő [1984] showed that the Lucchesi-Younger theorem implies that in a digraph $D = (V, A)$, the minimum size of a directed cut cover is equal to the maximum value of

$$(55.42) \qquad \sum_{i=1}^{k} \text{number of weak components of } D - V_i,$$

where $\emptyset \neq V_1 \subset V_2 \subset \cdots \subset V_k \subset V$ are such that no arc leaves any V_i and enters at most one of the V_i.

Frank and Tardos [1989] showed that a weakly connected digraph $D = (V, A)$ has a branching that intersects all directed cuts if and only if for each nonempty $U \subseteq V$, the number of weak components K of $D - U$ with $d^{\text{in}}(K) = 0$, is at most $|U|$.

Younger [1965] proved the Lucchesi-Younger theorem for digraphs having an arborescence, and, more generally, Younger [1979] proved it for source-sink connected digraphs (that is, each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc).

Tuza [1994] showed that for any planar directed graph $D = (V, A)$ and any collection $\mathcal{T}$ of directed triangles in D, the minimum number of arcs intersecting each triangle in $\mathcal{T}$ is equal to the maximum number of arc-disjoint triangles in $\mathcal{T}$.

Chapter 56

Minimum directed cuts and packing directed cut covers

A minimum-capacity directed cut can be found in strongly polynomial time, by applying the minimum-capacity $s-t$ cut algorithm, for all s, t, in some modified digraph.
As for packing directed cut covers it is unknown if it is polynomial-time tractable. Also it is unknown if the maximum number of disjoint directed cut covers is equal to the minimum size of a directed cut — this is Woodall's conjecture. But the capacitated version of it does not hold.
In this chapter we consider a few cases where Woodall's conjecture has been proved, in particular for the *source-sink connected* digraphs.

56.1. Minimum directed cuts and packing directed cut covers

The Lucchesi-Younger theorem states that in a digraph $D = (V, A)$, the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. Woodall [1978a,1978b] ventured the conjecture that this min-max relation would be maintained after interchanging the terms directed cut and directed cut cover:

Conjecture (Woodall's conjecture)**.** In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers.

This conjecture is open.

A capacitated version of Woodall's conjecture (conjectured by Edmonds and Giles [1977] and D.H. Younger) is however not true. Note that the Lucchesi-Younger theorem is equivalent to its weighted version, by replacing arcs by directed paths of length $l(a)$ if $l(a) \geq 1$, and contracting an arc a if $l(a) = 0$. We could attempt this approach to obtain an equivalent capacitated version from Woodall's conjecture, by replacing any arc a by $c(a)$ parallel arcs, but there is a problem here: if $c(a) = 0$, we delete a and can create new directed cuts.

A capacitated version with capacities 0 and 1 amounts to the statement that each directed cut k-cover can be partitioned into k directed cut covers.

Figure 56.1 gives a counterexample to this for the case $k = 2$ (Schrijver [1980a]). Note that the counterexample is planar, and that therefore the 'planar dual' assertion (on packing feedback arc sets) also does not hold.

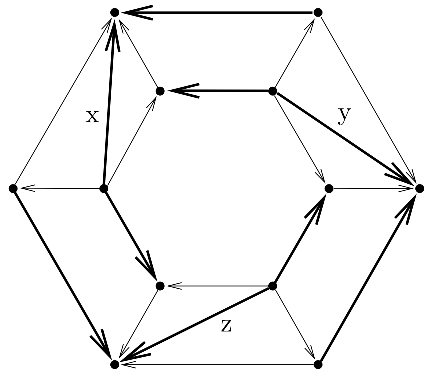

Figure 56.1
A directed cut 2-cover that cannot be split into two directed cut covers. Let C be the set of heavy arcs. Then C is a directed cut 2-cover, since for each arc $c \in C$, the set $C \setminus \{c\}$ is a directed cut cover, which is easy to check since up to symmetry there are only two types of arcs in C.
However, C cannot be split into directed cut covers C_1 and C_2. To see this, observe that each of these C_i must contain exactly one of the two arcs in C meeting any source or sink. Moreover, each C_i must contain at least one of the arcs labeled x, y, z, since the set of arcs from the inner hexagon to the outer hexagon forms a directed cut. Hence we may assume without loss of generality that C_1 contains the arcs x and y, but not z. But then C_1 does not intersect the directed cut of those arcs going from the right half of the figure to the left half.

To interpret this polyhedrally, consider the system:

(56.1)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_a \leq 1 & \text{for each } a \in A, \\ \text{(ii)} & x(B) \geq 1 & \text{for each directed cut cover } B. \end{array}$$

With Corollary 55.2b, the theory of blocking polyhedra gives that system (56.1) determines the convex hull of the incidence vectors of arc sets containing a directed cut. However, by the example in Figure 56.1, system (56.1) generally is not TDI, as total dual integrality amounts to the capacitated version of Woodall's conjecture.

In a number of special cases, Woodall's conjecture, and its capacitated extension, have been proved. In the remainder of this chapter we will consider such cases.

Two more counterexamples to the conjecture of Edmonds and Giles were given by Cornuéjols and Guenin [2002c], and they asked if, together with the example of Figure 56.1, these form all minimal counterexamples to the Edmonds-Giles conjecture.

56.2. Source-sink connected digraphs

Feofiloff and Younger [1987] and Schrijver [1982] showed that for source-sink connected digraphs, the min-max relation for packing directed cut covers does hold. Here a digraph is called *source-sink connected* if each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc. So an acyclic digraph is source-sink connected if each sink is reachable by a directed path from each source. We follow the proof of Schrijver [1982].

Theorem 56.1. *Let $D = (V, A)$ be a source-sink connected digraph and let $k \in \mathbb{Z}_+$. Then any directed cut k-cover C can be partitioned into k directed cut covers.*

Proof. Choose a counterexample with $|V| + |C|$ as small as possible. Then D is acyclic, since any strong component can be contracted to one vertex.

We may assume that if v is reachable in D from u and $v \neq u$, then $(u, v) \in A$. We first show:

(56.2) for any nonempty proper subset U of V with $\delta^{\text{out}}(U) = \emptyset$ and $|\delta^{\text{in}}_C(U)| = k$, one has $|U| = 1$ or $|U| = |V| - 1$.

Suppose not. Let $D' := D/U$ and $D'' := D/\overline{U}$ be the digraphs obtained from D by contracting U and $\overline{U} := V \setminus U$, respectively. Note that D' and D'' are source-sink connected again. Let C' be the set of arcs in C with tail in $\overline{U}$, and let C'' be the set of arcs in C with head in U.

Now each directed cut in D' intersects C' in at least k arcs, as it is a directed cut in D and hence intersects C in at least k arcs. So by the minimality of $|V|+|C|$, C' can be split into k directed cut covers $B'_1, \ldots, B'_k$ for D'. As $|\delta^{\text{in}}_C(U)| = k$, each B'_i has exactly one arc entering U. Similarly, C'' can be split into k directed cut covers $B''_1, \ldots, B''_k$ for D'', such that each B''_i has exactly one arc entering U. By choosing indices appropriately, we can assume that B'_i and B''_i have an arc entering U in common, for each $i = 1, \ldots, k$ (as $|\delta^{\text{in}}_C(U)| = k$).

Then each $B'_i \cup B''_i$ is a directed cut cover for D. For suppose that there is a nonempty proper subset W of V with $\delta^{\text{out}}(W) = \emptyset$ and $\delta^{\text{in}}(W)$ disjoint from $B'_i \cup B'_i$. Then $U \cap W \neq \emptyset$ and $U \not\subseteq W$, since otherwise $\delta^{\text{in}}(W)$ is a directed cut of D', and hence some arc in B'_i enters W. So some arc in B''_i enters $U \cap W$. Similarly, some arc in B'_i enters $U \cup W$. Since exactly one arc in $B'_i \cup B''_i$ enters U, it follows that at least one arc in $B'_i \cup B''_i$ enters W, contradicting our assumption.

So each $B'_i \cup B''_i$ is a directed cut cover for D. As they are disjoint, this contradicts our assumption, thus proving (56.2).

We next show the following. Let X be the set of sources of D and let Y be the set of sinks of D. Then:

(56.3) for each $a = (u, v) \in C$ we have $u \in X$ or $v \in Y$.

For suppose not. Then by (56.2), each directed cut of D intersects $C \setminus \{a\}$ in at least k arcs (as any directed cut intersecting C in exactly k arcs and containing a is equal to $\delta^{\text{in}}(\{v\})$ or $\delta^{\text{in}}(V \setminus \{u\})$, implying that v is a sink or u is a source). So by the minimality of $|V| + |C|$, we can split $C \setminus \{a\}$ into k directed cut covers. This implies that also C can be split into k directed cut covers, contradicting our assumption. This proves (56.3).

Next:

(56.4) if $a = (u, v) \in C$, $a' = (u', v') \in C$, and v is reachable from u', then $u' \in X$ or $v \in Y$.

For suppose not. By (56.3), $u \in X$ and $v' \in Y$, and hence (since D is source-sink connected), $a'' = (u, v') \in A$. Now $a \neq a'$, as $u \in X$ and $u' \notin X$. So $C' := (C \setminus \{a, a'\}) \cup \{a''\}$ is smaller than C. Moreover, C' is a directed cut k-cover. Indeed, let U be a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$. If $|U| = 1$ or $|U| = |V| - 1$, then $\delta^{\text{in}}_{C'}(U) = \delta^{\text{in}}_{C}(U)$, since then $U = \{r\}$ for some sink r or $U = V \setminus \{s\}$ for some source s. If $1 < |U| < |V| - 1$, then

(56.5) $|\delta^{\text{in}}_{C'}(U)| \geq |\delta^{\text{in}}_{C}(U)| - 1 \geq k$,

since if both a and a' enter U, then $u \notin U$ and $v' \in U$, and hence a'' enters U.

So C' is a directed cut k-cover, and hence, by the minimality of $|V| + |C|$, C' can be split into k directed cut covers. Let B be the directed cut cover containing a''. Then $B' := (B \setminus \{a''\}) \cup \{a, a'\}$ is a directed cut cover, since any directed cut $\delta^{\text{in}}(W)$ containing a'', contains at least one of a, a'. Indeed, otherwise $u, v \notin W$, $u', v' \in W$, but then $u' \neq v$ and arc (u', v) leaves W, contradicting the fact that W determines a directed cut.

So by replacing B by B' we obtain a splitting of C into k directed cut covers, contradicting our assumption. This proves (56.4).

This implies:

(56.6) V can be partitioned into sets R and S such that $\delta^{\text{in}}(R) = \emptyset$, $X \subseteq R$, $Y \subseteq S$, and if any $(u, v) \in C$ leaves R, then $u \in X$ and $v \in Y$.

For define

(56.7) $C' := \{(u, v) \in C \mid u \notin X \text{ or } v \notin Y\}$,
$R := \{v \in V \mid D' = (V, A \cup C^{-1}) \text{ has a directed } v - X \text{ path}\}$,
$S := V \setminus R$.

Then $X \subseteq R$, $\delta^{\text{in}}_A(R) = \emptyset$, and any $(u, v) \in C$ leaving R satisfies $u \in X$ and $v \in Y$. To see that $Y \subseteq S$, suppose to the contrary that D' has a directed $Y - X$ path P. Choose P shortest. Then by (56.3), P has of at most three arcs. Let (v', u') and (v, u) be the first and last arc of P. So $v' \in Y$ and $u \in X$. These arcs belong to C'^{-1}, and v is reachable from u' in D. So by (56.4), $u' \in X$ or $v \in Y$. This contradicts the definition of C'. This shows (56.6).

Fix R, S as in (56.6). Let $D' = (V, A')$ be the digraph arising from D by replacing any arc (u, v) of D by k parallel arcs from v to u. Then

(56.8) $$|\delta^{\text{in}}_{A' \cup C}(U)| \geq k$$

for each nonempty proper subset U of V. So by Theorem 54.11, $A' \cup C$ can be split into k $R - S$ bibranchings. Let $B_1, \ldots, B_k$ be the intersections of these bibranchings with C. We show that each B_i is a directed cut cover, contradicting our assumption, and therefore finishing the proof.

Suppose that say B_1 is not a directed cut cover. Let U be a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$, and suppose that no arc in B_1 enters U. Note that if U contains any source, it contains all sinks, since no arc of D leaves U. So U contains no sources or contains all sinks.

First assume that U contains no sources of D. As U contains at least one sink (as $U \neq \emptyset$ and $\delta^{\text{out}}(U) = \emptyset$), we know $U \not\subseteq R$. As $A' \cup B_1$ is an $R - S$ bibranching, we know that

(56.9) $$\delta^{\text{in}}_{A' \cup B_1}(U \cap S) \neq \emptyset.$$

As $\delta^{\text{out}}_A(U \cap S) = \emptyset$ (since $\delta^{\text{out}}_A(U) = \emptyset$ and $\delta^{\text{in}}_A(R) = \emptyset$), we have $\delta^{\text{in}}_{A'}(U \cap S) = \emptyset$. Hence some arc (u, v) in B_1 enters $U \cap S$. As by assumption (u, v) does not enter U, (u, v) enters S, and $u \in U$. However, by (56.6), u belongs to X. This contradicts our assumption that U contains no sources of D.

The case that U contains all sinks is symmetric, and leads again to a contradiction. ∎

A special case of Theorem 56.1 is Woodall's conjecture for source-sink connected digraphs:

Corollary 56.1a. *Let $D = (V, A)$ be a source-sink connected digraph. Then the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers.*

Proof. This is the case $C = A$ of Theorem 56.1. ∎

Also, a capacitated version can be derived from the theorem:

Corollary 56.1b. *Let $D = (V, A)$ be a source-sink connected digraph and let $c : A \to \mathbb{Z}_+$ be a capacity function. Then the minimum capacity of a directed cut is equal to the maximum number of directed cut covers such that no arc a is in more than $c(a)$ of these directed cut covers.*

Proof. Directly from Theorem 56.1, by adding, for any arc a of D, $c(a)$ arcs parallel to a, and by taking for C the set of newly added arcs. ∎

Equivalently, in TDI terms:

Corollary 56.1c. *If $D = (V, A)$ is a source-sink connected digraph, then system* (56.1) *is totally dual integral.*

Proof. This is a reformulation of Corollary 56.1b. ∎

Feofiloff [1983] gave a polynomial-time algorithm to find a maximum number of disjoint directed cut covers in a source-sink connected digraph. Also the proof above implies a polynomial-time algorithm.

A polynomial-time algorithm for the capacitated case can be derived from the ellipsoid method (cf. Grötschel, Lovász, and Schrijver [1988]). A semi-strongly polynomial-time algorithm also follows from Section 57.5 below.

Notes. Frank [1979b] showed the special case of Woodall's conjecture for digraphs having an arborescence. (Such digraphs are source-sink connected.) J. Edmonds observed that this can be derived from Edmonds' disjoint arborescences theorem (Corollary 53.1b): Let $D = (V, A)$ have an r-arborescence. Let k be the minimum size of a directed cut in D. Add to D, for each arc (u, v) of D, k parallel arcs from v to u. This makes the digraph $D' = (V, A')$ with $|\delta^{\text{in}}_{A'}(U)| \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$. Hence D' has k disjoint r-arborescences (by Edmonds' disjoint arborescences theorem). Now for any r-arborescence B in D', the set $B \cap A$ is a directed cut cover in D, since if U is a nonempty proper subset of V with $\delta^{\text{out}}_A(U) = \emptyset$, then $\delta^{\text{in}}_{A'}(U) = \delta^{\text{in}}_A(U)$, and hence $\delta^{\text{in}}_{B\cap A}(U) = \delta^{\text{in}}_B(U) \neq \emptyset$. So we obtain k disjoint directed cut covers in D.

56.3. Other cases where Woodall's conjecture is true

Another case where Woodall's conjecture holds is given in:

Theorem 56.2. *Let $D = (V, A)$ be a digraph arising from a directed tree $T = (V, A')$ such that $(u, v) \in A$ if and only if v is reachable in T from u. Let $c : A \to \mathbb{Z}_+$ be a capacity function. Then the minimum capacity of a directed cut is equal to the maximum number of directed cut covers such that each arc a is in at most $c(a)$ of them.*

Proof. The proof is by induction on the minimum capacity k of a directed cut. Then it suffices to show that there exists a directed cut cover B with $\chi^B \leq c$ and with $(c - \chi^B)(C) \geq k - 1$ for each directed cut C.

Let M be the $A' \times A$ network matrix generated by T and D (cf. Section 13.3). Then the rows of M are precisely the incidence vectors of inclusionwise minimal directed cuts. So it suffices to show that there exists an integer solution x of

(56.10) $\mathbf{0} \leq x \leq \mathbf{c}, Mx \geq \mathbf{1}, M(c-x) \geq (k-1)\mathbf{1},$

since for any such x there is a directed cut cover B satisfying $\chi^B \leq x$.

Since M is totally unimodular, it suffices to show that (56.10) has any solution. Define $x := \frac{1}{k}c$. Then x satisfies (56.10), since $Mc \geq k\mathbf{1}$ and hence $Mx \geq \mathbf{1}$ and $M(c-x) = \frac{k-1}{k}Mc \geq (k-1)\mathbf{1}$. ∎

The theorem can also be formulated in terms of partitioning directed cut k-covers:

Corollary 56.2a. *Let $D = (V, A)$ be a digraph such that A contains a directed spanning tree T with the property that for each arc (u, v) in A there exists a directed $u-v$ path in T. Then any directed cut k-cover in D can be partitioned into k directed cut covers.*

Proof. This follows from Theorem 56.2 by taking $c(u, v)$ equal to the number of times there is an arc from u to v in the directed cut k-cover. ∎

A. Frank also noted that Woodall's conjecture is true if the minimum size of a directed cut is at most 2:

Theorem 56.3. *Let $D = (V, A)$ be a digraph such that each directed cut has size at least two. Then there are two disjoint directed cut covers.*

Proof. As the underlying undirected graph is 2-edge-connected, it has a strongly connected orientation $D' = (V, A')$ (see Corollary 61.3a). Let B_1 be the set of arcs of D that have the same orientation in D' and let $B_2 := A \setminus B_1$. Then B_1 and B_2 are disjoint directed cut covers. ∎

Figure 56.1 shows that this cannot be generalized to each directed cut 2-cover being partitionable into two directed cut covers.

56.3a. Further notes

Karzanov [1985c] gave a strongly polynomial-time algorithm to find a minimum-*mean* capacity directed cut (cf. McCormick and Ervolina [1994]).

Chapter 57

Strong connectors

A *strong connector* is a set of new arcs whose addition to a given digraph D makes it strongly connected. If each potential new arc has been given a length, then finding a shortest strong connector is NP-complete, even if D has no arcs at all: finding a directed Hamiltonian circuit is a special case. (So even if each length is 0 or 1, the problem is NP-complete.)
However, there are a few cases where finding a shortest strong connector is tractable and where min-max relations and polyhedral characterizations hold — for instance, if D is source-sink connected. For these digraphs, packing strong connectors is similarly tractable. These results follow by reduction to directed cut covers discussed in the previous two chapters.

57.1. Making a directed graph strongly connected

Let (V, A) and (V, B) be digraphs. The set B is called a *strong connector for* D if the digraph $(V, A \cup B)$ is strongly connected.

Consider the following *strong connectivity augmentation problem*:

(57.1) Given a digraph $D = (V, A)$ and a cost function $k : V \times V \to \mathbb{Q}$, find a minimum-cost strong connector for D.

Theorem 57.1. *The strong connectivity augmentation problem is* NP*-complete, even if* $A = \emptyset$.

Proof. The problem of finding a Hamiltonian circuit in a digraph $D' = (V, A')$ is equivalent to the existence of a strong connector for $(V, \emptyset)$ of cost $|V|$, where $k(u, v) := 1$ if $(u, v) \in A'$, and $k(u, v) := 2$ otherwise. ∎

Eswaran and Tarjan [1976] showed that if the cost of each new arc equals 1, then there is an easy solution:

Theorem 57.2. *If* $D = (V, A)$ *is an acyclic digraph with at least* 2 *vertices, and with* ρ *sources and* σ *sinks, then the minimum size of a strong connector for* D *equals* $\max\{\rho, \sigma\}$.

Proof. To see that the minimum is at least $\max\{\rho, \sigma\}$, note that for each source r one should add at least one arc entering r; similarly, for each sink s one should add at least one arc leaving s.

That the bound can be attained is shown by induction on $\max\{\rho, \sigma\}$. If there is a pair of a source r and a sink s such that s is not reachable from r, add an arc (s, r). This reduces both ρ and σ by 1, while maintaining acyclicity, and induction gives the result.

So we can assume that each sink is reachable from each source. We can also assume that $\rho \geq \sigma$ (otherwise, reverse all orientations). Let $r_1, \ldots, r_\rho$ be the sources and let $s_1, \ldots, s_\sigma$ be the sinks. Then adding arcs (s_i, r_i) for $i = 1, \ldots, \sigma$, and arcs (s_i, r_1) for $i = \sigma+1, \ldots, \rho$ makes D strongly connected, proving the theorem. ∎

This implies for not necessarily acyclic digraphs:

Corollary 57.2a. *Let $D = (V, A)$ be a digraph which is not strongly connected, let ρ be the number of strong components K of D with $d^{\text{in}}(K) = 0$ and let σ be the number of strong components K of D with $d^{\text{out}}(K) = 0$. Then the minimum size of a strong connector for D equals* $\max\{\rho, \sigma\}$.

Proof. Apply Theorem 57.2 to the digraph obtained from D by contracting each strong component of D to one vertex. ∎

These proofs also give a polynomial-time algorithm to find a minimum-size strong connector. Eswaran and Tarjan [1976] describe a linear-time implementation.

57.2. Shortest strong connectors

Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs. Call a subset A' of A a *D_0-cut (in D)* if $A' = \delta^{\text{in}}_A(U)$ for some nonempty proper subset U of V with $\delta^{\text{in}}_{A_0}(U) = \emptyset$.

It is easy to see that a set B of arcs of D is a strong connector for D_0 if and only if B intersects each D_0-cut in D. The following consequence of the Lucchesi-Younger theorem was given in Schrijver [1982]. It gives a min-max relation for the minimum length of a strong connector, if the following condition holds for digraphs $D_0 = (V, A_0)$ and $D = (V, A)$:

(57.2) for each $(u, v) \in A$ there exist $u', v' \in V$ such that in D_0, u' is reachable from u and from v', and v from v'.

We mention two special cases where this condition is satisfied:

- A is a subset of A_0^{-1},
- D_0 is source-sink connected.

We derive from the Lucchesi-Younger theorem (Schrijver [1982]):

Theorem 57.3. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying (57.2) and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $l(a)$ of these cuts.*

Proof. We can assume that D_0 is acyclic, and that for any $u, v \in V$, if v is reachable in D_0 from u, then $(u, v) \in A_0$. (So $(v, v) \in A_0$ for each $v \in V$.)

We show the theorem by induction on the number τ of arcs $a = (u, v)$ of D for which $(v, u) \notin A_0$. If $\tau = 0$, the theorem is equivalent to Corollary 55.2a.

If $\tau > 0$, choose $(u, v) \in A$ with $(v, u) \notin A_0$. By assumption, there exist $u', v' \in V$ with $(u, u'), (v', u'), (v', v) \in A_0$. Introduce two new vertices, u'' and v'', and add arcs

(57.3) $\qquad (u, u''), (u'', u'), (v'', u''), (v'', v), (v', v'')$

to A_0. Moreover, replace arc (u, v) of A by (u'', v''), with length equal to that of the original arc (u, v). Let $\widetilde{D}_0 = (\widetilde{V}, \widetilde{A}_0)$ and $\widetilde{D} = (\widetilde{V}, \widetilde{A})$ denote the modified graphs.

This transformation decreases the number τ by 1. Moreover,

(57.4) $\qquad$ any subset C of A is a strong connector for D_0 if and only if the set $\widetilde{C} \subseteq \widetilde{A}$ is a strong connector for $\widetilde{D}_0$.

Here $\widetilde{C}$ arises from C by replacing (u, v) by (u'', v'') if $(u, v) \in C$. Proving (57.4) suffices, since it implies that the two numbers in the theorem are invariant under the transformation.

(57.4) can be seen as follows. Choose $C \subseteq A$. First let C be a strong connector for D_0. If $(u, v) \notin C$, then $\widetilde{C} = C$ is also strong connector for $\widetilde{D}_0$ (since in $\widetilde{D}_0$ the new vertex u'' is on a $u - u'$ path, and the new vertex v'' is on a $v' - v$ path). If $(u, v) \in C$, then $\widetilde{C} = (C \setminus \{(u, v)\}) \cup \{(u'', v'')\}$ is a strong connector for $\widetilde{D}_0$, since $A_0 \cup C$ contains the $u - v$ path (u, u''), (u'', v''), (v'', v).

Conversely, let $\widetilde{C}$ be a strong connector for $\widetilde{D}_0$. If $(u'', v'') \notin \widetilde{C}$, then $C = \widetilde{C}$ is also a strong connector for D_0, since any directed path in $\widetilde{A}_0 \cup \widetilde{C}$ connecting two vertices in V and traversing any of the new vertices u'', v'', can be shortcut to a path avoiding u'' and v''.

If $(u'', v'') \in \widetilde{C}$, then $C = (\widetilde{C} \setminus \{(u'', v'')\}) \cup \{(u, v)\}$ is a strong connector for D_0, since any directed path in $\widetilde{A}_0 \cup \widetilde{C}$ connecting two vertices in V and traversing arc (u'', v''), must traverse (u, u''), (u'', v''), and (v'', v), and hence gives a path in $A_0 \cup C$, by replacing this by (u, v). ∎

The proof gives also an algorithmic reduction to the problem of finding a minimum-length directed cut cover, and hence (by Theorem 55.5) a minimum-length strong connector for D_0 can be found in strongly polynomial time.

Theorem 57.3 includes several theorems considered earlier:

- $s, t \in V$ and $A_0 := \{(u,v) \mid u = t \text{ or } v = s\}$: max-potential min-work theorem (Theorem 8.3);
- V is the disjoint union of U and W, $A_0 := \{(u,w) \mid u \in U, w \in W\}$ and $A \subseteq \{(w,u) \mid w \in W, u \in U\}$: weighted version of the Kőnig-Rado edge cover theorem(Theorem 19.4);
- $A_0 = \{(v,r) \mid v \in V\}$ for some $r \in V$: optimum arborescence theorem(Theorem 52.3);
- V is the disjoint union of U and W and $A_0 := \{(u,w) \mid u \in U, w \in W\}$:optimum bibranching theorem(Corollary 54.8b);
- $A \subseteq \{(u,v) \mid (v,u) \in A_0\}$: Lucchesi-Younger theorem (Theorem 55.2).

A cardinality version of the previous theorem is:

Corollary 57.3a. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying* (57.2). *Then the minimum size of a strong connector in D for D_0 is equal to the maximum number of disjoint D_0-cuts in D.*

Proof. This is the case $l = \mathbf{1}$ of Theorem 57.3. ∎

We formulate this for the special case of source-sink connected digraphs. Recall that a digraph $D = (V, A)$ is called *source-sink connected* if each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc.

Corollary 57.3b. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $l(a)$ of these cuts.*

Proof. Directly from Theorem 57.3, since condition (57.2) is implied by the fact that D_0 is source-sink connected. ∎

The cardinality version is:

Corollary 57.3c. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Then the minimum size of a strong connector in D for D_0 is equal to the maximum number of disjoint D_0-cuts in D.*

Proof. This is the case $l = \mathbf{1}$ in Corollary 57.3b. ∎

57.3. Polyhedrally

Theorem 57.3 can be equivalently formulated in TDI terms:

Corollary 57.3d. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying* (57.2). *Then the system*

$$\text{(57.5)} \qquad \begin{array}{rll} \text{(i)} & x_a \geq 0 & a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}_A(U)) \geq 1 & U \subset V, U \neq \emptyset, \delta^{\text{in}}_{A_0}(U) = \emptyset, \end{array}$$

is TDI and determines the convex hull of the strong connectors of D_0.

Proof. This is a reformulation of Theorem 57.3. ∎

In fact, system (57.5) is box-TDI, as will follow from Theorem 60.3.

By the theory of blocking polyhedra, Corollary 57.3d implies:

Corollary 57.3e. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ satisfy* (57.2). *Then the system*

$$\text{(57.6)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_a \leq 1 & a \in A, \\ \text{(ii)} & x(B) \geq 1 & B \text{ strong connector for } D_0 \end{array}$$

determines the convex hull of subsets of A containing a D_0-cut.

Proof. System (57.6) determines the blocking polyhedron of the polyhedron determined by (57.5), and hence its vertices are the incidence vectors of subsets of A that intersect all strong connectors for D_0. These are precisely the sets of arcs in A containing a D_0-cut. ∎

System (57.6) generally is not TDI, by Figure 56.1. But if D_0 is source-sink connected, system (57.6) is totally dual integral, as is shown in the following section.

57.4. Disjoint strong connectors

Similarly to the derivation of Theorem 57.3 from the Lucchesi-Younger theorem, the following generalization can be derived as a consequence of Theorem 56.1 (Schrijver [1982]):

Theorem 57.4. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Then the minimum size of a D_0-cut in D is equal to the maximum number of disjoint strong connectors in D for D_0.*

Proof. The proof is similar to the derivation of Theorem 57.3 from the Lucchesi-Younger theorem. We can assume that for any $u, v \in V$, if v is reachable in D_0 from u, then $(u, v) \in A_0$.

We show the theorem by induction on the number τ of arcs (u,v) of D for which $(v,u) \notin A_0$. If $\tau = 0$, the theorem is equivalent to Theorem 56.1 (by taking $C := \{(u,v) \mid (v,u) \in A\}$).

If $\tau > 0$, choose $(u,v) \in A$ with $(v,u) \notin A_0$. Let u' be a sink of D_0 with $(u,u') \in A_0$ and let v' be a source of D_0 with $(v',v) \in A_0$. As D_0 is source-sink connected we know that $(v',u') \in A_0$. Now introduce two new vertices, u'' and v'', and add arcs

(57.7) $$(u,u''), (u'',u'), (v'',u''), (v'',v), (v',v'')$$

to A_0. Moreover, replace arc (u,v) of A by (u'',v''). Let $\widetilde{D}_0 = (\widetilde{V}, \widetilde{A}_0)$ and $\widetilde{D} = (\widetilde{V}, \widetilde{A})$ denote the modified graphs.

This transformation decreases τ by 1. Again (57.4) holds. This implies that the two numbers in the theorem are invariant under the transformation. Hence the theorem follows by induction. ∎

The condition given in this theorem cannot be relaxed to condition (57.2), as Figure 56.1 shows.

Theorem 57.4 has the following special cases:

- $s, t \in V$ and $A_0 := \{(u,v) \mid u = t \text{ or } v = s\}$: Menger's theorem (Corollary 9.1b);
- V is the disjoint union of U and W, $A_0 = \{(u,w) \mid u \in U, w \in W\}$ and $A \subseteq \{(w,u) \mid w \in W, u \in U\}$: Gupta's edge-colouring theorem (Theorem 20.5);
- $r \in V$ and $A_0 = \{(v,r) \mid v \in V\}$: Edmonds' disjoint arborescences theorem (Corollary 53.1b);
- V is the disjoint union of U and W and $A_0 = \{(u,w) \mid u \in U, w \in W\}$: disjoint bibranchings theorem(Theorem 54.11);
- $D_0 = (V, A_0)$ is source-sink connected and $A \subseteq \{(u,v) \mid (v,u) \in A_0\}$: Corollary 56.1b.

An equivalent capacitated version of Theorem 57.4 reads:

Corollary 57.4a. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the minimum capacity of a D_0-cut in D is equal to the maximum number of strong connectors in D for D_0 such that any arc a of D is in at most $c(a)$ of them.*

Proof. Directly from Theorem 57.4 by replacing any arc a of D by $c(a)$ parallel arcs. ∎

Equivalently, in TDI terms:

Corollary 57.4b. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Then system* (57.6) *is totally dual integral.*

Proof. This is a reformulation of Corollary 57.4a. ∎

57.5. Complexity

As for the complexity of finding disjoint strong connectors for a source-sink connected digraph, the proof of Theorem 57.4 gives a polynomial-time reduction to finding a maximum number of disjoint directed cut covers in a subset of the arcs of a source-sink connected graph. The latter problem is solvable in polynomial time by the methods of Section 56.2.

The capacitated case can be solved in semi-strongly polynomial time (that is, where rounding is taken as one arithmetic operation) with the ellipsoid method (cf. Grötschel, Lovász, and Schrijver [1988]). A combinatorial semi-strongly polynomial-time algorithm is as follows.

Let be given a source-sink connected digraph $D_0 = (V, A_0)$, a digraph $D = (V, A)$, and a capacity function $c : A \to \mathbb{Z}_+$. We show that an optimum *fractional* packing of strong connectors subject to c can be found in strongly polynomial time. Then an *integer* packing can be found by rounding (like in Section 51.4), thus yielding a semi-strongly polynomial-time algorithm.

Define $\mathcal{C} := \{U \mid \emptyset \neq U \subset V, d^{\text{in}}_{A_0}(U) = 0\}$. To find an optimum fractional packing, let κ be the minimum of $c(\delta^{\text{in}}_A(U))$ taken over sets $U \in \mathcal{C}$. (κ can be computed with a maximum flow algorithm.) We keep a subcollection $\mathcal{U}$ of $\mathcal{C}$ with $c(\delta^{\text{in}}_A(U)) = \kappa$ for each $U \in \mathcal{U}$.

Choose a strong connector $B \subseteq A$ for A_0 with $d^{\text{in}}_B(U) = 1$ for each $U \in \mathcal{U}$. (This can be found in strongly polynomial time, by finding a minimum-length strong connector for length function $l := \sum_{U \in \mathcal{U}} \chi^{\delta^{\text{in}}_B(U)}$. It exists by Theorem 57.4.)

If $c = \mathbf{0}$, we are done. If $c \neq \mathbf{0}$, determine a rational λ as follows. First set $\lambda := \min\{c(a) \mid a \in B\}$. Next, iteratively, find a $U \in \mathcal{C}$ minimizing

$$(57.8) \qquad (c - \lambda \cdot \chi^B)(\delta^{\text{in}}_A(U)).$$

If this minimum value is less than $\kappa - \lambda$, reset

$$(57.9) \qquad \lambda := \frac{c(\delta^{\text{in}}_A(U)) - \kappa}{d^{\text{in}}_B(U) - 1},$$

and iterate. If the minimum is equal to $\kappa - \lambda$, this ends the inner iterations. Then we reset $c := c - \lambda \cdot \chi^B$, $\kappa := \kappa - \lambda$, and $\mathcal{U} := \mathcal{U} \cup \{U\}$, and (outer) iterate.

In each outer iteration, the number of arcs a with $c(a) > 0$ decreases or the intersecting family generated by $\mathcal{U}$ increases (since for the U added we have $d^{\text{in}}_B(U) > 1$). Hence the number of outer iterations is bounded by $|A| + |V|^3$ (see the argument given in the proof of Theorem 53.9).

In each outer iteration, the number of inner iterations is at most $|B|$. To see this, consider any inner iteration, and denote by λ' and U' the objects λ and U in the next inner iteration. As U minimizes (57.8), we know

$$(57.10) \qquad (c - \lambda \cdot \chi^B)(\delta_A^{\text{in}}(U)) \leq (c - \lambda \cdot \chi^B)(\delta_A^{\text{in}}(U')).$$

Moreover, if the next iteration is not the last iteration, then

$$(57.11) \qquad (c - \lambda' \cdot \chi^B)(\delta_A^{\text{in}}(U')) < \kappa - \lambda' = (c - \lambda' \cdot \chi^B)(\delta_A^{\text{in}}(U))$$

(the equality follows from definition (57.9), replacing λ by λ'). Now (57.10) and (57.11) imply

$$(57.12) \qquad \begin{aligned} &\lambda'(d_B^{\text{in}}(U) - d_B^{\text{in}}(U')) < c(\delta_A^{\text{in}}(U)) - c(\delta_A^{\text{in}}(U')) \\ &\leq \lambda(d_B^{\text{in}}(U) - d_B^{\text{in}}(U')). \end{aligned}$$

Since $\lambda' < \lambda$ (as (57.8) is less than $\kappa - \lambda$), we have $d_B^{\text{in}}(U') < d_B^{\text{in}}(U)$. Hence the number of inner iterations is at most $|B|$.

57.5a. Crossing families

Theorem 57.4 and part of Theorem 57.3 were generalized by Schrijver [1983b]. Let $\mathcal{C}$ be a *crossing family* of subsets of a set V; that is:

(57.13) if $U, W \in \mathcal{C}$ and $U \cap W \neq \emptyset$ and $U \cup W \neq V$, then $U \cap W \in \mathcal{C}$ and $U \cup W \in \mathcal{C}$.

Let $D = (V, A)$ be a digraph. Call $B \subseteq A$ a *$\mathcal{C}$-cut* if $B = \delta^{\text{in}}(U)$ for some $U \in \mathcal{C}$. Call $B \subseteq A$ a *$\mathcal{C}$-cover* if B intersects each $\mathcal{C}$-cut.

Let $\mathcal{C}$ be a crossing family of nonempty proper subsets of a set V. In Schrijver [1983b] it is shown that the following are equivalent:

(57.14)
(i) for each digraph $D = (V, A)$, the minimum size of a $\mathcal{C}$-cut is equal to the maximum number of disjoint $\mathcal{C}$-covers;
(ii) for each digraph $D = (V, A)$ and each length function $l : A \to \mathbb{Z}_+$, the minimum length of a $\mathcal{C}$-cover is equal to the maximum number of $\mathcal{C}$-cuts such that no arc a is in more than $l(a)$ of these cuts;
(iii) there are no V_1, V_2, V_3, V_4, V_5 in $\mathcal{C}$ with $V_1 \subseteq V_3 \subseteq V_5$, $V_1 \subseteq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subseteq V_5$.

The configuration described in (iii) is depicted in Figure 57.1. As directed graphs may have parallel arcs, property (57.14)(i) is equivalent to its capacitated version. So condition (57.14)(i) is equivalent to the total dual integrality of

$$(57.15) \qquad \begin{array}{ll} x_a \geq 0 & \text{for } a \in A, \\ x(B) \geq 1 & \text{for each } \mathcal{C}\text{-cover } B \subseteq A, \end{array}$$

for each digraph (V, A). Similarly, condition (57.14)(ii) is equivalent to the total dual integrality of

$$(57.16) \qquad \begin{array}{ll} x_a \geq 0 & \text{for } a \in A, \\ x(B) \geq 1 & \text{for each } \mathcal{C}\text{-cut } B \subseteq A, \end{array}$$

for each digraph (V, A).

Frank [1979b] showed that (57.14)(i) holds if $\mathcal{C}$ is an intersecting family. (For any intersecting family $\mathcal{C}$, (iii) holds if $V \notin \mathcal{C}$, which we may assume without loss of generality.)

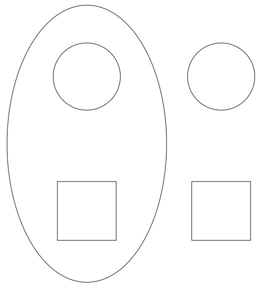

Figure 57.1
The configuration excluded in (57.14)(iii)**.** In this Venn-diagram, the collection is represented by the *interiors* of the ellipses and by the *exteriors* of the rectangles.

In (57.14)(i) and (ii) we require the min-max relation for cuts and covers to hold for *all* directed graphs on V. It is a more general problem to characterize pairs $(\mathcal{C}, D)$ of a crossing family $\mathcal{C}$ on V and a directed graph $D = (V, A)$ having the properties described in (57.14)(i) and (ii), respectively. For example, the Lucchesi-Younger theorem (Theorem 55.2), and its extension by Edmonds and Giles [1977], assert that if $\mathcal{C}$ is a crossing family on V and no arc of D leaves any set $U \in \mathcal{C}$, then $(\mathcal{C}, D)$ has the properties described in (ii). However, the example in Figure 56.1 shows that it generally does not have the property described in (57.14)(i). So for *fixed* graphs D, (57.14)(i) and (ii) are not equivalent.

Theorem 60.3 implies that a pair $(\mathcal{C}, D)$ has property (ii) if $\mathcal{C}$ is a crossing family and D a directed graph such that if $U_1, U_2, U_3 \in \mathcal{C}$ with $U_1 \subseteq V \setminus U_2 \subseteq U_3$, then no arc enters both U_1 and U_3. This generalizes the Lucchesi-Younger theorem.

We show the equivalence of (57.14)(ii) and (iii), for which we show a lemma indicating that condition (57.14)(iii) has a natural characterization in terms of total unimodularity.

For any collection $\mathcal{C}$ of subsets of a set V, let A be the collection of all ordered pairs of elements of V (making the complete directed graph $D = (V, A)$), and let $M_{\mathcal{C}}$ be the $\mathcal{C} \times A$ matrix with

(57.17) $$(M_{\mathcal{C}})_{U,a} := \begin{cases} 1 & \text{if } a \text{ enters } U, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 57.5α. *Let $\mathcal{C}$ be a cross-free collection of nonempty proper subsets of a set V. Then $M_{\mathcal{C}}$ is totally unimodular if and only if $\mathcal{C}$ satisfies* (57.14)(iii)*.*

Proof. To see necessity, let $M_{\mathcal{C}}$ be totally unimodular. Suppose that condition (57.14)(iii) is violated. So there exist V_1, V_2, V_3, V_4, V_5 in $\mathcal{C}$ with $V_1 \subseteq V_3 \subseteq V_5$, $V_1 \subseteq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subseteq V_5$. Choose $v_1 \in V_1$, $v_2 \in V \setminus V_2$, $v_4 \in V_4$, and $v_5 \in V \setminus V_5$. Define

(57.18) $$A_0 := \{(v_2, v_1), (v_4, v_1), (v_2, v_4), (v_5, v_4), (v_5, v_2)\}$$

(cf. Figure 57.2). Consider the submatrix of $M_{\mathcal{C}}$ with rows indexed by $V_1, \ldots, V_5$, and columns indexed by the arcs in A_0. Then, as one easily checks:

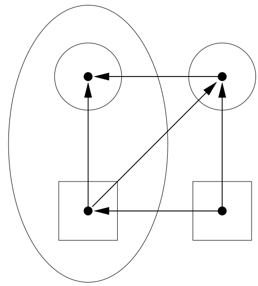

Figure 57.2

(57.19) each set in $\mathcal{C}_0$ is entered by exactly two arcs from A_0, and each arc in A_0 enters exactly two sets in $\mathcal{C}_0$.

So this submatrix has exactly two 1's in each row and each column, and hence is not totally unimodular.

To see sufficiency, let $\mathcal{C}$ satisfy (57.14)(iii). To prove that $M_\mathcal{C}$ is totally unimodular, we use the following characterization of Ghouila-Houri [1962b] (cf. Theorem 19.3 in Schrijver [1986b]): a matrix M is totally unimodular if and only if each collection R of rows of M can be partitioned into classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries $0, \pm 1$ only.

To check this condition, we can assume that we have chosen all rows of $M_\mathcal{C}$ (as any subset of the rows gives a matrix of the same type as $M_\mathcal{C}$). Make a digraph $D = (\mathcal{C}, A')$, where A' consists of all pairs (T, U) from $\mathcal{C}$ such that

(57.20) $T \subset U$, and there is no $W \in \mathcal{C}$ with $T \subset W \subset U$.

We show that the undirected graph underlying D' is bipartite, which will verify Ghouila-Houri's criterion: let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two colour classes; then any arc $a = (u, v)$ of D enters a chain of subsets in $\mathcal{C}$ (as $\mathcal{C}$ is cross-free), which subsets are alternatingly in $\mathcal{C}_1$ and $\mathcal{C}_2$. Hence the sum of the rows with index in $\mathcal{C}_1$ minus the sum of the rows with index in $\mathcal{C}_2$, has an entry 0 or ± 1 in position a.

To show that D' is bipartite, suppose that it has an (undirected) circuit of odd length. Since this circuit is odd, and since D' is acyclic, it follows that there are distinct $U_0, U_1, \ldots, U_k, U_{k+1}$ in $\mathcal{C}$, with $k \geq 3$, such that

(57.21) $(U_1, U_0), (U_1, U_2), (U_2, U_3), \ldots, (U_{k-1}, U_k), (U_{k+1}, U_k)$

belong to A'. So U_0 and U_2 are distinct minimal sets in $\mathcal{C}$ containing U_1 as a subset. As $\mathcal{C}$ is cross-free, $U_0 \cup U_2 = V$. Similarly, U_{k-1} and U_{k+1} are distinct maximal subsets of U_k, and hence $U_{k-1} \cap U_{k+1} = \emptyset$. As $U_2 \subseteq U_{k-1}$, it follows that $U_1 \subseteq U_0 \cap U_2$, $U_0 \cup U_2 = V$, $U_2 \cup U_{k+1} \subseteq U_k$, and $U_2 \cap U_{k+1} = \emptyset$. This contradicts (57.14)(iii). ∎

This gives the box-TDI result:

Theorem 57.5. *Let $\mathcal{C}$ be a crossing family of nonempty proper subsets of a set V satisfying* (57.14)(iii) *and let $D = (V, A)$ be a digraph. Then the system*

(57.22) $\quad x_a \geq 0 \quad$ for $a \in A$,
$\quad x(\delta^{\text{in}}(U)) \geq 1 \quad$ for $U \in \mathcal{C}$,

is box-TDI.

Proof. Let $w : A \to \mathbb{R}_+$. Consider the maximum value of

(57.23) $$\sum_{U \in \mathcal{C}} y_U$$

where $y : \mathcal{C} \to \mathbb{R}_+$ satisfies

(57.24) $$\sum_{U \in \mathcal{C}} y_U \chi^{\delta^{\text{in}}(U)} \leq w.$$

Choose $y : \mathcal{C} \to \mathbb{R}_+$ attaining the maximum, such that

(57.25) $$\sum_{U \in \mathcal{C}} y_U |U||V \setminus U|$$

is minimized. We show that the collection $\mathcal{F} := \{U \in \mathcal{C} \mid y_U > 0\}$ is cross-free; that is, for all $T, U \in \mathcal{F}$ one has

(57.26) $\quad T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$ or $T \cup U = V$.

Suppose that this is not true. Let $\alpha := \min\{y_T, y_U\}$. Decrease y_T and y_U by α, and increase $y_{T\cap U}$ and $y_{T\cup U}$ by α. Now (57.24) is maintained, and (57.23) did not change. However, (57.25) decreases (Theorem 2.1), contradicting our minimality assumption.

So $\mathcal{F}$ is cross-free. As $M_{\mathcal{F}}$ is totally unimodular by Lemma 57.5α, this gives the box-total dual integrality of (57.22) by Theorem 5.35. ∎

Condition (57.14)(iii) is necessary and sufficient for integrality of the polyhedron:

Corollary 57.5a. *For any crossing family $\mathcal{C}$ of nonempty proper subsets of a set V,* (57.22) *defines an integer polyhedron for each digraph $D = (V, A)$ if and only if condition* (57.14)(iii) *holds.*

Proof. Sufficiency follows from Theorem 57.5. To see necessity, suppose that (57.22)(iii) does not hold. Let $V_1, \ldots, V_5$ in $\mathcal{C}$ with $V_1 \subsetneq V_3 \subsetneq V_5$, $V_1 \subsetneq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subsetneq V_5$. Let $\mathcal{C}_0 := \{V_1, \ldots, V_5\}$ and $\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0$. Choose $v_1 \in V_1$, $v_2 \in V \setminus V_2$, $v_4 \in V_4$, $v_5 \in V \setminus V_5$. Let $D = (V, A)$ be a digraph, with $A = A_0 \cup A_1$, where A_0 is as defined in (57.18) and where

(57.27) $\quad A_1 := \{(u, v) \mid u, v \in V$ such that (u, v) enters no V_i $(i = 1, \ldots, 5)\}$.

Then

(57.28) $\quad$ each set in $\mathcal{C}_1$ is either entered by at least one arc in A_1 or by at least two arcs in A_0.

To see this, by definition of A_1, a subset U of V is entered by no arc in A_1 if and only if U belongs to the lattice generated by $\mathcal{C}_0$ (with respect to inclusion). This lattice consists of the sets

(57.29) $\emptyset, V, V_1, \ldots, V_5, V_1 \cup V_4, V_2 \cap V_3, (V_2 \cap V_3) \cup V_4, V_3 \cup V_4, V_2 \cap V_5,$

as (57.29) is closed under taking unions and intersections, and as each set in (57.29) is generated by taking unions and intersections from $\mathcal{C}_0$. Since each of the sets in (57.29), except $\emptyset$ and V, is entered by at least two arcs in A_0, we have (57.28).

(57.19) and (57.28) give:

(57.30) any $\mathcal{C}$-cover in A contains at least three arcs in A_0, and any $\mathcal{C}$-cut contains at least one arc in A_1 or at least two arcs in A_0.

Define $x : A \to \mathbb{Q}$ be $x := \chi^{A_1} + \frac{1}{2}\chi^{A_0}$ and a length function $l : A \to \mathbb{Z}$ by $l := \chi^{A_0}$. Then x satisfies (57.22) and $l^{\mathsf{T}} x = \frac{5}{2}$. However, $l(C) \geq 3$ for each $\mathcal{C}$-cover C. So (57.22) determines no integer polyhedron. ∎

Theorem 57.5 and Corollary 57.5a imply the equivalence of (57.14)(ii) and (iii). For the proof of the equivalence of (57.14)(i) and (iii), we refer to Schrijver [1983b].

Chapter 58

The traveling salesman problem

The traveling salesman problem (TSP) asks for a shortest Hamiltonian circuit in a graph. It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.

The problem shows up in practice not only in routing but also in various other applications like machine scheduling (ordering jobs), clustering, computer wiring, and curve reconstruction.

The traveling salesman problem is an NP-complete problem, and no polynomial-time algorithm is known. As such, the problem would not fit in the scope of the present book. However, the TSP is closely related to several of the problem areas discussed before, like 2-matching, spanning tree, and cutting planes, which areas actually were stimulated by questions prompted by the TSP, and often provide subroutines in solving the TSP.

Being NP-complete, the TSP has served as prototype for the development and improvement of advanced computational methods, to a large extent utilizing polyhedral techniques. The basis of the solution techniques for the TSP is branch-and-bound, for which good bounding techniques are essential. Here 'good' is determined by two, often conflicting, criteria: the bound should be *tight* and *fast* to compute. Polyhedral bounds turn out to be good candidates for such bounds.

58.1. The traveling salesman problem

Given a graph $G = (V, E)$, a *Hamiltonian circuit* in G is a circuit C with $VC = V$. The *symmetric traveling salesman problem* (TSP) is: given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

The directed version is as follows. Given a digraph $D = (V, A)$, a *directed Hamiltonian circuit*, or just a *Hamiltonian circuit*, in D is a directed circuit C with $VC = V$. The *asymmetric traveling salesman problem* (TSP or ATSP) is: given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

In the context of the traveling salesman problem, vertices are sometimes called *cities*, and a Hamiltonian circuit a *traveling salesman tour*. If the vertices are represented by points in the plane and each pair of vertices is connected by an edge of length equal to the Euclidean distance between the two points, one speaks of the *Euclidean traveling salesman problem*.

58.2. NP-completeness of the TSP

The problem of finding a Hamiltonian circuit and (hence) the traveling salesman problem are NP-complete. Indeed, in Theorem 8.11 and Corollary 8.11b we proved the NP-completeness of the directed and undirected Hamiltonian circuit problem. This implies the NP-completeness of the TSP, both in the undirected and the directed case:

Theorem 58.1. *The symmetric TSP and the asymmetric TSP are* NP-*complete.*

Proof. Given an undirected graph $G = (V, E)$, define $l(e) := 0$ for each edge e. Then G has a Hamiltonian circuit if and only if G has a Hamiltonian circuit of length ≤ 0. This reduces the undirected Hamiltonian circuit problem to the symmetric TSP.

One similarly shows the NP-completeness of the asymmetric TSP. ∎

This method also gives that the symmetric TSP remains NP-complete if the graph is complete and the length function satisfies the *triangle inequality*:

(58.1) $$l(uw) \leq l(uv) + l(vw) \text{ for all } u, v, w \in V.$$

Indeed, to test if a graph $G = (V, E)$ has a Hamiltonian circuit, define $l(uv) := 1$ if u and v are adjacent and $l(uv) := 2$ otherwise (for $u \neq v$). Then G has a Hamiltonian circuit if and only if there exists a traveling salesman tour of length $\leq |V|$.

Garey, Graham, and Johnson [1976] and Papadimitriou [1977a] showed that even the Euclidean traveling salesman problem is NP-complete. (Similarly for several other metrics, like l_1.) More on complexity can be found in Section 58.8b below.

58.3. Branch-and-bound techniques

The traveling salesman problem is NP-complete, and no polynomial-time algorithm is known. Most exact methods known are essentially enumerative, aiming at minimizing the enumeration. A general framework is that of *branch-and-bound.* The idea of branch-and-bound applied to the traveling salesman problem roots in papers of Tompkins [1956], Rossman and Twery [1958], and Eastman [1959]. The term 'branch and bound' was introduced by Little, Murty, Sweeney, and Karel [1963].

A rough, elementary description is as follows. Let $G = (V, E)$ be a graph and let $l : E \to \mathbb{R}$ be a length function. For any set $\mathcal{C}$ of Hamiltonian circuits, let $\mu(\mathcal{C})$ denote the minimum length of the Hamiltonian circuits in $\mathcal{C}$.

Keep a collection Γ of sets of Hamiltonian circuits and a function $\lambda : \Gamma \to \mathbb{R}$ satisfying:

(58.2) (i) $\bigcup \Gamma$ contains a shortest Hamiltonian circuit;
(ii) $\lambda(\mathcal{C}) \leq \mu(\mathcal{C})$ for each $\mathcal{C} \in \Gamma$.

A typical iteration is:

(58.3) Select a collection $\mathcal{C} \in \Gamma$ with $\lambda(\mathcal{C})$ minimal. Either find a circuit $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$ or replace $\mathcal{C}$ by (zero or more) smaller sets such that (58.2) is maintained.

Obviously, if we find $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$, then C is a shortest Hamiltonian circuit.

This method always terminates, but the method and its efficiency heavily depend on how the details in this framework are filled in: how to bound (that is, how to define and calculate $\lambda(\mathcal{C})$), how to branch (that is, which smaller sets replace $\mathcal{C}$), and how to find the circuit C.

As for branching, the classes $\mathcal{C}$ in Γ can be stored implicitly: for example, by prescribing sets B and F of edges such that $\mathcal{C}$ consists of all Hamiltonian circuits whose edge set contains B and is disjoint from F. Then we can split $\mathcal{C}$ by selecting an edge $e \in E \setminus (B \cup F)$ and replacing $\mathcal{C}$ by the classes determined by $B \cup \{e\}, F$ and by $B, F \cup \{e\}$ respectively.

As for bounding, one should choose $\lambda(\mathcal{C})$ that is fast to compute and close to $\mu(\mathcal{C})$. For this, polyhedral bounds seem good candidates, and in the coming sections we consider a number of them.

For finding the circuit $C \in \mathcal{C}$, a heuristic or exact method can be used. If it returns a circuit C with $l(C) > \lambda(\mathcal{C})$, we can delete all sets $\mathcal{C}'$ from Γ with $\lambda(\mathcal{C}') \geq l(C)$, thus saving computer space.

58.4. The symmetric traveling salesman polytope

The *(symmetric) traveling salesman polytope* of an undirected graph $G = (V, E)$ is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the Hamiltonian circuits. The TSP is equivalent to minimizing a function $l^{\mathsf{T}}x$ over the traveling salesman polytope. Hence this is NP-complete.

The NP-completeness of the TSP also implies that, unless NP=co-NP, no description in terms of inequalities of the traveling salesman polytope may be expected (Corollary 5.16a). In fact, as deciding if a Hamiltonian circuit exists is NP-complete, it is NP-complete to decide if the traveling salesman polytope is nonempty. Hence, if NP$\neq$co-NP, there exist no inequalities satisfied by the traveling salesman polytope such that their validity can be certified in polynomial time and such that they have no common solution.

58.5. The subtour elimination constraints

Polynomial-time computable lower bounds on the minimum length of a Hamiltonian circuit can be obtained by including the traveling salesman polytope in a larger polytope (a *relaxation*) over which $l^{\mathsf{T}}x$ can be minimized in polynomial time.

Dantzig, Fulkerson, and Johnson [1954a,1954b] proposed the following relaxation:

(58.4)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each edge } e, \\ \text{(ii)} & x(\delta(v)) = 2 & \text{for each vertex } v, \\ \text{(iii)} & x(\delta(U)) \geq 2 & \text{for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{array}$$

The integer solutions of (58.4) are precisely the incidence vectors of the Hamiltonian circuits. If (ii) holds, then (iii) is equivalent to:

(58.5) (iii') $x(E[U]) \leq |U| - 1$ for each $U \subseteq V$ with $\emptyset \neq U \neq V$.

These conditions are called the *subtour elimination constraints*.

It can be shown with the ellipsoid method that the minimum of $l^{\mathsf{T}}x$ over (58.4) can be found in strongly polynomial time (cf. Theorem 5.10). For this it suffices to show that the conditions (58.4) can be tested in polynomial time. This is easy for (i) and (ii). If (i) and (ii) are satisfied, we can test (iii) by taking x as capacity function, and test if there is a cut $\delta(U)$ of capacity less than 2, with $\emptyset \neq U \neq V$.

No combinatorial polynomial-time algorithm is known to minimize $l^{\mathsf{T}}x$ over (58.4). In practice, one can apply the simplex method to minimize $l^{\mathsf{T}}x$ over the constraints (i) and (ii), test if the solution satisfies (iii) by finding a cut $\delta(U)$ minimizing $x(\delta(U))$. If this cut has capacity at least 2, then x minimizes $l^{\mathsf{T}}x$ over (58.4). Otherwise, we can add the constraint $x(\delta(U)) \geq 2$ to the simplex tableau (a *cutting plane*), and iterate. (This method is implicit in Dantzig, Fulkerson, and Johnson [1954b].)

Branch-and-bound methods that incorporate such a cutting plane method to obtain bounds and that extend the cutting plane found to all other nodes of the branching tree to improve their bounds, are called *branch-and-cut*.

System (58.4) generally is not enough to determine the traveling salesman polytope: for the Petersen graph $G = (V, E)$, the vector x with $x_e = \frac{2}{3}$ for each $e \in E$ satisfies (58.4) but is not in the traveling salesman polytope of G (as it is empty).

Wolsey [1980] (also Shmoys and Williamson [1990]) showed that if G is complete and the length function l satisfies the triangle inequality, then the minimum of $l^{\mathsf{T}}x$ over (58.4) is at least $\frac{2}{3}$ times the minimum length of a Hamiltonian circuit. It is conjectured (cf. Carr and Vempala [2000]) that for any length function, a lower bound of $\frac{3}{4}$ holds (which is best possible). Related results are given by Papadimitriou and Vempala [2000] and Boyd and Labonté [2002] (who verified the conjecture for $n \leq 10$).

Maurras [1975] and Grötschel and Padberg [1979b] showed that, if G is the complete graph on V and $2 \leq |U| \leq |V|-2$, then the subtour elimination constraint (58.4)(iii) determines a facet of the traveling salesman polytope.

Chvátal [1989] showed the NP-completeness of recognizing if the bound given by the subtour elimination constraints is equal to the length of a shortest tour. He also showed that there is no nontrivial upper bound on the relative error of this bound.

58.6. 1-trees and Lagrangean relaxation

Held and Karp [1971] gave a method to find the minimum value of $l^{\mathsf{T}}x$ over (58.4), with the help of 1-*trees* and *Lagrangean relaxation*.

Let $G = (V, E)$ be a graph and fix a vertex, say 1, of G. A 1-*tree* is a subset F of E such that $|F \cap \delta(1)| = 2$ and such that $F \setminus \delta(1)$ forms a spanning tree on $V \setminus \{1\}$. So each Hamiltonian circuit is a 1-tree with all degrees equal to 2.

It is easy to find a shortest 1-tree F, as it consists of a shortest spanning tree of the graph $G - 1$, joined with the two shortest edges incident with vertex 1. Corollary 50.7c implies that the convex hull of the incidence vectors of 1-trees is given by:

(58.6)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(1)) = 2, & \\ \text{(iii)} & x(E[U]) \leq |U| - 1 & \text{for each nonempty } U \subseteq V \setminus \{1\}, \\ \text{(iv)} & x(E) = |V|. & \end{array}$$

Then (58.4) is equivalent to (58.6) added with (58.4)(ii).

The Lagrangean relaxation approach to find the minimum of $l^{\mathsf{T}}x$ over (58.4) is based on the following result. For any $y \in \mathbb{R}^V$ define

(58.7)
$$l_y(e) := l(e) - y_u - y_v$$

for $e = uv \in E$, and define

(58.8)
$$f(y) := 2y(V) + \min_F l_y(F),$$

where F ranges over all 1-trees. Christofides [1970] and Held and Karp [1970] observed that for each $y \in \mathbb{R}^V$:

(58.9)
$$f(y) \leq \text{the minimum length of a Hamiltonian circuit,}$$

since if C is a shortest Hamiltonian circuit, then $f(y) \leq 2y(V) + l_y(C) = l(C)$.

The function f is concave. Since a shortest 1-tree can be found fast, also $f(y)$ can be computed fast. Held and Karp [1970] showed:

Theorem 58.2. *The minimum value of* $l^{\mathsf{T}}x$ *over* (58.4) *is equal to the maximum value of* $f(y)$ *over* $y \in \mathbb{R}^V$.

Proof. This follows from general linear programming theory. Let $Ax = b$ be system (58.4)(ii) and let $Cx \geq d$ be system (58.6). As (58.4) is equivalent to $Ax = b$, $Cx \geq d$, we have, using LP-duality:

(58.10)
$$\begin{aligned}\min_{\substack{Ax=b\\ Cx\geq d}} l^{\mathsf{T}}x &= \max_{\substack{y,z\\ z\geq\mathbf{0}\\ y^{\mathsf{T}}A+z^{\mathsf{T}}C=l^{\mathsf{T}}}} y^{\mathsf{T}}b + z^{\mathsf{T}}d \\ &= \max_{y}(y^{\mathsf{T}}b + \max_{\substack{z\geq\mathbf{0}\\ z^{\mathsf{T}}C=l^{\mathsf{T}}-y^{\mathsf{T}}A}} z^{\mathsf{T}}d) = \max_{y}\left(y^{\mathsf{T}}b + \min_{Cx\geq d}(l^{\mathsf{T}}-y^{\mathsf{T}}A)x\right) \\ &= \max_{y} f(y).\end{aligned}$$

The last inequality holds as $Cx \geq d$ determines the convex hull of the incidence vectors of 1-trees. ∎

This translates the problem of minimizing $l^{\mathsf{T}}x$ over (58.4) to finding the maximum of the concave function f. We can find this maximum with a subgradient method (cf. Chapter 24.3 of Schrijver [1986b]). The vector y (the *Lagrangean multipliers*) can be used as a correction mechanism to urge the 1-tree to have degree 2 at each vertex. That is, if we calculate $f(y)$, and see that the 1-tree F minimizing $l_y(F)$ has degree more than 2 at a vertex v, we can increase l_y on $\delta(v)$ by decreasing y_v. Similarly, if the degree is less than 2, we can increase y_v. This method was proposed by Held and Karp [1970,1971].

The advantage of this approach is that one need not implement a linear programming algorithm with a constraint generation technique, but that instead it suffices to apply the more elementary tools of finding a shortest 1-tree and updating y. More can be found in Jünger, Reinelt, and Rinaldi [1995].

58.7. The 2-factor constraints

A strengthening of relaxation (58.4) is obtained by using the facts that each Hamiltonian circuit is a 2-factor and that the convex hull of the incidence vectors of 2-factors is known (Corollary 30.8a) (this idea goes back to Robinson [1949] for the asymmetric TSP and Bellmore and Malone [1971] for the symmetric TSP, and was used for the symmetric TSP by Grötschel [1977a] and Pulleyblank [1979b]):

(58.11)
(i) $0 \leq x_e \leq 1$ for each edge e,
(ii) $x(\delta(v)) = 2$ for each vertex v,
(iii) $x(\delta(U)) \geq 2$ for each $U \subseteq V$ with $\emptyset \neq U \neq V$,
(iv) $x(\delta(U) \setminus F) - x(F) \geq 1 - |F|$
for $U \subseteq V$, $F \subseteq \delta(U)$, F matching, $|F|$ odd.

Since a minimum-length 2-factor can be found in polynomial time, the inequalities (i), (ii), and (iv) can be tested in polynomial time (cf. Theorem 32.5). Hence the minimum of $l^{\mathsf{T}}x$ over (58.11) can be found in strongly polynomial time.

System (58.11) generally is not enough to determine the traveling salesman polytope, as can be seen, by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for each edge e.

Grötschel and Padberg [1979b] showed that, for complete graphs, each of the inequalities (58.11)(iv) determines a facet of the traveling salesman polytope (if $|F| \geq 3$). Boyd and Pulleyblank [1991] studied optimization over (58.11).

58.8. The clique tree inequalities

Grötschel and Pulleyblank [1986] found a large class of facet-inducing inequalities, the 'clique tree inequalities', that generalize the 'comb inequalities' (see below), which generalize both the subtour elimination constraints (58.4)(iii) and the 2-factor constraints (58.11)(iv). However, no polynomial-time test of clique tree inequalities is known.

A *clique tree inequality* is given by:

(58.12) $$\sum_{i=1}^{r} x(\delta(H_i)) + \sum_{j=1}^{s} x(\delta(T_j)) \geq 2r + 3s - 1,$$

where $H_1, \ldots, H_r$ are pairwise disjoint subsets of V and $T_1, \ldots, T_s$ are pairwise disjoint proper subsets of V such that

(58.13)
(i) no T_j is contained in $H_1 \cup \cdots \cup H_r$,
(ii) each H_i intersects an odd number of the T_j,
(iii) the intersection graph of $H_1, \ldots, H_r, T_1, \ldots, T_s$ is a tree.

(Here, the *intersection graph* is the graph with vertices $H_1, \ldots, H_r, T_1, \ldots, T_s$, two of them being adjacent if and only if they intersect. Each H_i is called a *handle* and each T_j a *tooth*.)

Theorem 58.3. *The clique tree inequality* (58.12) *is valid for the traveling salesman polytope.*

Proof. It suffices to show that each Hamiltonian circuit C satisfies:

(58.14) $$\sum_{i=1}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2r + 3s - 1.$$

We apply induction on r, the case $r = 0$ being easy (as it implies $s = 1$). For each $i = 1, \ldots, r$, let β_i be the number of T_j intersecting H_i.

If there is an i with $d_C(H_i) \geq \beta_i$, say $i = 1$, then, by parity, $d_C(H_1) \geq \beta_1 + 1$. The sets $H_2, \ldots, H_r, T_1, \ldots, T_s$ fall apart into β_1 collections of type (58.13), to which we can apply induction. Adding up the inequalities obtained, we get:

$$(58.15) \qquad \sum_{i=2}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2(r-1) + 3s - \beta_1.$$

Then (58.14) follows, as $d_C(H_1) \geq \beta_1 + 1$.

So we can assume that $d_C(H_i) \leq \beta_i - 1$ for each i. For all i, j, let $\alpha_{i,j} := 1$ if $T_j \cap H_i \neq \emptyset$ and C has no edge connecting $T_j \cap H_i$ and $T_j \setminus H_i$, and let $\alpha_{i,j} := 0$ otherwise. Then

$$(58.16) \qquad d_C(T_j) \geq 2 + 2\sum_{i=1}^{r} \alpha_{i,j},$$

since C restricted to T_j falls apart into at least $1 + \sum_{i=1}^{r} \alpha_{i,j}$ components (using (58.13)(i)).

Moreover, for each $i = 1, \ldots, r$, there exist at least $\beta_i - d_C(H_i)$ indices j with $\alpha_{i,j} = 1$. Hence

$$(58.17) \qquad \begin{aligned} \sum_{j=1}^{s} d_C(T_j) &\geq 2s + 2\sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_{i,j} \geq 2s + 2\sum_{i=1}^{r}(\beta_i - d_C(H_i)) \\ &\geq 2s + r + \sum_{i=1}^{r}(\beta_i - d_C(H_i)) = 2r + 3s - 1 - \sum_{i=1}^{r} d_C(H_i), \end{aligned}$$

since $\sum_{i=1}^{r} \beta_i = r + s - 1$, as the intersection graph of the H_i and the T_j is a tree with $r + s$ vertices, and hence with $r + s - 1$ edges.

(58.17) implies (58.14). ∎

Notes. Grötschel and Pulleyblank [1986] also showed that, if G is a complete graph, then any clique tree inequality determines a facet if and only if each H_i intersects at least three of the T_j.

The clique tree inequalities are not enough to determine the traveling salesman polytope, as is shown again by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for all $e \in E$.

The special case $r = 1$ of the clique tree inequality is called a *comb inequality*, and was introduced by Grötschel and Padberg [1979a] and proved to be facet-inducing (if G is complete and $s \geq 3$) by Grötschel and Padberg [1979b].

The special case of the comb inequality with $|H_1 \cap T_j| = 1$ for all $j = 1, \ldots, s$ is called a *Chvátal comb inequality*, introduced by Chvátal [1973b]. The special case of the Chvátal comb inequalities with $|T_j| = 2$ for each $j = 1, \ldots, s$ gives the 2-factor constraints (58.11)(iv) (since $2x(F) + \sum_{f \in F} x(\delta(f)) = 4|F|$).

No polynomial-time algorithm is know to test the clique tree inequalities, or the comb inequalities, or the Chvátal comb inequalities. Carr [1995,1997] showed that for each constant K, there is a polynomial-time algorithm to test the clique tree inequalities with at most K teeth and handles. (This can be done by first fixing intersection points of the $H_i \cap T_j$ (if nonempty) and points in $T_j \setminus (H_1 \cup \cdots \cup H_r)$, and next finding minimum-capacity cuts separating the appropriate sets of these points (taking x as capacity function). We can make them disjoint where necessary by the usual uncrossing techniques. As K is fixed, the number of vertices to be chosen is also bounded by a polynomial in $|V|$.)

Letchford [2000] gave a polynomial-time algorithm for testing a superclass of the comb inequalities in planar graphs. Related results are given in Carr [1996], Fleischer and Tardos [1996,1999], Letchford and Lodi [2002], and Naddef and Thienel [2002a,2002b].

58.8a. Christofides' heuristic for the TSP

Christofides [1976] designed the following algorithm to find a short Hamiltonian circuit in a complete graph $G = (V, E)$ (generally not the shortest however). It assumes a nonnegative length function l satisfying the following *triangle inequality*:

(58.18) $$l(uw) \leq l(uv) + l(vw)$$

for all $u, v, w \in V$.

First determine a shortest spanning tree T (with the greedy algorithm). Next, let U be the set of vertices that have odd degree in T. Find a shortest perfect matching M on U. Now $ET \cup M$ forms a set of edges such that each vertex has even degree. (If an edge occurs both in ET and in M, we take it as two parallel edges.) So we can make a closed path C such that each edge in $ET \cup M$ is traversed exactly once. Then C traverses each vertex at least once. By shortcutting we obtain a Hamiltonian circuit C' with $l(C') \leq l(C)$.

How far away is the length of C' from the minimum length μ of a Hamiltonian circuit?

Theorem 58.4. *$l(C') \leq \frac{3}{2}\mu$.*

Proof. Let C'' be a shortest Hamiltonian circuit. Then $l(T) \leq l(C'') = \mu$, since C'' contains a spanning tree. Also, $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$, since we can split C'' into two collections of paths, each having U as set of end vertices. They give two perfect matchings on U, of total length at most $l(C'')$ (by the triangle inequality (58.18)). Hence one of these matchings has length at most $\frac{1}{2}l(C'')$. So $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$.

Combining the two inequalities, we obtain

(58.19) $$l(C') \leq l(C) = l(T) + l(M) \leq \tfrac{3}{2}\mu,$$

which proves the theorem. ∎

The factor $\frac{3}{2}$ seems quite large, but it is the smallest factor for which a polynomial-time method is known. Don't forget moreover that it is a *worst-case* bound, and that in practice (or on average) the algorithm might have a much better performance.

Wolsey [1980] showed more strongly that (if l satisfies the triangle inequality) the length of the tour found by Christofides' algorithm, is at most $\frac{3}{2}$ times the lower bound based on the subtour elimination constraints (58.4). If all distances are 1 or 2, Papadimitriou and Yannakakis [1993] gave a polynomial-time algorithm with worst-case factor $\frac{7}{6}$. Hoogeveen [1991] analyzed the behaviour of Christofides' heuristic when applied to finding shortest Hamiltonian paths.

58.8b. Further notes on the symmetric traveling salesman problem

Adjacency of vertices of the symmetric traveling salesman polytope of a graph $G = (V, E)$ is co-NP-complete, as was shown by Papadimitriou [1978].

Norman [1955] remarked that the symmetric traveling salesman polytope of the complete graph K_n has dimension $\frac{1}{2}n(n-3) = \binom{n}{2} - n$ (if $n \geq 3$). Proofs were given by Maurras [1975] and Grötschel and Padberg [1979a].

The symmetric traveling salesman polytopes of K_n for small n were studied by Norman [1955], Boyd and Cunningham [1991], Christof, Jünger, and Reinelt [1991] ($n = 8$), and Naddef and Rinaldi [1992,1993]. Weinberger [1974a] showed that the up hull of the symmetric traveling salesman polytope of K_6 is not determined by inequalities with $0, 1$ coefficients only.

Rispoli and Cosares [1998] showed that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 4. Grötschel and Padberg [1985] conjecture that it is at most 2. (See Sierksma and Tijssen [1992] and Sierksma, Teunter, and Tijssen [1995] for supporting results.) Further work on the symmetric traveling salesman polytope includes Naddef and Rinaldi [1993], Queyranne and Wang [1993], Carr [2000], Cook and Dash [2001], and Naddef and Pochet [2001].

Rispoli [1998] showed that the monotonic diameter of the symmetric traveling salesman polytope of K_n is $\lfloor n/2 \rfloor - 1$ if $n \geq 6$. (The *monotonic diameter* of a polytope is the minimum λ such that for each linear function $l^{\mathsf{T}}x$ and each pair of vertices y, z such that $l^{\mathsf{T}}x$ is maximized over P at z, there is a $y - z$ path along vertices and edges of the polytope such that the function $l^{\mathsf{T}}x$ is monotonically nondecreasing and such that the number of edges in the path is at most λ.)

Sahni and Gonzalez [1976] showed that for any constant c, unless P=NP, there is no polynomial-time algorithm finding a Hamiltonian circuit of length at most c times the minimum length of a Hamiltonian circuit. Johnson and Papadimitriou [1985a] showed that unless P=NP there is no *fully polynomial approximation scheme* for the Euclidean traveling salesman problem (that is, there is no algorithm that gives for any $\varepsilon > 0$, a Hamiltonian circuit of length at most $1+\varepsilon$ times the minimum length of a Hamiltonian circuit, with running time bounded by a polynomial in the size of the problem and in $1/\varepsilon$).

However, Arora [1996,1997,1998] showed that for the Euclidean TSP there is a polynomial approximation scheme: there is an algorithm that gives, for any n vertices in the plane and any $\varepsilon > 0$, a Hamiltonian circuit of length at most $1 + \varepsilon$ times the minimum length of a Hamiltonian circuit, in $n^{O(1/\varepsilon)}$ time. The method also applies to several other metrics. Mitchell [1999] noticed that the methods of Mitchell [1996] imply similar results. Related work is reported in Trevisan [1997, 2000], Rao and Smith [1998], and Dumitrescu and Mitchell [2001]. Earlier work on plane TSP includes Karp [1977], Steele [1981], Moran [1984], Karloff [1989], and Clarkson [1991].

A polynomial-time approximation scheme for the traveling salesman problem where the length is determined by the shortest path metric in a weighted planar graph was given by Arora, Grigni, Karger, Klein, and Woloszyn [1998] (extending the unweighted case proved by Grigni, Koutsoupias, and Papadimitriou [1995]).

Yannakakis [1988,1991] showed that the traveling salesman problem on K_n cannot be expressed by a linear program of polynomial size that is invariant under the symmetric group on K_n. (A similar negative result was proved by Yannakakis for the perfect matching polytope.)

More valid inequalities for the symmetric traveling salesman polytope were given by Grötschel [1980a], Papadimitriou and Yannakakis [1984], Fleischmann [1988], Boyd and Cunningham [1991], Naddef [1992], Naddef and Rinaldi [1992], and Boyd, Cunningham, Queyranne, and Wang [1995].

Jünger, Reinelt, and Rinaldi [1995] gave a comparison of the values of various relaxations for several instances of the symmetric traveling salesman problem. Johnson, McGeoch, and Rothberg [1996] report on an 'asymptotic experimental analysis' of the Held-Karp bound. A probabilistic analysis of the Held-Karp bound for the Euclidean TSP was presented by Goemans and Bertsimas [1991].

A worst-case comparison of several classes of valid inequalities for the traveling salesman polytope was given by Goemans [1995]. Several integer programming formulations for the TSP were compared by Langevin, Soumis, and Desrosiers [1990]. Althaus and Mehlhorn [2000,2001] showed that the subtour elimination constraints solve traveling salesman problems coming from curve reconstruction, under appropriate sampling conditions.

Semidefinite programming was applied to the symmetric TSP by Cvetković, Čangalović, and Kovačević-Vujčić [1999a,1999b] and Iyengar and Çezik [2001].

Let $G = (V, E)$ be an undirected graph. The symmetric traveling salesman polytope of G is a face of the convex hull of all integer solutions of

$$\text{(58.20)} \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(U)) \geq 2 & \text{for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{array}$$

Fonlupt and Naddef [1992] characterized for which graphs G each vertex x of (58.20) is integer and has $x(\delta(v)) \equiv 0 \pmod 2$ for each vertex v of G.

Grötschel [1980a] studied the *monotone traveling salesman polytope* of a graph, which is the convex hull of the incidence vectors of subsets of Hamiltonian circuits.

Cornuéjols, Fonlupt, and Naddef [1985] considered the related problem of finding a shortest tour in a graph such that each vertex is traversed *at least* once, and the related polytope (cf. Naddef and Rinaldi [1991]). Further and related studies (also on shortest k-connected spanning subgraphs, on the 'Steiner network problem', and on the (equivalent) 'survivable network design problem') include Bienstock, Brickell, and Monma [1990], Grötschel and Monma [1990], Monma, Munson, and Pulleyblank [1990], Kelsen and Ramachandran [1991,1995], Barahona and Mahjoub [1992,1995], Chopra [1992,1994], Goemans and Williamson [1992,1995a], Grötschel, Monma, and Stoer [1992], Han, Kelsen, Ramachandran, and Tarjan [1992,1995], Khuller and Vishkin [1992,1994], Nagamochi and Ibaraki [1992a], Cheriyan, Kao, and Thurimella [1993], Gabow, Goemans, and Williamson [1993,1998], Garg, Santosh, and Singla [1993], Naddef and Rinaldi [1993], Queyranne and Wang [1993], Williamson, Goemans, Mihail, and Vazirani [1993,1995], Aggarwal and Garg [1994], Goemans, Goldberg, Plotkin, Shmoys, Tardos, and Williamson [1994], Khuller, Raghavachari, and Young [1994,1995a,1996], Mahjoub [1994,1997], Agrawal, Klein, and Ravi [1995], Khuller and Raghavachari [1995], Ravi and Williamson [1995, 1997], Cheriyan and Thurimella [1996a,2000], Didi Biha and Mahjoub [1996], Fernandes [1997,1998], Carr and Ravi [1998], Cheriyan, Sebő, and Szigeti [1998,2001], Auletta, Dinitz, Nutov, and Parente [1999], Czumaj and Lingas [1998,1999], Jain [1998,2001], Fonlupt and Mahjoub [1999], Fleischer, Jain, and Williamson [2001], Cheriyan, Vempala, and Vetta [2002], and Gabow [2002]. This problem relates to connectivity augmentation — see Chapter 63.

58.9. The asymmetric traveling salesman problem

We next consider the *asymmetric* traveling salesman problem. Let $D = (V, A)$ be a directed graph. The *(asymmetric) traveling salesman polytope* of D is the convex hull of the incidence vectors (in $\mathbb{R}^A$) of Hamiltonian circuits in D. Again, since the asymmetric traveling salesman problem is NP-complete, we know that unless NP=co-NP there is no system of linear inequalities that describes the traveling salesman polytope of a digraph such that their validity can be certified in polynomial time.

Again, we can obtain lower bounds on the minimum length of a Hamiltonian circuit in D by including the traveling salesman polytope in a larger polytope (a *relaxation*) over which $l^{\mathsf{T}}x$ can be minimized in polynomial time. The analogue of relaxation (58.4) for the directed case is:

(58.21)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_a \leq 1 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for } v \in V, \\ \text{(iii)} & x(\delta^{\text{out}}(v)) = 1 & \text{for } v \in V, \\ \text{(iv)} & x(\delta^{\text{in}}(U)) \geq 1 & \text{for } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{array}$$

With the ellipsoid method, the minimum of $l^{\mathsf{T}}x$ over (58.21) can be found in strongly polynomial time. However, no combinatorial polynomial-time algorithm is known. (The relaxation (i), (ii), (iii) is due to Robinson [1949].)

Grötschel and Padberg [1977] showed that each inequality (58.21)(iv) determines a facet of the traveling salesman polytope of the complete directed graph, if $2 \leq |U| \leq |U| - 2$. (This result was announced in Grötschel and Padberg [1975].)

(58.21) is not enough to determine the traveling salesman polytope, even not for digraphs on 4 vertices only. This is shown by Figure 58.1. Another example is obtained from the Petersen graph, by replacing each edge by two oppositely oriented edges and putting value $\frac{1}{3}$ on each arc.

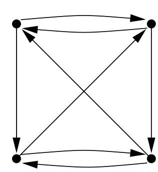

Figure 58.1

Setting $x_a := \frac{1}{2}$ for each arc a, we have a vector x satisfying (58.21) but not belonging to the traveling salesman polytope.

58.10. Directed 1-trees

As in the undirected case, Held and Karp [1970] showed that the minimum of $l^{\mathsf{T}}x$ over (58.21) can be obtained as follows.

Let $D = (V, A)$ be a digraph and fix a vertex 1 of D. Call a subset F of A a *directed* 1-*tree* if F contains exactly one arc, a say, entering 1 and if $F \setminus \{a\}$ is a directed 1-tree such that exactly one arc leaves 1.[7] Each Hamiltonian circuit is a directed 1-tree, and a minimum-length directed 1-tree can be found in strongly polynomial time (by adapting Theorem 52.1).

From Corollary 52.3b one may derive that the convex hull of the incidence vectors of directed 1-trees is determined by:

$$\text{(58.22)} \quad \begin{array}{rll} \text{(i)} & 0 \leq x_a \leq 1 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for each } v \in V, \\ \text{(iii)} & x(\delta^{\text{out}}(1)) = 1, & \\ \text{(iv)} & x(\delta^{\text{in}}(U)) \geq 1 & \text{for each nonempty } U \subseteq V \setminus \{1\}. \end{array}$$

Again, a *Lagrangean relaxation* approach can find the minimum of $l^{\mathsf{T}}x$ over (58.21), for $l \in \mathbb{R}^A$. For any $y \in \mathbb{R}^V$ define

$$\text{(58.23)} \quad l_y(a) := l(a) - y(u)$$

for any arc $a = (u, v) \in A$, and define

$$\text{(58.24)} \quad f(y) := \min_F l_y(F) + y(V),$$

where F ranges over directed 1-trees.

Then the minimum of $l^{\mathsf{T}}x$ over (58.21) is equal to the maximum of $f(y)$ over $y \in \mathbb{R}^V$. The proof is similar to that of Theorem 58.2.

58.10a. An integer programming formulation

The integer solutions of (58.21) are precisely the incidence vectors of Hamiltonian circuits, so it gives an integer programming formulation of the asymmetric traveling salesman problem. The system has exponentially many constraints. A.W. Tucker showed in 1960 (cf. Miller, Tucker, and Zemlin [1960]) that the asymmetric TSP can be formulated as the following integer programming problem, of polynomial size only. Set $n := |V|$, fix a vertex v_0 of D, and minimize $l^{\mathsf{T}}x$ where $x \in \mathbb{Z}^A$ and $z \in \mathbb{R}^V$ are such that

$$\text{(58.25)} \quad \begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for } v \in V, \\ \text{(ii)} & x(\delta^{\text{out}}(v)) = 1 & \text{for } v \in V, \\ \text{(iv)} & z_u - z_v + nx_a \leq n - 1 & \text{for } a = (u, v) \in A \text{ with } u, v \neq v_0. \end{array}$$

[7] Held and Karp used the term 1-*arborescence* for a directed 1-tree. To avoid confusion with r-arborescence (a slightly different notion), we have chosen for directed 1-tree.

The conditions (i), (ii), and (iii) and the integrality of x guarantee that x is the incidence vector of a set C of arcs forming directed circuits partitioning V. Then condition (iv) says the following. For any arc $a = (u, v)$ not incident with v_0, one has: if a belongs to C, then $z_u \leq z_v - 1$; if a does not belong to C, then $z_u - z_v \leq n - 1$. This implies that C contains no directed circuit disjoint from v_0. Hence C is a Hamiltonian circuit.

Conversely, for any incidence vector x of a Hamiltonian circuit, one can find $z \in \mathbb{R}^V$ satisfying (58.25).

Unfortunately, the linear programming bound one may derive from (58.25) is generally much worse than that obtained from (58.21).

58.10b. Further notes on the asymmetric traveling salesman problem

Bartels and Bartels [1989] gave a system of inequalities determining the traveling salesman polytope of the complete directed graph on 5 vertices (correcting Heller [1953a] and Kuhn [1955a]).

Padberg and Rao [1974] showed that the diameter of the asymmetric traveling salesman polytope of the complete directed graph on n vertices is equal to 1 if $3 \leq n \leq 5$, and to 2 if $n \geq 6$. Rispoli [1998] showed that the monotonic diameter of the asymmetric traveling salesman polytope of the complete directed graph on n vertices equals $\lfloor n/3 \rfloor$ if $n \geq 3$. (For the definition of monotonic diameter, see Section 58.8b.)

Adjacency of vertices of the asymmetric traveling salesman polytope of a graph $G = (V, E)$ is co-NP-complete, as was shown by Papadimitriou [1978][8]. The number of edges of the asymmetric traveling salesman polytope was estimated by Sarangarajan [1997].

H.W. Kuhn (cf. Heller [1953a], Kuhn [1955a]) claimed that the dimension of the asymmetric traveling salesman polytope of the complete directed graph on n vertices is equal to $n^2 - 3n + 1$ (if $n \geq 3$). A proof of this was supplied by Grötschel and Padberg [1977]. Further work on this polytope is reported in Kuhn [1991].

More valid inequalities for the asymmetric traveling salesman polytope were given by Grötschel and Padberg [1977], Grötschel and Wakabayashi [1981a,1981b], Balas [1989], Fischetti [1991,1992,1995], Balas and Fischetti [1993,1999], and Queyranne and Wang [1995].

A polytope generalizing the directed 1-tree polytope and the asymmetric traveling salesman polytope, the 'fixed-outdegree 1-arborescence polytope', was studied by Balas and Fischetti [1992]. Another polyhedron related to the asymmetric traveling salesman polytope was studied by Chopra and Rinaldi [1996].

Billera and Sarangarajan [1996] showed that each 0,1 polytope is affinely equivalent to the traveling salesman polytope of some directed graph.

Frieze, Karp, and Reed [1992,1995] investigated the tightness of the assignment bound (determined by (58.21)(i)-(iii)). Williamson [1992] compared the Held-Karp lower bound for the asymmetric TSP with the assignment bound.

Carr and Vempala [2000] related the relative error of the asymmetric TSP bound obtained from (58.21) to that of the symmetric TSP bound obtained from (58.4).

[8] Murty [1969] gave a characterization of adjacency that was shown to be false by Rao [1976].

Padberg and Sung [1991] compared different formulations of the asymmetric traveling salesman problem.

An analogue of Christofides' algorithm (Section 58.8a) for the asymmetric case is not known: no factor c and polynomial-time algorithm are known that give a Hamiltonian circuit in a digraph of length at most c times the length of a shortest Hamiltonian circuit, even not if the lengths satisfy the triangle inequality.

58.11. Further notes on the traveling salesman problem

58.11a. Further notes

There is an abundance of papers presenting algorithms, heuristics, and computational results for the traveling salesman problem. We give a short selection of it.

Milestones in solving large-scale symmetric traveling salesman problems were achieved by Dantzig, Fulkerson, and Johnson [1954b] (42 cities), Held and Karp [1962] (48 cities), Karg and Thompson [1964] (57 cities), Held and Karp [1971] (64 cities), Helbig Hansen and Krarup [1974] (80 cities), Camerini, Fratta, and Maffioli [1975] (100 cities), Grötschel [1980b] (120 cities), Crowder and Padberg [1980] and Padberg and Hong [1980] (318 cities), Padberg and Rinaldi [1987] (532 cities), Grötschel and Holland [1991] (666 cities), Padberg and Rinaldi [1990b,1991] (2392 cities), Applegate, Bixby, Chvátal, and Cook [1995] (7397 cities), and Applegate, Bixby, Chvátal, and Cook [1998] (13,509 cities). Although the complexity of a TSP instance is not simply a function of the number of cities, these papers represent substantial steps forward in developing computational techniques for the traveling salesman problem.

Dynamic programming approaches were proposed by Bellman [1962] and Held and Karp [1962]. Several methods were compared by computer experiments by Lin [1965]. The Lagrangean relaxation technique was introduced by Christofides [1970] and Held and Karp [1970,1971]. The Held-Karp method was implemented and extended by Helbig Hansen and Krarup [1974], Smith and Thompson [1977], and Volgenant and Jonker [1982,1983]. Related work includes Bazaraa and Goode [1977].

Miliotis [1976,1978] described a constraint generation approach, mixing subtour elimination constraints with Gomory cuts or with branching. Focusing on the asymmetric TSP are Little, Murty, Sweeney, and Karel [1963] (first reports on a branch-and-bound method), Bellmore and Malone [1971] (on the effect of the subtour elimination constraints), (cf. Garfinkel [1973], Smith, Srinivasan, and Thompson [1977], Lenstra and Rinnooy Kan [1978], Carpaneto and Toth [1980b], Zhang [1997a]), Balas and Christofides [1981] (a Lagrangean approach based on the assignment problem, solving randomly generated asymmetric TSP's with up to 325 cities), Miller and Pekny [1989,1991], Pekny and Miller [1992], and Carpaneto, Dell'Amico, and Toth [1995].

Further bounds for the symmetric and asymmetric TSP were given by Christofides [1972], Carpaneto, Fischetti, and Toth [1989] and Fischetti and Toth [1992].

Important heuristics (algorithms that yield a tour that is expected to be short, but not necessarily shortest) and local search techniques include the *nearest neighbour heuristic*: always go to the closest city not yet visited (Menger [1932a], Gavett

[1965], Bellmore and Nemhauser [1968]), the *Lin-Kernighan heuristic*: start with a Hamiltonian circuit and iteratively replace a limited number of edges by other edges as long as it makes the circuit shorter (Lin and Kernighan [1973]), and Christofides' heuristic discussed in Section 58.8a. From the further work on, and analyses of, heuristics and local search techniques we mention Christofides and Eilon [1972], Rosenkrantz, Stearns, and Lewis [1977], Cornuéjols and Nemhauser [1978], Frieze [1979], d'Atri [1980], Bentley and Saxe [1980], Ong and Moore [1984], Golden and Stewart [1985] (survey), Johnson and Papadimitriou [1985b] (survey), Karp and Steele [1985] (survey), Johnson, Papadimitriou, and Yannakakis [1988], Kern [1989], Bentley [1990,1992], Papadimitriou [1992] (showing that unless P=NP, any local search method taking polynomial time per iteration, can lead to a locally optimum tour that is arbitrarily far from the optimum), Fredman, Johnson, McGeoch, and Ostheimer [1993,1995], Chandra, Karloff, and Tovey [1994,1999], Tassiulas [1997], and Frieze and Sorkin [2001]. A survey and comparison of heuristics and local search techniques for the traveling salesman problem was given by Johnson and McGeoch [1997].

Polynomial-time solvable special cases of the traveling salesman problem were given by Gilmore and Gomory [1964a,1964b], Gilmore [1966], Lawler [1971a], Sysło [1973], Cornuéjols, Naddef, and Pulleyblank [1983], and Hartvigsen and Pulleyblank [1994]. Surveys of such problems were given by Gilmore, Lawler, and Shmoys [1985] and Burkard, Deĭneko, van Dal, van der Veen, and Woeginger [1998].

The standard reference book on the traveling salesman problem, covering a wide variety of aspects, was edited by Lawler, Lenstra, Rinnooy Kan, and Shmoys [1985]. In this book, Grötschel and Padberg [1985] considered the traveling salesman polytope, Padberg and Grötschel [1985] computation with the help of polyhedra, Johnson and Papadimitriou [1985a] the computational complexity of the TSP, and Balas and Toth [1985] branch-and-bound method methods. Computational methods and results are surveyed in the book by Reinelt [1994].

Survey articles on the traveling salesman problem were given by Gomory [1966], Bellmore and Nemhauser [1968], Gupta [1968], Tyagi [1968], Burkard [1979], Christofides [1979], Grötschel [1982] (also on other NP-complete problems), and Johnson and McGeoch [1997] (local search techniques). Introductions are given in the books by Minieka [1978], Sysło, Deo, and Kowalik [1983], Cook, Cunningham, Pulleyblank, and Schrijver [1998], and Korte and Vygen [2000]. An insightful survey of the computational methods for the symmetric TSP was given by Jünger, Reinelt, and Rinaldi [1995]. A framework for guaranteeing quality of TSP solutions was presented by Jünger, Thienel, and Reinelt [1994]. An early survey on branch-and-bound method techniques was given by Lawler and Wood [1966].

Barvinok, Johnson, Woeginger, and Woodroofe [1998] showed that there is a polynomial-time algorithm to find a *longest* Hamiltonian circuit in a complete graph with length determined by a polyhedral norm. Related work was done by Barvinok [1996]. More on the longest Hamiltonian circuit can be found in Fisher, Nemhauser, and Wolsey [1979], Serdyukov [1984], Kostochka and Serdyukov [1985], Kosaraju, Park, and Stein [1994], Hassin and Rubinstein [2000,2001], and Bläser [2002].

58.11b. Historical notes on the traveling salesman problem

Mathematically, the traveling salesman problem is related to, in fact generalizes, the question for a Hamiltonian circuit in a graph. This question goes back to Kirkman

[1856] and Hamilton [1856,1858] and was also studied by Kowalewski [1917b,1917a] — see Biggs, Lloyd, and Wilson [1976]. We restrict our survey to the traveling salesman problem in its general form.

The mathematical roots of the traveling salesman problem are obscure. Dantzig, Fulkerson, and Johnson [1954a] say:

> It appears to have been discussed informally among mathematicians at mathematics meetings for many years.

A 1832 manual

The traveling salesman problem has a natural interpretation, and Müller-Merbach [1983] detected that the problem was formulated in a 1832 manual for the successful traveling salesman, *Der Handlungsreisende — wie er sein soll und was er zu thun hat, um Aufträge zu erhalten und eines glücklichen Erfolgs in seinen Geschäften gewiß zu sein — Von einem alten Commis-Voyageur*[9] ('ein alter Commis-Voyageur' [1832]). (Whereas the politically correct nowadays prefer to speak of the traveling sales*person* problem, the manual presumes that the 'Handlungsreisende' is male, and it warns about the risks of women in or out of business.)

The booklet contains no mathematics, and formulates the problem as follows:

> Die Geschäfte führen die Handlungsreisenden bald hier, bald dort hin, und es lassen sich nicht füglich Reisetouren angeben, die für alle vorkommende Fälle passend sind; aber es kann durch eine zweckmäßige Wahl und Eintheilung der Tour, manchmal so viel Zeit gewonnen werden, daß wir es nicht glauben umgehen zu dürfen, auch hierüber einige Vorschriften zu geben. Ein Jeder möge so viel davon benutzen, als er es seinem Zwecke für dienlich hält; so viel glauben wir aber davon versichern zu dürfen, daß es nicht wohl thunlich sein wird, die Touren durch Deutschland in Absicht der Entfernungen und, worauf der Reisende hauptsächlich zu sehen hat, des Hin- und Herreisens, mit mehr Oekonomie einzurichten. Die Hauptsache besteht immer darin: so viele Orte wie möglich mitzunehmen, ohne den nämlichen Ort zweimal berühren zu müssen.[10]

The manual suggests five tours through Germany (one of them partly through Switzerland). In Figure 58.2 we compare one of the tours with a shortest tour, found with 'modern' methods. (Most other tours given in the manual do not qualify for 'die Hauptsache' as they contain subtours, so that some places are visited twice.)

Menger's Botenproblem 1930

K. Menger seems to be the first mathematician to have written about the traveling salesman problem. The root of his interest is given in his paper Menger [1928c]. In

---

[9] 'The traveling salesman — how he should be and what he has to do, to obtain orders and to be sure of a happy success in his business — by an old traveling salesman'

[10] Business brings the traveling salesman now here, then there, and no travel routes can be properly indicated that are suitable for all cases occurring; but sometimes, by an appropriate choice and arrangement of the tour, so much time can be gained, that we don't think we may avoid giving some rules also on this. Everybody may use that much of it, as he takes it for useful for his goal; so much of it however we think we may assure, that it will not be well feasible to arrange the tours through Germany with more economy in view of the distances and, which the traveler mainly has to consider, of the trip back and forth. The main point always consists of visiting as many places as possible, without having to touch the same place twice.

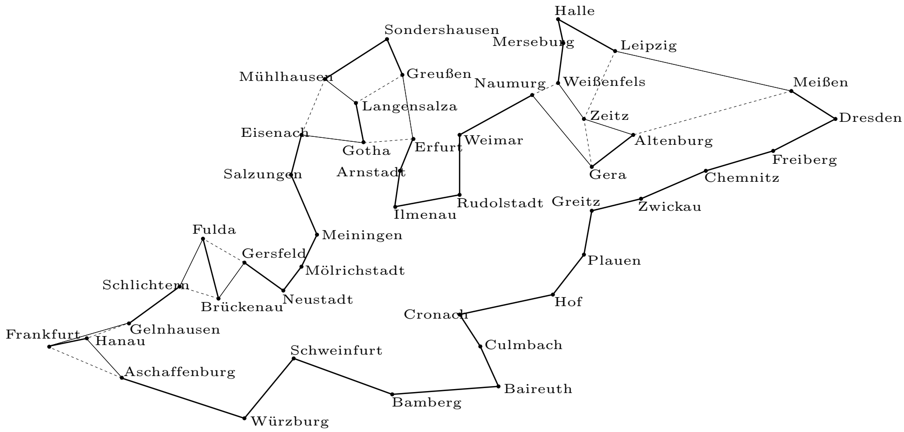

Figure 58.2

A tour along 45 German cities, as described in the 1832 traveling salesman manual, is given by the unbroken (bold and thin) lines (1285 km). A shortest tour is given by the unbroken bold and by the dashed lines (1248 km). We have taken geodesic distances — taking local conditions into account, the 1832 tour might be optimum.

this, he studies the *length* $l(C)$ of a simple curve C in a metric space S, which is, by definition,

$$l(C) := \sup \sum_{i=1}^{n-1} \operatorname{dist}(x_i, x_{i+1}), \tag{58.26}$$

where the supremum ranges over all choices of $x_1, \ldots, x_n$ on C *in the order determined by* C. What Menger showed is that we may relax this to finite subsets X of C and minimize over all possible orderings of X. To this end he defined, for any finite subset X of a metric space, $\lambda(X)$ to be the shortest length of a path through X (in graph terminology: a *Hamiltonian path*), and he showed that

$$l(C) = \sup_X \lambda(X), \tag{58.27}$$

where the supremum ranges over all finite subsets X of C. It amounts to showing that for each $\varepsilon > 0$ there is a finite subset X of C such that $\lambda(X) \geq l(C) - \varepsilon$.

Menger [1929a] sharpened this to:

$$l(C) = \sup_X \kappa(X), \tag{58.28}$$

where again the supremum ranges over all finite subsets X of C, and where $\kappa(X)$ denotes the minimum length of a *spanning tree* on X.

These results were reported also in Menger [1930]. In a number of other papers, Menger [1928b,1929b,1929a] gave related results on these new characterizations of the length function.

The parameter $\lambda(X)$ clearly is close to the practical interpretation of the traveling salesman problem. This relation was made explicit by Menger in the session of 5 February 1930 of his *mathematisches Kolloquium* in Vienna. Menger [1931a, 1932a] reported that he first asked if a further relaxation is possible by replacing $\kappa(X)$ by the minimum length of an (in current terminology) *Steiner tree* connecting X — a spanning tree on a superset of X in S. (So Menger toured along some basic combinatorial optimization problems.) This problem was solved for Euclidean spaces by Mimura [1933].

Next Menger posed the traveling salesman problem, as follows:

> Wir bezeichnen als *Botenproblem* (weil diese Frage in der Praxis von jedem Postboten, übrigens auch von vielen Reisenden zu lösen ist) die Aufgabe, für endlichviele Punkte, deren paarweise Abstände bekannt sind, den kürzesten die Punkte verbindenden Weg zu finden. Dieses Problem ist natürlich stets durch endlichviele Versuche lösbar. Regeln, welche die Anzahl der Versuche unter die Anzahl der Permutationen der gegebenen Punkte herunterdrücken würden, sind nicht bekannt. Die Regel, man solle vom Ausgangspunkt erst zum nächstgelegenen Punkt, dann zu dem diesem nächstgelegenen Punkt gehen usw., liefert im allgemeinen nicht den kürzesten Weg.[11]

So Menger asked for a shortest Hamiltonian path through the given points. He was aware of the complexity issue in the traveling salesman problem, and he realized that the now well-known nearest neighbour heuristic might not give an optimum solution.

Harvard, Princeton 1930-1934

Menger spent the period September 1930-February 1931 as visiting lecturer at Harvard University. In one of his seminar talks at Harvard, Menger presented his results (quoted above) on lengths of arcs and shortest paths through finite sets of points. According to Menger [1931b], a suggestion related to this was given by Hassler Whitney, who at that time did his Ph.D. research in graph theory at Harvard. This paper of Menger however does not mention if the practical interpretation was given in the seminar talk.

The year after, 1931-1932, Whitney was a National Research Council Fellow at Princeton University, where he gave a number of seminar talks. In a seminar talk, he mentioned the problem of finding the shortest route along the 48 States of America.

There are some uncertainties in this story. It is not sure if Whitney spoke about the 48 States problem during his 1931-1932 seminar talks (which talks he did give), or later, in 1934, as is said by Flood [1956] in his article on the traveling salesman problem:

> This problem was posed, in 1934, by Hassler Whitney in a seminar talk at Princeton University.

That memory can be shaky might be indicated by the following two quotes. Dantzig, Fulkerson, and Johnson [1954a] remark:

[11] We denote by *messenger problem* (since in practice this question should be solved by each postman, anyway also by many travelers) the task to find, for finitely many points whose pairwise distances are known, the shortest route connecting the points. Of course, this problem is solvable by finitely many trials. Rules which would push the number of trials below the number of permutations of the given points, are not known. The rule that one first should go from the starting point to the closest point, then to the point closest to this, etc., in general does not yield the shortest route.

> Both Flood and A.W. Tucker (Princeton University) recall that they heard about the problem first in a seminar talk by Hassler Whitney at Princeton in 1934 (although Whitney, recently queried, does not seem to recall the problem).

However, when asked by David Shmoys, Tucker replied in a letter of 17 February 1983 (see Hoffman and Wolfe [1985]):

> I cannot confirm or deny the story that I heard of the TSP from Hassler Whitney. If I did (as Flood says), it would have occurred in 1931-32, the first year of the old Fine Hall (now Jones Hall). That year Whitney was a postdoctoral fellow at Fine Hall working on Graph Theory, especially planarity and other offshoots of the 4-color problem. ... I was finishing my thesis with Lefschetz on n-manifolds and Merrill Flood was a first year graduate student. The Fine Hall Common Room was a very lively place — 24 hours a day.

(Whitney finished his Ph.D. at Harvard University in 1932.)

Another uncertainty is in which form Whitney has posed the problem. That he might have focused on finding a shortest route along the 48 states in the U.S.A., is suggested by the reference by Flood, in an interview on 14 May 1984 with Tucker [1984a], to the problem as the '48 States Problem of Hassler Whitney'. In this respect Flood also remarked:

> I don't know who coined the peppier name 'Traveling Salesman Problem' for Whitney's problem, but that name certainly has caught on, and the problem has turned out to be of very fundamental importance.

TSP, Hamiltonian paths, and school bus routing

Flood [1956] remembered that in 1937, A.W. Tucker pointed out to him the connections of the TSP with Hamiltonian games and Hamiltonian paths in graphs:

> I am indebted to A.W. Tucker for calling these connections to my attention, in 1937, when I was struggling with the problem in connection with a schoolbus routing study in New Jersey.

In the following quote from the interview by Tucker [1984a], Flood referred to school bus routing in a different state (West Virginia), and he mentioned the involvement in the TSP of Koopmans, who spent 1940-1941 at the Local Government Surveys Section of Princeton University ('the Princeton Surveys'):

> Koopmans first became interested in the "48 States Problem" of Hassler Whitney when he was with me in the Princeton Surveys, as I tried to solve the problem in connection with the work by Bob Singleton and me on school bus routing for the State of West Virginia.

1940

In 1940, some papers appeared that study the traveling salesman problem, in a different context. They seem to be the first containing mathematical results on the problem.

In the American continuation of Menger's *mathematisches Kolloquium*, Menger [1940] returned to the question of the shortest path through a given set of points in a metric space, followed by investigations of Milgram [1940] on the shortest Jordan curve that covers a given, not necessarily finite, set of points in a metric space. As the set may be infinite, a shortest curve need not exist.

Fejes [1940] investigated the problem of a shortest curve through n points in the unit square. In consequence of this, Verblunsky [1951] showed that its length is less than $2 + \sqrt{2.8n}$. Later work in this direction includes Few [1955], Beardwood, Halton, and Hammersley [1959], Steele [1981], Moran [1984], Karloff [1989], and Goddyn [1990].

Lower bounds on the expected value of a shortest path through n random points in the plane were studied by Mahalanobis [1940] in order to estimate the cost of a sample survey of the acreage under jute in Bengal. This survey took place in 1938 and one of the major costs in carrying out the survey was the transportation of men and equipment from one survey point to the next. He estimated (without proof) the minimum length of a tour along n random points in the plane, for Euclidean distance:

> It is also easy to see in a general way how the journey time is likely to behave. Let us suppose that n sampling units are scattered at random within any given area ; and let us assume that we may treat each such sample unit as a geometrical point. We may also assume that arrangements will usually be made to move from one sample point to another in such a way as to keep the total distance travelled as small as possible ; that is, we may assume that the path traversed in going from one sample point to another will follow a straight line. In this case it is easy to see that the mathematical expectation of the total length of the path travelled in moving from one sample point to another will be $(\sqrt{n} - 1/\sqrt{n})$. The cost of the journey from sample to sample will therefore be roughly proportional to $(\sqrt{n} - 1/\sqrt{n})$. When n is large, that is, when we consider a sufficiently large area, we may expect that the time required for moving from sample to sample will be roughly proportional to $\sqrt{n}$, where n is the total number of samples in the given area. If we consider the journey time per sq. mile, it will be roughly proportional to $\sqrt{y}$, where y is the density of number of sample units per sq. mile.

This research was continued by Jessen [1942], who estimated empirically a similar result for l_1-distance (Manhattan distance), in a statistical investigation of a sample survey for obtaining farm facts in Iowa:

> If a route connecting y points located at random in a fixed area is minimized, the total distance, D, of that route is[12]
>
> $$D = d\left(\frac{y-1}{\sqrt{y}}\right)$$
>
> where d is a constant.
> This relationship is based upon the assumption that points are connected by direct routes. In Iowa the road system is a quite regular network of mile square mesh. There are very few diagonal roads, therefore, routes between points resemble those taken on a checkerboard. A test wherein several sets of different members of points were located at random on an Iowa county road map, and the minimum distance of travel from a given point on the border of the county through all the points and to an end point (the county border nearest the last point on route), revealed that
>
> $$D = d\sqrt{y}$$
>
> works well. Here y is the number of randomized points (border points not included). This is of great aid in setting up a cost function.

[12] at this point, Jessen referred in a footnote to Mahalanobis [1940].

Marks [1948] gave a proof of Mahalanobis' bound. In fact he showed that $\sqrt{\frac{1}{2}A}(\sqrt{n}-1/\sqrt{n})$ is a lower bound, where A is the area of the region. Ghosh [1949] showed that this bound asymptotically is close to the expected value, by giving a heuristic for finding a tour, yielding an upper bound of $1.27\sqrt{An}$. He also observed the complexity of the problem:

> After locating the n random points in a map of the region, it is very difficult to find out *actually* the shortest path connecting the points, unless the number n is very small, which is seldom the case for a large-scale survey.

TSP, transportation, and assignment

As is the case for several other combinatorial optimization problems, the RAND Corporation in Santa Monica, California, played an important role in the research on the TSP. Hoffman and Wolfe [1985] write that

> John Williams urged Flood in 1948 to popularize the TSP at the RAND Corporation, at least partly motivated by the purpose of creating intellectual challenges for models outside the theory of games. In fact, a prize was offered for a significant theorem bearing on the TSP. There is no doubt that the reputation and authority of RAND, which quickly became the intellectual center of much of operations research theory, amplified Flood's advertizing.

(John D. Williams was head of the Mathematics Division of RAND at that time.)

At RAND, researchers considered the idea of transferring the successful methods for the transportation problem to the traveling salesman problem. Flood [1956] mentioned that this idea was brought to his attention by Koopmans in 1948. In the interview with Tucker [1984a], Flood remembered:

> George Dantzig and Tjallings Koopmans met with me in 1948 in Washington, D.C., at the meeting of the International Statistical Institute, to tell me excitedly of their work on what is now known as the linear programming problem and with Tjallings speculating that there was a significant connection with the Traveling Salesman Problem.

The issue was taken up in a RAND Report by Julia Robinson [1949], who, in an 'unsuccessful attempt' to solve the traveling salesman problem, considered, as a relaxation, the assignment problem, for which she found a cycle reduction method. The relation is that the assignment problem asks for an optimum permutation, and the TSP for an optimum *cyclic* permutation.

Robinson's RAND report might be the earliest mathematical reference using the term 'traveling salesman problem':

> The purpose of this note is to give a method for solving a problem related to the traveling salesman problem. One formulation is to find the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington. More generally, to find the shortest closed curve containing n given points in the plane.

Flood wrote (in a letter of 17 May 1983 to E.L. Lawler) that Robinson's report stimulated several discussions on the TSP of him with his research assistant at RAND, D.R. Fulkerson, during 1950-1952[13].

It was noted by Beckmann and Koopmans [1952] that the TSP can be formulated as a quadratic assignment problem, for which however no fast methods are known.

[13] Fulkerson started at RAND only in March 1951.

Dantzig, Fulkerson, Johnson 1954

Fundamental progress on the traveling salesman was made in a seminal paper by the RAND researchers Dantzig, Fulkerson, and Johnson [1954a] — according to Hoffman and Wolfe [1985] 'one of the principal events in the history of combinatorial optimization'. The paper introduced several new methods for solving the traveling salesman problem that are now basic in combinatorial optimization. In particular, it shows the importance of *cutting planes* for combinatorial optimization.

While the subtour elimination constraints (58.4)(iii) are enough to cut off the noncyclic permutation matrices from the polytope of doubly stochastic matrices (determined by (58.4)(i) and (ii)), they generally do not yield all facets of the traveling salesman polytope, as was observed by Heller [1953a]: there exist doubly stochastic matrices, of any order $n \geq 5$, that satisfy (58.4) but are not a convex combination of cyclic permutation matrices.

The subtour elimination constraints can nevertheless be useful for the TSP, since it gives a lower bound for the optimum tour length if we minimize over the constraints (58.4). This lower bound can be calculated with the simplex method, taking the (exponentially many) constraints (58.4)(iii) as *cutting planes* that can be added during the process when needed. In this way, Dantzig, Fulkerson, and Johnson were able to find the shortest tour along cities chosen in the 48 U.S. states and Washington, D.C. Incidentally, this is close to the problem mentioned by Julia Robinson in 1949 (and maybe also by Whitney in the 1930s).

The Dantzig-Fulkerson-Johnson paper gives no algorithm, but rather gives a tour and proves its optimality with the help of the subtour elimination constraints. This work forms the basis for most of the later work on large-scale traveling salesman problems.

Early studies of the traveling salesman polytope were reported by Heller [1953a,1953b,1955a,1955b,1956a,1956b], Kuhn [1955a], Norman [1955], and Robacker [1955b], who also made computational studies of the probability that a random instance of the traveling salesman problem needs the subtour elimination constraints (58.4)(iii) (cf. Kuhn [1991]). This made Flood [1956] remark on the intrinsic complexity of the traveling salesman problem:

> Very recent mathematical work on the traveling-salesman problem by I. Heller, H.W. Kuhn, and others indicates that the problem is fundamentally complex. It seems very likely that quite a different approach from any yet used may be required for succesful treatment of the problem. In fact, there may well be no general method for treating the problem and impossibility results would also be valuable.

Flood mentioned a number of other applications of the traveling salesman problem, in particular in machine scheduling, brought to his attention in a seminar talk at Columbia University in 1954 by George Feeney.

Other work on the traveling salesman problem in the 1950s was done by Morton and Land [1955] (a linear programming approach with a 3-exchange heuristic), Barachet [1957] (a graphic solution method), Bock [1958], Croes [1958] (a heuristic), and Rossman and Twery [1958]. In a reaction to Barachet's paper, Dantzig, Fulkerson, and Johnson [1959] showed that their method yields the optimality of Barachet's (heuristically found) solution.

In 1962, the soap company Proctor and Gamble run a contest, requiring to solve a traveling salesman problem along 33 U.S. cities. Little, Murty, Sweeney, and Karel [1963] report:

> The traveling salesman problem recently achieved national prominence when a soap company used it as the basis of a promotional contest. Prizes up to $10,000 were offered for identifying the most correct links in a particular 33-city problem. Quite a few people found the best tour. (The tie-breaking contest for these successful mathematicians was to complete a statement of 25 words or less on "I like...because...".) A number of people, perhaps a little over-educated, wrote the company that the problem was impossible—an interesting misinterpretation of the state of the art.

Chapter 59

Matching forests

Giles [1982a,1982b,1982c] introduced the concept of a *matching forest* in a mixed graph (V, E, A), which is a subset F of $E \cup A$ such that $F \cap A$ is a branching and $F \cap E$ is a matching only covering roots of the branching $F \cap A$. Equivalently, F contains no circuit (in the underlying undirected graph) and each $v \in V$ is head of at most one $e \in F$. (Here, for an undirected edge e, both ends of e are called head of e.)
Matching forests generalize both matchings in undirected graphs and branchings in directed graphs. Giles gave a polynomial-time algorithm to find a maximum-weight matching forest, yielding as a by-product a characterization of the matching forest polytope (the convex hull of the incidence vectors of matching forests).
Giles' results generalize the polynomial-time solvability and the polyhedral characterizations for matchings (Chapters 24–26) and for branchings (Chapter 52).

59.1. Introduction

A *mixed graph* is a triple (V, E, A), where (V, E) is an undirected graph and (V, A) is a directed graph. In this chapter, a graph can have multiple edges, but no loops. The *underlying* undirected graph of a mixed graph is the undirected graph obtained from the mixed graph by forgetting the orientations of the directed edges.

As usual, if an edge e is directed from u to v, then u is called the *tail* and v the *head* of e. In this chapter, if e is undirected and connects u and v, then both u and v will be called *head* of e.

A subset F of $E \cup A$ is called a *matching forest* if F contains no circuits (in the underlying undirected graph) and any vertex v is head of at most one edge in F. We call a vertex v a *root* of F if v is head of no edge in F. We denote the set of roots of F by $R(F)$.

It is convenient to consider the relations of matching forests with matchings in undirected graphs and branchings in directed graphs: M is a matching in an undirected graph (V, E) if and only if M is a matching forest in the mixed graph $(V, E, \emptyset)$. In this case, the roots of M are the vertices not covered by M. Similarly, B is a branching in a directed graph (V, A) if and only if B

is a matching forest in the mixed graph $(V, \emptyset, A)$. In this case, the concept of root of a branching and root of a matching forest coincide.

In turn, we can characterize matching forests in terms of matchings and branchings: for any mixed graph (V, E, A), a subset F of $E \cup A$ is a matching forest if and only if $F \cap A$ is a branching in (V, A) and $F \cap E$ is a matching in (V, E) such that $F \cap E$ only covers roots of $F \cap A$.

It will be useful to observe the following formulas, for any matching forest F in a mixed graph (V, E, A), setting $M := F \cap E$ and $B := F \cap A$:

(59.1) $\qquad R(F) = R(M) \cap R(B)$ and $V = R(M) \cup R(B)$.

In fact, for any matching M in (V, E) and any branching B in (V, A), the set $M \cup B$ is a matching forest if and only if $R(M) \cup R(B) = V$.

59.2. The maximum size of a matching forest

Giles [1982a] described a min-max formula for the maximum size of a matching forest. It can be derived from the Tutte-Berge formula with the following direct formula:

Theorem 59.1. *Let (V, E, A) be a mixed graph and let $\mathcal{K}$ be the collection of those strong components K of the directed graph (V, A) that satisfy $d_A^{\mathrm{in}}(K) = 0$. Consider the undirected graph H with vertex set $\mathcal{K}$, where two distinct $K, L \in \mathcal{K}$ are adjacent if and only if there is an edge in E connecting K and L. Then the maximum size of a matching forest in (V, E, A) is equal to*

(59.2) $\qquad \nu(H) + |V| - |\mathcal{K}|.$

Here $\nu(H)$ denotes the maximum size of a matching in H.

Proof. Let M' be a matching in H of size $\nu(H)$. Then M' yields a matching M of size $\nu(H)$ in (V, E), where each edge in M connects two components in $\mathcal{K}$. Now there exists a branching B in (V, A) such that B has exactly $|\mathcal{K}|$ roots, such that each $K \in \mathcal{K}$ contains exactly one root, and such that each vertex covered by M is a root of B. (To see that such a branching B exists, choose, for any $K \in \mathcal{K}$ not intersecting M, an arbitrary vertex in K. Let X be the set of chosen vertices together with the vertices covered by M. As X intersects each $K \in \mathcal{K}$, each vertex in V is reachable in (V, A) by a directed path from X. Hence there exists a branching B with root set X. This B has the required properties.)

Then $M \cup B$ is a matching forest, of size $\nu(H) + |V| - |\mathcal{K}|$ (as B has size $|V| - |\mathcal{K}|$).

To see that there is no larger matching forest, let F be any matching forest. Let $U := \bigcup \mathcal{K}$. Then F has at most $|V \setminus U|$ edges with at least one head in $V \setminus U$. Since no directed edge enters U, all other edges are contained in

U. So it suffices to show that F has at most $\nu(H)+|U|-|\mathcal{K}|$ edges contained in U.

Let N be the set of (necessarily undirected) edges in F connecting two different components in $\mathcal{K}$. For each $K \in \mathcal{K}$, let α_K be the number of edges in N incident with K. Then

(59.3) $$|N| - \sum_{K\in\mathcal{K}} \max\{0, \alpha_K - 1\} \leq \nu(H),$$

since by deleting, for each $K \in \mathcal{K}$, at most $\max\{0, \alpha_K - 1\}$ edges from N incident with K, we obtain a matching in the graph H defined above.

We have moreover that any $K \in \mathcal{K}$ spans at most $|K| - \max\{1, \alpha_K\}$ edges of F. With (59.3) this implies that the number of edges in F contained in U is at most

(59.4) $$|N| + \sum_{K\in\mathcal{K}} (|K| - \max\{1, \alpha_K\}) \leq \nu(H) + \sum_{K\in\mathcal{K}} (|K| - 1)$$
$$= \nu(H) + |U| - |\mathcal{K}|,$$

as required. ∎

The method described in this proof also directly implies that a maximum-size matching forest can be found in polynomial time (Giles [1982a]).

59.3. Perfect matching forests

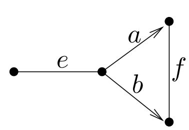

Figure 59.1
$\{e, f\}$ and $\{e, a, b\}$ are perfect matching forests.

A matching forest F is called *perfect* if each vertex is head of exactly one edge in F. (So a perfect matching forest need not be a maximum-size matching forest — cf. Figure 59.1.) The following is easy to see:

(59.5) A mixed graph (V, E, A) contains a perfect matching forest F if and only if the graph (V, E) contains a matching M such that each strong component K of (V, A) with $d^{\mathrm{in}}(K) = 0$ is intersected by at least one edge in M.

Indeed, if a perfect matching forest F exists, then $M := F \cap E$ is such a matching. Conversely, if such a matching M exists, any vertex is reachable by a directed path from at least one vertex covered by M; hence M can be augmented with directed arcs to a perfect matching forest.

This shows (59.5), which implies the following characterization for perfect matching forests of Giles [1982b]:

Theorem 59.2. *Let (V, E, A) be a mixed graph and let $\mathcal{K}$ be the collection of strong components K of (V, A) with $d_A^{\text{in}}(K) = 0$. Then (V, E, A) has a perfect matching forest if and only if for each $U \subseteq V$ and $\mathcal{L} \subseteq \mathcal{K}$ the graph $(V, E) - U$ has at most $|U| + \left|\bigcup \mathcal{L}\right| - |\mathcal{L}|$ odd components that are contained in $\bigcup \mathcal{L}$.*

Proof. Extend $G = (V, E)$ by, for each $K \in \mathcal{K}$, a clique C_K of size $|K| - 1$, such that each vertex in C_K is adjacent to each vertex in K. This makes the undirected graph H. Then (V, E, A) has a perfect matching forest if and only if graph H has a matching covering $\bigcup \mathcal{K}$. So we can apply Corollary 24.6a. ∎

This method also gives a polynomial-time algorithm to find a perfect matching forest.

59.4. An exchange property of matching forests

As a preparation for characterizing the matching forest polytope, we show an exchange property of matching forests. It generalizes the well-known and trivial exchange property of matchings in an undirected graph, based on considering the union of two matchings.

Lemma 59.3α. *Let F_1 and F_2 be matching forests in a mixed graph (V, E, A). Let $s \in R(F_2) \setminus R(F_1)$. Then there exist matching forests F_1' and F_2' such that $F_1' \cap F_2' = F_1 \cap F_2$, $F_1' \cup F_2' = F_1 \cup F_2$, $s \in R(F_1')$, and*

(59.6)
- (i) $|F_1'| < |F_1|$,
- or (ii) $|F_1'| = |F_1|$ and $|R(F_1')| > |R(F_1)|$,
- or (iii) $|F_1'| = |F_1|$, $R(F_1') = (R(F_1) \setminus \{t\}) \cup \{s\}$ for some $t \in R(F_1)$, and $|R(F_1' \cap A) \cap K| = |R(F_1 \cap A) \cap K|$ for each strong component K of the directed graph (V, A).

Proof. We may assume that F_1 and F_2 partition $E \cup A$, as we can delete edges that are not in $F_1 \cup F_2$, and add parallel edges to those in $F_1 \cap F_2$.

Define $M_i := F_i \cap E$ and $B_i := F_i \cap A$ for $i = 1, 2$. Let $\mathcal{K}$ be the collection of strong components K of the directed graph (V, A) with $\delta_A^{\text{in}}(K) = \emptyset$. Then each set in $\mathcal{K}$ intersects both $R(B_1)$ and $R(B_2)$, and $\{v\} \in \mathcal{K}$ for each $v \in R(B_1) \cap R(B_2)$.

So each $K \in \mathcal{K}$ with $|K| \geq 2$ intersects $R(B_1)$ and $R(B_2)$ in disjoint subsets. Hence we can choose for each such K

(59.7) a pair $e_K \subseteq K$ consisting of a vertex in $R(B_1) \setminus R(B_2)$ and a vertex in $R(B_2) \setminus R(B_1)$.

Let N be the set of pairs e_K for $K \in \mathcal{K}$ with $|K| \geq 2$. So N consists of disjoint pairs.

Then the undirected graph H on V with edge set

(59.8) $M_1 \cup M_2 \cup N$

consists of a number of vertex-disjoint paths and circuits, since no vertex in $R(B_1) \setminus R(B_2)$ is covered by M_2, and no vertex in $R(B_2) \setminus R(B_1)$ is covered by M_1.

Moreover, s has degree at most one in H. Indeed, s is not covered by M_2, as $s \in R(F_2) = R(M_2) \cap R(B_2)$. If s is covered by M_1, then $s \in R(B_1)$, and so $s \in R(B_1) \cap R(B_2)$, implying that s is not covered by N.

So s is the starting vertex of a path component P of H (possibly only consisting of s). Let Y be the set of edges in $M_1 \cup M_2$ occurring in P, and set

(59.9) $M_1' := M_1 \triangle Y$ and $M_2' := M_2 \triangle Y$

(where $\triangle$ denotes symmetric difference). Since Y is the union of the edge sets of some path components of the graph $(V, M_1 \cup M_2)$, we know that M_1' and M_2' are matchings again.

Then, obviously, $R(M_1')$ and $R(M_2')$ arise from $R(M_1)$ and $R(M_2)$ by exchanging these sets on VP; that is:

(59.10) $R(M_1') = (R(M_1) \setminus VP) \cup (R(M_2) \cap VP)$ and
$R(M_2') = (R(M_2) \setminus VP) \cup (R(M_1) \cap VP)$.

We show that a similar operation can be performed with respect to B_1 and B_2; that is, we show that there exist disjoint branchings B_1' and B_2' in (V, A) satisfying

(59.11) $R(B_1') = (R(B_1) \setminus VP) \cup (R(B_2) \cap VP)$ and
$R(B_2') = (R(B_2) \setminus VP) \cup (R(B_1) \cap VP)$.

By Lemma 53.2α, it suffices to show that each $K \in \mathcal{K}$ intersects both sets in (59.11). If $|K| = 1$, then K is contained in both $R(B_1)$ and $R(B_2)$, and hence in both sets in (59.11). If $|K| \geq 2$, then e_K intersects both $R(B_1)$ and $R(B_2)$. Since e_K is either contained in VP or disjoint from VP, e_K intersects both sets in (59.11). Hence, as $e_K \subseteq K$, also K intersects both sets in (59.11). Therefore, branchings B_1' and B_2' satisfying (59.11) exist.

(59.10) and (59.11) imply:

(59.12) $F_1' := M_1' \cup B_1'$ and $F_2' := M_2' \cup B_2'$ are matching forests.

To see this, we must show that $R(M_1') \cup R(B_1') = V$ and $R(M_2') \cup R(B_2') = V$. Since $R(M_1) \cup R(B_1) = V$ and $R(M_2) \cup R(B_2) = V$, this follows directly from (59.10) and (59.11). This shows (59.12).

Since $R(F) = R(M) \cap R(B)$ for any matching forest F (with $M := F \cap E$ and $B := F \cap A$), (59.10) and (59.11) imply that also $R(F_1')$ and $R(F_2')$ arise from $R(F_1)$ and $R(F_2)$ by swapping on P; that is:

(59.13) $R(F_1') = (R(F_1) \setminus VP) \cup (R(F_2) \cap VP)$ and
$R(F_2') = (R(F_2) \setminus VP) \cup (R(F_1) \cap VP)$.

This implies:

(59.14) $s \in R(F_1') \setminus R(F_2')$,

since $s \in VP$ and $s \in R(F_2) \setminus R(F_1)$.

We study the effects of the exchanges (59.10) and (59.11), to show that one of the alternatives (59.6) holds. It is based on the following observations on the sizes of M_1' and B_1'. Let t be the last vertex of P (possible $t = s$).

Suppose that none of the alternatives (59.6) hold. If $s = t$, then s is not covered by M_1, and so $M_1' = M_1$ and $R(B_1') = R(B_1) \cup \{s\}$, implying $|F_1'| < |F_1|$, which is alternative (59.6)(i). So $s \neq t$.

By the exchanges made, $|M_1| - |M_1'| = |M_1 \cap EP| - |M_2 \cap EP|$ and $|R(F_1)| - |R(F_1')| = |R(F_1) \cap VP| - |R(F_2) \cap VP|$. This gives, as $|F_1'| \geq |F_1|$, since alternative (59.6)(i) does not hold:

(59.15) $|M_1 \cap EP| - |M_2 \cap EP| + |R(F_1) \cap VP| - |R(F_2) \cap VP|$
$= |M_1| + |R(F_1)| - |M_1'| - |R(F_1')| = |F_1'| - |F_1| \geq 0.$

(The last equality holds as $|F_i'| = |V| - |M_i'| - |R(F_i')|$ for $i = 1, 2$, since $|F_i'| + |M_i'|$ is the number of heads of edges in F_i'.)

We next note:

(59.16) no intermediate vertex v of P belongs to $R(F_1) \cup R(F_2)$.

For suppose that $v \in R(F_1)$. Then (as v is an intermediate vertex of P) v is covered by M_2 and some $e_K \in N$. Hence $v \in R(B_2)$, and therefore $v \notin R(B_1)$ (by (59.7)), contradicting the fact that $v \in R(F_1)$. One similarly shows that $v \notin R(F_2)$, proving (59.16).

As $s \in R(F_2) \setminus R(F_1)$, (59.16) implies that

(59.17) $|R(F_1) \cap VP| \leq |R(F_2) \cap VP|$, with equality if and only if $t \in R(F_1) \setminus R(F_2)$.

With (59.15) this gives that $|M_1 \cap EP| \geq |M_2 \cap EP|$.

Let k be the number of edges in $M_1 \cup M_2$ on P. Note that the edges in $M_1 \cup M_2$ occur along P alternatingly in M_1 and M_2, as any intermediate $e_K \in N$ on P connects an edge in M_1 and an edge in M_2 (as by (59.7), $e_K \in N$ consists of a vertex not in $R(B_2)$ and a vertex not in $R(B_1)$).

Suppose that k is odd. Then $|M_1 \cap EP| = |M_2 \cap EP| + 1$. So the last edge in $M_1 \cup M_2$ along P (seen from s) belongs to M_1. Moreover, one has that $t \notin R(F_1)$. For if $t \in R(F_1)$, then t is not covered by M_1, and hence t belongs to some $e_K = \{v, t\} \in N$ with v covered by M_1. Hence $v \in R(B_1)$, and hence $t \notin R(B_1)$ (by (59.7)), contradicting the fact that $t \in R(F_1)$. So $t \notin R(F_1)$.

Then (59.17) implies that $|R(F_2) \cap VP| > |R(F_1) \cap VP|$. This implies with (59.15) that $|F_1'| = |F_1|$ (as $|M_1 \cap EP| = |M_2 \cap EP| + 1$), and with (59.13) that $|R(F_1')| > |R(F_1)|$. So (59.6)(ii) holds, a contradiction.

So k is even, and hence $|M_1 \cap EP| = |M_2 \cap EP|$, which implies with (59.13), (59.15), and (59.17) that $|R(F_1)| = |R(F_2)|$ and $t \in R(F_1) \setminus R(F_2)$. Therefore, $|F_1'| = |F_1|$ (by (59.16)) and $R(F_1') = (R(F_1) \setminus \{t\}) \cup \{s\}$.

Finally, $|R(B_1') \cap K| = |R(B_1) \cap K|$ for each strong component K of D. This follows directly (with (59.11)) from the fact that for any $v \in K \cap VP$ one has either $K = \{v\}$ (if $|K| = 1$) or $v \in e_K$ (if $|K| \geq 2$). For suppose that $v \in VP$ is incident with no $e_K \in N$. We show that $v \in R(B_1) \cap R(B_2)$, implying $\{v\} \in \mathcal{K}$. If v is an intermediate vertex of P, then v is covered by M_1 and M_2 and hence v belongs to $R(B_1)$ and $R(B_2)$. If $v = s$, then $v \in R(F_2)$ (so $v \in R(B_2)$) and v is covered by M_1, so $v \in R(B_1)$. If $v = t$, then $v \in R(F_1)$ (so $v \in R(B_1)$) and v is covered by M_2, so $v \in R(B_2)$. ∎

59.5. The matching forest polytope

The *matching forest polytope* of a mixed graph (V, E, A) is the convex hull of the incidence vectors of the matching forests. So the matching forest polytope is a polytope in $\mathbb{R}^{E \cup A}$.

Giles [1982b] showed that the matching forest polytope is determined by the following inequalities:

(59.18)
(i) $x_e \geq 0$ for each $e \in E \cup A$,
(ii) $x(\delta^{\text{head}}(v)) \leq 1$ for each $v \in V$,
(iii) $x(\gamma(\mathcal{L})) \leq \lfloor |\bigcup \mathcal{L}| - \frac{1}{2}|\mathcal{L}| \rfloor$ for each subpartition $\mathcal{L}$ of V with $|\mathcal{L}|$ odd and all classes nonempty.

Here we use the following notation and terminology. $\delta^{\text{head}}(v)$ denotes the set of edges with head v. A *subpartition* of V is a collection of disjoint subsets of V. As usual, $\bigcup \mathcal{L}$ denotes the union of the sets in $\mathcal{L}$. For each subpartition $\mathcal{L}$, we define:

(59.19) $\gamma(\mathcal{L}) :=$ the set of undirected edges spanned by $\bigcup \mathcal{L}$ and directed edges spanned by any set in $\mathcal{L}$.

The inequalities (i) and (ii) in (59.18) are trivially valid for the incidence vector of any matching forest F. To see that (iii) is valid, we can assume that $F \subseteq \gamma(\mathcal{L})$ and that $V = \bigcup \mathcal{L}$. Then $|R(F \cap A)| \geq |\mathcal{L}|$, since each set in $\mathcal{L}$ contains at least one root of $F \cap A$ (since no directed edge enters any set in $\mathcal{L}$). Moreover, $|F \cap E| \leq \lfloor \frac{1}{2}|R(F \cap A)| \rfloor$, since $F \cap E$ is a matching on a subset of $R(F \cap A)$. As $|F \cap A| = |V| - |R(F \cap A)|$, this gives:

(59.20)
$$|F| = |F \cap E| + |F \cap A| \leq \lfloor \tfrac{1}{2}|R(F \cap A)| \rfloor + (|V| - |R(F \cap A)|) = \lfloor |V| - \tfrac{1}{2}|R(F \cap A)| \rfloor \leq \lfloor |\bigcup \mathcal{L}| - \tfrac{1}{2}|\mathcal{L}| \rfloor,$$

as required.

Each integer solution x of (59.18) is the incidence vector of a matching forest. Indeed, as x is a $0,1$ vector by (i) and (ii), we know that $x = \chi^F$ for some $F \subseteq E \cup A$. By (ii), each vertex is head of at most one edge in F. Hence, if F would contain a circuit (in the underlying undirected graph), it is a directed circuit C. But then for $\mathcal{L} := \{VC\}$, condition (iii) is violated. So F is a matching forest.

We show that system (59.18) is totally dual integral. This implies that it determines an integer polytope, which therefore is the matching forest polytope.

The proof method is a generalization of the method in Section 25.3a for proving the Cunningham-Marsh formula, stating that the matching constraints are totally dual integral.

The total dual integrality of (59.18) is equivalent to the following. For any weight function $w : E \cup A \to \mathbb{Z}$, let ν_w denote the maximum weight of a matching forest. Call a matching forest F *w-maximal* if $w(F) = \nu_w$. Let Λ be the set of subpartitions $\mathcal{L}$ of V with $|\mathcal{L}|$ odd and with all classes nonempty.

Then the total dual integrality of (59.18) is equivalent to: for each weight function $w : E \cup A \to \mathbb{Z}$, there exist $y : V \to \mathbb{Z}_+$ and $z : \Lambda \to \mathbb{Z}_+$ satisfying

$$\text{(59.21)} \qquad \sum_{v \in V} y_v + \sum_{\mathcal{L} \in \Lambda} z(\mathcal{L}) \left\lfloor |\bigcup \mathcal{L}| - \tfrac{1}{2}|\mathcal{L}| \right\rfloor \leq \nu_w$$

and

$$\text{(59.22)} \qquad \sum_{v \in V} y_v \chi^{\delta^{\text{head}}(v)} + \sum_{\mathcal{L} \in \Lambda} z(\mathcal{L}) \chi^{\gamma(\mathcal{L})} \geq w.$$

Now we can derive (Schrijver [2000b]):

Theorem 59.3. *For each mixed graph* (V, E, A)*, system* (59.18) *is totally dual integral.*

Proof. We must prove that for each mixed graph (V, E, A) and each function $w : E \cup A \to \mathbb{Z}$, there exist y, z satisfying (59.21) and (59.22).

In proving this, we can assume that w is nonnegative. For suppose that w has negative entries, and let w' be obtained from w by setting all negative entries to 0. As $\nu_{w'} = \nu_w$ and $w' \geq w$, any y, z satisfying (59.21) and (59.22) with respect to w', also satisfy (59.21) and (59.22) with respect to w.

Suppose that the theorem is not true. Choose a counterexample (V, E, A) and $w : E \cup A \to \mathbb{Z}_+$ with $|V| + |E \cup A| + \sum_{e \in E \cup A} w(e)$ as small as possible.

Then the underlying undirected graph of (V, E, A) is connected, since otherwise one of the components will form a smaller counterexample. Moreover, $w(e) \geq 1$ for each edge e, since otherwise we can delete e to obtain a smaller counterexample.

Next:

(59.23) for each $v \in V$, there exists a w-maximal matching forest F with $v \in R(F)$.

For suppose that no such matching forest exists. For any edge e, let $w'(e) := w(e) - 1$ if v is head of e and $w'(e) := w(e)$ otherwise. Then $\nu_{w'} = \nu_w - 1$. By the minimality of w, there exist y, z satisfying (59.21) and (59.22) with respect to w'. Replacing y_v by $y_v + 1$ we obtain y, z satisfying (59.21) and (59.22) with respect to w, contradicting our assumption. This proves (59.23).

This implies:

(59.24) each weak component of the directed graph (V, A) is strongly connected.

To see this, it suffices to show that each directed edge $e = (u, v)$ is contained in some directed circuit. By (59.23) there exists a w-maximal matching forest F with $v \in R(F)$. Then the weak component of F containing v is an arborescence rooted at v. As F has maximum weight, $F \cup \{e\}$ is not a matching forest, and hence $F \cap A$ contains a directed $v-u$ path. This makes a directed circuit containing e, and proves (59.24).

Let $\mathcal{K}$ denote the collection of strong components of (V, A). Define $w'(e) := w(e) - 1$ for each edge e. The remainder of this proof consists of showing that $|\mathcal{K}|$ is odd (so $\mathcal{K} \in \Lambda$), and that

(59.25) $\nu_w \geq \nu_{w'} + \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor$.

This is enough, since, by the minimality of w, there exist y, z satisfying (59.21) and (59.22) with respect to w'. Replacing $z(\mathcal{K})$ by $z(\mathcal{K}) + 1$ we obtain y, z satisfying (59.21) and (59.22) with respect to w (note that $\gamma(\mathcal{K}) = E \cup A$), contradicting our assumption.

To show (59.25), choose a w'-maximal matching forest F of maximum size $|F|$. Under this condition, choose F such that it maximizes $|R(F)|$.

We show that for each $s \in V$ the following holds, where r is the root of the arborescence[14] in $F \cap A$ containing s:

(59.26) there exist a $t \in R(F)$ and a w'-maximal matching forest F' satisfying $|F'| = |F|$, $R(F') = (R(F) \setminus \{t\}) \cup \{s\}$, and $|R(F' \cap A) \cap K| = R(F \cap A) \cap K|$ for each strong component K of (V, A); if $r \in R(F)$, then moreover $t = r$ and $R(F' \cap A) = (R(F \cap A) \setminus \{r\}) \cup \{s\}$.

Let $F_1 := F$ and let F_2 be a w-maximal forest with $s \in R(F_2)$ (which exists by (59.23)). We first find F_1' and F_2' as follows.

If $r \notin R(F)$, then $s \notin R(F) = R(F_1)$ (since otherwise s is a root of $F \cap A$, and hence $r = s \in R(F)$). Applying Lemma 59.3α to F_1 and F_2 yields the matching forests F_1' and F_2'.

If $r \in R(F)$, then $s \notin R(F \cap A)$. Apply Theorem 53.2 to $B_1 := F_1 \cap A$ and $B_2 := F_2 \cap A$. It yields branchings B_1' and B_2' in (V, A) satisfying $B_1' \cap B_2' =$

[14] An *arborescence in* a branching B is a weak component of (V, B), or just the arc set of it.

$B_1 \cap B_2$, $B_1' \cup B_2' = B_1 \cup B_2$, and $R(B_1') = R(B_1) \cup \{s\}$ or $R(B_1') = (R(B_1) \setminus \{r\}) \cup \{s\}$. This implies $R(B_2') = R(B_2) \setminus \{s\}$ or $R(B_2') = (R(B_2) \setminus \{s\}) \cup \{r\}$. Now define $F_i' := (F_i \cap E) \cup B_i'$ for $i = 1, 2$. Then the F_i' are matching forests, since $r \in R(F_1 \cap E)$ and $s \in R(F_2 \cap E)$.

In both constructions, $|F_1'| \leq |F_1|$, and if $|F_1'| = |F_1|$, then $|R(F_1')| \geq |R(F_1)|$. Moreover,

$$\chi^{F_1'} + \chi^{F_2'} = \chi^{F_1} + \chi^{F_2}, \tag{59.27}$$

which implies that $w(F_1') + w(F_2') = w(F_1) + w(F_2)$. Hence

$$\begin{aligned} w'(F_1') + w(F_2') &= w(F_1') + w(F_2') - |F_1'| \geq w(F_1) + w(F_2) - |F_1| \\ &= w'(F_1) + w(F_2). \end{aligned} \tag{59.28}$$

Therefore, since F_1 is a w'-maximal matching forest and F_2 is a w-maximal matching forest, we have equality throughout in (59.28). So F_1' is w'-maximal and $|F_1'| = |F_1|$. Hence $|R(F_1')| \geq |R(F_1)|$. Then, by the maximality of $|R(F)|$, we know that $|R(F_1')| = |R(F_1)|$.

Set $F' := F_1'$. If $r \notin R(F)$, we know that (59.6)(iii) holds, which gives (59.26). If $r \in R(F)$, then (59.26) holds for $t := r$, and $R(B_1') = (R(B_1) \setminus \{t\}) \cup \{s\}$ or $R(B_2') = (R(B_2) \setminus \{s\}) \cup \{t\}$ (since $|F_1'| = |F_1|$, $F_1' \cap E = F_1 \cap E$, $|F_2'| = |F_2|$, $F_2' \cap E = F_2 \cap E$). Moreover, s and t belong to the same strong component of (V, A): as $r = t$ is the root of the arborescence in $F_1 \cap A$ containing s, there exists a $t - s$ path in (V, A); since each weak component of (V, A) is a strong component (by (59.24)), there is a directed $s - t$ path in (V, A). This implies (59.26).

Note that (59.26) implies in particular that $R(F) \neq \emptyset$. Suppose $|R(F)| \geq 2$. Choose F under the additional condition that the minimum distance in (V, E, A) between distinct vertices $u, v \in R(F)$ is as small as possible. Here, the distance in (V, E, A) is the length of a shortest $u - v$ path in the underlying undirected graph.

Necessarily, this distance is at least two, since otherwise we can extend F by an edge connecting u and v, thereby maintaining w'-maximality but increasing the size. This contradicts the maximality of $|F|$.

So we can choose an intermediate vertex s on a shortest $u - v$ path. Let F' be the matching forest described in (59.26), with $t \in R(F)$. By symmetry of u and v we can assume that $t \neq u$. So $u, s \in R(F')$, contradicting the choice of F, as the distance of u and s is smaller than that of u and v.

This implies that $|R(F)| = 1$. Let $R(F) = \{r\}$ and let K be the strong component of (V, A) containing r. We choose F (and r) under the additional constraint that $|R(F \cap A) \cap K|$ is as large as possible.

Suppose $|R(F \cap A) \cap K| \geq 2$. Choose F under the additional constraint that r has minimal distance in (V, A) from some root u of $F \cap A$ in $K \setminus \{r\}$. In this case, the distance in (V, A) from u to r is the length of a shortest directed $u - r$ path. (Such a path exists, since K is strongly connected.)

Let T be the arborescence in $F \cap A$ containing r. Let s be the first vertex on a shortest directed $u - r$ path Q in (V, A) that belongs to T. Necessarily

$s \neq r$, since otherwise we can extend F by the last edge of Q, contradicting the maximality of $|F|$.

Let F' be the matching forest described in (59.26). Then $s \in R(F')$ and $R(F' \cap A) = (R(F \cap A) \setminus \{r\}) \cup \{s\}$. Hence u remains a root of $F' \cap A$, while the distance in (V, A) from u to s is shorter than that from u to r. This contradicts our choice of F (replacing K, r by L, s).

So $|R(F \cap A) \cap K| = 1$. Suppose that there exists a component L of (V, A) with $|R(F \cap A) \cap L| \geq 2$. Choose s in L arbitrarily. Let F' be the matching forest described in (59.26). Then $s \in R(F')$ while $|R(F' \cap A) \cap L| \geq 2$, contradicting the choice of F.

So no such component L exists; that is, each $L \in \mathcal{K}$ contains exactly one root of $F \cap A$. So $|F \cap A| = |V| - |\mathcal{K}|$. Moreover, as $|R(F)| = 1$, $|\mathcal{K}|$ is odd and $|F \cap E| = \lfloor \frac{1}{2}|\mathcal{K}| \rfloor$. So $|F| = |F \cap A| + |F \cap E| = \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor$. Hence

(59.29) $$\nu_w \geq w(F) = w'(F) + |F| = \nu_{w'} + |F| = \nu_{w'} + \lfloor |V| - \tfrac{1}{2}|\mathcal{K}| \rfloor,$$

thus proving (59.25). ∎

We remark that the optimum dual solution y, z constructed in this proof has the following additional property: if $\mathcal{K}, \mathcal{L} \in \Lambda$ and $z(\mathcal{K}), z(\mathcal{L}) > 0$, then $\mathcal{K}$ and $\mathcal{L}$ are 'laminar' in the following sense:

(59.30) $$\begin{aligned} &\forall K \in \mathcal{K} \;\; \exists L \in \mathcal{L} : K \subseteq L, \\ &\text{or } \forall L \in \mathcal{L} \;\; \exists K \in \mathcal{K} : L \subseteq K, \\ &\text{or } \forall K \in \mathcal{K} \;\; \forall L \in \mathcal{L} : K \cap L = \emptyset. \end{aligned}$$

Theorem 59.3 implies the characterization of the matching forest polytope of Giles [1982b]:

Corollary 59.3a. *For each mixed graph* (V, E, A)*, the matching forest polytope is determined by* (59.18)*.*

Proof. By Theorems 59.3 and 5.22, the vertices of the polytope determined by (59.18) are integer. Since the integer solutions of (59.18) are the incidence vectors of matching forests, this proves the corollary. ∎

59.6. Further results and notes

59.6a. Matching forests in partitionable mixed graphs

Call a mixed graph $G = (V, E, A)$ *partitionable (into R and S)* if V can be partitioned into classes R and S such that each undirected edge connects R and S, while each directed arc is spanned by R or by S.

Trivially, a mixed graph is partitionable if and only if each circuit has an even number of undirected edges. That is, by contracting all directed arcs we obtain a bipartite graph. (Another characterization is: the incidence matrix is totally unimodular.)

In a different form, we have studied matching forests in partitionable mixed graphs before. Let $G = (V, E, A)$ be a mixed graph partitionable into R and S. Orient the edges in E from R to S, and turn the orientation of any arc in A spanned by R. We obtain a directed graph $D' = (V, A')$. Then it is easy to see that:

(59.31) a set of edges and arcs of G is a matching forest $\Longleftrightarrow$ the corresponding arcs in D' form an $R-S$ bifurcation.

This implies that a number of theorems on matching forests in a partitionable mixed graph can be obtained from those on $R-S$ bifurcations. First we have:

Theorem 59.4. *Let $G = (V, E, A)$ be a partitionable mixed graph. Then the maximum size of a matching forest in G is equal to the minimum size of $|V| - |\mathcal{L}|$, where $\mathcal{L}$ is a collection of strong components K of the directed graph $D = (V, A)$ with $d_D^{\text{in}}(K) = 0$ such that no edge in E connects two components in $\mathcal{L}$.*

Proof. This is equivalent to Theorem 54.9. ∎

We similarly obtain a min-max relation for the maximum weight of a matching forest in a partitionable mixed graph, by the total dual integrality of the following system:

(59.32)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E \cup A, \\ \text{(ii)} & x(\delta^{\text{head}}(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(A[U]) \leq |U| - 1 & \text{for each nonempty } U \text{ with } U \subseteq R \text{ or } U \subseteq S. \end{array}$$

Here $\delta^{\text{head}}(v)$ is the set of edges and arcs having v as head.

Theorem 59.5. *If G is a mixed graph partitionable into R and S, then* (59.32) *is TDI and determines the matching forest polytope.*

Proof. This is equivalent to Corollary 54.10a. ∎

For covering by matching forests in partitionable mixed graphs we have:

Theorem 59.6. *Let $G = (V, E, A)$ be a mixed graph partitionable into R and S. Then $E \cup A$ can be covered by k matching forests if and only if*

(59.33)
$$\begin{array}{rl} \text{(i)} & |\delta^{\text{head}}(v)| \leq k \textit{ for each } v \in V; \\ \text{(ii)} & |A[U]| \leq k(|U| - 1) \textit{ for each nonempty subset } U \textit{ of } R \textit{ or } S. \end{array}$$

Proof. This is equivalent to Corollary 54.11c. ∎

The case $A = \emptyset$ is Kőnig's edge-colouring theorem (Theorem 20.1).

An equivalent, polyhedral way of formulating Theorem 59.6 is:

Corollary 59.6a. *If G is a partitionable mixed graph, then the matching forest polytope has the integer decomposition property.*

Proof. Directly from Theorem 59.6. ∎

59.6b. Further notes

The facets of the matching forest polytope are characterized in Giles [1982c].

Matching forests form a special case of matroid matching. Let $G = (V, E, A)$ be a mixed graph. Consider the space $\mathbb{R}^V \times \mathbb{R}^V$. Associate with any undirected edge $e = uv \in E$, the pair $(\chi^u, 0)$, $(\chi^v, 0)$ of vectors in $\mathbb{R}^V \times \mathbb{R}^V$. Associate with any directed arc $a = (u, v) \in A$, the pair $(\chi^v, 0)$, $(0, \chi^u - \chi^v)$ of vectors in $\mathbb{R}^V \times \mathbb{R}^V$. One easily checks that $M \subseteq E \cup A$ is a matching forest if and only if its associated pairs form a matroid matching. Thus matroid matching theory implies a min-max relation and a polynomial-time algorithm for the maximum size of a matching forest. However, as we saw in Section 59.2, there is an easy direct method for this.

Chapter 60

Submodular functions on directed graphs

At two structures we came across the proof technique of making a collection of subsets cross-free: at submodular functions (like in polymatroid intersection) and at directed graphs (like in the proof of the Lucchesi-Younger theorem).
Edmonds and Giles [1977] combined the two structures into one general framework, consisting of a submodular function defined on the vertex set of a directed graph. Johnson [1975a] and Frank [1979b] designed a variant of Edmonds and Giles' framework, containing the polymatroid intersection theorem and the optimum arborescence theorem as special cases.
We first describe the results of Edmonds and Giles, and after that we present a variant, from which the results of Frank can be derived. At the base is the method of Edmonds and Giles to represent any cross-free family by a directed tree (the *tree-representation*) and to derive a network matrix if the family consists of subsets of the vertex set of a directed graph — see Section 13.4.

60.1. The Edmonds-Giles theorem

Let $D = (V, A)$ be a digraph and let $\mathcal{C}$ be a crossing family of subsets of V (that is, if $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$ and $T \cup U \neq V$, then $T \cap U, T \cup U \in \mathcal{C}$). A function $f : \mathcal{C} \to \mathbb{R}$ is called *submodular on crossing pairs*, or *crossing submodular*, if for all $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$ and $T \cup U \neq V$ one has

(60.1) $$f(T) + f(U) \geq f(T \cap U) + f(T \cup U).$$

Given such $D, \mathcal{C}, f$, a *submodular flow* is a function $x \in \mathbb{R}^A$ satisfying:

(60.2) $$x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq f(U) \text{ for each } U \in \mathcal{C}.$$

The set P of all submodular flows is called the *submodular flow polyhedron*.

Equivalently, P is equal to the set of all vectors x in $\mathbb{R}^A$ with the property that the 'gain' vector of x is in the extended polymatroid EP_f. (The *excess function* of x equals Mx where M is the $V \times A$ incidence vector of D.)

Then Edmonds and Giles [1977] showed:

Theorem 60.1 (Edmonds-Giles theorem). *If f is crossing submodular, then* (60.2) *is box-TDI.*

Proof. Choose $w \in \mathbb{R}^A$, and let y be an optimum solution to the dual of maximizing $w^{\mathsf{T}}x$ over (60.2):

$$(60.3) \qquad \min\{\sum_{U\in\mathcal{C}} y(U)f(U) \mid y \in \mathbb{R}_+^{\mathcal{C}}, \sum_{U\in\mathcal{C}} y(U)(\chi^{\delta^{\text{in}}(U)} - \chi^{\delta^{\text{out}}(U)}) = w\}.$$

Choose y such that

$$(60.4) \qquad \sum_{U\in\mathcal{C}} y(U)|U||V\setminus U|$$

is as small as possible. Let $\mathcal{C}_0 := \{U \in \mathcal{C} \mid y(U) > 0\}$. We first prove that $\mathcal{C}_0$ is cross-free.

Suppose to the contrary that $T, U \in \mathcal{C}_0$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, $T \cup U \neq V$. Let $\alpha := \min\{y(T), y(U)\} > 0$. Then decreasing $y(T)$ and $y(U)$ by α, and increasing $y(T \cap U)$ and $y(T \cup U)$ by α, maintains feasibility of z, u, y, while its value is not increased (hence it remains optimum). However, sum (60.4) decreases (by Theorem 2.1). This contradicts the minimality of (60.4).

As $\mathcal{C}_0$ is cross-free, the submatrix formed by the constraints corresponding to $\mathcal{C}_0$ is totally unimodular (by Corollary 13.21a). Hence, by Theorem 5.35, (60.2) is box-TDI. ∎

Note that the proof also yields that the solution y in (60.3) can be taken such that the collection $\{U \in \mathcal{C} \mid y(U) > 0\}$ is cross free.

Box-TDI implies primal integrality (a polyhedron P is *box-integer* if $P \cap \{x \mid d \leq x \leq c\}$ is integer for all integer vectors d, c):

Corollary 60.1a. *If f is integer, the polyhedron determined by* (60.2) *is box-integer.*

Proof. By Theorem 60.1, $\max\{w^{\mathsf{T}}x \mid x \in P\}$ is achieved by an integer solution x, for each vector w. ∎

Complexity. The algorithmic results on polymatroid intersection of Cunningham and Frank [1985] and Fujishige, Röck, and Zimmermann [1989] imply that the optimization problem associated with the Edmonds-Giles theorem can be solved in strongly polynomial time.

Indeed, let $D = (V, A)$ be a digraph, let $\mathcal{C}$ be a crossing family, let $f : \mathcal{C} \to \mathbb{Q}$ be crossing submodular, and let $c, d, l : A \to \mathbb{Q}$. If we want to find a submodular flow x with $d \leq x \leq c$ minimizing $l^{\mathsf{T}}x$, we can assume that all arcs in A are vertex-disjoint. Moreover, we can assume that for each arc $a = (u, v) \in A$ we have $f(\{v\}) = c(a)$ and $f(\{u\}) = -d(a)$. Hence we can ignore d and c, and assume that we want to find a submodular flow x minimizing $l^{\mathsf{T}}x$.

Now define $\mathcal{C}_2 := \{\{u,v\} \mid (u,v) \in A\}$ and $f_2(\{u,v\}) := 0$, $w(v) := l(u,v)$, and $w(u) := 0$, for each $(u,v) \in A$. Then the problem is equivalent to finding a vector x in $EP_f \cap EP_{f_2}$ with $x(V) = 0$ and minimizing $w^{\mathsf{T}}x$. This can be solved in strongly polynomial time by Theorem 49.9.

(Frank [1982b] gave a strongly polynomial-time algorithm for the special case if f is integer, $c = \mathbf{1}$, and $d = \mathbf{0}$.)

A similar reduction of submodular flows to polymatroid intersection was given by Kovalev and Pisaruk [1984].

60.1a. Applications

Network flows. If we take $\mathcal{C} := \{\{v\} \mid v \in V\}$ and $f = \mathbf{0}$, then (60.2) determines circulations, and Theorem 60.1 passes into a theorem on minimum-cost circulations. It may be specialized easily to several other results on flows in networks, e.g., to the max-flow min-cut theorem (Theorem 10.3; take $d = \mathbf{0}, c \geq \mathbf{0}$, and $w(a) = 0$ for $a \neq (s,r)$ and $w((s,r)) = 1$) and to Hoffman's circulation theorem (Theorem 11.2).

Lucchesi-Younger theorem. Let $D = (V,A)$ be a digraph and define

(60.5) $\quad \mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \neq V \text{ and } d_A^{\text{out}}(U) = 0\}.$

So $\mathcal{C}$ consists of all sets U such that the collection of arcs entering U forms a directed cut. Taking $f := -\mathbf{1}$, $c := \mathbf{0}$, $d := -\boldsymbol{\infty}$, and $w := \mathbf{1}$, Theorem 60.1 passes into the Lucchesi-Younger theorem (Theorem 55.2, cf. Corollary 55.2b): the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. For arbitrary w we obtain a weighted version.

Polymatroid intersection. Let f_1 and f_2 be nonnegative submodular set function on S. Let S' and S'' be two disjoint copies of S, let $V = S' \cup S''$, and define $\mathcal{C}$ by

(60.6) $\quad \mathcal{C} := \{U' \mid U \subseteq S\} \cup \{S' \cup U'' \mid U \subseteq S\}$

where U' and U'' denote the sets of copies of elements of U in S' and S''. Define $f : \mathcal{C} \to \mathbb{R}_+$ by

(60.7)
$$\begin{aligned} f(U') &:= f_1(U) && \text{for } U \subseteq S, \\ f(V \setminus U'') &:= f_2(U) && \text{for } U \subseteq S, \\ f(S') &:= \min\{f_1(S), f_2(S)\}. \end{aligned}$$

Then $\mathcal{C}$ and f satisfy (60.1). If we take $d = \mathbf{0}$ and $c = \boldsymbol{\infty}$, Theorem 60.1 passes into the polymatroid intersection theorem (Corollary 46.1a, cf. Theorem 46.1).

Frank and Tardos [1989] showed that also Theorem 44.7 (a generalization of Lovász [1970a] of Kőnig's matching theorem) fits into the Edmonds-Giles model. For applications of the Edmonds-Giles theorem to graph orientation, see Chapter 61.

60.1b. Generalized polymatroids and the Edmonds-Giles theorem

The Edmonds-Giles theorem (Theorem 60.1) also comprises the total dual integrality of the system defining the intersection of two generalized polymatroids (Section 49.11b). Indeed, let S be a finite set, let, for $i = 1,2$, $\mathcal{C}_i$ and $\mathcal{D}_i$ be collections of subsets of S, and let $f_i : \mathcal{C}_i \to \mathbb{R}$ and $g_i : \mathcal{D}_i \to \mathbb{R}$ form a paramodular pair (f_i, g_i). Then the system

(60.8)
$$\begin{aligned} x(U) &\leq f_1(U) && \text{for } U \in \mathcal{C}_1,\\ x(U) &\geq g_1(U) && \text{for } U \in \mathcal{D}_1,\\ x(U) &\leq f_2(U) && \text{for } U \in \mathcal{C}_2,\\ x(U) &\geq g_2(U) && \text{for } U \in \mathcal{D}_2, \end{aligned}$$

is box-totally dual integral, which is Corollary 49.12b.

To see this as a special case of the Edmonds-Giles theorem, let S_1 and S_2 be disjoint copies of S, and let $V := S_1 \cup S_2$. For each $s \in S$, let a_s be the arc (s_2, s_1), where s_1 and s_2 are the copies of s in S_1 and S_2 respectively. Let $A := \{a_s \mid s \in S\}$.

Let

(60.9) $\mathcal{C} := \{U_1 \mid U \in \mathcal{C}_1\} \cup \{V \setminus U_1 \mid U \in \mathcal{D}_1\} \cup \{V \setminus U_2 \mid U \in \mathcal{C}_2\} \cup \{U_2 \mid U \in \mathcal{D}_2\}$,

where U_i denotes the set of copies of the elements in U in S_i $(i = 1, 2)$. It is easy to see that $\mathcal{C}$ is a crossing family.

Define $f : \mathcal{C} \to \mathbb{R}$ by:

(60.10)
$$\begin{aligned} f(U_1) &:= f_1(U) && \text{for } U \in \mathcal{C}_1,\\ f(V \setminus U_1) &:= -g_1(U) && \text{for } U \in \mathcal{D}_1,\\ f(V \setminus U_2) &:= f_2(U) && \text{for } U \in \mathcal{C}_2,\\ f(U_2) &:= -g_2(U) && \text{for } U \in \mathcal{D}_2. \end{aligned}$$

(In case that $f(S_1)$ or $f(S_2)$ would be defined more than once, we take the smallest of the values.) Then f is submodular on crossing pairs. Now the system (in $x \in \mathbb{R}^A$)

(60.11) $x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq f(U)$ for $U \in \mathcal{C}$

is the same as (60.8) (after renaming each variable $x(s)$ to $x(a_s)$). So the box-total dual integrality of (60.8) follows from the Edmonds-Giles theorem.

Frank [1984b] showed that, conversely, the solution set of the 'Edmonds-Giles' system (60.2) is the projection of the intersection of two generalized polymatroids.

60.2. A variant

We now give a theorem similar to Theorem 60.1, which includes as special cases again the Lucchesi-Younger theorem and the polymatroid intersection theorem, and moreover theorems on optimum arborescences, bibranchings, and strong connectors.

For any digraph $D = (V, A)$ and any family $\mathcal{C}$ of subsets of V, define the $\mathcal{C} \times A$ matrix M by

(60.12)
$$M_{U,a} := \begin{cases} 1 & \text{if } a \text{ enters } U,\\ 0 & \text{otherwise,} \end{cases}$$

for $U \in \mathcal{C}$ and $a \in A$.

This matrix is totally unimodular if $\mathcal{C}$ is cross-free and the following condition holds:

(60.13) if $X, Y, Z \in \mathcal{C}$ with $X \subseteq V \setminus Y \subseteq Z$, then no arc of D enters both X and Z.

Theorem 60.2. *If $\mathcal{C}$ is cross-free and* (60.13) *holds, then M is totally unimodular.*

Proof. Let $T = (W, B)$ and $\pi : V \to W$ form a tree-representation for $\mathcal{C}$. For any arc $a = (u, v)$ of D, the set of forward arcs in the undirected $\pi(u) - \pi(v)$ path in T is contiguous, that is, forms a directed path, say from u' to v'. This follows from the fact that there exist no arcs b, c, d in this order on the path with b and d forward and c backward, by (60.13).

Define $a' := (u', v')$, and let $D' = (W, A')$ be the digraph with $A' := \{a' \mid a \in A\}$. Then M is equal to the network matrix generated by T and D' (identifying $b \in B$ with the set X_b in $\mathcal{C}$ determined by b). Hence by Theorem 13.20, M is totally unimodular. ∎

Recall that a function g on a crossing family $\mathcal{C}$ is called *supermodular on crossing pairs*, or *crossing supermodular*, if for all $T, U \in \mathcal{C}$:

(60.14) $\quad$ if $T \cap U \neq \emptyset$ and $T \cup U \neq V$, then $g(T) + g(U) \leq g(T \cap U) + g(T \cup U)$.

Consider the polyhedron P determined by:

(60.15)
$$\begin{array}{ll} x_a \geq 0 & \text{for } a \in A, \\ x(\delta^{\text{in}}(U)) \geq g(U) & \text{for } U \in \mathcal{C}. \end{array}$$

Theorem 60.3. *If g is crossing supermodular and* (60.13) *holds, then system* (60.15) *is box-TDI.*

Proof. Let $w \in \mathbb{R}^A$ and let y achieve the maximum in the dual of minimizing $w^{\mathsf{T}} x$ over (60.15):

(60.16)
$$\max\{\sum_{U \in \mathcal{C}} y(U) g(U) \mid y \in \mathbb{R}_+^{\mathcal{C}}, \sum_{U \in \mathcal{C}} y(U) \chi^{\delta^{\text{in}}(U)} \geq w\},$$

in such a way that

(60.17)
$$\sum_{U \in \mathcal{C}} y(U) |U| |V \setminus U|$$

is as small as possible. Define

(60.18)
$$\mathcal{C}_0 := \{U \in \mathcal{C} \mid y(U) > 0\}.$$

We first show that $\mathcal{C}_0$ is cross-free. Suppose to the contrary that there are T, U in $\mathcal{C}$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, and $T \cup U \neq V$. Let $\alpha := \min\{y(T), y(U)\}$. Now decrease $y(T)$ and $y(U)$ by α, and increase $y(T \cap U)$ and $y(T \cup U)$ by α. Then y remains feasible and optimum, while sum (60.17) decreases (Theorem 2.1), a contradiction.

Since $\mathcal{C}_0$ determines a totally unimodular submatrix by Theorem 60.2, by Corollary 5.20b system (60.15) is box-TDI. ∎

Note that the proof yields that (60.16) has a solution y with $\{U \in \mathcal{C} \mid y(U) > 0\}$ cross-free. Condition (60.13) cannot be deleted, as is shown by Figure 60.1.

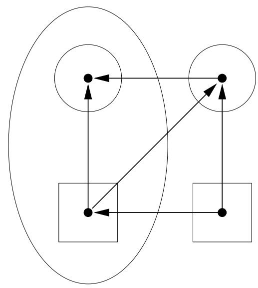

Figure 60.1
A collection and a digraph showing that condition (60.13) **cannot be deleted in Theorem 60.3.** In this Venn-diagram, the collection is represented by the *interiors* of the ellipses and by the *exteriors* of the rectangles.

Again, there is the following standard corollary for primal integrality:

Corollary 60.3a. *If g is integer, the polyhedron determined by* (60.15) *is box-integer.*

Proof. As before. ∎

Notes. Johnson [1975a] proved Theorem 60.3 for the special case that $\mathcal{C}$ is the collection of all nonempty subsets of $V \setminus \{r\}$ (where r is a fixed element of V), and Frank [1979b] extended this result to the case where $\mathcal{C}$ is any intersecting family of subsets of $V \setminus \{r\}$. Note that in this case condition (60.13) is trivially satisfied.

60.2a. Applications

We list some applications of Theorem 60.3, which may be compared with the applications of the Edmonds-Giles theorem (Section 60.1a).

Kőnig-Rado edge cover theorem. Let $G = (V, E)$ be a bipartite graph, with colour classes V_1 and V_2. Let $D = (V, A)$ be the digraph arising from G by orienting all edges from V_2 to V_1. Define $\mathcal{C} := \{\{v\} \mid v \in V_1\} \cup \{V \setminus \{v\} \mid v \in V_2\}$ and let $d := \mathbf{0}$, $c := \infty$, $g := \mathbf{1}$, $w := \mathbf{1}$. Then Theorem 60.3 gives the Kőnig-Rado edge cover theorem (Theorem 19.4): the minimum size of an edge cover in a bipartite graph is equal to the maximum size of a stable set. Taking w arbitrary gives a weighted version.

Optimum arborescence theorem. Let $D = (V, A)$ be a digraph, let $r \in V$, and let $\mathcal{C}$ be the collection of all nonempty subsets of $V \setminus \{r\}$. Let $g := \mathbf{1}$, $d := \mathbf{0}$, $c := \infty$, and let $w : A \to \mathbb{Z}_+$. Theorem 60.3 now gives the optimum arborescence theorem (Theorem 52.3): the minimum weight of an r-arborescence is equal to the maximum number of r-cuts such that no arc a is in more than $w(a)$ of these r-cuts.

Optimum bibranching theorem. Let $D = (V, A)$ be a digraph and let V be split into sets R and S. Define $\mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \subseteq S$ or $S \subseteq U \subset V\}$ and $d := \mathbf{0}$, $c := \infty$, $g := \mathbf{1}$, and let $w : A \to \mathbb{Z}_+$. Then Theorem 60.3 gives Corollary 54.8b: the minimum weight of a bibranching is equal to the maximum number of subsets in $\mathcal{C}$ such that no arc a enters more than $w(a)$ of these subsets.

Lucchesi-Younger theorem. Let $D = (V, A)$ be a digraph and let $\mathcal{C}$ be the collection of all nonempty proper subsets U of V with $\delta^{\text{out}}_A(U) = \emptyset$. Let $g := \mathbf{1}$, $d := \mathbf{0}$, $c := \infty$, and $w := \mathbf{1}$. Then Theorem 60.3 gives the Lucchesi-Younger theorem (Theorem 55.2): the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. Taking w arbitrary, gives a weighted version.

Strong connectors. Suppose that $g = \mathbf{1}$, $d = \mathbf{0}$, and $c = \infty$, and that for all $V_1, V_2 \in \mathcal{C}$ we have: if $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cap V_2 \in \mathcal{C}$, and if $V_1 \cup V_2 \neq V$, then $V_1 \cup V_2 \in \mathcal{C}$. Then Theorem 60.3 is equivalent to Theorem 57.3.

Indeed, let $D = (V, A)$ and $D_0 = (V, A_0)$ be digraphs such that for each arc $a = (u, v)$ of D there are vertices u' and v' such that D_0 contains directed paths from u to u', from v' to v, and from v' to u'. Let $w : A \to \mathbb{Z}_+$. Then the minimum weight of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $w(a)$ of these D_0-cuts.

This can be derived from Theorem 60.3 by taking $\mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \neq V, \delta^{\text{in}}_{A_0}(U) = \emptyset\}$. Conversely, if $\mathcal{C}$ satisfies the condition given above, we can take $A_0 := \{(u, v) \mid u, v \in V, (u, v)$ enters no $U \in \mathcal{C}\}$.

Polymatroid intersection. Let g_1 and g_2 be integer supermodular nondecreasing set functions on S with $g_1(\emptyset) = g_2(\emptyset) = 0$. Then

(60.19)
$$\min\{x(S) \mid x \in \mathbb{Z}^S_+, x(U) \geq g_i(U) \text{ for } U \subseteq S, i = 1, 2\} = \max_{U \subseteq S}(g_1(U) + g_2(S \setminus U)).$$

This follows by taking disjoint copies S' and S'' of S, and setting $V := S' \cup S''$, $\mathcal{C} := \{T \subseteq V \mid T \subseteq S'$ or $S' \subseteq T\}$, $A := \{(s'', s') \mid s \in S\}$, $g(U') := g_1(U)$ and $g(V \setminus U'') := g_2(U)$ for $U \subseteq S$ (without loss of generality, $g_1(S) = g_2(S)$), $d := \mathbf{0}$, $c := \infty$, $w := \mathbf{1}$.

By taking d, c, w arbitrary, several other (contra)polymatroid intersection theorems follow.

60.3. Further results and notes

60.3a. Lattice polyhedra

In a series of papers, Hoffman [1976a,1978] and Hoffman and Schwartz [1978] developed a theory of 'lattice polyhedra', which extends results of Johnson [1975a]. This theory has much in common with the theories described above.

Let $(L, \leq)$ be a partially ordered set and let $\wedge : L \times L \to L$ be a function such that

(60.20) for all $a, b \in L$: $a \wedge b \leq a$ and $a \wedge b \leq b$.

Let S be a finite set and let $\phi : L \to \mathcal{P}(S)$ be such that

(60.21) if $a < b < c$, then $\phi(a) \cap \phi(c) \subseteq \phi(b)$

for a, b, c in L. Let $\vee : L \times L \to L$ and let $f : L \to \mathbb{R}_+$ satisfy:

(60.22) $f(a \wedge b) + f(a \vee b) \leq f(a) + f(b)$

for all a, b in L. So f is, in a sense, *submodular*.

Define

(60.23) $S' := \{u \in S \mid \forall a, b \in L : \chi^{\phi(a\wedge b)}(u) + \chi^{\phi(a\vee b)}(u) \leq \chi^{\phi(a)}(u) + \chi^{\phi(b)}(u)\}$ and
$S'' := \{u \in S \mid \forall a, b \in L : \chi^{\phi(a\wedge b)}(u) + \chi^{\phi(a\vee b)}(u) \geq \chi^{\phi(a)}(u) + \chi^{\phi(b)}(u)\}$.

The polyhedron determined by:

(60.24)
$$\begin{array}{ll} x_u \geq 0 & (u \in S \setminus S'), \\ x_u \leq 0 & (u \in S \setminus S''), \\ x(\phi(a)) \leq f(a) & (a \in L). \end{array}$$

is called a *lattice polyhedron*. Hoffman and Schwartz [1978] showed that system (60.24) is box-totally dual integral.

Theorem 60.4. *System* (60.24) *is box-TDI.*

Proof. Choose $w \in \mathbb{R}^S_+$. Consider the dual of maximizing $w^\mathsf{T} x$ over (60.24):

(60.25) $\min\{y^\mathsf{T} f \mid y \in \mathbb{R}^L_+, \sum_{a\in L} y_a \chi^{\phi(a)} \leq w(u)$ if $u \in S'$ and $\sum_{a\in L} y_a \chi^{\phi(a)} \geq w(u)$ if $u \in S''\}$.

Order the elements of L as $a_1, \ldots, a_n$ such that if $a_i \leq a_j$, then $i \leq j$. Let y attain (60.25), such that $y(L)$ is minimal, and, under this condition, such that

(60.26) $(y(a_1), \ldots, y(a_n))$

is lexicographically maximal.

Then the collection $C := \{a \in L \mid y_a > 0\}$ is a chain in L. For suppose to the contrary that $a, b \in C$ with $a \not\leq b \not\leq a$. Let $\alpha := \min\{y_a, y_b\}$. Reset y by decreasing y_a and y_b by α, and increasing $y(a \wedge b)$ and $y(a \vee b)$ by α. One easily checks, using (60.22) and (60.23), that the new y again attains the minimum (60.25), and moreover that $(y(a_1), \ldots, y(a_n))$ lexicographically increases, contradicting our assumption.

By (60.21) for each u in S, the set of a in C with $u \in \phi(a)$ forms an interval in C. So the linear inequalities corresponding to C make up a totally unimodular matrix (as it is a network matrix generated by a directed path and a directed graph (Theorem 13.20)). Therefore, by Theorem 5.35, system (60.24) is box-TDI. ∎

We give some applications of Theorem 60.4 (more applications are in Hoffman [1976a], Hoffman and Schwartz [1978], and Gröflin [1984,1987]).

Shortest paths (Johnson [1975a]). Let $D = (V, A)$ be a digraph and let $s, t \in V$. Let $L := \{U \subseteq V \mid s \in U, t \notin U\}$ and let $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$. Let $S := A$ and let for each $U \in L$, $\phi(U) := \delta^{\text{out}}(U)$. These data satisfy (60.20) and (60.21), where $S' := S$. If $f = -\mathbf{1}$, Theorem 60.4 gives: the minimum length of an $s-t$ path is equal to the maximum number of $s-t$ cuts such that no arc a is in more than $c(a)$ of these $s-t$ cuts — the max-potential min-work theorem (Theorem 7.1).

Matroid intersection (Hoffman [1976a]). Let $(S, \mathcal{I})$ and $(S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2 and assume $r_1(S) = r_2(S)$. Let S' and S'' be two disjoint copies of S and let $V := S' \cup S''$. Let $L := \{U \subseteq V \mid U \subseteq S' \text{ or } S' \subseteq U\}$. Let $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$. Define for $T \subseteq S$:

(60.27) $$\begin{aligned} f(T') &:= r_1(T), & \phi(T') &:= T, \\ f(V \setminus T'') &:= r_2(T), & \phi(V \setminus T'') &:= T. \end{aligned}$$

As these data satisfy (60.20), (60.21), and (60.22), Theorem 60.4 yields the matroid intersection theorem. Polymatroid intersection can be included similarly.

Chains and antichains in partially ordered sets (Hoffman and Schwartz [1978]). Let $(V, \preceq)$ be a partially ordered set and let L be the collection of lower ideals of V (a subset Y of V is a *lower ideal* if $y \preceq x \in V$ implies $y \in V$). Define $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$.

First, let $S := V$. For $Y \in L$, let $\phi(Y)$ be the collection of maximal elements of Y. These data satisfy (60.20) and (60.21), and $S' = S'' = S$.

Theorem 60.4 with $f(Y) := k$ for each $Y \in L$ then gives the theorem of Greene [1976] (Corollary 14.10b) that the maximum size of the union of k chains is equal to the minimum value of $|V \setminus Y| + k \cdot c_1(Y)$, where Y ranges over all subsets of V and where $c_1(Y)$ denotes the maximum size of a chain contained in Y.

Indeed, Theorem 60.4 gives the total dual integrality of

(60.28) $$\begin{aligned} &0 \leq x_v \leq 1 && \text{for } v \in V, \\ &x(A) \leq k && \text{for each antichain } A. \end{aligned}$$

Hence the maximum size of the union of k chains is, by Dilworth's decomposition theorem, equal to (where $\mathcal{A}$ denotes the collection of antichains in V)

(60.29) $$\begin{aligned} &\max\{\mathbf{1}^{\mathsf{T}} x \mid x \in \{0,1\}^V, x(A) \leq k \text{ for } A \in \mathcal{A}\} \\ &= \min\{k \sum_{A \in \mathcal{A}} y_A + z(V) \mid y \in \mathbb{Z}_+^{\mathcal{A}}, z \in \mathbb{Z}_+^V, \sum_{A \in \mathcal{A}} y_A \chi^A + z \geq \mathbf{1}\} \\ &= \min_{Y \subseteq V} (|V \setminus Y| + k \cdot (\text{minimum number of antichains covering } Y)) \\ &= \min_{Y \subseteq V} (|V \setminus Y| + k \cdot c_1(Y)). \end{aligned}$$

Also the dual result (exchanging ‘chain’ and ‘antichain’) due to Greene and Kleitman [1976] (Corollary 14.8b) can be derived. Let $L, \leq, \wedge, \vee$ be as above and let $S := V \cup \{w\}$, where w is some new element. For $Y \in L$, let $\phi(Y)$ be the collection of maximal elements of Y together with w and let $f(Y) := -|\phi(Y)|$. These data again satisfy (60.20) and (60.21), and $S' = S'' = S$.

Then Theorem 60.4 gives the box-total dual integrality of the system

(60.30) $\quad x(A) + \lambda \leq -|A|$ for each antichain A,

and hence of the system

(60.31) $\quad x(A) + \lambda \geq |A|$ for each antichain A.

Then the maximum union of k antichains is equal to

(60.32)
$$\begin{aligned} &\max\{\sum_{A\in\mathcal{A}} y_A|A| \mid y \in \mathbb{Z}_+^{\mathcal{A}}, \sum_{A\in\mathcal{A}} y_A\chi^A \leq \mathbf{1}, \sum_{A\in\mathcal{A}} y_A = k\} \\ &= \min\{x(V) + k\cdot\lambda \mid x \in \mathbb{Z}_+^V, \lambda \in \mathbb{Z}, x(A) + \lambda \geq |A| \text{ for each } A \in \mathcal{A}\} \\ &\geq \min_{Y\subseteq V}(|V \setminus Y| + k\cdot(\text{maximum size of an antichain contained in } Y)). \end{aligned}$$

The equality follows from the box-total dual integrality of (60.31). The inequality follows by taking $Y := \{v \in V \mid x_v = 0\}$. Then λ is at least the maximum size of an antichain contained in Y, since for any antichain $A \subseteq Y$: $\lambda = x(A) + \lambda \geq |A|$.

Common base vectors in two polymatroids (Gröflin and Hoffman [1981]). Let f_1 and f_2 be submodular set functions on S. The polymatroid intersection theorem gives:

(60.33)
$$\begin{aligned} f(T) &:= \max\{x(T) \mid x(U) \leq f_i(U) \text{ for } U \subseteq S \text{ and } i = 1, 2\} \\ &= \min_{U\subseteq T}(f_1(U) + f_2(T \setminus U)), \end{aligned}$$

for $T \subseteq S$. Gröflin and Hoffman [1981] showed that Theorem 46.4 follows from Theorem 60.4 above as follows. (The proof of Theorem 46.4 was modelled after the proof of Theorem 60.4.)

Let L be the set of all pairs (T, U) of subsets of S with $T \cap U = \emptyset$, partially ordered by $\leq$ as follows:

(60.34) $\quad (T, U) \leq (T', U')$ and only if $T \subseteq T'$ and $U \supseteq U'$.

Then $(L, \leq)$ is a lattice with lattice operations $\wedge$ and $\vee$ (say). Define $\phi(T, U) := |S \setminus (T \cup U)|$ and $f(T, U) := f_1(T) + f_2(U) - f(S)$. As these data satisfy (60.20), (60.21), and (60.22), Theorem 60.4 applies. We have $S' = S'' = S$. Hence the system

(60.35) $\quad x(S \setminus (T \cup U)) \geq f_1(T) + f_2(U) - f(S)$ for $(T, U) \in L$

is box-TDI. With the definition of f, this implies the box-total dual integrality of

(60.36) $\quad x(T) \leq f(S \setminus T) - f(S)$ for $T \subseteq S$,

and (equivalently) of

(60.37) $\quad x(T) \geq f(S) - f(S \setminus T)$ for $T \subseteq S$.

That is, we have Theorem 46.4.

Convex sets in partially ordered sets (Gröflin [1984]). Let $(S, \leq)$ be a partially ordered set. A subset C of S is called *convex* if $a, b \in C$ and $a \leq x \leq b$ imply $x \in C$. Then the system

(60.38) $\quad x(C) \leq 1$ for each convex subset C of S,

is box-TDI. Note that this system describes the polar of the convex hull of the incidence vectors of convex sets.

To see the box-total dual integrality of (60.38), define

(60.39) $\quad L := \{(A, B) \mid A \text{ lower ideal and } B \text{ upper ideal in } S \text{ with } A \cup B = S\}.$

(An *upper ideal* is a subset B such that if $b \in B$ and $x \geq b$, then $x \in B$. Similarly, a *lower ideal* is a subset B such that if $b \in B$ and $x \leq b$, then $x \in B$.) Make L to a lattice by defining a partial order $\preceq$ on L by:

(60.40) $\quad (A, B) \preceq (A', B') \iff A \subseteq A', B \supseteq B'.$

Define $f : L \to \mathbb{R}$ and $\phi : L \to \mathcal{P}(S)$ by: $f(A, B) := 1$ and $\phi(A, B) := A \cap B$, for $(A, B) \in L$. Applied to this structure, Theorem 60.4 gives the box-total dual integrality of (60.38).

('Greedy' algorithms for some lattice polyhedra problems were investigated by Kornblum [1978].)

An extension of lattice polyhedra, to handle rooted-connectivity augmentation of a digraph, was given by Frank [1999b].

60.3b. Polymatroidal network flows

Hassin [1978,1982] and Lawler and Martel [1982a,1982b] gave the following 'polymatroidal network flow' model equivalent to that of Edmonds and Giles. Let $D = (V, A)$ be a digraph. For each $v \in V$, let $\mathcal{C}_v^{\text{out}}$ and $\mathcal{C}_v^{\text{in}}$ be intersecting families of subsets of $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$, respectively, and let $f_v^{\text{out}} : \mathcal{C}_v^{\text{out}} \to \mathbb{R}$ and $f_v^{\text{in}} : \mathcal{C}_v^{\text{in}} \to \mathbb{R}$ be submodular on intersecting pairs. Then the system

(60.41)
$$\begin{array}{ll} x(\delta^{\text{out}}(v)) = x(\delta^{\text{in}}(v)) & \text{for } v \in V, \\ x(B) \leq f_v^{\text{in}}(B) & \text{for each } v \in V \text{ and } B \in \mathcal{C}_v^{\text{in}}, \\ x(B) \leq f_v^{\text{out}}(B) & \text{for each } v \in V \text{ and } B \in \mathcal{C}_v^{\text{out}}, \end{array}$$

is box-TDI. Frank [1982b] showed that this can be derived from the Edmonds-Giles theorem (Theorem 60.1) as follows. Make a digraph $D' = (V', A')$, where A' consists of disjoint arcs $a' := (u_a, v_a)$ for each $a \in A$. Let $\mathcal{C}$ consist of all subsets U of V' such that there exists a $v \in V$ satisfying:

(60.42)
$$\begin{array}{l} U = \{v_a \mid a \in \delta^{\text{in}}(v)\} \cup \{u_a \mid a \in \delta^{\text{out}}(v)\}, \\ \text{or } \exists B \in \mathcal{C}_v^{\text{in}} : U = \{v_a \mid a \in B\}, \\ \text{or } \exists B \in \mathcal{C}_v^{\text{out}} : U = V' \setminus \{u_a \mid a \in B\}. \end{array}$$

Define $f(U) := 0$, $f(U) := f_v^{\text{in}}(B)$, and $f(U) := f_v^{\text{in}}(B)$, respectively. Then the box-total dual integrality of (60.41) is equivalent to that of

(60.43) $\quad x(\delta_{A'}^{\text{in}}(U)) - x(\delta_{A'}^{\text{out}}(U)) \leq f(U)$ for $U \in \mathcal{C}$, which follows from Theorem 60.1.

Lawler [1982] showed that, conversely, the Edmonds-Giles model is a special case of the polymatroidal network flow model. To see this, let $D = (V, A)$ be a digraph, let $\mathcal{C}$ be a crossing family of subsets of V, and let $f : \mathcal{C} \to \mathbb{R}$ be crossing submodular. Let $\hat{\mathcal{C}}$ be the collection of all sets $U = U_1 \cap \cdots \cap U_t$ with $U_1, \ldots, U_t \in \mathcal{C} \setminus \{V\}$ such that $U_i \cup U_j = V$ for all i, j with $1 \le i < j \le t$. Define $\hat{f} : \hat{\mathcal{C}} \to \mathbb{R}$ by

(60.44) $$\hat{f}(U) := \min(f(U_1) + \cdots + f(U_t)),$$

where the minimum ranges over sets $U_1, \ldots, U_t$ as above. Then $\hat{\mathcal{C}}$ is an intersecting family and $\hat{f}$ is intersecting submodular (Theorem 49.6).

Now extend D by a new vertex r, and arcs (v, r) for $v \in V$, thus making the digraph $D' = (V \cup \{r\}, A')$. Let $\mathcal{C}_r^{\text{in}}$ consist of all subsets B of $\delta_{A'}^{\text{in}}(r)$ for which there is a $U \in \hat{\mathcal{C}}$ satisfying

(60.45) $$B = \{(v, r) \mid v \in U\}.$$

Define $f_r^{\text{in}}(B) := \hat{f}(U)$. Then

(60.46) $$\begin{aligned} x(\delta_{A'}^{\text{out}}(v)) &= x(\delta_{A'}^{\text{in}}(v)) && \text{for } v \in V,\\ x(\delta_{A'}^{\text{in}}(r)) &= 0, &&\\ x(B) &\le f_r^{\text{in}}(B) && \text{for } B \in \mathcal{C}_r^{\text{in}}, \end{aligned}$$

is a special case of (60.41). Moreover, the box-total dual integrality of (60.46) implies the box-total dual integrality of

(60.47) $$x(\delta_A^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \le f(U) \text{ for } U \in \mathcal{C},$$

since in (60.46) we can restrict B to those B for which there exists a $U \in \mathcal{C}$ with $B = \{(v, r) \mid v \in U\}$ (since $x(\delta_{A'}^{\text{in}}(r)) = 0$). Then

(60.48) $$x(B) = \sum_{v \in U} \big(x(\delta_A^{\text{in}}(v)) - x(\delta_A^{\text{out}}(v)) \big) = x(\delta_A^{\text{in}}(U)) - x(\delta_A^{\text{out}}(U)).$$

So it implies the Edmonds-Giles theorem (Theorem 60.1).

60.3c. A general model

In Schrijver [1984a] the following general framework was given. Let S be a finite set, let $n \in \mathbb{Z}_+$, let $\mathcal{C}$ be a collection of subsets of S, let $b, c \in (\mathbb{R} \cup \{\pm\infty\})^n$, and let $f : \mathcal{C} \to \mathbb{R}$ and $h : \mathcal{C} \to \{0, \pm 1\}^n$ satisfy:

(60.49)
(i) if $\{T_1, T_2, T_3\}$ is a cross-free subcollection of $\mathcal{C}$, then for each $j = 1, \ldots, n$, there exist $u, v \in S$ such that for $i = 1, 2, 3$: $h(T_i)_j = +1$ if and only if (u, v) enters T_i, and $h(T_i)_j = -1$ if and only if (u, v) leaves T_i;
(ii) if T and U are crossing sets in $\mathcal{C}$, then there exist $T', U' \in \mathcal{C}$ such that $T' \subset T$ and
$$f(T) + f(U) - f(T') - f(U') \ge (h(T) + h(U) - h(T') - h(U'))x$$
for each x with $b \le x \le c$.

Then the system (in $x \in \mathbb{R}^n$)

(60.50) $$\begin{aligned} & b \leq x \leq c, \\ & h(T)x \leq f(T) \quad \text{for } T \in \mathcal{C}, \end{aligned}$$

is box-TDI. This contains the Edmonds-Giles theorem (Theorem 60.1) and Theorems 60.3 and 60.4 as special cases.

A proof of the box-total dual integrality of (60.50) can be sketched as follows. If we maximize a linear functional $w^{\mathsf{T}}x$ over (60.50), condition (60.49)(ii) implies that there exists an optimum dual solution whose active constraints correspond to a cross-free subfamily of $\mathcal{C}$. Next, condition (60.49)(i) implies that these constraints form a network matrix, hence a totally unimodular matrix, proving the box-total dual integrality of (60.50) with Theorem 5.35.

60.3d. Packing cuts and Győri's theorem

Let $D = (V, A)$ be a digraph and let $g : \mathcal{P}(V) \to \mathbb{Z}_+$ satisfy the supermodular inequality

(60.51) $$g(U) + g(W) \leq g(U \cap W) + g(U \cup W)$$

for all $U, W \subseteq V$ such that $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(W) \neq \emptyset$ and $g(U) > 0$, $g(W) > 0$.

The following was shown by Frank and Jordán [1995b] (in the terminology of bisupermodular functions — see Corollary 60.5a):

Theorem 60.5. *Let $D = (V, A)$ be a digraph satisfying:*

(60.52) *V can be partitioned into two sets S and T such that A consists of all arcs from S to T.*

Let g be as above, with $g(U) = 0$ if $\delta^{\text{in}}(U) = \emptyset$. Then the minimum of $x(A)$ taken over all $x : A \to \mathbb{Z}_+$ satisfying

(60.53) *$x(\delta^{\text{in}}(U)) \geq g(U)$ for each $U \subseteq V$,*

is equal to the maximum value of $\sum_{U \in \mathcal{B}} g(U)$, where $\mathcal{B}$ is a collection of subsets U such that the $\delta^{\text{in}}(U)$ for $U \in \mathcal{B}$ are disjoint.

Proof. Let $\tau(g)$ and $\nu(g)$ denote the minimum and maximum value, respectively.

Then $\nu(g) \leq \tau(g)$, since, if $x : A \to \mathbb{Z}_+$ satisfies (60.53) and $\mathcal{B}$ is as described, then

(60.54) $$x(A) \geq \sum_{U \in \mathcal{B}} x(\delta^{\text{in}}(U)) \geq \sum_{U \in \mathcal{B}} g(U).$$

The reverse inequality $\tau(g) \leq \nu(g)$ is shown by induction on $\nu(g)$. If $\nu(g) = 0$, then $g(U) = 0$ for all $U \subseteq V$, and hence $\tau(g) = 0$. Now let $\nu(g) \geq 1$.

For each $a \in A$, define a function g^a by

(60.55) $$g^a(U) := \begin{cases} g(U) - 1 & \text{if } a \in \delta^{\text{in}}(U) \text{ and } g(U) \geq 1, \\ g(U) & \text{otherwise,} \end{cases}$$

for $U \subseteq V$. In other words:

(60.56) $$g^a(U) = \max\{g(U) - d^{\text{in}}_{\{a\}}(U), 0\}.$$

Then

(60.57) g^a again satisfies (60.51).

Indeed, if $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(W) \neq \emptyset$, and $g^a(U) > 0$ and $g^a(W) > 0$, then $g(U) > 0$ and $g(W) > 0$, and $g^a(U) = g(U) - d^{\text{in}}_{\{a\}}(U)$ and $g^a(W) = g(W) - d^{\text{in}}_{\{a\}}(W)$. Hence

(60.58)
$$\begin{aligned} g^a(U) + g^a(W) &= g(U) + g(W) - d^{\text{in}}_{\{a\}}(U) - d^{\text{in}}_{\{a\}}(W) \\ &\leq g(U \cap W) + g(U \cup W) - d^{\text{in}}_{\{a\}}(U \cap W) - d^{\text{in}}_{\{a\}}(U \cup W) \\ &\leq g^a(U \cap W) + g^a(U \cup W). \end{aligned}$$

So g^a satisfies (60.51).

The following is the key of the proof:

(60.59) there exists an arc a with $\nu(g^a) \leq \nu(g) - 1$.

Suppose to the contrary that $\nu(g^a) = \nu(g)$ for all $a \in A$. As $\nu(g) \geq 1$, there exists a $W \subseteq V$ with $g(W) \geq 1$. For each $a \in \delta^{\text{in}}(W)$, as $\nu(g^a) = \nu(g)$, there exists a collection $\mathcal{B}^a$ such that any arc of D enters at most one $U \in \mathcal{B}^a$, such that $g^a(\mathcal{B}^a) = \nu_g$, and such that $g(U) > 0$ for each $U \in \mathcal{B}^a$. As $g(\mathcal{B}^a) \leq g^a(\mathcal{B}^a)$, a enters no $U \in \mathcal{B}^a$.

Now for each $U \subseteq V$, let $w(U)$ be the number of times U occurs among the $\mathcal{B}^a$ (over all $a \in \delta^{\text{in}}(W)$). Reset $w(W) := w(W) + 1$. Then w has the following properties:

(60.60)
(i) $\sum_{U \subseteq V} w(U)\chi^{\delta^{\text{out}}(U)} \leq |\delta^{\text{in}}(W)| \cdot \mathbf{1}$ and
(ii) $\sum_{U \subseteq V} w(U)g(U) > |\delta^{\text{in}}(W)|\nu(g)$.

Moreover, $g(U) \geq 1$ whenever $w(U) > 0$.

Now as long as there exist $U, U' \subseteq V$ with $w(U) > 0$ and $w(U') > 0$ and not satisfying:

(60.61) $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(U') = \emptyset$ or $U \subseteq U'$ or $U' \subseteq U$,

decrease $w(U)$ and $w(U')$ by 1, and increase $w(U \cap U')$ and $w(U \cup U')$ by 1. This operation maintains (60.60) and decreases

(60.62)
$$\sum_{U \in \mathcal{P}(V)} w(U)|U||V \setminus U|$$

(by Theorem 2.1). So after a finite number of these operations, w satisfies (60.60) and all U, U' with $w(U) > 0$ and $w(U') > 0$ satisfy (60.61).

Let $\mathcal{F}$ be the collection of $U \subseteq V$ with $w(U) > 0$. We apply the length-width inequality for partially ordered sets (Theorem 14.5) to $(\mathcal{F}, \subseteq)$. By (60.60)(i), the maximum of $w(\mathcal{C})$ taken over chains in $\mathcal{F}$ is at most $|\delta^{\text{in}}(W)|$, since by (60.52), there is an arc $a \in A$ entering all $U \in \mathcal{C}$ (as $\delta^{\text{in}}(U) \neq \emptyset$, since $g(U) \geq 1$, for each $U \in \mathcal{F}$). Moreover, the maximum of $g(\mathcal{B})$ taken over antichains $\mathcal{B}$ in $\mathcal{F}$ is at most $\nu(g)$, since the elements in $\mathcal{F}$ satisfy (60.61), and therefore $\mathcal{B}$ gives a collection of disjoint cuts. But then (60.60)(ii) contradicts the length-width inequality. This proves (60.59).

We now can apply induction, since trivially $\tau(g) \leq \tau(g^a) + 1$, as increasing x_a by 1 for any x satisfying (60.53) with respect to g^a, gives an x satisfying (60.53) with respect to g. So $\tau(g) \leq \tau(g^a) + 1 = \nu(g^a) + 1 \leq \nu(g)$. ∎

This theorem can be equivalently formulated as follows. Let S and T be finite sets. Consider functions $h : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathbb{R}$ satisfying

(60.63) $h(X_1 \cap X_2, Y_1 \cup Y_2) + h(X_1 \cup X_2, Y_1 \cap Y_2) \geq h(X_1, Y_1) + h(X_2, Y_2)$ for all $X_1, X_2 \subseteq S$ and $Y_1, Y_2 \subseteq T$ with $X_1 \cap X_2 \neq \emptyset$, $Y_1 \cap Y_2 \neq \emptyset$, $h(X_1, Y_1) > 0$, and $h(X_2, Y_2) > 0$.

Call a collection $\mathcal{F} \subseteq \mathcal{P}(S) \times \mathcal{P}(T)$ *independent* if $X_1 \cap X_2 = \emptyset$ or $Y_1 \cap Y_2 = \emptyset$ for all distinct (X_1, Y_1), (X_2, Y_2) in $\mathcal{F}$. So $\mathcal{F}$ is independent if the sets $X \times Y$ for $(X, Y) \in \mathcal{F}$ are disjoint.

As usual,

(60.64) $$h(\mathcal{F}) := \sum_{(X,Y)\in\mathcal{F}} h(X, Y).$$

Moreover, if $z : S \times T \to \mathbb{R}$, denote

(60.65) $$z(X \times Y) := \sum_{(x,y)\in X\times Y} z(x, y)$$

for $X \subseteq S$ and $Y \subseteq T$.

Corollary 60.5a. *Let $h : \mathcal{P}(S)\times\mathcal{P}(T) \to \mathbb{Z}_+$ satisfy* (60.63)*, such that $h(X, Y) = 0$ if $X = \emptyset$ or $Y = \emptyset$. Then the minimum value of $z(S \times T)$ where $z : S \times T \to \mathbb{Z}_+$ satisfies*

(60.66) $z(X \times Y) \geq h(X, Y)$ *for all* $X \subseteq S$, $Y \subseteq T$,

is equal to the maximum value of $h(\mathcal{F})$ where $\mathcal{F}$ is independent.

Proof. We can assume that S and T are disjoint. Let $V := S \cup T$, and define a set function g on V by:

(60.67) $g(U) := h(S \setminus U, T \cap U)$

for $U \subseteq V$. Let $D = (V, A)$ be the digraph with A consisting of all arcs from S to T. Then D and g satisfy the condition of Theorem 60.5, and the min-max equality proved in Theorem 60.5 is equivalent to the min-max equality described in the present corollary. ∎

Frank and Jordán showed that this theorem implies the following 'minimax theorem for intervals' of Győri [1984]. Let $\mathcal{I}$ and $\mathcal{J}$ be collections of sets. Then $\mathcal{J}$ is said to *generate* $\mathcal{I}$ if each set in $\mathcal{I}$ is a union of sets in $\mathcal{J}$. Győri's theorem characterizes the minimum size of a collection of intervals generating a given finite collection $\mathcal{I}$ of intervals. For this, we can take an 'interval' to be a finite, contiguous set of integers.

To describe the min-max equality, consider the undirected graph $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}})$ with

(60.68) $V_{\mathcal{I}} := \{(s, I) \mid s \in I \in \mathcal{I}\}$,

where two distinct pairs (s, I) and (s', I') are adjacent if and only if $s \in I'$ and $s' \in I$. Call a subset C of $V_{\mathcal{I}}$ *stable* if any two elements of C are nonadjacent (in other words, C is a stable set in $G_{\mathcal{I}}$).

Corollary 60.5b (Győri's theorem)**.** *Let $\mathcal{I}$ be a finite collection of intervals. Then the minimum size of a collection of intervals generating $\mathcal{I}$ is equal to the maximum size of a stable subset of $V_{\mathcal{I}}$.*

Proof. To see that the minimum is not less than the maximum, observe that if $\mathcal{J}$ generates $\mathcal{I}$ and C is a stable subset of $V_{\mathcal{I}}$, then for any $J \in \mathcal{J}$, there is at most one $(s, I) \in C$ with $s \in J \subseteq I$, while for any $(s, I) \in C$ there is at least one such $J \in \mathcal{J}$. So $|\mathcal{J}| \geq |C|$.

Equality is shown with Corollary 60.5a. Let S be the union of the intervals in $\mathcal{I}$. Define a function $h : \mathcal{P}(S) \times \mathcal{P}(S) \to \{0, 1\}$ by

(60.69) $h(X, Y) = 1$ if and only if X and Y are nonempty intervals such that $\max X = \min Y$ and $X \cup Y \in \mathcal{I}$, and such that there is no $I \in \mathcal{I}$ with $X \cap Y \subseteq I \subset X \cup Y$,

for $X, Y \subseteq S$. (Here $\max Z$ and $\min Z$ denote the maximum and minimum element of Z, respectively.)

Then h satisfies (60.63). To see this, note first that each (X, Y) with $h(X, Y) = 1$ is characterized by a point $s \in S$ and an inclusionwise minimal interval $I \in \mathcal{I}$ containing s (inclusionwise minimal among all intervals in $\mathcal{I}$ containing s). The relation is that $\{s\} = X \cap Y$ and $I = X \cup Y$.

To see that h satisfies (60.63), let $h(X_1, Y_1) = 1$ and $h(X_2, Y_2) = 1$ and $X_1 \cap X_2 \neq \emptyset$ and $Y_1 \cap Y_2 \neq \emptyset$. We show $h(X_1 \cap X_2, Y_1 \cup Y_2) = 1$ (then $h(X_1 \cup X_2, Y_1 \cap Y_2) = 1$ follows by symmetry).

In fact, this is straightforward case-checking. Let $X_i = [a_i, b_i]$, $Y_i = [b_i, c_i]$, and $I_i := X_i \cup Y_i$ for $i = 1, 2$. As $X_1 \cap I_2 \neq \emptyset \neq Y_1 \cap I_2$, we know that $b_1 \in [a_2, c_2]$, and similarly $b_2 \in [a_1, c_1]$. By symmetry, we can assume that $a_1 \leq a_2$. Hence, by the minimality of $X_1 \cup Y_1$ as an interval containing b_1, $c_1 \leq c_2$. Now, if $b_2 \leq b_1$, we have $X_1 \cap X_2 = [a_2, b_2] = X_2$ and $Y_1 \cup Y_2 = [b_2, c_2] = Y_2$, and therefore $h(X_1 \cap X_2, Y_1 \cup Y_2) = h(X_2, Y_2) = 1$. If $b_1 < b_2$, then $X_1 \cap X_2 = [a_2, b_1]$ and $Y_1 \cup Y_2 = [b_1, c_2]$. Suppose that there is an $I \in \mathcal{I}$ with $b_1 \in I \subset [a_2, c_2]$. By the minimality of $[a_2, c_2]$ as an interval containing b_2, we know $b_2 \notin I$. Hence $I \subset [a_1, c_1]$, contradicting the minimality of $[a_1, c_1]$ as an interval containing b_1. Therefore, no such I exists, and hence we have $h(X_1 \cap X_2, Y_1 \cup Y_2) = 1$.

So Corollary 60.5a applies (taking $T := S$). Let z and $\mathcal{F}$ attain the minimum and maximum respectively. Let $\mathcal{J}$ be the collection of intervals $[s, t]$ with $z(s, t) \geq 1$ and $s \leq t$. Let C be the collection of pairs (s, I) with $s \in S$ and $I \in \mathcal{I}$ such that there is an $(X, Y) \in \mathcal{F}$ with $h(X, Y) = 1$, $X \cap Y = \{s\}$, and $X \cup Y = I$. Then $\mathcal{J}$ generates $\mathcal{I}$, since $z(X \times Y) \geq h(X, Y)$ for all X, Y. Moreover, C is stable as $\mathcal{F}$ is independent. Finally, $|\mathcal{J}| \leq z(S \times S) = h(\mathcal{F}) = |C|$. ∎

(Frank [1999a] gave an alternative, algorithmic proof.)

Győri's theorem in fact states that the colouring number of the complementary graph of $G_{\mathcal{I}}$ is equal to its clique number. It has the following consequence, proved by Chaiken, Kleitman, Saks, and Shearer [1981] and conjectured by V. Chvátal. Let P be a rectilinear polygon in $\mathbb{R}^2$ (where each side horizontal or vertical), such that the intersection of P with each horizontal or vertical line is convex. Then the minimum number of rectangles contained in P needed to cover P, is equal to the maximum number of points in P no two of which are contained in any rectangle contained in P.

Franzblau and Kleitman [1984] gave an $O(|\mathcal{I}|^2)$-time algorithm to find the optima in Győri's theorem, with a proof of equality as by-product.

Győri's theorem was extended by Lubiw [1991a] to a weighted version. She noted that a fully weighted version of the theorem does not hold; that is, taking integer weights $w(s, I)$ on any pair (s, I), the maximum weight of a stable set need not be equal to the minimum size of a family $\mathcal{J}$ of intervals such that for any (s, I) there are at least $w(s, I)$ intervals J in $\mathcal{J}$ satisfying $s \in J \subseteq I$. (In other words, $G_{\mathcal{I}}$ need not be perfect.)

However, as Lubiw showed, these two optima are equal if $w(s, I)$ only depends on s; that is, if $w(s, I) = w(s)$ for some $w : S \to \mathbb{Z}_+$. Also this can be derived from Frank and Jordán's theorem: instead of defining $h(X, Y) := 1$ in (60.69), define $h(X, Y) := w(s)$ where $\{s\} = X \cap Y$.

As Frank and Jordán observed, their method extends Győri's theorem to the case where we take 'interval' to mean: interval on the circle (instead of just the real line).

Other applications of Theorem 60.5 are to vertex-connectivity augmentation — see Theorem 63.11.

60.3e. Further notes

For another model equivalent to that of Edmonds and Giles, based on distributive lattices, see Gröflin and Hoffman [1982] — cf. Schrijver [1984b]. Grishuhin [1981] gave a lattice model requiring total unimodularity as a condition.

Further algorithms for minimum-cost submodular flow were given by Fujishige [1978a,1987], Zimmermann [1982b,1992], Barahona and Cunningham [1984], Cui and Fujishige [1988], Chung and Tcha [1991], Gabow [1993a], McCormick and Ervolina [1993], Iwata, McCormick, and Shigeno [1998,1999,2000,2002], Wallacher and Zimmermann [1999], Fleischer and Iwata [2000], and Fleischer, Iwata, and McCormick [2002]. A survey on algorithms for submodular flows was presented by Fujishige and Iwata [2000].

Zimmermann [1982a,1982b,1985] considered group-valued extensions of some of the models. Federgruen and Groenevelt [1988] extended some models to more general objective functions. Zimmermann [1986] considered duality for balanced submodular flows. Qi [1988a] and Murota [1999] gave generalizations of submodular flows. Convex-cost submodular flows were considered by Iwata [1996,1997].

An algorithm for a model comprising the Edmonds-Giles and the lattice polyhedron model (Section 60.3a) was given by Karzanov [1983]. For 0,1 problems it is polynomial-time.

The effectivity of uncrossing techniques is studied by Hurkens, Lovász, Schrijver, and Tardos [1988] and Karzanov [1996].

The facets of submodular flow polyhedra were studied by Giles [1975].

For a comparison of models, see Schrijver [1984b], and for a survey, including results on the dimension of faces of submodular flow polyhedra, see Frank and Tardos [1988]. A survey of submodular functions and flows is given by Murota [2002].

Chapter 61

Graph orientation

Orienting an undirected graph so as to obtain a k-arc-connected directed graph is the object of study in this chapter. Recall that a directed graph D is called an *orientation* of an undirected graph G if G is the underlying undirected graph of D.

Central result is a deep theorem of Nash-Williams showing that each undirected graph has an orientation that keeps at least half of the connectivity (rounded down) between any two vertices.

It implies that a $2k$-edge-connected undirected graph has a k-arc-connected orientation. This can be proved alternatively and easier with the help of submodular functions (cf. Section 61.4).

61.1. Orientations with bounds on in- and outdegrees

We first consider orientations obeying bounds on the indegrees and/or outdegrees. The results follow quite directly from bipartite matching or (equivalently) flow theory.

Hakimi [1965] considered lower bounds on the indegrees:

Theorem 61.1. *Let $G = (V, E)$ be an undirected graph and let $l : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) \geq l(v)$ for each $v \in V$ if and only if each $U \subseteq V$ is incident with at least $l(U)$ edges.*

Proof. Let $\mathcal{A}$ be the family of subsets of E obtained by taking set $\delta(v)$ with multiplicity $l(v)$, for each $v \in V$. Then the existence of an orientation as required is equivalent to the existence of a transversal of $\mathcal{A}$. By Hall's marriage theorem (Theorem 22.1), this is equivalent to the condition in the theorem. ∎

A direct consequence is:

Corollary 61.1a. *Let $G = (V, E)$ be an undirected graph and let $l : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) = l(v)$ for each $v \in V$ if and only if $l(V) = |E|$ and each $U \subseteq V$ is incident with at least $l(U)$ edges.*

Proof. Directly from Theorem 61.1. ∎

Another consequence concerns upper bounds:

Corollary 61.1b. *Let $G = (V, E)$ be an undirected graph and let $u : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) \leq u(v)$ for each $v \in V$ if and only if each $U \subseteq V$ spans at most $u(U)$ edges.*

Proof. For each $v \in V$, define $l(v) := \deg(v) - u(v)$. (We may assume that, for each $v \in V$, $u(v) \leq \deg(v)$, since otherwise resetting $u(v) := \deg(v)$ does not change the conditions in the theorem.)

Now G has an orientation with $\deg^{\text{in}}(v) \leq u(v)$ for each v if and only if G has an orientation with $\deg^{\text{in}}(v) \geq l(v)$ for each v (just by reversing the orientation of all edges). By Theorem 61.1, the latter is equivalent to: each $U \subseteq V$ is incident with at least $l(U)$ edges; that is: $E[U]| + |\delta_E(U)| \geq l(U)$. Since

(61.1) $$l(U) = \sum_{v \in U} (\deg(v) - u(v)) = 2|E[U]| + |\delta_E(U)| - u(U),$$

it is equivalent to: $|E[U]| \leq u(U)$, as required. ∎

Frank and Gyárfás [1978] gave a characterization for the case of lower bounds on both indegrees and outdegrees:

Theorem 61.2. *Let $G = (V, E)$ be an undirected graph and let $l, u : V \to \mathbb{Z}_+$ with $l \leq u$. Then G has an orientation $D = (V, A)$ with $l(v) \leq \deg^{\text{in}}_A(v) \leq u(v)$ for each $v \in V$ if and only if each $U \subseteq V$ is incident with at least $l(U)$ edges and spans at most $u(U)$ edges.*

Proof. The condition trivially being necessary, we prove sufficiency. Let $D = (V, A)$ be an arbitrary orientation of G. It suffices to show that there exists a function $x : A \to \{0, 1\}$ such that for each $v \in V$:

(61.2) $$l(v) \leq \deg^{\text{in}}_A(v) - x(\delta^{\text{in}}_A(v)) + x(\delta^{\text{out}}_A(v)) \leq u(v),$$

since reversing the orientation of the arcs a with $x(a) = 1$ then gives an orientation as required. Condition (61.2) is equivalent to:

(61.3) $$\deg^{\text{in}}_A(v) - u(v) \leq x(\delta^{\text{in}}_A(v)) - x(\delta^{\text{out}}_A(v)) \leq \deg^{\text{in}}_A(v) - l(v).$$

By Corollary 11.2i, such an x exists if and only if

(61.4) $$|\delta^{\text{in}}_A(U)| \geq \max\{\sum_{v \in U} (\deg^{\text{in}}_A(v) - u(v)), \sum_{v \in V \setminus U} (l(v) - \deg^{\text{in}}_A(v))\}$$

for each $U \subseteq V$. Since $|\delta^{\text{in}}_A(U)| + \sum_{v \in V \setminus U} \deg^{\text{in}}_A(v)$ is equal to the number of edges incident with $V \setminus U$ and since $\sum_{v \in U} \deg^{\text{in}}_A(v) - |\delta^{\text{in}}_A(U)|$ is equal to the number of edges spanned by U, this is equivalent to the condition given in the theorem. ∎

Ford and Fulkerson [1962] observed that the undirected edges of a mixed graph (V, E, A) can be oriented so as to obtain an Eulerian directed graph if and only if

(61.5) (i) $\deg_E(v) + \deg_A^{\text{in}}(v) + \deg_A^{\text{out}}(v)$ is even for each $v \in V$,

(ii) $d_A^{\text{out}}(U) - d_A^{\text{in}}(U) \leq d_E(U)$ for each $U \subseteq V$.

This can be proved similarly.

61.2. 2-edge-connectivity and strongly connected orientations

Each $2k$-edge-connected undirected graph has a k-arc-connected orientation, which will be seen in Section 61.3. In the present section we consider the special case $k = 2$, which goes back to a theorem of Robbins [1939]. Tarjan [1972] showed that depth-first search is the tool behind. We follow his approach.

Theorem 61.3. *Given an undirected graph $G = (V, E)$ we can find an orientation D of G, in linear time, such that for each $u, v \in V$, if G has two edge-disjoint $u - v$ paths, then D has a directed $u - v$ path.*

Proof. Choose $s \in V$ arbitrarily, and consider a depth-first search tree T starting at s. Orient each edge in T away from s. For each remaining edge $e = uv$, there is a directed path in T that connects u and v. Let the path run from u to v. Then orient e from v to u. This gives the orientation D of G.

Then any edge not in T belongs to a directed circuit in D. Moreover, any edge f in T that is not a cut edge, belongs to a directed circuit in D (since there is an edge $e \notin T$ connecting the two components of $T - f$). This implies that D is as required. ∎

This implies the theorem of Robbins [1939] on strongly connected orientations:

Corollary 61.3a (Robbins' theorem)**.** *An undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected.*

Proof. Necessity is easy, and sufficiency follows from Theorem 61.3. ∎

(The proof by Robbins [1939] uses the fact that each 2-edge-connected graph has an 'ear-decomposition' — cf. Section 15.5a.)

The above proof also shows that a strongly connected orientation can be found in linear time:

Corollary 61.3b. *Given a 2-edge-connected graph G, a strongly connected orientation of G can be found in linear time.*

Proof. Directly from Theorem 61.3. ∎

Robbins' theorem (Corollary 61.3a) extends to the following result of Frank [1976a] and Boesch and Tindell [1980] for mixed graphs.

Theorem 61.4. *Let $G = (V, E)$ be a graph in which part of the edges is oriented. Then the remainder of the edges can be oriented so as to obtain a strongly connected digraph if and only if G is 2-edge-connected and there is no nonempty proper subset U of V such that all edges in $\delta(U)$ are oriented from U to $V \setminus U$.*

Proof. Necessity being easy, we show sufficiency. Let G be a counterexample with a minimum number of undirected edges. Then there is at least one undirected edge, say $e = uv$. By the minimality assumption, orienting e from s to t violates the condition. That is, there exists a $U \subseteq V$ with $u \in U$, $v \in V \setminus U$, such that each edge $\neq e$ in $\delta(U)$ is oriented from U to $V \setminus U$. Similarly, there exists a $T \subseteq V$ with $v \in T$, $u \in V \setminus T$, such that each edge $\neq e$ $\delta(T)$ is oriented from T to $V \setminus T$.

Then each edge in $\delta(U \cap T)$ is oriented from $U \cap T$ to $V \setminus (U \cap T)$, and hence $U \cap T = \emptyset$. Similarly, $U \cup T = V$. Hence $\delta(U) = \{e\}$, a contradiction. ∎

The graph $K_{2,3}$ shows that a 2-edge-connected graph need not have an orientation in which each two vertices belong to a directed circuit; that is an orientation such that for each two vertices s, t there exists an arc-disjoint pair of an $s - t$ and a $t - s$ path.

Chung, Garey, and Tarjan [1985] gave a linear-time algorithm to find an orientation as described in Theorem 61.4.

61.2a. Strongly connected orientations with bounds on degrees

Robbins' theorem (Corollary 61.3a) states that an undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected. Frank and Gyárfás [1978] extended this to the case where upper and lower bounds are prescribed on the indegrees of the orientation.

Let $\kappa(G)$ denote the number of its components of any graph G.

Theorem 61.5. *Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a strongly connected orientation $D = (V, A)$ satisfying $l(v) \leq \deg_A^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if for each $U \subseteq V$:*

(61.6) (i) $|E[U]| + \kappa(G - U) \leq u(U)$,
(ii) $|E[U]| + |\delta(U)| - \kappa(G - U) \geq l(U)$.

Proof. It is easy to see that condition (61.6) is necessary. To see sufficiency, let (61.6) hold. Let $D = (V, A)$ be a strongly connected orientation of G with

(61.7) $$\sum_{v \in V} \max\{0, \deg_A^{\text{in}}(v) - u(v), l(v) - \deg_A^{\text{in}}(v)\}$$

as small as possible. (A strongly connected orientation exists by Corollary 61.3a.)

If sum (61.7) is 0, we are done, so assume that it is positive. Then there exists a vertex v_0 with $\deg_A^{\text{in}}(v_0) > u(v_0)$ or $l(v_0) > \deg_A^{\text{in}}(v_0)$. Suppose that $\deg_A^{\text{in}}(v_0) > u(v_0)$. Let U be the set of vertices v for which D has two arc-disjoint $v - v_0$ paths. Then $\deg_A^{\text{in}}(v) \geq u(v)$ for each $v \in U$, since otherwise we can reverse the orientation on the arcs of one of the two arc-disjoint $v - v_0$ paths, thereby keeping the orientation strongly connected while decreasing sum (61.7).

We claim that U violates (61.6)(i). To this end, we show that

(61.8) each component K of $G - U$ is left by exactly one arc of D.

This can be seen as follows. For each $v \in K$, there exists a $U_v \subseteq V$ with $d_A^{\text{out}}(U_v) = 1$ and $v \in U_v$, $v_0 \notin U_v$ (as there exist no two arc-disjoint $v_0 - v$ paths). We choose each U_v inclusionwise maximal.

It suffices to show that

(61.9) $U_v = K$ for each $v \in K$.

To see this, note first that, for each $v \in K$, we have $U_v \subseteq K$. Indeed, $U_v \cap U = \emptyset$, since if say $v_1 \in U \cap U_v$, then there exist no two arc-disjoint $v_1 - v_0$ paths in D, contradicting the definition of U. If U_v would intersect another component K' of $G - U$, then $d_A^{\text{out}}(U_v) = d_A^{\text{out}}(U_v \cap K) + d_A^{\text{out}}(U_v \cap K') \geq 2$ — a contradiction.

Moreover, if $v \neq v'$ and $U_v \cap U_{v'} \neq \emptyset$, then $U_v = U_{v'}$. This follows from:

(61.10) $$1 \leq d_A^{\text{out}}(U_v \cup U_{v'}) \leq d_A^{\text{out}}(U_v) + d_A^{\text{out}}(U_{v'}) - d_A^{\text{out}}(U_v \cap U_{v'}) \leq 1 + 1 - 1 = 1.$$

So, if $U_v \neq U_{v'}$, we would increase U_v or $U_{v'}$ by replacing it by $U_v \cup U_{v'}$ — a contradiction.

So the U_v partition K. Now if $U_v \neq K$, there exist v and v' such that $U_v \neq U_{v'}$ and such that G has an edge connecting U_v and $U_{v'}$. We can assume that it is oriented from $U_{v'}$ to U_v. So it is the unique edge leaving $U_{v'}$. Hence $d_A^{\text{out}}(U_v \cup U_{v'}) \leq d_A^{\text{out}}(U_v) = 1$. So replacing U_v by $U_v \cup U_{v'}$ would increase U_v — a contradiction. This shows (61.9), and hence (61.8).

So $d_A^{\text{out}}(K) = 1$ for each component K of $G - U$. Therefore

(61.11) $$|E[U]| + \kappa(G - U) = \sum_{v \in U} \deg_A^{\text{in}}(v) > u(U).$$

Thus U violates (61.6)(i).

One similarly shows that $\deg_A^{\text{in}}(v_0) < l(v_0)$ implies violation of (61.6)(ii). ∎

This implies an alternative characterization:

Corollary 61.5a. *Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a strongly connected orientation $D = (V, A)$ satisfying $l(v) \leq \deg_A^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if G has strongly connected orientations $D' = (V, A')$ and $D'' = (V, A'')$ with $l(v) \leq \deg_{A'}^{\text{in}}(v)$ and $\deg_{A''}^{\text{in}}(v) \leq u(v)$ for each $v \in V$.*

Proof. Directly from Theorem 61.5, as (61.6)(i) is void if $u = \infty$ and as (61.6)(ii) is void if $l = \mathbf{0}$ (since $\kappa(G - U) \leq d_G(U)$). ∎

For further results, see Theorem 61.7.

61.3. Nash-Williams' orientation theorem

The result of Robbins [1939] was extended deeply by Nash-Williams [1960]. Before stating and proving it, we give a useful lemma of Nash-Williams [1960]. Let $\phi : V \times V \to \mathbb{R}$ be a symmetric function (that is, $\phi(u, v) = \phi(v, u)$ for all $u, v \in V$). Define a set function R on V by:

(61.12) $$R(U) := \max_{u \in U, v \in V \setminus U} \phi(u, v) \text{ if } \emptyset \subset U \subset V, \text{ and}$$
$$R(\emptyset) := R(V) := 0.$$

Lemma 61.6α. *For all $T, U \subseteq V$:*

(61.13) $$R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$$
$$\text{or } R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T).$$

Proof. Suppose not. Then $\emptyset \neq T \neq V$ and $\emptyset \neq U \neq V$. Choose $s \in T$, $t \in V \setminus T$, $u \in U$, $v \in V \setminus U$ such that $R(T) = \phi(s, t)$ and $R(U) = \phi(u, v)$. By symmetry, we can assume that $R(T) \leq R(U)$ and $u \in T$. So $u \in T \cap U$, and hence $T \cap U$ splits[15] $\{u, v\}$. This implies that $T \cup U$ splits neither $\{s, t\}$, nor $\{u, v\}$, as otherwise the first inequality in (61.13) holds (as $\phi(s, t) \leq \phi(u, v)$).

Hence $t, v \in T \cup U$, and so $t \in U \setminus T$ and $v \in T \setminus U$. Then $T \setminus U$ splits $\{u, v\}$, and $U \setminus T$ splits $\{s, t\}$, implying the second inequality in (61.13). ∎

For any undirected graph $G = (V, E)$ and $s, t \in V$, let $\lambda_G(s, t)$ denote the maximum number of edge-disjoint $s - t$ paths in G. Similarly, for any directed graph $D = (V, A)$ and $s, t \in V$, let $\lambda_D(s, t)$ denote the maximum number of arc-disjoint $s - t$ paths in D.

Theorem 61.6 (Nash-Williams' orientation theorem)**.** *Any undirected graph $G = (V, E)$ has an orientation $D = (V, A)$ with*

(61.14) $$\lambda_D(s, t) \geq \lfloor \tfrac{1}{2} \lambda_G(s, t) \rfloor$$

for all $s, t \in V$.

Proof. Call a partition of a set T into pairs, a *pairing* of T. For any number k, define

(61.15) $$k^* := 2 \lfloor \tfrac{1}{2} k \rfloor.$$

For any subset U of V, define

[15] Set X *splits* set Y if both $Y \cap X$ and $Y \setminus X$ are nonempty.

(61.16) $$r(U) := \max_{u\in U, v\notin U} \lambda_G(u,v),$$

setting $r(U) := 0$ if $U = \emptyset$ or $U = V$. Let T be the set of vertices of odd degree of G.

I. It suffices to show that T has a pairing P such that

(61.17) $d_G(U) - d_P(U) \geq r(U)^*$ for each $U \subseteq V$.

To see that this is sufficient, let $G' = (V, E \cup P)$. That is, G' is the graph obtained from G by adding all pairs in P as new edges (possibly in parallel). Then all degrees in G' are even, and hence G' has an Eulerian orientation $D' = (V, A')$. So

(61.18) $$\deg^{\text{out}}_{D'}(v) = \deg^{\text{in}}_{D'}(v) = \tfrac{1}{2}\deg_{G'}(v)$$

for each $v \in V$. This implies that, for each $U \subseteq V$,

(61.19) $$d^{\text{out}}_{D'}(U) = \tfrac{1}{2} d_{G'}(U).$$

Let A be the restriction of A' to the original edges of G and let $D = (V, A)$. We claim that D is an orientation of G as required. Indeed, by (61.17), for each $U \subseteq V$,

(61.20) $$\begin{aligned} d^{\text{out}}_{D}(U) &\geq d^{\text{out}}_{D'}(U) - d_P(U) = \tfrac{1}{2} d_{G'}(U) - d_P(U) \\ &= \tfrac{1}{2}(d_G(U) - d_P(U)) \geq \lfloor \tfrac{1}{2} r(U) \rfloor. \end{aligned}$$

Hence, for any $u, v \in V$, if $U \subseteq V$ with $u \in U$, $v \notin U$, and $\lambda_D(u,v) = d^{\text{out}}_D(U)$, then

(61.21) $$\lambda_D(u,v) = d^{\text{out}}_D(U) \geq \lfloor \tfrac{1}{2} r(U) \rfloor \geq \lfloor \tfrac{1}{2} \lambda_G(u,v) \rfloor.$$

II. We now prove the theorem. Define for any $Y \subseteq V$ and $U \subseteq V$,

(61.22) $$r_Y(U) := \max_{u\in Y\cap U, v\in Y\setminus U} \lambda_G(u,v),$$

setting $r_Y(U) := 0$ if $Y \cap U = \emptyset$ or $Y \subseteq U$.

By Lemma 61.6α, for any $U, W \subseteq V$,

(61.23) $$\begin{aligned} &r_Y(U)^* + r_Y(W)^* \leq r_Y(U \cap W)^* + r_Y(U \cup W)^* \\ &\text{or } r_Y(U)^* + r_Y(W)^* \leq r_Y(U \setminus W)^* + r_Y(W \setminus U)^*. \end{aligned}$$

This follows from Lemma 61.6α by taking $\phi(u,v) := \lambda_G(u,v)^*$ if $u, v \in Y$ and $\phi(u,v) := 0$ otherwise.

Suppose that there exist graphs G for which T has no pairing P satisfying (61.17). Choose G with $|V| + |E|$ minimal.

Choose $Y \subseteq V$ such that T has no pairing P satisfying

(61.24) $d_G(U) - d_P(U) \geq r_Y(U)^*$ for each $U \subseteq V$,

with $|Y|$ as small as possible. Then

(61.25) For any subset X of V splitting Y and satisfying $d_G(X)^* = r_Y(X)^*$, one has $|X| = 1$ or $|X| = |V| - 1$.

For suppose $1 < |X| < |V| - 1$. Consider the graph $G_1 = (V_1, E_1)$ obtained from G by contracting $V \setminus X$ to one vertex, v_1 say. Let T_1 be the set of vertices of odd degree of G_1. By the minimality of $|V|+|E|$, T_1 has a pairing P_1 such that for each subset U of X:

(61.26) $$d_{G_1}(U) - d_{P_1}(U) \geq r(U)^*.$$

(Note that $r(U) \leq \max_{u \in U, v \in V \setminus U} \lambda_{G_1}(u, v)$.)

Similarly, consider the graph $G_2 = (V_2, E_2)$ obtained from G by contracting X to one vertex, v_2 say. Let T_2 be the set of vertices of odd degree of G_2. Again by the minimality of $|V| + |E|$, T_2 has a pairing P_2 such that for each subset U of $V \setminus X$ and for each $u \in U$, $v \in V_2 \setminus U$:

(61.27) $$d_{G_2}(U) - d_{P_2}(U) \geq r(U)^*.$$

Now define a pairing P of T as follows. (Observe that $v_1 \in T_1$ if and only if $v_2 \in T_2$.) If $v_1 \notin T_1$ and $v_2 \notin T_2$, let $P := P_1 \cup P_2$. If $v_1 \in T_1$ and $v_2 \in T_2$, let $u_1 \in X$ and $u_2 \in V \setminus X$ be such that $u_1v_1 \in P_1$ and $u_2v_2 \in P_2$. Then define

(61.28) $$P := (P_1 \setminus \{u_1v_1\}) \cup (P_2 \setminus \{u_2v_2\}) \cup \{u_1u_2\}.$$

We claim that P satisfies (61.24). To show this, we may assume by (61.23) that $r_Y(U \cap X)^* + r_Y(U \cup X)^* \geq r_Y(U \setminus X)^* + r_Y(X \setminus U)^*$. (Otherwise, replace U by $V \setminus U$.)

Set $U_1 := U \cap X$ and $U_2 := V \setminus (U \cup X)$. Then

(61.29) $$\begin{aligned} d_G(U) + d_G(X) &\geq d_G(U_1) + d_G(U_2) = d_{G_1}(U_1) + d_{G_2}(U_2) \\ &\geq r(U_1)^* + d_{P_1}(U_1) + r(U_2)^* + d_{P_2}(U_2) \geq r_Y(U)^* + r_Y(X)^* + d_P(U) \\ &\geq r_Y(U)^* + d_G(X) + d_P(U) - 1, \end{aligned}$$

using (61.26) and (61.27). As $d_G(U) + d_P(U)$ is even, (61.24) follows. This contradicts our assumption, showing (61.25).

We next show that

(61.30) each edge of G intersects Y.

For assume that G has an edge $e = st$ disjoint from Y. By (61.25), there is no U splitting Y with $d_G(U)^* = r_Y(U)^*$ and $s \in U$, $t \notin U$. So deleting edge e, changes no $r_Y(U)^*$. Let G' be the graph obtained from G by deleting e. Let T' be the set of vertices of G' of odd degree. (So $T' = T \triangle \{s, t\}$.) Then, by the minimality of $|V| + |E|$, we know that T' has a pairing P' such that, for each $U \subseteq V$,

(61.31) $$d_{G'}(U) - d_{P'}(U) \geq r(U)^* \geq r_Y(U)^*.$$

It is not difficult to transform pairing P' of T' to a pairing P of T with the property that $|P \setminus P'| \leq 1$.[16] Then (61.24) holds. Indeed, $d_G(U) \geq d_{G'}(U)$

[16] If $s, t \in T'$ and $st \in P'$, define $P := P' \setminus \{st\}$. If $s, t \in T'$ and $st \notin P'$, let s' and t' be such that $ss' \in P'$ and $tt' \in P'$, and define $P := (P' \setminus \{ss', tt'\}) \cup \{s't'\}$. If $s \in T'$ and $t \notin T'$, let s' be such that $ss' \in P'$, and define $P := (P' \setminus \{ss'\}) \cup \{ts'\}$. If $s \notin T'$ and $t \in T'$, let t' be such that $tt' \in P'$, and define $P := (P' \setminus \{tt'\}) \cup \{st'\}$. If $s \notin T'$ and $t \notin T'$, define $P := P' \cup \{st\}$.

and $d_P(U) \leq d_{P'}(U)+1$ (as $|P \setminus P'| \leq 1$). Hence (61.24) follows from (61.31), with parity. This contradiction proves (61.30).

Next:

(61.32) $\quad |Y| \geq 2.$

For suppose that $|Y| \leq 1$. In G there exist $\frac{1}{2}|T|$ edge-disjoint paths such that each vertex in T occurs exactly once as an end vertex of one of these paths. (This can be seen by taking an arbitrary pairing Q of T, and considering an Eulerian tour C in the graph $G = (V, E \cup Q)$. Then removing Q from C decomposes C into paths as required.) Let P be the set of pairs of end vertices of these paths. Then $d_G(U) \geq d_P(U)$ for each $U \subseteq V$, and (61.24) follows, contradicting our assumption. So we know (61.32).

Choose a set X splitting Y with $d_G(X)$ minimal. Then $d_G(X) = r_Y(X)$. By (61.25), we may assume that $X = \{x\}$ for some $x \in Y$. So $r_Y(U) = d_G(x)$ for any U splitting Y, since for any $y \in Y \setminus U$ we have

(61.33) $\quad d_G(x) \leq d_G(U) \leq r_Y(U) \leq \lambda_G(x, y) \leq d_G(x).$

.

Define $Y' := Y \setminus \{x\}$. Then, by the minimality of $|Y|$, T has a pairing P such that for each $U \subseteq V$,

(61.34) $\quad d_G(U) - d_P(U) \geq r_{Y'}(U)^*.$

We show that (61.24) holds, which forms a contradiction. To this end, choose $U \subseteq V$.

First assume that U splits Y'. Then $r_{Y'}(U) \geq r_Y(U)$, since $\lambda_G(x, y) = d_G(X) \leq r_{Y'}(U)$ for each $y \in Y'$ (by the minimality of $d_G(X)$, since any splitting of Y' also splits Y). This implies (61.34).

So we can assume that U splits Y but does not split Y'; that is, $U \cap Y = \{x\}$. Consider any $u \in U \setminus \{x\}$. Let α_u denote the number of edges connecting u and x and let β_u denote the number of edges connecting u and $Y \setminus \{x\}$. By (61.30), $\alpha_u + \beta_u = \deg_G(u)$. Since $X = \{x\}$ splits Y with $d_G(X)$ minimum, we have $d_G(\{x, u\}) \geq \deg_G(x)$. Hence $\beta_u \geq \alpha_u$, with strict inequality if $u \in T$ (since then $\alpha_u + \beta_u$ is odd).

Therefore, setting $U' := U \setminus \{x\}$ and $\lambda :=$ number of edges connecting x and $V \setminus U$,

(61.35)
$$\begin{aligned} d_G(U) &= \lambda + \sum_{u \in U'} \beta_u \geq \lambda + \sum_{u \in U'} \alpha_u + |U' \cap T| = \deg_G(x) + |U' \cap T| \\ &= r_Y(U) + |U' \cap T| \geq r_Y(U) + |U \cap T| - 1 \geq r_Y(U) + d_P(U) - 1. \end{aligned}$$

Hence, with parity, we have (61.24). ∎

(This is the original proof of Nash-Williams [1960]. Mader [1978a] and Frank [1993a] gave alternative proofs.)

An orientation satisfying the condition described in Theorem 61.6 is called *well-balanced*. Nash-Williams [1969] (giving an introduction to the proof above) remarks that with methods similar to those used in the proof of Theorem 61.6, one can prove that for any graph G and any subgraph H of G, there is a well-balanced orientation of G such that the restriction to H is well-balanced again.

61.4. k-arc-connected orientations of $2k$-edge-connected graphs

Nash-Williams' orientation theorem directly implies:

Corollary 61.6a. *An undirected graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

Proof. Directly from Theorem 61.6. ∎

A direct proof of this corollary, based on total dual integrality, was given by Frank [1980b] and Frank and Tardos [1984b], and is as follows.

Orient the edges of G arbitrarily, yielding the directed graph $D = (V, A)$. Consider the system

(61.36) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq d^{\text{in}}(U) - k$ for each nonempty $U \subset V$.

By the Edmonds-Giles theorem (Theorem 60.1), this system is TDI, and hence determines an integer polytope P. If G is $2k$-edge-connected, then P is nonempty, since the vector $x := \frac{1}{2} \cdot \mathbf{1}$ belongs to P.

As P is nonempty and integer, (61.36) has an integer solution x. Then G has a k-arc-connected orientation D': reversing the orientation of the arcs a of D with $x_a = 1$ gives a k-arc-connected orientation D', since

(61.37) $$d^{\text{in}}_{D'}(U) = d^{\text{in}}_{D}(U) - x(\delta^{\text{in}}_{D}(U)) + x(\delta^{\text{out}}_{D}(U)) \geq k$$

for any nonempty proper subset U of V.

Notes. The total dual integrality of (61.36) implies also the following result of Frank, Tardos, and Sebő [1984] (denoting the number of (weak) components of a (di)graph G by $\kappa(G)$). Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $U \subseteq V$. Then the minimum number of arcs entering U over all strongly connected orientations of G is equal to the maximum of

(61.38) $$\sum_{T \in \mathcal{P}} \kappa(G - T),$$

taken over partitions $\mathcal{P}$ of U into nonempty classes such that no edge connects different classes of $\mathcal{P}$.

This implies another result of Frank, Tardos, and Sebő [1984]: Let $D = (V, A)$ be a digraph and let $C = \delta^{\text{in}}(U)$ be a directed cut. Then the minimum of $|B \cap C|$ where B is a directed cut cover is equal to the maximum of

$$(61.39) \qquad \sum_{T \in \mathcal{P}} \kappa(D - T),$$

taken over partitions $\mathcal{P}$ of U into nonempty classes such that no arc of D connects distinct classes of $\mathcal{P}$.

As A. Frank (personal communication 2002) observed, the proof above yields a stronger result of Nash-Williams [1969]: let $G = (V, E)$ be a $2k$-edge-connected graph and let $F \subseteq E$ have an Eulerian orientation; then the remaining edges have an orientation so as to obtain a k-arc-connected digraph. This follows by taking for A any orientation extending the orientation of F, and by setting $x_a := 0$ for each arcs in F, and $x_a := \frac{1}{2}$ for all other arcs a. Then x satisfies (61.36), and the result follows as above.

61.4a. Complexity

By the results in Section 60.1 on the complexity of the Edmonds-Giles problem, one can find a k-arc-connected orientation of a $2k$-edge-connected undirected graph in polynomial time; more generally, one can find a minimum-length k-arc-connected orientation in strongly polynomial time, if we are given a length for each orientation of each edge.

A direct method of finding a minimum-length k-arc-connected orientation can be based on weighted matroid intersection, similarly to the method described in Section 55.5 to find a minimum-length directed k-cover in a directed graph (such that the k-arc-connected orientations form the common bases of two matroids).

61.4b. k-arc-connected orientations with bounds on degrees

Frank [1980b] extended Corollary 61.6a to the case where lower and upper bounds on the indegrees of the vertices are prescribed:

Theorem 61.7. *Let $G = (V, E)$ be a $2k$-connected undirected graph and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a k-arc-connected orientation D with $l(v) \leq \deg_D^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if*

$$(61.40) \qquad |E[W]| + |\delta(\mathcal{P})| \geq k|\mathcal{P}| + \max\{\sum_{v \in W} l(v), \sum_{v \in W} (\deg_G(v) - u(v))\}$$

for each subpartition $\mathcal{P}$ of V with nonempty classes, where $W := V \setminus \bigcup \mathcal{P}$.

Proof. It is not difficult to see that the condition is necessary. To show sufficiency, by Corollary 61.6a, G has a k-arc-connected orientation D. Choose D such that

$$(61.41) \qquad \sum_{v \in V} \max\{0, \deg_D^{\text{in}}(v) - u(v), l(v) - \deg_D^{\text{in}}(v)\}$$

is as small as possible. If sum (61.41) is 0 we are done, so assume that it is positive. By symmetry, we can assume that there is a vertex r with $\deg_D^{\text{in}}(r) < l(r)$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal nonempty subsets U of $V \setminus \{r\}$ with $d^{\text{in}}(U) = k$, and let $W := V \setminus \bigcup \mathcal{P}$.

Then the sets in $\mathcal{P}$ are disjoint. For let $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$(61.42) \qquad 2k \leq d_D^{\text{in}}(U \cap W) + d_D^{\text{in}}(U \cup W) \leq d_D^{\text{in}}(U) + d_D^{\text{in}}(W) = 2k,$$

implying $d_D^{\text{in}}(U \cup W) = k$, and so $U = W = U \cup W$.

Suppose that there exists a vertex $s \in W$ with $\deg_D^{\text{in}}(s) > l(s)$. Then reversing the orientations of the arcs of any $r - s$ path in D gives again a k-arc-connected orientation (since there is no $U \subseteq V$ with $d_D^{\text{in}}(U) = k$ and $s \in U$, $r \notin U$), but decreases sum (61.41). This contradicts our minimality assumption.

So $\deg_D^{\text{in}}(v) \leq l(v)$ for each $v \in W$, with strict inequality for at least one $v \in W$ (namely for r). Now each edge of G that is spanned by no set in $\mathcal{P}$, either enters some $U \in \mathcal{P}$, or has its head in W. So the number of such edges is

(61.43) $$k|\mathcal{P}| + \sum_{v \in W} \deg_D^{\text{in}}(v), \text{ which is less than } k|\mathcal{P}| + \sum_{v \in W} l(v).$$

This contradicts the condition. ∎

This has an alternative characterization as consequence:

Corollary 61.7a. *Let $G = (V, E)$ be an undirected graph, let $k \in \mathbb{Z}_+$, and let $l, u \in \mathbb{Z}_+^E$ with $l \leq u$. Then G has a k-arc-connected orientation D with $l(v) \leq \deg_D^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if G has k-arc-connected orientations D' and D'' with $l(v) \leq \deg_{D'}^{\text{in}}(v)$ and $\deg_{D''}^{\text{in}}(v) \leq u(v)$ for each $v \in V$.*

Proof. This follows from the fact that the condition in Theorem 61.7 can be decomposed into a condition on l and one on u. ∎

61.4c. Orientations of graphs with lower bounds on indegrees of sets

Let $G = (V, E)$ be an undirected graph and let $l : \mathcal{P}(V) \to \mathbb{Z}_+$ be such that

(61.44) $$l(T) + l(U) - d(T, U) \leq l(T \cap U) + l(T \cup U), \text{ for all } T, U \subseteq V \text{ with } T \cap U \neq \emptyset \text{ and } T \cup U \neq V,$$

where $d(T, U)$ denotes the number of edges connecting $T \setminus U$ and $U \setminus T$.

Frank [1980b] showed with submodularity theory:

Theorem 61.8. *Let $G = (V, E)$ be a graph and let $l : \mathcal{P}(V) \to \mathbb{Z}_+$ satisfy* (61.44). *Then G has an orientation $D = (V, A)$ with*

(61.45) $$d_A^{\text{in}}(U) \geq l(U)$$

for each $U \subseteq V$ if and only if

(61.46) $$|\delta(\mathcal{P})| \geq \max\{\sum_{U \in \mathcal{P}} l(U), \sum_{U \in \mathcal{P}} l(V \setminus U)\}$$

for each partition $\mathcal{P}$ of V into nonempty proper subsets, where $\delta(\mathcal{P})$ denotes the set of edges of G connecting different classes of $\mathcal{P}$.

Proof. The necessity of the condition is obvious. To prove sufficiency, let $D = (V, A)$ be an arbitrary orientation of G. Define for each nonempty proper subset U of V

(61.47) $$f(U) := d_A^{\text{in}}(U) - l(U).$$

One easily checks, using (61.44), that f is crossing submodular. Moreover, if $x : A \to \{0,1\}$ is such that

(61.48) $$x(\delta^{\text{in}}_A(U)) - x(\delta^{\text{out}}_A(U)) \le f(U)$$

for each nonempty proper subset U of V, then the digraph $D' = (V, A')$ obtained from $D = (V, A)$ by reversing the direction of the arcs a with $x_a = 1$, has indegrees as required by (61.45), since

(61.49) $$d^{\text{in}}_{A'}(U) = d^{\text{in}}_A(U) - x(\delta^{\text{in}}_A(U)) + x(\delta^{\text{out}}_A(U)) \ge d^{\text{in}}_A(U) - f(U) = l(U).$$

Hence it suffices to show that (61.48) has an integer solution x with $\mathbf{0} \le x \le \mathbf{1}$.

Consider x as a transshipment. The 'excess function' $\text{excess}_x \in \mathbb{R}^V$ of x is given by:

(61.50) $$\text{excess}_v := x(\delta^{\text{in}}_A(v)) - x(\delta^{\text{out}}_A(v))$$

for $v \in V$. Then (61.48) is equivalent to

(61.51) $$y(U) \le f(U).$$

Now y is the excess function of some $x \in \{0,1\}^A$ if and only if x is an integer y-transshipment with $\mathbf{0} \le x \le \mathbf{1}$. So, by Corollary 11.2f, y is the excess function of some $x \in \{0,1\}^A$ if and only if y is integer, $y(V) = 0$, and

(61.52) $$y(U) \le d^{\text{in}}_A(U)$$

for each $U \subseteq V$. Since $l(U) \ge 0$, (61.52) is implied by (61.51).

So it suffices to show that (61.51) has an integer solution y with $y(V) = 0$. By Theorem 49.10, y exists if and only if

(61.53) $$\sum_{U \in \mathcal{P}} f(U) \ge 0$$

for each partition or copartition $\mathcal{P}$ of V, where each set in $\mathcal{P}$ is a nonempty proper subset of V. (A *copartition* of V is a collection of sets whose complements form a partition of V.) This is equivalent to the condition given in the present theorem. ∎

61.4d. Further notes

Frank [1980b] observed that Edmonds' disjoint arborescences theorem implies:

Corollary 61.8a. *Let $G = (V, E)$ be an undirected graph and $r \in V$. Then G has an orientation such that each nonempty subset U of $V \setminus \{r\}$ is entered by at least k arcs if and only if G contains k edge-disjoint spanning trees.*

Proof. Necessity follows from the fact that if the orientation $D = (V, A)$ as required exists, then by Edmonds' disjoint arborescences theorem (Corollary 53.1b), D has k disjoint r-arborescences. Hence G has k edge-disjoint spanning trees.

Sufficiency follows from the fact that we can orient each spanning tree in G so as to become an r-arborescence. Orienting the remaining edges arbitrarily, we obtain an orientation as required. ∎

Frank [1993c] gave a direct proof of the existence of this orientation from the conditions given in the Tutte-Nash-Williams disjoint trees theorem (Corollary

51.1a), yielding (with Edmonds' disjoint arborescences theorem) a proof of the Tutte-Nash-Williams disjoint trees theorem.

Frank [1982a] showed that each k-arc-connected orientation of an undirected graph can be obtained from any other by reversing iteratively directed paths and circuits, without destroying k-arc-connectivity. This can be derived from a result of L. Lovász that two k-arc-connected orientations are adjacent on the polytope determined by (61.36) if and only if they differ on a directed circuit or on a collection of vertex-disjoint directed paths. Frank [1982b] showed that a minimum-cost k-arc-connected orientation can be found in strongly polynomial time, by reduction to the Edmonds-Giles model. Accelerations were given by Gabow [1993a,1993b,1994, 1995c].

Frank, Jordán, and Szigeti [1999,2001] and Frank and Király [1999,2002] studied graph orientations that satisfy parity and connectivity conditions. Orientations preserving prescribed shortest paths are considered by Hassin and Megiddo [1989].

Chvátal and Thomassen [1978] showed that each 2-edge-connected graph of radius r has a strongly connected orientation of radius at most $r^2 + r$. This was extended to mixed graphs by Chung, Garey, and Tarjan [1985].

For surveys on applying submodularity to orientation problems, see Frank [1993a,1996b].

Chapter 62

Network synthesis

The network synthesis problem asks for a graph having prescribed connectivity properties, with a minimum number of edges. If the edges have costs, a minimum total cost is required.
The problem can be seen as the special case of the connectivity augmentation problem where the input graph is edgeless. Connectivity augmentation in general will be discussed in Chapter 63.

62.1. Minimal k-(edge-)connected graphs

We first consider the easy problem of finding a graph of given connectivity, with a minimal number of edges. First, vertex-connectivity:

Theorem 62.1. *Let k and n be positive integers with $n \geq 2$. The minimum number of edges of a k-vertex-connected graph with n vertices is $n-1$ if $k = 1$, $\lceil \frac{1}{2}kn \rceil$ if $1 < k < n$, and $\frac{1}{2}n(n-1)$ otherwise.*

Proof. Since any k-vertex-connected graph contains a spanning tree, has minimum degree at least k if $k < n$, and is a complete graph if $k \geq n$, the values given are lower bounds. Moreover, if $k = 1$ or $k \geq n$, the bound is tight. So we can assume $1 < k < n$, and it suffices to show that there exists a k-vertex-connected graph $G = (V, E)$ with $|V| = n$ and $|E| = \lceil \frac{1}{2}kn \rceil$.

Let $V := \{1, \ldots, n\}$ and let C be the circuit on V with edge set $\{\{i, i+1\} \mid i \in V\}$, taking addition mod n. Let G be the graph on V with edges all pairs of vertices at distance at most $\frac{1}{2}k$ in C.

First assume that k is even. Then G has $\frac{1}{2}kn$ edges. We show that G is k-vertex-connected. Suppose to the contrary that G has a vertex-cut U of size less than k. There are at least two components K of $C[U]$ such that the two neighbours of K in C belong to different components of $G - U$ (as $G - U$ is disconnected). In particular, the two neighbours have distance more than $\frac{1}{2}k$ in C, and so these components each have size at least $\frac{1}{2}k$. This contradicts the fact that $|U| < k$.

Next, if k is odd, we add to G $\lceil \frac{1}{2}n \rceil$ edges $\{i, j\}$, where i and j have distance $\lfloor \frac{1}{2}n \rfloor$ in C, and such that these edges cover all vertices in V. So G has $\lceil \frac{1}{2}kn \rceil$ edges. We show that G is k-vertex-connected.

Suppose that G has a vertex-cut U of size less than k. By the above, $C[U]$ consists of two components of size $l := \frac{1}{2}(k-1)$ each. We can assume that $U = [1, l] \cup [s+1, s+l]$ for some s with $l < s$ and $s+l < n$. Now n is adjacent to no vertex in $[l+1, s]$, while n is adjacent to at least one of $\lfloor \frac{1}{2}n \rfloor$ and $\lceil \frac{1}{2}n \rceil$. So $\lfloor \frac{1}{2}n \rfloor < l+1$ or $\lceil \frac{1}{2}n \rceil > s$, implying $n > 2s$ (as $k < n$). By symmetry of the two components we similarly have $n > 2(n-s)$, that is $n < 2s$, a contradiction. ∎

For edge-connectivity the answer is almost the same:

Theorem 62.2. *Let k and n be positive integers with $n \geq 2$. The minimum number of edges of a k-edge-connected graph with n vertices is $n-1$ if $k = 1$, and $\lceil \frac{1}{2}kn \rceil$ otherwise. If $k \leq n-1$ the minimum is attained by a simple graph.*

Proof. Again, the values are lower bounds, as a k-edge-connected graph contains a spanning tree and has each degree at least k. Clearly the lower bound can be attained if $k = 1$, so assume $k \geq 2$. Let C be a graph on $V := \{1, \ldots, n\}$ with edges all pairs $\{i, i+1\}$ for $i \in V$ (with addition mod n). Let G be the graph obtained from C by replacing each edge by $\lfloor \frac{1}{2}k \rfloor$ parallel edges.

If k is even, then G is k-edge-connected as required. If k is odd, add $\lceil \frac{1}{2}n \rceil$ edges $\{i, j\}$ to G, where i and j have distance $\lfloor \frac{1}{2}n \rfloor$ in C, and such that these edges cover all vertices in V. So G has $\lceil \frac{1}{2}kn \rceil$ edges. We show that G is k-edge-connected. Suppose that $d_G(U) < k$ for some nonempty proper subset U of V. By symmetry, we can assume that $|U| \geq \frac{1}{2}n$. Now $C[U]$ is connected (as otherwise $d_G[U] \geq 4\lfloor \frac{1}{2}k \rfloor \geq k$, since $k > 1$). So we can assume that $U = [1, s]$, with $s \geq \lceil \frac{1}{2}n \rceil$. However, $n \in V \setminus U$ is adjacent to at least one of $\lfloor \frac{1}{2}n \rfloor$ and $\lceil \frac{1}{2}n \rceil$. As both of these vertices belong to U, we have $d_G(U) \geq 2\lfloor \frac{1}{2}k \rfloor + 1 = k$, a contradiction.

Finally, if $k \leq n-1$, the minimum is attained by a simple graph. Indeed, by Theorem 62.1, there is a k-vertex-connected graph $G = (V, E)$ with n vertices and $\lceil \frac{1}{2}kn \rceil$ edges. Necessarily, G is simple. We show that G is k-edge-connected. Suppose that there is a nonempty $U \subset V$ with $d_G(U) < k$. Then $|U||V \setminus U| \geq n-1 \geq k$, and hence there exist $s \in U$ and $t \in V \setminus U$ that are not adjacent. Hence G has k internally vertex-disjoint $s-t$ paths, and therefore k edge-disjoint $s-t$ paths. This contradicts the fact that $d_G(U) < k$. ∎

The directed case is even simpler. For vertex-connectivity one has:

Theorem 62.3. *Let k and n be positive integers with $n \geq 2$. Then the minimum number of arcs of a k-vertex-connected directed graph with n vertices is kn if $k \leq n-1$, and $n(n-1)$ otherwise.*

Proof. Since each vertex should have at least $\min\{k, n-1\}$ outneighbours, the values are lower bounds. Trivially it is attained if $k \geq n$.

If $k \leq n-1$, let D be the directed graph on $V := \{1, \ldots, n\}$ with arcs all pairs $(i, i+l)$ with $i \in V$ and $1 \leq l \leq k$, taking addition mod n. Then D has kn arcs. To see that D is k-vertex-connected, let U be a vertex-cut. Choose $i, j \in V \setminus U$ with j not reachable from i in $D - U$. We may assume that $1 \leq i < j \leq n$ and that $j - i$ is as small as possible. Then $j - i > k$ and $i+1, \ldots, j-1$ belong to U. So $|U| \geq k$. ∎

Finally, for arc-connectivity (Fulkerson and Shapley [1971]):

Theorem 62.4. *Let k and n be positive integers with $n \geq 2$. Then the minimum number of arcs of a k-arc-connected directed graph with n vertices is kn. If $k \leq n-1$, the minimum is attained by a simple directed graph.*

Proof. Since each vertex should be left by at least k arcs, kn is a lower bound. It is attained by the directed graph obtained from a directed circuit on n vertices, by replacing any arc by k parallel arcs.

If $k \leq n-1$, the minimum is attained by a simple directed graph. Indeed, by Theorem 62.3, there is a k-vertex-connected directed graph $D = (V, A)$ with n vertices and kn arcs. Necessarily, D is simple. We show that D is k-arc-connected. Suppose that there is a nonempty $U \subset V$ with $d_D^{\text{out}}(U) < k$. Then $|U||V \setminus U| \geq n-1 \geq k$, and hence there exist $s \in U$ and $t \in V \setminus U$ with $(s,t) \notin A$. Hence D has k internally vertex-disjoint $s-t$ paths, and therefore k arc-disjoint $s-t$ paths. This contradicts the fact that $d_D^{\text{out}}(U) < k$. ∎

Notes. Edmonds [1964] showed that for each simple graph with all degrees at least k, there exists a k-edge-connected simple graph with the same degree-sequence.

62.2. The network synthesis problem

Let V be a finite set and let $r : V \times V \to \mathbb{R}_+$. A *realization* of r is a pair of a directed graph $D = (V, A)$ and a capacity function $c : A \to \mathbb{R}_+$ such that for all $s, t \in V$, each $s-t$ cut in G has capacity at least $r(s,t)$. The pair D, c is called an *exact realization* if for all $s, t \in V$ with $s \neq t$, the minimum capacity of an $s-t$ cut in D as equal to $r(s,t)$.

Obviously, any function r has a realization. We say that r is *exactly realizable* if it has an exact realization. The *network synthesis problem* is the problem to find an exact or cheapest realization for a given r (or to decide that none exist).

The following theorem due to Gomory and Hu [1961] characterizes the exactly realizable *symmetric*[17] functions. It also shows that if $r : V \times V \to \mathbb{R}$ is exactly realizable and symmetric, then r has an *undirected* exact realization

[17] A function $r : V \times V \to \mathbb{R}$ is called *symmetric* if $r(u,v) = r(v,u)$ for all $u, v \in V$.

(more precisely, an exact realization D, c where for each arc $a = (u, v)$ of D, also (v, u) is an arc, with $c(u, v) = c(v, u)$).

Theorem 62.5. *A symmetric function $r : V \times V \to \mathbb{R}_+$ is exactly realizable if and only if*

(62.1) $$r(u, w) \geq \min\{r(u, v), r(v, w)\}$$

for all distinct $u, v, w \in V$. If r is exactly realizable, there is a tree that gives an exact realization of r.

Proof. Necessity being easy, we show sufficiency. Let $T = (V, E)$ be a tree on V maximizing

(62.2) $$\sum_{uv \in E} r(u, v).$$

Taking $c(uv) := r(u, v)$ for each edge $uv \in E$ gives an exact realization of r. To see this, note that for all s, t, the minimum capacity of an $s - t$ cut is equal to $\min_{uv \in EP} r(u, v)$, where P is the $s - t$ path in T. By (62.1) we know that $r(s, t)$ is not smaller than this minimum. To show equality, suppose to the contrary that $r(u, v) < r(s, t)$ for some $uv \in P$. Then replacing T by $(T - uv) \cup st$ gives a tree with larger sum (62.2). ∎

Notes. Obviously, condition (62.1) remains necessary for exact realizability of non-symmetric functions. Resh [1965] claimed that (62.1) also remains sufficient, but a counterexample is given by the function $r : V \times V \to \mathbb{R}_+$ with $V = \{1, 2, 3, 4\}$, and $r(1, 2) = r(1, 3) = r(1, 4) = r(2, 4) = r(3, 4) = 1$, and $r(s, t) = 0$ for all other s, t (cf. Mayeda [1962]).

62.3. Minimum-capacity network design

Theorem 62.5 yields a tree as an exact realization of a given function $r : V \times V \to \mathbb{R}_+$. A tree is a most economical realization in the sense of having a minimum number of edges with nonzero capacity. It generally gives no exact realization for which the sum of the capacities is minimum. Such an exact realization has been characterized by Chien [1960] (extending Mayeda [1960]), while Gomory and Hu [1961] showed that if r is integer, there is a half-integer optimum exact realization.

As a preparation, we first show the following lemma of Gomory and Hu [1961] ($\lambda_G(s, t)$ denotes the maximum number of edge-disjoint $s - t$ paths in G):

Lemma 62.6α. *Let $r : V \times V \to \mathbb{R}_+$ be symmetric and let T be a spanning tree on V maximizing $r(T)$. Then any graph $G = (V, E)$ satisfies*

(62.3) $$\lambda_G(s, t) \geq r(s, t)$$

for all $s, t \in V$ if and only if (62.3) *is satisfied for each edge st of T.*

Proof. Necessity being trivial, we show sufficiency. Let $s, t \in V$ and let P be the $s-t$ path in T. By the maximality of $r(T)$, we know that $r(s,t) \leq r(e)$ for each edge e on P. Hence

$$\text{(62.4)} \qquad \lambda_G(s,t) \geq \min_{e=uv \in EP} \lambda_G(u,v) \geq \min_{e \in EP} r(e) \geq r(s,t),$$

as required. ∎

We also use the following lemma:

Lemma 62.6β. *If $r : V \times V \to \mathbb{R}_+$ is symmetric and exactly realizable, then there exists a spanning tree T on V that maximizes $r(T)$ over all spanning trees, and that moreover is a Hamiltonian path.*

Proof. Let T maximize $r(T)$. Choose T and k such that T contains a path $v_1, \ldots, v_k$, with k as large as possible. Choose T, k moreover such that the vector $(\deg_T(v_1), \ldots, \deg_T(v_k))$ is lexicographically minimal. If T is not a path, there is a j with $1 < j < k$ and $\deg_T(v_j) \geq 3$. Let $v_j u$ be an edge of T incident with v_j, with $u \neq v_{j-1}, v_{j+1}$. If $r(v_{j+1}, u) \geq r(v_j, u)$, we can replace edge $v_j u$ of T by $v_{j+1} u$, contradicting the lexicographic minimality. So $r(v_j, u) > r(v_{j+1}, u)$, and so $r(v_j, v_{j+1}) \leq r(v_{j+1}, u)$, since $r(v_{j+1}, u) \geq \min\{r(v_j, v_{j+1}), r(v_j, u)\}$ by (62.1). Hence replacing edge $v_j v_{j+1}$ of T by $v_{j+1} u$ would give a tree with a longer path, contradicting our assumption. ∎

Now we can formulate and prove the theorem. For any $r : V \times V \to \mathbb{R}$ and $u \in V$ define

$$\text{(62.5)} \qquad R(u) := \max_{v \neq u} r(u,v).$$

Theorem 62.6. *Let $r : V \times V \to \mathbb{R}_+$ be symmetric and exactly realizable. Then the minimum value of $\sum_{e \in E} c(e)$ where $G = (V, E)$ and c form an (undirected) exact realization of r, is equal to*

$$\text{(62.6)} \qquad \tfrac{1}{2} \sum_{u \in V} R(u).$$

Moreover, if r is integer, the minimum is attained by a half-integer exact realization c.

Proof. We may assume that r is integer. (62.6) indeed is a lower bound, since for each exact realization $G = (V, E), c$ of r one has

$$\text{(62.7)} \qquad \sum_{e \in E} c(e) = \tfrac{1}{2} \sum_{u \in V} \sum_{e \in \delta(u)} c(e) \geq \tfrac{1}{2} \sum_{u \in V} R(u).$$

To see that the lower bound is attained by a half-integer exact realization, let T be a spanning tree on V maximizing $r(T)$. By Lemma 62.6β, we can assume that T is a path $v_1, \ldots, v_n$.

Let $k := \max_{u,v} r(u, v)$. For $i = 0, \ldots, k$, let E_i be the set of edges e of T with $r(e) \leq i$, and for each nonsingleton component P of $T - E_i$, make a circuit consisting of edges parallel to P and one edge connecting the end vertices of P. Let $G = (V, E)$ arise by taking the edge-disjoint union of these circuits. Let $c(e) := \frac{1}{2}$ for each $e \in E$. Then

$$(62.8) \qquad \lambda_G(v_j, v_i) = 2r(v_j, v_i)$$

for all i, j with $1 \leq j < i \leq n$.

Indeed, in proving $\geq$, we can assume that $j = i - 1$ (by Lemma 62.6α). As v_{i-1}, v_i are contained in $r(v_{i-1}, v_i)$ edge-disjoint circuits, we have $\lambda_G(v_{i-1}, v_i) \geq 2r(v_{i-1}, v_i)$.

Conversely, the inequality $\leq$ in (62.8) follows from

$$(62.9) \qquad \lambda_G(v_j, v_i) \leq \min_{j<h\leq i} 2r(v_{h-1}, v_h) \leq 2r(v_j, v_i).$$

The first inequality here follows from the fact that for each h with $j < h \leq i$, the number of edges connecting $\{v_1, \ldots, v_{h-1}\}$ and $\{v_h, \ldots, v_n\}$ is equal to $2r(v_{h-1}, v_h)$. The second inequality follows from (62.1). ∎

Notes. Note that also any nonexact realization has size at least (62.6), and therefore, the theorem also characterizes the minimum size of any realization.

Wing and Chien [1961] observed that a minimum-capacity realization can be found by linear programming, and Gomory and Hu [1962,1964] showed that also the weighted case can be solved by linear programming. Indeed, the polyhedron P of all realizations of a given function $r : V \times V \to \mathbb{Q}_+$ can be described as follows. Let E is the collection of all unordered pairs of elements of V. Then P is determined by:

$$(62.10) \qquad \begin{array}{ll} x_e \geq 0 & \text{for all } e \in E, \\ x(\delta(U)) \geq R(U) & \text{for all nonempty } U \subset V, \end{array}$$

where $R(U) := \max_{u\in U, v\in V\setminus U} r(u, v)$.

This formulation was given by Gomory and Hu [1962] and applied to finding a minimum-cost realization with linear programming, by a row-generating implementation of the simplex method (thus avoiding listing the exponential number of constraints). Bland, Goldfarb, and Todd [1981] observed that description (62.10) implies polynomial-time solvability with the ellipsoid method, since the constraints (62.10) can be tested in polynomial time.

A direct, polynomial-size linear programming formulation was given by Gomory and Hu [1964], by extending the number of variables. Indeed, P consists of those $x \in \mathbb{R}^E_+$ such that for all distinct $s, t \in V$, there exists an $s - t$ flow $f_{s,t} : E \to \mathbb{R}^E_+$ with $f \leq x$ and of value $r(s, t)$.

The latter description implies that a minimum-weight realization can be determined in polynomial time, by solving an explicit linear programming problem — in fact, in strongly polynomial time, with the method of Tardos [1986].

Note that the *exact* realizations do not form a convex set; for instance, if $V = \{u, v, w\}$ and $r(s,t) = 1$ for all $s, t \in V$, then $x(uv) = x(vw) = x(uw) = \frac{2}{3}$ is a convex combination of exact realizations, but is not itself an exact realization.

62.4. Integer realizations and r-edge-connected graphs

In Section 62.3, the fractional version of the minimum-capacity network design problem was discussed. We now consider the case where all capacities are required to be integer. It relates to: given $r : V \times V \to \mathbb{Z}_+$, find an r-edge-connected undirected graph $G = (V, E)$ with a minimum number of edges. Here a graph $G = (V, E)$ is called *r-edge-connected* if $\lambda_G(s,t) \geq r(s,t)$ for all $s, t \in V$ with $s \neq t$.

Eswaran and Tarjan [1976] observed that the weighted version of the integer realization problem is NP-complete, as finding a Hamiltonian circuit in an undirected graph can be reduced to it. (So even if $r = \mathbf{2}$ and all weights belong to $\{0, 1\}$, it is NP-complete.)

Chou and Frank [1970] gave a polynomial-time algorithm for finding a minimum-size integer realization, implying the following characterization of the minimum number $\gamma(r)$ of edges of an r-edge-connected graph[18].

To this end we can assume that r is symmetric and exactly realizable, that is, satisfies (62.1) (since resetting $r(s,t)$ to the maximum of $\min_{e=uv \in EP} r(u,v)$ over all $s - t$ paths P, does not change the problem).

Again, define for each $u \in V$,

$$R(u) := \max_{v \neq u} r(u,v). \tag{62.11}$$

Theorem 62.7. *Let $r : V \times V \to \mathbb{Z}_+$ be symmetric and satisfy* (62.1).
(i) *If $R(u) = 1$ for some $u \in V$, then $\gamma(r) = \gamma(r') + 1$, where r' is the restriction of r to $(V \setminus \{u\}) \times (V \setminus \{u\})$.*
(ii) *If $R(u) \neq 1$ for all $u \in V$, then*

$$\gamma(r) = \lceil \tfrac{1}{2} \sum_{u \in V} R(u) \rceil. \tag{62.12}$$

Proof. We first show (i). The inequality $\gamma(r) \leq \gamma(r') + 1$, is easy, since an r-edge-connected graph can be obtained from an r'-edge-connected graph by adding one edge connecting u with some $v \neq u$ with $r(u,v) = 1$.

To see the reverse inequality, let $G = (V, E)$ be an r-edge-connected graph with $|E| = \gamma(r)$. As $R(u) = 1$, we have $\deg_G(u) \geq 1$. Let ut be an edge incident with u. Let H be the graph obtained from G by contracting ut. Then H is r'-edge-connected and has $|E| - 1$ edges, showing $\gamma(r') \leq |E| - 1 = \gamma(r) - 1$.

[18] The construction of Chou and Frank [1970] is lacunary, and does not apply, e.g., to the case where $r(u,v) = 3$ for all u, v and $|V|$ is odd.

We next show (ii). Trivially, for any r-edge-connected graph $G = (V, E)$:

(62.13) $$|E| = \tfrac{1}{2}\sum_{u\in V} \deg_G(u) \geq \tfrac{1}{2}\sum_{u\in V} R(u).$$

This proves $\geq$ in (62.12). To prove $\leq$, order the vertices as $v_1, \ldots, v_n$ such that $R(v_1) \geq R(v_2) \geq \cdots \geq R(v_n)$. Note that $R(v_1) = R(v_2)$.

Let $k := \lfloor \frac{1}{2}R(v_1)\rfloor$. Let W be the set of vertices v with $R(v)$ odd. Let M be a set of $\lceil \frac{1}{2}|W|\rceil$ edges covering W such that if $v_1, v_2 \in W$, then $v_1v_2 \in M$.

For $i = 1, \ldots, k$, let C_i be a circuit on $\{v \in V \mid R(v) \geq 2i\}$. So C_1 is a Hamiltonian circuit. We choose C_1 in such a way that the components of $C_1 - v_1 - v_2$ span no edge in M. Let H be the (edge-disjoint) union of M and C_1. Then for any $U \subseteq V$:

(62.14) if $d_H(U) = 2$ and $U \cap \{v_1, v_2\} = \emptyset$, then $U \cap W = \emptyset$.

Indeed, if $d_H(U) = 2$, then U induces a path on C_1. As $U \cap \{v_1, v_2\} = \emptyset$, U is contained in a component of $C_1 - v_1 - v_2$. Hence each edge in M incident with U belongs to $d_H(U)$. As $d_H(U) = 2$, it follows that no edge in M is incident with U. So $U \cap W = \emptyset$, proving (62.14).

Let G be the (edge-disjoint) union of $M, C_1, \ldots, C_k$. Note that the number of edges of G is equal to

(62.15) $$\begin{aligned}|M| + \sum_{i=1}^{k} |EC_i| &= |M| + \sum_{u\in V} \lfloor \tfrac{1}{2}R(u)\rfloor = \lceil \tfrac{1}{2}|W|\rceil + \sum_{u\in V} \lfloor \tfrac{1}{2}R(u)\rfloor \\ &= \lceil \tfrac{1}{2}\sum_{u\in V} R(u)\rceil.\end{aligned}$$

We finally show that G is r-connected, for which it suffices to show that for $i = 2, \ldots, n$:

(62.16) $$\lambda_G(v_{i-1}, v_i) \geq R(v_i).$$

To see that this is sufficient, note that for $h < j$ one has

(62.17) $$\begin{aligned}\lambda_G(v_h, v_j) &\geq \min_{h<i\leq j} \lambda_G(v_{i-1}, v_i) \geq \min_{h<i\leq j} R(v_i) \\ &= R(v_j) \geq r(v_h, v_j).\end{aligned}$$

To prove (62.16), choose the smallest $i \geq 2$ for which it is not true. Then G has a cut $\delta(U)$ with $v_i \in U$, $v_{i-1} \notin U$, and $d_G(U) < R(v_i)$. By the minimality of i, $\delta(U)$ separates no pair among $v_1, \ldots, v_{i-1}$, and hence $v_1, \ldots, v_{i-1} \notin U$. Now, setting $l := \lfloor \frac{1}{2}R(v_i)\rfloor$:

(62.18) $$2l + 1 \geq R(v_i) > d_G(U) \geq d_H(U) + \sum_{j=2}^{l} d_{C_j}(U) \geq 2l$$

(as C_j covers v_{i-1} and v_i for $j = 1, \ldots, l$). Hence $d_H(U) = 2$ and $R(v_i)$ is odd. So $U \cap W \neq \emptyset$. Hence, by (62.14), $i = 2$. Then $v_1v_2 \in M$, and so $d_H(U) \geq 3$, a contradiction. ∎

Notes. In fact, by choosing M in this proof in such a way that $\deg_M(u) = 1$ if $u \in W \setminus \{v_1\}$, and $\deg_M(u) = 0$ if $u \in (V \setminus W) \setminus \{v_1\}$, we obtain a graph G with $\deg_G(u) = R(u)$ for each $u \neq v_1$. Hence $\lambda_G(u, v) = \min\{R(u), R(v)\}$ for all distinct $u, v \in V$.

The construction can be extended to obtain a strongly polynomial-time algorithm that, for given integer function r, finds a minimum-capacity integer realization c (Sridhar and Chandrasekaran [1990,1992]).

(Chou and Frank [1970] claim to give an algorithm to find a minimum-size *exact* realization, but their construction fails when taking $r(1, 2) := r(3, 4) := r(4, 5) := 5$, $r(2, 3) := r(5, 6) := 3$, and $r(i, j) := \min_{i<h\leq j} r(h-1, h)$ for $i < j$. The construction gives 15 edges, while there is an exact realization with 14 edges only.)

Frank and Chou [1970] announced a polynomial-time algorithm for the problem: given a symmetric $r : V \times V \to \mathbb{Z}_+$, find a *simple* r-edge-connected graph $G = (V, E)$ (if any) with $|E|$ minimal.

Wang and Kleitman [1973] characterized the degree-sequences of k-vertex-connected simple undirected graphs.

Chapter 63

Connectivity augmentation

This last chapter of Part V is devoted to the connectivity augmentation problem: given a graph, find the minimum number of edges to be added to make it k-connected. There is an undirected and a directed variant, and a vertex-connectivity and an edge/arc-connectivity variant. Thus we will come across:

- making a directed graph k-arc-connected (Section 63.1),
- making an undirected graph k-edge-connected (Section 63.3),
- making a directed graph k-vertex-connected (Section 63.5),
- making an undirected graph k-vertex-connected (Section 63.6).

For the first three problems, min-max relations and polynomial-time algorithms have been found. The core is formed by fundamental theorems of Frank and Jordán. As for the fourth problem, only for fixed k the polynomial-time solvability has been proved. The complexity for general k is open.

Two special cases of connectivity augmentation have been considered before: making a digraph 1-arc-connected — that is, strongly connected (Chapter 57), and making an edge- or arcless (di)graph k-vertex- or edge/arc-connected — the network synthesis problem (Chapter 62).

63.1. Making a directed graph k-arc-connected

Let (V, A) and (V, B) be directed graphs. The set B is called a *k-arc-connector for D* if the directed graph $(V, A \cup B)$ is k-arc-connected (where in $A \cup B$ arcs are taken parallel if they occur both in A and in B). So 1-arc-connectors are precisely the strong connectors, which we discussed in Chapter 57.

Frank [1990a,1992a] characterized the minimum size of a k-arc-connector for a directed graph, with the help of the following result of Mader [1982] (we follow the proof of Frank [1992a]).

Lemma 63.1α. *Let $D = (V, A)$ be a directed graph, let $k \in \mathbb{Z}_+$, and let $x, y : V \to \mathbb{Z}_+$. Then D has a k-arc-connector B with $\deg^{\text{in}}_B(v) = x_v$ and $\deg^{\text{out}}_B(v) = y_v$ for each $v \in V$ if and only if $x(V) = y(V)$ and*

$$(63.1) \qquad x(U) \geq k - d^{\text{in}}_A(U) \text{ and } y(U) \geq k - d^{\text{out}}_A(U)$$

for each nonempty proper subset U of V.

Proof. Necessity is easy, since for each nonempty $U \subset V$,

(63.2) $$k \leq d^{\text{in}}_{A\cup B}(U) = d^{\text{in}}_{B}(U) + d^{\text{in}}_{A}(U) \leq x(U) + d^{\text{in}}_{A}(U),$$

and similarly for y.

To see sufficiency, choose a counterexample with $x(V)$ minimal. Trivially, $x(V) \geq 1$. Let $\mathcal{X}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U)+d^{\text{in}}(U) = k$, and let $\mathcal{Y}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $y(U) + d^{\text{out}}(U) = k$. (We set d^{in} and d^{out} for d^{in}_{A} and d^{out}_{A}.)

Let $R := \{v \in V \mid x_v \geq 1\}$ and $S := \{v \in V \mid y_v \geq 1\}$. Then

(63.3) for all $r \in R$ and $s \in S$, there exists a $U \in \mathcal{X} \cup \mathcal{Y}$ with $r, s \in U$.

Otherwise, we could augment D by a new arc (s, r) and decrease both x_r and y_s by 1. Then (63.1) is maintained, and we obtain a smaller counterexample, contradicting our assumption. This shows (63.3).

Now note that for each $U \in \mathcal{X}$:

(63.4) $$y(V \setminus U) \geq k - d^{\text{out}}(V \setminus U) = k - d^{\text{in}}(U) = x(U).$$

This implies, for each $U \in \mathcal{X}$:

(63.5) if $S \subseteq U$, then $U \cap R = \emptyset$; if $R \subseteq U$, then $U \cap S = \emptyset$.

Indeed, if $S \subseteq U$, then $y(V \setminus U) = 0$, implying (with (63.4)) that $x(U) = 0$, that is, $U \cap R = \emptyset$. Similarly, if $R \subseteq U$, then $x(U) = x(V)$, implying (with (63.4)) that $y(V \setminus U) = y(V)$, that is, $U \cap S = \emptyset$. This proves (63.5).

Now choose $r \in R$, $s \in S$, and let $U \in \mathcal{X} \cup \mathcal{Y}$ with $r, s \in U$. By symmetry, we may assume that $U \in \mathcal{X}$. By (63.5), $S \not\subseteq U$. Choose $t \in S \setminus U$. Let $T \in \mathcal{X} \cup \mathcal{Y}$ contain r and t.

First assume that $T \in \mathcal{X}$. Then $T \cup U = V$, by the maximality of T and U and the submodularity of the set function $x(W) + d^{\text{in}}(W)$. This implies (using (63.4)):

(63.6) $$\begin{aligned} y(V) &\geq y(V \setminus U) + y(V \setminus T) \geq x(U) + x(T) = x(T \cup U) + x(T \cap U) \\ &> x(V) = y(V) \end{aligned}$$

(since $V \setminus U$ and $V \setminus T$ are disjoint, and since $r \in T \cap U$), a contradiction.

So $T \in \mathcal{Y}$. But then

(63.7) $$\begin{aligned} 2k &= x(T) + d^{\text{in}}(T) + y(U) + d^{\text{out}}(U) \\ &\geq x(T \setminus U) + d^{\text{in}}(T \setminus U) + y(U \setminus T) + d^{\text{out}}(U \setminus T) + x(T \cap U) \\ &\quad + y(T \cap U) \geq 2k, \end{aligned}$$

implying equality throughout. Hence $x(T \cap U) = 0$, contradicting the fact that $r \in T \cap U$. ∎

From this, the min-max relation for minimum-size k-arc-connectors of Frank [1990a,1992a] (generalizing Corollary 57.2a) easily follows:

Theorem 63.1. *Let $D = (V, A)$ be a directed graph and let $k, \gamma \in \mathbb{Z}_+$. Then D has a k-arc-connector of size at most γ if and only if*

$$(63.8) \qquad \gamma \geq \sum_{X \in \mathcal{P}} (k - d^{\mathrm{in}}(X)) \text{ and } \gamma \geq \sum_{X \in \mathcal{P}} (k - d^{\mathrm{out}}(X))$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity follows since for each nonempty subset X of V, at least $k - d^{\mathrm{in}}(X)$ arcs entering X must be in any k-arc-connector, and at least $k - d^{\mathrm{out}}(X)$ arcs leaving X must be in any k-arc-connector. As any new arc can enter at most one set in $\mathcal{P}$, we have (63.8).

To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying $x(U) \geq k - d^{\mathrm{in}}(U)$ for each nonempty $U \subset V$, with $x(V)$ as small as possible.

We show $x(V) \leq \gamma$. Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d^{\mathrm{in}}(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise $V \setminus T$ and $V \setminus U$ are disjoint, and we obtain the contradiction

$$(63.9) \qquad \begin{aligned} \gamma &\geq k - d^{\mathrm{out}}(V \setminus T) + k - d^{\mathrm{out}}(V \setminus U) = 2k - d^{\mathrm{in}}(T) - d^{\mathrm{in}}(U) \\ &= x(T) + x(U) \geq x(T \cup U) = x(V) > \gamma. \end{aligned}$$

Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

$$(63.10) \qquad \begin{aligned} x(T)+x(U) &= 2k - d^{\mathrm{in}}(T) - d^{\mathrm{in}}(U) \leq 2k - d^{\mathrm{in}}(T \cap U) - d^{\mathrm{in}}(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned}$$

by the maximality of T.

Now each $v \in V$ with $x_v \geq 1$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

$$(63.11) \qquad x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d^{\mathrm{in}}(U)) \leq \gamma.$$

Hence $x(V) \leq \gamma$. Similarly, there exists a $y : V \to \mathbb{Z}_+$ satisfying $y(U) \geq k - d^{\mathrm{out}}(U)$ for each nonempty proper subset U of V and $y(V) \leq \gamma$. We can assume that $x(V) = y(V) = \gamma$. So we can apply Lemma 63.1α, which gives the theorem. ∎

The proof yields a polynomial-time algorithm, as the proof reduces to a polynomial-time number of tests if a given $x : V \to \mathbb{Z}_+$ satisfies

$$(63.12) \qquad x(U) \geq k - d^{\mathrm{in}}(U) \text{ for each nonempty } U \subset V.$$

(Similarly for y.) This can be done by maximum flow calculations: add a new vertex s and for each $v \in V$, add x_v (parallel) arcs from s to v and k parallel arcs from v to s. Then (63.12) is satisfied if and only if in the extended graph there exist k arc-disjoint $u - v$ paths, for all $u, v \in V$.

Thus (Frank [1990a,1992a]):

Theorem 63.2. *Given a directed graph $D = (V, A)$ and $k \in \mathbb{Z}_+$, a minimum-size k-arc-connector can be found in time bounded by a polynomial in the size of D and in k.*

Proof. See above. ∎

63.1a. k-arc-connectors with bounds on degrees

Frank [1990a,1992a] derived similarly characterizations of the existence of k-arc-connectors of given size and satisfying given lower and upper bounds on the in- and outdegrees:

Theorem 63.3. *Let $D = (V, A)$ be an undirected graph, let $k, \gamma \in \mathbb{Z}_+$, and let $l^{\text{in}}, l^{\text{out}}, u^{\text{in}}, u^{\text{out}} \in \mathbb{Z}_+^V$ with $l^{\text{in}} \leq u^{\text{in}}$ and $l^{\text{out}} \leq u^{\text{out}}$. Then D has a k-arc-connector B of size at most γ satisfying $l^{\text{in}}(v) \leq \deg_B^{\text{in}}(v) \leq u^{\text{in}}(v)$ and $l^{\text{out}}(v) \leq \deg_B^{\text{out}}(v) \leq u^{\text{out}}(v)$ for each $v \in V$ if and only if $\gamma \leq u^{\text{in}}(V)$, $\gamma \leq u^{\text{out}}(V)$,*

(63.13) $$k - d^{\text{in}}(U) \leq u^{\text{in}}(U) \text{ and } k - d^{\text{out}}(U) \leq u^{\text{out}}(U)$$

for each nonempty proper subset U of V, and

(63.14) $$\gamma \geq l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{X \in \mathcal{P}} (k - d^{\text{in}}(X)) \text{ and}$$
$$\gamma \geq l^{\text{out}}(V \setminus \bigcup \mathcal{P}) + \sum_{X \in \mathcal{P}} (k - d^{\text{out}}(X))$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity is easy. To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying $l^{\text{in}} \leq x \leq u^{\text{in}}$, $x(V) \geq \gamma$, and $x(U) \geq k - d^{\text{in}}(U)$ for each nonempty $U \subset V$, with $x(V)$ as small as possible.

We show $x(V) \leq \gamma$. Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d^{\text{in}}(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise $V \setminus T$ and $V \setminus U$ are disjoint, and we obtain the contradiction

(63.15) $$\gamma \geq k - d^{\text{out}}(V \setminus T) + k - d^{\text{out}}(V \setminus U) = 2k - d^{\text{in}}(T) - d^{\text{in}}(U)$$
$$= x(T) + x(U) \geq x(T \cup U) \geq x(V) > \gamma.$$

Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

(63.16) $$x(T) + x(U) = 2k - d^{\text{in}}(T) - d^{\text{in}}(U) \leq 2k - d^{\text{in}}(T \cap U) - d^{\text{in}}(T \cup U)$$
$$< x(T \cap U) + x(T \cup U) = x(T) + x(U),$$

by the maximality of T.

Now each $v \in V$ with $x_v > l^{\text{in}}(v)$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

(63.17) $$x(V) = l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} x(U) = l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq \gamma.$$

Hence $x(V) = \gamma$. Similarly, there exists a $y : V \to \mathbb{Z}_+$ satisfying $l^{\text{out}} \leq y \leq u^{\text{out}}$, $y(V) = \gamma$, and $y(U) \geq k - d^{\text{out}}(U)$ for each nonempty $U \subset V$. So we can apply Lemma 63.1α, which gives the theorem. ∎

Again, this proof yields a polynomial-time algorithm to find a minimum-size k-arc-connector satisfying prescribed bounds on the in- and outdegrees.

Notes. The following problem is NP-complete: given a directed graph $D = (V, A)$, a function $r : V \times V \to \mathbb{Z}_+$, and a cost function $k : V \times V \to \mathbb{Q}_+$, find a minimum-cost set of new arcs whose addition to D makes the graph r-arc-connected. Frank [1990a,1992a] showed that if there are functions $k', k'' : V \to \mathbb{Q}_+$ with $k(u, v) = k'(u) + k''(v)$ for all $u, v \in V$, then this problem is solvable in polynomial time.

Gusfield [1987a] gave a linear-time algorithm to find a minimum number of directed arcs to be added to a mixed graph such that it becomes strongly connected (that is, for all vertices u, v there is a $u - v$ path traversing directed edges in the right direction only).

Frank and Jordán [1995b] gave an alternative proof of Theorem 63.1 based on bisubmodular functions, and showed a number of related results.

Frank [1993c] gave some further methods for the problems discussed in these sections.

Kajitani and Ueno [1986] showed that the minimum size of a k-arc-connector for a directed tree $D = (V, A)$ is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$.

Frank [1990a,1992a] gave a polynomial-time algorithm for: given directed graph $D = (V, A)$, $r \in V$, and $k \in \mathbb{Z}_+$, find a minimum number of arcs to be added to D such that for each $s \in V$ there exist k arc-disjoint $r - s$ paths in the augmented graph. The complexity was improved by Gabow [1991b].

63.2. Making an undirected graph 2-edge-connected

Let (V, E) and (V, F) be undirected graphs. The set F is called a *k-edge-connector* for G if the graph $(V, E \cup F)$ is k-edge-connected (where in $E \cup F$ edges are taken parallel if they occur both in E and in F).

The minimum size of a 1-edge-connector of a graph G trivially is one less than the number of components of G. Eswaran and Tarjan [1976] and Plesník [1976] characterized the minimum size of a 2-edge-connector, by first showing:

Theorem 63.4. *Let $G = (V, E)$ be a forest with at least two vertices and with p vertices of degree 1 and q isolated vertices. Then the minimum size of a 2-edge-connector for G equals $\lceil \frac{1}{2}p \rceil + q$.*

Proof. Each vertex of degree 1 should be incident with at least one new edge, and each vertex of degree 0 should be incident with at least two new edges. So any 2-edge-connector has size at least $\frac{1}{2}p + q$.

To see that $\lceil\frac{1}{2}p\rceil + q$ can be attained, first assume that G is not connected. Choose vertices u and v in different components, with $\deg(u) \leq 1$ and $\deg(v) \leq 1$. Adding edge uv, reduces $\frac{1}{2}p + q$ by 1, as one easily checks.

So we can assume that G is a tree. If $p \leq 3$, the graph is a path or a subdivision of $K_{1,3}$, and the theorem is easy.

If $p \geq 4$, there is a pair of end vertices u, v such that at least two edges of G leave the $u - v$ path P in G. Let G' be the tree obtained from G by contracting P to one vertex. Then G' has $p - 2$ end vertices. Applying induction shows that G' has a 2-edge-connector F of size $\lceil\frac{1}{2}p\rceil - 1$. By adding edge uv we obtain a 2-edge-connector of G, proving the theorem. ∎

This implies, for not necessarily forests:

Corollary 63.4a. *Let $G = (V, E)$ be a non-2-edge-connected undirected graph. For $i = 0, 1$, let p_i be the number of 2-edge-connected components K with $d_E(K) = i$. Then the minimum size of a 2-edge-connector equals $\lceil\frac{1}{2}p_1\rceil + p_0$.*

Proof. Directly from Theorem 63.4, by contracting each 2-edge-connected component to one vertex. ∎

These proofs give polynomial-time algorithms to find a minimum-size 2-edge-connector for a given undirected graph. Eswaran and Tarjan [1976] gave a linear-time algorithm.

63.3. Making an undirected graph k-edge-connected

Watanabe and Nakamura [1987] gave a min-max formula and a polynomial-time algorithm for the minimum size of a k-edge-connector for any undirected graph. Cai and Sun [1989] and Frank [1992a] showed that the min-max relation can be derived from the following lemma (given, in a different, 'vertex-splitting' terminology, by Mader [1978a] and Lovász [1979a] (Problem 6.53); the proof below follows Frank [1992a]):

Lemma 63.5α. *Let $G = (V, E)$ be a graph, let $k \in \mathbb{Z}_+$, with $k \geq 2$, and let $x : V \to \mathbb{Z}_+$. Then G has a k-edge-connector F with $\deg_F(v) = x_v$ for each $v \in V$ if and only if $x(V)$ is even and*

(63.18) $$x(U) \geq k - d_E(U)$$

for each nonempty proper subset U of V.

Proof. Necessity is easy, since for each nonempty $U \subset V$,

(63.19) $$k \leq d_{E \cup F}(U) = d_F(U) + d_E(U) \leq x(U) + d_E(U).$$

To see sufficiency, choose a counterexample with $x(V)$ minimal. Trivially, $x(V) \geq 2$.

Let $S := \{v \in V \mid x_v \geq 1\}$, and fix $s \in S$. Let $\mathcal{U}$ be the collection of inclusionwise maximal sets $U \subset V$ containing s and satisfying $x(U)+d_E(U) \leq k+1$. Note that

(63.20) $\quad x(U) \leq \frac{1}{2}x(V)$ for each $U \in \mathcal{U}$,

since otherwise $x(V \setminus U) \leq x(U) - 2$, implying the contradiction $k \leq x(V \setminus U) + d_E(V \setminus U) \leq x(U) - 2 + d_E(U) \leq k-1$.

Moreover,

(63.21) $\quad$ for all $t \in S \setminus \{s\}$, there exists a $U \in \mathcal{U}$ containing t.

Otherwise, we could augment G by a new edge st and decrease both x_s and x_t by 1. Then (63.18) is maintained, and we obtain a smaller counterexample, contradicting our assumption. This shows (63.21).

Next:

(63.22) $\quad$ for any two distinct $T, U \in \mathcal{U}$, G has an edge leaving $T \cap U$, and no edge connecting $T \cap U$ and $V \setminus (T \cup U)$.

Consider:

(63.23)
$$\begin{aligned}
&2(k+1) \geq x(T) + d_E(T) + x(U) + d_E(U)\\
&= x(T \setminus U) + d_E(T \setminus U) + x(U \setminus T) + d_E(U \setminus T)\\
&+2|E[T \cap U, V \setminus (T \cup U)]| + 2x(T \cap U) \geq 2k+2,
\end{aligned}$$

implying equality throughout. So $x(T \cap U) = 1$ and $|E[T \cap U, V \setminus (T \cup U)] = \emptyset$. Since $d_E(T \cap U) \geq k - x(T \cap U) = k-1 \geq 1$, this proves (63.22).

Now by (63.21) and (63.20) we can choose three sets $T, U, W \in \mathcal{U}$. Then

(63.24) $\quad T \cap U = T \cap W = U \cap W.$

Indeed, by symmetry it suffices to prove that $T \cap U \cap W = U \cap W$. Let $M := U \cap W$. Suppose $T \cap M \neq M$; so $M \not\subseteq T$, and hence $T \cup M \neq T$. Defining $\phi(X) := x(X) + d_E(X)$ for $X \subseteq V$, we obtain the contradiction

(63.25)
$$\begin{aligned}
&k \leq \phi(T \cap M) \leq \phi(T) + \phi(M) - \phi(T \cup M)\\
&\leq \phi(T) + \phi(U) + \phi(W) - \phi(U \cup W) - \phi(T \cup M)\\
&\leq 3(k+1) - 2(k+2) = k-1,
\end{aligned}$$

since the maximality of T, U, and W gives $\phi(U \cup W) \geq k+2$ and $\phi(T \cup M) \geq k+2$. This shows (63.24).

Now by (63.22), G has an edge leaving $T \cap U$, while the other end should be in each of $T \cup U$, $T \cup W$, and $U \cup W$, and hence in $T \cap U$, a contradiction. ∎

From this, the min-max result for minimum-size k-edge-connectors of Watanabe and Nakamura [1987] follows:

Theorem 63.5. *Let $G = (V, E)$ be an undirected graph and let $k, \gamma \in \mathbb{Z}_+$, with $k \geq 2$. Then G has a k-edge-connector of size at most γ if and only if*

$$(63.26) \qquad 2\gamma \geq \sum_{U \in \mathcal{P}} (k - d(U))$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity follows since for each nonempty proper subset U of V, at least $k - d(U)$ edges entering U must be in any k-edge-connector. As any new edge can enter at most two sets in $\mathcal{P}$, we have (63.26).

To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying (63.18), with $x(V)$ as small as possible. By Lemma 63.5α it suffices to show that $x(V) \leq 2\gamma$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal subsets U of V satisfying $x(U) = k - d(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise we obtain the contradiction

$$(63.27) \qquad \begin{aligned} 2\gamma &< x(V) = x(T \cup U) \leq x(T) + x(U) = k - d(T) + k - d(U) \\ &= k - d(V \setminus T) + k - d(V \setminus U) \leq 2\gamma, \end{aligned}$$

using (63.26). Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

$$(63.28) \qquad \begin{aligned} x(T) + x(U) &= 2k - d(T) - d(U) \leq 2k - d(T \cap U) - d(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned}$$

by the maximality of T. Now each $v \in V$ with $x_v \geq 1$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

$$(63.29) \qquad x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d(U)) \leq 2\gamma,$$

which proves the theorem. ∎

Similarly to the directed case, the proof implies that a minimum-size k-edge-connector can be found in polynomial time (Watanabe and Nakamura [1987]):

Theorem 63.6. *Given an undirected graph G and $k \in \mathbb{Z}_+$, a minimum-size k-edge-connector can be found in strongly polynomial time.*

Proof. The proof method reduces to a polynomially bounded number of tests of (63.18), which can be performed in strongly polynomial time by reducing it to maximum flow computations. ∎

Notes. Also Naor, Gusfield, and Martel [1990,1997], and Gabow [1991b] gave polynomial-time algorithms to find a minimum-size k-edge-connector. Frank [1990a, 1992a] and Benczúr [1994,1999] gave strongly polynomial-time algorithms for the (integer) capacitated version of the problem. More and related results can be found in Gabow [1994], Benczúr [1995], Nagamochi and Ibaraki [1995,1996,1997,1999c], Nagamochi, Shiraki, and Ibaraki [1997], Benczúr and Karger [1998,2000], Nagamochi and Eades [1998], Bang-Jensen and Jordán [2000], and Nagamochi, Nakamura, and Ibaraki [2000].

63.3a. k-edge-connectors with bounds on degrees

Frank [1990a,1992a] derived similarly characterizations of the existence of k-edge-connectors of given size and satisfying given lower and upper bounds on the degrees:

Theorem 63.7. *Let $G = (V, E)$ be an undirected graph, let $k, \gamma \in \mathbb{Z}_+$, with $k \geq 2$, and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a k-edge-connector F of size at most γ satisfying $l(v) \leq \deg_F(v) \leq u(v)$ for each $v \in V$ if and only if $2\gamma \leq u(V)$,*

(63.30) $$k - d_E(U) \leq u(U)$$

for each nonempty proper subset U of V, and

(63.31) $$2\gamma \geq l(V \setminus \bigcup\mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d_E(U)).$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. The conditions can be easily seen to be necessary.

To see sufficiency, let $x : V \to \mathbb{Z}_+$ satisfy $l \leq x \leq u$, $x(V) \geq 2\gamma$, $x(U) \geq k - d_E(U)$ for each nonempty $U \subset V$, and with $x(V)$ as small as possible. Such an x exists, by (63.30). By Lemma 63.5α, it suffices to show that $x(V) \leq 2\gamma$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise we obtain the contradiction

(63.32) $$\begin{aligned} 2\gamma < x(V) = x(T \cup U) \leq x(T) + x(U) = k - d(T) + k - d(U) \\ = k - d(V \setminus T) + k - d(V \setminus U) \leq 2\gamma, \end{aligned}$$

by (63.31). Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

(63.33) $$\begin{aligned} x(T) + x(U) = 2k - d(T) - d(U) \leq 2k - d(T \cap U) - d(T \cup U) \\ < x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned}$$

by the maximality of T. Now each $v \in V$ with $x_v > l(v)$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

(63.34) $$x(V) = l(V \setminus \bigcup\mathcal{P}) + \sum_{U \in \mathcal{P}} x(U) = l(V \setminus \bigcup\mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d(U)) \leq 2\gamma,$$

as required. ∎

Notes. T. Jordán (cf. Bang-Jensen and Jordán [1997]) showed that finding a minimum number of edges that makes a given simple graph k-edge-connected and keeps it simple, is NP-complete. On the other hand, Bang-Jensen and Jordán [1997,1998] gave, for any fixed k, an $O(n^4)$-time algorithm for this problem. Taoka, Watanabe, and Takafuji [1994] gave an $O(m + n \log n)$-time algorithm for $k = 4$ and an $O(n^2 + m)$-time algorithm for $k = 5$ (assuming the input graph is $k-1$-edge-connected). Other fast algorithms for undirected edge-connectivity augmentation were given by Benczúr [1999].

Ueno, Kajitani, and Wada [1988] gave a polynomial-time algorithm for finding a minimum-size k-edge-connector for a tree.

63.4. r-edge-connectivity and r-edge-connectors

Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$. G is called *r-edge-connected* if for all $u, v \in V$ there exist $r(u, v)$ edge-disjoint paths connecting u and v. So, by Menger's theorem, G is r-edge-connected if and only if $d_E(U) \geq r(u, v)$ for all $U \subseteq V$ and $u \in U$, $v \in V \setminus U$.

An *r-edge-connector* for G is a set F of edges on V such that the graph $G' := (V, E \cup F)$ satisfies $\lambda_{G'}(u, v) \geq r(u, v)$ for all $u, v \in V$. (Again, $E \cup F$ is the disjoint union, allowing parallel edges.) Define $\gamma(G, r)$ as the minimum size of an r-edge-connector for G.

Given an undirected graph $G = (V, E)$, a function $r : V \times V \to \mathbb{Z}_+$, and a cost function $k : V \times V \to \mathbb{Q}_+$, it is NP-complete to find a minimum-cost r-edge-connector (since for $E = \emptyset$, $r = \mathbf{2}$, it is the traveling salesman problem).

Frank [1990a,1992a] gave a polynomial-time algorithm and a min-max formula for the cardinality case: given a graph $G = (V, E)$ and $r : V \times V \to \mathbb{Z}_+$, find the minimum number of edges to be added to make G r-edge-connected. We describe the method in this section.

It is based on the following theorem of Mader [1978a] (conjectured by Lovász [1976a]; we follow the proof of Frank [1992b]):

Lemma 63.8α. *Let $G = (V \cup \{s\}, E)$ be an undirected graph, where s has even and positive degree, and s is not incident with a bridge of G. Then s has two neighbours u and v such that the graph G' obtained from G by replacing su and sv by one new edge uv satisfies*

(63.35) $$\lambda_{G'}(x, y) = \lambda_G(x, y)$$

for all $x, y \in V$.

Proof. By induction on $|V| + \deg(s)$. For any $U \subseteq V$ with $\emptyset \neq U \neq V$, define

(63.36) $$R(U) := \max_{u \in U, v \in V \setminus U} \lambda_G(u, v),$$

and set $R(\emptyset) := R(V) := 0$. So $R(U) \leq d(U)$ for each $U \subseteq V$.

Let $\mathcal{P}$ be the collection of nonempty proper subsets U of V with $d(U) = R(U)$, and let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $d(U) \leq R(U) + 1$ (hence $\mathcal{P} \subseteq \mathcal{U}$).

Note that $u, v \in N(s)$ are as required in the lemma if and only if there is no $U \in \mathcal{U}$ containing both u and v. So we can assume that

(63.37) for each pair $u, v \in N(s)$ there is a $U \in \mathcal{U}$ containing u and v.

We first show:

(63.38) $|T| = 1$ for each $T \in \mathcal{P}$.

Suppose not. Consider the graph G/T obtained from G by contracting T (where T also denotes the vertex obtained by contracting T). By induction,

G/T has two edges su' and sv such that for the graph H obtained from G/T by replacing su' and sv by a new edge $u'v$, one has

(63.39) $$\lambda_H(x,y) = \lambda_{G/T}(x,y)$$

for all $x,y \in V(G/T) \setminus \{s\}$. By symmetry of u' and v, we may assume that $v \neq T$, that is, $v \in V \setminus T$. Let $u := u'$ if $u' \neq T$ and choose $u \in T \cap N(s)$ if $u' = T$.

Then for all $Z \in \mathcal{U}$:

(63.40) if $T \subseteq Z$ or $T \cap Z = \emptyset$ then $u \notin Z$ or $v \notin Z$.

Indeed, as $R(Z) \geq d(Z) - 1$, there exist $x,y \in V$ such that Z splits x,y and $\lambda_G(x,y) \geq d(Z) - 1$. Since $T \subseteq Z$ or $T \cap Z = \emptyset$, we may assume that $x \notin T$. Define $y' := y$ if $y \notin T$, and $y' := T$ if $y \in T$. Define $Z' := Z$ if $T \cap Z = \emptyset$, and $Z' := (Z \setminus T) \cup \{T\}$ if $T \subseteq Z$. Suppose now $u,v \in Z$. Then $d_H(Z') = d_G(Z) - 2$. This gives the contradiction

(63.41) $$\lambda_{G/T}(x,y') \geq \lambda_G(x,y) \geq d_G(Z) - 1 > d_H(Z') \geq \lambda_H(x,y') \\ = \lambda_{G/T}(x,y'),$$

proving (63.40).

Now let $U \in \mathcal{U}$ contain u and v. By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$.

If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then

(63.42) $$d(T) + d(U) \geq d(T \cap U) + d(T \cup U) \geq R(T \cap U) + R(T \cup U) \\ \geq R(T) + R(U) \geq d(T) + d(U) - 1,$$

implying $R(T \cup U) \geq d(T \cup U) - 1$. So $T \cup U \in \mathcal{U}$ and $u,v \in T \cup U$, contradicting (63.40).

So $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$. Hence

(63.43) $$d(T) + d(U) \geq d(T \setminus U) + d(U \setminus T) \geq R(T \setminus U) + R(U \setminus T) \\ \geq R(T) + R(U) \geq d(T) + d(U) - 1.$$

So $d(T) + d(U) = d(T \setminus U) + d(U \setminus T)$, and hence $T \cap U$ contains no neighbours of s. So $u' \neq T$ (otherwise $u \in T \cap U \cap N(s)$). Hence $u' = u \in U \setminus T$. By (63.43) we also know $R(U \setminus T) \geq d(U \setminus T) - 1$. So $U \setminus T \in \mathcal{U}$ and $u,v \in U \setminus T$, contradicting (63.40). This proves (63.38).

Note that (63.38) implies

(63.44) $\lambda_G(u,v) = \min\{\deg(u), \deg(v)\}$ for all $u,v \in V$,

since $\lambda_G(u,v) = d_E(U)$ for some $U \subseteq V$ splitting $\{u,v\}$. So $U \in \mathcal{P}$, and hence $|U| = 1$, implying (63.44).

Choose a vertex $t \in N(s)$ of minimum degree. Let $\mathcal{U}'$ be a minimal collection of inclusionwise maximal sets in $\mathcal{U}$ containing t such that $\bigcup \mathcal{U}' \supseteq N(s)$ (this exists by (63.37)). Note that for each $U \in \mathcal{U}$ one has $|E[U,s]| \leq \frac{1}{2}\deg(s)$, since otherwise $d(V \setminus U) \leq d(U) - 2$ (since $\deg(s)$ is even), and hence

(63.45) $$R(V \setminus U) \leq d(V \setminus U) \leq d(U) - 2 \leq R(U) - 1 = R(V \setminus U) - 1,$$

a contradiction. Hence $|\mathcal{U}'| \geq 3$ (as $t \in U$ for each $U \in \mathcal{U}'$). Moreover,

(63.46) for each $U \in \mathcal{U}'$, there is a $v \in N(s)$ such that U is the only set in $\mathcal{U}'$ containing v.

(Otherwise we could delete U from $\mathcal{U}'$.)

Also,

(63.47) $R(U \setminus \{t\}) \geq R(U)$ for each $U \in \mathcal{U}'$.

Indeed, choose $x \in U$ and $y \in V \setminus U$ with $R(U) = \lambda_G(x, y)$. If $x \neq t$, then $R(U \setminus \{t\}) \geq \lambda_G(x, y) = R(U)$, as required. If $x = t$, then for any $u \in N(s) \cap (U \setminus \{t\})$,

(63.48) $$R(U) = \lambda_G(t, y) = \min\{\deg(t), \deg(y)\} \leq \min\{\deg(u), \deg(y)\} = \lambda_G(u, y) \leq R(U \setminus \{t\}).$$

This shows (63.47).

Moreover, for any distinct $X, Y \in \mathcal{U}'$ one has

(63.49) $R(X) + R(Y) \leq R(X \setminus Y) + R(Y \setminus X)$.

Suppose not. Then, by Lemma 61.6α we know $R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y)$, and by symmetry we can assume that $R(X) > R(X \setminus Y)$. Hence by (63.47), $X \cap Y \neq \{t\}$, and hence by (63.38), $X \cap Y \notin \mathcal{P}$. By the maximality of X and Y, $R(X \cup Y) \leq d(X \cup Y) - 2$. This gives the contradiction

(63.50) $$\begin{aligned} R(X \cap Y) + R(X \cup Y) &\leq (d(X \cap Y) - 1) + (d(X \cup Y) - 2) \\ &\leq d(X) + d(Y) - 3 \leq R(X) + R(Y) - 1 \\ &\leq R(X \cap Y) + R(X \cup Y) - 1, \end{aligned}$$

proving (63.49).

This implies that for any distinct $X, Y \in \mathcal{U}'$ one has

(63.51) $|X \setminus Y| = |Y \setminus X| = 1$, and st is the only edge connecting $X \cap Y$ and $(V \cup \{s\}) \setminus (X \cup Y)$.

Indeed, by (63.49) (as st connects $X \cap Y$ and $X \cup Y$),

(63.52) $$\begin{aligned} &d(X) + d(Y) \\ &= d(X \setminus Y) + d(Y \setminus X) + 2|E[X \cap Y, (V \cup \{s\}) \setminus (X \cup Y)]| \\ &\geq d(X \setminus Y) + d(Y \setminus X) + 2 \geq R(X \setminus Y) + R(Y \setminus X) + 2 \\ &\geq R(X) + R(Y) + 2 \geq d(X) + d(Y). \end{aligned}$$

So we have equality throughout. Hence $X \setminus Y, Y \setminus X \in \mathcal{P}$, and therefore, by (63.38), $|X \setminus Y| = |Y \setminus X| = 1$. Moreover, $d(X \cap Y, V \setminus (X \cup Y)) = 1$, proving (63.51).

Now choose $X, Y, Z \in \mathcal{U}'$. Then (63.51) and (63.46) imply that $X \cap Y = X \cap Z = Y \cap Z$. So st is the only edge leaving $X \cap Y$, and hence st is a bridge. This contradicts the condition given in this lemma. ∎

Note that

(63.53) the graph G' arising in Lemma 63.8α again has no bridge incident with s,

as for any two neighbours x, y of s in G' one has $\lambda_{G'}(x,y) = \lambda_G(x,y) \geq 2$. The lemma therefore can be applied iteratively to yield:

Theorem 63.8. *Let $G = (V,E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ be symmetric. Let $x : V \to \mathbb{Z}_+$ be such that $x(K) \neq 1$ for each component K of G. Then G has an r-edge-connector F satisfying $\deg_F(v) = x_v$ for each $v \in V$ if and only if $x(V)$ is even and*

(63.54) $$x(U) + d_E(U) \geq r(u,v)$$

for all $U \subseteq V$ and all $u \in U$, $v \in V \setminus U$.

Proof. Necessity of (63.54) follows from the fact that $x(U) + d_E(U) \geq d_F(U) + d_E(U) \geq r(u,v)$. To see sufficiency, extend V by a new vertex s and, for each $v \in V$, x_v edges connecting s and v (parallel if $x_v \geq 2$). Let H be the extended graph. Then (63.54) implies

(63.55) $$d_H(U) \geq r(u,v)$$

for all $U \subseteq V$ and all $u \in U$, $v \in V \setminus U$. Hence, for all $u, v \in V$,

(63.56) $$\lambda_H(u,v) \geq r(u,v).$$

Now by iteratively splitting s as in Lemma 63.8α (cf. (63.53)), we obtain a set F of new edges such that adding F to G, the new graph G' satisfies

(63.57) $$\lambda_{G'}(u,v) = \lambda_H(u,v) \geq r(u,v)$$

for all $u, v \in V$. As moreover $\deg_F(v) = x_v$ for each $v \in V$, F is an r-edge-connector as required. ∎

The condition that $x(K) \neq 1$ for each component K cannot be deleted, as can be seen by taking $G = (V, \emptyset)$, $r := \mathbf{1}$, $x := \mathbf{1}$, with $|V| \geq 4$.

We next give the theorem of Frank [1990a,1992a] characterizing the minimum size $\gamma(G,r)$ of an r-edge-connector. To this end we can assume that r satisfies:

(63.58) $$\begin{aligned} &\text{(i) } r(u,v) = r(v,u) \geq \lambda_G(u,v) \text{ for all } u,v \in V;\\ &\text{(ii) } r(u,w) \geq \min\{r(u,v), r(v,w)\} \text{ for all } u,v,w \in V. \end{aligned}$$

Define

(63.59) $$\begin{aligned} &R(U) := \max_{u \in U, v \in V \setminus U} r(u,v) \text{ if } \emptyset \subset U \subset V, \text{ and}\\ &R(\emptyset) := R(V) := 0. \end{aligned}$$

Call a component K of G *marginal* if $K \neq V$, $r(u,v) = \lambda_G(u,v)$ for all $u, v \in K$, and $r(u,v) \leq 1$ for all $u \in K$ and $v \in V \setminus K$.

Theorem 63.9. *Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ satisfy* (63.58).

(i) *If K is a marginal component of G, then*

(63.60) $$\gamma(G, r) = \gamma(G - K, r') + R(K),$$

where r' is the restriction of r to $(V \setminus K) \times (V \setminus K)$.

(ii) *If G has no marginal components, then $\gamma(G, r)$ is equal to the maximum value of*

(63.61) $$\lceil \tfrac{1}{2} \sum_{U \in \mathcal{P}} (R(U) - d_E(U)) \rceil$$

taken over all collections $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. We first show (i). Let K be a marginal component of G and define $\alpha := R(K)$. As K is marginal, $\alpha \leq 1$. The inequality

(63.62) $$\gamma(G, r) \leq \gamma(G - K, r') + \alpha$$

is easy, since an r-edge-connector for G can be obtained from an r'-edge-connector F for $G - K$: if $\alpha = 0$, then F is an r-edge-connector, and if $\alpha = 1$, we obtain an r-edge-connector by adding to F one edge connecting some pair $u \in K$, $v \in V \setminus K$ with $r(u, v) = 1$.

To see the reverse inequality, let F be a minimum-size r-edge-connector for G. Let $G' := (V, E \cup F)$. So G' is r-edge-connected.

If F contains no edges connecting K and $V \setminus K$, then $\alpha = 0$ and F contains an r'-edge-connector for $G - K$. Hence $\gamma(G, r) = |F| \geq \gamma(G - K, r') = \gamma(G - K, r') + \alpha$.

If F contains an edge uv with $u \in K$, $v \in V \setminus K$, then the graph H obtained from G' by contracting $(K \cup \{v\})$ to one vertex, is r'-edge-connected. Since edge $uv \in F$ is contracted, it implies that $G - K$ has an r'-edge-connector of size at most $|F| - 1$. So $\gamma(G - K, r') \leq |F| - 1 \leq \gamma(G, r) - \alpha$.

We next show (ii). Let G have no marginal components. Choose $x : V \to \mathbb{Z}_+$ such that $x(U) + d_E(U) \geq R(U)$ for each $U \subseteq V$, with $x(V)$ as small as possible. Let μ be the maximum value of (63.61). It suffices to show that $x(V) \leq 2\mu$, since then we can apply Theorem 63.8 (after increasing $x(v)$ by 1 for some $v \in V$ if $x(V)$ is odd). So assume $x(V) > 2\mu$. As $\mu > 0$ (otherwise $x = \mathbf{0}$), we know $x(V) > 2$.

Then

(63.63) $$x(K) \neq 1 \text{ for each component } K \text{ of } G.$$

For suppose $x(K) = 1$. We show that K is marginal, which is a contradiction. First, $K \neq V$, since $x(V) > 2$. Second, for each $u \in K$, $v \in V \setminus K$, we have $r(u, v) \leq x(K) + d_E(K) = x(K) \leq 1$. Third, to prove that $r(u, v) = \lambda_G(u, v)$ for $u, v \in K$, there is a subset U of K with $|U \cap \{u, v\}| = 1$, $\lambda_G(u, v) = d_E(U)$, and $x(U) = 0$. Then $r(u, v) \leq x(U) + d_E(U) = d_E(U) = \lambda_G(u, v)$. So K is marginal, contradicting our assumption. This proves (63.63).

By the minimality of x, there exists a collection $\mathcal{P}$ of nonempty proper subsets U of V satisfying $x(U) = R(U) - d_E(U)$, such that $\mathcal{P}$ covers $\{v \mid x_v \geq 1\}$. Choose $\mathcal{P}$ such that

(63.64) $$\sum_{U \in \mathcal{P}} |U|$$

is as small as possible. Then

(63.65) $T \cap U = \emptyset$ for distinct $T, U \in \mathcal{P}$.

For suppose $T \cap U \neq \emptyset$. Note that $T \not\subseteq U \not\subseteq T$, by the minimality of (63.64). Observe also that $T \cup U \neq V$, since otherwise we obtain the contradiction

(63.66) $$\begin{aligned} 2\mu &< x(V) = x(T \cup U) \leq x(T) + x(U) \\ &= R(T) - d(T) + R(U) - d(U) \\ &= R(V \setminus T) - d(V \setminus T) + R(V \setminus U) - d(V \setminus U) \leq 2\mu. \end{aligned}$$

(by definition of μ, since $V \setminus T$ and $V \setminus U$ are disjoint).

By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$.

If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then

(63.67) $$\begin{aligned} x(T) + x(U) &= R(T) - d(T) + R(U) - d(U) \\ &\leq R(T \cap U) - d(T \cap U) + R(T \cup U) - d(T \cup U) \\ &\leq x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned}$$

and hence we have equality throughout. This implies that $x(T \cup U) = R(T \cup U) - d_E(T \cup U)$, and hence replacing T and U by $T \cup U$ would decrease (63.64), a contradiction.

If $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$, then

(63.68) $$\begin{aligned} x(T) + x(U) &= R(T) - d(T) + R(U) - d(U) \\ &\leq R(T \setminus U) - d(T \setminus U) + R(U \setminus T) - d(U \setminus T) \\ &\leq x(T \setminus U) + x(U \setminus T) \leq x(T) + x(U), \end{aligned}$$

implying equality throughout. This implies that $x(T \setminus U) = R(T \setminus U) - d_E(T \setminus U)$, and hence replacing T by $T \setminus U$ would decrease (63.64), a contradiction.

This proves (63.65), yielding the contradiction

(63.69) $$2\mu < x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (R(U) - d(U)) \leq 2\mu,$$

which proves the theorem. ∎

Frank [1990a,1992a] also gave a polynomial-time algorithm to find a minimum-cost r-edge-connector if the cost of any new edge uv is given by $k(u) + k(v)$, for some function $k : V \to \mathbb{Q}_+$. This is done with the help of the following auxiliary result:

Theorem 63.10. *Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ be symmetric. Define $R(U)$ as in* (63.59). *Then*

(63.70) $$Q := \{x \in \mathbb{R}_+^V \mid x(U) \geq R(U) - d_E(U) \text{ for all } U \subseteq V\}$$

is a contrapolymatroid, with associated supermodular function given by, for $X \subseteq V$*:*

(63.71) $$g(X) := \max_{\mathcal{U}} \sum_{U \in \mathcal{U}} (R(U) - d_E(U)),$$

where the maximum ranges over collections $\mathcal{U}$ *of disjoint nonempty subsets of* X*.*

Proof. Clearly, for any $x \in \mathbb{R}_+^V$ one has $x \in Q$ if and only if $x(U) \geq g(X)$ for each $X \subseteq V$.

To see that g is supermodular, choose $X, Y \subseteq V$. Let

(63.72) $$g(X) = \sum_{U \in \mathcal{U}} (R(U) - d(U)) \text{ and } g(Y) = \sum_{T \in \mathcal{T}} (R(T) - d(T)),$$

where $\mathcal{U}$ and $\mathcal{T}$ are collections of disjoint nonempty subsets of X and of Y, respectively. The collections $\mathcal{U}$ and $\mathcal{T}$ together form a family $\mathcal{S}$ of nonempty subsets of V satisfying

(63.73) $$\sum_{S \in \mathcal{S}} \chi^S \leq \chi^{X \cap Y} + \chi^{X \cup Y} \text{ and } g(X) + g(Y) \leq \sum_{S \in \mathcal{S}} (R(S) - d(S)).$$

We now choose $\mathcal{S}$ such that (63.73) is satisfied and such that

(63.74) $$\sum_{S \in \mathcal{S}} |S|(|V \setminus S| + 1)$$

is as small as possible.

We claim that $\mathcal{S}$ is laminar; that is,

(63.75) if $T, U \in \mathcal{S}$, then $T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$.

Suppose not. By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$. If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then replacing T and U by $T \cap U$ and $T \cup U$ maintains (63.73) but decreases (63.74) (by Theorem 2.1), contradicting the minimality assumption. If $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$, then replacing T and U by $T \setminus U$ and $U \setminus T$ maintains (63.73) but decreases (63.74) (again by Theorem 2.1), again contradicting the minimality condition. This proves (63.75).

Now let $\mathcal{P}$ be the collection of inclusionwise maximal elements in $\mathcal{S}$ and let $\mathcal{Q}$ be the collection of remaining sets in $\mathcal{S}$. (If a set occurs twice in $\mathcal{S}$, it is both in $\mathcal{P}$ and in $\mathcal{Q}$.) Then each set in $\mathcal{P}$ is contained in $X \cup Y$, and each set in $\mathcal{Q}$ is contained in $X \cap Y$. Moreover, both $\mathcal{P}$ and $\mathcal{Q}$ are collections of disjoint sets. Hence

(63.76) $$\begin{aligned} g(X \cup Y) + g(X \cap Y) &\geq \sum_{P \in \mathcal{P}} (R(P) - d(P)) + \sum_{Q \in \mathcal{Q}} (R(Q) - d(Q)) \\ &= \sum_{S \in \mathcal{S}} (R(S) - d(S)) \geq g(X) + g(Y); \end{aligned}$$

that is, g is supermodular. ∎

With this theorem, also good characterizations and polynomial-time algorithms can be obtained for the minimum size of an r-edge-connector satisfying prescribed lower and upper bounds on its degrees — see Frank [1990a,1992a].

Bang-Jensen, Frank, and Jackson [1995] extended these results to mixed graphs.

63.5. Making a directed graph k-vertex-connected

Let (V, A) and (V, B) be directed graphs. The set B is called a *k-vertex-connector* for D if the directed graph $(V, A \cup B)$ is k-vertex-connected. (Note that parallel edges will not help the vertex-connectivity.)

Since, for directed graphs, 1-vertex-connectors and 1-arc-connectors coincide, the problem of finding a minimum-size 1-vertex-connector for a given directed graph is addressed in Section 57.1.

Frank and Jordán [1995b] showed the following min-max relation for minimum-size k-vertex connector in directed graphs (which is a special case of Frank and Jordán's Theorem 60.5 above).

Call a pair (X, Y) of subsets of V a *good pair* if $X \neq \emptyset$, $Y \neq \emptyset$, $X \cap Y = \emptyset$, and D has no arc from X to Y. Call a collection $\mathcal{F}$ of good pairs a *good collection* if $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$ for all distinct (X, Y), $(X', Y') \in \mathcal{F}$.

Theorem 63.11. *Let $D = (V, A)$ be a directed graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a k-vertex-connector for D is equal to the maximum value of*

(63.77) $$\sum_{(X,Y)\in\mathcal{F}} (k - |V \setminus (X \cup Y)|),$$

where $\mathcal{F}$ ranges over good collections of good pairs.

Proof. Let γ be the maximum value. The minimum is not less than γ, since for any $(X, Y) \in \mathcal{F}$, at least $k - |V \setminus (X \cup Y)|$ arcs from X to Y should be added, while such arcs do not run from X' to Y' for any other pair (X', Y') in $\mathcal{F}$ (as $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$).

To see equality, we can assume that D is not k-vertex-connected. Then there exist disjoint nonempty subsets T and U of V such that D has no arc from T to U and such that $|V \setminus (T \cup U)| < k$.

If there exist $t \in T$ and $u \in U$ such that augmenting D with the arc (t, u), the maximum decreases, we are done by induction. So we can assume that no such pair t, u exists. Hence for each $t \in T$ and $u \in U$, there exists a good collection $\mathcal{F}_{t,u}$ of good pairs, with

(63.78) $$\sum_{(X,Y)\in\mathcal{F}_{t,u}} (k - |V \setminus (X \cup Y)|) = \gamma,$$

and with $t \notin X$ or $u \notin Y$ for all $(X,Y) \in \mathcal{F}_{t,u}$.

Concatenating these collections $\mathcal{F}_{t,u}$ for all $t \in T$, $u \in U$, and adding the pair (T,U), we obtain a family $\mathcal{G}$ of good pairs satisfying:

(63.79) (i) for each $x \in T$, $y \in U$, there are at most $|T||U|$ pairs (X,Y) in $\mathcal{G}$ with $x \in X$ and $y \in Y$;
(ii) $\sum_{(X,Y)\in\mathcal{G}} (k - |V \setminus (X \cup Y)|) > \gamma |T||U|$.

Among all families $\mathcal{G}$ satisfying (63.79), we choose one minimizing

(63.80) $$\sum_{(X,Y)\in\mathcal{G}} (|X| + |V \setminus Y|)(|Y| + |V \setminus X|).$$

Then

(63.81) for all $(X,Y), (X',Y') \in \mathcal{G}$ one has $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$ or $X \subseteq X', Y' \subseteq Y$ or $X' \subseteq X, Y \subseteq Y'$.

Suppose not. Replace (X,Y) and (X',Y') by $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$. This maintains (63.79), while (63.80) decreases[19], contradicting our assumption. This proves (63.81).

Now consider the partial order $\leq$ on pairs (X,Y) of subsets of V, defined by $(X,Y) \leq (X',Y')$ if $X \subseteq X', Y' \subseteq Y$. For each pair (X,Y), let its 'weight' $w(X,Y)$ be the number of times (X,Y) occurs in $\mathcal{G}$, and let its 'length' $l(X,Y)$ be equal to $k - |V \setminus (X \cup Y)|$. Then by (63.79)(i), any chain has weight at most $|T||U|$. By (63.79)(ii), the sum of $l(X,Y)w(X,Y)$ over $(X,Y) \in \mathcal{G}$ is more than $\gamma|T||U|$. Hence, by the length-width inequality for partially ordered sets (Theorem 14.5), $\mathcal{G}$ contains an antichain $\mathcal{F}$ of length more than γ. Then $\mathcal{F}$ is a good collection by (63.81). This contradicts the definition of γ. ∎

The theorem implies that the minimum size of a k-vertex-connector for a given directed graph $D = (V,A)$ is equal to the minimum value of

(63.82) $$\sum_{u,v\in V} x_{u,v}$$

subject to

(63.83) (i) $x_{u,v} \geq 0$ for all $u, v \in V$,
(ii) $\sum_{u\in X}\sum_{v\in Y} x_{u,v} \geq k - |V \setminus (X \cup Y)|$ for all disjoint nonempty $X, Y \subseteq V$ with no arc from X to Y.

[19] This can be seen with Theorem 2.1: Make a copy $\widetilde{V}$ of V, and let $\widetilde{Y}$ be the set of copies of elements of Y. Define $Z_{X,Y} := X \cup (\widetilde{V} \setminus \widetilde{Y})$. Then $|X| + |V \setminus Y| = |Z_{X,Y}|$ and $|Y| + |V \setminus X| = |(V \cup \widetilde{V}) \setminus Z_{X,Y}|$. Moreover, for (X,Y) and (X',Y') we have $Z_{X\cap X',Y\cup Y'} = Z_{X,Y} \cap Z_{X',Y'}$ and $Z_{X\cup X',Y\cap Y'} = Z_{X,Y} \cup Z_{X',Y'}$. So the replacements decrease (63.80) by Theorem 2.1.

This can be seen by observing that Theorem 63.11 implies that this LP-problem has integer primal and dual solutions of equal value.

As Frank and Jordán [1995b] pointed out, this implies that a minimum-size k-vertex-connector can be found in polynomial time with the ellipsoid method, as follows.

Since the conditions (63.83) can be checked in polynomial time, the ellipsoid method (as we discuss below) implies that the minimum size of a k-arc-connector can be determined in polynomial time. Then an explicit minimum-size k-vertex-connector can be found by testing, for each pair $u, v \in V$, whether augmenting D by the new arc (u, v) decreases the minimum size of a k-arc-connector. If so, we add (u, v) to D and iterate.

The ellipsoid method applies, since given $x_{u,v} \geq 0$ $(u, v \in V)$, we can test if (63.83)(ii) holds. Indeed, let B be a set of new arcs forming a complete directed graph on V. Define a capacity function c on $A \cup B$ by: $c(a) := \infty$ for each $a \in A$ and $c(b) = x_{u,v}$ for each arc $b \in B$ from u to v. Then (63.83)(ii) is equivalent to: for each $s, t \in V$ there is an $s-t$ flow $f_{s,t}$ in $(V, A \cup B)$ subject to c of value k, such that for any vertex $v \neq s, t$, the amount of flow traversing v is at most 1 (since the set of arcs in B from X to Y, together with the vertices in $V \setminus (X \cup Y)$, form a mixed arc/vertex-cut separating s and t). As this can be tested in polynomial time, we have a polynomial-time test for (63.83).

In fact, we can transform the problem into a linear programming problem of polynomial size, by including the flow variables $f_{s,t}(a)$ (for $s, t \in V$ and $a \in A \cup B$), into the LP-problem. Thus the minimum size of a k-arc-connector can be described as the solution of a linear programming problem of polynomial size.

There is no combinatorial polynomial-time algorithm known to find a minimum-size k-vertex-connector for a given directed graph. (Frank and Jordán [1995a] describe a combinatorial polynomial-time algorithm for finding a minimum-size 2-vertex-connector for a strongly connected directed graph. Frank and Jordán [1999] extended it to a polynomial-time algorithm (for any fixed k) to find a minimum-size k-vertex-connector.)

Notes. Frank and Jordán [1995b] also showed that a directed graph $D = (V, A)$ has a k-vertex-connector B with all in- and outdegrees at most $k - \kappa(D)$ (where $\kappa(D)$ denotes the vertex-connectivity of D).

Frank [1994a] gave the following conjecture:

(63.84) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a k-vertex-connector for D is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$. (?)

An $O(kn)$-time algorithm finding a minimum-size k-vertex-connector for a rooted tree was given by Masuzawa, Hagihara, and Tokura [1987]. Frank [1994a] observed that this result easily extends to branchings.

Approximation algorithms for the minimum size of a k-vertex-connector for a directed graph were given by Jordán [1993a].

63.6. Making an undirected graph k-vertex-connected

Let (V, E) and (V, F) be undirected graphs. The set F is called a *k-vertex-connector* for G if the graph $(V, E \cup F)$ is k-vertex-connected.

Trivially, the minimum size of a 1-vertex-connector for an undirected graph G is equal to one less than the number of components of G.

The minimum size of a 2-vertex-connector for undirected graphs was given by Eswaran and Tarjan [1976] and Plesník [1976]. To this end, call a block *pendant* if it contains exactly one cut vertex of G. Moreover, call a block *isolated* if it contains no cut vertex of G. So an isolated block is a component of G.

Theorem 63.12. *Let $G = (V, E)$ be a non-2-vertex-connected undirected graph, with p pendant blocks and q isolated blocks. Let d be the maximum number of components of $G - v$, maximized over $v \in V$. Then the minimum size of a 2-vertex-connector for G is equal to*

(63.85) $$k := \max\{d - 1, \lceil \tfrac{1}{2}p \rceil + q\}.$$

Proof. One needs at least $d-1$ edges, since for any $v \in V$, after deleting v the augmented graph should be connected. Any block containing no cut vertex should be incident with at least two new edges, and any block containing one cut vertex should be incident with at least one new edges. Hence the number of new edges is at least $\frac{1}{2}p + q$, and hence at least k.

To show that k can be attained, choose a counterexample G with k minimal. Then G is connected. Otherwise, we can choose two blocks B, B' from different components of G such that each of B, B' is pendant or isolated. We can choose a non-cut vertex from each of B, B', and connect them by a new edge to obtain graph G'. After that, k has decreased by exactly 1, and we can apply induction to G', implying the theorem.

So G is connected, and hence $q = 0$ (as G is not 2-vertex-connected). Moreover, $k \geq 2$, since otherwise $p \leq 2$, and we can add one edge to make G 2-vertex-connected.

Let U be the set of vertices v for which $G-v$ has at least three components and let W be the set of vertices v for which $G - v$ has $k + 1$ components. So $W \subseteq U$. Moreover, $|U| \geq 2$, since otherwise we can add $d - 1$ edges to make G connected.

We show:

(63.86) there exist two distinct pendant blocks B, B' such that each $B-B'$ path traverses all vertices in W and at least two vertices in U.

If $|W| \leq 1$, this is trivial. So we may assume that $|W| \geq 2$. Then, as $W \subseteq U$, it suffices to show that there exists a path traversing all vertices in W. If such a path would not exist, there exists a subset X of W with $|X| = 3$ that is not on a path. Then for each $v \in X$, one component K of $G - v$ contains $X \setminus \{v\}$. So for each $v \in X$, $G - v$ has k components disjoint from

X. Moreover, for distinct $v, v' \in X$, if K and K' are components of $G - v$ and $G - v'$ (respectively) each disjoint from X, then $K \cap K' = \emptyset$. Since for each $v \in X$ and each component K of $G - v$, $K \cup \{v\}$ contains at least one pendant block, we know $p \geq 3k \geq 3\lceil \frac{1}{2}p \rceil$, contradicting the fact that $p > 0$.

This shows (63.86). Now augment G by an edge connecting non-cut vertices in B and B', giving graph G'. As this augmentation decreases k (by the conditions given in (63.86)), we would obtain a counterexample with k smaller. ∎

This proof directly gives a polynomial-time algorithm to find a minimum-size 2-vertex-connector for G. Eswaran and Tarjan [1976] mention that a linear-time implementation of this algorithm was communicated to them in 1973 by R. Pecherer and A. Rosenthal — see Rosenthal and Goldner [1977]. (See also Hsu and Ramachandran [1991,1993].)

An equivalent form of Theorem 63.12 is:

Corollary 63.12a. *Let $G = (V, E)$ be a non-2-vertex-connected graph. Then G has a 2-vertex-connector of size at most γ if and only if for each vertex v, $G - v$ has at most $\gamma + 1$ components and*

(63.87) $$\sum_{U \in \mathcal{P}} (2 - |N(U)|) \leq 2\gamma$$

for each collection $\mathcal{P}$ of disjoint nonempty subsets U of V with $|U| \leq |V| - 3$.

Proof. Directly from Theorem 63.12. ∎

Jackson and Jordán [2001] showed that for each fixed k, a minimum-size k-vertex-connector for an undirected graph can be found in polynomial time.

Notes. Watanabe and Nakamura [1988,1993] give a characterization of the minimum size of a 3-vertex-connector, and Watanabe and Nakamura [1993] describe an $O(n(n+m)^2)$-time algorithm (for a sketch, see Watanabe and Nakamura [1988, 1990]). Hsu and Ramachandran [1991] gave a linear-time algorithm for this problem. Hsu [1992,2000] gave an almost-linear-time algorithm to find a minimum-size 4-vertex-connector for a 3-connected undirected graph.

Note that the natural extensions of Corollary 63.12a does not hold for k-vertex-connectors with $k \geq 4$, as is shown by the complete bipartite graph $K_{3,3}$.

For approximation algorithms, see Jordán [1993b,1995,1997a], Khuller and Thurimella [1993], Cheriyan and Thurimella [1996b,1999], Nutov and Penn [1997], Penn and Shasha-Krupnik [1997], and Jackson and Jordán [2000].

63.6a. Further notes

Corollary 53.6b implies the following characterization for connectivity augmentation, due to Frank [1979b].

Theorem 63.13. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$ be such that D contains k disjoint r-arborescences. Moreover, let $D' = (V, A')$ and $l \in \mathbb{Z}_+^{A'}$. Then the minimum of $l(C)$ where $C \subseteq A'$ such that the digraph $(V, A \cup C)$ (taking arcs multiple) has $k + 1$ disjoint r-arborescences is equal to the maximum size t of a family of nonempty subsets $U_1, \ldots, U_t$ of $V \setminus \{r\}$ such that $d_D^{\text{in}}(U_j) = k$ for $j = 1, \ldots, t$ and such that each arc a of D' enters at most $l(a)$ of the U_j.*

Proof. Consider the digraph $D'' = (V, A'')$ with $A'' := A \cup A'$ (taking multiple arcs for arcs occurring both in A and in A'). Now the minimum in this corollary is equal to the minimum of $\sum_{a \in A'} l(a)x_a$ where $x \in \mathbb{Z}^{A''}$ satisfies

(63.88)
$$0 \leq x_a \leq 1 \text{ if } a \in A',$$
$$x(\delta_{D'}^{\text{in}}(U)) \geq k + 1 - d_D^{\text{in}}(U) \text{ for each nonempty } U \subseteq V \setminus \{r\}.$$

Since (63.88) is TDI by Corollary 53.6b, this minimum is equal to the maximum described in the present corollary. ∎

The problem of making a bipartite directed graph strongly connected while preserving bipartiteness is considered by Gabow and Jordán [1999,2000a]. Augmenting the arc-connectivity while preserving bipartiteness is studied by Gabow and Jordán [2000b]. Making a bipartite undirected graph k-edge-connected while preserving bipartiteness, and, more generally, edge-connectivity augmentation with partition constraints, is studied by Bang-Jensen, Gabow, Jordan, and Szigeti [1998, 1999].

For the 'successive augmentation problem', see Cheng and Jordán [1999]. For NP-completeness and approximation results for connectivity augmentation, see Frederickson and Ja'Ja' [1981,1982]. Frank and Király [2001] studied problems that combine graph orientation and connectivity augmentation.

Planar graph connectivity augmentation was considered by Provan and Burk [1999].

Ishii, Nagamochi, and Ibaraki [1997,1998b,1998a,1999,2000,2001] considered the problem of making an undirected graph both k-vertex- and l-edge-connected.

For surveys on connectivity augmentation, see Frank [1993a,1994a], Jordán [1994,1997b], and Nagamochi [2000].

Part VI

Cliques, Stable Sets, and Colouring

Part VI: Cliques, Stable Sets, and Colouring

We now arrive at a class of problems that are in general NP-complete: finding a maximum-size clique or stable set or a minimum vertex-colouring in an undirected graph. These problems relate to each other: a stable set in a graph is a clique in the complementary graph, a colouring is a partitioning of the vertex set into stable sets, and the maximum size of a clique is a lower bound for the minimum number of colours.

Graph colouring was motivated originally by the four-colour conjecture formulated in the 1850s, stating that each planar map can be coloured with at most four colours — since 1977 a theorem of Appel and Haken. Later, colouring turned out to have several other applications, like in school scheduling, timetabling, and warehouse planning and in bungalow, terminal, platform, and frequency assignment. Finding optimum cliques of stable sets again can be used in frequency assignment, and in set packing problems, which show up for instance in crew scheduling.

While these problems are in general NP-complete, some are polynomial-time solvable for special classes of graphs: perfect graphs, t-perfect graphs, claw-free graphs. They form the body of this part.

Perfect graphs carry one of the deepest theorems in graph theory, the strong perfect graph theorem — recently proved by Chudnovsky, Robertson, Seymour, and Thomas. The proof is highly complicated, and we cannot give it in this book.

We refer to Part III for stable sets in and colouring of *line graphs* — equivalently, matchings and edge-colouring.

Chapters:

Chapter 64

Cliques, stable sets, and colouring

This chapter studies cliques, stable sets, and colouring for general graphs: complexity, polyhedra, fractional solutions, weighted versions.
In studying later chapters of this part, one can do largely without the results of the present chapter. Only some elementary definitions and terminology will be needed. It suffices to use this chapter just for reference.
In this chapter, all graphs can be assumed to be simple.

64.1. Terminology and notation

Let $G = (V, E)$ be an undirected graph. A *clique* is a set of vertices any two of which are adjacent. The maximum size of a clique in G is the *clique number* of G, and is denoted by $\omega(G)$.

A *stable set* is a set of vertices any two of which are nonadjacent. The maximum size of a stable set in G is called the *stable set number* of G, and is denoted by $\alpha(G)$.

A *vertex cover* is a set of vertices intersecting all edges. The minimum size of a vertex cover in G is called the *vertex cover number* of G, and is denoted by $\tau(G)$.

A *(vertex-)colouring* of G is a partition of V into stable sets $S_1, \ldots, S_k$. The sets $S_1, \ldots, S_k$ are called the *colours* of the colouring. The minimum number of colours in a vertex-colouring of G is called the *(vertex-)colouring number* of G, denoted by $\chi(G)$. A graph G is called k-*(vertex-)colourable* if $\chi(G) \leq k$, and k-*chromatic* if $\chi(G) = k$. A *minimum (vertex-)colouring* is a colouring with $\chi(G)$ colours. A k-*(vertex-)colouring* is a colouring with k colours.

A *clique cover* of G is a partition of V into cliques. The minimum number of cliques in a clique cover of G is called the *clique cover number* of G, and is denoted by $\overline{\chi}(G)$. A *minimum clique cover* is a clique cover with $\overline{\chi}(G)$ cliques.

The following relations between these parameters are immediate:

(64.1) $$\alpha(G) = \omega(\overline{G}),\ \overline{\chi}(G) = \chi(\overline{G}),\ \omega(G) \leq \chi(G),\ \alpha(G) \leq \overline{\chi}(G),$$
$$\tau(G) = |V| - \alpha(G).$$

64.2. NP-completeness

It is NP-complete to find a maximum-size stable set in a graph. To be more precise, the *stable set problem*: given a graph G and a natural number k, decide if $\alpha(G) \geq k$, is NP-complete (according to Karp [1972b] this is implicit in the work of Cook [1971] and was also known to R. Reiter):

Theorem 64.1. *Determining the stable set number is* NP*-complete.*

Proof. We reduce the satisfiability problem to the stable set problem. Let $C_1 \wedge \cdots \wedge C_k$ be a Boolean expression, where each C_i is of the form $y_1 \vee \cdots \vee y_m$, with $y_1, \ldots, y_m \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$. Call $x_1, \neg x_1, \ldots, x_n, \neg x_n$ the *literals*. Consider the graph $G = (V, E)$ with $V := \{(\sigma, i) \mid \sigma$ is a literal in $C_i\}$ and $E := \{\{(\sigma, i), (\tau, j)\} \mid i = j$ or $\sigma = \neg\tau\}$. Then the expression is satisfiable if and only if G has a stable set of size k. ∎

It can be shown that the stable set problem remains NP-complete if the graphs are restricted to 3-regular planar graphs (Garey, Johnson, and Stockmeyer [1976]) or to triangle-free graphs (Poljak [1974]).

Since a subset U of VG is a vertex cover if and only if $VG \setminus U$ is a stable set, we also have:

Corollary 64.1a. *Determining the vertex cover number is* NP*-complete.*

Proof. By Theorem 64.1, since the vertex cover number of a graph G is equal to $|VG|$ minus the stable set number. ∎

A subset C of VG is a clique in a graph G if and only if C is a stable set in the complementary graph $\overline{G}$. So finding a maximum-size clique in G is equivalent to finding a maximum-size stable set in $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$. Hence, as determining $\alpha(G)$ is NP-complete, also determining $\omega(G)$ is NP-complete.

Also, it is NP-complete to decide if a graph is k-colourable (Karp [1972b]):

Theorem 64.2. *Determining the vertex-colouring number is* NP*-complete.*

Proof. We show that the stable set problem can be reduced to the vertex-colouring problem. Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. We want to decide if $\alpha(G) \geq k$. To this end, let V' be a copy of V and let C be a set of size k, where V, V', and C are disjoint. Make a graph H with vertex set $V \cup V' \cup C$ as follows. A pair of vertices in V is adjacent in H if and only if it is adjacent in G. The sets V' and C are cliques in H. Each vertex in V is adjacent to each vertex in $V' \cup C$, except to its copy in V'. No vertex in V' is adjacent to any vertex in C.

This defines the graph H. Then $\alpha(G) \geq k$ if and only if $\chi(H) \leq |V| + 1$. ∎

Well-known is the *four-colour conjecture* (or *4CC*), stating that $\chi(G) \leq 4$ for each loopless planar graph G. This conjecture was proved by Appel and Haken [1977] and Appel, Haken, and Koch [1977], and is now called the *four-colour theorem*. (A shorter proof was given by Robertson, Sanders, Seymour, and Thomas [1997], leading to an $O(n^2)$-time 4-colouring algorithm for planar graphs (Robertson, Sanders, Seymour, and Thomas [1996]).)

However, it is NP-complete to decide if a planar graph is 3-colourable, even if the graph has maximum degree 4 (Garey, Johnson, and Stockmeyer [1976]). Moreover, determining the colouring number of a graph G with $\alpha(G) \leq 4$ is NP-complete (cf. Garey and Johnson [1979]). Holyer [1981] showed that deciding if a 3-regular graph is 3-edge-colourable is NP-complete (see Section 28.3). Note that one can decide in polynomial time if a graph G is 2-colourable, since bipartiteness can be checked in polynomial time.

These NP-completeness results imply that if NP$\neq$co-NP, then one may not expect a min-max relation characterizing the stable set number $\alpha(G)$, the vertex cover number $\tau(G)$, the clique number $\omega(G)$, or the colouring number $\chi(G)$ of a graph G.

64.3. Bounds on the colouring number

A lower bound on the colouring number is given by the clique number:

(64.2) $\qquad \omega(G) \leq \chi(G).$

This is easy, since in any clique all vertices should have different colours.

There are several graphs which have strict inequality in (64.2). We mention the odd circuits C_k, with k odd and ≥ 5: then $\omega(C_k) = 2$ and $\chi(C_k) = 3$. Moreover, for the complement $\overline{C}_k$ of any such graph we have: $\omega(\overline{C}_k) = \lfloor k/2 \rfloor$ and $\chi(\overline{C}_k) = \lceil k/2 \rceil$.

It was a conjecture of Berge [1963a] that these graphs are crucial. In May 2002, M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced that they have found a proof of this conjecture.

Strong perfect graph theorem: Each graph G with $\omega(G) < \chi(G)$ has C_k or $\overline{C}_k$ as induced subgraph for some odd $k \geq 5$.

It is convenient to define a *hole* of a graph G to be an induced subgraph of G isomorphic to C_k for some $k \geq 4$. Moreover, an *antihole* is an induced subgraph isomorphic to $\overline{C}_k$ for some $k \geq 4$. A hole or antihole is *odd* if it has an odd number of vertices. Then the strong perfect graph theorem can be formulated as: each graph G with $\omega(G) < \chi(G)$ has an odd hole or odd antihole.

For more on this we refer to Chapter 65.

64.3a. Brooks' upper bound on the colouring number

There is a trivial upper bound on the colouring number:

(64.3) $$\chi(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum degree of G. This bound follows by colouring the vertices 'greedily' one by one: at any stage, at least one colour (out of $\Delta(G)+1$ colours) is not used by the neighbours.

Brooks [1941] sharpened this inequality as follows. We follow the proof given by Lovász [1975d].

Theorem 64.3 (Brooks' theorem). *For any connected graph G one has $\chi(G) \leq \Delta(G)$, except if G is a complete graph or an odd circuit.*

Proof. We can assume that G is 2-connected, since otherwise we can apply induction. Moreover, we can assume that $\Delta(G) \geq 3$. Let $k := \Delta(G)$.

I. First assume that G has nonadjacent vertices u and w with $G - u - w$ disconnected. Let V_1 and V_2 be proper subsets of V such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \{u, w\}$, and no edge connects $V_1 \setminus \{u, w\}$ and $V_2 \setminus \{u, w\}$. Let $G_1 := G[V_1]$ and $G_2 := G[V_2]$.

For $i = 1, 2$, we know by induction that $\chi(G_i) \leq k$, since G_i is not complete (as u and w are nonadjacent), and since $\Delta(G_i) \leq k$ and $k \geq 3$. By symmetry of G_1 and G_2, we can assume that each k-colouring of G_1 gives u and w the same colour (otherwise G_1 and G_2 have k-colourings that coincide on u and w, yielding a k-colouring of G). This implies that both u and w have degree at least $k - 1$ in G_1. Hence they have degree at most 1 in G_2. Therefore, as $k \geq 3$, G_2 has a k-colouring giving u and w the same colour. So G is k-colourable.

II. Now choose a vertex v of maximum degree. As G is not a complete graph, v has two nonadjacent neighbours, say u and w. By part I, we can assume that $G - u - w$ is connected. Hence it has a spanning tree T. Orient T so as to obtain a rooted tree, rooted at v. Hence we can order the vertices of G as $v_1, \ldots, v_n$ such that $v_1 = v$, $v_{n-1} = u$, $v_n = w$, and such that each v_i with $i > 1$ is adjacent to some v_j with $j < i$. Give u and w colour 1. Next successively for $i = n-2, n-1, \ldots, 1$, we can give a colour from $1, \ldots, k$ to v_i different from the colours given to the neighbours v_j of v_i with $j > i$. Such a colour exists, since if $i > 1$, there are less than k neighbours v_j of v_i with $j > i$; and if $i = 1$, there are k such neighbours, but neighbours u and w have the same colour. ∎

(A related proof was given by Ponstein [1969], and a strengthening of Brooks' theorem by Reed [1999a]. For another proof of Brooks' theorem, see Melnikov and Vizing [1969].)

64.3b. Hadwiger's conjecture

Another upper bound on the colouring number is conjectured by Hadwiger [1943]. Since there exist graphs with $\omega(G) < \chi(G)$, it is not true that if $\chi(G) \geq k$, then G contains the complete graph K_k on k vertices as a subgraph. However, Hadwiger conjectured the following, where a graph H is called a *minor* of a graph G if H arises from some subgraph of G by contracting some (possibly none) edges.

Hadwiger's conjecture: If $\chi(G) \geq k$, then G contains K_k as a minor.

In other words, for each k, the graph K_k is the only graph G with the property that G is not $(k-1)$-colourable and each proper minor of G is $(k-1)$-colourable.

Hadwiger's conjecture is trivial for $k = 1, 2, 3$, and was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]):

Theorem 64.4. *If G has no K_4 minor, then $\chi(G) \leq 3$.*

Proof. One may assume that $G = (V, E)$ is not a forest or a circuit. Then G has a circuit C not covering all vertices of G. Choose $v \in V \setminus VC$. If G is 3-connected, there are three paths from v to VC, disjoint except for v. This creates a K_4 minor, a contradiction.

So G is not 3-connected, that is, G has a vertex-cut of size less than 3. Then $\chi(G) \leq 3$ follows by induction: if G is disconnected or has a 1-vertex-cut, this is trivial, and if G is 2-connected and has a 2-vertex-cut $\{u, w\}$, we can apply induction to the graphs $G - K$ after adding an edge uw, for each component K of $G - u - w$. ∎

(For another proof, see Woodall [1992].)

As planar graphs contain no K_5 minor, Hadwiger's conjecture for $k = 5$ implies the four-colour theorem. In fact, Wagner [1937a] showed that his decomposition theorem (Theorem 3.3) implies that Hadwiger's conjecture for $k = 5$ is equivalent to the four-colour conjecture. (Young [1971] gave a 'quick' proof of this equivalence.) The four-colour conjecture was proved by Appel and Haken [1977] and Appel, Haken, and Koch [1977]. (Robertson, Sanders, Seymour, and Thomas [1997] gave a shorter proof.)

Robertson, Seymour, and Thomas [1993] showed that Hadwiger's conjecture is true also for $k = 6$, by reducing it again to the four-colour theorem. For $k \geq 7$, Hadwiger's conjecture is unsettled.

Halin [1964] has proved that if G has no K_k minor, then $\chi(G) \leq 2^{k-2}$ (Wagner [1964] gave a short proof). Further progress on Hadwiger's conjecture was made by Wagner [1960], Mader [1968], Jakobsen [1971], Duchet and Meyniel [1982], Kostochka [1982], Fernandez de la Vega [1983], Thomason [1984], and Reed and Seymour [1998].

Hajós' conjecture. G. Hajós[1] conjectured (more strongly than Hadwiger) that any k-chromatic graph contains a subdivision of K_k as subgraph. For $k \leq 4$, Hajós' conjecture is equivalent to Hadwiger's conjecture.

Hajós' conjecture was refuted by Catlin [1979] for $k = 8$. He showed that the line graph $L(G)$ of the graph G obtained from the 5-circuit C_5 by replacing each

[1] According to Toft [1996], Hajós considered the conjecture already in the 1940s in connection with the four-colour conjecture, but he never published it. (The paper Hajós [1961] commonly referred to, does not give Hajós' conjecture.) An early written record of the conjecture is in the review of Tutte [1961b], in the January 1961 issue of *Mathematical Reviews*, of the book *Färbungsprobleme auf Flächen und Graphen* (Colouring Problems on Surfaces and Graphs) by Ringel [1959]. This book itself however does not mention the conjecture.

edge by three parallel edges, has colouring number 8 (as $L(G)$ has 15 vertices and stable set number 2), but contains no subdivision of K_8.

Catlin in fact gave a counterexample to Hajós' conjecture for each $k \geq 7$. Erdős and Fajtlowicz [1981] showed that almost all graphs are counterexamples to Hajós' conjecture.

Related is the following result of Hajós [1961]: any graph G with $\chi(G) \geq k$ can be obtained from the complete graph K_k by a series of the following operations on graphs (each preserving $\chi \geq k$):

(64.4)
- (i) add vertices or edges;
- (ii) identify two nonadjacent vertices;
- (iii) take two disjoint graphs G_1 and G_2, choose edges $e_1 = u_1v_1$ of G_1 and $e_2 = u_2v_2$ of G_2, identify u_1 and u_2, delete e_1 and e_2, and add edge v_1v_2.

64.4. The stable set, clique, and vertex cover polytope

The *stable set polytope* $P_{\text{stable set}}(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the stable sets in G. Since finding a maximum-size stable set is NP-complete, one may not expect a polynomial-time checkable system of linear inequalities describing the stable set polytope (Corollary 5.16a). More precisely, if NP≠co-NP, then there do not exist inequalities satisfied by the stable set polytope such that their validity can be certified in polynomial time and such that the inequality $\mathbf{1}^{\mathsf{T}}x \leq \alpha(G)$ is a nonnegative linear combination of them.

The *clique polytope* $P_{\text{clique}}(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of cliques. Trivially

(64.5) $$P_{\text{clique}}(G) = P_{\text{stable set}}(\overline{G}).$$

Hence, similar observations hold for the clique polytope.

Another related polytope is the *vertex cover polytope* $P_{\text{vertex cover}}(G)$ of G, being the convex hull of the incidence vectors of vertex covers in G. Since a subset U of V is a vertex cover if and only if $V \setminus U$ is a stable set, we have

(64.6) $$x \in P_{\text{vertex cover}}(G) \iff \mathbf{1} - x \in P_{\text{stable set}}(G).$$

This shows that problems on the two types of polytopes can be reduced to each other.

64.4a. Facets and adjacency on the stable set polytope

Padberg [1973] (for facets induced by odd circuits) and Nemhauser and Trotter [1974] observed that

(64.7) each facet of the stable set polytope of an induced subgraph $G[U]$ of G, is the restriction to U of some unique facet of $P_{\text{stable set}}(G)$.

More precisely, for each facet F of $P_{\text{stable set}}(G[U])$ there is a unique facet F' of $P_{\text{stable set}}(G)$ with the property that $F = \{x \in \mathbb{R}^U \mid x' \in F'\}$, where $x'_v = x_v$ if $v \in U$ and $x'_v = 0$ if $v \in V \setminus U$.

To prove (64.7), it suffices to prove it for $U = V \setminus \{v\}$ for some $v \in V$. Let F be a facet of $P_{\text{stable set}}(G - v)$. We can consider F as a face of codimension 2 of $P_{\text{stable set}}(G)$ (by extending F with a 0 at coordinate v). Define $H := \{x \in \mathbb{R}^V \mid x_v = 0\}$. As F is on the facet $F'' := P_{\text{stable set}}(G) \cap H$ of $P_{\text{stable set}}(G)$, there is a unique facet F' of $P_{\text{stable set}}(G)$ with $F = F'' \cap F'$. This implies $F = F' \cap H$, since

(64.8) $$F = F' \cap F'' = F' \cap P_{\text{stable set}}(G) \cap H = F' \cap H.$$

Suppose now that $P_{\text{stable set}}(G)$ has another facet F''' with $F = F''' \cap H$. Then $F \subseteq F''' \cap F'' \subseteq F''' \cap H = F$, and hence $F = F'' \cap F'''$, contradicting the unicity of F'. This proves (64.7).

Padberg [1973] also showed the following:

Theorem 64.5. *Let $G = (V, E)$ be a graph and let $a \in \mathbb{Z}_+^V$. Then the inequality*

(64.9) $$a^{\mathsf{T}} x \leq 1$$

is valid for the stable set polytope of G if and only if a is the incidence vector of a clique C. Moreover, (64.9) *determines a facet if and only if C is an inclusionwise maximal clique.*

Proof. Trivially, inequality (64.9) is valid if $a = \chi^C$ for some clique C. Conversely, if (64.9) is valid, then a is a $0, 1$ vector, and hence the incidence vector of a subset C of V. Then C is a clique, since otherwise C contains a stable set S of size 2, implying that (64.9) is not valid for $x := \chi^S$.

In proving the second statement, we can assume that $a = \chi^C$ for some clique C. Suppose that (64.9) determines a facet, and that C is not an inclusionwise maximal clique. Then there is a clique C' properly containing C. Hence for each $x \in P_{\text{stable set}}(G)$, if $x(C) = 1$, then $x(C') = 1$. This implies that the inequality $x(C) \leq 1$ is not facet-inducing, a contradiction.

Finally suppose that C is an inclusionwise maximal clique. To see that (64.9) determines a facet, let $a^{\mathsf{T}} x = \beta$ be satisfied by all x in the stable set polytope with $x(C) = 1$. So $a(S) = \beta$ for each stable set S with $|S \cap C| = 1$. Then $a_v = \beta$ for each $v \in C$, as $S := \{v\}$ is stable. Also, $a_u = 0$ for each $u \in V \setminus C$, since by the maximality of C, there is a vertex $v \in C$ that is not adjacent to u. So $S := \{u, v\}$ is stable, and hence $a_u + a_v = \beta$. So $a_u = 0$. Concluding, $a^{\mathsf{T}} x = \beta$ is some multiple of $x(C) = 1$, and hence $x(C) \leq 1$ determines a facet. ∎

Graphs for which the nonnegativity and clique inequalities determine all facets, are precisely the *perfect* graphs — see Chapter 65.

Trivially, the vertices of the stable set polytope are precisely the incidence vectors of the stable sets. Chvátal [1975a] characterized adjacency:

Theorem 64.6. *The incidence vectors of two different stable sets R, S are adjacent vertices of the stable set polytope if and only if $R \triangle S$ induces a connected subgraph of G.*

Proof. To see necessity, if $G[R \triangle S]$ is not connected, then (as it is bipartite) it has two colour classes U and W with $\{U, W\} \neq \{R \setminus S, S \setminus R\}$. Let $U' := U \cup (R \cap S)$ and

$W' := W \cup (R \cap S)$. Then U' and W' are stable sets and $\frac{1}{2}(\chi^R + \chi^S) = \frac{1}{2}(\chi^{U'} + \chi^{W'})$, contradicting the adjacency of χ^R and χ^S.

To see sufficiency, if χ^R and χ^S are not adjacent, then there exist stable sets U and W, and $\lambda, \mu \in (0,1)$ such that $\lambda\chi^R + (1-\lambda)\chi^S = \mu\chi^U + (1-\mu)\chi^W$ and $\{U, W\} \neq \{R, S\}$. So $U \cap W = R \cap S$. Hence $U \setminus W, W \setminus U$ forms a bipartition of $G[R \triangle S]$ different from the bipartition $R \setminus S, S \setminus R$. This contradicts the connectedness of $G[R \triangle S]$. ∎

64.5. Fractional stable sets

The incidence vectors of stable sets in an undirected graph $G = (V, E)$ are precisely the integer vectors $x \in \mathbb{R}^V$ satisfying

(64.10)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & x_u + x_v \leq 1 & \text{for } \{u, v\} \in E. \end{array}$$

(The inequalities (64.10)(ii) are called the *edge inequalities*.) Any (not necessarily integer) solution x of (64.10) is called a *fractional stable set*. By definition, its *size* is equal to $x(V)$.

The maximum size of a fractional stable set is called the *fractional stable set number* and is denoted by $\alpha^*(G)$. By linear programming duality, $\alpha^*(G)$ is equal to the *fractional edge cover number* $\rho^*(G)$ (assuming that G has no isolated vertices), which is the minimum value of $y(E)$ over all $y \in \mathbb{R}^E$ satisfying

(64.11)
$$\begin{array}{lll} \text{(i)} & 0 \leq y_e \leq 1 & \text{for } e \in E, \\ \text{(ii)} & y(\delta(v)) \geq 1 & \text{for } v \in V. \end{array}$$

Any solution y of (64.11) is called a *fractional edge cover*.

This was also discussed in Section 30.11, where it was shown that each vertex of the polytope determined by (64.11) (the fractional edge cover polytope) is half-integer. A similar result holds for the *fractional stable set polytope*, which is the polytope determined by (64.10) (the result is implicit in Balinski [1965]):

Theorem 64.7. *Each vertex of the fractional stable set polytope P is half-integer.*

Proof. Let x be a vertex of P. Let $U := \{v \in V \mid 0 < x_v < \frac{1}{2}\}$ and let $W := \{v \in V \mid \frac{1}{2} < x_v < 1\}$. Then there is an $\varepsilon > 0$ such that both $x + \varepsilon(\chi^U - \chi^W)$ and $x - \varepsilon(\chi^U - \chi^W)$ belong to P. As x is a vertex, it follows that $\chi^U - \chi^W = 0$. So $U = W = \emptyset$. ∎

(This proof was provided to Nemhauser and Trotter [1974] by a referee of their paper.)

The theorem also follows from the observation of Balinski [1965] that each nonsingular submatrix of the incidence matrix of a graph has a half-integer inverse.

Theorem 64.7 implies that $\alpha^*(G) = \frac{1}{2}\alpha_2(G)$, where $\alpha_2(G)$ is the maximum size of a 2-*stable set*, which is an integer vector $x \in \mathbb{R}^V$ satisfying

(64.12) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x_u + x_v \leq 2$ for $\{u, v\} \in E$

(cf. Section 30.9).

Moreover, it implies a characterization of the 2-*stable set polyhedron*, which is the convex hull of the 2-stable sets:

Corollary 64.7a. *The* 2*-stable set polyhedron is determined by* (64.12).

Proof. Directly from Theorem 64.7. ∎

With the following construction, the problem of finding a maximum-weight fractional stable set (and similarly, a maximum-weight 2-stable set), can be reduced to the problem of finding a maximum-weight stable set in a bipartite graph. The latter problem is strongly polynomial-time solvable, by Theorem 21.10.

Let $G = (V, E)$ be a graph. Let V' be a copy of V. For any $v \in V$, let v' denote the copy of V in V'. Define $\widetilde{V} := V \cup V'$. Let $\widetilde{E}$ be the set of pairs $u'v$ and uv', over all edges uv of G. Then $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ is a bipartite graph.

For any weight function $w : V \to \mathbb{R}_+$, define $\tilde{w} : \widetilde{V} \to \mathbb{R}_+$ by $\tilde{w}(v) := \tilde{w}(v') := w(v)$ for $v \in V$. Then any stable set S in $\widetilde{G}$ maximizing $\tilde{w}(S)$ gives a 2-stable set x in G maximizing $w^\mathsf{T} x$, by defining $x_v := |S \cap \{v, v'\}|$. Indeed, for any 2-stable set x' in G we can define a stable set S' in $\widetilde{G}$ by

(64.13) $$S' := \{v \in V \mid x'_v \geq 1\} \cup \{v \in V' \mid x'_v \geq 2\}.$$

Then $w^\mathsf{T} x' = \tilde{w}(S') \leq \tilde{w}(S) = w^\mathsf{T} x$. (Here we assume without loss of generality that G has no isolated vertices.)

64.5a. Further on the fractional stable set polytope

Nemhauser and Trotter [1974] characterized the vertices of the fractional stable set polytope:

Theorem 64.8. *A vector $x \in \mathbb{R}^V$ is a vertex of the fractional stable set polytope P of G if and only if $x = \chi^{U_2} + \frac{1}{2}\chi^{U_1}$, where U_2 is a stable set of G, where U_1 is disjoint from $U_2 \cup N(U_2)$, and where each component of $G[U_1]$ is nonbipartite.*

Proof. *Necessity.* Let x be a vertex of P, and define $U_2 := \{v \in V \mid x_v = 1\}$ and $U_1 := \{v \in V \mid x_v = \frac{1}{2}\}$. Then U_2 is a stable set and no vertex in U_1 is adjacent to any vertex in U_2. So U_1 is disjoint from $U_2 \cup N(U_2)$.

If some component of $G[U_1]$ would be bipartite, say with colour classes S and T, then $x \pm \varepsilon(\chi^S - \chi^T)$ would belong to P for some $\varepsilon \neq 0$. This contradicts the fact that x is a vertex of P.

Sufficiency. Suppose that x satisfies the condition, and that x is not a vertex of P. Then there is a nonzero vector y such that both $x + y$ and $x - y$ belong to P. Necessarily, $y_v = 0$ if $v \notin U_1$. Moreover, for each edge uw of $G[U_1]$ one has $y_u + y_w = 0$, since $x_u + x_w = 1$. As each component of $G[U_1]$ contains an odd circuit, this implies $y_v = 0$ for each $v \in U_1$. So $y = \mathbf{0}$, a contradiction. ∎

A useful condition was given by Nemhauser and Trotter [1975]:

Theorem 64.9. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let $S \subseteq V$ be a stable set. If S is a maximum-weight stable set in the subgraph of G induced by $S \cup N(S)$, then S is contained in some maximum-weight stable set of G.*

Proof. Let T be a maximum-weight stable set of G. Define $U := (S \cup T) \setminus N(S)$. Trivially, U is stable. Also, $w(N(S) \cap T) \leq w(S \setminus T)$, since $w((S \cup N(S)) \cap T) \leq w(S)$, as S has maximum weight in $G[S \cup N(S)]$. Hence

(64.14) $$w(U) = w(T) + w(S \setminus T) - w(N(S) \cap T) \geq w(T),$$

implying that U is a maximum-weight stable set in G. ∎

This implies (Nemhauser and Trotter [1975]):

Corollary 64.9a. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let x be a maximum-weight fractional stable set. Then $S := \{v \mid x_v = 1\}$ is contained in a maximum-weight stable set.*

Proof. This follows from Theorem 64.9, since S is a maximum-weight stable set in $G[S \cup N(S)]$. For if T would be a stable set in $G[S \cup N(S)]$ with $w(T) > w(S)$, then $x + \varepsilon(\chi^T - \chi^S)$ would belong to the fractional stable set polytope for some $\varepsilon > 0$, while it has weight larger than x, a contradiction. ∎

Picard and Queyranne [1977] showed that, for any graph $G = (V, E)$ and any weight function $w : V \to \mathbb{R}$, there is a unique minimal subset of vertices that has fractional values in some optimum fractional stable set (solving a problem posed by Nemhauser and Trotter [1975]):

Theorem 64.10. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let x and y be maximum-weight fractional stable sets. Then there is a maximum-weight fractional stable set z such that, for each vertex v, z_v is integer if x_v or y_v integer.*

Proof. We can assume that x and y are half-integer (as we can assume that x and y are vertices of the fractional stable set polytope). For $i = 0, 1, 2$, let $X_i := \{v \mid x_v = i/2\}$ and $Y_i := \{v \mid y_v = i/2\}$. Then

(64.15) $$w(Y_0 \cap X_2) \leq w(X_0 \cap Y_2),$$

since

$$(64.16) \qquad y + \tfrac{1}{2}(\chi^{Y_0 \cap X_2} - \chi^{X_0 \cap Y_2})$$

is a fractional stable set. Otherwise, since X_2 is stable, there is an edge uv with $y_u + y_v = 1$, $u \in Y_0 \cap X_2$, and $v \notin X_0 \cap Y_2$. So $y_u = 0$, and hence $y_v = 1$. Also, $x_u = 1$, and hence $x_v = 0$. So $v \in X_0 \cap Y_2$, a contradiction. This shows (64.15).

Moreover,

$$(64.17) \qquad w(X_0 \setminus Y_0) \leq w(X_2 \setminus Y_2),$$

since

$$(64.18) \qquad x + \tfrac{1}{2}(\chi^{X_0 \setminus Y_0} - \chi^{X_2 \setminus Y_2})$$

is a fractional stable set. Otherwise there is an edge uv with $x_u + x_v = 1$, $u \in X_0 \setminus Y_0$, and $v \notin X_2 \setminus Y_2$. So $x_u = 0$, and hence $x_v = 1$. Also, $y_u > 0$, and hence $y_v < 1$. So $v \in X_2 \setminus Y_2$, a contradiction. This shows (64.17).

(64.15) and (64.17) imply that

$$(64.19) \qquad \begin{aligned} w(Y_1 \cap X_2) &= w(X_2 \setminus Y_2) - w(X_2 \cap Y_0) \geq w(X_0 \setminus Y_0) - w(Y_2 \cap X_0) \\ &= w(Y_1 \cap X_0). \end{aligned}$$

Hence

$$(64.20) \qquad z := y + \tfrac{1}{2}(\chi^{Y_1 \cap X_2} - \chi^{Y_1 \cap X_0})$$

has weight at least that of y. Moreover, z is a fractional stable set. Otherwise, as X_2 is stable, there is an edge uv with $y_u + y_v = 1$, $u \in Y_1 \cap X_2$ and $v \notin Y_1 \cap X_0$. So $y_u = y_v = \frac{1}{2}$, $x_u = 1$, hence $x_v = 0$. So $v \in Y_1 \cap X_0$, a contradiction. Hence z is a fractional stable set as required. ∎

Nemhauser and Trotter [1975] and Picard and Queyranne [1977] gave a polynomial-time algorithms to find a half-integer maximum-weight fractional stable set attaining the minimum number of fractional values. (This can be derived from the uniqueness of the minimal set of fractional vertices: just try $x_v = 0$ and $x_v = 1$ for each $v \in V$, and see if the fractional stable set number drops.)

Pulleyblank [1979a] and Bourjolly and Pulleyblank [1989] characterized the minimal set of fractional values. Related results were given by Grimmett [1986].

64.6. Fractional vertex covers

Similar results hold for *fractional vertex covers*, which are vectors $x \in \mathbb{R}^V$ satisfying

$$(64.21) \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & x_u + x_v \geq 1 & \text{for } \{u, v\} \in E. \end{array}$$

Trivially, a vector x is a fractional vertex cover if and only if $\mathbf{1} - x$ is a fractional stable set.

The minimum size of a fractional vertex cover is called the *fractional vertex cover number*, and is denoted by $\tau^*(G)$. So

(64.22) $\tau^*(G) + \alpha^*(G) = |V|$.

Again, by linear programming duality, $\tau^*(G)$ is equal to the *fractional matching number* $\nu^*(G)$, which is the maximum value of $y(E)$ over all $y \in \mathbb{R}^E$ satisfying

(64.23)
$$\begin{array}{lll} \text{(i)} & 0 \leq y_e \leq 1 & \text{for } e \in E, \\ \text{(ii)} & y(\delta(v)) \leq 1 & \text{for } v \in V. \end{array}$$

Any solution y of (64.23) is called a *fractional matching*. This was also discussed in Section 30.3, where it was shown that each vertex of the polytope determined by (64.23) (the fractional matching polytope) is half-integer. A similar result holds for the *fractional vertex cover polytope*, which is the polytope determined by (64.21):

Theorem 64.11. *Each vertex of the fractional vertex cover polytope P is half-integer.*

Proof. Directly from Theorem 64.7, since x belongs to the fractional vertex cover polytope if and only if $\mathbf{1}-x$ belongs to the fractional stable set polytope. ∎

Theorem 64.11 implies that $\tau^*(G) = \frac{1}{2}\tau_2(G)$, where $\tau_2(G)$ is the minimum size of a 2-*vertex cover*, which is an integer vector $x \in \mathbb{R}^V$ satisfying

(64.24)
$$\begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x_u + x_v \geq 2 & \text{for } \{u,v\} \in E \end{array}$$

(cf. Section 30.10).

It also implies a characterization of the 2-*vertex cover polyhedron*, which is the convex hull of the 2-vertex covers:

Corollary 64.11a. *The 2-vertex cover polyhedron is determined by* (64.24).

Proof. Directly from Theorem 64.11. ∎

By the results on fractional stable sets and 2-stable sets given in Section 64.5, and using the reductions described above, a minimum-weight fractional vertex cover and a minimum-weight 2-vertex cover can be found in strongly polynomial time.

Notes. Corollary 64.9a and Theorem 64.10 have direct analogues for vertex covers: given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}$,

(64.25) for each minimum-weight fractional vertex cover x there is a minimum-weight vertex cover contained in $\{v \mid x_v \neq 0\}$,

and

(64.26) for any two minimum-weight fractional vertex covers x and y there is a minimum-weight fractional vertex cover z such that for each $v \in V$: $x_v \in \mathbb{Z}$ or $y_v \in \mathbb{Z} \Rightarrow z_v \in \mathbb{Z}$.

These statements can be derived from Corollary 64.9a and Theorem 64.10 by again observing that a vector x is a (fractional) stable set if and only if $\mathbf{1} - x$ is a (fractional) vertex cover. Similarly, Theorem 64.8 implies a characterization of the vertices of the fractional vertex cover polytope.

64.6a. A bound of Lorentzen

The fractional stable set and vertex cover numbers give upper and lower bounds on the stable set and vertex cover number, respectively. These bounds are computable in polynomial time. A better polynomial-time computable bound was given by Lorentzen [1966]:

Theorem 64.12. *For each graph $G = (V, E)$:*

(64.27) $$2\nu^*(G) - \nu(G) \leq \tau(G).$$

Proof. Since $\nu^*(G) = 2\nu_2(G)$ (cf. Section 30.2), there is a half-integer fractional matching $x : E \to \mathbb{R}_+$ with $x(E) = \nu^*(G)$, such that the support of x is the disjoint union of a matching and a number t of odd circuits. We can assume that each edge of G belongs to the support of x (as deleting edges increases neither $\nu(G)$ nor $\tau(G)$). Also we can assume that G has no isolated vertices. Then $\nu(G) = \frac{1}{2}(|V| - t)$, $\tau(G) = \frac{1}{2}(|V| + t)$, and $\nu^*(G) = \frac{1}{2}|V|$. ∎

Bound (64.27) is generally a better lower bound on $\tau(G)$ than $\tau^*(G)$ (for example, for $G = K_3$). It implies an upper bound for $\alpha(G)$, generally better than $\alpha(G) \leq \rho^*(G)$:

Corollary 64.12a. *For each graph $G = (V, E)$ without isolated vertices:*

(64.28) $$\alpha(G) \leq 2\rho^*(G) - \rho(G).$$

Proof. Using Theorem 30.9, we have $\alpha(G) = |V| - \tau(G) \leq |V| - 2\nu^*(G) + \nu(G) = 2(|V| - \nu^*(G)) - (|V| - \nu(G)) = 2\rho^*(G) - \rho(G)$. ∎

64.7. The clique inequalities

A set of constraints stronger than the edge inequalities (64.10)(ii) is obtained by the 'clique inequalities'. Let $P(G)$ be the polytope in $\mathbb{R}^V$ determined by

(64.29) $$\begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for each } v \in V, \\ \text{(ii)} & x(C) \leq 1 & \text{for each clique } C. \end{array}$$

The inequalities (64.29)(ii) are called the *clique inequalities.*

Since the integer solutions of (64.29) are exactly the incidence vectors of stable sets, the stable set polytope of G is equal to the integer hull of $P(G)$ (the convex hull of the integer vectors in $P(G)$).

We call any vector x satisfying (64.29) a *strong fractional stable set*. We denote

(64.30) $\alpha^{**}(G)$:= *strong fractional stable set number* := the maximum size of a strong fractional stable set.

Since each strong fractional stable set is a fractional stable set, we know

(64.31) $\alpha(G) \leq \alpha^{**}(G) \leq \alpha^*(G)$.

So $\alpha^{**}(G)$ gives a better upper bound on $\alpha(G)$ than $\alpha^*(G)$ gives — however, $\alpha^{**}(G)$ is generally more difficult to compute.

Note that $P(G)$ is the antiblocking polyhedron of the clique polytope of G:

(64.32) $P(G) = A(P_{\text{clique}}(G))$.

(For background on antiblocking polyhedra, see Section 5.9.)

64.8. Fractional and weighted colouring numbers

For any graph $G = (V, E)$, the *fractional colouring number* $\chi^*(G)$ is the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.33) $\lambda_1 \chi^{S_1} + \cdots + \lambda_k \chi^{S_k} = \mathbf{1}$.

So if the λ_i are required to be integer, we have the colouring number.

By linear programming duality, the fractional colouring number is equal to the maximum of $\mathbf{1}^{\mathsf{T}} x$ over the polytope $\overline{P}(G)$ in $\mathbb{R}^V_+$ determined by

(64.34)
$$\begin{array}{ll} x_v \geq 0 & \text{for each } v \in V, \\ x(S) \leq 1 & \text{for each stable set } S. \end{array}$$

(So $\overline{P}(G) = P(\overline{G})$ and $\overline{P}(G) = A(P_{\text{stable set}}(G))$.) Hence we have:

(64.35) $\chi^*(\overline{G}) = \alpha^{**}(G)$.

We denote

(64.36) $\overline{\chi}^*(G) := \chi^*(\overline{G})$,

which is called the *fractional clique cover number* of G.

No polynomial-time algorithm is known to calculate $\chi^*(G)$. Note that the separation problem for $\overline{P}(G)$ is NP-complete, since the optimization problem over $P_{\text{stable set}}(G)$ is NP-complete.

Given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Z}_+$, the *weighted colouring number* $\chi_w(G)$ is the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.37) $\lambda_1 \chi^{S_1} + \cdots + \lambda_k \chi^{S_k} = w$.

So if $w = \mathbf{1}$, then $\chi_w(G)$ is equal to the colouring number $\chi(G)$ of G. Hence determining $\chi_w(G)$ is NP-complete.

For $w : V \to \mathbb{Z}_+$, let graph G^w arise from G by replacing each vertex by a clique C_v of size $w(v)$, two vertices in different cliques C_u,C_v being adjacent if and only if u and v are adjacent. Then

(64.38) $\qquad \chi_w(G) = \chi(G^w).$

We denote

(64.39) $\qquad \overline{\chi}_w(G) := \chi_w(\overline{G}),$

called the *weighted clique cover number* of G.

The fractional version is the *fractional weighted colouring number* $\chi_w^*(G)$, defined as the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.40) $\qquad \lambda_1\chi^{S_1} + \cdots + \lambda_k\chi^{S_k} = w.$

This value is equal to the maximum value of $w^{\mathsf{T}}x$ over the antiblocking polytope $A(P_{\text{stable set}}(G))$ pf $P_{\text{stable set}}(G)$. Since the optimization problem over $P_{\text{stable set}}(G)$ is NP-complete, determining $\chi_w^*(G)$ is NP-hard.

We denote

(64.41) $\qquad \overline{\chi}_w^*(G) := \chi_w^*(\overline{G}),$

called the *fractional weighted clique cover number* of G.

The complexity results above can be specialized to classes of graphs. By the results of Grötschel, Lovász, and Schrijver [1981,1984c]:

(64.42) For any collection $\mathcal{G}$ of graphs: there is a polynomial-time algorithm to find the fractional weighted colouring number for any graph in $\mathcal{G}$ and any weight function if and only there is a polynomial-time algorithm to find a maximum-weight stable set in any graph in $\mathcal{G}$ and for any weight function.

Since the problem of determining $\alpha(G)$ is NP-complete even if we restrict ourselves to planar cubic graphs, determining $\chi_w^*(G)$ for such graphs is NP-hard. As noticed in Grötschel, Lovász, and Schrijver [1981], determining $\chi_w^*(G)$ and $\chi(G)$ seem incomparable with respect to complexity. For cubic graphs G, $\chi(G)$ can be easily found in polynomial time (using Brooks' theorem (Theorem 64.3)), while determining $\chi_w^*(G)$ is NP-hard. In contrast to this, for the line graph G of a cubic graph H, $\chi(G)$ is NP-complete to compute by Holyer's theorem that 3-edge colourability is NP-complete (see Section 28.3), whereas $\chi_w^*(G)$ can be computed in polynomial time, since the separation problem over $A(P_{\text{stable set}}(G))$ is polynomial-time solvable, as the optimization problem over $P_{\text{stable set}}(G)$ is polynomial-time solvable (as it amounts to finding a maximum-weight matching in H).

64.8a. The ratio of $\chi(G)$ and $\chi^*(G)$

For later purposes we prove the following upper bound for the colouring number in terms of the fractional colouring number, obtained by applying a greedy-type algorithm (Johnson [1974a], Lovász [1975c]):

Theorem 64.13. *For any graph $G = (V, E)$:*

(64.43) $$\chi(G) \leq (1 + \ln \alpha(G))\chi^*(G).$$

Proof. Set $k := \alpha(G)$. Iteratively choose a maximum-size stable set S in G and reset G to $G - S$. We stop if VG is empty.

The stable sets found form a colouring $\mathcal{C}$ of the (original) vertex set V. So $\chi(G) \leq |\mathcal{C}|$.

For each $v \in V$, define

(64.44) $$x_v := \frac{1}{|S|},$$

where S is the set in $\mathcal{C}$ containing v. Then $x(V) = |\mathcal{C}|$, and hence

(64.45) $$\chi(G) \leq x(V).$$

Consider any stable set S' of G. Let S' consist of vertices $v_1, \ldots, v_k$, in the order by which they are covered by stable sets S in the algorithm. Then for each $i = 1, \ldots, k$, we have

(64.46) $$x_{v_i} \leq \frac{1}{k - i + 1}.$$

Indeed, let v_i be covered by $S \in \mathcal{C}$. When we selected S, the vertices $v_i, v_{i+1}, \ldots, v_k$ were uncovered yet. As we chose S, we know $|S| \geq |\{v_i, v_{i+1}, \ldots, v_k\}| = k - i + 1$, implying (64.46).

(64.46) implies

(64.47) $$x(S') \leq \sum_{i=1}^{k} \frac{1}{k - i + 1} = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k \leq 1 + \ln \alpha(G).$$

So $(1 + \ln \alpha(G))^{-1} \cdot x$ satisfies (64.34), and hence

(64.48) $$(1 + \ln \alpha(G))^{-1} \cdot x(V) \leq \chi^*(G).$$

Together with (64.45), this implies (64.43). ∎

This theorem will be used in proving Theorem 67.17.

64.8b. The Chvátal rank

In Section 36.7a we defined the polyhedron P' for any rational polyhedron P and the notion of the Chvátal rank of a polyhedron P.

Let $P(G)$ denote the polytope of strong fractional stable sets, that is, the polytope determined by (64.29) (the nonnegativity and clique constraints). For any polyhedron P, let P_{I} denote the *integer hull* of P, that is, the convex hull of the integer vectors in P.

Chvátal [1973a] showed that there is no fixed t such that $P(G)^{(t)} = P(G)_{\mathrm{I}}$ for each graph G, even if we restrict G to graphs with $\alpha(G) = 2$. Chvátal, Cook, and Hartmann [1989] showed that t can be at least $\frac{1}{3}\log n$ for such graphs (where n is the number of vertices).

We will see in Corollary 65.2e that the class of graphs G with $P(G)_{\mathrm{I}} = P(G)$ is exactly the class of perfect graphs. By Edmonds' matching polytope theorem (Corollary 25.1a) if G is the line graph of some graph H, then $P(G)' = P(G)_{\mathrm{I}}$, which is the matching polytope of H.

The smallest t for which $P(G)^{(t)} = P(G)_{\mathrm{I}}$ might be an indication of the computational complexity of the stable set number $\alpha(G)$. For each fixed t, the stable set problem for graphs with $P(G)^{(t)} = P(G)_{\mathrm{I}}$ belongs to NP∩co-NP. Chvátal [1973a] raised the question whether it belongs to P. (A negative indication is the result of Eisenbrand [1999] that given a polytope P by linear inequalities and given x, deciding if x belongs to P' is co-NP-complete.)

Another (weaker, but easier to compute) relaxation is: $Q(G)$ is the polytope of fractional stable sets; that is, the polytope in $\mathbb{R}^V$ determined by

(64.49) (i) $x_v \geq 0$ for each $v \in V$,
(ii) $x_v + x_w \leq 1$ for each $vw \in E$.

Again $Q(G)_{\mathrm{I}} = P_{\text{stable set}}(G)$. Since $Q(G) \supseteq P(G)$, there is no fixed t with $Q(G)^{(t)} = Q(G)_{\mathrm{I}}$ for each graph G. Chvátal [1973a] noticed that for $G = K_n$ the smallest t with $Q(G)^{(t)} = P_{\text{stable set}}(G)$ is about $\log n$.

It is not difficult to see that $Q(G)'$ is the polytope determined by (64.49) together with

(64.50) (iii) $x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor$ for each odd circuit C.

Graphs G with $Q(G)' = P_{\text{stable set}}(G)$ are called *t-perfect*. More on t-perfect graphs can be found in Chapter 68.

Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph G with n vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from system (64.49) by adding at most $p(n)$ cutting planes. (That is, a list of at most $p(n)$ inequalities $a_i^{\mathsf{T}}x \leq \lfloor \beta_i \rfloor$ such that, for each i, a_i is an integer vector and the inequality $a_i^{\mathsf{T}}x \leq \beta_i$ is a nonnegative combination of inequalities from (64.49) and inequalities occurring earlier in the list.)

Chvátal, Cook, and Hartmann [1989] showed that the Chvátal rank of the following relaxation of the stable set polytope:

(64.51) $x_v \geq 0$ for $v \in V$,
$x(U) \leq \alpha(G[U])$ for $U \subseteq V$,

is $\Omega((n/\log n)^{\frac{1}{2}})$, where G is a graph with n vertices. This relaxation is stronger than the polytope determined by just the nonnegativity and clique constraints.

64.9. Further results and notes

64.9a. Graphs with polynomial-time stable set algorithm

In the remaining chapters of this part we will see that a maximum-weight stable set can be found in strongly polynomial time in perfect graphs and their complements, in t-perfect graphs, and in claw-free graphs. In perfect graphs and their complements, also a minimum vertex-colouring can be found in polynomial time. In this section we list some other classes of graphs where a maximum-size stable set or a minimum vertex-colouring can be found in polynomial time.

A graph is a *circular-arc graph* if it is the intersection graph of a set of intervals on a circle. Gavril [1974a] gave polynomial-time algorithms for finding a maximum-size clique, a maximum-size stable set, and a minimum clique cover in these graphs. Karapetyan [1980] showed that $\chi(G) \leq \frac{3}{2}\omega(G)$ for any circular-arc graph G (proving a conjecture of Tucker [1975]). More on circular-arc graphs can be found in Klee [1969], Tucker [1971,1974,1975,1978,1980], Trotter and Moore [1976], Garey, Johnson, Miller, and Papadimitriou [1980], Golumbic [1980], Orlin, Bonuccelli, and Bovet [1981], Gupta, Lee, and Leung [1982], Skrien [1982], Leung [1984], Hsu [1985, 1995], Teng and Tucker [1985], Apostolico and Hambrusch [1987], Golumbic and Hammer [1988], Masuda and Nakajima [1988], Spinrad [1988], Shih and Hsu [1989a, 1989b], Bertossi and Moretti [1990], Hell, Bang-Jensen, and Huang [1990], Hsu and Tsai [1991], Deng, Hell, and Huang [1992,1996], Eschen and Spinrad [1993], Hsu and Spinrad [1995], Bhattacharya, Hell, and Huang [1996], Bhattacharya and Kaller [1997], Hell and Huang [1997], Feder, Hell, and Huang [1999], and McConnell [2001]. See also Section 65.6d.

A graph is a *circle graph* if its vertex set is a set of chords of the circle, two chords being adjacent if they intersect or cross. For these graphs, Gavril [1973] gave polynomial-time algorithms for finding a maximum-size clique and a maximum-size stable set. Bouchet [1985,1987b,1994], Naji [1985], and Gabor, Supowit, and Hsu [1989] showed that circle graphs can be recognized in polynomial time; this was improved to quadratic time by Spinrad [1994]. (Related results can be found in Fournier [1978], Garey, Johnson, Miller, and Papadimitriou [1980], Golumbic [1980], Rotem and Urrutia [1981], de Fraysseix [1984], Hsu [1985], Naji [1985], Dagan, Golumbic, and Pinter [1988], Gabor, Supowit, and Hsu [1989], Masuda, Nakajima, Kashiwabara, and Fujisawa [1990], Felsner, Müller, and Wernisch [1994], Ma and Spinrad [1994], Spinrad [1994], and Elmallah and Stewart [1998]. See also Section 65.6d.)

The weighted stable set problem was shown to be polynomial-time solvable for graphs without $K_5 - e$ minor by Barahona and Mahjoub [1994b]. (The graph $K_5 - e$ is obtained from K_5 by deleting one edge.) Descriptions of the corresponding polytopes are given by Barahona and Mahjoub [1994b,1994c].

Hsu, Ikura, and Nemhauser [1981] gave, for each fixed k, a polynomial-time algorithm for the weighted stable set problem for graphs without odd circuits of length larger than $2k + 1$. A 'nice class for the vertex packing problem' (obtained from bipartite graphs and claw-free graphs by repeated substitution) was given by Bertolazzi, De Simone, and Galluccio [1997]. Another nice class was given by De Simone [1993].

In Section 60.3d (Corollary 60.5b) we gave a proof of Győri's theorem (Győri [1984]), stating that the following class of graphs G satisfies $\alpha(G) = \overline{\chi}(G)$. Let A be a $\{0, 1\}$ matrix such that the 1's in each row form a contiguous interval. Then G has vertex set $\{(i, j) \mid a_{i,j} = 1\}$, where two pairs (i, j) and (i', j') are adjacent if and only if $a_{i,j'} = a_{i',j} = 1$. The method of Frank and Jordán [1995b] also yields a polynomial-time algorithm to find a maximum-size stable set and a minimum clique cover. Frank [1999a] gave an alternative algorithmic proof. This class of graphs is not closed under taking induced subgraphs, and they need not be perfect.

Hammer, Mahadev, and de Werra [1985], Balas, Chvátal, and Nešetřil [1987], Balas and Yu [1989], De Simone and Sassano [1993], Hertz and de Werra [1993], Hertz [1995,1997], Brandstädt and Hammer [1999], Mosca [1999], and Lozin [2000] gave further classes of graphs for which the maximum-size or maximum-weight clique problem is polynomial-time solvable.

64.9b. Colourings and orientations

Let $D = (V, A)$ be an orientation of an undirected graph $G = (V, E)$. The following was shown by Gallai [1968a] and Roy [1967] (referring to conjectures by P. Erdős and C. Berge, respectively):

(64.52) $\chi(G) \leq \lambda(D)$,

where

(64.53) $\lambda(D) :=$ the maximum number of vertices on a directed path in D.

To see this, consider an inclusionwise maximal subset A' of A with the property that $D' = (V, A')$ is acyclic. For any $v \in V$, let $h(v)$ be the number of vertices in a longest directed path in D' ending at v. If $h(u) = h(v)$ for distinct vertices u and v, then u and v are nonadjacent, since otherwise we can add the arc joining u and v to A'. So h defines a colouring of V, with no more colours than the number of vertices in a longest directed path in D'.

This proves (64.52). Note that (64.52) implies that each tournament ($\equiv$ orientation of a complete graph) has a Hamiltonian path (a theorem of Rédei [1934] (Corollary 14.14a)).

Roy [1967] also observed that each undirected graph $G = (V, E)$ has an acyclic orientation in which the number of vertices in a longest directed path is equal to the colouring number of G. (This follows by colouring the vertices with colours $1, \ldots, \chi(G)$, and orienting any edge from u to v if the colour of u is smaller than that of v, which gives a digraph D with $\lambda(D) \leq \chi(G)$.)

This result is equivalent to the fact that for any undirected graph $G = (V, E)$:

(64.54) $\chi(G) = \min\limits_{D} \lambda(D)$,

where D ranges over all acyclic orientations of G.

These results are essentially based on the (easy) fact that the minimum number of antichains needed to cover a partially ordered set is equal to the size of a maximum chain (Theorem 14.1).

Minty [1967] showed that for each graph $G = (V, E)$:

(64.55) $\chi(G) \leq k \iff G$ has an orientation such that each undirected circuit has at least $\frac{1}{k}|VC|$ forward arcs.

Necessity follows by colouring the vertices with colours $1, \ldots, k$, and orienting any edge from u to v if $\text{colour}(u) < \text{colour}(v)$. To see sufficiency, let D be an orientation as described. Give each arc a length $k - 1$, and add an arc in the reverse direction of length -1. Then each directed circuit in the extended digraph has nonnegative length. Hence there is a 'potential' $p : V \to \mathbb{Z}$ with $1 \leq p(v) - p(u) \leq k - 1$ for each arc (u, v) of D. Reducing p mod k gives a k-colouring as required.

Note that each orientation as in (64.55) is acyclic, and that any orientation D with $\lambda(D) \leq k$ is as in (64.55). The equivalence (64.55) gives a vertex-free description of the colouring number, and implies that $\chi(G)$ only depends on the cycle matroid of G.

Deming [1979a] showed that dual statements can be derived from Dilworth's decomposition theorem (Theorem 14.2), where 'chain' and 'antichain' are interchanged.

First one has, as a dual to (64.52), that for any orientation $D = (V, A)$ of an undirected graph $G = (V, E)$:

(64.56) $\qquad \alpha(G) \geq \xi(D),$

where

(64.57) $\qquad \xi(D) :=$ the minimum number of directed paths in D needed to cover V.

To see (64.56), again consider an inclusionwise maximal subset A' of A with $D' = (V, A')$ acyclic. By Dilworth's decomposition theorem, V has a subset U of size $\xi(D)$ such that no two vertices in U are connected by a directed path in D. Then U is a stable set in G, since if two distinct $u, v \in U$ are adjacent in G, say $(u, v) \in A$, then $(u, v) \notin A'$, and hence $A' \cup \{(u, v)\}$ is not acyclic. But then A' contains a directed path from v to u, a contradiction.

This shows (64.56). Deming [1979a] showed also a dual form of (64.54):

(64.58) $\qquad \alpha(G) = \max_{D} \xi(D),$

where D ranges over all acyclic orientations of G. Indeed, $\geq$ in (64.58) follows from (64.56). To see $\leq$, let U be a maximum-size stable set in G. Let D be any acyclic orientation of G in which each vertex in U is a source. Then $\xi(D) \geq |U| = \alpha(G)$.

64.9c. Algebraic methods

Lovász [1994] gave the following relations between stable sets, cliques, and colourings, using Hilbert's Nullstellensatz (extending Li and Li [1981] and unpublished work of D.J. Kleitman and L. Lovász). For any graph $G = (V, E)$, define the polynomial p_G in the variables x_v ($v \in V$) by:

(64.59) $\qquad p_G := \prod_{uv \in E} (x_u - x_v)$

(fixing some orientation of the edges). Then $\alpha(G) \leq k$ if and only if there are graphs $H_1, \ldots, H_t$ on V satisfying

(64.60) $\qquad p_G = p_{H_1} + \cdots + p_{H_t},$

with $\overline{\chi}(H_i) \leq k$ for $i = 1, \ldots, t$. The number t can be exponentially large — hence (64.60) gives no good characterization for the stable set number. Similarly, $\chi(G) \geq k$ if and only if there are graphs satisfying (64.60) with $\omega(H_i) \geq k$ for $i = 1, \ldots, t$.

Let $G = (V, E)$ be a (simple) graph, with adjacency matrix A_G. Motzkin and Straus [1965] showed that the maximum value of $x^{\mathsf{T}} A_G x$ over $x : V \to \mathbb{R}_+$ satisfying $x(V) = 1$, is equal to $1 - \omega(G)^{-1}$.

The proof of this is easy: for any two nonadjacent vertices u, v with $x_u > 0$ and $x_v > 0$, we can reset $x_u := x_u + \varepsilon$, $x_v := x_v - \varepsilon$ for some $\varepsilon \neq 0$ without decreasing $x^{\mathsf{T}} A_G x$. Hence the maximum value is attained by a vector x whose support is a clique C. As x takes the maximum value, we should have $x_v = 1/|C|$ for each $v \in C$. Then $x^{\mathsf{T}} A_G x$ is maximized if C is a maximum-size clique.

Motzkin and Straus' theorem implies the result of Korn [1968] that the minimum value of $x^{\mathsf{T}}(I + A_G)x$ over $x : V \to \mathbb{R}_+$ with $x(V) = 1$, is equal to $\alpha(G)^{-1}$. Indeed,

(64.61) $$\min_x x^{\mathsf{T}}(I + A_G)x = \min_x x^{\mathsf{T}}(J - A_{\overline{G}})x = 1 - \max_x x^{\mathsf{T}} A_{\overline{G}} x = \omega(\overline{G})^{-1} = \alpha(G)^{-1}.$$

More on this can be found in Gibbons, Hearn, Pardalos, and Ramana [1997].

Lovász [1982,1994] gave surveys of algebraic, topological, and other methods for the stable set and the vertex colouring problem.

64.9d. Approximation algorithms

Lund and Yannakakis [1993,1994] showed that unless NP=P, there do not exist a constant c and a polynomial-time algorithm that finds a vertex-colouring of any graph G using at most $c\chi(G)$ colours. (This was proved for $c < 2$ by Garey and Johnson [1976].)

More generally, Lund and Yannakakis [1993,1994] showed that there exists an $\varepsilon > 0$ such that, unless NP=P, there is no polynomial-time algorithm to find the colouring number of a graph up to a factor of n^{ε} (where n is the number of vertices).

A similar result for maximum-size stable sets was proved by Arora, Lund, Motwani, Sudan, and Szegedy [1992,1998]. Håstad [1996,1999] showed that, if NP$\neq$P, then there is no $\varepsilon > 0$ and a polynomial-time algorithm that finds a clique that is maximum-size up to a factor $n^{1/2-\varepsilon}$. Under a slightly stronger complexity assumption (NP$\neq$ZPP), Håstad proved a factor of $n^{1-\varepsilon}$.

For background, see Johnson [1992] and Papadimitriou [1994]. Related results can be found in Hochbaum [1983a], Wigderson [1983], Berger and Rompel [1990], Feige, Goldwasser, Lovász, Safra, and Szegedy [1991,1996], Berman and Schnitger [1992], Boppana and Halldórsson [1992], Bellare, Goldwasser, Lund, and Russell [1993], Khanna, Linial, and Safra [1993,2000], Bellare and Sudan [1994], Feige and Kilian [1994,1996,1998a,1998b,2000], Karger, Motwani, and Sudan [1994,1998], Bellare, Goldreich, and Sudan [1995,1998], Feige [1995,1997], Fürer [1995], Håstad [1996,1999], Alon and Kahale [1998], Arora and Safra [1998], Engebretsen and Holmerin [2000], Srinivasan [2000], and Khot [2001].

In contrast, there is an easy algorithm to obtain a vertex cover in a graph $G = (V, E)$ of size at most $2\tau(G)$ (F. Gavril 1974 (cf. Garey and Johnson [1979])): choose any inclusionwise maximal matching M (greedily); then the set of vertices covered by M is a vertex cover of size $2|M|$. Since $\tau(G) \geq |M|$, this is a vertex cover as described.

No polynomial-time algorithm yielding a factor better than 2 is known. Håstad [1997,2001] showed that, if NP$\neq$P, no factor better than $\frac{7}{6}$ is achievable in polynomial time.

See also Section 67.4f below.

64.9e. Further notes

Yannakakis [1988,1991] showed that the stable set polytope of the line graph $L(K_n)$ of a complete graph K_n cannot be represented as the projection of a polytope in higher dimensions that is symmetric under the automorphism group of $L(K_n)$. Cao and Nemhauser [1998] characterized line graphs as those graphs whose stable set polytope is determined by the inequalities corresponding to the matching polytope constraints.

Euler, Jünger, and Reinelt [1987] extended results of Padberg [1973] on facets of the stable set polytope, to more general 'independence' polytopes.

More on the stable set polytope can be found in Fulkerson [1971a], Chvátal [1973a,1975a,1985a], Padberg [1973,1974b,1977,1979,1980,1984], Nemhauser and Trotter [1974], Trotter [1975], Wolsey [1976], Balas and Zemel [1977], Naddef and Pulleyblank [1981a], Sekiguchi [1983], Ikura and Nemhauser [1985], Grötschel, Lovász, and Schrijver [1986], Lovász and Schrijver [1989,1991], Cheng and Cunningham [1995,1997], Cánovas, Landete, and Marín [2000], Lipták and Lovász [2000, 2001], and Cheng and de Vries [2002a,2002b].

The convex hull of the incidence vectors of the stable sets of size at most a given k is studied by Janssen and Kilakos [1999]. Generalizations of the stable set polytope to more general $0, \pm 1$ programming and satisfiability problems were studied by Johnson and Padberg [1982], Hooker [1996], and Sewell [1996].

Methods for and computational results on the stable set problem (or the equivalent clique, vertex cover, and set packing problems) are given by Balas and Samuelsson [1977], Chvátal [1977], Houck and Vemuganti [1977], Tarjan and Trojanowski [1977], Geoffroy and Sumner [1978], Gerhards and Lindenberg [1979], Hansen [1980b], Bar-Yehuda and Even [1981,1982,1985], Billionnet [1981], Chiba, Nishizeki, and Saito [1982] (planar graphs), Hochbaum [1982,1983a], Loukakis and Tsouros [1982], Baker [1983,1994], Clarkson [1983], Monien and Speckenmeyer [1983,1985], Balas and Yu [1986], Jian [1986], Robson [1986], Shindo and Tomita [1988], Hurkens and Schrijver [1989], Carraghan and Pardalos [1990], Nemhauser and Sigismondi [1992], Balas and Xue [1991,1996], Boppana and Halldórsson [1992], Pardalos and Rodgers [1992], Paschos [1992], Khuller, Vishkin, and Young [1993,1994], Berman and Fürer [1994], Mannino and Sassano [1994], Halldórsson [1995], Balas, Ceria, Cornuéjols, and Pataki [1996], Bourjolly, Laporte, and Mercure [1997], Halldórsson and Radhakrishnan [1997], Alon and Kahale [1998], Arkin and Hassin [1998], Feige and Kilian [1998a], Kleinberg and Goemans [1998], Chandra and Halldórsson [1999, 2001], Nagamochi and Ibaraki [1999b], Bar-Yehuda [2000], Halperin [2000,2002], Krivelevich and Vu [2000], Chen, Kanj, and Jia [2001], Krivelevich, Nathaniel, and Sudakov [2001a,2001b], and Guha, Hassin, Khuller, and Or [2002].

Methods for graph colouring are proposed and investigated by Christofides [1971], Brown [1972], Matula, Marble, and Isaacson [1972], Corneil and Graham [1973], Johnson [1974b], Wang [1974], Lawler [1976a], McDiarmid [1979], Matula and Beck [1983], Sysło, Deo, and Kowalik [1983], Wigderson [1983], Edwards [1986], Berger and Rompel [1990], Hertz [1991], Halldórsson [1993], Blum [1994], Demange, Grisoni, and Paschos [1994], Karger, Motwani, and Sudan [1994,1998], Schiermeyer [1994], Beigel and Eppstein [1995], Blum and Karger [1997], Krivelevich and Vu [2000], Eppstein [2001], Halperin, Nathaniel, and Zwick [2001], Molloy and Reed [2001], Stacho [2001], Alon and Krivelevich [2002], and Charikar [2002].

For computational results on clique, stable set, and colouring problems, consult also Johnson and Trick [1996].

A survey of graph colouring algorithms was given by Matula, Marble, and Isaacson [1972]. Chiba, Nishizeki, and Saito [1981], Thomassen [1994], and Robertson, Sanders, Seymour, and Thomas [1996] gave linear-time 5-colouring algorithms for planar graphs. The worst-case behaviour of graph colouring algorithms was investigated by Johnson [1974b].

Mycielski [1955] showed that triangle-free graphs can have arbitrarily large colouring number. King and Nemhauser [1974] and Gyárfás [1987] and Fouquet, Giakoumakis, Maire, and Thuillier [1995] studied classes of graphs for which the colouring number can be bounded by a function of the clique number.

Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\chi(G) \leq g(\omega(G))$ for each graph G without odd holes. Equivalently, for $\omega \in \mathbb{Z}_+$, let $g(\omega)$ be the maximum colouring number of a graph without odd holes and cliques of size $> \omega$. Then Gyárfás' conjectures that g is finite. It is easy to see that $g(2) = 2$. N. Robertson, P.D. Seymour, and R. Thomas announced that they proved $g(3) = 4$ (this was conjectured by G. Ding).

Upper bounds for the stable set number of a graph in terms of the degrees were presented by Hansen [1979,1980b]. Relations between the colouring number and the fractional colouring number are investigated by Kilakos and Marcotte [1997]. Reed [1998] discussed bounding the chromatic number of a graph by a convex combination of its clique number and its maximum degree plus 1. Gerke and McDiarmid [2001a,2001b] investigated the ratio of the weighted colouring and the weighted clique number.

A theorem of Turán [1941] implies that any simple graph G with n vertices and m edges satisfies:

(64.62) $$\alpha(G) \geq \frac{n^2}{n + 2m}.$$

Bondy [1978] showed that $m \geq 2\tau(G) - 1$ if G is connected. A study of the relations between several parameters derived from stability and colouring was given by Hell and Roberts [1982].

A survey on the stable set problem is given by Padberg [1979], on approximation methods for the stable set problem by Halldórsson [1998], and on colourings by Jensen and Toft [1995] and Toft [1995]. Colouring is also discussed in most graph theory books mentioned in Chapter 1.

Chapter 65

Perfect graphs: general theory

In this and the next two chapters, we consider the 'perfect' graphs, introduced by C. Berge in the 1960s. They turn out to unify several results in combinatorial optimization, in particular, min-max relations and polyhedral characterizations.
Berge proposed two conjectures, the *weak* and the *strong* perfect graph conjecture. The second implies the first.
The weak perfect graph conjecture says that perfection is maintained under taking the complementary graph. This was proved by Lovász [1972c]: the perfect graph theorem.
The strong perfect graph conjecture characterizes perfect graphs by excluding odd holes and odd antiholes. A proof of this was announced in May 2002 by Chudnovsky, Robertson, Seymour, and Thomas, resulting in the strong perfect graph theorem. The announced proof is highly complicated, and we cannot give it here.
Many of the results described in this and the next chapter follow directly as a consequence of the strong perfect graph theorem (while some of them are used in the proof). Where possible and appropriate, we give direct proofs of these consequences.
In this chapter, we give general theory, in Chapter 66 we discuss classes of perfect graphs, and in Chapter 67 we show the polynomial-time solvability of the maximum-weight clique and minimum colouring problems for perfect graphs.

65.1. Introduction to perfect graphs

As we saw before, the clique number $\omega(G)$ and the colouring number $\chi(G)$ of a graph $G = (V, E)$ are related by the inequality

(65.1) $\qquad \omega(G) \leq \chi(G).$

Strict inequality can occur, for instance, for any odd circuit of length at least five, and its complement.

Having equality in (65.1) does not say that much about the internal structure of a graph: any graph $G = (V, E)$ can be extended to a graph $G' = (V', E')$ satisfying $\omega(G') = \chi(G')$, simply by adding to G a clique of size $\chi(G)$, disjoint from V.

However, the condition becomes much more powerful if we require that equality in (65.1) holds for each induced subgraph of G. The idea for this was formulated by Berge [1963a]. He defined a graph $G = (V, E)$ te be *perfect* if $\omega(G') = \chi(G')$ holds for each induced subgraph G' of G.

Various classes of graphs could be shown to be perfect, like the bipartite graphs (trivially) and the line graphs of bipartite graphs (by Kőnig's edge-colouring theorem).

Berge [1960a,1963a] observed the important phenomenon that for several of these classes, also the complementary graphs are perfect. Berge therefore conjectured that the complement of any perfect graph is perfect again — the *weak perfect graph conjecture*. This conjecture was proved by Lovász [1972c], by proving an equivalent form of the conjecture given by Fulkerson [1972a] (the replication lemma — see Corollary 65.2c below).

As mentioned, obvious examples of imperfect graphs are the odd circuits of length at least five, and their complements. Berge [1963a] and P.C. Gilmore (cf. Berge [1966]) made the conjecture that this characterizes perfect graphs, which is the strong perfect graph conjecture. A proof was announced in May 2002 by Chudnovsky, Robertson, Seymour, and Thomas.

To simplify formulation, it is convenient to introduce the notions of 'hole' and 'antihole'. A *hole* in a graph G is an induced subgraph of G isomorphic to a circuit with at least four vertices. An *antihole* in G is an induced subgraph of G isomorphic to the complement of a circuit with at least four vertices. A hole or antihole is *odd* if it has an odd number of vertices.

Theorem 65.1 (Strong perfect graph theorem)**.** *A graph G is perfect if and only if G contains no odd hole and no odd antihole.*

A graph containing no odd hole or odd antihole is called a *Berge graph*[2]. So the strong perfect graph theorem says that Berge graphs are precisely the perfect graphs.

An alternative formulation is in terms of minimally imperfect graphs. A *minimally imperfect* (or *critically imperfect*) graph is an imperfect graph such that each proper induced subgraph is perfect. Then the strong perfect graph theorem says that the only minimally imperfect graphs are the odd circuits of length at least five, and their complements.

It is (as yet) unknown if perfection of a graph can be tested in polynomial time. (Lovász [1986] 'would guess' that such an algorithm exists.) The clique number of a perfect graph can be determined in polynomial time, with the help of the ellipsoid method — see Chapter 67. However, no combinatorial polynomial-time algorithm is known.

We will next discuss perfect graph theory in greater detail (although we cannot give a proof of the strong perfect graph theorem). Let us make a useful observation:

[2] This term was introduced by Chvátal and Sbihi [1987].

(65.2) any minimally imperfect graph $G = (V, E)$ has no stable set S with $\omega(G - S) < \omega(G)$.

Otherwise, $\omega(G) \geq \omega(G - S) + 1 = \chi(G - S) + 1 \geq \chi(G)$, since we can use S as colour.

Similarly, for any class $\mathcal{G}$ of graphs closed under taking induced subgraphs:

(65.3) each graph $G \in \mathcal{G}$ is perfect $\iff$ each graph $G \in \mathcal{G}$ with $VG \neq \emptyset$ has a stable set S with $\omega(G - S) < \omega(G)$.

Here necessity follows from the fact that we can take for S any of the colours in a minimum colouring of G. Sufficiency follows by induction on $|VG|$: $\chi(G) \leq \chi(G - S) + 1 = \omega(G - S) + 1 \leq \omega(G)$.

65.2. The perfect graph theorem

Lovász [1972a] proved the weak perfect graph conjecture in the following stronger form (suggested by A. Hajnal), which we show with the elegant linear-algebraic proof found by Gasparian [1996].

Theorem 65.2. *A graph G is perfect if and only if $\omega(G')\alpha(G') \geq |VG'|$ for each induced subgraph G' of G.*

Proof. Necessity is easy, since if G is perfect, then $\omega(G') = \chi(G')$ for each induced subgraph G' of G, and since $\chi(G')\alpha(G') \geq |VG'|$ for any graph G'.

To see sufficiency, it suffices to show that each minimally imperfect graph G satisfies $|VG| \geq \alpha(G)\omega(G) + 1$. We can assume that $VG = \{1, \ldots, n\}$. Define $\omega := \omega(G)$ and $\alpha := \alpha(G)$.

We first construct

(65.4) stable sets $S_0, \ldots, S_{\alpha\omega}$ such that each vertex is covered by exactly α of the S_i.

Let S_0 be a stable set in G of size α. By the minimality of G, we know that for each $v \in S_0$, the graph $G - v$ is perfect, and that hence $\chi(G - v) = \omega(G - v) \leq \omega$. Therefore, $V \setminus \{v\}$ can be partitioned into ω stable sets. Doing this for each $v \in S_0$, we obtain stable sets as in (65.4).

Now for each $i = 0, \ldots, \alpha\omega$, there exists a clique C_i of size ω with $C_i \cap S_i = \emptyset$ (by (65.2)). Then, for distinct i, j with $0 \leq i, j \leq \alpha\omega$, we have $|C_i \cap S_j| = 1$. This follows from the fact that C_i has size ω and intersects each S_j in at most one vertex, and hence, by (65.4), it intersects $\alpha\omega$ of the S_j. As $C_i \cap S_i = \emptyset$, we have that $|C_i \cap S_j| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega+1) \times n$ incidence matrices M and N of $S_0, \ldots, S_{\alpha\omega}$ and $C_0, \ldots, C_{\alpha\omega}$ respectively. So M and N are $\{0, 1\}$ matrices, with $M_{i,j} = 1 \iff j \in S_i$, and $N_{i,j} = 1 \iff j \in C_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^{\mathsf{T}} = J - I$, where J is the $(\alpha\omega+1) \times (\alpha\omega+1)$ all-1 matrix, and I the $(\alpha\omega+1)\times(\alpha\omega+1)$ identity matrix. As $J-I$ has rank $\alpha\omega+1$, we have $n \geq \alpha\omega+1$. ∎

Theorem 65.2 implies (Lovász [1972c]):

Corollary 65.2a (perfect graph theorem)**.** *The complement of a perfect graph is perfect again.*

Proof. Directly from Theorem 65.2, as the condition given in it is invariant under taking the complementary graph. ∎

As was observed by Cameron [1982], Theorem 65.2 implies that the question 'Given a graph, is it perfect?' belongs to co-NP. Indeed, to certify imperfection of a graph, it is sufficient, and possible, to specify:

(65.5) (i) an induced subgraph $G=(V,E)$,
(ii) integers $\alpha,\omega \geq 2$ with $|V|=\alpha\omega+1$, and
(iii) for each $v \in V$, an ω-colouring of $G-v$ and an α-colouring of $\overline{G}-v$.

This is possible, since, by Theorem 65.2, a minimally imperfect subgraph G has these properties for $\omega := \omega(G)$ and $\alpha := \alpha(G)$. It is also sufficient to certify imperfection, since (65.5)(iii) implies that $\omega(G) \leq \omega$ and $\alpha(G) \leq \alpha$, and hence by (65.5)(ii), that G is not perfect.

Theorem 65.2 implies:

Corollary 65.2b. *Each minimally imperfect graph G satisfies*

(65.6) $$|VG| = \alpha(G)\omega(G)+1.$$

Proof. We have $|VG| \leq \alpha(G)\omega(G)+1$, since for any vertex v of G, the graph $G-v$ is perfect, and hence

(65.7) $$|VG|-1 = |V(G-v)| \leq \alpha(G-v)\omega(G-v) \leq \alpha(G)\omega(G).$$

Conversely, $|VG| \geq \alpha(G)\omega(G)+1$, since if $|VG| \leq \alpha(G)\omega(G)$, then $|VG'| \leq \alpha(G')\omega(G')$ for each induced subgraph G' of G (by the minimal imperfection of G). This implies with Theorem 65.2 that G is perfect, a contradiction. ∎

65.3. Replication

Let $G=(V,E)$ be a graph and let $v \in V$. Extend G with some new vertex, v' say, which is adjacent to v and to all vertices adjacent to v in G. In this way we obtain a new graph H, which we say is obtained from G by *duplicating* v. Repeated duplicating a vertex is called *replicating*. *Replicating* a vertex v *by a factor* k means duplicating v $k-1$ times if $k \geq 1$, and deleting v if $k=0$.

Corollary 65.2c (replication lemma)**.** *Let H arise from G by duplicating vertex v. Then if G is perfect, also H is perfect.*

Proof. By the perfect graph theorem, it suffices to show that $\overline{H}$ is perfect, and hence (as we can apply induction) that $\omega(\overline{H}) = \chi(\overline{H})$.

By the perfect graph theorem, if G is perfect, then $\overline{G}$ is perfect. Hence $\omega(\overline{H}) = \omega(\overline{G}) = \chi(\overline{G}) = \chi(\overline{H})$. ∎

Repeated application of Corollary 65.2c implies the following (the weighted colouring number is defined in Section 64.8):

Corollary 65.2d. *Let G be a perfect graph and let $w : V \to \mathbb{Z}_+$ be a 'weight' function. Then the maximum weight of a clique is equal to the weighted colouring number $\chi_w(G)$ of G.*

Proof. Let G^w be the graph arising from G by replicating any vertex v by a factor $w(v)$. By Corollary 65.2c, G^w is perfect, and so $\omega(G^w) = \chi(G^w)$. Since $\omega(G^w)$ is equal to the maximum weight of a clique in G and since $\chi(G^w) = \chi_w(G)$, the corollary follows. ∎

65.4. Perfect graphs and polyhedra

The *clique polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the cliques. Clearly, any vector x in the clique polytope satisfies:

(65.8) (i) $x_v \geq 0$ for each $v \in V$,
(ii) $x(S) \leq 1$ for each stable set S.

Fulkerson [1972a] and Chvátal [1975a] showed that Corollary 65.2d implies a polyhedral characterization of perfect graphs:

Corollary 65.2e. *A graph G is perfect if and only if its clique polytope is determined by* (65.8).

Proof. *Necessity.* Let G be perfect. To prove that the clique polytope is determined by (65.8), it suffices to show that for each weight function $w : V \to \mathbb{Z}_+$, the maximum weight t of a clique in G is not less than the maximum of $w^{\mathsf{T}}x$ over (65.8). By Corollary 65.2d, there exist stable sets $S_1, \ldots, S_t$ with

(65.9) $$w = \chi^{S_1} + \cdots + \chi^{S_t}.$$

Hence for each x satisfying (65.8) we have

(65.10) $$w^{\mathsf{T}}x = x(S_1) + \cdots + x(S_t) \leq t.$$

Sufficiency. Let the clique polytope of G be determined by (65.8). Suppose that G is not perfect. Choose a minimal set U with $\omega(G[U]) < \chi(G[U])$. Let

$w := \chi^U$. The function $w^{\mathsf{T}}x$ is maximized over $P_{\text{clique}}(G)$ by the incidence vector of each maximum-size clique of $G[U]$. Moreover, by linear programming duality, there exists a stable set S with $x(S) = 1$ for each optimum solution x. So S intersects each maximum-size clique of $G[U]$, and hence

(65.11) $$\omega(G[U \setminus S]) \leq \omega(G[U]) - 1 < \chi(G[U]) - 1 \leq \chi(G[U \setminus S]),$$

contradicting the minimality of U. ∎

Corollary 65.2e is equivalent to: G is perfect if and only if $P_{\text{clique}}(G) = A(P_{\text{stable set}}(G))$. (Here $A(P)$ is the antiblocking polyhedron of P.) Hence it implies the perfect graph theorem (using the theory of antiblocking polyhedra (cf. Section 5.9)):

(65.12) $$\begin{aligned} & G \text{ is perfect } \iff P_{\text{clique}}(G) = A(P_{\text{stable set}}(G)) \\ & \iff P_{\text{stable set}}(G) = A(P_{\text{clique}}(G)) \\ & \iff P_{\text{clique}}(\overline{G}) = A(P_{\text{stable set}}(\overline{G})) \iff \overline{G} \text{ is perfect.} \end{aligned}$$

Corollary 65.2d also implies that perfect graphs can be characterized by total dual integrality:

Corollary 65.2f. *A graph G is perfect if and only if system* (65.8) *is totally dual integral.*

Proof. Directly from Corollaries 65.2d and 65.2e. ∎

So for any graph G we have that (65.8) determines an integer polytope if *and only if* it is totally dual integral.

65.4a. Lovász's proof of the replication lemma

The proof of Lovász [1972c] of the weak perfect graph theorem is based on proving the 'replication lemma' (Corollary 65.2c above), as follows.

By (65.2), it suffices to find a stable set S in H intersecting all maximum-size cliques of H, since any induced subgraph of H is an induced subgraph of G or arises from it by replication.

Consider an $\omega(G)$-colouring of G, and let S be the colour containing v. Then S intersects each maximum-size clique C of H. Indeed, if $v' \notin C$, then C is a maximum-size clique of G, and so it intersects S. If $v' \in C$, then also $v \in C$ (as $C \cup \{v\}$ is a clique), and so C intersects S.

This proves the replication lemma, which by repeated application gives Corollary 65.2d. Since the proof of Corollary 65.2e given above only uses Corollary 65.2d, this shows (with (65.12)) that the replication lemma implies the perfect graph theorem. This is Fulkerson's proof of the equivalence of the replication lemma and the weak perfect graph conjecture ($\equiv$ perfect graph theorem).

65.5. Decomposition of Berge graphs

The proof of the strong perfect graph conjecture is based on a decomposition theorem of Berge graphs, stating that each Berge graph can be decomposed into 'basic' graphs: bipartite graphs and their complements, and line graphs of bipartite graphs and their complements. We formulate the decomposition rules.

Let $G = (V, E)$ be a graph. A *2-join* of G is a partition of V into sets V_1 and V_2 such that for $i = 1, 2$, $|V_i| \geq 3$ and V_i contains disjoint nonempty subsets A_i, B_i with the property that for all $v_1 \in V_1$ and $v_2 \in V_2$:

(65.13) v_1 and v_2 are adjacent $\iff$ $v_1 \in A_1$, $v_2 \in A_2$, or $v_1 \in B_1$, $v_2 \in B_2$.

A *skew partition* of G is a partition V_1, V_2 of V such that $G[V_1]$ and $\overline{G}[V_2]$ are disconnected. An *homogeneous pair* of G is a pair A, B of disjoint subsets of V such that $3 \leq |A| + |B| \leq |V| - 2$ and such that for all $x, y \in A \cup B$ and $z \in V \setminus (A \cup B)$, if $xz \in E$ and $yz \notin E$, then x and y belong to distinct sets A, B.

M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced in May 2002 that they proved the following[3]:

Theorem 65.3. *Let G be a Berge graph. Then G or $\overline{G}$ is bipartite or the line graph of a bipartite graph, or has a 2-join, a skew partition, or a homogeneous pair.*

Unfortunately, we cannot give the (highly complicated) proof of this theorem. M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas also showed that any minimum-size imperfect Berge graph has no skew partition[4]. Since such a graph has no 2-join (Cornuéjols and Cunningham [1985] and Kapoor [1994] — see Corollary 65.7a below) and no homogeneous pair (Chvátal and Sbihi [1987]), and since bipartite graphs and their line graphs are perfect (Kőnig [1916] — see Section 66.1), this implies:

Theorem 65.4 (strong perfect graph theorem)**.** *A graph is perfect if and only if it is a Berge graph.*

65.5a. 0- and 1-joins

A *0-join* of a graph $G = (V, E)$ is a partition of V into nonempty sets V_1 and V_2 such that no edge connects V_1 and V_2. Let $G_1 := G[V_1]$ and $G_2 = G[V_2]$. Then G is called the *0-join* of G_1 and G_2. Trivially:

---

[3] This was conjectured by M. Conforti, G. Cornuéjols, N. Robertson, P.D. Seymour, R. Thomas, and K. Vušković (cf. Cornuéjols [2002]). It builds on work of Roussel and Rubio [2001], and it was stimulated by interaction with concurrent work of Conforti, Cornuéjols, Vušković, and Zambelli [2002] and Conforti, Cornuéjols, and Zambelli [2002b].

[4] conjectured by Chvátal [1985c].

Theorem 65.5. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. This follows from the facts that $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$ and $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$, and that induced subgraphs of G arise by the same construction from induced subgraphs of G_1 and G_2. ∎

Hence

(65.14) no minimally imperfect graph has a 0-join.

A 1-*join* (or *join*) of a graph $G = (V, E)$ is a partition of V into subsets V_1 and V_2 such that $|V_1| \geq 2$, $|V_2| \geq 2$, and such that there exist nonempty $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ with the property that for all $v_1 \in V_1$ and $v_2 \in V_2$:

(65.15) v_1 and v_2 are adjacent $\iff$ $v_1 \in A_1$ and $v_2 \in A_2$.

Choose $v_1 \in A_1$ and $v_2 \in A_2$, and define $G_1 := G[V_1 \cup \{v_2\}]$ and $G_2 := G[V_2 \cup \{v_1\}]$. Then G is called the 1-*join* of G_1 and G_2.

Bixby [1984] proved (generalizing a result of Lovász [1972c]):

Theorem 65.6. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. Necessity follows from the fact that G_1 and G_2 are induced subgraphs of G. To prove sufficiency, it suffices to show $\omega(G) = \chi(G)$, since each induced subgraph of G arises by the same construction, or by a 0-join from induced subgraphs of G_1 and G_2. Let $\omega := \omega(G)$ and $a_i := \omega(G[A_i])$ for $i = 1, 2$. It suffices to show that for each $i = 1, 2$,

(65.16) $G[V_i]$ has an ω-colouring such that A_i uses a_i colours only,

since then we can assume that we use different colours for A_1 and A_2 (as $a_1 + a_2 \leq \omega$), yielding an ω-colouring of G.

To prove (65.16), we may assume that $i = 1$. Let G_1' be the graph obtained from G_1 by replicating v_2 by a factor $\omega - a_1$. So $\omega(G_1') = \omega$. By the replication lemma, G_1' is perfect. Hence $\omega(G_1') = \chi(G_1')$. As the clique of vertices obtained from v_2 has size $\omega - a_1$, we use only a_1 colours for A_1, as required. ∎

An alternative proof follows from Cunningham [1982b]. Cunningham [1982a] described an $O(n^3)$-time algorithm to find a 1-join (if any).

Theorem 65.6 implies:

(65.17) no minimally imperfect graph has a 1-join,

since it has no 0-join, and hence G_1 and G_2 as above are proper induced subgraphs of G, implying that they are perfect. Therefore, by Theorem 65.6, G is perfect, a contradiction.

65.5b. The 2-join

We next show that a minimally imperfect graph has no 2-join, except if it is an odd circuit. This was shown by Cornuéjols and Cunningham [1985] (for a special case) and Kapoor [1994].

The proof uses the following 'special replication lemma' (Cornuéjols and Cunningham [1985]). Let $e = uv$ be an edge of a graph G. Let G' be the graph obtained from replicating v and deleting edge uv', where v' is the new vertex.

Lemma 65.7α (special replication lemma)**.** *If G is perfect and uv is not contained in a triangle of G, then G' is again perfect.*

Proof. It suffices to show that G' has a stable set S' such that $\omega(G' - S') < \omega(G')$. If $\omega(G') > \omega(G)$, we can take $S' = \{v'\}$. So we may assume $\omega(G') = \omega(G)$. Let S be the colour of an $\omega(G)$-colouring of G with $v \in S$. Let $S' := (S \setminus \{v\}) \cup \{v'\}$. Then S' is a stable set in G'. If $\omega(G' - S') < \omega(G')$ we are done. So assume that $\omega(G'-S') = \omega(G')$. Let C be a clique in $G'-S'$ of size $\omega(G)$. Since $\omega(G-S) < \omega(G)$ and since $G - S = G' - S' - v$, we know $v \in C$. Since $\omega(G') = \omega(G)$, we know $u \in C$. Hence, $C = \{u, v\}$ (since uv is not in a triangle). So $\omega(G') = 2$, and hence vv' is not contained in a triangle of G'. But then v' has degree 1 in G', implying $\chi(G') = \chi(G) = \omega(G) = \omega(G')$. ∎

Next we consider a *special 2-join*, namely where the sets A_i and B_i in the definition of 2-join are connected by a path in $G[V_i]$ (for $i = 1, 2$). For $i = 1, 2$, let P_i be a shortest $A_i - B_i$ path in $G[V_i]$. Define $G_1 := G[V_1 \cup VP_2]$ and $G_2 := G[V_2 \cup VP_1]$.

Theorem 65.7. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. Necessity follows from the fact that G_1 and G_2 are induced subgraphs of G. To prove sufficiency, it is enough to prove $\omega(G) = \chi(G)$, since each induced subgraph of G arises by the same construction, or by 1- or 0-joins, from induced subgraphs of G_1 and G_2. Define $\omega := \omega(G)$, and

(65.18) $\quad a_i := \omega(G[A_i])$ and $b_i := \omega(G[B_i])$ for $i = 1, 2$.

Note that perfection of G_1 implies that $|EP_1| \equiv |EP_2| \pmod 2$, since $VP_1 \cup VP_2$ induces a hole in G_1.

For any colouring ϕ of a graph and any set X of vertices, let $\phi(X)$ denote the set of colours used by X. We show that, for each $i = 1, 2$, $G[V_i]$ has an ω-colouring $\phi : V \to \{1, \ldots, \omega\}$ such that

(65.19)
(i) $\phi(A_i) = \{1, \ldots, a_i\}$;
(ii) if $|EP_i|$ is even, then $\phi(B_i) = \{1, \ldots, b_i\}$;
(iii) if $|EP_i|$ is odd, then $\phi(B_i) = \{\omega - b_i + 1, \ldots, \omega\}$.

This yields an ω-colouring of G, by replacing the colour, i say, of any vertex in V_2 by $\omega - i + 1$. (The correctness follows from $a_1 + a_2 \leq \omega$ and $b_1 + b_2 \leq \omega$.)

To prove the existence of a colouring satisfying (65.19), we may assume $i = 1$. Let $v_0, v_1, \ldots, v_k$ be the vertices (in order) of the $A_2 - B_2$ path P_2.

First assume that $k > 1$ or $a_1 + b_1 \geq \omega$. Let G_1' be the graph arising from G_1 by replicating v_j by a factor

(65.20)
$$\begin{array}{ll} \omega - a_1 & \text{if } j < k-1 \text{ and } j \text{ is even,} \\ a_1 & \text{if } j < k-1 \text{ and } j \text{ is odd,} \\ \min\{\omega - a_1, b_1\} & \text{if } j = k-1 \text{ and } j \text{ is even,} \\ \min\{a_1, b_1\} & \text{if } j = k-1 \text{ and } j \text{ is odd,} \\ \omega - b_1 & \text{if } j = k. \end{array}$$

Then $\omega(G_1') = \omega$, and any ω-colouring of G_1' yields a colouring satisfying (65.19). Indeed, if k is even, then (65.20) implies that the set of colours used by the copies of v_0 and the set of colours used by the copies of v_k are comparable[5]. If k is odd, then (65.20) implies that the set of colours used by the copies of v_0 and the set of colours *not* used by the copies of v_k are comparable.

Next assume that $k = 1$ and $a_1 + b_1 < \omega$. Extend G_1 by a new vertex v', adjacent to all vertices in B_1 and to v_1. By the special replication lemma (Lemma 65.7α), the new graph G_1'' is again perfect. Let G_1' be the graph arising from G_1'' by replicating v_0 by a factor $\omega - a_1$, v_1 by a factor a_1, and v' by a factor $\omega - a_1 - b_1$. Again, $\omega(G_1') = \omega$, and any ω-colouring of G_1' yields a colouring satisfying (65.19). ∎

This implies:

Corollary 65.7a. *Any minimally imperfect graph having a 2-join is an odd circuit.*

Proof. Let G be a minimally imperfect graph, and let V_i, A_i, B_i (for $i = 1, 2$) be as in the definition of 2-join. If for some i, the graph $G[V_i]$ has no $A_i - B_i$ path, then G has a 0- or 1-join, contradicting (65.14) or (65.17). So we can assume that, for $i = 1, 2$, $G[V_i]$ has an $A_i - B_i$ path. Let P_i be a shortest such path.

By Theorem 65.7 and by symmetry, we may assume that $G[V_1 \cup VP_2]$ is not perfect. Hence, by the minimal imperfection of G, $G = G[V_1 \cup VP_2]$.

We first show $\omega(G) = 2$. Choose an internal vertex u on P_2. (This exists, since $|V_2| \geq 3$.) Choose $v \in V \setminus \{u\}$. By the minimal imperfection of G, we know $\chi(\overline{G} - v) = \alpha(G - v)$. Therefore, $VG \setminus \{v\}$ can be partitioned into $\alpha(G - v)$ cliques. Since $|VG| = \alpha(G)\omega(G) + 1$ (by (65.6)), each of these cliques has size $\omega(G)$. In particular, u is in a clique of size $\omega(G)$. Hence, since u is an internal vertex of P_2, $\omega(G) = 2$.

As $\omega(G) = 2$, $\chi(G - v) \leq 2$ for each $v \in VG$; that is, $G - v$ is bipartite for each $v \in VG$. So each odd circuit is Hamiltonian. As G is not bipartite, G has an odd circuit. This circuit has no chords, as otherwise there exists a shorter odd circuit. ∎

Cornuéjols and Cunningham [1985] gave an $O(n^2m^2)$-time algorithm to find a 2-join in a given graph (if any).

65.6. Pre-proof work on the strong perfect graph conjecture

In this section we survey research done on the strong perfect graph conjecture before it was proved in general. Many of the results follow as a consequence of the strong perfect graph theorem. Since the proof of this theorem is very complicated, we will include proofs not based on the strong perfect graph theorem.

[5] Sets X and Y are called *comparable* if $X \subseteq Y$ or $Y \subseteq X$.

65.6a. Partitionable graphs

The strong perfect graph theorem implies that each minimally imperfect graph is a circuit or its complement, and hence is highly symmetric. Before the strong perfect graph theorem was proved, several regularity properties of minimally imperfect graphs were shown, initiated by the work of Padberg [1974a].

A graph $G = (V, E)$ is called *partitionable* if $|V| = \alpha(G)\omega(G)+1$ and $\chi(G-v) = \omega(G)$ and $\overline{\chi}(G - v) = \alpha(G)$ for each $v \in V$. By Corollary 65.2b, each minimally imperfect graph is partitionable. As each partitionable graph is imperfect, the strong perfect graph theorem is equivalent to: each partitionable graph has an odd hole or odd antihole.

Partitionable graphs are characterized as follows[6]. Our proof of necessity is based on Gasparian [1996] (and is similar to the proof of Theorem 65.2).

Theorem 65.8. *A graph G is partitionable if and only if $|VG| = \alpha(G)\omega(G) + 1$ and each vertex is contained in exactly $\alpha(G)$ stable sets of size $\alpha(G)$ and in exactly $\omega(G)$ cliques of size $\omega(G)$.*

Proof. Define $n := |VG|$, $\alpha := \alpha(G)$, and $\omega := \omega(G)$.

I. To see necessity, let G be partitionable. Then the proof method of Theorem 65.2 applies: We again construct

(65.21) stable sets $S_0, \ldots, S_{\alpha\omega}$ such that each vertex is covered by exactly α of the S_i.

Indeed, let S_0 be a stable set in G of size α. For each vertex v, as G is partitionable, we know $\chi(G - v) = \omega$. Therefore, $VG \setminus \{v\}$ can be partitioned into ω stable sets. Doing this for each $v \in S_0$, we obtain stable sets as in (65.21).

Next, for each $i = 0, \ldots, \alpha\omega$, there exists a clique C_i of size ω with $C_i \cap S_i = \emptyset$. To see this, choose $v \in S_i$. As G is partitionable, $\chi(\overline{G}-v) = \alpha$, and hence $VG\setminus\{v\}$ can be partitioned into α cliques. Since $n = \alpha\omega + 1$, each clique has size ω. Since $|S_i \setminus \{v\}| \leq \alpha - 1$, at least one of these cliques is disjoint from S_i.

Then, for distinct i, j with $0 \leq i, j \leq \alpha\omega$, we have $|C_i \cap S_j| = 1$. This follows from the fact that C_i has size ω and intersects each S_j in at most one vertex, and hence, by (65.21), C_i intersects $\alpha\omega$ of the S_j. As $C_i \cap S_i = \emptyset$, we have that $|C_i \cap S_j| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega+1) \times n$ incidence matrices M and N of $S_0, \ldots, S_{\alpha\omega}$ and $C_0, \ldots, C_{\alpha\omega}$ respectively. So M and N are $\{0, 1\}$ matrices, with $M_{i,j} = 1 \iff j \in S_i$, and $N_{i,j} = 1 \iff j \in C_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^{\mathsf{T}} = J - I$, where J is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ all-1 matrix, and I the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ identity matrix. So M and N are nonsingular.

It then suffices (by symmetry) to show that each maximum-size clique C occurs among $C_0, \ldots, C_n$. Now $(M\chi^C)_i$ is 1 if $|C \cap S_i| = 1$, and is 0 otherwise. As $|C| = \omega$ and as each $v \in V$ belongs to exactly α of the S_i, C intersects precisely $\alpha\omega$ of the S_i. That is, there is exactly one, say S_j, disjoint from C. Hence $M\chi^C = M\chi^{C_j}$, and therefore $C = C_j$, as M is nonsingular.

[6] Necessity of the condition for minimally imperfect graphs was shown by Padberg [1974a], and for partitionable graphs in general by Bland, Huang, and Trotter [1979]. As to sufficiency, Cameron [1982] referred to private communication with A. Lubiw in 1981, and Whitesides [1982] called it ‘well known’.

II. To see sufficiency, let G satisfy the condition. As each vertex of G is in exactly α stable sets of size α, there are exactly n maximum-size stable sets. Similarly, there are exactly n maximum-size cliques.

Let M and N be the incidence matrix of the maximum-size stable sets and maximum-size cliques, respectively. We can order the rows such that $MN^{\mathsf{T}} = J - I$, where J is the all-one $n \times n$-matrix, and I the identity matrix of order n. To see this, each maximum-size stable set S intersects precisely $\alpha\omega$ maximum-size cliques, since $|S| = \alpha$ and each vertex $v \in S$ is in precisely ω maximum-size cliques. Hence there is a unique maximum-size clique C disjoint from S. Similarly, for each maximum-size clique C there is a unique maximum-size stable set S disjoint from C.

So $MN^{\mathsf{T}} = J - I$, implying

(65.22)
$$\begin{aligned} M(J-I)N^{\mathsf{T}} &= MJN^{\mathsf{T}} - MN^{\mathsf{T}} = \alpha JN^{\mathsf{T}} - (J-I) = \alpha\omega J - J + I \\ &= nJ - 2J + I = (J-I)(J-I) = MN^{\mathsf{T}}MN^{\mathsf{T}}. \end{aligned}$$

Since M and N^{T} are nonsingular, this implies $N^{\mathsf{T}}M = J - I$.

Now choose $v \in V$. As $N^{\mathsf{T}}M = J - I$, for each $u \in V \setminus \{v\}$ there exists a unique pair of a maximum-size clique C_u and a maximum-size stable set S_u with $u \in C_u$, $v \in S_u$, and $C_u \cap S_u = \emptyset$. Then for each $w \in C_u$ we have $C_w = C_u$, since $w \in C_u$ and $v \in S_u$. So the C_u partition $V \setminus \{v\}$, and hence $\chi(\overline{G} - v) = \alpha$. Then also $\chi(G - v) = \omega$ by symmetry. ∎

A partitionable graph G with $\alpha(G) = \alpha$ and $\omega(G) = \omega$, is also called an (α, ω)-*graph.*

The proof of Theorem 65.8 also implies the following further properties of partitionable graphs (properties (i)-(iii) were proved for minimally imperfect graphs by Padberg [1974a] and for partitionable graphs by Bland, Huang, and Trotter [1979]; property (iv) was shown by Whitesides [1982]):

Theorem 65.9. *Let G be a partitionable graph with n vertices. Then:*

(65.23)
(i) *G contains exactly n maximum-size cliques and exactly n maximum-size stable sets;*
(ii) *the matrix N formed by the incidence vectors of the maximum-size cliques is nonsingular, and the matrix M formed by the incidence vectors of the maximum-size stable sets is nonsingular;*
(iii) *each maximum-size clique intersects all but one maximum-size stable sets, and each maximum-size stable set intersects all but one maximum-size cliques;*
(iv) *for any two distinct vertices u, v of G there is a unique pair of a maximum-size clique C and a maximum-size stable set S with $u \in C$, $v \in S$, and $C \cap S = \emptyset$.*

Proof. See the proof of Theorem 65.8. ∎

Notes. One may show that $|\det M| = \alpha(G)$ and $|\det N| = \omega(G)$ for any partitionable graph. Indeed, since $M\mathbf{1} = \alpha(G) \cdot \mathbf{1}$, we have that $M^{-1}\mathbf{1} = \alpha(G)^{-1} \cdot \mathbf{1}$. Hence $\alpha(G)$ divides $\det M$ (as $(\det M) \cdot M^{-1}$ is an integer matrix). Similarly, $\omega(G)$ divides $\det N$. Now $|\det M \cdot \det N| = |\det(MN^{\mathsf{T}})| = |\det(J - I)| = |VG| - 1 = \alpha(G)\omega(G)$. So $|\det M| = \alpha(G)$ and $|\det N| = \omega(G)$.

Shepherd [1994b] showed that a graph G is partitionable if and only if for some $p, q \geq 2$ with $|VG| = pq + 1$: (i) G has a family of $|VG|$ stable sets of size p such that each vertex is in precisely p of them, and (ii) G has no stable set S of size p that intersects each clique of size q. A polynomial-time recognition algorithm of partitionable graphs was given by Shepherd [2001].

65.6b. More characterizations of perfect graphs

It is not difficult to show that for any partitionable graph G one has:

(65.24) $$\chi^*(G) = \omega(G) + \frac{1}{\alpha(G)}.$$

Indeed, let $n := |VG|$, $\alpha := \alpha(G)$, $\omega := \omega(G)$, $\chi^* := \chi^*(G)$. To see $\geq$, observe that the vector $\alpha^{-1} \cdot \mathbf{1}$ satisfies all stable set inequalities (64.34), and hence $\chi^* \geq n\alpha^{-1} = \omega + \alpha^{-1}$. To see $\leq$, give each stable set of size α a value α^{-1}. This gives a fractional colouring of size $\omega + \alpha^{-1}$. So $\chi^* \leq \omega + \alpha^{-1}$, proving (65.24).

Hence perfect graphs can be characterized by:

Theorem 65.10. *A graph G is perfect $\iff$ $\chi^*(G')$ is an integer for each induced subgraph G' of G.*

Proof. See above. ∎

Berge [1973a] gave the following further characterization of perfect graphs. For any graph $G = (V, E)$, let $\chi_2(G)$ denote the *bicolouring number* of G, being the minimum number of stable sets $S_1, \ldots, S_t$ such that each vertex is in two of the S_i. Alternatively, it is the minimum number of colours such that we can assign to each vertex a pair of colours in such a way that any two adjacent vertices get two disjoint pairs of colours.

Theorem 65.11. *A graph G is perfect if and only if $\chi_2(G') = 2\chi(G')$ for each induced subgraph G' of G.*

Proof. To see necessity, we have $2\omega(G) \leq \chi_2(G) \leq 2\chi(G)$ for each graph G. Hence if G is perfect, then $\omega(G) = \chi(G)$, and hence $\chi_2(G) = 2\chi(G)$. As perfection is closed under taking induced subgraphs, necessity of the condition follows.

To see sufficiency, let G be a minimally imperfect graph. Consider two nonadjacent vertices u and v. Then $\chi_2(G) \leq \chi(G - u) + \chi(G - v) + 1$ (as we can take $\{u, v\}$ as a colour). Since, by the condition, $\chi_2(G) = 2\chi(G)$, we can assume, by symmetry, that $\chi(G) \leq \chi(G - u)$. Hence $\chi(G) \leq \chi(G - u) \leq \omega(G - u) \leq \omega(G)$, contradicting the fact that G is minimally imperfect. ∎

(This proof does not use the perfect graph theorem.)

65.6c. The stable set polytope of minimally imperfect graphs

The following theorem of Padberg [1976] is a direct consequence of the strong perfect graph conjecture, but we give a direct proof (we adapt the proof of Seymour [1990b]):

Theorem 65.12. *Let $G = (V, E)$ be a minimally imperfect graph. Then the polytope determined by*

(65.25) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(C) \leq 1$ for each clique C,

has precisely one noninteger vertex, namely $\omega^{-1} \cdot \mathbf{1}$, where $\omega := \omega(G)$.

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let x^* be a noninteger vertex of the polytope determined by (65.25). Set $n := |V|$.

First we have that

(65.26) $x^*_v > 0$ for each vertex v.

For suppose that $x^*_v = 0$. Then $x^*|V \setminus \{v\}$ is a noninteger vertex of the polytope (65.25) for $G - v$, contradicting the perfection of $G - v$ (by Corollary 65.2e). This proves (65.26).

Let $\mathcal{C}$ be a collection of cliques C such that $x^*(C) = 1$ for each $C \in \mathcal{C}$ and such that $\{\chi^C \mid C \in \mathcal{C}\}$ is a set of n linearly independent vectors. For $v \in V$, let $\mathcal{C}_v$ denote the collection of $C \in \mathcal{C}$ with $v \in C$. Then:

(65.27) $|\mathcal{C}_v| \leq \omega$ for each $v \in V$.

To see this, consider any $v \in V$ and any $C \in \mathcal{C} \setminus \mathcal{C}_v$. Since $G - v$ is perfect, the vector $x^*|V \setminus \{v\}$ is a convex combination $\sum_S \lambda_S \chi^S$ of stable sets S in $G - v$. For each $u \in C$, choose a stable set S_u with $u \in S_u$ and $\lambda_{S_u} > 0$. Then $|C' \cap S_u| = 1$ for each $C' \in \mathcal{C} \setminus \mathcal{C}_v$ (since $(x^*|V \setminus \{v\})(C') = 1$). So the incidence vectors χ^{S_u} for $u \in C$ are linearly independent. This implies that the vectors $\chi^{S_u} - x^*$ for $u \in C$ have rank at least $|C| - 1$. As each of these vectors is orthogonal to $\chi^{C'}$ for each $C' \in \mathcal{C} \setminus \mathcal{C}_v$, we have

(65.28) $|\mathcal{C} \setminus \mathcal{C}_v| \leq (n - 1) - (|C| - 1) = n - |C|$.

Let U be the set of vertices not covered by all cliques in $\mathcal{C}$. Then:

(65.29)
$$\begin{aligned} n &= \sum_{C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} \sum_{v \in V \setminus C} \frac{1}{n - |C|} = \sum_{v \in U} \sum_{C \in \mathcal{C} \setminus \mathcal{C}_v} \frac{1}{n - |C|} \\ &\leq \sum_{v \in U} \sum_{C \in \mathcal{C} \setminus \mathcal{C}_v} \frac{1}{|\mathcal{C} \setminus \mathcal{C}_v|} = \sum_{v \in U} 1 = |U| \leq n. \end{aligned}$$

So we have equality throughout; that is, $U = V$ and $|\mathcal{C} \setminus \mathcal{C}_v| = |V| - |C|$ for each $v \in V$ and each $C \in \mathcal{C} \setminus \mathcal{C}_v$. This gives equality in (65.28). So $|\mathcal{C}_v| = |C| \leq \omega$, proving (65.27).

Let $\mathcal{C}'$ denote the collection of maximum-size cliques in G. By Theorem 65.9, each $v \in V$ is in precisely ω sets in $\mathcal{C}'$. Hence

(65.30)
$$\begin{aligned} n &= \sum_{C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} x^*(C) = \sum_{v \in V} |\mathcal{C}_v| x^*_v \leq \omega \sum_{v \in V} x^*_v = \sum_{C \in \mathcal{C}'} x^*(C) \\ &\leq \sum_{C \in \mathcal{C}'} 1 = n. \end{aligned}$$

Hence we have equality throughout. Therefore, x^* satisfies $x^*(C) = 1$ for each maximum-size clique C. Hence $x^* = \omega^{-1} \cdot \mathbf{1}$. ∎

By the antiblocking relation, Theorem 65.12 implies that the stable set polytope of a minimally imperfect graph has precisely one facet not given by the clique and nonnegative constraints:

Corollary 65.12a. *Let $G = (V, E)$ be a minimally imperfect graph. Then the stable set polytope of G is determined by:*

(65.31)
$$\begin{aligned} &x_v \geq 0 &&\text{for } v \in V,\\ &x(C) \leq 1 &&\text{for each clique } C,\\ &x(V) \leq \alpha(G). && \end{aligned}$$

Proof. Directly from Theorem 65.12 applied to the antiblocking polytope of the polytope determined by (65.25) (for $\overline{G}$). ∎

Shepherd [1990,1994b] calls a graph *near-perfect* if its stable set polytope is determined by (65.31), and he showed that a graph G is minimally imperfect if and only if both G and its complement $\overline{G}$ are near-perfect.

65.6d. Graph classes

Before a proof of the strong perfect graph theorem in general was announced in 2002, it had been proved for several classes of graphs. Next to the classes to be discussed in Chapter 66, it was shown for (among other):

- *claw-free graphs*, that is, graphs not having $K_{1,3}$ (a *claw*) as induced subgraph (Parthasarathy and Ravindra [1976] (cf. Tucker [1979] and Maffray and Reed [1999], and Giles and Trotter [1981] for a simpler proof)).
  It follows that the line graph $L(G)$ of a graph G is perfect if and only if G contains no odd circuit with at least five vertices as (not necessarily induced) subgraph. So these graphs have edge-colouring number $\chi'(G)$ equal to the maximum degree $\Delta(G)$ (if $\Delta(G) \geq 3$); moreover, the matching number $\nu(G)$ is equal to the minimum number of stars and triangles needed to cover the edges of G; this extends Kőnig's edge-colouring and matching theorems (cf. Trotter [1977] and de Werra [1978]).
  Sbihi [1978,1980] and Minty [1980] showed that the weighted stable set problem is solvable in strongly polynomial time for claw-free graphs (see Chapter 69). A combinatorial polynomial-time algorithm for the colouring problem for claw-free perfect graphs was given by Hsu [1981], and for the weighted clique and clique cover problems by Hsu and Nemhauser [1981,1982,1984].
  Chvátal and Sbihi [1988] gave a polynomial-time algorithm to recognize claw-free perfect graphs, based on decomposition (cf. Maffray and Reed [1999]). Koch [1979] gave a polynomial-time algorithm which for any claw-free graph either finds a maximum-size stable set and a minimum-size clique cover of equal cardinalities, or else finds an odd hole or odd antihole.
  Perfection of line graphs was also studied by Cao and Nemhauser [1998]. The validity of the strong perfect graph conjecture for claw-free graphs was extended to 'pan-free' graphs by Olariu [1989b].
- *K_4-free graphs* — graphs not having K_4 as subgraph (Tucker [1977b], cf. Tucker [1979,1987a]).

- *diamond-free graphs* (Tucker [1987b][7]) — graphs not having $K_4 - e$ (a *diamond*) as induced subgraph (where $K_4 - e$ is the graph obtained from K_4 by deleting an edge) (Conforti [1989] gave an alternative proof). Tucker [1987b] gave an $O(n^2m)$-time algorithm to colour such graphs optimally. Fonlupt and Zemirline [1987] and Conforti and Rao [1993] gave polynomial-time perfection tests for diamond-free graphs. Related results can be found in Conforti and Rao [1989, 1992a,1992b] and Fonlupt and Zemirline [1992,1993].
- *paw-free graphs* — graphs not having a *paw* (a K_4 with two incident edges deleted) as induced subgraph. This follows from the perfection of Meyniel graphs (Theorem 66.6). It also follows from a characterization of Olariu [1988e] of paw-free graphs.
- *square-free graphs* (Conforti, Cornuéjols, and Vušković [2002]) — graphs not having a C_4 (a *square*) as induced subgraph.
- *bull-free graphs* (Chvátal and Sbihi [1987]) — graphs not having a *bull* as induced subgraph, where a bull is the (self-complementary) graph on five vertices a, b, c, d, e and edges ab, ac, bc, bd, ce (see Figure 65.1). Reed and Sbihi [1995] gave a polynomial-time perfection test for bull-free graphs. More on bull-free graphs can be found in de Figueiredo [1995], de Figueiredo, Maffray, and Porto [1997,2001], and Hayward [2001].
- *chair-free graphs* (Sassano [1997]) — graphs not having a *chair* as induced subgraph, where a chair is the graph on five vertices a, b, c, d, e and edges ab, bc, cd, be (see Figure 65.1).
- *dart-free graphs* (Sun [1991]) — graphs not having a *dart* as induced subgraph (a dart is a graph with vertices a, b, c, d, e and edges ab, ac, ad, ae, bc, cd (see Figure 65.1)); Chvátal, Fonlupt, Sun, and Zemirline [2000,2002] gave a polynomial-time recognition algorithm for dart-free perfect graphs.
- graphs having neither P_5 nor K_5 as induced subgraph (Maffray and Preissmann [1994], Barré and Fouquet [1999]).
- *circular-arc graphs* (Tucker [1975]) — these are the intersection graphs of families of intervals on a circle (cf. Section 64.9a).
- *circle graphs* (Buckingham and Golumbic [1984]) — these are the intersection graphs of families of chords of a circle (cf. Section 64.9a).
- planar graphs (Tucker [1973b]). Tucker [1984b] showed that this can be derived (without appealing to the four-colour theorem) from the validity of the strong perfect graph conjecture for K_4-free graphs: a K_4 subgraph in a planar graph $G \neq K_4$ contains a triangle that is a vertex-cut of G; hence one can apply induction to find a 4-colouring of G.
  Tucker and Wilson [1984] gave an $O(n^2)$ algorithm for finding a minimum colouring of a planar perfect graph, Hsu [1987b] gave an $O(n^3)$-time perfection test for planar graphs, and Hsu [1988] described strongly polynomial-time algorithms for the maximum-weight stable set, the weighted colouring, and the weighted clique cover problems for planar perfect graphs.
- graphs embeddable in the torus or in the Klein bottle (Grinstead [1980,1981]).
- *checked graphs* (Gurvich and Temkin [1992]) — graphs whose vertex set is a subset of $\mathbb{R}^2$, two vertices being adjacent if and only the line segment connecting them is horizontal or vertical.

[7] An partial proof was given by Parthasarathy and Ravindra [1979], cf. Tucker [1987b].

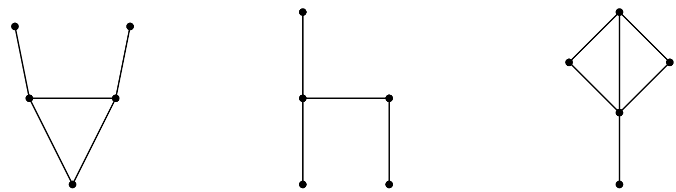

Figure 65.1

The perfect graph theorem implies that the strong perfect graph conjecture is true also for the classes of graphs complementary to those listed above.

Since C_k and $\overline{C}_k$ (for odd $k \geq 5$) are claw-free, the result of Parthasarathy and Ravindra [1976] implies that, to show the strong perfect graph theorem, it suffices to show that each minimally imperfect graph is claw-free.

Other classes of graphs for which the strong perfect graph conjecture holds were found by Rao and Ravindra [1977], Olariu [1988d], Jamison and Olariu [1989b], Carducci [1992], Galeana-Sánchez [1993], Lê [1993b], De Simone and Galluccio [1994], Maire [1994b], Maffray and Preissmann [1995], Xue [1995,1996], Ait Haddadene and Gravier [1996], Maffray, Porto, and Preissmann [1996], Aït Haddadène and Maffray [1997], Kroon, Sen, Deng, and Roy [1997], Babaĭtsev [1998], Hoàng and Le [2000b, 2001], and Gerber and Hertz [2001].

65.6e. The P_4-structure of a graph and a semi-strong perfect graph theorem

V. Chvátal noticed that the collection of 4-sets inducing the 4-vertex path P_4 as a subgraph, provides a useful tool in studying perfection. (Note that $\overline{P_4}$ is isomorphic to P_4.) It led Chvátal [1984a] to conjecture the following 'semi-strong perfect graph theorem', which was proved by Reed [1987] (announced in Reed [1985]).

Call two graphs G and H, with $VG = VH$, *P_4-equivalent* if for each $U \subseteq V$ one has: U induces a P_4-subgraph of G if and only if U induces a P_4-subgraph of H. Then Reed's theorem states that

(65.32) if G and H are P_4-equivalent and G is perfect, then H is perfect.

This theorem implies the perfect graph theorem, since G and $\overline{G}$ are P_4-equivalent. On the other hand, the theorem is implied by the strong perfect graph theorem, since any graph P_4-equivalent to an odd circuit of length at least 5 is equal to that circuit or to its complement.

Other relations between the P_4-structure and perfection were proved by Chvátal and Hoang [1985] and Chvátal, Lenhart, and Sbihi [1990]. Let $G = (V, E)$ be a graph and let V be partitioned into classes V_0 and V_1, with both $G[V_0]$ and $G[V_1]$ perfect. For each word $x = x_1x_2x_3x_4$ of length 4 over the alphabet $\{0, 1\}$, let $\mathcal{Q}_x$ denote the set of chordless paths in G with vertices v_1, v_2, v_3, v_4 (in order) with $v_i \in V_{x_i}$ for $i = 1, 2, 3, 4$. Then G is perfect if:

(65.33) (i) $\mathcal{Q}_{1000} = \mathcal{Q}_{0100} = \mathcal{Q}_{0111} = \mathcal{Q}_{1011} = \emptyset$, or

(ii) $\mathcal{Q}_{0000} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(iii) $\mathcal{Q}_{1000} = \mathcal{Q}_{0100} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(iv) $\mathcal{Q}_{1000} = \mathcal{Q}_{0101} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(v) $\mathcal{Q}_{1000} = \mathcal{Q}_{0101} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{0111} = \emptyset$, or
(vi) $\mathcal{Q}_{1000} = \mathcal{Q}_{1001} = \mathcal{Q}_{1011} = \emptyset$.

Sufficiency of (i) was shown by Chvátal and Hoang [1985], and of the other cases by Chvátal, Lenhart, and Sbihi [1990] ((ii) also by Gurvich [1993a,1993b]), who also showed that (65.33) essentially covers all cases where perfection of G follows from perfection of its constituents and the 'colouring' of the P_4-subgraphs. In fact, (65.33) and its symmetrical cases (interchanging V_0 and V_1 and/or replacing G by $\overline{G}$) describe exactly the cases excluded by G or $\overline{G}$ being an odd chordless circuit of length ≥ 5 or its complement.

A theorem of Seinsche [1974] states that each graph without an induced P_4 subgraph is perfect. (This follows from the perfection of Meyniel graphs (Theorem 66.6).[8])

Hence, case (65.33)(ii) implies the result of Hoang [1985] that any graph is perfect if there is a set U of vertices having an odd intersection with each chordless path with 4 vertices. More generally, it implies perfection of any graph if there is a set U of vertices such that each induced P_4 subgraph has exactly one of its two middle vertices in U or has exactly one of its ends in U.

Related (and more general than the results of Chvátal and Hoang quoted above) is the following theorem of Chvátal [1987a]. Let $G = (V, E)$ be a graph and let V be partitioned into two classes X and Y such that there are no $x \in X$, $y \in Y$, and $U \subseteq V$ such that both $U \cup \{x\}$ and $U \cup \{y\}$ induce a P_4 subgraph. Then G is perfect if and only if $G[X]$ and $G[Y]$ are perfect.

More work on the P_4-structure related to perfection is reported in Jung [1978], Jamison and Olariu [1989c,1992a,1992b,1995a,1995b], Hayward and Lenhart [1990], Hoàng [1990,1995,1999], Olariu [1991], Ding [1994], Rusu [1995a,1999b], Giakoumakis [1996], Hoàng, Hougardy, and Maffray [1996], Hougardy [1996b,1997,1999, 2001], Babel and Olariu [1997,1998,1999], Giakoumakis, Roussel, and Thuillier [1997], Giakoumakis and Vanherpe [1997], Hougardy, Le, and Wagler [1997], Babel [1998a,1998b], Babel, Brandstädt, and Le [1999], Brandstädt and Le [1999,2000], Roussel, Rusu, and Thuillier [1999], Brandstädt, Le, and Olariu [2000], Hoàng and Le [2000a,2001], Barré [2001], and Hayward, Hougardy, and Reed [2002].

65.6f. Further notes on the strong perfect graph conjecture

Markosyan and Karapetyan [1984] showed that the strong perfect graph conjecture is equivalent to: each minimally imperfect graph G is regular of degree $2\omega(G) - 2$.

For $k, n \in \mathbb{Z}_+$, let $C_{n,k}$ be the graph obtained from the circuit C_n (with n vertices) by adding all edges connecting two vertices at distance less than k. If $n \equiv 1 \pmod{k+1}$ and $n \geq 2k + 3$, then $C_{n,k}$ is partitionable. Chvátal [1976] showed that the strong perfect graph conjecture is equivalent to: each minimally imperfect graph G has $C_{|VG|,\omega(G)}$ as *spanning* subgraph (not necessarily induced).

[8] Arditti and de Werra [1976] claimed that Seinsche's result also follows from the 'fact' that any graph without induced P_4 subgraph is the comparability graph of a branching, therewith overlooking C_4.

Bland, Huang, and Trotter [1979] and Chvátal, Graham, Perold, and Whitesides [1979] gave examples of partitionable graphs G not containing $C_{|VG|,\omega(G)}$ as a spanning subgraph. Sebő [1996a] and Bacsó, Boros, Gurvich, Maffray, and Preissmann [1998] showed that these constructions give no counterexamples to the strong perfect graph conjecture. Related results are given by Chvátal [1984b]. A computer search for partitionable graphs was reported by Lam, Swiercz, Thiel, and Regener [1980].

Call a pair of vertices u, v in a graph an *even pair* if each induced $u - v$ path has even length. Meyniel [1987] showed that a minimally imperfect graph has no even pair. (This was extended to partitionable graphs by Bertschi and Reed [1987].) Hougardy [1996a] proved that the strong perfect graph conjecture is equivalent to: each properly induced subgraph of a minimally imperfect graph has an even pair or is a clique. Bienstock [1991] showed that it is NP-complete to test if a graph has no even pair. More on even pairs can be found in Hoàng and Maffray [1989], Bertschi [1990], Hougardy [1995], Rusu [1995c], Everett, de Figueiredo, Linhares-Sales, Maffray, Porto, and Reed [1997] (survey), Linhares Sales, Maffray, and Reed [1997], Linhares Sales and Maffray [1998], de Figueiredo, Gimbel, Mello, and Szwarcfiter [1999], Rusu [2000], and Everett, de Figueiredo, Linhares Sales, Maffray, Porto, and Reed [2001] (survey).

Prömel and Steger [1992] showed that 'almost all Berge graphs are perfect': the ratio of the number of n-vertex perfect graphs and the number of n-vertex Berge graphs, tends to 1 if $n \to \infty$.

The role of uniquely colourable perfect graphs for the strong perfect graph conjecture was investigated by Tucker [1983b]. Bacsó [1997] studied the conjecture that a uniquely colourable perfect graph G is either a clique or contains two maximum-size cliques intersecting each other in $\omega(G) - 1$ vertices. This is implied by the strong perfect graph theorem. Related work was given by Sakuma [2000].

Corneil [1986] investigated families of graphs 'complete' for the strong perfect graph conjecture (that is, proving the conjecture for these graphs suffices to prove the conjecture in general).

Equivalent versions of the strong perfect graph conjecture were given by Olaru [1972,1973b], Ravindra [1975], Markosyan [1981], Markosyan and Gasparyan [1987], Olariu [1990b], Huang [1991], Markosian, Gasparian, and Markosian [1992], Galeana-Sánchez [1993], De Simone and Galluccio [1994], Lonc and Zaremba [1995], and Padberg [2001].

Giles, Trotter, and Tucker [1984], Hsu [1984], Fonlupt and Sebő [1990], Croitoru and Radu [1992b], Sebő [1992], Panda and Mohanty [1997], and Rusu [1997] gave further techniques for proving the strong perfect graph conjecture.

Several other properties of minimally imperfect and partitionable graphs were derived by Olaru [1969,1972,1973a,1973b,1977,1980,1993,1998], Sachs [1970], Padberg [1974a,1974b,1975,1976], Parthasarathy and Ravindra [1976], Tucker [1977b, 1983a], Olaru and Suciu [1979], Markosyan [1981,1985], Sridharan and George [1982], Whitesides [1982], Buckingham and Golumbic [1983], Chvátal [1984c,1985c], Grinstead [1984], Olaru and Sachs [1984], Chvátal and Sbihi [1987], Meyniel [1987], Olariu [1988b,1988c,1990a,1991], Meyniel and Olariu [1989], Preissmann [1990], Sebő [1992,1996a,1996b], Cornuéjols and Reed [1993], Hougardy [1993], Maffray [1993], Olariu and Stewart [1993], Hayward [1995], Hoàng [1996c], Perz and Zaremba [1996], Fouquet, Maire, Rusu, and Thuillier [1997], Gasparyan [1998], Barré and Fouquet [1999,2001], Croitoru [1999], de Figueiredo, Klein, Kohayakawa, and Reed [2000], Barré [2001], Roussel and Rubio [2001], and Conforti, Cornuéjols, Gasparyan, and Vušković [2002]. Surveys were given by Preissmann and Sebő [2001] and Cornuéjols [2002].

65.7. Further results and notes

65.7a. Perz and Rolewicz's proof of the perfect graph theorem

An interesting proof of the perfect graph theorem was given by Perz and Rolewicz [1990]. It does not use the replication lemma, and is based on linear algebra, in a manner different from the proof of Gasparian given in Section 65.2, namely on the value of determinants.

In fact, Perz and Rolewicz [1990] show (in a different but equivalent terminology) that a graph $G = (V, E)$ is perfect if and only if $P_{\text{stable set}}(G)$ and $P_{\text{clique}}(G)$ form an antiblocking pair of polytopes. They prove sufficiency in a way similar to the proof of Fulkerson given for sufficiency in Corollary 65.2e above.

They proved necessity as follows. Choose a counterexample with $|V|$ minimal. So G is perfect, and $P_{\text{stable set}}(G)$ and $P_{\text{clique}}(G)$ do not form an antiblocking pair. Hence there exist $x \in A(P_{\text{stable set}}(G))$ and $y \in A(P_{\text{clique}}(G))$ with $x^{\mathsf{T}}y > 1$. Choose such x, y with $x^{\mathsf{T}}y$ maximal. Let $\nu := x^{\mathsf{T}}y$.

We first show

(65.34) $$\nu \leq \frac{n}{n-1},$$

where $n := |V|$. Indeed, for each $u \in V$, deleting the uth component of x and y, we obtain vectors in $A(P_{\text{stable set}}(G - u))$ and $A(P_{\text{clique}}(G - u))$, respectively. By the minimality of G, we have $\sum_{v \neq u} x_v y_v \leq 1$. Hence

(65.35) $$\nu = \sum_v x_v y_v = \frac{1}{n-1} \sum_u \Big(\sum_{v \neq u} x_v y_v \Big) \leq \frac{n}{n-1}.$$

This proves (65.34). By the minimality of G, we also have $x_v > 0$ and $y_v > 0$ for each v.

Now $\nu^{-1} \cdot x \in P_{\text{clique}}(G)$, since otherwise there is a $z \in A(P_{\text{clique}}(G))$ with $\nu^{-1}x^{\mathsf{T}}z > 1$, contradicting the maximality of $x^{\mathsf{T}}y$. So there exist cliques $C_1, \ldots, C_n$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

(65.36) $$x = \sum_{i=1}^{n} \lambda_i \chi^{C_i} \text{ and } \sum_{i=1}^{n} \lambda_i = \nu.$$

Similarly, there exist stable sets $S_1, \ldots, S_n$ and $\mu_1, \ldots, \mu_n > 0$ such that

(65.37) $$y = \sum_{j=1}^{n} \mu_j \chi^{S_j} \text{ and } \sum_{j=1}^{n} \mu_j = \nu.$$

Then $y(C_i) = 1$ for $i = 1, \ldots, n$, since $y(C_i) \leq 1$ (as $y \in A(P_{\text{clique}}(G))$), and

(65.38) $$\nu = x^{\mathsf{T}}y = \sum_{i=1}^{n} \lambda_i y(C_i) \leq \sum_{i=1}^{n} \lambda_i = \nu.$$

Similarly, $x(S_j) = 1$ for $j = 1, \ldots, n$.

Consider, for some $i = 1, \ldots, n$,

(65.39) $$1 = y(C_i) = \sum_{j=1}^{n} \mu_j |C_i \cap S_j| \leq \sum_{j=1}^{n} \mu_j = \nu.$$

So the inequality is strict, and hence there is at least one j with $C_i \cap S_j = \emptyset$. Then

(65.40) $$1 = x(S_j) = \sum_{i'} \lambda_{i'} |C_{i'} \cap S_j| = \sum_{i' \neq i} \lambda_{i'} |C_{i'} \cap S_j| \leq \sum_{i' \neq i} \lambda_{i'} = \nu - \lambda_i.$$

Hence with (65.34),

(65.41) $$n \leq \sum_{i} (\nu - \lambda_i) = n\nu - \nu \leq n,$$

implying equality in (65.40) for each i. So if $C_i \cap S_j = \emptyset$, then $C_{i'} \cap S_j \neq \emptyset$ for each $i' \neq i$. Hence for each j, there is exactly one i with $C_i \cap S_j = \emptyset$, and conversely. We can assume that $C_i \cap S_j = \emptyset$ if and only if $i = j$.

Let M and N be the incidence matrices of $S_1, \ldots, S_n$ and of $C_1, \ldots, C_n$ respectively. So $MN^{\mathsf{T}} = J - I$. Hence $|\det M \det N| = |\det(J - I)| = n - 1$. Since $y(S_i) = 1$ for each i, we have $My = \mathbf{1}$. So $y' := |\det M| \cdot y$ is a positive integer vector. Similarly, $x' := |\det N| \cdot x$ is a positive integer vector. Then

(65.42) $$(x')^{\mathsf{T}} y' = |\det M \det N| x^{\mathsf{T}} y = |\det(J - I)| \nu = n.$$

The kernel of the argument now is that this implies that x' and y' are the all-one vectors, and therefore x and y each are scalar multiples of the all-one vector.

As $x \in A(P_{\text{stable set}}(G))$, $x(S) \leq 1$ for any stable set S, and hence $x = \alpha'^{-1} \cdot \mathbf{1}$ for some $\alpha' \geq \alpha(G)$. Similarly, $y = \omega'^{-1} \cdot \mathbf{1}$ for some $\omega' \geq \omega(G)$. As G is perfect, $\alpha'\omega' \geq \alpha(G)\omega(G) \geq n$. Hence $\nu = x^{\mathsf{T}} y = (\alpha'\omega')^{-1} n \leq 1$, a contradiction.

65.7b. Kernel solvability

The following generalization of the Gale-Shapley theorem on stable matchings was conjectured by Berge and Duchet [1986,1988a][9] and proved by Boros and Gurvich [1996], using a technique from game theory due to Scarf [1967]. With the strong perfect graph theorem it characterizes perfect graph by being kernel solvable.

Call a graph $G = (V, E)$ *kernel solvable* if the following holds: if for each clique C of G we have a total order $<_C$ of C, then there exists a stable set S such that for each $v \in V$ there is an $s \in S$ and a clique C such that $v, s \in C$ and $v \leq_C s$. Berge and Duchet conjectured that kernel solvable graphs are precisely the perfect graphs. With Theorem 65.14 below, this conjecture is implied by the strong perfect graph theorem.

Kernel solvability can be formulated equivalently in terms of kernels of digraphs. A *kernel* of a directed graph $D = (V, A)$ is a subset S of V such that S spans no arc of D and such that for each $v \in V \setminus S$ there is a $u \in S$ with $(v, u) \in A$.

For any graph $G = (V, E)$, a directed graph $D = (V, A)$ is called a *superorientation* of G if $E = \{\{u, v\} \mid (u, v) \in A\}$. (So $\{u, v\}$ is an edge of $G \iff$ at least

[9] Berge and Duchet [1986] refer to 'Séminaire du Lundi, MSH, Paris, Janvier 1983' (Monday Seminar, MSH, Paris, January 1983). See Jensen and Toft [1995] p. 140 for further references to the history of this conjecture.

one of (u, v) and (v, u) belongs to A.) Then a graph $G = (V, E)$ is kernel solvable if and only if any superorientation D of G has a kernel if each clique C of G induces a subgraph of D with a kernel.

Kernel solvability is closed under taking induced subgraphs: if $U \subseteq V$, and each clique C of $G[U]$ has a total order $<_C$, we can choose for each clique C of G a total order which coincides with $<_{C\cap U}$ on $C \cap U$ and which has $C \cap U$ as upper ideal.

Since neither C_k nor $\overline{C}_k$ is kernel solvable for odd $k \geq 5$, the strong perfect graph theorem implies that each kernel solvable graph is perfect.

Boros and Gurvich [1996] proved that a graph G is perfect if and only if each graph H arising from G by replicating vertices is kernel solvable. It implies that the strong perfect graph theorem is equivalent to: each Berge graph is kernel solvable (since the class of Berge graphs is closed under replicating vertices).

To show that each perfect graph is kernel solvable, we follow the proof method of Aharoni and Holzman [1998]. We first prove the following results of Scarf [1967].

Let M and N be disjoint finite nonempty sets, and for each $i \in M$ let $<_i$ be a total order of N. For any U, write $y <_i U$ if $y <_i u$ for each $u \in U$. Define $K := M \cup N$.

Call a subset L of K *light* if for each $j \in N$ there is an $i \in M\setminus L$ with $j \leq_i L\setminus M$. So any subset of a light set is light again. Let $m := |M|$ and define

(65.43) $$\mathcal{S} := \{M\} \cup \{L \mid L \text{ light}, |L| = m\}.$$

Note that M is not light.

Now Scarf first proved:

Lemma 65.13α. *Any light set L with $|L| = m - 1$ is contained in precisely two sets in $\mathcal{S}$.*

Proof. Extend each $<_i$ to a total order on K, with $i <_i j <_i i'$ for all $j \in N$ and all $i' \in M \setminus \{i\}$. Then

(65.44) any subset L of K is light if and only if for each $k \in K$ there is an $i \in M$ with $k \leq_i L$.

To see necessity in (65.44), let $L \subseteq K$ be light and let $k \in K$. If $k \in M$, then $k \leq_k L$. If $k \in N$, then there is an $i \in M \setminus L$ with $k \leq_i L \setminus M$. As $i \notin L$, we have also $k \leq_i L \cap M$ (since $k \leq_i M \setminus \{i\}$). So $k \leq_i L$.

To see sufficiency in (65.44), suppose $\forall k \in K \exists i \in M : k \leq_i L$. Let $j \in N$. Then $\exists i \in M : j \leq_i L$. Then $i \notin L$ (as otherwise $j \leq_i i$). Moreover, $j \leq_i L \setminus M$. This proves (65.44).

For any $i \in M$ and any nonempty $U \subseteq K$, let $\min_i U$ and $\max_i U$ denote the minimal and maximal element of U with respect to $<_i$.

First assume that $L \subseteq M$, say $L = M \setminus \{i\}$. Then L is contained in M, which belongs to $\mathcal{S}$. Moreover, $z := \max_i N$ is the unique element with $L \cup \{z\}$ light.[10] This proves the lemma.

So henceforth we can assume that $L \not\subseteq M$. Define $\pi : M \to L$ by $\pi(i) := \min_i L$. Then π is onto, since, as L is light, for each $r \in L$ there is an $i \in M$ with $r \leq_i L$. So $r = \min_i L = \pi(i)$.

[10] For let $x \in N$. Then $L \cup \{x\}$ is light $\iff$ $\forall j \in N \exists i \in M \setminus L : j \leq_i (L \cup \{x\}) \setminus M$ $\iff$ $\forall j \in N : j \leq_i x$ $\iff$ $x = \max_i N$.

Hence, as $|L| = |M| - 1$, there exist distinct $i_1, i_2 \in M$ with $\pi(i_1) = \pi(i_2)$, while all other values of π are mutually distinct and different from $\pi(i_1)$.

For $h = 1, 2$, define

(65.45) $$R_h := \{k \in K \mid k \not\leq_i L \text{ for all } i \neq i_h\}.$$

Then $R_h \neq \emptyset$, since $i_h \in R_h$, as there is an $r \in L \setminus M$ (as $L \not\subseteq M$), hence if $i \neq i_h$, then $i_h \not\leq_i r$. Also, $R_h \cap L = \emptyset$, since if $r \in L$, there is an $i \neq i_h$ with $\pi(i) = r$, so $\min_i L = r$, implying $r \leq_i L$, and hence $r \notin R_h$. Moreover, $R_1 \cap R_2 = \emptyset$, since otherwise there is a $k \in K$ with $k \not\leq_i L$ for all $i \in M$, contradicting the fact that L is light.

Define for $h = 1, 2$:

(65.46) $$r_h := \max_{i_h} R_h.$$

We first show that $L \cup \{r_1\}$ and $L \cup \{r_2\}$ are light. Suppose that (say) $L \cup \{r_1\}$ is not light. So there is a $k \in K$ with $k \not\leq_i L \cup \{r_1\}$ for each $i \in M$. Since $r_1 \not\leq_i L$ for each $i \neq i_1$ (by definition of R_1), it follows that $k \not\leq_i L$ for each $i \neq i_1$. Hence $k \in R_1$. However, $r_1 <_{i_1} L$, since $r_1 \in R_1$ and $r_1 \leq_i L$ for some $i \in M$. So $k \leq_{i_1} r_1 <_{i_1} L$, and therefore $k \leq_{i_1} L \cup \{r_1\}$, a contradiction. So $L \cup \{r_1\}$ and $L \cup \{r_2\}$ are light.

Finally we show that for any $s \in K \setminus L$, if $L \cup \{s\}$ is light, then $s = r_1$ or $s = r_2$. So let $L \cup \{s\}$ be light. Then the function $\pi' : M \to L \cup \{s\}$ defined by $\pi'(i) := \min_i(L \cup \{s\})$ is onto (as $L \cup \{s\}$ is light), implying that it is one-to-one (as $|M| = |L \cup \{s\}|$). Hence π' coincides with π on all but one element of M. Necessarily the exceptional element belongs to $\{i_1, i_2\}$. Say $\pi'(i) = \pi(i)$ for each $i \neq i_1$, while $\pi'(i_1) = s$. So $\min_i L = \pi(i) = \pi'(i) <_i s$ for each $i \neq i_1$; that is, $s \in R_1$. Suppose $s \neq r_1$. So $s <_{i_1} r_1$. Then $r_1 \not\leq_i L \cup \{s\}$ for each $i \in M$, contradicting the fact that $L \cup \{s\}$ is light. ∎

From this, Scarf derived:

Theorem 65.13 (Scarf's lemma). *Let A be a nonnegative $m \times n$ matrix and let $b \in \mathbb{R}^m_+$ be such that the polytope $P := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is nonempty and bounded. For each $i = 1, \ldots, m$, let $<_i$ be a total order on $\{1, \ldots, n\}$. Then P has a vertex x such that*

(65.47) *for each $j \in \{1, \ldots, n\}$ there is an $i \in \{1, \ldots, m\}$ such that $a_i^{\mathsf{T}} x = b_i$ and such that $x_k = 0$ for each $k <_i j$.*

Proof. We can assume, by slightly perturbing b, that for each vertex x of P there are precisely n constraints among $x \geq \mathbf{0}$, $Ax \leq b$ satisfied with equality. Add n to each index i of $<_i$. (So $x <_{n+i} y$ in the new notation $\iff$ $x <_i y$ in the old notation.) Let $N := \{1, \ldots, n\}$, $M := \{n+1, \ldots, n+m\}$, and $K := N \cup M$. For each face f of P define

(65.48) $K_f := \{k \in K \mid$ the kth constraint in $x \geq \mathbf{0}$, $Ax \leq b$ is not tight at some point in $f\}$.

So $|K_v| = m$ for any vertex v and $|K_e| = m + 1$ for any edge e. Call an edge e of P *good* if $1 \in K_e$ and the set $K_e \setminus \{1\}$ belongs to $\mathcal{S}$ (cf. (65.43)).

Now $\mathbf{0}$ is incident with precisely one good edge. Hence there is a vertex $v \neq \mathbf{0}$ incident with an odd number of good edges. We show that $K_v \in \mathcal{S}$, and hence K_v is light (since $K_v \neq M$, as $v \neq \mathbf{0}$), implying that v satisfies (65.47).

Let e be a good edge incident with v. Then $K_e = K_v \cup \{k\}$ for some k. As e is good, we know that $1 \in K_e$ and $K_e \setminus \{1\} \in \mathcal{S}$.

If $1 \notin K_v$, then $k = 1$ and hence $K_v \in \mathcal{S}$. So we can assume that $1 \in K_v$. Applying Lemma 65.13α to the light set $K_v \setminus \{1\}$, there is precisely one $j \notin K_v \setminus \{1\}$ with $j \neq k$ and $K_v \setminus \{1\} \cup \{j\} \in \mathcal{S}$. If $j \neq 1$, then v is incident with precisely two good edges, a contradiction. So $j = 1$, and hence $K_v \in \mathcal{S}$. ∎

This implies the theorem of Boros and Gurvich [1996]:

Corollary 65.13a. *A perfect graph is kernel solvable.*

Proof. Let $G = (V, E)$ be a perfect graph, and for each clique C, let $<_C$ be a total order on C. We must prove that

(65.49) there exist a stable set S such that for each $v \in V$ there is a clique C and an element $s \in C \cap S$ with $v \in C$ and $v \leq_C s$.

Extend each $<_C$ to a total order on V with $w <_C v$ for each $w \in C$, $v \in V \setminus C$. Then by Theorem 65.13, the polytope in $\mathbb{R}^V$ determined by $x \geq \mathbf{0}$, $x(C) \leq 1$ (C clique), has a vertex x such that for each $v \in V$ there is a clique C with $x(C) = 1$ and such that $x_u = 0$ for each $u <_C v$. By Corollary 65.2e, x is the incidence vector of some stable set S. So for each $v \in V$ there is a clique C with $|C \cap S| = 1$ and with $u \notin S$ if $u <_C v$. Therefore, for the vertex s in $C \cap S$ we have $v \leq_C s$, and hence $v \in C$. This shows (65.49). ∎

It was conjectured by Berge and Duchet that conversely, each kernel solvable graph is perfect. This follows from the strong perfect graph theorem, since kernel solvability is closed under taking induced subgraphs and since odd circuits of length at least five and their complements are not kernel solvable.

It implies the following theorem found by Boros and Gurvich [1996], for which we give a direct proof. A graph H is called a *blow-up* of a graph G, if H arises from G by replicating vertices (replacing vertices by cliques).

Theorem 65.14. *A graph G is perfect if and only if each blow-up of G is kernel solvable.*

Proof. Since each blow-up of a perfect graph is perfect again (by the replication lemma (Corollary 65.2c)), necessity follows from Corollary 65.13a.

Sufficiency is shown by proving that each graph $G = (V, E)$ with $|V| \geq \alpha(G)\omega(G) + 1$ has a blow-up that is not kernel solvable. (This is sufficient by Theorem 65.2.)

Let $\mathcal{C}$ be the collection of cliques in G, and for each vertex v, let $\mathcal{C}_v$ be the collection of cliques containing v. Let $n := |V|$ and define

(65.50) $Y := \{y : \mathcal{C} \to \mathbb{Z}_+ \mid y(\mathcal{C}) \leq n|\mathcal{C}|\}.$

For each $y \in Y$, we choose a vertex v_y of G with

(65.51) $y(\mathcal{C}_{v_y}) \leq \omega(G)|\mathcal{C}|.$

This is possible since

(65.52) $$\sum_{v\in V} y(\mathcal{C}_v) = \sum_{C\in\mathcal{C}} |C| y_C \leq \omega(G) \sum_{C\in\mathcal{C}} y_C \leq \omega(G) n |\mathcal{C}|.$$

Let H be the graph with vertex set Y, two distinct vertices $y, z \in Y$ being adjacent if $v_y = v_z$ or v_y and v_z are adjacent in G. So H is a blow-up of G. We show that H is not kernel solvable.

For each clique $K \subseteq Y$ of H, the set $C := \{v_y \mid y \in K\}$ is a clique of G. Then choose a total order $<_K$ on K such that for all $y, z \in K$:

(65.53) if $y_C < z_C$, then $y <_K z$.

Assume that H is kernel solvable. Then H has a stable set $Z \subseteq Y$ such that for each $y \in Y$ there is a $z \in Z$ and a clique K of H with $y, z \in K$ and $y \leq_K z$. As Z is stable in H, the v_z for $z \in Z$ are distinct and form a stable set S in G. So for each clique C of G there is at most one $z \in Z$ with $v_z \in C$. Define $y : \mathcal{C} \to \mathbb{Z}_+$ by:

(65.54) $$y_C := \begin{cases} z_C + 1 & \text{if } v_z \in C, \text{ for } z \in Z, \\ 0 & \text{if } C \cap S = \emptyset. \end{cases}$$

Then y belongs to Y, since

(65.55) $$\begin{aligned} y(\mathcal{C}) &= \sum_{z\in Z} \sum_{C\in\mathcal{C}_{v_z}} (z_C + 1) = \sum_{z\in Z} (z(\mathcal{C}_{v_z}) + |\mathcal{C}_{v_z}|) \leq |Z|\omega(G)|\mathcal{C}| + |\mathcal{C}| \\ &\leq (\alpha(G)\omega(G) + 1)|\mathcal{C}| \leq n|\mathcal{C}|. \end{aligned}$$

(The first inequality follows from (65.51).) Hence there exist a $z \in Z$ and a clique K of H with $y, z \in K$ and $y \leq_K z$. So for $C := \{v_x \mid x \in K\}$ we have, by (65.53), $y_C \leq z_C$. Since $v_z \in C$ (as $z \in K$), this contradicts (65.54). ∎

Before the strong perfect graph conjecture was settled, and hence the conjecture of Berge and Duchet, partial and related results on the latter conjecture were obtained by Blidia [1986], Maffray [1986,1992], Duchet [1987], Berge and Duchet [1988b,1990], Champetier [1989], Blidia and Engel [1992], Blidia, Duchet, and Maffray [1993,1994], Chilakamarri and Hamburger [1993], and Galeana-Sánchez [1995, 1996,1997].

65.7c. The amalgam

A composition generalizing the 1-join, the *amalgam*, was shown to preserve perfection by Burlet and Fonlupt [1984]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be perfect graphs such that $K := V_1 \cap V_2$ is a clique in both graphs. For $i = 1, 2$, let $v_i \in V_i \setminus K$ be such that each vertex in K is adjacent to v_i and to each neighbour of v_i. Let H be the graph on $(V_1 \setminus \{v_1\}) \cup (V_2 \setminus \{v_2\})$ obtained from the union of $G_1 - v_1$ and $G_2 - v_2$ by adding all edges between $N(v_1) \setminus K$ and $N(v_2) \setminus K$.

Theorem 65.15. *If G_1 and G_2 are perfect, then H is perfect.*

Proof. It suffices to show that $\omega(H) = \chi(H)$, since each induced subgraph of H arises by the same construction.

For $i = 1, 2$, let $p_i := \omega(G_i[N(v_i)])$ and let G_i' be the graph obtained from G_i by replicating v_i by a factor $\omega(H) - p_i$. So $\omega(G_i') = \omega(H)$. By the replication lemma, G_i' is perfect. Hence $\omega(H) = \chi(G_i')$.

Consider colourings of G_1' and G_2' with colours $1, \ldots, \omega(H)$. So $N(v_i)$ uses precisely p_i colours. As $p_1 + p_2 - |K| \leq \omega(H)$, we have $(\omega(H) - p_1) + (\omega(H) - p_2) \geq \omega(H) - |K|$. Hence we can assume that in G_1 and G_2 the colourings of K are the same, and that all colours are used by the replication vertices of v_1 and v_2 and by K. Then $N(v_1) \setminus K$ and $N(v_2) \setminus K$ have no colours in common. Hence we obtain an $\omega(H)$-colouring of H. ∎

Cornuéjols and Cunningham [1985] gave an $O(n^2m)$-time algorithm to decide if a graph is the amalgam of smaller graphs.

Perfection is trivially closed under 'clique sums', that is, identifying two cliques in two graphs. Whitesides [1981] gave an $O(nm)$ algorithm to find a clique cut in a graph, that is, a vertex-cut that is a clique. Tarjan [1985] gave an $O(nm)$-time algorithm to find for any graph a decomposition by clique cuts.

Fonlupt and Uhry [1982] gave conditions under which identification of two vertices in a graph maintains perfection. Ravindra and Parthasarathy [1977], Ravindra [1978], Mândrescu [1991], and Kwaśnik and Szelecka [1997] investigated the behaviour of perfection under taking (various) products of graphs.

More on the (de)composition of perfect graphs can be found in Hsu [1986,1987a], Conforti and Rao [1992a,1992b], Corneil and Fonlupt [1993], Burlet and Fonlupt [1994], and Conforti, Cornuéjols, Kapoor, and Vušković [1995].

65.7d. Diperfect graphs

Berge [1982a] introduced a directed variant of perfect graphs. In fact, there are two symmetric variants, as no complementary phenomenon holds in the directed case.

A *stable set* or *clique* in a directed graph is a stable set of clique in the underlying undirected graph. A directed graph $D = (V, A)$ is called *α-diperfect* if for every induced subgraph $D' = (V', A')$ of D and for each maximum-size stable set S in D' there is a partition of V' into directed paths each intersecting S in exactly one vertex.

Then:

(65.56) if the underlying undirected graph G of D is perfect, then D is α-diperfect.

Indeed, if G is perfect, there is a maximum-size stable set S and a partition of V into cliques each intersecting S. Each clique C gives a tournament on C in D, and hence, by Rédei's theorem (Corollary 14.14a), it contains a directed path spanning C.

Another class of α-diperfect digraphs is formed by the *symmetric* digraphs: directed graphs $D = (V, A)$ such that if $(u, v) \in A$, then $(v, u) \in A$:

(65.57) each symmetric digraph is α-diperfect.

To see this, let S be a maximum-size stable set in D, and let D' arise from D by deleting all arcs entering S. By the Gallai-Milgram theorem (Theorem 14.14), V can be partitioned into $|S|$ directed paths in D'. These paths are as required.

Berge offered the following conjecture characterizing α-diperfect digraphs:

(65.58) (?) A directed graph $D = (V, A)$ is α-diperfect if and only if D has no induced subgraph C whose underlying undirected graph is a chordless odd circuit of length ≥ 5, say with vertices $v_1, \ldots, v_{2k+1}$ (in order) such that each of $v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k}$ is a source or a sink. (?)

The odd circuit described is not α-diperfect, since $\{v_1, v_4, v_6, v_8, \ldots, v_{2k}\}$ is a maximum-size stable set, but there are no directed paths as required.

A 'dual' concept is that of a χ-*diperfect* graph, which is a digraph $D = (V, A)$ such that for each induced subgraph $D' = (V', A')$ of D and for each minimum vertex-colouring (in the underlying undirected graph of D') there exists a directed path intersecting each colour exactly once.

Again one has:

(65.59) if the underlying undirected graph G of D is perfect, then D is χ-diperfect.

Indeed, any maximum-size clique C intersects each colour in each minimum vertex-colouring, and, again by Rédei's theorem (Corollary 14.14a), there is a path spanning C.

Also:

(65.60) any symmetric digraph is χ-diperfect.

To see this, let $S_1, \ldots, S_k$ be an optimum vertex-colouring. Let D' be the graph obtained from D by deleting all arcs from S_j to S_i for all $j > i$. By the theorem of Gallai and Roy (see (64.52)), D' has a directed path of length k. Necessarily, it intersects each S_i exactly once.

One may show that the odd undirected circuit described in (65.58) is not χ-diperfect. So conjecture (65.58) would imply that each χ-diperfect digraph is α-diperfect.

In fact, any odd undirected circuit that contains three consecutive vertices v_1, v_2, v_3 that are sources or sinks, is not χ-diperfect (since there is an optimum 3-vertex-colouring where $\{v_2\}$ is one of the colours — hence v_2 should belong to a directed path with 3 vertices). In particular, the undirected circuit with vertices $v_1, \ldots, v_7$ and arcs

(65.61) $(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_1, v_7)$

is α-diperfect but not χ-diperfect (cf. Figure 65.2).

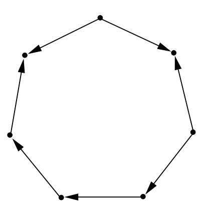

Figure 65.2

65.7e. Further notes

Cameron, Edmonds, and Lovász [1986] showed that if the edges of a complete graph are coloured with three colours such that no triangle gets three different colours, and two of these colours form perfect graphs, then so does the third. (This generalizes the perfect graph theorem.) A generalization and a related characterization of perfection in terms of decomposition was given by Cameron and Edmonds [1997].

Markosyan and Karapetyan [1976] characterize perfection with the help of *critical edges* (edges e with $\alpha(G-e) > \alpha(G)$) and *essential edges* (edges e with $\chi(G-e) > \chi(G)$). More on such edges can be found in Markosyan [1975], Karapetyan [1976], Sebő [1996a], and Markossian, Gasparian, Karapetian, and Markosian [1998]. Edge-minimal perfect graphs are studied by Wagler [1999], colouring perfect 'degenerate' graphs by Aït Haddadène and Maffray [1997], 'Gallai graphs' and 'anti-Gallai graphs' by Le [1993a,1996b], and 'edge perfect graphs' by Müller [1996].

An alternative polyhedral characterization of perfection of graphs was given by Zaremba and Perz [1982]. Related is the work of Zaremba [1991] and Hujter [1999].

Chandrasekaran and Tamir [1984] and Cook, Fonlupt, and Schrijver [1986] showed that, for any perfect graph $G = (V, E)$ and any weight $w : V \to \mathbb{Z}_+$, the weighted colouring number is attained by a weighted colouring using at most $|V|$ different stable sets.

Von Rimscha [1983] showed that if $G = (V, E)$ and $H = (V, F)$ are graphs with $G - v$ isomorphic to $H - v$ for each $v \in V$, then G is perfect if and only if H is perfect.

Bienstock [1991] showed that it is NP-complete to decide if a given graph has an odd hole containing a prescribed vertex. More on the complexity of finding odd holes can be found in Reed [1990]. A survey on forbidding holes and antiholes was given by Hayward and Reed [2001].

Le [1996a] showed that if a graph G is imperfect and has no odd hole, then the intersection graph of the edge sets of chordless circuits in G has an odd hole. Akiyama and Chvátal [1990] characterized for which graphs $G = (V, E)$ the intersection graph of the triples spanning at least two edges, is perfect. Olaru and Mândrescu [1992] considered perfection of products of graphs, and de Werra and Hertz [1999] perfection of sums of graphs. Hertz [1998] characterized the graphs for which all graphs obtained by 'switching' are perfect.

Variants of the notion of perfect graph were studied by Kőrner [1973], Duchet [1980], Galeana-Sánchez [1982,1986,1988], Duchet and Meyniel [1983], Galeana-Sánchez and Neumann-Lara [1986,1991a,1991b,1994,1996,1998], Lehel and Tuza [1986], Conforti, Corneil, and Mahjoub [1987], Cameron [1989], Brown, Corneil, and Mahjoub [1990], Markosyan and Gasparyan [1990], Reed [1990], Scheinerman and Trenk [1990,1993], Berge [1992b,1992a,1995], Kőrner, Simonyi, and Tuza [1992], Lehel [1994], Trenk [1995], Cai and Corneil [1996], Markossian, Gasparian, and Reed [1996], Tamura [1997,2000], Gutin and Zverovich [1998], De Simone and Kőrner [1999], Huang and Guo [1999], Fachini and Kőrner [2000], and de Figueiredo and Vušković [2000].

Introductions to and surveys of perfect graphs are given by Berge [1973b,1975, 1986], Golumbic [1980], Lovász [1983b], Berge and Chvátal [1984] (a collection of papers on perfect graphs), Chvátal [1985b,1987b], Jensen and Toft [1995], Toft [1995], Ravindra [1997], Brandstädt, Le, and Spinrad [1999], and Ramírez Alfonsín and Reed [2001] (a collection of survey papers on perfect graphs). The latter reference includes a bibliography on perfect graphs by Chvátal [2001]. Applications of perfect graphs to graph entropy were surveyed by Simonyi [2001] (cf. Simonyi [1995]). Algorithmic aspects are discussed in Golumbic [1984].

We refer for historical remarks on perfect graphs to Section 67.4g.

Chapter 66

Classes of perfect graphs

In this chapter we consider classes of perfect graphs. The phenomenon observed by Berge that clique number and colouring number are equal for bipartite graphs, their line graphs, comparability graphs, and chordal graphs, and for their complements, formed the motivation for him to raise the conjecture that the complement of a perfect graph is perfect again ($\equiv$ perfect graph theorem).
The perfection of the graphs considered in this chapter follows directly from the strong perfect graph theorem. However, since its proof is highly complicated, we will give direct proofs of the perfection of several of these graphs.

66.1. Bipartite graphs and their line graphs

The perfect graph theorem can be used to prove several min-max relations on bipartite graphs: Kőnig's matching theorem, the Kőnig-Rado edge cover theorem, and Kőnig's edge colouring theorem.

We start from the trivial observation that:

Theorem 66.1. $\omega(G) = \chi(G)$ *for each bipartite graph* G.

Proof. Trivial. ∎

Since the class of bipartite graphs is closed under taking induced subgraphs, this gives:

Corollary 66.1a. *Each bipartite graph is perfect.*

Proof. See above. ∎

Hence, by the perfect graph theorem, also the complements of bipartite graphs are perfect. This amounts to the Kőnig-Rado edge cover theorem (Theorem 19.4):

Corollary 66.1b (Kőnig-Rado edge cover theorem)**.** *For any bipartite graph* G, $\alpha(G) = \overline{\chi}(G)$. *Equivalently, the stable set number of any bipartite graph (without isolated vertices) is equal to its edge cover number.*

Proof. Directly from the perfect graph theorem, since by Theorem 66.1, any bipartite graph is perfect. Note that if G is a bipartite graph, then its cliques have size at most 2; hence $\overline{\chi}(G)$ is equal to the edge cover number of G if G has no isolated vertices. ∎

We saw in Section 16.2 that by Gallai's theorem (Theorem 19.1), the Kőnig-Rado edge cover theorem implies Kőnig's matching theorem (Theorem 16.2), saying that the matching number of a bipartite graph G is equal to its vertex cover number. That is, the stable set number of the line graph $L(G)$ of G is equal to the minimum number of cliques of $L(G)$ that cover all vertices of $L(G)$; in notation:

(66.1) $$\alpha(L(G)) = \overline{\chi}(L(G)).$$

As this is true for any induced subgraph of $L(G)$ we know that the complement $\overline{L(G)}$ of the line graph $L(G)$ of any bipartite graph G is perfect.

Hence with the perfect graph theorem we know:

Corollary 66.1c. *The line graph of any bipartite graph is perfect.*

Proof. See above. ∎

This amounts to Kőnig's edge-colouring theorem (Theorem 20.1):

Corollary 66.1d (Kőnig's edge-colouring theorem)**.** *If G is the line graph of a bipartite graph, then $\omega(G) = \chi(G)$. Equivalently, the edge-colouring number of any bipartite graph is equal to its maximum degree.*

Proof. Again directly from Kőnig's matching theorem and the perfect graph theorem. ∎

Complexity. In Part II on bipartite matching and covering, we saw that the optimization problems corresponding to the perfect graph parameters are solvable in polynomial time, and their weighted versions are solvable in strongly polynomial time, mainly by utilizing network flow techniques. We review the results.

The maximum-weight clique and the minimum colouring problem for bipartite graphs are trivially solvable in strongly polynomial time. Also the weighted colouring problem for bipartite graphs is easily solvable in strongly polynomial time.

A maximum-size stable set and a minimum clique cover in a bipartite graph can be found in polynomial time (cf. Corollary 19.3a). Note that in bipartite graphs, the minimum clique cover problem amounts to the minimum-size edge cover problem. Also the weighted versions are solvable in strongly polynomial time by max-flow techniques (cf. Corollary 21.25a). In bipartite graphs, the minimum weighted clique cover problem amounts to the minimum-size b-edge cover problem.

A bipartite graph is easily recognized, by checking if there is no odd circuit.

In *line graphs of bipartite graphs*, finding a maximum-weight clique is trivial (by checking all stars of the graph). In Sections 20.1 and 20.2 we saw that a minimum weighted colouring can be found in strongly polynomial time.

Finding a maximum clique and a minimum colouring in the complement of the line graph of a bipartite graph G amounts to finding a maximum-size matching and a minimum-size vertex cover in G, which can be found in polynomial time (cf. Theorem 16.3 and Corollary 16.6a). Their weighted versions can be found in strongly polynomial time with the methods for the assignment and the minimum-cost flow problems (cf. Theorems 17.4 and 17.6).

Van Rooij and Wilf [1965] showed that line graphs of bipartite graphs can be recognized in polynomial time, and that the corresponding bipartite graph can be reconstructed in polynomial time.

66.2. Comparability graphs

Also Dilworth's decomposition theorem (Theorem 14.2) can be derived from the perfect graph theorem. Let $(V, \leq)$ be a partially ordered set. Let $G = (V, E)$ be the graph with:

(66.2) $uv \in E$ if and only if $u < v$ or $v < u$.

Any graph G obtained in this way is called a *comparability graph*.

In Theorem 14.1 we saw the following easy 'dual' form of Dilworth's decomposition theorem:

Theorem 66.2. *In any partially ordered set $(V, \leq)$, the maximum size of a chain is equal to the minimum number of antichains needed to cover V.*

Proof. For any $v \in V$ define the *height* of v as the maximum size of a chain in V with maximum element v. Let k be the maximum height of the elements of V. For $i = 1, \ldots, k$, let A_i be the set of elements of height i. Then $A_1, \ldots, A_k$ are antichains covering V, and moreover, there is a chain of size k, since there is an element of height k. ∎

Equivalently, we have $\omega(G) = \chi(G)$ for any comparability graph. As the class of comparability graphs is closed under taking induced subgraphs we have:

Corollary 66.2a. *Each comparability graph is perfect.*

Proof. Directly from Theorem 66.2. ∎

Hence, by the perfect graph theorem, also the complement of a comparability graph is perfect. This implies:

Corollary 66.2b (Dilworth's decomposition theorem)**.** *In any partially ordered set $(V, \leq)$, the maximum size of an antichain is equal to the minimum number of chains needed to cover V.*

Proof. Directly from Corollary 66.2a. ∎

Complexity. The optimization problems corresponding to the perfect graph parameters for comparability graphs can be solved in strongly polynomial time by path and flow techniques, as we saw in Chapter 14. With a greedy method, one can find a maximum-weight clique in a comparability graph $G = (V, E)$ with weight function $w : V \to \mathbb{Q}_+$ (if the underlying partial order $\leq$ is given): if all weights are 0, the problem is trivial; if there exist vertices of positive weights, find the set S of minimal elements of positive weight, let $\alpha := \min_{v \in S} w(v)$, reset $w(v) := w(v) - \alpha$ for $v \in S$, and find recursively a maximum-weight clique C for the new weights. Then we can assume that $C \cap S \neq \emptyset$. Hence C is also a maximum-weight clique for the original weight function.

This method also solves the weighted colouring problem in strongly polynomial time. An $O(n^2)$ algorithm for the weighted colouring problem for comparability graphs was given by Hoàng [1994]. The weighted stable set and clique cover problems can be solved in strongly polynomial time with flow techniques (see Chapter 14).

Trivially, recognizing comparability graphs belongs to NP (by giving the underlying partial order), and membership of co-NP follows from the characterizations of Ghouila-Houri [1962a,1964] and Gilmore and Hoffman [1964]. A method of Gallai [1967] implies that the problem in fact is polynomial-time solvable (cf. Pnueli, Lempel, and Even [1971], Golumbic [1977], Spinrad [1985], Muller and Spinrad [1989], and McConnell and Spinrad [1994,1997,1999] (the latter paper gives a linear-time recognition algorithm)).

Golumbic, Rotem, and Urrutia [1983] and Lovász [1983b] characterized complements of comparability graphs as those graphs that are the intersection graph of a family of continuous functions $f : (0, 1) \to \mathbb{R}$. (Here f and g *intersect* if $f(x) = g(x)$ for some $x \in (0, 1)$.)

Permutation graphs. A *permutation graph* is a graph on $\{1, \ldots, n\}$ for which there exists a permutation π of $\{1, \ldots, n\}$ such that $i, j \in \{1, \ldots, n\}$ are adjacent if and only if $(i-j)(\pi(i)-\pi(j)) > 0$. A graph G is (isomorphic to) a permutation graph if and only if both G and $\overline{G}$ are comparability graphs (Dushnik and Miller [1941] (also Even, Pnueli, and Lempel [1972])). McConnell and Spinrad [1997] showed that permutation graphs can be recognized in linear time (improving McConnell and Spinrad [1994]). Another characterization was given by Baker, Fishburn, and Roberts [1972].

The books by Even [1973] and Golumbic [1980] devote chapters to comparability graphs and to permutation graphs.

66.3. Chordal graphs

We next consider a further class of perfect graphs, the 'chordal graphs' (or 'rigid circuit graphs' or 'triangulated graphs'). A graph G is called *chordal* if each circuit in G of length at least 4 has a chord. (A *chord* is an edge connecting two vertices of the circuit that are nonadjacent in the circuit.) Equivalently, a graph is chordal if it has no hole.

For any set U of vertices, let $N(U)$ denote the set of vertices not in U that are adjacent to at least one vertex in U. Call a vertex v *simplicial* if $N(\{v\})$ is a clique in G.

Dirac [1961] showed the following basic property of chordal graphs:

Theorem 66.3. *Each chordal graph G contains a simplicial vertex.*

Proof. We may assume that G has at least two nonadjacent vertices a, b. Let U be a maximal nonempty subset of V with $G[U]$ connected and with $U \cup N(U) \neq V$. Such a subset U exists as $U := \{a\}$ induces a connected subgraph of G and as $\{a\} \cup N(\{a\}) \neq V$.

Let $W := V \setminus (U \cup N(U))$. Then each vertex v in $N(U)$ is adjacent to each vertex in W, since otherwise we could increase U by v. Moreover, $N(U)$ is a clique, for suppose that $u, w \in N(U)$ are nonadjacent. Choose $v \in W$. Let P be a shortest path in $U \cup N(U)$ connecting u and w. Then $P \cup \{u, v, w\}$ would form a chordless circuit of length at least 4, a contradiction.

Now inductively we know that $G[W]$ contains a vertex v that is simplicial in $G[W]$. Since $N(U)$ is a clique and since each vertex in W is adjacent to each vertex in $N(U)$, v is also simplicial in G. ∎

(The proof of Theorem 66.3 implies that, in a chordal graph, each vertex v that is nonadjacent to at least one vertex $w \neq v$, is nonadjacent to at least one simplicial vertex $w \neq v$. Hence each noncomplete chordal graph contains at least two nonadjacent simplicial vertices.)

As was observed by Fulkerson [1972a], Theorem 66.3 implies a result of Berge [1963a] (the result was announced (with partial proof) in Berge [1960a]):

Theorem 66.4. *Any chordal graph G satisfies $\omega(G) = \chi(G)$.*

Proof. By Theorem 66.3, G has a simplicial vertex v. By induction we have $\omega(G - v) = \chi(G - v)$. In particular, $G - v$ has an $\omega(G)$-vertex-colouring. As $|N(v)| \leq \omega(G) - 1$ (since $\{v\} \cup N(v)$ is a clique), we can extend this to an $\omega(G)$-vertex-colouring of G. ∎

As the class of chordal graphs is closed under taking induced subgraphs, this implies:

Corollary 66.4a. *Each chordal graph is perfect.*

Proof. Directly from Theorem 66.4. ∎

With the perfect graph theorem, this implies the following result of Hajnal and Surányi [1958] (which also can be derived directly from Theorem 66.3):

Corollary 66.4b. *For any chordal graph G, $\alpha(G) = \overline{\chi}(G)$.*

Proof. Directly from Corollary 66.4a and the perfect graph theorem (Corollary 65.2a). ∎

Complexity. Dirac's theorem (Theorem 66.3) can be used to obtain strongly polynomial-time algorithms for the basic optimization problems for chordal graphs. The proof of Theorem 66.4 yields such an algorithm to find an optimum colouring and clique, also for the weighted versions. Similarly, the strong polynomial-time solvability of the weighted stable set and clique cover problems can be derived (Gavril [1972], Frank [1976b]). $O(n^2)$ algorithms for minimum weighted colouring for chordal graphs were given by Balas and Xue [1991] and Hoàng [1994].

Dirac's theorem also directly gives a polynomial-time recognition algorithm for chordal graphs: iteratively find and delete a simplicial vertex until the graph is empty. Linear-time algorithms were given by Lueker [1974], Rose and Tarjan [1975], Rose, Tarjan, and Lueker [1976], and Tarjan and Yannakakis [1984]. (Gavril [1974b] gave another polynomial-time algorithm.)

Dirac's theorem also implies the following other characterizations of chordal graphs (Dirac [1961] (stated explicitly by Fulkerson and Gross [1965] and Rose [1970])):

(66.3) A graph $G = (V, E)$ is chordal $\iff$ each induced subgraph has a simplicial vertex $\iff$ G has an acyclic orientation $D = (V, A)$ such that if (u, v), $(u, w) \in A$, then $\{v, w\} \in E$.

Dirac [1961] moreover showed that a graph is chordal if and only if each inclusionwise minimal vertex-cut is a clique.

Interval graphs. An *interval graph* is the intersection graph G of a family $\mathcal{C}$ of nonempty intervals on the real line[11]. Trivially, such a graph is the complement of a comparability graph: define $I < J \iff i < j$ for all $i \in I$, $j \in J$. This gives a partial order, and the corresponding comparability graph is equal to $\overline{G}$.

Perfection of the complements of interval graphs was observed by T. Gallai (see Hajnal and Surányi [1958]) — that is, the maximum number of disjoint intervals in $\mathcal{C}$ is equal to the minimum number of points intersecting all intervals in $\mathcal{C}$. This is not hard to prove, and can be proved similarly to the easy dual of Dilworth's decomposition theorem (Theorem 14.1). In fact, a graph is an interval graph if and only if it is chordal and its complement is a comparability graph.

The clique, stable set, colouring, and clique cover problem and their weighted versions can be solved in strongly polynomial time with the methods for comparability graphs described above. If the intervals are given in the order of their maximal elements, and we consecutively assign to each interval the smallest available colour (numbering the colours $1, 2, \ldots$), we obtain an optimum colouring. (Kierstead [1988] showed that if we get the intervals in an *arbitrary* order and we assign to any given interval the smallest possible colour ('on-line'), then we need at most $40\chi(G)$ colours.)

In fact, for any clique C in G there is a point x such that all intervals in C contain x (by Helly's theorem: a family of pairwise intersecting intervals has a nonempty intersection). So finding a maximum-weight clique is trivial. A maximum-size stable

[11] The *intersection graph* of a family $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$, two sets in $\mathcal{C}$ being adjacent if and only if they intersect.

set can be found by a greedy method: first find an interval $I \in \mathcal{C}$ with $\sup I$ minimal. Next find recursively a maximum-size stable set S among the intervals in $\mathcal{C}$ disjoint from I. Then $S \cup \{I\}$ is a maximum-size stable set in G.

In reply to questions of Hajós [1957] and Benzer [1959], interval graphs have been characterized by Lekkerkerker and Boland [1962] (cf. Halin [1982]), Gilmore and Hoffman [1964], and Fulkerson and Gross [1965]. The latter paper gives a polynomial-time recognition algorithm. A linear-time recognition algorithm was given by Booth and Lueker [1975,1976]. This was simplified by Korte and Möhring [1987], Corneil, Olariu, and Stewart [1998], Hsu and Ma [1999], and Hsu [2002].

More on interval graphs can be found in the books by Golumbic [1980], Skrien [1982], Fishburn [1985], and Brandstädt, Le, and Spinrad [1999], and in the survey article by Golumbic [1985].

Split graphs. A *split graph* is a graph $G = (V, E)$ where V can be partitioned into a clique C and a stable set S. Trivially, a split graph is perfect, since C is contained in a maximum-size clique; hence we can assume that C is a maximum-size clique; so for each $s \in S$ there is a $c \in C$ nonadjacent to s; this yields a $|C|$-vertex-colouring of G.

A graph G is a split graph if and only if both G and $\overline{G}$ are chordal graphs (Foldes and Hammer [1977], Hammer and Simeone [1981]). The book by Golumbic [1980] devotes a chapter to split graphs.

Trivially perfect graphs. Golumbic [1978] calls a graph *trivially perfect* if for each induced subgraph, the stability number is equal to the number of inclusion-wise maximal cliques. Trivially, each trivially perfect graph is perfect. Choudom, Parthasarathy, and Ravindra [1975] and Golumbic [1978] showed that a graph is trivially perfect if and only if it has no induced subgraph equal to the path P_4 or the circuit C_4 (each with 4 vertices). This implies (by a theorem of Wolk [1962] (proof simplified in Wolk [1965])) that a graph is trivially perfect if and only if it is the comparability graph coming from a branching. Another characterization of trivially perfect graphs was given by Alexe and Olaru [1997].

Threshold graphs. A *threshold graph* is a graph on vertex set V given by a function $w : V \to \mathbb{R}$, such that two distinct vertices u, v are adjacent if and only if $w(u)+w(v) > 0$. Chvátal and Hammer [1977] showed that a graph G is a threshold graph if and only if neither G nor $\overline{G}$ has an induced subgraph equal to the path P_4 or the circuit C_4 (each with 4 vertices) — that is, both G and $\overline{G}$ are trivially perfect.

Each threshold graph is a split graph (trivially) and a permutation graph (order the vertices as $v_1, \ldots, v_n$ such that $w(v_1) \leq w(v_2) \leq \cdots \leq w(v_n)$, and let π be the permutation given by ordering $|w(v_1)|, |w(v_2)|, \ldots, |w(v_n)|$). However, the path P_4 with 4 vertices is both a split graph and a permutation graph, but no threshold graph.

The book by Mahadev and Peled [1995] focuses on threshold graphs, and the book by Golumbic [1980] devotes a chapter to threshold graphs. A related class of graphs was described by Wang [1995,1996].

'Strongly chordal' graphs have been studied by Farber [1983,1984] and Kaplan and Shamir [1994], and an analogue of chordal graphs for bipartite graphs by Golumbic and Goss [1978].

66.3a. Chordal graphs as intersection graphs of subtrees of a tree

Chordal graphs can be characterized as intersection graphs of subtrees of a tree, as was shown by L. Surányi (see Gyárfás and Lehel [1970]) and also by Walter [1972, 1978], Buneman [1974], and Gavril [1974c].

Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. The *intersection graph* of $\mathcal{S}$ is the graph with vertex set $\mathcal{S}$, where two vertices S, S' are adjacent if and only if S and S' have at least one vertex in common.

The class of graphs obtained in this way coincides with the class of chordal graphs. To see this, we first show the following elementary lemma:

Lemma 66.5α. *Let $\mathcal{S}$ be a collection of pairwise intersecting subtrees of a tree T. Then there is a vertex of T contained in all subtrees in $\mathcal{S}$.*

Proof. By induction on $|VT|$. If $|VT| = 1$ the lemma is trivial, so assume $|VT| \geq 2$. Let t be an end vertex of T. If there exists a subtree in $\mathcal{S}$ consisting only of t, the lemma is trivial. Hence we may assume that each subtree in $\mathcal{S}$ containing t also contains the neighbour of t. So deleting t from T and from all subtrees in $\mathcal{S}$ gives the lemma by induction. ∎

Then we have the subtree characterization of chordal graphs:

Theorem 66.5. *A graph is chordal if and only if it is isomorphic to the intersection graph of a collection of subtrees of some tree.*

Proof. *Necessity.* Let $G = (V, E)$ be chordal. By Theorem 66.3, G contains a simplicial vertex v. By induction, the graph $G - v$ is the intersection graph of a collection $\mathcal{S}$ of subtrees of some tree T. Let $\mathcal{S}'$ be the subcollection of $\mathcal{S}$ corresponding to the set N of neighbours of v in G. As N is a clique, $\mathcal{S}'$ consists of pairwise intersecting subtrees. Hence, by Lemma 66.5α, these subtrees have a vertex t of T in common. Now we extend T and all subtrees in $\mathcal{S}'$ with a new vertex s and a new edge st. Moreover, we introduce a new subtree $\{s\}$ representing v. In this way we obtain a subtree representation for G.

Sufficiency. Let G be the intersection graph of some collection $\mathcal{S}$ of subtrees of some tree T. By (66.3) it suffices to show that G has a simplicial vertex. Let s be an end vertex of T. If $\mathcal{S}$ contains a subtree only consisting of s, it gives a simplicial vertex in G. If $\mathcal{S}$ contains no such subtree, then each subtree in $\mathcal{S}$ containing s also contains the neighbour t (say) of s. So deleting s from T and from all subtrees in $\mathcal{S}$, does not modify the graph G. Hence we are done by induction. ∎

This theorem enables us to interpret the perfection of chordal graphs in terms of trees:

Corollary 66.5a. *Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. Then the maximum number of pairwise vertex-disjoint trees in $\mathcal{S}$ is equal to the minimum number of vertices of T intersecting each tree in $\mathcal{S}$.*

Proof. Directly from Corollary 66.4b and Theorem 66.5, using Lemma 66.5α. ∎

(This result was also stated by Cockayne, Hedetniemi, and Slater [1979].)

Similarly we have:

Corollary 66.5b. *Let $\mathcal{S}$ be a collection of subtrees of a tree T. Let k be the maximum number of times that any vertex of T is covered by trees in $\mathcal{S}$. Then $\mathcal{S}$ can be partitioned into subcollections $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that each $\mathcal{S}_i$ consists of pairwise vertex-disjoint trees.*

Proof. Directly from Theorems 66.4 and 66.5, again using Lemma 66.5α. ∎

Variations of the problem of packing and covering a tree by subtrees were studied by Bárány, Edmonds, and Wolsey [1986]. More characterizations of chordal graphs were offered by Benzaken, Crama, Duchet, Hammer, and Maffray [1990]. More on chordal graphs can be found in the book of Golumbic [1980] and in Skrien [1982], Leung [1984], Seymour and Weaver [1984] (a generalization of chordal graphs), Lubiw [1987], Wallis and Wu [1995], and Nakamura and Tamura [2000] (a generalization to bidirected graphs).

66.4. Meyniel graphs

Markosyan and Karapetyan [1976] and Meyniel [1976] showed the perfection of graphs in which each odd circuit of length at least five has at least two chords (*Meyniel graphs*). This was conjectured by Olaru [1969,1972].

It implies the perfection of *Gallai graphs* — graphs in which each odd circuit of length at least five has two *noncrossing* chords (Gallai [1962], cf. Surányi [1968] for a shorter proof [12]), *parity graphs* — graphs in which each odd circuit of length at least five has two crossing chords (Olaru [1969,1972, 1977], cf. Sachs [1970]), and graphs that have no path P_4 as induced subgraph (Seinsche [1974]).

We follow the proof given by Lovász [1983b] (which is a simplification of Meyniel's original proof).

Theorem 66.6. *Each Meyniel graph is perfect.*

Proof. I. We first show that in a Meyniel graph $G = (V, E)$:

(66.4) for each odd circuit C and each vertex v on C, C has a chord disjoint from v or each vertex of $C - v$ is adjacent to v.

[12] Gallai [1962] published a proof that $\alpha(G) = \overline{\chi}(G)$ for graphs in which each odd circuit of length 5 has two noncrossing chords. Berge [1997] wrote that Gallai informed him in a letter that he knew that also $\omega(G) = \chi(G)$ holds for these graphs.

Let C have no chord disjoint from v. Then the subgraph of G induced by VC is outerplanar, with C as boundary. As each odd circuit of size at least five has a chord we know that each odd bounded face is a triangle. (A face is *odd* (*even*) if its is incident with an odd (even) number of edges.)

Moreover, as C is odd, there is at least one odd bounded face. So if v is not adjacent to all vertices of $C - v$, there is an even bounded face, neighbouring an odd bounded face. But then the union of these two faces forms an odd circuit with only one chord, contradicting the condition.

II. We now prove the theorem. It suffices to show that $\chi(G) = \omega(G)$ for any Meyniel graph $G = (V, E)$, as the class of Meyniel graphs is closed under taking induced subgraphs. We may assume that $V = \{1, \ldots, n\}$. Let $k := \chi(G)$.

For each colouring $\phi : V \to \{1, \ldots, k\}$, let the (ordered) clique $K_\phi = (v_1, \ldots, v_t)$ be obtained recursively as follows. If $v_1, \ldots, v_i$ have been determined (for $i \geq 0$), then v_{i+1} is the largest vertex of colour $i+1$ that is adjacent to each of $v_1, \ldots, v_i$. If no such vertex exists, we stop, setting $t := i$.

Let ϕ be a k-colouring with $K_\phi = (v_1, \ldots, v_t)$ lexicographically minimal. If $t = k$ we are done, so assume $t < k$. Consider the subgraph of G induced by the vertices coloured t and $t + 1$, and let H be its component containing v_t. Let ψ be the colouring obtained from ϕ by interchanging colours t and $t+1$ in H. We show that K_ψ is lexicographically less than K_ϕ, contradicting our assumption.

Trivially, $v_1, \ldots, v_{t-1}$ belong to K_ψ (since we did not change any of the colours $1, \ldots, t-1$). If no other vertex is in K_ψ we are done, so we can assume that K_ψ contains a vertex w with $\psi(w) = t$.

Then $w \neq v_t$, since $\psi(v_t) = t + 1$. If $w < v_t$ we are done, so we can assume that $w > v_t$. If $\phi(w) = t$, this contradicts the choice of $v_t \in K_\phi$. So $\phi(w) = t + 1$, and H contains a shortest path P from v_t to w. Necessarily, this path is odd, and has no chords.

Let u be the second vertex on P. So $\phi(u) = t + 1$. Since v_t is the last vertex in K_ϕ we know that there is an $i \in \{1, \ldots, t-1\}$ with v_i not adjacent to u. Let C be the circuit made by P, $v_i v_t$, and $v_i w$. As P has no chords, by (66.4) v_i is adjacent to u, a contradiction. ∎

Ravindra [1982] showed that each Meyniel graph is strongly perfect (see Section 66.5a below). This was extended by Hoàng [1987b], who showed that Meyniel graphs are precisely those graphs with the property that for each induced subgraph H and each vertex v of H, there exists a stable set in H containing v and intersecting all inclusionwise maximal cliques of H. (This was conjectured by Meyniel.)

Complexity. Burlet and Fonlupt [1984] showed that the class of Meyniel graphs is closed under amalgamation (see Section 65.7c) and that each Meyniel graph arises by amalgamation from chordal graphs and bipartite graphs added with one vertex connected to all vertices of the bipartite graph. They showed that it yields a polynomial-time recognition algorithm (speeded up by Roussel and Rusu [1999b]).

Hoàng [1987b] gave an $O(n^8)$-time algorithm to find a minimum colouring and a maximum clique. An $O(n^3)$ algorithm was given by Hertz [1990a].

(Conforti, Cornuéjols, Kapoor, and Vušković [1999] consider an extension by decomposing *cap-free* graphs (where a *cap* is a circuit with exactly one chord, connecting two vertices at distance two in the circuit) — a (not necessarily perfect) generalization of Meyniel graphs.)

Gallai graphs. As mentioned, these are graphs in which each odd circuit of length ≥ 5 has two noncrossing chords. Polynomial-time recognition algorithms were given by Burlet and Fonlupt [1984], Whitesides [1984], and Cicerone and Di Stefano [1999b] (linear-time). The latter paper also gives a linear-time maximum-weight clique algorithm. A linear-time colouring algorithm was found by Roussel and Rusu [1999a].

Parity graphs. As mentioned, these are graphs in which each odd circuit of length ≥ 5 has two crossing chords. Parity graphs can be characterized alternatively as those graphs such that for each pair u, v of vertices, all chordless $u - v$ paths have the same parity.

Combinatorial strongly polynomial-time algorithms to solve the weighted clique, stable set, colouring, and clique cover problems in parity graphs were given by Burlet and Uhry [1982], who also gave a polynomial-time recognition algorithm (by decomposition of the graph into smaller parity graphs).

The parity graphs include the *line-perfect graphs*, which are graphs whose line graph is perfect. They were characterized by Trotter [1977] — see the claw-free graphs in Section 65.6d. More on parity graphs can be found in Adhar and Peng [1990], Bandelt and Mulder [1991], Przytycka and Corneil [1991], Rusu [1995b], Jansen [1998], and Cicerone and Di Stefano [1999a].

66.5. Further results and notes

66.5a. Strongly perfect graphs

Following Berge and Duchet [1984], a graph $G = (V, E)$ is *strongly perfect* if each induced subgraph H has a stable set intersecting all inclusionwise maximal cliques of H. Each strongly perfect graph is perfect (by (65.2)). Berge and Duchet showed that comparability graphs, chordal graphs, and complements of chordal graphs are strongly perfect. Ravindra [1982] showed that Meyniel graphs are strongly perfect, and Chvátal [1984d] that perfectly orderable graphs are strongly perfect.

Berge and Duchet also showed that the recognition problem for strongly perfect graphs belongs to co-NP. No combinatorial polynomial-time algorithms are known for the optimization problems for strongly perfect graphs.

Olaru [1996] showed that the graphs that are both minimally strongly imperfect and imperfect are precisely the odd circuits of length at least five and their complements. Hence to prove the strong perfect graph theorem it suffices to show that each minimally imperfect graph is also minimally strongly imperfect.

More on strongly perfect graphs can be found in Ravindra [1981,1999], Berge [1983], Basavayya and Ravindra [1985,1987], Preissmann [1985], Preissmann and de Werra [1985], Olaru and Mîndrescu [1986a,1986b], Olaru [1987,1993], Ravindra and Basavayya [1988,1992,1994,1995], Olariu [1989a], Włoch [1995], Blidia, Duchet, and Maffray [1996], Szelecka and Włoch [1996], and Alexe and Olaru [1997].

66.5b. Perfectly orderable graphs

A graph $G = (V, E)$ is a *perfectly orderable graph* if it has an acyclic orientation $D = (V, A)$ such that if four vertices v_1, v_2, v_3, v_4 induce a chordless path in G with edges v_1v_2, v_2v_3, v_3v_4, then $(v_1, v_2) \in A$ or $(v_4, v_3) \in A$. Chvátal [1984d] showed that perfectly orderable graphs are perfect — in fact, strongly perfect:

Theorem 66.7. *Each perfectly orderable graph is strongly perfect.*

Proof. We can assume that $V = \{1, \ldots, n\}$ and that if $(i, j) \in A$, then $i < j$. Let S be the stable set with $\sum(2^{-i} \mid i \in S)$ maximal. Then each $v \notin S$ has a neighbour $u \in S$ with $u < v$, since otherwise $(S \setminus N(v)) \cup \{v\}$ is better than S.

We show that each inclusionwise maximal clique K intersects S. Suppose $K \cap S = \emptyset$. For $s \in S$, let K_s be the set of neighbours $v \in K$ with $s < v$. Choose $s \in S$ with $\sum(2^i \mid i \in K_s)$ maximal. As K is a maximal clique, s is nonadjacent to some $v \in K$. Let $u \in S$ be a neighbour of v with $u < v$. So $v \in K_u \setminus K_s$. By the choice of s, there is a vertex $i \in K_s \setminus K_u$ with $i > v$. So $u < v < i$, and hence u and i are nonadjacent (otherwise $i \in K_u$). As u and s are nonadjacent (since $u, s \in S$) and v and i are adjacent (since $v, i \in K$), u, v, i, s induce a P_4 subgraph with $(u, v), (v, i), (s, i) \in A$, a contradiction. ∎

(Another proof, and a generalization, of this was given by Duchet and Olariu [1991].)

Note that the set S in this proof can be found by a greedy method. So we can find an optimum colouring in polynomial time. Given an orientation as above, also a maximum-size clique can be found in a greedy way — see Chvátal [1984d]. Hoàng [1994] gave $O(nm)$-time algorithms, also for the weighted versions. Middendorf and Pfeiffer [1990a] showed that it is NP-complete to decide if a graph is perfectly orderable.

Comparability graphs, chordal graphs, and complements of chordal graphs are perfectly orderable.

More on perfectly orderable graphs can be found in Cochand and de Werra [1986], Preissmann, de Werra, and Mahadev [1986], Chvátal, Hoàng, Mahadev, and de Werra [1987], Lehel [1987], Hertz [1988,1990b], Hoàng and Khouzam [1988], Olariu [1988a,1993], Bielak [1989], Hoàng and Mahadev [1989], Hoàng and Reed [1989a,1989b], Jamison and Olariu [1989a], Chvátal [1990,1993], Hoàng, Maffray, and Preissmann [1991], Croitoru and Radu [1992a], Hoàng, Maffray, Olariu, and Preissmann [1992], Gavril, Toledano Laredo, and de Werra [1994], Arikati and Peled [1996], Giakoumakis [1996], Hoàng [1996a,1996b,2001], Rusu [1996], Hayward [1997a], Hoàng, Maffray, and Noy [1999], and Hoàng and Tu [2000].

More classes of graphs based on orienting or colouring edges are given by Hoàng [1987a].

66.5c. Unimodular graphs

A graph $G = (V, E)$ is *unimodular* if the following matrix M is totally unimodular: the columns are indexed by V and the rows are the incidence vectors of all inclusionwise maximal cliques of G. Any induced subgraph of a unimodular graph is unimodular again, since for each $v \in V$ and for each maximal clique C of $G - v$, either C or $C \cup \{v\}$ is a maximal clique of G.

Unimodular graphs include bipartite graphs, line graphs of bipartite graphs, and interval graphs.

Perfection of unimodular graphs and their complements was shown by Berge [1963a]. Perfection of the complements of unimodular graphs follows from the Hoffman-Kruskal theorem (Hoffman and Kruskal [1956]), since

(66.5) $$\alpha(G) = \max\{\mathbf{1}^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq \mathbf{1}\} = \chi(\overline{G}),$$

as the LP-optima are attained by integer vectors x and y.

The perfection of a unimodular graph $G = (V, E)$ can also be derived from the Hoffman-Kruskal theorem, with an idea which Berge [1963a] attributed to M.H. McAndrew. It suffices to find a stable set that intersects all maximum-size cliques. Let M' be the submatrix of M corresponding to the maximum-size cliques. The system $\mathbf{0} \leq x \leq \mathbf{1}$, $Mx \leq \mathbf{1}$, $M'x \geq \mathbf{1}$ has a solution (namely $x = \omega(G)^{-1}\mathbf{1}$). Hence, as M is total unimodular, it has an integer solution x. This is the incidence vector of a stable set as required.

By a result of Heller [1957] (cf. Theorem 21.4 in Schrijver [1986b]), a unimodular graph has at most $|V|(|V|+1)$ inclusionwise maximal cliques. As W.H. Cunningham (cf. Grötschel, Lovász, and Schrijver [1988]) observed, this gives a polynomial-time method to enumerate all maximal cliques: Choose $v \in V$. Enumerate the maximal cliques $C_1, \ldots, C_t$ of $G - v$ (recursively). Then the maximal cliques of G are among the cliques C_i ($i = 1, \ldots, t$), and $(C_i \cap N(v)) \cup \{v\}$ ($i = 1, \ldots, t$). We can select the maximal cliques among these cliques in polynomial time. Since $t \leq |V|(|V| + 1)$, this gives a polynomial-time method.

This directly gives a strongly polynomial-time method to find a maximum-weight clique. It also implies that the weighted versions of the stable set, colouring, and clique cover problems can be solved in strongly polynomial time, by solving an explicit linear programming problem (using Tardos [1986]). The colouring problem can be solved recursively by first finding (with LP-techniques) a $0, 1$ vector x satisfying $x(C) \leq 1$ for each maximal clique C and $x(C) = 1$ for each maximum-size clique C, and next colouring $G - S$ recursively (where $x = \chi^S$). The weighted version can be solved similarly.

Since by a theorem of Seymour [1980a], totally unimodular matrices can be recognized in polynomial time, this also yields a polynomial-time method to recognize a unimodular matrix.

Ghouila-Houri [1962b] showed that a graph $G = (V, E)$ is unimodular if and only if each nonempty subset U of V contains two disjoint sets U_1 and U_2 such that $U_1 \cup U_2 \neq \emptyset$ and such that each maximal clique C of G with $|C \cap U|$ even, satisfies $|C \cap U_1| = |C \cap U_2|$.

66.5d. Further classes of perfect graphs

Weakly chordal graphs. A graph $G = (V, E)$ is called *weakly chordal* (or *weakly triangulated*) if neither G nor its complement contains a chordless circuit of length at least 5. Hayward [1985] showed that weakly chordal graphs are perfect. Polynomial-time algorithms for the optimization problems related to weakly chordal graphs were given by Hayward, Hoàng, and Maffray [1989], Spinrad and Sritharan [1995], and Hayward, Spinrad, and Sritharan [2000], and polynomial-time recognition algorithms by Spinrad and Sritharan [1995] and Hayward, Spinrad, and Sritharan [2000]. The class of weakly chordal graphs contains both the chordal graphs and their complements.

More on weakly chordal graphs is given in Hoàng, Maffray, Olariu, and Preissmann [1992], Hayward [1996,1997a,1997b], and McMorris, Wang, and Zhang [1998]. Weakly chordal comparability graphs were studied by Eschen, Hayward, Spinrad, and Sritharan [1999].

Quasi-parity graphs. A graph $G = (V, E)$ is a *quasi-parity graph* if each induced subgraph H that is not a clique has two vertices that are not connected by a chordless path of odd length. Meyniel [1987] showed that these graphs are perfect, and that they include the Meyniel graphs and the perfectly orderable graphs. (A short proof of this last is given by Hertz and de Werra [1988].)

Berge [1986] showed that the class of quasi-parity graphs can be enlarged to those graphs in which for each induced subgraph H with at least two vertices, there exist two vertices such that in H or $\overline{H}$ there is no chordless odd-length path connecting them.

Edmonds-Giles graphs. Let $D = (V, A)$ be a directed graph and let $\mathcal{C}$ be a crossing collection of subsets of V with $\delta^{\text{out}}(U) = \emptyset$ for each $U \in \mathcal{C}$. Make an undirected graph G with vertex set A, two arcs a, a' being adjacent if and only if there is a $U \in \mathcal{C}$ such that both a and a' enter U. In Schrijver [1983a] such a graph is called an *Edmonds-Giles graph*. Each such graph is perfect, as can be seen as follows.

A special case of the Edmonds-Giles theorem (Theorem 60.1) is that the system (in $x \in \mathbb{R}^A$)

$$\text{(66.6)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x(a) \leq 1 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(U)) \leq 1 & \text{for } U \in \mathcal{C}, \end{array}$$

is totally dual integral. Hence it determines an integer polytope. Now the integer vectors x satisfying (66.6) are exactly the incidence vectors of the stable sets of G. Each inequality (66.6)(ii) is a clique inequality. The stable set polytope of G therefore is determined by the clique inequalities, and hence G is perfect (Corollary 65.2e). It in particular implies that each clique of G is contained in $\delta^{\text{in}}(U)$ for some $U \in \mathcal{C}$.

A special case of Edmonds-Giles graphs was given by Kahn [1984], where $D = (V, A)$ is a directed graph and $\mathcal{C}$ is the collection of nonempty proper subsets U of V with $\delta^{\text{out}}(U) = \emptyset$ and $|\delta^{\text{in}}(U)|$ minimal. With the perfect graph theorem this implies that the arcs of a digraph can be coloured in such a way that each minimum-size directed cut contains each colour exactly once.

p-comparability graphs. Cameron and Edmonds [1992] showed perfection of the following graphs. Let $D = (V, A)$ be a directed graph and let $U \subseteq V$ be such that each directed circuit of D has precisely one vertex in U. Let G be the undirected graph on $V \setminus U$ with any two $u, v \in V \setminus U$ adjacent if and only if there is a directed circuit containing u and v. Cameron and Edmonds [1992] call such graphs *p-comparability graphs*. Any comparability graph is a p-comparability graph, but not conversely.

Each such graph G is perfect. The proof is by reduction to minimum-cost flow, using the facts that each clique of G is contained in some directed circuit of D and that by Theorem 65.11 it suffices to show that $\overline{\chi}^*(G) = \overline{\chi}(G)$. (The class of p-comparability graphs is closed under taking induced subgraphs, since adding all arcs (u, v) for which there is a directed $u-v$ path avoiding U, maintains the above property of D.)

Now a minimum fractional clique cover of G corresponds to a minimum fractional covering of $V \setminus U$ by directed circuits. By the integer flow theorem, this last is attained by an integer covering of $V \setminus U$ by directed circuits. Hence, the minimum fractional clique cover in G is attained by an integer clique cover. This amounts to $\overline{\chi}^*(G) = \overline{\chi}(G)$.

Polyominoes. A *polyomino* is a union of unit squares in the plane. (A unit square is a square with integer coordinates and area 1.)

Given a polyomino P, make a graph G with vertices all unit squares contained in P, two of them being adjacent if and only if P contains a rectangle (with horizontal and vertical sides) containing both squares. Győri [1984] showed that if P is horizontally convex, then $\alpha(G) = \overline{\chi}(G)$ (see Section 60.3d). (P is *horizontally convex* if each horizontal line has a convex intersection with P.) This extends a result of Chaiken, Kleitman, Saks, and Shearer [1981], who proved $\alpha(G) = \overline{\chi}(G)$ if P is orthogonally convex. (P is *orthogonally convex* if each horizontal or vertical line has a convex intersection with P.) The latter paper also mentions that E. Szemerédi gave an example that one cannot delete orthogonal convexity, and it gives an example of F.R.K. Chung (1979) showing that one cannot relax it to simple connectivity.

Saks [1982] showed that if P is orthogonally convex, then the subgraph of G induced by the boundary squares is perfect. (A *boundary square* of P is a unit square having a neighbouring square not in P.) (This was proved for the subset of corner squares by Chaiken, Kleitman, Saks, and Shearer [1981]. (A *corner square* of P is a unit square having at least two neighbouring squares not in P.))

Shearer [1982] showed that also the following graph G arising from a *simply connected* polyomino P is perfect: the vertices of G are the rectangles contained in P, where two of them are adjacent if and only if they have a unit square in common.

Motwani, Raghunathan, and Saran [1989] showed that the visibility graph of a horizontally convex polyomino is perfect; in fact, a permutation graph. More on this and related problems can be found in Berge, Chen, Chvátal, and Seow [1981], Győri [1985], Motwani, Raghunathan, and Saran [1988,1990], and Maire [1994a].

66.5e. Further notes

Hayward [1990] showed that graphs containing neither C_5 nor P_6 nor $\overline{P_6}$ as induced subgraphs, are perfect. Other classes of perfect graphs were studied by Ravindra

[1976], Payan [1983], Golumbic, Monma, and Trotter [1984], Hammer and Mahadev [1985], Monma, Reed, and Trotter [1988], Hertz [1989a,1989b,1989c], Hoàng and Maffray [1989,1992], Bertschi [1990], Lubiw [1991b], Sun [1991], Croitoru and Radu [1993], Gurvich, Temkin, Udalov, and Shapovalov [1993], Thomas [1993], Maire [1994b,1996], Rusu [1995b,1999c,1999a], Cheah and Corneil [1996], Gyárfás, Kratsch, Lehel, and Maffray [1996] Giakoumakis [1997], Giakoumakis and Rusu [1997], and Maffray and Preissmann [1999]. Le [2000] gave conjectures on the perfection of certain classes of graphs. A survey of several classes of perfect graphs and their recognition and interrelations, is given in the book by Brandstädt, Le, and Spinrad [1999]. The book of Simon [1992] studies efficient algorithms for classes some of perfect graphs.

Conforti, Cornuéjols, Kapoor, and Vušković [1997] investigated 'universally signable' graphs, a generalization of chordal graphs.

Hammer and Maffray [1993] introduced 'preperfect' graphs, and showed that each preperfect graph is perfect, and that preperfect graphs include the Gallai and the parity graphs (cf. Section 66.4).

Corneil and Stewart [1990] studied the complexity of finding minimum-size dominating sets in several classes of perfect graphs. (A *dominating set* is a set U of vertices with $U \cup N(U) = V$.)

Berge and Las Vergnas [1970] showed that a graph G is perfect if for each odd circuit C and each maximal clique K, the intersection of C and K does not consist of two vertices that form an edge of C.

Vertex cuts in perfect and minimally imperfect graphs were surveyed by Rusu [2001]. A characterization of perfect total graphs was given by Rao and Ravindra [1977].

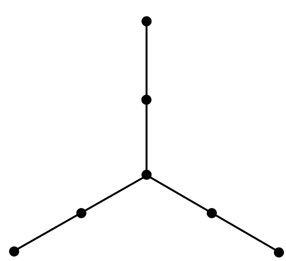

Figure 66.1

Lovász [1983b] calls a graph *k-perfect* if for each induced subgraph $G = (V, E)$ one has:

(66.7) $$\omega_k(G) = \min_{U \subseteq V} (k\chi(G - U) + |U|)$$

where $\omega_k(G)$ is the maximum size of a union of k cliques. By the results of Greene and Kleitman (Corollaries 14.8a and 14.10a), comparability graphs and their complements are k-perfect for each k. Also, complements of line graphs of bipartite graphs are k-perfect, by Corollary 21.4b. On the other hand, the line graph of the bipartite graph in Figure 66.1 is not 2-perfect (Greene [1976]). Related results were given by Berge [1989b,1992a,1992b] and Cameron [1989].

A.J. Hoffman and E.L. Johnson (cf. Golumbic [1980]) proposed the following sharpening of perfection. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$. A k-*interval colouring* is an assignment to each vertex v of an open subinterval of $[0, k]$ of length $w(v)$ such that adjacent vertices obtain disjoint intervals. Let $\chi_{\text{int}}(G, w)$ denote the minimum value of k for which G has a k-interval colouring. If $w(v) = 1$ for each vertex v, then $\chi_{\text{int}}(G, w) = \chi(G)$. Call G *superperfect* if $\chi_{\text{int}}(G, w)$ is equal to the maximum of $w(K)$ over all cliques K in G. As Hoffman observed, each comparability graph is superperfect (this can be derived from Dilworth's decomposition theorem), but none of the other known interesting classes of perfect graphs have this property.

A survey on subclasses of 'classical' perfect graphs (comparability graphs and chordal graphs) was given by Duchet [1984]. More examples and applications of perfect graphs were given by Shannon [1956], Berge [1967], and Tucker [1973a].

Chapter 67

Perfect graphs: polynomial-time solvability

In this chapter we show that a maximum-weight stable set and a minimum weighted clique cover in a perfect graph can be found in strongly polynomial time. This was shown by Grötschel, Lovász, and Schrijver [1981,1988] with the help of the ellipsoid method and of the function $\vartheta(G)$, introduced by Lovász [1979d] as upper bound on the Shannon capacity of a graph G. No combinatorial polynomial-time algorithms for these problems are known.

We should stress that the naive approach of applying the ellipsoid method to the stable set polytope of a perfect graph using the clique inequalities does not work: it reduces the problem of finding a maximum-weight stable set to deciding for any $x \in \mathbb{R}_+^V$ if there is a clique C satisfying $x(C) > 1$. This is equivalent to finding a maximum-weight clique, which is equivalent to finding a maximum-weight stable set in the complementary graph, which is perfect again. So this would give nothing but a reduction to itself.

In this chapter, all graphs can be assumed to be simple.

67.1. Optimum clique and colouring in perfect graphs algorithmically

Lovász [1979d] introduced the following real number $\vartheta(G)$ for any graph $G = (V, E)$. Let $\mathcal{M}_G$ be the collection of symmetric $V \times V$ matrices satisfying $M_{u,v} = 0$ for any two distinct adjacent vertices u and v and satisfying $\mathrm{Tr}M = 1$. Here $\mathrm{Tr}M$ is the trace of M (sum of diagonal elements). Define

(67.1) $$\vartheta(G) := \max\{\mathbf{1}^{\mathsf{T}} M \mathbf{1} \mid M \in \mathcal{M}_G \text{ positive semidefinite}\}.$$

Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^V$.

$\vartheta(G)$ has two important properties: it can be calculated (at least, approximated) in polynomial time, and it gives an, often close, upper bound on the stable set number $\alpha(G)$ (Lovász [1979d]).

First we show (where $\overline{\chi}^*(G)$ denotes the fractional clique cover number — cf. Section 64.8):

Theorem 67.1. *For any graph $G = (V, E)$:*

$$\alpha(G) \leq \vartheta(G) \leq \overline{\chi}^*(G). \tag{67.2}$$

Proof. To see $\alpha(G) \leq \vartheta(G)$, let S be a maximum-size stable set and let M be the matrix given by:

$$M := \frac{1}{|S|}\chi^S(\chi^S)^\mathsf{T}. \tag{67.3}$$

Here χ^S is the incidence vector of S in $\mathbb{R}^V$. Then M belongs to $\mathcal{M}_G$ and is positive semidefinite. Hence $\alpha(G) = |S| = \mathbf{1}^\mathsf{T}M\mathbf{1} \leq \vartheta(G)$.

To see $\vartheta(G) \leq \overline{\chi}^*(G)$, let M attain the maximum in (67.1). Consider cliques $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ with

$$\sum_{i=1}^k \lambda_i \chi^{C_i} = \mathbf{1} \text{ and } \sum_{i=1}^k \lambda_i = \overline{\chi}^*(G). \tag{67.4}$$

Then, setting $\gamma := \overline{\chi}^*(G)$:

$$\begin{aligned} 0 &\leq \sum_{i=1}^k \lambda_i(\gamma\cdot\chi^{C_i} - \mathbf{1})^\mathsf{T}M(\gamma\cdot\chi^{C_i} - \mathbf{1}) \\ &= \gamma^2\sum_{i=1}^k \lambda_i(\chi^{C_i})^\mathsf{T}M\chi^{C_i} - 2\gamma\sum_{i=1}^k \lambda_i(\chi^{C_1})^\mathsf{T}M\mathbf{1} + \gamma\mathbf{1}^\mathsf{T}M\mathbf{1} \\ &= \gamma^2\mathrm{Tr}M - 2\gamma\mathbf{1}^\mathsf{T}M\mathbf{1} + \gamma\mathbf{1}^\mathsf{T}M\mathbf{1} = \gamma^2 - \gamma\vartheta(G), \end{aligned} \tag{67.5}$$

since $\mathrm{Tr}M = 1$, $\mathbf{1}^\mathsf{T}M\mathbf{1} = \vartheta(G)$, and $M_{u,v} = 0$ if $u \neq v$ and $u, v \in C_i$ for some i.

(67.5) implies that $\vartheta(G) \leq \gamma = \overline{\chi}^*(G)$. ∎

Moreover, $\vartheta(G)$ can be approximated in polynomial time (Grötschel, Lovász, and Schrijver [1981]):

Theorem 67.2. *There is an algorithm that for any given graph $G = (V, E)$ and any $\varepsilon > 0$, returns a rational closer than ε to $\vartheta(G)$, in time bounded by a polynomial in $|V|$ and $\log(1/\varepsilon)$.*

Proof. This is a consequence of Corollary (4.3.12) in Grötschel, Lovász, and Schrijver [1988], stating that we can solve a convex optimization problem approximatively in polynomial time, if we know a ball contained in the feasible region and a ball containing the feasible region, and if we can test membership of the feasible region in polynomial time. These conditions are satisfied, if we restrict ourselves to the affine space $\mathcal{M}_G$. The convex body of all positive semidefinite matrices in $\mathcal{M}_G$ contains the ball with center $(1/|V|)\cdot I$ and radius $1/|V|^2$, and is contained in the ball with center the all-zero matrix and radius $|V|^2$. Membership can be tested in polynomial time, since we can test positive semidefiniteness in polynomial time. ∎

The two theorems above imply:

Corollary 67.2a. *For any graph G satisfying $\alpha(G) = \overline{\chi}(G)$, the stable set number can be found in polynomial time.*

Proof. Theorem 67.1 implies $\alpha(G) = \vartheta(G) = \overline{\chi}(G)$, and by Theorem 67.2 we can find a number closer than $\frac{1}{2}$ to $\vartheta(G)$ in time polynomial in $|V|$. Rounding to the closest integer yields $\alpha(G)$. ∎

To obtain an explicit maximum-size stable set, we need perfection of the graph:

Corollary 67.2b. *A maximum-size stable set in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph. Iteratively, for each $v \in V$, replace G by $G - v$ if $\alpha(G - v) = \alpha(G)$. By the perfection of G, we can calculate these values in polynomial time, by Corollary 67.2a.

We end up with a graph that forms a maximum-size stable set in the original graph. ∎

As perfection is closed under taking complements, also a maximum-size clique in a perfect graph can be found in polynomial time.

The method described in the proof of Corollary 67.2b applies to all graphs G for which $\alpha(H) = \vartheta(H)$ holds for each induced subgraph H of G; but, as we shall see in Corollary 67.14a, these are precisely the perfect graphs.

From Corollary 67.2b one can derive that a minimum colouring of a perfect graph can also be found in polynomial time (we follow the method given in Grötschel, Lovász, and Schrijver [1988]):

Corollary 67.2c. *A minimum colouring in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph. It suffices to find a stable set S intersecting each maximum-size clique in G; applying recursion to $G - S$ does the rest.

Starting with $t = 0$, we iteratively extend a list of maximum-size cliques $K_1, \ldots, K_t$ as follows. First, find a stable set S intersecting each of $K_1, \ldots, K_t$. This can be done by considering

(67.6) $$c := \chi^{K_1} + \cdots + \chi^{K_t},$$

and finding a stable set S maximizing $c(S)$. This can be found by replacing each vertex v by $c(v)$ nonadjacent vertices (adjacent to the new vertices that replace vertices adjacent to v), and finding a maximum-size stable set in the new graph. This gives a stable set S in the original graph maximizing $c(S)$.

Necessarily, $c(S) = t$, since G has a stable set intersecting each maximum-size clique (as G is perfect). So S intersects each K_i.

If $\omega(G-S) < \omega(G)$, then S intersects each maximum-size clique in G, and we are done. If $\omega(G-S) = \omega(G)$, we find a maximum-size clique K_{t+1} in $G-S$, add it to our list, and iterate.

The number of iterations is bounded by $|V|$, since in each iteration the dimension of the space L_t of vectors $x \in \mathbb{R}^V$ with $x(K_i) = 1$ for each i drops, as for the S found we have $\chi^S \in L_t$ and $\chi^S \notin L_{t+1}$. ∎

67.2. Weighted clique and colouring algorithmically

In a straightforward way, the results of the previous section can be extended to the weighted case. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$ be a weight function. Let G_w be the graph obtained from G by replacing each vertex v by a stable set S_v of size $w(v)$, where vertices in distinct S_u and S_v are adjacent if and only if u and v are adjacent in G. So the maximum *weight* of a stable set in G is equal to the maximum *size* of a stable set in G_w. Define:

(67.7) $$\alpha_w(G) := \alpha(G_w),\ \vartheta_w(G) := \vartheta(G_w),\ \overline{\chi}_w(G) := \overline{\chi}(G_w),$$ $$\overline{\chi}^*_w(G) := \overline{\chi}^*(G_w).$$

So $\alpha_w(G)$ is equal to the maximum weight of a stable set in G. The definitions of $\overline{\chi}_w(G)$ and $\overline{\chi}^*_w(G)$ agree with those in Section 64.8.

Theorem 67.1 gives the following inequalities:

Theorem 67.3. *For any graph $G = (V, E)$ and $w : V \to \mathbb{R}_+$:*

(67.8) $$\alpha_w(G) \leq \vartheta_w(G) \leq \overline{\chi}^*_w(G).$$

Proof. Directly from Theorem 67.1 and (67.7). ∎

In order to calculate $\vartheta_w(G)$, we need not construct G_w and calculate $\vartheta(G_w)$. This would not be a polynomial-time method. We can calculate $\vartheta_w(G)$ more concisely as follows.

Define $\sqrt{w} : V \to \mathbb{R}_+$ by:

(67.9) $$\sqrt{w}(v) := \sqrt{w(v)}$$

for $v \in V$. Then:

Theorem 67.4. *For any graph G and $w : VG \to \mathbb{Z}_+$:*

(67.10) $$\vartheta_w(G) = \max\{\sqrt{w}^{\mathsf{T}} M \sqrt{w} \mid M \in \mathcal{M}_G \text{ positive semidefinite}\}.$$

Proof. We may assume that $w > \mathbf{0}$. Let D be the $VG_w \times VG$ matrix defined by

(67.11) $$D_{u,v} := \begin{cases} w(v)^{-\frac{1}{2}} & \text{if } u \in S_v, \\ 0 & \text{if } u \notin S_v, \end{cases}$$

for $u \in VG_w$ and $v \in VG$.

First let M attain the maximum in (67.10). Then $M' := DMD^{\mathsf{T}}$ is positive semidefinite, and, moreover, belongs to $\mathcal{M}_{G_w}$. Indeed, for adjacent vertices u, u' of G_w, say $u \in S_v$ and $u' \in S_{v'}$, with v and v' adjacent vertices of G, we have $M_{v,v'} = 0$, and hence

(67.12) $$M'_{u,u'} = (DMD^{\mathsf{T}})_{u,u'} = \sum_{t,t' \in VG} D_{u,t} M_{t,t'} D_{u',t'} = w(v)^{-\frac{1}{2}} w(v')^{-\frac{1}{2}} M_{v,v'} = 0.$$

Also (setting $v_u := v$ if $u \in S_v$):

(67.13) $$\mathrm{Tr} M' = \mathrm{Tr}(DMD^{\mathsf{T}}) = \sum_{u \in VG_w} \sum_{v,v' \in VG} D_{u,v} D_{u,v'} M_{v,v'} = \sum_{u \in VG_w} w(v_u)^{-1} M_{v_u,v_u} = \sum_{v \in VG} w(v)^{-1} w(v) M_{v,v} = \mathrm{Tr} M = 1.$$

So $M' \in \mathcal{M}_{G_w}$. Hence

(67.14) $$\vartheta_w(G) = \vartheta(G_w) \geq \mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \mathbf{1}^{\mathsf{T}} (DMD^{\mathsf{T}}) \mathbf{1} = \sqrt{w}^{\mathsf{T}} M \sqrt{w}.$$

This shows $\geq$ in (67.10).

To see the reverse inequality, let M' be a positive semidefinite matrix in $\mathcal{M}_{G_w}$ with $\mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \vartheta(G_w)$. Then $M := D^{\mathsf{T}} M' D$ is positive semidefinite, and, moreover, belongs to $\mathcal{M}_G$. Indeed, for adjacent $v, v' \in VG$ we have

(67.15) $$M_{v,v'} = (D^{\mathsf{T}} M D)_{v,v'} = \sum_{u,u' \in VG_w} D_{u,v} D_{u',v'} M'_{u,u'} = \sum_{u \in S_v} \sum_{u' \in S_{v'}} w(v)^{-\frac{1}{2}} w(v')^{-\frac{1}{2}} M'_{u,u'} = 0.$$

Also:

(67.16) $$\mathrm{Tr} M = \sum_{v \in VG} \sum_{u,u' \in VG_w} D_{u,v} D_{u',v} M'_{u,u'} = \sum_{v \in VG} \sum_{u \in S_v} \sum_{u' \in S_v} w(v)^{-1} M'_{u,u'} \leq \sum_{v \in VG} \sum_{u \in S_v} M'_{u,u} = \mathrm{Tr} M' = 1.$$

The inequality holds as for any positive semidefinite matrix A one has: $\mathbf{1}^{\mathsf{T}} A \mathbf{1} \leq \mathbf{1}^{\mathsf{T}} \mathbf{1} \cdot \mathrm{Tr} A$, since the largest eigenvalue of A is at most $\mathrm{Tr} A$. This is applied to the $S_v \times S_v$ submatrix of M, for each $v \in V$.

Hence the matrix $\widetilde{M} := (\mathrm{Tr} M)^{-1} \cdot M$ belongs to $\mathcal{M}_G$, and so the maximum in (67.10) is at least $\sqrt{w}^{\mathsf{T}} \widetilde{M} \sqrt{w}$, and hence at least

(67.17) $$\sqrt{w}^{\mathsf{T}} M \sqrt{w} = \sqrt{w}^{\mathsf{T}} D^{\mathsf{T}} M' D \sqrt{w} = \mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \vartheta_w(G).$$ ∎

This implies that $\vartheta_w(G)$ can be approximated in polynomial time:

Theorem 67.5. *There is an algorithm that for any given graph $G = (V, E)$, any $w : V \to \mathbb{Z}_+$, and any $\varepsilon > 0$, returns a rational closer than ε to $\vartheta_w(G)$, in time bounded by a polynomial in $|V|$, $\log \|w\|_\infty$, and $\log(1/\varepsilon)$.*

Proof. Similar to the proof of Theorem 67.2. ∎

The two theorems above imply:

Corollary 67.5a. *For any graph G and weight function $w : V \to \mathbb{Z}_+$ satisfying $\alpha_w(G) = \overline{\chi}_w(G)$, the weighted stable set number can be found in polynomial time.*

Proof. Theorem 67.3 implies $\alpha_w(G) = \vartheta_w(G) = \overline{\chi}_w(G)$, and by Theorem 67.5 we can find a number closer than $\frac{1}{2}$ to $\vartheta_w(G)$ in time polynomial in $|V|$. Rounding to the closest integer yields $\alpha_w(G)$. ∎

To obtain a maximum-weight stable set explicitly, we again need perfection of the graph:

Corollary 67.5b. *A maximum-weight stable set in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph and $w : V \to \mathbb{Z}_+$. Iteratively, for each $v \in V$, replace G by $G - v$ if $\alpha_w(G - v) = \alpha_w(G)$. By the perfection of G, we can calculate these values in polynomial time, by Corollary 67.5a.

We end up with a graph that forms a maximum-weight stable set in the original graph. ∎

As perfection is closed under taking complements, also a maximum-weight clique in a perfect graph can be found in polynomial time. So for any $w : V \to \mathbb{Z}_+$, we can determine

(67.18) $\quad \omega_w(G) :=$maximum of $w(C)$ over cliques C of G

in polynomial time.

Moreover, a minimum weighted colouring of a perfect graph can be found in polynomial time (again, we follow the method given in Grötschel, Lovász, and Schrijver [1988]):

Corollary 67.5c. *Given a perfect graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Z}_+$, a minimum weighted colouring can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph and let $w : V \to \mathbb{Z}_+$. As in the proof of Corollary 67.2c, we can find a stable set S intersecting each maximum-weight clique in G, as follows. Starting with $t = 0$, we iteratively extend a list of maximum-weight cliques $K_1, \ldots, K_t$. First find a stable set S intersecting each of $K_1, \ldots, K_t$. This can be done by considering

$$(67.19) \qquad c := \chi^{K_1} + \cdots + \chi^{K_t},$$

and finding a stable set S maximizing $c(S)$. This can be found by replacing each vertex v by $c(v)$ nonadjacent vertices (adjacent to the new vertices that replace vertices adjacent to v), and finding a maximum-size stable set in the new graph. This gives a stable set S maximizing $c(S)$.

Necessarily, $c(S) = t$, since G has a stable set intersecting each maximum-weight clique (as G_w is perfect). So S intersects each K_i.

If $\omega_w(G - S) < \omega_w(G)$, then S intersects each maximum-weight clique in G, and we have the required S. If $\omega_w(G - S) = \omega_w(G)$, we find a maximum-weight clique K_{t+1} in $G - S$, add it to our list, and iterate.

The number of iterations is bounded by $|V|$, since in each iteration the dimension of the space L_t of vector $x \in \mathbb{R}^V$ with $x(K_i) = 1$ for each i drops, since for the S found we have $\chi^S \in L_t$ and $\chi^S \notin L_{t+1}$.

This describes the method to find a stable set intersecting all maximum-weight cliques. To find a minimum weighted colouring, we iteratively find stable sets $S_1, \ldots, S_i$, $\lambda_1, \ldots, \lambda_i \in \mathbb{Z}_+$, and a weight function w_i as follows. Set $w_1 := w$. Next iteratively for $i = 1, 2, \ldots$, as long as $w_i \neq \mathbf{0}$, find a stable set S_i intersecting all cliques C maximizing $w_i(C)$, calculate

$$(67.20) \qquad \lambda_i := \omega_{w_i}(G) - \omega_{w_i}(G - S_i),$$

and set $w_{i+1} := w_i - \lambda_i \chi^{S_i}$.

Then the λ_i, S_i form a minimum weighted colouring, since

$$(67.21) \qquad \sum_i \lambda_i \chi^{S_i} = w \text{ and } \sum_i \lambda_i = \omega_w(G) = \chi_w(G).$$

To prove this, we first show:

$$(67.22) \qquad \omega_{w_{i+1}}(G) = \omega_{w_{i+1}}(G - S_i) = \omega_{w_i}(G - S_i) = \omega_{w_i}(G) - \lambda_i.$$

Here the second equality is trivial (since w_i and w_{i+1} coincide outside S_i). The third inequality follows from definition (67.20) of λ_i. For the first equality, $\geq$ is trivial. To see $\leq$, consider a clique C intersecting S_i. Then $w_{i+1}(C) = w_i(C) - \lambda_i |C \cap S_i| \leq \omega_{w_i}(G) - \lambda_i$. This proves (67.22), which implies the second equality in (67.21).

Moreover, the number of iterations is at most $|V|$, since in each iteration the face of the clique polytope spanned by the cliques C maximizing $w_i(C)$, increases in dimension: each clique C in G maximizing $w_i(C)$ also maximizes $w_{i+1}(C)$ (since $w_{i+1}(C) \geq w_i(C) - \lambda_i = \omega_{w_i}(G) - \lambda_i = \omega_{w_{i+1}}(G)$, by (67.22)), and there is a clique C maximizing $w_{i+1}(C)$ but not $w_i(C)$ (namely any clique C of $G - S_i$ maximizing $w_i(C)$, since $w_{i+1}(C) = w_i(C) = \omega_{w_i}(G - S_i) = \omega_{w_{i+1}}(G)$). ∎

67.3. Strong polynomial-time solvability

In the previous section we showed the polynomial-time solvability of the weighted versions of the stable set and colouring problems in perfect graphs. By Theorem 5.11 of Frank and Tardos [1985,1987], this can be strengthened to *strong* polynomial-time solvability.

Theorem 67.6. *A maximum-weight clique and a minimum weighted colouring in a perfect graph can be found in strongly polynomial time.*

Proof. A maximum-weight clique can be found in strongly polynomial time by Theorem 5.11, since the class of clique polytopes of perfect graphs is polynomial-time solvable by Corollary 67.5b.

Next, a minimum weighted colouring can be found with the method described in the proof of Corollary 67.5c: it is strongly polynomial-time because we can find (by the above) a maximum-weight clique in strongly polynomial time. ∎

This implies:

Corollary 67.6a. *A maximum-weight stable set and a minimum-weight vertex cover in a perfect graph can be found in strongly polynomial time.*

Proof. Directly from Theorem 67.6, since stable sets in a perfect graph are precisely the cliques in the complementary graph, which is again perfect. Moreover, the vertex covers are precisely the complements of stable sets. ∎

67.4. Further results and notes

67.4a. Further on $\vartheta(G)$

In this section we give some further results on the function $\vartheta(G)$, and we consider the related convex body $\mathrm{TH}(G)$. We use the following notation, for vector $a, b \in \mathbb{R}_+^V$:

(67.23) b/a is the vector in $\mathbb{R}^V$ with vth entry $b(v)/a(v)$,
$\sqrt{b} = b^{\frac{1}{2}}$ is the vector in $\mathbb{R}^V$ with vth entry $b(v)^{\frac{1}{2}}$,
$b^{-\frac{1}{2}}$ is the vector in $\mathbb{R}^V$ with vth entry $b(v)^{-\frac{1}{2}}$,
Δ_b is the $V \times V$ diagonal matrix with diagonal b.

We set $(b/a)_v := 0$ if $a_v = 0$ and $(b^{-\frac{1}{2}})_v := 0$ if $b_v = 0$. (This will turn out not to harm the consistency.)

Moreover, we define, for any graph $G = (V, E)$ and any symmetric matrix M:

(67.24) $\mathcal{L}_G :=$ the set of symmetric $V \times V$ matrices A with $A_{u,v} = 0$ if $u = v$ or u and v are nonadjacent;
$\Lambda(M) :=$ the largest eigenvalue of M,
$\mathrm{PSD} :=$ the set of symmetric positive semidefinite matrices.

We usually restrict PSD to appropriate dimensions, like $V \times V$. We define for any two matrices X, Y (of equal dimensions) the 'inner product' $X \bullet Y$ by

(67.25) $$X \bullet Y := \mathrm{Tr}(XY^{\mathsf{T}}).$$

So if $X \in \mathcal{M}_G$ and $Y \in \mathcal{L}_G$, then $X \bullet Y = 0$.

A min-max relation for $\vartheta_w(G)$

$\vartheta_w(G)$ is defined as a maximum. Applying convex duality, we can describe $\vartheta_w(G)$ alternatively as a minimum (Lovász [1979d]):

Theorem 67.7. *For each* $w \in \mathbb{R}_+^V$*:*

(67.26) $$\vartheta_w(G) = \min\{\Lambda(W + A) \mid A \in \mathcal{L}_G\},$$

where $W := \sqrt{w}\sqrt{w}^{\mathsf{T}}$.

Proof. Let M maximize $\sqrt{w}^{\mathsf{T}} M \sqrt{w}$ over $\mathrm{PSD} \cap \mathcal{M}_G$. So $\vartheta_w(G) = \sqrt{w}^{\mathsf{T}} M \sqrt{w}$.

To prove $\leq$ in (67.26), let $A \in \mathcal{L}_G$ attain the minimum in (67.26) and let $\lambda := \Lambda(W + A)$. Then $Y := \lambda I - W - A$ is positive semidefinite, and hence

(67.27) $$0 \leq Y \bullet M = (\lambda I - W - A) \bullet M = \lambda \mathrm{Tr} M - W \bullet M = \lambda - \sqrt{w}^{\mathsf{T}} M \sqrt{w} = \Lambda(W + A) - \vartheta_w(G).$$

To prove $\geq$ in (67.26), we use convexity theory. Since M maximizes $W \bullet M$ over the intersection of the convex sets PSD and $\mathcal{M}_G$, there exist supporting hyperplanes $\{X \mid C \bullet X = \gamma\}$ of PSD and $\{X \mid D \bullet X = \delta\}$ of $\mathcal{M}_G$ such that

(67.28) $$\mathrm{PSD} \subseteq \{X \mid C \bullet X \geq \gamma\},\ \mathcal{M}_G \subseteq \{X \mid D \bullet X \geq \delta\},\ C \bullet M = \gamma,\ D \bullet M = \delta, \text{ and } W = C + D.$$

Since PSD and $\mathcal{M}_G$ consist of symmetric matrices only, we can assume that C and D are symmetric (we can replace them by $\frac{1}{2}(C + C^{\mathsf{T}})$ and $\frac{1}{2}(D + D^{\mathsf{T}})$).

Since PSD is a convex cone, we have $\gamma = 0$. Then $C \in \mathrm{PSD}$, as $xx^{\mathsf{T}} \in \mathrm{PSD}$ for each $x \in \mathbb{R}^V$, hence $x^{\mathsf{T}} C x = C \bullet (xx^{\mathsf{T}}) \geq 0$.

Since $\mathcal{M}_G$ is an affine space and since $D \bullet M = \delta$, we have $\mathcal{M}_G \subseteq \{X \mid D \bullet X = \delta\}$. This implies that $D = \delta \cdot I - A$ for some $A \in \mathcal{L}_G$ (since each symmetric $0,1$ matrix containing precisely one 1 belongs to $\mathcal{M}_G$; the matrix remains to belong to $\mathcal{M}_G$ after putting a nonzero entry in any nonadjacent position and its transpose). So

(67.29) $$\delta = D \bullet M = (W - C) \bullet M = W \bullet M.$$

As C is positive semidefinite, $\delta \cdot I - W - A$ is positive semidefinite. Hence

(67.30) $$\Lambda(W + A) \leq \delta = W \bullet M = \vartheta_w(G).$$ ∎

The product $\vartheta(G)\vartheta(\overline{G})$ is at least $|V|$

For perfect graphs $G = (V, E)$, we have $\alpha(G)\omega(G) \geq |V|$, and hence $\vartheta(G)\vartheta(\overline{G}) \geq |V|$. The latter inequality holds for any graph G. To prove it, we use the following fact from matrix theory:

(67.31) If X and Y are symmetric positive semidefinite $n \times n$ matrices, then also $X * Y$ is positive definite,

where $X * Y$ is the $n \times n$ matrix given by: $(X * Y)_{i,j} = X_{i,j} Y_{i,j}$. (67.31) follows from the fact that there exist vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ with $X_{i,j} = u_i^{\mathsf{T}} u_j$ and $Y_{i,j} = v_i^{\mathsf{T}} v_j$ for all i, j. Hence $(X * Y)_{i,j} = (u_i \circ v_i)^{\mathsf{T}} (u_j \circ v_j)$ for all i, j, where $\circ$ denotes tensor product[13]. So $X * Y$ is positive semidefinite.

Theorem 67.8. *$\vartheta(G)\vartheta(\overline{G}) \geq |V|$ for each graph $G = (V, E)$.*

Proof. By (67.26), there exist $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_{\overline{G}}$ with

(67.32) $$\vartheta(G) = \Lambda(J + A) \text{ and } \vartheta(\overline{G}) = \Lambda(J + B).$$

So $C := \vartheta(G) \cdot I - J - A$ and $D := \vartheta(\overline{G}) \cdot I - J - B$ are positive semidefinite. Now

(67.33) $$\begin{aligned} C * D + C * J + J * D &= (C + J) * (D + J) - J * J \\ &= (\vartheta(G) \cdot I - A) * (\vartheta(\overline{G}) \cdot I - B) - J = \vartheta(G)\vartheta(\overline{G}) \cdot I - J \end{aligned}$$

(as $A * I = I * B = A * B$ is the all-zero matrix). By (67.31), the first matrix in (67.33) is positive semidefinite, hence also the last. So

(67.34) $$0 \leq \mathbf{1}^{\mathsf{T}} (\vartheta(G)\vartheta(\overline{G}) \cdot I - J)\mathbf{1} = \vartheta(G)\vartheta(\overline{G})|V| - |V|^2,$$

implying the theorem. ∎

The convex body TH(G)

The function $\vartheta_w(G)$ is related to a convex body TH(G) defined in Grötschel, Lovász, and Schrijver [1986]. The following equivalent representation of TH(G) was given by Lovász and Schrijver [1991].

For any symmetric matrix A, define the matrix $R(A)$ by:

(67.35) $$R(A) := \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix},$$

where $a := \text{diag}A$ (the diagonal vector of A; that is, $a_i = A_{i,i}$ for each coordinate i).

Given a graph $G = (V, E)$, consider the collection $\mathcal{R}_G$ of symmetric $V \times V$ matrices A with $R(A)$ positive semidefinite and with $A_{u,v} = 0$ for distinct adjacent u, v. Then define:

(67.36) $$\text{TH}(G) = \{\text{diag}A \mid A \in \mathcal{R}_G\}.$$

Theorem 67.9. TH(G) *is convex and down-monotone in* $\mathbb{R}_+^V$.

Proof. TH(G) is convex, as it is a projection of the convex set $\mathcal{R}_G$. Moreover, if $a \in \text{TH}(G)$ and $\mathbf{0} \leq b \leq a$, then $b \in \text{TH}(G)$. Indeed, since $a \in \text{TH}(G)$, there exists a matrix $A \in \mathcal{R}_G$ with $a = \text{diag}A$. Then the matrix

(67.37) $$\begin{pmatrix} 1 & 0 \\ 0 & \Delta_{b/a} \end{pmatrix} \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta_{b/a} \end{pmatrix} = \begin{pmatrix} 1 & b^{\mathsf{T}} \\ b & \Delta_{b/a} A \Delta_{b/a} \end{pmatrix}$$

[13] The *tensor product* of vectors $x \in \mathbb{R}^U$ and $y \in \mathbb{R}^V$ is the vector $x \circ y$ in $\mathbb{R}^{U \times V}$ defined by: $(x \circ y)_{(u,v)} := x_u y_v$ for $u \in U$ and $v \in V$.

is positive semidefinite. As the vth entry on the diagonal of $\Delta_{b/a} A \Delta_{b/a}$ is equal to $b(v)^2/a(v)$ (or 0 if $a(v) = 0$), which is at most $b(v)$, we have that

(67.38) $$\Delta_{b/a} A \Delta_{b/a} + (\Delta_{b-b^2/a})$$

belongs to $\mathcal{R}_G$ and has diagonal equal to b. This proves that $b \in \mathrm{TH}(G)$, and hence $\mathrm{TH}(G)$ is down-monotone. ∎

To obtain a relation of $\mathrm{TH}(G)$ with the function $\vartheta_w(G)$, we first show the following, where for $x, y \in \mathbb{R}^V$, $x * y$ is the vector in $\mathbb{R}^V$ defined by:

(67.39) $$(x * y)_v := x_v y_v \text{ for } v \in V.$$

Theorem 67.10. *Let M maximize $\sqrt{w}^{\mathsf{T}} M \sqrt{w}$ over $\mathrm{PSD} \cap \mathcal{M}_G$. Then*

(67.40) $$M\sqrt{w} = \vartheta_w(G) \cdot b * w^{-\frac{1}{2}},$$

where $b := \mathrm{diag} M$.

Proof. The maximum of

(67.41) $$\sqrt{w}^{\mathsf{T}} \Delta_x M \Delta_x \sqrt{w}$$

over $x \in \mathbb{R}^V$ satisfying $x^{\mathsf{T}} \Delta_b x = 1$, is attained by $x = \mathbf{1}$. (Otherwise we can replace M by $\Delta_x M \Delta_x$ to increase $\sqrt{w}^{\mathsf{T}} M \sqrt{w}$.) Now (67.41) is equal to

(67.42) $$x^{\mathsf{T}} \Delta_{\sqrt{w}} M \Delta_{\sqrt{w}} x.$$

So the maximum of (67.42) over $x \in \mathbb{R}^V$ satisfying $x^{\mathsf{T}} \Delta_b x = 1$, is attained by $x = \mathbf{1}$. Hence, by Lagrange's theorem, there exists a $\mu \in \mathbb{R}$ with

(67.43) $$\Delta_{\sqrt{w}} M \Delta_{\sqrt{w}} \mathbf{1} = \mu \cdot \Delta_b \mathbf{1} = \mu \cdot b.$$

Then

(67.44) $$\vartheta_w(G) = \sqrt{w}^{\mathsf{T}} M \sqrt{w} = \mathbf{1}^{\mathsf{T}} \Delta_{\sqrt{w}} M \Delta_{\sqrt{w}} \mathbf{1} = \mu \mathbf{1}^{\mathsf{T}} b = \mu \mathrm{Tr} M = \mu.$$

(67.43) and (67.44) give

(67.45) $$M\sqrt{w} = M \Delta_{\sqrt{w}} \mathbf{1} = \mu \cdot w^{-\frac{1}{2}} * b = \vartheta_w(G) \cdot b * w^{-\frac{1}{2}},$$

which is (67.40). ∎

Now the relation of $\mathrm{TH}(G)$ with $\vartheta_w(G)$ is:

Theorem 67.11. *For each $w \in \mathbb{R}^V_+$:*

(67.46) $$\vartheta_w(G) = \max\{w^{\mathsf{T}} x \mid x \in \mathrm{TH}(G)\}.$$

Proof. I. We first show $\leq$ in (67.46). Let M be a matrix maximizing $\sqrt{w}^{\mathsf{T}} M \sqrt{w}$ over the positive semidefinite matrices $M \in \mathcal{M}_G$. It suffices to show that the matrix

(67.47) $$A := \vartheta_w(G) \cdot \Delta_{w^{-\frac{1}{2}}} M \Delta_{w^{-\frac{1}{2}}}$$

belongs to $\mathcal{R}_G$, since $w^{\mathsf{T}} \mathrm{diag} A = \vartheta_w(G) \mathrm{Tr} M = \vartheta_w(G)$.

Trivially, $A_{u,v} = 0$ for distinct adjacent u, v (since $M_{u,v} = 0$ for distinct adjacent u, v). To see that $R(A)$ is positive semidefinite, write $a := \mathrm{diag} A$, $b := \mathrm{diag} M$, and $\vartheta := \vartheta_w(G)$. By (67.40) we have $M\sqrt{w} = \vartheta \cdot b * w^{-\frac{1}{2}}$. So $\Delta_{w^{-\frac{1}{2}}} M \sqrt{w} = \vartheta \cdot \Delta_{w^{-\frac{1}{2}}} (b * w^{-\frac{1}{2}}) = \vartheta \cdot (b/w)$. Hence

(67.48) $$R(A) = \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} = \begin{pmatrix} 1 & \vartheta \cdot (b/w)^{\mathsf{T}} \\ \vartheta \cdot (b/w) & \vartheta \cdot \Delta_{w^{-\frac{1}{2}}} M \Delta_{w^{-\frac{1}{2}}} \end{pmatrix}$$
$$= \begin{pmatrix} \vartheta^{-1}\sqrt{w}^{\mathsf{T}} M \sqrt{w} & \sqrt{w}^{\mathsf{T}} M \Delta_{w^{-\frac{1}{2}}} \\ \Delta_{w^{-\frac{1}{2}}} M \sqrt{w} & \vartheta \cdot \Delta_{w^{-\frac{1}{2}}} M \Delta_{w^{-\frac{1}{2}}} \end{pmatrix} = \vartheta^{-1} \cdot U^{\mathsf{T}} M U,$$

where U is the matrix given by

(67.49) $$U := (\sqrt{w} \quad \vartheta \cdot \Delta_{w^{-\frac{1}{2}}}).$$

So $R(A)$ is positive semidefinite.

II. To see $\geq$ in (67.46), let $A \in \mathcal{R}_G$ maximize $w^{\mathsf{T}}\mathrm{diag}A$. Define $a := \mathrm{diag}A$, $\eta := w^{\mathsf{T}}a$, and

(67.50) $$M := \eta^{-1} \cdot \Delta_{w^{\frac{1}{2}}} A \Delta_{w^{\frac{1}{2}}}.$$

Trivially, M is positive semidefinite and belongs to $\mathcal{M}_G$. Also

(67.51) $$0 \leq (\eta, -w^{\mathsf{T}}) \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} \begin{pmatrix} \eta \\ -w \end{pmatrix} = \eta^2 - 2\eta \cdot w^{\mathsf{T}} a + w^{\mathsf{T}} A w$$
$$= \eta \sqrt{w}^{\mathsf{T}} M \sqrt{w} - \eta^2.$$

Therefore $\sqrt{w}^{\mathsf{T}} M \sqrt{w} \geq \eta$, which proves $\geq$ in (67.46). ∎

(67.46) implies that $\vartheta_w(G)$ is a convex function of w and that

(67.52) $$\mathrm{TH}(G) = \{x \in \mathbb{R}^V_+ \mid w^{\mathsf{T}} x \leq \vartheta_w(G) \text{ for each } w \in \mathbb{R}^V_+\}.$$

By (67.8),

(67.53) $$\alpha_w(G) \leq \vartheta_w(G) \leq \overline{\chi}^*_w(G).$$

This gives:

Corollary 67.11a. *For each graph $G = (V, E)$:*

(67.54) $$P_{\text{stable set}}(G) \subseteq \mathrm{TH}(G) \subseteq A(P_{\text{clique}}(G)).$$

Proof. This follows directly from Theorem 67.11 with the inequalities (67.53), since for each $w \in \mathbb{R}^V_+$:

(67.55) $$\alpha_w(G) = \max\{w^{\mathsf{T}} x \mid x \in P_{\text{stable set}}(G)\},$$
$$\vartheta_w(G) = \max\{w^{\mathsf{T}} x \mid x \in \mathrm{TH}(G)\},$$
$$\overline{\chi}^*_w(G) = \max\{w^{\mathsf{T}} x \mid x \in A(P_{\text{clique}}(G))\}.$$ ∎

The antiblocking body of TH(G)

It turns out that taking the antiblocking body $A(\mathrm{TH}(G))$ of $\mathrm{TH}(G)$ corresponds to replacing G by its complement (Grötschel, Lovász, and Schrijver [1986]). We first observe that

(67.56) $$A(\mathrm{TH}(G)) = \{w \in \mathbb{R}^V_+ \mid \vartheta_w(G) \leq 1\},$$

since for each $w : V \to \mathbb{R}_+$: $w \in A(\mathrm{TH}(G)) \iff \max\{w^\top x \mid x \in \mathrm{TH}(G)\} \le 1$ $\iff \vartheta_w(G) \le 1$.

Theorem 67.12. *$A(\mathrm{TH}(G)) = \mathrm{TH}(\overline{G})$.*

Proof. I. We first show $A(\mathrm{TH}(G)) \subseteq \mathrm{TH}(\overline{G})$. Let $w \in A(\mathrm{TH}(G))$; that is (by (67.56)), $\vartheta_w(G) \le 1$. To show that w belongs to $\mathrm{TH}(\overline{G})$ we should show by (67.52) that

$$w^\top a \le \vartheta_a(\overline{G}) \tag{67.57}$$

for each $a \in \mathbb{R}^V_+$.

By (67.26), there exist $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_{\overline{G}}$ such that

$$\vartheta_w(G) = \Lambda(\sqrt{w}\sqrt{w}^\top + A) \text{ and } \vartheta_a(\overline{G}) = \Lambda(\sqrt{a}\sqrt{a}^\top + B). \tag{67.58}$$

So $C := \vartheta_w(G) \cdot I - \sqrt{w}\sqrt{w}^\top - A$ and $D := \vartheta_a(\overline{G}) \cdot I - \sqrt{a}\sqrt{a}^\top - B$ are positive semidefinite. Therefore, the matrix

$$\vartheta_w(G)\vartheta_a(\overline{G})\cdot I - \sqrt{w*a}\sqrt{w*a}^\top = C*D + C*(\sqrt{a}\sqrt{a}^\top) + (\sqrt{w}\sqrt{w}^\top)*D \tag{67.59}$$

is positive semidefinite by (67.31) (note that $A*I = I*B = A*B$ is the all-zero matrix). Hence

$$\begin{aligned} 0 &\le \sqrt{w*a}^\top(\vartheta_w(G)\vartheta_a(\overline{G})\cdot I - \sqrt{w*a}\sqrt{w*a}^\top)\sqrt{w*a} \\ &= \vartheta_w(G)\vartheta_a(\overline{G})\sqrt{w*a}^\top\sqrt{w*a} - \sqrt{w*a}^\top\sqrt{w*a}\sqrt{w*a}^\top\sqrt{w*a} \\ &= \vartheta_w(G)\vartheta_a(\overline{G})w^\top a - (w^\top a)^2, \end{aligned} \tag{67.60}$$

implying (67.57).

II. To prove $\mathrm{TH}(\overline{G}) \subseteq A(\mathrm{TH}(G))$, let $w \in \mathrm{TH}(\overline{G})$. By (67.56) we should prove $\vartheta_w(G) \le 1$.

Let B maximize $\sqrt{w}^\top B\sqrt{w}$ over $\mathrm{PSD} \cap \mathcal{M}_G$. Let $b := \mathrm{diag}B$ and define

$$C := \Delta_{\sqrt{w/b}} B \Delta_{\sqrt{w/b}}. \tag{67.61}$$

Then, with (67.40),

$$C\sqrt{b} = \Delta_{\sqrt{w/b}} B\sqrt{w} = \mu \cdot \Delta_{\sqrt{w/b}} b * w^{-\frac{1}{2}} = \mu \cdot \sqrt{b}, \tag{67.62}$$

where $\mu := \vartheta_w(G)$. So C has $\sqrt{b}$ as eigenvector, with eigenvalue μ. Since C is positive semidefinite, also the matrix

$$C - \mu(\sqrt{b}\sqrt{b}^\top) \tag{67.63}$$

is positive semidefinite. Hence the matrix

$$\Delta_{w^{-\frac{1}{2}}}(C - \mu\cdot\sqrt{b}\sqrt{b}^\top)\Delta_{w^{-\frac{1}{2}}} = \Delta_{b^{-\frac{1}{2}}} B \Delta_{b^{-\frac{1}{2}}} - \mu\cdot\sqrt{b/w}\sqrt{b/w}^\top \tag{67.64}$$

is positive semidefinite.

Define $A := I - \Delta_{b^{-\frac{1}{2}}} B \Delta_{b^{-\frac{1}{2}}}$ and $z := b/w$. So $A \in \mathcal{L}_{\overline{G}}$ and $\mu\cdot\sqrt{z}\sqrt{z}^\top + A$ has largest eigenvalue at most 1. Hence $\vartheta_z(\overline{G}) \le \mu^{-1}$, and so

$$\vartheta_w(G)\vartheta_z(\overline{G}) = \mu\vartheta_z(\overline{G}) \le 1 = \mathrm{Tr}B = b^\top\mathbf{1} = w^\top z \le \vartheta_z(\overline{G}), \tag{67.65}$$

where the last inequality holds as $w \in \mathrm{TH}(\overline{G})$. Hence $\vartheta_w(G) \le 1$. ∎

Facets of TH(G)

A subset F of TH(G) is called a *facet* of TH(G) if there is an inequality $c^\mathsf{T}x \leq \gamma$ (with $c \neq \mathbf{0}$) which is valid for TH(G), such that F is the set of vectors in TH(G) having equality and such that F has dimension $|V|-1$. Then (Grötschel, Lovász, and Schrijver [1986]):

Theorem 67.13. *For each graph $G=(V,E)$, each facet F of* TH(G) *is determined by an inequality $x_v \geq 0$ for some $v \in V$ or by $x(C) \leq 1$ for some clique C of G.*

Proof. Let F be determined by the inequality $c^\mathsf{T}x \leq \gamma$. If there is a $v \in V$ with $x_v = 0$ for each $x \in F$, then F is determined by the inequality $x_v \geq 0$. So we can assume that $x > 0$ for some $x \in F$. Since $\mathrm{TH}(G) = A(\mathrm{TH}(\overline{G}))$, there is a $w \in \mathbb{R}_+^V$ with $\vartheta_w(\overline{G}) = 1$ and F is determined by $w^\mathsf{T}x \leq 1$. So $w \in \mathrm{TH}(\overline{G})$, and therefore there is a matrix $A \in \mathcal{R}_{\overline{G}}$ with diag$A = w$. As $A \in \mathcal{R}_{\overline{G}}$, the matrix $R(A)$ is positive semidefinite. Hence there exist linearly independent vectors $\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix}$ $(i=1,\ldots,k)$ such that

(67.66) $$\begin{pmatrix}1 & w^\mathsf{T}\\ w & A\end{pmatrix} = R(A) = \sum_{i=1}^{k}\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix}(\alpha_i, a_i^\mathsf{T}).$$

We can assume that $\alpha_i \geq 0$ for each $i=1,\ldots,k$. Now

(67.67) $a_i^\mathsf{T}x = \alpha_i$ for each $x \in F$ and each $i=1,\ldots,k$.

To see this, choose $x \in F$. As $x \in \mathrm{TH}(G)$, there is a matrix $B \in \mathcal{R}_G$ with diag$B = x$. Since $R(B)$ is positive semidefinite, also the matrix

(67.68) $$B' := \begin{pmatrix}1 & -x^\mathsf{T}\\ -x & B\end{pmatrix}$$

is positive semidefinite. We therefore have (where again $X \bullet Y := \mathrm{Tr}(XY^\mathsf{T})$):

(67.69) $$\sum_{i=1}^{k}(\alpha_i, a_i^\mathsf{T})B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = R(A)\bullet B' = 1-2w^\mathsf{T}x+A\bullet B = 1-2w^\mathsf{T}x+w^\mathsf{T}x = 0.$$

(Here $A \bullet B = w^\mathsf{T}x$ follows from the fact that $A \in \mathcal{R}_G$, $B \in \mathcal{R}_{\overline{G}}$, diag$A = w$, and diag$B = x$.)

Since B' is positive semidefinite, (67.69) implies that, for each $i=1,\ldots,k$:

(67.70) $$(\alpha_i, a_i^\mathsf{T})B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = 0,$$

and therefore

(67.71) $$B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = \mathbf{0}.$$

In particular,

(67.72) $$(1, -x^\mathsf{T})\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = 0,$$

that is, $a_i^{\mathsf{T}} x = \alpha_i$, proving (67.67).

Since F is a facet, and since the $\begin{pmatrix} \alpha_i \\ a_i \end{pmatrix}$ are linearly independent, we know $k = 1$. So

$$(67.73) \qquad \begin{pmatrix} 1 & w^{\mathsf{T}} \\ w & A \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ a_1 \end{pmatrix} (\alpha_1, a_1^{\mathsf{T}}).$$

Since $\alpha_1 \geq 0$, this implies $\alpha_1 = 1$ and $a_1 = w$. Since diag$A = w$, we know $w(v)^2 = w(v)$ for each $v \in V$, and so $w \in \{0,1\}^V$. Hence $A = \chi^C (\chi^C)^{\mathsf{T}}$ for some $C \subseteq V$. As $A_{u,v} = 0$ for distinct nonadjacent u, v, we know that C is a clique. ∎

This gives as consequence:

Corollary 67.13a. TH(G) *is a polytope if and only if G is perfect.*

Proof. If G is perfect, we have

$$(67.74) \qquad P_{\text{stable set}}(G) \subseteq \text{TH}(G) \subseteq A(P_{\text{clique}}(G)) = P_{\text{stable set}}(G),$$

implying that TH$(G) = P_{\text{stable set}}(G)$, and therefore is a polytope.

To see the reverse implication, if TH(G) is a polytope, by (67.54) and Theorem 67.13, TH(G) is fully determined by the nonnegativity and clique inequalities; that is,

$$(67.75) \qquad \text{TH}(G) = A(P_{\text{clique}}(G)).$$

Since also $A(\text{TH}(G)) = \text{TH}(\overline{G})$ is a polytope, we know similarly that $\text{TH}(\overline{G}) = A(P_{\text{clique}}(\overline{G}))$. Hence

$$(67.76) \qquad \text{TH}(G) = A(\text{TH}(\overline{G})) = P_{\text{clique}}(\overline{G}) = P_{\text{stable set}}(G).$$

(67.75) and (67.76) imply that $P_{\text{stable set}}(G) = A(P_{\text{clique}}(G))$, and therefore G is perfect by Corollary 65.2e. ∎

Characterizing perfection by $\vartheta(G)$

Lovász [1983b] showed that perfection can be characterized by the function $\vartheta(G)$. To this end, Lovász first proved:

Theorem 67.14. *If G is a partitionable graph, then*

$$(67.77) \qquad \alpha(G) < \vartheta(G) < \overline{\chi}^*(G).$$

Proof. Let M be the incidence matrix of the maximum-size stable sets in G and let N be the incidence matrix of the maximum-size cliques of G. Define $n := |VG|$, $\alpha := \alpha(G)$, and $\omega := \omega(G)$. We first show the second inequality.

Let λ be the smallest eigenvalue of $N^{\mathsf{T}}N$. Since N is nonsingular (Theorem 65.9), we know $\lambda > 0$, and since $\text{Tr}(N^{\mathsf{T}}N) = n\omega$ and $N^{\mathsf{T}}N\mathbf{1} = \omega^2 \cdot \mathbf{1}$, we know $\lambda < \omega$ (otherwise $\text{Tr}(N^{\mathsf{T}}N) \geq \omega^2 + (n-1)\omega > n\omega$). So

$$(67.78) \qquad N^{\mathsf{T}}N - \lambda I - \frac{\omega^2 - \lambda}{n} J$$

is positive semidefinite, and therefore

$$(67.79)\qquad \frac{n(\omega-\lambda)}{\omega^2-\lambda}I - J + \frac{n}{\omega^2-\lambda}(N^{\mathsf{T}}N-\omega I)$$

is positive semidefinite. So (using (67.26) and (65.24))

$$(67.80)\qquad \vartheta(G) \le \Lambda(J - \frac{n}{\omega^2-\lambda}(N^{\mathsf{T}}N-\omega I)) \le \frac{n(\omega-\lambda)}{\omega^2-\lambda} < \frac{n}{\omega} = \chi^*(\overline{G}).$$

So we have the second inequality in (67.77), which implies the first, since:

$$(67.81)\qquad \vartheta(G) \ge \frac{n}{\vartheta(\overline{G})} > \frac{n}{\chi^*(G)} = \alpha,$$

by (65.24) and Theorem 67.8. ∎

This implies a characterization of perfect graphs:

Corollary 67.14a. *For any graph G, the following are equivalent:*

(67.82)
(i) *G is perfect,*
(ii) *$\alpha(H) = \vartheta(H)$ for each induced subgraph H of G,*
(iii) *$\vartheta(H) = \overline{\chi}^*(H)$ for each induced subgraph H of G,*
(iv) *$\vartheta(H)$ is an integer for each induced subgraph H of G.*

Proof. Directly from Theorem 67.14, using (65.24). ∎

67.4b. The Shannon capacity $\Theta(G)$

Shannon [1956] introduced the following parameter $\Theta(G)$, now called the Shannon capacity of a graph G.

The *strong product* $G \cdot H$ of graphs G and H is the graph with vertex set $VG \times VH$, with two distinct vertices (u, v) and (u', v') adjacent if and only if u and u' are equal or adjacent in G and v and v' are equal or adjacent in H.

The strong product of k copies of G is denoted by G^k. Then the *Shannon capacity* $\Theta(G)$ of G is defined by:

$$(67.83)\qquad \Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}.$$

(The interpretation is that if V is an alphabet, and adjacency means 'confusable', then $\alpha(G^k)$ is the maximum number of k-letter words any two of which have unequal and inconfusable letters in at least one position. Then $\Theta(G)$ is the maximum possible 'information rate'.)

Since $\alpha(G^{k+l}) \ge \alpha(G^k)\alpha(G^l)$, we know by Fekete's lemma (Corollary 2.2a) that

$$(67.84)\qquad \Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}.$$

Guo and Watanabe [1990] showed that there exist graphs G for which $\Theta(G)$ is not achieved by a finite product (that is, $\sqrt[k]{\alpha(G^k)} < \Theta(G)$ for each k).

Since $\alpha(G^k) \ge \alpha(G)^k$, we have

$$(67.85)\qquad \alpha(G) \le \Theta(G),$$

while strict inequality may hold: the 5-circuit C_5 has $\alpha(C_5) = 2$ and $\alpha(C_5^2) = 5$. (If C_5 has vertices $1, \ldots, 5$ and edges 12, 23, 34, 45, and 51, then $(1,1)$, $(2,3)$, $(3,5)$, $(4,2)$, $(5,4)$ is a stable set in C_5^2.) So $\Theta(C_5) \geq \sqrt{5}$, and Shannon [1956] raised the question if equality holds here. Shannon proved $\Theta(C_5) \leq \frac{5}{2}$; more generally, he proved, for any graph G:

(67.86) $$\Theta(G) \leq \overline{\chi}^*(G),$$

where $\overline{\chi}^*(G)$ is the fractional clique cover number. This bound can be proved by showing that

(67.87) $$\overline{\chi}^*(G \cdot H) \leq \overline{\chi}^*(G)\overline{\chi}^*(H).$$

This follows from the fact that if C and D are cliques of G and H respectively, then $C \times D$ is a clique of $G \cdot H$; hence if $\lambda : \mathcal{C} \to \mathbb{R}_+$ and $\mu : \mathcal{D} \to \mathbb{R}_+$ are minimum fractional clique covers for G and H respectively, where $\mathcal{C}$ and $\mathcal{D}$ denote the collections of cliques of G and H respectively, then (where $\circ$ denotes tensor product — see footnote on page 1161, and $\mathbf{1}_U$ denotes the all-one vector in $\mathbb{R}^U$, for any set U)

(67.88) $$\sum_{C\in\mathcal{C}}\sum_{D\in\mathcal{D}} \lambda_C\mu_D\chi^{C\times D} = \sum_{C\in\mathcal{C}}\sum_{D\in\mathcal{D}} \lambda_C\mu_D(\chi^C \circ \chi^D)$$
$$= \Big(\sum_{C\in\mathcal{C}} \lambda_C\chi^C\Big) \circ \Big(\sum_{D\in\mathcal{D}} \mu_D\chi^D\Big) = \mathbf{1}_{VG} \circ \mathbf{1}_{VH} = \mathbf{1}_{VG\times VH}$$

and hence

(67.89) $$\overline{\chi}^*(G \cdot H) \leq \sum_{C\in\mathcal{C}}\sum_{D\in\mathcal{D}} \lambda_C\mu_D = \Big(\sum_{C\in\mathcal{C}} \lambda_C\Big)\Big(\sum_{D\in\mathcal{D}} \mu_D\Big) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

This proves (67.87) (in (67.112) we show equality).

(67.87) implies (67.86), since

(67.90) $$\sqrt[k]{\alpha(G^k)} \leq \sqrt[k]{\overline{\chi}^*(G^k)} \leq \sqrt[k]{\overline{\chi}^*(G)^k} = \overline{\chi}^*(G).$$

This bound was improved by Lovász [1979d] as follows (which will imply that $\Theta(C_5) = \sqrt{5}$):

Theorem 67.15. *$\Theta(G) \leq \vartheta(G)$ for each graph G.*

Proof. Since $\alpha(G) \leq \vartheta(G)$, it suffices to show that for each k: $\alpha(G^k) \leq \vartheta(G)^k$. For this it suffices to show that

(67.91) $$\vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H)$$

for any graphs G and H.

By (67.26), there exist matrices $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_H$ such that

(67.92) $$\vartheta(G) = \Lambda(J_{VG} + A) \text{ and } \vartheta(H) = \Lambda(J_{VH} + B),$$

where J_U denotes the $U \times U$ all-one matrix, for any set U. Hence the matrices

(67.93) $$C := \vartheta(G) \cdot I_{VG} - J_{VG} - A \text{ and } D := \vartheta(H) \cdot I_{VH} - J_{VH} - B$$

are positive semidefinite, where I_U denotes the $U \times U$ identity matrix, for any set U.

Therefore, also the following matrix[14] is positive semidefinite:

[14] The *tensor product* of a $W \times X$ matrix M and a $Y \times Z$ matrix N (where W, X, Y, Z are sets), is the $(W \times Y) \times (X \times Z)$ matrix $M \circ N$ defined by

(67.94) $$C \circ D + C \circ J_{VH} + J_{VG} \circ D = (C + J_{VG}) \circ (D + J_{VH}) - J_{VG} \circ J_{VH} \\ = (\vartheta(G) \cdot I_{VG} - A) \circ (\vartheta(H) \cdot I_{VH} - B) - J_{VG \times VH} \\ = \vartheta(G)\vartheta(H) \cdot I_{VG \times VH} - J_{VG \times VH} - M,$$

where $M := \vartheta(G) \cdot I_{VG} \circ B + \vartheta(H) A \circ I_{VH} - A \circ B$. Since $I_{VG} \circ B$, $A \circ I_{VH}$, and $A \circ B$ belong to $\mathcal{L}_{G \cdot H}$,[15] also M belongs to $\mathcal{L}_{G \cdot H}$. Therefore,

(67.95) $$\vartheta(G \cdot H) \leq \Lambda(J_{VG \times VH} + M) \leq \vartheta(G)\vartheta(H),$$

giving (67.91). ∎

This proof consists of showing the inequality (67.91) for any two graphs G and H. In fact, equality holds (Lovász [1979d]):

(67.96) $$\vartheta(G \cdot H) = \vartheta(G)\vartheta(H).$$

Indeed, let M and N attain the maximum in definition (67.1) for $\vartheta(G)$ and $\vartheta(H)$ respectively. Then $M \circ N \in \mathcal{M}_{G \cdot H}$, and hence

(67.97) $$\vartheta(G \cdot H) \geq \mathbf{1}_{VG \times VH}^{\mathsf{T}} (M \circ N) \mathbf{1}_{VG \times VH} = (\mathbf{1}_{VG}^{\mathsf{T}} M \mathbf{1}_{VG})(\mathbf{1}_{VH}^{\mathsf{T}} N \mathbf{1}_{VH}) \\ = \vartheta(G)\vartheta(H).$$

Theorem 67.15 implies that $\Theta(C_5) = \sqrt{5}$. One may give an explicit construction to prove this, but it also follows from the following general result (Lovász [1979d]):[16]

Theorem 67.16. *For each graph $G = (V, E)$: $\vartheta(G)\vartheta(\overline{G}) \geq |V|$, with equality if G is vertex-transitive.*

Proof. The inequality is Theorem 67.8. If G is vertex-transitive, then $\mathbf{1}^{\mathsf{T}} x$ is maximized over $\mathrm{TH}(G)$ at a vector $x = \mu \cdot \mathbf{1}$ for some $\mu \in \mathbb{R}$, since if it is maximized at x we can replace it by

(67.98) $$\frac{1}{|\Gamma|} \sum_{P \in \Gamma} Px,$$

where Γ is the group of permutation matrices representing automorphisms of G. (This follows from the fact that $Px \in \mathrm{TH}(G)$ and $\mathbf{1}^{\mathsf{T}} Px = \mathbf{1}^{\mathsf{T}} x$.)

As the maximum value is equal to $\vartheta := \vartheta(G)$, we know $\mathbf{1}^{\mathsf{T}} x = \vartheta$, and so $\mu = \vartheta/n$, where $n := |V|$. Since $x \in \mathrm{TH}(G) = A(\mathrm{TH}(\overline{G}))$ (by Theorem 67.12), we have $\vartheta_x(\overline{G}) \leq 1$; hence (as $x = \mu \cdot \mathbf{1}$) $\vartheta(\overline{G}) \leq \mu^{-1} = n/\vartheta$. This shows $\vartheta(G)\vartheta(\overline{G}) \leq n$. ∎

$$(M \circ N)_{(w,y),(x,z)} := M_{w,x} N_{y,z}$$

for $w \in W$, $x \in X$, $y \in Y$, $z \in Z$. If M and N are symmetric positive semidefinite matrices, then $M \circ N$ is symmetric and positive semidefinite again, since if $M = U^{\mathsf{T}} U$ and $N = V^{\mathsf{T}} V$, then $M \circ N = (U \circ V)^{\mathsf{T}} (U \circ V)$.

[15] To see this, let (u, v) and (u', v') be equal or nonadjacent. Then (by definition of $G \cdot H$) $u = u'$ and $v = v'$, or $u \neq u'$ and u and u' are nonadjacent, or $v \neq v'$ and v and v' are nonadjacent. Hence $(I_{VG})_{u,u'} = 0$ or $B_{v,v'} = 0$, and $A_{u,u'} = 0$ or $(I_{VH})_{v,v'} = 0$, and $A_{u,u'} = 0$ or $B_{v,v'} = 0$.

[16] An *automorphism* of a graph $G = (V, E)$ is a permutation $\pi : V \to V$ with $E = \{\{\pi(u), \pi(v)\} \mid \{u, v\} \in E\}$. The graph G is *vertex-transitive* if for all $u, v \in V$ there exists an automorphism π with $\pi(u) = v$.

Since $\overline{C}_5$ is isomorphic to C_5, Theorem 67.16 gives $\vartheta(C_5) = \sqrt{5}$. So $\Theta(G) \leq \sqrt{5}$. As $\Theta(G) \geq \sqrt{\alpha(C_5^2)} = \sqrt{5}$, one has $\Theta(G) = \sqrt{5}$.

Another consequence of Theorem 67.16 is that for any vertex-transitive graph G: $\Theta(G \cdot \overline{G}) = |VG|$, since the pairs (v, v) for $v \in VG$ form a stable set in $G \cdot \overline{G}$ (so $\Theta(G \cdot \overline{G}) \geq |VG|$), and since $\Theta(G \cdot \overline{G}) \leq \vartheta(G \cdot \overline{G}) = \vartheta(G)\vartheta(\overline{G}) = |VG|$. If moreover G is self-complementary (like C_5), then $\Theta(G) = \sqrt{|VG|}$.

For graphs that are not vertex-transitive, $\vartheta(G)\vartheta(\overline{G}) > |VG|$ may hold, even $\alpha(G)\alpha(\overline{G}) > |VG|$, for instance for $G = K_{1,2}$.

Lovász [1979d] also gave the value of $\vartheta(C_n)$ for any odd circuit C_n:

(67.99) $$\vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} \text{ for odd } n.$$

For odd $n \geq 7$, it is unknown if this is the value of $\Theta(C_n)$. Since each C_n is vertex-transitive, by Theorem 67.16 we can derive from (67.99) the value of $\vartheta(\overline{C}_n)$ for odd n.

Lovász asked the question if $\Theta(G) = \vartheta(G)$ for each graph G. This was answered in the negative by Haemers [1979], by giving the following alternative upper bound on the Shannon capacity of a graph $G = (V, E)$. Let $\eta(G)$ be the minimum rank of a $V \times V$ matrix M (over any field) such that $M_{v,v} = 1$ for each $v \in V$ and $M_{u,v} = 0$ for distinct nonadjacent u and v. Then

(67.100) $$\Theta(G) \leq \eta(G).$$

This follows from the facts that $\alpha(G) \leq \eta(G)$ (since any stable set S in G gives an $S \times S$ identity submatrix of M), and that $\eta(G \cdot H) \leq \eta(G)\eta(H)$ (since $\text{rank}(M \circ N) = \text{rank}(M)\text{rank}(N)$ for any two matrices (over the same field)). Moreover, one has $\eta(G) \leq \overline{\chi}(G)$ (by considering, for any clique cover of G, the $\{0, 1\}$ matrix M with $M_{u,v} = 1$ if and only if u and v belong to some clique in the clique cover).

Haemers gave a graph G on 27 vertices (the complement of the 'Schläfli graph') with $\eta(G) \leq 7$ and $\vartheta(G) = 9$, implying $\Theta(G) \leq 7 < \vartheta(G)$. Since $\vartheta(\overline{G}) = 3$, this also gives an example of a graph G satisfying $\Theta(G)\Theta(\overline{G}) < |VG|$ and (hence) $\Theta(G)\Theta(\overline{G}) < \Theta(G \cdot \overline{G})$. (This disproves the conjecture of Shannon [1956] that $\Theta(G)\Theta(H) = \Theta(G \cdot H)$ for all graphs G, H, and answering to the negative the question of Lovász [1979d] whether $\Theta(G)\Theta(\overline{G}) \geq |VG|$ for all graphs G.)

It is unknown if Haemers' bound $\eta(G)$ can be computed in polynomial time. (Peeters [1996] reports results on this. More work on Haemers' bound in Haemers [1981].)

The following bound follows with a method of Rosenfeld [1967]:

(67.101) $$\alpha(G \cdot H) \leq \overline{\chi}^*(G)\alpha(H).$$

To see this, let $C_1, \ldots, C_k$ be cliques in G and $\lambda_1, \ldots, \lambda_k \geq 0$ be such that

(67.102) $$\lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k} = \mathbf{1}_{VG} \text{ and } \lambda_1 + \cdots + \lambda_k = \overline{\chi}^*(G).$$

Let $S \subseteq VG \times VH$ be a stable set in $G \cdot H$ of size $\alpha(G \cdot H)$. For each $u \in VG$, let $S_u := \{v \in VH \mid (u, v) \in S\}$. Then S_u is a stable set of H, and if u and u' are adjacent vertices of G, then $S_u \cap S_{u'} = \emptyset$. For each $i = 1, \ldots, k$, let

(67.103) $$T_i := \{v \in VH \mid \exists u \in C_i : (u, v) \in S\} = \bigcup_{u \in C_i} S_u.$$

Since C_i is a clique in G, T_i is a stable set in H, and $|T_i| = \sum_{u \in C_i} |S_u|$. Hence

(67.104) $$|S| = \sum_{u \in VG} |S_u| = \sum_{i=1}^{k} \lambda_i \sum_{u \in C_i} |S_u| = \sum_{i=1}^{k} \lambda_i |T_i| \le \sum_{i=1}^{k} \lambda_i \alpha(H) = \overline{\chi}^*(G)\alpha(H).$$

This shows (67.101).

Rosenfeld [1967] showed that for each graph G:

(67.105) $\alpha(G \cdot H) = \alpha(G)\alpha(H)$ for each graph $H \iff \alpha(G) = \overline{\chi}^*(G)$.

Here $\Longleftarrow$ follows from (67.101). To see $\Longrightarrow$, let $x \in \mathbb{Q}_+^{VG}$ be a vector satisfying $x(C) \le 1$ for each clique C, and $\mathbf{1}^\mathsf{T} x = \overline{\chi}^*(G)$. Let K be a positive integer such that $w := K \cdot x$ is integer. Let G^w be the graph obtained from G by replacing each vertex u by a clique C_u of size $w(u)$ (where vertices in distinct $C_u, C_{u'}$ are adjacent if and only if u and u' are adjacent). Then $\omega(G^w) \le K$. Hence for $H := \overline{G^w}$ we have $\alpha(H) \le K$.

Now let

(67.106) $S := \{(u, v) \mid u \in VG, v \in C_u\}$.

Then S is a stable set in $G \cdot H$, since if (u, v) and (u', v') are distinct elements in S, then, if $u = u'$, v and v' belong to C_u and hence are nonadjacent in H, and, if $u \ne u'$, u and u' are nonadjacent in G or v and v' are nonadjacent in H.

So $|S| \le \alpha(G \cdot H) = \alpha(G)\alpha(H)$. Hence

(67.107) $$\overline{\chi}^*(G) = \mathbf{1}^\mathsf{T} x = \frac{1}{K}\mathbf{1}^\mathsf{T} w = \frac{1}{K}|S| \le \frac{1}{K}\alpha(G)\alpha(H) \le \alpha(G).$$

Hence $\overline{\chi}^*(G) = \alpha(G)$.

More results on the stable set number of products of graphs are given by Vizing [1963], Barnes and Mackey [1978], and Jha and Slutzki [1994].

The stable set number of products of circuits

The following equality was given by Baumert, McEliece, Rodemich, Rumsey, Stanley, and Taylor [1971] and Markosyan [1971]:

(67.108) $\alpha(C_{2k+1}^2) = k^2 + \lfloor \frac{1}{2}k \rfloor$.

$\le$ directly follows from (67.101), since $\alpha(C_{2k+1}) = k$ and $\overline{\chi}^*(C_{2k+1}) = k + \frac{1}{2}$. To see $\ge$, we may assume that the vertices of C_{2k+1} are $0, 1, \ldots, 2k$, in order. Then the pairs $(2i, \lfloor 2i/k \rfloor)$, for $i = 1, \ldots, k^2 + \lfloor \frac{1}{2}k \rfloor$, where we take integers mod $2k+1$, form a stable set of size $k^2 + \lfloor \frac{1}{2}k \rfloor$ in C_{2k+1}^2.

Baumert, McEliece, Rodemich, Rumsey, Stanley, and Taylor [1971] showed moreover the following inequalities (next to several other estimates for $\alpha(C_n^k)$):

(67.109) $$\alpha(C_{n+2}^k) \ge 1 + \frac{(n+2)^k - 2^k}{n^k}\alpha(C_n^k),$$
$$\alpha(C_n^k) \le \frac{n^k - n^{k-1}}{2^k},$$
$$\alpha(C_5^3) = 10,\ \alpha(C_5^4) = 25,\ \alpha(C_7^3) = 33.$$

Hales [1973] extended (67.108) to:

(67.110) $\alpha(C_{2k+1} \cdot C_{2l+1}) = kl + \lfloor \frac{1}{2}\min\{k, l\} \rfloor$.

Related results on the stable set number of products of circuits are given by Sonnemann and Krafft [1974], Stein [1977], Hell and Roberts [1982], Mead and Narkiewicz [1982], Vesel [1998], and Vesel and Žerovnik [1998].

67.4c. Clique cover numbers of products of graphs

As for the analogue of the Shannon capacity for clique cover numbers, McEliece and Posner [1971] showed that it gives no new parameter. We follow the proof of Lovász [1975c].

Theorem 67.17. *For any graph G:*

$$(67.111)\qquad \inf_k \sqrt[k]{\overline{\chi}(G^k)} = \lim_{k\to\infty} \sqrt[k]{\overline{\chi}(G^k)} = \overline{\chi}^*(G).$$

Proof. We first show that for any two graphs G, H:

$$(67.112)\qquad \overline{\chi}^*(G\cdot H) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

Here $\leq$ follows from (67.87). To see $\geq$, choose vectors $x : VG \to \mathbb{R}_+$ with $x(C) \leq 1$ for each clique, and with $x(VG) = \overline{\chi}^*(G)$, and $z : VH \to \mathbb{R}_+$ with $z(C) \leq 1$ for each clique, and with $z(VH) = \overline{\chi}^*(H)$. Define $y : VG \times VH \to \mathbb{R}_+$ by

$$(67.113)\qquad y(u, v) := x(u)z(v)$$

for $(u, v) \in VG \times VH$. Then $y(C) \leq 1$ for each clique C of $G \cdot H$, since there are cliques C' and C'' of G and H, respectively, such that $C \subseteq C' \times C''$; then $y(C) \leq y(C' \times C'') = x(C')z(C'') \leq 1$.

Hence

$$(67.114)\qquad \overline{\chi}^*(G\cdot H) \geq y(VG \times VH) = x(VG)z(VH) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

This proves (67.112).

To prove (67.111), the first equality follows from Fekete's lemma (Corollary 2.2a), since $\overline{\chi}(G^{k+l}) = \overline{\chi}(G^k)\cdot\overline{\chi}(G^l)$. Also we have by (67.112):

$$(67.115)\qquad \inf_k \sqrt[k]{\overline{\chi}(G^k)} \geq \inf_k \sqrt[k]{\overline{\chi}^*(G^k)} = \overline{\chi}^*(G),$$

So it suffices to prove the reverse inequality in (67.115). Since $\omega(G^k) = \omega(G)^k$ and since $\overline{\chi}^*(G^k) = \overline{\chi}^*(G)^k$, we have by Theorem 64.13 (applied to $\overline{G^k}$):

$$(67.116)\qquad \inf_k \sqrt[k]{\overline{\chi}(G^k)} \leq \inf_k \sqrt[k]{(1+\ln\omega(G^k))\overline{\chi}^*(G^k)}$$
$$= \inf_k \sqrt[k]{(1+k\ln\omega(G))}\overline{\chi}^*(G) = \overline{\chi}^*(G),$$

as required. ∎

An alternative proof was given by Hell and Roberts [1982]. A related information-theoretic characterization of perfect graphs was given by Csiszár, Kőrner, Lovász, Marton, and Simonyi [1990] (proving a conjecture of Kőrner and Marton [1988]). More on the colouring number of products of graphs can be found in Borowiecki [1972], Greenwell and Lovász [1974], Vesztergombi [1980,1981], Turzík [1983], Duffus, Sands, and Woodrow [1985], El-Zahar and Sauer [1985], Puš [1988], Soukop [1988], Linial and Vazirani [1989], and Klavžar [1996] (survey).

Hales [1973] showed that for all graphs G, H:

$$(67.117)\qquad \overline{\chi}(G\cdot H) \geq \overline{\chi}^*(G)\overline{\chi}(H),$$

and

$$(67.118)\qquad \overline{\chi}(C_{2k+1}\cdot C_{2l+1}) = (k+1)(l+1) - \lceil \tfrac{1}{2}\min\{k, l\}\rceil.$$

McEliece and Taylor [1973] showed that $\overline{\chi}(C_{n,t}^2) = \lceil n/t\lceil n/t\rceil\rceil$, where $C_{n,t}$ is the graph obtained from the circuit C_n by adding all chords connecting vertices at distance less than t in C_n.

67.4d. A sharper upper bound $\vartheta'(G)$ on $\alpha(G)$

McEliece, Rodemich, and Rumsey [1978] and Schrijver [1979a] gave the following sharper bound $\vartheta'(G)$ on the stable set number $\alpha(G)$, generally sharper than $\vartheta(G)$. Again, let $\mathcal{M}_G$ be the collection of symmetric $V \times V$ matrices satisfying $M_{u,v} = 0$ for any two distinct adjacent vertices u and v, and $\mathrm{Tr}M = 1$. (Here $\mathrm{Tr}M$ is the trace of M (sum of diagonal elements).) Then define

(67.119) $\vartheta'(G) := \max\{\mathbf{1}^{\mathsf{T}} M \mathbf{1} \mid M \in \mathcal{M}_G$ nonnegative and positive semidefinite$\}$.

Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^V$. Similarly to $\vartheta(G)$, the value of $\vartheta'(G)$ can be calculated in polynomial time. Moreover

(67.120) $\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$

for each graph G. The first inequality is proved similarly to the proof of the first inequality in Theorem 67.1, while the second inequality follows from the fact that the range of the maximization problem for $\vartheta'(G)$ is contained in that for $\vartheta(G)$.

$\vartheta'(G)$ indeed can be a sharper upper bound on the stable set number than $\vartheta(G)$, as M.R. Best (cf. Schrijver [1979a]) found the following example of a graph G with $\vartheta'(G) < \vartheta(G)$. The vertex set is $\{0,1\}^6$, two vectors being adjacent if and only if their Hamming distance[17] is at most 3. Then $\vartheta'(G) = 4$ whereas $\vartheta(G) = 16/3$.

Schrijver [1979a] gave relations of $\vartheta'(G)$ with the linear programming bound for codes of Delsarte [1973]. Related work can be found in Schrijver [1981a] and Miklós [1996]. (The polynomial-time computable upper bound for $\alpha(G)$ given by Luz [1995] is at least $\vartheta'(G)$ for all graphs G.)

67.4e. An operator strengthening convex bodies

The matrix method describing $\mathrm{TH}(G)$ given in Section 67.4a can be seen as a special case of a method of improving approximations of the stable set polytope — in fact, of any polytope with $\{0,1\}$ vertices (Lovász and Schrijver [1989,1991]).

Let K be a convex set, let $R(A)$ be defined as in (67.35), and define

(67.121) $\mathcal{N}_K :=$ the collection of symmetric $n \times n$ matrices A with $R(A)$ positive semidefinite, and with $A_i \in A_{i,i} \cdot K$ and $\mathrm{diag}A - A_i \in (1 - A_{i,i}) \cdot K$ for each $i = 1, \ldots, n$,

where A_i denotes the ith column of A.

Define the following new convex set $N_+(K)$:

(67.122) $N_+(K) := \{\mathrm{diag}A \mid A \in \mathcal{N}_K\}$.

Then $N_+(K) \subseteq [0,1]^n$, since $R(A)$ is positive semidefinite. The ellipsoid method gives, for any collection $\mathcal{K}$ of convex sets:

(67.123) if the optimization problem over K is polynomial-time solvable for each $K \in \mathcal{K}$, then also the optimization problem over $N_+(K)$ is polynomial-time solvable for each $K \in \mathcal{K}$.

[17] The *Hamming distance* of two vectors of equal dimension is equal to the number of coordinates in which they differ.

Indeed, if the optimization problem over K is polynomial-time solvable, then the membership problem over K is polynomial-time solvable. Hence the membership problem over $\mathcal{N}_K$ is polynomial-time solvable, implying that the optimization problem over $\mathcal{N}_K$ is polynomial-time solvable. Therefore, the optimization problem over $N_+(K)$ is polynomial-time solvable. (Cf. Chapter 4 of Grötschel, Lovász, and Schrijver [1988].)

Before proving further properties of the operator N_+, we note that it commutes with the following reflection. Define $r : \mathbb{R}^n \to \mathbb{R}^n$ by $r(x)_1 := 1 - x_1$ and $r(x)_i := x_i$ for $i = 2, \ldots, n$, for $x \in \mathbb{R}^n$: Then

(67.124) $$N_+(r(K)) = r(N_+(K)).$$

To see this, let, for any $n \times n$ matrix A, the matrix A' be defined by:

(67.125) $$A'_{1,1} := 1 - A_{1,1};\ A'_{1,i} := A'_{i,1} := A_{i,i} - A_{i,1} \text{ for } i = 2, \ldots, n;$$
$$A'_{i,j} := A_{i,j} \text{ for } i, j = 2, \ldots, n.$$

Then $R(A)$ is positive semidefinite if and only if $R(A')$ is positive semidefinite, since

(67.126) $$R(A') = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & I \end{pmatrix} R(A) \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Moreover, $A \in \mathcal{N}_K \iff A' \in \mathcal{N}_{r(K)}$ and $\text{diag}A' = r(\text{diag}A)$. This gives (67.124).

From this one can derive, if K is compact and convex and intersects $[0,1]^n$:

(67.127) $$N_+(K) \subseteq K.$$

For let $A \in \mathcal{N}_K$ and define $a := \text{diag}A$. If $a \notin K$, there exists a $w \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ with $w^\mathsf{T} x \leq \beta$ for each $x \in K$ and $w^\mathsf{T} a > \beta$. Since by (67.124) we can flip signs if necessary, we can assume $w \geq \mathbf{0}$. Then, since for each i the vector A_i belongs to $A_{i,i} \cdot K$,

(67.128) $$w^\mathsf{T} A w = \sum_i w_i \big(\sum_j w_j A_{i,j}\big) = \sum_i w_i (w^\mathsf{T} A_i) \leq \sum_i w_i A_{i,i} \beta = \beta w^\mathsf{T} a.$$

Hence

(67.129) $$0 \leq (w^\mathsf{T} a, -w^\mathsf{T}) \begin{pmatrix} 1 & a^\mathsf{T} \\ a & A \end{pmatrix} \begin{pmatrix} w^\mathsf{T} a \\ -w \end{pmatrix} = (w^\mathsf{T} a)^2 - 2(w^\mathsf{T} a)^2 + w^\mathsf{T} A w$$
$$\leq -(w^\mathsf{T} a)^2 + \beta \cdot w^\mathsf{T} a = (\beta - w^\mathsf{T} a) w^\mathsf{T} a < 0,$$

since $\beta - w^\mathsf{T} a < 0$ and $w^\mathsf{T} a > \beta \geq 0$ (since $\beta \geq w^\mathsf{T} x \geq 0$ for any $x \in K \cap [0,1]^n$). This is a contradiction, showing (67.127).

Moreover, if $K \subseteq [0,1]^n$, then $N_+(K)$ remains to contain the integer hull of K:

(67.130) $$(N_+(K))_\mathrm{I} = K_\mathrm{I}.$$

To see this, it suffices to show that $x \in N_+(K)$ for each 0,1 vector x in K. Obviously, $A := xx^\mathsf{T}$ belongs to $\mathcal{N}_K$. Hence $x = \text{diag}A$ belongs to $N_+(K)$. This proves (67.130).

Finally, if $K \subseteq [0,1]^n$, then repeated application of the N_+ operator gives the integer hull K_I of K. In fact, one has:

(67.131) $$N_+^n(K) = K_\mathrm{I}.$$

This follows from the fact that for each $j = 1, \ldots, n$:

(67.132) $\quad N_+(K) \subseteq \text{conv.hull}\{x \in K \mid x_j \in \{0,1\}\}$.

To see this, we may assume that $j = n$. Let $a \in N_+(K)$, with $a = \text{diag}A$ and $A \in \mathcal{N}_K$. Then $A_n \in a_n \cdot K$ and $(a - A_n) \in (1 - a_n) \cdot K$. If $a_n \in \{0,1\}$, then a belongs to the right-hand side of (67.132). So we can assume that $0 < a_n < 1$. Set

(67.133) $\quad a' := \frac{1}{a_n} A_n$ and $a'' := \frac{1}{1-a_n}(a - A_n)$.

Then a' and a'' belong to K, and $a'_n = 1$, $a''_n = 0$. As $a = a_n \cdot a' + (1 - a_n) \cdot a''$, we have that a belongs to the right-hand side of (67.132). This proves (67.132).

(67.131) implies that, when starting with $K := \text{TH}(G)$, we can obtain better and better approximations of $P_{\text{stable set}}(G)$ by applying the N_+ operator. After any fixed number of iterations, we can optimize over the convex body in polynomial time, by (67.123).

Stephen and Tunçel [1999] showed that for the line graph $G = L(K_{2n+1})$ of the complete graph K_{2n+1}, when starting with the polytope determined by the nonnegativity and edge constraints ((64.10) in Section 64.5), the number of iterations is precisely n. Related results were given by Cook and Dash [2001].

Leaving out the positive semidefiniteness condition in $\mathcal{N}_K$ yields a weaker operator $N(K)$, which however still satisfies a number of the above properties, including (67.131). The operator $N(K)$ is a special case of a more general operator introduced by Sherali and Adams [1990].

Results relating a related operator to perfection of graphs were given by Aguilera, Escalante, and Nasini [2002].

67.4f. Further notes

Juhász [1982] showed that for a random graph G on n vertices, $\vartheta(G)$ is of the order $\sqrt{n}$, while $\Theta(G)$ is 'likely' to be of the order $\log n$. Knuth [1994] asked if there is a constant c such that $\vartheta(G) \leq c\sqrt{n}\alpha(G)$ for each graph G. This was answered negatively by Feige [1995,1997], who showed that there is a constant $c > 0$ such that

(67.134) $\quad \vartheta(G) > \alpha(G)n/2^{c\sqrt{\log n}}$

for infinitely many graphs G (where $n := |VG|$).

The results of Kashin and Konyagin [1981] and Konyagin [1981] imply that if $\alpha(G) \leq 2$, then $\vartheta(G) \leq 2^{\frac{2}{3}} n^{\frac{1}{3}}$ and (in the worst case) $\vartheta(G) = \Omega(n^{\frac{1}{3}}/\sqrt{\log n})$.

Karger, Motwani, and Sudan [1994,1998] showed the existence of a constant $c > 0$ such that

(67.135) $\quad \overline{\chi}(G) \leq n^{1 - \frac{c}{\vartheta(G)}}$

for each graph G (where $n := |VG|$). More on approximating $\alpha(G)$ or $\overline{\chi}(G)$ by $\vartheta(G)$ can be found in Szegedy [1994] and Charikar [2002].

Kleinberg and Goemans [1998] observed that for any graph G:

(67.136) $\quad \tau(G) \leq 2(|V| - \vartheta(G)) \leq 2\tau(G)$

(where $\tau(G)$ is the vertex cover number of G), and they showed that the factor 2 cannot be improved. Thus the factor 2 as relative error of $\nu(G)$ for approximating $\tau(G)$ is not improved by $2(|V| - \vartheta(G))$.

Fast practical algorithms to compute $\vartheta(G)$, based on interior-point methods, were developed by Alizadeh [1991,1995]. The latter paper also gives a survey on applying semidefinite programming to combinatorial optimization.

A colouring algorithm for perfect graphs based on decomposition was described by Hsu [1986]. An on-line colouring algorithm for perfect graphs (not necessarily yielding an optimum colouring) was given by Kierstead and Kolossa [1996]. An algorithm for colouring some perfect graphs was given by Aït Haddadène, Gravier, and Maffray [1998]. Kratochvil and Sebő [1997] studied the complexity of colouring a perfect graph if some vertices are pre-coloured. Brandstädt [1987] showed the NP-completeness of several optimization problems for special classes of perfect graphs, like finding a minimum feedback vertex set or a minimum dominating set.

Introductory surveys were given by Knuth [1994] and Goemans [1997] on $\vartheta(G)$, by Grötschel, Lovász, and Schrijver [1984c] on polynomial-time algorithms for clique and colouring problems in perfect graphs, and by Reed [2001a] on semi-definite programming in relation to perfect graphs. Another characterization of perfection in terms of $\mathrm{TH}(G)$ was given by Shepherd [2001].

A generalization of $\vartheta(G)$ was given by Narasimhan and Manber [1990]. A generalization of the Shannon capacity to directed graphs was studied by Bidamon and Meyniel [1985]. An analogue of the Shannon capacity based on the 'independent domination number' of a graph, was investigated by Farber [1986]. The Shannon capacity of probabilistic graphs was investigated by Marton [1993].

Further investigations of eigenvalue methods to bound the Shannon capacity are reported by Haemers [1995] and Fiol [1999]. Further convex programming duality phenomena for perfect graphs were found by Wei [1988].

67.4g. Historical notes on perfect graphs

Shannon

As Berge [1997] mentioned, the perfect graph conjectures root in work of Shannon [1956] concerning the 'zero error capacity of a noisy channel'. It amounts to a study of what we now call the Shannon capacity of a graph. Shannon gave the example of C_5 where $\alpha(C_5) = 2$ and $\alpha(C_5^2) = 5$, implying $\Theta(C_5) \geq \sqrt{5} > \alpha(C_5)$. Denoting the logarithm of the Shannon capacity by C_0, Shannon remarked:

> No method has been found for determining C_0 for the general discrete channel, and this we propose as an interesting problem in coding theory.

Shannon proved the following lower and upper bounds on the Shannon capacity $\Theta(G)$ of a graph $G = (V, E)$. First:

$$\text{(67.137)} \qquad \max_p \left(\sum (p_u p_v \mid u, v \in V, u = v \text{ or } uv \in E)\right)^{-1} \leq \Theta(G),$$

where p ranges over all $p \in \mathbb{R}_+^V$ with $\sum_{v \in V} p(v) = 1$. It was observed by Korn [1968] that this lower bound (and also the lower bound given by Gallager [1965]) is equal to the stable set number $\alpha(G)$: if $p_u > 0$ and $p_v > 0$ for two adjacent vertices u and v, either resetting $p_u := p_u + p_v$ and $p_v := 0$, or resetting $p_v := p_u + p_v$ and $p_u := 0$, would increase the value in (67.137), a contradiction. So the set $S := \{v \mid p_v > 0\}$ is a stable set. Then the value in (67.137) is maximized by taking $p_v := 1/|S|$ for $v \in S$. (As we saw in Section 64.9c, this also follows from a theorem of Motzkin and Straus [1965].)

The upper bound given by Shannon [1956] amounts to:

(67.138) $\Theta(G) \leq \overline{\chi}^*(G)$.

Shannon formulated and proved this upper bound in terms of information theory as follows. Let V be an alphabet, let Σ be a set of 'signals', and for $v \in V$ and $\sigma \in \Sigma$, let $p_{v,\sigma}$ be the probability that when transmitting symbol v, signal σ is received. So $\sum_{\sigma \in \Sigma} p_{v,\sigma} = 1$ for each $v \in V$. Let G be the graph on V where two elements $u, v \in V$ are adjacent if and only if there is a signal σ with $p_{u,\sigma} > 0$ and $p_{v,\sigma} > 0$. For each $\sigma \in \Sigma$, define the clique $K_\sigma := \{v \in V \mid p_{v,\sigma} > 0\}$ and the real number $\lambda_\sigma := \max\{p_{v,\sigma} \mid v \in V\}$. So

(67.139) $$\sum_{\sigma \in \Sigma} \lambda_\sigma \chi^{K_\sigma} \geq \mathbf{1}.$$

Hence, by definition of $\overline{\chi}^*(G)$,

(67.140) $$\overline{\chi}^*(G) \leq \sum_{\sigma \in \Sigma} \lambda_\sigma.$$

Moreover, for any fixed G, the minimum of the right-hand side in (67.140) is equal to the left-hand side.

For any $v = (v_1, \ldots, v_k) \in V^k$ and $s = (s_1, \ldots, s_k) \in \Sigma^k$ define

(67.141) $$p_{v,s} := \prod_{i=1}^{k} p_{v_i,s_i} \text{ and } \lambda_s := \prod_{i=1}^{k} \lambda_{s_i}.$$

So $p_{v,s}$ is the probability that transmitted word v is received as word s.

Now consider any nonempty 'code' $C \subseteq V^k$. The 'error probability' of C is equal to

(67.142) $$q(C) := \min_{\phi} \frac{1}{|C|} \sum_{v \in C} \sum (p_{v,s} \mid s \in \Sigma^k, \phi(s) \neq v),$$

where ϕ ranges over all functions $\phi : \Sigma^k \to C$. So it is the minimum error probability taken over all possible 'decoding schemes' ϕ. Trivially, this minimum is attained by the function ϕ with $\phi(s)$ equal to any $v \in C$ maximizing $p_{v,s}$ over $v \in C$. So

(67.143) $$1 - q(C) = \frac{1}{|C|} \sum_{s \in \Sigma^k} \max_{v \in C} p_{v,s} \leq \frac{1}{|C|} \sum_{s \in \Sigma^k} \lambda_s = \frac{1}{|C|} \Big(\sum_{\sigma \in \Sigma} \lambda_\sigma \Big)^k.$$

Therefore,

(67.144) $$\sqrt[k]{|C|} \leq \frac{\sum_{\sigma \in \Sigma} \lambda_\sigma}{\sqrt[k]{1 - q(C)}}.$$

Now $q(C) = 0$ if and only if C is stable in G^k. Minimizing over all Σ and probability distributions $p_{v,\sigma}$ then yields

(67.145) $$\sqrt[k]{|C|} \leq \overline{\chi}^*(G).$$

So this gives (67.138).

Shannon next observed that if a graph $G = (V, E)$ has a function $f : V \to V$ such that $f(u) \neq f(v)$ for any distinct nonadjacent vertices u and v, and such that $f(V)$ is a stable set, then $\Theta(G) = \alpha(G)$. The condition clearly is equivalent to: $\alpha(G) = \overline{\chi}(G)$. Shannon noticed that this yields the value of $\Theta(G)$ for all graphs G

with at most 5 vertices, except for C_5, for which he derived $\sqrt{5} \leq \Theta(C_5) \leq \frac{5}{2}$ from (67.138). Shannon observed that on 6 vertices all but four graphs have $\alpha(G) = \overline{\chi}(G)$, and that the Shannon capacity of these four graphs can be expressed in terms of $\Theta(C_5)$. On 7 vertices, he stated that 'at least one new situation arises', namely C_7.

Shannon proved that if G and H are disjoint graphs, then $\Theta(G + H) \geq \Theta(G)+\Theta(H)$ and $\Theta(G\cdot H) \geq \Theta(G)\cdot\Theta(H)$, and that equality holds if $\alpha(G) = \overline{\chi}(G)$. Moreover, he conjectured equality for all G, H, but for the product this was disproved by Haemers [1979], and for the sum by Alon [1998].

Berge

As remarked, in developing the concept of perfect graph Berge was motivated by Shannon's problem on the capacity of graphs. We quote from the article 'Motivations and history of some of my conjectures' of Berge [1997]:

> *June 1957:* When he heard that I was writing a book on graph theory, my friend M.P. Schützenberger drew my attention on an interesting paper of Shannon [51] which was presented at a meeting for engineers and statisticians, but which could have been missed by mathematicians working in algebra or combinatorics.

(Berge's reference [51] is Shannon [1956].)

In his book 'Théorie des graphes' (Theory of Graphs), Berge [1958b] called a function $\sigma : VG \to VG$ a *preserving function* ('application préservante'), if for any two distinct nonadjacent vertices u, v, also $\sigma(u)$ and $\sigma(v)$ are distinct and nonadjacent. Then, like Shannon, he considered graphs G having a preserving function σ mapping VG to a stable subset of VG. Clearly, these are exactly the graphs with $\alpha(G) = \overline{\chi}(G)$.

Berge [1958b] also mentioned that M.P. Schützenberger conjectured that

$$\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}, \tag{67.146}$$

which was shown by Lyubich [1964] to follow directly from Fekete's lemma (Corollary 2.2a).

According to Berge [1997], the problem of finding the minimal graphs G with $\alpha(G) < \Theta(G)$ was discussed in January 1960 at the Seminar of R. Fortet, where he asked (prompted by graphs found by A. Ghouila-Houri) if it is true that each graph G not having an odd hole or odd antihole satisfies $\alpha(G) = \Theta(G)$:

> This conjecture, somewhat weaker than the Perfect Graph Conjecture, was motivated by the remark that for the most usual channels, the graphs representing the possible confusions between a set of signals (in particular the interval graphs) have no odd holes and no odd antiholes, and are *optimal in the sense of Shannon.*

At the first international meeting on graph theory held at Dobogókő (Hungary) in October 1959, Hajnal and Surányi [1958] presented the result that $\alpha(G) = \chi(\overline{G})$ for each chordal graph G. This motivated Berge to show that the same holds for complements of chordal graphs. This result was announced, with partial proof, in the paper Berge [1960a], which moreover mentions that several known results yield other classes of graphs G with $\omega(G) = \chi(G)$. In particular, it is observed that theorems of Kőnig imply that $\omega(G) = \chi(G)$ if G or $\overline{G}$ is the line graph of a bipartite graph — 'propriétés remarquables' (remarkable properties) according to Berge.

These results were presented at the Second International Symposium on Graph Theory at the Martin-Luther-Universität in Halle an der Saale (German Democratic Republic) in April 1960. In his memoirs, Berge [1997] mentioned that[18]

> At that time, we were pretty sure that there were no other minimal obstructions; for that reason, at the end of my talk in Halle, I proposed the following open problem: *If a graph G and its complement are semi-Gallai graphs, is it ture that* $\gamma(G) = \omega(G)$*?*

where a graph is *semi-Gallai* if it has no odd hole, and where $\gamma(G)$ is Berge's notation for the colouring number of G.

So, according to Berge, the strong perfect graph conjecture was stated in 1960 in Halle. It seems however that Berge was hesitating in putting the conjecture in print. It is not quoted in the written abstract of the talk (Berge [1961]), which in this respect only says that

> Angesichts einer solchen Menge von Beispielen könnte man vermuten, daß für jeden semi-Gallaischen Graphen G die Beziehung $\omega(G) = \gamma(G)$ gilt. Aber das stimmt nicht, wie das folgende, von einem unserer Schüler, Herrn GHOUILA-HOURI, angegebene Gegenbeispiel zeigt:
> G ist ein Graph mit den Knoten a, b, c, d, e, f, g und den Kanten ac, ad, ae, af, bd, be, bf, bg, ce, cf, cg, df, dg, eg. Man kann leicht zeigen, daß G ein semi-Gallaischer Graph ist mit $\omega(G) = 3$, aber $\gamma(G) = 4$ (siehe Abbildung 1).[19]

(This example ($\overline{C}_7$) was also given by Shannon [1956].) Incidentally, in this paper, Berge called graphs G satisfying $\alpha(G) = \overline{\chi}(G)$ *perfect graphs of Shannon* ('vollkommenen Graphen von Shannon').

About the strong perfect graph conjecture, Berge and Chvátal [1984] wrote:

> An early effort of Alain Ghouila-Houri failed to produce a counterexample to this conjecture. Despite this encouraging sign, Berge felt that the conjecture might be too ambitious. Therefore he restricted himself to a weaker conjecture in the hope that it might be easier to settle.

According to Berge and Chvátal [1984] (where a triangulated graph is a chordal graph),

> After the meeting at Halle an der Saale in 1960, the Strong Perfect Graph Conjecture received the enthusiastic support of G. Hajós and T. Gallai. In fact, Gallai provided further evidence in support of the conjecture by strengthening the results on triangulated graphs: he proved that a graph is α-perfect and γ-perfect whenever each of its odd cycles of length at least five has at least two non-crossing chords.

In Gallai [1962], only a proof of $\alpha(G) = \overline{\chi}(G)$ is given, for graphs G in which any odd circuit of length at least 5 has two noncrossing chords. Berge [1997] reported that Gallai informed him in a letter that he knew that also $\omega(G) = \chi(G)$ holds for such graphs. However, Gallai's paper does not mention this, and no reference is made to Berge's conjectures.

Berge and Chvátal [1984] continued:

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[18] As we aim at verbatim quotations, we leave the typo unchanged.

[19] In view of such a multitude of examples one could conjecture that for each semi-Gallai graph G the relation $\omega(G) = \gamma(G)$ holds. But that does not hold, as the following counterexample, presented by one of our students, Mr GHOUILA-HOURI, shows:
G is a graph with nodes a, b, c, d, e, f, g and edges ac, ad, ae, af, bd, be, bf, bg, ce, cf, cg, df, dg, eg. One can easily show, that G is a semi-Gallai graph with $\omega(G) = 3$, but $\gamma(G) = 4$ (see Figure 1).

> Nevertheless, Berge still felt that the weak conjecture was more promising. At a conference at Rand Corporation in the summer of 1961, he had fruitful discussions with Alan Hoffman, Ray Fulkerson and others. Later on, discussions between Alan Hoffman and Paul Gilmore led Gilmore to a rediscovery of the Strong Perfect Graph Conjecture and to an attempt to axiomatize the relevant properties of cliques in perfect graphs.

Berge [1997] wrote that the discussions at the RAND Corporation with Alan Hoffman encouraged him to write a paper 'in English'. This paper might have been the first version of the paper 'Some classes of perfect graphs' (Berge [1963a]), published in a booklet 'Six Papers on Graph Theory' by the Indian Statistical Institute in Calcutta, which Berge visited in March-April 1963 and where he gave a series of lectures. The booklet contains no year of publication, and the preface mentions that it is intended for private circulation, and that the papers will be given for publication by journals.

The paper contains as new results that $\omega(G) = \chi(G)$ for unimodular graphs and their complements, and also a full proof that it holds for chordal graphs (announced earlier). The paper seems to be the first written account of the concept of perfect graph, and of the perfect graph conjectures, in the last section of the paper:

> V. CONJECTURES
>
> The problem of characterizing α-perfect and γ-perfect graphs seems difficult, but the preceding results enable us to state several conjectures. For instance :
>
> Conjecture 1. A graph is α-perfect if and only if it is γ-perfect
>
> Conjecture 2. A graph is γ-perfect if and only if it does not contain an elementary odd cycle of one of the following types :
>
> type 1 : the cycle is of length greater than 3 and does not possess any chord ;
>
> type 2 : the cycle is of length greater than 3, and does not possess any triangular chord, but possesses all its non-triangular chords (a chord is triangular if it determines a triangle with the edges of the cycle)
>
> Conjecture 3. A graph is α-perfect if and only if it does not contain an elementary odd cycle of type 1 or 2.
>
> It is easy to show that conjecture 2 is equivalent to conjecture 3, and implies conjecture 1. It is also easy to show that if a graph is γ-perfect (or α-perfect), then it does not contain an elementary odd cycle of type 1 or 2.

At the General Assembly of the U.R.S.I. (Union Radio Scientifique Internationale) in Tokyo in September 1963, Berge developed further on the relations between perfection and optimum codes in the sense of Shannon. We quote the abstract (Berge [1963b]):

> 3. Claude Berge : *Sur une conjecture relative au problème des codes optimaux de Shannon*, on considère un émetteur qui peut émettre un ensemble de signaux, par suite du bruit chaque signal peut donner plusieurs interprétations à la réception. On trace le graphe dont les sommets représentent les différents signaux, deux points étant liés par une arête si les signaux correspondants peuvent être confondus à la réception. Le problème essentiel est de caractériser les graphes que l'on peut enrichir, on aboutit ainsi à une conjecture que l'on démontre pour certaines classes particulières.[20]

[20] 3. Claude Berge : *On a conjecture related to the problem of the optimal codes of Shannon*, we consider a transmitter that can transmit a set of signals, as a consequence of noise each signal can give several interpretations at the reception. We make the graph

Berge [1997] wrote that the paper Berge [1963a] was distributed to all participants of the U.R.S.I. meeting in 1963, and that a French version of it was published as Berge [1966], added with some new results and an appendix with some results proved in Berge [1967], in order to make the conjecture more plausible and more interesting.

The paper Berge [1966] is more descriptive, but gives more relations to the Shannon problem, and also mentions the strong perfect graph conjecture, attributing it jointly to P.C. Gilmore. After remarking that $\alpha(G) \neq \overline{\chi}(G)$ for odd circuits of length at least 5 and their complements, the paper states:

> Nous nous sommes proposés de voir si la réciproque était vraie, et sommes arrivés à la conjecture suivante avec P. GILMORE:
> **Conjecture.** *Soit G un graphe de signaux; il est parfait si et seulement s'il ne contient pas un cycle impair sans cordes (de longueur > 3), ni le complémentaire d'un cycle impair sans cordes (de longueur > 3).*[21]

Berge [1966] also claimed, without proof, that $\Theta(G) = \alpha(G)$ if and only if $\overline{\chi}(G) = \alpha(G)$:

> On voit aussi *que la condition nécessaire et suffisante pour que la capacité du graphe de signaux G soit égale à $\alpha(G)$ est que $\alpha(G) = \theta(G)$.*[22]

(Italics of Berge, who denoted the clique cover number $\overline{\chi}(G)$ of G by $\theta(G)$.) However, the line graph $L(K_6)$ of K_6 is a counterexample to this (it has $\alpha = \Theta = \overline{\chi}^* = 3$ and $\overline{\chi} = 4$).

The paper 'Some classes of perfect graphs' was published again in a book on *Graph Theory and Theoretical Physics* edited by F. Harary (Berge [1967]). According to Berge [1997], this paper is 'a final version' of the manuscript, with suggestions by Hoffman, and was handed over to Harary at the end of a NATO Advanced Study Institute on Graph Theory in Frascati, Italy in March-April 1964. Compared with Berge [1963a], the paper contains no new results, and moreover the last section with the perfect graph conjectures (quoted above) has been omitted.

This paper was published also in the Proceedings of a Conference on Combinatorial Mathematics and Its Applications at the University of North Carolina at Chapel Hill, 10–14 April 1967. It is followed by a 'Discussion on Professor Berge's Paper' by M.E. Watkins stating that 'it seems likely that G is perfect if and only if $\overline{G}$ is perfect'. Berge [1996] mentioned that this addendum

> contributed to make the perfect graph conjecture popular. Before the Chapel Hill conference, I did not get much interest for my problems from the mathematics community; the first symposium lecture about perfect graphs from other mathematicians was delivered by Horst Sachs [20] at the Calgary conference in 1969.

(Berge's reference [20] is Sachs [1970].)

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the vertices of which represent the different signals, two points being connected by an edge if the corresponding signals can be confused at the reception. The essential problem is of characterizing the graphs that one can enrich, we arrive this way at a conjecture that we prove for certain particular classes.

[21] We have resolved to see if the reverse would be true, and have arrived at the following conjecture with P. GILMORE:

Conjecture. *Let G be a graph of signals; it is perfect if and only if it neither contains an odd circuit without chords (of length > 3), nor the complement of an odd circuit without chords (of length > 3).*

[22] One also sees *that the necessary and sufficient condition for that the capacity of the graph of signals G is equal to $\alpha(G)$ is that $\alpha(G) = \theta(G)$.*

Fulkerson

The results on perfect graphs obtained until then being restricted to specific classes of graphs, the first serious dent in solving the perfect graph conjectures in general was made by Fulkerson in a RAND Report of 1970 on antiblocking polyhedra. They led Fulkerson to prove a 'pluperfect graph theorem', but also to doubt the validity of the weak perfect graph conjecture, which blocked him finishing it off.

The RAND Report (Fulkerson [1970c]) was published as Fulkerson [1972a], and the results were presented at the Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications at the University of North Carolina at Chapel Hill in May 1970 (Fulkerson [1970d]), and at the 7th International Mathematical Programming Symposium in 1970 in The Hague, for which a survey paper on blocking and antiblocking pairs of polyhedra was written (Fulkerson [1970a,1971a]).

Fulkerson called a graph G *γ-pluperfect* if $\chi(H) = \omega(H)$ for each graph H obtained from G by deleting and replicating vertices. In particular, if G is γ-pluperfect, then G is γ-perfect.

What Fulkerson [1970a,1971a] proved is that:

(67.147) $$G \text{ is } \gamma\text{-pluperfect} \iff \overline{G} \text{ is } \gamma\text{-pluperfect}.$$

The proof is not hard, but is based on a series of pioneering observations and general polyhedral insights that are now fundamental in polyhedral combinatorics. It uses the linear programming duality equality

(67.148) $$\max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq w^{\mathsf{T}}\},$$

where M is the incidence matrix of the stable sets of G and where $w : V \to \mathbb{R}_+$. Then:

(67.149) $G = (V, E)$ is γ-pluperfect
$\overset{1}{\iff}$ $\forall w : V \to \mathbb{Z}_+$, both optima in (67.148) are attained by integer solutions x and y
$\overset{2}{\iff}$ $\forall w : V \to \mathbb{Z}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{3}{\iff}$ $\forall w : V \to \mathbb{Q}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{4}{\iff}$ $\forall w : V \to \mathbb{R}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{5}{\iff}$ each vertex of the polytope $\{x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\}$ is integer
$\overset{6}{\iff}$ the clique polytope of G is determined by the nonnegativity and stable set constraints.

The first equivalence in (67.149) follows by observing that a weight $w(v)$ of a vertex v corresponds to replacing v by a clique of size $w(v)$; this is equivalent to duplicating v $w(v) - 1$ times, or, if $w(v) = 0$, deleting v. The second equivalence can be derived by considering, for any $w : V \to \mathbb{Z}_+$ an inequality $x(S) \leq 1$ in $Mx \leq \mathbf{1}$ satisfied with equality by all optimum solutions; hence replacing w by $w - \chi^S$ the maximum decreases, hence by at least 1 (as it has an integer value); as the minimum decreases by at most 1, we obtain an integer optimum dual solution by induction. The third and fourth equivalences follow by scaling w and by continuity. The fifth equivalence is general polyhedral theory, and the sixth one follows by observing that the integer solutions of $x \geq \mathbf{0}$, $Mx \leq \mathbf{1}$ are precisely the incidence vectors of cliques.

Now by Fulkerson's theory of antiblocking polyhedra, the last statement in (67.149) is invariant under interchanging 'clique' and 'stable set'; that is, under replacing G by the complementary graph $\overline{G}$. Hence the same holds for the first statement.

Fulkerson [1970c,1970a,1971a,1972a] gave another, symmetrical characterization of γ-pluperfect graphs:

(67.150) a graph $G = (V, E)$ is γ-pluperfect if and only if for all $l, w : V \to \mathbb{Z}_+$, the maximum of $l(S)w(C)$ over all stable sets S and cliques C is at least $\sum_v l(v)w(v)$.

For this, Fulkerson was inspired by the length-width inequality for blocking pairs of hypergraphs given in a 1965 preprint of Lehman [1965,1979].

The weak perfect graph conjecture implies that each perfect graph G is γ-pluperfect, since trivially if $\chi(\overline{H}) = \omega(\overline{H})$ for each induced subgraph H of G, then $\chi(\overline{H}) = \omega(\overline{H})$ for each H obtained from G by deleting and replicating vertices. (Note that $\chi(\overline{H}) = \chi(\overline{G})$ and $\omega(\overline{H}) = \omega(\overline{G})$ if H arises from G by duplicating a vertex.)

So, as Fulkerson [1970a,1971a] remarked ('theorem 14'), the perfect graph conjecture is equivalent to: each γ-perfect graph is γ-pluperfect; or: γ-perfection is maintained under duplicating vertices (later called the *replication lemma*):

> Thus to prove the perfect graph conjecture, it would suffice to prove that γ-perfection implies γ-pluperfection. For this it would suffice to show that if G is γ-perfect, and if we duplicate an arbitrary vertex v in G and join v to its duplicate vertex, the new graph G' is again γ-perfect.

Another way of stating it is: if for each $w : V \to \{0, 1\}$ both optima in (67.148) have integer solutions, then likewise for each $w : V \to \mathbb{Z}_+$. This might seem too strong from a general polyhedral point of view, and it made Fulkerson [1970a,1971a] mistrust the conjecture:

> It is our feeling that theorem 14 casts some doubt on the validity of the perfect graph conjecture.

Lovász

The weak perfect graph conjecture was finally proved by Lovász [1972c], stating:

> Fulkerson [5] reduced the problem to the following conjecture, using the theory of antiblocking polyhedra:
> *Duplicating an arbitrary vertex of a perfect graph and joining the obtained two vertices by an edge, the arising graph is perfect.*
> In §1 we prove a theorem which contains this conjecture.

(Reference [5] is Fulkerson [1972a].) Lovász also wrote:

> It should be pointed out that thus the proof consists of two steps and the more difficult second step was done first by Fulkerson.

With respect to this, Fulkerson [1973] remarked in his comments 'On the perfect graph theorem':

> Concerning this proof, Lovász states: "It should be pointed out that thus the proof consists of two steps, and the most difficult second step was done first by Fulkerson." I would be less than candid if I did not say that I agree with this remark, at least in retrospect. But the fact remains that, while part of my aim in developing the anti-blocking theory had been to settle the perfect graph conjecture, and that while I had succeeded via this theory in reducing the conjecture to a simple lemma about graphs [3,4] (the "replication lemma", a proof of which is given in this paper) and had developed other seemingly more complicated equivalent versions of the conjecture [3,4,5], I eventually began to feel that the conjecture was probably false and thus spent several fruitless months trying to construct a counterexample. It is not altogether clear to me now just why I felt the conjecture was false, but I think it was due mainly to one equivalent version I had found [4,5], a version that does not explicitly mention graphs at all.

(The references [3,4,5] correspond to Fulkerson [1972a,1971a,1970d].)

In the preprint of this article, Fulkerson [1972b] wrote moreover, after stating the replication lemma:

> Actually I knew more: Namely that the truth or falsity of the perfect graph conjecture rested entirely on the truth or falsity of the replication lemma. I tried for awhile to prove this lemma, without success, and then, as was mentioned earlier, became convinced on other grounds that the perfect graph conjecture was probably false, and began to look for a graph that was perfect but not pluperfect. (I knew that it would do no good to look at known classes of perfect graphs, since I had been able to prove that all of these were pluperfect.) The fact is that such graphs don't exist, of course. After some months of sporadic effort along these lines, I quit working on the perfect graph conjecture, thinking that I would come back to it later. There were other aspects of anti-blocking pairs of polyhedra, and of blocking pairs of polyhedra, that I wanted to study, and, in any event, I felt that the pluperfect graph theorem was a beautiful result in its own right.
> In the spring of 1971 I received a postcard from Berge, who was then visiting the University of Waterloo, saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it.

After having given a simple proof of the replication lemma, Fulkerson [1972b] continued:

> As can be seen, there is nothing deep or complicated about the proof of this lemma. Perhaps the fact that I saw a proof of it only after knowing it had to be true may say something about the psychology of invention (or, better yet, anti-invention) in mathematics, at least for me.

This is indeed an instructive illustration that believing a conjecture may help in proving it.

In a subsequent paper, Lovász [1972a] proved more strongly that a graph G is perfect if and only if $\alpha(H)\omega(H) \geq |VH|$ for each induced subgraph H of G. This generalizes the perfect graph theorem, and was suggested by A. Hajnal. It also sharpens Fulkerson's result (67.150), implying that one may restrict l and w to $\{0,1\}$-valued functions with $l = w$.

The problem of Shannon [1956] concerning the Shannon capacity of C_5 was solved by Lovász [1979d].

In May 2002, M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced that they found a proof of the strong perfect graph conjecture, by proving a number of deep results, and building on and inspired by earlier results of, among others, V. Chvátal, M. Conforti, G. Cornuéjols, W.H. Cunningham, A. Kapoor, F. Roussel, P. Rubio, N. Sbihi, K. Vušković, and G. Zambelli.

More historical notes are given by Berge and Ramírez Alfonsín [2001] and Reed [2001b].

Chapter 68

T-perfect graphs

The class of t-perfect graphs is defined polyhedrally: the stable set polytope should be determined by the nonnegativity, edge, and odd circuit constraints. It implies that a maximum-weight stable set in such graphs can be found in polynomial time. LP duality gives a min-max relation for the maximum-weight of a stable set in t-perfect graphs.
A characterization of t-perfect graphs is not known. The widest class of t-perfect graphs known consists of those not containing certain subdivisions of K_4 as subgraph.

68.1. T-perfect graphs

A graph $G = (V, E)$ is called *t-perfect*[23] if the stable set polytope of G is determined by

(68.1)
$$\begin{array}{rll} \text{(i)} & 0 \le x_v \le 1 & \text{for each } v \in V, \\ \text{(ii)} & x_u + x_v \le 1 & \text{for each edge } uv \in E, \\ \text{(iii)} & x(VC) \le \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C. \end{array}$$

A prominent non-t-perfect graph is K_4. Below we shall see that, on the other hand, if K_4 does not occur in a graph in a certain way, then the graph is t-perfect. But no exact characterization of t-perfection is known.

A motivation for studying t-perfection is algorithmic, since the definition implies:

Theorem 68.1. *A maximum-weight stable set in a t-perfect graph can be found in strongly polynomial time.*

Proof. By Theorems 5.10 and 5.11, it suffices to show that the separation problem over the stable set polytope is polynomial-time solvable. Conditions (i) and (ii) in (68.1) can be tested one by one. If they are satisfied, define a function $y : E \to \mathbb{R}_+$ by:

(68.2)
$$y_e := 1 - x_u - x_v$$

for each $e = uv \in E$. Then condition (iii) is equivalent to:

[23] t stands for 'trou' (French for 'hole').

(68.3) $y(EC) \geq 1$ for each odd circuit C

(since $y(EC) = |EC| - 2x(VC)$). The latter condition can be checked in polynomial time: Consider y as a length function, and for each $u \in V$, find an odd circuit C through u with $y(EC)$ minimal. This can be done by replacing each vertex v by two vertices v', v'', and each edge $e = vw$ by two edges $v'w''$ and $v''w'$, each of length y_e; then a shortest path from u' to u'' gives the required circuit.

If $y(EC) < 1$, we have a violated inequality. ∎

A combinatorial polynomial-time algorithm to find the stable set number of a t-perfect graph was given by Eisenbrand, Funke, Garg, and Könemann [2002]. It is based on finding (by a greedy method similar to that used in the proof of Theorem 64.13) an approximative fractional dual solution to the problem of maximizing $\mathbf{1}^{\mathsf{T}}x$ over (68.1). with relative error less than $1/|V|$. Rounding then gives the stable set number. Applying this iteratively gives an explicit maximum-size stable set.

Notes. The construction given in the proof of Theorem 68.1 shows that the maximum-weight stable set problem in a t-perfect graph can be described by a 'compact' linear programming: the stable set polytope is the projection of a polytope whose dimension and number of facets are polynomially bounded. To see this, introduce, next to the variables $y \in \mathbb{R}_+^E$, a variable $z_{u,v}$ for each $u, v \in V$. Requiring:

(68.4)
$$\begin{aligned} & z_{v,v} \geq 1 && \text{for each } v \in V, \\ & z_{u,v} \leq y_{uv} && \text{for each edge } uv \in E, \\ & z_{t,w} \leq z_{t,u} + y_{uv} + y_{vw} && \text{for all } t, u, v, w \in V \text{ with } uv, vw \in E, \end{aligned}$$

is equivalent to the odd circuit constraints. (In fact, one can do without the variables y_e, as they can be expressed in the x_v.) So a maximum-weight stable set in a t-perfect graph can be found in polynomial time with any polynomial-time linear programming algorithm.

T-perfection can also be characterized in terms of the vertex cover polytope:

Theorem 68.2. *A graph $G = (V, E)$ is t-perfect if and only if the vertex cover polytope of G is determined by:*

(68.5)
$$\begin{aligned} &\text{(i)} && 0 \leq x_v \leq 1 && \text{for each } v \in V, \\ &\text{(ii)} && x_u + x_v \geq 1 && \text{for each edge } uv \in E, \\ &\text{(iii)} && x(VC) \geq \lceil \tfrac{1}{2}|VC| \rceil && \text{for each odd circuit } C. \end{aligned}$$

Proof. System (68.5) arises from (68.1) by the reflection $x \to \mathbf{1} - x$. So integrality of the two polytopes is equivalent. ∎

68.2. Strongly t-perfect graphs

A graph $G = (V, E)$ is called *strongly t-perfect* if system (68.1) is totally dual integral. So each strongly t-perfect graph is t-perfect (Theorem 5.22). It is unknown if the reverse implication holds:

(68.6) Is every t-perfect graph strongly t-perfect?

Strong t-perfection can be characterized by the weighted version of the stable set number and a certain weighted 'edge and circuit' cover number. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$. In this chapter, a *w-cover* is a family of vertices, edges, and odd circuits covering each vertex v at least $w(v)$ times. By definition, the *cost* of a vertex or edge is 1, and the *cost* of an odd circuit C is $\lfloor \frac{1}{2}|VC| \rfloor$. The *cost* of a w-cover $\mathcal{F}$ is the sum of the costs of the elements of $\mathcal{F}$. Define

(68.7) $\alpha_w(G) :=$ the maximum weight of a stable set in G,
$\tilde{\rho}_w(G) :=$ the minimum cost of a w-cover.

Obviously, $\alpha_w(G) \leq \tilde{\rho}_w(G)$ for any graph G. Moreover:

(68.8) G is strongly t-perfect $\iff$ $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w : V \to \mathbb{Z}_+$.

This follows directly from a combinatorial interpretation of total dual integrality.

Notes. W.R. Pulleyblank (cf. Gerards [1989a]) observed that, even for $w = \mathbf{1}$, determining $\tilde{\rho}_w(G)$ is NP-complete, since the vertex set of a graph G can be partitioned into triangles if and only if $\tilde{\rho}_w(G) = \frac{1}{3}|V|$ where $w = \mathbf{1}$. The problem of partitioning a graph into triangles is NP-complete. Since partitioning into triangles remains to be NP-complete for planar graphs (Dyer and Frieze [1986]), even determining $\tilde{\rho}_w(G)$ for planar graphs is NP-complete.

Again, strong t-perfection is equivalent to the total dual integrality of the vertex cover constraints (68.5).

68.3. Strong t-perfection of odd-K_4-free graphs

K_4 is the smallest graph that is not t-perfect. Gerards and Schrijver [1986] showed that any graph not containing an 'odd K_4-subdivision' is t-perfect — in fact, as Gerards [1989a] showed, strongly t-perfect. We will prove this in this section (with a method inspired by Geelen and Guenin [2001]).

Call a subdivision of K_4 *odd* if each triangle of K_4 has become an odd circuit — equivalently, if the evenly subdivided edges of K_4 form a cut of K_4. We say that a graph *contains no* odd K_4-subdivision if it has no subgraph which is an odd K_4-subdivision.

Theorem 68.3. *A graph containing no odd K_4-subdivision is strongly t-perfect.*

Proof. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimum. Then G has no isolated vertices. So we can assume that any minimum-cost w-cover contains no vertices (for any w).

For any weight function $w : V \to \mathbb{Z}_+$, denote $\alpha_w := \alpha_w(G)$ and $\tilde{\rho}_w := \tilde{\rho}_w(G)$. As G is a counterexample, there exists a $w : V \to \mathbb{Z}_+$ with $\alpha_w < \tilde{\rho}_w$.

For any such w we have, for each edge $e = uv$,

(68.9) if S maximizes $w(S)$ over stable sets S of $G - e$, then S contains u and v.

Otherwise, S is a stable set of G, implying that (by the minimality of $|V| + |E|$):

(68.10) $$\alpha_w(G) \geq \alpha_w(G - e) = \tilde{\rho}_w(G - e) \geq \tilde{\rho}_w(G),$$

a contradiction.

This implies

(68.11) $$w \geq \mathbf{1},$$

since if $w(v) = 0$ for some vertex v, then for any edge e incident with v there is a stable set S of $G - e$ maximizing $w(S)$ and not containing v (since deleting v from S does not decrease $w(S)$). This contradicts (68.9).

We next show that we can assume w to have some additional properties (for an edge $e = uv$, χ^e is the incidence vector of the set $\{u, v\}$, that is, it is the $0, 1$ vector in $\mathbb{R}^V$ having 1's in positions u and v):

Claim 1. There exist $w : V \to \mathbb{Z}_+$ and $f \in E$ such that

(68.12) $$\tilde{\rho}_{w+\chi^f} = \alpha_w + 1 = \tilde{\rho}_w = \alpha_{w+\chi^f}$$

and such that

(68.13) $$\alpha_{w-\chi^{VC}} = \tilde{\rho}_{w-\chi^{VC}}$$

for each odd circuit C.

Proof of Claim 1. As G is not bipartite (by Theorem 19.7) and not just an odd circuit (as this is trivially strongly t-perfect), we know that H has a chordless odd circuit C_0 that has at least one vertex of degree at least 3. Let v be such a vertex, and let e be an edge incident with v but which is not on C_0.

Let $B := VC_0 \setminus \{v\}$. We choose w such that $w(V \setminus B)$ is minimal. There exists a $k \in \mathbb{Z}_+$ such that for $w' := w + k \cdot \chi^B$, each stable set S of $G - e$ maximizing $w'(S)$ satisfies $|S \cap B| = \frac{1}{2}|B|$. Hence no such set S contains v, and therefore, by (68.9), $\alpha_{w'} = \tilde{\rho}_{w'}$.

Now let M be the perfect matching in $C_0 - v$. For $y : M \to \mathbb{Z}_+$ define

(68.14) $$w^y := w + \sum_{f \in M} y_f \chi^f.$$

As $\alpha_{w'} = \tilde{\rho}_{w'}$, there exists a $y : M \to \mathbb{Z}_+$ such that

(68.15) $$\alpha_{w^y} = \tilde{\rho}_{w^y}.$$

We choose such a y with $\sum_{f\in M} y_f$ minimal. Since $\alpha_w < \tilde{\rho}_w$, there exists an $f \in M$ with $y_f \geq 1$. Then, by the minimality of y, we have $\alpha_{w^y-\chi^f} < \tilde{\rho}_{w^y-\chi^f}$. So we can assume that $y_f = 1$ and $y_{f'} = 0$ for each $f' \in M \setminus \{f\}$. We show that w and f are as required.

To show (68.12), we have $\alpha_{w+\chi^f} \leq \alpha_w + 1$, since any stable set S satisfies $(w + \chi^f)(S) = w(S) + |f \cap S| \leq w(S) + 1$. This implies

(68.16) $$\alpha_w + 1 \leq \tilde{\rho}_w \leq \tilde{\rho}_{w+\chi^f} = \alpha_{w+\chi^f} \leq \alpha_w + 1,$$

implying (68.12).

Next, consider any odd circuit C in G. Then $(w-\chi^{VC})(V\setminus B) < w(V\setminus B)$, since VC is not contained in B. Therefore, by the choice of w, we have (68.13).

End of Proof of Claim 1

As from now we fix w and f satisfying (68.12) and (68.13). Let f connect vertices u and u'. Since by the minimality of G, G has no isolated vertices, there exists a minimum-cost $w+\chi^f$-cover $\mathcal{F}$ consisting only of edges and odd circuits, say, $e_1, \ldots, e_t$, $C_1, \ldots, C_k$. We choose them such that

(68.17) $$|VC_1| + \cdots + |VC_k|$$

is as small as possible. Then:

(68.18) at least two of the C_i traverse f.

To see this, let $G' := G - f$ (the graph obtained by deleting edge f). If $\alpha_w(G') = \alpha_w(G)$, then by the minimality of G, G' has a w-cover of cost α_w. As this is a w-cover in G as well, this would imply $\alpha_w = \tilde{\rho}_w$, a contradiction.

So $\alpha_w(G') > \alpha_w(G)$. That is, there exists a stable set S in G' with $w(S) > \alpha_w$. Necessarily, S contains both u and u'. Then for any circuit C traversing f:

(68.19) $$|VC \cap S| \leq \lfloor \tfrac{1}{2}|VC| \rfloor + 1.$$

Also, f is not among $e_1, \ldots, e_t$, since otherwise $\mathcal{F} \setminus \{f\}$ is a w-cover of cost $\tilde{\rho}_{w+\chi^f} - 1 = \tilde{\rho}_w - 1$, contradicting the definition of $\tilde{\rho}_w$. Setting l to the number of C_i traversing f, we obtain:

(68.20) $$\begin{aligned}\tilde{\rho}_{w+\chi^f} &\leq \alpha_w + 1 \leq w(S) = (w + \chi^f)(S) - 2 \\ &\leq -2 + \sum_{j=1}^{t} |e_j \cap S| + \sum_{i=1}^{k} |VC_i \cap S| \leq -2 + t + \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|VC_i| \rfloor + l \\ &= \tilde{\rho}_{w+\chi^f} + l - 2.\end{aligned}$$

So $l \geq 2$, which is (68.18).

By (68.18) we can assume that C_1 and C_2 traverse f. It is convenient to assume that $EC_1 \setminus \{f\}$ and $EC_2 \setminus \{f\}$ are disjoint; this can be achieved by adding parallel edges. So $EC_1 \cap EC_2 = \{f\}$.

Then:

(68.21) if C is an odd circuit with $EC \subseteq EC_1 \cup EC_2$, then $f \in EC$ and $EC_1 \triangle EC_2 \triangle EC$ is again an odd circuit.

Indeed, as $EC_1 \triangle EC_2 \triangle EC$ is an odd cycle, it can be decomposed into circuits $C'_2, \ldots, C'_p$, with $C'_2, \ldots, C'_q$ odd and $C'_{q+1}, \ldots, C'_p$ even ($q \geq 2$). Then

$$(68.22) \quad \begin{aligned} \sum_{i=2}^{p} |EC'_i| &= |EC_1 \triangle EC_2 \triangle EC| \\ &= |EC_1| + |EC_2| - |EC| - 2|\{f\} \setminus EC|. \end{aligned}$$

Choose for each $i = q+1, \ldots, p$ a perfect matching M_i in C'_i. Let $e'_1, \ldots, e'_r$ be the edges in the matchings M_i and in $\{f\} \setminus EC$. Then, defining $C'_1 := C$,

$$(68.23) \quad \chi^{VC_1} + \chi^{VC_2} = \sum_{i=1}^{q} \chi^{VC'_i} + \sum_{j=1}^{r} \chi^{e'_j}$$

and (using (68.22))

$$(68.24) \quad \begin{aligned} &\lfloor \tfrac{1}{2}|VC_1| \rfloor + \lfloor \tfrac{1}{2}|VC_2| \rfloor = \tfrac{1}{2}|EC_1| + \tfrac{1}{2}|EC_2| - 1 \\ &= -1 + |\{f\} \setminus EC| + \tfrac{1}{2}\sum_{i=1}^{p} |EC'_i| = -1 + r + \tfrac{1}{2}\sum_{i=1}^{q} |EC'_i| \\ &\geq r + \sum_{i=1}^{q} \lfloor \tfrac{1}{2}|VC'_i| \rfloor. \end{aligned}$$

So replacing C_1, C_2 by $C'_1, \ldots, C'_q$ and adding $e'_1, \ldots, e'_r$ to $e_1, \ldots, e_t$, gives again a $w + \chi^f$-cover of cost at most $\tilde{\rho}_{w+\chi^f}$. This also implies $q = 2$, since otherwise we have strict inequality in (68.24), and we would obtain a w-cover of cost less than $\tilde{\rho}_w$.

If $f \notin EC$, then f is among $e'_1, \ldots, e'_r$. Hence deleting f gives a w-cover of cost at most $\tilde{\rho}_{w+\chi^f} - 1 \leq \alpha_w$, contradicting (68.12). So $f \in EC$. As this is true for any odd circuit in $EC_1 \cup EC_2$ we know that $f \in EC'_i$ for $i = 1, 2$.

If $p \geq 3$ or $r \geq 1$, then $|EC'_1| + |EC'_2| < |EC_1| + |EC_2|$, contradicting the minimality of (68.17). So $p = q = 2$ and $r = 0$, which proves (68.21).

First, it implies

(68.25) a circuit in $EC_1 \cup EC_2$ is odd if and only if it traverses f.

A second consequence is as follows. Let P_i be the $u-u'$ path $C_i \setminus \{f\}$. Orient the edges occurring in the path $P_i := C_i \setminus \{f\}$ in the direction from u to u', for $i = 1, 2$. Then

(68.26) the orientation is acyclic.

For suppose that it contains a directed circuit C. Then $(EC_1 \cup EC_2) \setminus EC$ contains a directed $u-u'$ path, and hence an odd circuit C'. Hence by (68.21), $EC_1 \triangle EC_2 \triangle EC'$ is an odd circuit, however containing the even circuit EC, a contradiction.

Define

(68.27) $\quad W := VP_1 \cup VP_2$ and $F := EP_1 \cup EP_2$.

Consider the graph (W, F). It is bipartite, as it contains no odd circuits by (68.25). Moreover, u and u' belong to the same colour class. Let A and B be the colour classes of (W, F), such that $u, u' \in A$. So

(68.28) $\quad A := \{v \in W \mid$ there exists an even-length directed $u - v$ path$\}$,
$B := \{v \in W \mid$ there exists an odd-length directed $u - v$ path$\}$.

(Here and below, when speaking of a directed path, it is assumed to use only the edges in $EP_1 \cup EP_2$.) Define

(68.29) $\quad X := VP_1 \cap VP_2$ and

$$U := \{v \in V \mid w(v) = \sum_{j=1}^{t} |e_j \cap \{v\}| + \sum_{j=1}^{k} |VC_j \cap \{v\}|\}.$$

So $u, u' \notin U$, $u, u' \in X$, and $X \setminus \{u, u'\}$ is the set of vertices in W having degree 4 in the graph (W, F).

We next show the following technical, but straightforward to prove, claim:

Claim 2. Let $z \in A$, let Q be an even-length directed $u - z$ path, and let S be a stable set in G. Then

(68.30) $\quad (w - \chi^{VQ})(S) \geq \alpha_w - \lfloor \frac{1}{2}|VQ| \rfloor + 1$

if and only if

(68.31)
(i) $|e_j \cap S| = 1$ *for each* $j = 1, \ldots, t$,
(ii) $|VC_j \cap S| = \lfloor \frac{1}{2}|VC_j| \rfloor$ *for* $j = 3, \ldots, k$,
(iii) $S \subseteq U$,
(iv) S *contains* $B \setminus VQ$ *and is disjoint from* $A \setminus VQ$,
(v) S *contains* $B \cap X$ *and is disjoint from* $A \cap X$.

Proof of Claim 2. By rerouting C_1 and C_2, we can assume that $EQ \subseteq EC_1$. Define $Z := VC_1 \setminus VQ$. So $|Z|$ is even. Consider the following sequence of (in)equalities:

(68.32)
$$\begin{aligned}
(w - \chi^{VQ})(S) &= w(S) - |VQ \cap S| \\
&\leq \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=1}^{k} |VC_j \cap S| - |VQ \cap S| \\
&= \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=2}^{k} |VC_j \cap S| + |Z \cap S| \leq t + \sum_{j=2}^{k} \lfloor \tfrac{1}{2}|VC_j| \rfloor + |Z \cap S| \\
&= \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + |Z \cap S| \leq \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + \tfrac{1}{2}|Z| \\
&= \alpha_w + 1 - \lfloor \tfrac{1}{2}|VQ| \rfloor.
\end{aligned}$$

Hence (68.30) holds if and only if equality holds throughout in (68.32), which is equivalent to (68.31). Note that (68.31)(iv) and (v) are equivalent to: S contains $VC_2 \cap B$ and is disjoint from $VC_2 \cap A$, and S contains $Z \cap B$ and is disjoint from $Z \cap A$. Hence it is equivalent to (as $u, u' \notin S$ by (68.31)(iii)): $|VC_2 \cap S| = \lfloor \frac{1}{2}|VC_2| \rfloor$ and $|Z \cap S| = \frac{1}{2}|Z|$. *End of Proof of Claim 2*

By (68.26), we can order the vertices in X as $x_0 = u, x_1, \ldots, x_s = u'$ such that both P_1 and P_2 traverse them in this order. For $j = 0, \ldots, s$, let $\mathcal{P}_j$ be the collection of directed $u - x$ paths, where $x = x_j$ if $x_j \in A$, and x is an inneighbour of x_j if $x_j \in B$. So $x \in A$ and each path in each $\mathcal{P}_j$ has even length.

Let j be the largest index for which there exists a path $Q \in \mathcal{P}_j$ with

(68.33) $$\alpha_{w-\chi^{VQ}} \leq \alpha_w - \lfloor \tfrac{1}{2}|VQ| \rfloor.$$

Such a j exists, since (68.33) holds for the trivial directed $u - u$ path, as $\alpha_{w-\chi^u} \leq \alpha_w$. Also, $j < s$, since otherwise $VQ = VC$ for some odd circuit C, and hence, with (68.13) we have

(68.34) $$\tilde{\rho}_w \leq \tilde{\rho}_{w-\chi^{VC}} + \lfloor \tfrac{1}{2}|VC| \rfloor = \alpha_{w-\chi^{VC}} + \lfloor \tfrac{1}{2}|VC| \rfloor \leq \alpha_w,$$

contradicting (68.12).

Let Q_1 and Q_2 be the two paths in $\mathcal{P}_{j+1}$ that extend Q. By the maximality of j, we know

(68.35) $$\alpha_{w-\chi^{VQ_i}} \geq \alpha_w - \lfloor \tfrac{1}{2}|VQ_i| \rfloor + 1.$$

So there exist stable sets S_1 and S_2 with

(68.36) $$(w - \chi^{VQ_i})(S_i) \geq \alpha_w - \lfloor \tfrac{1}{2}|VQ_i| \rfloor + 1$$

for $i = 1, 2$. So for $i = 1, 2$, (68.31) holds for Q_i, S_i. By (68.31)(iv), S_1 and S_2 coincide on $W \setminus (VQ_1 \cup VQ_2)$, and they coincide on X. In other words:

(68.37) $$(S_1 \triangle S_2) \cap W \subseteq (VQ_1 \cup VQ_2) \setminus X.$$

Let H be the subgraph of G induced by $S_1 \triangle S_2$. So H is a bipartite graph, with colour classes $S_1 \setminus S_2$ and $S_2 \setminus S_1$. Define

(68.38) $$Y_i := VQ_i \setminus VQ$$

for $i = 1, 2$. Then

(68.39) H contains a path connecting Y_1 and Y_2.

For suppose not. Let K be the union of the components of H that intersect Y_1. So K is disjoint from Y_2. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that Q, S satisfy (68.31). Hence (68.30) holds, contradicting (68.33). This proves (68.39).

Let C be the (even) circuit formed by the two directed $x_j - x_{j+1}$ paths. So Y_1 and Y_2 are subsets of VC. Let R be a shortest path in H that connects Y_1 and Y_2; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$.

Since $y_1, y_2 \in S_1 \triangle S_2$, we know by (68.37) that $y_1, y_2 \notin X$. By (68.31)(iv), if $y_1 \in S_1 \setminus S_2$, then $y_1 \in A$ (since if $y_1 \in B$, then $y_1 \in B \setminus VQ_2$, and so $y_1 \in S_2$), and if $y_1 \in S_2 \setminus S_1$, then $y_1 \in B$ (since if $y_1 \in A$, then $y_1 \in A \setminus VQ_2$, and so $y_1 \notin S_2$). Similarly, if $y_2 \in S_2 \setminus S_1$, then $y_2 \in A$ and if $y_2 \in S_1 \setminus S_2$, then $y_2 \in B$.

So if R is even, then y_1 and y_2 belong to the same set among $S_1 \setminus S_2$, $S_2 \setminus S_1$, and hence they belong to different sets A, B. Similarly, if R is odd, then y_1 and y_2 belong to the same set among A, B. Hence R forms with part of C an odd circuit.

By (68.37), there exist a directed $u - x_j$ path N' and a directed $x_{j+1} - u'$ path N'' that are (vertex-)disjoint from $S_1 \triangle S_2$. Concatenating N', f, and N'' makes an $x_{j+1} - x_j$ path N. Then N, R, and C make an odd K_4-subdivision, with 3-valent vertices x_j, x_{j+1}, y_1, y_2. ∎

(The above proof of Claim 1 was given by D. Gijswijt.)

Notes. Theorem 68.3 includes the t-perfection of series-parallel graphs (conjectured by Chvátal [1975a], and proved by M.J. Clancy in 1977 and by Mahjoub [1988]), the strong t-perfection of series-parallel graphs (Boulala and Uhry [1979], who also gave a polynomial-time algorithm to find a maximum-weight stable set in series-parallel graphs), the t-perfection of almost bipartite graphs — graphs G having a vertex v with $G - v$ bipartite (Fonlupt and Uhry [1982]), the strong t-perfection of almost bipartite graphs (this is implicit in Sbihi and Uhry [1984]), and the t-perfection of odd-K_4-free graphs (Gerards and Schrijver [1986]).

68.4. On characterizing t-perfection

The problem if a given graph $G = (V, E)$ is t-perfect, belongs to co-NP: non-t-perfection can be certified by a noninteger vertex x^* of the polytope determined by (68.1), together with a nonsingular system of constraints that are tight for x^*. One must check that x^* satisfies all constraints among (68.1) — this can be done in polynomial time by the methods described in the proof of Theorem 68.1. A polynomial-time algorithm for, or a combinatorial certificate of, non-t-perfection is not known.

T-perfection and strong t-perfection are not closed under taking subgraphs, as is shown by Figure 68.1. However, t-perfection is closed under taking induced subgraphs. This is easy to check, as well as that it is closed under the following operation:

(68.40) choose a vertex v with $N(v)$ a stable set, and contract all edges in $\delta(v)$.

So one may ask for the minimally non-t-perfect graphs — minimal with respect to taking induced subgraphs and applying operation (68.40). Known minimal graphs include the wheels[24] with an even number of vertices and the graphs consisting of a circuit of length $4k$ and all chords connecting a

[24] A *wheel* is a graph obtained from a circuit C by adding a new vertex, adjacent to all vertices in C.

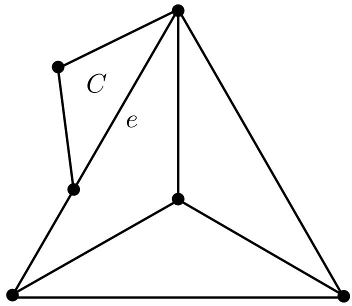

Figure 68.1
A strongly t-perfect graph G with $G - e$ not t-perfect. The strong t-perfection of G can be derived from the fact that each inclusionwise maximal stable set intersects the triangle C. Hence for any integer weight function, subtracting the incidence vector of VC, reduces the maximum weight of a stable set by 1. We therefore can assume that at least one of the vertices of C has weight 0, and hence we can delete it. We are left with a graph containing no odd K_4-subdivision — hence being strongly t-perfect (Theorem 68.3).

vertex with its opposite vertex ($k \geq 1$). Also strong t-perfection is closed under taking induced subgraphs and the operation (68.40). So one may ask a similar question for strong t-perfection.

A characterization that has been achieved is of those graphs for which each, also noninduced, subgraph is t-perfect. Here subdivisions of K_4 play a role. Call a subdivision of K_4 *bad* if it is not t-perfect.

It has been shown by Gerards and Shepherd [1998] that any graph without bad K_4-subdivision is t-perfect. Hence, each subgraph of a graph G is t-perfect if and only if G contains no bad K_4-subdivision. This was extended to: any graph without bad K_4-subdivision is strongly t-perfect (Schrijver [2002b]). So each subgraph of a graph is t-perfect if and only if each subgraph is strongly t-perfect.

The K_4-subdivisions that are bad have been characterized by Barahona and Mahjoub [1994c]. They showed that a K_4-subdivision is not t-perfect if and only if it is an odd K_4-subdivision such that the following does *not* hold: the edges of K_4 that have become an even path, form a 4-cycle in K_4, while the two other edges of K_4 are not subdivided. One may check that this is equivalent to the fact that one cannot obtain K_4 by the operations (68.40). So necessity in this characterization follows from the closedness of t-perfection under operation (68.40).

68.5. A combinatorial min-max relation

A subdivision of K_4 is called *totally odd* if it arises from K_4 be replacing each edge by an odd-length path. So a totally odd K_4-subdivision is an odd K_4-subdivision. A graph containing no totally odd K_4-subdivision need not be t-perfect (see Figure 68.2, from Chvátal [1975a]). However, Sewell and Trotter [1990,1993] showed that for weight function $w = \mathbf{1}$, the min-max relation is maintained for totally odd K_4-free graphs.

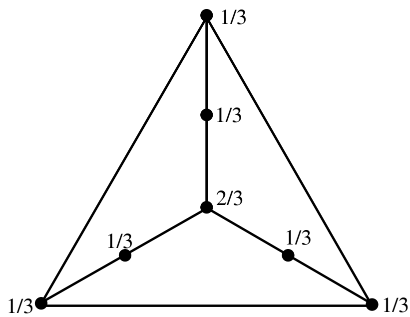

Figure 68.2
A graph containing no totally odd K_4-subdivision and not being t-perfect. The values at the vertices represent a vector satisfying (68.1) but not belonging to the stable set polytope.

This can be formulated in terms of the nonweighted version $\tilde{\rho}(G)$ of $\tilde{\rho}_w(G)$ defined in (68.7):

(68.41) $\tilde{\rho}(G) :=$ the minimum cost of a family of vertices, edges, and odd circuits covering V.

One easily checks that the minimum is attained by a vertex-disjoint family.

Obviously, for any graph G,

(68.42) $\alpha(G) \leq \tilde{\rho}(G)$.

So Sewell and Trotter [1990,1993] showed that equality holds for graphs without totally odd K_4-subdivision (generalizing a result of Gerards [1989a], who proved it for graphs without odd K_4-subdivision — a consequence of Theorem 68.3; Chvátal [1975a] proved it for series-parallel graphs).

Theorem 68.4. *For any graph G containing no totally odd K_4-subdivision, the stable set number $\alpha(G)$ is equal to $\tilde{\rho}(G)$.*

Proof. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimal. Set $\alpha := \alpha(G)$. Then G is connected, and

(68.43) $\alpha(G - v) = \alpha$ for each $v \in V$ and $\alpha(G - e) > \alpha$ for each $e \in E$,

since otherwise $G - v$ or $G - e$ would be a smaller counterexample.

Hence for each vertex v, there exists a vertex-disjoint collection of vertices, edges, and odd circuits, covering $V \setminus \{v\}$ and of cost α. Let F_v be the set of edges contained in this collection or in one of the circuits in it. Let G_v be the graph $(V \setminus \{v\}, F_v)$. So

(68.44) $\alpha(G_v) = \alpha$,

and G_v has maximum degree at most 2. Moreover, the minimality of G implies:

(68.45) $F_u \cup F_v = E \setminus \{uv\}$ for each edge uv.

To see this, trivially, $uv \notin F_u \cup F_v$. Suppose that $e \neq uv$ is an edge not contained in $F_u \cup F_v$. As $\alpha(G - e) > \alpha$, $G - e$ has a stable set S of size $\alpha + 1$. By symmetry, we can assume that $v \notin S$. Then S is a stable set in the graph G_v, contradicting (68.44).

This proves (68.45), which gives:

(68.46) for each edge uv, each edge $e \neq uv$ incident with u belongs to F_v.

This follows directly from (68.45), since $e \notin F_u$ (as $u \in e$).

This is used in proving:

(68.47) G is 3-regular.

For if vertex v has degree 1, with neighbour u, then $\alpha(G - v - u) < \alpha$; since moreover $\tilde{\rho}(G) \leq \tilde{\rho}(G - u - v) + 1$ (since we can add edge uv to any collection attaining the minimum for $G - u - v$), we have $\alpha(G) \geq \alpha(G - u - v) + 1 = \tilde{\rho}(G - u - v) + 1 \geq \tilde{\rho}(G)$. This contradicts the fact that G is a counterexample.

If v has degree 2, let G' be the graph obtained by contracting the edges incident with v. Then G' contains no totally odd K_4-subdivision. Moreover, it is straightforward to check that $\alpha(G) \geq \alpha(G') + 1$ and $\tilde{\rho}(G) \leq \tilde{\rho}(G') + 1$. As G' is smaller than G, we have $\tilde{\rho}(G') = \alpha(G')$. Hence $\alpha(G) \geq \alpha(G') + 1 = \tilde{\rho}(G') + 1 \geq \tilde{\rho}(G)$. Again, this contradicts the fact that G is a counterexample.

So v has degree at least 3. Let u be one of its neighbours. Then $\delta(v) \subseteq F_u \cup \{uv\}$ by (68.46). As G_u has maximum degree at most 2, we have $|\delta(v)| = 3$. This proves (68.47).

By (68.45) and (68.47),

(68.48) for each edge uv of G, u is traversed by an odd circuit in F_v.

Moreover:

(68.49) Let uv be an edge of G and let C be the odd circuit in F_v traversing u. Consider any edge $e = xy$ on C with $e \notin F_u$. Then both x and y have even distance from u along $C - e$.

Let S be a stable set of $G-e$ of size $\alpha+1$. So $x, y \in S$. Moreover, $u \in S$, since otherwise $\alpha(G_u) > \alpha$ (since $e \notin F_u$), contradicting (68.44). So $v \notin S$, and hence $S \setminus \{y\}$ is a maximum-size stable set of $G-v$. Hence, $S \setminus \{y\}$ intersects C in $\lfloor \frac{1}{2}|VC| \rfloor$ vertices. Therefore, S intersects C in $\lceil \frac{1}{2}|VC| \rceil$ vertices. As $x, y \in S$, (68.49) follows.

Now choose a vertex, r say, and its neighbours, u_1, u_2, u_3 say. For each $i \in \{1,2,3\}$, F_{u_i} contains an odd circuit C_i traversing r (by (68.48)), and hence traversing the edges ru_{i+1} and ru_{i+2} (taking indices mod 3). We will construct a totally odd K_4-subdivision from them, which contradicts the condition of the theorem.

For $i = 1,2,3$, let P_i be the path in C_i from u_{i+1} to u_{i+2} obtained by deleting vertex r from C_i. Since $\alpha(G - ru_i) > \alpha$, G has a stable set S_i of size α, intersecting $\{r, u_1, u_2, u_3\}$ precisely in $\{u_i\}$. Then for all distinct $i, j \in \{1,2,3\}$:

(68.50) S_j contains all vertices along P_i at even distance from u_j.

To see this, we may assume that $i = 1$, $j = 2$. Since S_2 is a maximum-size stable set in $G - u_1$, it intersects C_1 in $\lfloor \frac{1}{2}|VC_1| \rfloor$ vertices. Since $r, u_3 \notin S_2$, S_2 contains all vertices along P_1 at even distance from u_2, proving (68.50).

This implies, for distinct $i, j, k \in \{1,2,3\}$:

(68.51) $VP_i \subseteq S_j \triangle S_k$.

One similarly shows, for distinct $i, j, k \in \{1,2,3\}$:

(68.52) P_i contains an edge that splits P_i into two even-length paths $P_{i,j}$ (containing u_j) and $P_{i,k}$ (containing u_k), in such a way that S_i contains all vertices along $P_{i,j}$ at odd distance from u_j and all vertices along $P_{i,k}$ at odd distance from u_k.

To prove this, we may assume that $i = 1$, $j = 2$, $k = 3$. Since $S := S_1 \setminus \{u_1\} \cup \{r\}$ is a maximum-size stable set in $G - u_1$, it intersects C_1 in $\lfloor \frac{1}{2}|VC_1| \rfloor$ vertices. Since S contains r, there is precisely one edge on P_1 not intersected by S. This gives the edge as required in (68.52).

This implies, for distinct $i, j \in \{1,2,3\}$:

(68.53) $VP_{i,j} = VP_i \cap (S_i \triangle S_j)$,

and hence, for distinct $i, j, k \in \{1,2,3\}$:

(68.54) $VP_i \cap VP_j = VP_{i,k} \cap VP_{j,k}$,

since

(68.55) $VP_{i,k} \cap VP_{j,k} = VP_i \cap (S_i \triangle S_k) \cap VP_j \cap (S_j \triangle S_k) = VP_i \cap VP_j$

(using (68.51)).

For each $i = 1,2,3$, vertex u_{i+2} is on P_i and P_{i+1}. Hence there is a first vertex v_i on P_i (starting from u_{i+1}), that also belongs to P_{i+1}. By (68.54), v_i occurs after v_{i+2} along P_i (seen from u_{i+1}), since $v_i \in VP_i \cap VP_{i+1} \subseteq VP_{i,i+2}$ and $v_{i+2} \in VP_{i+2} \cap VP_i \subseteq VP_{i,i+1}$. Moreover, (68.56) v_i has even distance from u_{i+2} along P_i and along P_{i+1}.

To prove this, we may assume that $i = 1$. Suppose that v_1 has odd distance from u_3 along P_1. Let f and e be the previous and next edge along P_1 (seen from u_2) and let g be the third edge incident with v_1. Since v_1 is the first vertex along P_1 belonging to P_2, we know that f is not on P_2. So $f \notin F_{u_2}$, and hence (by (68.45)) $f \in F_r$. Since g is not on P_1, we have $g \notin F_{u_1}$, and hence (again by (68.45)) $g \in F_r$. Therefore (as F_r has maximum degree at most degree 2), $e \notin F_r$. Then (68.49) implies that v_1 has even distance from u_3 along P_1. Hence $v_1 \in S_3$, and so v_1 has also even distance from u_3 along P_2 (by (68.50)). This proves (68.56).

For $i = 1, 2, 3$, let Q_i be the $u_{i+1} - v_i$ part of P_i. Then Q_i and Q_{i+1} intersect each other only in v_i (since v_i is the first vertex along P_i that is on P_{i+1}). This implies that Q_1, Q_2, Q_3 together with the edges ru_1, ru_2, and ru_3, form a totally odd K_4-subdivision, a contradiction. ∎

Recall that a graph is bipartite if and only if for each subgraph H, the stable set number $\alpha(H)$ is equal to the edge cover number $\rho(H)$. An extension of this is implied by the theorem above:

Corollary 68.4a. *A graph G contains no totally odd K_4-subdivision if and only $\alpha(H) = \tilde{\rho}(H)$ for each subgraph H of G.*

Proof. Necessity follows from Theorem 68.4. Sufficiency follows from the fact that if G is a totally odd K_4-subdivision, then $\alpha(G) < \tilde{\rho}(G)$. This can be seen by induction on $|VG|$. If $|VG| = 4$, then $G = K_4$, and $\alpha(G) = 1$, $\tilde{\rho}(G) > 1$. If $|VG| > 4$, G has a vertex v of degree 2. Let G' arise by contracting the two edges incident with v. Then, using the induction hypothesis, $\alpha(G) \leq \alpha(G') + 1 < \tilde{\rho}(G') + 1 \leq \tilde{\rho}(G)$. ∎

Theorem 68.4 also implies (in fact, is equivalent to) the following. A graph $G = (V, E)$ is called *α-critical* if $\alpha(G - e) > \alpha(G)$ for each $e \in E$. Then each connected α-critical graph is either K_1, or K_2, or an odd circuit, or contains a totally odd K_4-subdivision (answering a question of Chvátal [1975a]).

We note that Theorem 68.4 implies that the stable set number $\alpha(G)$ of a graph G without totally odd K_4-subdivision can be determined in polynomial time, as $\alpha(G)$ is equal to the maximum of $\mathbf{1}^\mathsf{T}x$ over (68.1) (since the separation problem is polynomial-time solvable — see Theorem 68.1). This implies that an explicit maximum-size stable set can be found in polynomial time (just by deleting vertices as long as the stable set number does not decrease).

The vertex cover number. Another consequence of Theorem 68.4 concerns the vertex cover number $\tau(G)$ of a graph $G = (V, E)$. Trivially, $\tau(G) + \alpha(G) = |V|$. Define the *profit* of an edge to be 1, and the *profit* of a circuit C to be $\lceil \frac{1}{2}|VC| \rceil$. The *profit* of a family of edges and circuits is equal to the sum of the profits of its elements. Let $\tilde{\nu}(G)$ denote the maximum profit of a collection of pairwise vertex-disjoint edges and odd circuits in G. Then there is the following analogue to Gallai's theorem (Theorem 19.1):

Theorem 68.5. *For any graph $G = (V, E)$: $\tilde{\nu}(G) + \tilde{\rho}(G) = |V|$.*

Proof. Define the profit of any vertex to be 0. Then $\tilde{\nu}(G)$ is equal to the maximum profit of a collection of vertices, edges, and circuits partitioning V. Similarly, $\tilde{\rho}(G)$ is equal to the minimum cost of a collection of vertices, edges, and circuits partitioning V. Now for any collection $\mathcal{C}$ of vertices, edges, and circuits partitioning V we have $\text{cost}(\mathcal{C})+\text{profit}(\mathcal{C}) = |V|$. Hence the minimum cost over all such collections equals $|V|$ minus the maximum profit over all such collections. This gives the required equality. ∎

With Theorem 68.4, this implies a min-max relation for the vertex cover number of totally-odd-K_4-free graphs:

Corollary 68.5a. *For any graph G containing no totally odd K_4-subdivision, the vertex cover number $\tau(G)$ is equal to $\tilde{\nu}(G)$.*

Proof. Directly from Theorems 68.4 and 68.5, and from the fact that $\alpha(G)+\tau(G) = |V|$ for any graph G. ∎

68.6. Further results and notes

68.6a. The w-stable set polyhedron

The t-perfection of odd-K_4-free graphs can be extended to apply to w-stable sets. Given a graph $G = (V, E)$ and a function $w : E \to \mathbb{Z}_+$, a *w-stable set* is a function $x : V \to \mathbb{Z}_+$ such that $x_u + x_v \leq w_e$ for each edge $e = uv$. So if $w = \mathbf{1}$ and G has no isolated vertices, w-stable sets are the incidence vectors of stable sets. The *w-stable set polyhedron* is the convex hull of the w-stable sets.

Theorem 68.3 implies a characterization of the w-stable set polyhedron of odd-K_4-free graphs. Consider the following system:

(68.57)
$$\begin{array}{rll} \text{(i)} & x_v \geq 0 & \text{for each } v \in V, \\ \text{(ii)} & x(e) \leq w_e & \text{for each } e \in E, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2} w(EC) \rfloor & \text{for each odd circuit } C, \end{array}$$

where $x(e) = x_u + x_v$ for $e = uv$.

Theorem 68.6. *For any graph $G = (V, E)$ containing no odd K_4-subdivision and for any $w : E \to \mathbb{Z}_+$, system* (68.57) *determines the w-stable set polyhedron.*

Proof. We show that (68.57) determines an integer polyhedron, and hence is equal to the w-stable set polyhedron. Let x be a noninteger vertex of P. By resetting $w_e := w_e - \lfloor x_u \rfloor - \lfloor x_v \rfloor$ for $e = uv \in E$ and $x_v := x_v - \lfloor x_v \rfloor$ for $v \in V$, x remains a noninteger vertex of the new P. So we can assume that $0 \leq x_v < 1$ for each $v \in V$.

Let E' be the set of edges e of G with $w_e = 1$. Then $G' = (V, E')$ contains no odd K_4-subdivision, and hence is t-perfect (Theorem 68.3). So x is a convex combination of incidence vectors of stable sets of G'. As each such incidence vector satisfies (i) and (ii) of (68.57) (since $x_u + x_v \le 1+1 = 2 \le w_e$ for each edge $e = uv$ in $E \setminus E'$), it also satisfies (iii) (as it is integer). Hence x is a convex combination of integer solutions of (68.57). So P is integer. ∎

It was shown by Gijswijt and Schrijver [2002] that system (68.57) is totally dual integral for each $w : E \to \mathbb{Z}_+$ if and only if G contains no bad K_4-subdivision.

68.6b. Bidirected graphs

We saw bidirected graphs before in Chapter 36. We recall some definitions and terminology. A *bidirected graph* is a triple $G = (V, E, \sigma)$, where (V, E) is an undirected graph and where σ assigns to each $e \in E$ and each $v \in e$ a 'sign' $\sigma_{e,v} \in \{+1, -1\}$. The graph (V, E) may have loops, but we will assume that the 'two' ends of the loop have the same sign. (Other loops will be meaningless in our discussion.)

The edges e for which $\sigma_{e,v} = 1$ for each $v \in e$ are called the *positive edges*, those with $\sigma_{e,v} = -1$ for each $v \in e$ are the *negative edges*, and the remaining edges are the *directed edges*.

Clearly, undirected graphs and directed graphs can be considered as special cases of bidirected graphs. Graph terminology extends in an obvious way to bidirected graphs. The undirected graph (V, E) is called the *underlying undirected graph* of G. We also will need the *underlying signed graph* $G = (V, E, \Sigma)$, where Σ is the family of positive and negative edges. We call a circuit C in (V, E) *odd* or *even*, if $|EC \cap \Sigma|$ is odd or even, respectively.

A signed graph $G = (V, E, \Sigma)$ is called an *odd K_4-subdivision* if (V, E) is a subdivision of K_4 such that each triangle has become an odd circuit (with respect to Σ). A bidirected graph is called an *odd K_4-subdivision* if its underlying signed graph is an odd K_4-subdivision.

The $E \times V$ *incidence matrix* of a bidirected graph $G = (V, E, \sigma)$ is the $E \times V$ matrix M defined by, for $e \in E$ and $v \in V$:

(68.58)
$$M_{e,v} := \begin{cases} \sigma_{e,v} & \text{if } e \text{ is not a loop,} \\ 2\sigma_{e,v} & \text{if } e \text{ is a loop,} \end{cases}$$

setting $\sigma_{e,v} := 0$ if $v \notin e$.

For $b \in \mathbb{Z}^E$, we consider integer solutions of the system $Mx \le b$. To this end, define for any circuit C (in the undirected graph (V, E)) and any vertex v:

(68.59)
$$a_{C,v} := \tfrac{1}{2} \sum_{e \in EC} M_{e,v} \text{ and } d_C := \lfloor \tfrac{1}{2} \sum_{e \in EC} b_e \rfloor.$$

As C is a circuit, $a_{C,v}$ is an integer. Hence each integer solution x of $Mx \le b$ satisfies

(68.60)
$$\sum_{v \in V} a_{C,v} x_v = \tfrac{1}{2} \sum_{e \in EC} \sum_{v \in V} M_{e,v} x_v \le \lfloor \tfrac{1}{2} \sum_{e \in EC} b_e \rfloor = d_C.$$

Therefore, each integer solution of $Mx \le b$ satisfies:

(68.61)
(i) $Mx \le b$,
(ii) $\sum_{v \in V} a_{C,v} x_v \le d_C$ for each odd circuit C.

(Again, 'odd' is with respect to Σ.) Then Theorem 68.6 implies:

Corollary 68.6a. *If a bidirected graph G contains no odd K_4-subdivision, then system* (68.61) *determines an integer polyhedron.*

Proof. Make from the bidirected graph $G = (V, E, \sigma)$ the following auxiliary undirected graph $G' = (V', E')$. For each $e \in E$ which is not a positive loop, let $c_e := 1$ if e is positive, $c_e := 2$ if e is directed, and $c_e := 3$ if e is negative. Then replace e by a path P_e of length c_e connecting the two vertices in V incident with e. Let $\tilde{e}$ be the unique edge on P_e that is not incident with a vertex v of G with $\sigma_{e,v} = -1$.

If e is a positive loop at v, make a circuit P_e of length 3 starting and ending at v. Let $\tilde{e}$ be one of the two edges on P_e incident with v.

Let F be the set of edges f of G' that are on P_e for some $e \in E$ and satisfy $f \neq \tilde{e}$. As G has no odd K_4-subdivision (as a bidirected graph), G' has no odd K_4-subdivision (as an undirected graph). Hence by Theorem 68.6, the following system (in $x \in \mathbb{R}^{V'}$) determines an integer polyhedron:

$$\text{(68.62)} \qquad \begin{array}{rll} \text{(i)} & x(\tilde{e}) \leq b_e & \text{for each edge } e \in E, \\ \text{(ii)} & x(f) = 0 & \text{for each edge } f \in F, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C \text{ in } G'. \end{array}$$

(Here 'odd' refers to the length of the circuit. As usual, $x(f) := x_u + x_v$ where u and v are the ends of f for $f \in E'$.) This implies that system (68.61) determines an integer polyhedron, since the conditions (68.62)(ii) allow elimination of the variables x_v for $v \in V' \setminus V$. ∎

This theorem may be used to characterize odd-K_4-free bidirected graphs. Let $G = (V, E, \sigma)$ be a bidirected graph, with $E \times V$ incidence matrix M. For $a, b \in \mathbb{Z}^E$ consider integer solutions of

$$\text{(68.63)} \qquad a \leq Mx \leq b.$$

As the matrix

$$\text{(68.64)} \qquad \begin{pmatrix} M \\ -M \end{pmatrix}$$

is again the incidence matrix of some bidirected graph, we can consider the inequalities (68.61)(ii) corresponding to matrix (68.64). They amount to:

$$\text{(68.65)} \qquad \sum_{v \in V} \tfrac{1}{2}\Big(\sum_{e \in F} M_{e,v} - \sum_{e \in EC \setminus F} M_{e,v}\Big)x_v \leq \Big\lfloor \tfrac{1}{2}\Big(\sum_{e \in F} b_e - \sum_{e \in EC \setminus F} a_e\Big)\Big\rfloor \text{ for each odd circuit } C \text{ and each } F \subseteq EC.$$

To describe the characterization, we define 'minor' of a signed graph $G = (V, E, \Sigma)$. For $e \in E$, *deletion* of e means resetting E and Σ to $E \setminus \{e\}$ and $\Sigma \setminus \{e\}$. *Deletion* of a vertex v means deleting all edges incident with v, and deleting v from V. If e is not a loop, *contraction* of e means the following. Let e have ends u and v. If $e \in \Sigma$, reset $\Sigma := \Sigma \triangle \delta(u)$. Otherwise, let Σ be unchanged. Next contract e in (V, E). This definition depends on the choice of the end u of e, but for the application below this will be irrelevant. A *resigning* means choosing $U \subseteq V$ and resetting Σ to $\Sigma \triangle \delta(U)$. A signed graph H is called a *minor* of a signed graph G if

H arises from G by a series of deletions of edges and vertices, contractions of edges, and resignings.

Then we have the following characterization (Gerards and Schrijver [1986]), where odd-K_4 stands for the signed graph (VK_4, EK_4, EK_4).

Corollary 68.6b. *For any bidirected graph G the following are equivalent:*

(68.66) (i) *G contains no odd K_4-subdivision as subgraph;*
(ii) *the signed graph underlying G has no odd-K_4 minor;*
(iii) *for all integer vectors a, b, system* (68.63)(68.65) *determines a box-integer polyhedron.*

Proof. The implication (ii)$\Rightarrow$(i) follows from the easy fact that any odd K_4-subdivision in G would yield an odd-K_4 minor of the signed graph underlying G.

The implication (i)$\Rightarrow$(iii) can be derived from Corollary 68.6a as follows. Replace any 'box' constraint $d_v \leq x_v \leq c_v$ by $2d_v \leq 2x_v \leq 2c_v$, and incorporate it into M, by adding loops at v. Then the constraint (68.65) corresponding to such a loop C at v is $x_v \leq c_v$ or $-x_v \leq -d_v$. This gives a reduction to Corollary 68.6a.

To see the implication (iii)$\Rightarrow$(ii), note that (iii) is invariant under deleting rows of M and under multiplying rows or columns by -1. It is also closed under contractions of any edge e, as it amounts to taking $a_e = b_e = 0$ in (68.63). So, in proving (iii)$\Rightarrow$(ii), if the signed graph underlying G has an odd-K_4 minor, we may assume that it is odd-K_4. By multiplying rows and columns of M by -1, we may assume that M is nonnegative. Then we do not have an integer polytope for $a = \mathbf{0}$, $b = \mathbf{1}$, $d = \mathbf{0}$, $c = \mathbf{1}$. ∎

In other words, the bidirected graphs without odd K_4-subdivision are precisely those whose $E \times V$ incidence matrix has strong Chvátal rank at most 1 (cf. Section 36.7a, where it is shown that the transpose of *each* such matrix has strong Chvátal rank at most 1).

68.6c. Characterizing odd-K_4-free graphs by mixing stable sets and vertex covers

A similar characterization can be formulated in terms of just undirected graphs, by mixing stable sets and vertex covers. Call a graph H an *odd minor* of a graph G if H arises from G by deleting edges and vertices, and by contracting all edges in some cut $\delta(U)$ (in the graph without the deleted edges). The following is easy to show:

(68.67) A graph G contains an odd K_4-subdivision $\iff$ G contains K_4 as odd minor.

For a graph $G = (V, E)$ and $F \subseteq E$, a subset U of V is called F-*stable* if U is a stable set of the graph (V, F). U is called an F-*cover* if U is a vertex cover of (V, F). Let F_1 and F_2 be disjoint subsets of E, and consider the system:

$$(68.68)\qquad \begin{array}{ll} 0 \leq x_v \leq 1 & \text{for } v \in V, \\ x(e) \leq 1 & \text{for } e \in F_1, \\ x(e) \geq 1 & \text{for } e \in F_2, \\ \displaystyle\sum_{e \in EC \cap F_1} x(e) - \sum_{e \in EC \cap F_2} x(e) \leq |EC \cap F_1| - |EC \cap F_2| - 1 & \\ & \text{for each odd circuit } C \text{ with } EC \subseteq F_1 \cup F_2 \end{array}$$

(where $x(e) := x_u + x_v$ for $e = uv \in E$).

Corollary 68.6c. *For any graph $G = (V, E)$ the following are equivalent:*

(68.69) (i) *G contains no odd K_4-subdivision;*
(ii) *for all disjoint $F_1, F_2 \subseteq E$, the convex hull of the incidence vectors of the F_1-stable F_2-covers is determined by* (68.68).

Proof. The implication (i)⇒(ii) follows from Corollary 68.6a. To see (ii)⇒(i), we first show that (ii) is maintained under taking odd minors. Maintenance under deletion of edges or vertices is trivial. To see that it is maintained under contraction of cuts, let $U \subseteq V$ and let $G' = (V', E')$ be the contracted graph. Let F_1' and F_2' be disjoint subsets of E', and let x' satisfy (68.68) for G', F_1', F_2'. Define $x : V \to \mathbb{R}$ as follows, where, for $v \in V$, v' denotes the vertex of G' to which v is contracted:

$$(68.70)\qquad x_v := \begin{cases} x'_{v'} & \text{if } v \in U, \\ 1 - x'_{v'} & \text{if } v \in V \setminus U. \end{cases}$$

Moreover, define F_1 and F_2 by:

$$(68.71)\qquad \begin{array}{l} F_1 := (F_1' \cap E[U]) \cup (F_2' \cap E[V \setminus U]) \cup \delta(U), \\ F_2 := (F_2' \cap E[U]) \cup (F_1' \cap E[V \setminus U]). \end{array}$$

Then x satisfies (68.68) with respect to G, F_1, F_2. Hence x is a convex combination of integer solutions of (68.68). Applying the construction in reverse to (68.70), we obtain x' as a convex combination of integer solutions of (68.68) with respect to G', F_1', F_2'.

This shows that (68.69)(ii) is maintained under taking odd minors. Moreover, K_4 violates the condition (taking $F_1 := E$, $F_2 := \emptyset$, $x_v := \frac{1}{3}$ for each $v \in V$). This shows sufficiency of the condition. ∎

68.6d. Orientations of discrepancy 1

A directed graph $D = (V, A)$ is said to have *discrepancy k* if for each (undirected) circuit, the number of forward arcs differs by at most k from the number of backward arcs.

The proof of Gerards [1989a] of the strong t-perfection of odd-K_4-free graphs (Theorem 68.3) is by showing that each such graph can be decomposed into graphs that have an orientation of discrepancy 1, using a characterization of Gerards [1994] of orientability of discrepancy 1 and a decomposition theorem of Gerards, Lovász, Schrijver, Seymour, Shih, and Truemper [1993] (cf. Gerards [1990]). As the graphs having an orientation of discrepancy 1 can be shown to be strongly t-perfect with minimum-cost flow techniques (see Theorem 68.7 below), and as the composition maintains total dual integrality of (68.1), the required result follows.

It is not difficult to show that the underlying undirected graph of any digraph of discrepancy 1, contains no odd K_4-subdivision. So, by Theorem 68.3, any undirected graph having an orientation of discrepancy 1, is strongly t-perfect. Gerards gave a direct proof of the strong t-perfection of such graphs, based on the following minimum-cost circulation argument:

Lemma 68.7α. *Let $D = (V, A)$ be a directed graph and let $b : A \to \mathbb{Z}_+$. Then the following system is totally dual integral:*

(68.72) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(VC) \leq b(AC)$ for each directed circuit C in D.

Proof. Choose $w : V \to \mathbb{Z}_+$. We must show that the dual of maximizing $w^\mathsf{T} x$ over (68.72) has an integer optimum solution.

Make another directed graph $\widetilde{D} = (\widetilde{V}, \widetilde{A})$ as follows. For each vertex v of D, make two vertices v', v'' and an arc (v', v''), and for each arc (u, v) of D, make an arc (u'', v'). This defines $\widetilde{D}$.

Define $g, f : \widetilde{A} \to \mathbb{Z}_+$ by:

(68.73) $g(v', v'') := w(v)$ and $f(v', v'') := 0$ for $v \in V$,
$g(u'', v') := 0$ and $f(u'', v') := b(u, v)$ for $(u, v) \in A$.

Then the maximum of $w^\mathsf{T} x$ over (68.72) is equal to the maximum of $g^\mathsf{T} z$ where $z : \widetilde{A} \to \mathbb{R}_+$ satisfies

(68.74) $z(A\widetilde{C}) \leq f(A\widetilde{C})$ for each directed circuit $\widetilde{C}$ in $\widetilde{D}$.

So if we consider $f - z$ as length function on $\widetilde{A}$, then (68.74) says that each directed circuit in $\widetilde{D}$ has nonnegative length. Hence, by Theorem 8.2, the maximum is equal to the maximum of $g^\mathsf{T} z$ over $z : \widetilde{A} \to \mathbb{R}_+$ for which there exists a $p : \widetilde{V} \to \mathbb{R}$ such that

(68.75) $z(\tilde{a}) + p(t) - p(s) \leq f(\tilde{a})$ for each $\tilde{a} = (s, t) \in \widetilde{A}$.

The latter system has a totally unimodular constraint matrix, and hence the LP has integer optimum primal and dual solutions. The dual asks for the minimum of $y^\mathsf{T} f$ where $y : \widetilde{A} \to \mathbb{Z}_+$ satisfies

(68.76) $y(\tilde{a}) \geq g(\tilde{a})$ for each $\tilde{a} \in \widetilde{A}$,
$y(\delta^{\text{in}}(\tilde{v})) = y(\delta^{\text{out}}(\tilde{v}))$ for each $\tilde{v} \in \widetilde{V}$.

So y is a circulation in $\widetilde{D}$. Hence y is a nonnegative integer combination of incidence vectors of directed circuits $\widetilde{C}$ in $\widetilde{D}$:

(68.77) $$y = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} \chi^{A\widetilde{C}}.$$

For each directed circuit $\widetilde{C}$ in $\widetilde{D}$, let C denote the corresponding directed circuit in D (obtained by contracting all arcs (v', v'') occurring in $\widetilde{C}$). Then

(68.78) $$y^\mathsf{T} f = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} (\chi^{A\widetilde{C}})^\mathsf{T} f = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} f(A\widetilde{C}) = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} b(AC)$$

and

(68.79) $$\sum_{\widetilde{C}} \lambda_{\widetilde{C}} \chi^{VC} \geq w.$$

Hence we have obtained an integer dual solution for the problem of maximizing $w^{\mathsf{T}}x$ over (68.72). ∎

This lemma implies:

Theorem 68.7. *Let $G = (V, E)$ be an undirected graph having an orientation D of discrepancy 1. Then G is strongly t-perfect.*

Proof. Let $D' = (V, A')$ be the digraph obtained from D by adding a reverse arc (v, u) for each arc (u, v) of D, defining $b(u, v) := 1$ and $b(v, u) := 0$. Then the total dual integrality of (68.1) follows directly from the total dual integrality of (68.72). Note that each directed circuit C' in D' gives an undirected circuit C in D, with $b(AC')$ equal to the number of forward arcs in C. As D has discrepancy 1, $\lfloor \frac{1}{2}|VC| \rfloor$ is equal to the minimum value of $b(AC')$ and $b(AC'^{-1})$. ∎

This immediately implies the strong t-perfection of *almost bipartite* graphs — graphs having a vertex v with $G - v$ bipartite, since they have an orientation of discrepancy 1, as one easily checks.

68.6e. Colourings and odd K_4-subdivisions

Zang [1998] and Thomassen [2001] showed that any graph G without totally odd K_4-subdivision satisfies $\chi(G) \leq 3$.[25] We may interpret this in terms of the integer decomposition and rounding properties. Consider the antiblocking polytope Q of the stable set polytope of a graph $G = (V, E)$:

(68.80)
$$\begin{array}{ll} x_v \geq 0 & \text{for each } v \in V, \\ x(S) \leq 1 & \text{for each stable set } S. \end{array}$$

If G is t-perfect, the vertices of Q are: the origin, the unit base vectors, the incidence vectors of the edges, and the vectors $\chi^{VC}/\lfloor \frac{1}{2}|VC| \rfloor$ where C is an odd circuit. (This follows from the definition of t-perfection with antiblocking polyhedra theory.) Hence the fractional colouring number $\chi^*(G)$ of G, which is equal to the maximum of $\mathbf{1}^{\mathsf{T}}x$ over (68.80) (cf. Section 64.8), is equal to

(68.81) $$\max\{2, \max\{\frac{|VC|}{\lfloor \frac{1}{2}|VC| \rfloor} \mid C \text{ odd circuit}\}\}$$

(assuming $E \neq \emptyset$). For nonbipartite graphs, this value is equal to 3. So for graphs G without totally odd K_4-subdivision, the colouring number $\chi(G)$ is equal to the round-up $\lceil \chi^*(G) \rceil$ of the fractional colouring number.

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[25] This was conjectured by Toft [1975], and extends results of Hadwiger [1943] that a 4-chromatic graph contains a K_4-subdivision, of Catlin [1979] that it contains an odd K_4-subdivision, and of Gerards and Shepherd [1998] that it contains a bad K_4-subdivision. Zeidl [1958] showed that any vertex of a minimally 4-chromatic graph lies in a subdivided K_4 that contains an odd circuit. Other partial and related results were found by Krusensjterna-Hafstrøm and Toft [1980], Thomassen and Toft [1981], and Jensen and Shepherd [1995].

A.M.H. Gerards (personal communication 2001) showed that system (68.80) has the integer rounding property if G has no odd K_4-subdivision. It implies that the corresponding stable set polytope has the integer decomposition property. This is equivalent to:

(68.82) $\chi_w(G) = \lceil \chi_w^*(G) \rceil$

for each odd-K_4-free graph G and each $w : VG \to \mathbb{Z}_+$.

This does not hold for any t-perfect graph: M. Laurent and P.D. Seymour showed in 1994 that the complement of the line graph of a prism (complement of C_6) is t-perfect, but is not 3-colourable; hence its stable set polytope does not have the integer decomposition property.

68.6f. Homomorphisms

Let G and H be simple graphs. A *homomorphism* $G \to H$ is a function $\phi : VG \to VH$ such that if $uv \in EG$, then $\phi(u)\phi(v) \in EH$ (in particular, $\phi(u) \neq \phi(v)$). Obviously, if there exists a homomorphism $G \to H$, then $\chi(G) \leq \chi(H)$.

For any k, let $K_4^{(k)}$ be the graph obtained from K_4 by replacing each edge by a path of length k. Then one may check that for odd k there is no homomorphism $K_4^{(k)} \to C_{2k+1}$.

Catlin [1985] showed that this is essentially the only counterexample: if G is a connected graph of maximum degree 3 and $k \in \mathbb{Z}_+$, such that any two vertices of G of degree 3 have distance at least k, and such that there is no homomorphism $G \to C_{2k+1}$, then k is odd and $G = K_4^{(k)}$. (This extends Brooks' theorem (Theorem 64.3) for $k = 1$.)

Gerards [1988] extended this to: if a nonbipartite graph G has no odd minor equal to K_4 or to the graph obtained from the triangle by adding for each edge a new vertex adjacent to the ends of the edge, then there is a homomorphism $G \to C_t$, where t is the shortest length of an odd circuit of G. Further results are given by Catlin [1988].

68.6g. Further notes

Sbihi and Uhry [1984] call a graph $G = (V, E)$ *h-perfect*[26] if the stable set polytope is determined by

(68.83)
$$\begin{array}{rll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(C) \leq 1 & \text{for each clique } C, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C. \end{array}$$

So perfect graphs and t-perfect graphs are h-perfect. Sbihi and Uhry showed that substituting bipartite graphs for edges of a series-parallel graph preserves h-perfection.

The t-perfection of line graphs, and classes of graphs that are h-perfect but not t-perfect, were studied by Cao and Nemhauser [1998]. Gerards [1990] gave a survey on signed graphs without odd K_4-subdivision.

[26] h stands for 'hole' (English for 'trou').

Chapter 69

Claw-free graphs

Claw-free graphs are graphs not having $K_{1,3}$ as induced subgraph. We show the result of Minty and Sbihi that a maximum-size stable set in a claw-free graph can be found in strongly polynomial time, and the extension of Minty to the weighted case.

69.1. Introduction

A graph $G = (V, E)$ is called *claw-free* if no induced subgraph of G is isomorphic to $K_{1,3}$. Minty [1980] and Sbihi [1980] showed that a maximum-size stable set in a claw-free graph can be found in polynomial time. Since the line graph of any graph is claw-free, this generalizes Edmonds' polynomial-time algorithm for finding a maximum-size matching in a graph.

Sbihi's algorithm is an extension of Edmonds' blossom shrinking technique, while Minty gave a reduction to the maximum-size matching problem. Minty [1980] also indicated that his algorithm can be extended to the weighted case by reduction to Edmonds' weighted matching algorithm. The final argument for this was given by Nakamura and Tamura [2001].

In Section 69.2, we describe Minty's method for finding a maximum-size stable set in claw-free graphs, and in Section 69.3 we describe the extension to the weighted case.

69.2. Maximum-size stable set in a claw-free graph

An important property of claw-free graphs is that any vertex has at most two neighbours in any stable set. This enables us to augment stable sets by S-augmenting paths, which we define now.

Let $G = (V, E)$ be a graph and let S be a stable set in G. A walk $P = (v_0, v_1, \ldots, v_k)$ (given by its vertex-sequence) is called *S-alternating* if precisely one of v_{i-1}, v_i belongs to S, for each $i = 1, \ldots, k$. It is an *S-augmenting path* if moreover P is a path, $v_0, v_k \notin S$, and $(S \setminus \{v_1, v_3, \ldots, v_{k-1}\}) \cup \{v_0, v_2, \ldots, v_k\}$ is stable. This implies that (if $k \geq 2$) each of v_0 and v_k has precisely one neighbour in S, and each of $v_2, v_4, \ldots, v_{k-2}$ precisely two.

It is easy to see that if G is claw-free, then there is a stable set larger than S if and only if there exists an S-augmenting path. Indeed, sufficiency follows from the definition of S-augmenting path. To see necessity, let S' be a stable set larger than S. Then the subgraph of G induced by $S \triangle S'$ has a component K with more vertices in S' than in S. Since G is claw-free, this subgraph has maximum degree 2, and hence K forms an S-augmenting path.

So in order to find a maximum-size stable set, it suffices to have a polynomial-time algorithm to find for given S, an S-augmenting path, if any. For this, it suffices to describe a polynomial-time algorithm to find an S-augmenting $a-b$ path for prescribed $a, b \in V \setminus S$ (if any). Varying over all $a, b \in V \setminus S$, we find an S-augmenting path (if any).

Therefore, from now on we fix $a, b \in V \setminus S$. Then we can assume:

(69.1) $a \neq b$; a and b have degree 1, each with neighbour in S, say s_a and s_b; $s_a \neq s_b$; each $v \in V \setminus S$ with $v \neq a, b$ has precisely two neighbours in S; for each $s \in S$ with $s \neq s_a, s_b$ there are at least two vertices in S at distance two from s; G is connected.

Indeed, otherwise finding an S-augmenting path is trivial, or it does not exist; moreover, we can delete all neighbours of a or b distinct from s_a or s_b, and all vertices in $S \setminus \{s_a, s_b\}$ that have less than two vertices in S at distance two.

The assumptions (69.1) imply that any S-augmenting path connects a and b. Consider an S-alternating path

(69.2) $P = (v_0, s_1, v_1, \ldots, s_k, v_k)$

(given by its vertex-sequence), with $v_0 = a$ and $v_k = b$. So $s_1 = s_a$ and $s_k = s_b$. Then (under the assumptions (69.1)):

Lemma 69.1α. *P is S-augmenting if and only if v_{i-1} and v_i are nonadjacent for each $i = 2, \ldots, k-1$.*

Proof. Necessity being trivial, we show sufficiency. It suffices to show that $(S \setminus \{s_1, \ldots, s_k\}) \cup \{v_0, \ldots, v_k\}$ is a stable set. Any two vertices in S are nonadjacent. All neighbours in S of any v_i are among $s_1, \ldots, s_k$. Finally, suppose that any v_i, v_j are adjacent, with $i < j$. Then $j \geq i + 2$, since v_i and v_{i+1} are nonadjacent by the condition. But then v_i is adjacent to the three pairwise nonadjacent vertices s_i, s_{i+1}, and v_j. This contradicts the claw-freeness of G. ∎

We next prove a basic lemma of Minty [1980]. Define, for $u, v \in V \setminus S$:

(69.3) $u \sim v \iff N(u) \cap S = N(v) \cap S.$

Clearly, $\sim$ is an equivalence relation. We call any equivalence class a *similarity class*, and if $u \sim v$ we say that u and v are *similar*. So for each $s \in S$, $N(s)$ is a union of similarity classes.

We call a vertex $s \in S$ *splittable* if $N(s)$ can be partitioned into two classes X, Y such that

(69.4) $\qquad uv \in E \iff u, v \in X$ or $u, v \in Y$

for all $u, v \in N(s)$ with $u \not\sim v$. If s is splittable, we call X and Y the *classes* of s. Define

(69.5) $\qquad S' := \{s \in S \mid s$ is splittable$\}$ and $S'' := S \setminus S'$.

Then $s_a, s_b \in S'$, since $N(s_a) \setminus \{a\}$ is a clique, as a has no neighbours in $N(s_a)$ (by assumption (69.1)) and as G is claw-free — similarly for s_b. Moreover:

Lemma 69.1β. *Each vertex $s \in S$ having at least three vertices in S at distance two, belongs to S'.*

Proof. Since $s_a, s_b \in S'$, we may assume that $s \neq s_a, s_b$. Let $G' = (N(s), F)$ be the subgraph of G with

(69.6) $\qquad F := \{uv \in E \mid u, v \in N(s), u \not\sim v\}$.

Then

(69.7) $\qquad$ each component of G' induces a clique of G.

Suppose not. Let $P = (v_0, v_1, \ldots, v_k)$ be a shortest path in G' with $v_0v_k \notin E$. If $k = 2$, then $v_0 \not\sim v_1 \not\sim v_2$, and hence v_1 has a neighbour $t \in S$ which is not a neighbour of v_0 or v_2. But then v_1 is adjacent to the pairwise nonadjacent t, v_0, v_2, contradicting the claw-freeness of G.

If $k = 3$, then as P is shortest, $v_0v_2, v_1v_3 \in E \setminus F$. So $v_0 \sim v_2$ and $v_1 \sim v_3$. Choose a vertex p with $p \not\sim v_0$ and $p \not\sim v_1$. (This is possible since $N(s)$ contains at least three similarity classes.) Then p has a neighbour t in S which is not a neighbour of any of v_0, v_1, v_2, v_3. Since $N(s)$ contains no three pairwise nonadjacent vertices (as G is claw-free), we know that $v_0p \in E$ or $v_3p \in E$. By symmetry, we can assume that $v_0p \in E$, and hence $v_0p \in F$. Then, by the minimality of k, we know successively that $v_1p \in F$, $v_2p \in F$, and $v_3p \in F$. But then v_0p and pv_3 are in F, and hence, by the minimality of k, $v_0v_3 \in E$.

If $k \geq 4$, then $v_0v_2, v_0v_3 \in E$, hence (since $v_2 \not\sim v_3$) $v_0v_2 \in F$ or $v_0v_3 \in F$, contradicting the minimality of k. This proves (69.7).

Since G is claw-free, G' has at least one component, X say, that intersects at least two of the similarity classes. If G' has at most two components, or if X contains all but at most one similarity class, we are done, taking $Y := N(s) \setminus X$. If G' has at least three components and $N(s) \setminus X$ intersects at least two similarity classes, then G' has two other components Y, Z for which there exist $x \in X$, $y \in Y$, and $z \in Z$ with $x \not\sim y \not\sim z \not\sim x$, as one easily checks[27]. But then s is adjacent to the three pairwise nonadjacent vertices x, y, z, contradicting the claw-freeness of G. ∎

[27] Let $y \in N(s) \setminus X$ be such that there exist $x', x'' \in X$ with $y \not\sim x' \not\sim x'' \not\sim y$. Let Y be the component of G' containing y. Let Z be a third component, if possible containing

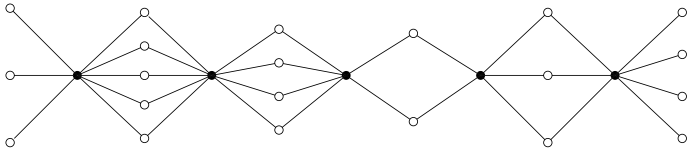


Figure 69.1
A typical bone

Now consider the subgraph

(69.8) $\qquad (V \setminus S', \delta(S''))$

of G. It is a bipartite graph, with colour classes S'' and $V \setminus S$. We call each component of this graph a *bone*. (A typical bone is depicted in Figure 69.1.) By Lemma 69.1β, each $s \in S''$ has at most two vertices in S at distance two. Hence any bone B consists of a series of vertices $s_1, \ldots, s_k$ in S'', together with disjoint nonempty sets $V_0, V_1, \ldots, V_k$ of vertices such that s_i is incident with each vertex in $V_{i-1} \cup V_i$ for each $i = 1, \ldots, k$. Moreover, B has two neighbours in S', say s and t, where s is adjacent to all vertices in V_0 and t is adjacent to all vertices in V_k. (It might be that $s = t$.) The degenerate case is that $k = 0$, where B is a singleton vertex in $V \setminus S$ with two neighbours in S'.

The relevance of bones is that if we leave out from any S-augmenting path the vertices that belong to S', we are left with a number of subpaths, each of which is an S''-augmenting path contained in some bone. So in constructing or analyzing an S-augmenting path, we can decompose it into S''-augmenting paths, glued together at vertices in S'. Here the classes of the vertices in S' come in, since the ends of the two subpaths glued together at $s \in S'$ should belong to different classes of s. This motivates the following graph H (called the *Edmonds graph* by Minty [1980])[28].

H has vertex set

(69.9) $\qquad \{(s, X) \mid s \in S',\ X \text{ class of } s\} \setminus \{(s_a, \{a\}), (s_b, \{b\})\}$

and the following edges:

(69.10) $\qquad$ (i) $\{(s, X), (s, Y)\}$ for $s \in S' \setminus \{s_a, s_b\}$ and X, Y the classes of s;

a vertex nonsimilar to y. Then, if Z contains a vertex $z \not\sim y$, we can take for x one of x', x''. If Z contains no such vertex, let $z \in Z$. Then Y contains a vertex $y' \not\sim z$. As $z \not\sim x'$ and $z \not\sim x''$, we are done again.

[28] We note that in constructing H we could restrict S' to those vertices in S that have at least 3 vertices in S at distance two, together with s_a and s_b. However, for the extension to the weighted case, we need S' as defined above (namely, by being splittable).

(ii) $\{(s,X),(t,Y)\}$ for vertices $(s,X),(t,Y)$ of H such that there exists an S''-augmenting $X-Y$ path P.

So the path P is contained in the bone B containing x and y for some $x \in X$ and some $y \in Y$. Its existence can be checked as follows. Let $V_0, V_1, \ldots, V_k$ be as above. Make the digraph D on $V_0 \cup V_1 \cup \ldots \cup V_k$ with an arc from $u \in V_{i-1}$ to $v \in V_i$ if $uv \notin E$ (for $i = 1, \ldots, k$). Then a directed $X-Y$ path in D gives a path P as required, and conversely.

Let M be the matching of edges in (69.10)(i). So M covers all vertices of H, except the vertices $(s_a, N(s_a) \setminus \{a\})$ and $(s_b, N(s_b) \setminus \{b\})$. Then (under the assumptions (69.1)):

Lemma 69.1γ. *G has an S-augmenting path $\Longleftrightarrow$ H has an M-augmenting path. We can obtain one from the other in polynomial time.*

Proof. Let $P = (v_0, s_1, v_1, \ldots, s_k, v_k)$ be an S-augmenting path in G, with $v_0 = a$ and $v_k = b$. Let $s_{i_1}, \ldots, s_{i_t}$ be those vertices in P that belong to S' (in order). So $i_1 = 1$ and $i_t = k$. For $j = 1, \ldots, t$, let X_j and Y_j be the classes of s_{i_j} that contain $v_{i_j - 1}$ and v_{i_j}, respectively. Then $X_j \neq Y_j$, since v_{i_j-1} and v_{i_j} are nonsimilar and nonadjacent. Moreover, the subpath of P between any two s_{i_j} and $s_{i_{j+1}}$ forms an S''-augmenting $Y_j - X_{j+1}$ path. Hence

(69.11) $$((s_{i_1}, Y_1), (s_{i_2}, X_2), (s_{i_2}, Y_2), \ldots, (s_{i_{t-1}}, X_{t-1}), (s_{i_{t-1}}, Y_{t-1}), (s_{i_t}, X_t))$$

is an M-augmenting path in H.

We can reverse this construction. Indeed, any M-augmenting path Q yields an S-alternating $a-b$ walk P in G, by inserting appropriate S''-augmenting paths.

In fact, P is a path. For suppose that P traverses some vertex u of G more than once. Then u belongs to two of the inserted paths. Necessarily, they belong to the same bone B. Hence B has a neighbour in S' that is traversed more than once. But then Q traverses some matching edge more than once, a contradiction.

So P is a path. Moreover, any two vertices at distance two in P are nonadjacent, by construction of P. So P is S-augmenting, by Lemma 69.1α. ∎

Concluding, we have obtained the result of Minty [1980] and Sbihi [1980]:

Theorem 69.1. *A maximum-size stable set in a claw-free graph can be found in polynomial time.*

Proof. From Lemma 69.1γ, since finding an M-augmenting path in H is equivalent to finding a perfect matching in M. The latter problem is polynomial-time solvable by Corollary 24.4a. ∎

69.3. Maximum-weight stable set in a claw-free graph

There is an obvious way of extending the above construction to the weighted case, but there is a catch in it. The idea was noted by Minty [1980], and finalized by Nakamura and Tamura [2001].

Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{R}_+$ be a weight function. Call a stable set S *extreme* if it has maximum weight among all stable sets of size $|S|$. It suffices to describe an algorithm to derive from any extreme stable set S, an extreme set of size $|S| + 1$, if any (since then we can start with $S := \emptyset$, enumerate extreme stable sets of all possible sizes, and choose one of maximum weight among them).

The following observations are basic:

Lemma 69.2α. *Let $G = (V, E)$ be a claw-free graph, let $w : V \to \mathbb{R}_+$, and let S be an extreme stable set. Then:*

(69.12) (i) *each S-alternating chordless circuit satisfies $w(VC \setminus S) \leq w(VC \cap S)$;*
(ii) *if P is an S-augmenting path maximizing $w(VP \setminus S) - w(VP \cap S)$, then $S \triangle VP$ is an extreme stable set of size $|S| + 1$.*

Proof. (i) follows from the fact that $S \triangle VC$ is a stable set of size $|S|$, and hence $w(S) \geq w(S \triangle VC) = w(S) + w(VC \setminus S) - w(VC \cap S)$.

(ii) can be seen as follows. Let $\widetilde{S}$ be an extreme stable set of size $|S|+1$. The subgraph induced by $S \triangle \widetilde{S}$ has a component K with $|K \cap \widetilde{S}| > |K \cap S|$. Since G is claw-free, K has maximum degree at most 2. So K is an S-augmenting path, and hence $|K \cap \widetilde{S}| = |K \cap S| + 1$. Let $L := (S \triangle \widetilde{S}) \setminus K$. Then $S \triangle L$ and $\widetilde{S} \triangle L$ are stable sets of size $|S|$ and $|S| + 1$ respectively. Since S is extreme, $w(L \cap \widetilde{S}) \leq w(L \cap S)$. Hence $w(\widetilde{S} \triangle L) \geq w(\widetilde{S})$. So $\widetilde{S} \triangle L$ is extreme again. Hence we can assume that $L = \emptyset$. Then, since K is an S-augmenting path:

(69.13)
$$\begin{aligned} w(S \triangle VP) &= w(S) + w(VP \setminus S) - w(VP \cap S) \\ &\geq w(S) + w(K \setminus S) - w(K \cap S) = w(\widetilde{S}). \end{aligned}$$

So $S \triangle VP$ is extreme. ∎

Statement (ii) of Lemma 69.2α implies that, to find an extreme stable set of size $|S| + 1$, it suffices to find an S-augmenting path P maximizing $w(VP \setminus S) - w(VP \cap S)$. By enumerating over all pairs $a, b \in V \setminus S$, it suffices to find, for each fixed $a, b \in V \setminus S$, an S-augmenting $a - b$ path P maximizing $w(VP \setminus S) - w(VP \cap S)$ (if any). Then we can make again the assumptions (69.1), and construct the graph H. Define a weight function ω on the edges of H (following the items in (69.10)) as follows:

(69.14) (i) $\omega(\{(s, X), (s, Y)\}) := w(s)$,
(ii) $\omega(\{(s, X), (t, Y)\}) :=$ the maximum of $w(VP \setminus S'') - w(VP \cap S'')$ over all S''-augmenting $X - Y$ paths P.

The maximum in (69.14)(ii) can be found in strongly polynomial time, since it amounts to finding a longest directed $X-Y$ path in the acyclic digraph D described just after (69.10).

Now a maximum-weight perfect matching in H need not yield a maximum-weight stable set in G, as was pointed out by Nakamura and Tamura [2001], since there might exist M-alternating circuits that increase the weight of M, while they do not correspond to a chordless S-alternating circuit. However, this can be avoided by preprocessing as follows.

We can assume that for each $v \in V \setminus S$ with $v \neq a, b$:

(69.15) (i) there exist s, t, x, y such that (x, s, v, t, y) is a chordless S-alternating path and such that $N(x) \cap N(y) \cap S = \emptyset$;
(ii) there exist $s, t \in S'$ and classes X of s and Y of t such that there exists an S''-augmenting $X-Y$ path and such that each S''-augmenting $X-Y$ path P attaining the maximum in (69.14)(ii), traverses v.

Otherwise v is on no maximum-weight S-augmenting path, and hence we can delete v. The conditions (69.15) can be tested in strongly polynomial time (for (ii) using digraph D). Hence the deletions take strongly polynomial time only.

Fix for each edge e of H in (69.14)(ii), a path P_e attaining the maximum. Then we can transform any M-alternating path or circuit to an S-alternating walk or closed walk, by replacing each such edge e by P_e. We call this the *corresponding* walk or closed walk in G.

Lemma 69.2β. *Under the assumptions* (69.15)*, each M-alternating circuit C in H satisfies $\omega(EC \setminus M) \leq \omega(EC \cap M)$.*

Proof. Suppose not. Choose C maximizing $\omega(EC \setminus M) - \omega(EC \cap M)$. Let Γ be the corresponding S-alternating closed walk in G. Then Γ is not a chordless circuit, since otherwise

(69.16) $$w(V\Gamma \setminus S) - w(V\Gamma \cap S) = \omega(EC \setminus M) - \omega(EC \cap M) > 0,$$

which contradicts (i) of Lemma 69.2α.

Since each P_e is simple and chordless, it follows that $EC \setminus M$ contains distinct edges e, f for which there exist $u \in VP_e$ and $v \in VP_f$ with $u = v$ or $uv \in E$. This implies that C has length 4, and that e and f are the only edges in $EC \setminus M$. So P_e and P_f are in the same bone B. Let s and t be the neighbours of B in S'. Let s have classes Y, Z and t have classes W, X such that P_e connects Y and W and P_f connects Z and X. Write

(69.17) $$P_e = (u_0, s_1, u_1, \ldots, s_k, u_k) \text{ and } P_f = (v_0, s_1, v_1, \ldots, s_k, v_k)$$

for some $k \geq 0$ and $s_1, \ldots, s_k \in S''$, where $u_0 \in Y$, $u_k \in W$, $v_0 \in Z$, $v_k \in X$. We define $s_0 := s$ and $s_{k+1} := t$. Now

(69.18) for each $i = 1, \ldots, k$: $u_{i-1}v_i \in E$ or $v_{i-1}u_i \in E$.

Otherwise, we can 'switch' P_e and P_f at s_i to obtain the S''-augmenting paths

(69.19) $$Q := (u_0, s_1, \ldots, u_{i-1}, s_i, v_i, \ldots, s_k, v_k) \text{ and}$$
$$R := (v_0, s_1, \ldots, v_{i-1}, s_i, u_i, \ldots, s_k, u_k).$$

Hence H has edges $\{(s,Y),(t,X)\}$ and $\{(s,Z),(t,W)\}$, and

(69.20) $$\omega(\{(s,Y),(t,X)\}) + \omega(\{(s,Z),(t,W)\})$$
$$\geq \omega(\{(s,Y),(t,W)\}) + \omega(\{(s,Z),(t,X)\}).$$

By the choice of C, we have equality, and hence the paths Q and R attain the corresponding maxima in (69.14)(ii). It implies, by assumption (69.15)(ii), that u_{i-1}, v_{i-1}, u_i, and v_i are the only neighbours of s_i. Since none of u_{i-1}, v_{i-1} are adjacent to any of u_i, v_i, we have that s_i is splittable, that is, $s_i \in S'$, a contradiction. This proves (69.18).

Next

(69.21) $$u_0 v_0 \notin E \text{ and } u_k v_k \notin E.$$

For suppose that (say) $u_0 v_0 \in E$. By (69.15)(i), there exist $x, y \in V \setminus S$ such that (x, s, u_0, s_1, y) is a chordless path and such that $N(x) \cap N(y) \cap S = \emptyset$. As x is nonadjacent to u_0, and as $u_0 \in X$, we have $x \in Y$, and so (as $v_0 \in Y$) $x v_0 \in E$.

If $k = 0$, we have similarly $y v_0 \in E$. Then v_0 is adjacent to the pairwise nonadjacent x, u_0, y, a contradiction.

So $k \geq 1$. Then $y \sim u_1$ and $N(y) \cap S = \{s_1, s_2\}$. So $x s_1, x s_2 \notin E$ (since $N(x) \cap N(y) \cap S = \emptyset$). This implies $x u_1 \notin E$, since otherwise u_1 is adjacent to the pairwise nonadjacent s_1, s_2, x. Hence $v_0 u_1 \notin E$, since otherwise v_0 is adjacent to the pairwise nonadjacent x, u_0, u_1. By symmetry, also $u_0 v_1 \notin E$. This contradicts (69.18), and hence proves (69.21).

Moreover,

(69.22) there is an i with $0 \leq i \leq k$ and $u_i v_i \in E$,

as otherwise each circuit $(s_i, u_i, s_{i+1}, v_i, s_i)$ is S-alternating and chordless, which implies $w(u_i) + w(v_i) - w(s_i) - w(s_{i+1}) \leq 0$ by Lemma 69.2α. This gives the contradiction

(69.23) $$0 < \omega(EC \setminus M) - \omega(EC \cap M)$$
$$= w(VP_e \setminus S'') - w(VP_e \cap S'') + w(VP_f \setminus S'') - w(VP_f \cap S'')$$
$$- w(s) - w(t) = \sum_{i=0}^{k} (w(u_i) + w(v_i) - w(s_i) - w(s_{i+1})) \leq 0,$$

proving (69.22).

Now let i be the smallest index with $u_i v_i \in E$. By (69.21), we know $1 \leq i \leq k-1$. By (69.18) and by symmetry we can assume that $v_i u_{i+1} \in E$. Since s_i is adjacent to u_{i-1}, v_{i-1}, and v_i, and since $u_{i-1} v_{i-1} \notin E$ and $v_{i-1} v_i \notin E$, we know $u_{i-1} v_i \in E$. Then v_i is adjacent to the pairwise nonadjacent u_{i-1}, u_i, and u_{i+1}, a contradiction. ∎

Now find a maximum-weight perfect matching N in H, with the maximum-weight perfect matching algorithm (Chapter 26). By Lemma 69.2β, we can assume that $N = M \triangle EQ$ for some M-augmenting path Q in H (since if $N \triangle M$ contains a circuit C, then $N \triangle EC$ again is a maximum-weight perfect matching in H). Then Q maximizes $\omega(EQ \setminus M) - \omega(EQ \cap M)$ over all M-augmenting paths. Let P be the corresponding path in G. Then P is an S-augmenting path in G maximizing $w(VP \setminus S) - w(VP \cap S)$, as required.

We conclude:

Theorem 69.2. *A maximum-weight stable set in a claw-free graph can be found in strongly polynomial time.*

Proof. See above. ∎

69.4. Further results and notes

69.4a. On the stable set polytope of a claw-free graph

The polynomial-time solvability of the maximum-weight stable set problem for claw-free graphs implies that the optimization problem over the stable set polytope $P_{\text{stable set}}(G)$ of a claw-free graph $G = (V, E)$ is polynomial-time solvable. Hence also the separation problem is polynomial-time solvable (with the ellipsoid method (Theorem 5.10)). It implies (cf. Theorem 5.11) that, given a vector $x \in \mathbb{Q}^V$, one can decide in strongly polynomial time if x belongs to $P_{\text{stable set}}(G)$, and if not, find a facet-inducing inequality violated by x.

So in this respect, the stable set polytope of a claw-free graph is under control. However, no explicit description is known of a system that determines $P_{\text{stable set}}(G)$. As we saw in Section 25.2, such a description is known for the special case where G is the line graph of some graph H — that is, for the matching polytope of H. In this special case, each facet can be described by an inequality with coefficients in $\{0, 1\}$.

The latter fact does not generalize to claw-free graphs. Giles and Trotter [1981] showed that for each $k \in \mathbb{Z}_+$ there exists a claw-free graph such that its stable set polytope has a facet that is described by a linear inequality with coefficients k and $k + 1$. (This refutes a conjecture of Sbihi [1978].)

Galluccio and Sassano [1997] characterized those facets of the stable set polytope of a claw-free graph that can be described by an inequality with all coefficients in $\{0, 1\}$ (the *rank facets*).

More on facets of the stable set polytope of special classes of claw-free graphs can be found in Ben Rebea [1981] and Oriolo [2002] (for graphs such that for each vertex v, the graph induced by $N(v)$ is the complement of a bipartite graph) and Pulleyblank and Shepherd [1993] (for claw-free graphs such that no vertex has three pairwise nonadjacent vertices at distance two).

69.4b. Further notes

Minty [1980] observed that finding a maximum-size stable set in a graph without induced $K_{1,4}$ is NP-complete. This follows from the fact that the 3-dimensional assignment problem can be reduced to it (its intersection graph has no induced $K_{1,4}$).

Poljak [1974] showed that finding a maximum-size stable set in a triangle-free graph is NP-complete. It implies that finding a maximum-size clique in a claw-free graph is NP-complete.

Shepherd [1995] characterized the stable set polytope of *near-bipartite graphs*, that is, graphs with $G - N(v)$ bipartite for each $v \in VG$. They include the complements of line graphs, and the complement of any near-bipartite graph is claw-free.

Ben Rebea [1981] showed that each connected claw-free graph G with $\alpha(G) \geq 3$ not containing an induced C_5, contains no odd antihole. This was extended by Fouquet [1993] who showed that each connected claw-free graph G with $\alpha(G) \geq 4$ contains no odd antihole with at least 7 vertices.

Lovász and Plummer [1986] gave a variant of Minty's reduction of the maximum-size stable set problem in claw-free graphs to the maximum-size matching problem.

Beineke [1970] (for simple graphs), N. Robertson (unpublished), Hemminger [1971] (abstract only), and Bermond and Meyer [1973] characterized line graphs by means of forbidden induced subgraphs (six graphs next to $K_{1,3}$).

The polynomial-time solvability of the weighted stable set problem for claw-free graphs was extended to claw-free bidirected graphs by Nakamura and Tamura [1998]. A linear-time algorithm for 'triangulated' bidirected graphs was given by Nakamura and Tamura [2000].

Part VII

Multiflows and Disjoint Paths

Part VII: Multiflows and Disjoint Paths

The problem of finding a maximum flow from one source s to one sink t in a directed graph is highly tractable. There is a very efficient algorithm, which outputs an integer maximum flow if all capacities are integer. Moreover, the maximum flow value is equal to the minimum capacity of a cut separating s and t. If all capacities are equal to 1, the problem reduces to finding arc-disjoint paths. Some direct transformations give similar results for vertex capacities, for vertex-disjoint paths, and for undirected graphs.

Often in practice however, one is not interested in connecting only one pair of source and sink by a flow or by paths, but several pairs of sources and sinks simultaneously. One may think of a large communication or transportation network, where several messages or goods must be transmitted all at the same time over the same network, between different pairs of terminals. Also railway circulation with different types of rolling stock gives a multicommodity flow problem. A recent application is the design of *very large-scale integrated* (VLSI) circuits, where several pairs of pins must be interconnected by wires on a chip, in such a way that the wires follow a given grid and that the wires connecting different pairs of pins do not intersect each other. This leads to the area of multicommodity flows (briefly: multiflows) and disjoint paths. Most polyhedral and polynomial-time methods for 1-commodity flows and paths do not extend to multicommodity flows and paths. Yet a number of cases can be solved efficiently, in particular when the terminals have a special structure or when the graph is planar or, more generally, can be embedded in a specific surface.

Chapters:

Chapter 70

Multiflows and disjoint paths

We discuss basic, general facts and terminology on multiflows and disjoint paths. In particular, we study general interrelations between fractional multiflows, integer multiflows, disjoint paths, the 'cut condition', and the 'Euler condition'.

70.1. Directed multiflow problems

Given two directed graphs, a *supply digraph* $D = (V, A)$ and a *demand digraph* $H = (T, R)$ with $T \subseteq V$, a *multiflow* is a function f on R where f_r is an $s-t$ flow in D for each $r = (s, t) \in R$.[1] In this context, each pair in R is called a *net*, and each vertex covered by R is called a *terminal*.

For $k := |R|$, we also speak of a *k-commodity flow*. Occasionally, we will list the nets as $(s_1, t_1), \ldots, (s_k, t_k)$. Then for $r = (s_i, t_i)$ we denote f_r also by f_i. The indices $1, \ldots, k$ are called the *commodities*.

The *value* of f is the function $\phi : R \to \mathbb{R}_+$ where ϕ_r is the value of f_r. The *total value*, or (if no confusion may arise) just the *value*, is $\sum_{r \in R} \phi_r$.

Given a 'capacity' function $c : A \to \mathbb{R}_+$, we say that a multiflow f is *subject to* c if

(70.1) $$\sum_{r \in R} f_r(a) \leq c(a)$$

for each arc a.

The *multiflow problem* or *k-commodity flow problem* (for $k := |R|$) is:

(70.2) given: a supply digraph $D = (V, A)$, a demand digraph $H = (T, R)$ with $T \subseteq V$, a capacity function $c : A \to \mathbb{R}_+$, and a demand function $d : R \to \mathbb{R}_+$,
find: a multiflow subject to c of value d.

Given c and d, a multiflow subject to c of value d is called a *feasible multiflow*, or just a *multiflow* if no confusion is expected to arise. We call the problem *feasible* if there exists a feasible multiflow.

[1] Throughout, we use the terms 'multicommodity flow' and 'multiflow' as synonyms.

If we require each f_r to be an integer flow, the problem is called the *integer multiflow problem* or the *integer k-commodity flow problem*. Similarly for *half-integer*, *quarter-integer*, etc. For clarity, we sometimes add the adjective *fractional* if no integrality is required.

Related is the *maximum-value multiflow problem* or *maximum-value k-commodity flow problem*:

(70.3) given: a supply digraph $D = (V, A)$, a demand digraph $H = (T, R)$ with $T \subseteq V$, and a capacity function $c : A \to \mathbb{R}_+$,
find: a multiflow subject to c, of maximum total value.

Again we add *integer* (*half-integer*, etc) if we require the f_r to be integer (half-integer, etc.).

We can reduce a multiflow problem with demands $d_1, \ldots, d_k$ to a maximum-value multiflow problem, by extending the graph by an arc from a new vertex s_i' to s_i of capacity d_i (for $i = 1, \ldots, k$). Then the multiflow problem in the original graph is feasible if and only if the maximization problem in the new graph, with nets (s_i', t_i), has maximum total value equal to $d_1 + \cdots + d_k$.

70.2. Undirected multiflow problems

The problems described above have a natural analogue for undirected graphs. Let be given two undirected graphs, a *supply graph* $G = (V, E)$ and a *demand graph* $H = (T, R)$ with $T \subseteq V$. Again, each pair in R is called a *net*, and each vertex covered by R is called a *terminal*.

For $s, t \in V$, a function $f : E \to \mathbb{R}_+$ is called an *$s - t$ flow* if there exists an orientation (V, A) of G such that f is an $s - t$ flow in D.

A *multiflow* is a function f on R such that f_r is an $s - t$ flow for each $r = st \in R$. For $k := |R|$, the multiflow is also called a *k-commodity flow*. Again, occasionally we will list the nets as $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$.

The *value* of a multiflow f is the function $\phi : R \to \mathbb{R}_+$ where ϕ_r is the value of f_r. The *total value*, or just the *value*, is $\sum_{r \in R} \phi_r$.

Given a capacity function $c : E \to \mathbb{R}_+$, we say that a multiflow f is *subject to c* if

(70.4) $$\sum_{r \in R} f_r(e) \leq c(e)$$

for each edge e. Note that generally for each $r = st \in R$, there is a different orientation D_r of G that makes f_r into an $s - t$ flow in D_r. So in (70.4), the sum of the flows through both orientations of a given edge e are bounded above by $c(e)$.

In this way we obtain the *undirected multiflow problem* or *undirected k-commodity flow problem*, and the *undirected maximum-value multiflow problem* or *undirected maximum-value k-commodity flow problem*. Again, we add *integer* (*half-integer*, etc.) if we require the f_r to be integer (half-integer, etc.) flows. We skip the adjective 'undirected' if it is clear from the context.

70.3. Disjoint paths problems

If all capacities and demands are equal to 1, the integer multiflow problem is equivalent to the *(k) arc-* or *edge-disjoint paths problem*:

(70.5) given: a directed (or undirected) graph $D = (V, A)$ and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G,
find: arc- (or edge-)disjoint paths $P_1, \ldots, P_k$ where P_i is an $s_i - t_i$ path $(i = 1, \ldots, k)$.

For undirected graphs, the pairs s_i, t_i need not be ordered.

A *fractional solution* (*half-integer solution* respectively) of the arc- or edge-disjoint paths problem is a fractional (half-integer respectively) multiflow for all capacities and demands 1.

Related is the *(vertex-)disjoint paths problem* (or k *(vertex-)disjoint paths problem*):

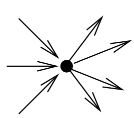
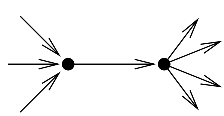
(70.6) given: a (directed or undirected) graph $D = (V, A)$ and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G,
find: vertex-disjoint paths $P_1, \ldots, P_k$ where P_i is an $s_i - t_i$ path $(i = 1, \ldots, k)$.

70.4. Reductions

Above we mentioned two versions of the multiflow problem: directed and undirected, and four versions of the disjoint paths problem: directed vertex-disjoint, directed arc-disjoint, undirected vertex-disjoint, and undirected edge-disjoint. There are a number of constructions that reduce versions among them.

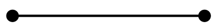
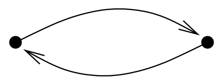
First, the *undirected edge-disjoint* paths problem can be reduced to the *undirected vertex-disjoint* paths problem by replacing the graph by its line graph. Similarly, the *directed arc-disjoint* paths problem can be reduced to the *directed vertex-disjoint* paths problem by replacing the digraph by its line digraph.

Conversely, the *directed vertex-disjoint* paths problem can be reduced to the *directed arc-disjoint* paths problem by replacing each vertex

(70.7)  by  .

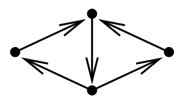
So far, these reductions do not maintain planarity.

The *undirected vertex-disjoint* paths problem can be reduced to the *directed vertex-disjoint* paths problem by replacing each edge

(70.8)  by  .

Trivially, this construction maintains planarity.

Finally, there is the following reduction of the *undirected edge-disjoint* paths problem to the *directed arc-disjoint* paths problem: replace each edge

(70.9) by  .

This reduction also applies to (integer, half-integer, fractional) multiflow problems. Again, this construction maintains planarity.

We represent these reductions in the following diagram, where a double arrow means a reduction maintaining planarity:

(70.10)
$$\begin{array}{ccc} \boxed{\text{undirected edge-disjoint}} & \longrightarrow & \boxed{\text{undirected vertex-disjoint}} \\ \Downarrow & & \Downarrow \\ \boxed{\text{directed arc-disjoint}} & \longleftrightarrow & \boxed{\text{directed vertex-disjoint}} \end{array}$$

Notes. These reductions maintain the set of nets and the demand values. Even, Itai, and Shamir [1975,1976] gave an interesting construction reducing the *directed arc-disjoint* paths problem to the *undirected edge-disjoint* paths problem. It reduces the directed arc-disjoint paths problem with k commodities of demands $d_1, \ldots, d_k$ in a digraph $D = (V, A)$, to the undirected edge-disjoint paths problem with k commodities of demands $d_1 + |A|, \ldots, d_k + |A|$. The construction does not maintain planarity.

70.5. Complexity of the disjoint paths problem

In Section 70.6 we shall see that the fractional multiflow problem is solvable in strongly polynomial time, since it is a linear programming problem.

The integer multiflow problem is NP-complete, even the disjoint paths problem is NP-complete, in any mode (directed/undirected, vertex/edge-disjoint), even for planar graphs. In some cases, however, the problem is polynomial-time solvable if we fix the number k of commodities. We survey the complexity results in the following table:

	directed		*undirected*	
	arc-disjoint	*vertex-disjoint*	*edge-disjoint*	*vertex-disjoint*
general	NP-complete[2]	NP-complete[2]	NP-complete[3]	NP-complete[2]
planar	NP-complete[4]	NP-complete[5]	NP-complete[4]	NP-complete[5]
		for fixed k:		
general	NP-complete[6]	NP-complete[6]	polynomial-time[7]	polynomial-time[7]
planar	?[8]	polynomial-time[9]	polynomial-time[7]	polynomial-time[7]

Complexity of the k disjoint paths problem

By the reduction described at the end of Section 70.1, the NP-completeness of the integer multiflow and disjoint paths problems implies that also the corresponding maximization problems are NP-complete.

70.6. Complexity of the fractional multiflow problem

The fractional multiflow problem can easily be described as one of solving a system of linear inequalities in the variables $f_i(a)$ for $i = 1, \ldots, k$ and $a \in A$. The constraints are the flow conservation laws and the demand constraint for each flow f_i separately, together with the capacity constraints (70.1). Therefore, the fractional multiflow problem can be solved in polynomial time with any polynomial-time linear programming algorithm. Tardos [1986] showed that the fractional multiflow problem is solvable in strongly polynomial time,

[2] D.E. Knuth, 1974 (cf. Karp [1975]), who proved the NP-completeness of the undirected vertex-disjoint version. It implies the NP-completeness of the directed vertex-disjoint case (by reduction (70.8)), which in turn implies the NP-completeness of the directed arc-disjoint version (by reduction (70.7)). Even, Itai, and Shamir [1975,1976] showed that the directed arc-disjoint paths problem is NP-complete even if the digraph is acyclic and $s_2 = \cdots = s_k$ and $t_2 = \cdots = t_k$.

[3] Even, Itai, and Shamir [1975,1976] — NP-complete even if $|\{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}| = 2$; equivalently, the integer 2-commodity flow problem is NP-complete even if all capacities are 1.

[4] Kramer and van Leeuwen [1984], who proved the NP-completeness of the planar undirected edge-disjoint paths problem, implying the NP-completeness of the planar directed arc-disjoint paths problem, by reduction (70.9). Kramer and van Leeuwen showed NP-completeness even if the graphs are restricted to rectangular grids.

[5] Lynch [1975], who proved the NP-completeness of the planar undirected vertex-disjoint paths problem. It implies the NP-completeness of the planar directed vertex-disjoint paths problem, by reduction (70.8). The problems remain NP-complete for cubic planar graphs (Richards [1984]), and also if the graph together with the nets is planar and cubic (Middendorf and Pfeiffer [1993]).

[6] Fortune, Hopcroft, and Wyllie [1980] — NP-complete even for $k = 2$ opposite nets (s, t) and (t, s).

[7] Robertson and Seymour [1995], who proved the polynomial-time solvability of the k vertex-disjoint paths problem in undirected graphs, for any fixed k. By replacing a graph by its line graph, it implies the polynomial-time solvability of the k edge-disjoint paths problem in undirected graphs, for any fixed k.

[8] unknown also if $k = 2$ and the two nets are opposite.

[9] Schrijver [1994a].

by proving that any linear programming problem with $\{0, \pm 1\}$ constraint matrix is solvable in strongly polynomial time.

Onaga [1970] gave the following good characterization for the feasibility of the fractional multiflow problem, which can be derived (as Iri [1971] observed) from Farkas' lemma ($\mathrm{dist}_l(s,t)$ denotes the length of a shortest $s-t$ path with respect to a length function l):

Theorem 70.1. *The (directed or undirected) fractional multiflow problem* (70.2) *has a solution if and only if*

$$(70.11) \qquad \sum_{i=1}^{k} d_i \cdot \mathrm{dist}_l(s_i, t_i) \leq \sum_{a \in A} l(a)c(a)$$

for each length function $l : A \to \mathbb{Z}_+$.

Proof. For $i = 1, \ldots, k$, let $\mathcal{P}_i$ denote the collection of arc sets of $s_i - t_i$ paths. Then there is a feasible multiflow if and only if there exist $\lambda_{i,P} \geq 0$ (for $i = 1, \ldots, k$ and $P \in \mathcal{P}_i$), such that

$$(70.12) \qquad \begin{aligned} &\sum_{P \in \mathcal{P}_i} \lambda_{i,P} = d_i && \text{for } i = 1, \ldots, k, \\ &\sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} \lambda_{i,P} \chi^P(a) \leq c(a) && \text{for } a \in A. \end{aligned}$$

By Farkas' lemma, this is equivalent to: for all $b_1, \ldots, b_k \in \mathbb{R}$ and $l : A \to \mathbb{R}_+$, if

$$(70.13) \qquad b_i \leq \sum_{a \in P} l(a) \text{ for } i = 1, \ldots, k \text{ and } P \in \mathcal{P}_i,$$

then

$$(70.14) \qquad \sum_{i=1}^{k} b_i d_i \leq \sum_{a \in A} l(a)c(a).$$

Now we may assume that each b_i is chosen maximal such that it satisfies (70.13). Then b_i is equal to the minimum of $\sum_{a \in P} l(a)$ taken over all $P \in \mathcal{P}_i$, that is, to $\mathrm{dist}_l(s_i, t_i)$. Hence the condition is equivalent to (70.11). ∎

(Onaga and Kakusho [1971] gave an alternative proof. If we restrict l to $\{0,1\}$-valued functions, we obtain a necessary condition (a 'multicut condition'), as was observed by Naniwada [1969], who raised the question if the above theorem may hold.)

A min-max relation for the maximum-value multiflow problem can be derived similarly from LP-duality (cf. Lomonosov [1978a]):

Theorem 70.2. *Let $D = (V, A)$ be a directed or undirected graph, let $(s_1, t_1), \ldots, (s_k, t_k)$ be nets, and let $c : A \to \mathbb{R}_+$ be a capacity function. Then* *the maximum total value of a multiflow subject to c is equal to the minimum value of $\sum_{a \in A} l(a)c(a)$ taken over all $l : A \to \mathbb{R}_+$ satisfying*

(70.15) $\quad \operatorname{dist}_l(s_i, t_i) \geq 1$ *for each* $i = 1, \ldots, k$.

Proof. Let $\mathcal{P}$ denote the collection of arc sets of paths running from s_i to t_i for some $i = 1, \ldots, k$. Then the maximum total value of a multiflow is equal to the maximum of $\sum_{P \in \mathcal{P}} \lambda_P$, where $\lambda_P \geq 0$ for $P \in \mathcal{P}$, such that

(70.16) $$\sum_{P \in \mathcal{P}} \lambda_P \chi^P(a) \leq c(a) \text{ for } a \in A.$$

By LP-duality, this value is equal to the minimum value of $\sum_{a \in A} l(a)c(a)$ where $l : A \to \mathbb{R}_+$ such that

(70.17) $$\sum_{a \in P} l(a) \geq 1 \text{ for each } P \in \mathcal{P}.$$

As (70.17) is equivalent to (70.15), we have the theorem. ∎

70.7. The cut condition for directed graphs

In Theorem 70.1 we saw a good characterization for the feasibility of the fractional multiflow problem. In some cases, it can be replaced by a weaker condition, the *cut condition*:

(70.18) $$c(\delta_A^{\text{out}}(U)) \geq d(\delta_R^{\text{out}}(U)) \text{ for each } U \subseteq V.$$

The cut condition indeed is a direct consequence of condition (70.14) described in Theorem 70.1. For define $l(a) := 1$ if $a \in \delta^{\text{out}}(U)$, and $l(a) := 0$ otherwise. Then (70.11) implies:

(70.19) $$c(\delta_A^{\text{out}}(U)) = \sum_{a \in A} l(a)c(a) \geq \sum_{i=1}^{k} d_i \cdot \operatorname{dist}_l(s_i, t_i) \geq d(\delta_R^{\text{out}}(U)).$$

However, the cut condition is in general not sufficient, even not in the two simple cases given in Figure 70.1.

For directed graphs, the cut condition is known to be sufficient for the existence of a fractional multiflow only if $s_1 = \cdots = s_k$ or $t_1 = \cdots = t_k$ (this follows from the (one-commodity) max-flow min-cut theorem). In a sense, this is the only case:

Theorem 70.3. *Let $H = (T, R)$ be a demand digraph, where R contains no loops. Then for each supply digraph $D = (V, A)$ with $V \supseteq T$, the cut condition* (70.18) *is sufficient for the existence of a fractional multiflow if and only if all arcs of H have a common head, or they all have a common tail.*

Proof. Let $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$.

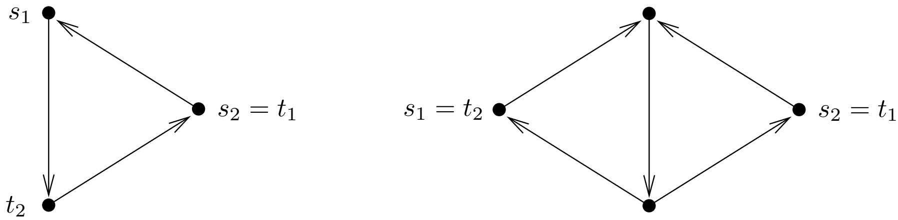


Figure 70.1
Two digraphs where the cut condition holds, but no fractional multiflow exists (taking all capacities and demands equal to 1). The nonexistence of a fractional multiflow can be shown with Theorem 70.1, by taking $l(a) := 1$ for each arc a.

To see sufficiency, by symmetry we can assume that $s_1 = \cdots = s_k$. Let $s := s_1$, and let t be a new vertex. For $i = 1, \ldots, k$, add a new arc (t_i, t), with capacity d_i. Then, by the max-flow min-cut theorem, the cut condition implies that the extended graph has an $s-t$ flow of value $d_1+\cdots+d_k$, subject to the capacity. Restricted to the original graph, we can decompose the flow into a feasible k-commodity flow of values $d_1, \ldots, d_k$.

To see necessity, if the condition is not met, then there exist nets (s_i, t_i) and (s_j, t_j) with $s_i \neq s_j$ and $t_i \neq t_j$. We can assume that $i = 1$, $j = 2$. Then $\{s_1, t_2\}$ is disjoint from $\{s_2, t_1\}$, and then the second example in Figure 70.1 can be adapted to obtain an example with net set R, and where the cut condition holds but no fractional multiflow exists. ∎

As for maximizing the total value of a multiflow, in a directed triangle, with as nets the opposites of all arcs and all capacities equal to 1, the maximum total value is $\frac{3}{2}$, while the minimum capacity of an arc set disconnecting all nets is 2.

70.8. The cut condition for undirected graphs

Similarly, one can formulate the cut condition in the undirected case:

(70.20) $\quad c(\delta_E(U)) \geq d(\delta_R(U))$ for each $U \subseteq V$.

In the special case of the edge-disjoint paths problem (where all capacities and demands are equal to 1), the cut condition amounts to:

(70.21) $\quad d_E(U) \geq d_R(U)$ for each $U \subseteq V$.

As was observed by Tang [1965], in the undirected case the cut condition is equivalent to the 'disconnecting set condition':

(70.22) $\quad c(F) \geq d(\mathrm{disc}_R(F))$ for each $F \subseteq E$, where $\mathrm{disc}_R(F)$ denotes the family of nets st where s and t are in different components of $G - F$.

Indeed, trivially, the cut condition is implied by (70.22). To see the reverse implication, let $\mathcal{K}$ be the set of components of $G-F$. Then the cut condition implies

$$(70.23)\qquad c(F) \geq \tfrac{1}{2}\sum_{K\in\mathcal{K}} c(\delta_E(K)) \geq \tfrac{1}{2}\sum_{K\in\mathcal{K}} d(\delta_R(K)) = d(\mathrm{disc}_R(F)),$$

which is (70.22).

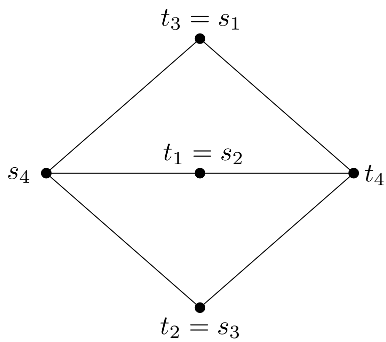


Figure 70.2
An undirected graph where the cut condition holds, but no fractional multiflow exists (taking all capacities and demands equal to 1). This last can be shown with Theorem 70.1, by taking $l(e) := 1$ for each edge e.

Figure 70.2 shows that, also in the undirected case, the cut condition is not sufficient[10]. Hu [1963] showed that, in the undirected case, if $k = 2$, then the cut condition is sufficient for the existence of a fractional multiflow. This is *Hu's 2-commodity flow theorem* (Theorem 71.1b). In Section 70.11, we will list more cases where the cut condition is sufficient for the existence of a fractional multiflow.

Hu's 2-commodity flow theorem implies the *max-biflow min-cut theorem* (Corollary 71.1d): in the undirected case with $k = 2$, the maximum value of a 2-commodity flow is equal to the minimum capacity of a cut separating both s_1 and t_1 and s_2 and t_2.

[10] Hakimi [1962b] and Tang [1962] claimed erroneously to give proofs that the cut condition is sufficient for any number k of commodities. According to Hu [1963], a counterexample was first found by L.R. Ford, Jr.

A strengthening of the cut condition that Hu [1964] claimed to be necessary (and conjectured to be sufficient) for the existence of a fractional multiflow, was shown to be not necessary by Tang [1965].

A similar ‘maximum-triflow min-cut theorem’ does not hold, even not if the three nets form a triangle: take $K_{1,3}$ and all pairs of end vertices as nets, all capacities being 1; then the minimum number of edges disconnecting each commodity is equal to 2, while the maximum total value of a fractional multiflow is equal to $\frac{3}{2}$ (example of Ford and Fulkerson [1954,1956b]).

Anyway, if the nets form a triangle, finding a minimum-size set of edges disconnecting each net, is NP-complete (Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [1992,1994]).

It will be useful to note that the cut condition only needs to be required for cuts with both sides connected (if G is connected):

Theorem 70.4. *Let $G = (V, E)$ an $H = (T, R)$ be a supply and demand graph, with G connected. Let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. If the cut condition* (70.20) *is violated, then it is violated by some $U \subseteq V$ for which both $G[U]$ and $G[V \setminus U]$ are connected.*

Proof. Let U violate the cut condition; that is, $c(\delta_E(U)) < d(\delta_R(U))$. Choose U such that $|\delta_E(U)|$ is as small as possible. We show that $G[U]$ and $G[V \setminus U]$ are connected. By symmetry, it suffices to show that $G[U]$ is connected. Let $K_1, \ldots, K_t$ be the components of $G[U]$. Suppose $t \geq 2$. Then

(70.24) $$\sum_{j=1}^{t} c(\delta_E(K_j)) = c(\delta_E(U)) < d(\delta_R(U)) \leq \sum_{j=1}^{t} d(\delta_R(K_j)).$$

So $c(\delta_E(K_j)) < d(\delta_R(K_j))$ for at least one j. As $|\delta_E(K_j)| < |\delta_E(U)|$ (by the connectivity of G), this contradicts the minimality of $|\delta_E(U)|$. ∎

Notes. Călinescu, Fernandes, and Reed [1998] gave a polynomial-time approximation algorithm for finding a minimum multicut in an unweighted graph of bounded degree and bounded ‘tree-width’. More on the minimum multicut problem can be found in Klein, Agrawal, Ravi, and Rao [1990], Garg, Vazirani, and Yannakakis [1993a,1993b,1996,1997], Tardos and Vazirani [1993], Klein, Rao, Agrawal, and Ravi [1995], and Naor and Zosin [1997,2001].

70.9. Relations between fractional, half-integer, and integer solutions

There are the following implications for the multiflow problem:

(70.25) $\exists$ integer multiflow $\Longrightarrow$ $\exists$ half-integer multiflow $\Longrightarrow$ $\exists$ fractional multiflow.

As the existence of a fractional multiflow can be tested in strongly polynomial time, it yields a useful necessary condition for the existence of an integer multiflow.

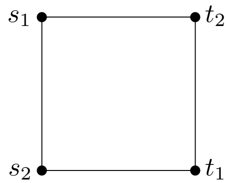


Figure 70.3
There is a half-integer, but no integer multiflow (where all capacities and demands are 1).

As has been discussed in Chapter 10, for 1-commodity flow problems with integer capacities, we can turn all implications around in (70.25). For general multiflow problems, however, this is not the case. For undirected graphs, Figure 70.3 shows that a half-integer multiflow does not imply the existence of an integer multiflow (for integer capacities and demands). Middendorf and Pfeiffer [1993] showed that the half-integer multiflow problem in undirected graphs is NP-complete, even if all capacities and demands are equal to 1.

For undirected 2-commodity flows, Hu [1963] showed that the existence of a fractional multiflow implies the existence of a half-integer multiflow, if all capacities and demands are integer. Figure 70.3 shows that an integer multiflow need not exist. In fact, the undirected integer 2-commodity flow problem is NP-complete (Even, Itai, and Shamir [1975,1976]).

Hu's theorem prompted Jewell [1967] to conjecture that if a k-commodity flow problem with integer capacities and demands has a fractional solution, then it has a $1/p$-integer solution for some $p \leq k$. More strongly, Seymour [1981d] conjectured that a fractional multiflow implies the existence of a half-integer multiflow (for integer capacities and demands).

This was disproved by a series of examples of Lomonosov [1985], which even imply that there is no integer p such that each undirected 3-commodity flow problem has a $1/p$-integer solution when it has a fractional solution (for integer capacities and demands). A simplified version of Lomonosov's example is given in Figure 70.4. It consists of an integer-capacitated 3-commodity flow problem with demands 1, $2k$, and $2k$, such that each feasible multiflow has $\frac{1}{2k}$ among its values.

A simpler counterexample to Seymour's conjecture was given by Pfeiffer [1990] — see Figure 70.5, showing that a quarter-integer multiflow need not imply the existence of a half-integer multiflow (for integer capacities and demands).

With construction (70.9) we obtain similar results for directed graphs.

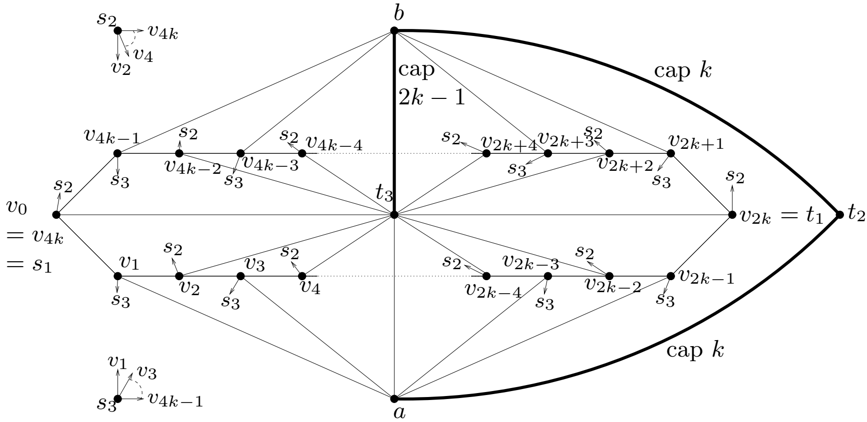


Figure 70.4

A feasible integer-capacitated 3-commodity flow problem with demands 1, $2k$, and $2k$, such that each feasible multiflow has $\frac{1}{2k}$ among its values. The nets are the pairs s_1t_1, s_2t_2, and s_3t_3. The graph consists of a circuit C of length $4k$, with vertices $v_1, \ldots, v_{4k}$ (in order), vertices s_2 and t_3 adjacent to each v_i with i even, a vertex s_3 adjacent to each v_i with i odd, a vertex a adjacent to t_3 and to each v_i with i odd and $0 < i < 2k$, a vertex b adjacent to t_3 and to each v_i with i odd and $2k < i < 4k$, and a vertex t_2 adjacent to a and b. Set $s_1 := v_0 := v_{4k}$ and $t_1 := v_{2k}$. Let P and Q be the paths $v_0, v_1, \ldots, v_{2k}$ and $v_{4k}, v_{4k-1}, \ldots, v_{2k}$, respectively.

Edges t_2a and t_2b have capacity k, and edge bt_3 capacity $2k-1$. All other edges have capacity 1. Let $d(s_1t_1) := 1$ and $d(s_2t_2) := d(s_3t_3) := 2k$.

To see that there exists a feasible multiflow, reset (temporarily) the capacities of at_3 and bt_3 to 0 and $2k$ respectively. Then a feasible multiflow (f_1, f_2, f_3) is given as follows. Flow f_1 consists of the incidence vector of path Q. Flow f_2 takes value 1 on the edges t_3v_i for $i = 2k+2, 2k+4, \ldots, 4k-2$, on av_i for $i = 1, 3, \ldots, 2k-1$, and on s_2v_i for $i = 2, 4, \ldots, 4k$, value $\frac{1}{2}$ on the edges of P, and on t_3v_0 and t_3v_{2k}, value k on t_3b, t_2a, and t_2b, and value 0 on all other edges. Flow f_3 takes value 1 on t_3v_i for $i = 2, 4, \ldots, 2k-2$, on s_3v_i for $i = 1, 3, \ldots, 4k-1$, and on bv_i for $i = 2k+1, 2k+3, \ldots, 4k-1$, value $\frac{1}{2}$ on the edges of P, and on t_3v_0 and t_3v_{2k}, and value k on bt_3. By symmetry, also after resetting the capacities of at_3 and bt_3 to $2k$ and 0 respectively, there exists a feasible solution. Hence also the original capacity function (which is a convex combination of the modified capacity functions) has a feasible solution.

To see that any feasible multiflow contains a value $\frac{1}{2k}$, note that the $s_1 - t_1$ flow can only use edges on the circuit C: each edge leaving C is in a tight cut (= a cut having equality in the cut condition) not separating s_1 and t_1 (consider the cuts determined by $\{s_2\}$, $\{s_3\}$, and $\{t_2, t_3, a, b\}$). So in any feasible multiflow, the $s_1 - t_1$ flow f_1 is a convex combination of the incidence vectors of P and Q. Consider now the cut determined by $U := \{t_2, s_3, a, v_1, v_3, \ldots, v_{2k-1}\}$. It has capacity $4k+1$ and demand $4k$, it does not split s_1 and t_1, and contains all edges of P. Hence the capacity left for f_1 is at most 1. As P has length $2k$, it implies that f_1 can send a flow of value at most $\frac{1}{2k}$ along P. Similarly, the cut determined by

$U := \{t_2, s_3, b, v_{2k+1}, v_{2k+3}, \ldots, v_{4k-1}\}$ has capacity $6k-1$ and demand $4k$, it does not split s_1 and t_1, and contains all edges of Q. Hence the capacity left for f_1 is at most $2k-1$. As Q has length $2k$, it implies that f_1 can send a flow of value at most $1-\frac{1}{2k}$ along Q. Concluding, f_1 sends $\frac{1}{2k}$ flow along P and $1-\frac{1}{2k}$ flow along Q.

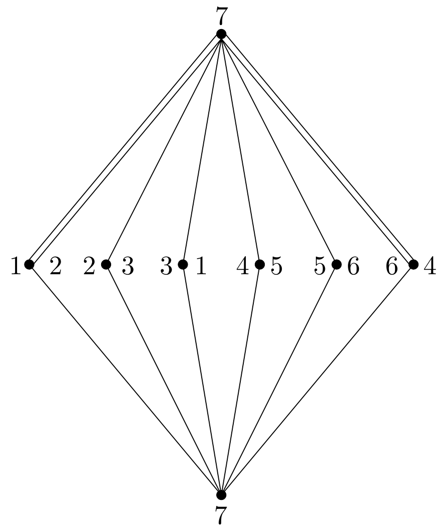


Figure 70.5

There is a quarter-integer, but no half-integer multiflow. The 7 nets are indicated by indices $1, \ldots, 7$ at the terminals. All capacities and demands are equal to 1.

In fact, there is a unique fractional multiflow. Since the distance between the terminals in any net is 2 and since there are 14 edges, any multiflow uses the capacity of each edge fully, and each of the flows is a convex combination of incidence vectors of paths of length 2. Also, the edges incident with any vertex v of degree 2 can only be used by the nets that have a terminal at v.

Let β be the fraction of flow for net 7 that traverses the leftmost vertex. For $i = 1, \ldots, 6$, let α_i be the fraction of flow for net i that traverses the topmost vertex. Then $\alpha_2 + \alpha_3 = 1$ and $\alpha_1 + \alpha_3 = 1$, and hence $\alpha_1 = \alpha_2$. So $\beta + \alpha_1 + \alpha_2 = 2$ and $\beta + (1-\alpha_1) + (1-\alpha_2) = 1$, hence $\alpha_1 + \alpha_2 = 1 + \beta = 2 - \beta$. So $\beta = \frac{1}{2}$ and $\alpha_1 = \alpha_2 = \frac{3}{4}$.

70.10. The Euler condition

In some cases adding the following *Euler condition* turns out to be of help:

(70.26) $\qquad c(\delta_E(v)) + d(\delta_R(v))$ is even, for each vertex v.

In case all capacities and demands are equal to 1, that is, for the edge-disjoint paths problem, the Euler condition is equivalent to

(70.27) the graph $G + H = (V, E \cup R)$ is Eulerian

(taking multiplicities into account).

If $k = 2$ and the capacities and demands are integer and satisfy the Euler condition, then the cut condition implies the existence of an integer multiflow. This result, also due to Rothschild and Whinston [1966a], implies Hu's 2-commodity flow theorem, as mentioned in Section 70.8 (by multiplying all capacities and demands by 2, so as to achieve the Euler condition).

We will see several other cases where the existence of a half-integer multiflow, together with the Euler condition, implies the existence of an integer multiflow. But it is not sufficient in general, as otherwise a quarter-integer multiflow would always imply the existence of a half-integer multiflow (by multiplying all capacities and demands by 2), and to this we saw the counterexample of Pfeiffer [1990] in Figure 70.5[11]. The NP-completeness of the half-integer multiflow problem, with all capacities and demands equal to 1 (Middendorf and Pfeiffer [1993]), implies that the edge-disjoint paths problem is NP-complete even if the Euler condition holds.

Fractional and integer multiflows for digraphs. As for the *directed* case, Figure 70.5 implies with construction (70.9) that a quarter-integer multiflow does not imply the existence of a half-integer multiflow. The graph in Figure 70.6 (Hurkens, Schrijver, and Tardos [1988]) shows that a half-integer multiflow does not imply the existence of an integer multiflow, even if the directed analogue of the Euler condition holds (the graph obtained from the supply digraph and the reverse of the demand digraph is an Eulerian digraph). Note that in Figure 70.6 the union $D+H$ of D and H is planar.

70.11. Survey of cases where a good characterization has been found

Let $G = (V, E)$ be an undirected graph and let $R = \{s_1t_1, \ldots, s_kt_k\}$ be a family of nets. Let $c : E \to \mathbb{R}_+$ be a capacity function and let $d_1, \ldots, d_k$ be demands (so $d(s_it_i) := d_i$).

In the following cases of the undirected multiflow problem, the cut condition has been proved to imply the existence of a fractional multiflow; if moreover the capacities and demands are integer, there is a half-integer multiflow; if moreover the Euler condition holds, there is an integer solution[12]:

(70.28) (i) if there exist two vertices u, v such that each s_it_i intersects uv (Hu [1963], E.A. Dinits — see Corollary 71.1b),

[11] A more complicated (planar) example satisfying the Euler condition and where a half-integer but no integer multiflow exists, was given by Hurkens, Schrijver, and Tardos [1988]. Earlier, a nonplanar example with these properties was given by P.D. Seymour (personal communication).

[12] For graphs $G = (V, E)$ and $H = (T, R)$, $G + H$ is the graph $(V \cup T, E \cup R)$, where $E \cup R$ is the disjoint union (as families).

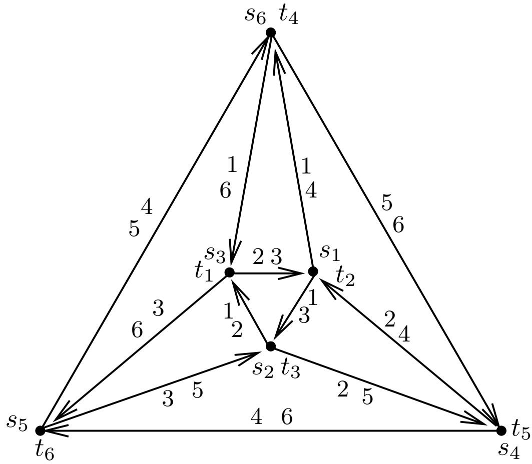


Figure 70.6

A directed example where the Euler condition holds, with $D + H$ is planar, and where a half-integer, but no integer multiflow exists. All capacities and demands are 1. The half-integer multiflow is indicated by the indices of the nets: index i at arc a means $f_i(a) = \frac{1}{2}$.

(ii) if $|\{s_1, t_1, \ldots, s_k, t_k\}| \leq 4$ or $s_1t_1, \ldots, s_kt_k$ form a five-circuit (Papernov [1976], Lomonosov [1976,1985], Seymour [1980c] — see Section 72.1),

(iii) if $G + H$ has no K_5 minor, in particular if $G + H$ is planar (Seymour [1981a] — see Sections 74.2 and 75.6),

(iv) if G is planar and there exist two faces F_1 and F_2 such that for each $i = 1, \ldots, k$: $s_i, t_i \in \text{bd}(F_1)$ or $s_i, t_i \in \text{bd}(F_2)$ (Okamura and Seymour [1981], Okamura [1983] — see Theorems 74.1 and 74.4),

(v) if G is planar and has a vertex r on the outer boundary such that for each i either both s_i and t_i are on the outer boundary, or $r \in \{s_i, t_i\}$ (Okamura [1983] — Theorem 74.5).

(vi) if G is planar and has two bounded faces F_1 and F_2 such that $s_1, \ldots, s_k$ occur clockwise around $\text{bd}(F_1)$ and $t_1, \ldots, t_k$ occur clockwise around $\text{bd}(F_2)$ (Schrijver [1989b] — cf. Section 74.3b).

Here $\text{bd}(F)$ denotes the boundary of F.

In particular, in each of these cases, if the capacities and demands are integer and satisfy the Euler condition, the existence of a fractional multiflow implies the existence of an integer multiflow. Next to the cases listed in (70.28), this property has been proved in the following cases (extending (70.28)(ii) and (iv)):

(70.29) (i) if $|\{s_1, t_1, \ldots, s_k, t_k\}| \leq 5$ (Karzanov [1987a] — see Section 72.2a),
(ii) if G is planar and there exist faces F_1, F_2, F_3 such that for each $i = 1, \ldots, k$ there is a $j \in \{1, 2, 3\}$ such that $s_i, t_i \in \mathrm{bd}(F_j)$ (Karzanov [1994b] — see Section 74.3c).

This in particular implies that if the capacities and demands are integer and there exists a fractional multiflow, then there exists a half-integer multiflow.

In the following case (extending (70.29)(ii)), it has been proved that if c and d are integer and a fractional multiflow exists, then a quarter-integer multiflow exists; if moreover the Euler condition holds, then a half-integer solution exists:

(70.30) if G is planar and there are four faces such that each net is spanned by one of these faces (Karzanov [1995] — see Section 74.3c).

(We say that a pair of vertices is *spanned* by a face F if it is spanned by the boundary of F.)

In Section 73.1c we shall see that an integer multiflow can be found in polynomial time also if the nets form a triangle (no Euler condition is required).

70.12. Relation between the cut condition and fractional cut packing

As was noted by Karzanov [1984] and Seymour [1979b], if the cut condition is sufficient for the existence of a fractional multiflow, one can derive an interesting polarity relation between multiflows and fractional packing of cuts.

Let $G = (V, E)$ and $H = (V, R)$ be graphs. Consider the cone K in $\mathbb{R}^R \times \mathbb{R}^E$ generated by the vectors[13]

(70.31) $(\chi^r; \chi^{EP})$ for $r \in R$ and r-path P in G,
$(\mathbf{0}; \chi^e)$ for $e \in E$.

Here EP denotes the set of edges of P. For any $r = st \in R$, an *r-path* is a path connecting s and t. χ^r and χ^e denote the rth and eth unit base vectors in $\mathbb{R}^R$ and $\mathbb{R}^E$, respectively.

For any $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the existence of a feasible multiflow subject to c and of value d is equivalent to the fact that $(d; c)$ belongs to K. So we have that the property:

[13] We write $(x; y)$ for $\begin{pmatrix} x \\ y \end{pmatrix}$.

(70.32) for each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a feasible multiflow,

is equivalent to the fact that K consists of all vectors $(d; c) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying:

(70.33)
$$\begin{array}{ll} d(\delta_R(U)) \leq c(\delta_E(U)) & \text{for } U \subseteq V, \\ d_r \geq 0 & \text{for } r \in R, \\ c_e \geq 0 & \text{for } e \in E. \end{array}$$

Let K^* be the polar cone of K (cf. Section 5.7). Then (70.32) is equivalent to $-K^*$ being generated by the vectors:

(70.34)
$$\begin{array}{ll} (-\chi^{\delta_R(U)}; \chi^{\delta_E(U)}) & \text{for } U \subseteq V, \\ (\chi^r; \mathbf{0}) & \text{for } r \in R, \\ (\mathbf{0}; \chi^e) & \text{for } e \in E. \end{array}$$

Also, by definition of K, $-K^*$ consists of all vectors $(m; l) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying:

(70.35)
$$\begin{array}{ll} m_r + l(EP) \geq 0 & \text{for } r \in R \text{ and } r\text{-path } P \text{ in } G, \\ l_e \geq 0 & \text{for } e \in E. \end{array}$$

This implies the following theorem relating the cut condition to distances and fractional packings of cuts:

Theorem 70.5. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs. Then for each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a feasible fractional multiflow if and only if for each length function $l : E \to \mathbb{R}_+$ there exist $\lambda_U \geq 0$ for $U \subseteq V$ such that*

(70.36)
$$\sum_U \lambda_U \chi^{\delta_E(U)} \leq l$$

and

(70.37)
$$\mathrm{dist}_l(s, t) = \sum_U \lambda_U \chi^{\delta_R(U)}(r),$$

for each $r = st \in R$. Here $\mathrm{dist}_l(s, t)$ denotes the minimum length of an $s - t$ path in G, with respect to l.

Proof. As we saw above, (70.32) is equivalent to the fact that $-K^*$ is generated by the vectors (70.34). It is equivalent to: each $(m; l) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying (70.35) is a nonnegative combination of vectors (70.34). Since $(\chi^r; \mathbf{0})$ is one of the vectors (70.34), we can restrict the $(m; l)$ to those for which m_r is smallest so as to satisfy (70.35). That is, we can assume that $m_r = -\mathrm{dist}_l(s, t)$ where $r = st$. Hence (70.32) is equivalent to: for each $l : E \to \mathbb{R}_+$, the vector $(-\mathrm{dist}_l; l)$ is a nonnegative combination of vectors (70.34). This is equivalent to the condition stated in the theorem. ∎

This is based on interpreting *feasibility* of multiflows in terms of *cones*. We next consider an interpretation of the *maximization* of multiflows in terms of *polyhedra*.

Let $\mathcal{P}$ be the collection of r-paths for all $r \in R$. Let $\mathcal{B}$ be the collection of subsets of E that intersect each path in $\mathcal{P}$.

Consider the inequality system:

$$(70.38) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(EP) \geq 1 & \text{for } P \in \mathcal{P}. \end{array}$$

Then by the theory of blocking polyhedra:

Theorem 70.6. *The up hull of the incidence vectors of the sets in $\mathcal{B}$ is determined by (70.38) if and only if the up hull of the incidence vectors of paths in $\mathcal{P}$ is determined by*

$$(70.39) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(B) \geq 1 & \text{for } B \in \mathcal{B}. \end{array}$$

Proof. Directly from the theory of blocking polyhedra. ∎

In terms of flows this is equivalent to:

Corollary 70.6a. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs. For each $c : E \to \mathbb{R}_+$, the maximum total value of a multiflow subject to c is equal to the minimum capacity of a set in $\mathcal{B}$ if and only if for each length function $l : E \to \mathbb{R}_+$ satisfying $\mathrm{dist}_l(s, t) \geq 1$ for each $r = st \in R$, there exist $\lambda_B \geq 0$ for $B \in \mathcal{B}$ such that*

$$(70.40) \qquad \sum_{B \in \mathcal{B}} \lambda_B = 1 \textit{ and } \sum_{B \in \mathcal{B}} \lambda_B \chi^B \leq l.$$

Proof. The first statement is equivalent to the fact that the up hull of the incidence vectors of sets in $\mathcal{B}$ is determined by (70.38). The second statement is equivalent to the fact that the up hull of the incidence vectors of paths in $\mathcal{P}$ is determined by (70.39). The equivalence is stated by Theorem 70.6. ∎

70.12a. Sufficiency of the cut condition sometimes implies an integer multiflow

As was also noted by Karzanov [1984,1987a] and Seymour [1979b], in certain collections of graphs+nets, if the cut condition implies the existence of a fractional multiflow, we can derive integrality of solutions. This can be made explicit as follows.

Consider an Eulerian graph $G = (V, E)$, and let e and f be distinct edges incident with a vertex v of degree ≥ 4. We describe the operation of separating e and f at v: introduce a new vertex v', rejoin half of the edges incident with v to

v', such that e remains incident with v and f becomes incident with v', and add $\frac{1}{2}\deg_E(v) - 2$ parallel edges connecting v and v'.

Call any graph G' arising in this way a *splitting* of G *separating* e *and* f *at* v. Note that, if $G' = (V', E')$ denotes the new graph, then $\deg_{E'}(v) = \deg_{E'}(v') = \deg_E(v) - 2$.

Let $\mathcal{I}$ be a collection of pairs (G, R) of an Eulerian graph $G = (V, E)$ and a subset R of E, with the following property:

(70.41) for each $(G, R) \in \mathcal{I}$, for each vertex v of G of degree at least 4 with $\deg_{E \setminus R}(v) > \deg_R(v)$, and for each two edges e and f of G incident with v, not both in R, $\mathcal{I}$ contains a pair (G', R') where G' is a splitting of G separating e and f at v, and where R' is the set of edges arising from R by this splitting.

As examples we can take for $\mathcal{I}$ the set of all pairs (G, R) consisting of an Eulerian graph $G = (V, E)$ and $R \subseteq E$ such that one of the following holds (the first four examples follow from the fact that for each fixed graph $H = (T, R)$, the class of pairs (G, R) with $G = (V, E)$ Eulerian and $R \subseteq E$ satisfies (70.41)):

(70.42)
(i) R consists of two parallel classes of edges;
(ii) there are two vertices intersecting all edges in R;
(iii) R covers at most four vertices;
(iv) the edges in R form a pentagon, with parallel edges added;
(v) (V, E) is planar;
(vi) $(V, E \setminus R)$ is planar, such that all vertices covered by R are on the outer boundary of $(V, E \setminus R)$;
(vii) $(V, E \setminus R)$ is planar, such that it has two faces with the property that each edge in R is spanned by one of these faces.

As we shall see in later chapters, in each of these cases the premise, and hence the conclusion, of the following theorem hold. The theorem applies to the multiflow problem with supply graph $(V, E \setminus R)$ and demand graph (V, R), with all capacities and demands equal to 1 (so to the edge-disjoint paths problem):

Theorem 70.7. *Let* $\mathcal{I}$ *satisfy* (70.41) *and have the property that for each* $(G, R) \in \mathcal{I}$*, the cut condition implies the existence of a fractional multiflow. Then, for each* $(G, R) \in \mathcal{I}$*, the cut condition implies the existence of an integer multiflow.*

Proof. Consider a counterexample $(G, R) \in \mathcal{I}$ with

(70.43) $$\sum_{v \in V} 2^{\deg_G(v)}$$

minimal. So the cut condition holds, and hence there is a fractional multiflow. It implies that there is a collection $\mathcal{C}$ of circuits in G, each intersecting R in exactly one edge, and, for each $C \in \mathcal{C}$, there is a $\lambda_C > 0$ such that for each edge e:

(70.44) $$\sum_{C \in \mathcal{C}} \lambda_C \chi^C(e) \leq 1,$$

with equality if $e \in R$. Here we consider circuits as edge sets.

Note that:

(70.45) for each $C \in \mathcal{C}$ and each $U \subseteq V$ with $|\delta_R(U)| = |\delta_{E\setminus R}(U)|$, if U splits at least one edge of C, then U splits the edge in $C \cap R$ and exactly one edge in $C \setminus R$.

(Here U *splits* e if $e \in \delta(U)$.) This follows from the fact that if U splits an edge in C, then it splits at least one edge in $C \setminus R$, and hence

(70.46) $$|\delta_R(U)| = \sum_C \lambda_C \chi^C(\delta_R(U)) \leq \sum_C \lambda_C \chi^C(\delta_{E\setminus R}(U)) \leq |\delta_{E\setminus R}(U)|.$$

Here we use $|C \cap \delta_R(U)| \leq |C \cap \delta_{E\setminus R}(U)|$, since $|C \cap \delta_R(U)| \leq 1$ and $|C \cap \delta_E(U)|$ is even. Equality throughout in (70.46) implies (70.45).

Now it suffices to show that

(70.47) for any two $C, D \in \mathcal{C}$, if $(C \setminus R) \cap (D \setminus R) \neq \emptyset$, then $C \setminus R = D \setminus R$.

That this is sufficient follows from the following. (70.47) implies that for each parallel class in R consisting of (say) μ edges connecting s and t, there are at least μ different $s-t$ paths among the $C \setminus R$ for $C \in \mathcal{C}$. Since they are edge-disjoint (by (70.47)), there exists an obvious integer solution.

To prove (70.47), suppose to the contrary that $C \setminus R$ and $D \setminus R$ have an edge in common and that $C \setminus R \neq D \setminus R$. Then (possibly after exchanging C and D), there is a vertex v on the paths made by $C \setminus R$ and $D \setminus R$ such that $C \setminus R$ and $D \setminus R$ have an edge f incident with v in common and such that $C \setminus R$ contains another edge, e say, incident with v with $e \notin D$. Let g be the edge in D incident with v and satisfying $g \neq f$. So $g \neq e$. Possibly $g \in R$.

So v has degree at least 4. Moreover, $\deg_{E\setminus R}(v) > \deg_R(v)$, by (70.45), since C contains two edges in $E \setminus R$ incident with v. Let $G' = (V', E')$ be a splitting of e and g at v with $(G', R') \in \mathcal{I}$, where R' is the set of edges arising from R by this splitting. By symmetry, we can assume that, in G', edge f is incident with v'. (We leave open which of e and g is incident with v'.) Then (G', R') has no integer multiflow, as it would give an integer multiflow in (G, R) (by contracting the new edges). Hence, as for G' the sum (70.43) is reduced, the cut condition is violated for (G', R'). Let $U \subseteq V'$ violate the cut condition. That is, $|\delta_{R'}(U)| > |\delta_{E'\setminus R'}(U)|$. Then, as G' is Eulerian, $|\delta_{R'}(U)| \geq |\delta_{E'\setminus R'}(U)| + 2$. Also, U separates v and v', since otherwise it would give a cut violating the cut condition for G, R. So we can assume that $v \in U$ and $v' \notin U$. Hence $U \subseteq V$.

Let G' have γ parallel edges connecting v and v'. So $\deg_{E'}(v') = 2\gamma + 2$. Let $\alpha := \deg_{R'}(v')$. Then:

(70.48) $$\begin{aligned} &|\delta_R(U)| \leq |\delta_{E\setminus R}(U)| = |\delta_{E'\setminus R'}(U \cup \{v'\})| \\ &\leq |\delta_{E'\setminus R'}(U)| + \deg_{E'\setminus R'}(v') - 2\gamma \\ &= |\delta_{E'\setminus R'}(U)| + \deg_{E'}(v') - 2\gamma - \deg_{R'}(v') = |\delta_{E'\setminus R'}(U)| + 2 - \alpha \\ &\leq |\delta_{R'}(U)| - \alpha \leq |\delta_R(U)|. \end{aligned}$$

Hence we have equality throughout. In particular, $|\delta_R(U)| = |\delta_{E\setminus R}(U)|$ (as the first inequality is an equality), U splits all edges in $E \setminus R$ that become incident in G' with v' (as the second inequality becomes equality), and U splits no edge in R that becomes incident in G' with v' (as the last inequality becomes equality). In particular, U splits f.

Now one of e and g is (in G') incident with v'. If e is incident with v', then U splits e, and we have a contradiction with (70.45) for circuit C. If g is incident with v', then if $g \in E \setminus R$, U splits g, and if $g \in R$, U does not split g; in both cases we have a contradiction with (70.45) for circuit D. ∎

70.12b. The cut condition and integer multiflows in directed graphs

Nagamochi and Ibaraki [1989] showed that for directed graphs, if the cut condition implies the existence of a fractional multiflow, and if this holds in a certain hereditary way, then it implies the existence of an integer multiflow:

Theorem 70.8. *Let $D = (V, A)$ and $H = (V, R)$ be a supply and demand digraph, respectively. Suppose that for each $c : A \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the cut condition implies the existence of a fractional multiflow. Then it implies the existence of an integer multiflow.*

Proof. Let c and d be such that the cut condition holds, but no integer multiflow exists. Choose such c, d with $c(A) + d(R)$ minimal. By assumption, there exists a fractional multiflow $(f_r : A \to \mathbb{R}_+ \mid r \in R)$. Then for each $a \in A$ we have

$$c(a) = \sum_{r\in R} f_r(a), \tag{70.49}$$

for otherwise we have, for any $U \subseteq V$ with $a \in \delta^{\text{out}}_A(U)$:

$$\begin{aligned} c(\delta^{\text{out}}_A(U)) &= \sum_{a\in\delta^{\text{out}}_A(U)} c(a) > \sum_{a\in\delta^{\text{out}}_A(U)} \sum_{r\in R} f_r(a) \geq \sum_{r\in\delta^{\text{out}}_R(U)} d(r) \\ &= d(\delta^{\text{out}}_R(U)). \end{aligned} \tag{70.50}$$

Hence (by integrality of c and d), we can replace $c(a)$ by $c(a) - 1$ without violating the cut condition, and obtain a smaller counterexample — a contradiction.

This proves (70.49). It implies that the directed analogue of the Euler condition holds, since for any vertex v:

$$\begin{aligned} c(\delta^{\text{out}}_A(v)) - c(\delta^{\text{in}}_A(v)) &= \sum_{r\in R}(f_r(\delta^{\text{out}}_A(v)) - f_r(\delta^{\text{in}}_A(v))) \\ &= d(\delta^{\text{out}}_R(v)) - d(\delta^{\text{in}}_R(v)). \end{aligned} \tag{70.51}$$

The latter equality holds as each f_r is a flow.

Now consider any $r' \in R$ with $d(r') \geq 1$, say $r' = (s, t)$. Replacing $d(r')$ by $d(r') - 1$, the cut condition is maintained. Hence (by the minimality of c, d) there is an integer multiflow $(f'_r \mid r \in R)$ satisfying the new demands. Consider the capacity function

$$c' := c - \sum_{r\in R} f'_r. \tag{70.52}$$

By (70.51), there is at least one $s - t$ path traversing only arcs a with $c'(a) \geq 1$. Hence we can increase $f'_{r'}$ along this path by 1, to obtain an integer multiflow satisfying the original demands. ∎

70.13. Further results and notes

70.13a. Fixing the number of commodities in undirected graphs

Robertson and Seymour [1995] showed that for each fixed k, the k vertex-disjoint paths problem in undirected graphs is polynomial-time solvable.[14] Describing the algorithm would require more space than fits within the limits of this book. The methods are quite different in nature from the more polyhedral methods discussed here, and are based on the deep graph minors techniques developed by Robertson and Seymour. For an outline of the disjoint paths algorithm, see Robertson and Seymour [1990].

The running time of Robertson and Seymour's algorithm is bounded by $O(n^3)$ (where the constant depends (heavily) on k). It implies a polynomial-time algorithm for the k edge-disjoint paths problem for fixed k, by considering the line graph.

More generally, Robertson and Seymour gave for each fixed k an $O(n^3)$-time algorithm for the *vertex-disjoint trees problem*:

(70.53) given: a graph $G = (V, E)$ and subsets $W_1, \ldots, W_p$ of V,
find: vertex-disjoint subtrees $T_1, \ldots, T_p$ in G such that T_i spans W_i, for $i = 1, \ldots, p$,

taking $k := |W_1 \cup \cdots \cup W_p|$.

For *planar* graphs, Reed, Robertson, Schrijver, and Seymour [1993] gave a linear-time algorithm for the disjoint trees problem, fixing $|W_1 \cup \cdots \cup W_p|$. Moreover, Schrijver [1991c] showed that for each fixed q, there is a polynomial-time algorithm for the disjoint trees problem in planar graphs such that $W_1 \cup \cdots \cup W_p$ can be covered by the boundary of at most q faces. The method is based on enumerating homotopy classes (see Section 76.7a), and here the degree of the polynomial depends on q.

Sebő [1993c] showed that for each fixed k, if $G + H$ is planar and $|VH| \leq k$, then the integer multiflow problem is polynomial-time solvable. (The demands and capacities can be arbitrarily large, so there is no reduction to the edge-disjoint paths problem for a fixed number of paths.) Sebő showed this by proving a more general result on the complexity of packing T-cuts for fixed $|T|$.

Related is the following. For any k, let $f(k)$ be the smallest number such that in any $f(k)$-connected graph, any instance of the k vertex-disjoint paths problem has a solution. Jung [1970] showed that $f(k) \leq 2^{3k}$ (a larger bound was shown by Larman and Mani [1970]). Thomassen [1980] proved $f(2) = 6$ and conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

For any k, let $g(k)$ be the smallest number such that in each $g(k)$-edge-connected graph, any instance of the k edge-disjoint paths problem has a solution. This value is finite — in fact, $g(k) \leq 2k$, since a $2k$-edge-connected graph has k edge-disjoint spanning trees (by the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a)). These trees contain k edge-disjoint paths as required.

Trivially $g(k) \geq k$. Moreover, for even k one has $g(k) \geq k + 1$, as is shown by replacing each edge of the circuit C_{2k} by $\frac{1}{2}k$ parallel edges, taking as nets the

[14] The correctness of the algorithm depends on a lemma proved in the preprint Robertson and Seymour [1992], which did not appear yet.

k pairs of opposite vertices. Cypher [1980] and Thomassen [1980] conjecture that $g(k) = k$ if k is odd and $g(k) = k + 1$ if k is even.

It is known that $g(2) = 3$ (as follows from a result of Dinits and Karzanov [1979] and Seymour [1980b] (see Section 71.4a)), $g(3) = 3$ (Okamura [1984a]), $g(4) = 5$ (Mader [1985], Hirata, Kubota, and Saito [1984], H. Enomoto and A. Saito (cf. Hirata, Kubota, and Saito [1984])), $g(k) \leq k + 1$ if k is odd, and $g(k) \leq k + 2$ if k is even (Huck [1991]).

Earlier, partial results were obtained by Cypher [1980] showing that $g(k) \leq k+2$ for $k \leq 5$, Hirata, Kubota, and Saito [1984] showing that $g(k) \leq 2k - 3$ if $k \geq 4$, and Okamura [1987,1988,1990]. Related results can be found in Enomoto and Saito [1984] and Huck [1992]. See also the notes in Section 72.2b.

The corresponding result for *directed* graphs has been shown for any k — see Section 70.13b.

70.13b. Fixing the number of commodities in directed graphs

For directed graphs, Fortune, Hopcroft, and Wyllie [1980] showed that deciding if two given vertices of a digraph belong to a directed circuit, is NP-complete. It implies that the arc-disjoint paths problem is NP-complete for $k = 2$ commodities, even if the nets are 'opposite' (that is, $s_2 = t_1$ and $t_2 = s_1$). It also implies that the directed vertex-disjoint paths problem is NP-complete (as the arc-disjoint problem can be reduced to vertex-disjoint by considering the line digraph).

Shiloach [1979a] observed that Edmonds' disjoint arborescences theorem implies that in any k-arc-connected digraph the k arc-disjoint problem always has a solution. (This can be shown by adding a new vertex r and new arcs (r, s_i) for each beginning terminal s_i. As the original digraph is k-arc-connected, by Edmonds' disjoint arborescences theorem (Corollary 53.1b) the new digraph has k arc-disjoint r-arborescences. They contain paths as required.)

If we restrict ourselves to *planar* digraphs, then for each fixed k, the k vertex-disjoint paths problem is polynomial-time solvable (Schrijver [1994a]). The method is based again on enumerating homotopy types of paths. (The polynomial-time solvability for $k = 2$ opposite nets (requiring only *internally* vertex-disjoint paths), was shown by Seymour [1991].)

It can be extended to the polynomial-time solvability, for any fixed q, of the problem of finding vertex-disjoint rooted subarborescences in a planar graph, with prescribed roots and terminals to be covered, provided that these roots and terminals can be covered by the boundaries of at most q faces.

An open problem is the complexity of the k *arc*-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$. This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

For *acyclic* digraphs, the k vertex-disjoint paths problem is polynomial-time solvable for each fixed k. This was shown by Fortune, Hopcroft, and Wyllie [1980] (extending an earlier result for $k = 2$ of Perl and Shiloach [1978]) — see Section 70.13c. By considering line digraphs, it implies the polynomial-time solvability of the k arc-disjoint paths problem in acyclic digraphs for each fixed k.

70.13c. Disjoint paths in acyclic digraphs

Fortune, Hopcroft, and Wyllie [1980] showed that the vertex-disjoint paths problem is NP-complete for digraphs, even when fixing the number of paths to $k = 2$. Moreover, Even, Itai, and Shamir [1975,1976] showed that the arc-disjoint paths problem in acyclic digraphs is NP-complete, even if the nets form two parallel classes. By taking the line digraph, it implies that the vertex-disjoint paths problem is NP-complete for acyclic digraphs. Vygen [1995] showed that the arc-disjoint paths problem in acyclic digraphs remains NP-complete, even if the nets form three parallel classes and the Euler condition holds; and also if the digraphs are restricted to acyclic and planar.

On the other hand, Fortune, Hopcroft, and Wyllie [1980] proved that for each fixed k, the k vertex-disjoint paths problem in acyclic digraphs can be solved in polynomial time. (This was proved for $k = 2$ by Perl and Shiloach [1978].)

Theorem 70.9. *For each fixed k, there exists a polynomial-time algorithm for the k vertex-disjoint paths problem for acyclic digraphs.*

Proof. Let $D = (V, A)$ be an acyclic digraph and let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of vertices of D (the nets), all distinct. To solve the disjoint paths problem we may assume that each s_i is a source of D and each t_i is a sink of D.

Make an auxiliary digraph $D' = (V', A')$ as follows. The vertex set V' consists of all k-tuples $(v_1, \ldots, v_k)$ of distinct vertices of D. In D' there is an arc from $(v_1, \ldots, v_k)$ to $(w_1, \ldots, w_k)$ if and only if there exists an $i \in \{1, \ldots, k\}$ such that:

(70.54) (i) $v_j = w_j$ for all $j \neq i$;
(ii) (v_i, w_i) is an arc of D;
(iii) for each $j \neq i$ there is no directed path in D from v_j to v_i.

Now the following holds:

(70.55) D contains vertex-disjoint directed paths $P_1, \ldots, P_k$ such that P_i runs from s_i to t_i $(i = 1, \ldots, k)$ $\Longleftrightarrow$ D' contains a directed path P from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$.

To see $\Longrightarrow$, let P_i follow the vertices $v_{i,0}, v_{i,1}, \ldots, v_{i,p_i}$ for $i = 1, \ldots, k$. So $v_{i,0} = s_i$ and $v_{i,p_i} = t_i$ for each i. Choose $j_1, \ldots, j_k$ such that $0 \leq j_i \leq p_i$ for each i and such that:

(70.56) (i) D' contains a directed path from $(s_1, \ldots, s_k)$ to $(v_{1,j_1}, \ldots, v_{k,j_k})$,
(ii) $j_1 + \cdots + j_k$ is as large as possible.

Let $I := \{i \mid j_i < p_i\}$. If $I = \emptyset$ we are done, so assume $I \neq \emptyset$. Then by the definition of D' and the maximality of $j_1 + \cdots + j_k$ there exists for each $i \in I$ an $i' \neq i$ such that there is a directed path in D from $v_{i',j_{i'}}$ to v_{i,j_i}. Since $t_{i'}$ is a sink we know that $v_{i',j_{i'}} \neq t_{i'}$ and that hence i' belongs to I. So each vertex in $\{v_{i,j_i} \mid i \in I\}$ is end vertex of a directed path in D starting at another vertex in $\{v_{i,j_i} \mid i \in I\}$. This contradicts the fact that D is acyclic.

To see $\Longleftarrow$ in (70.55), let P be a directed path from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$ in D'. Let P follow the vertices $(v_{1,j}, \ldots, v_{k,j})$ for $j = 0, \ldots, p$. So $v_{i,0} = s_i$ and $v_{i,p} = t_i$ for $i = 1, \ldots, k$. For each $i = 1, \ldots, k$, let P_i be the path in D following $v_{i,j}$ for $j = 0, \ldots, p$, taking repeated vertices only once. So P_i is a directed path from s_i to t_i.

Then $P_1, \ldots, P_k$ are vertex-disjoint. For suppose that P_1 and P_2 (say) have a vertex in common. That is $v_{1,j} = v_{2,j'}$ for some $j \neq j'$. Without loss of generality, $j < j'$ and $v_{1,j} \neq v_{1,j+1}$. By definition of D', there is no directed path in D from $v_{2,j}$ to $v_{1,j}$. However, this contradicts the facts that $v_{1,j} = v_{2,j'}$ and that there exists a directed path in D from $v_{2,j}$ to $v_{2,j'}$. ∎

One can derive from Theorem 70.9 that for fixed k also the k *arc*-disjoint paths problem is solvable in polynomial time for acyclic digraphs (by considering the line digraph).

Similarly to the proof of Theorem 70.9, one can prove that for each fixed k, the following problem is solvable in polynomial time: given an acyclic digraph $D = (V, A)$, pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices, and subsets $A_1, \ldots, A_k$ of A, find arc-disjoint directed paths $P_1, \ldots, P_k$, where P_i runs from s_i to t_i and traverses only arcs in A_i $(i = 1, \ldots, k)$.

Thomassen [1985] characterized the solvability of the 2 vertex-disjoint paths problem for acyclic digraphs, similarly to characterization (71.26). (Metzlar [1993] gave a generalization.)

70.13d. A column generation technique for multiflows

The (fractional) multiflow problem is a linear programming problem, and hence can be solved with linear programming techniques (in strongly polynomial time). Ford and Fulkerson [1958a] suggested a different LP-formulation of the multiflow problem, and a column generation technique to solve it with the simplex method.

As we saw in Section 70.1, the multiflow (feasibility) problem can be reduced to the maximum-value multiflow problem. This is equivalent to the following LP-problem. Let $D = (V, A)$ be a digraph, let $(s_1, t_1), \ldots, (s_k, t_k)$ be nets, and let $c : A \to \mathbb{R}_+$ be a capacity function. Let $\mathcal{P}$ denote the collection of all $s_i - t_i$ paths for all $i = 1, \ldots, k$ (taken as arc sets). Then the maximum-value multiflow problem can be formulated as:

$$\text{(70.57)} \qquad \begin{array}{lll} \text{maximize} & \displaystyle\sum_{P \in \mathcal{P}} z_P & \\ \text{subject to} & \text{(i)} \quad z_P \geq 0 & (P \in \mathcal{P}). \\ & \text{(ii)} \quad \displaystyle\sum_{P \in \mathcal{P}} z_P \chi^P(a) \leq c(a) & (a \in A). \end{array}$$

This is a linear programming problem with an exponential number of variables. Ford and Fulkerson [1958a] showed that this large number of variables can be avoided when solving the problem with the simplex method. The variables can be handled implicitly by using a *column generation technique* as follows.

When solving (70.57) with the simplex method we first should add a slack variable z_a for each $a \in A$. Let M denote the $A \times \mathcal{P}$ matrix with the incidence vectors of all paths in $\mathcal{P}$ as its columns and let w be the vector in $\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^E$ with $w_P := 1$ for $P \in \mathcal{P}$ and $w_a := 0$ for $a \in A$. Then (70.57) is equivalent to:

$$\text{(70.58)} \qquad \begin{array}{l} \text{maximize } w^{\mathsf{T}} z \\ \text{subject to } [M \;\; I] z = c, \\ \qquad\qquad\quad z \geq \mathbf{0}. \end{array}$$

If we solve (70.58) with the simplex method, each simplex tableau is completely determined by the set of variables in the current base. So it is determined by subsets $\mathcal{P}'$ of $\mathcal{P}$ and A' of A, giving the indices of variables in the base. This is enough to know implicitly the whole tableau. Note that $|\mathcal{P}'| + |A'| = |A|$. So although the tableau is exponentially large, it can be represented in a concise way.

Let B be the matrix consisting of those columns of $[M \;\, I]$ corresponding to $\mathcal{P}'$ and A'. So the rows of B are indexed by A and the columns by $\mathcal{P}' \cup A'$. The basic solution corresponding to B is easily computed: the vector $B^{-1}c$ gives the values for z_P if $P \in \mathcal{P}'$ and for z_a if $a \in A'$, while we set $z_P := 0$ if $P \notin \mathcal{P}'$ and $z_a := 0$ if $a \notin A'$. Initially, $B = I$, that is $\mathcal{P}' = \emptyset$ and $A' = A$, implying $z_P = 0$ for all $P \in \mathcal{P}$ and $z_a = c(a)$ for all $a \in A$.

Now we describe pivoting (that is, finding variables leaving and entering the base) and checking optimality. Interestingly, it turns out that this can be done by solving a set of shortest path problems.

First consider the dual variable corresponding to an arc a. It has value (in the current tableau):

(70.59) $$w_B^{\mathsf{T}} B^{-1} \chi^a - w_a = w_B^{\mathsf{T}} (B^{-1})_a,$$

where, as usual, w_B denotes the part of vector w corresponding to B (that is, corresponding to $\mathcal{P}'$ and A') and where χ^a denotes the ath unit base vector in $\mathbb{R}^A$ (which is the column corresponding to a in $[M \;\, I]$). Note that the columns of B^{-1} are indexed by A; then $(B^{-1})_a$ is the column corresponding to a. Note also that $w_a = 0$ by definition.

Similarly, the dual variable corresponding to a path P in $\mathcal{P}$ has value:

(70.60) $$w_B^{\mathsf{T}} B^{-1} \chi^P - w_P = \big(\sum_{a \in P} w_B^{\mathsf{T}} (B^{-1})_a\big) - 1.$$

In order to pivot, we should find a negative dual variable. To this end, we first check if (70.59) is negative for some arc a. If so, we choose such an arc a and take z_a as the variable entering the base. Selecting the variable leaving the base now belongs to the standard simplex routine. For that, we only have to consider that part of the tableau corresponding to $\mathcal{P}', A'$, and a. We select an element f in $\mathcal{P}' \cup A'$ for which the quotient $z_f/(B^{-1})_{f,a}$ has positive denominator and is as small as possible. Then z_f is the variable leaving the base.

Suppose next that (70.59) is nonnegative for each arc a. We consider $w_B^{\mathsf{T}}(B^{-1})_a$ as the length $l(a)$ of a. Then for any path P,

(70.61) $$\sum_{a \in P} w_B^{\mathsf{T}} (B^{-1})_a$$

is equal to the length $\sum_{a \in P} l(a)$ of P. Hence, finding a dual variable (70.60) of negative value is the same as finding a path in $\mathcal{P}$ of length less than 1.

Such a path can be found by applying a shortest path algorithm: for each $i = 1, \ldots, k$, we find a shortest $s_i - t_i$ path (with respect to l). If each of these shortest paths has length at least 1, we know that all dual variables have nonnegative value, and hence the current basic solution is optimum.

If we find some $s_i - t_i$ path P of length less than 1, we choose z_P as variable entering the base. Again selecting a variable leaving the base is standard: we select an element f in $\mathcal{P}' \cup A'$ for which the quotient $z_f/(B^{-1}\chi^P)_f$ has positive denominator and is as small as possible.

This describes pivoting. In order to avoid cycling and to guarantee termination, a lexicographic rule can be incorporated for selecting the variable leaving the base as usual. (This only requires ordering A.)

The length function l in the final tableau has the properties described in Theorem 70.2.

70.13e. Approximate max-flow min-cut theorems for multiflows

In general, the cut condition is not sufficient for the existence of a feasible multiflow. Leighton and Rao [1988,1999] gave an upper bound (only depending on the number of vertices) on the relative error in case each pair of vertices forms a net, with all demands equal.

Let $G = (V, E)$ and $H = (V, R)$ be a supply and a demand graph, and let $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ be a capacity and a demand function. Let λ be the maximum value for which there exists a multiflow subject to c of demand $\lambda \cdot d$. By the cut condition,

$$\lambda \leq \mu := \min_U \frac{c(\delta_E(U))}{d(\delta_R(U))}, \tag{70.62}$$

where the minimum is taken over all subsets U of V with $d(\delta_R(U)) > 0$.

Leighton and Rao proved that if R is the collection of *all* pairs from V and d is constant, then $\mu/\lambda = O(\log n)$ where $n := |V|$. They also showed that $O(\log n)$ is best possible, and that a set U attaining the minimum in (70.62) up to a factor $O(\log n)$ can be found in polynomial time.

Klein, Agrawal, Ravi, and Rao [1990] (cf. Klein, Rao, Agrawal, and Ravi [1995]) showed that if H is any demand graph, then $\mu/\lambda = O(\log C \log D)$, where C and D denote the sum of the capacities and demands, respectively. This was improved to $O(\log n \log D)$ by Tragoudas [1996], to $O(\log |R| \log D)$ by Garg, Vazirani, and Yannakakis [1993a,1996], and to $O(\log^2 |R|)$ by Plotkin and Tardos [1993,1995]. These papers also give polynomial-time algorithms to find a subset U attaining the minimum (70.62) up to the corresponding factor.

For planar graphs, Klein, Plotkin, and Rao [1993] gave a bound of $O(\log D)$, improved to $O(\log |R|)$ by Plotkin and Tardos [1993,1995], and of $O(1)$ if R consists of all pairs of vertices.

More results on approximate multiflows are given by Raghavan and Thompson [1987], Klein, Stein, and Tardos [1990], Shahrokhi and Matula [1990], Leighton, Makedon, Plotkin, Stein, Tardos, and Tragoudas [1991,1995], Goldberg [1992], Klein, Plotkin, and Rao [1993], Leong, Shor, and Stein [1993], Tardos and Vazirani [1993], Awerbuch and Leighton [1994], Klein, Plotkin, Stein, and Tardos [1994], Kamath and Palmon [1995], Linial, London, and Rabinovich [1995], Radzik [1995,1997], Aumann and Rabani [1998], Garg and Könemann [1998], Fleischer [1999a,2000a], Guruswami, Khanna, Rajaraman, Shepherd, and Yannakakis [1999], Leighton and Rao [1999], Baveja and Srinivasan [2000], Srivastav and Stangier [2000], Cheriyan, Karloff, and Rabani [2001], Fleischer and Wayne [2002], Günlük [2002], Karakostas [2002], and Kolman and Scheideler [2002]. A survey is given by Shmoys [1997]. Approximation algorithms for Steiner and directed multicuts are given by Klein, Plotkin, Rao, and Tardos [1997].

For approximating minimum-cost multiflows, see Plotkin, Shmoys, and Tardos [1991,1995], Kamath, Palmon, and Plotkin [1995], Karger and Plotkin [1995], Grigoriadis and Khachiyan [1996b,1996a], Garg and Könemann [1998], Goldberg, Oldham, Plotkin, and Stein [1998], and Karakostas [2002].

The 'quickest multicommodity flow problem' was investigated by Fleischer and Skutella [2002].

For surveys on approximation algorithms, see Shmoys [1995] and the book by Vazirani [2001].

70.13f. Further notes

Ford and Fulkerson [1958a] designed a (non-polynomial-time) algorithm for the fractional multiflow problem, based on the simplex method, with column generation — see Section 70.13d. Jewell [1958,1966] described a primal-dual simplex method, Sakarovitch [1966] gave a labeling algorithm solving a sequence of one-commodity flow problems after allocating the total capacity of each arc to each net, and Saigal [1967] developed an algorithm based on an arc-circuit formulation, using a column generation technique to handle the circuits. Dantzig-Wolfe decomposition was applied to multiflow problems by Chen and DeWald [1974]. Kapoor and Vaidya [1986,1996] and Kamath and Palmon [1995] study the complexity of applying interior point algorithms to multiflows.

Grinold [1968,1969] described a primal-dual algorithm for the maximum-value multiflow problem, based on allocating capacities to commodities and iteratively adapt the allocation. A simplex-based algorithm for minimum-cost and maximum-value multiflow problems was given by Hartman and Lasdon [1972]. Also Tomlin [1966], Wollmer [1972], Dragan [1974], and Nagamochi, Fukushima, and Ibaraki [1990] studied minimum-cost multiflows. A 'partitioning' algorithm for the multiflow problem was given by Grigoriadis and White [1972]. Related work was done by Kennington [1977], Farvolden, Powell, and Lustig [1993], and Hadjiat, Maurras, and Vaxes [2000]. Jarvis [1969] noticed the equivalence of vertex-arc and arc-chain formulations of the multiflow problem.

Bellmore, Greenberg, and Jarvis [1970] and Jarvis and Tindall [1972] described algorithms to find a minimum-capacity set disconnecting all nets in a directed multiflow problem.

Swoveland [1973] studied a generalization of the multiflow problem, where upper bounds can be prescribed for the sum of the flows of subsets of the nets on arcs. Ferland [1974] and Klessig [1974] studied nonlinear costs.

Computational work on multiflows is reported by Minoux [1975], Ulrich [1975], Helgason and Kennington [1977a], Kennington [1977,1978] (also minimum-cost), Kennington and Shalaby [1977], Ali, Helgason, Kennington, and Lall [1980], Kennington and Helgason [1980], Ali, Barnett, Farhangian, Kennington, Patty, Shetty, McCarl, and Wong [1984], Saviozzi [1986], Boland and Mees [1990], Nagamochi, Fukushima, and Ibaraki [1990], Barnhart [1993], Leong, Shor, and Stein [1993], Bienstock and Günlük [1995], Barnhart, Hane, and Vance [1996], Castro and Nabona [1996], Barnhart, Hane, and Vance [1997], McBride and Mamer [1997], McBride [1998], and Frangioni and Gallo [1999].

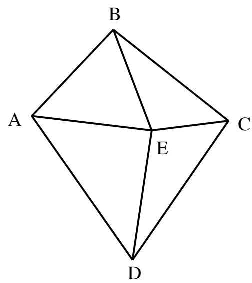
Surveys on multiflows were given by Hu [1969], Frank and Frisch [1971], Assad [1978], Kennington [1978], Phillips and Garcia-Diaz [1981], Gondran and Minoux [1984], Bazaraa, Jarvis, and Sherali [1990], Ahuja, Magnanti, and Orlin [1993], and Korte and Vygen [2000], on disjoint paths by Frank [1990e,1993a,1995], and on maximum-value multiflows by Karzanov [1991]. A bibliography on network optimization, including multicommodity flows, was compiled by Golden and Magnanti [1977].

70.13g. Historical notes on multicommodity flows

We review a few papers on multicommodity flows that are of historical interest.

In his monograph *Mathematical Methods of Organizing and Planning Production*, Kantorovich [1939] introduced linear programming methods for the multicommodity flow problem, giving as example the problem of a railroad network on which several connections have to be made simultaneously:

> Let us mention still another problem of different character which, although it does not lead directly to questions A, B, and C, can still be solved by our methods. That is the choice of transportation routes.



> Let there be several points A, B, C, D, E (Fig. 1) which are connected to one another by a railroad network. It is possible to make the shipments from B to D by the shortest route BED, but it is also possible to use other routes as well: namely BCD, BAD. Let there also be given a schedule of freight shipments; that is, it is necessary to ship from A to B a certain number of carloads, from D to C a certain number, and so on. The problem consists of the following. There is given a maximum capacity for each route under the given conditions (it can of course change under new methods of operation in transportation). It is necessary to distribute the freight flows among the different routes in such a way as to complete the necessary shipments with a minimum expenditure of fuel, under the condition of minimizing the empty runs of freight cars and taking account of the maximum capacities of the routes. As was already shown, this problem can also be solved by our methods.

A problem analogous to the multicommodity flow problem, the multi-index transportation problem, was considered by Motzkin [1952] and Schell [1955].

It was noted by Ford and Fulkerson [1954,1956b] that the max-flow min-cut theorem does not extend to maximum multiflows:

> It is worth pointing out that the minimal cut theorem is not true for networks with several sources and corresponding sinks, where shipment is restricted to be from a source to its sink.

Ford and Fulkerson give the example of the graph $K_{1,3}$, with nets all pairs of vertices of degree 1.

Robacker [1956a] observed that the following 'decomposition theorem' applies: for a graph $G = (V, E)$, nets $\{s_1, t_1\}, \dots, \{s_k, t_k\}$, and a capacity function $c : E \to \mathbb{R}_+$, the maximum total value of a multiflow subject to c is equal to

(70.63) $$\max_{c_1,\ldots,c_k} \sum_{i=1}^{k} \min_{C\in\mathcal{C}_i} c_i(C).$$

Here the maximum ranges over all k-tuples of vectors $c_1, \ldots, c_k$ in $\mathbb{R}_+^E$ with $c_1 + \cdots + c_k = c$. Moreover, $\mathcal{C}_i$ denotes the set of all $s_i - t_i$ cuts and $c_i(C)$ denotes the capacity of cut C with respect to the capacity function c_i.

So the theorem decomposes the maximum multicommodity flow problem into k maximum single-commodity flow problems. The problem is reduced to finding the optimum decomposition of the capacity function c into k functions $c_1, \ldots, c_k$. Robacker [1956a] remarked:

> At present there are no computational techniques other than those of linear programming for determining maximal flow through multicommodity networks. It is hoped, however, that the decomposition theorem may lead to new methods as did the minimum-cut, maximum-flow theorem for single-commodity networks.

Kalaba and Juncosa [1956] described applications of the multicommodity flow problem to telecommunication networks. In particular they mention:

> In a system such as the Western Union System, which has some 15 regional switching centers all connected to each other, an optimal routing problem of this type would have about 450 conditions and involve around 3000 variables. If solved using the simplex method in its most general form, this would be at the threshhold of the capacity of modern large-scale computers and would require several hours for solution.

They express the expectation that developments in computer technology and possible extensions of the combinatorial methods for one-commodity flows, will improve the situation greatly.

It turned out, however, that the combinatorial techniques that made the single-commodity flow problem so tractable, do not extend to multicommodity flows. Ford and Fulkerson [1958a] suggested a variant of the simplex method based on a column-generation technique, where each simplex step consists of determining a shortest path. Although they did not carry out computations, they expected that their method is more practicable than the direct simplex method, at least in space required. A primal-dual algorithm for multiflows was designed by Jewell [1958] (cf. Jewell [1966]).

Hu [1963] gave a combinatorial algorithm for the 2-commodity flow problem, but doubted whether it could be extended to general multicommodity flows:

> Although the algorithm for constructing maximum bi-flow is very simple, it is unlikely that similar techniques can be developed for constructing multicommodity flows. The linear programming approach used by Ford and Fulkerson to construct maximum multicommodity flows in a network is the only tool now available.

For remarks on the early history of multicommodity flows, see Jewell [1966].

Chapter 71

Two commodities

The integer 2-commodity flow problem is NP-complete, even if all capacities are 1 (Even, Itai, and Shamir [1975,1976]). Equivalently, the edge-disjoint paths problem in undirected graphs is NP-complete, even if the nets form two parallel classes.
However, if we add the Euler condition, the problem has a good characterization and can be solved in polynomial time. It is a generalization of Hu's 2-commodity flow theorem, stating that the cut condition implies the existence of a half-integer multiflow (for integer capacities and demands). This and related results form the topic of this chapter.
Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. In fact, often in this chapter, $k = 2$, so $R = \{s_1t_1, s_2t_2\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) = d_i$. We denote $G+H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

71.1. The Rothschild-Whinston theorem and Hu's 2-commodity flow theorem

It is a basic theorem of Hu [1963], that for 2-commodity flow problems in undirected graphs, the cut condition implies the existence of a feasible 2-commodity flow. Recall that the cut condition (in the undirected case) states that

(71.1) $$c(\delta_E(U)) \geq d(\delta_R(U))$$

for each $U \subseteq V$. This theorem, 'Hu's 2-commodity flow theorem', will be shown below as Corollary 71.1b.

Hu also showed that if moreover all capacities are integer, there is a half-integer 2-commodity flow. Generally, an integer multiflow need not exist, as is shown by Figure 70.3. In fact, the undirected integer 2-commodity flow problem is NP-complete (Even, Itai, and Shamir [1975,1976]).

Rothschild and Whinston [1966a] extended Hu's theorem by showing that adding the Euler condition guarantees the existence of an integer 2-commodity flow. We recall that the Euler condition states that

(71.2) $c(\delta_E(v)) + d(\delta_R(v))$ is even for each $v \in V$.

Theorem 71.1 (Rothschild-Whinston theorem)**.** *Let $G = (V, E)$ be a graph, let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be pairs of vertices of G, and let $c : E \to \mathbb{Z}_+$ and $d_1, d_2 \in \mathbb{Z}_+$ satisfy the Euler condition. Then there exists an integer 2-commodity flow subject to c and with value d_1, d_2 if and only if the cut condition is satisfied.*

Proof. Necessity being trivial, we show sufficiency. Suppose that the cut condition holds. Orient the edges of G arbitrarily, yielding the digraph $D = (V, A)$. For any $a \in A$, we denote by $c(a)$ the capacity of the underlying undirected edge. For $i = 1, 2$, define $p_i : V \to \mathbb{Z}$ by

(71.3) $p_i := d_i \cdot (\chi^{t_i} - \chi^{s_i})$.

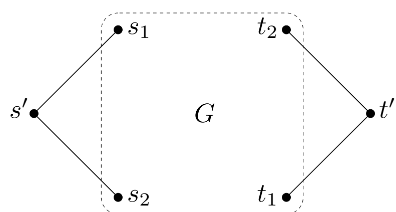


Figure 71.1

Extend G by two new vertices, s' and t', and new edges $s's_1$ and t_1t', each of capacity d_1, and new edges $s's_2$ and t_2t', each of capacity d_2 (Figure 71.1). This gives the graph G'.

By the max-flow min-cut theorem, G' contains an integer $s' - t'$ flow g of value $d_1 + d_2$, since by the cut condition the minimum capacity of an $s' - t'$ cut in G' is equal to $d_1 + d_2$. By the Euler condition we can assume that $g(e) \equiv c(e) \pmod 2$ for each $e \in E$: the edges e with $g(e) \not\equiv c(e) \pmod 2$ form an Eulerian graph; that is, each vertex is incident with an even number of such edges. Hence we can add a unit flow along a circuit, so as to decrease the number of such edges e.

Now in D, g gives a function $g' : A \to \mathbb{Z}$ satisfying

(71.4) $g'(a) \equiv c(a) \pmod 2$ and $|g'(a)| \leq c(a)$ for each $a \in A$, and $\text{excess}_{g'} = p_1 + p_2$.

(Here $\text{excess}_{g'}(v) := g'(\delta^{\text{in}}(v)) - g'(\delta^{\text{out}}(v))$ for $v \in V$.)

Similarly, by extending G by two new vertices, s'' and t'', and new edges $s''s_1$ and t_1t'', each of capacity d_1, and $s''t_2$ and s_2t'', each of capacity d_2 (Figure 71.2), we obtain a function $g'' : A \to \mathbb{Z}$ satisfying

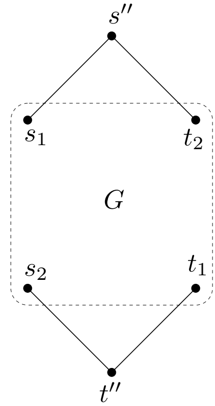


Figure 71.2

(71.5) $g''(a) \equiv c(a) \pmod 2$ and $|g''(a)| \leq c(a)$ for each $a \in A$, and $\text{excess}_{g''} = p_1 - p_2$.

Now define $f_1 := \frac{1}{2}(g' + g'')$ and $f_2 := \frac{1}{2}(g' - g'')$. Then f_1 and f_2 form a 2-commodity flow as required. Indeed, since $g' \equiv c \equiv g'' \pmod 2$, we know that f_1 and f_2 are integer. Moreover, $|f_1(a)| + |f_2(a)| = \frac{1}{2}(|g'(a)| + |g''(a)|) \leq c(a)$ for each $a \in A$. Finally, $\text{excess}_{f_i} = p_i$ for $i = 1, 2$, as follows directly from (71.4) and (71.5). ∎

This method of proof was given by Rothschild and Whinston [1966a] (similar proofs were given by Sakarovitch [1973] and Seymour [1978]).

A combinatorial form of Theorem 71.1 is:

Corollary 71.1a. *Let $G = (V, E)$ be a graph, let $s_1, t_1, s_2, t_2 \in V$, and let $d_1, d_2 \in \mathbb{Z}_+$, such that each vertex $v \neq s_1, t_1, s_2, t_2$ has even degree, while $\deg_G(s_i) \equiv \deg_G(t_i) \equiv d_i \pmod 2$ for $i = 1, 2$. Then there exist d_1 $s_1 - t_1$ paths and d_2 $s_2 - t_2$ paths, all edge-disjoint if and only if the cut condition* (70.21) *is satisfied.*

Proof. Directly from Theorem 71.1 by taking all capacities equal to 1. ∎

Conversely, Theorem 71.1 follows from Corollary 71.1a by replacing each edge e by $c(e)$ parallel edges.

Theorem 71.1 also implies a half-integer 2-commodity flow theorem, given by Hu [1963][15]:

Corollary 71.1b (Hu's 2-commodity flow theorem)**.** *Let $G = (V, E)$ be a graph, let s_1, t_1 and s_2, t_2 be pairs of vertices of G, let $c : E \to \mathbb{R}_+$, and*

[15] Hakimi [1962b] gave an erroneous proof of this theorem.

let $d_1, d_2 \in \mathbb{R}_+$. Then there exists a 2-commodity flow subject to c and with value d_1, d_2 if and only if the cut condition is satisfied. If all capacities and demands are integer, then we can take the flow half-integer.

Proof. By continuity and compactness, we can assume that c and the d_i are rational-valued, and hence, by scaling, even-integer-valued. So the Euler condition holds. Let the cut condition be satisfied. Then Theorem 71.1 gives the existence of a 2-commodity flow.

If c and the d_i are integer-valued, multiplying them by 2 and applying Theorem 71.1 gives an integer 2-commodity flow, and hence a half-integer multiflow for the original c and d_i. ∎

Notes. The proof of Theorem 71.1 yields a strongly polynomial-time algorithm to find a feasible integer 2-commodity flow if the Euler condition holds, of the same time order as that of finding a maximum one-commodity integer flow. It implies a strongly polynomial-time algorithm to find a half-integer 2-commodity flow, for integer capacities and demands.

Also Cherkasskiĭ [1973] gave a strongly polynomial-time ($O(n^2m)$) algorithm to find a feasible half-integer 2-commodity flow. Hu [1963] gave a combinatorial algorithm, which Itai [1978] showed to have a strongly polynomial-time implementation ($O(n^3)$). A similar algorithm was described by Arinal [1969].

71.1a. Nash-Williams' proof of the Rothschild-Whinston theorem

An alternative simple proof of the Rothschild-Whinston theorem was given by C.St.J.A. Nash-Williams (cf. Lovász [1979a] p. 289). We give the proof for the equivalent Corollary 71.1a. As necessity is easy, we show sufficiency.

By Menger's theorem (undirected version), G has d_1+d_2 edge-disjoint $\{s_1, s_2\}-\{t_1, t_2\}$ paths such that d_i of them start at s_i, and d_i of them end at t_i, for $i = 1, 2$. (But the paths starting at s_1 may end at t_2, and those starting at s_2 may end at t_1.) Hence G has an orientation $D = (V, A)$ with $(V, A \cup B)$ Eulerian, where B consists of d_i parallel arcs from t_i to s_i, for $i = 1, 2$.

Then Menger's theorem (directed version) implies that D has d_1 arc-disjoint directed $s_1 - t_1$ paths. Indeed, consider any $U \subseteq V$ with $s_1 \in U$, $t_1 \notin U$. We show $d_A^{\text{out}}(U) \geq d_1$. As $(V, A \cup B)$ is Eulerian, we have

$$(71.6) \qquad d_A^{\text{out}}(U) + d_B^{\text{out}}(U) = d_{A\cup B}^{\text{out}}(U) = d_{A\cup B}^{\text{in}}(U) = d_A^{\text{in}}(U) + d_B^{\text{in}}(U).$$

If $d_B^{\text{out}}(U) = 0$, this gives $d_A^{\text{out}}(U) \geq d_B^{\text{in}}(U) \geq d_1$. If $d_B^{\text{out}}(U) > 0$, then $t_2 \in U$, $s_2 \notin U$, hence $d_B^{\text{in}}(U) = d_1$ and $d_B^{\text{out}}(U) = d_2$. So

$$(71.7) \qquad d_A^{\text{out}}(U) = \tfrac{1}{2}(d_A^{\text{out}}(U) + d_A^{\text{in}}(U) + d_1 - d_2) = \tfrac{1}{2}(d_E(U) + d_1 - d_2) \geq d_1,$$

since $d_E(U) \geq d_1 + d_2$.

So D contains d_1 arc-disjoint $s_1 - t_1$ paths. Now delete from $(V, A \cup B)$ all arcs occurring in these paths, and delete the d_1 parallel arcs from t_1 to s_1. We are left with an Eulerian digraph, and hence the d_2 parallel arcs from t_2 to s_2 belong to d_2 arc-disjoint directed circuits. This gives the d_2 paths from s_2 to t_2 as required.

71.2. Consequences

E.A. Dinits (cf. Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975]) observed that Hu's 2-commodity flow theorem and the Rothschild-Whinston theorem imply:

Corollary 71.1c. *Let $G = (V, E)$ be an undirected graph and let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be pairs of vertices, such that there exist a two vertices intersecting each $\{s_i, t_i\}$. Let $c : E \to \mathbb{R}_+$ and $d_1, \ldots, d_k \in \mathbb{R}_+$. Then the cut condition implies the existence of a feasible multiflow. If c and the d_i are integer, there exists a half-integer multiflow. If moreover the Euler condition holds, there exists an integer multiflow.*

Proof. We can assume that $s_i = s$ for $i = 1, \ldots, l$, and that $t_i = t'$ for $i = l+1, \ldots, k$. Let t and s' be two new vertices. For each $i = 1, \ldots, l$, add a new edge connecting t_i and t, of capacity d_i. For each $i = l+1, \ldots, k$, add a new edge connecting s' and s_i, of capacity d_i. This makes the graph H. Define $d := d_1 + \cdots + d_l$ and $d' := d_{l+1} + \cdots + d_k$.

Then the cut condition for G implies that each cut $\delta_H(U)$ in H has capacity at least $d + d'$ if it is both $s - t$ and $s' - t'$ separating; at least d if it separates s and t; and at least d' if it separates s' and t'. Hence, by Hu's 2-commodity flow theorem, H has a feasible 2-commodity flow. Restriction to G gives a feasible multiflow.

The last two statement of this corollary follow similarly. ∎

Another consequence of Theorem 71.1 is what Hu called the *max-biflow min-cut theorem*[16]:

Corollary 71.1d (max-biflow min-cut theorem)**.** *Let $G = (V, E)$ be a graph, let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be pairs of vertices, and let $c : E \to \mathbb{R}_+$. Then the maximum total value M of a 2-commodity flow subject to c is equal to the minimum capacity m of a cut which is both $s_1 - t_1$ and $s_2 - t_2$ separating. If c is integer, the maximum is attained by a half-integer multiflow. If c is integer and $c(\delta(v))$ is even for each vertex $v \neq s_1, t_1, s_2, t_2$, the maximum is attained by an integer multiflow.*

Proof. By continuity, compactness, and scaling, we can assume that c is integer and that $c(\delta(v))$ is even for each $v \neq s_1, t_1, s_2, t_2$. By replacing edges by parallel edges, we can assume that $c(e) = 1$ for each $e \in E$. So M is equal to the maximum number of edge-disjoint paths, each connecting either s_1 and t_1, or s_2 and t_2. As trivially $M \leq m$, it suffices to prove $M \geq m$. We can assume that $m > 0$.

[16] due to Hu [1963] and (the last statement) to Rajagopalan [1994] (who also showed a hole in the proof by Sakarovitch [1973] of this); it sharpens a result of Rothschild and Whinston [1966b], who required that $c(\delta(v))$ is even for *all* vertices v.

First assume that $\deg_G(s_1) \equiv \deg_G(t_1) \pmod 2$ and (hence) $\deg_G(s_2) \equiv \deg_G(t_2) \pmod 2$. For $i = 1, 2$, let m_i be the minimum size of an $s_i - t_i$ cut. We show that

(71.8) there exists $d_1, d_2 \in \mathbb{Z}_+$ such that $d_1 \leq m_1$, $d_2 \leq m_2$, $d_1 + d_2 = m$, $d_1 \equiv \deg_G(s_1) \pmod 2$, and $d_2 \equiv \deg_G(s_2) \pmod 2$.

To see this, note that $m \leq m_1 + m_2$ (since the union of an $s_1 - t_1$ cut and an $s_2 - t_2$ cut separates both s_1 and t_1, and s_2 and t_2), $m_1 \leq m$, and $m \equiv \deg_G(s_1) + \deg_G(s_2) \pmod 2$. As $m > 0$, by symmetry we may assume that $m_2 > 0$. If $m_1 \equiv \deg_G(s_1) \pmod 2$, then we can take $d_1 := m_1$ and $d_2 := m - m_1$. If $m_1 \not\equiv \deg_G(s_1) \pmod 2$, then we can take $d_1 := m_1 - 1$ and $d_2 := m - m_1 + 1$. Indeed, as $m_1 \not\equiv \deg_G(s_1) \pmod 2$, any minimum-size $s_1 - t_1$ cut also separates s_2 and t_2. So $m = m_1$. Hence $d_1 \geq 0$ (as $m > 0$) and $d_2 = 1 \leq m_2$ (as $m_2 > 0$).

This shows (71.8). By Corollary 71.1a, there exist d_1 $s_1 - t_1$ paths and d_2 $s_2 - t_2$ paths, any two of which are edge-disjoint. So $M \geq d_1 + d_2 = m$.

Next assume that $\deg_G(s_1) \not\equiv \deg_G(t_1) \pmod 2$ and (hence) $\deg_G(s_2) \not\equiv \deg_G(t_2) \pmod 2$. By symmetry, we may assume that m is attained by a cut with s_1, s_2 at one side and t_1, t_2 at the other side. So the size of any cut with s_1, t_2 at one side and t_1, s_2 at the other side, has parity different from that of m; hence its size is at least $m+1$. Therefore, adding a new edge connecting s_2 and t_1 increases the minimum m by 1. Moreover, the maximum M increases by at most 1. In the new situation, the degrees of s_1 and t_1 have the same parity, and similarly for s_2 and t_2. Hence the first part of this proof applies, showing $M \geq m$. ∎

(An alternative proof was given by Lovász [1976b].)

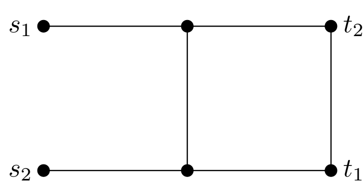


Figure 71.3
The maximum total value of a 2-commodity flow (subject to capacity **1**) is equal to 2, but the maximum total value of an integer 2-commodity flow is equal to 1.

The graph in Figure 71.3 shows that in the max-biflow min-cut theorem (Corollary 71.1d) we cannot delete the parity conditions (example of Rothschild and Whinston [1966b]). This example is critical, as is shown by the following result, which is a special case of a general hypergraph theorem of Seymour [1977b] (Theorem 80.1).

Theorem 71.2. *Let $G = (V, E)$ be a graph and $s_1, t_1, s_2, t_2 \in V$. Then for each capacity function $c : E \to \mathbb{Z}_+$, the maximum total value of an integer 2-commodity flow is equal to the minimum capacity of a cut separating both s_1 and t_1, and s_2 and t_2, if and only if G has no subgraph contractible to the graph of Figure 71.3, up to exchanging s_1 and t_1, and s_2 and t_2.*

Here we assume that the subgraph contains the s_i and t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure. For a proof, we refer to Section 80.5a.

A similar result for feasibility can be derived (Seymour [1981a]):

(71.9) Let $G = (V, E)$ be a graph and $s_1, t_1, s_2, t_2 \in V$. Then for each capacity function $c : E \to \mathbb{Z}_+$ and each demands $d_1, d_2 \in \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow if and only if the graph of Figure 70.3 is not a minor of G.

We derive this result from Theorem 71.2. By taking $c(e)$ large one can see that the property described is closed under contractions of edges. As Figure 70.3 satisfies the cut condition but has no integer multiflow (for $c = \mathbf{1}$, $d = \mathbf{1}$), we have necessity of the condition in (71.9).

To derive sufficiency from Theorem 71.2, let G have no subgraph contractible to Figure 70.3 and let $c : E \to \mathbb{Z}_+$ and $d_1, d_2 \in \mathbb{Z}_+$ satisfy the cut condition. Let s'_1 and s'_2 be two new vertices, and let $s'_1 s_1$ and $s'_2 s_2$ be two new edges, of capacity d_1 and d_2 respectively. Then the extended graph G' has no subgraph contractible to the graph of Figure 71.3, with s_i replaced by s'_i $(i = 1, 2)$, up to exchanging s'_1 and t_1, and s'_2 and t_2. Also, the minimum capacity of a cut in G' separating both s'_1 and t_1, and s'_2 and t_2, is equal to $d_1 + d_2$. Hence by Theorem 71.2, G' has an integer multiflow of total value $d_1 + d_2$. Restricted to G this gives a multiflow satisfying d_1, d_2.

Notes. For the case where $G + H$ is planar, Lomonosov [1983] characterized for *fixed* integer capacity function c, when the maximum and minimum in Theorem 71.2 are equal. He also showed that if $G + H$ is planar, the maximum and minimum differ by at most 1.

71.3. 2-commodity cut packing

By Theorem 70.5, Hu's 2-commodity flow theorem implies that if $G = (V, E)$ is an undirected graph, $s_1, t_1, s_2, t_2 \in V$, and $l : E \to \mathbb{R}_+$, then there exist $\lambda_U \geq 0$ for $U \subseteq V$ such that

(71.10) $$\sum_U \lambda_U \chi^{\delta_E(U)} \leq l$$

and

(71.11) $$\mathrm{dist}_l(s_i, t_i) = \sum_U \lambda_U \chi^{\delta_R(U)}(s_i t_i),$$

for $i = 1, 2$, where $R := \{s_1t_1, s_2t_2\}$. (Here $\text{dist}_l(s,t)$ is the distance of s and t, taking l as length function.)

We shall see that if l is integer, we can take the λ_U half-integer. More precisely, and more strongly:

(71.12) if l is integer such that each circuit in G has even length, then we can take the λ_U integer.

This was proved by Seymour [1978]. Equivalently (by replacing each edge e by a path of length $l(e)$; $\text{dist}_G(s,t)$ denotes the distance of s and t in G (for length function $\mathbf{1}$)):

Theorem 71.3. *Let $G = (V,E)$ be a bipartite graph and let $s_1, t_1, s_2, t_2 \in V$. Then there exist disjoint cuts such that s_i and t_i are separated by $\text{dist}_G(s_i,t_i)$ of these cuts, for $i = 1, 2$.*

Proof. We may assume that G is connected. Denote $d(u,v) := \text{dist}_G(u,v)$ for $u, v \in V$. Define for each vertex v:

(71.13)
$$\varphi(v) := \tfrac{1}{2}(d(s_1,v) + d(s_2,v) - d(s_1,s_2)),$$
$$\psi(v) := \tfrac{1}{2}(d(s_1,v) - d(s_2,v) + d(s_1,s_2)).$$

These numbers are nonnegative and integer, by the triangle inequality and by the fact that each circuit in G has even length.

If u and v are adjacent vertices of G, then either $\varphi(u) = \varphi(v)$ and $|\psi(v) - \psi(u)| = 1$, or $\psi(u) = \psi(v)$ and $|\varphi(v) - \varphi(u)| = 1$, since $d(s_1,v) - d(s_1,u) = \pm 1$ and $d(s_2,v) - d(s_2,u) = \pm 1$. Let A_i be the set of edges uv with $\varphi(v) = i-1$ and $\varphi(u) = i$. Let B_i be the set of edges uv with $\psi(v) = i-1$ and $\psi(u) = i$. So the sets $A_1, A_2, \ldots, B_1, B_2, \ldots$ are cuts partitioning E.

Now $\varphi(s_1) = \psi(s_1) = 0$ and $\varphi(t_1) + \psi(t_1) = d(s_1,t_1)$. So there exist $d(s_1,t_1)$ cuts among $A_1, A_2, \ldots, B_1, B_2, \ldots$ that separate s_1 and t_1.

Moreover

(71.14)
$$\begin{aligned}&|\varphi(t_2) - \varphi(s_2)| + |\psi(t_2) - \psi(s_2)| = \\ &\tfrac{1}{2}|d(s_1,t_2) + d(s_2,t_2) - d(s_1,s_2)| + \tfrac{1}{2}|d(s_1,t_2) - d(s_2,t_2) - d(s_1,s_2)| \\ &= d(s_2,t_2).\end{aligned}$$

This implies that there exist $d(s_2,t_2)$ cuts among $A_1, A_2, \ldots, B_1, B_2, \ldots$ that separate s_2 and t_2. ∎

A consequence of Theorem 71.3 is a min-max relation for the maximum number of disjoint cuts that are both $s_1 - t_1$ and $s_2 - t_2$ separating, in a bipartite graph (Seymour [1978]):

Corollary 71.3a. *Let $G = (V,E)$ be a bipartite graph and let $s_1, t_1, s_2, t_2 \in V$. Then the maximum number of disjoint cuts each separating both s_1 and t_1, and s_2 and t_2, is equal to the minimum of $\text{dist}_G(s_1,t_1)$ and $\text{dist}_G(s_2,t_2)$.*

Proof. We may assume that G is connected. Let $d(u,v) := \text{dist}_G(u,v)$ for $u,v \in V$. Let $k := \min\{d(s_1,t_1), d(s_2,t_2)\}$. Let $C_1,\ldots,C_t$ be the cuts described in Theorem 71.3. At least k of these cuts separate s_1 and t_1, and at least k of these cuts separate s_2 and t_2. If C_i separates s_1 and t_1 and C_j separates s_2 and t_2, then $C_i \cup C_j$ contains a cut separating both s_1 and t_1, and s_2 and t_2. Thus by properly combining the C_i, we obtain k disjoint cuts as required.

(To see this, we can assume that $C_1,\ldots,C_k$ separate s_1 and t_1, and that $C_{l+1},\ldots,C_{l+k}$ separate s_2 and t_2, where $0 \leq l \leq k$. Then each of the (disjoint) sets $C_1 \cup C_{k+1},\ldots,C_l \cup C_{l+k}, C_{l+1},\ldots,C_k$ contains a cut separating both s_1 and t_1, and s_2 and t_2.) ∎

Let $G=(V,E)$ be a graph and let $s_1,t_1,s_2,t_2 \in V$. Let $\mathcal{C}$ be the collection of all cuts that are both s_1-t_1 and s_2-t_2 separating. Consider a length function $l : E \to \mathbb{R}_+$. Corollary 70.6a applied to the max-biflow min-cut theorem gives:

(71.15) $\min\{\text{dist}_l(s_1,t_1), \text{dist}_l(s_2,t_2)\}$ is equal to the maximum value of $\sum_{C\in\mathcal{C}} y(C)$, where $y : \mathcal{C} \to \mathbb{R}_+$ is such that $\sum_{C\in\mathcal{C}} y(C)\chi^C \leq l$.

Then Corollary 71.3a implies (Seymour [1978], Pevzner [1979b]):

Corollary 71.3b. *If l is integer-valued, we can take y half-integer valued.*

Proof. Replace each edge e by a path of length $2l(e)$. This makes the bipartite graph H. Applying Corollary 71.3a to H does the rest. ∎

Bipartiteness is necessary in Corollary 71.3a, since otherwise the graph in Figure 71.4 (Seymour [1977b], cf. Hu [1973]) would yield a contradiction. (This answers a question of Fulkerson [1971a].)

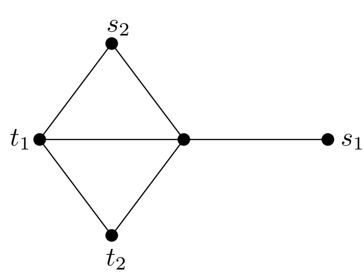


Figure 71.4
The minimum of the distances of s_1 and t_1 and of s_2 and t_2 is equal to 2, but there exist no two disjoint cuts each separating both s_1 and t_1, and s_2 and t_2.

From a more general hypergraph result of Seymour [1977b] (Theorem 80.1), it follows that Figure 71.4 is critical for the existence of an integer-valued packing of cuts:

Theorem 71.4. *Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. Then for each $l : E \to \mathbb{Z}_+$, the maximum in* (71.15) *is attained by an integer-valued y if and only if G has no subgraph contractible to the graph in Figure* 71.4*, up to permuting indices and permuting s_1 and t_1.*

Again, we assume that the subgraph contains the s_i and t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure. For a proof, we refer to Section 80.5a.

Notes. Seymour [1981a] showed the following related result: let $G = (V, E)$ be a bipartite graph, and choose $s_1, t_1, s_2, t_2 \in V$, with s_1, s_2 in one colour class and t_1, t_2 in the other. Choose odd integers $d_1 \leq \text{dist}_G(s_1, t_1)$ and $d_2 \leq \text{dist}_G(s_2, t_2)$ such that $d_1 + d_2 \leq \text{dist}_G(s_1, s_2) + \text{dist}_G(t_1, t_2)$ and $d_1 + d_2 \leq \text{dist}_G(s_1, t_2) + \text{dist}_G(t_1, s_2)$. Then there exist disjoint cuts, d_1 of which separate s_1 and t_1 and not s_2 and t_2, and d_2 of which separate s_2 and t_2 and not s_1 and t_1.

Let $\mathcal{S}$ be a collection of nonempty proper subsets of a finite set T. Let $G = (V, E)$ be a graph with $V \supseteq T$. Let $\mathcal{A}$ be the collection of subsets U of V with $U \cap T \in \mathcal{S}$.

Consider any length function $l : E \to \mathbb{R}_+$ and any $d : \mathcal{S} \to \mathbb{R}_+$. The multicut analogue of the multiflow problem asks for a function $y : \mathcal{A} \to \mathbb{R}_+$ such that

(71.16) $$\sum_{U \in \mathcal{A}} y(U) \chi^{\delta_E(U)} \leq l$$

and such that

(71.17) $$\sum (y(U) \mid U \in \mathcal{A}, U \cap T = X) = d(X)$$

for each $X \in \mathcal{S}$. A necessary condition for the existence of y is that

(71.18) $$\text{dist}_l(s, t) \geq \sum (d(X) \mid X \in \mathcal{S}, X \text{ splits } s, t)$$

for all distinct $s, t \in T$. (Here X *splits* s, t if X contains precisely one of s, t.) Karzanov [1984] showed that this condition is sufficient for each graph G and each l if and only if $\mathcal{S}$ contains no three pairwise crossing sets. If moreover l and d are integer, there is a half-integer y. If moreover $l(C)$ is even for each circuit C and both sides of (71.18) have the same parity for all s, t, then there is an integer y. Karzanov [1984] also gave a polynomial-time greedy-type algorithm to find y.

As for the corresponding maximization problem, consider any length function $l : E \to \mathbb{R}_+$ and the problem

(71.19) $$\min\{l^{\mathsf{T}} x \mid x \in \mathbb{R}_+^E : x(\delta_E(U)) \geq 1 \text{ for each } U \in \mathcal{A}\}.$$

By linear programming duality, this minimum is equal to the maximum value of

(71.20) $$\sum_{U \in \mathcal{A}} y(U),$$

where $y : \mathcal{A} \to \mathbb{R}_+$ satisfies (71.16). Karzanov [1984] showed the following. Let $\mathcal{S}$ have the following property: for any three pairwise crossing sets A_1, A_2, A_3 in $\mathcal{S}$, there exist $\gamma_1, \gamma_2, \gamma_3 \geq 0$ and $z : \mathcal{S} \to \mathbb{R}_+$ such that

(71.21) $$\sum_{i=1}^{3} \gamma_i \leq \sum_{U \in \mathcal{S}} z_U \text{ and } \sum_{i=1}^{3} \gamma_i \chi^{\delta_R(A_i)} > \sum_{U \in \mathcal{S}} z_U \chi^{\delta_R(U)},$$

where $R := \{st \mid s, t \in T, s \neq t\}$. Then the maximum value of (71.20) is equal to the minimum value of

(71.22) $$\sum_{r \in R} \beta(r) \mathrm{dist}_l(r),$$

where $\beta : R \to \mathbb{R}_+$ satisfies $\beta(\delta_R(U)) \geq 1$ for each $U \in \mathcal{S}$. (We write $\mathrm{dist}_l(r)$ for $\mathrm{dist}_l(s, t)$ if $r = st$.) This specifies the above $x : E \to \mathbb{R}_+$ as a function

(71.23) $$x = \sum_{r \in R} \beta(r) \chi^{P_r},$$

where P_r is a shortest r-path with respect to l, since

(71.24) $$l^{\mathsf{T}} x = \sum_{r \in R} \beta(r) l(P_r) \text{ and } x(\delta_E(U)) \geq \sum_{r \in \delta_R(U)} \beta(r) = \beta(\delta_R(U)) \geq 1$$

for each $U \in \mathcal{S}$. (An *r-path* is a path connecting the vertices in r.) Again this characterization is tight.

This has as special cases theorems on packing $s - t$ cuts (Theorem 6.1), 2-commodity cuts (Theorem 71.3), and T-cuts (Corollary 29.9a).

These cases are further characterized by the following result of Karzanov [1985a]. Let T be a finite set and let $\mathcal{S}$ be a collection of nonempty proper subsets of T such that (i) if $U \in \mathcal{S}$, then $T \setminus U \in \mathcal{S}$, (ii) for each $t \in T$ there is a $U \in \mathcal{S}$ with $t \in U$ and $U \setminus \{t\} \notin \mathcal{S}$, (iii) for all distinct $s, t \in T$ there is a $U \in \mathcal{S}$ separating s and t. Let G be the complete graph on vertex set V with $V \supset T$ and $|V| \geq |T| + 2$. Then minimum (71.19) is attained by an integer optimum solution x for each $l : E \to \mathbb{R}_+$ if and only if:

(71.25)
(i) there exist $s, t \in T$ such that each set in $\mathcal{S}$ contains exactly one of s and t, and such that the collection of sets in $\mathcal{S}$ containing s, is closed under unions and intersections,
or (ii) $T = \{s_1, t_1, s_2, t_2\}$ and $\mathcal{S} = \{\{s_1, s_2\}, \{s_1, t_2\}, \{t_1, s_2\}, \{t_1, t_2\}\}$,
or (iii) $\mathcal{S}$ is equal to the collection of odd-size subsets of T, where $|T|$ is even.

71.4. Further results and notes

71.4a. Two disjoint paths in undirected graphs

The polynomial-time solvability of the 2 vertex-disjoint paths problem in undirected graphs was shown by Seymour [1980b], Shiloach [1980b], and Thomassen [1980]. As was observed by Seymour [1980b], this can be derived from the following characterization of Seymour [1980b] and Thomassen [1980] (as usual, $N(K)$ denotes the set of vertices not in K adjacent to at least one vertex in K):

Theorem 71.5. *Let $G = (V, E)$ be a graph and let s_1, t_1, s_2, t_2 be distinct vertices. Then G has disjoint paths P_1 and P_2, where P_i connects s_i and t_i ($i = 1, 2$), if and only if there is no subset U of V such that:*

(71.26) (i) $s_1, t_1, s_2, t_2 \in U$,
(ii) $|N(K)| \leq 3$ *for each component* K *of* $G - U$;
(iii) *the graph* H *obtained from* $G[U]$ *by adding, for each component* K *of* $G - U$ *and each distinct* $u, v \in N(K)$, *an edge connecting* u *and* v, *is planar, with* s_1, s_2, t_1, t_2 *in this order cyclically on the outer boundary of* H.

In fact, condition (ii) is superfluous. (Theorem 71.5 was proved for 4-connected graphs by Jung [1970], generalizing Watkins [1968] who proved it for 4-connected graph containing a subdivision of K_5.)

The polynomial-time solvability of the 2 vertex-disjoint paths problem can be derived by observing that we can reduce the problem if there is a $K \subseteq V \setminus \{s_1, t_1, s_2, t_2\}$ with $|N(K)| \leq 3$ (remove K and add edges as in (71.26)(iii)). So we can assume that no such K exists. Hence, by the characterization, if no paths as required exist, the graph should be planar with the terminals in the cyclic order s_1, s_2, t_1, t_2 along the outer boundary — this can be tested in polynomial time. (Khuller, Mitchell, and Vazirani [1992] gave a parallel implementation.)

A related result for the 2 edge-disjoint paths problem was given by Dinits and Karzanov [1979] and Seymour [1980b]:

(71.27) Let $G = (V, E)$ be a connected graph and let $s_1, t_1, s_2, t_2 \in V$. Then G has edge-disjoint paths P_1 and P_2, where P_i connects s_i and t_i $(i = 1, 2)$ if and only if the cut condition holds and there is no $F \subseteq E$ such that the graph G/F, obtained from G by contracting all edges in F, is connected and planar and has maximum degree ≤ 3, while s_1, s_2, t_1, t_2 are distinct, all have degree at most 2, and occur in this order around the outer boundary of G/F.

This implies in particular that if G is 3-edge-connected, then the 2 edge-disjoint paths problem has a solution, for any choice of two nets.

71.4b. A directed 2-commodity flow theorem

Frank [1989] observed that a directed version of the 2-commodity flow theorem holds:

Theorem 71.6. *Let* $D = (V, A)$ *be as digraph, and let* R *consist of two parallel classes of arcs, with* $(V, A \cup R^{-1})$ *Eulerian. Then the cut condition is necessary and sufficient for the solvability of the arc-disjoint paths problem.*

Proof. Let R consist of k_i parallel arcs from s_i to t_i, for $i = 1, 2$. With Menger's theorem, the cut condition implies that there exist k_1 arc-disjoint $s_1 - t_1$ paths in D. After deleting the arcs of these paths from D, the remainder has k_2 arc-disjoint $s_2 - t_2$ paths, as adding k_2 parallel $t_2 - s_2$ arcs makes the remainder Eulerian. ∎

This proof also gives a polynomial-time algorithm. We should note that in the directed case, Eulericity is rather prohibitive: unlike in the undirected case we cannot make a digraph Eulerian by some simple doubling argument.

Frank, Ibaraki, and Nagamochi [1995,1998] gave a characterization and polynomial-time algorithm for the problem: given an Eulerian digraph $D = (V, A)$ and

$s_1, t_1, s_2, t_2 \in V$, find two arc-disjoint directed paths P_1 and P_2, where P_i connects s_i and t_i, in one way or the other ($i = 1, 2$). The characterization is analogous to Theorem 71.5.

It implies a characterization and algorithm of Ibaraki and Poljak [1991] for the 3 arc-disjoint paths problem if the Euler condition holds. For let $D = (V, A)$ be a digraph, and let $s_1, t_1, \ldots, s_3, t_3 \in V$, such that for $R := \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, the digraph $(V, A \cup R^{-1})$ is Eulerian. Extend D by four new vertices x_1, y_1, x_2, y_2 and arcs (t_1, x_2), (x_2, s_2), (t_3, y_1), (y_1, s_1), (t_2, y_2), (y_2, x_1), (x_1, s_3). Then the new digraph is Eulerian. Moreover, it has arc-disjoint directed $x_i - y_i$ paths (for $i = 1, 2$) if and only if D has arc-disjoint $s_i - t_i$ paths (for $i = 1, 2, 3$).

71.4c. Kleitman, Martin-Löf, Rothschild, and Whinston's theorem

Let G be an undirected graph. Suppose that we have four disjoint sets S_1, T_1, S_2, T_2 of vertices, and that we want to know the maximum number of edge-disjoint paths, each connecting either S_1 and T_1, or S_2 and T_2. Generally it is *not true* that the maximum number of such paths is equal to the minimum number of edges intersecting each such path. This even is not the case if the graph is Eulerian, as is shown by the graph in Figure 71.5 (cf. Rothschild and Whinston [1966b]). (One

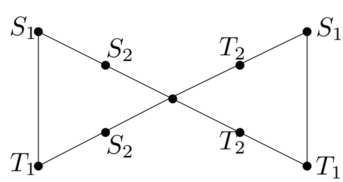


Figure 71.5
The maximum number of edge-disjoint paths each connecting vertices labeled S_i and T_i for some i, is equal to 4, whereas the minimum size of an edge set intersecting each such path is equal to 5. Note that the graph is Eulerian.

could think of a proof method based on adding 4 new vertices s_1, t_1, s_2, t_2, adjacent, by a large number of parallel edges, to all vertices in S_1, T_1, S_2, T_2 respectively, and then applying Corollary 71.1d. But this procedure can create new paths, for instance, from S_1 to T_1 via s_2.)

However, if S_1, T_1, S_2, T_2 partition the vertex set, such a generalization holds, as was shown by Kleitman, Martin-Löf, Rothschild, and Whinston [1970]. In fact, they showed a more general result, that can be proved with the help of the following theorem equivalent to (the edge-disjoint undirected version of) Menger's theorem (which is the special case where A and B are stars).

If $G = (V, E)$ a graph and $A, B \subseteq E$, we say that a path *connects* A and B if it traverses at least one edge in A and at least one edge in B.

Theorem 71.7. *Let $G = (V, E)$ be a graph and let $A, B \subseteq E$. Then the maximum number of edge-disjoint paths each connecting A and B is equal to the minimum number of edges intersecting each such path.*

Proof. We can assume that $A \cap B = \emptyset$, since deleting any edge in $A \cap B$ reduces both optima by 1.

Construct a new graph H as follows. Add two new vertices s and t. For each edge $e \in A \cup B$, put a new vertex v_e on e, and connect it to s if $e \in A$ and to t if $e \in B$.

We apply Menger's theorem to the $s-t$ paths in H. Let $Q_1, \ldots, Q_k$ be a maximum number of edge-disjoint $s-t$ paths in H. Consider any of these paths Q_j. We can assume that the second vertex, v_a say, and the one but last vertex, v_b say, of Q_j are the only two vertices on Q_j that belong to $\{v_e \mid e \in A \cup B\}$ (otherwise we can shortcut Q_j, since each vertex v_e has degree 3). Replacing the first two edges of Q_j by edge a of G, and the last two edges of Q_j by edge b of G, we obtain a path P_j in G connecting A and B.

This gives k edge-disjoint paths in G each connecting A and B. By Menger's theorem, there exists a set D of k edges of H intersecting each $s-t$ path. For $e \in A \cup B$, replacing (in D) any edge sv_e and any of the split-offs of e, by e, we obtain a set C of at most k edges in G that intersects each path connecting A and B. Indeed, consider any path P in G connecting A and B. We can assume that it intersects A and B only at its ends. So we can transform P to an $s-t$ path Q in H, by deviating the end edges towards s and t. Then Q intersects D, implying that P intersects C. This shows the theorem. ∎

This implies the theorem of Kleitman, Martin-Löf, Rothschild, and Whinston [1970]:

Corollary 71.7a. *Let $G = (V, E)$ be a graph, let $S_1, T_1, \ldots, S_k, T_k$ be subsets of V, with $S_i \cap T_i = \emptyset$ for $i = 1, \ldots, k$, and define $U_i := V - S_i - T_i$ for $i = 1, \ldots, k$. If $U_1, \ldots, U_k$ are disjoint, then the maximum number of edge-disjoint paths among $\{P \mid \exists i : P \text{ is an } S_i - T_i \text{ path}\}$ is equal to the minimum number of edges intersecting each such path.*

Proof. We can assume that the U_i partition V, since we can add an extra pair S_0, T_0 with $S_0 := V - U_1 - \cdots - U_k$ and $T_0 := \emptyset$. We can also assume that, for any i, no edge connects S_i and T_i, since deleting it reduces both optima by 1.

Let R be the set of edges connecting distinct sets among $U_1, \ldots, U_k$. Then for each i, any inclusionwise minimal $S_i - T_i$ path has its end edges in R and has no other edges in R (since all internal vertices belong to U_i). Let A be the set of edges e in R such that e is disjoint from an even number of $S_1, \ldots, S_k$, and let $B := R - A$.

The sets A and B have the following property. Let P be a path with only its end edges in R. Then:

(71.28) P connects S_i and T_i for some i if and only if P connects A and B.

With Theorem 71.7, this immediately proves the present corollary.

To prove (71.28), let u and w be the first and last vertex of P, let I be the set of internal vertices of P, and let c and d be the first and last edge of P. Since only the end edges of P are in R, we know by definition of R that there exists an i such that each internal edge of P only meets U_i and such that $u, v \notin U_i$. In other words, $I \subseteq U_i$ and $u, w \in S_i \cup T_i$.

Consider any $j \neq i$. As $I \subseteq U_i$ and $U_i \cap U_j = \emptyset$, we know $I \subseteq S_j \cup T_j$. As no edge connects S_j and T_j, either $I \subseteq S_j$ and $u, w \in U_j \cup S_j$, or $I \subseteq T_j$ and $u, w \in U_j \cup T_j$. So $c \cap S_j = \emptyset$ if and only if $d \cap S_j = \emptyset$. Hence, by definition of A and B:

(71.29) P connects A and B $\iff$ precisely one of $c \cap S_i$ and $d \cap S_i$ is nonempty $\iff$ precisely one of u, w belongs to S_i, the other belongs to T_i $\iff$ P connects S_i and T_i.

So we have (71.28). ∎

The proof method directly gives an algorithmic reduction to the (one-commodity) disjoint paths problem. (Kleitman [1971] and Kant [1974] describe other methods.)

71.4d. Further notes

Itai and Zehavi [1984] showed that if $G = (V, E)$ is a graph and $s_1, t_1, s_2, t_2 \in V$ are such that for $i = 1, 2$, there exist k edge-disjoint $s_i - t_i$ paths, then for each choice of d_1, d_2 with $d_1 + d_2 = k$, there exist d_1' and d_2' with $d_1' + d_2' = k$, $d_1 \leq d_1' \leq d_1 + 1$, and a collection of edge-disjoint paths such that d_i' of them connect s_i and t_i $(i = 1, 2)$.

The integer 2-commodity flow problem is solvable in polynomial time if $G + H$ is planar — see Section 74.2b.

Rebman [1974] studied a generalization of totally unimodular matrices appropriate for 2-commodity flows.

Chapter 72

Three or more commodities

Hu's 2-commodity theorem concerns multiflows where the demand graph H consists of two edges — whatever the supply graph is. In this chapter we consider to which extent Hu's theorem can be generalized to other demand graphs. That is, we study for which graphs $H = (T, R)$ it is true that for *each* graph $G = (V, E)$ with $V \supseteq T$ and each capacity and demand functions the phenomena described in the previous chapter are maintained (sufficiency of the cut condition, existence of a half-integer multiflow, sufficiency of the Euler condition to obtain an integer multiflow).

Results of Papernov [1976], Lomonosov [1976,1985], and Seymour [1980c] give an answer to this question: the graphs H are those containing neither of the two graphs in Figure 72.1 below as a subgraph. These are exactly the graphs H that are the union of two stars or are equal to K_4 or C_5 (up to adding isolated vertices, loops, and parallel edges).

Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) = d_i$. We denote $G + H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

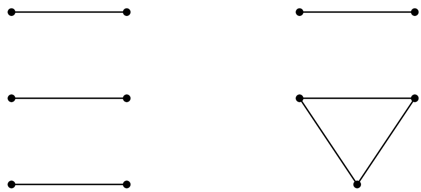
72.1. Demand graphs for which the cut condition is sufficient

Consider for any graph $H = (T, R)$ the following property:

(72.1) H has neither of the two graphs of Figure 72.1 as a subgraph.

Theorem 72.1. *Let $H = (T, R)$ be a simple graph without isolated vertices. Then H satisfies* (72.1) *if and only if $H = K_4$, or $H = C_5$, or H is the union of two stars.*

Proof. Sufficiency is direct. Necessity is shown by induction on $|R|$. If all degrees of H are at most 2, the theorem is easy. Assume now that H has a vertex u of degree at least 3. For any edge $e = uw$ incident with u, if $H - e$ is K_4 or C_5 (after deleting any isolated vertex), then H contains one of the



(a) (b)

Figure 72.1

graphs in Figure 72.1. So $H - e \neq K_4$ and $H - e \neq C_5$. Hence, by induction, there exist two vertices s, t such that each edge of $H - e$ intersects $\{s, t\}$. If $u \in \{s, t\}$, then H is the union of two stars. So we can assume that $u \notin \{s, t\}$. Hence each neighbour v of u with $v \neq w$ belongs to $\{s, t\}$. So u has degree 3, and each edge f of H not incident with u connects two neighbours of u (as any neighbour of u can serve as w). So $H = K_4$. ∎

The following theorem extends Theorem 71.1 and Corollary 71.1c, and was proved by Lomonosov [1976,1985] and Seymour [1980c] for $H = K_4$, and by Lomonosov [1976,1985] for $H = C_5$. The proof below is inspired by the direct proof given by Frank [1990e]. (Here $G + H$ is the graph $(V, E \cup R)$, taking multiplicities of edges into account.)

Theorem 72.2. *Let $G = (V, E)$ and $H = (T, R)$ be supply and demand graphs, with $T \subseteq V$. Let $H = (T, R)$ satisfy* (72.1)*, with $G + H$ Eulerian. Then there exist edge-disjoint paths P_r (for $r \in R$) such that P_r connects the vertices in r if and only if the cut condition holds.*

Proof. Necessity being easy, we show sufficiency. Let G, H form a counterexample with $|E| + |R|$ minimal. Then G is connected. Also, there is no net $r \in R$ parallel to an edge $e \in E$, since otherwise deleting r and e would give a smaller counterexample.

Call a subset U of V *tight* if $d_E(U) = d_R(U)$.[17] By the minimality of the counterexample we have[18]

(72.2) for each pair of edges e and f incident with a vertex v there is a tight set splitting both e and f.

Otherwise we can replace $e = uv$ and $f = wv$ by a new edge uw to obtain a smaller counterexample.

Another observation is:[19]

[17] As usual, $d_E(U) = |\delta_E(U)|$ and $d_R(U) = |\delta_R(U)|$.

[18] A set X *splits* a pair uv if X contains exactly one of u and v.

[19] $F[X, Y]$ denotes the set of pairs xy in F with $x \in X$ and $y \in Y$.

(72.3) for each tight set X and each $v \in V \setminus X$ we have $|E[X,v]| - |R[X,v]| \leq \frac{1}{2}(\deg_E(v) - \deg_R(v))$,

since, setting $X' := X \cup \{v\}$ we have $d_E(X') = d_E(X) + \deg_E(v) - 2|E[X,v]|$ and $d_R(X') = d_R(X) + \deg_R(v) - 2|R[X,v]|$. Then $d_E(X) = d_R(X)$ and $d_E(X') \geq d_R(X')$ give (72.3).

The following is also useful to observe:

(72.4) if X and Y are tight, and no net connects $X \setminus Y$ and $Y \setminus X$, then $X \cap Y$ and $X \cup Y$ are tight again, and no edge connects $X \setminus Y$ and $Y \setminus X$.

To see this, consider:

(72.5)
$$\begin{aligned} & d_R(X) + d_R(Y) = d_E(X) + d_E(Y) \\ & = d_E(X \cap Y) + d_E(X \cup Y) + 2|E[X \setminus Y, Y \setminus X]| \\ & \geq d_E(X \cap Y) + d_E(X \cup Y) \geq d_R(X \cap Y) + d_R(X \cup Y) \\ & = d_R(X) + d_R(Y). \end{aligned}$$

So we have equality throughout, proving (72.4).

This implies:

(72.6) let B be a set of vertices intersecting all nets, and let X and Y be tight sets with $X \cap B = Y \cap B$. Then $X \cap Y$ and $X \cup Y$ are tight.

Otherwise, by (72.4) there is a net connecting $X \setminus Y$ and $Y \setminus X$, and hence not intersecting B, a contradiction.

We next show that for each terminal t:[20]

(72.7) $\deg_E(t) = \deg_R(t)$.

Assume $\deg_E(t) > \deg_R(t)$. Let $\mathcal{X}$ be the collection of inclusionwise maximal tight subsets of $V \setminus \{t\}$. For each edge or net p, let $\mathcal{X}_p$ denote the set of $U \in \mathcal{X}$ splitting p.

We have $|\mathcal{X}_e| \geq 2$ for each edge $e \in \delta_E(t)$, since by (72.2), each pair of edges incident with t is split by some $U \in \mathcal{X}$, and since no tight set X splits all edges incident with t simultaneously, as it would imply

(72.8)
$$\begin{aligned} & d_E(X \cup \{t\}) = d_E(X) - \deg_E(t) = d_R(X) - \deg_E(t) \\ & < d_R(X) - \deg_R(t) \leq d_R(X \cup \{t\}). \end{aligned}$$

Also we have $|\mathcal{X}_r| \leq 2$ for each $r \in \delta_R(t)$, and we have $|\mathcal{X}| \leq 4$. Indeed, let $r = st$. By (72.1), there exists a vertex u such that each net intersects $B := \{s, t, u\}$. Therefore, by (72.6), any two sets in $\mathcal{X}$ have a different intersection with B (as otherwise their union is tight, contradicting their maximality). As no set in $\mathcal{X}$ contains t, we have the required inequalities.

This gives with (72.3):

[20] A *terminal* is a vertex covered by at least one net.

$$
\begin{aligned}
(72.9) \qquad 2(\deg_E(t)-\deg_R(t)) &\leq \sum_{e\in\delta_E(t)} |\mathcal{X}_e| - \sum_{r\in\delta_R(t)} |\mathcal{X}_r| \\
&= \sum_{U\in\mathcal{X}} (|E[U,t]| - |R[U,t]|) \leq \tfrac{1}{2}|\mathcal{X}|(\deg_E(t)-\deg_R(t)) \\
&\leq 2(\deg_E(t)-\deg_R(t)).
\end{aligned}
$$

Hence equality holds throughout in (72.9). So $|\mathcal{X}| = 4$, $|\mathcal{X}_e| = 2$ for each $e \in \delta_E(t)$, and $|\mathcal{X}_r| = 2$ for each $r \in \delta_R(t)$. Hence the $\mathcal{X}_e$ form a graph on the vertex set $\mathcal{X}$, such that any two of its edges intersect (by (72.2)), but no vertex is in all edges (since no tight set splits all edges in $\delta_E(t)$). So there is a $U \in \mathcal{X}$ contained in no $\mathcal{X}_e$; that is, $E[U,t] = \emptyset$. Since we have equality in (72.3) (as we have equality throughout in (72.9)), it follows that $\deg_E(t) - \deg_R(t) = 0$, that is, we have (72.7).

(72.7) implies that

(72.10) no two terminals s and t are adjacent,

since otherwise $d_E(\{s,t\}) < \deg_E(s) + \deg_E(t) = \deg_R(s) + \deg_R(t) = d_R(\{s,t\})$, contradicting the cut condition.

Now choose $st \in R$. By (72.1), there is a vertex $u \notin \{s,t\}$ such that each commodity disjoint from st intersects u. We can assume that u is a terminal, as otherwise s and t are the only terminals, in which case the theorem follows from Menger's theorem. Since, by (72.7), $V \setminus \{u\}$ is tight, there exists a tight subset Z which is inclusionwise minimal under the conditions that $s,t \in Z$ and $u \notin Z$.

Then s has a neighbour $v \in Z$. Otherwise we have

(72.11) $d_E(Z) = \deg_E(s) + d_E(Z\setminus\{s\}) \geq \deg_R(s) + d_R(Z\setminus\{s\}) > d_R(Z)$

(as $s,t \in Z$), contradicting the tightness of Z.

Let $\mathcal{Y}$ be the collection of all inclusionwise maximal tight subsets of $V\setminus\{v\}$ containing s. By (72.10), v is not a terminal. Hence, by (72.3), $|E[Y,v]| \leq \frac{1}{2}\deg_E(v)$ for each $Y \in \mathcal{Y}$. Therefore, since (by (72.2)) each edge incident with v is split by at least one $Y \in \mathcal{Y}$, we have $|\mathcal{Y}| \geq 3$.

Then by (72.6), the sets in $\mathcal{Y}$ all have different intersections with $\{t,u\}$. (By definition, each set in $\mathcal{Y}$ contains s.) Moreover, $Y \cap \{t,u\} \neq \{t\}$ for each $Y \in \mathcal{Y}$, since otherwise also $Y \cap Z$ is tight (by (72.6), as $Z \cap \{t,u\} = \{t\}$), contradicting the minimality of Z (note that $v \notin Y \cap Z$).

So $|\mathcal{Y}| = 3$ and the sets in $\mathcal{Y}$ intersect $\{t,u\}$ in $\emptyset$, $\{u\}$, and $\{t,u\}$ — denote these sets by S, U, and W, respectively (cf. Figure 72.2).

By the maximality of S, U, and W, $S \cup U$ and $U \cup W$ are not tight. Hence, by (72.4), there is a net γ connecting $S \setminus U$ and $U \setminus S$, and a net δ connecting $W \setminus U$ and $U \setminus W$. Then γ and δ intersect $\{s,t,u\}$. As $s,t \notin \gamma$ (since $s,t \in S \cap U$) and $u \notin S \setminus U$, we know $\gamma = uw$ for some $w \in S \setminus U$. As $s,u \notin \delta$ (since $s \in S \cap W$ and $u \notin S \cup W$) and $t \notin U \setminus W$, we know $\delta = tx$ for some $x \in U \setminus W$. As st and tx are disjoint from uw, each net disjoint from uw contains t.

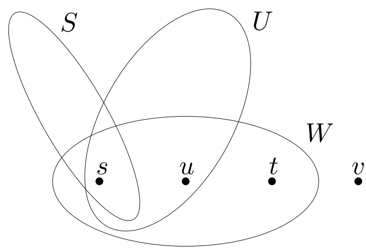


Figure 72.2

However, as edge sv connects $W \cap S$ and $V \setminus (W \cup S)$, by (72.4) (applied to $X := W$ and $Y := V \setminus S$), there is a net sa connecting these two sets. Then $a \neq u, w, t$, and therefore sa is disjoint from u, w, t, a contradiction. ∎

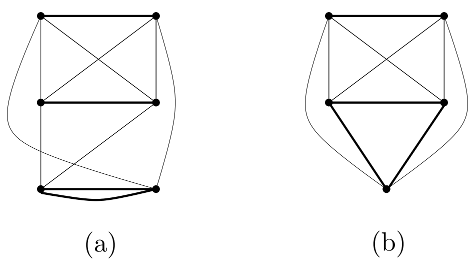


Figure 72.3
Examples where the cut and Euler conditions hold, but no fractional multiflow exists. The heavy lines are the nets and the other lines the edges. All capacities and demands are equal to 1.

Theorem 72.2 also holds if H consists of three disjoint edges — see Theorem 72.3. The examples in Figure 72.3 (from Papernov [1976]) show that the condition on the demand graph H in Theorem 72.2 is close to tight. This is made more precise in the following characterization implied by Theorem 72.2 (the equivalence (i)⇔(iv)⇔(v) is due to Papernov [1976], the other equivalences to Lomonosov [1976,1985] and (for K_4) to Seymour [1980c]):

Corollary 72.2a. *For each simple graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.12) (i) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a fractional multiflow;*

(ii) *for each graph* $G = (V, E)$ *with* $V \supseteq T$*, and each* $c : E \to \mathbb{Z}_+$ *and* $d : R \to \mathbb{Z}_+$*, the cut condition implies the existence of a half-integer multiflow;*

(iii) *for each graph* $G = (V, E)$ *with* $V \supseteq T$*, and each* $c : E \to \mathbb{Z}_+$ *and* $d : R \to \mathbb{Z}_+$*, the cut and Euler conditions imply the existence of an integer multiflow;*

(iv) H *contains none of the graphs in Figure* 72.1 *as subgraph;*

(v) $H = K_4$*, or* $H = C_5$*, or* H *is the union of two stars.*

Proof. The equivalence (iv)$\Leftrightarrow$(v) was shown in Theorem 72.1. The implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) are general multiflow theory, while (i)$\Rightarrow$(iv) follows from the examples of Figure 72.3. The implication (iv)$\Rightarrow$(iii) follows from Theorem 72.2, by replacing each edge of G by $c(e)$ parallel edges and each edge of H by $d(e)$ parallel edges. ∎

Karzanov [1979b] gave a strongly polynomial-time algorithm finding a half-integer multiflow as required if H satisfies (72.12)(iv) (or finding a cut violating the cut condition).

72.2. Three commodities

An important case excluded by the theorems in the previous sections is that of a demand graph consisting of three disjoint edges, with $d = \mathbf{1}$.

Theorem 72.3. *Let* $G = (V, E)$ *and* $H = (T, R)$ *be graphs, with* $T \subseteq V$*, such that* $G + H$ *is Eulerian and such that* R *consist of three disjoint edges. Then there exist edge-disjoint paths* P_r *(for* $r \in R$*) such that* P_r *connects the vertices in* r *if and only if the cut condition holds.*

Proof. Let $R = \{r_1, r_2, r_3\}$. Let G be a counterexample with a minimum number of edges. Then G is connected, and each vertex of G has degree at least two. Call $U \subseteq V$ *tight* if $d_E(U) = d_R(U)$. Also:

(72.13) $d_R(U) = 3$ for each tight nonempty proper subset U of V.

To see this, let U be a counterexample with $d_E(U)$ smallest. Then $G[U]$ and $G - U$ are connected. (Otherwise we could replace U by one of the components K of $G[U]$ or $G - U$, while $d_E(K) < d_E(U)$.) Also, $d_R(U) = d_E(U) \geq 1$ as G is connected. So we can assume that $r_1 \in \delta_R(U)$ and that $V \setminus U$ spans r_2. Contract U to obtain graph G/U. As the cut condition remains to hold, and as G/U is smaller than G (since $|U| \geq 2$, as $d_E(v) \geq 2 > d_R(v)$ for each $v \in V$), G/U contains edge-disjoint paths Q_1 and Q_2 where Q_i connects (the contractions of) the vertices in r_i ($i = 1, 2$). As $G[U]$ is connected, $G[U]$ contains a path connecting the vertex in $r_1 \cap U$ and the end of the edge in $\delta_E(U)$ that is traversed by Q_1. It follows that G contains two edge-disjoint paths P_1 and P_2 where P_i connects the vertices in r_i ($i = 1, 2$). Removing the edges of P_1 and P_2 from G, we are left with a graph with exactly two vertices of odd degree, namely the vertices in the pair r_3. Hence this graph contains a path P_3 connecting the vertices in r_3. Then P_1, P_2, and P_3 are as required. This is a contradiction, proving (72.13).

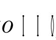
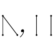
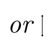
Consider now any $r = st \in R$ and any edge tu of G incident with t. Let $R' := (R \setminus \{st\}) \cup \{su\}$. Let $G' = (V, E')$ be the graph obtained from G by deleting edge tu. If the cut condition holds for G', R', we obtain (by induction) three paths in G' that directly yields paths as required in G. So we can assume that there is a subset U of V with $d_{R'}(U) > d_{E'}(U)$ and $t \notin U$. As $G' + H'$ is Eulerian, we know $d_{R'}(U) \geq d_{E'}(U) + 2$. Then

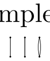
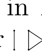
$$(72.14) \qquad d_R(U) \geq d_{R'}(U) - 1 \geq d_{E'}(U) + 1 \geq d_E(U) \geq d_R(U).$$

So we have equality throughout, and hence $d_E(U) = d_R(U)$, and $s \notin U$, $u \in U$ (otherwise $d_{R'}(U) \geq d_R(U)$). So $d_R(U) \leq 2$, contradicting (72.13). ∎

With Theorem 72.2, this implies the following characterization:

Corollary 72.3a. *For any loopless graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.15)
(i) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut and Euler condition (with respect to H), the edge-disjoint paths problem has a solution;*
(ii) *T has two vertices intersecting all pairs in R, or $|T| \leq 4$, or H is C_5 with parallel edges added, or R consists of three disjoint edges;*
(iii) *H has no subgraph equal to* , , , , *or* .

Proof. The implication (iii)⇒(ii) follows from Theorem 72.1. The implication (ii)⇒(i) follows from Theorems 72.2 and 72.3. The implication (i)⇒(iii) follows from the examples in Figure 72.3, since from each of graphs given in (iii) we can obtain  or , by identifying some vertices. Then from the examples in Figure 72.3 we can obtain examples for the graphs in (iii) by adding two parallel edges between any pair of identified vertices. ∎

Notes. For $|R| = 3$, Okamura [1984a] showed that the cut condition implies the existence of a half-integer solution for the edge-disjoint paths problem. (This seems not to follow from Theorem 72.3. On the other hand, having Okamura's result, to prove Theorem 72.3 it suffices to show that if the Euler condition holds and a half-integer solution exists, there is an integer solution.)

This implies the following characterization, extending Corollary 72.3a.

Theorem 72.4. *For any loopless graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.16)
(i) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut condition, the edge-disjoint paths problem has a fractional solution;*

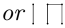
(ii) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut condition, the edge-disjoint paths problem has a half-integer solution;*
(iii) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut and Euler condition, the edge-disjoint paths problem has a solution;*
(iv) *T has two vertices intersecting all pairs in R, or $|T| \leq 4$, or H is C_5 with parallel edges added, or R consists of three disjoint pairs;*
(v) *H has no subgraph equal to* , , , , *or* .

Figure 70.4 shows that there is no integer p such that if a 3-commodity problem, with integer capacities and demands, has a fractional solution, then it has a $1/p$-integer solution. More precisely, for each integer $k \geq 2$, there is a graph $G = (V, E)$ and a collection R of three disjoint pairs from V, such that for $c : E \to \mathbb{Z}_+$ defined by $c(e) = 1$ for each edge e and $d : R \to \mathbb{Z}_+$ with values $1, 2k, 2k$ respectively, there is a fractional multiflow, but each feasible solution has some of its values equal to $1/2k$.

By doubling capacities and demands, one obtains an example of a 3-commodity flow problem satisfying the Euler condition, where a fractional but no half-integer multiflow exists. A variant of the example gives a 3-commodity flow problem satisfying the Euler condition, where a half-integer but no integer solution exists.

M. Middendorf and F. Pfeiffer (cf. Pfeiffer [1990]) showed that it is NP-complete to decide if the edge-disjoint paths problem has a half-integer solution, even if the nets consist of three disjoint parallel classes of edges. This implies a result of Vygen [1995] that it is NP-complete to decide if the edge-disjoint paths problem has a solution, even if the nets consist of three disjoint parallel classes of edges and the Euler condition holds.

Let H_6 be the graph obtained from $K_{3,3}$ by adding in each of the two colour class one new edge (cf. Figure 72.3(a)). Seymour [1981a] showed for each graph $G = (V, E)$:

(72.17) G has no H_6 minor if and only if for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \setminus R \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ satisfying the Euler condition, the cut condition implies the existence of an integer multiflow (where $(V, E \setminus R)$ is the supply graph).

The proof is by showing that each 3-connected graph without H_6 minor is K_5 or has no K_5 minor, and hence can be decomposed into planar graphs and copies of V_8 (Wagner's theorem (Theorem 3.3)).

72.2a. The $K_{2,3}$-metric condition

Karzanov [1987a] showed that a strengthened form of the cut condition, the '$K_{2,3}$-metric condition', is sufficient for having a fractional multiflow for a class of demand graphs larger than described in Section 72.1.

This is described as follows. Let Γ be a graph and let V be a finite set. A metric μ on V is called a *Γ-metric* if there is a function $\phi : V \to V\Gamma$ with

(72.18) $\mu(u, v) = \mathrm{dist}_\Gamma(\phi(u), \phi(v))$

for all $u, v \in V$. (Here $\mathrm{dist}_\Gamma(x, y)$ denotes the distance of x and y in Γ.)

Γ-metrics give rise to the following necessary condition, the *Γ-metric condition*, for the existence of a feasible fractional multiflow:

(72.19) $$\sum_{r=st\in R} d(r)\mu(s,t) \leq \sum_{e=uv\in E} c(e)\mu(u,v) \text{ for each } \Gamma\text{-metric } \mu \text{ on } V.$$

This is a specialization of condition (70.11). Since each cut gives a K_2-metric, and hence a $K_{2,3}$-metric, condition (72.19) includes the cut condition.

Karzanov [1987a] showed:

Theorem 72.5. *Let $G = (V, E)$ be a graph and let $H = (T, R)$ be a complete graph with $|T| = 5$ and $T \subseteq V$. Let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. Then there exists a fractional multiflow if and only if the $K_{2,3}$-metric condition holds. If moreover c and d are integer, there is a half-integer multiflow. If moreover the Euler condition holds, there is an integer multiflow.*

Theorem 72.5 implies the following characterization:

Corollary 72.5a. *For each simple graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.20)
- (i) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the existence of a fractional solution implies the existence of a half-integer solution;*
- (ii) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the Euler condition and the existence of a fractional solution imply the existence of an integer solution;*
- (iii) *H has no three disjoint edges and no two disjoint triangles;*
- (iv) *$|VH| = 5$, or H is the union of a triangle and a star, or H is the union of two stars.*

That (72.20)(iv) implies (72.20)(i) follows from Theorem 72.5, as we can replace a star with center s by an edge sw, where w is a new vertex, with the construction of Dinits given in the proof of Corollary 71.1c. Conversely, (72.20)(i) implies (72.20)(iii). It requires giving a counterexample if H consists of three disjoint edges, and one if H consists of two disjoint triangles. If H consists of three disjoint edges, a counterexample was given in Figure 70.4. If H consists of two disjoint triangles, a counterexample follows (by doubling all capacities and demands) from Figure 72.4 (A.V. Karzanov, personal communication 2000), where c and d are integer, and where a quarter-integer, but no half-integer solution exists.

Karzanov [1991] conjectures that if R consists of two disjoint triangles and c and d are integer and satisfy the Euler condition, then the existence of a fractional solution implies the existence of a half-integer solution[21]. This would imply that for each fixed graph $H = (T, R)$ the following equivalences holds:

(72.21) (?) there is an integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{k}$-integer multiflow
$\Longleftrightarrow$ for each graph $G = (V, E)$ with $V \supseteq T$ and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{4}$-integer multiflow
$\Longleftrightarrow$ H has no three disjoint edges. (?)

[21] A proof of this was announced in Karzanov [1987a], but A.V. Karzanov communicated to me that the proof failed.

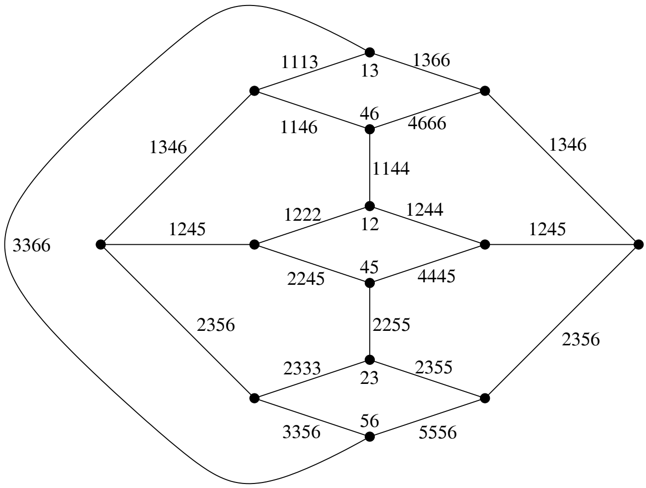


Figure 72.4

A quarter-integer multiflow exists, but no half-integer multiflow. The nets (indexed by $1, \ldots, 6$) are indicated by indices at vertices. All capacities and demands are 1. The quarter-integer multiflow is indicated by indices at the edges: k times index i at edge e means $f_i(e) = \frac{k}{4}$.

The nonexistence of a half-integer multiflow can be seen as follows. Give each of the (three) edges that connect terminals length 2, and any other edge length 1. Then the distance between the two terminals in any net is 4. Also, the sum of the lengths of the edges equals 24. So any flow f_i in a half-integer multiflow, can be decomposed as half of the sum of two flows following $s_i - t_i$ paths $P_{i,1}$ and $P_{i,2}$ of length 4. Moreover, on each edge, the capacity is fully used. Hence, each vertex v of degree 3 not being a terminal, is traversed by three paths $P_{i,j}$, each using a different pair of edges incident with v. One easily checks that this is not possible.

Karzanov [1998d] studied the existence of an integer multiflow if the nets form a disjoint union of a triangle and an edge.

72.2b. Six terminals

Okamura [1987] showed the following. Let $G = (V, E)$ and $H = (T, R)$ be a supply and a demand graph. If $|T| \leq 6$ and $k := |R|$ is odd, and if moreover G has k edge-disjoint $s - t$ paths, for each $st \in R$, then there exists a family $(P_r \mid r \in R)$ of edge-disjoint paths in G, where P_r connects the ends of r (for $r \in R$). (For $|T| \leq 5$ this was proved in Okamura [1984b].)

Okamura [1998] showed that if $|T| \leq 6$ and G is l-edge-connected, where

(72.22) $$l := \max_{U \subseteq V} d_R(U),$$

then the edge-disjoint paths problem has a half-integer solution (that is, for each $r \in R$ there exist paths P'_r and P''_r connecting the ends of r, such that each edge of G is in at most two of the paths P'_r, P''_r (over all $r \in R$)). She conjectures that here the condition $|T| \leq 6$ can be deleted.

72.3. Cut packing

By Theorem 70.5, Corollary 72.2a implies a fractional cut packing theorem. A stronger (integer) version of it was proved by Karzanov [1985b], which generalizes Theorem 71.3 (we follow the proof given in Schrijver [1991e]):

Theorem 72.6. *Let $G = (V, E)$ be a connected bipartite graph and let $H = (T, R)$ be a simple graph satisfying* (72.1)*, with $T \subseteq V$. Then G has disjoint cuts such that for each $st \in R$, $\mathrm{dist}_G(s, t)$ is equal to the number of these cuts separating s and t.*

Proof. Let $G = (V, E)$ be a counterexample with $|E|$ as small as possible. Define $d(u, v) := \mathrm{dist}_G(u, v)$ for $u, v \in V$. We first show:

(72.23) for each nonempty cut C there exist a pair $st \in R$ and an $s - t$ path P with $|EP \setminus C| \leq d(s, t) - 2$.

If not, contract all edges in C, giving graph G'. Then for all $st \in R$ we have

$$(72.24)\qquad d'(s', t') = \begin{cases} d(s, t) - 1 & \text{if } C \text{ separates } \{s, t\}, \\ d(s, t) & \text{if } C \text{ does not separate } \{s, t\}. \end{cases}$$

(Here and below, v' denotes the image of v in G, and d' denotes the distance function of G'.) As G is a smallest counterexample, G' has disjoint cuts $C_1, \ldots, C_t$ such that $d'(s', t')$ is equal to the number of cuts separating s' and t', for each $st \in R$. Together with C this gives, in the original graph G, cuts as required, by (72.24). This proves (72.23).

From (72.23) we derive:

(72.25) for all $u, w \in V$, there exists a pair $st \in R$ such that $\{s, t\} \cap \{u, w\} = \emptyset$ and such that
$$d(s, t) + d(u, w) \geq d(s, w) + d(u, t) \text{ and}$$
$$d(s, t) + d(u, w) \geq d(s, u) + d(w, t).$$

To prove this, let X be the set of vertices that are on at least one shortest $u - w$ path.

First, suppose that $X = V$. By (72.23), there exist $st \in R$ and an $s - t$ path P with $|EP \setminus \delta(u)| \leq d(s, t) - 2$. So P is a shortest $s - t$ path traversing u, and $u \neq s, t$. To see that $w \neq s, t$, suppose $w = t$, say. Then, as $d(u, w) = d(u, s) + d(s, w)$ (since $s \in X$),

$$\begin{aligned}(72.26)\qquad |EP \setminus \delta(u)| &= |EP| - 2 = d(s, u) + d(u, t) - 2 \\ &= d(s, u) + d(u, w) - 2 = 2d(s, u) + d(s, w) - 2 > d(s, w) - 2 \\ &= d(s, t) - 2,\end{aligned}$$

a contradiction. So $\{s,t\} \cap \{u,w\} = \emptyset$. Moreover,

(72.27) $$d(s,t) + d(u,w) = d(s,t) + d(u,s) + d(s,w) \geq d(s,w) + d(u,t).$$

One similarly shows the second inequality in (72.25).

Second, suppose that $X \neq V$. Let $C := \delta(X)$, and let G' be the graph obtained from G by contracting all edges in C. Then for each vertex x:

(72.28) $$d'(u',x') \geq d(u,x) - 1 \text{ and } d'(w',x') \geq d(w,x) - 1.$$

To see the first inequality, let P be a $u-x$ path in G with $|EP \setminus C| = d'(u',x')$. Choose P with $|EP \cap C|$ smallest. If the first inequality does not hold, then $|EP \cap C| \geq 2$. Then we can split P as $P'P''$ such that $|EP' \cap C| = 2$. Let P' connect u and v. As $|EP' \cap C| = 2$ and $u \in X$ we know $v \in X$. Since P' is not fully contained in X, we know that $|EP'| \geq d(u,v) + 2$. Let Q be a shortest $u-v$ path in G. Then $|EQ| = d(u,v) \leq |EP'| - 2$, and Q is fully contained in X. Let R be the concatenation QP''. Then $|ER \setminus C| \leq |EP \setminus C|$ and $|ER \cap C| = |EP \cap C| - 2$, contradicting the minimality of $|EP \cap C|$. This shows the first inequality in (72.28); the second inequality is proved similarly.

By (72.23), there exists $st \in R$ such that $d'(s',t') \leq d(s,t) - 2$. Then (72.28) implies $\{s,t\} \cap \{u,w\} = \emptyset$. Moreover, there exist a $v \in X$ and a shortest $s'-t'$ path in G' traversing v'. Hence

(72.29) $$\begin{aligned} d(s,t) + d(u,w) &\geq d'(s',t') + d(u,w) + 2 \\ &= d'(s',v') + d'(v',t') + d(u,v) + d(v,w) + 2 \\ &\geq d'(s',v') + d'(v',t') + d'(u',v') + d'(v',w') + 2 \\ &\geq d'(s',w') + d'(u',t') + 2 \geq d(s,w) + d(u,t). \end{aligned}$$

This gives the first inequality in (72.25); the second inequality is proved similarly.

(72.25) implies that for each pair $\{u,w\}$ of vertices of G there exists an $st \in R$ disjoint from $\{u,w\}$. So H is not the union of two stars, and hence $H = K_4$ or $H = C_5$ (up to isolated vertices, which we can ignore).

If $H = K_4$, let $T = \{r_1, r_2, r_3, r_4\}$. Then by (72.25):

(72.30) $$\begin{aligned} d(r_1,r_2) + d(r_3,r_4) &\geq d(r_1,r_3) + d(r_2,r_4) \geq d(r_1,r_4) + d(r_2,r_3) \\ &\geq d(r_1,r_2) + d(r_3,r_4). \end{aligned}$$

Hence we have equality throughout, that is

(72.31) $$d(t,u) + d(v,w) = d(t,v) + d(u,w) \text{ for all distinct } t,u,v,w \in T.$$

This implies that there exists a function $\phi : T \to \mathbb{R}_+$ such that $d(u,v) = \phi(u) + \phi(v)$ for each two distinct $u,v \in T$. (Indeed, let $\phi(v) := \frac{1}{2}(d(u,v) + d(v,w) - d(u,w))$ for arbitrary u,w with $v \neq u \neq w \neq v$. That this is independent of the choice of u,w follows from (72.31).)

Since all vertices are distinct, $d(u,v) > 0$ for all distinct $u,v \in T$, and so $\phi(v) > 0$ for at least one $v \in T$. By (72.23), there exist $st \in R$ and an $s-t$ path P such that $|EP \setminus \delta(v)| \leq d(s,t) - 2$. So P traverses v, and $|EP| = d(s,t) = \phi(s) + \phi(t)$. However,

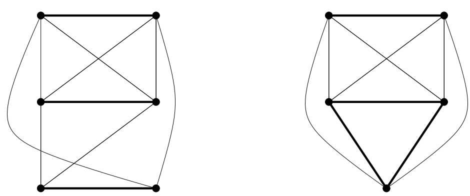
$$|EP| \geq d(s,v) + d(v,t) = \phi(s) + 2\phi(v) + \phi(t) > \phi(s) + \phi(t), \tag{72.32}$$

a contradiction.

If $H = C_5$, let $T = \{r_1, \ldots, r_5\}$ and $R = \{r_i r_{i+1} \mid i = 1, \ldots, 5\}$, taking indices mod 5. Applying (72.25) to $u := r_i$ and $w := r_{i+2}$, we obtain $st = r_{i+3}r_{i+4}$ (as it is the unique pair in R disjoint from $\{u, w\}$), and hence

$$\begin{aligned} d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) &\geq d(r_i, r_{i+3}) + d(r_{i+2}, r_{i+4}) \\ &\qquad (i = 1, \ldots, 5), \\ d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) &\geq d(r_i, r_{i+4}) + d(r_{i+2}, r_{i+3}) \\ &\qquad (i = 1, \ldots, 5). \end{aligned} \tag{72.33}$$

Adding up these ten inequalities, we obtain the same sum at both sides of the inequality sign. So we have equality in each of (72.33). This is equivalent to (72.31), and we obtain a contradiction in the same way as above. ∎



(a) (b)

Figure 72.5

The heavy lines are the edges of H, the other lines those of G. In both cases, G has no disjoint cuts such that for any edge r in H, the distance in G between the vertices in r is equal to the number of cuts separating them.

We cannot delete condition (72.1) in Theorem 72.6, as is shown by the examples given in Figure 72.5.

Notes. Karzanov [1985b] gave an $O(n^3)$ algorithm to find the cut packings of Theorem 72.6 (also for the weighted case). Theorem 72.2 can also be derived from Theorem 72.6, with the help of Theorems 70.5 and 70.7.

Karzanov [1990b] extended these cut packing results to packing $K_{2,3}$-metrics (cf. Section 72.2a):

(72.34) Let $G = (V, E)$ be a bipartite graph and let $T \subseteq V$ with $|T| = 5$. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\mathrm{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if $u, v \in T$.

Chapter 73

T-paths

We now go over to the problem of finding a maximum number of disjoint paths whose ends are two different vertices in a given set T of vertices — the T-paths. (So the nets are all pairs of distinct vertices in T.) Fundamental theorems of Mader imply min-max relations for this.

73.1. Disjoint T-paths

Let $G = (V, E)$ be a graph and let $T \subseteq V$. A path is called an *T-path* if its ends are distinct vertices in T and no internal vertex belongs to T.

Mader [1978c] gave a min-max formula for the maximum number of internally vertex-disjoint T-paths. It generalizes the undirected, vertex-disjoint version of Menger's theorem (by taking $|T| = 2$) and the Tutte-Berge formula (by adding to each vertex v of a graph G a copy v' of v and an edge vv'; taking for T the set of new vertices, the maximum number of internally vertex-disjoint T-paths is equal to the matching number of G).

As in Schrijver [2001], we derive Mader's theorem from a theorem of Gallai [1961], which we derive (as Gallai did) from matching theory (the Tutte-Berge formula):

Theorem 73.1 (Gallai's disjoint T-paths theorem)**.** *Let $G = (V, E)$ be an undirected graph and let $T \subseteq V$. The maximum number of disjoint T-paths is equal to the minimum value of*

$$(73.1) \qquad |U| + \sum_K \lfloor \tfrac{1}{2}|K \cap T| \rfloor$$

taken over $U \subseteq V$, where K ranges over the components of $G - U$.

Proof. The maximum is at most the minimum, since for each $U \subseteq V$, each T-path intersects U or has its ends in $K \cap T$ for some component K of $G - U$.

To see equality, let μ be equal to the minimum value of (73.1). Let the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ arise from G by adding a disjoint copy G' of $G - T$, and making the copy v' of each $v \in V \setminus T$ adjacent to v and to all neighbours of v in G. By the Tutte-Berge formula (Theorem 24.1), $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. To see this, we must prove that for any $\widetilde{U} \subseteq \widetilde{V}$:

$$|\widetilde{U}| + \sum_{\widetilde{K}} \lfloor \tfrac{1}{2} |\widetilde{K}| \rfloor \geq \mu + |V \setminus T|, \tag{73.2}$$

where $\widetilde{K}$ ranges over the components of $\widetilde{G} - \widetilde{U}$. Now if for some $v \in V \setminus T$ exactly one of v, v' belongs to $\widetilde{U}$, then we can delete it from $\widetilde{U}$, thereby not increasing the left-hand side of (73.2).

So we can assume that for each $v \in V \setminus T$, either $v, v' \in \widetilde{U}$ or $v, v' \notin \widetilde{U}$. Define $U := \widetilde{U} \cap V$. Then each component K of $G - U$ is equal to $\widetilde{K} \cap V$ for some component $\widetilde{K}$ of $\widetilde{G} - \widetilde{U}$. Hence

$$|\widetilde{U}| + \sum_{\widetilde{K}} \lfloor \tfrac{1}{2} |\widetilde{K}| \rfloor = |U| + \sum_{K} \lfloor \tfrac{1}{2} |K \cap T| \rfloor + |V \setminus T| \geq \mu + |V \setminus T|, \tag{73.3}$$

where K ranges over the components of $G - U$. Thus we have (73.2).

So $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. Let N be the matching $\{vv' \mid v \in V \setminus T\}$ in $\widetilde{G}$. As $|M| = \mu + |V \setminus T| = \mu + |N|$, the union $M \cup N$ has at least μ components with more edges in M than in N. Each such component is a path connecting two vertices in T. Then contracting the edges in N yields μ disjoint T-paths in G. ∎

We now derive Mader's theorem. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be a collection of disjoint subsets of V. A path in G is called an *$\mathcal{S}$-path* if it connects two different sets in $\mathcal{S}$ and has no internal vertex in any set in $\mathcal{S}$. Denote $T := \bigcup \mathcal{S}$.

Theorem 73.2 (Mader's disjoint $\mathcal{S}$-paths theorem)**.** *The maximum number of disjoint $\mathcal{S}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2} |B_i| \rfloor, \tag{73.4}$$

taken over all partitions $U_0, \ldots, U_n$ of V such that each $\mathcal{S}$-path intersects U_0 or traverses some edge spanned by some U_i. Here B_i denotes the set of vertices in U_i that belong to T or have a neighbour in $V \setminus (U_0 \cup U_i)$.

Proof. Let μ be the minimum value of (73.4). Trivially, the maximum number of disjoint $\mathcal{S}$-paths is at most μ, since any $\mathcal{S}$-path disjoint from U_0 and traversing an edge spanned by U_i, traverses at least two vertices in B_i.

Fixing V, choose a counterexample E, $\mathcal{S}$ minimizing

$$|E| - |\{\{x, y\} \mid x, y \in V, \exists X, Y \in \mathcal{S} : x \in X, y \in Y, X \neq Y\}|. \tag{73.5}$$

Then each $X \in \mathcal{S}$ is a stable set of G, since deleting any edge e spanned by X does not change the maximum and minimum value in Mader's theorem (as no $\mathcal{S}$-path traverses e and as deleting e does not change any set B_i), while it decreases (73.5).

If $|X| = 1$ for each $X \in \mathcal{S}$, the theorem reduces to Gallai's disjoint T-paths theorem (Theorem 73.1): we can take for U_0 any set U minimizing (73.1), and for $U_1, \ldots, U_n$ the components of $G - U$.

So $|X| \geq 2$ for some $X \in \mathcal{S}$. Choose $s \in X$. Define

$$\mathcal{S}' := (\mathcal{S} \setminus \{X\}) \cup \{X \setminus \{s\}, \{s\}\}. \tag{73.6}$$

Replacing $\mathcal{S}$ by $\mathcal{S}'$ decreases (73.5), but it does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}'$-path and as $\bigcup \mathcal{S}' = T$). Hence there exists a collection $\mathcal{P}$ of μ disjoint $\mathcal{S}'$-paths.

Necessarily, there is a path $P_0 \in \mathcal{P}$ connecting s with another vertex in X (otherwise $\mathcal{P}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{P}$ are $\mathcal{S}$-paths. Let u be an internal vertex of P_0 (u exists, since X is a stable set). Define

$$\mathcal{S}'' := (\mathcal{S} \setminus \{X\}) \cup \{X \cup \{u\}\}. \tag{73.7}$$

Replacing $\mathcal{S}$ by $\mathcal{S}''$ decreases (73.5), but it does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}''$-path and as $\bigcup \mathcal{S}'' \supseteq T$). So there exists a collection $\mathcal{Q}$ of μ disjoint $\mathcal{S}''$-paths. Choose $\mathcal{Q}$ such that $\mathcal{Q}$ uses a minimal number of edges not used by $\mathcal{P}$.

Necessarily, u is an end of some path $Q_0 \in \mathcal{Q}$ (otherwise $\mathcal{Q}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{Q}$ are $\mathcal{S}$-paths. As $|\mathcal{P}| = |\mathcal{Q}|$ and as u is not an end of any path in $\mathcal{P}$, there exists an end r of some path $P \in \mathcal{P}$ that is not an end of any path in $\mathcal{Q}$.

Then P intersects some path in $\mathcal{Q}$ (otherwise $(\mathcal{Q} \setminus \{Q_0\}) \cup \{P\}$ would form μ disjoint $\mathcal{S}$-paths). So when following P starting from r, there is a first vertex w that is on some path in $\mathcal{Q}$, say on $Q \in \mathcal{Q}$. Let Q be split at w into subpaths Q' and Q'' say (possibly of length 0). Let P' be the $r - w$ part of P.

If $EQ' \not\subseteq EP$ and $EQ'' \not\subseteq EP$, we may assume that r is not in the same class of $\mathcal{S}''$ as the end of Q' is. Then after replacing part Q'' of Q by P', Q remains an $\mathcal{S}''$-path disjoint from the other paths in $\mathcal{Q}$. This contradicts our minimality assumption on $\mathcal{Q}$.

So we can assume that $EQ' \subseteq EP$. If $P \neq P_0$, then after resetting Q to P, Q remains an $\mathcal{S}''$-path disjoint from the other paths in $\mathcal{Q}$. Again this contradicts our minimality assumption on $\mathcal{Q}$.

So $P = P_0$, and hence (since $EQ' \subseteq EP$) we have $Q = Q_0$. Then replacing part Q' of Q by P', we obtain μ disjoint $\mathcal{S}$-paths as required. ∎

(The case splitting finishing this proof is due to A. Frank (personal communication 2002).)

Theorem 73.2 is equivalent to the original form of Mader's theorem on internally vertex-disjoint T-paths (instead of fully disjoint $\mathcal{S}$-paths), which reads as follows.

For any graph G, let $B_G(U)$ denote the set of vertices in U having a neighbour that is not in U.

Corollary 73.2a (Mader's internally disjoint T-paths theorem). *Let $G = (V, E)$ be a graph and let T be a stable subset of V. Then the maximum number of internally vertex-disjoint T-paths is equal to the minimum value of*

$$(73.8) \qquad |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2} |B_{G-U_0}(U_i)| \rfloor,$$

where $U_0, U_1, \ldots, U_n$ partition $V \setminus T$ such that each T-path intersects U_0 or traverses some edge spanned by some U_i.

Proof. Trivially, the maximum is not more than the minimum (since each T-path not intersecting U_0 traverses at least two vertices in some U_i, hence it traverses at least two vertices in U_i that have a neighbour out of $U_i \cup U_0$).

To see equality, we can assume that no two vertices in T have a common neighbour v. Otherwise we can apply induction by deleting v, which reduces both the maximum and the minimum by 1.

Now the present corollary follows from Theorem 73.2 applied to $G - T$ and the collection $\mathcal{S} := \{N(s) \mid s \in T\}$. ∎

Mader's internally disjoint T-paths theorem in turn implies the edge-disjoint version, proved by Mader [1978b]:

Corollary 73.2b (Mader's edge-disjoint T-paths theorem). *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the maximum number of edge-disjoint T-paths is equal to the minimum value of*

$$(73.9) \qquad \tfrac{1}{2}\Big(\sum_{s \in T} d_E(X_s) - \kappa\Big),$$

where the X_s are disjoint sets with $s \in X_s$ (for $s \in T$), and where κ denotes the number of components K of the graph $G - \bigcup_{s \in T} X_s$ with $d_E(K)$ odd.

Proof. Let t be the maximum number of edge-disjoint T-paths. It is easy to see that t cannot exceed the minimum value of (73.9).

To see equality, first observe that, if G has an edge e such that by deleting e, the maximum drops by 1, we can apply induction on $|E|$, since the minimum drops by at most 1.

So we can assume that no such edge exists. We make an auxiliary graph $G' = (V', E')$ as follows. For each $u \in V \setminus T$, let W_u be a stable set of size $3t + 1$. For each edge $e \in E$, let v_e be a new vertex. Let v_e be adjacent to all vertices in W_u if $u \in e$, and to $s \in T$ if $s \in e$. This defines the graph G' (with vertex set $V' = T \cup \{v_e \mid e \in E\} \cup \bigcup_{u \in V \setminus T} W_u$).

Then t is equal to the maximum number of disjoint T-paths in G'. Hence, by Corollary 73.2a, there exist disjoint subsets $U_0, \ldots, U_n$ of $V' \setminus T$ such that

(73.10) each T-path in G' intersects U_0 or traverses an edge spanned by some U_i, and such that

$$(73.11) \qquad t \geq |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor,$$

where B_i is the set of vertices in U_i having a neighbour in $V' \setminus (U_0 \cup U_i)$.

By our assumption that the maximum does not drop by deleting any edge e, we know that U_0 contains no vertex v_e.

We may assume that $|B_i| \geq 2$ for each i, since if $|B_i| \leq 1$, we can delete U_i, as no T-path in G' avoiding U_0 traverses any edge spanned by U_i. This implies that $|B_i| \leq 3\lfloor \frac{1}{2}|B_i| \rfloor$, and hence

$$(73.12) \qquad |U_0| + \sum_{i=1}^{n} |B_i| \leq 3(|U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor) < 3t + 1.$$

So for each $u \in V \setminus T$, there exists a $w_u \in W_u$ with $w_u \notin U_0 \cup B_1 \cup \cdots \cup B_n$. For $u \in T$, let $w_u := u$.

For each $i = 1, \ldots, n$, let $Y_i := \{u \in V \setminus T \mid w_u \in U_i\}$ and let E_i be the set of edges $e \in E$ with $v_e \in B_i$. Then

$$(73.13) \qquad \delta_E(Y_i) \subseteq E_i$$

for each $i = 1, \ldots, n$. To see this, let $e \in \delta_E(Y_i)$, with $e = uv$ and $u \in Y_i$ and $v \notin Y_i$. Then $u \in Y_i$ implies $w_u \in U_i$. Hence $w_u \in U_i \setminus B_i$, implying $v_e \in U_i$. As $v \notin Y_i$, we know $w_v \notin U_i$. Hence v_e has a neighbour out of $U_0 \cup U_i$, and so $v_e \in B_i$. Therefore, $e \in E_i$, proving (73.13).

Hence no edge of G connects two different Y_i and Y_j (since $E_i \cap E_j = \emptyset$). Suppose now that $G - Y_1 - \cdots - Y_n$ contains a T-path P. Route P as a T-path P' in G', by replacing each edge of P by v_e and any $u \in V \setminus T$ by w_u. Then P' is disjoint from U_0. So P' traverses an edge spanned by some U_i. Then P' traverses a vertex $w_u \in U_i$ for some $u \in V \setminus T$. Hence P traverses Y_i, a contradiction.

For $s \in T$, let X_s be the set of vertices of G reachable in $G - Y_1 - \cdots - Y_n$ from s. Then we have

$$(73.14) \qquad t \geq \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|E_i| \rfloor \geq \tfrac{1}{2}\big(\sum_{s \in T} d_E(X_s) - \kappa\big),$$

where κ is the number of components K of $G - \bigcup_{s \in T} X_s$ with $d_E(K)$ odd. ∎

(This min-max formula was proved also in an unpublished manuscript of Lomonosov [1978b].)

73.1a. Disjoint T-paths with the matroid matching algorithm

As Lovász [1980a] showed, Mader's theorem can be derived from matroid matching theory, and also a polynomial-time algorithm to find a maximum packing of T-paths follows from it. We restrict ourselves to deriving polynomial-time solvability, and consider the equivalent problem of finding a maximum packing of $\mathcal{S}$-paths.

Let $G = (V, E)$ be a graph and let $S_1, \ldots, S_k$ be disjoint subsets of V. Let $\mathcal{S} := \{S_1, \ldots, S_k\}$ and $T := S_1 \cup \cdots \cup S_k$.

We can assume that each S_i is a stable set. Consider the linear space $(\mathbb{R}^2)^V$, considered as the set of functions $x : V \to \mathbb{R}^2$. For each edge $e = uw$ of G, let L_e be the linear subspace of $(\mathbb{R}^2)^V$ given by:

(73.15) $$L_e := \{x \in (\mathbb{R}^2)^V \mid x(v) = \mathbf{0} \text{ for each } v \in V \setminus \{u, w\}, x(u) + x(w) = \mathbf{0}\}.$$

So $\dim L_e = 2$.

Choose distinct 1-dimensional subspaces $l_1, \ldots, l_k$ of $\mathbb{R}^2$. For each $v \in V$, let $L_v := l_i$ if $v \in S_i$ for some i, and $L_v := \{\mathbf{0}\}$ otherwise. Define

(73.16) $$Q := \{x \in (\mathbb{R}^2)^V \mid \forall v \in V : x(v) \in L_v\}.$$

Let $\mathcal{E}$ be the collection of subspaces L_e/Q (for $e \in E$) of $(\mathbb{R}^2)^V/Q$. Then $\dim(L_e/Q) = 2$ for each edge e, since e connects no two vertices in the same S_i (so $L_e \cap Q = \{\mathbf{0}\}$).

For any $F \subseteq E$, let $\mathcal{L}_F$ denote the corresponding collection of lines in $\mathcal{E}$:

(73.17) $$\mathcal{L}_F := \{L_e/Q \mid e \in F\}.$$

We show that for each $F \subseteq E$:

(73.18) $\mathcal{L}_F$ is a matching if and only if F is a forest such that each component of (V, F) has at most two vertices in common with T, and at most one with each S_i.

Let $X := \sum(L_e \mid e \in F)$. Then one easily checks that X consists of all $x : V \to \mathbb{R}^2$ with $\sum_{v \in K} x(v) = \mathbf{0}$ for each component K of (V, F). So $\dim(X) = 2(|V| - \kappa)$, where κ is the number of components of (V, F). Also, $\dim(X \cap Q) = 0$ if and only if each component of (V, F) has at most two vertices in common with T, and at most one with each S_i. Now

(73.19) $$\dim(\mathcal{L}_F) = \dim(X/Q) = \dim(X) - \dim(X \cap Q) \leq \dim(X) \leq 2|F|.$$

Hence $\mathcal{L}_F$ is a matching if and only if $\dim(X) = 2|F|$ and $\dim(X \cap Q) = 0$. By the previous this gives (73.18).

(73.18) then implies the following relation to $\mathcal{S}$-paths:

(73.20) if G is connected, the maximum number of disjoint $\mathcal{S}$-paths is equal to $\nu(\mathcal{E}) - |V| + |T|$.

To see this, let t be the maximum number of disjoint $\mathcal{S}$-paths. Let Π form a packing of t $\mathcal{S}$-paths and let F' be the set of edges contained in these paths. Extend F' to a forest F such that each component of (V, F) contains either a unique path in Π or a unique vertex in T. Then F satisfies the condition given in (73.18), and $|F| = t + |V| - |T|$. So $\mathcal{L}_F$ forms a matching of size $t + |V| - |T|$, and hence $\nu(\mathcal{E}) \geq t + |V| - |T|$.

Conversely, let $\mathcal{F} \subseteq \mathcal{E}$ be a matching of size $\nu(\mathcal{E})$. Then $\mathcal{F} = \mathcal{L}_F$ for some forest $F \subseteq E$ satisfying the condition in (73.18). Let t be the number of components of (V, F) intersecting T twice. Then deleting t edges from F, we obtain a forest such that each component intersects T at most once. So $|F| - t \leq |V| - |T|$, and hence $t \geq \nu(\mathcal{E}) - |V| + |T|$. This shows (73.20).

Theorem 43.4 implies with (73.20) the polynomial-time solvability of finding a maximum packing of $\mathcal{S}$-paths:

(73.21) Given a graph $G = (V, E)$ and a collection $\mathcal{S}$ of disjoint subsets of V, a maximum number of disjoint $\mathcal{S}$-paths can be found in polynomial time.

73.1b. Polynomial-time findability of edge-disjoint T-paths

J.C.M. Keijsper, R.A. Pendavingh, and L. Stougie (personal communication 2000) showed that with the ellipsoid method one can derive from Mader's edge-disjoint T-paths theorem (Corollary 73.2b) that a maximum number of edge-disjoint T-paths can be found in polynomial time.

To see this, let $G = (V, E)$ be a graph and let $T \subseteq V$. Consider the polyhedron P in $\mathbb{R}^E$ consisting of all $x \in \mathbb{R}^E$ satisfying

(73.22)
$$\begin{array}{rlll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(U)) \leq |\delta(U)| - 1 & \text{for each } U \subseteq V \setminus T \text{ with } |\delta(U)| \text{ odd}, \\ \text{(iii)} & x(\delta(s)) \leq x(\delta(X)) & \text{for each } s \in T \text{ and } X \subseteq V \\ & & \text{with } X \cap T = \{s\}. \end{array}$$

These conditions can be tested in polynomial time: (i) is easy (one by one). To test (ii), let G' be the graph obtained from G by contracting T to one vertex. Moreover, define $T' := \{v \in VG' \mid \deg_{G'}(v) \text{ odd}\}$. Define a capacity function c by $c_e := 1 - x_e$ for $e \in E$. Then (ii) is valid if and only if the minimum capacity of a T'-cut in G' is at least 1. This can be tested in polynomial time (Corollary 29.6a). Finally, testing (iii) amounts to finding a cut separating s and $T \setminus \{s\}$ of minimum capacity, taking x as capacity function.

So by the ellipsoid method, we can optimize any linear function over P in polynomial time. Now the maximum value λ of

(73.23)
$$\tfrac{1}{2} \sum_{s \in T} x(\delta(s))$$

over $x \in P$ is equal to the maximum number μ of edge-disjoint T-paths in G.

The inequality $\lambda \geq \mu$ follows from the fact that the incidence vectors of μ edge-disjoint T-paths sum up to a vector x satisfying (73.22) and having (73.23) equal to μ.

To see equality, by Corollary 73.2b there exist disjoint sets X_s ($s \in T$) such that $s \in X_s$ and such that

(73.24)
$$\mu = \tfrac{1}{2}\big(\sum_{s \in T} d_E(X_s) - \kappa\big),$$

where κ denotes the number of components K of the graph $G' := G - \bigcup_{s \in T} X_s$ with $d(K)$ odd. This implies a dual solution of the linear program defining λ, of value at most μ. Indeed, let x attain the maximum value of (73.23) over P. Let $W := VG'$, let $\mathcal{K}$ be the collection of components of G', and let F be the set of edges connecting different sets X_s. Then

(73.25)
$$\begin{aligned} 2\lambda &= \sum_{s \in T} x(\delta(s)) \leq \sum_{s \in T} x(\delta(X_s)) = 2x(F) + x(\delta(W)) \\ &= 2x(F) + \sum_{K \in \mathcal{K}} x(\delta(K)) \leq 2|F| + \sum_{K \in \mathcal{K}} x(\delta(K)) \\ &\leq 2|F| + \sum_{K \in \mathcal{K}} 2\lfloor \tfrac{1}{2} d_E(K) \rfloor = \sum_{s \in T} d_E(X_s) - \kappa = 2\mu. \end{aligned}$$

Concluding, we have $\lambda = \mu$.

This implies that μ can be determined in polynomial time. The paths can be found explicitly by iteratively deleting edges if it does not reduce μ. Similarly, we can replace pairs of adjacent edges uv, vw by one edge uw, if it does not reduce μ. We end up with a graph with μ edges spanned by T. Working our way back, we find the required paths in the original graph.

This approach can be extended to obtain a strongly polynomial-time algorithm for the capacitated case, where each edge e has an integer capacity $c(e)$ and we want to find a maximum number of T-paths such that each edge e is contained in at most $c(e)$ of them.

73.1c. A feasibility characterization for integer K_3-flows

Seymour [1980b] showed that Corollary 73.2b also implies the following feasibility characterization for integer K_3-flows (we follow the formulation and proof given by Frank [1990e]). The cut condition is applied to $R = \{s_1s_2, s_1s_3, s_2s_3\}$ with demand $d(s_is_j) = d_{i,j}$, and capacity $\mathbf{1}$:

Corollary 73.2c. *Let $G = (V, E)$ be a graph, let $s_1, s_2, s_3 \in V$, and let $d_{1,2}, d_{1,3}, d_{2,3} \in \mathbb{Z}_+$. Then there exists a collection of edge-disjoint paths such that $d_{i,j}$ of them connect s_i and s_j ($1 \leq i < j \leq 3$), if and only if the cut condition holds and*

(73.26) $$s(U_1) + s(U_2) + s(U_3) \geq \kappa$$

for each choice of disjoint sets U_1, U_2, U_3 with $s_i \in U_i$ ($i = 1, 2, 3$). Here $s(X) := |\delta_E(X)| - d(\delta_R(X))$ and κ is number of components K of $G - U_1 - U_2 - U_3$ with $|\delta_E(K)|$ odd.

Proof. Necessity is easy: we have

(73.27) $$d(\delta_R(U_1))+d(\delta_R(U_2))+d(\delta_R(U_3)) \leq |\delta_E(U_1)|+|\delta_E(U_2)|+|\delta_E(U_3)|-\kappa,$$

since from any component K of $G - U_1 - U_2 - U_3$ with $|\delta_E(K)|$ odd, we cannot use all edges of $\delta_E(K)$.

Next we show sufficiency. We can assume that G is connected. For $i = 1, 2, 3$, extend G by a new vertex r_i and $k_i := \deg_R(s_i)$ parallel edges connecting r_i and s_i. Let $G' = (V', E')$ be the extended graph, and let $T := \{r_1, r_2, r_3\}$. It suffices to show that G' has $d_{1,2} + d_{1,3} + d_{2,3} = \frac{1}{2}(k_1 + k_2 + k_3)$ edge-disjoint T-paths (since then $\frac{1}{2}(k_1 + k_2 - k_3) = d_{1,2}$ of them connect r_1 and r_2; similarly for $d_{1,3}$ and $d_{2,3}$). For this we can invoke Corollary 73.2b. Hence suppose to the contrary that there exist three disjoint subsets X_1, X_2, X_3 of V' with $r_i \in X_i$ ($i = 1, 2, 3$) such that

(73.28) $$\sum_{i=1}^{3} |\delta_{E'}(X_i)| - \kappa < k_1 + k_2 + k_3,$$

where κ denotes the number of components K of the graph $G' - X_1 - X_2 - X_3$ with $d_{E'}(K)$ odd.

We can assume that each X_i induces a connected subgraph of G'. For suppose that L is a component of $G'[X_1]$ not containing r_1. Let $X_1' := X_1 \setminus L$, and let κ' be the number of components K of $G' - X_1' - X_2 - X_3$ with $|\delta_{E'}(K)|$ odd. Then $\kappa' \geq \kappa - |\delta_{E'}(L)|$, and hence

(73.29) $$|\delta_{E'}(X_1')| = |\delta_{E'}(X_1)| - |\delta_{E'}(L)| \leq |\delta_{E'}(X_1)| - \kappa + \kappa'.$$

So replacing X_1 by X'_1 preserves (73.28).

If $s_i \in X_i$ for $i = 1, 2, 3$, then $U_i := X_i \setminus \{r_i\}$ for $i = 1, 2, 3$ would violate (73.26). So we can assume that $s_3 \notin X_3$, and so $X_3 = \{r_3\}$.

Then we can assume that $G - X_1 - X_2 - X_3$ has only one component. Otherwise it has a component L not containing s_3, and so L is not connected to X_3. We can assume that $|E'[L, X_1]| \geq |E'[L, X_2]|$. Let $X'_1 := X_1 \cup L$ and let κ' be the number of components K of $G' - X'_1 - X_2 - X_3$ with $|\delta_{E'}(K)|$ odd. Then $|\delta_{E'}(X'_1)| \leq |\delta_{E'}(X_1)| - \kappa + \kappa'$, and so replacing X_1 by X'_1 preserves (73.28).

So we may assume $\kappa \leq 1$, and hence, as by parity the left-hand side in (73.28) is even, we obtain the contradiction

(73.30) $$k_1 + k_2 + k_3 \geq \sum_{i=1}^{3} |\delta_{E'}(X_i)| - \kappa + 2 > \sum_{i=1}^{3} |\delta_{E'}(X_i)| \geq k_1 + k_2 + k_3,$$

where the last inequality follows from the cut condition. ∎

A polynomial-time algorithm to find a circuit traversing three prescribed vertices in an undirected graph, was given by LaPaugh and Rivest [1978,1980].

73.2. Fractional packing of T-paths

If all vertices not in T have even degree, Mader's edge-disjoint T-paths theorem (Corollary 73.2b) reduces to the following result of Cherkasskiĭ [1977b] and Lovász [1976b] (thus answering a question of Kupershtokh [1971]):

Corollary 73.2d. *Let $G = (V, E)$ be a graph and let $T \subseteq V$, with $\deg_G(v)$ even for each $v \in V \setminus T$. Then the maximum number of edge-disjoint T-paths in G is equal to*

(73.31) $$\tfrac{1}{2} \sum_{s \in T} \gamma_G(s).$$

Here $\gamma_G(s)$ denotes the minimum size of a cut in G separating s and $T \setminus \{s\}$.

Proof. Directly from Corollary 73.2b, since $\kappa = 0$. ∎

This corollary has the following consequence on multiflows, also due to Cherkasskiĭ [1977b] (the fractional version was stated, with incorrect proof, by Kupershtokh [1971]):

Corollary 73.2e. *Let $G = (V, E)$ be a graph, let $T \subseteq V$, and let $c : E \to \mathbb{R}_+$ be a capacity function. Then the maximum total value of a multiflow for the nets $\{st \mid s, t \in T, s \neq t\}$ is equal to*

(73.32) $$\tfrac{1}{2} \sum_{s \in T} \gamma_c(s),$$

where $\gamma_c(s)$ denotes the minimum capacity of a cut separating s and $T \setminus \{s\}$. If all capacities are integer there is a half-integer maximum-value multiflow. If moreover $c(\delta(v))$ is even for each $v \in V \setminus T$, there is an integer maximum-value multiflow.

Proof. By continuity and compactness, we can assume that c is integer and that $c(\delta(v))$ is even for each $v \in V \setminus T$.

Replacing each edge e by $c(e)$ parallel edges we obtain a graph to which we can apply Corollary 73.2d. The paths obtained in the new graph give an integer multiflow as required in the original graph. ∎

Notes. Karzanov [1979a] gave an $O(\mathrm{MF}(n,m) \cdot \log |T|)$ algorithm to find a half-integer maximum-value multiflow for integer c. ($\mathrm{MF}(n,m)$ is the time needed to find a maximum flow in a digraph with n vertices and m arcs.) Ibaraki, Karzanov, and Nagamochi [1998] extended this algorithm to obtain an integer solution if $c(\delta_E(v))$ is even for each $v \in V \setminus T$. They also gave an extension to directed graphs.

Lovász [1976b] mentioned the following consequence of Corollary 73.2b:

(73.33) Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{Z}_+$ be a capacity function with $c(\delta(v))$ even for each $v \in V$. Then for each $u \in V$, the maximum number of circuits in G that traverse u, such that no edge e is in more than $c(e)$ of these circuits, is equal to half of the minimum capacity of a family of edges meeting each circuit through u at least twice.

To prove this, let $s_1, \ldots, s_d$ be the neighbours of u. Replace u by d new vertices $u_1, \ldots, u_d$, and for each $i = 1, \ldots, d$, add $c(us_i)$ parallel edges connecting u_i and s_i. Moreover, replace each edge e of G not containing u by $c(e)$ parallel edges. Then the assertion follows from Corollary 73.2b applied to the new graph and to $T := \{u_1, \ldots, u_d\}$.

73.2a. Direct proof of Corollary 73.2d

Let G, T form a counterexample with $|V| + |E|$ as small as possible. Let μ be equal to (73.31). Then:

(73.34) for any $s \in T$ and any minimum-size cut $\delta(U)$ separating s and $T \setminus \{s\}$, with $U \cap T = \{s\}$, one has $U = \{s\}$.

To see this, suppose $U \neq \{s\}$. Contract U to one vertex, s' say, obtaining graph G'. Let $T' := (T \setminus \{s\}) \cup \{s'\}$. By the minimality of G, G' contains μ' edge-disjoint T'-paths, where μ' equals (73.31) for G', T'. Each edge in $\delta(U)$ belongs to one of these T'-paths (as in G' it is a minimum-size cut separating s' and $T' \setminus \{s'\}$). Let G'' be the graph obtained from G by contracting $V \setminus U$ to one new vertex, u say. By the minimality of $\delta(U)$, G'' contains $d_E(U)$ edge-disjoint $s - u$ paths (by Menger's theorem). By concatenation, we find μ' edge-disjoint T-paths in G. As $\mu' \geq \mu$, this contradicts the fact that G is a counterexample. This proves (73.34).

As G, T form a counterexample, there is at least one vertex $v \in V \setminus T$ with at least two different neighbours. Let uv and vw be two of the edges incident with v, with $u \neq w$. Replacing these two edges by one new edge uw, we obtain a graph G'''. As G is a counterexample, G''' has no μ edge-disjoint T-paths. As G''' is smaller than G, it is no counterexample, and so there is an $s \in T$ with $\gamma_{G'''}(s) < \gamma_G(s)$. Hence there is a $U \subseteq V$ with $U \cap T = \{s\}$ and $d_{G'''}(U) < \gamma_G(s)$. Then, by parity, $d_{G'''}(U) \leq \gamma_G(s) - 2$, and hence $d_G(U) \leq \gamma_G(s)$. So by (73.34), $U = \{s\}$. Hence $d_G(U) = d_{G'''}(U) < \gamma_G(s)$, a contradiction.

By similar methods one may prove an analogous result for *directed graphs*, due to M.V. Lomonosov (cf. Karzanov [1979b]) and Frank [1989]: Given a digraph $D = (V, A)$ and $T \subseteq V$, call a directed path P an *T-path* if its end vertices are distinct and belong to T, and no internal vertex of P belongs to T. Then, if D is Eulerian, the maximum number of edge-disjoint T-paths in G is equal to the minimum value of

(73.35) $$\sum_{s\in T} d_A^{\text{out}}(X_s),$$

taken over disjoint sets $X_s \subseteq V$ with $s \in X_s$ for $s \in T$.

73.3. Further results and notes

73.3a. Further notes on Mader's theorem

In general it is not true that given any subset T of the vertex set of a graph, the maximum number M of edge-disjoint T-paths is equal to the minimum size m of an edge set intersecting each T-path: the complete bipartite graph $K_{t,n}$, with t odd and T the colour class with t vertices, has $M = \frac{1}{2}n(t-1)$ and $m = n(t-1)$. Mader's edge-disjoint T-paths theorem (Corollary 73.2b) implies the conjecture of Gallai [1961] (cf. Lovász [1976b]) that $M \geq \frac{1}{2}m$ for any graph. (Lovász [1976b] showed that $M \geq \frac{1}{4}m$, and P.D. Seymour (personal communication 1977) that $M \geq \frac{1}{3}m$.)

For Eulerian graphs, Corollary 73.2d implies the sharper inequality

(73.36) $$M \geq \frac{t}{2(t-1)}m,$$

where $t := |T|$. Indeed, for each $s \in T$, let E_s be a minimum-size $s - T \setminus s$ cut. Let E_t have the largest size among them. Then $\bigcup_{s\neq t} E_s$ intersects each T-path. Hence

(73.37) $$m \leq |\bigcup_{s\neq t} E_s| \leq \sum_{s\neq t} |E_s| \leq (1 - \frac{1}{t})\sum_{s\in T} |E_s| = \frac{t-1}{t}2M.$$

This proves Gallai's conjecture that $M \geq \frac{1}{2}m$ for Eulerian graphs.

The graph which arises from the complete bipartite graph $K_{t,n}$ by replacing each edge by two parallel edges, with T the colour class with t elements, has $M = tn$ and $m = 2(t-1)n$. So inequality (73.36) is sharp for Eulerian graphs.

Gallai [1961] derived, from matching theory, the following on edge-disjoint paths with both ends in T (not necessarily distinct). Let $G = (V, E)$ be a graph and $T \subseteq V$. Call a path a *weak T-path* if it has length at least 1, and connects two (not necessarily distinct) vertices in T, while no internal vertex belongs to T. For any $U \subseteq V$, let $\mathcal{K}_U$ denote the set of components of $G - U$. Then the maximum number of edge-disjoint weak T-paths is equal to the minimum value of

(73.38) $$|E[U]| + \sum_{K\in\mathcal{K}_U} \lfloor\frac{d_E(K)}{2}\rfloor,$$

over U with $T \subseteq U \subseteq V$. The maximum number of internally vertex-disjoint weak T-paths is equal to the minimum value of

$$(73.39) \qquad |E[U]| + |W \setminus U| + \sum_{K \in \mathcal{K}_W} \lfloor \frac{d_E(K)}{2} \rfloor,$$

over U, W satisfying $T \subseteq U \subseteq W \subseteq V$.

A min-max relation and a polynomial-time algorithm for the minimum cost of a maximum collection of edge-disjoint T-paths were given by Karzanov [1993,1997]. A corresponding polyhedron was described by Burlet and Karzanov [1998].

Nash-Williams [1961a] gave necessary and sufficient conditions for a graph $G = (V, E)$ and a function $g : V \to \mathbb{Z}_+$ such that the edges of G can be partitioned into (nonclosed) paths such that $g(v)$ of these paths end at v, for each $v \in V$.

More on Mader's theorem can be found in Mader [1989], and on Gallai's theorem in Mader [1980].

73.3b. A generalization of fractionally packing T-paths

The following theorem was announced by Karzanov and Lomonosov [1978] and proved by Karzanov [1985d,1987d] and Lomonosov [1985] (the latter paper does not consider the parity case). Taking H to be a complete graph we obtain Corollary 73.2d.

Theorem 73.3. *Let $G = (V, E)$ and $H = (T, R)$ be graphs, where H is the complement of the line graph of some triangle-free graph H_0. Let $c : E \to \mathbb{Z}_+$ be a capacity function. Then there exists a quarter-integer maximum-value multiflow. If H_0 is bipartite, there exists a half-integer maximum-value multiflow. If $c(\delta(v))$ is even for each $v \in V \setminus T$, there exists a half-integer maximum-value multiflow. If $c(\delta(v))$ is even for each $v \in V$ and H_0 is bipartite, there exists an integer maximum-value multiflow.*

(For the special case where H is the union of two complete bipartite graphs H' and H'' such that $VH' \subseteq VH''$ or such that $H' = K_2$, Cherkasskiĭ [1976] showed that the maximum multiflow is attained by a half-integer multiflow (for integer capacities).)

Related is the following characterization of the maximum value of a multiflow, announced by Karzanov and Lomonosov [1978], and proved by Karzanov [1979b, 1985d,1987d] and Lomonosov [1985]:

Theorem 73.4. *Let $G = (V, E)$ and $H = (T, R)$ be graphs, where H is the complement of the line graph of some triangle-free graph. Let $c : E \to \mathbb{R}_+$ be a capacity function. Let $\mathcal{U}$ denote the collection of subsets U of V such that $U \cap T$ is a stable set of H. Then the maximum total value of a multiflow subject to c is equal to the minimum value of*

$$(73.40) \qquad \sum_U \lambda_U c(\delta_G(U))$$

taken over $\lambda : \mathcal{U} \to \mathbb{R}_+$ satisfying

$$(73.41) \qquad \sum_U \lambda_U \chi^{\delta_R(U)} \geq \mathbf{1}_R.$$

It implies that if H is the complement of the line graph of some triangle-free graph, then in Theorem 70.2 one can restrict the length functions l to nonnegative combinations of cut functions. Karzanov and Pevzner [1979] showed that if H is not the line graph of a triangle-free graph, then Theorem 73.4 does not hold for some G, c.

Karzanov [1987b,1989] proved that for any graph $H = (T, R)$:

(73.42) if there exists an integer $k \geq 1$ such that for any graph $G = (V, E)$ with $T \subseteq V$ and any $c : E \to \mathbb{Z}_+$, there is a $\frac{1}{k}$-integer maximum-value multiflow, then any three pairwise intersecting inclusionwise maximal stable sets A, B, C of H satisfy $A \cap B = A \cap C = B \cap C$.

Karzanov [1991] conjectured that the reverse implication holds and that $k = 4$ will do. (Karzanov [1987a] announced a proof of this, but the proof failed.) The techniques of Karzanov [1987d] yield a strongly polynomial-time algorithm for the problems in Theorems 73.3 and 73.4.

73.3c. Lockable collections

Let T be a set and let $H = (T, R)$ be the complete graph on T. A collection $\mathcal{A}$ of subsets of T is called *lockable* if for each (supply) graph $G = (V, E)$ with $V \supseteq T$ and for each capacity function $c : E \to \mathbb{R}_+$, there is a multiflow for demand graph H such that

(73.43) for each $U \in \mathcal{A}$, the sum of the flow values of those nets split by U is equal to the minimum of $c(\delta_E(X))$ taken over $X \subseteq V$ satisfying $X \cap T = U$.

(Here U *splits* a pair of vertices if precisely one of them is in U.)

The following characterization of lockable collections was proved jointly by Karzanov [1979b,1984] and Lomonosov [1985] (announced in Lomonosov [1979b]). Recall that two subsets X, Y of T are called *crossing* if each of $X \cap Y$, $X \setminus Y$, $Y \setminus X$, and $T \setminus (X \cup Y)$ is nonempty. (The 1979 references did not consider the Euler condition.) A short proof was given by Frank, Karzanov, and Sebő [1992, 1997].

(73.44) A collection $\mathcal{A}$ is lockable if and only if $\mathcal{A}$ contains no three pairwise crossing sets. If $\mathcal{A}$ is lockable and c is integer, there is a half-integer multiflow satisfying (73.43). If moreover $c(\delta(v))$ is even for each $v \in V \setminus T$, there is an integer multiflow satisfying (73.43).

We show that this generalizes two results proved earlier. First we show that Corollary 73.2d can be derived. Let $G = (V, E)$ be a graph and let $T \subseteq V$ be such that each vertex $v \in V \setminus T$ has even degree. Let $\mathcal{A} := \{\{v\} \mid v \in T\}$. Then $\mathcal{A}$ contains no three pairwise crossing sets, and hence (73.44) applies. Let $c := \mathbf{1}$. By (73.44), there exists a collection $\mathcal{P}$ of edge-disjoint T-paths such that for each $v \in T$, there are $\gamma_G(v)$ paths in $\mathcal{P}$ with end vertex v. So $|\mathcal{P}| = \frac{1}{2} \sum_{v \in T} \gamma_G(v)$, and we have Corollary 73.2d.

Second we derive Theorem 72.2 for $H = C_5$. Let $G = (V, E)$ be a graph and let $H = (T, R)$ be the graph C_5, with $T \subseteq V$. Let $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ be a capacity and demand function satisfying the Euler and cut conditions. We show that there is a feasible integer multiflow. To this end we can assume that

(73.45) $\quad c(\delta_E(v)) = d(\delta_R(v))$ for each $v \in T$.

If this is not the case, add an edge $e = v'v$, where v' is a new vertex, define $c(e) := d(\delta_R(v))$, and replace v by v' in H. This does not violate the Euler and cut conditions.

Define

(73.46) $\quad \mathcal{A} := \{\{v\} \mid v \in T\} \cup \{\{u,v\} \mid u,v \in T, u \neq v, uv \notin R\}$.

Then no three sets in $\mathcal{A}$ are pairwise crossing, and so (73.44) applies; that is, there is an integer multiflow $(x_r \mid r \in R')$ such that (73.43) holds, where

(73.47) $\quad R' := \{st \in T \mid s,t \in T, s \neq t\}$.

. (So x_r is an $s-t$ flow in G for $r = st \in R'$.) We show that the value of x_r is equal to d_r for each $r \in R$, and is equal to 0 for each $r \in R' \setminus R$, as required.

Let b_r be the value of x_r, for $r \in R'$. By (73.43) and the cut condition,

(73.48) $\quad b(\delta_{R'}(U)) \geq d(\delta_R(U))$ for each $U \in \mathcal{A}$,

since there exists an $X \subseteq V$ with $X \cap T = U$ and

(73.49) $\quad b(\delta_{R'}(U)) = c(\delta_E(X)) \geq d(\delta_R(X)) = d(\delta_R(U))$.

Moreover, equality holds if $|U| = 1$, since for $v \in T$ we have $b(\delta_{R'}(v)) \leq c(\delta_E(v)) = d(\delta_R(v))$, by (73.45). Now add up all inequalities (73.48) for those $U \in \mathcal{A}$ with $|U| = 1$. Similarly, add up all inequalities (73.48) for those $U \in \mathcal{A}$ with $|U| = 2$. Both sums have the same terms at the right-hand sides of the inequality sign. But the first sum has more terms at the left-hand side than the second sum has. As the first one has equality, the second one also has equality, and the terms are equal. That is, equality holds in (73.48) for each $U \in \mathcal{A}$. This implies that $b_r = d_r$ for each $r \in R$ and $b_r = 0$ for each $r \in R' \setminus R$, as (73.48) yields a nonsingular system of equations.

This shows Theorem 72.2 for the case $H = C_5$. Lomonosov [1985] argued how also the case $H = K_4$ can be derived from (73.44).

Pevzner [1987] studied the maximum size of a collection of sets no three of which are pairwise crossing. For a short proof, see Fleiner [2001b]. More on lockable collections and related structures can be found in Ibaraki, Karzanov, and Nagamochi [1998], Ilani, Korach, and Lomonosov [2000], and Ilani and Lomonosov [2000].

73.3d. Mader matroids

The exchange phenomenon for $\mathcal{S}$-paths used in the proof of Mader's disjoint $\mathcal{S}$-paths theorem (Theorem 73.2) gives rise to a matroid as follows.

Let $G = (V, E)$ be an undirected graph and let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of disjoint subsets of V. Define $T := S_1 \cup \cdots \cup S_k$. Let $\mathcal{I}$ be the collection of all subsets I of T with the property that there exists a collection $\mathcal{P}$ of disjoint $\mathcal{S}$-paths with $I \subseteq \text{ends}(\mathcal{P})$. Here ends$(\mathcal{P})$ denotes the set of ends of the paths in $\mathcal{P}$.

Theorem 73.5. *$M = (T, \mathcal{I})$ is a matroid.*

Proof. $\mathcal{I}$ trivially is nonempty and closed under taking subsets. To see that it gives a matroid, we apply Theorem 39.1. For any collection $\mathcal{R}$ of paths, let $E\mathcal{R}$ denote the set of edges traversed by the paths in $\mathcal{R}$. Choose $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. We show that $I + j \in \mathcal{I}$ for some $j \in J \setminus I$. The proof of this is by induction on $|E\mathcal{Q} \setminus E\mathcal{P}|$, where $\mathcal{P}$ and $\mathcal{Q}$ are collections of disjoint $\mathcal{S}$-paths with $I \subseteq \text{ends}(\mathcal{P})$ and $J \subseteq \text{ends}(\mathcal{Q})$.

Let $I \setminus J = \{r\}$. Let r be an end of path $P \in \mathcal{P}$, and let r belong to S_i say. If P is disjoint from all paths in $\mathcal{Q}$, then $J + r \subseteq \text{ends}(\mathcal{Q} \cup \{P\})$, and hence $I + j \in \mathcal{I}$ for each $j \in J \setminus I$.

If P intersects some path in $\mathcal{Q}$, follow path P starting at r, until we meet, at vertex v say, a path in $\mathcal{Q}$, Q say. Let Q have ends s and t. Let Q^s and Q^t be the $s - v$ and $t - v$ part of Q. By symmetry, we may assume that

(73.50) $\qquad EQ^s \not\subseteq EP$ and $t \notin S_i$.

Indeed, if $EQ^s \not\subseteq EP$ and $EQ^t \not\subseteq EP$, then by symmetry we can assume $t \notin S_i$ (as $s \notin S_i$ or $t \notin S_i$). If $EQ^s \subseteq EP$ or $EQ^t \subseteq EP$, then by symmetry we can assume $EQ^t \subseteq EP$, hence $EQ^s \not\subseteq EP$ (as $Q \neq P$) and $t \notin S_i$ (as t is the other end of P and as $r \in S_i$). So we may assume (73.50).

Let Q' be the path obtained by concatenating Q^t and the $v - r$ part of P. Then Q' is an $\mathcal{S}$-path disjoint from all paths in $\mathcal{Q} \setminus \{Q\}$. Define $\mathcal{Q}' := (\mathcal{Q} \setminus \{Q\}) \cup \{Q'\}$ and $J' := J - s + r$. So $J' \subseteq \text{ends}(\mathcal{Q}')$. Hence $J' \in \mathcal{I}$.

If $s \notin I$, we are done, since then there is a $j \in J \setminus I$ with $I + j \subseteq J - s + r$. If $s \in I$, then $J' \setminus I = J \setminus I$, and we can apply the induction hypothesis, since

(73.51) $\qquad |E\mathcal{Q}' \setminus E\mathcal{P}| < |E\mathcal{Q} \setminus E\mathcal{P}|$.

Hence, by induction, there is a $j \in J \setminus I$ with $I + j \in \mathcal{I}$ as required. ∎

We call a matroid $M = (T, \mathcal{I})$ obtained in this way a *Mader matroid*. If $k = 2$, we call the Mader matroid also a *Menger matroid*. The *matching matroids* (cf. Section 39.4a) are the special case of Mader matroids where $\mathcal{S} = \{\{v\} \mid v \in V\}$.

The question is how Mader matroids relate to known classes of matroids. The class of gammoids seems close to Mader matroids. Hence the question:

(73.52) $\qquad$ Is each Mader matroid a gammoid?

What can be proved is that each *Menger* matroid is a gammoid. More precisely:

Theorem 73.6. *A matroid is a gammoid if and only if it is a contraction of a Menger matroid.*

Proof. To see necessity, each gammoid is the contraction of a transversal matroid (Corollary 39.5a). Hence it suffices to show that each transversal matroid $M = (T, \mathcal{I})$ is a contraction of a Menger matroid. We can assume that the transversal matroid is obtained from a bipartite graph G with colour classes S and T, such that the independent sets of M are the subsets of T covered by some matching in G, and such that G has a matching of size $|S|$. Let M' be the Menger matroid on $S \cup T$ obtained from G by taking $\mathcal{S} := \{S, T\}$. Then contracting S in M' gives M. So M is the contraction of a Menger matroid.

To prove sufficiency, it suffices to show that each Menger matroid is a gammoid (as the class of gammoids is closed under contractions). Let $G = (V, E)$ be an undirected graph and let S_1 and S_2 be disjoint subsets of V. Define $\mathcal{S} := \{S_1, S_2\}$. Let M be the Menger matroid obtained this way. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum-size collection of disjoint $\mathcal{S}$-paths in G such that B is the set of ends of these paths. We can assume that neither S_1 nor S_2 spans an edge of G (as it is not in any $\mathcal{S}$-path).

Let $D = (V, A)$ be the directed graph obtained from G by orienting each edge incident with S_1 away from S_1 and by orienting each edge incident with S_2 towards S_2, and by replacing each remaining edge e by two oppositely oriented arcs connecting the ends of e. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum-size collection of disjoint directed paths in D from S_1 to S_2, such that B is the set of ends of these paths.

Derive an undirected graph $\widetilde{G}$ from D as follows. Replace each vertex $v \notin S_1 \cup S_2$ by two vertices, v' and v''. For $v \in S_1$ define $v' := v$, and for $v \in S_2$ define $v'' := v$. Replace each arc (u, v) of D by an edge $u'v''$ of $\widetilde{G}$. Moreover, for each $v \in V \setminus (S_1 \cup S_2)$, make an edge $v'v''$ of $\widetilde{G}$. This makes the undirected graph $\widetilde{G}$. Then for any subset I of $S_1 \cup S_2$ one has, by a well-known argument (cf. Theorem 39.5):

(73.53) D contains disjoint directed paths from S_1 to S_2, such that I is the collection of the ends of these paths $\iff$ $\widetilde{G}$ contains a matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus I$.

So a subset B of $S_1 \cup S_2$ is a base of M if and only if $\widetilde{G}$ has a maximum-size matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus B$. So M is the matroid obtained from the matching matroid of $\widetilde{G}$ by contracting all vertices in $V \setminus (S_1 \cup S_2)$. As each matching matroid is a transversal matroid (cf. Section 39.4a), this proves that each Menger matroid is the contraction of a transversal matroid, and hence is a gammoid (Corollary 39.5a). ∎

By the results in Section 39.4a, the class of gammoids is also equal to the class of contractions of matching matroids. So contractions of Menger matroids and those of matching matroids (two special cases of Mader matroids) coincide.

A question related to (73.52) is:

(73.54) Is each Mader matroid linear?

As gammoids are representable over all large enough fields, a positive answer to question (73.52) implies a positive answer to question (73.54). The constructions given in Section 73.1a suggest a positive answer to (73.54).

73.3e. Minimum-cost maximum-value multiflows

Karzanov [1979d] showed that if H is a complete graph and all capacities are integer, there exists a half-integer minimum-cost maximum-value multiflow (and he gave a pseudo-polynomial-time algorithm to find it). This can be directly extended to the case where H is a complete multipartite graph.

A short proof, together with a strongly polynomial-time algorithm, was given by Karzanov [1994a], where also the existence of a half-integer optimum dual solution was shown. Other algorithms (based on scaling) were given by Goldberg and Karzanov [1997].

On the other hand, Karzanov [1987b] showed that if $H = (T, R)$ is not a complete multipartite graph (that is, H contains two intersecting inclusionwise maximal stable sets), then there is no fixed integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and each integer capacity function and each cost function, there is a $\frac{1}{k}$-integer minimum-cost maximum-value multiflow.

73.3f. Further notes

Lomonosov [1985] (announced in Lomonosov [1979a]) gave a min-max formula for the maximum total value of a multiflow if H is the union of two (not necessarily disjoint) cliques.

Karzanov and Manoussakis [1996] showed: Let $G = (V, E)$ and $H = (T, R)$ be graphs, with $T \subseteq V$, where $H = K_{2,r}$, and where $\deg_G(v)$ is even for each $v \in V \setminus T$. For any T-path P, let $\alpha(P)$ denote the distance in H between the ends of P. Then the maximum value of

(73.55) $$\sum_{P \in \mathcal{P}} \alpha(P),$$

where $\mathcal{P}$ is a collection of edge-disjoint T-paths, is equal to the minimum value of

(73.56) $$\sum_{u,v \in T} |E[X_u, X_v]| \cdot \mathrm{dist}_H(u, v),$$

where $(X_u \mid u \in T)$ is a partition of V with $u \in X_u$ for $u \in T$. (As usual, $E[X, Y]$ is the set of edges connecting X and Y.) Extensions and related results are given by Karzanov [1998a,1998b,1998c].

Rothfarb and Frisch [1969] showed that the maximum total value of a 3-commodity flow equals the minimum capacity of a set of edges disconnecting all nets, if $|V| = 3$.

Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [1992,1994] showed that it is NP-complete to find a minimal number of edges disconnecting any two vertices among three given vertices in an undirected graph. Chopra and Rao [1991] studied the corresponding polyhedron.

Chapter 74

Planar graphs

Finding disjoint paths in *planar* graphs is of interest not only for planar communication or transportation networks, but especially also for the design of VLSI-circuits. The routing of the wires should follow certain channels on layers of the chip. On each layer, these channels form a planar graph.

Even for planar graphs, disjoint paths problems are in general hard. However, for some special cases, polynomial-time algorithms and good characterizations have been found. In this chapter we discuss some of these cases.

Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) := d_i$. We denote $G + H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

Recall that the *Euler condition* states that $G + H$ is Eulerian.

74.1. All nets spanned by one face: the Okamura-Seymour theorem

The complexity of the edge-disjoint paths problem for planar graphs with all nets on the outer boundary, is open. However, Okamura and Seymour [1981] showed that if the Euler condition holds, the edge-disjoint paths problem is polynomial-time solvable, and the cut condition is sufficient for solvability. We follow their method of proof.

Theorem 74.1 (Okamura-Seymour theorem)**.** *Let $G = (V, E)$ be a planar graph and let $H = (T, R)$ be a graph where T is the set of vertices of G incident with the unbounded face of G. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity of the cut condition being trivial, we show sufficiency. The cut condition implies that $|R| \leq |E|$ (assuming that each $r \in R$ consists of two distinct vertices), since

$$2|R| = \sum_{v\in V} \deg_R(v) \le \sum_{v\in V} \deg_E(v) = 2|E|. \tag{74.1}$$

So we can consider a counterexample with $2|E| - |R|$ minimal. Then

(74.2) G is 2-connected.

Indeed, if G is disconnected, we can deal with the components separately. Suppose next that G is connected and has a cut vertex v. We may assume that for no $r = st \in R$, the vertices s and t belong to different components of $G - v$, since otherwise we can replace r by sv and vt, without violating the Euler or cut condition. For any component K of $G - v$ consider the graph induced by $K \cup \{v\}$. Again, the Euler and cut conditions hold (with respect to those nets contained in $K \cup \{v\}$). So by the minimality of $2|E| - |R|$, we know that paths as required exist in $K \cup \{v\}$. As this is the case for each component of $G - v$, we have paths as required in G. This proves (74.2).

Let C be the circuit formed by the outer boundary of G. If some $r \in R$ has the same ends as some edge e of G, we can delete e from G and r from R, and obtain a smaller counterexample. So no such r exists.

Call a subset X of V *tight* if $d_E(X) = d_R(X)$. Then

(74.3) there exists a tight subset X of V such that $\delta_E(X)$ intersects EC in precisely two edges.

Indeed, if there is no tight set X with $\emptyset \ne X \ne V$, we can choose an edge $e \in EC$, and replace E and R by $E \setminus \{e\}$ and $R \cup \{e\}$. This does not violate the cut condition, and hence would give a smaller counterexample.

So there exists a tight proper nonempty subset X of V. Choose X with $|\delta_E(X)|$ minimal. Then $G[X]$ and $G - X$ are connected. For suppose that (say) $G[X]$ is not connected. Let K be a component of $G[X]$. Then

$$\begin{aligned} |\delta_E(K)| + |\delta_E(X \setminus K)| &\ge |\delta_R(K)| + |\delta_R(X \setminus K)| \ge |\delta_R(X)| \\ &= |\delta_E(X)| = |\delta_E(K)| + |\delta_E(X \setminus K)|. \end{aligned} \tag{74.4}$$

So K is tight, while $|\delta_E(K)| < |\delta_E(X)|$, contradicting the minimality assumption. Hence $G[X]$ and $G - X$ are connected, implying (74.3).

Choose a set X as in (74.3) with $|X|$ minimal. Let e be one of the two edges in EC that belong to $\delta_E(X)$. Say $e = uw$ with $u \notin X$ and $w \in X$.

Since $d_R(X) = d_E(X) \ge 2$, we know $\delta_R(X) \ne \emptyset$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Choose $r \in \delta_R(X)$ such that t_r is as close as possible to u in the graph $C - X$.

Since s_r and t_r are nonadjacent, we know that $\{s_r, t_r\} \ne \{u, w\}$. So we can choose $v \in \{u, w\} \setminus \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, v t_r\}$. Trivially the Euler condition is maintained. We show that also the cut condition is maintained, yielding a contradiction as $2|E| - |R'| < 2|E| - |R|$ and as a solution for R' yields a solution for R.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

(74.5) $\qquad d_E(Y) < d_{R'}(Y).$

By Theorem 70.4, we can take Y such that $G[Y]$ and $G - Y$ are connected. So $\delta_E(Y)$ has two edges on C. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.5) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

(74.6) $\qquad d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y).$

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and vt_r, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

By the choice of r, there is no pair r' in R connecting $X \setminus Y$ and $Y \setminus X$ (since then $t_{r'} \in Y \setminus X$, and hence $t_{r'}$ is closer than t_r to u in $C - X$). So (using Theorem 3.1)

(74.7) $\qquad d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y).$

Moreover,

(74.8) $\qquad d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y).$

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.8), and therefore $X \cap Y$ is tight. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.8). ∎

For multiflows, the Okamura-Seymour theorem implies the following result of H. Okamura (cf. note on p. 80 of Okamura and Seymour [1981]):

Corollary 74.1a. *Let $G = (V, E)$ be a planar graph, let R be a set of pairs of vertices on the outer boundary of G, and let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. Then there exists a feasible multiflow if and only if the cut condition holds. If moreover c and d are integer, there is a half-integer multiflow.*

Proof. By compactness, continuity, and scaling, we can assume that c and d are integer. Replacing any edge e by $2c(e)$ parallel copies, and any pair $r \in R$ by $2d(r)$ parallel nets, we can apply the Okamura-Seymour theorem. The paths in the new graph give the multiflow in the original graph as required. ∎

Notes. The proof of Theorem 74.1 yields a polynomial-time algorithm for finding the edge-disjoint paths, since we can determine a minimum-size cut containing e' and e'', for any pair of edges e', e'' on the outer boundary of G (by finding a shortest path in the dual graph). Frank [1985] outlined that it in fact leads to an $O(n^3 \log n)$-time algorithm. (As P.D. Seymour observed, also the splitting-off technique used by Lins [1981] to prove Corollary 74.1b below yields a polynomial-time algorithm to find paths as required.)

74.1a. Complexity survey

Complexity survey for the disjoint paths problem in planar graphs with all terminals on the outer boundary and satisfying the Euler condition ($*$ indicates an asymptotically best bound in the table):

	$O(n^4)$	Hassin [1984] (also capacitated case)
	$O(n^3 \log n)$	Frank [1985] (also capacitated case)
	$O(n^2 \log^* n)$	Matsumoto, Nishizeki, and Saito [1985]: feasibility test (also capacitated case)
	$O(tn\sqrt{\log n})$	Matsumoto, Nishizeki, and Saito [1985]: feasibility test (also capacitated case)
	$O(kn + n^2\sqrt{\log n})$	Matsumoto, Nishizeki, and Saito [1985] (also capacitated case)
$*$	$O(tn + n\sqrt{t \log n})$	Frederickson [1987b]: feasibility test (also capacitated case)
	$O(n^2)$	Becker and Mehlhorn [1986]
	$O(tn)$	Becker and Mehlhorn [1986]: feasibility test
	$O(n^{5/3}(\log\log n)^{1/3})$	Kaufmann and Klär [1991]
	$O(kn + n\sqrt{\log n})$	Weihe [1993] (also capacitated case)
$*$	$O(n)$	Wagner and Weihe [1993,1995]
$*$	$O(kn)$	Weihe [1997c] (using Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]): capacitated case

Here $k := |R|$, t is the number of vertices that belong to at least one pair in R, and $\log^* n$ is the minimum l such that $\log^{(l)} n \leq 1$, where $\log^{(l)} n$ is obtained from n by taking l times the logarithm.

For sketches of the linear-time method of Wagner and Weihe [1993,1995], see Wagner [1993] or Ripphausen-Lipa, Wagner, and Weihe [1995].

Research problem. Is the undirected edge-disjoint paths problem polynomial-time solvable for planar graphs with all nets on the outer boundary? Is it NP-complete?

74.1b. Graphs on the projective plane

The Okamura-Seymour theorem is equivalent to a theorem of Lins [1981] on Eulerian graphs embedded in the projective plane. A closed curve in the projective plane is called *orientation-reversing* if after one turn the meaning of 'left' and 'right' is flipped. If a graph is embedded in a space S, we identify G with its image in S.

Corollary 74.1b (Lins' theorem)**.** *Let $G = (V, E)$ be an Eulerian graph embedded in the projective plane P^2. Then the maximum number of edge-disjoint orientation-*

reversing circuits in G is equal to the minimum number of intersections with G of any orientation-reversing closed curve in $P^2 \setminus V$.

Proof. Since any two orientation-reversing closed curve in the projective plane intersect, the maximum does not exceed the minimum. To see equality, let D be an orientation-reversing closed curve in $P^2 \setminus V$ having a minimum number, k say, of intersections with G. Necessarily, any intersection of D with G is a crossing of D and an edge of G. Let R be the set of edges of G intersected by D and let $G' := (V, E \setminus R)$. Then G' is a planar graph, embedded in the open sphere obtained from P^2 by deleting D. Each pair in R connects two vertices on the outer boundary of G'. It suffices to show that G' contains edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r. Then the $P_r \cup \{r\}$ for $r \in R$ form a set of k edge-disjoint orientation-reversing circuits in G as required.

To show that the paths P_r exist, we can apply the Okamura-Seymour theorem. To this end, we must test the cut condition for G', R. Note that the pairs in R can be ordered as $r_1, \ldots, r_k$ such that when following the boundary of $P^2 \setminus D$, in one round we first meet $r_1, \ldots, r_k$ consecutively, and next we meet again $r_1, \ldots, r_k$ consecutively.

Let $X \subseteq V$, with $G'[X]$ and $G' - X$ connected, and with $d_R(X) > 0$. Then $\delta_{E \setminus R}(X)$ contains exactly two edges on the outer boundary of G'. Hence we can find an orientation-reversing closed curve in P^2 intersecting the edges of G' in $\delta_{E \setminus R}(X)$ and those in $R \setminus \delta_R(X)$. Hence

$$(74.9) \qquad d_{E \setminus R}(X) + |R| - d_R(X) \geq k = |R|,$$

that is, $d_{E \setminus R}(X) \geq d_R(X)$. So the cut condition holds for G' and R. ∎

In turn, the Okamura-Seymour theorem can be derived from Lins' theorem. To this end, we first show that in the Okamura-Seymour theorem one can make a number of assumptions that do not restrict the generality. Let $G = (V, E)$ be a planar graph, and let R be a set of pairs of vertices on the outer boundary of G, such that the Euler condition and the cut condition hold.

First one can assume that the pairs in R are disjoint: if $r = st$ and $r' = st'$ are two pairs in R, we can add a new vertex s' in the outer face, and a new edge $s's$, and reset $r' := s't'$. Second one may assume that any two pairs $r = st$, $r' = s't'$ in R are 'crossing' around the outer boundary of G; that is, s, s', t, t' occur in this order cyclically around the outer boundary. If this is not the case, there exist two pairs $r = st$ and $r' = s't'$ such that s, s', t', t occur in this order cyclically around the outer boundary and such that no vertex between s and s' (along the outer boundary) belongs to any pair in R. Now we can add three new vertices, q, q' and p, and edges qp, $q'p$, ps, ps', and reset $r := qt$ and $r' := q't'$ (Figure 74.1).

Let $G' = (V', E')$ be the new graph, and let R' be the new set of pairs. This construction maintains the cut condition. To see this, let $X \subseteq V \cup \{q, q', p\}$. Without loss of generality, $p \in X$. Suppose $d_{E'}(X) < d_{R'}(X)$. Then (using parity)

$$(74.10) \qquad d_{R'}(X) - 2 \geq d_{E'}(X) \geq d_E(X \cap V) \geq d_R(X \cap V) \geq d_{R'}(X) - 2,$$

and hence we have equality throughout. In particular, none of the new edges belong to $\delta_{E'}(X)$, and so $s, s', q, q' \in X$. But then $d_R(X \cap V) = d_{R'}(X)$, a contradiction.

So the cut condition is maintained. Also, any edge-disjoint pair of a $q - t$ path P and a $q' - t'$ path P' in G' contains an edge-disjoint pair of an $s - t$ path Q and

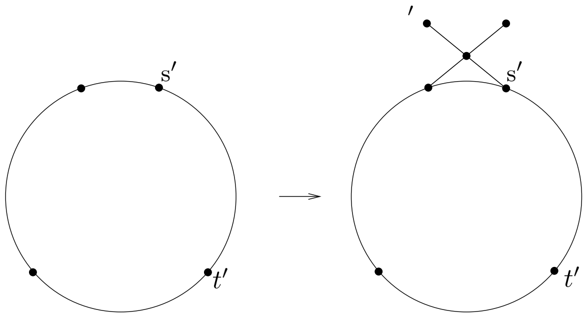


Figure 74.1

an $s' - t'$ path Q': if P traverses s and P' traverses s', this is trivial; if P traverses s' and P' traverses s, then P and P' intersect necessarily in V (as the pairs $s't$ and st' cross), and hence we can exchange P and P' at this intersection to obtain Q and Q' as required.

As we can embed G' such that q, q', t, t' occur in this order cyclically around the outer boundary of G', we have decreased the number of noncrossing pairs in R. Repeating this we can assume that all pairs in R are crossing.

Now, assuming that G is embedded in $\mathbb{R}^2$, we can embed $\mathbb{R}^2$ in the projective plane P^2. Then $P^2 \backslash \mathbb{R}^2$ is a 'cross-cap' (Möbius strip). We can extend the embedding of G to an embedding of the Eulerian graph $G + H = (V, E \cup R)$, by embedding any $r \in R$ as an edge over the cross-cap. (Since any two nets cross in $\mathbb{R}^2$, they can be drawn disjoint in $P^2 \setminus \mathbb{R}^2$.)

We derive from Lins' theorem that $G + H$ has $|R|$ edge-disjoint orientation-reversing circuits: this gives paths as required for the Okamura-Seymour theorem, as each of the circuits must contain at least one edge traversing the cross-cap, and hence at least one edge in R. As there are $|R|$ circuits, each contains exactly one edge in R, and so deleting the edges in R we obtain paths as required in the Okamura-Seymour theorem.

In order to apply Lins' theorem, we must show that each orientation-reversing closed curve D in $P^2 \setminus V$ has at least $|R|$ intersections with $G + H$. To show this, we can assume that D traverses any face of $G + H$ at most once (otherwise we can shortcut D). As D is orientation-reversing, it traverses the cross-cap an odd number of times. Between any two traversals of D over the cross-cap, we can reroute D (in $\mathbb{R}^2$) such that instead of intersecting edges of G, it intersects edges in R, in such a way that the number of new intersections with R is not more than the number of deleted intersections with E (this follows from the cut condition in the Okamura-Seymour theorem). Doing this between any two traversals of the cross-cap, we obtain an orientation-reversing closed curve only intersecting edges in R. As each of the edges in R must be intersected (since D is orientation-reversing), we see that D has at least $|R|$ intersections with $G + H$. This shows that we can apply Lins' theorem.

74.1c. If only inner vertices satisfy the Euler condition

Frank [1985] showed an interesting extension of the Okamura-Seymour theorem, to the case where the parity condition is only required for the vertices not on the outer boundary. The proof amounts to appropriately pairing those vertices v on the outer boundary for which $\deg_E(v) + \deg_R(v)$ is odd. To this end, Frank first showed the following 'pairing lemma'. We say that a pair u, v of vertices of a circuit C *crosses* a pair e, f of edges of C, if u and v are in different components of the graph $C - e - f$.

For any set X let $\binom{X}{2}$ denote the collection of unordered pairs from X. A *pairing* of a set is a partition into pairs.

Lemma 74.2α (pairing lemma)**.** *Let $C = (V, E)$ be a circuit with $|V|$ even, and let $s : \binom{E}{2} \to \mathbb{Z}$ be such that, for each $x \in \binom{E}{2}$, $s(x)$ has the same parity as the size of any of the two components of $C - x$. Then V has a pairing M such that each $x \in \binom{E}{2}$ is crossed by at most $s(x)$ pairs in M if and only if*

(74.11) $$\sum_{x \in \mathcal{B}} s(x) \geq \tfrac{1}{2} q$$

for each collection $\mathcal{B}$ consisting of disjoint pairs in $\binom{E}{2}$. Here q denotes the number of odd components of the graph G obtained from the complete graph on V by deleting all edges crossing at least one pair in $\mathcal{B}$.

Proof. For any $x \in \binom{E}{2}$ and any $R \subseteq \binom{V}{2}$, let $\mathrm{cr}_R(x)$ denote the number of pairs in R crossing x.

Necessity of the condition is easy: if M as required exists, let $N \subseteq M$ be the set of those pairs in M leaving at least one odd component of G. Since each odd component of G is left by at least one pair in N, we have $|N| \geq \frac{1}{2}q$. On the other hand, each pair in N crosses at least one pair in $\mathcal{B}$, and so

(74.12) $$\tfrac{1}{2} q \leq |N| \leq \sum_{x \in \mathcal{B}} \mathrm{cr}_N(x) \leq \sum_{x \in \mathcal{B}} s(x),$$

proving (74.11).

To see sufficiency, first assume that $s(x) > 0$ for each $x \in \binom{E}{2}$. Let M be any of the two perfect matchings in C. Then for any $x \in \binom{E}{2}$, $\mathrm{cr}_M(x)$ is at most 2 and has the same parity as $s(x)$; therefore $\mathrm{cr}_M(x) \leq s(x)$.

Hence we can assume that there is a $y \in \binom{E}{2}$ with $s(y) = 0$. Let K be a component of $C - y$. Among all such y, K, choose y, K such that K is smallest. Let $V' := V \setminus K$, let u and w be the end vertices of the path $C - K$, and let $C' = (V', E')$ be the circuit obtained from $C - K$ by adding the new edge $f = uw$. As $s(y)$ is even, both K and V' have even size. Let N be the unique perfect matching in the path $C[K]$.

For each $x \in \binom{E'}{2}$, define $s'(x)$ by: $s'(x) := s(x)$ if $f \notin x$, and

(74.13) $$s'(x) := \min\{\min\{s(\{e, g\}) \mid g \in E \setminus E', g \notin N\}, \min\{s(\{e, g\}) - 1 \mid g \in N\}\}$$

if $x = \{e, f\}$. Trivially, $s'(x)$ has the same parity as any component of $C' - x$ for each $x \in \binom{E'}{2}$.

We show that condition (74.11) holds for the smaller structure. That is, for any collection $\mathcal{B}'$ of disjoint pairs in $\binom{E'}{2}$ one has

$$(74.14) \qquad \sum_{x \in \mathcal{B}'} s'(x) \geq \tfrac{1}{2} q',$$

where q' is the number of odd components of the graph G' obtained from the complete graph on V' by deleting all edges crossing at least one pair in $\mathcal{B}'$.

If $f \notin x$ for each $x \in \mathcal{B}'$, let $\mathcal{B} := \mathcal{B}'$. Then (74.14) follows from (74.11), as $s'(x) = s(x)$ for each $x \in \mathcal{B}$ and as $q' = q$.

If $f \in x$ for some $x \in \mathcal{B}'$, this x is unique. Let $z = \{e, g\} \in \binom{E}{2}$ attain the minimum in (74.13). If $g \notin N$, let $\mathcal{B} := (\mathcal{B}' \setminus \{x\}) \cup \{z\}$. Again (74.14) follows from (74.11), as $q' = q$.

If $g \in N$, let $\mathcal{B} := (\mathcal{B}' \setminus \{x\}) \cup \{y, z\}$. Then $q = q' + 2$ (as each component of G is a component of G' or is one of the odd components of $C[K] - g$). Also $s'(a) = s(a)$ for each $a \in \mathcal{B}' \setminus \{x\}$, while $s(z) = s'(x) - 1$ and $s(y) = 0$. Hence by (74.11) we have (74.14).

Hence, by (74.14), there exists a pairing M' of V' such that for each $x \in \binom{E'}{2}$, $\text{cr}_{M'}(x) \leq s'(x)$. Then $M := N \cup M'$ is a pairing of V. We show that $\text{cr}_M(z) \leq s(z)$ for each $z \in \binom{E}{2}$. If $z \in \binom{E'}{2}$, then $\text{cr}_M(z) = \text{cr}_{M'}(z) \leq s'(z) = s(z)$. If $z \notin \binom{E'}{2}$, let $z = \{e, g\}$ with $g \in E \setminus E'$. If $e \in E'$, let $x := \{e, f\}$. If $g \notin N$, then $\text{cr}_M(z) = \text{cr}_{M'}(x) \leq s'(x) \leq s(z)$. If $g \in N$, then $\text{cr}_M(z) = \text{cr}_{M'}(x) + 1 \leq s'(x) + 1 \leq s(z)$. Finally, if $e \in E \setminus E'$, then $\text{cr}_M(z) = \text{cr}_N(z) \leq s(z)$, since, by the choice of y, $s(z)$ is positive and has the same parity as $\text{cr}_N(z)$, while $\text{cr}_N(z) \leq 2$. ∎

The proof gives a polynomial-time algorithm to find the pairing: iteratively one finds a pair x with $s(x) = 0$ and applies the reduction described in the proof; if no pair x with $s(x) = 0$ exists, one takes any perfect matching in C.

The pairing lemma implies (Frank [1985]):

Theorem 74.2. *Let $G = (V, E)$ be a planar graph such that each vertex not on the outer boundary has even degree. Let R be a set of pairs of vertices on the outer boundary of G. Then there exist edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r, if and only if*

$$(74.15) \qquad \sum_{j=1}^{l} (d_E(X_j) - d_R(X_j)) \geq \tfrac{1}{2} q$$

for each collection of subsets $X_1, \ldots, X_l$. Here q denotes the number of components K of $G' := G - \delta_E(X_1) - \cdots - \delta_E(X_l)$ with $d_E(K) + d_R(K)$ odd.

Proof. Call a vertex v or a subset X of V *odd* if $\deg_E(v) - \deg_R(v)$ or $d_E(X) - d_R(X)$ is odd.

Necessity of (74.15) is easy: let E' be the set of edges not used by the P_r. Then for any set X, $d_{E'}(X) \leq d_E(X) - d_R(X)$, while on the other hand $d_{E'}(X) \geq 1$ if X is odd. Thus at least $\frac{1}{2}q$ edges from $\bigcup_j \delta_E(X_j)$ belong to E'. So

$$(74.16) \qquad \tfrac{1}{2} q \leq \Big|\bigcup_j \delta_{E'}(X_j)\Big| \leq \sum_j d_{E'}(X_j) \leq \sum_j (d_E(X_j) - d_R(X_j));$$

that is, we have (74.15).

Sufficiency follows from the pairing lemma (Lemma 74.2α) and the Okamura-Seymour theorem. Indeed, let $v_1, \ldots, v_{2n}$ be the odd vertices, in cyclic order along the outer boundary. Let C be the circuit with vertices $v_1, \ldots, v_{2n}$ and edges $v_{i-1}v_i$ for $i = 1, \ldots, 2n$, setting $v_0 := v_{2n}$. For each pair x of edges e, e' of C, define

(74.17) $$s(x) := \min\{d_E(U) - d_R(U) \mid U \subseteq V, \delta_F(U) = \{e, e'\}\}.$$

Then s satisfies the conditions described in the pairing lemma. Indeed, the parity condition is easily checked. To see (74.11), let $\mathcal{B}$ be a collection of disjoint pairs from EC. For each $x \in \mathcal{B}$, let U_x attain the minimum (74.17). Let $G' := G - \bigcup_{x\in\mathcal{B}} \delta_E(U_x)$. Let H be the graph obtained from the complete graph on VC by deleting all edges crossed by at least one pair in $\mathcal{B}$. Then for each component K of G' one has: the odd vertices in K are contained in some component of H (since $K \cap VC \subseteq U_x$ or $K \cap VC \subseteq V \setminus U_x$ for each $x \in \mathcal{B}$; so no two vertices in $K \cap VC$ cross any x in $\mathcal{B}$). Hence the number of odd components of G' is at least the number of odd components of H. So the condition in the pairing lemma follows from condition (74.15).

Applying the pairing lemma, we obtain a matching M of the odd vertices with $d_M(U) \leq d_E(U) - d_R(U)$ for each $U \subseteq V$. Also we have that $\deg_E(v) + \deg_R(v) + \deg_M(v)$ is even for each $v \in V$. So for $R' := R \cup M$, we can apply the Okamura-Seymour theorem, to obtain in G for each $r = st \in R'$ an $s-t$ path P_r, such that the P_r are edge-disjoint. Restriction to R gives paths as required. ∎

In the theorem one can assume that $l \leq |E|$, since for each edge e of G we need at most one X_i splitting e. So the theorem is a good characterization.

As the pairing lemma is polynomial-time constructive, one can find edge-disjoint paths as required if the condition is met — similarly for the capacitated case. Frank [1985] showed that under the conditions of Theorem 74.2, the edge-disjoint paths problem, and its capacitated version, can be solved in $O(n^3 \log n)$ time. Also Becker and Mehlhorn [1986] showed that this problem is polynomial-time solvable, and they gave a time bound of $O(tn + T(n))$, where $T(n)$ is the time needed to solve a problem where the Euler condition holds, and where t is the number of vertices on the outer boundary. Weihe [1999] finally gave a linear-time algorithm.

The special case where G is a rectangular grid was solved by Frank [1982c], showing that condition (74.15) can be simplified in this case.

74.1d. Distances and cut packing

With planar duality one may derive another, dual result of the Okamura-Seymour theorem, that relates distances to packings of cuts in planar graphs (Hurkens, Schrijver, and Tardos [1988]):

Corollary 74.2a. *Any planar bipartite graph G contains disjoint cuts such that any two vertices s, t on the outer boundary of G are separated by $\mathrm{dist}_G(s,t)$ of these cuts.*

Proof. Let $\mathcal{X}$ be the set of pairs e, e' of edges along the outer boundary of G such that if $e = st$ and $e' = s't'$ where s, t, s', t' occur in this order cyclically around the outer boundary, then

(74.18) $$\mathrm{dist}_G(s,s') + \mathrm{dist}_G(t,t') - \mathrm{dist}_G(s,t') - \mathrm{dist}_G(s',t) = 2.$$

(Note that for any e, e', the left-hand side equals 0 or 2, by the triangle inequality, and by the fact that each $s-s'$ path intersect each $t-t'$ path.)

We say that a pair e, e' of edges along the outer boundary *crosses* a pair u, v of vertices along the outer boundary if any $u - v$ path along the outer boundary traverses exactly one of e and e'. We show that for any two vertices u, v on the outer boundary of G:

(74.19) $\quad \text{dist}_G(u, v) = \text{number of pairs in } \mathcal{X} \text{ that cross } u, v.$

To see this, assume that $v_1, \ldots, v_n$ are the vertices of G cyclically along the outer boundary, and let $u = v_n$ and $v = v_k$. Then (setting $v_0 := v_n$):

(74.20)
$$\begin{aligned}&\text{number of pairs in } \mathcal{X} \text{ that cross } u, v\\ &= \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n}(\text{dist}_G(v_{i-1}, v_{j-1}) + \text{dist}_G(v_i, v_j) - \text{dist}_G(v_{i-1}, v_j)\\ &-\text{dist}_G(v_i, v_{j-1})) =\\ &\tfrac{1}{2}\sum_{i=1}^{k}\big(\text{dist}_G(v_{i-1}, v_k) - \text{dist}_G(v_{i-1}, v_n) + \text{dist}_G(v_i, v_n) - \text{dist}_G(v_i, v_k)\big)\\ &= \tfrac{1}{2}\text{dist}_G(v_0, v_k) + \tfrac{1}{2}\text{dist}_G(v_k, v_n) = \text{dist}_G(u, v)\end{aligned}$$

(by cancellation).

This shows (74.19), which implies that it suffices to show that we can find disjoint cuts C_π for $\pi \in \mathcal{X}$, such that C_π intersects the outer boundary of G in the two edges in π. To show that these cuts exist, we can apply the Okamura-Seymour theorem to a modification of the planar dual graph G^* of G. Indeed, we must show that there exist edge-disjoint circuits D_π in G^*, for $\pi \in \mathcal{X}$, such that D_π traverses the two edges of G^* dual to the edges of G in π. The existence of these circuits follows from the Okamura-Seymour theorem applied to the graph G' obtained from G^* by deleting the vertex of G^* dual to the unbounded face of G, and all edges incident with it. Let R be the set of pairs of vertices of G' that are ends of pairs of edges dual to $\pi \in \mathcal{X}$. Then (74.19) implies that the cut condition holds, and that paths in G', and hence circuits in G^*, as required exist. ∎

This corollary is related to the Okamura-Seymour theorem by two different forms of duality: by planar duality and by polarity. As for planar duality, this is shown in the proof of this corollary. For polarity, this can be seen with Theorem 70.5, which gives that there exist $\lambda_U \in \mathbb{R}_+$ for $U \subseteq V$ such that

(74.21)
$$\begin{aligned}&\sum_U \lambda_U \chi^{\delta_R(U)}(r) \geq \text{dist}_G(s, t) \text{ for each } r = st \in R \text{ and}\\ &\sum_U \lambda_U \chi^{\delta_E(U)}(e) \leq 1 \text{ for each } e \in E.\end{aligned}$$

Now Corollary 74.2a asserts that the λ_U can be taken integer if G is bipartite.

74.1e. Linear algebra and distance realizability

As for the results on distances and cut packings discussed in Section 74.1d, the following further observations were made by Hurkens, Schrijver, and Tardos [1988]. Let $C = (V, E)$ be a circuit with n vertices and n edges, say:

(74.22) $\quad V = \{v_1, \ldots, v_n\},\ E = \{e_1 = v_0v_1, \ldots, e_n = v_{n-1}v_n\},$

where $v_0 := v_n$. Again, let $\binom{V}{2}$ and $\binom{E}{2}$ denote the sets of unordered pairs of distinct elements from V and E, respectively. Let M be the $\binom{V}{2} \times \binom{E}{2}$ matrix given by:

(74.23) $$M_{\{v_i,v_j\},\{e_g,e_h\}} := \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ and } \{e_g, e_h\} \text{ cross,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{v_i, v_j\}$ and $\{e_g, e_h\}$ are said to *cross* if v_i and v_j belong to different components of the graph $C - e_g - e_h$. Then the matrix M is nonsingular, with $\binom{E}{2} \times \binom{V}{2}$ inverse N given by:

(74.24) $$N_{\{e_g,e_h\},\{v_i,v_j\}} := \begin{cases} +\frac{1}{2} & \text{if } \{i,j\} = \{g,h\} \text{ or } \{i,j\} = \{g-1,h-1\}, \\ -\frac{1}{2} & \text{if } \{i,j\} = \{g,h-1\} \text{ or } \{i,j\} = \{g-1,h\}, \\ 0 & \text{otherwise.} \end{cases}$$

To see

(74.25) $$N = M^{-1},$$

choose $\{e_g, e_h\}, \{e_a, e_b\} \in \binom{E}{2}$. Then

(74.26) $$(NM)_{\{e_g,e_h\},\{e_a,e_b\}} = \tfrac{1}{2}M_{\{v_g,v_h\},\{e_a,e_b\}} + \tfrac{1}{2}M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_{g-1},v_h\},\{e_a,e_b\}}.$$

If $\{g, h\} = \{a, b\}$, then it is easy to see that this last expression is equal to 1. If $\{g, h\} \neq \{a, b\}$, then without loss of generality $g \notin \{a, b\}$. Then

(74.27) $$M_{\{v_g,v_h\},\{e_a,e_b\}} = M_{\{v_{g-1},v_h\},\{e_a,e_b\}} \text{ and } M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} = M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}},$$

which implies that (74.26) equals 0. This proves (74.25). (It can be shown that $|\det M| = 2^{\binom{n-1}{2}}$.)

(74.25) implies that for any function $d : \binom{V}{2} \to \mathbb{R}$ there is a unique $b : \binom{E}{2} \to \mathbb{R}$ such that

(74.28) $$d(\{v_i, v_j\}) = \sum(b(\{e_g, e_h\}) \mid \{e_g, e_h\} \in \tbinom{E}{2} \text{ where } \{e_g, e_h\} \text{ crosses } \{v_i, v_j\}).$$

Indeed, (74.28) is equivalent to: $d = Mb$. Hence $b := Nd$ is the unique b satisfying (74.28).

This can be applied to $d = \text{dist}_G$ for some bipartite planar graph $G = (V', E')$ with $C = (V, E)$ being the outer boundary of G. Consider the collection $\mathcal{X}$ of pairs of edges on the outer boundary of G defined in the proof of Corollary 74.2a. ($\mathcal{X}$ is a partition of E into pairs.) Then the uniqueness of b in (74.28) yields that $\mathcal{X}$ is the unique collection of pairs of edges on the boundary of G with the property that for any two vertices s, t on the outer boundary of G, the distance $\text{dist}_G(s, t)$ is equal to the number of pairs in $\mathcal{X}$ crossing $\{s, t\}$.

Another consequence of (74.25) is as follows. Consider again the circuit $C = (V, E)$ given by (74.22). Call a function $m : \binom{V}{2} \to \mathbb{R}_+$ *realizable as the distance function of a planar graph with boundary* C, or briefly *realizable*, if there exists a planar graph $G = (V', E')$, with $V' \supseteq V$, $E' \supseteq E$ such that $v_1, \ldots, v_n$ occur in this order cyclically around the outer boundary, and a length function $l : E' \to \mathbb{R}_+$ such that for all $s, t \in V$, $m(\{s, t\}) = \text{dist}_G(s, t)$. Then

(74.29) a function $m : \binom{V}{2} \to \mathbb{R}_+$ is realizable if and only if for all $i, j = 1, \ldots, n$ we have

$$m(\{v_i, v_j\}) + m(\{v_{i-1}, v_{j-1}\}) \geq m(\{v_i, v_{j-1}\}) + m(\{v_{i-1}, v_j\}),$$

setting $m(\{v_i, v_i\}) := 0$ for all i.

Necessity of the condition is trivial, since any $v_i - v_j$ path in G crosses any $v_{i-1} - v_{j-1}$ path in G. To see sufficiency, we construct a graph G as follows. Let $w_1, \ldots, w_n$ be points on the unit circle, in this cyclic order. Set $w_0 := w_n$. Add all line-segments $\overline{w_g w_h}$ $(g, h = 1, \ldots, n;\ g \neq h)$. The figure now forms a planar graph H, with vertices the points that are on two or more of these line segments. Let H^* be the dual graph. Put a new point v_i on the edge of H^* dual to the edge $w_i w_{i+1}$ of H $(i = 0, \ldots, n-1)$. Next delete the vertex of H^* dual to the outer face of H and delete all edges incident with it. This makes the graph $G = (V', E')$.

Let $d := Nm$. By the condition given in (74.29), $d \geq \mathbf{0}$. For each edge e of G, define $l(e) := d(\{e_g, e_h\})$ if e is dual to an edge of H which is on the line segment $\overline{w_g w_h}$. Using the fact that $Md = m$ it is easy to see that this gives a realization as required.

Also the 'pairing lemma' (Lemma 74.2α) can be interpreted in terms of the matrix M: it characterizes when there exists an $x : \binom{V}{2} \to \mathbb{Z}_+$ with $x(\delta(v))$ odd for each $v \in V$ and with $x^{\mathsf{T}} M \leq s$ for some given $s : \binom{E}{2} \to \mathbb{Z}_+$.

74.1f. Directed planar graphs with all terminals on the outer boundary

It was observed by Diaz and de Ghellinck [1972] that if the supply graph is directed and planar, and all terminals are on the outer boundary in the order $s_1, \ldots, s_k, t_k, \ldots, t_1$, then the integer multicommodity flow problem is solvable in polynomial time, and the cut condition suffices. This follows by a reduction to a minimum-cost circulation problem: add arcs from t_i to s_i for $i = 1, \ldots, k$.

Related, and more difficult, is the following result of Nagamochi and Ibaraki [1990]. Let the supply digraph $D = (V, A)$ be planar and acyclic, and let the demand digraph $H = (T, R)$ have all terminals on the outer boundary of D. Then for each $c : A \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ satisfying the directed analogue of the Euler condition (that is, $(V, A \cup R^{-1})$ is Eulerian), if there is a fractional multiflow, there is an integer multiflow.

Nagamochi and Ibaraki also gave a polynomial-time algorithm to find the integer multiflow. Moreover, they extended the results to the case where the set of vertices that violate the Euler condition, all lie on the outer boundary of D, in such a way that the vertices v with $c(\delta_A^{\text{out}}(v)) - d(\delta_R^{\text{in}}(v)) > c(\delta_A^{\text{in}}(v)) - d(\delta_R^{\text{out}}(v))$ can be separated (on the outer boundary of D) by two vertices from those vertices v where the opposite strict inequality holds.

74.2. $G + H$ planar

Seymour [1981d] gave another tractable case of the planar edge-disjoint paths problem if the Euler condition holds: the case where the graph together with all its nets (taken as edges) is a planar graph; that is, if $G = (V, E)$ and $H = (V, R)$ are graphs with

(74.30) $\qquad G + H := (V, E \cup R)$

planar (where $E \cup R$ is the disjoint union, respecting multiplicities in E and R). This case can be handled with the help of matching theory (more specifically, minimum-size T-joins and disjoint T-cuts).

Theorem 74.3. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs with $G+H$ planar and Eulerian. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity being trivial, we show sufficiency. Let the cut condition be satisfied. Consider the dual graph $(G+H)^*$ of $G+H$. Let R^* be the family of edges of $(G+H)^*$ dual to those in R. Let T be the set of vertices of $(G+H)^*$ which are incident with an odd number of edges in R^*. So R^* is a T-join in $(G+H)^*$.

In fact, R^* is a minimum-size T-join in $(G+H)^*$. For suppose not. Then there exist $E_0 \subseteq E$ and $R_0 \subseteq R$ such that $E_0^* \cup R_0^*$ is a T-join and $|E_0|+|R_0| < |R|$. As $E_0^* \cup R_0^*$ is a T-join, each vertex of $(G+H)^*$ is incident with an even number of edges in

(74.31) $$(E_0^* \cup R_0^*) \triangle R^* = E_0^* \cup (R \setminus R_0)^*.$$

Hence $E_0 \cup (R \setminus R_0)$ forms a cut in $G+H$. Since $|E_0| < |R \setminus R_0|$, this contradicts the cut condition.

So R^* is a minimum-size T-join in $(G+H)^*$. As $(G+H)^*$ is bipartite, by Theorem 29.2, there exist disjoint cuts $D_1, \ldots, D_t$ in $(G+H)^*$ such that (i) each cut D_j intersects R^* in exactly one element and (ii) each edge of R^* is in exactly one of the D_j. Condition (i) implies that the dual C_j of each D_j is a circuit in $G+H$ containing exactly one edge in R. Hence the C_j give edge-disjoint paths in G as required. ∎

Notes. The reduction to matching theory given in this proof implies that feasibility can be tested, and edge-disjoint paths can be found, in strongly polynomial time (also for the capacitated case). Matsumoto, Nishizeki, and Saito [1986] showed that feasibility can be tested in $O(n^{3/2} \log n)$ time, and edge-disjoint paths can be found in $O(n^{5/2} \log n)$ time (also for the capacitated case). The latter bound was improved by Barahona [1990] to $O(n^{3/2} \log n)$.

With the help of Wagner's theorem (Theorem 3.3), Theorem 74.3 can be extended to the case where $G+H$ has no K_5 minor. We derive this result from Guenin's theorem in Section 75.6.

The fractional version of Theorem 74.3 was published in Seymour [1979b].

74.2a. Distances and cut packing

By Theorem 70.5, Theorem 74.3 implies that if $G = (V, E)$ and $H = (V, R)$ are graphs with $G+H$ planar, then there is a fractional packing of cuts in G such that for any $r = st \in R$, s and t are separated by $\mathrm{dist}_G(s, t)$ of these cuts. A.V. Karzanov (personal communication 1986) observed that in fact from a theorem of Seymour [1979b] the existence of a half-integer packing can be derived. More generally:

(74.32) Let $G = (V, E)$ and $H = (V, R)$ be graphs with G bipartite and $G+H$ planar. Then there exist disjoint cuts in G such that for each $r = st \in R$, s and t are separated by $\text{dist}_G(s,t)$ of these cuts.

This can be derived from Theorem 29.3 above (of Seymour [1979b]), saying:

(74.33) Let $G = (V, E)$ be a planar graph and let $p : E \to \mathbb{Z}_+$. Then p is a nonnegative integer sum of incidence vectors of circuits of G if and only if $p(\delta(v))$ is even for each $v \in V$ and $p(e) \leq p(D \setminus \{e\})$ for each cut D of G and each $e \in D$.

Applying planar duality, (74.33) becomes:

(74.34) Let $G = (V, E)$ be a planar graph and let $p : E \to \mathbb{Z}_+$. Then p is a nonnegative integer sum of incidence vectors of cuts of G if and only if $p(C)$ is even for each circuit C of G and $p(e) \leq p(C \setminus \{e\})$ for each circuit C of G and each $e \in C$.

(Here we consider circuits as edge sets.) We apply this to the graph $G+H$, where G is bipartite and $G + H$ is planar. Define $p(e) := 1$ for $e \in E$ and $p(r) := \text{dist}_G(s,t)$ for $r = st \in R$. Then $p(C)$ is even for each circuit C of $G+H$ and $p(e) \leq p(C \setminus \{e\})$ for each circuit of $G + H$ and each $e \in E$. (The latter property is trivial if $e \in E$. If $e = st \in R$, we can replace any occurrence of an r in $C \setminus \{e\}$ with $r = uv \in R$, by a shortest $u - v$ path in G. This does not increase $p(C \setminus \{e\})$. Repeating this, we can assume that $C \cap R = \{e\}$, and so $C \setminus \{e\}$ is an $s - t$ path in G, implying $p(e) = \text{dist}_G(s,t) \leq |C \setminus \{e\}| = p(C \setminus \{e\})$.)

Therefore, by (74.34), p is a nonnegative integer sum of incidence vectors of cuts of $G + H$. By definition of p, this gives edge-disjoint cuts in G as required in (74.32).

74.2b. Deleting the Euler condition if $G + H$ is planar

Middendorf and Pfeiffer [1993] showed that if $G + H$ is planar (but not necessarily Eulerian), then the edge-disjoint paths problem is NP-complete. (With construction (70.9), it implies the same result for the directed case.) In fact they showed that if $G + H$ is planar and cubic, then the edge-disjoint paths problem is NP-complete. Hence, also the vertex-disjoint paths problem is NP-complete if $G+H$ is planar and cubic. (Assuming P≠NP, this disproves a conjecture of Schrijver [1990b].) Middendorf and Pfeiffer [1990b,1993] showed that, on the other hand, if $G + H$ is planar and the edges of H belong to a bounded number of faces of G, then the edge-disjoint paths problem is polynomial-time solvable.

Middendorf and Pfeiffer [1990b,1993] also presented a counterexample to a conjecture of A. Frank (cf. Sebő [1988a]) that if $G+H$ is planar, then the edge-disjoint paths problem has a solution if and only if G contains a fractional packing of paths as required and of a T'-join, where T' is the set of vertices having odd degree in $G + H$.

Korach and Penn [1992] showed that if $G + H$ is planar and the cut condition holds, then there is an 'almost complete' packing of paths as required: there is at most one edge in R on each bounded face of G such that leaving out these edges from R, the problem has a solution. A generalization of this was given by Frank and Szigeti [1995]. (Related work can be found in Granot and Penn [1992,1993,1996].)

Seymour [1981d] also showed the following:

(74.35) Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs such that $G + H$ is planar and such that R consists of two classes of parallel edges. Then there exist edge-disjoint paths if and only if the cut condition holds and we cannot contract edges of G to obtain a graph G' with at most four vertices in which the corresponding edge-disjoint paths do not exist.

Frank [1990d] observed that the latter condition can be formulated as:

(74.36) $d_{E\cup R}(X \cap Y)$ is even for any two tight sets $X, Y \subseteq V$,

which Frank called the *intersection criterion.* (A subset X of Y is called *tight* if $d_E(X) = d_R(X)$.)

The intersection criterion is a necessary condition for the existence of edge-disjoint paths: if paths as required exists, then for each tight X all edges in $\delta_E(X)$ are used by these paths; hence if X and Y are tight, all edges in $\delta_E(X \cap Y)$ are used; hence $d_E(X \cap Y) \equiv d_R(X \cap Y) \pmod 2$, that is, $d_{E\cup R}(X \cap Y)$ is even.

In other words, Frank observed that (74.35) is equivalent to:

(74.37) Let $G = (V, E)$ and $H = (V, R)$ be graphs such that $G+H$ is planar and such that R consists of two classes of parallel edges. Then there exist edge-disjoint paths if and only if the cut condition and the intersection criterion hold.

This was extended by Frank [1990d] to:

(74.38) Let $G = (V, E)$ and $H = (V, R)$ be graphs such that $G + H$ is planar and such that the edges of H are on at most two of the faces of G. Then there exist edge-disjoint paths if and only if the cut condition and the intersection criterion hold.

Lomonosov [1983] proved a maximization version of (74.35), which Frank [1990e] showed to follow from (74.35). Korach and Penn [1993] gave an $O(n\sqrt{\log n})$-time algorithm for the edge-disjoint paths problem if $G + H$ is planar and H consists of two parallel classes of nets.

Sebő [1993c] showed that for each fixed k, if $G + H$ is planar and $|VH| \leq k$, then the integer multiflow problem is polynomial-time solvable. (The demands can be arbitrarily large, so there is no reduction to the edge-disjoint paths problem for a fixed number of paths. It was shown for $k = 3$ by Korach [1982].) Sebő showed this by proving a more general result on the complexity of packing T-cuts for fixed $|T|$.

It is an open question if one may relax this condition to H being spanned by a fixed number of faces of G. (For demand $d = \mathbf{1}$ this was shown by Middendorf and Pfeiffer, as mentioned above.)

Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if $G + H$ is embeddable in the torus and there is a quarter-integer solution. He gave the example of Figure 70.5 with 8 vertices to show that this generally does not hold if $G + H$ is embeddable in the double torus.

Pfeiffer [1994] showed that the half-integer multiflow problem is NP-complete if $G + H$ is apex. (An *apex graph* is a graph having a vertex whose deletion makes the graph planar.) Pfeiffer also showed that the half-integer multiflow problem is NP-complete if the supply and demand digraphs form a directed planar graph.

74.3. Okamura's theorem

Okamura [1983] gave the following extension of the Okamura-Seymour theorem. We follow the proof found in 1984 by G. Tardos (cf. Frank [1990e]). The first half of the proof below is similar to the proof of the Okamura-Seymour theorem (Theorem 74.1).

Theorem 74.4 (Okamura's theorem)**.** *Let $G = (V, E)$ be a planar graph and let F_1 and F_2 be two of its faces. Let R be a set of pairs of vertices of G such that each $r = st \in R$ satisfies $s, t \in \mathrm{bd}(F_1)$ or $s, t \in \mathrm{bd}(F_2)$. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity of the cut condition being trivial, we show sufficiency. The cut condition implies that $|R| \leq |E|$ (assuming that each $r \in R$ consists of two distinct vertices), since

(74.39) $$2|R| = \sum_{v \in V} \deg_R(v) \leq \sum_{v \in V} \deg_E(v) = 2|E|.$$

So we can consider a counterexample with $2|E| - |R|$ minimal. Then

(74.40) G is 2-connected.

Indeed, if G is disconnected, we can deal with the components separately. Suppose next that G is connected and has a cut vertex v. We may assume that for no $r = st \in R$, the vertices s and t belong to different components of $G - v$, since otherwise we can replace r by sv and vt, without violating the Euler or cut condition. For any component K of $G - v$ consider the graph induced by $K \cup \{v\}$. Again, the Euler and cut conditions hold (with respect to those nets contained in $K \cup \{v\}$). So by the minimality of $2|E| - |R|$ we know that paths as required exist in $K \cup \{v\}$. As this is the case for each component of $G - v$, we have paths as required in G. This proves (74.40).

If some $r \in R$ is parallel to an edge of G we can delete this edge from G, and r from R, to obtain a smaller counterexample. Hence such r, e do not exist.

Call a subset X of V *tight* if $d_E(X) = d_R(X)$. Let C_1 and C_2 be the circuits forming the boundaries of F_1 and F_2 respectively. Then

(74.41) Each tight set X with $|\delta_E(X) \cap EC_1| = 2$ intersects VC_2.

For suppose that $X \cap VC_2 = \emptyset$. Choose such a set X with $|X|$ minimal. Let e be one of the two edges in $\delta_E(X) \cap EC_1$. Say $e = uw$ with $u \notin X$ and $w \in X$.

Since $d_R(X) = d_E(X) \geq 2$, we know $\delta_R(X) \neq \emptyset$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Choose $r \in \delta_R(X)$ such that t_r is as close as possible to u in the graph $C_1 - X$.

Since $\{u, w\} \neq \{s_r, t_r\}$, we can choose $v \in \{u, w\}$ with $v \notin \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, v t_r\}$. Trivially the Euler condition is maintained. We prove that also the cut condition is maintained, which is a contradiction as $2|E| - |R'| < 2|E| - |R|$ and as a solution for R' yields a solution for R.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

$$(74.42) \qquad d_E(Y) < d_{R'}(Y).$$

By Theorem 70.4, we can take Y such that $G[Y]$ and $G - Y$ are connected. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.42) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

$$(74.43) \qquad d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y).$$

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and vt_r, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

As Y and $V \setminus Y$ intersect VC_1 and as $G[Y]$ and $G - Y$ are connected, we know $|\delta_E(Y) \cap EC_1| = 2$. By the choice of r, there is no pair r' in R connecting $X \setminus Y$ and $Y \setminus X$ (otherwise, $t_{r'} \in Y \setminus X$ and hence $t_{r'}$ would be closer than t_r to u in $C_1 - X$). So (using Theorem 3.1)

$$(74.44) \qquad d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y).$$

Moreover,

$$(74.45) \qquad d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y).$$

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.45), and therefore $X \cap Y$ is tight. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.45). This proves (74.41).

Now choose $r = st \in R$. By symmetry of F_1 and F_2, we may assume that $s, t \in VC_1$. Let P and Q be the two $s - t$ paths along C_1. Deleting the edges of P from G and r from R, must violate the cut condition (as the Euler condition is maintained, and as for the new data there is no solution, since with P it gives a solution for the original data). So $|\delta_{E \setminus EP}(K)| < |\delta_{R \setminus \{r\}}(K)|$ for some $K \subseteq V$, with $G[K]$ and $G - K$ connected (by Theorem 70.4 taking $c := \chi^{E \setminus EP}$ and $d := \chi^{R \setminus \{r\}}$). Since $G[K]$ and $G - K$ are connected, and using (74.41), $|\delta_E(K) \cap EC_i| = 2$ for $i = 1, 2$. Moreover, K is tight, $\delta_E(K)$ contains two edges of P, and K does not split r. So we may assume that $s, t \notin K$. Similarly, there is a tight subset L of V such that $|\delta_E(L) \cap EC_i| = 2$ for $i = 1, 2$, such that $\delta_E(L)$ contains two edges of Q, and such that $s, t \notin L$.

As each of K, $V \setminus K$, L, and $V \setminus L$ intersects VC_2, each $s - t$ path in G intersects $K \cup L$ (since K contains a path from VP to VC_2 and L contains a path from VQ to VC_2). Hence we can partition $V \setminus (K \cup L)$ into sets M and N, with $s \in M$, $t \in N$, and $E[M, N] = \emptyset$. (Here and below, $E[X, Y]$ and $R[X, Y]$ denote the set of pairs xy in E and R respectively with $x \in X$ and $y \in Y$.)

We can assume by symmetry that $R[M, K \cap L] = \emptyset$. For suppose that $R[M, K \cap L] \neq \emptyset$ and $R[N, K \cap L] \neq \emptyset$. Since $K \cap L$ does not intersect VC_1, it would follow that both M and N intersect VC_2. However, this implies $K \cap L = \emptyset$, and hence $R[M, K \cap L] = \emptyset$.

Then we have the contradiction

$$
\begin{aligned}
(74.46)\qquad & d_R(K) + d_R(L) = d_E(K) + d_E(L) \\
& = (d_E(K \cup M) + |E[M, K]| - |E[M, L \setminus K]| - |E[M, N]|) \\
& +(d_E(L \cup M) + |E[M, L]| - |E[M, K \setminus L]| - |E[M, N]|) \\
& \geq d_E(K \cup M) + d_E(L \cup M) \geq d_R(K \cup M) + d_R(L \cup M) \\
& = (d_R(K) - |R[M, K]| + |R[M, L \setminus K]| + |R[M, N]|) \\
& + (d_R(L) - |R[M, L]| + |R[M, K \setminus L]| + |R[M, N]|) \\
& > d_R(K) + d_R(L).
\end{aligned}
$$

This follows from a straightforward count of edges, and from the facts that $E[M, L \setminus K] \subseteq E[M, L]$, $E[M, K \setminus L] \subseteq E[M, K]$, $E[M, N] = \emptyset$, $R[M, K] = R[M, K \setminus L]$ (as $R[M, K \cap L] = \emptyset$), $R[M, L] = R[M, L \setminus K]$ (similarly), and $R[M, N] \neq \emptyset$ (as $st \in R[M, N]$). ∎

Notes. Suzuki, Nishizeki, and Saito [1985b,1989] gave an $O(kn + nt_1 \cdot \mathrm{SP}_+(n))$-time algorithm for finding the edge-disjoint paths in this case (similarly for the capacitated case), where $k := |R|$, t_1 is the number of vertices on the boundary of F_1, and $\mathrm{SP}_+(n)$ is any upper bound on the time needed to find a shortest path in a planar n-vertex graph with nonnegative edge lengths.

The example of Figure 70.2 shows that Okamura's theorem cannot be extended to more than two selected faces, and also is not maintained if we allow 'mixed pairs'; that is, nets that connect the two selected faces. Under certain conditions one can allow such pairs — see (74.55) and (76.50) below.

74.3a. Distances and cut packing

By Theorem 70.5, Okamura's theorem implies that for any planar graph $G = (V, E)$ and any choice of two faces F_1 and F_2, there is a fractional packing of cuts such that any two vertices s, t that are either both incident with F_1 or both incident with F_2, are separated by $\mathrm{dist}_G(s, t)$ of these cuts. In fact, there is a half-integer packing, as follows from the following result of Schrijver [1989a], generalizing Corollary 74.2a:

(74.47) Let $G = (V, E)$ be a bipartite planar graph and let F_1 and F_2 be two of its faces. Then there exist edge-disjoint cuts such that any two vertices s, t with $s, t \in \mathrm{bd}(F_1)$ or $s, t \in \mathrm{bd}(F_2)$ are separated by $\mathrm{dist}_G(s, t)$ of these cuts.

Karzanov [1990a] gave an alternative proof of this, yielding a strongly polynomial-time algorithm for finding the cuts, also for the weighted case (that is, for length function $l : E \to \mathbb{Z}_+$ with $l(C)$ even for each circuit C of G, finding an integer packing of cuts).

74.3b. The Klein bottle

In Schrijver [1989b] the following relation between Okamura's theorem and graphs embedded in the Klein bottle is given. It generalizes the relation between the Okamura-Seymour theorem and graphs embedded in the projective plane, as described in Section 74.1b.

We can represent the Klein bottle as obtained from the 2-sphere by adding two cross-caps. A closed curve C on the Klein bottle is called *orientation preserving* if after one turn of C the meaning of 'left' and 'right' is unchanged. Otherwise, it is called *orientation-reversing.*

Thus a closed curve is orientation-preserving if and only if it traverses the cross-caps an even number of times. It is orientation-reversing if and only if it traverses the cross-caps an odd number of times. So, if $G = (V, E)$ is a graph embedded in the Klein bottle, there is a subset R of E such that a circuit in G is orientation-reversing if and only if it traverses the edges in R an odd number of times.

Let $G = (V, E)$ be a graph embedded in the Klein bottle. Define

(74.48) $\mathcal{C}$:= collection of orientation-reversing circuits in G;
$\mathcal{D}$:= collection of edge sets intersecting each orientation-reversing circuit of G.

(Here we take circuits as edge sets.)

In Schrijver [1989b], the following is derived from (74.47):

(74.49) Let $G = (V, E)$ be a bipartite graph embedded in the Klein bottle. Then the minimum length of an orientation-reversing circuit in G is equal to the maximum number of disjoint sets in $\mathcal{D}$.

(In fact it suffices to require, instead of bipartiteness, that each face of G is surrounded by an even number of edges.)

(74.49) implies that the up hull of the incidence vectors of sets in $\mathcal{C}$ is determined by:

(74.50)
$$\begin{aligned} x_e &\geq 0 && \text{for } e \in E, \\ x(D) &\geq 1 && \text{for } D \in \mathcal{D}. \end{aligned}$$

This follows from the fact that for any $l : E \to \mathbb{Z}_+ \setminus \{0\}$, the minimum value of

(74.51)
$$\sum_{e \in E} l(e)x_e$$

over (74.50) is achieved by an integer vector x. To see this, we may assume that $l(e)$ is even for each $e \in E$. Now replace each edge e of G by a path of length $l(e)$. We obtain a bipartite graph G'. Let C' be a minimum-length orientation-reversing circuit in G'. By (74.49), there exist disjoint edge sets $D'_1, \ldots, D'_t$ in G' each intersecting all orientation-reversing circuits in G', such that t is equal to the number of edges in C'. Let $C, D_1, \ldots, D_t$ be the edge sets in G corresponding to $C', D'_1, \ldots, D'_t$. So $D_1, \ldots, D_t \in \mathcal{D}$. Then

(74.52)
$$\sum_{e \in E} l(e)\chi^C(e) = t = \sum_{i=1}^{t} 1 \text{ and } \sum_{i=1}^{t} \chi^{D_i} \leq l.$$

So $D_1, \ldots, D_t$ give a dual solution to minimizing (74.51) over (74.50) of value t, and hence $x := \chi^C$ is an optimum solution.

So the vertices of the polyhedron determined by (74.50) are incidence vectors of orientation-reversing circuits. By the theory of blocking polyhedra, this implies that the up hull of the incidence vectors of the sets in $\mathcal{D}$ is determined by:

(74.53) $x_e \geq 0$ for $e \in E$,
$x(C) \geq 1$ for $C \in \mathcal{C}$.

From this the following stronger property has been derived in Schrijver [1989b], generalizing Lins' theorem (Corollary 74.1b):

(74.54) Let $G = (V, E)$ be an Eulerian graph embedded in the Klein bottle. Then the maximum number of edge-disjoint orientation-reversing circuits is equal to the minimum number of edges intersecting all orientation-reversing circuits.

This result cannot be extended to compact surfaces with more than two cross-caps, as we can embed K_5 in such a surface in such a way that the orientation-reversing circuits are exactly the odd-size circuits of K_5. Then the maximum number of edge-disjoint orientation-reversing circuits is equal to 2, while at least 4 edges are needed to intersect all orientation-reversing circuits.

From (74.54) one can derive Okamura's theorem (Theorem 74.4) and also another disjoint paths theorem for planar graphs (Schrijver [1989b]):

(74.55) Let $G = (V, E)$ be a planar graph, and let $H = (V, R)$ be a graph, with $R = \{s_1t_1, \ldots, s_kt_k\}$, such that G has two bounded faces F_1 and F_2 with the property that $s_1, \ldots, s_k$ occur in clockwise order along $\mathrm{bd}(F_1)$ and $t_1, \ldots, t_k$ occur in clockwise order along $\mathrm{bd}(F_2)$. Let $G+H$ be Eulerian. Then there exist edge-disjoint paths P_r, where P_r is an r-path for $r \in R$, if and only if the cut condition holds.

(Here $G+H$ is the graph $(V, E \cup R)$, taking multiplicities of edges into account. An *r-path* is a path connecting the vertices in r.) To see this, we can extend the plane to a Klein bottle, by adding a cylinder between the boundaries of F_1 and F_2. (That is, we first make the plane to a sphere, next take out the interiors of the faces F_1 and F_2, and then add the cylinder, in such a way that we obtain a nonorientable surface.) By the condition on the orders of the s_i and t_i along the boundaries of F_1 and F_2, we can extend the embedding of G to an embedding of $G + H$ in the Klein bottle, by embedding the edges s_it_i over the cylinder. Then a circuit in $G + H$ is orientation-reversing if and only if it contains an odd number of edges in R. So it suffices to show that $G + H$ contains k orientation-reversing circuits.

By (74.54) one must show that each set D of edges of $G + H$ intersecting all orientation-reversing circuits has size at least k. We may assume that D is a minimal set of edges in $G+H$ intersecting all orientation-reversing circuits in $G+H$. This implies that for each circuit C of $G + H$, $|D \cap C|$ is odd if and only if C is orientation-reversing. (Indeed, for each $e \in D \cap C$ there is an orientation-reversing circuit C_e disjoint from $D \setminus \{e\}$ (by the minimality of D). As C_e intersects D we know $e \in C_e$. Hence the symmetric difference X of C and the C_e for $e \in D \cap C$ is disjoint from D. So X contains no orientation-reversing circuit. Therefore, X is the symmetric difference of an even number of orientation-reversing circuits. So C is orientation-reversing if and only if $|D \cap C|$ is odd.)

In particular, $|D \cap C|$ is even for each circuit C in G. So $D \cap E$ is a cut $\delta_E(X)$ in G. Then for each $i = 1, \ldots, k$:

(74.56) if X does not separate s_i and t_i, then $s_i t_i \in D$.

Indeed, if X does not separate s_i and t_i, then there is an $s_i - t_i$ path P in G containing an even number of edges in D. As $P \cup \{s_i t_i\}$ is an orientation-reversing circuit, it intersects D an odd number of times, and hence $s_i t_i \in D$.

(74.56) implies $|D \cap R| \geq |R \setminus \delta_R(X)|$. Hence

(74.57) $$|D| = |D \cap E| + |D \cap R| \geq |\delta_E(X)| + |R \setminus \delta_R(X)| \geq |R| = k,$$

since $|\delta_E(X)| \geq |\delta_R(X)|$ by the cut condition. So $|D| \geq k$ as required.

One can similarly derive Okamura's theorem. First one may assume, without loss of generality, that $R = \{s_1 t_1, \ldots, s_k t_k\}$ such that $s_1, \ldots, s_l, t_1, \ldots, t_l$ occur cyclically around $\mathrm{bd}(F_1)$ and $s_{l+1}, \ldots, s_k, t_{l+1}, \ldots, t_k$ occur cyclically around $\mathrm{bd}(F_2)$. This can be achieved with the construction described in Section 74.1b (cf. Figure 74.1).

Now we can obtain a Klein bottle by adding a cross-cap in the interior of F_1 and a cross-cap in the interior of F_2 (assuming that G is embedded in the 2-sphere). We can extend the embedding of G to an embedding of $G + H$, by adding edges $s_i t_i$ for $i = 1, \ldots, l$ over the first cross-cap, and adding edges $s_i t_i$ for $i = l+1, \ldots, k$ over the second cross-cap. Applying (74.54), we obtain Okamura's theorem.

74.3c. Commodities spanned by three or more faces

Karzanov [1994c,1994b] showed that Okamura's theorem and the dual cut packing result (74.47) can be extended in a certain way to planar graphs where the nets are on three or more faces. These results can be compared to those in Section 72.2a.

We repeat the definition of Γ-metric. Let Γ be a graph, and let V be a finite set. A metric μ on V is called a *Γ-metric* if there is a function $\phi : V \to V\Gamma$ with

(74.58) $$\mu(u, v) = \mathrm{dist}_\Gamma(\phi(u), \phi(v))$$

for all $u, v \in V$. (Here $\mathrm{dist}_\Gamma(x, y)$ denotes the distance of x and y in Γ.)

The *Γ-metric condition*, a necessary condition for the existence of a feasible multiflow in a supply graph $G = (V, E)$ with demand graph $H = (V, R)$, capacities $c : E \to \mathbb{R}_+$ and demands $d : R \to \mathbb{R}_+$, reads:

(74.59) $$\sum_{r=st \in R} d(r)\mu(s,t) \leq \sum_{e=uv \in E} c(e)\mu(u,v) \text{ for each } \Gamma\text{-metric } \mu \text{ on } V.$$

The $K_{2,3}$-metric condition generalizes the cut condition.

For the edge-disjoint paths problem, Karzanov [1994b] showed that for extending Okamura's theorem to three faces, adding the $K_{2,3}$-metric condition suffices:

(74.60) Let $G = (V, E)$ be a planar graph, let F_1, F_2, and F_3 be three of its faces, and let $H = (V, R)$ be a graph such that for each $r = st \in R$ there is an $i = 1, 2, 3$ with s and t on the boundary of F_i. Let $G + H$ be Eulerian. Then there exist edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r, if and only if the $K_{2,3}$-metric condition holds.

In particular, if a fractional solution exists, then an integer solution exists.

Karzanov [1994b] derived (74.60) from a dual result on packing cuts and $K_{2,3}$-metrics, proved in Karzanov [1994c]:

(74.61) Let $G = (V, E)$ be a bipartite planar graph and let $\mathcal{F}$ be a set of three of its faces. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an $F \in \mathcal{F}$ with both u and v incident with F.

Sebő [1993a] showed that a related result on surfaces with three cross-caps also holds (in the same way as the results on the Klein bottle above relate to Okamura's theorem (Theorem 74.4)). Let S be the compact surface with three cross-caps. Let $G = (V, E)$ be a graph embedded in S, and consider the system:

(74.62)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E, \\ x(C) \geq 1 & \text{for each orientation-reversing circuit } C. \end{array}$$

Sebő showed that the polyhedron determined by (74.62) has half-integer vertices only. Moreover, if Z denotes the set of minimal $\{0, \frac{1}{2}, 1\}$ solutions of (74.62), then the system

(74.63)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E, \\ z^{\mathsf{T}} x \geq 1 & \text{for each 0,1 vector } z \in Z, \\ 2z^{\mathsf{T}} x \geq 2 & \text{for each } z \in Z \end{array}$$

(which determines the blocking polyhedron of (74.62)) is totally dual half-integral. More strongly, for each $c : E \to \mathbb{Z}_+$ with $c(C)$ even for each circuit C of G, the dual of minimizing $c^{\mathsf{T}} x$ over (74.63) has an integer optimum solution.

From this, Sebő derived a result related to (74.61), in the same was as (74.54) is related to Okamura's theorem (Theorem 74.4):

(74.64) Let $G = (V, E)$ be a bipartite planar graph and let F_1, F_2, F_3 be three of its faces, with F_1 and F_2 bounded. Let $s_1, \ldots, s_k$ occur clockwise along $\text{bd}(F_1)$ and let $t_1, \ldots, t_k$ occur clockwise along $\text{bd}(F_2)$. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an i with $u = s_i$, $v = t_i$, or if both u and v are incident with F_3.

For the extension of (74.61) to four or more faces, there is not a finite collection $\mathcal{G}$ of graphs such that in (74.60) and (74.61) one can consider Γ-metrics for Γ in $\mathcal{G}$. However, for four faces, Karzanov [1994c] proved:

(74.65) Let $G = (V, E)$ be a bipartite planar graph and let $\mathcal{F}$ be a set of four of its faces. Then there exists a collection of metrics $\mu_1, \ldots, \mu_k$ such that each μ_i is a Γ-metric for some bipartite planar graph Γ with four faces, and such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an $F \in \mathcal{F}$ with both u and v incident with F.

This implies, with the usual polarity argument, that if $G = (V, E)$ is a planar graph, $\mathcal{F}$ a set of four of its faces, $H = (V, R)$ a graph such that for each $r = st \in F$ there is an $F \in \mathcal{F}$ with s and t incident with F, $c : E \to \mathbb{R}_+$, and $d : R \to \mathbb{R}_+$, then there is a feasible multiflow if and only if the Γ-metric condition (74.59) holds for each planar bipartite graph Γ with four faces.

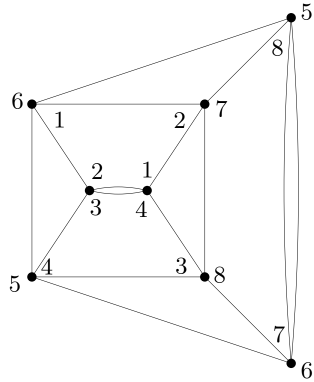


Figure 74.2
An example of a planar graph where each commodity is spanned by one of the four 4-sided faces and where there exists a half-integer, but no integer multiflow, while the Euler condition holds. The nets are indicated by pairs of indices at the vertices. All capacities and demands are 1. The half-integer multiflow is obtained by putting, for each index i, a flow of value $\frac{1}{2}$ along each of the two paths along the boundary of the (unique) face incident with both vertices i.

However, if $G + H$ is Eulerian, an integer solution (for $c = \mathbf{1}$, $d = \mathbf{1}$) need not exist, as is shown in Karzanov [1994b]. In fact, Karzanov gave an example where $G + H$ is Eulerian and where a half-integer solution exists, but no integer solution (Figure 74.2). Karzanov [1995] however showed that if c and d are integer and satisfy the Euler condition, then the existence of a fractional multiflow implies the existence of a half-integer multiflow. Hence, if c and d are integer (but not necessarily satisfy the Euler condition), then the existence of a fractional multiflow implies the existence of a quarter-integer multiflow.

Karzanov [1994c] showed that (74.65) cannot be extended to a set $\mathcal{F}$ of five faces by adding Γ-metrics for planar bipartite graphs Γ with five faces.

74.4. Further results and notes

74.4a. Another theorem of Okamura

Next to Theorem 74.4 ('Okamura's theorem'), Okamura [1983] gave another generalization of the Okamura-Seymour theorem:

Theorem 74.5. *Let $G = (V, E)$ be a planar graph. Let R be a set of nets such that there is a vertex q on the outer boundary of G with the property that each net is spanned by the outer boundary of G or it contains q. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity being trivial, we show sufficiency. As in the proof of Theorem 74.4 we consider a counterexample with $2|E| - |R|$ minimal. It again implies that G is 2-connected, and that no $r \in R$ is parallel to an edge of G. Moreover, R contains at least one pair r with $q \notin r$, as otherwise the theorem follows easily from Menger's theorem.

Let C be the circuit formed by the outer boundary of G. Consider any pair $g = xy$ in R with $q \notin g$ (so $x, y \in VC$), such that the $x - y$ path P along C not containing q, is as short as possible. Deleting the edges in P from G, and net g from R, the cut condition is not maintained (as otherwise we have a smaller counterexample). As in the proof of Theorem 74.4 it implies that there exists a tight X with $x, y \notin X$ and such that X intersects C in a subpath of P. Choose X with $|X|$ minimal. Note that by the choice of g, X spans no pair in R.

If $\delta_R(X)$ contains no pair $r = st$ with both ends on C, it contains only pairs qv with $v \in X$. Hence we can contract X to one vertex and obtain a smaller counterexample (note that $|X| \geq 2$, since any net in $\delta_R(X)$ is equal to $qv \in R$ for some $v \in X$ with $v \notin VC$).

So we can assume that $\delta_R(X)$ contains a pair with both ends on C. Let e be one of the (two) edges in EC that belong to $\delta_E(X)$. We choose e such that there is a pair $r = st$ in $\delta_R(X)$ such that $s, t \in VC$ and such that the $s - t$ path along C containing e does not traverse q except possibly at its ends. Let $e = uw$ with $u \notin X$ and $w \in X$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Since $q \notin X$, we know that each such t_r is on C. Choose $r \in \delta_R(X)$ such that s_r belongs to VC, such that the $s_r - t_r$ path along C containing e does not traverse q except possibly at its ends, and such that t_r is as close as possible to u when following $C - X$. By the choice of e, such an r exists.

Since s_r and t_r are nonadjacent, we know that $\{s_r, t_r\} \neq \{u, w\}$. So we can choose $v \in \{u, w\}$ with $v \notin \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, v t_r\}$. Trivially the Euler condition is maintained. We show that also the cut condition is maintained, contradicting the minimality of the counterexample.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

(74.66) $$d_E(Y) < d_{R'}(Y).$$

By Theorem 70.4, we can assume that $G[Y]$ and $G - Y$ are connected. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.66) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

(74.67) $$d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y).$$

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and $v t_r$, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

By the choice of r, there is no pair in R connecting $X \setminus Y$ and $Y \setminus X$. So (using Theorem 3.1)

(74.68) $$d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y).$$

Moreover,

(74.69) $$d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y).$$

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.69), and therefore $d_E(X \cap Y) = d_R(X \cap Y)$. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.69). ∎

Suzuki, Nishizeki, and Saito [1985a,1985b] gave an $O(t^2 n + n \cdot \mathrm{SP}_+(n))$-time algorithm for finding the edge-disjoint paths in this case (similarly for the capacitated case), where t is the number of vertices on the outer boundary, and where $\mathrm{SP}_+(n)$ is any upper bound on the time needed to find a shortest path in a planar n-vertex graph with nonnegative edge lengths.

With Theorem 70.5, Theorem 74.5 implies that for any planar graph $G = (V, E)$ and any vertex q on the outer boundary, there is a fractional cut packing such that any pair s, t of vertices, with s, t both on the outer boundary or $s = q$, is separated by $\mathrm{dist}_G(s, t)$ of these cuts. It seems to be open if the corresponding integer packing theorem for bipartite planar graphs holds.

74.4b. Some other planar cases where the cut condition is sufficient

It was announced by Gerards [1993] that if $G = (V, E)$ is a bipartite planar graph and $s, t \in V$, then there exist disjoint cuts such that for each $u, v \in V$ with u, v both on the outer boundary, or with $u = s$, $v = t$, the distance of u and v is equal to the number of cuts separating u and v. By Theorem 70.5, this implies that the cut condition implies the existence of a fractional multiflow, if each net is spanned by the outer boundary or is equal to some fixed pair $\{s, t\}$ of vertices.

Gerards [1993] also announced that if G is a graph embedded in the Möbius strip, and if $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ are nets such that the terminals are either in the order $s_1, \ldots, s_k, t_1, \ldots, t_k$ along the boundary, or in the order or $s_1, \ldots, s_k, t_k, \ldots, t_1$, then the cut condition and the Euler condition imply the existence of an integer multiflow.

74.4c. Vertex-disjoint paths in planar graphs

Let $G = (V, E)$ be a planar graph, embedded in the plane $\mathbb{R}^2$ and let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be disjoint pairs of vertices (the 'nets'). Robertson and Seymour [1986] observed that there is an easy greedy-type algorithm for the vertex-disjoint paths problem if all vertices $s_1, t_1, \ldots, s_k, t_k$ belong to the outer boundary of G. That is, there exists a polynomial-time algorithm for the following problem:

(74.70) given: a planar graph $G = (V, E)$ and disjoint pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ of vertices on the outer boundary of G,
find: vertex-disjoint paths $P_1, \ldots, P_k$ in G, where P_i connects s_i and t_i $(i = 1, \ldots, k)$.

We describe the simple intuitive idea of the method. (Pinter [1983] attributed this idea to C.P. Hsu (1982), and applied it to the vertex-disjoint paths problem in rectangular grids.)

We say that two disjoint pairs $\{s, t\}$ and $\{s', t'\}$ cross (around G) if there exist no disjoint curves in the unbounded face, connecting s and t, and connecting s' and t'. The following *noncrossing condition* is a necessary condition for (74.70) to have a solution:

(74.71) No two distinct nets $\{s_i, t_i\}, \{s_j, t_j\}$ cross.

The noncrossing condition implies that there exists an i such that at least one of the two $s_i - t_i$ paths along $\text{bd}(F)$ contains no s_j or t_j for $j \neq i$: just choose i such that the shortest $s_i - t_i$ path along the outer boundary is shortest among all $i = 1, \ldots, k$.

Without loss of generality, $i = k$. Let Q be a shortest $s_k - t_k$ path along the outer boundary. Let $G' := G - VQ$. Next solve the vertex-disjoint paths problem for input G', $\{s_1, t_1\}, \ldots, \{s_{k-1}, t_{k-1}\}$. If this gives a solution $P_1, \ldots, P_{k-1}$, then $P_1, \ldots, P_{k-1}, Q$ forms a solution to the original problem (trivially).

If the reduced problem turns out to have no solution, then the original problem also has no solution. This follows from the fact that if $P_1, \ldots, P_{k-1}, P_k$ would be a solution to the original problem, we may assume without loss of generality that $P_k = Q$, since we can 'push' P_k 'against' the outer boundary. Hence $P_1, \ldots, P_{k-1}$ would form a solution to the reduced problem. This intuitive idea is the basis of a polynomial-time algorithm for problem (74.70):

Theorem 74.6. *The vertex-disjoint paths problem is polynomial-time solvable for planar graphs with all terminals on the outer boundary.*

Proof. See above. ∎

Linear-time implementations were given by Suzuki, Akama, and Nishizeki [1988c,1990] and Liao and Sarrafzadeh [1991].

The method implies moreover a characterization by means of a cut condition for the existence of a solution to (74.70). A *simple closed curve* C in $\mathbb{R}^2$ is by definition a one-to-one continuous function from the unit circle to $\mathbb{R}^2$. We will identify the function C with its image.

We say that C *separates* the pair $\{s, t\}$ if each curve connecting s and t intersects C. (In particular, if s or t is on C.) Now the following *cut condition* clearly is necessary for the existence of a solution to the vertex-disjoint paths problem in planar graphs:

(74.72) each simple closed curve in $\mathbb{R}^2$ intersects G at least as often as it separates pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$.

Robertson and Seymour [1986] showed with the method above:

Theorem 74.7. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$ and let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be pairs of vertices on the outer boundary of G. Then there exist vertex-disjoint paths $P_1, \ldots, P_k$ where P_i connects s_i and t_i ($i = 1, \ldots, k$) if and only if the noncrossing condition* (74.71) *and the cut condition* (74.72) *hold.*

Proof. Necessity of the conditions is trivial. We show sufficiency by induction on k, the case $k = 0$ being trivial. Let $k \geq 1$ and let (74.71) and (74.72) be satisfied. Suppose that paths $P_1, \ldots, P_k$ as required do not exist. Trivially, $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ are disjoint (otherwise there would exist a simple closed curve C with $|C \cap G| = 1$ and intersecting two nets, thus violating the cut condition).

We may assume that G is connected, as we can decompose G into its components. (If some s_i and t_i would belong to different components, there trivially exists a closed curve C violating the cut condition.) We can also assume that there is no cut vertex v such that $G-v$ has a component K containing no terminal (otherwise we could delete K from G without violating the cut condition).

Now there exists an i and a simple $s_i - t_i$ path P_i such that P_i follows the outer boundary and traverses no other terminals than s_i and t_i. We can assume that $i = k$. Let $G' := G - VP_k$.

Then G' contains no vertex-disjoint $s_i - t_i$ paths ($i = 1, \ldots, k-1$), since otherwise G contains vertex-disjoint $s_i - t_i$ paths ($i = 1, \ldots, k$). Hence, by the induction hypothesis, there exists a simple closed curve C with $|C \cap G'|$ smaller than the number of pairs $\{s_1, t_1\}, \ldots, \{s_{k-1}, t_{k-1}\}$ separated by C.

We can assume that C traverses the unbounded face of G' exactly once and that it intersects G only in vertices of G. We choose C such that it has a minimum number of intersections with P_k. Then C intersects P_k at most once. If C does not intersect P_k, then $|C \cap G| = |C \cap G'|$, and C violates the cut condition also for G. If C intersects P_k, then $|C \cap G| = |C \cap G'| + 1$ and C separates s_k and t_k, and so again C violates the cut condition for G. ∎

It is easy to extend the algorithm and Theorem 74.7 to the directed case, and also to the following *vertex-disjoint trees problem*:

(74.73) given: a planar graph $G = (V, E)$ and sets $S_1, \ldots, S_k$ of vertices on the outer boundary of G,
find: vertex-disjoint subtrees $T_1, \ldots, T_k$ of G such that T_i covers S_i ($i = 1, \ldots, k$).

More generally, with similar techniques, Ding, Schrijver, and Seymour [1992] generalized Theorem 74.7 (and the polynomial-time algorithm) as follows.

Theorem 74.8. *Let $D = (V, A)$ be a directed planar graph, let B be a family of ordered pairs of vertices on the outer boundary of D (with $s \neq t$ if $(s, t) \in B$), for each $b \in B$ let $A_b \subseteq A$, and let H be a set of unordered pairs from B. Then there exist paths P_b for $b \in B$ such that:*

(74.74) (i) *for $b = (s, t) \in B$, P_b is a directed $s - t$ path in (V, A_b),*
(ii) *P_b and P_c are vertex-disjoint for each $\{b, c\} \in H$,*

if and only if the following two conditions hold: the 'noncrossing condition':

(74.75) *if $\{(r, s), (t, u)\} \in H$, then (r, s) and (t, u) are disjoint and do not cross,*

and the 'cut condition':

(74.76) *for each curve C starting and ending in the unbounded face and not intersecting any s, t with $(s, t) \in B$ and for each choice of $b_1, \ldots, b_n \in B$ satisfying:*
- *$\{b_j, b_{j+1}\} \in H$ for $j = 1, \ldots, n-1$,*
- *$f, x_1, \ldots, x_n, l, y_n, \ldots, y_1$ are all distinct and occur in this order clockwise around the outer boundary, where x_j and y_j are such that $b_j = (x_j, y_j)$ or $b_j = (y_j, x_j)$, and where f and l denote the first and last point of intersection of C with D,*

there exist distinct points $p_1, \ldots, p_n$ traversed by C in this order such that for each $j = 1, \ldots, n$:

- *p_i is on the image of D in $\mathbb{R}^2$, if $b_j = (x_j, y_j)$, then some arc in A_{b_j} is entering C at p_j from the left and some arc in A_{b_j} is leaving C at p_j from the right,*
- *if $b_j = (y_j, x_j)$, then some arc in A_{b_j} is entering C at p_j from the right and some arc in A_{b_j} is leaving C at p_j from the left.*

(The points p_i can be vertices of D or be on arcs of D.)

Theorem 74.8 implies an even more general characterization and algorithm for disjoint rooted subarborescences. Let $D = (V, A)$ be a planar digraph, let B be a collection of ordered pairs (r, S) where r is a vertex on the outer boundary of D, and S is a set of vertices on the outer boundary of D with $r \notin S$. For each $b \in B$, let $A_b \subseteq A$, and let H be a set of unordered pairs from B. Then Theorem 74.8 implies necessary and sufficient conditions for the existence of rooted subarborescences T_b in D (for $b \in B$), with the property that

(74.77) (i) for $b = (r, S) \in B$, T_b is rooted at r, covers S, and is contained in A_b,
(ii) T_b and T_c are vertex-disjoint for each $\{b, c\} \in H$.

The reduction to Theorem 74.8 is by replacing each pair (r, S) in B by the pairs (r, s) for $s \in S$, and reset H to all pairs $\{(r, s), (r', s')\}$ coming from pairs $\{(r, S), (r', S')\}$ in the original H.

Notes. Suzuki, Akama, and Nishizeki [1988c,1990] and Liao and Sarrafzadeh [1991] gave linear-time algorithms for problem (74.73). For a description, see also Wagner [1993].

Theorem 74.6 implies that the vertex-disjoint paths problem is polynomial-time solvable for outerplanar graphs. This was generalized to series-parallel graphs by Korach and Tal [1993].

Takahashi, Suzuki, and Nishizeki [1992] gave an $O(n \log n)$-time algorithm to find pairwise noncrossing paths of minimum total length, connecting prescribed terminals in a planar graph with all terminals on two specified face boundaries.

74.4d. Grid graphs

Grid graphs form a class of planar graphs that are of special interest for disjoint paths problem, as they arise in the design of VLSI-circuits, in particular in routing the wires on the layers of a chip.

Any finite subgraph of the 2-dimensional rectangular grid is called a *grid graph*. So its vertex set is a finite subset of $\mathbb{Z}^2$, and any two adjacent vertices have Euclidean distance 1. (It is not required conversely that any two vertices at Euclidean distance 1 are adjacent; so the subgraph need not be an *induced* subgraph.)

Kramer and van Leeuwen [1984] showed that both the vertex-disjoint and the edge-disjoint paths problems are NP-complete even when restricted to grid graphs. Pinter [1983] showed that the vertex-disjoint paths problem remains NP-complete for grid graphs in which all faces are bounded by a rectangle (including a square).

A *rectangular grid* is a grid graph whose outer boundary is a rectangle and whose bounded faces all are unit squares. The *channel routing problem* is the vertex-disjoint paths problem in a rectangular grid, where all nets connect a vertex on the upper horizontal border with one on the lower horizontal border. A criterion for the feasibility of the channel routing problem was given by Dolev, Karplus, Siegel, Strong, and Ullman [1981], while Rivest, Baratz, and Miller [1981] gave a heuristic algorithm approximating the minimal height of the rectangle, given the positions of the terminals (cf. Preparata and Lipski [1984] and Mehlhorn, Preparata, and Sarrafzadeh [1986]). A linear-time algorithm for channel-routing, allowing also multiterminal nets, was given by Greenberg and Maley [1992].

The feasibility criterion was extended by Pinter [1983] to *switchboxes*, which are rectangular grids in which the terminals can be anywhere along the outer boundary. For the vertex-disjoint paths problem in switchboxes, Pinter showed Theorem 74.7 and described the corresponding greedy-type algorithm. He attributes the idea to C.P. Hsu (1982).

Algorithms for the *edge*-disjoint paths problem in a switchbox were given by Frank [1982c] ($O(n \log n)$) and Mehlhorn and Preparata [1986] ($O(u \log u)$, where u is the circumference of the rectangle — note that this is sufficient to specify the graph). Frank also showed that solvability only depends on horizontal and vertical cuts.

A *generalized switchbox* is a grid graph with all bounded faces being unit squares. Nishizeki, Saito, and Suzuki [1985] gave an $O(n^2)$-time algorithm for routing in generalized switchboxes for which any two vertices on the outer boundary are connected by a path with at most one bend; all terminals are on the outer boundary. They also showed that in this case one may restrict the cuts to those that are either horizontal or vertical, if the global Euler condition holds. (A correction and generalization was given by Lai and Sprague [1987].)

Kaufmann and Mehlhorn [1986] described an $O(n \log^2 n + q^2)$-time algorithm for the edge-disjoint paths problem in a generalized switchbox, with all terminals on the outer boundary. Here q denotes the number of vertices v with $\deg_G(v)+\deg_H(v)$ odd. So if the Euler condition holds, the time bound is $O(n \log^2 n)$.

Kaufmann and Mehlhorn [1986] also showed that in a generalized switchbox satisfying the Euler condition and such that no vertex is end point of more than two curves, the cut condition holds whenever it holds for all 1-bend cuts. (A cut is called a 1-*bend cut* if it is the set of edges crossed by the union of some horizontal and some vertical halfline with one common end vertex.)

Kaufmann and Klär [1993] gave an $O(u \log^2 u)$-time algorithm for generalized switchboxes, whose outer boundary is simple and has no 'rectilinearly visible corners'. (Two corners p and q of the outer boundary are called *rectilinearly visible* if the (unique) rectangle of which p and q are opposite vertices, has a nonempty interior and intersects the outer boundary only in p and q.)

Wagner and Weihe [1993,1995] showed that for such problems, if $G + H$ is Eulerian, then there is even a linear-time algorithm, even for general planar graphs. (This improves earlier results of Becker and Mehlhorn [1986] and Kaufmann [1990].)

If G is a rectangle with one rectangular hole, and all nets join two vertices either on the outer rectangle or on the inner rectangle, and if the Euler condition holds, Suzuki, Ishiguro, and Nishizeki [1990] gave a linear-time algorithm. Related results are given in Frank, Nishizeki, Saito, Suzuki, and Tardos [1992].

Takahashi, Suzuki, and Nishizeki [1993] gave a polynomial-time algorithm for the minimum-length 'noncrossing' paths problem in certain grid graphs.

The problem of finding edge-disjoint trees connecting specified sets of vertices on the outer boundary of a rectangle is NP-complete (Sarrafzadeh [1987b]). More on channel routing can be found in Preparata and Sarrafzadeh [1985], Sarrafzadeh and Preparata [1985], Mehlhorn, Preparata, and Sarrafzadeh [1986], Sarrafzadeh [1987a], Formann, Wagner, and Wagner [1991,1993], Greenberg and Shih [1995, 1996], and Chan and Chin [1997,2000]. Surveys on disjoint paths problems in grid graphs are given by Kaufmann and Mehlhorn [1990] and in the book by Lengauer [1990].

74.4e. Further notes

The Lucchesi-Younger theorem (Theorem 55.2) implies the following. Let $D = (V, A)$ and $H = (V, R)$ be digraphs with D acyclic and $(V, A \cup R)$ planar. Then D has arc-disjoint paths P_r for $r \in R$, where P_r runs from s to t if $r = (s, t)$, if and only if for each $B \subseteq A$:

(74.78) $|B| \geq$ number of $r = (s, t) \in R$ such that B intersects each $s-t$ path in D.

Trivially, this condition is necessary. The derivation of sufficiency from the Lucchesi-Younger theorem is as follows. Consider the planar digraph $Q = (V, A \cup R^{-1})$. We need to show that if (74.78) holds for each $B \subseteq A$, then Q contains $|R|$ arc-disjoint directed circuits. Equivalently, the planar dual Q^* contains $|R|$ disjoint directed cuts. Applying the Lucchesi-Younger theorem to Q^* yields for Q that we should show that (74.78) implies that each set C of arcs of Q intersecting each directed circuit of Q has size at least $|R|$. Set $B := C \cap A$ and $R' := C^{-1} \cap R$. Then for each $r = (s, t) \in R \setminus R'$, each $s-t$ path in D intersects B. So by (74.78), $|B| \geq |R \setminus R'|$, and hence $|C| = |B| + |R'| \geq |R|$.

Similarly the polynomial-time solvability of the corresponding arc-disjoint paths problem follows (using Theorem 55.7).

Korte, Prömel, and Steger [1990] showed that the edge-disjoint trees problem is NP-complete, even if we ask for two disjoint trees in a planar graph, where the trees should cover two prescribed sets of vertices.

Surveys on linear-time methods for disjoint paths problems in planar graphs were given by Wagner [1993] and Ripphausen-Lipa, Wagner, and Weihe [1995]. For extensions to nets spanned by a fixed number of faces, see Section 76.7a.

Chapter 75

Cuts, odd circuits, and multiflows

Minimum-size cuts in a graph are well under control from an algorithmic point of view, as we saw in Parts I and V. Finding a *maximum*-size cut is however an NP-complete problem.
The complement of a maximum-size cut is a minimum-size odd circuit cover — a set of edges intersecting all odd circuits. By duality, this relates to maximum collections of edge-disjoint odd circuits. This in turn relates to multiflows.
Weakly bipartite graphs are those graphs where the polyhedral approach works. It makes that the maximum cut problem is polynomial-time solvable for these graphs.
Key result in this chapter is a theorem of Guenin characterizing weakly bipartite graphs, and its extension by Geelen and Guenin to evenly bipartite graphs. These results turn out to unify several multiflow and odd circuit packing theorems.

75.1. Weakly and strongly bipartite graphs

Let $G = (V, E)$ be an undirected graph. Call a subset B of E *bipartite* if (V, B) is bipartite; equivalently, if B does not contain the edge set of any odd circuit; equivalently, if B is contained in some cut C. So finding a maximum-size bipartite set of edges is equivalent to finding a maximum-size cut, and hence it is NP-complete (cf. Section 75.1a).

The *bipartite subgraph polytope* $P_{\text{bipartite subgraph}}(G)$ of G is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of bipartite subsets B of E:

(75.1) $$P_{\text{bipartite subgraph}}(G) := \text{conv.hull}\{\chi^B \mid B \subseteq E \text{ bipartite}\}.$$

Any vector x in the bipartite subgraph polytope satisfies

(75.2) $$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(C) \leq |C| - 1 & \text{for each odd circuit } C. \end{array}$$

In general, these constraints are not enough to determine the bipartite subgraph polytope: for the complete graph K_5, the vector x with $x_e = \frac{2}{3}$ for each edge e satisfies (75.2), but does not belong to the bipartite subgraph polytope (since the largest bipartite subgraph has 6 edges, while $10 \cdot \frac{2}{3} > 6$).

Following Grötschel and Pulleyblank [1981], a graph G is called *weakly bipartite* if its bipartite subgraph polytope is determined by (75.2). An equivalent characterization is in terms of odd circuit covers. An *odd circuit cover* in an undirected graph $G = (V, E)$ is a set of edges intersecting all odd circuits. The *odd circuit cover polytope* is the convex hull of the incidence vectors of odd circuit covers. It is contained in the polytope determined by

(75.3) $\quad 0 \leq x_e \leq 1 \quad$ for each $e \in E$,
$\quad\quad x(C) \geq 1 \quad$ for each odd circuit C.

Then a graph is weakly bipartite if and only the odd circuit polytope is determined by (75.3). This follows directly from the facts that a set of edges is an odd circuit cover if and only if its complement is bipartite, and that x satisfies (75.2) if and only if $\mathbf{1} - x$ satisfies (75.3).

The relevance of weakly bipartite graphs comes from the fact that a maximum-capacity cut in these graphs can be found in strongly polynomial time, with the ellipsoid method, since the separation problem over the polytopes (75.2) is polynomial-time solvable (cf. Section 5.11). Indeed, checking (75.2) is equivalent to checking (75.3). One can check the constraints in (75.3)(i) one by one, and so one may assume that $\mathbf{0} \leq x \leq \mathbf{1}$. Next, considering x as length function, one checks if there is an odd circuit of length < 1 (like in Theorem 68.1). If so, we find a violated constraint. If not, x satisfies (75.3).

Weakly bipartite graphs were characterized by Guenin [1998a,2001a], proving a conjecture of Seymour [1981a]. This characterization also holds for the more general structure of *signed graphs*, for which it is easier to prove as it allows a finer contraction operation — see Sections 75.2 and 75.5. For just undirected graphs the characterization can be formulated as follows.

Call a graph H an *odd minor* of a graph G if H arises from G by deleting edges and vertices and by contracting all edges in a cut. The class of weakly bipartite graphs is closed under taking odd minors. To see this, it is easily seen that this class is closed under deleting edges and vertices. To see that it is closed under contracting a cut, let $G = (V, E)$ be a weakly bipartite graph, let $U \subseteq V$ and $G' = G/\delta(U)$, and take $x \in \mathbb{R}^{E'}$, where $E' = E \setminus \delta(U)$ is the edge set of G'. Let x satisfy (75.3) with respect to G'. Define $x_e := 0$ for each $e \in \delta(U)$. Then the extended x satisfies (75.3) with respect to G. So the extended x belongs to the odd circuit cover polytope of G, implying that the original x belongs to the odd circuit cover polytope of G'.

Now Guenin's characterization reads for undirected graphs:

(75.4) $\quad$ an undirected graph G is weakly bipartite $\iff$ K_5 is not an odd minor of G.

Related is a characterization of those graphs for which (75.2) is totally dual integral. These graphs are called *strongly bipartite*[22]. A general hypergraph theorem of Seymour [1977b] implies a characterization of strongly bipartite graphs. They are precisely the graphs containing no odd K_4-subdivision — equivalently, the graphs not having K_4 as odd minor. Again, this is easier to handle in the context of signed graphs — see Section 75.4.

75.1a. NP-completeness of maximum cut

In this section we show (Karp [1972b]):

Theorem 75.1. *Finding the maximum size of a cut in an undirected graph is* NP-*complete.*

Proof. We reduce the problem of finding the minimum size of a vertex cover in a graph $G = (V, E)$ to the maximum-size cut problem. This is sufficient, since the first problem is NP-complete by Corollary 64.1a.

We can assume that G has no isolated vertices, since they will not occur in any minimum-size vertex cover. Extend G by a new vertex u and, for each $v \in V$, by $\deg_G(v) - 1$ parallel edges connecting v and u. Let G' be the extended graph. Then

(75.5) the minimum size of a vertex cover in G is equal to $2|E|$ minus the maximum size of a cut in G'.

To see this, we have for any $U \subseteq V$:

$$(75.6)\qquad \begin{aligned} |\delta_{G'}(U)| &= |\delta_G(U)| + \sum_{v \in U} (\deg_G(v) - 1) \\ &= 2|\{e \in E \mid e \text{ intersects } U\}| - |U|. \end{aligned}$$

Hence, if U is a minimum-size vertex cover of G, then $|\delta_{G'}(U)| = 2|E| - |U|$, proving $\geq$ in (75.5).

To see the reverse inequality, choose a subset U of V that determines a maximum-size cut $\delta_{G'}(U)$ in G'. Then U is a vertex cover of G. Otherwise, $V \setminus U$ spans an edge e of G. Then extending U by one of the ends of e increases (75.6), a contradiction. So U is a vertex cover and $|U| = 2|E| - |\delta_{G'}(U)|$, proving $\leq$ in (75.5). ∎

75.1b. Planar graphs

Although we do not use these results in later sections, we first show that planar graphs are weakly bipartite, as it gives an interesting relation with T-joins (Barahona [1980]):

Theorem 75.2. *A planar graph is weakly bipartite.*

Proof. Consider the dual graph $G^* = (V^*, E^*)$. An odd circuit in G corresponds to an odd-size cut in G^*, that is, to a T-cut, where T is the set of vertices of G^* of

[22] A strongly bipartite graph need not be bipartite, as is shown by K_3.

odd degree. For G it means that an odd circuit cover in G corresponds to a set of edges of G^* containing a T-join. By Corollary 29.2b, the convex hull of these edge sets in G^* is determined by

$$(75.7)\qquad \begin{array}{ll} 0 \leq x(e^*) \leq 1 & \text{for } e^* \in E^*, \\ x(C) \geq 1 & \text{for each } T\text{-cut } C \text{ in } G^*. \end{array}$$

Hence the odd circuit cover polytope of G is determined by (75.3). ∎

With the help of the decomposition theorem of Wagner [1937a] (Theorem 3.3), this result can be extended to graphs without K_5 minor (Fonlupt, Mahjoub, and Uhry [1992]). We will however derive this from Guenin's more general characterization of weakly bipartite graphs.

75.2. Signed graphs

Guenin's characterization of weakly bipartite graph is valid, and easier to prove, in the more general context of signed graphs. In this section we collect some general terminology and facts on signed graphs.

A *signed graph* is a triple $G = (V, E, \Sigma)$, where (V, E) is an undirected graph and $\Sigma \subseteq E$. The graph (V, E) is called the *underlying graph* and Σ is called a *signing*.

Call a set of edges, or a path, or a circuit *odd* (*even*, respectively) if it contains an odd (even, respectively) number of edges in Σ. An *odd circuit cover* is a set of edges intersecting all odd circuits.

It is easy to show that, for any undirected graph (V, E),

(75.8) Two signings Σ and Σ' give the same collection of odd circuits $\iff$ $\Sigma \triangle \Sigma'$ is a cut of (V, E).

If $\Sigma \triangle \Sigma'$ is a cut, we call the two signed graphs, or the two signings, *equivalent*. The following is an important observation: for any signed graph $G = (V, E, \Sigma)$,

(75.9) the collection of inclusionwise minimal odd circuit covers of G is equal to the collection of inclusionwise minimal signings equivalent to Σ.

Indeed, any signing Σ' equivalent to Σ intersects each odd circuit in an odd number of edges, and hence is an odd circuit cover. Conversely, any inclusionwise minimal odd circuit cover B intersects each odd circuit C in an odd number of edges: by the minimality of B, for each $e \in B \cap C$ there exists an odd circuit C_e disjoint from $B \setminus \{e\}$. If $|B \cap C|$ is even, the symmetric difference of C and the C_e gives an odd cycle disjoint from B, a contradiction.

(75.9) has several consequences. The inclusionwise minimal sets among $\Sigma \triangle \delta(U)$ (for $U \subseteq V$) are precisely the inclusionwise minimal odd circuit covers. For any two inclusionwise minimal odd circuit covers B_1, B_2 there exists a subset U of V with

(75.10) $B_1 \triangle B_2 = \delta(U)$

(since $B_1 = \Sigma \triangle \delta(U_1)$ and $B_2 = \Sigma \triangle \delta(U_2)$ for some $U_1, U_2 \subseteq V$, hence $B_1 \triangle B_2 = \delta(U_1) \triangle \delta(U_2) = \delta(U_1 \triangle U_2)$).

(75.9) also implies that for each inclusionwise minimal odd circuit cover B of G, the set $B \triangle \Sigma$ is a cut. (We recall that, by definition, the empty set is also a cut.)

We can define the concepts of deletion, contraction, subgraph, and minor in a signed graph $G = (V, E, \Sigma)$. *Deleting* an edge e means replacing G by $G - e := (V, E \setminus \{e\}, \Sigma \setminus \{e\})$. Similarly, *deleting* a vertex v means deleting v in V and deleting in E and Σ all edges incident with v.

Contracting a (nonloop) edge e means: if $e \notin \Sigma$, replacing G by $G/e := (\widetilde{V}, \widetilde{E}, \Sigma)$, where $(\widetilde{V}, \widetilde{E})$ is obtained from (V, E) by contracting e; if $e \in \Sigma$, choose $v \in e$, replace Σ by $\Sigma \triangle \delta(v)$, and apply the previous operation. So the operation of contraction is not uniquely defined, but the outcome is unique up to equivalence of signings. This is sufficient for our purposes.

A *subgraph* of a signed graph is obtained by a series of deletions of vertices and edges. A *minor* is obtained by a series of deletions of vertices and edges and contractions of edges, and by replacing the signing by an equivalent signing.

For any complete graph K_n, let *odd-K_n* be the signed graph

(75.11) $\text{odd-}K_n := (VK_n, EK_n, EK_n).$

A signed graph (V, E, Σ) is called an *odd K_4-subdivision* if (V, E) is a subdivision of K_4 such that each triangle has become an odd circuit (with respect to Σ). It is not difficult to show that:

(75.12) a signed graph contains an odd K_4-subdivision if and only if it has odd-K_4 as minor.

75.3. Weakly, evenly, and strongly bipartite signed graphs

In an obvious way, the notions of weakly and strongly bipartite graphs can be lifted to signed graphs. A signed graph $G = (V, E, \Sigma)$ is *weakly bipartite* if each vertex of the polyhedron (in $\mathbb{R}^E$) determined by:

(75.13)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each edge } e, \\ \text{(ii)} & x(C) \geq 1 & \text{for each odd circuit } C, \end{array}$$

is integer, that is, the incidence vector of an odd circuit cover.

System (75.13) gives rise to two stronger properties. First, a signed graph is called *strongly bipartite* if (75.13) is totally dual integral. Equivalently, for each function $w : E \to \mathbb{Z}_+$ the minimum of $w^\mathsf{T} x$ over (75.13) has integer primal and dual optimum solutions. Or: for each weight function $w : E \to \mathbb{Z}_+$, the minimum weight of an odd circuit cover is equal to the maximum size of a family of odd circuits such that each edge e is in at most $w(e)$ of them.

We also define an intermediate property (only seemingly intermediate, since it will turn out to be equivalent to weakly bipartite). A signed graph $G = (V, E, \Sigma)$ is called *evenly bipartite* if for each $w : E \to \mathbb{Z}_+$ with $w(\delta(v))$ even for each $v \in V$, the minimum of $w^\mathsf{T}x$ over (75.13) is attained by integer primal and dual optimum solutions. Equivalently, for each weight function $w : E \to \mathbb{Z}_+$ with $w(\delta(v))$ even for all $v \in V$, the minimum weight of an odd circuit cover is equal to the maximum size of a family of odd circuits such that each edge e is in at most $w(e)$ of them.

There are the following direct implications:

(75.14) strongly bipartite $\Longrightarrow$ evenly bipartite $\Longrightarrow$ weakly bipartite.

It is easy to check that the classes of weakly, evenly, and strongly bipartite signed graphs are closed under taking minors. So each class can be characterized by forbidden minors.

Now a theorem of Seymour [1977b] implies that a signed graph G is strongly bipartite if and only if it has no odd-K_4 minor (Corollary 75.3a below). Guenin [1998a,2001a] showed that G is weakly bipartite if and only if it has no odd-K_5 minor. This was sharpened by Geelen and Guenin [2001], who proved that G is evenly bipartite if and only if G has no odd-K_5 minor (Corollary 75.4a below). So weakly and evenly bipartite are equivalent.

75.4. Characterizing strongly bipartite signed graphs

A general hypergraph theorem of Seymour [1977b] (Theorem 80.1) implies a characterization of strongly bipartite signed graphs. This will be derived from the following equivalent result, which we prove with a method of Geelen and Guenin [2001]:

Theorem 75.3. *In a signed graph G without odd-K_4 minor, the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.*

Proof. For any signed graph $G = (V, E, \Sigma)$, let $\pi(G)$ denote the minimum size of an odd circuit cover and let $\mu(G)$ denote the maximum number of edge-disjoint odd circuits. We must show $\mu(G) = \pi(G)$ for any signed graph G without odd-K_4 minor.

Suppose that this is not true. Choose a counterexample $G = (V, E, \Sigma)$, with $\pi(G)$ minimum, $|V|$ minimum, and $|E|$ maximum, in this order of priority. Such a graph exists, since if there are more than $\pi(G)$ parallel edges connecting two vertices, we can contract them to obtain a counterexample with $|V|$ smaller.

Define $\pi := \pi(G)$. Fix an edge $e = xy$ not contained in every minimum-size odd circuit cover. By adding a parallel edge connecting x and y, we do not change $\pi(G)$ or $|V|$, but we increase $|E|$. Hence in the extended graph there exist π edge-disjoint odd circuits. This means that in the original graph G there exist odd circuits $C_1, \ldots, C_\pi$ with $e \in C_1 \cap C_2$ and with $C_1 \setminus \{e\}$, $C_2 \ldots, C_\pi$ disjoint. (Here we take circuits as edge sets.) We choose the C_i with $|C_1 \cup C_2|$ minimal.

For $i = 1, 2$, let P_i be the $x - y$ path $C_i \setminus \{e\}$, for $i = 1, 2$. (Also the paths are taken as edge sets.) Then

(75.15) $\quad P_1 \cup P_2$ contains no odd circuit C.

Otherwise, replacing C_1 and C_2 by C and $C_1 \triangle C_2 \triangle C$ gives π edge-disjoint odd circuits, a contradiction.

Moreover, let $x = v_0, v_1, \ldots, v_k = y$ be the common vertices of P_1 and P_2, in the order on which they occur along P_1. Then

(75.16) $\quad v_0, v_1, \ldots, v_k$ occur in this order also along P_2.

Indeed, orient P_1 and P_2 from x to y. Then we create no directed circuit, since otherwise there exist circuits $C_1', C_2' \subseteq C_1 \cup C_2$ with $C_1' \cap C_2' = \{e\}$ and $|C_1' \cup C_2'| < |C_1 \cup C_2|$. Then C_1' and C_2' are odd (since otherwise $C_1 \triangle C_1'$ is odd, and hence contains an odd circuit, contradicting (75.15)). This contradicts the minimality of $|C_1 \cup C_2|$.

Now choose j with $0 \leq j \leq k$ such that

(75.17) $\quad \pi(G - (P \cup \{e\})) \leq \pi - 2$

for each $v_j - y$ path P in $P_1 \cup P_2$ and such that j is as large as possible. Such a j exists, as (75.17) holds for each $x - y$ path P in $P_1 \cup P_2$ (otherwise, $G - (P \cup \{e\})$ contains $\pi - 1$ disjoint odd circuits; hence, with $P \cup \{e\}$ it gives π disjoint odd circuits in G as required).

Since $\pi(G) = \pi$ we know $\pi(G - \{e\}) \geq \pi - 1$, and hence $j < k$. By the maximality of j, there is a $v_{j+1} - y$ path R in $P_1 \cup P_2$ such that

(75.18) $\quad \pi(G - (R \cup \{e\})) \geq \pi - 1.$

Let Q_1 and Q_2 be the two $v_j - v_{j+1}$ paths in $P_1 \cup P_2$. By (75.17) we know

(75.19) $\quad \pi(G - (Q_i \cup R \cup \{e\})) \leq \pi - 2$

for $i = 1, 2$. Hence for each $i = 1, 2$ there exists an inclusionwise minimal odd circuit cover B_i with $|B_i \setminus (Q_i \cup R \cup \{e\})| \leq \pi - 2$. So B_i contains one edge of $C_3, \ldots, C_\pi$ each, and consists for the rest of edges in $Q_i \cup R \cup \{e\}$. As $C_1 \cup C_2$ contains an odd circuit disjoint from $Q_i \cup R$, we know $e \in B_i$.

Since B_1 and B_2 are minimal odd circuit covers, there exists a subset U of V with

(75.20) $\quad B_1 \triangle B_2 = \delta(U)$

and U disjoint from e (as $e \notin B_1 \triangle B_2$). So U is disjoint from all $x - v_j$ paths and from the $v_{j+1} - y$ path R' in $P_1 \cup P_2$ edge-disjoint from R (since B_1 and B_2 are edge-disjoint from these paths).

As G has no odd-K_4 minor, there is no path contained in U that connects VQ_1 and VQ_2 and that consists only of edges out of B_1. (It creates with Q_1, Q_2, R', e and any $x - v_j$ path in $P_1 \cup P_2$ an odd K_4-subdivision, as B_1 can serve as a signing.) So U has a subset X such that $VQ_1 \cap U \subseteq X$ and $X \cap VQ_2 = \emptyset$ and such that each edge connecting X and $U \setminus X$ belongs to B_1. So $\delta(X) \subseteq B_1 \cup \delta(U) \subseteq B_1 \cup B_2$. Define

(75.21) $\qquad B := B_1 \triangle \delta(X).$

Then B is an odd circuit cover. We show that $|B \setminus (R \cup \{e\})| \leq \pi - 2$, contradicting (75.18).

Since $U \cap VQ_1 \subseteq X \subseteq U$, we know that $B_1 \cap Q_1 \subseteq \delta(U) \cap Q_1 \subseteq \delta(X)$, and hence $B \cap Q_1 = \emptyset$. Also $B \cap Q_2 = \emptyset$, as $\delta(X)$ contains no edge of Q_2, since X is disjoint from VQ_2.

As $\delta(X) \subseteq B_1 \cup B_2$, we know that $B \subseteq B_1 \cup B_2$. As $|B_i \cap C_h| = 1$ for each $h = 3, \ldots, \pi$, this implies that $|B \cap C_h| \leq 2$, and hence $|B \cap C_h| = 1$ (as it is odd). So we have $|B \setminus (R \cup \{e\})| \leq \pi - 2$. This contradicts (75.18). ∎

This theorem implies a characterization of strongly bipartite graphs:

Corollary 75.3a. *A signed graph G is strongly bipartite if and only if G has no odd-K_4 minor.*

Proof. Necessity follows from the fact that odd-K_4 is not strongly bipartite. To see sufficiency, let $G = (V, E, \Sigma)$ be a signed graph without odd-K_4 minor. Let $w : E \to \mathbb{Z}_+$. We must show that minimizing $w^\mathsf{T} x$ over (75.13) has an integer optimum dual solution.

Let G' arise from G by replacing (in E and in Σ) any edge e by $w(e)$ parallel edges. Then the minimum value of $w^\mathsf{T} x$ over integer vectors x satisfying (75.13) is equal to the minimum size of an odd circuit cover in G'. As G' has no odd-K_4 minor, by Theorem 75.3 this is equal to the maximum number of edge-disjoint odd circuits in G'. This gives an integer optimum dual solution to minimizing $w^\mathsf{T} x$ over (75.13). ∎

To interpret this characterization for (nonsigned) undirected graphs, and to get some subtleties straight, it is good to realize that for any undirected graph $G = (V, E)$ one has:

(75.22) $\qquad$ the signed graph (V, E, E) has odd-K_4 as minor $\iff$ the undirected graph (V, E) has K_4 as odd minor $\iff$ the undirected graph (V, E) contains an odd K_4-subdivision.

(Recall that an undirected graph H is an *odd minor* of an undirected graph G if H arises from G by deleting edges and vertices and contracting all edges in some cut. A subdivision of K_4 is called *odd* if each triangle of K_4 becomes an odd circuit.)

Hence:

Corollary 75.3b. *An undirected graph is strongly bipartite if and only if it contains no odd K_4-subdivision as subgraph.*

Proof. See above. ∎

For multiflows, Seymour's theorem implies (where we take $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, and where $c|E \setminus R$ and $c|R$ are the capacity and demand function, respectively):

Corollary 75.3c. *Let $G = (V, E)$ be a graph and let $R \subseteq E$ be such that the signed graph (V, E, R) has no odd-K_4 minor. Then for each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow.*

Proof. Let $c : E \to \mathbb{Z}_+$ satisfy the cut condition. So for each cut D we have $c(D \cap R) \leq c(D \setminus R)$. Hence for each cut D:

(75.23) $$c(D \triangle R) = c(D \setminus R) + c(R \setminus D) \geq c(D \cap R) + c(R \setminus D) = c(R).$$

So R minimizes $c(R)$ over all odd circuit covers. Therefore, as (V, E, R) has no odd-K_4 minor, by Corollary 75.3a, there exist odd circuits $C_1, \ldots, C_k$ such that each edge e is in at most $c(e)$ of the C_i and such that $k = c(R)$. Hence

(75.24) $$\sum_{i=1}^{k} |C_i \cap R| \leq c(R) = k.$$

This implies, since each $|C_i \cap R|$ is odd, that $|C_i \cap R| = 1$ for each i, and hence we have equality in (75.24). This gives the required multiflow. ∎

75.5. Characterizing weakly and evenly bipartite signed graphs

Guenin [1998a,2001a] showed that odd-K_5 is the only minor-minimal signed graph that is not weakly bipartite (unique up to resigning). It proves a special case of a hypergraph conjecture of Seymour [1977b] (cf. Section 78.3). We prove Guenin's theorem using shortenings of his proof found by Geelen and Guenin [2001] (yielding a similar characterization of evenly bipartite graphs) and Schrijver [2002a].

We use the following lemma on undirected graphs. (Recall that a K_4-subdivision is called *odd* if each triangle of K_4 has become a circuit with an odd number of edges.)

Lemma 75.4α. *Let $G = (V, E)$ be a graph, let u be a vertex of G, and let v_1, v_2, and v_3 be three of its neighbours. Let S_1, S_2, and S_3 be disjoint stable sets in G, with $v_i \in S_i$ for $i = 1, 2, 3$. Suppose that for all distinct $i, j \in \{1, 2, 3\}$, the subgraph induced by $S_i \cup S_j$ contains a $v_i - v_j$ path. Then G contains an odd K_4-subdivision containing the edges uv_1, uv_2, and uv_3.*

Proof. Consider a counterexample with $|V| + |E|$ minimal. So $V = S_1 \cup S_2 \cup S_3 \cup \{u\}$ and E consists of the edges uv_1, uv_2, and uv_3, and of the edges contained in the paths as described. Hence for distinct i, j, there is a unique path $P_{i,j}$ from v_i to v_j contained in $S_i \cup S_j$. Then

(75.25) for distinct i, j: $S_i \cup S_j = VP_{i,j}$.

For if (say) $v \in S_1 \setminus VP_{1,2}$, then v is only on $P_{1,3}$, and hence has degree 2. Then we can contract the two edges incident with v to obtain a smaller counterexample, a contradiction.

(75.25) implies $|S_1| = |S_2| = |S_3|$. If $|S_1| = 1$, $G = K_4$ and we are done. So we can assume that each $|S_i| \geq 2$. Hence each path $P_{i,j}$ has length at least 3. Let v_2' be the second vertex along $P_{1,2}$, v_3' the second vertex along $P_{2,3}$, and v_1' the second vertex along $P_{3,1}$. Contract the edges incident with u. The new vertex u' is adjacent to v_1', v_2', and v_3'. For $i = 1, 2, 3$, let $S_i' := S_i \setminus \{v_i\}$. So S_i' contains v_i' and is a stable set in the contracted graph G'. Moreover,

(75.26) for distinct i, j, $S_i' \cup S_j'$ contains a $v_i' - v_j'$ path.

To prove this, we can assume $i = 1$, $j = 2$. By (75.25), since $v_1' \in S_1$, we know that v_1' is on $P_{1,2}$. Since also v_2' is on $P_{1,2}$, $S_1 \cup S_2$ contains a $v_1' - v_2'$ path avoiding v_1 and v_2. This proves (75.26).

As G' is smaller than G, G' contains an odd K_4-subdivision containing $u'v_1'$, $u'v_2'$, and $u'v_3'$. By decontracting we obtain an odd K_4-subdivision in G as required. ∎

(The proof implies that the odd K_4-subdivision found in fact is a *bad* K_4-subdivision (cf. Section 68.4).)

This lemma is used in the characterization of Geelen and Guenin [2001] of evenly bipartite signed graphs. The following is the kernel of this characterization (a signed graph is called *Eulerian* if its underlying graph is Eulerian):

Theorem 75.4. *In an Eulerian signed graph without odd-K_5 minor, the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.*

Proof. For any signed graph $G = (V, E, \Sigma)$, let $\pi(G)$ denote the minimum size of an odd circuit cover and let $\mu(G)$ denote the maximum number of edge-disjoint odd circuits. It suffices to show $\mu(G) = \pi(G)$ for any Eulerian signed graph G without odd-K_5 minor.

Suppose that this is not true. Choose a counterexample $G = (V, E, \Sigma)$, with $\pi(G)$ minimum, $|V|$ minimum, and $|E|$ maximum, in this order of priority. Such a graph exists, since if there are more than $\pi(G)$ parallel edges connecting two vertices, we can contract them to obtain a counterexample with $|V|$ smaller.

Fix an edge $e = xy$ not contained in every minimum-size odd circuit cover. By adding two parallel edges connecting x and y, we do not change $\pi(G)$ or $|V|$, but we increase $|E|$. Hence in the extended graph there exist $\pi(G)$ edge-disjoint odd circuits. This means that in the original graph G

(75.27) there exist odd circuits $C_1, \ldots, C_{\pi(G)}$ with $e \in C_1 \cap C_2 \cap C_3$ and with $C_1 \setminus \{e\}$, $C_2 \setminus \{e\}$, C_3, $C_4, \ldots, C_{\pi(G)}$ disjoint

(describing circuits by edge sets). $G, C_1, \ldots, C_{\pi(G)}$ moreover satisfy:

(75.28) $\pi(G - C) \leq \pi(G) - 3$ for each odd circuit $C \subseteq C_1 \cup C_2 \cup C_3$ such that $((C_1 \cup C_2 \cup C_3) \setminus C) \cup \{e\}$ contains an odd circuit.

Otherwise, by the minimality of $\pi(G)$, $G - C$ contains disjoint odd circuits $C'_1, \ldots, C'_{\pi(G)-2}$. Then $E' := E \setminus (C \cup C'_1 \cup \cdots \cup C'_{\pi(G)-2})$ contains an odd circuit C'', since E is Eulerian and since G has a minimum-size odd circuit cover B of size $\pi(G)$; so, as B is an equivalent signing of G, $|E' \cap B|$ is odd. Hence $C, C'', C'_1, \ldots, C'_{\pi(G)-2}$ form $\pi(G)$ disjoint odd circuits in G, contradicting our assumption. This proves (75.28).

We show that for signed Eulerian graphs G, conditions (75.27) and (75.28) imply that G has an odd-K_5 minor, which finishes the proof.

We delete our earlier minimality assumptions, and now choose a counterexample to this with $|E|$ minimal and (secondly) $|C_1 \cup C_2 \cup C_3|$ minimal. Let P_i be the $x - y$ path $C_i \setminus \{e\}$ for $i = 1, 2, 3$ (describing paths by edge sets). Then:

Claim 1. P_1, P_2, P_3 are internally vertex-disjoint.

Proof of Claim 1. Suppose not. Define $F := P_1 \cup P_2 \cup P_3$. We first show:

(75.29) F contains no odd circuit.

To see this, first observe that any $P_i \cup P_j$ contains no odd circuit, since otherwise, for the third path P_k there exist $\pi(G) - 2$ disjoint odd circuits in $G - (P_k \cup \{e\})$, contradicting (75.28).

Hence there exists an inclusionwise minimal odd circuit cover B disjoint from $P_1 \cup P_2$. Then for each vertex v in VP_3 that is also in $VP_1 \cup VP_2$, the $x - v$ part of P_3 has an even number of edges in B (as it forms with part of P_1 or P_2 an even cycle). Hence between two contacts of P_3 with $VP_1 \cup VP_2$, P_3 has an even number of edges in B. This implies (75.29).

Orient the edges in $C_1 \cup C_2 \cup C_3$ by orienting each P_i from x to y, and by orienting edge e from y to x. Then

(75.30) F contains no directed circuit C, for otherwise $F \setminus C$ contains three edge-disjoint $x-y$ paths. They yield odd circuits C_1', C_2', C_3' avoiding C, with $e \in C_1' \cap C_2' \cap C_3'$ and with $C_1' \setminus \{e\}$, $C_2' \setminus \{e\}$, $C_3' \setminus \{e\}$ disjoint. This contradicts the minimality of $C_1 \cup C_2 \cup C_3$.

So F is acyclic, and hence there exists a total order $\leq$ on V with $s < t$ for each arc (s,t) in F. So all vertices v in $VP_1 \cup VP_2 \cup VP_3$ have $x \leq v \leq y$.

Then for each undirected $x-y$ path P in F:

(75.31) P is a directed path $\iff$ $(C_1 \cup C_2 \cup C_3) \setminus P$ contains an odd circuit.

To prove $\Rightarrow$, let P be a directed path. Then there exists a directed $x-y$ path edge-disjoint from Q. Hence $Q \cup \{e\}$ is an odd circuit disjoint from $(C_1 \cup C_2 \cup C_2) \setminus P$.

To prove $\Leftarrow$, let C be an odd circuit in $(C_1 \cup C_2 \cup C_3) \setminus P$. Then $C \setminus \{e\}$ is an $x-y$ path Q edge-disjoint from P. If P is not directed, there is a vertex v such that P traverses two arcs entering v. Now there exist precisely three arcs (s,t) with $s < v \leq t$. Hence P contains all three, and nothing is left for Q, a contradiction.

This proves (75.31). So any circuit C qualifies for (75.28) if and only if it is a directed circuit.

Let W be the set of vertices that are in at least two of the P_i. Since P_1, P_2, and P_3 are not internally vertex-disjoint by assumption, we know $|W| \geq 3$.

Call a directed path in F a *link* if it connects two distinct vertices in W, while each internal vertex is not in W. Then:

(75.32) there exist vertices u, v and a $u-v$ link Q such that $u \neq x$ and such that there is at least one directed $u-v$ path edge-disjoint from Q and such that each directed $x-u$ path is a link.

To see this, first observe that there is a directed $x-y$ path P traversing all vertices in W. Indeed, for all $s,t \in W$ with $s < t$, there is a directed $s-t$ path. This follows from the fact that at least two of the P_i leave s and at least two of the P_i enter t, and that hence at least one of the P_i leaves s and enters t.

Now to prove (75.32), let u be the smallest vertex in W with $u \neq x$. Then each directed $x-u$ path is a link. Let Q be a link leaving u which is not on P. Taking for v the end vertex of Q, we obtain (75.32).

Let X be the set of edges that are on directed $u-v$ paths $\neq Q$. We may assume that if C_i intersects X, then C_i traverses both u and v. So X consists of one or two $u-v$ paths. Then

(75.33) $(C_1 \cup C_2 \cup C_3) \setminus (Q \cup X)$ contains no arc leaving u or no arc entering v.

Otherwise, by (75.32), F has three arcs leaving u and three arcs entering v. So each C_i contains a $u-v$ path, which hence is in $Q \cup X$. This proves (75.33).

Consider $G' := G/Q$ and $C'_i := C_i \setminus (Q \cup X)$ for $i = 1, 2, 3$. Then (75.27) is maintained for G', C'_1, C'_2, C'_3. Hence, by the minimality of $C_1 \cup C_2 \cup C_3$, there is a directed circuit C' in G' with

$$\text{(75.34)} \qquad \pi(G' - C') \geq \pi(G') - 2.$$

Now $\pi(G') \geq \pi(G)$ (as this is true for any contraction of G). If C' is also a directed circuit in G, we have $\pi(G - C') \leq \pi(G) - 3$ by (75.28), and hence $G - C'$ has an odd circuit cover B of size $\leq \pi(G) - 3$. By (75.27), B does not intersect Q. Hence B is an odd circuit cover of $G - C'/Q = G' - C'$ of size $\leq \pi(G) - 3$, a contradiction.

So C' is not a directed circuit in G. Then $C' \cup Q$ forms a circuit in G, and, by (75.33), it is a directed circuit. Hence C' contains a link R entering u. As $u \in W$, there is another link, S say, entering u.

Consider $G'' := (G - R)/S$ and $C''_i := C_i \setminus (R \cup S)$ for $i = 1, 2, 3$. Then (75.27) is maintained. Moreover, $\pi(G'') \geq \pi(G)$. For suppose that G'' has an odd circuit cover B of size $\pi(G'') \leq \pi(G) - 1$. By (75.27), $|B| = \pi(G) - 2$ (since it intersects each C_i in an odd number of edges), B does not intersect Q, and contains e. Hence $\pi((G - (R \cup \{e\}))/Q) \leq \pi(G) - 3$. This implies (since $R \cup \{e\} \subseteq C'$):

$$\text{(75.35)} \qquad \pi(G' - C') \leq \pi(G' - (R \cup \{e\})) = \pi(G - (R \cup \{e\})/Q) \leq \pi(G) - 3,$$

contradicting (75.34). This proves that $\pi(G'') \geq \pi(G)$.

Now, by the minimality of G, C_1, C_2, C_3, (75.28) is not maintained. So there is a directed circuit C'' in G'' with $\pi(G'' - C'') \geq \pi(G) - 2$. Then $C'' \cup S$ contains an odd circuit of G, hence also $C'' \cup R$ contains an odd circuit of G (since R and S are parallel links). So (by (75.28) for G) $\pi(G - (C'' \cup R)) \leq \pi(G) - 3$. Hence $G - (C'' \cup R)$ has an odd circuit cover B of size $\pi(G) - 3$, which by (75.27) is disjoint from $F \cup \{e\}$. Then B is an odd circuit cover of $G - (C'' \cup R)/S = G'' - C''$, and so $\pi(G'' - C'') \leq \pi(G) - 3$, contradicting our assumption. *End of Proof of Claim 1*

Set $\pi := \pi(G)$. Since by (75.28), for each $i = 1, 2, 3$, $\pi(G - C_i) \leq \pi - 3$, there is an inclusionwise minimal odd circuit cover B_i of G with $|B_i \setminus C_i| \leq \pi - 3$. By (75.27), we know that $B_i \cap P_j = \emptyset$ for $j \leq 3$ with $j \neq i$, and that $|B_i \cap C_j| = 1$ for $j \geq 4$. Since B_i intersects each of C_1, C_2, C_3, we have $e \in B_i$.

By (75.10), there exist $U_1, U_2, U_3 \subseteq V$ such that

$$\text{(75.36)} \qquad B_j \triangle B_k = \delta(U_i)$$

for distinct $i, j, k \in \{1, 2, 3\}$. We can assume that each U_i is disjoint from e, since $e \notin B_j \triangle B_k$ (as $e \in B_j \cap B_k$). Moreover, we can assume that $U_3 = U_1 \triangle U_2$ — otherwise, just reset $U_3 := U_1 \triangle U_2$. (This works, since $\delta(U_1 \triangle U_2) = \delta(U_1) \triangle \delta(U_2) = (B_2 \triangle B_3) \triangle (B_1 \triangle B_3) = B_1 \triangle B_2$.)

Define

$$\text{(75.37)} \qquad S_i := U_j \cap U_k$$

for distinct $i, j, k \in \{1, 2, 3\}$. So S_1, S_2, S_3 are disjoint and

(75.38) $\quad U_i = S_j \cup S_k$

for distinct $i, j, k \in \{1, 2, 3\}$ (since $U_1 \triangle U_2 \triangle U_3 = \emptyset$). Define

(75.39) $\quad S_0 := V \setminus (S_1 \cup S_2 \cup S_3)$.

Then

(75.40) $\quad$ each $f \in E \setminus (B_1 \cup B_2 \cup B_3)$ is spanned by S_0, S_1, S_2, or S_3.

Otherwise, f belongs to some $\delta(U_i)$, and hence to some B_j, by (75.36).

Moreover,

(75.41) $\quad VP_i \subseteq S_0 \cup S_i$

for each $i \in \{1, 2, 3\}$, since $VP_i \cap \delta(U_i) = \emptyset$ by (75.36) and since $x, y \notin U_i$.

We in fact have for each $i \in \{1, 2, 3\}$:

(75.42) $\quad C_i \subseteq B_i$.

For suppose that $f \in C_1 \setminus B_1$. Then G/f again satisfies (75.27) and (75.28), for $C'_1 := C_1 \setminus \{f\}$, $C'_2 := C_2$, $C'_3 := C_3$. Indeed, each odd circuit C of G/f contained in $C'_1 \cup C'_2 \cup C'_3$ is equal to one of the C'_i, and moreover

(75.43) $\quad \pi((G/f) - C'_i) \leq |B_i \setminus C_i| \leq \pi(G) - 3 \leq \pi(G/f) - 3$.

This contradicts the minimality of $|E|$. So we have (75.42).

Similarly,

(75.44) $\quad$ each $f \in E \setminus (B_1 \cup B_2 \cup B_3)$ is spanned by $VP_1 \cup VP_2 \cup VP_3$.

Otherwise, we can contract f to obtain a smaller example satisfying (75.27) and (75.28) (by (75.43) for $C'_i := C_i$).

Now let E' be the set of edges in $B_1 \triangle B_2 \triangle B_3$ that are in $C_1 \cup C_2 \cup C_3$ or connect two distinct sets among S_1, S_2, S_3. So $C_1 \cup C_2 \cup C_3 \subseteq E'$. As $E' \subseteq B_1 \triangle B_2 \triangle B_3$ and as $B_1 \triangle B_2 \triangle B_3$ is a signing equivalent to Σ, it suffices to show that the undirected graph $G' = (V, E')$ has K_5 as odd minor.

By definition of E', for each $i \in \{1, 2, 3\}$:

(75.45) $\quad S_i$ is a stable set of G'.

Moreover, for all distinct $i, j \in \{1, 2, 3\}$,

(75.46) $\quad G'$ has a path contained in $S_i \cup S_j$ and connecting VP_i and VP_j.

To see this, we may assume $i = 1$, $j = 2$. Suppose that no such path exists. Then U_3 ($= S_1 \cup S_2$) has a subset X such that $S_1 \cap VP_1 \subseteq X$ and $X \cap VP_2 = \emptyset$ and such that no edge of G' connects X and $U_3 \setminus X$. So

(75.47) $\quad \delta_{E'}(X) \subseteq \delta(U_3)$.

Then

(75.48) $\quad \delta_E(X) \subseteq B_1 \cup B_2 \cup B_3$.

Indeed, let $f \in \delta_E(X) \setminus (B_1 \cup B_2 \cup B_3)$. By (75.44), f is spanned by $VP_1 \cup VP_2 \cup VP_3$. Moreover, by (75.40), as f is incident with X, f is spanned by S_1 or S_2. So f is spanned by $S_1 \cap VP_1$ or by $S_2 \cap VP_2$, contradicting the fact that f leaves X. This proves (75.48).

Define

(75.49) $$B := B_1 \triangle \delta_E(X).$$

Then B is an odd circuit cover of G. So $|B| \geq \pi$. Since $S_1 \cap VP_1 \subseteq X$ and $P_1 \subseteq B_1$, we know that $P_1 \subseteq \delta(X)$, and so B is disjoint from P_1. For $i = 2, 3$, B is disjoint from P_i, as $\delta(X)$ contains no edge of P_i, since X is disjoint from VP_i. Hence, as $B \subseteq B_1 \cup B_2 \cup B_3$ by (75.48), we know that $|B \cap C_j| \geq 2$ for some $j = 4, \ldots, \pi$. Then $|B \cap C_j| \geq 3$. As $|B_1 \cap C_j| = 1$ and $|B_2 \cap C_j| = 1$, it follows that there exists an edge $f \in B \cap C_j$ with $f \notin B_1 \cup B_2$. So $f \in \delta_E(X)$, hence $f \in B_3$, therefore $f \in \delta(U_1) \cap \delta(U_2)$, and so $f \in E'$. Therefore, $f \in \delta_{E'}(X)$, and hence by (75.47), $f \in \delta(U_3)$, contradicting (75.36). This proves (75.46).

Consider the minor H of G' obtained by contracting, for each $i = 1, 2, 3$, $VC_i \setminus \{x, y\}$ to one vertex, z_i say. By Lemma 75.4α, $H - y$ has an odd K_4-subdivision containing the edges xz_1, xz_2, and xz_3. Since y is adjacent to x, z_1, z_2, and z_3, H has K_5 as odd minor. ∎

A consequence of this is a characterization of weakly and evenly bipartite graphs. (The equivalence of (i) and (iii) is Guenin's theorem (Guenin [1998a,2001a]), and the equivalence with (ii) was found by Geelen and Guenin [2001].)

Corollary 75.4a. *For any signed graph G the following are equivalent:*

(75.50) (i) *G is weakly bipartite;*
(ii) *G is evenly bipartite;*
(iii) *G has no odd-K_5 minor.*

Proof. The implications (ii)⇒(i)⇒(iii) follow from (75.14) and from the facts that weak bipartiteness is closed under taking minors and that odd-K_5 is not weakly bipartite.

The implication (iii)⇒(ii) follows from Theorem 75.4. Let $G = (V, E, \Sigma)$ be a signed graph without odd-K_5 minor and let $c : E \to \mathbb{Z}_+$ be such that $c(\delta(v))$ is even for each $v \in V$. We must show that the dual of minimizing $c^{\mathsf{T}}x$ over (75.13) has an integer optimum dual solution.

Let G' arise from G by replacing (in E and in Σ) any edge e by $c(e)$ parallel edges. So G' is Eulerian. Then the minimum value of $c^{\mathsf{T}}x$ over integer vectors x satisfying (75.13) is equal to the minimum size of an odd circuit cover in G'. As G' has no odd-K_5 minor, by Theorem 75.4 this is equal to the maximum number of edge-disjoint odd circuits in G'. This gives an integer optimum dual solution to minimizing $c^{\mathsf{T}}x$ over (75.13). ∎

For (nonsigned) undirected graphs, this characterization can be described in terms of odd minors as follows. (Recall that an undirected graph H is an *odd minor* of an undirected graph G if H arises from G by deleting edges and vertices and contracting all edges in some cut.) Then for any undirected graph $G = (V, E)$:

(75.51) the signed graph (V, E, E) has odd-K_5 as minor $\iff$ the undirected graph (V, E) has K_5 as odd minor.

(This is a simple exercise.) Hence:

Corollary 75.4b. *An undirected graph G is weakly bipartite if and only if K_5 is not an odd minor of G.*

Proof. See above. ∎

Notes. Special cases of the equivalence of (i) and (iii) in Corollary 75.4a were shown by Barahona [1980] (for planar graphs; cf. Theorem 75.2), Fonlupt, Mahjoub, and Uhry [1992] (for graphs without K_5 minor), Barahona [1983a] (for graphs G such that $G - u - v$ is bipartite for two of its vertices u, v), and Gerards [1992a] (for graphs G such that $G - v$ is planar with at most two odd faces, for some vertex v).

75.6. Applications to multiflows

Geelen and Guenin's theorems also have consequences for multiflows (where again we take $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, and where $c|E \setminus R$ and $c|R$ are the capacity and demand function, respectively):

Corollary 75.4c. *Let $G = (V, E)$ be a graph and let $R \subseteq E$ be such that the signed graph (V, E, R) has no odd-K_5 minor. Then for each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a fractional multiflow. If moreover c is integer, there is a half-integer multiflow. If moreover the Euler condition holds, there is an integer multiflow.*

Proof. By Corollary 75.4a, (V, E, R) is weakly bipartite. Let c satisfy the cut condition. So for each cut D we have $c(D \cap R) \leq c(D \setminus R)$. Hence for each cut D:

(75.52) $$c(D \triangle R) = c(D \setminus R) + c(R \setminus D) \geq c(D \cap R) + c(R \setminus D) = c(R).$$

So R minimizes $c(R)$ over all odd circuit covers. Therefore, as (V, E, R) is weakly bipartite, there exist odd circuits $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k > 0$ with

(75.53) $$\sum_{i=1}^{k} \lambda_i \chi^{C_i} \leq c \text{ and } \sum_{i=1}^{k} \lambda_i = c(R).$$

Hence

(75.54) $$\sum_{i=1}^{k} \lambda_i |C_i \cap R| \leq c(R) = \sum_{i=1}^{k} \lambda_i.$$

This implies, since each $|C_i \cap R|$ is odd, that $|C_i \cap R| = 1$ for each i, and that we have equality in (75.54). This gives the required multiflow.

The integrality results follow from Theorem 75.4. ∎

This implies the following generalization of Theorem 74.3, due to Seymour [1981a] (who derived it from Theorem 74.3 by using Wagner's theorem on the decomposition of K_5-free graphs (Theorem 3.3)):

Corollary 75.4d. *A graph $G = (V, E)$ has no K_5 minor if and only if for each $R \subseteq E$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow. Moreover, if G has no K_5 minor and c is integer, the cut condition implies the existence of a half-integer multiflow. If moreover the Euler condition holds, then it implies the existence of an integer multiflow.*

Proof. Directly from Corollary 75.4c. ∎

For planar graphs, these integrality results can be derived also from results on packing T-cuts (Theorem 29.2), using duality like in Theorem 75.2 (cf. Theorem 74.3).

75.7. The cut cone and the cut polytope

Let $G = (V, E)$ be an undirected graph. Recall that a subset C of E is called a *cut* if $C = \delta(U)$ for some $U \subseteq V$. The *cut polytope* $P_{\text{cut}}(G)$ of G is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of cuts in G:

(75.55) $$P_{\text{cut}}(G) := \text{conv.hull}\{\chi^C \mid C \text{ cut in } G\}.$$

As $\emptyset$ is a cut, the cut polytope contains the origin.

Since a set of edges is bipartite if and only if it is contained in a cut, the bipartite subgraph polytope can be expressed in terms of the cut polytope:

(75.56) $$P_{\text{bipartite subgraph}}(G) = \{x \in \mathbb{R}_+^E \mid \exists y \geq x : y \in P_{\text{cut}}(G)\}.$$

Any vector x in the cut polytope of G satisfies

(75.57)
(i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(F) - x(C \setminus F) \leq |F| - 1$ for each circuit C and $F \subseteq C$ with $|F|$ odd.

A full characterization is known of those graphs for which (75.57) determines the cut polytope: they are the graphs without K_5 minor (Seymour [1981a], Barahona [1983b]). This can be deduced from the characterization of weakly bipartite graphs.

This characterization can be formulated equivalently in terms of the *cut cone* of a graph $G = (V, E)$, which is the convex cone generated by the incidence vectors of the cuts. Necessary conditions for its elements are:

$$(75.58) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x_f \leq x(C \setminus \{f\}) & \text{for each circuit } C \text{ and } f \in C. \end{array}$$

The graph K_5 shows that these conditions generally are not sufficient: fix distinct $u, v \in VK_5$; then $x := \mathbf{2} - \chi^{\delta(\{u,v\})}$ satisfies (75.58). However, x does not belong to the cut cone of K_5, since the incidence vector z of any cut satisfies $2z(\delta(\{u, v\})) \geq z(EK_5)$.

K_5 is the only minor-minimal example, as Seymour [1981a] showed:

Corollary 75.4e. *The cut cone is determined by* (75.58) *if and only if G has no K_5 minor.*

Proof. Necessity is shown by the example above, and by the closedness of the property under taking minors. If G has the property, and we contract an edge e, then any x satisfying (75.58) for G/e can be extended to a vector x satisfying (75.58) for G by defining $x_e := 0$. Then the extended x is in the cut cone of G, and hence the original x is in the cut cone of G/e.

If we delete e, extend x satisfying (75.58) for $G-e$ by defining x_e to be the distance in $G - e$ between the end vertices of e, taking x as length function. Again, the extended x is in the cut cone of G, and hence the original x is in the cut cone of $G - e$. This shows necessity.

To see sufficiency, let G have no K_5 minor. Let $c^{\mathsf{T}}x \geq 0$ be a valid inequality for the cut cone. Define $R := \{e \in E \mid c(e) < 0\}$. Taking $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, the cut condition holds for capacity function $c|E \setminus R$ and demand function $-c|R$. By Corollary 75.4d, there exists a multiflow subject to $c|E \setminus R$ and of value $-c|R$. It means that c is a nonnegative combination of vectors $-\chi^f + \chi^{C \setminus \{f\}}$ where C is a circuit and $f \in C$, and of vectors χ^e where $e \in E \setminus R$. Hence the inequality $c^{\mathsf{T}}x \geq 0$ is a nonnegative linear combination of the inequalities (75.58). ∎

Using the symmetry of the cut polytope (as observed by Barahona and Grötschel [1986]), Corollary 75.4e has as a consequence (Barahona [1983b]):

Corollary 75.4f. *The cut polytope of a graph $G = (V, E)$ is determined by* (75.57) *if and only if G has no K_5 minor.*

Proof. By Corollary 75.4e, it suffices to show that the cut polytope is determined by (75.57) if and only if the cut cone is determined by (75.58).

First assume that the cut polytope is determined by (75.57). Since the origin belongs to the cut polytope, the cut cone is determined by the inequalities among (75.57) with right-hand side 0 — that is, by (75.58).

Conversely, assume that the cut cone is determined by (75.58). Then (75.58) determines the cut polytope in the neighbourhood of the origin.

Consider now any vertex χ^D of the cut polytope, where D is a cut of G. For each $x \in \mathbb{R}^E$, define $\tilde{x} \in \mathbb{R}^E$ by:

$$(75.59) \qquad \tilde{x}_e := \begin{cases} 1 - x_e & \text{if } e \in D, \\ x_e & \text{if } e \notin D. \end{cases}$$

The function $x \to \tilde{x}$ brings the cut polytope to itself (since $D' \triangle D$ is a cut for any cut D' and $\tilde{x} = \chi^{D' \triangle D}$ if $x = \chi^{D'}$), and χ^D to $\mathbf{0}$. Since the cut cone is determined by (75.58), it implies that in the neighbourhood of χ^D, the cut polytope is determined by the inequalities (75.58) applied to $\tilde{x}$:

$$(75.60) \qquad \begin{array}{lll} \text{(i)} & \tilde{x}_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & \tilde{x}_f \leq \tilde{x}(C \setminus \{f\}) & \text{for each circuit } C \text{ and } f \in C. \end{array}$$

Now inequality (75.60)(i) follows from (75.57)(i). To see inequality (75.60)(ii), we first consider the case $f \notin D$. Define $F := (C \cap D) \cup \{f\}$. Then, using (75.57)(ii), (75.60)(ii) follows from

$$(75.61) \qquad \begin{aligned} \tilde{x}_f = x_f = x(F) - x(C \cap D) &\leq x(C \setminus F) - x(C \cap D) + |F| - 1 \\ &= \tilde{x}(C \setminus F) + \tilde{x}(C \cap D) = \tilde{x}(C \setminus \{f\}). \end{aligned}$$

If $f \in D$, define $F := (C \cap D) \setminus \{f\}$. Then, again using (75.57)(ii), (75.60)(ii) follows from

$$(75.62) \qquad \begin{aligned} \tilde{x}_f = 1 - x_f = 1 + x(F) - x(C \cap D) &\leq x(C \setminus F) - x(C \cap D) + |F| \\ &= x(C \setminus D) + x_f - x(C \cap D) + |C \cap D| - 1 = \tilde{x}(C \setminus \{f\}). \end{aligned} \qquad \blacksquare$$

Notes. By Corollary 75.4f, the cut polytope of any planar graph is determined by (75.57). As cuts in planar graphs correspond to $\emptyset$-joins ($\equiv$ cycles) in the dual graph (Orlova and Dorfman [1972]), one may also derive this from Corollary 29.2e on the T-join polytope.

Hadlock [1975] showed in a similar way that a maximum-capacity cut in a planar graph can be found in strongly polynomial time. Using the decomposition of graphs without K_5 minors into planar graphs and copies of V_8 (Theorem 3.3), Barahona [1983b] derived from this a combinatorial strongly polynomial-time algorithm to find a maximum-capacity cut in graphs without K_5 minor.

Poljak [1992] showed that for each graph G, the polytope determined by (75.2) is the down hull of the polytope determined by (75.57).

Karzanov [1985b] showed that the separation problem for the cut cone is co-NP-complete.

Barahona and Mahjoub [1986] showed that the separation problem over (75.57) is polynomial-time solvable, hence any linear objective function can be optimized over (75.57) in strongly polynomial time (with the ellipsoid method).

Integer decomposition. What about integer decomposition in the cut cone? A theorem of Chvátal [1980] implies that it is NP-complete to decompose a given metric as a nonnegative integer sum of incidence vectors of cuts. Let $\mathcal{H}$ be the class of graphs such that each integer vector x in the cut cone with $x(C)$ even for each circuit C, is a nonnegative integer combination of incidence vectors of cuts. (Equivalently, the incidence vectors of the cuts form a Hilbert base.)

By (74.34), each planar graph belongs to $\mathcal{H}$. This was extended (using Wagner's theorem (Theorem 3.3)) by Fu and Goddyn [1999] who showed that each graph without K_5 minor belongs to $\mathcal{H}$.

Goddyn [1993] also conjectured that each graph not having the Petersen graph as minor, belongs to $\mathcal{H}$. However, Laurent [1996b] showed that K_6 does not belong to $\mathcal{H}$. She also showed that all proper subgraphs of K_6 belong to $\mathcal{H}$. (More on this can be found in Laburthe [1995] and in the survey by Goddyn [1993].)

Fu and Goddyn [1999] asked: is $\mathcal{H}$ closed under taking minors?

Metrics and hypermetrics. The following *metric inequalities* are valid for the vectors in the cut cone of the complete graph K_V on a vertex set V:

(75.63) (i) $x_{uv} \geq 0$ for distinct $u, v \in V$,
(ii) $x_{uv} + x_{vw} \geq x_{uw}$ for distinct $u, v, w \in V$.

The cone determined by these inequalities is called the *metric cone*.

Tylkin [1960,1962] (= M.M. Deza) introduced a stronger set of valid inequalities, the *hypermetric inequalities*:

(75.64) (i) $x_{uv} \geq 0$ for distinct $u, v \in V$,
(ii) $\displaystyle\sum_{\substack{u,v \in V \\ u \neq v}} c_u c_v x_{uv} \leq 0$ for each $c : V \to \mathbb{Z}$ with $c(V) = 1$.

These inequalities are valid for the vectors in the cut cone, since for each cut $\delta(U)$ one has (setting $x := \chi^{\delta(U)}$):

(75.65)
$$\begin{aligned}\sum_{\substack{u,v \in V \\ u \neq v}} c_u c_v x_{uv} &= 2 \sum_{u\in U} \sum_{v \in V\setminus U} c_u c_v = 2c(U)c(V \setminus U) \\ &= \tfrac{1}{2}(c(U) + c(V \setminus U))^2 - \tfrac{1}{2}(c(U) - c(V \setminus U))^2 \\ &= \tfrac{1}{2} - \tfrac{1}{2}(c(U) - c(V \setminus U))^2 \leq 0,\end{aligned}$$

since $|c(U) - c(V \setminus U)| \geq 1$, as $c(V) = 1$ and c is integer.

Hypermetric inequalities generalize the metric inequalities, since (75.63)(ii) is equivalent to taking $c := \chi^u + \chi^w - \chi^v$ in (75.64)(ii).

The cone determined by (75.64) is called the *hypermetric cone*. Deza, Grishukhin, and Laurent [1993] showed that this cone is polyhedral (despite that there are infinitely many inequalities in (75.64)(ii)).

Avis and Grishukhin [1993] showed that it is co-NP-complete to decide if a given vector is in the hypermetric cone. Relations with the geometry of numbers are given in Deza, Grishukhin, and Laurent [1995]. More on the metric cone can be found in Avis [1980b,1980c], Grishukhin [1992], Laurent and Poljak [1992,1995b], Lomonosov and Sebő [1993], and Laurent [1996a], and on metrics and hypermetrics in the book by Deza and Laurent [1997].

75.8. The maximum cut problem and semidefinite programming

The maximum-capacity cut problem has a natural semidefinite relaxation. Let V be a finite set and denote

(75.66) $\mathcal{M}_V :=$ the set of all symmetric positive semidefinite $V \times V$ matrices M with $M_{v,v} = 1$ for each $v \in V$.

Let $c : V \times V \to \mathbb{R}_+$ be a 'capacity' function, with $c(u,v) = c(v,u)$ for all $u, v \in V$. Consider c as a capacity function on the complete graph K_V on V. Let C be the $V \times V$ matrix with (u,v)th entry equal to $c(u,v)$. The maximum-capacity cut problem asks for the maximum of

(75.67) $$\sum_{u \in U} \sum_{v \in V \setminus U} c(u,v).$$

Theorem 75.1 implies that this is an NP-complete problem.

A relaxation is to maximize

(75.68) $$\tfrac{1}{4}\mathrm{Tr} C(J - M)$$

over $M \in \mathcal{M}_V$. If we restrict M to matrices of rank 1 (so $M = xx^{\mathsf{T}}$ for some $\{-1,+1\}$ vector x in $\mathbb{R}^V$), we have the maximum-capacity cut problem.

Goemans and Williamson [1994,1995b] showed the following surprising bound (surprising also since the proof is very simple). Define

(75.69) $$\alpha := \min_{0 < \phi \leq \pi} \frac{\phi}{1 - \cos\phi} \frac{2}{\pi} = 0.87856....$$

(The latter estimate results from a numerical computation.)

Theorem 75.5. *Let μ be the maximum capacity of a cut and let ν be the maximum value of* (75.68). *Then*

(75.70) $$\alpha\nu \leq \mu \leq \nu.$$

Proof. The inequality $\mu \leq \nu$ was shown above. To see the first inequality, let M maximize (75.68). As M is positive semidefinite, there exist vectors $x_v \in \mathbb{R}^n$ for $v \in V$ such that $x_u^{\mathsf{T}} x_v = M_{u,v}$ for all $u, v \in V$. (Here $n := |V|$.) So $\|x_v\| = 1$ for each $v \in V$.

For any hyperplane H in $\mathbb{R}^n$ with $\mathbf{0} \in H$, let D be the set of edges uv of K_V with u and v at different sides of H. Choosing H at random, the set D is a cut, with probability 1. Any edge uv of K_V is in D with probability

(75.71) $$\frac{\angle(x_u, x_v)}{\pi}.$$

($\angle(a,b)$ is the angle of a and b.) This follows from the fact that (75.71) is the probability that x_u and x_v are at different sides of H.

So the expected value of the capacity of D is equal to

(75.72) $$\sum_{uv \in EK_V} \frac{\angle(x_u, x_v)}{\pi} c(u,v).$$

Now if $\phi = \angle(x_u, x_v)$, then $x_u^{\mathsf{T}} x_v = \cos\phi$. Hence, by definition of α, (75.73) $$\frac{\angle(x_u, x_v)}{\pi} = \frac{\phi}{\pi} \geq \tfrac{1}{2}\alpha(1 - \cos\phi) = \tfrac{1}{2}\alpha(1 - x_u^{\mathsf{T}} x_v).$$

Hence (75.72) is at least

(75.74) $$\sum_{uv \in EK_V} \alpha \cdot \tfrac{1}{2}c(u,v)(1 - x_u^{\mathsf{T}} x_v) = \tfrac{1}{4}\alpha \mathrm{Tr} C(J - M) = \alpha\nu.$$

Concluding, there exists a cut of capacity at least $\alpha\nu$. So $\mu \geq \alpha\nu$. ∎

Since the separation problem over $\mathcal{M}_V$ is solvable in polynomial time, in a certain approximation model (cf. Grötschel, Lovász, and Schrijver [1988]), with the ellipsoid method, one can optimize any linear objective function over $\mathcal{M}_V$ in strongly polynomial time, or rather approximate the optimum. Hence the value of ν can be approximated in polynomial time. As Goemans and Williamson [1994,1995b] pointed out, this gives a randomized polynomial-time algorithm to find a cut of capacity at least $\alpha\nu \geq 0.87856\nu$: choosing a random hyperplane H as above gives a random cut of expected capacity as required. By derandomization, such a cut can in fact be found deterministically in polynomial-time (Mahajan and Ramesh [1995,1999]).

This approach also gives a *relaxation* ($\equiv$ superset) of the cut polytope. Indeed, let $G = (V, E)$ be an undirected graph. For any $M \in \mathcal{M}_V$, define $x_M : E \to \mathbb{R}_+$ by:

(75.75) $$x_M(e) := \tfrac{1}{2}(1 - M_{u,v})$$

for $e = uv \in E$. Then

(75.76) $$P_{\mathrm{cut}}(G) \subseteq K := \{x_M \mid M \in \mathcal{M}_V\},$$

since for each cut $\delta(U)$, the matrix M given by

(75.77) $$M := (\mathbf{1} - 2\chi^U)(\mathbf{1} - 2\chi^U)^{\mathsf{T}}$$

belongs to $\mathcal{M}_V$ and satisfies $x_M = \chi^{\delta(U)}$.

So K is a relaxation of the cut polytope. With the ellipsoid method, one can optimize over $\mathcal{M}_V$, and hence over K in polynomial time. What Goemans and Williamson's theorem tells is that for nonnegative $c : E \to \mathbb{R}_+$, maximizing $c^{\mathsf{T}}x$ over K has only a small relative error compared to maximizing over $P_{\mathrm{cut}}(G)$. In other words:

(75.78) $$K \subseteq \alpha^{-1} \cdot P_{\text{bipartite subgraph}}(G).$$

Here we use that $P_{\text{bipartite subgraph}}(G)$ is the down hull in $\mathbb{R}_+^E$ of $P_{\mathrm{cut}}(G)$.

Feige and Schechtman [2001,2002] showed that for each $\varepsilon > 0$ there is a graph for which the ratio of the semidefinite programming bound ν and the maximum cut-size is no better than $\alpha + \varepsilon$.

Notes. Before Goemans and Williamson found their theorem, only a factor of 2 was known to be achievable in polynomial time, by just taking $c(E)$ as upper bound. This gives a factor 2, since a random cut has expected capacity $\frac{1}{2}c(EK_V)$, as each edge has probability $\frac{1}{2}$ to be in the random cut (Johnson and Lafuente [1970] and Sahni and Gonzalez [1976]).

Håstad [1997,2001] showed that if NP≠P, then there is no polynomial-time algorithm that finds a cut of capacity more than $\frac{16}{17}$ of the maximum cut-capacity (cf. Trevisan, Sorkin, Sudan, and Williamson [1996,2000]).

Related work can be found in Bellare, Goldreich, and Sudan [1995,1998], Karloff [1996,1999], Zwick [1999b], Alon and Sudakov [2000], and Alon, Sudakov, and Zwick [2001,2002].

Earlier eigenvalue methods for the maximum cut problem include Poljak [1992] and Delorme and Poljak [1993a,1993b,1993c].

Surveys on semidefinite methods for the maximum cut problem (and more generally in combinatorial optimization) are given by Goemans [1997], Reed [2001a], and Laurent and Rendl [2002]. Alizadeh [1995] gives a survey of applying interior-point methods to semidefinite programming in combinatorial optimization. More on the semidefinite relaxation of the cut polytope can be found in Laurent and Poljak [1995a,1996a,1996b]. Other approximation algorithms for the maximum cut problem were given by Arora, Karger, and Karpinski [1995,1999], Fernandez de la Vega [1996], Frieze and Kannan [1996,1999], Fernandez de la Vega and Kenyon [1998,2001], Feige and Langberg [2001], and Halperin, Livnat, and Zwick [2002].

An extension of the semidefinite programming bound to 3-cuts was given by Goemans and Williamson [2001]. For extensions to directed graphs, see Feige and Goemans [1995], Matuura and Matsui [2001], and Lewin, Livnat, and Zwick [2002].

For a survey on approximation algorithms, see Shmoys [1995] and the book by Vazirani [2001].

75.9. Further results and notes

75.9a. Cuts and stable sets

The vertex cover polytope of a graph $G = (V, E)$ can be considered as a face of the cut polytope of the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ obtained from G by adding one new vertex u adjacent to all vertices of G. Since the stable set polytope can be expressed in terms of the vertex cover polytope (as S is a stable set if and only if $V \setminus S$ is a vertex cover), this gives a relation between cuts and stable sets.

To see the relation between $P_{\text{vertex cover}}(G)$ and $P_{\text{cut}}(\widetilde{G})$, first note that each $x \in P_{\text{cut}}(\widetilde{G})$ satisfies

(75.79) $\qquad x(T) \leq 2$ for each triangle $T \subseteq \widetilde{E}$.

Therefore, the set of vectors x in $P_{\text{cut}}(\widetilde{G})$ satisfying

(75.80) $\qquad x(T) = 2$ for each triangle $T = \{uv, uw, vw\}$ containing u

(so vw is an edge of G), forms a face F of $P_{\text{cut}}(\widetilde{G})$.

Now we project $\mathbb{R}^{\widetilde{E}}$ on $\mathbb{R}^{\widetilde{E}\setminus E}$ by deleting the coordinates indexed by E. Moreover, we identify any edge uv in $\widetilde{E} \setminus E$ with vertex v of G. This brings F one-to-one to the vertex cover polytope of G.

More precisely, define a projection $\pi : \mathbb{R}^{\widetilde{E}} \to \mathbb{R}^V$ by $\pi(x)_v := x_{uv}$ for $v \in V$ and $x \in \mathbb{R}^{\widetilde{E}}$. Then:

Theorem 75.6. *$\pi|F$ is a bijection between F and $P_{\text{vertex cover}}(G)$.*

Proof. First, $\pi|F$ is injective, since if $\pi(x) = \pi(y)$ for $x, y \in F$, then for each $v \in V$, $x_{uv} = y_{uv}$. Hence, by (75.80), for each $vw \in E$, $x_{vw} = 2 - x_{uv} - x_{uw} = 2 - y_{uv} - y_{uw} = y_{vw}$. So $x = y$.

To see that $\pi(F) \subseteq P_{\text{vertex cover}}(G)$, let C be a cut in $\widetilde{G}$ with $\chi^C \in F$ (that is, χ^C is a vertex of F). Then for each edge vw of G, precisely two of the edges uv, uw, vw belong to C. Hence at least one of uv, uw belongs to C. So $\pi(\chi^C)$ is the incidence vector of a vertex cover of G.

Conversely, to see $P_{\text{vertex cover}}(G) \subseteq \pi(F)$, let U be a vertex cover of G. So $U \subseteq V$. Let C be the cut in $\widetilde{G}$ determined by U. Then χ^C belongs to F, since for each edge vw of G we have that precisely two of uv, uw, vw belong to C. Moreover, $\pi(\chi^C) = \chi^U$, since for each $v \in V$: $v \in U \iff uv \in C$. ∎

The relation given in this theorem can be useful when we have a good description of the cut polytope for certain classes of graphs. The description then can be transferred to the vertex cover polytope, hence to the stable set polytope, for certain derived classes of graphs.

In particular, we can derive from Guenin's theorem the t-perfection of graphs without odd K_4-subdivision (a consequence of Theorem 68.3 (Gerards and Schrijver [1986])):

Theorem 75.7. *A graph G without odd K_4 subdivision as subgraph is t-perfect.*

Proof. Let $G = (V, E)$ be a graph without odd K_4-subdivision as subgraph. By (75.22), G has no K_4 as odd minor. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be the graph obtained from G by adding a new vertex u, adjacent to all vertices in V.

Then $\widetilde{G}$ has no K_5 as odd minor. For suppose it has. Then by deleting the vertex from the K_5 to which u has been contracted (if any) we obtain a graph being K_4 or K_5. It implies that G has K_4 as odd minor, a contradiction.

Now let $y \in \mathbb{R}^V$ satisfy

$$(75.81) \qquad \begin{array}{ll} 0 \leq y_v \leq 1 & \text{for } v \in V, \\ y_u + y_v \leq 1 & \text{for } uv \in E, \\ y(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C. \end{array}$$

Define $x \in \mathbb{R}^{\widetilde{E}}$ by

$$(75.82) \qquad \begin{array}{ll} x(vw) := y_v + y_w & \text{for each edge } vw \text{ of } G, \\ x(uv) := 1 - y_v & \text{for each } v \in V. \end{array}$$

Then x satisfies (75.2) with respect to $\widetilde{G}$. Indeed, (75.2)(i) trivially holds. To see (ii), we can restrict ourselves to chordless odd circuits C. If C traverses u, it is a triangle containing u, and we have $x(EC) = 2 = |VC| - 1$. If C does not traverse u, then $x(EC) = 2y(VC) \leq |VC| - 1$.

So by Corollary 75.4a, x is a convex combination of incidence vectors of bipartite subgraphs B:

$$(75.83) \qquad x = \sum_B \lambda_B \chi^B.$$

Since $x(C) = 2$ for each triangle C containing u, those B with $\lambda_B > 0$ intersect each such C in precisely two edges. Hence (since each circuit in $\widetilde{G}$ is a symmetric difference of triangles containing u) B intersects each circuit in an even number of edges. So B is a cut $\delta(U)$ of $\widetilde{G}$. We can assume that $u \notin U$. Then $V \setminus U$ is a stable set of G, and

(75.84) $$y = \sum_U \lambda_{\delta(U)} \chi^{V \setminus U}$$

describes y as a convex combination of incidence vectors of stable sets. ∎

It should be noted that the face F of the cut polytope described above is at the same time a face of the (larger) bipartite subgraph polytope (while the cut polytope need not be a face of the bipartite subgraph polytope). Indeed, also the bipartite subgraph polytope satisfies (75.79). Moreover, any set B of edges having even intersection with the triangle $\{uv, uw, vw\}$ for each edge vw of G, has even intersection with each circuit of $\widetilde{G}$, as it is a binary sum of such triangles. So B is a cut.

Laurent, Poljak, and Rendl [1997] showed how the set TH(G) (defined in Section 67.4a) can be derived as an affine image from the convex body K in Section 75.8.

75.9b. Further notes

Chvátal, Cook, and Hartmann [1989] showed that the Chvátal rank of system (75.2) is at least $\frac{1}{4}|V| - 1$ for complete graphs $G = (V, E)$.

Conforti and Gerards [2000] described (by forbidden odd minors) another class of Eulerian graphs for which the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.

Barahona [1983b] showed that the maximum-size cut problem is NP-complete for *apex graphs*, that is graphs G having a vertex v with $G - v$ planar. (More strongly, Barahona proved NP-completeness if $G - v$ is cubic and planar.)

Grötschel and Nemhauser [1984] showed that for each fixed k there is a polynomial-time algorithm to solve the maximum-capacity cut problem for graphs without odd circuits of length $\geq k$.

Facets of the bipartite subgraph polytope were studied by Barahona, Grötschel, and Mahjoub [1985] (cf. Gerards [1985]), and facets of the cut polytope and cut cone by Barahona and Mahjoub [1986], De Simone [1989,1990], Deza and Laurent [1990, 1992a,1992b], Deza, Laurent, and Poljak [1992], and Laurent and Poljak [1996a]. Compositions in the bipartite subgraph polytope were given by Fonlupt, Mahjoub, and Uhry [1992].

Conforti and Rao [1987] showed that a minimum-weight odd circuit cover can be found in strongly polynomial time, if its weight is less than the minimum weight of a nonempty cut.

For more geometric background on the cut cone and the cut polytope, see the book by Deza and Laurent [1997]. Gerards [1990] gave a survey on signed graphs without odd K_4-subdivision. For more background on the relations between odd circuits and multicommodity flows, see Sebő [1990a] and Gerards [1993]. For surveys on maximum cut and the cut cone, see Deza, Grishukhin, and Laurent [1995] (hypermetrics) and Poljak and Tuza [1995]. For related work, see Conforti, Rao, and

Sassano [1990a,1990b], Jerrum and Sorkin [1993,1998], Feige and Goemans [1995], Frieze and Jerrum [1995,1997], Ageev and Sviridenko [1999], Ageev, Hassin, and Sviridenko [2001], Feige and Langberg [2001], Halperin and Zwick [2001a,2001b, 2002], Ye [2001], Han, Ye, and Zhang [2002], and Lewin, Livnat, and Zwick [2002].

Chapter 76

Homotopy and graphs on surfaces

As we saw in Chapter 74, disjoint paths and multiflow problems are generally hard even for planar graphs. In some special cases, these problems are polynomial-time solvable.
If we require the paths (or flows) to have certain homotopies, the range of polynomial-time solvable problems can be extended. By enumerating homotopies, it sometimes implies polynomial-time solvability for nonhomotopic versions of the problems.
This can be extended to general surfaces and yield polyhedral characterizations for circulations, flows, and paths of prescribed homotopies.

76.1. Graphs, curves, and their intersections: terminology and notation

Let S be a compact surface. A *closed curve* on S is a continuous function $C : S^1 \to S$, where S^1 is the unit circle in $\mathbb{C}$. It is *simple* if C is one-to-one.

Two closed curves C and C' are called *freely homotopic*, in notation $C \sim C'$, if there exists a continuous function bringing C to C'; that is, a continuous function $\Phi : S^1 \times [0,1] \to S$ with $\Phi(z,0) = C(z)$ and $\Phi(z,1) = C'(z)$ for each $z \in S^1$.

For any pair of closed curves C, D on S, let $\mathrm{cr}(C,D)$ denote the number of intersections of C and D, counting multiplicities:

$$(76.1) \qquad \mathrm{cr}(C,D) := |\{(w,z) \in S^1 \times S^1 \mid C(w) = D(z)\}|.$$

Moreover, $\mathrm{mincr}(C,D)$ denotes the minimum of $\mathrm{cr}(C',D')$ where C' and D' range over closed curves freely homotopic to C and D, respectively:

$$(76.2) \qquad \mathrm{mincr}(C,D) := \min\{\mathrm{cr}(C',D') \mid C' \sim C, D' \sim D\}.$$

Similarly, $\mathrm{cr}(C)$ denotes the number of self-intersections of C:

$$(76.3) \qquad \mathrm{cr}(C) := \tfrac{1}{2}|\{(w,z) \in S^1 \times S^1 \mid C(w) = C(z), w \neq z\}|,$$

and $\mathrm{mincr}(C)$ denotes the minimum of $\mathrm{cr}(C')$ where C' ranges over all closed curves freely homotopic to C:

(76.4) $\quad \text{mincr}(C) := \min\{\text{cr}(C') \mid C' \sim C\}.$

As is well-known, mincr(C, D) and mincr(C) are finite numbers.

Let $G = (V, E)$ be an undirected graph embedded in S. We identify G with its topological graph, and with its embedding in S.

For any closed curve D on S, cr(G, D) denotes the number of intersections of G and D (counting multiplicities):

(76.5) $\quad \text{cr}(G, D) := |\{z \in S^1 \mid D(z) \in G\}|.$

Moreover, mincr(G, D) denotes the minimum of cr(G, D') where D' ranges over all closed curves freely homotopic to D and not intersecting V:

(76.6) $\quad \text{mincr}(G, D) := \min\{\text{cr}(G, D') \mid D'$ is a closed curve in $S \setminus V$ freely homotopic to $D\}$.

(It would seem more consistent with definition (76.2) if we would also allow to shift G over S so as to obtain G' and minimize cr(G', D'), where G' is possibly not one-to-one mapped in S. However, Theorem 76.1 below implies that this would not change the minimum value.)

We say that a closed curve C is *in* a graph G if $C : S^1 \to G$.

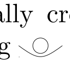
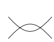
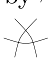
76.2. Making curves minimally crossing by Reidemeister moves

The proof of Theorem 76.1 below is based on the following result of de Graaf and Schrijver [1997b]. Let $C_1, \ldots, C_k$ be closed curves on S. Call $C_1, \ldots, C_k$ *minimally crossing* if

(76.7) (i) cr(C_i) = mincr(C_i) for each $i = 1, \ldots, k$;
(ii) cr(C_i, C_j) = mincr(C_i, C_j) for all $i, j = 1, \ldots, k$ with $i \neq j$.

Call $C_1, \ldots, C_k$ *regular* if $C_1, \ldots, C_k$ have only a finite number of (self-)intersections, each being a crossing of only two curve parts. (That is, each point of S traversed twice by the $C_1, \ldots, C_k$ has a disk-neighbourhood on which the curve parts are topologically homeomorphic to two crossing straight lines.)

In de Graaf and Schrijver [1997b] the following was shown:

(76.8) Any regular system of closed curves on a compact surface S can be transformed to a minimally crossing system by a series of *Reidemeister moves*: replacing  by  (*type 0*); replacing  by (*type I*); replacing  by (*type II*); replacing  by  (*type III*).

The pictures in (76.8) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S — no other curve parts than those shown intersect this disk.

It is important to note that in (76.8) we do not allow to apply the operations in the reverse direction — otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation, and would not be powerful enough for our purposes. The Reidemeister moves given in (76.8) do not increase the number of intersections.

76.3. Decomposing the edges of an Eulerian graph on a surface

We first show a homotopic analogue of the theorems in previous chapters relating distances and cut packings. It will be used to derive that the cut condition is sufficient for the existence of a fractional packing of circuits of prescribed homotopies in a graph on a surface (analogous to the line of proof developed in Section 70.12).

A graph is called *Eulerian* if each vertex has even degree. (We do not assume connectedness of the graph.) Moreover, *decomposing* the edges into closed curves $C_1, \ldots, C_k$ means that $C_1, \ldots, C_k$ are closed curves in G such that each edge is traversed by exactly one C_i, and by that C_i exactly once.

We now give the theorem, due to de Graaf and Schrijver [1997a], which was proved for the projective plane by Lins [1981] (Corollary 74.1b above) and for compact orientable surfaces by Schrijver [1991a].

Theorem 76.1. *Let $G = (V, E)$ be an Eulerian graph embedded in a compact surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that*

$$\text{(76.9)} \qquad \text{mincr}(G, D) = \sum_{i=1}^{k} \text{mincr}(C_i, D)$$

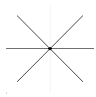
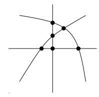
for each closed curve D on S.

Proof. First note that the inequality $\geq$ in (76.9) trivially holds, for *any* decomposition of the edges into closed curves $C_1, \ldots, C_k$: by definition of $\text{mincr}(G, D)$, there exists a closed curve $D' \sim D$ in $S \setminus V$ with $\text{mincr}(G, D) = \text{cr}(G, D')$, and hence

$$\text{(76.10)} \qquad \text{mincr}(G, D) = \text{cr}(G, D') = \sum_{i=1}^{k} \text{cr}(C_i, D') \geq \sum_{i=1}^{k} \text{mincr}(C_i, D).$$

The content of the theorem is that there exists a decomposition attaining equality.

To prove this, we may assume that each vertex v of G has degree at most 4. If v would have a degree larger than 4, we can replace G in a neighbourhood of v like

 by  .

This modification does not change the value of $\text{mincr}(G, D)$ for any D. Moreover, closed curves decomposing the edges of the modified graph satisfying (76.9), directly yield closed curves decomposing the edges of the original graph satisfying (76.9).

For any graph G embedded in S with each vertex having degree 2 or 4, we define the *straight decomposition* of G as the regular system of closed curves $C_1, \ldots, C_k$ such that $G = C_1 \cup \cdots \cup C_k$. So each vertex of G of degree 4 represents a (self-)crossing of $C_1, \ldots, C_k$.

Up to some trivial operations, such a decomposition is unique, and conversely, it uniquely describes G. Moreover, any Reidemeister move applied to $C_1, \ldots, C_k$ carries over a modification of G. So we can speak of Reidemeister moves applied to G. Then straightforwardly:

(76.11) if G' arises from G by one Reidemeister move of type III, then $\text{mincr}(G', D) = \text{mincr}(G, D)$ for each closed curve D.

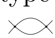
Call any graph $G = (V, E)$ that is a counterexample to the theorem such that each vertex has degree at most 4 and such that it has a minimum number of faces, a *minimal counterexample.* (A *face* is a connected component of $S \setminus G$.)

From (76.11) it directly follows that:

(76.12) if G' arises from a minimal counterexample G by one Reidemeister move of type III, then G' is a minimal counterexample again.

Moreover:

(76.13) if G is a minimal counterexample, then no Reidemeister move of type 0, I or II can be applied to G.

For suppose that a Reidemeister move of type II can be applied to G. Then G contains the following subconfiguration: . Replacing this by  would give a smaller counterexample (since the function $\text{mincr}(G, D)$ does not change by this operation), contradicting the minimality of G. One similarly sees that no Reidemeister move of type I can be applied. No Reidemeister move of type 0 can be applied, as otherwise we can delete the circuit to obtain a smaller counterexample. This proves (76.13).

The proof now is finished by showing that the straight decomposition $C_1, \ldots, C_k$ of any minimal counterexample G satisfies (76.9) — contradicting the fact that G is a counterexample.

Choose a closed curve D. We may assume that $D, C_1, \ldots, C_k$ form a regular system. By (76.8) we can apply Reidemeister moves so as to obtain a minimally crossing system $D', C_1', \ldots, C_k'$. Let G' be the graph formed by $C_1', \ldots, C_k'$.

By (76.12) and (76.13) we did not apply Reidemeister moves of type 0, I or II to $C_1, \ldots, C_k$. Hence, by (76.11), $\text{mincr}(G', D) = \text{mincr}(G, D)$. So

(76.14) $$\begin{aligned}\text{mincr}(G, D) &= \text{mincr}(G', D) = \text{mincr}(G', D') \leq \text{cr}(G', D') \\ &= \sum_{i=1}^{k} \text{cr}(C_i', D') = \sum_{i=1}^{k} \text{mincr}(C_i', D') = \sum_{i=1}^{k} \text{mincr}(C_i, D).\end{aligned}$$

Since the converse inequality holds by (76.10), we have (76.9). ∎

The theorem can be sharpened to include compact surfaces with holes, just by replacing holes by handles.

76.4. A corollary on lengths of closed curves

Using surface duality we derive the following consequence of Theorem 76.1 (Schrijver [1991a], de Graaf and Schrijver [1997a]). If G is a graph embedded in a surface S and C is a closed curve in G, then $\text{minlength}_G(C)$ denotes the minimum length of any closed curve $C' \sim C$ in G. Here the *length* $\text{length}_G(C')$ of C' is the number of edges traversed by C', counting multiplicities. So

(76.15) $$\text{minlength}_G(C) = \min\{\text{length}_G(C') \mid C' \sim C, C' \text{ in } G\}.$$

Corollary 76.1a. *Let $G = (V, E)$ be a bipartite graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves in G. Then there exist closed curves $D_1, \ldots, D_t$ in $S \setminus V$ such that each edge of G is crossed by exactly one D_j and by this D_j only once, and such that*

(76.16) $$\text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j)$$

for each $i = 1, \ldots, k$.

Proof. Let

(76.17) $$d := \max\{\text{minlength}_G(C_i) \mid i = 1, \ldots, k\}.$$

We can extend G to a bipartite graph L embedded in S, such that each face of L is an open disk. By inserting d new vertices on each edge of L not occurring in G, we obtain a bipartite graph H satisfying $\text{minlength}_H(C_i) = \text{minlength}_G(C_i)$ for each $i = 1, \ldots, k$ (since the new edges cannot be used to obtain a closed curve shorter than $\text{minlength}_G(C_i)$).

Consider a surface dual graph H^* of H. Then for each $i = 1, \ldots, k$,

(76.18) $$\text{mincr}(H^*, C_i) = \text{minlength}_H(C_i) = \text{minlength}_G(C_i).$$

Since H is bipartite, H^* is Eulerian. Hence by Theorem 76.1, the edges of H^* can be decomposed into closed curves $D_1, \ldots, D_t$ such that

(76.19) $$\text{mincr}(H^*, C) = \sum_{j=1}^{t} \text{mincr}(D_j, C)$$

for each closed curve C. With (76.18), this gives (76.16). ∎

Notes. This proof also implies that we can replace $C_1, \ldots, C_k$ by the set of *all* closed curves on S if G is *cellularly embedded* (i.e., each face is an open disk) — in that case we need not extend G to L and H.

It is not difficult to see that this also holds for not-cellularly embedded bipartite graphs in the torus, since then there is essentially only one closed curve C in G to consider.

This is not true for the double torus (a surface with two handles), as is shown by the example of Figure 76.1 (from Schrijver [1991a]).

76.5. A homotopic circulation theorem

By linear programming duality (Farkas' lemma) we derive from Corollary 76.1a the following 'homotopic circulation theorem' — a fractional packing theorem for closed curves of given homotopies in a graph on a compact surface.

Let $G = (V, E)$ be a graph embedded in a compact surface S. For any closed curve C in G define the vector tr^C in $\mathbb{Z}_+^E$ by:

(76.20) $$\text{tr}^C(e) := \text{number of times } C \text{ traverses } e,$$

for $e \in E$.

Let C_0 be a closed curve on S. Call a function $f : E \to \mathbb{R}$ a *circulation freely homotopic to* C_0 (of *value* 1) if f is a convex combination of functions tr^C, where the C are closed curves in G freely homotopic to C_0.

Corollary 76.1b (homotopic circulation theorem)**.** *Let $G = (V, E)$ be an undirected graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Then there exist circulations $f_1, \ldots, f_k$ freely homotopic to $C_1, \ldots, C_k$ respectively, such that*

(76.21) $$f_1(e) + \cdots + f_k(e) \leq 1$$

for each edge e, if and only if

(76.22) $$\text{cr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D)$$

for each closed curve D in $S \setminus V$.

Proof. *Necessity.* First note that if f is a circulation freely homotopic to a closed curve C_0, then

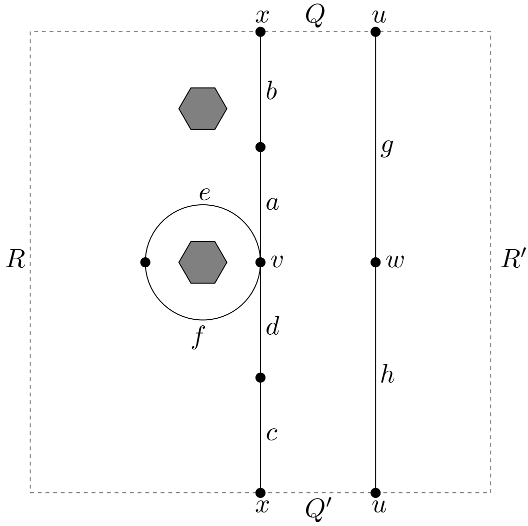


Figure 76.1
A not-cellularly embedded bipartite graph G in the double torus S for which Corollary 76.1a is not true if we replace $C_1, \ldots, C_k$ by all closed curves on S. The double torus is obtained from the square by identifying R and R' and identifying Q and Q' (so as to obtain the torus) and next deleting the interiors of the two hexagons and identifying their boundaries (so as to obtain the double torus).
For $i = 0, 1, 2, \ldots$ let C_i be the closed curve in G which, starting at v, follows e and f once, and next follows i times the closed curve a, b, c, d. Then $\text{minlength}_G(C_i) = 4i+2$. Suppose now that $D_1, \ldots, D_t$ are closed curves as described in Corollary 76.1a. Choose an arbitrary curve P from v to w. Then C_i is homotopic to the closed curve $\widetilde{C}_i$ obtained by, starting at v, first following e and f, next following P, then following i times the closed curve g, h, and finally following P back from w to v. Hence for each i (where B is the closed curve from w to w following g and h):

$$
\begin{aligned}
&4i + 2 = \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j) \\
&\leq \sum_{j=1}^{t} \text{cr}(\widetilde{C}_i, D_j) = \sum_{j=1}^{t} (\text{cr}(C_0, D_j) + 2 \cdot \text{cr}(P, D_j) + i \cdot \text{cr}(B, D_j)) \\
&= \sum_{j=1}^{t} (\text{cr}(C_0, D_j) + 2 \cdot \text{cr}(P, D_j)) + 2i.
\end{aligned}
$$

As the first term in the last sum is independent of i, this is a contradiction.

$$\sum_{e\in E} f(e)\mathrm{cr}(e,D) \geq \mathrm{mincr}(C_0,D). \tag{76.23}$$

for each closed curve D in $S\setminus V$ (denoting by $\mathrm{cr}(e,D)$ the number of times D intersects edge e). This follows from the fact that (76.23) holds for $f := \mathrm{tr}^C$ for each C freely homotopic to C_0 as

$$\sum_{e\in E} \mathrm{tr}^C(e)\mathrm{cr}(e,D) = \mathrm{cr}(C,D) \geq \mathrm{mincr}(C_0,D), \tag{76.24}$$

and hence also for any convex combination of such functions.

Suppose now that there exist circulations $f_1,\ldots,f_k$ as required. Let D be a closed curve in $S\setminus V$. Then, using (76.23):

$$\begin{aligned} \mathrm{cr}(G,D) &= \sum_{e\in E}\mathrm{cr}(e,D) \geq \sum_{e\in E}\mathrm{cr}(e,D)\sum_{i=1}^k f_i(e) \\ &= \sum_{i=1}^k\sum_{e\in E} f_i(e)\mathrm{cr}(e,D) \geq \sum_{i=1}^k \mathrm{mincr}(C_i,D). \end{aligned} \tag{76.25}$$

Sufficiency. Suppose that (76.22) is satisfied for each closed curve D in $S\setminus V$. Let $I := \{1,\ldots,k\}$ and let K be the convex cone in $\mathbb{R}^I\times\mathbb{R}^E$ generated by the vectors[23]

$$\begin{array}{ll} (\chi^i;\mathrm{tr}^C) & (i\in I; C \text{ closed curve in } G \text{ with } C\sim C_i), \\ (\mathbf{0}_I;\chi^e) & (e\in E). \end{array} \tag{76.26}$$

Here χ^i denotes the ith unit base vector in $\mathbb{R}^I$ and χ^e denotes the eth unit base vector in $\mathbb{R}^E$. Moreover, $\mathbf{0}_I$ denotes the all-zero vector in $\mathbb{R}^I$.

Although generally there are infinitely many vectors (76.26), K is finitely generated. This can be seen by observing that, for each $i\in I$, we can restrict the vectors $(\chi^i;\mathrm{tr}^C)$ in the first line of (76.26) to those that are minimal with respect to the usual partial order $\leq$ on $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ (with $(x;y)\leq(x';y') \iff x_i\leq x_i'$ for all $i\in I$ and $y_e\leq y_e'$ for all $e\in E$). They form an 'antichain' in $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ (i.e., a set of pairwise incomparable vectors). Since each antichain in $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ is finite, K is finitely generated.

We must show that the vector $(\mathbf{1}_I;\mathbf{1}_E)$ belongs to K. Here $\mathbf{1}_I$ and $\mathbf{1}_E$ denote the all-one vectors in $\mathbb{R}^I$ and $\mathbb{R}^E$, respectively. By Farkas' lemma, it suffices to show that each vector $(d;l)\in\mathbb{Q}^I\times\mathbb{Q}^E$ having nonnegative inner product with each of the vectors (76.26), also has nonnegative inner product with $(\mathbf{1}_I;\mathbf{1}_E)$. Thus let $(d;l)\in\mathbb{Q}^I\times\mathbb{Q}^E$ have nonnegative inner product with each vector among (76.26). This is equivalent to:

$$\begin{array}{lll} \text{(i)} & d_i + \sum_{e\in E} l(e)\mathrm{tr}^C(e) \geq 0 & (i\in I; C \text{ closed curve in } G \\ & & \text{with } C\sim C_i), \\ \text{(ii)} & l(e)\geq 0 & (e\in E). \end{array} \tag{76.27}$$

[23] We write $(x;y)$ for $\binom{x}{y}$.

Suppose now that $(d;l)^{\mathsf{T}}(\mathbf{1}_I;\mathbf{1}_E) < 0$. By increasing l slightly, we may assume that $l(e) > 0$ for each $e \in E$. Next, by multiplying $(d;l)$ appropriately, we may assume that each entry in $(d;l)$ is an even integer.

Let G' be the graph arising from G by replacing each edge e of G by a path of length $l(e)$. That is, we insert $l(e)-1$ new vertices on e. Then by (76.27)(i),

$$(76.28) \qquad -d_i \leq \operatorname{minlength}_{G'}(C_i)$$

for each $i \in I$. Since G' is bipartite, by Corollary 76.1a there exist closed curves $D_1, \ldots, D_t$ intersecting no vertex of G' such that each edge of G' is intersected by exactly one D_j and only once by that D_j and such that

$$(76.29) \qquad \operatorname{minlength}_{G'}(C_i) = \sum_{j=1}^{t} \operatorname{mincr}(C_i, D_j)$$

for each $i \in I$. So

$$(76.30) \qquad l(e) = \sum_{j=1}^{t} \operatorname{cr}(e, D_j)$$

for each edge e of G. Hence (76.22), (76.28) and (76.29) give

$$(76.31) \qquad \begin{aligned} \sum_{e \in E} l(e) &= \sum_{j=1}^{t} \sum_{e \in E} \operatorname{cr}(e, D_j) = \sum_{j=1}^{t} \operatorname{cr}(G, D_j) \\ &\geq \sum_{j=1}^{t} \sum_{i=1}^{k} \operatorname{mincr}(C_i, D_j) = \sum_{i=1}^{k} \sum_{j=1}^{t} \operatorname{mincr}(C_i, D_j) \\ &= \sum_{i=1}^{k} \operatorname{minlength}_{G'}(C_i) \geq -\sum_{i=1}^{k} d_i. \end{aligned}$$

So $(d;l)^{\mathsf{T}}(\mathbf{1}_I;\mathbf{1}_E) \geq 0$. ∎

This corollary has an equivalent capacitated version. Let C_0 be a closed curve on S. Call a function $f : E \to \mathbb{R}$ a *circulation freely homotopic to* C_0 of *value* d if f is a nonnegative linear combination of functions tr^C, where the C are closed curves in G freely homotopic to C_0 and where the scalars add up to d.

Corollary 76.1c. *Let $G = (V, E)$ be an undirected graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Let $c : E \to \mathbb{R}_+$ and $d_1, \ldots, d_k \in \mathbb{R}_+$. Then there exist circulations $f_1, \ldots, f_k$ freely homotopic to $C_1, \ldots, C_k$ respectively and of values $d_1, \ldots, d_k$ respectively, such that*

$$(76.32) \qquad \sum_{i=1}^{k} f_i(e) \leq c(e)$$

for each edge e *if and only if*

$$(76.33) \qquad \sum_{e\in E} c(e)\mathrm{cr}(e,D) \geq \sum_{i=1}^{k} d_i \cdot \mathrm{mincr}(C_i,D)$$

for each closed curve D *in* $S \setminus V$.

Proof. Using the argument on the finite generation of the convex cone K in the proof of Corollary 76.1b, we can assume that c and the d_i are rational, and hence integer. Replace each edge e of G by $c(e)$ parallel edges, and replace any C_i by d_i copies of C_i. Then the present corollary follows from Corollary 76.1b. ∎

Notes. Frank and Schrijver [1992] showed that if S is the torus and each C_i is a simple closed curve, then there exist half-integer circulations in Corollary 76.1b — that is, where the scalars of the tr^C are $\frac{1}{2}$ (similarly, in Corollary 76.1c if c and the d_i are integer). More generally, it is shown that there are integer circulations if the following Euler condition holds:

(76.34) for each closed curve D in $S \setminus V$, the number of crossings of D with G has the same parity as the number of crossings with $C_1, \ldots, C_k$.

This condition in particular implies that each vertex of G has even degree. This result can be formulated equivalently as:

(76.35) Let $G = (V,E)$ be a graph embedded in the torus S and let $C_1, \ldots, C_k$ be simple closed curves on S such that the Euler condition (76.34) holds. Then G has edge-disjoint closed walks $C'_1, \ldots, C'_k$ (each traversing no edge more than once) with C'_i freely homotopic to C_i for $i = 1, \ldots, k$, if and only if condition (76.22) holds.

The C'_i need not be simple; they may have self-intersections at vertices. (See Schrijver [1992] for a survey on disjoint circuits in graphs on the torus.)

Figures 76.2 and 76.3 show that we cannot delete in (76.35) the Euler condition or the condition that the C_i are simple. Moreover, Figure 76.4 shows that (76.35) does not extend to the double torus (a surface with two handles).

76.6. Homotopic paths in planar graphs with holes

As was shown in Schrijver [1991a], Corollary 76.1b gives a 'homotopic flow-cut theorem', stating that a homotopic cut condition implies the existence of a fractional solution for the planar edge-disjoint paths problem, if the paths have prescribed homotopies in the surface obtained from the plane by deleting the interiors of certain faces covering all terminals. (This answers a question of C.St.J.A. Nash-Williams.)

Before formulating the result, we introduce some notation and terminology. Fix some subset T of $\mathbb{R}^2$. A *curve* in T is a continuous function $D : [0,1] \to T$. The points $D(0)$ and $D(1)$ are the *end points* of D.

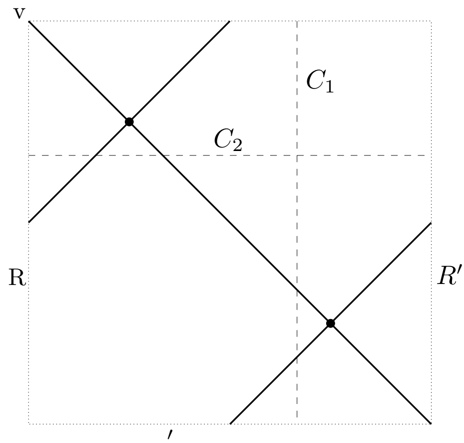


Figure 76.2

A graph G and curves C_1, C_2 on the torus satisfying the cut condition (76.22) (but not the Euler condition (76.34)), where G has no edge-disjoint circuits C_1' and C_2' with C_i' freely homotopic to C_i ($i = 1, 2$).

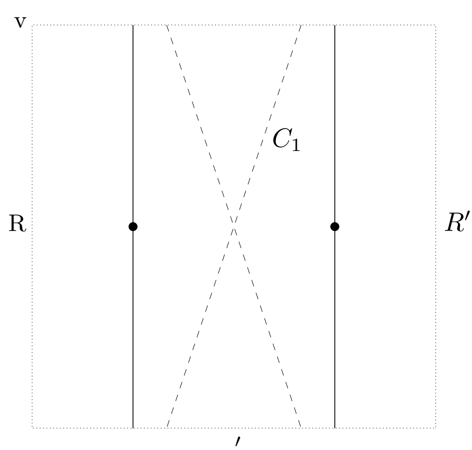


Figure 76.3

A graph G and a *nonsimple* curve C_1 on the torus satisfying the cut condition (76.22) and the Euler condition (76.34), where G has no closed curve C_1' freely homotopic to C_1 such that C_1' traverses any edge of G at most once.

Two curves D, D' are called *homotopic* (*in* T), denoted by $D \sim D'$, if there exists a continuous function $\Phi : [0,1] \times [0,1] \to T$ with $\Phi(x, 0) = D(x)$,

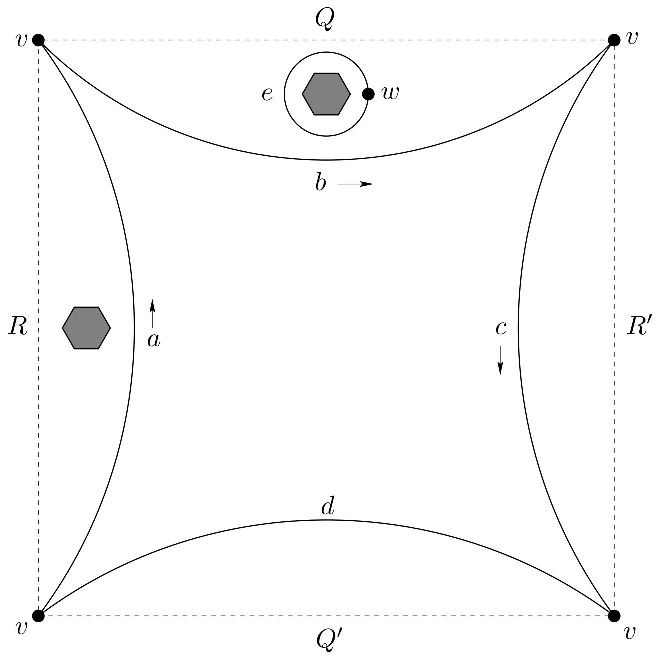


Figure 76.4

A graph G and curves C_1, C_2 on the double torus satisfying the cut condition (76.22) and Euler condition (76.34), but where no integer feasible circulations exist. The double torus is obtained from the square by identifying R and R' and identifying Q and Q' (so as to obtain the torus) and next deleting the interiors of the two hexagons and identifying their boundaries (so as to obtain the double torus). The graph G has two vertices, v and w, and four loops, a, b, c, d, at v, and one loop, e, at w. Curve C_1 follows the edges a and b, and curve C_2 follows the edges b and c — in the directions indicated. The cut condition (76.22) and the Euler condition (76.34) hold, but G has no edge-disjoint closed walks freely homotopic to C_1 and C_2 respectively. (The cut condition follows from the existence of a fractional solution.)

$\Phi(x,1) = D'(x)$, $\Phi(0,x) = D(0)$, $\Phi(1,x) = D(1)$ for each $x \in [0,1]$. (It follows that $D(0) = D'(0)$ and $D(1) = D'(1)$.)

If C and D are curves in T, then we denote:

(76.36) $$\mathrm{cr}(C,D) := |\{(x,y) \in [0,1] \times [0,1] \mid C(x) = D(y)\}|,$$
$$\mathrm{mincr}(C,D) := \min\{\mathrm{cr}(C',D') \mid C' \sim C, D' \sim D\}.$$

Let $G = (V,E)$ be a graph embedded in T. For any curve D in T and any $e \in E$, let

(76.37) $$\mathrm{cr}(e,D) := |\{x \in [0,1] \mid D(x) \in e\}|,$$

and

$$(76.38) \qquad \mathrm{cr}(G, D) = \sum_{e \in E} \mathrm{cr}(e, D).$$

For any walk P in G, let tr^P be the vector in $\mathbb{Z}_+^E$ defined by

$$(76.39) \qquad \mathrm{tr}^P(e) := \text{number of times } P \text{ traverses } e,$$

for $e \in E$. For any curve C in T, a *flow homotopic to* C (*of value* 1) is a convex combination of functions tr^P where P is a walk in G being (as a curve) homotopic to C in T.

Corollary 76.1d. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$. Let $F_1, \ldots, F_p$ be (the interiors of) some of the faces of G, including the unbounded face. Let $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$. Let $C_1, \ldots, C_k$ be curves in T with ends points being vertices of G on the boundary of T. Then there exist flows $f_1, \ldots, f_k$ homotopic to $C_1, \ldots, C_k$ respectively, each of value* 1*, such that*

$$(76.40) \qquad f_1(e) + \cdots + f_k(e) \leq 1$$

for each edge e of G if and only if

$$(76.41) \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$$

for each curve D in $T \setminus V$ with end points on the boundary of T.

Proof. Necessity is shown similarly as in the proof of Corollary 76.1b. To see sufficiency, let the condition hold. We construct a compact orientable surface S. First embed $\mathbb{R}^2$ in the 2-dimensional sphere S^2. Next for each $i = 1, \ldots, k$ make a handle H_i between the faces among $F_1, \ldots, F_k$ incident with the end points of C_i. This yields S.

Let G' be the graph obtained from G by adding, for each $i = 1, \ldots, k$, an edge f_i between the end points of C_i, by routing f_i over H_i. This can be done in such a way that the new edges do not intersect each other, and do no intersect the edges of G. Each curve C_i now can be extended to a closed curve C_i' by adding f_i.

We apply Corollary 76.1b to G' and S. The circulations described in Corollary 76.1b give flows as required in the present corollary. So it suffices to check condition (76.22) for G' and S. That is, for any closed curve D in $S \setminus V$ we must show

$$(76.42) \qquad \mathrm{cr}(G', D) \geq \sum_{i=1}^{k} \mathrm{mincr}'(C_i', D).$$

Here mincr' denotes the function mincr with respect to S. To show (76.42), we distinguish three cases.

Case 1: D is contained in T. Let y be some point on D, let z be some point on the boundary of T with $z \notin V$, and let R be some curve in T connecting z and y, such that R does not intersect V, and intersects G only a finite number of times. For $n \in \mathbb{Z}_+$, let Q_n be the curve from z to z which follows R from z to y, then follows n times the closed curve D, and next returns from y to z over R. Let $r := \mathrm{cr}(G, R)$. Let D^n be the closed curve that follows n times D. Then for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}(76.43)\qquad n \cdot \sum_{i=1}^{k} \mathrm{mincr}'(C'_i, D) &= \sum_{i=1}^{k} \mathrm{mincr}'(C'_i, D^n) \leq \sum_{i=1}^{k} \mathrm{mincr}(C_i, Q_n)\\ &\leq \mathrm{cr}(G, Q_n) = 2r + n \cdot \mathrm{cr}(G', D).\end{aligned}$$

Here the first equality is a general relation for curves on compact orientable surfaces (see Proposition 5 in Schrijver [1991a]). The first inequality holds as any curve homotopic to Q_n is equal to a *closed* curve *freely* homotopic to D^n. The second inequality follows from (76.41). The last equality follows from the definition of Q_n. Since (76.43) holds for each n, while r is fixed, we have (76.42).

Case 2: D does not intersect T. Then

$$(76.44)\qquad \mathrm{cr}(G', D) = \sum_{i=1}^{k} \mathrm{cr}(C'_i, D) \geq \sum_{i=1}^{k} \mathrm{mincr}'(C'_i, D).$$

Case 3: D intersects both T and $S \setminus T$. Set $H := S \setminus T$. Then we can split D into curves $D_1, D_2, \ldots, D_{2q}$, such that for odd i, D_i is contained in T and connects two points on the boundary of T, while for even i, D_i is contained in H, except for its end points. Then we have:

$$\begin{aligned}(76.45)\qquad \mathrm{cr}(G', D) &= \sum_{j=1}^{q} \mathrm{cr}(G, D_{2j-1}) + \sum_{j=1}^{q}\sum_{i=1}^{k} \mathrm{cr}(f_i, D_{2j})\\ &\geq \sum_{j=1}^{q}\sum_{i=1}^{k} \mathrm{mincr}(C_i, D_{2j-1}) + \sum_{j=1}^{q}\sum_{i=1}^{k} \mathrm{cr}(f_i, D_{2j})\\ &= \sum_{i=1}^{k}\sum_{j=1}^{q} (\mathrm{mincr}(C_i, D_{2j-1}) + \mathrm{cr}(f_i, D_{2j})) \geq \sum_{i=1}^{k} \mathrm{mincr}'(C'_i, D).\end{aligned}$$

The first inequality follows from (76.41). The last inequality can be derived as follows. Fix $i = 1, \ldots, k$. Then there exist curves $\widetilde{C}_i \sim C_i$ and $\widetilde{D}_{2j-1} \sim D_{2j-1}$ ($j = 1, \ldots, q$) with $\mathrm{mincr}(C_i, D_{2j-1}) = \mathrm{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1})$ for $j = 1, \ldots, q$. (This can be derived, for instance, from (76.8).) Hence $\widetilde{C}_i$ attains the minimum simultaneously for all D_{2j-1}. So

$$(76.46)\qquad \sum_{j=1}^{q} \mathrm{mincr}(C_i, D_{2j-1}) = \sum_{j=1}^{q} \mathrm{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1}).$$

Hence, where $\widetilde{D}$ is the concatenation of $\widetilde{D}_1, D_2, \widetilde{D}_3, D_4, \ldots, \widetilde{D}_{2q-1}, D_{2q}$, and $\widetilde{C}_i'$ is the concatenation of $\widetilde{C}_i$ and f_i,

$$(76.47) \qquad \sum_{j=1}^{q}(\text{mincr}(C_i, D_{2j-1}) + \text{cr}(f_i, D_{2j})) \\ = \sum_{j=1}^{q}(\text{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1}) + \text{cr}(f_i, D_{2j})) = \text{cr}(\widetilde{C}_i', \widetilde{D}) \geq \text{mincr}'(C_i', D),$$

proving the last inequality in (76.45). ∎

Notes. Related is the following *homotopic edge-disjoint paths problem*:

(76.48) given: a planar graph $G = (V, E)$, a subcollection $F_1, \ldots, F_p$ of the faces of G (including the unbounded face), curves $C_1, \ldots, C_k$ in $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, with end points in vertices of G on the boundary of T,
find: edge-disjoint walks $P_1, \ldots, P_k$, such that P_i traverses any edge at most once and is homotopic to C_i in T ($i = 1, \ldots, k$).

In this context, the faces $F_1, \ldots, F_p$ are called the *holes*.

Clearly, the *homotopic cut condition* (76.41) is a necessary condition for the feasibility of (76.48), while Figure 70.3 shows that it is generally not sufficient. By Corollary 76.1d, it is equivalent to the existence of a *fractional* solution of (76.48).

We can add the following *Euler condition* (or *local Euler condition*):

(76.49) for each vertex $v \in V$, the degree of v in G has the same parity as the number of times v is end point of the C_i (counting for 2 if C_i begins and ends at v).

By the Okamura-Seymour theorem, if $p = 1$ the homotopic cut and local Euler conditions are sufficient for the feasibility of (76.48). This was extended to $p = 2$ by van Hoesel and Schrijver [1990]:

(76.50) if $p = 2$ and the local Euler condition (76.49) holds, then the homotopic edge-disjoint paths problem (76.48) has a solution if and only if the homotopic cut condition (76.41) holds.

It implies that if $p = 2$, we can take the flows in Corollary 76.1d half-integer.

(76.50) cannot be extended to $p = 3$, as is shown by Figure 76.5. In fact, Kaufmann and Maley [1993] showed that it is NP-complete to solve the homotopic edge-disjoint paths problem even if the local Euler condition (76.49) holds and the graph is a grid graph (with as holes all faces enclosed by more than four edges).

We can consider a stronger parity condition, the *global Euler condition*:

(76.51) $\text{cr}(G, D) \equiv \sum_{i=1}^{k} \text{mincr}(C_i, D) \pmod 2$ for each curve D in $T \setminus V$ with end points on $\text{bd}(T)$ and having no touchings with G.

Kaufmann and Mehlhorn [1992] showed that if G is a grid graph and the holes are those faces enclosed by more than four edges, and if the global Euler condition holds, then the homotopic edge-disjoint paths problem has a solution if and only if

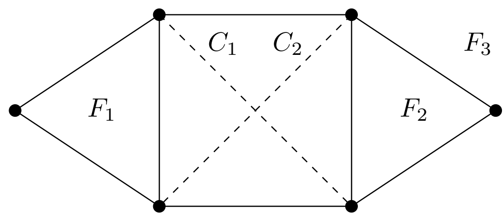


Figure 76.5
This graph G with curves C_1 and C_2 satisfies the cut condition (76.41) (as there is a fractional solution) and the (local) Euler condition, but G has no edge-disjoint walks $P_1 \sim C_1$ and $P_2 \sim C_2$ (in $\mathbb{R}^2 \setminus (F_1 \cup F_2 \cup F_3)$).

the homotopic cut condition holds. Kaufmann and Mehlhorn also gave an $O(n^2)$-time algorithm to find the paths. This was improved to a linear-time algorithm by Kaufmann [1987] and Kaufmann and Mehlhorn [1994].

Other types of grids, like the hexagonal and the octo-square grid, were considered by Kaufmann [1987]. A generalization to 'straight-line planar graphs' was given by Schrijver [1991d]. A *straight-line planar graph* is a planar graph G such that each edge is a straight line segment, where $F_1, \ldots, F_p$ are such that for each edge e of G and each vertex v on e, when extending the line segment forming e slightly at v we arrive either in another edge of G or in one of the faces F_i. In this case, if the global Euler condition holds, then the homotopic edge-disjoint paths problem has a solution if and only if the homotopic cut condition holds. Moreover, the problem is solvable in polynomial time in this case.

76.7. Vertex-disjoint paths and circuits of prescribed homotopies

76.7a. Vertex-disjoint circuits of prescribed homotopies

As for the vertex-disjoint analogue of the results studied above, the existence of vertex-disjoint circuits of prescribed homotopies in a graph on a compact surface can be fully characterized.

Let $G = (V, E)$ be a graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be pairwise disjoint simple closed curves on S. We say that a closed curve D on a surface S is *doubly odd* (with respect to G, $C_1, \ldots, C_k$), if D is the concatenation of two closed curves D_1 and D_2, with common end point not on G, such that

(76.52) $$\operatorname{cr}(G, D_j) \not\equiv \sum_{i=1}^{k} \operatorname{mincr}(C_i, D_j) \pmod 2 \text{ for } j = 1, 2.$$

Then the following was shown in Schrijver [1991b] (conjectured by L. Lovász and P.D. Seymour):

Theorem 76.2. *There exist disjoint circuits $C'_1, \ldots, C'_k$ in G where C'_i is freely homotopic to C_i $(i = 1, \ldots, k)$ if and only if for each closed curve D on S one has*

(76.53) $$\mathrm{cr}(G,D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D),$$

with strict inequality if D is doubly odd.

We only show necessity of the condition. To that end, we can assume that $C_1, \ldots, C_k$ are disjoint circuits in G. Then the inequality follows from

(76.54) $$\mathrm{cr}(G,D) \geq \sum_{i=1}^{k} \mathrm{cr}(C_i, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).$$

Moreover, if D is doubly odd, let D_1 and D_2 be as above. Then:

(76.55) $$\begin{aligned}\mathrm{cr}(G,D) = \mathrm{cr}(G,D_1) + \mathrm{cr}(G,D_2) &> \sum_{i=1}^{k} (\mathrm{cr}'(C_i, D_1) + \mathrm{cr}'(C_i, D_2)) \\ = \sum_{i=1}^{k} \mathrm{cr}'(C_i, D) &\geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).\end{aligned}$$

Here $\mathrm{cr}'(C,D)$ counts the number of crossing (and not touchings) of C and D. The strict inequality holds as

(76.56) $$\mathrm{cr}(G,D_j) \not\equiv \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_j) \equiv \sum_{i=1}^{k} \mathrm{cr}'(C_i, D_j) \pmod 2$$

for $j = 1, 2$.

For the proof of sufficiency, based on solving a system of linear inequalities in integers, we refer to Schrijver [1991b]. The proof also implies a polynomial-time algorithm to find disjoint circuits as required in Theorem 76.2.

For the torus, the condition in Theorem 76.2 on the strictness of the inequality is superfluous, and the characterization can be formulated as:

(76.57) Let G be a graph embedded in the torus T, and let C be a simple closed curve on T. Then G contains k disjoint circuits each freely homotopic to C if and only if

$$\mathrm{cr}(G,D) \geq k \cdot \mathrm{mincr}(C,D)$$

for each closed curve D on T.

This was extended to directed graphs by Seymour [1991] (including polynomial-time solvability). A shorter proof of this, together with an extension to the Klein bottle, was given by Ding, Schrijver, and Seymour [1993]. A survey is given in Schrijver [1992].

76.7b. Vertex-disjoint homotopic paths in planar graphs with holes

In a similar way one can prove (or derive from Theorem 76.2 as in the proof of Corollary 76.1d for the fractional edge-disjoint case) results on vertex-disjoint homotopic paths in a planar graph with holes.

Consider the following *disjoint homotopic paths problem*:

(76.58) given: A planar graph $G = (V, E)$, faces $F_1, \ldots, F_p$ of G, including the unbounded face, disjoint curves $C_1, \ldots, C_k$ in $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, each with end points in vertices of G on the boundary of T,
find: disjoint paths $P_1, \ldots, P_k$ in G, where P_i is homotopic to C_i in T $(i = 1, \ldots, k)$.

Frank and Schrijver [1990] and Schrijver [1991c] gave polynomial-time algorithms for this problem, and gave the following characterization (the first paper gives an algorithm using the ellipsoid method, the second paper a combinatorial algorithm):

Theorem 76.3. *Problem* (76.58) *has a solution if and only if for each curve D in T with end points on* $\mathrm{bd}(T)$ *we have*

(76.59) $$\mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D),$$

and for each doubly odd closed curve D in T traversing no fixed point of any C_i we have

(76.60) $$\mathrm{cr}(G, D) > \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).$$

Here a point p is called a *fixed point of* C if each curve homotopic to C traverses p. (In particular, the ends points of C are fixed points of C.)

Figure 76.6 (due to L. Lovász, cf. Robertson and Seymour [1986]) shows that condition (76.60) cannot be deleted in Theorem 76.3.

Theorem 76.3 was proved by Cole and Siegel [1984] for the special case where G is a grid graph (a subgraph of the rectangular grid), and the F_i are precisely the faces that are not surrounded by exactly four edges of the grid, and the boundary of each face F_i is a rectangle. In this case, condition (76.60) is superfluous. Cole and Siegel [1984] also gave a polynomial-time ($O(n \log n)$) algorithm for this case (answering a question of Pinter [1983]), using an oracle to test homotopy of curves. A polynomial-time algorithm for such graphs, not using a homotopy testing oracle, was given by Leiserson and Maley [1985]. Maley [1987] gave an $O(n^2 \log n)$-time algorithm (where the solution has the additional property that each of the paths found is shortest among all possible solutions), while Maley [1996] gave an $O(n \log n)$-time algorithm to test routability (not constructing the solution), under some mild conditions on the routing rules and the input layout.

Theorem 76.3 and the polynomial-time solvability of (76.58) was proved for $p \leq 2$ by Robertson and Seymour [1986], where again condition (76.60) is superfluous. (A linear-time algorithm for $p = 2$ was given by Ripphausen-Lipa, Wagner, and Weihe [1993a], if at least one of the curves C_i connects F_1 and F_2.) A short proof for the case $p = 2$ was given by Frank [1990c], which also extends to the directed case, implying a result of Seymour [1991].

The polynomial-time solvability of (76.58) implies the following for *nonhomotopic* disjoint paths (Schrijver [1991c]):

(76.61) for each fixed p, the vertex-disjoint paths problem is polynomial-time solvable for planar graphs if the terminals can be covered by the boundaries of at most p faces.

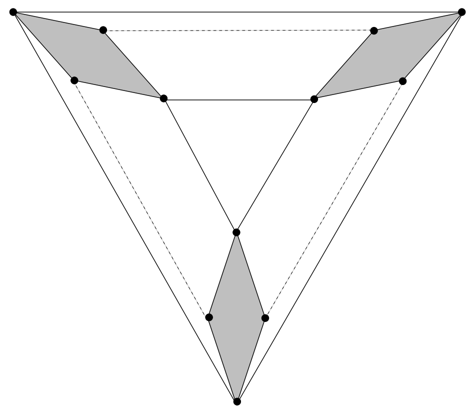


Figure 76.6
The three holes are indicated by grey regions, and the curves by dashed lines. We assume that the graph is embedded in the 2-sphere, such that there is no unbounded face.
The cut condition (76.59) holds, as there is a fractional solution, but no vertex-disjoint paths homotopic to the given curves exist.

(This was conjectured by Robertson and Seymour [1986], who proved it for $p \leq 2$ (see Section 74.4c for $p = 1$). For $p \leq 2$, Suzuki, Akama, and Nishizeki [1988a, 1988b,1988c,1990] gave an $O(n \log n)$-time algorithm, improved to linear-time by Ripphausen-Lipa, Wagner, and Weihe [1993a,1993b]. (The algorithm of Suzuki, Akama, Nishizeki is linear-time if each net is spanned by F_1 or by F_2.) For $p = 3$, a linear-time algorithm if each net is spanned by F_1, F_2, or F_3 was announced by H. Suzuki, T. Kumagai, and T. Nishizeki (1993; cf. Ripphausen-Lipa, Wagner, and Weihe [1995]).)

The idea of proof of (76.61) is that for each net r we choose a curve C_r connecting the points in r such that the C_r are disjoint, and next try to find paths as in (76.58); it can be proved that we need to consider only a polynomially bounded number of homotopy classes of curves C_r (for fixed p), which gives the required result.

In Schrijver [1993] this was extended, by similar methods, to the directed case:

(76.62) for each fixed p, the disjoint paths problem is polynomial-time solvable for directed planar graphs if the terminals can be covered by the boundaries of at most p faces.

This remains the case if we prescribe for each net (s_i, t_i) a subset A_i of the arc set that path P_i is allowed to use.

For a sketch of the method for (76.58), see Schrijver [1990b].

Robertson and Seymour [1995] proved that if the number of terminals is fixed, the vertex-disjoint paths problem in undirected graphs is $O(n^3)$-time solvable, also for nonplanar graphs. If moreover the graph is planar, Reed, Robertson, Schrijver, and Seymour [1993] gave a *linear*-time algorithm. This connects to the results described in Section 70.13a.

76.7c. Disjoint trees

The polynomial-time solvability of finding paths (76.58) can be generalized to disjoint trees (Schrijver [1991c]). The following problem is solvable in polynomial time:

(76.63) given: a planar graph G, faces $F_1, \ldots, F_p$ of G (including the unbounded face), curves $C_{1,1}, \ldots, C_{1,t_1}, \ldots, C_{k,1}, \ldots, C_{k,t_k}$ in the space $S := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, with end points in vertices of G on $\mathrm{bd}(S)$, such that for each $i = 1, \ldots, k$, $C_{i,1}, \ldots, C_{i,t_i}$ have the same starting vertex;
find: disjoint subtrees $T_1, \ldots, T_k$ of G such that for each $i = 1, \ldots, k$ and $j = 1, \ldots, t_i$, T_i contains a path homotopic to $C_{i,j}$ in S.

Again, by enumerating homotopy classes, it can be derived that, for each fixed p, the problem

(76.64) given: a graph $G = (V, E)$ and disjoint subsets $W_1, \ldots, W_k$ of V;
find: disjoint subtrees $T_1, \ldots, T_k$ of G such that T_i spans W_i for $i = 1, \ldots, k$,

is polynomial-time solvable if G is planar and $W_1, \ldots, W_k$ can be covered by the boundaries of at most p faces of G. (For $p \leq 2$, Suzuki, Akama, and Nishizeki [1988a, 1988b,1988c,1990] gave an $O(n \log n)$-time algorithm, improved to linear-time by Ripphausen-Lipa, Wagner, and Weihe [1993a,1993b].)

Robertson and Seymour [1995] showed that for each fixed p, (76.64) is $O(n^3)$-time solvable for any graph if $|W_1 \cup \cdots \cup W_k| \leq p$. If moreover the graph is planar, Reed, Robertson, Schrijver, and Seymour [1993] gave a *linear*-time algorithm.

For minimum-length homotopic routing in grid graphs, see Ho, Suzuki, and Sarrafzadeh [1993]. Surveys of homotopic routing methods are given by Schrijver [1990b,1994b], and of applications of polyhedral combinatorics to multiflows on surfaces by Schrijver [1990a]. 'Gridless' homotopic routing (that is, routing in the plane (not in a graph), observing mutual distances between curves) was studied by Tompa [1981] and Maley [1988].

Part VIII

Hypergraphs

Part VIII: Hypergraphs

Hypergraphs form a framework in which many of the min-max relations discussed before can be formulated. This is not to say that they all can be *derived* from general hypergraph theory. Rather, hypergraph theory yields relations between different min-max relations, for instance through the blocking and antiblocking relations of hypergraphs and of polyhedra. Moreover, certain hereditary min-max relations can be characterized by equivalent but weaker conditions. This can be helpful in proving min-max relations for special classes of hypergraphs.

The material in this part is grouped by the hypergraph generalizations of four notions that also played a central role in the earlier parts on graphs: matching, vertex cover, edge cover, and stable set. Among the landmarks of this part are theorems of Lehman on minimally nonideal hypergraphs and of Seymour characterizing binary Mengerian hypergraphs.

Chapters:

Chapter 77

Packing and blocking in hypergraphs: elementary notions

Packing in hypergraphs asks for a maximum number of disjoint edges. *Blocking* concerns the minimum number of vertices intersecting each edge. In this chapter we give basic concepts of hypergraphs, in particular those related to packing and blocking.

77.1. Elementary hypergraph terminology and notation

We start with some elementary definitions and notation on hypergraphs. A *hypergraph*[1] is a pair $H = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a family of subsets of V. Any element of V is called a *vertex* of H and any set in $\mathcal{E}$ an *edge* of H. We sometimes denote the vertex set and the edge set of H by VH and EH respectively. In our discussions, we can assume without loss of generality that $\mathcal{E}$ is a collection of subsets (rather than a family, with multiplicities).

Graphs are special cases of hypergraphs: they are the hypergraphs that have all its edges of size 2.

The $\mathcal{E} \times V$ *incidence matrix* of H is the $\mathcal{E} \times V$ matrix M with $M_{F,v} = 1$ if $v \in F$ and $M_{F,v} = 0$ if $v \notin F$ (for $v \in V$, $F \in \mathcal{E}$). For most of our purposes, studying a hypergraph is equivalent to studying its incidence matrix. Any result on hypergraphs is simultaneously a result on $0, 1$ matrices, and conversely. We will go back and forth between both interpretations and often choose the most appropriate one.

The *dual hypergraph* H^* of a hypergraph $H = (V, \mathcal{E})$ is the hypergraph with vertex set $\mathcal{E}$ and edges all sets $\{E \in \mathcal{E} \mid v \in E\}$ for $v \in V$. So the incidence matrix of H^* is the transpose of the incidence matrix of H.

For any hypergraph $H = (V, \mathcal{E})$ we denote

$$r_{\min}(H) := \min\{|E| \bigm| E \in \mathcal{E}\} \text{ and } r_{\max}(H) := \max\{|E| \bigm| E \in \mathcal{E}\}. \tag{77.1}$$

[1] Berge [1996] said that the name 'hypergraph' was invented in 1969 by J.-M. Pla, after earlier attempts to call it 'graphoid' (e.g. Berge [1969]).

77.2. Deletion, restriction, and contraction

We describe two operations on a hypergraph $H = (V, \mathcal{E})$, deletion and contraction. Let $v \in V$, and define:

(77.2) $$\mathcal{E} \setminus v := \{E \in \mathcal{E} \mid v \notin E\}, \quad H \setminus v := (V \setminus \{v\}, \mathcal{E} \setminus v),$$
$$\mathcal{E}/v := \{E \setminus \{v\} \mid E \in \mathcal{E}\}, \quad H/v := (V \setminus \{v\}, \mathcal{E}/v).$$

Replacing H by $H \setminus v$ is called *deleting* v and replacing H by H/v is called *contracting* v. We say that H' is a *restriction* of H if it arises by a series of deletions, and a *contraction* of H if it arises by a series of contractions. The *restriction to* $U \subseteq V$ is $H \setminus (V \setminus U)$.

Deletions and contractions commute in the ways one may expect: for distinct $u, v \in V$ one has

(77.3) $(H/u)/v = (H/v)/u$, $(H \setminus u) \setminus v = (H \setminus v) \setminus u$, and $(H/u) \setminus v = (H \setminus v)/u$.

Deletion of an edge E means replacing $\mathcal{E}$ by $\mathcal{E} \setminus \{E\}$. A hypergraph H' is called a *minor* of H, if H' arises from H by a series of deletions and contractions of vertices, and deletions of edges that are not inclusionwise minimal edges.

77.3. Duplication and parallelization

Let $H = (V, \mathcal{E})$ be a hypergraph and let $v \in V$. *Duplicating* v means extending V by a new vertex, v' say, and replacing $\mathcal{E}$ by

(77.4) $$\mathcal{E} \cup \{(E \setminus \{v\}) \cup \{v'\} \mid v \in E \in \mathcal{E}\}.$$

A hypergraph obtained from H by a sequence of deletions and duplications of vertices, is called a *parallelization* of H. If $w : V \to \mathbb{Z}_+$, we denote by H^w the result of deleting any vertex v with $w(v) = 0$, and duplicating any vertex v $w(v) - 1$ times, if $w(v) \geq 2$. So restrictions correspond to functions $w : V \to \{0, 1\}$. In a certain sense, contractions correspond to functions $w : V \to \{1, \infty\}$.

77.4. Clutters

For any hypergraph $H = (V, \mathcal{E})$, define

(77.5) $$H^{\min} := (V, \{F \in \mathcal{E} \mid \text{there is no } E \in \mathcal{E} \text{ with } E \subset F\}) \text{ and}$$
$$H^{\uparrow} := (V, \{F \subseteq V \mid \text{there is an } E \in \mathcal{E} \text{ with } E \subseteq F\}).$$

A hypergraph $H = (V, \mathcal{E})$ is called a *clutter* if no two sets in $\mathcal{E}$ are contained in each other[2]. So for any hypergraph, $H^{\min}$ is a clutter.

[2] The term 'clutter' was introduced by Edmonds and Fulkerson [1970].

77.5. Packing and blocking

Let $H = (V, \mathcal{E})$ be a hypergraph. The following notions generalize the corresponding notions defined for graphs.

A *vertex cover* is a set of vertices intersecting each edge of H. A *matching* is a collection of pairwise disjoint edges of H. Define

(77.6) $\tau(H) :=$ the minimum size of a vertex cover in H,
$\nu(H) :=$ the maximum size of a matching in H.

Determining these numbers is NP-complete, since determining $\tau(G)$ and (the stability number) $\alpha(G)$ of a graph $G = (V, E)$ is NP-complete (cf. Theorem 64.1), and since $\alpha(G) = \nu(G^*)$.

We should note that replacing H by $H^{\min}$ or $H^{\uparrow}$ does not change the value of $\tau(H)$ or $\nu(H)$. So $\tau(H) = \tau(H^{\uparrow}) = \tau(H^{\min})$ and $\nu(H) = \nu(H^{\uparrow}) = \nu(H^{\min})$.

There is the following straightforward inequality:

(77.7) $\nu(H) \leq \tau(H)$.

In the previous parts we met several classes of hypergraphs where equality holds in (77.7), and the purpose of this and the coming chapters is to treat them in a unifying and clarifying framework.

77.6. The blocker

For any hypergraph $H = (V, \mathcal{E})$, the *blocking hypergraph*, or *blocker*, of H is the hypergraph $b(H) = (V, \mathcal{B})$ where $\mathcal{B}$ is the collection of all inclusionwise minimal vertex covers of H. So $b(H)$ is a clutter and

(77.8) $\tau(H) = r_{\min}(b(H))$.

Moreover, $b(H)^{\uparrow}$ is the collection of vertex covers.

The following important duality relation was noticed by Lawler [1966] (also by Edmonds and Fulkerson [1970]):

Theorem 77.1. *For any hypergraph $H = (V, \mathcal{E})$, $b(b(H)) = H^{\min}$. In particular, if H is a clutter, then $b(b(H)) = H$.*

Proof. It suffices to show $b(b(H))^{\uparrow} = H^{\uparrow}$. If $U \in H^{\uparrow}$, then U intersects each set in $b(H)$. Hence U is a vertex cover of $b(H)$, and so $U \in b(b(H))^{\uparrow}$.

Conversely, if $U \notin H^{\uparrow}$, then $V \setminus U$ is a vertex cover of H. So $V \setminus U \in b(H)$. Hence U is not a vertex cover of $b(H)$. So $U \notin b(b(H))^{\uparrow}$. ∎

One may check that the operations of deletion and contraction interchange when passing to the blocker. More precisely, for any vertex v of a hypergraph H one has:

(77.9) $b(H/v) = b(H) \setminus v$ and $b(H \setminus v) = (b(H)/v)^{\min}$.

77.7. Fractional matchings and vertex covers

Let $H = (V, \mathcal{E})$ be a hypergraph. A *fractional vertex cover* is a function $x : V \to \mathbb{R}_+$ satisfying

(77.10) $$\sum_{v \in F} x_v \geq 1 \text{ for each } F \in \mathcal{E}.$$

A *fractional matching* is a function $y : \mathcal{E} \to \mathbb{R}_+$ satisfying

(77.11) $$\sum_{F \ni v} y_F \leq 1 \text{ for each } v \in V.$$

(Here and below, F ranges over the edges of H.) Let $\tau^*(H)$ denote the minimum size of a fractional vertex cover and let $\nu^*(H)$ denote the maximum size of a fractional matching (where the *size* of a vector is the sum of its components).

We can describe $\tau^*(H)$ and $\nu^*(H)$ by linear programs[3]:

(77.12) $$\tau^*(H) = \min\{\mathbf{1}^{\mathsf{T}} x \mid x \in \mathbb{R}^V_+, Mx \geq \mathbf{1}\},$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. Similarly,

(77.13) $$\nu^*(H) = \max\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}^{\mathcal{E}}_+, y^{\mathsf{T}} M \leq \mathbf{1}^{\mathsf{T}}\}.$$

As these linear programs are each others dual, this gives:

(77.14) $$\nu^*(H) = \tau^*(H).$$

77.8. k-matchings and k-vertex covers

There is an alternative interpretation of the parameters $\nu^*(H)$ and $\tau^*(H)$, in terms of 'k-vertex covers' and 'k-matchings'.

A *k-vertex cover* is a function $x : V \to \mathbb{Z}_+$ such that

(77.15) $$\sum_{v \in F} x_v \geq k \text{ for each } F \in \mathcal{F}.$$

Let $\tau_k(H)$ denote the minimum size of a k-vertex cover. Since (minimal) 1-vertex covers are precisely the incidence vectors of the vertex covers, we have $\tau_1(H) = \tau(H)$.

A *k-matching* is a function $y : \mathcal{E} \to \mathbb{Z}_+$ such that

(77.16) $$\sum_{F \ni v} y_F \leq k \text{ for each } v \in V.$$

Let $\nu_k(H)$ denote the maximum size of a k-matching in H. As 1-matchings are the incidence vectors of the matchings, we have $\nu_1(H) = \nu(H)$.

Then for any $k \in \mathbb{Z}_+$:

[3] $\mathbf{1}$ stands for all-one column vectors of appropriate sizes.

(77.17) $\nu_k(H) \leq \tau_k(H)$,

since for any k-vertex cover x and any k-matching y:

(77.18) $$\sum_F y_F \leq \frac{1}{k}\sum_F y_F \sum_{v\in F} x_v = \frac{1}{k}\sum_v x_v \sum_{F\ni v} y_F \leq \sum_v x_v.$$

More extensively, one has for each $k \geq 1$:

(77.19) $$\nu(H) \leq \frac{\nu_k(H)}{k} \leq \nu^*(H) = \tau^*(H) \leq \frac{\tau_k(H)}{k} \leq \tau(H).$$

The first two inequalities follow from the facts that if y is a 1-matching, then $k \cdot y$ is a k-matching, and that if y is a k-matching, then $k^{-1} \cdot y$ is a fractional matching. The last two inequalities are shown similarly.

We will investigate classes of hypergraphs where some or all of the inequalities in (77.19) are satisfied with equality. Obviously, if $\nu(H) = \tau(H)$, then all terms in (77.19) are equal.

$\nu^*(H)$ can be described in terms of the $\nu_k(H)$ (Lovász [1974]):

(77.20) $$\nu^*(H) = \max_k \frac{\nu_k(H)}{k} = \lim_{k\to\infty} \frac{\nu_k(H)}{k}.$$

Here the left-hand side equality holds as the maximum in (77.13) is attained by a rational optimum solution y. If k is the common denominator of the components of y, then $k \cdot y$ is a k-matching, and hence $k \cdot \nu^*(H) \leq \nu_k(H)$; so equality follows by (77.19).

The right-hand side equality follows from Fekete's lemma (Theorem 2.2), using the fact that for all $k, l \geq 1$:

(77.21) $\nu_{k+l}(H) \geq \nu_k(H) + \nu_l(H)$,

since if y' and y'' are a k- and an l-matching respectively, then $y' + y''$ is a $k + l$-matching.

Similarly we have:

(77.22) $$\tau^*(H) = \min_k \frac{\tau_k(H)}{k} = \lim_{k\to\infty} \frac{\tau_k(H)}{k},$$

using (77.12) and the fact that for all $k, l \geq 1$:

(77.23) $\tau_{k+l}(H) \leq \tau_k(H) + \tau_l(H)$.

77.9. Further results and notes

77.9a. Bottleneck extrema

Edmonds and Fulkerson [1970] showed that for any clutter $H = (V, \mathcal{E})$, its blocker $(V, \mathcal{B})$ is the unique clutter with the property that for each $f : V \to \mathbb{R}$ the following equality holds:

(77.24) $$\min_{E\in\mathcal{E}} \max_{x\in E} f(x) = \max_{B\in\mathcal{B}} \min_{y\in B} f(y).$$

(These extrema are called *bottleneck extrema.*) To see that the collection $\mathcal{B}$ of minimal vertex covers of H has this property, let $E \in \mathcal{E}$ and $x \in E$ attain the first minimum and first maximum. Then each $F \in \mathcal{E}$ contains a vertex z with $f(z) \geq f(x)$. Hence $\{z \in V \mid f(z) \geq f(x)\}$ contains a set B in $\mathcal{B}$. So $f(x) \leq \min_{y \in B} f(y)$. This shows $\leq$ in (77.24). Moreover, for any $B \in \mathcal{B}$, as E intersects B, we have $f(x) \geq \min_{y \in B} f(y)$. This gives $\geq$ in (77.24).

To see that this property characterizes the blocker, let $\mathcal{B}$ be any clutter satisfying (77.24) for each $f : V \to \mathbb{R}$. Then for each $B \in \mathcal{B}$ and $E \in \mathcal{E}$ we have $B \cap E \neq \emptyset$, since otherwise we can define f such that $f(x) < 0$ for all $x \in E$ and $f(y) > 0$ for each $y \in B$, giving $<$ in (77.24), a contradiction.

Finally, each vertex cover B of $\mathcal{E}$ contains a set in $\mathcal{B}$. If not, we can define f such that $f(x) > 0$ for each $x \in B$ and $f(y) < 0$ for each $y \in V \setminus B$. Then we have $>$ in (77.24), again a contradiction.

77.9b. The ratio of τ and τ^*

The following theorem of Johnson [1974a] and Lovász [1975c] bounds $\tau(H)$ in terms of $\tau^*(H)$ and the maximum degree of H. (The *degree* of a vertex v is the number of edges containing v. The *maximum degree* of H is the maximum of the degrees of its vertices.) The method is similar to that of Theorem 64.13.

Theorem 77.2. *For any hypergraph $H = (V, \mathcal{E})$ of maximum degree d one has:*

(77.25) $$\tau(H) \leq (1 + \ln d)\tau^*(H).$$

Proof. Iteratively choose vertices $v_1, v_2, \ldots$, where, for each $i = 1, 2, \ldots$, vertex v_i is chosen such that it is contained in a maximum number of edges not intersecting $\{v_1, \ldots, v_{i-1}\}$. We stop if the set $\{v_1, \ldots, v_k\}$ of chosen vertices is a vertex cover. So $\tau(H) \leq k$.

For each $i = 1, \ldots, k$, let d_i be the number of edges containing v_i but not intersecting $\{v_1, \ldots, v_{i-1}\}$. For each $F \in \mathcal{E}$, define

(77.26) $$y_F := \frac{1}{d_i},$$

where i is the smallest index with $v_i \in F$. Then

(77.27) $$\sum_{F \in \mathcal{E}} y_F = \sum_{i=1}^{k} d_i \frac{1}{d_i} = k,$$

and hence

(77.28) $$\tau(H) \leq \sum_{F \in \mathcal{E}} y_F.$$

We next show that $(1 + \ln d)^{-1} \cdot y$ is a fractional matching. To this end, consider any vertex v. Let $F_1, \ldots, F_t$ be the edges of H containing v, in the order by which they are intersected by $v_1, \ldots, v_k$. Then for each $j = 1, \ldots, t$, we have

(77.29) $$y_{F_j} \leq \frac{1}{t - j + 1}.$$

For let i be the smallest index with $v_i \in F_j$. So $y_{F_j} = 1/d_i$. Moreover, $d_i \geq t-j+1$, since v is contained in at least $t-j+1$ edges not intersected by $\{v_1, \ldots, v_{i-1}\}$. This proves (77.29).

Hence

(77.30) $$\sum_{j=1}^{t} y_{F_j} \leq \sum_{j=1}^{t} \frac{1}{t-j+1} = \sum_{j=1}^{t} \frac{1}{j} \leq 1 + \ln t \leq 1 + \ln d.$$

As this holds for each vertex v, $(1+\ln d)^{-1} \cdot y$ is a fractional matching.

This implies

(77.31) $$\tau(H) \leq \sum_{F \in \mathcal{E}} y_F \leq (1+\ln d)\nu^*(H) = (1+\ln d)\tau^*(H),$$

as required. ∎

(Related work can be found in Balas [1984].)

The proof shows that one can find a vertex cover of size less than $(1+\ln d)\tau^*$, by iteratively selecting a vertex of maximum degree and deleting it.

The proof method of Theorem 67.17 gives that for any hypergraph $H = (V, \mathcal{E})$:

(77.32) $$\tau^*(H) = \lim_{k \to \infty} \sqrt[k]{\tau(H^k)},$$

where H^k is the hypergraph on V^k with edges all sets $E_1 \times \cdots \times E_k$ with $E_1, \ldots, E_k \in \mathcal{E}$.

77.9c. Further notes

Füredi, Kahn, and Seymour [1993] showed that each hypergraph $H = (V, \mathcal{E})$ has a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

(77.33) $$\sum_{F \in \mathcal{M}} (|F| - 1 + \frac{1}{|F|}) \geq \nu^*(H).$$

In particular, for any hypergraph H:

(77.34) $$\nu(H) \geq \frac{r_{\max}}{r_{\max}^2 - r_{\max} + 1} \nu^*(H),$$

where $r_{\max} := r_{\max}(H)$ (the maximum edge size of H). (For *uniform* hypergraphs H (that is, all edges of H have the same size), this was proved by Füredi [1981] (confirming a conjecture of L. Lovász (cf. Füredi [1988])).)

Füredi, Kahn, and Seymour [1993] conjecture the following weighted extension of (77.33):

(77.35) (?) For each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

$$\sum_{F \in \mathcal{M}} \big(|F| - 1 + \frac{1}{|F|}\big) w(F) \geq \nu_w^*(H),$$

where $\nu_w^*(H)$ is the maximum weight $w^\mathsf{T} y$ of a fractional matching $y : \mathcal{E} \to \mathbb{R}_+$. Füredi, Kahn, and Seymour [1993] proved this conjecture for uniform hypergraphs, and also for hypergraphs H with $\nu(H) = 1$.

Related work on the relations between fractional and integer packing and covering was reported by Chvátal [1979], Dobson [1982], Fisher and Wolsey [1982], Aharoni, Erdős, and Linial [1985,1988], Raghavan [1988], Feige [1996,1998], and Slavík [1996,1997].

Lovász [1975b] showed that for each choice of $\nu, \tau \in \mathbb{Z}_+$ and $r \in \mathbb{Q}_+$ satisfying $1 \leq \nu \leq r \leq \tau$ and $r > 1$, there exists a hypergraph H with $\nu(H) = \nu$, $\tau^*(H) = r$, and $\nu(H) = \nu$. Chung, Füredi, Garey, and Graham [1988] showed that for each rational number r, there exists a 3-uniform hypergraph H with $\tau^*(H) \equiv r \pmod 1$. (For each 2-uniform hypergraph (= graph) H, $\tau^*(H)$ belongs to $\frac{1}{2}\mathbb{Z}$ (cf. Section 64.6).)

Saks [1986] studied the behaviour of the parameters τ and ν under taking unions of edges and vertex covers.

The hypergraph analogue of matching augmenting paths in graphs was studied by Edmonds [1962].

Seymour [1977a] gave a forbidden minor characterization of those clutters H that come from an undirected graph $G = (V, E)$ and $s, t \in V$, by taking as edges of H all edge sets of $s - t$ paths. (Related work can be found in Novick and Sebő [1995].)

Determining the vertex cover number $\tau(H)$ of a hypergraph H is equivalent to the set covering problem. In Section 82.6b we give further references for this problem. Determining the matching number $\nu(H)$ of H is equivalent to the vertex packing (equivalently, the set packing) problem. In Section 64.9e we gave further references for this problem.

Connectivity augmentation for hypergraphs was studied by Bang-Jensen and Jackson [1999], Benczúr [1999], Benczúr and Frank [1999], Cheng [1999], and Szigeti [1999].

The problems of finding a maximum-size matching and a minimum-size vertex cover in a hypergraph are equivalent to finding a maximum-size stable set in a graph and a minimum-size edge cover in a hypergraph. For references to general methods for these problems, we refer to Sections 64.9e and 82.6b, respectively.

Extensions of Gallai's theorem (Theorem 19.1) to hypergraphs were given by Tuza [1991], and generalizations of Kőnig's and Hall's theorems to hypergraphs by Aharoni and Haxell [2000] and Aharoni, Berger, and Ziv [2002].

Surveys on packing and covering in hypergraphs were given by Berge [1973b, 1973c,1978a,1979b,1989a], Schrijver [1979b], Füredi [1988], and Cornuéjols [2001].

Chapter 78

Ideal hypergraphs

Ideal hypergraphs are those hypergraphs for which the convex hull of the vertex covers is given by the edge inequalities. They therefore form a class of hypergraphs where polyhedral methods apply. Since the relations of blocking hypergraphs and of blocking polyhedra coincide in this case, the class of ideal hypergraphs is closed under taking blockers.
The class of ideal hypergraphs is also closed under taking minors. A characterization of ideal hypergraphs in terms of forbidden minors is not known, but a theorem of Lehman gives powerful properties of minimally nonideal hypergraphs.

78.1. Ideal hypergraphs

For any hypergraph $H = (V, \mathcal{E})$, let P_H be the set of all fractional vertex covers; that is, P_H is the solution set of

(78.1) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(F) \geq 1$ for $F \in \mathcal{E}$.

A hypergraph $H = (V, \mathcal{E})$ is called *ideal* if P_H is integer[4]. Obviously, H is ideal $\iff$ $H^{\min}$ is ideal $\iff$ $H^{\uparrow}$ is ideal.

Note that each integer vertex of P_H is a 0,1 vector, and hence the incidence vector of some vertex cover of H. So H is ideal if and only if (78.1) determines the up hull of the incidence vectors of the vertex covers of H. By Theorem 5.19, H is ideal if and only if the convex hull of the incidence vectors of the vertex covers of H is determined by

(78.2) (i) $0 \leq x_v \leq 1$ for $v \in V$,
(ii) $x(F) \geq 1$ for $F \in \mathcal{E}$.

By the theory of blocking polyhedra (cf. Theorem 5.8), H is ideal if and only if each vertex of the polyhedron determined by

[4] Alternatively, such hypergraphs are called *Fulkersonian*, or said to satisfy the *length-width inequality* or the *width-length inequality*, or to have the *max-flow min-cut property*, the $\mathbb{Q}_+$-*max-flow min-cut property*, shortly the *MFMC property* or the $\mathbb{Q}_+$-*MFMC property*. Sakarovitch [1975,1976] used the term *quasi-balanced*.

(78.3) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(B) \geq 1$ for $B \in b(H)$

is integer — that is, is the incidence vector of an edge of H. This gives the following important theorem of Fulkerson [1970b,1971a]:

Theorem 78.1. *A hypergraph H is ideal $\iff$ its blocker $b(H)$ is ideal.*

Proof. See above. ∎

The class of ideal hypergraphs is also closed under taking minors (Lehman [1965,1979]):

Theorem 78.2. *Any minor of an ideal hypergraph is ideal again.*

Proof. Let $H = (V, \mathcal{E})$ be ideal and let $v \in V$. Choose $x \in P_{H/v}$. Let $\tilde{x} \in \mathbb{R}^V$ be defined by $\tilde{x}_u := x_u$ for $u \in V \setminus \{v\}$ and $\tilde{x}_v := 0$. Then $\tilde{x} \in P_H$. Hence $\tilde{x}$ is a convex combination of integer vectors z in P_H. Each of these vectors z satisfies $z_v = 0$. Hence we obtain x as a convex combination of integer vectors in $P_{H/v}$.

Next choose $x \in P_{H\backslash v}$. Now let $\tilde{x} \in \mathbb{R}^V$ be defined by $\tilde{x}_u := x_u$ for $u \in V \setminus \{v\}$ and $\tilde{x}_v := 1$. Then $\tilde{x} \in P_H$. Hence $\tilde{x}$ is a convex combination of integer vectors z in P_H. Now deleting the vth component from any such z, we obtain an integer vector in $P_{H\backslash v}$. Hence we obtain x as a convex combination of integer vectors in $P_{H\backslash v}$. ∎

78.2. Characterizations of ideal hypergraphs

We will give several characterizations of ideal hypergraphs — albeit not by forbidden minors, since such a characterization is not known.

In the present section we discuss some equivalent properties each characterizing ideality. In Section 78.4, we show Lehman's theorem, which gives properties of minimally nonideal hypergraphs. From this, some further characterizations of ideality will be derived.

The definition of ideal hypergraph can be stated equivalently as:

(78.4) H is ideal if and only if for each $w : V \to \mathbb{R}_+$, the minimum of $w^{\mathsf{T}}x$ over (78.1) is attained by an integer vector x.

As we can restrict ourselves to rational-valued w, and hence to integer-valued w, we have equivalently:

(78.5) H is ideal if and only if for each $w : V \to \mathbb{Z}_+$, the minimum of $w^{\mathsf{T}}x$ over (78.1) is attained by an integer vector x.

We can formulate this in terms of the 'parallelization' H^w (defined in Section 77.3). To this end, it is good to observe that, for any 'weight' function $w : V \to \mathbb{Z}_+$:

(78.6) $\tau(H^w) =$ the minimum weight of a vertex cover of H

and

(78.7) $\nu(H^w) =$ the maximum number t of edges $E_1, \ldots, E_t$ of H such that each $v \in V$ is in at most $w(v)$ of the E_i.

The values of $\tau^*(H^w)$ and $\nu^*(H^w)$ can be described by dual linear programs:

(78.8)
$$\tau^*(H^w) = \min\{w^\mathsf{T} x \mid x \in \mathbb{R}_+^V, Mx \geq \mathbf{1}\}$$
$$= \min\{y^\mathsf{T}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{E}}, y^\mathsf{T} M \leq w^\mathsf{T}\} = \nu^*(H^w),$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. So we have:

(78.9) H is ideal if and only if $\tau^*(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$.

The following further characterizations were found[5]:

Theorem 78.3. *For any hypergraph $H = (V, \mathcal{E})$ the following are equivalent:*

(78.10)
(i) *H is ideal, that is $\tau^*(H') = \tau(H')$ for each parallelization H' of H;*
(ii) *$\tau^*(H')$ is an integer for each parallelization H' of H;*
(iii) *$b(H)$ is ideal;*
(iv) *$\tau^*(b(H)')$ is an integer for each parallelization $b(H)'$ of $b(H)$;*
(v) *P_H and $P_{b(H)}$ form a pair of blocking polyhedra;*
(vi) *$\tau(H^w)\tau(b(H)^l) \leq w^\mathsf{T} l$ for all $w, l : V \to \mathbb{Z}_+$.*

Proof. The implications (i)⇒(ii) and (iii)⇒(iv) are trivial. The equivalences (i) $\iff$ (iii) $\iff$ (v) were shown above.

The implication (ii)⇒(i) is shown as follows[6]. We must show that (ii) implies that each vertex x^* of the polyhedron P_H defined by (78.1) is integer. Suppose not. Choose $v \in V$ with x_v^* not integer. As x^* is a vertex, there is a weight function $w : V \to \mathbb{R}_+$ such that the minimum of $w^\mathsf{T} x$ over P_H is attained uniquely by x^*. By scaling, we can assume that w is integer and that for $\tilde{w} := w + \chi^v$, also the minimum of $\tilde{w}^\mathsf{T} x$ over P_H is attained at x^*. So $w^\mathsf{T} x^*$ and $\tilde{w}^\mathsf{T} x^*$ are integers (by (ii)), and hence $x_v^* = \tilde{w}^\mathsf{T} x^* - w^\mathsf{T} x^*$ is an integer, contradicting our assumption.

This proves (ii)⇒(i) and similarly (iv)⇒(iii). So conditions (i), (ii), (iii), (iv), and (v) are equivalent. We finally consider condition (vi).

Necessity of (vi) can be seen as follows. Choose $w, l : V \to \mathbb{Z}_+$. Let $\alpha := \tau(H^w) = \tau^*(H^w)$ and $\beta := \tau(b(H)^l) = \tau^*(b(H)^l)$. So $w(B) \geq \alpha$ for each edge B of $b(H)$ (= minimal vertex cover of H), and hence $\alpha^{-1} \cdot w(B) \geq 1$

[5] (i)⇔(vi)⇔(iii) was shown by Lehman [1965,1979], (i)⇔(v) by Fulkerson [1970b,1971a], and (i)⇔(ii) and (iii)⇔(iv) by Lovász [1977b].

Lehman called condition (i) the *max-flow min-cut property*, and condition (vi) the *width-length inequality* (motivated by work of Moore and Shannon [1956] who proved this inequality for the width (minimum cut-capacity) and length (shortest path) of a network).

[6] It also follows directly from general polyhedral theory (Theorem 5.18).

for each edge of $b(H)$. So $\alpha^{-1} \cdot w \in P_{b(H)}$. Similarly, $\beta^{-1} \cdot l$ belongs to P_H. As $P_{b(H)}$ is the blocking polyhedron of P_H, we have $(\alpha^{-1} \cdot w)^{\mathsf{T}}(\beta^{-1} \cdot l) \geq 1$, that is $w^{\mathsf{T}} l \geq \alpha\beta$, as required.

To see sufficiency of (vi), suppose that P_H has a noninteger vertex x^*. Then there is a hyperplane separating x^* from the integer vectors in P_H. So there is a $y \in \mathbb{Q}_+^V$ with $y^{\mathsf{T}} x^* < 1$ while $y^{\mathsf{T}} x \geq 1$ for each integer vector x in P_H. Let $\alpha > 0$ and $\beta > 0$ be such that $w := \alpha \cdot y$ and $l := \beta \cdot x^*$ are integer vectors. As $y^{\mathsf{T}} x \geq 1$ for each integer vector x in P_H, we have $w^{\mathsf{T}} x \geq \alpha$ for each integer vector x in P_H, and so $w(B) \geq \alpha$ for each vertex cover B of H; that is $\tau(H^w) \geq \alpha$. Since x^* belongs to P_H, we have that $x^*(F) \geq 1$ for each $F \in \mathcal{E}$, and hence $l(F) \geq \beta$ for each $F \in \mathcal{E}$; that is $\tau(b(H)^l) \geq \beta$. This implies

(78.11) $$\tau(H^w)\tau(b(H)^l) \geq \alpha\beta > \alpha\beta \cdot y^{\mathsf{T}} x^* = w^{\mathsf{T}} l,$$

contradicting (vi). ∎

78.3. Minimally nonideal hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *minimally nonideal* if H is nonideal and each proper minor of H is ideal. In particular, H is a clutter.

So being ideal can be characterized by not having a minimally nonideal hypergraph as a minor. Since the class of ideal hypergraphs is closed under taking the blocker, the blocker of any minimally nonideal hypergraph is minimally nonideal again.

There turn out to be infinitely many minimally nonideal hypergraphs. Known examples are[7]:

(78.12)
- (i) for each $n \geq 3$: $J_n :=$ the hypergraph with vertex set $\{1, \ldots, n\}$ and edges $\{2, \ldots, n\}$, $\{1, 2\}, \ldots, \{1, n\}$;
- (ii) the odd circuits C_{2k+1} and their blockers $b(C_{2k+1})$ $(k \geq 1)$;
- (iii) $F_7 :=$ the hypergraph with vertex set the points of the projective plane of order 2, and edges all lines (the *Fano hypergraph*)[8];
- (iv) $\mathcal{O}(K_5) :=$ the hypergraph with vertex set EK_5 and edges all odd circuits of K_5, and its blocker $b(\mathcal{O}(K_5))$ (having edges the complements of the nonempty cuts of K_5);
- (v) the hypergraph with vertex set EK_5 and edges all triangles of K_5, and its blocker;

[7] Examples (i), (ii), (iii) were given by Lehman [1965,1979], example (iv) by Seymour [1977b], examples (v) and (vi) by Cornuéjols and Novick [1994], example (vii) by P.D. Seymour (cf. Ding [1993]), $\mathcal{C}_5^3$ and $\mathcal{C}_7^4$ by Lehman [1965,1979] (they are the blockers of the circuits C_5 and C_7), $\mathcal{C}_8^3$ by Cornuéjols and Novick [1994] and Ding [1993], $\mathcal{C}_9^5$ and $\mathcal{C}_{11}^6$ by Qi [1989], and the other $\mathcal{C}_n^k$ by Cornuéjols and Novick [1994].

[8] Equivalently, $VF_7 = \{1, \ldots, 7\}$ and $EF_7 = \{\{i, i+1, i+3\} \mid i = 1, \ldots, 7\}$, taking addition mod 7.

(vi) the hypergraph with vertex set EK_5 and edges the complements of maximum-size cuts, and its blocker;

(vii) the hypergraph $\mathcal{D}_8$ with vertex set $\{1,\ldots,8\}$ and edges $\{1,2,6\}$, $\{2,3,5\}$, $\{3,4,8\}$, $\{4,5,7\}$, $\{2,5,6\}$, $\{1,6,7\}$, $\{4,7,8\}$, and $\{1,3,8\}$, and its blocker $b(\mathcal{D}_8)$;

(viii) the hypergraphs $\mathcal{C}_5^3$, $\mathcal{C}_8^3$, $\mathcal{C}_{11}^3$, $\mathcal{C}_{14}^3$, $\mathcal{C}_{17}^3$, $\mathcal{C}_7^4$, $\mathcal{C}_{11}^4$, $\mathcal{C}_9^5$, $\mathcal{C}_{11}^6$, $\mathcal{C}_{13}^7$ (where $\mathcal{C}_n^k$ has vertex set VC_n and edges all consecutive k-tuples from VC_n), and their blockers.

Note that $b(F_7) = F_7$ and $b(J_n) = J_n$ for each n. The hypergraphs given in (viii) are all the minimally nonideal hypergraphs of the form $\mathcal{C}_n^k$ with $k \geq 3$. This was proved by Cornuéjols and Novick [1994], who also gave several thousands of other minimally nonideal hypergraphs. A 'catalogue' of minimally nonideal hypergraphs was given by Lütolf and Margot [1998].

Seymour [1981a][9] conjectures that $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 are the only *binary* minimally nonideal hypergraphs (see Chapter 80).

We saw in Section 75.5 that $\mathcal{O}(K_5)$ is the unique minimally nonideal hypergraph among the hypergraphs obtained from a signed graph $G = (V, E, \Sigma)$ by taking EG as vertex set and the circuits C in G with $|C \cap \Sigma|$ odd as edges.

To see that F_7 is nonideal, the vector $x : VF_7 \to \mathbb{R}_+$ defined by $x_v := \frac{1}{3}$ for each $v \in VF_7$, is a fractional vertex cover of size $\frac{7}{3}$, but F_7 has no vertex cover of size $\leq \frac{7}{3}$. Moreover, F_7 is minimally nonideal: if we contract any vertex $v \in VF_7$, we obtain the hypergraph Q_6 isomorphic to the hypergraph $\mathcal{O}(K_4)$ (the hypergraph with vertex set EK_4 and edges all triangles). As this is a proper minor of $\mathcal{O}(K_5)$, it is ideal. Since $b(F_7) = F_7$, also deleting any vertex of F_7 results in an ideal hypergraph.

78.4. Properties of minimally nonideal hypergraphs: Lehman's theorem

A full list of minimally nonideal hypergraphs is not known, but the following theorems of Lehman [1990] show that minimally nonideal hypergraphs different from J_n ($n \geq 3$) are remarkably regular (shorter proofs were given by Padberg [1993] and Seymour [1990b] — we follow the latter):

Theorem 78.4. *Let $H = (V, \mathcal{E})$ be a minimally nonideal hypergraph with $H \neq J_n$ for $n := |V|$. Then P_H has a unique noninteger vertex, namely $r^{-1} \cdot \mathbf{1}$, where $r := r_{\min}(H)$. Moreover, H has precisely n edges of size r, and each vertex of H is contained in precisely r of them.*

Proof. Let x be a noninteger vertex of P_H. Then

(78.13) $\quad 0 < x_v < 1$ for each $v \in V$.

[9] Seymour [1981a] said that this conjecture was presented in Seymour [1977b], but the latter paper presents the three hypergraphs only as minimally nonideal hypergraphs.

For suppose first that $x_v = 0$. Then $x|V \setminus \{v\}$ is a noninteger vertex of $P_{H/v}$, contradicting the minimality of H. Similarly, if $x_v = 1$, then $x|V \setminus \{v\}$ is a noninteger vertex of $P_{H \setminus v}$, again contradicting the minimality of H. This proves (78.13).

Let $\mathcal{F}$ be the collection of sets $F \in \mathcal{E}$ having equality for x in (78.1)(ii). As x is a vertex, $\mathcal{F}$ has dimension n. (Here and below, the *dimension* of a collection of subsets of V, is the dimension of the collection of incidence vectors of these subsets.) Let $\mathcal{F}_v$ and $\mathcal{F} \setminus v$ be the collections of sets in $\mathcal{F}$ containing v and not containing v, respectively. Then

(78.14) (i) For each $F \in \mathcal{F}$ and $v \in V \setminus F$: $\dim(\mathcal{F} \setminus v) \leq n - |F|$;
(ii) for each $F \in \mathcal{F}$ and $v \in F$: $\dim(\mathcal{F}_v) \leq n - |F| + 1$.

To see (i), choose $F \in \mathcal{F}$ and $v \in V \setminus F$. Since $H \setminus v$ is ideal, $x|V \setminus \{v\}$ is a convex combination of incidence vectors of vertex covers of $H \setminus v$. For each $u \in F$, since $x_u > 0$, there is a vertex cover B_u of $H \setminus v$ having positive scalar in this convex decomposition and with $u \in B_u$. So $B_u \cap F = \{u\}$ (as $x(F) = 1$). Hence the incidence vectors χ^{B_u} for $u \in F$ are linearly independent. This implies that the vectors $\chi^{B_u} - x$ for $u \in F$ have dimension at least $|F| - 1$. As each of these vectors is orthogonal to $\chi^{F'}$ for each $F' \in \mathcal{F} \setminus v$, we have $\dim(\mathcal{F} \setminus v) \leq (n-1) - (|F| - 1) = n - |F|$, proving (78.14)(i).

We prove (ii) similarly. Choose $F \in \mathcal{F}$ and $v \in F$. Define $z := (1 - x_v)^{-1} \cdot x|V \setminus \{v\}$. Then $z \in P_{H/v}$, since $x(F' \setminus \{v\}) \geq 1 - x_v$ for each $F' \in \mathcal{F}$. Hence, since H/v is ideal, z is a convex combination of incidence vectors of vertex covers of H/v. For each $u \in F \setminus \{v\}$, since $z_u > 0$, there is a vertex cover B_u of H/v having positive scalar in this convex decomposition and with $u \in B_u$. So $B_u \cap F = \{u\}$ (since $z(F \setminus \{v\}) = 1$). Hence the incidence vectors χ^{B_u} for $u \in F \setminus \{v\}$ are linearly independent. This implies that the vectors $\chi^{B_u} - x$ for $u \in F$ have affine dimension at least $|F| - 1$. As each of these vectors is orthogonal to $\chi^{F'}$ for each $F' \in \mathcal{F}_v$, we have $\dim(\mathcal{F}_v) \leq (n-1) - (|F \setminus \{v\}| - 1) = n - |F| + 1$, proving (78.14)(ii).

Now (78.14)(i) implies:

(78.15) $|\mathcal{F}| = n$ and $|\mathcal{F} \setminus v| = n - |F|$ for each $v \in V$ and $F \in \mathcal{F} \setminus v$.

Indeed, let $\mathcal{F}'$ be a subcollection of $\mathcal{F}$ of dimension and size n. By (78.14)(i), $|\mathcal{F}' \setminus v| \leq n - |F|$ for each $v \in V$ and each $F \in \mathcal{F} \setminus v$. Let U be the set of $v \in V$ not covered by all sets in $\mathcal{F}'$. Then:

(78.16)
$$n = \sum_{F \in \mathcal{F}'} 1 = \sum_{F \in \mathcal{F}'} \sum_{v \in V \setminus F} \frac{1}{n - |F|} = \sum_{v \in U} \sum_{F \in \mathcal{F}' \setminus v} \frac{1}{n - |F|}$$
$$\leq \sum_{v \in U} \sum_{F \in \mathcal{F}' \setminus v} \frac{1}{|\mathcal{F}' \setminus v|} = \sum_{v \in U} 1 = |U| \leq n.$$

So we have equality throughout; that is, $U = V$ and $|\mathcal{F}' \setminus v| = n - |F|$ for each $v \in V$ and each $F \in \mathcal{F}' \setminus v$.

We deduce that $\mathcal{F}' = \mathcal{F}$. For suppose that there exists an $F \in \mathcal{F} \setminus \mathcal{F}'$. Then there is an $F' \in \mathcal{F}'$ such that $\mathcal{F}'' := (\mathcal{F}' \setminus \{F'\}) \cup \{F\}$ has dimension n. Choose $v \in F \setminus F'$ and $F'' \in \mathcal{F}'' \setminus v$. So $F'' \neq F$ and hence $F'' \in \mathcal{F}' \setminus v$. Hence $|\mathcal{F}'' \setminus v| = n - |F''| = |\mathcal{F}' \setminus v|$, contradicting the fact that $v \in F \setminus F'$. Concluding, $\mathcal{F}' = \mathcal{F}$ and we have (78.15).

(78.15) and (78.14)(ii) imply:

(78.17) $\quad |F| + |F'| \leq n + 1$ for any two distinct $F, F' \in \mathcal{F}$.

For choose $v \in F' \setminus F$. Then

(78.18) $$n = |\mathcal{F}| = |\mathcal{F}\setminus v| + |\mathcal{F}_v| \leq n - |F| + n - |F'| + 1 = 2n - |F| - |F'| + 1,$$

implying (78.17).

Let G be the graph on V where distinct $u, v \in V$ are adjacent if there is an $F \in \mathcal{F}$ with $u, v \notin F$. So by (78.15), $|\mathcal{F} \setminus u| = |\mathcal{F} \setminus v|$ for adjacent u, v. Hence, if G is connected, then $|\mathcal{F} \setminus v|$ is independent of v, and hence by (78.15), all sets in $\mathcal{F}$ have the same size, p say. Hence $x = p^{-1} \cdot \mathbf{1}$ and $p \geq r$ (as r is the minimum size of the sets in $\mathcal{E}$). On the other hand, the inequality $x(E) \geq 1$ for any minimum-size $E \in \mathcal{E}$, gives that $r \geq p$. So $p = r$, and the theorem follows.

So we can assume that G is not connected. Then there exists a partition of V into nonempty sets V_1, V_2 with $V_1 \subseteq F$ or $V_2 \subseteq F$ for each $F \in \mathcal{F}$. Let $\mathcal{F}_i$ be the collection of sets $F \in \mathcal{F}$ with $V_i \subseteq F$ (for $i = 1, 2$). So $\mathcal{F}_1, \mathcal{F}_2$ partition $\mathcal{F}$. By (78.17) we can assume that $\mathcal{F}_1 \subseteq \{V_1\}$ (since $|V_1| + |V_2| = n$). Then (78.17) gives moreover that $\mathcal{F}_2 \subseteq \{V_2 \cup \{v\} \mid v \in V_1\}$ (as $|\mathcal{F}| = n \geq 3$, so $\mathcal{F} \neq \{V_1, V_2\}$). Since $|\mathcal{F}_1| + |\mathcal{F}_2| = n$, it follows that $|V_1| = n - 1$, and $(V, \mathcal{F}) = J_n$. Since any subset of V is contained in or contains one of the sets in $\mathcal{F}$, we know that $H = J_n$, a contradiction. ∎

This theorem implies:

Corollary 78.4a. *Let H be a minimally nonideal hypergraph. Define $n := |VH|$, $r := r_{\max}(H)$, and $s := \tau(H)$. Then $\tau(H) - 1 < \tau^*(H) < \tau(H)$. If moreover $H \neq J_n$, then $rs > n$ and $\tau^*(H) = n/r$.*

Proof. First assume that $H = J_n$. Then $\tau(H) = 2$ and $\tau^*(H) = (2n - 3)/(n-1) = 2 - \frac{1}{n-1}$ as one easily checks. So we can assume that $H \neq J_n$.

Consider a pair $x \in P_H$ and $y \in P_{b(H)}$ minimizing $x^{\mathsf{T}}y$. So $x^{\mathsf{T}}y < 1$ (since P_H and $P_{b(H)}$ form no blocking pair of polyhedra). We can assume that x and y are vertices of P_H and $P_{b(H)}$ respectively. Moreover, x and y are noninteger, for if, say, x is integer, it is the incidence vector of a vertex cover of H, and hence $x^{\mathsf{T}}y \geq 1$, since $y \in P_{b(H)}$.

As H and $b(H)$ are minimally nonideal, we know by Theorem 78.4 that $x = r^{-1} \cdot \mathbf{1}$ and $y = s^{-1} \cdot \mathbf{1}$. Then $x^{\mathsf{T}}y < 1$ implies $rs > n$.

Let z minimize $\mathbf{1}^T z$ over $z \in P_H$, where z is a vertex of P_H. So z is a minimum-size vertex cover, and hence $\mathbf{1}^{\mathsf{T}}z = \tau^*(H)$. If z is integer, then

$\mathbf{1}^{\mathsf{T}}z \geq s > n/r$. If z is noninteger, then $z = r^{-1} \cdot \mathbf{1}$ by Theorem 78.4, and hence $\mathbf{1}^{\mathsf{T}}z = n/r$. So $\tau^*(H) = n/r$.

As $rs > n$, we have $n/r < s$ and so $\tau^*(H) < \tau(H)$. Moreover, for any $v \in VH$ we have $\tau^*(H \setminus v) \leq (n-1)/r$, since $r^{-1} \cdot \mathbf{1}_{V \setminus \{v\}}$ is a fractional vertex cover of H. Hence

$$(78.19) \qquad \tau^*(H) = \frac{n}{r} > \frac{n-1}{r} \geq \tau^*(H \setminus v) = \tau(H \setminus v) \geq \tau(H) - 1,$$

as required. ∎

Corollary 78.4a implies a number of further characterizations of ideal hypergraphs, partly sharpening Theorem 78.3 (Lehman [1990], Padberg [1993], Seymour [1990b]; the equivalence (i)⇔(iv) answers a question of P.D. Seymour (personal communication 1976)):

Corollary 78.4b. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(78.20)
- (i) *H is ideal, that is, $\tau(H^w) = \tau^*(H^w)$ for each $w : V \to \mathbb{Z}_+$;*
- (ii) *$H' \neq J_n$ (for all $n \geq 3$) and $\tau(H')r_{\min}(H') \leq |VH'|$, for each minor H' of H;*
- (iii) *$\tau^*(H') \in \mathbb{Z}$ for each minor H' of H;*
- (iv) *$\tau(H') = \tau^*(H')$ for each minor H' of H;*
- (v) *$\tau(H^w) = \tau^*(H^w)$ for each $w : V \to \{0, 1, |V|\}$.*

Proof. Condition (i) implies each of (ii)-(v), since ideality is closed under taking minors and parallelization. The implication (iv)⇒(iii) is direct. The implications (ii)⇒(i) and (iii)⇒(i) follow from Corollary 78.4a: if H is not ideal, it has a minor H' that is minimally nonideal; then Corollary 78.4a contradicts (ii) and (iii). So it suffices to show (v)⇒(iv).

Let (v) hold. Let H' be a minor obtained from H by contracting the vertices in a set U and deleting the vertices in a set W. Define $w(v) := 0$ if $v \in W$, $w(v) := |V|$ if $v \in U$, and $w(v) := 1$ otherwise. We assume that $\tau(H')$ is finite (so $\emptyset \in EH'$). We show

$$(78.21) \qquad \tau(H') \leq \tau(H^w) = \tau^*(H^w) \leq \tau^*(H'),$$

which implies (iv).

We first show the first inequality in (78.21). If $\tau(H') \geq |V|$, then $\tau(H') \geq |VH'|$, and hence each singleton is an edge of H'. So $\tau(H') = |VH'|$, and hence $\tau(H^w) \geq |VH'| = \tau(H')$. So we can assume that $\tau(H') < |V|$. Then $\tau(H') \leq \tau(H^w)$, since otherwise $\tau(H^w) < |V|$, and hence H has a vertex cover B contained in $V \setminus U$ with $|B \setminus W| = \tau(H^w)$. So $\tau(H') \leq |B \setminus W| = \tau(H^w)$. This proves the first inequality in (78.21).

To see the second inequality, let $x \in \mathbb{R}^{V \setminus (U \cup W)}$ be a minimum-size fractional vertex cover of H'. We can extend x to a fractional vertex cover $\tilde{x} \in \mathbb{R}^V$ of H by defining $\tilde{x}_v := 0$ if $v \in U$ and $\tilde{x}_v := 1$ if $v \in W$. Then

(78.22) $$\tau^*(H^w) \leq w^\mathsf{T}\tilde{x} = \mathbf{1}^\mathsf{T}x = \tau^*(H').$$

Hence $\tau^*(H^w) \leq \tau^*(H')$, proving (78.21). ∎

With Theorem 78.4 some more properties of minimally nonideal hypergraphs can be derived (Lehman [1990]), where J denotes an all-one matrix:

Theorem 78.5. *Let $H = (V, \mathcal{E})$ be a minimally nonideal hypergraph with $H \neq J_n$ where $n := |V|$. Let $r := r_{\min}(H)$ and $s = \tau(H)$. Let $\mathcal{F}$ and $\mathcal{C}$ be the collections of minimum-size edges of H and $b(H)$ respectively. Let M and N be the $\mathcal{F} \times V$ and $\mathcal{C} \times V$ incidence matrices of $\mathcal{F}$ and $\mathcal{C}$ respectively. Then the rows of M can be ordered such that*

(78.23) $$MN^\mathsf{T} = J + (rs - n)I = N^\mathsf{T}M.$$

Proof. For each $B \in \mathcal{C}$ we have:

(78.24) $$\sum_{F \in \mathcal{F}} |F \cap B| = rs,$$

since $|B| = s$ and since each $v \in V$ is in exactly r sets in $\mathcal{F}$ (by Theorem 78.4).

As $|F \cap B| \geq 1$ for each $F \in \mathcal{F}$, (78.24) gives:

(78.25) $|F \cap B| \leq rs - n + 1$ for each $F \in \mathcal{F}$, and $|F \cap B| \geq 2$ for at least one $F \in \mathcal{F}$.

Choose for each $B \in \mathcal{C}$ a set $F_B \in \mathcal{F}$ with $|B \cap F_B| \geq 2$. Then

(78.26) for each $v \in V$ there are at least $rs - n + 1$ sets $B \in \mathcal{C}$ with $v \in B \cap F_B$.

To see this, consider $H \setminus v$ and the vector $x := r^{-1} \cdot \mathbf{1}$ in $\mathbb{R}^{V \setminus \{v\}}$. Then x satisfies (78.1) for $H \setminus v$. As $H \setminus v$ is ideal, there exist distinct $B_1, \ldots, B_m \in b(H)$ and $\lambda_1, \ldots, \lambda_m > 0$ with

(78.27) $$x \geq \sum_{i=1}^{m} \lambda_i \chi^{B_i \setminus \{v\}} \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

We can assume that $v \in B_i \in \mathcal{C}$ holds for $i = 1, \ldots, k$, and $v \notin B_i$ or $B_i \notin \mathcal{C}$ for $i > k$. So $|B_i \setminus \{v\}| \geq s$ for $i > k$. Then (78.27) implies

(78.28) $$\begin{aligned} \frac{n-1}{r} &= x^\mathsf{T}\mathbf{1} \geq \sum_{i=1}^{m} \lambda_i |B_i \setminus \{v\}| \geq \sum_{i=1}^{k} \lambda_i (s-1) + \sum_{i=k+1}^{m} \lambda_i s \\ &= s - \sum_{i=1}^{k} \lambda_i \geq s - \frac{k}{r}, \end{aligned}$$

since $\lambda_i \leq 1/r$ for each i, by (78.27). (78.28) implies $k \geq rs - n + 1$. Now for each $i \leq k$, we have $v \in F_{B_i}$, since otherwise $x(F_{B_i}) = 1$, implying $|B_i \cap F_{B_i}| = 1$ (by (78.27)), a contradiction. So we have (78.26).

This implies

$$(78.29)\qquad \begin{aligned} n(rs-n+1) &\geq \sum_{B\in\mathcal{C}} |B\cap F_B| \\ &= \sum_{v\in V}(\text{number of } B\in\mathcal{C} \text{ with } v\in B\cap F_B) \geq n(rs-n+1), \end{aligned}$$

and hence we have equality throughout. So for each $B\in\mathcal{C}$ we have $|B\cap F_B| = rs-n+1$ and $|B\cap F|=1$ for each $F\in\mathcal{F}$ with $F\neq F_B$. By symmetry we have, for each $F\in\mathcal{F}$, that $|B\cap F|=1$ for all but one $B\in\mathcal{C}$, which has $|B\cap F| = rs-n+1$. So the set of pairs (B,F) with $|B\cap F| = rs-n+1$ forms a perfect matching covering $\mathcal{C}$ and $\mathcal{F}$. Hence we can reorder the rows of M such that $MN^\mathsf{T} = J+(rs-n)I$. In particular, M and N are nonsingular.

This implies

$$(78.30)\qquad \begin{aligned} MN^\mathsf{T}MN^\mathsf{T} &= (J+(rs-n)I)(J+(rs-n)I) \\ &= (n+2(rs-n))J+(rs-n)^2I = rsJ+(rs-n)(J+(rs-n)I) \\ &= MJN^\mathsf{T}+(rs-n)MN^\mathsf{T} = M(J+(rs-n)I)N^\mathsf{T}. \end{aligned}$$

So $N^\mathsf{T}M = J+(rs-n)I$ (as M and N are nonsingular). ∎

Notes. Seymour [1990b] asked the following related questions. Suppose that $H = (V,\mathcal{E})$ is a hypergraph without J_n minor ($n\geq 3$). Let $l,w : V\to\mathbb{Z}_+$ be such that

$$(78.31)\qquad \tau(H^w)\cdot\tau(b(H)^l) > l^\mathsf{T}w.$$

Is there a minor H' of H and $l',w' : VH' \to \{0,1\}$ such that

$$(78.32)\qquad \tau((H')^{w'})\cdot\tau(b(H')^{l'}) > l'^\mathsf{T}w'$$

and such that $\tau((H')^{w'}) \leq \tau(H^w)$ and $\tau(b(H')^{l'}) \leq \tau(b(H)^l)$?

A second question of Seymour is: Let $H=(V,\mathcal{E})$ be a nonideal hypergraph. Is the minimum of $\tau(H')$ over all parallelizations and minors H' of H with $\tau^*(H') < \tau(H')$ attained by a minor of H?

78.4a. Application of Lehman's theorem: Guenin's theorem

Lehman's theorem can be used as a tool in proving the characterization of Guenin [1998a,2001a] of weakly bipartite graphs (Corollary 75.4a). We follow the derivation as given in Schrijver [2002a].

Recall that a signed graph $G=(V,E,\Sigma)$ is called *weakly bipartite* if each vertex of the polyhedron (in $\mathbb{R}^E$) determined by:

$$(78.33)\qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for each edge } e, \\ \text{(ii)} & x(C)\geq 1 & \text{for each odd circuit } C, \end{array}$$

is integer, that is, the incidence vector of an odd circuit cover. Equivalently, if the hypergraph with vertex set E and edge set all odd circuits of G, is ideal.

Again, let odd-K_5 be the signed graph (VK_5, EK_5, EK_5). Then:

Theorem 78.6 (Guenin's theorem)**.** *A signed graph is weakly bipartite if and only if it has no odd-K_5 minor.*

Proof. Necessity follows from the fact that weak bipartition is closed under taking minors and that odd-K_5 is not weakly bipartite.

To see sufficiency, let $G = (V, E, \Sigma)$ be a minimally non-weakly bipartite signed graph (minimal under taking minors). We show that $G = (V, E, \Sigma)$ contains an odd-K_5 minor. Note that the operations of deletion and contraction in the signed graph G correspond to deletion and contraction in the hypergraph defined above.

Let $n := |E|$, let r be the minimum size of an odd circuit, and let s be the minimum size of an odd circuit cover. Let M (N, respectively) be the matrix whose rows are the incidence vectors of the minimum-size odd circuits (minimum-size odd circuit covers, respectively). By Lehman's theorem (Theorem 78.5), we know that both M and N have precisely n rows, that $rs > n$, and that the rows of M can be ordered such that

(78.34) $$MN^{\mathsf{T}} = J + (rs - n)I = N^{\mathsf{T}}M.$$

This implies that we can index the minimum-size odd circuits as $C_1, \ldots, C_n$ and the minimum-size odd circuit covers as $B_1, \ldots, B_n$ in such a way that for all $i, j = 1, \ldots, n$:

(78.35) $$|C_i \cap B_j| = 1 \text{ if } i \neq j \text{ and } |C_i \cap B_j| = q \text{ if } i = j,$$

where $q := rs - n + 1$. Since $q = |C_1 \cap B_1|$ is odd and ≥ 2 (as $rs > n$), we have $q \geq 3$.

The fact that $N^{\mathsf{T}}M = J + (rs - n)I$ is equivalent to:

(78.36)
(i) for each $e \in E$ there are precisely q indices i with $e \in C_i \cap B_i$,

(ii) for all distinct $e, f \in E$ there is precisely one index i with $e \in B_i$ and $f \in C_i$.

Then for all distinct $i, j = 1, \ldots, n$:

(78.37) the only odd circuits contained in $C_i \cup C_j$ are C_i and C_j; the only odd circuit covers contained in $B_i \cup B_j$ are B_i and B_j.

For let C be an odd circuit contained in $C_i \cup C_j$. Then $C_i \triangle C_j \triangle C$ contains an odd circuit, C' say. This implies that $C \cup C' \subseteq C_i \cup C_j$ and $C \cap C' \subseteq C_i \cap C_j$ (for if $e \in C \cap C'$, then $e \notin C_i \triangle C_j$). Hence $|C| + |C'| \leq |C_i| + |C_j|$. So also C and C' are minimum-size odd circuits and $C \cup C' = C_i \cup C_j$. As $|C_i \cap B_i| \geq 3$ we have $|C \cap B_i| \geq 2$ or $|C' \cap B_i| \geq 2$. Therefore, C or C' is equal to C_i, and the other is equal to C_j. The proof for odd circuit covers is analogous. This shows (78.37).

We now construct an odd-K_5 minor. Fix an edge $e \in E$, with ends v_1 and v_2, say. By (78.36)(i) we can assume that e is contained in $C_i \cap B_i$ for $i = 1, \ldots, q$. Then, by (78.36):

(78.38) any two sets among $C_1 \setminus \{e\}, \ldots, C_q \setminus \{e\}, B_1 \setminus \{e\}, \ldots, B_q \setminus \{e\}$ are disjoint, except that $|(C_i \setminus \{e\}) \cap (B_i \setminus \{e\})| = q - 1$ for $i = 1, \ldots, q$.

To see this, choose distinct $i, j = 1, \ldots, q$. Then $C_i \cap B_j = \{e\}$, as $|C_i \cap B_j| = 1$. Moreover, $C_i \cap C_j = \{e\}$, for suppose $f \in C_i \cap C_j$ with $f \neq e$. Then $f \in C_i \cap C_j$ and $e \in B_i \cap B_j$, contradicting (78.36)(ii). One similarly shows that $B_i \cap B_j = \{e\}$. This proves (78.38).

(78.37) implies:

(78.39) $VC_i \cap VC_j = \{v_1, v_2\}$ for distinct $i, j = 1, \ldots, q$.

Otherwise $(C_i \cup C_j) \setminus \{e\}$ contains a path P from v_1 to v_2 different from $C_i \setminus \{e\}$ and $C_j \setminus \{e\}$. By (78.37), $(C_i \cup C_j) \setminus \{e\}$ contains no odd circuit. Hence P and $C_i \setminus \{e\}$ have the same parity (with respect to Σ), and so $P \cup \{e\}$ is an odd circuit in $C_i \cup C_j$, contradicting (78.37). This proves (78.39).

Since $B_i \triangle \Sigma$ is a cut for each $i = 1, 2, 3$, there exist $U_1, U_2, U_3 \subseteq V$ such that

(78.40) $\delta(U_i) = B_j \triangle B_k = (B_j \cup B_k) \setminus \{e\}$

for all distinct $i, j, k \in \{1, 2, 3\}$. As $e \notin B_j \triangle B_k$, we can assume $v_1, v_2 \notin U_i$. Also

(78.41) U_i induces a connected subgraph of G.

If not, there is a $K \subseteq U_i$ such that $\delta(K)$ is a nonempty proper subset of $\delta(U_i)$. Then $B_j \triangle \delta(K)$ is an odd circuit cover contained in $B_j \cup B_k$, distinct from B_j and B_k, contradicting (78.37).

By (78.40), $\delta(U_1 \triangle U_2 \triangle U_3) = \delta(U_1) \triangle \delta(U_2) \triangle \delta(U_3) = \emptyset$, and hence $U_1 \triangle U_2 \triangle U_3$ $=\emptyset$ (as G is connected). So there exist pairwise disjoint sets V_1, V_2, V_3 of vertices with $U_i = V_j \cup V_k$ for all distinct $i, j, k \in \{1, 2, 3\}$. Define $V_0 := V \setminus (V_1 \cup V_2 \cup V_3)$.

(78.38) and (78.40) imply that $\delta(U_j) \cap \delta(U_k) = B_i \setminus \{e\}$ for distinct i, j, k. Hence $B_i \setminus \{e\}$ is the set of edges connecting either V_i and V_0, or V_j and V_k. So any edge not in $(B_1 \cup B_2 \cup B_3) \setminus \{e\}$ is spanned by one of the sets V_0, V_1, V_2, V_3.

Let $\{i, j, k\} = \{1, 2, 3\}$. Since C_i contains no edge in $(B_j \cup B_k) \setminus \{e\} = \delta(U_i)$, the set VC_i is disjoint from $U_i = V_j \cup V_k$. As $|C_i \cap B_i| \geq 3$ we know that VC_i intersects V_i.

We can reset Σ to an equivalent signing

(78.42) $\Sigma := B_1 \triangle B_2 \triangle B_3 \triangle \delta(V_0)$.

So Σ consists of e and all edges connecting distinct sets among V_1, V_2, V_3. For each $i = 1, 2, 3$ and $k = 1, 2$, let $e_{i,k}$ be the first edge along the path $C_i \setminus \{e\}$ that belongs to B_i, when starting from vertex v_k. So both $e_{i,1}$ and $e_{i,2}$ connect V_0 and V_i.

Let H be the minor of G obtained by deleting all edges except those in $C_1 \cup C_2 \cup C_3$ and those spanned by $V_1 \cup V_2 \cup V_3$, and contracting all remaining edges that are not in $\Sigma \cup \{e_{i,k} \mid i = 1, 2, 3; k = 1, 2\}$.

H can be described as follows. H contains the edge e, connecting the vertices $\overline{v}_1$ and $\overline{v}_2$ to which v_1 and v_2 are contracted (we have $\overline{v}_1 \neq \overline{v}_2$ by (78.39)). For each $i = 1, 2, 3$, the part of the path $C_i \setminus \{e\}$ that is between $e_{i,1}$ and $e_{i,2}$ belongs to one contracted vertex of H, call it z_i. This vertex z_i is adjacent to $\overline{v}_1$ and $\overline{v}_2$ by the edges $e_{i,1}$ and $e_{i,2}$. For each $i = 1, 2, 3$, V_i has been contracted to z_i and a number of other vertices, together forming the stable set S_i (say) in H. Any further edge of H connects S_i and S_j for some distinct $i, j \in \{1, 2, 3\}$.

By (78.41), the subgraph of H induced by $S_i \cup S_j$ is connected (for all distinct $i, j = 1, 2, 3$). So by Lemma 75.4α, the graph $H - \overline{v}_2$ has an odd K_4-subdivision as subgraph, containing the edges $\overline{v}_1 z_1$, $\overline{v}_1 z_2$, and $\overline{v}_1 z_3$. As $\overline{v}_2$ is adjacent to $\overline{v}_1$, z_1, z_2, and z_3, it follows that H has an odd-K_5 minor. ∎

78.4b. Ideality is in co-NP

Seymour [1990b] showed (upon a suggestion of J. Edmonds) that Lehman's theorem (Theorem 78.5) implies that the question 'Given a hypergraph, is it ideal?' belongs to co-NP.

In this, we should be careful in the way the hypergraph $(V, \mathcal{E})$ is given. In most classes of examples, the number of edges is exponential in the number of vertices, and we have no full list of all edges at hand. We can however assume that we have an oracle telling us, for any subset U of V, if U contains an edge of H; that is, if $U \in H^{\uparrow}$. This gives us a polynomial-time test if a subset belongs to $H^{\min}$, and also a polynomial-time test if a subset B is a vertex cover (since B is a vertex cover if and only if $V \setminus B \notin H^{\uparrow}$). So if we have such an oracle for H, we can derive one for its blocker $b(H)$, and conversely.

Moreover, for any $v \in V$, an oracle for H gives oracles for H/v and $H \backslash v$. Indeed, for any $U \subseteq V \setminus \{v\}$: $U \in (H/v)^{\uparrow} \iff U \cup \{v\} \in H^{\uparrow}$ and $U \in (H \setminus v)^{\uparrow} \iff U \in H^{\uparrow}$.

Now to certify that a hypergraph is nonideal, it is sufficient and possible to specify *either* a minor H with $H = J_n$ for $n := |VH|$, *or* a minor H together with numbers r, s, edges $F_1, \ldots, F_n$, and vertex covers $B_1, \ldots, B_n$ (where $n := |VH|$) of H such that

(78.43) (i) $rs > n$,
(ii) $|F_i| = r$, $|B_i| = s$, and $|B_i \cap F_i| = rs - n + 1$ for each $i = 1, \ldots, n$;
(iii) each $v \in VH$ is in precisely r of the F_i and in precisely s of the B_i.

This is possible by Theorem 65.2. If $H = J_n$, this can be tested easily with the oracle. If $H \neq J_n$, then the sets F_i (B_i respectively) can be taken to be minimal edges of H ($b(H)$ respectively); the oracle can tell us that they belong to H ($b(H)$ respectively).

It is also sufficient to certify nonideality: (78.43) implies that $\tau(H) \geq s$: a vertex cover B of H intersects at most $r|B|$ of the F_i, and hence $r|B| \geq n$, implying $|B| \geq s$ (since otherwise $(s-1)r \geq n$ and hence $rs - n + 1 > r$, contradicting (78.43)(ii)). Similarly, (78.43) implies that $r_{\min}(H) \geq r$. As $rs > n$, this implies that H is nonideal.

78.5. Further results and notes

78.5a. Composition of clutters

Billera [1971] described the following composition of hypergraphs. Let $H' = (V', \mathcal{E}')$ and $H'' = (V'', \mathcal{E}'')$ be hypergraphs with V' and V'' disjoint, and choose $v \in V'$. Let $V := (V' \setminus \{v\}) \cup V''$, and define $\mathcal{E}$ by:

(78.44) $$\mathcal{E} := \{E' \in \mathcal{E}' \mid v \notin E'\} \cup \{(E' \setminus \{v\}) \cup E'' \mid E' \in \mathcal{E}', v \in E', E'' \in \mathcal{E}''\}.$$

Let $H = (V, \mathcal{E})$. Then H is ideal if and only if H' and H'' are ideal. (The 'only if' part was shown by Billera [1971] and the 'if' part by Bixby [1971].)

Related results were reported by Chopra [1995]. An extension of these results to *clutter amalgam* was given by Nobili and Sassano [1993a] (cf. Nobili and Sassano [1993b]).

78.5b. Further notes

Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs H with $r_{\min}(H) > 2$ and $\tau(H) > 2$. This would confirm the question of Ding [1993] whether there exists a number t such that each minimally nonideal hypergraph H satisfies $r_{\min}(H) \leq t$ or $\tau(H) \leq t$.

Since by Lehman's theorem, each minimally nonideal hypergraph $H \neq J_n$ satisfies $\tau^*(H) = r^{-1}\tau_r(H) < \tau(H)$, where $r := r_{\min}(H)$, the existence of such a t would imply that the following property characterizes ideality of a hypergraph H:

(78.45) H contains no J_n minor ($n \geq 3$) and satisfies $\tau_k(H') = k \cdot \tau(H')$ and $\tau_k(b(H')) = k \cdot \tau(b(H'))$ for each minor H' of H and each $k \leq t$.

Ding wondered if $t = 3$ would do.

Ding [1993] conjectures that for each fixed $k \geq 2$, each minor-minimal hypergraph H with $\tau_k(H) < k \cdot \tau(H)$, contains some J_n minor ($n \geq 3$) or satisfies the regularity conditions of Lehman's theorems (Theorem 78.4 and 78.5). Ding [1993] proved this for $k = 2$: if H is minor-minimal with the property $\tau_2 < 2\tau$ and if H has no J_n minor ($n \geq 3$), then the minimum-size vertex covers form an odd circuit on VH.

A $\{0, \pm 1\}$ matrix M is called *ideal* if the polytope

(78.46) $$\{x \mid \mathbf{0} \leq x \leq \mathbf{1}, Mx \geq \mathbf{1} - b\}$$

is integer, where b is the vector with b_i equal to the number of -1's in the ith row of M. These matrices generalize the incidence matrices of ideal hypergraphs. Guenin [1998b] and Nobili and Sassano [1995,1998] showed that they can be characterized in terms of ideal hypergraphs.

Related work on ideal hypergraphs was reported by Novick and Sebő [1996]. A survey on ideal hypergraphs was given by Cornuéjols and Guenin [2002b].

Chapter 79

Mengerian hypergraphs

Mengerian hypergraphs form a subclass of the ideal hypergraphs. They are characterized by the total dual integrality of the edge inequalities (where ideal hypergraphs require only totally *primal* integrality). So Mengerian hypergraphs satisfy min-max relations that are combinatorial at both optima.

This chapter gives a few characterizations of Mengerity. No characterization in terms of forbidden minors is known. In Chapter 80 we will give Seymour's forbidden minor characterization of *binary* Mengerian hypergraphs.

79.1. Mengerian hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *Mengerian* if $\nu(H') = \tau(H')$ for each parallelization H' of H.[10] Equivalently:

(79.1) $\quad H$ is Mengerian $\iff$ system (78.1) is totally dual integral.

By (77.19) (or by the theory of total dual integrality), each Mengerian hypergraph is ideal. Like ideal hypergraphs, the class of Mengerian hypergraphs is closed under taking minors:

Theorem 79.1. *Any minor of a Mengerian hypergraph is Mengerian again.*

Proof. As restriction is a special case of parallelization, any restriction of a Mengerian hypergraph is again Mengerian. As for contraction, let $H = (V, \mathcal{E})$ be a Mengerian hypergraph and let $v \in V$ and $w : V \setminus \{v\} \to \mathbb{Z}_+$. Define $w' : V \to \mathbb{Z}_+$ by $w'(u) := w(u)$ if $u \in V \setminus \{v\}$ and $w'(v) := \tau((H/v)^w)$. Then

(79.2) $$\tau((H/v)^w) \leq \tau(H^{w'}) = \nu(H^{w'}) \leq \nu((H/v)^w).$$

So $\tau((H/v)^w) = \nu((H/v)^w)$. Concluding, H/v is Mengerian. ∎

Unlike ideal hypergraphs, the class of Mengerian hypergraphs is not closed under taking blockers, as we shall see in Section 79.2.

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[10] Alternatively, such hypergraphs are said to have the $\mathbb{Z}_+$-*max-flow min-cut property*, shortly the $\mathbb{Z}_+$-*MFMC property*.

Theorem 78.3 implies some characterizations of Mengerian hypergraphs (Lovász [1975a] showed (i)⇔(ii), and Lovász [1976c] (i)⇔(iii); the equivalence (i)⇔(ii) also follows from a more general theorem of Hoffman [1974]):

Theorem 79.2. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(79.3)
(i) *H is Mengerian, that is, $\nu(H') = \tau(H')$ for each parallelization H' of H;*
(ii) *$\nu^*(H') = \nu(H')$ for each parallelization H' of H;*
(iii) *$\nu_2(H') = 2\nu(H')$ for each parallelization H' of H.*

Proof. The equivalence (i) $\iff$ (ii) follows from Theorem 78.3, since (79.3)(ii) implies that $\nu^*(H')$ is an integer and since $\nu^*(H') = \tau^*(H')$. The implications (ii)⇒(iii) follows from (77.19), since $\nu^*(H') \geq \frac{1}{2}\nu_2(H') \geq \nu(H')$. So it suffices to prove (iii)⇒(ii).

First observe that for each $w : V \to \mathbb{Z}_+$ and all $j, k \in \mathbb{Z}_+$ we have $\nu_{jk}(H^w) = \nu_k(H^{jw})$. Hence, if (79.3)(iii) holds, then for each $w : V \to \mathbb{Z}_+$ and each i:

(79.4) $$\nu_{2^{i+1}}(H^w) = \nu_2(H^{2^i w}) = 2\nu(H^{2^i w}) = 2\nu_{2^i}(H^w).$$

So by induction on i we find that for all i:

(79.5) $$\nu_{2^i}(H^w) = 2^i \nu(H^w), \text{ that is, } \frac{\nu_{2^i}(H^w)}{2^i} = \nu(H^w).$$

As

(79.6) $$\nu^*(H^w) = \lim_{k\to\infty} \frac{\nu_k(H^w)}{k}$$

(by (77.20)), this gives (79.3)(ii). ∎

Another characterization, in terms of the blocker, is:

Theorem 79.3. *Let $H = (V, \mathcal{E})$ be a hypergraph. Then the blocker $b(H)$ of H is Mengerian if and only if for each natural number k, each k-vertex cover is the sum of k 1-vertex covers.*

Proof. By definition, $b(H)$ is Mengerian if and only if $\nu(b(H)^l) = \tau(b(H)^l)$ for each $l : V \to \mathbb{Z}_+$. Now $\tau(b(H)^l)$ is equal to the minimum value of $l(E)$ for $E \in \mathcal{E}$ (by Theorem 77.1). In other words, $\tau(b(H)^l)$ is equal to the maximum number k for which l is a k-vertex cover.

Moreover, $\nu(b(H)^l)$ is equal to the maximum number k of vertex covers $B_1, \ldots, B_k$ with

(79.7) $$\chi^{B_1} + \cdots + \chi^{B_k} \leq l.$$

So $\nu(b(H)^l) = \tau(b(H)^l)$ for each $l : V \to \mathbb{Z}_+$ if and only if for each k, each k-vertex cover l is the sum of k 1-vertex covers. ∎

Note that the right-hand side of the equivalence in this theorem directly implies, by definition of τ_k, that $\tau_k(H) = k \cdot \tau(H)$ for each k; that is, $\tau^*(H) = \tau(H)$.

79.1a. Examples of Mengerian hypergraphs

Bipartite graphs. Let $G = (V, E)$ be a bipartite graph. It is very easy to show that $\nu_2(G) = 2\nu(G)$. (It amounts to the fact that each bipartite graph G of maximum degree at most 2 has a matching of size at least $\frac{1}{2}|EG|$.)

Since the class of bipartite graphs is closed under parallelization, Theorem 79.2 gives $\nu(G) = \tau(G)$; that is, the matching number of G is equal to the vertex cover number of G. This is Kőnig's matching theorem (Theorem 16.2).

Network flows. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. Let $\mathcal{P}$ be the collection of arc sets of $s-t$ paths. Consider the hypergraph $H = (A, \mathcal{P})$. Then $b(H)$ is the hypergraph with edge set all inclusionwise minimal $s-t$ cuts.

Now $\nu(b(H)) = \tau(b(H))$, since the minimum size k of an $s-t$ path is equal to the maximum number of pairwise disjoint $s-t$ cuts. This is an easy result, by considering the cuts $\delta^{\text{out}}(V_i)$ for $i = 1, \ldots, k$, where V_i is the set of vertices at distance $< i$ from s.

Since the class of hypergraphs $b(H)$ obtained in this way is closed under parallelization (it corresponds to replacing arcs by paths), $b(H)$ is Mengerian. Hence $b(H)$ is ideal, and hence H is ideal. That is, for each weight function $w : A \to \mathbb{Z}_+$ we have $\tau(H^w) = \tau^*(H^w) = \nu^*(H^w)$. This gives that the minimum weight of an $s-t$ cut is equal to the maximum of $\sum_{P \in \mathcal{P}} \lambda_P$ where $\lambda : \mathcal{P} \to \mathbb{R}_+$ with $\sum_{P \in \mathcal{P}} \lambda_P \chi^P \leq w$. That is, we have the max-flow min-cut theorem.

By Menger's theorem, we even know that $\nu(H) = \tau(H)$. As the class of these hypergraphs is closed under parallelization (it corresponds to adding parallel arcs to arcs), we know that H is Mengerian. By Theorem 79.2, to prove the existence of an integer maximum flow, it suffices to show that $\nu_2(H) = 2\nu(H)$, since this class of hypergraphs is closed under parallelization.

Arborescences. Let $D = (V, A)$ be a directed graph and let $r \in V$. Recall that a subset B of A is called an *r-arborescence* if (V, B) is a rooted tree with root r. An r-cut is a set $\delta^{\text{in}}(U)$ of arcs, where U is a nonempty subset of $V \setminus \{r\}$. Let H be the hypergraph with vertex set A and edges all r-arborescences. So the blocker $b(H)$ of H has edges all inclusionwise minimal r-cuts.

Since this class of hypergraphs is closed under parallelization, Edmonds' disjoint arborescences theorem (Theorem 53.1b) implies that H is Mengerian. By the optimum arborescence theorem (Theorem 52.3) also $b(H)$ is Mengerian.

Directed cuts. Let $D = (V, A)$ be a directed graph. Recall that a *directed cut* is a set of arcs of the form $\delta^{\text{in}}(U)$ where U is a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting all directed cuts. Let H be the hypergraph with vertex set A and edges all directed cuts. So the blocker $b(H)$ of H has edges all inclusionwise minimal directed cut covers.

One may show that $\nu_2(H) = 2\nu(H)$, as was done in the proof of the Lucchesi-Younger theorem (Theorem 55.2). Since again this class of hypergraphs is closed under parallelization, Theorem 79.2 implies that H is Mengerian. This is the Lucchesi-Younger theorem.

So H is ideal, and hence also $b(H)$ is ideal. The example of Figure 56.1 in Section 56.1 shows that $b(H)$ in general is not Mengerian.

79.2. Minimally non-Mengerian hypergraphs

By Theorem 79.1, the class of Mengerian hypergraphs is closed under taking minors. It is not closed under taking blockers, since the hypergraph

(79.8) $$Q_6 := \mathcal{O}(K_4)$$

(the hypergraph with vertex set EK_4 and edges all triangles of K_4) is non-Mengerian, while its blocker is Mengerian: Q_6 is non-Mengerian, since $\nu(Q_6) = 1$ (K_4 has no two edge-disjoint triangles), while $\tau(Q_6) = 2$ (no edge is contained in all triangles). Its blocker $H := b(Q_6)$ has edges all complements of nonempty cuts of K_4. To see that it is Mengerian, we show that $\nu(H^l) = \tau(H^l)$ for each 'length' function $l : EK_4 \to \mathbb{Z}_+$. Then $\tau(H^l)$ is the minimum length of a triangle in K_4. To calculate $\nu(H^l)$, observe that the edges of H are the triangles and the perfect matchings of K_4. Consider any perfect matching M of K_4 with $l(e) > 0$ for both edges $e \in M$. Then replacing l by $l - \chi^M$ reduces $\tau(H^l)$ by 1 and $\nu(H^l)$ by at least 1. So inductively we can assume that each perfect matching of K_4 contains an edge e with $l(e) = 0$. So l is 0 on all edges of a triangle, in which case $\tau(H^l) = 0 \leq \nu(H^l)$, or on all edges of a star, in which case both $\tau(H^l)$ and $\nu(H^l)$ are equal to the minimum length of the edges of the complementary triangle.

Call a hypergraph $H = (V, \mathcal{E})$ *minimally non-Mengerian* if H is a non-Mengerian hypergraph and each proper minor of H is Mengerian.

The hypergraph Q_6 is minimally non-Mengerian. Indeed, choose a vertex e of Q_6. The restriction $Q_6 \setminus e$ has only two edges, and hence is trivially Mengerian. The contraction Q_6/e is isomorphic to $b(Q_6) \setminus e$, and hence is Mengerian, as we saw above.

In Section 80.5 we will see that Q_6 is the only *binary* minimally nonideal hypergraph (binary is defined in Chapter 80). We list this and other examples of minimally non-Mengerian hypergraphs ((i) was given by Lovász [1974], and (ii)-(vi) by Seymour [1977b]):

(79.9)
- (i) $Q_6 = \mathcal{O}(K_4)$, the hypergraph with vertex set EK_4, and edges all triangles;
- (ii) any odd circuit;
- (iii) the blocker of any odd circuit;
- (iv) J_n for $n \geq 3$ (cf. (78.12));
- (v) the circuit on $1, 2, 3, 4, 5, 6, 7, 9$ (in order) added with the edge $\{3, 6, 9\}$;
- (vi) the blocker of the hypergraph in (v);

(vii) the hypergraph with vertex set $\{0,1,2,3\}$ and edges $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$, and its blocker.

Example (vii) shows that a minimally non-Mengerian hypergraph H can satisfy $\nu(H) = \tau(H)$.

Notes. Seymour [1977b] conjectured that Q_6 is the only minimally non-Mengerian hypergraph with Mengerian blocker. However, the example of Figure 56.1 gives a minimally non-Mengerian hypergraph on 9 vertices. Its blocker is Mengerian (by the Lucchesi-Younger theorem). Two other examples of hypergraphs consisting of directed cut covers in a directed graph were given by Cornuéjols and Guenin [2002c] and yield two more minimally non-Mengerian hypergraphs with Mengerian blockers.

Seymour [1977b] indicated by a construction that it might be hard to characterize all minimally non-Mengerian hypergraphs.

79.3. Further results and notes

79.3a. Packing hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *packing* if $\nu(H') = \tau(H')$ for each minor H' of H. So we have for any hypergraph H (using Theorem 79.1 and Corollary 78.4b):

(79.10) $\qquad H$ Mengerian $\Rightarrow$ H packing $\Rightarrow$ H ideal.

Q_6 is an example which is ideal but not packing, but no example is known of a non-Mengerian packing hypergraph. In fact, Conforti and Cornuéjols [1993] conjecture that both concepts coincide. Cornuéjols, Guenin, and Margot [1998,2000] proved this for *dyadic hypergraphs*, that is, hypergraphs H with $|E \cap B| \leq 2$ for each edge E of $H^{\min}$ and each edge B of $b(H)$.

The definition of packing implies that it is closed under taking minors. Call a hypergraph *minimally nonpacking* if it is nonpacking, but each proper minor is packing. So it is a minor-minimal hypergraph satisfying $\nu < \tau$.

Cornuéjols, Guenin, and Margot [1998,2000] showed that if a hypergraph is both minimally nonideal and minimally nonpacking, then $H = J_n$ for some $n \geq 3$ or $r_{\min}(H)\tau(H) = |VH| + 1$. They conjecture that, conversely, each minimally nonideal hypergraph H with $r_{\min}(H)\tau(H) = |VH| + 1$ is minimally nonpacking. By a computer program, this conjecture was verified for all hypergraphs with ≤ 14 vertices.

Another conjecture of Cornuéjols, Guenin, and Margot [1998,2000] is that $\tau(H) = 2$ for each ideal minimally nonpacking hypergraph H. This implies the above conjecture of Conforti and Cornuéjols that each packing hypergraph is Mengerian. For suppose that $H = (V, \mathcal{E})$ is packing and minimally non-Mengerian. Since H is non-Mengerian, there is a $w : V \to \mathbb{Z}_+$ with H^w nonpacking. Choose w with $w(V)$ minimal. Then H^w is minimally nonpacking. So by the second conjecture, $\tau(H^w) = 2$. As H is packing, $w(v) \geq 2$ for some $v \in V$. Now $\tau(H^w/v) \geq \tau(H^w) = 2$. Hence $\nu(H^w/v) \geq 2$. So there exist edges E_1, E_2 of H with $\chi^{E_1 \setminus \{v\}} + \chi^{E_2 \setminus \{v\}} \leq w$. Since $w(v) \geq 2$, this implies $\chi^{E_1} + \chi^{E_2} \leq w$, and hence $\nu(H^w) \geq 2$. This contradicts the fact that H^w is minimally nonpacking.

For a survey, see Cornuéjols and Guenin [2002b].

79.3b. Restrictions instead of parallelizations

It is tempting to conjecture that a hypergraph H is Mengerian if and only if $\nu(H') = \tau(H')$ for each restriction H' of H (instead of for each parallelization H' of H). Similarly, one may speculate that H is ideal if and only if $\tau^*(H') = \tau(H)$ for each restriction H' of H. But these characterizations are not valid, as is shown by the hypergraph of (79.9)(vii): then $\nu(H') = \tau(H')$ for each restriction H' of H (as soon as we delete any vertex we obtain a hypergraph with at most one edge); but H is nonideal, since if we duplicate vertex 0, we obtain a hypergraph with $\tau = 2$ and $\tau^* = \frac{3}{2}$.

So in Theorem 78.3, we cannot restrict H' to restrictions of H instead of parallelizations. But the equivalence of (78.10)(i) and (ii) is maintained if we restrict H' to restrictions of H:

Theorem 79.4. *For any hypergraph, $H = (V, \mathcal{E})$, the following are equivalent:*

(79.11) (i) $\tau(H') = \tau^*(H')$ *for each restriction H' of H;*
(ii) $\tau^*(H')$ *is an integer for each restriction H' of H.*

Proof. Since obviously (i)⇒(ii), we prove (ii)⇒(i). Choose a counterexample with $|V|$ smallest. Let x be a fractional vertex cover of size $\tau^*(H)$. Choose a vertex v with $x_v > 0$. As $x|V \setminus \{v\}$ is a fractional vertex cover of $H \setminus v$, we know:

(79.12) $$\tau^*(H \setminus v) \leq x(V \setminus \{v\}) < x(V) = \tau^*(H).$$

As $\tau^*(H)$ and $\tau^*(H \setminus v)$ are integer, this implies that $\tau^*(H \setminus v) \leq \tau^*(H) - 1$. By the minimality of V we know $\tau^*(H \setminus v) = \tau(H \setminus v)$. Therefore,

(79.13) $$\tau(H) \leq 1 + \tau(H \setminus v) = 1 + \tau^*(H \setminus v) \leq \tau^*(H),$$

and so $\tau(H) = \tau^*(H)$. ∎

As a direct consequence one has (Lovász [1974]):

Corollary 79.4a. *For any hypergraph, $H = (V, \mathcal{E})$, the following are equivalent:*

(79.14) (i) $\tau(H') = \nu(H')$ *for each restriction H' of H;*
(ii) $\nu^*(H') = \nu(H')$ *for each restriction H' of H.*

Proof. Directly from Theorem 79.4. ∎

(Lovász [1974] called hypergraphs with the properties (79.14) *seminormal*.)

Symmetry suggests the question if we can replace in Theorem 79.4 or Corollary 79.4a 'restriction' by 'contraction'.

79.3c. Equivalences for k-matchings and k-vertex covers

Some of the equivalences in Theorems 78.3 and 79.2 can be generalized as follows (Lovász [1977b] ($k \leq 2$), Schrijver and Seymour [1979] (general k)).

Theorem 79.5. *For any hypergraph $H = (V, \mathcal{E})$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(79.15) (i) $k \cdot \tau^*(H') = \tau_k(H')$ *for each parallelization* H' *of* H*;*
(ii) $k \cdot \tau^*(H')$ *is an integer for each parallelization* H' *of* H*.*

Proof. Similar to the proof of Theorem 78.3. ∎

This implies a result proved by Schrijver and Seymour [1979] (just by adapting the proof methods of Lovász [1975a] for the equivalence (i)⇔(ii) for $k \leq 2$ and of Lovász [1977b] for the equivalence (i)⇔(iii) for $k = 1$):

Corollary 79.5a. *For any hypergraph* $H = (V, \mathcal{E})$ *and any* $k \in \mathbb{Z}_+$*, the following are equivalent:*

(79.16) (i) $\nu_k(H') = \tau_k(H')$ *for each parallelization* H' *of* H*;*
(ii) $k \cdot \nu^*(H') = \nu_k(H')$ *for each parallelization* H' *of* H*;*
(iii) $\nu_{2k}(H') = 2\nu_k(H')$ *for each parallelization* H' *of* H*.*

Proof. Similar to the proof of Theorem 79.2. ∎

As an application, let $G = (V, E)$ be an undirected graph. Then $\nu_4(G) = 2\nu_2(G)$ is not difficult to show. Since the class of graphs is closed under parallelization, Corollary 79.5a implies that $\nu_2(G) = \tau_2(G)$, which is Theorem 30.1.

79.3d. A general technique

The following general result (derived with a method given by Lovász [1976c]) gives some more equivalences:

Theorem 79.6. *Let* $H = (V, \mathcal{E})$ *be a hypergraph and* $w : V \to \mathbb{Z}_+$*. Let* $f : \mathbb{Z}_+^V \longrightarrow \mathbb{R}_+$ *satisfy*

(79.17) (i) $f(x + y) \geq f(x) + f(y)$ *for all* $x, y \in \mathbb{Z}_+^V$*;*
(ii) *if* $u \leq w$*, then* $f(u) \in \mathbb{Z}_+$*;*
(iii) *if* $x \leq w + \mathbf{1}$*, then* $f(2x) = 2f(x)$*;*
(iv) $f(\chi^U) > 0$ *for each* $U \in \mathcal{E}$*.*

Then $\tau(H^w) \leq f(w)$*.*

Proof. By induction on $\mathbf{1}^\mathsf{T} w$, the case where $\tau(H^w) = 0$ being trivial. Assume $\tau(H^w) > 0$. That is, the support U of w contains an edge of H. Choose $x \in \mathbb{Z}_+^V$ with $w \leq x \leq w + \chi^U$ such that $f(x) = f(w)$ and such that $\mathbf{1}^\mathsf{T} x$ is as large as possible. Then $x \neq w + \chi^U$, since $f(w + \chi^U) \geq f(w) + f(\chi^U) > f(w)$, by (i) and (iv) (note that f is monotone by (i)). So $x_v < w_v + 1$ for some $v \in U$. By the maximality of x we know that $f(x + \chi^v) > f(x)$. Moreover, by induction, $\tau(H^{w-\chi^v}) \leq f(w - \chi^v)$, and hence

(79.18) $$\begin{aligned}\tau(H^w) &\leq 1 + \tau(H^{w-\chi^v}) \leq 1 + f(w - \chi^v) \leq 1 + f(x - \chi^v) \\ &\leq 1 + f(2x) - f(x + \chi^v) = 1 + 2f(x) - f(x + \chi^v) < 1 + f(x) = 1 + f(w),\end{aligned}$$

and hence, since $f(w) \in \mathbb{Z}$ we have $\tau(H^w) \leq f(w)$. ∎

This gives the following equivalent form of Theorem 79.4:

Corollary 79.6a. *Let $H = (V, \mathcal{E})$ be a hypergraph and let $w \in \mathbb{Z}_+^V$ be such that $\tau^*(H^x) \in \mathbb{Z}$ for each $x \leq w$. Then $\tau(H^w) = \tau^*(H^w)$.*

Proof. Define $f(x) := \tau^*(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6. ∎

We also obtain a generalization of Theorem 79.2:

Corollary 79.6b. *Let $H = (V, \mathcal{E})$ be a hypergraph and let $w \in \mathbb{Z}_+^V$ be such that $\nu(H^x) = \frac{1}{2}\nu_2(H^x)$ for each $x \leq w + \mathbf{1}$. Then $\tau(H^w) = \nu(H^w)$.*

Proof. Define $f(x) := \nu(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6. ∎

A special case of this is:

Corollary 79.6c. *Let $H = (V, \mathcal{E})$ be a hypergraph with $\nu_2(H^w) = 2\nu(H^w)$ for each $w : V \to \{0, 1, 2\}$. Then $\nu(H) = \tau(H)$.*

Proof. This follows by taking $w = \mathbf{1}$ in Corollary 79.6b. ∎

This gives a generalization to arbitrary k (instead of $k = 2$), since if $\nu(H) = \frac{1}{k}\nu_k(H)$ for some $k \geq 2$, then $\nu(H) = \frac{1}{k-1}\nu_{k-1}(H)$. This follows from

(79.19) $$\nu_{k-1}(H) \leq \nu_k(H) - \nu(H) = k \cdot \nu(H) - \nu(H) = (k-1)\nu(H).$$

Hence $\nu(H) = \frac{1}{2}\nu_2(H)$.

Another consequence of Theorem 79.6 is:

Corollary 79.6d. *For any hypergraph $H = (V, \mathcal{E})$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(79.20) (i) *$\tau_k(H') = k \cdot \tau(H')$ for each restriction H' of H;*
(ii) *$\frac{1}{k}\tau_k(H') \in \mathbb{Z}$ for each restriction H' of H.*

Proof. Define $f(x) := \frac{1}{k}\tau_k(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6 to $w = \mathbf{1}$. ∎

79.3e. Further notes

Seymour [1979a] gave the following example of an ideal hypergraph H with $\tau(H) \neq \frac{1}{2}\nu_2(H)$. Replace each edge of the Petersen graph by a path of length 2, making the graph G. Let $T := VG \setminus \{v\}$, where v is an arbitrary vertex of v of degree 3. Let $\mathcal{E}$ be the collection of T-joins. Let $T_{30} := (EG, \mathcal{E})$. Then $\tau(H') = \nu^*(H')$ for each parallelization H' of T_{30}, by Theorem 29.5. On the other hand, $\tau(T_{30}) = 2$ while G has no T-joins J_1, J_2, J_3, J_4 containing each edge of G at most twice. Otherwise, the sets $J_1 \triangle J_2$, $J_1 \triangle J_3$, and $J_1 \triangle J_4$ are cycles, together containing every edge of G precisely twice. Hence their complements give a 3-edge-colouring of the Petersen graph. This is not possible.

In this example, T_{30} is not only ideal, but also satisfies $\tau(b(T_{30})') = \frac{1}{2}\nu_2(b(T_{30})')$ for each parallelization $b(T_{30})'$ of $b(T_{30})$ (by Corollary 29.2a). Seymour [1981a] conjectures that T_{30} is the unique minor-minimal binary ideal hypergraph with the property $\nu_2 < 2\tau$.

P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph H one has $\nu_k(H) = k \cdot \tau(H)$ where k is some power of 2. He also asks if $k = 4$ would do in all cases. Moreover, Seymour [1979a] conjectures that for each ideal hypergraph H, the g.c.d. of those k with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

In Schrijver and Seymour [1979] it is shown that, for each hypergraph H there is an integer k such that $\nu_k(H') = \tau_k(H')$ for each parallelization H' of H.

Chapter 80

Binary hypergraphs

Several hypergraphs coming from graphs are binary. Binary hypergraphs are hypergraphs such that the symmetric difference of any odd number of edges contains an edge as subset.
Binary hypergraphs have a convenient algebraic structure, that enables to handle packing and blocking problems better than for general hypergraphs. Key result of this chapter is Seymour's characterization of binary Mengerian hypergraphs.

80.1. Binary hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *binary* if

(80.1) for all odd s and $E_1, \ldots, E_s \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with $E \subseteq E_1 \triangle \cdots \triangle E_s$.

Trivially, for each binary hypergraph $H = (V, \mathcal{E})$, the hypergraph $H^{\min}$ is again binary.

In previous chapters we have seen several examples of binary hypergraphs: given an undirected graph $G = (V, E)$, binary hypergraphs on E are formed by the odd circuits, by the complements of cuts, by the $s-t$ paths, by the $s-t$ cuts (given $s, t \in V$), by the T-joins, by the T-cuts (given $T \subseteq V$), and by the paths that connect either s_1 and t_1, or s_2 and t_2 (given $s_1, t_1, s_2, t_2 \in V$).

It is not difficult to show that the class of binary hypergraphs is closed under taking minors, parallelizations, and blockers (see also Section 80.3 below).

80.2. Binary hypergraphs and binary matroids

Binary hypergraphs have a strong linear algebraic structure over the field GF(2), and are strongly related to binary matroids. It will be good to understand these relations.

For a binary hypergraph $H = (V, \mathcal{E})$, a *cycle* is the symmetric difference of any number of edges of H. Call the cycle *odd* (*even*, respectively), if it is the symmetric difference of an odd (even, respectively) number of edges of H.

The odd cycles of H form again a binary hypergraph, say H'. By definition of binarity, the inclusionwise minimal edges of H' coincide with the inclusionwise minimal edges of H. So from a packing and blocking point of view there is no difference in considering any of the binary hypergraphs H, H', or $H^{\min}$.

If $\emptyset \not\in \mathcal{E}$, there is no cycle that is both odd and even. The even cycles form a subspace of the boolean space $\mathcal{P}(V)$, and (if $\emptyset \not\in \mathcal{E}$) the odd cycles form a cospace of it.

A clutter $H = (V, \mathcal{E})$ is binary if and only if there is a binary matroid $M = (V, \mathcal{I})$ such that, for some $B \subseteq V$:

(80.2) $\mathcal{E}$ is equal to the collection of circuits C of M with $|C \cap B|$ odd.

The matroid M is unique and is equal to the binary matroid whose circuits are the minimal nonempty cycles of H. The set B (generally) is not unique: any set B qualifies for it if and only if $|B \cap E|$ is odd for each $e \in E$.

Another way of obtaining a binary hypergraph H from a binary matroid $M = (V, \mathcal{I})$ is by choosing a vertex $v \in V$, and taking as edges of H the sets $C \setminus \{v\}$ where C is a circuit of M containing v. This hypergraph will be denoted by $H_{M,v}$ and is called a *matroid port*. Each binary clutter can be obtained in this way.

80.3. The blocker of a binary hypergraph

The following is an important observation:

(80.3) The blocker $b(H)$ of a binary hypergraph $H = (V, \mathcal{E})$ is equal to the collection of all inclusionwise minimal sets B satisfying $|B \cap E|$ odd for each $E \in \mathcal{E}$.

To see this, if $|B \cap E|$ is odd for each $E \in \mathcal{E}$, then B is a vertex cover, and hence it contains a set in $b(H)$. Conversely, if $B \in b(H)$, then $|B \cap E|$ is odd for each $E \in \mathcal{E}$. For suppose that $|B \cap E|$ is even. As B is a minimal vertex cover, for each $v \in B \cap E$ there is an $E_v \in \mathcal{E}$ with $E_v \cap B = \{v\}$. Then by (80.1) the symmetric difference of E and the sets E_v for $v \in B \cap E$ contains a set $F \in \mathcal{E}$. Then $F \cap B = \emptyset$, a contradiction.

This proves (80.3), which implies that:

(80.4) if H is binary, then $b(H)$ is binary; if H is a clutter, then: H is binary $\iff$ $b(H)$ is binary.

The second statement follows from the fact that $b(b(H)) = H$ if H is a clutter.

If $H = H_{M,v}$ for some binary matroid M and $v \in VM$, then the blocker satisfies $b(H_{M,v}) = H_{M^*,v}$.

80.3a. Further characterizations of binary clutters

Lehman [1964] and Seymour [1976b] gave further characterizations of binary clutters. They showed that the following are equivalent for any clutter $H = (V, \mathcal{E})$:

(80.5)
(i) H is binary, that is, satisfies (80.1);
(ii) for all $E_1, E_2, E_3 \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with $E \subseteq E_1 \triangle E_2 \triangle E_3$;
(iii) $|B \cap E|$ is odd for all $E \in H$ and $B \in b(H)$;
(iv) $|B \cap E| \neq 2$ for all $E \in H$ and $B \in b(H)$;
(v) H has no minor equal to $P_4 := (\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\}\})$ or to J_n for any $n \geq 3$ (defined in (78.12)).

(The equivalence of (i) and (iii) was shown by Lehman [1964], the equivalence of (i) and (ii) by A. Lehman (unpublished) and Seymour [1976b], and the other equivalences by Seymour [1976b].)

80.4. On characterizing binary ideal hypergraphs

Since the class of binary ideal hypergraphs is closed under taking minors, it can be characterized by specifying the collection of binary minimally nonideal hypergraphs. Seymour [1981a] offered the conjecture that this collection consists precisely of $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 (see (78.12)).

This conjecture is still open. As we saw in Sections 75.5 and 78.4a, the conjecture has been proved for the class of hypergraphs formed by the odd circuits of signed graphs, by Guenin [1998a,2001a]. For this class, only $\mathcal{O}(K_5)$ is a forbidden minor, since $b(\mathcal{O}(K_5))$ and F_7 do not arise in this way.

By Corollary 29.2b, the conjecture is also true for the class of hypergraphs of T-joins since neither of the three proposed forbidden minors comes from T-joins.

This was extended by Cornuéjols and Guenin [2002a] to binary hypergraphs without Q_6^+ or Q_7^+ minor. Here for any hypergraph $H = (V, \mathcal{E})$, the hypergraph H^+ arises by adding a new vertex u to V and by taking as edges all sets $E \cup \{u\}$ with $E \in \mathcal{E}$. The hypergraph Q_7 arises from Q_6^+ by adding as edges the perfect matchings of K_4.

This implies that for any regular matroid $M = (V, \mathcal{I})$ and any $\Sigma \subseteq V$, the collection of circuits C of M with $|C \cap \Sigma|$ odd, form a hypergraph for which Seymour's conjecture holds. Other cases where Seymour's conjecture holds were given by Guenin [2001c,2002c].

Adding an Eulerian condition. In Section 79.3e we saw that the following ideal hypergraph H does not satisfy $\nu_2(H) = 2\tau(H)$. Let G be obtained from the Petersen graph $\mathbf{P}_{10}$ by replacing each edge by a path of length 2. Let $T := VG \setminus \{v\}$ for some degree-3 vertex v of G. Let T_{30} be the hypergraph of T-joins on EG.

Seymour [1981a] conjectured that any binary ideal hypergraph $H = (V, \mathcal{E})$ without T_{30} minor satisfies $\frac{1}{2}\nu_2(H) = \tau(H)$. If moreover all edges of $b(H)$ have the same parity, then $\nu(H) = \tau(H)$. However, as A.M.H. Gerards and B. Guenin observed, the Petersen graph gives a simpler counterexample to the second conjecture: the hypergraph T_{15} of $V\mathbf{P}_{10}$-joins in $\mathbf{P}_{10}$ is a binary ideal hypergraph without T_{30} minor and having all edges of $b(T_{15})$ odd, while $\nu(T_{15}) = 2 < 3 = \tau(T_{15})$. This suggests the question if for each binary hypergraph $H = (V, \mathcal{E})$:

(80.6) (?) $\nu(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$ with $w(B)$ even for all $B \in b(H) \iff \nu_2(H^w) = 2\tau(H^w)$ for each $w : V \to \mathbb{Z}_+ \iff H$ has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, F_7, or T_{15} minor. (?)

80.5. Seymour's characterization of binary Mengerian hypergraphs

For binary Mengerian hypergraphs, the forbidden minors are known: Seymour [1977b] showed that the only binary minimally non-Mengerian hypergraph is $Q_6 = \mathcal{O}(K_4)$. In this section we give a proof based on Seymour [1977b] and on the short proof by Guenin [2002a].

Theorem 80.1. *A binary hypergraph is Mengerian if and only if it has no Q_6 minor.*

Proof. We have seen necessity in Section 79.2. We prove sufficiency.

Call a hypergraph $H = (V, \mathcal{E})$ *critical* if each vertex is contained in a vertex cover of size $\tau(H)$. Call a subset of V a *cycle* if it is a symmetric difference of edges, a *k-cycle* if it is a symmetric difference of k edges, an *even cycle* if it is a k-cycle for some even k, and a *circuit* if it is a minimal nonempty cycle. Call two vertices x, y *parallel* if $\{x, y\}$ is a 2-cycle. By the definition of binary hypergraph, for each cycle C and each edge E, the set $C \triangle E$ contains an edge. Hence, if x and y are parallel, then for any inclusionwise minimal edge F of H with $x \in F$, one has $y \notin F$ and $(F \setminus \{x\}) \cup \{y\}$ is again an edge. So any minimal vertex cover containing x also contains y.

To see sufficiency, it suffices to show that $\nu(H) = \tau(H)$ for each binary hypergraph without Q_6 minor (since the class of binary hypergraphs without Q_6 minor is closed under parallelization). Choose a counterexample $H = (V, \mathcal{E})$ to this with $|V|$ minimal. So H has no Q_6 minor while $\nu(H) < \tau(H)$. Choose H moreover such that the number of pairs of parallel elements is as large as possible.

Note that the minimality of V implies that H is critical (as any vertex that belongs to no minimum-size vertex cover can be deleted to obtain a smaller counterexample). Define $\tau := \tau(H)$ and, for each $v \in V$ define:

(80.7) $\beta_v := \{B \mid B \text{ vertex cover}, |B| = \tau, v \in B\}$.

Define U to be the set of vertices u with β_u inclusionwise minimal:

(80.8) $U := \{u \mid \text{there is no } v \in V \text{ with } \beta_v \subset \beta_u\}$.

So U is nonempty. Note that if u and v are parallel, then $\beta_u = \beta_v$. Let M be the set of pairs of nonparallel elements u, v in U with $\beta_u = \beta_v$. Then:

(80.9) each element $u \in U$ is contained in a pair in M.

Suppose not. By the minimality of V, $\nu(H \setminus u) = \tau(H \setminus u) = \tau - 1$ (since $\tau - 1 \leq \tau(H \setminus u) = \nu(H \setminus u) \leq \nu(H) < \tau$). So $V \setminus \{u\}$ contains a subset Y that is the union of $\tau - 1$ disjoint edges of H. So Y is a $(\tau - 1)$-cycle.

Let K be the parallel class of u. By the minimality of V, $\nu(H/K) = \tau(H/K) \geq \tau$. So $V \setminus K$ contains a collection $\mathcal{F}$ of disjoint edges of H/K with $|\mathcal{F}| = \tau$. Then $\mathcal{F}$ partitions $V \setminus K$, since for each $v \in V \setminus K$ there exists a $B \in \beta_v \setminus \beta_u$ (since $\beta_v \not\subseteq \beta_u$, as u is contained in no pair in M, by assumption). So $u \notin B$, hence $B \cap K = \emptyset$. As $|B| = \tau$, B intersects each edge in $\mathcal{F}$ precisely once. Hence $\mathcal{F}$ covers v. Concluding, $\mathcal{F}$ partitions $V \setminus K$.

This implies that $V \setminus K$ is contained in some τ-cycle L. So $L \triangle Y$ is a $(2\tau - 1)$-cycle, and hence contains a minimal edge E of H. As K is a parallel class, $|E \cap K| \leq 1$. If $E \cap K = \emptyset$ let $E' := E$; if $E \cap K \neq \emptyset$, let $E' := (E \setminus K) \cup \{u\}$. Then E' is disjoint from Y, so $\nu(H) \geq \tau$, contradicting our assumption. This proves (80.9).

Next:

(80.10) each pair $e \in M$ contains a vertex u such that H has edges $E_1, \ldots, E_\tau$ with $E_1 \cap E_2 = \{u\}$ and with $E_1 \setminus \{u\}, E_2 \setminus \{u\}, E_3, \ldots, E_\tau$ partitioning $V \setminus e$.

Indeed, let $e = \{u, v\}$ be such that u has at least as many parallel elements as v has. Let $\widetilde{H}$ be obtained from H by deleting v and adding an extra parallel element to u (that is, we duplicate u). This increases the number of pairs of parallel elements. So, by the choice of H, $\nu(\widetilde{H}) = \tau(\widetilde{H})$. Moreover, since $\beta_u = \beta_v$, we have that $\tau(\widetilde{H}) = \tau$ and $\widetilde{H}$ is critical. So $V\widetilde{H}$ can be partitioned into τ edges of $\widetilde{H}$. This gives (80.10).

A consequence of (80.10) is that $V \setminus e$ is a τ-cycle of H. Hence

(80.11) $e \triangle f$ is an even cycle of H, for all $e, f \in M$,

since $e \triangle f = (V \setminus e) \triangle (V \setminus f)$.

Now fix a pair $e \in M$, and let $u, E_1, \ldots, E_\tau$ be as in (80.10). Then

(80.12) $E_1 \triangle E_2$ contains no edge E of H,

since otherwise replacing E_1 and E_2 by E and $E_1 \triangle E_2 \triangle E$ would show $\nu(H) \geq \tau$.

Let H' be a smallest minor of H such that $H' = H \setminus Y / X$ for some disjoint subsets X, Y of $E_1 \triangle E_2$ and such that, defining $C := (E_1 \triangle E_2) \setminus (X \cup Y)$:

(80.13) (i) $X \cup Y$ is a union of circuits of H;
(ii) C is a cycle of H';
(iii) $C \cup \{u\}$ contains an edge of H';
(iv) $\tau(H') \geq \tau$;

(v) each edge E of H' contained in $C \cup \{u\}$ satisfies $\tau(H' \setminus E) \leq \tau - 2$.

Such an H' exists, since H has these properties. Then:

Claim 1. C is a circuit of H'.

Proof of Claim 1. Suppose that C is not a circuit of H'. Then, as C is a cycle, C can be partitioned into two nonempty cycles C_1 and C_2 of H'. By (80.12), C_1 and C_2 are even cycles.

Define

(80.14) $\quad H_1 := H'/C_2.$

We show that H_1 satisfies (80.13)(i)-(iv).

To see (80.13)(i) for H_1, $X \cup Y \cup C_2$ is a union of cycles of H, since $C_2 = C' \setminus (X \cup Y)$ for some cycle C' of H. To see (80.13)(ii) for H_1, C_1 is a cycle of H', hence also of H_1. To see (80.13)(iii) for H_1, $C \cup \{u\}$ contains an edge of H', hence $C_1 \cup \{u\} = (C \cup \{u\}) \setminus C_2$ contains an edge of H_1. To see (80.13)(iv) for H_1, we have $\tau(H'/C_2) \geq \tau(H') \geq \tau$.

So H_1 satisfies (80.13)(i)-(iv). Hence, by the minimality of H', H_1 has an edge $E \subseteq C_1 \cup \{u\}$ with $\tau(H_1 \setminus E) \geq \tau - 1$. Define $P := E \setminus \{u\}$, $Q = C_1 \setminus E$, and

(80.15) $\quad H_2 := H' \setminus P/Q.$

We show that H_2 satisfies (80.13), which contradicts the minimality of H'. To see (80.13)(i) for H_2, $X \cup Y \cup C_1$ is a union of circuits of H, since $C_1 = C' \setminus (X \cup Y)$ for some cycle C' of H. To see (80.13)(ii) for H_2, C_2 is a cycle of H', hence also of H_2. To see (80.13)(iii) for H_2, $E = E' \setminus C_2$ for some edge E' of H'. Then $E' \triangle C_1$ contains an edge E'' of H'. Then $E'' \cap P = \emptyset$, since $E' \triangle C_1$ is disjoint from P (as $P \subseteq E' \cap C_1$). So $E'' \setminus Q = E'' \setminus C_1 \subseteq (E' \triangle C_1) \setminus C_1 \subseteq C_2 \cup \{u\}$. This proves (80.13)(iii) for H_2.

To see (80.13)(iv) for H_2, suppose to the contrary that B is a minimum-size vertex cover of H_2 of size $\leq \tau - 1$. Then B intersects each of $E_3, \ldots, E_\tau$ at least once, and hence does not intersect C_2 (as $|B \cap C_2|$ is even). So B is a vertex cover of H_2/C_2. As $C_2 \cup \{u\}$ contains an edge of H_2, we also know that $u \in B$. So $B \setminus \{u\}$ is a vertex cover of $H_2/C_2 \setminus \{u\} = H' \setminus E/(C_2 \cup Q) = H_1 \setminus E/Q$, and hence of $H_1 \setminus E$. This contradicts the fact that $\tau(H_1 \setminus E) \geq \tau - 1$. So H_2 satisfies (80.13)(iv).

To see (80.13)(v) for H_2, let F be an edge of H_2 contained in $C_2 \cup \{u\}$. As $H_2 = H' \setminus P/Q$, there exists a $Q' \subseteq Q$ such that $F \cup Q'$ is an edge of H'.

Suppose that $Q' \neq Q$. Choose $r \in Q \setminus Q'$. Let B be a vertex cover of H of size τ containing r. Then B intersects each of $E_3, \ldots, E_\tau$ at least once. As $r \in B$, B intersects $E_1 \triangle E_2$ at least once, hence at least twice (as $E_1 \triangle E_2$ is an even cycle). Hence, as $|B| = \tau$, $u \notin B$ and B intersects each E_i in precisely one element. Moreover, B is disjoint from $C_2 \cup X \cup Y$, as this last set is a union of circuits of H, implying that if B intersects $C_2 \cup X \cup Y$ at least once, then at least twice; since $r \notin C_2 \cup X \cup Y$, this is not possible. So B is a vertex cover of H_1. Hence $B \cap E \neq \emptyset$. So B contains a second element in C_1, say s. As $F \cup Q' \cup X$ contains an edge of H, it intersects B. So $s \in Q'$. This would mean that B is disjoint from E, a contradiction.

So $Q' = Q$. Then $R := F \cup P = (F \cup Q) \triangle C_1$ contains an edge of H'. By (80.13)(v), $H' \setminus R$ has a vertex cover B of size $\tau - 2$. As B intersects each of $E_3, \ldots, E_\tau$, B is disjoint from Q. So B is a vertex cover of $H' \setminus R/Q = H_2 \setminus F$, proving $\tau(H_2 \setminus F) \leq \tau - 2$. So H_2 satisfies (80.13)(v), contradicting the minimality of H'. *End of Proof of Claim 1*

By (80.13)(ii)(iii), H' has edges F_1, F_2 with $F_1 \cap F_2 = \{u\}$ and $F_1 \cup F_2 = C \cup \{u\}$. By (80.13)(v), there exist vertex covers B_1 and B_2 of H' with $|B_i \setminus F_i| \leq \tau - 2$. So $B_1 \cap F_2 = B_2 \cap F_1 = \{u\}$. Then

(80.16) H' has an edge F_3 disjoint from $(B_1 \setminus F_1) \cup (B_2 \setminus F_2) \cup \{u\}$.

Otherwise, the latter set contains a minimal vertex cover B of H'. Now each B_i intersects each of the edges $E_3, \ldots, E_\tau$ precisely once. So B intersects each of these E_i at most twice, hence precisely once. Therefore $|B| \leq \tau - 1$, contradicting (80.13)(iv). This proves (80.16).

Choose F_3 in (80.16) with $F_3 \setminus C$ minimal. Let H'' arise from H' by deleting all vertices not in $F_1 \cup F_2 \cup F_3$. Then $\tau(H'') \geq 2$, since $F_1 \cap F_2 \cap F_3 = \{u\} \cap F_3 = \emptyset$. Moreover, $\nu(H'') = 1$, for suppose that H'' has disjoint edges F and F', with $u \notin F$. By the minimality of $F_3 \setminus C$ we know that $F \setminus C = F_3 \setminus C$. So $F' \subseteq C \cup \{u\}$. But then, since C is a circuit, $F' = F_1$ or $F' = F_2$ (otherwise $F' \triangle F_1$ is a nonempty cycle properly contained in C). However, F intersects B_1 and B_2, hence F intersects $B_1 \cap F_1$ and $B_2 \cap F_2$, and hence it intersects F_1 and F_2, a contradiction. So $\nu(H'') = 1 < \tau(H'')$.

The minimality of H implies $H'' = H' = H$ and $\tau = 2$. The equality $\tau = 2$ implies that $U = V$ and M forms a perfect matching on V: if $v \in V \setminus U$, then $\beta_u \subset \beta_v$ for some $u \in U$, and hence (using (80.9)) any minimum-size vertex cover containing u has at least three elements — a contradiction, since $\tau = 2$; similarly, if $u \in V$ would be in two pairs in M, there is a minimum-size vertex cover of size ≥ 3 — again a contradiction.

Also, $V \setminus e = E_1 \triangle E_2 = F_1 \triangle F_2$, whence it is a circuit (by Claim 1). As any two pairs from M form an even cycle (by (80.11)), we know $|V \setminus e| = 4$. So $|V| = 6$, $|M| = 3$, giving $H = Q_6$. ∎

Notes. Tseng and Truemper [1986] gave a decomposition theorem for binary Mengerian hypergraphs. It implies that the property of being Mengerian belongs to NP for binary hypergraphs. Shorter proofs of the decomposition result were given by Bixby and Rajan [1989] and Truemper [1987]. The latter paper also gives polynomial-time algorithms for testing Mengerity of a binary hypergraph and for finding a minimum-weight vertex cover and a maximum packing of edges subject to a weight function in binary Mengerian hypergraphs. A description of this algorithm was given in Bixby and Cunningham [1995]. Also Hartvigsen and Wagner [1988] gave a polynomial-time algorithm to test Mengerity of a binary hypergraph.

More background can be found in the book of Truemper [1992].

80.5a. Applications of Seymour's theorem

We describe a number of applications of Theorem 80.1, some of which we have seen in previous parts of this book. Except for those in the first two applications below, the theorems are due to Seymour [1977b].

$s-t$ cuts. Let $G = (V, E)$ be a graph and let $s, t \in V$. The collection of $s-t$ cuts forms a binary hypergraph on E, without Q_6 minor. Hence Theorem 80.1 implies the edge-disjoint version of (the easy) Theorem 6.1: the maximum number of edge-disjoint $s-t$ cuts is equal to the minimum length of an $s-t$ path (the max-potential min-work theorem).

$s-t$ paths. Let $G = (V, E)$ be a graph and let $s, t \in V$. The collection of $s-t$ paths forms a binary hypergraph on E, without Q_6 minor. Hence Theorem 80.1 implies the edge-disjoint undirected version of Menger's theorem (Corollary 9.1b): the maximum number of edge-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ cut.

***T*-cuts.** Let $G = (V, E)$ be a graph and let $T \subseteq V$. The collection of T-cuts forms a binary hypergraph on E. If it is Q_6, then $G = K_4$ and $T = VK_4$. Hence Theorem 80.1 implies Corollary 29.9a: If K_4, VK_4 is not a minor of G, T (in the sense of Section 29.11b), then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts.

***T*-joins.** Let $G = (V, E)$ be a graph and let $T \subseteq V$. The collection of T-joins forms a binary hypergraph on E. If it is Q_6, then $G = K_{2,3}$ and $T = VK_{2,3} \setminus \{u\}$, where u is a vertex of degree 3. Hence Theorem 80.1 implies Theorem 29.10: If $K_{2,3}, VK_{2,3} \setminus \{u\}$ is not a minor of G, T (in the sense of Section 29.11b), then the minimum size of a T-cut is equal to the maximum number of disjoint T-joins.

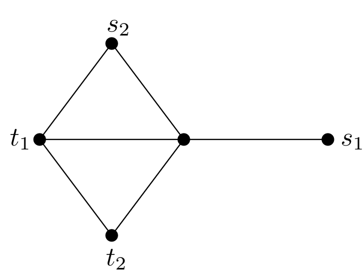

Figure 80.1
s_1 and t_1, and s_2 and t_2, have distance 2, but there exist no two disjoint cuts each separating both s_1 and t_1, and s_2 and t_2.

***$s_1 - t_1$* and *$s_2 - t_2$* cuts.** Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. The collection of cuts that separate *both* s_1 and t_1, *and* s_2 and t_2 forms a binary hypergraph on E. If it is Q_6, then G is the graph in Figure 80.1 up to permuting indices and exchanging s_1 and t_1. Hence Theorem 80.1 implies Theorem 71.4: if G has no subgraph contractible to the graph in Figure 80.1 up to permuting indices and exchanging s_1 and t_1, then the minimum length of a path connecting either s_1 and t_1, or s_2 and t_2 is equal to the maximum number of pairwise disjoint cuts each separating both s_1 and t_1, and s_2 and t_2. (Here we assume that the subgraph contains the s_i, t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure.)

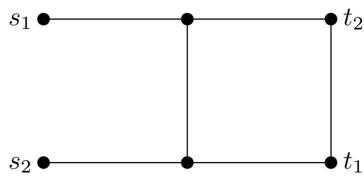

Figure 80.2
The maximum total value of a 2-commodity flow (subject to capacity 1) is equal to 2, but the maximum total value of an integer 2-commodity flow is equal to 1.

***$s_1 - t_1$* and *$s_2 - t_2$* paths.** Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. The collection of paths that connect *either* s_1 and t_1, *or* s_2 and t_2 forms a binary hypergraph on E. If it is Q_6, then it is the graph of Figure 80.2 up to exchanging s_1 and t_1, and s_2 and t_2. Hence Theorem 80.1 implies Theorem 71.2: If G has no subgraph contractible to the graph of Figure 80.2 up to exchanging s_1 and t_1, and s_2 and t_2, then the maximum number of edge-disjoint paths, each connecting either s_1 and t_1, or s_2 and t_2, is equal to the minimum size of a cut separating both s_1 and t_1, and s_2 and t_2.

Odd circuits. Let $G = (V, E, \Sigma)$ be a signed graph; that is $G = (V, E)$ is an undirected graph and $\Sigma \subseteq E$. Call a circuit C *odd* if $|C \cap \Sigma|$ is odd. The collection of odd circuits forms a binary hypergraph on E. If it is Q_6, then $G = (V, E, \Sigma)$ is the odd-K_4; that is, $V = VK_4$ and $E = \Sigma = EK_4$. Hence Theorem 80.1 implies Corollary 75.3a: if $G = (V, E, \Sigma)$ has no odd-K_4 minor, then the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover. In other words, if $G = (V, E, \Sigma)$ has no odd-K_4 minor, then G is strongly bipartite.

Odd circuit covers. Let $G = (V, E, \Sigma)$ be a signed graph. The collection of inclusionwise minimal odd circuit covers forms a binary hypergraph on E. If it is Q_6, then $G = K_3^2$ and $\Sigma = \Delta$, where Δ is a triangle in K_3^2. Here K_3^2 is the graph with three vertices, each pair of which connected by two parallel edges. Hence Theorem 80.1 implies: if $G = (V, E, \Sigma)$ has no (VK_3^2, EK_3^2, Δ) as minor, then the maximum number of edge-disjoint odd circuit covers is equal to the minimum length of an odd circuit.

Notes. Gan and Johnson [1989] developed a framework that includes the above examples on T-joins, T-cuts, odd circuits, and odd circuit covers, and derived algorithms for the corresponding optimization problems.

80.6. Mengerian matroids

Seymour's characterization of binary Mengerian hypergraphs implies a full characterization of matroids that have the corresponding matroidal Mengerian property. In this case, we need not restrict the characterization to binary matroids, since binarity of the matroid *follows* from the Mengerian property.

Again, for any matroid $M = (V, \mathcal{I})$ and any $v \in V$, let $H_{M,v}$ be the hypergraph on $V \setminus \{v\}$ with edges all sets $C \setminus \{v\}$, where C is a circuit of M containing v. Call M *Mengerian* if $H_{M,v}$ is a Mengerian hypergraph for each $v \in V$.

Theorem 80.1 implies the following conjecture of Th. Chang of the late 1960s (cf. Seymour [1977b]):

Corollary 80.1a. *A matroid is Mengerian if and only if it is binary and has no F_7^* minor.*

Proof. As the 2-uniform matroid U_4^2 on 4 elements and F_7^* are not Mengerian (since $H_{M,v} = K_3$ or $H_{M,v} = Q_6$ for these matroids), necessity follows (using the fact that each matroid without U_4^2 minor is binary (Theorem 39.4)). To see sufficiency, observe that, if $M = (V, \mathcal{I})$ is a binary matroid, then for each $v \in V$, the hypergraph $H_{M.v}$ is binary, and that if M contains no F_7^* minor (in the matroidal sense), then $H_{M,v}$ contains no Q_6 minor (in the hypergraphical sense). ∎

Notes. As was outlined by Seymour [1980a,1981a] and Bixby [1982], there is an easier direct proof of this corollary, based on the fact that binary matroids without F_7^* minor can be decomposed into regular matroids (coming from totally unimodular matrices) and copies of F_7. (This follows from the 'splitter theorem' of Seymour [1980a].)

Mengerity of regular matroids follows from the total unimodularity of the matrix representing the matroid, as was shown by Gallai [1959b] (and also by Minty [1966] (an alternative proof was given by Fulkerson [1968])).

Bixby [1982] described that this decomposition gives a polynomial-time algorithm finding the optima in Corollary 80.1a. For background we refer to the book of Truemper [1992].

80.6a. Oriented matroids

Matroids generalize *undirected* graphs, and one may ask for an extension of matroid theory to include directed structures, in order to investigate the max-flow min-cut theorem in greater generality. Bland and Las Vergnas [1978] and Folkman and Lawrence [1978], following work of Minty [1966], Fulkerson [1968], Rockafellar [1969], and Lawrence [1975], developed a theory of oriented matroids. It may be seen as the abstraction of a linear subspace of $\mathbb{R}^n$; the abstraction of any vector is a $\{0,+1,-1\}$ vector, having 0, +1, or -1 in the positions where the original vector has a zero, positive, or negative entry, respectively. If we have a digraph $D = (V, A)$ and we take as $\{0,+1,-1\}$ vectors all $x \in \{0,\pm 1\}^A$ for which there is an undirected circuit C with $x_a = 1$ for forward arcs a of C, $x_a = -1$ for backward arcs a of C, and $x_a = 0$ for all other arcs a, then we obtain an oriented matroid. Again, one may define the Mengerian property for oriented matroids; its characterization by excluded minors is unsolved.

More on oriented matroids can be found in Bachem and Kern [1992] and Björner, Las Vergnas, Sturmfels, White, and Ziegler [1993].

A different approach to extending Menger's theorem to matroids was given by Tutte [1965b] — see Section 41.5a.

80.7. Further results and notes

80.7a. $\tau_2(H) = 2\tau(H)$ for binary hypergraphs H

Lovász [1975a] showed:

Theorem 80.2. *$\tau_2(H) = 2\tau(H)$ for each binary hypergraph H.*

Proof. Let x be a minimum-size 2-vertex cover of H. Let $U := \{v \in V \mid x_v = 0\}$ and $W := \{v \in V \mid x_v = 2\}$. Let $H' := H/U \setminus W$ and $V' := V \setminus (U \cup W)$. Then H' is binary and each edge of H' has size at least 2, since for any edge F of H' there is an edge E of H with $E \cap W = \emptyset$ and $E \setminus U = F$. Then $|F| = x(E) \geq 2$.

As $r_{\max}(H') \geq 2$, for each $v \in V'$ there is a $B \in b(H')$ with $v \notin B$. Consider now the cospace

(80.17) $$\mathcal{C} := \{B \subseteq V' \bigm| |B \cap F| \text{ is odd for each edge } F \text{ of } H'\}.$$

Then for each $v \in V'$ there is a $B \in \mathcal{C}$ with $v \notin B$. As $\mathcal{C}$ is a cospace, it follows that v is in at most half of the sets in $\mathcal{C}$. As this is true for each $v \in V'$, $\mathcal{C}$ contains a set B of size at most $\frac{1}{2}|V'|$. Then $W \cup B$ is a vertex cover of H of size at most $\frac{1}{2}x(V) = \frac{1}{2}\tau_2(H)$. So $\tau(H) \leq \frac{1}{2}\tau_2(H)$ as required. ∎

Lovász [1975a] showed more generally:

Theorem 80.3. *Let $H = (V, \mathcal{E})$ be a hypergraph such that*

(80.18) *if $X, Y, Z \in \mathcal{E}$, $y \in (X \cap Y) \setminus Z$ and $z \in (X \cap Z) \setminus Y$, then there is an $F \in \mathcal{E}$ satisfying $F \subseteq (X \cup Y \cup Z) \setminus \{y, z\}$.*

Then $\tau_2(H) = 2\tau(H)$.

Proof. Consider a counterexample with $|V|$ minimal. Let x be a minimum-size 2-vertex cover of H. Then

(80.19) $$x_v = 1 \text{ for each } v \in V.$$

For if $x_v = 0$, then $x|V \setminus \{v\}$ is a 2-vertex cover of H/v, and hence, since H/v again satisfies (80.18):

(80.20) $$2\tau(H) \leq 2\tau(H/v) = \tau_2(H/v) \leq x(V \setminus \{v\}) = x(V) = \tau_2(H),$$

contradicting the fact that H is a counterexample. Similarly, if $x_v = 2$, then $x|V\setminus\{v\}$ is a 2-vertex cover of $H \setminus v$, and hence, since $H \setminus v$ again satisfies (80.18):

(80.21) $$2\tau(H) \leq 2\tau(H\setminus v)+2 = \tau_2(H\setminus v)+2 \leq x(V\setminus\{v\})+2 = x(V) = \tau_2(H),$$

again contradicting the fact that H is a counterexample.

This proves (80.19). Hence $|E| \geq 2$ for each $E \in \mathcal{E}$ and we must show that there is a vertex cover of size $\leq \frac{1}{2}|V|$. By the minimality of x, there is an edge X of size 2 (otherwise we can reset $x_v := 0$ for some $v \in V$), say $X = \{y, z\}$. Then $\tau_2(H \setminus \{y, z\}) \leq |V| - 2 = \tau_2(H) - 2$, and so, by the minimality of $|V|$:

(80.22) $$\tau(H \setminus \{y, z\}) = \tfrac{1}{2}\tau_2(H \setminus \{y, z\}) \leq \tfrac{1}{2}\tau_2(H) - 1 < \tau(H) - 1,$$

and hence $\tau(H \setminus \{y, z\}) \leq \tau(H) - 2$. Let $U \subseteq V \setminus \{y, z\}$ be a minimum-size vertex cover of $H \setminus \{y, z\}$. Since $U \cup \{y\}$ and $U \cup \{z\}$ are not vertex covers of H (since $\tau(H) \geq |U| + 2$), there are edges Y and Z in H disjoint from $U \cup \{z\}$ and $U \cup \{y\}$ respectively. As $U \cup \{y, z\}$ does intersect all edges, we know $y \in Y$ and $z \in Z$. Then X, Y, Z contradict (80.18). ∎

Condition (80.18) is closed under taking minors. The hypergraph $(\{1, 2\}, \{\{1\}, \{2\}, \{1, 2\}\})$ is the unique minor-minimal hypergraph violating (80.18).

80.7b. Application: T-joins and T-cuts

Let $G = (V, E)$ be an undirected graph and let $T \subseteq V$ with $|T|$ even. Let $\mathcal{C}$ be the collection of T-cuts. Then $H = (E, \mathcal{C})$ is a binary hypergraph, and its blocker consists of the minimal T-joins.

We will derive

(80.23) $\nu_2(H) = 2\tau(H)$, and $\nu(H) = \tau(H)$ if G is bipartite,

from general hypergraph theory and from the result that

(80.24) $\nu_2(H) = 2\nu(H)$ if G is bipartite

(Seymour [1981d]).

We first give Seymour's proof of (80.24). Let $U_1, \ldots, U_t$ be subsets of V with each $|U_i \cap T|$ odd such that each edge of G is in at most two of the $\delta(U_i)$ and such that $t = \nu_2(H)$. Such U_i exist by the definition of $\nu_2(H)$. Choose them such that

(80.25) $$\sum_{i=1}^{t} |U_i||V \setminus U_i|$$

is as small as possible. Then the U_i are *cross-free*, that is, for all $i, j = 1, \ldots, t$ one has

(80.26) $$U_i \subseteq U_j \text{ or } U_j \subseteq U_i \text{ or } U_i \cap U_j = \emptyset \text{ or } U_i \cup U_j = V.$$

If this would not hold, we can replace U_i and U_j either by $U_i \cap U_j$ and $U_i \cup U_j$ (if $|U_i \cap U_j \cap T|$ is odd) or by $U_i \setminus U_j$ and $U_j \setminus U_i$ (otherwise), therewith decreasing the sum (80.25) — a contradiction.

So (80.26) holds. By symmetry, we can assume that $|U_i| \leq |V \setminus U_i|$ for each i. If each U_i is a singleton, then the U_i form a 2-stable set in the subgraph $G[T]$ induced by T, and hence $G[T]$ has a stable set of size at least $\frac{1}{2}t$ (as $G[T]$ is bipartite). This implies $\nu(H) \geq \frac{1}{2}\nu_2(H)$.

If some U_i is a singleton and $U_i = U_j$ for some $j \neq i$, we can contract $\delta(U_i)$ and obtain a bipartite graph $G' = (V', E')$ and $T' \subseteq V'$, and a 2-packing of T'-cuts of G' of size $t-2$. Hence, inductively, G' has a packing of T-cuts of size at least $\frac{1}{2}(t-2)$. With $\delta(U_i)$ this gives a packing of T cuts in G, of size at least $\frac{1}{2}t$.

So we can assume that each singleton occurs at most once among the U_i and that not each U_i is a singleton. Then we can assume that U_1 is a minimal nonsingleton set among the U_i. Let $U_2, \ldots, U_r$ be the sets properly contained in U_1. So $U_2, \ldots, U_r$ are singletons from $T \cap U_1$. Hence $r - 1 \leq |T \cap U_1|$. As $|T \cap U_1|$ is odd and G is bipartite, there is a stable set $S \subseteq T \cap U_i$ with $2|S| \geq |T \cap U_i| + 1 \geq r$. Replacing $U_1, \ldots, U_r$ by twice the singletons from S, gives a 2-packing of t T-cuts with smaller sum (80.25) — a contradiction. This proves (80.24).

Now (80.24) implies:

(80.27) $\qquad \nu_4(H) = 2\nu_2(H)$ for any graph G.

Indeed, replace each edge by a path of length 2, thus obtaining the bipartite graph $G' = (V', E')$, with $T \subseteq V'$. Let H' be the corresponding hypergraph of T-cuts. Then by (80.24):

(80.28) $$\nu_4(H) = \nu_2(H') = 2\nu(H') = 2\nu_2(H),$$

which is (80.27).

As the class of hypergraphs H obtained in this way from graphs is closed under parallelization (since it corresponds to replacing edges by paths), Corollary 79.5a then implies $\nu_2(H) = \tau_2(H)$. Hence, with Theorem 80.2 we obtain $\nu_2(H) = 2\tau(H)$, and, using (80.24), we have (80.23).

80.7c. Box-integrality of $k \cdot P_H$

A polyhedron P is called *box-integer* if for all $c, d \in \mathbb{Z}^V$, the polytope

(80.29) $$P \cap \{x \in \mathbb{R}^V \mid d \leq x \leq c\}$$

is integer. Gerards and Laurent [1995] showed that the following are equivalent for any binary hypergraph $H = (V, \mathcal{E})$, where P_H is defined by (78.1):

(80.30)
(i) $k \cdot P_H$ is box-integer for each $k \geq 1$;
(ii) $k \cdot P_H$ is box-integer for some $k \geq 2$;
(iii) H has no Q_6 or $b(Q_6)^+$ minor.

As in Section 80.4, the hypergraph H^+ arises from a hypergraph $H = (V, \mathcal{E})$ by adding a new vertex, u say, and taking as edges all sets $F \cup \{u\}$ for $F \in \mathcal{E}$.

This characterization extends results of Laurent and Poljak [1995b] for the bipartite subgraph polytope.

Chapter 81

Matroids and multiflows

Corollary 80.1a gives a forbidden minor characterization of matroids for which the corresponding generalization of the integer max-flow min-cut theorem holds. Seymour [1981a] showed that several theorems on multiflows can be generalized similarly to the level of matroids. We give a survey of these results, without proofs.

81.1. Multiflows in matroids

Let $M = (E, \mathcal{I})$ be a matroid, let $R \subseteq E$, and let $c : E \to \mathbb{R}_+$. Let $\mathcal{C}_R$ be the collection of circuits C of M with $|C \cap R| = 1$.

The *multiflow problem* in M asks for a function $y : \mathcal{C}_R \to \mathbb{R}_+$ satisfying

(81.1)
$$\sum_{\substack{C \in \mathcal{C}_R \\ e \in C}} y_C \geq c_e \quad \text{if } e \in R,$$
$$\sum_{\substack{C \in \mathcal{C}_R \\ e \in C}} y_C \leq c_e \quad \text{if } e \in E \setminus R.$$

We call any y satisfying (81.1) a *multiflow* in M (relative to R and c). So R plays the role of the 'demand edges', and $E \setminus R$ the role of the 'supply edges'.

The corresponding *cut condition* is:

(81.2) (cut condition) $c(D \cap R) \leq c(D \setminus R)$ for each cocircuit D of M.

This condition is necessary for the existence of a multiflow y, since

(81.3)
$$c(D \cap R) \leq \sum_{C \in \mathcal{C}_R} y_C |C \cap D \cap R| \leq \sum_{C \in \mathcal{C}_R} y_C |C \cap D \setminus R| \leq c(D \setminus R).$$

Here we use that $|C \cap D| \neq 1$ for any circuit C and cocircuit D, implying $|C \cap D \cap R| \leq |C \cap D \setminus R|$ if $|C \cap R| = 1$.

In this terminology, Corollary 80.1a can be formulated as:

(81.4) for each $R \subseteq E$ with $|R| = 1$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow $\iff$ M has no U_4^2 or F_7^* minor.

So Corollary 80.1a concerns integer 1-commodity flows in a matroid.

Let $k \in \mathbb{Z}_+$. Seymour [1981a] called a matroid $M = (V, \mathcal{I})$ *k-flowing* if

(81.5) for each $R \subseteq E$ with $|R| \leq k$ and for each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow y.

The matroid is *integer k-flowing* if for integer c we can take y integer.

As was the case for multiflows in graphs, the following *Euler condition* (for $c : E \to \mathbb{Z}_+$) will turn out to be helpful:

(81.6) (Euler condition) $c(D)$ is even for each cocircuit D.

M is called *k-cycling* if

(81.7) for each $R \subseteq E$ with $|R| \leq k$ and for each $c : E \to \mathbb{Z}_+$, the cut and Euler condition implies the existence of an integer multiflow y.

For each k, there are the following direct implications:

(81.8) integer k-flowing $\Longrightarrow$ k-cycling $\Longrightarrow$ k-flowing.

As Seymour [1981a] showed, for each fixed $k \geq 2$ the concepts of k-cycling and k-flowing are equivalent.

M is called *∞-flowing*, *integer ∞-flowing*, *∞-cycling*, respectively, if M is k-flowing, integer k-flowing, k-cycling, respectively, for each k. Seymour [1981a] showed that the concepts of 4-flowing, 4-cycling, ∞-flowing, and ∞-flowing are equivalent.

We will now discuss Seymour's results in some greater detail.

81.2. Integer k-flowing

By definition, a matroid is integer 1-flowing if and only if M is Mengerian (Section 80.6). Corollary 80.1a therefore characterizes integer 1-flowing matroids, by forbidding U_4^2 and F_7^* as minors.

Also for other values of k, a forbidden minor characterization of binary integer k-flowing matroids is known. In fact, Seymour [1981a] proved that for binary matroids, the concepts of integer ∞-flowing and integer 2-flowing coincide:

Theorem 81.1. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.9) (i) *M is integer ∞-flowing, that is, for each $R \subseteq E$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow;*

(ii) *M is integer 2-flowing, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow;*

(iii) *M has no $M(K_4)$ minor.*

Restricted to graphic and cographic matroids, this bears upon series-parallel graphs.

In Theorem 81.1, the implications (i)⇒(ii)⇒(iii) are easy. The proof of (iii)⇒(i) is based on showing that each binary matroid without $M(K_4)$ minor can be decomposed into matroids with at most 3 elements.

81.3. 1-flowing and 1-cycling

As before, for any matroid $M = (V, \mathcal{I})$ and any $v \in V$, let $H_{M,v}$ be the hypergraph on $V \setminus \{v\}$ with edges all sets $C \setminus \{v\}$, where C is a circuit of M containing v (like in Section 80.6). Then M is 1-flowing if and only if $H_{M,v}$ is ideal for each $v \in V$ (cf. Chapter 78). Since no forbidden minor characterization of ideal hypergraphs is known, we cannot infer a characterization of 1-flowing matroids. While the latter characterization yet might be easier to prove, no such characterization is known. Similarly, no characterization of 1-cycling matroids is known. Seymour [1981a] conjectures that for binary matroids both concepts are equivalent; in fact, that for any binary matroid M:

(81.10) (?) M is 1-cycling $\Longleftrightarrow$ M is 1-flowing $\Longleftrightarrow$ M has no AG(3,2), T_{11}, or T_{11}^* minor. (?)

Here T_{11} is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, AG(3,2) is the matroid with 8 elements, obtained from the 3-dimensional affine geometry over GF(2); equivalently, AG(3,2) is the binary matroid represented by the columns of the matrix[11]:

(81.11)
$$\begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix}.$$

The second equivalence in conjecture (81.10) is a consequence of Seymour's conjecture that $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 are the only binary minimally nonideal hypergraphs.

81.4. 2-flowing and 2-cycling

The next theorem of Seymour [1981a] lifts Hu's 2-commodity flow theorem to matroids. It shows that for binary matroids, the concepts of 2-flowing and 2-cycling coincide.

[11] Seymour [1981a] used the notation AG(2,3) instead of the (more standard) AG(3,2) (for the 3-dimensional affine geometry over GF(2)).

Theorem 81.2. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.12) (i) *M is 2-cycling, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*
(ii) *M is 2-flowing, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*
(iii) *M has no* AG(3,2) *or S_8 minor.*

Here S_8 is the binary matroid represented by the columns of the matrix

(81.13) $$\begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&1 \end{pmatrix}.$$

Since AG(3,2) and S_8 are self-dual, this describes a self-dual property. These matroids are nongraphic and (hence) noncographic. For graphic matroids, Theorem 81.2 amounts to the results on 2-commodity flows described in Chapter 71. For cographic matroids, it amounts to the theorem mentioned in the Notes at the end of Section 71.3.

In Theorem 81.2, the implications (i)⇒(ii)⇒(iii) are easy. The proof of (iii)⇒(i) is based on showing that each binary matroid without AG(3,2) or S_8 minor can be decomposed into regular matroids and copies of F_7 and F_7^*.

81.5. 3-flowing and 3-cycling

Also the concepts of 3-flowing and 3-cycling are equivalent, as follows from the following characterization, again of Seymour [1981a]:

Theorem 81.3. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.14) (i) *M is 3-cycling, that is, for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*
(ii) *M is 3-flowing, that is, for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*
(iii) *M has no F_7, R_{10}, or $M(H_6)$ minor.*

Here H_6 is the graph obtained from $K_{3,3}$ by adding in each colour class one additional edge.

For graphic matroids, Theorem 81.3 gives a theorem on 3-commodity flows. For cographic matroids, this gives nothing new compared with Theorem 81.4 below.

In Theorem 81.3, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are easy. The proof of (iii)$\Rightarrow$(i) is based on showing that each binary matroid without F_7, R_{10}, or $M(H_6)$ minor, can be decomposed into cographic matroids and copies of $M(K_5)$.

81.6. 4-flowing, 4-cycling, ∞-flowing, and ∞-cycling

Trivially, one has the implications:

(81.15) ∞-cycling $\Longrightarrow$ ∞-flowing $\Longrightarrow$ 4-flowing.

Seymour [1981a] showed that these implications can be reversed for binary matroids, and gave the following characterization:

Theorem 81.4. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.16) (i) *M is ∞-cycling, that is, for each $R \subseteq E$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*

(ii) *M is ∞-flowing, that is, is for each $R \subseteq E$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*

(iii) *M is 4-flowing, that is, for each $R \subseteq E$ with $|R| \leq 4$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*

(iv) *M has no F_7, R_{10}, or $M(K_5)$ minor.*

The matroid R_{10} is the matroid on EK_5 with all minimally nonempty even cycles of K_5 as circuits. (Equivalently, the circuits of R_{10} are the even circuits of K_5 and their complements.) An alternative characterization is that R_{10} is the binary matroid represented by all vectors in $\{0,1\}^5$ with precisely three 1's. So R_{10} arises from T_{11} by deleting one element.

For graphic matroids, Theorem 81.4 implies Corollary 75.4d on multiflows if the underlying graph added with the demand edges has no K_5 minor. For cographic matroids, this gives Theorem 29.2 that in bipartite graphs the minimum-size of a T-join is equal to the maximum number of disjoint T-cuts.

In Theorem 81.4, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are easy. The proof of (iv)$\Rightarrow$(i) is based on showing that each binary matroid without F_7, R_{10}, or $M(K_5)$ minor can be decomposed into cographic matroids and copies of F_7^* and of $M(V_8)$ (Figure 3.2).

Notes. Schwärzler and Sebő [1993] extended Theorem 81.4 so as to include Karzanov's Theorem 72.5 that characterizes when the $K_{2,3}$-metric condition suffices for the existence of a multiflow. Related work can also be found in Marcus and Sebő [2001].

81.7. The circuit cone and cycle polytope of a matroid

Circuits in matroids generalize both circuits and cuts in graphs. Hence studying the cone generated by the circuits in a matroid, bears on the circuit cone of a graph considered in Section 29.7 (sums of circuits) and on the cut cone of a graph considered in Section 75.7.

Studying the circuit cone of a matroid relates to multiflows, as it concerns the question under which conditions equality holds in the inequalities (81.1).

Let $M = (E, \mathcal{I})$ be a matroid. The *circuit cone* is the convex cone generated by the incidence vectors of the circuits. Each vector x in the circuit cone satisfies:

$$\begin{array}{lll} (81.17) & x_e \geq 0 & \text{for } e \in E, \\ & x_f \leq x(D \setminus \{f\}) & \text{for each cocircuit } D \text{ and each } f \in D. \end{array}$$

Indeed, if $x = \chi^C$ for some circuit C, then

$$(81.18) \qquad x(D \setminus \{f\}) = \sum_{C \in \mathcal{C}} |(C \cap D) \setminus \{f\}| \geq \sum_{C \in \mathcal{C}} |C \cap \{f\}| = x_f,$$

since $C \cap D \neq \{f\}$.

Seymour [1981a] says that M has the *sums of circuits property* if the circuit cone is determined by (81.17). He derived from Theorem 81.4 the following characterization of this property:

Corollary 81.4a. *For any matroid M the following are equivalent:*

(81.19) (i) *M has the sums of circuits property;*
(ii) *M is binary and ∞-flowing;*
(iii) *M is binary and has no F_7^*, R_{10}, or $M^*(K_5)$ minor.*

Since none of these forbidden minors are graphic, this generalizes Corollary 29.2f (due to Seymour [1979b]). For cographic matroids, this generalizes Corollary 75.4e.

The derivation of Corollary 81.4a from Theorem 81.4 is similar to the derivation of Corollary 75.4e from Corollary 75.4d.

The *cycle polytope* of a binary matroid $M = (E, \mathcal{I})$ is the convex hull of the incidence vectors of cycles. (A *cycle* is the disjoint union of circuits.)

For each x in the cycle polytope the following is necessary:

$$\begin{array}{lll} (81.20) & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ & x(F) - x(D \setminus F) \leq |F| - 1 & \text{for each cocircuit } D \text{ and each} \\ & & F \subseteq D \text{ with } |F| \text{ odd.} \end{array}$$

Barahona and Grötschel [1986] showed that Corollary 81.4a gives (in fact, is equivalent to) the following cycle polytope result:

Corollary 81.4b. *For any binary matroid M the following are equivalent:*

(81.21) (i) *the cycle polytope of M is determined by* (81.20)*;*
(ii) *M is ∞-flowing;*
(iii) *M has no F_7^*, R_{10}, or $M^*(K_5)$ minor.*

The derivation of this from Corollary 81.4a is similar to the derivation of Corollary 75.4f from Corollary 75.4e.

Barahona and Grötschel [1986] characterized adjacency and facets of the cycle polytope of matroids with the sums of circuits property. Grötschel and Truemper [1989] gave an alternative proof of Corollary 81.4b.

81.8. The circuit space and circuit lattice of a matroid

The *circuit space* and the *circuit lattice* of $M = (E, \mathcal{I})$ are the linear space and the lattice, respectively, generated by the incidence vectors of the circuits of M.

Barahona and Grötschel [1986] showed that for any matroid $M = (E, \mathcal{I})$, a vector $x \in \mathbb{R}^E$ belongs to the circuit space of M if and only if

(81.22) $x_e = 0$ if e is a bridge; $x_e = x_f$ if e and f are in series.

The proof is based on an idea of Seymour [1981a]. Necessity being direct, we prove sufficiency, by induction on $|E|$. We may assume that M has no bridges. For each series class P of M, the vector $\mathbf{1}_{E\setminus P}$ belongs to the circuit space of $M \setminus P$. (Here $\mathbf{1}_X$ denotes the all-one vector in $\mathbb{R}^X$.) This follows by induction, as $M \setminus P$ has no bridges. Hence $\mathbf{1}_E - \chi^P$ belongs to the circuit space of M. As this is true for each series class P, we have the theorem.

Now let M be binary. Then each vector x in the circuit lattice satisfies (81.22) and the *Euler condition*:

(81.23) $x(D)$ is even for each cocircuit D.

Lovász and Seress [1993] showed that for any binary matroid M this is enough to characterize the circuit lattice if and only if M^* has no restriction that is a binary sum of copies of the Fano matroid F_7. In particular, if M has no F_7^* minor, then the circuit lattice is characterized by (81.22) and (81.23). (Further work on this in Goddyn [1993], Lovász and Seress [1995], and Fleiner, Hochstättler, Laurent, and Loebl [1999].)

81.9. Nonnegative integer sums of circuits

A necessary condition that a vector x is a nonnegative *integer* combination of incidence vectors of circuits is that x is integer and satisfies the Euler condition (81.23). This is not sufficient, as is shown by the cycle matroid $M(\mathbf{P}_{10})$ of the Petersen graph $\mathbf{P}_{10}$ (which is graphic, and hence has the sums of circuits property): choose a perfect matching N in $\mathbf{P}_{10}$ and let x be 2 on the edges of N, and 1 on the other edges.

Fu and Goddyn [1999] characterized when this necessary condition is sufficient, thus proving a conjecture of Seymour [1981a]:

Theorem 81.5. *For any matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.24) (i) *each vector $x \in \mathbb{Z}_+^E$ satisfying* (81.17) *and* (81.23) *is a nonnegative integer combination of incidence vectors of circuits;*
(ii) *M is binary and has no F_7^*, R_{10}, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor.*

For graphic matroids, this reduces to Theorem 29.4 of Alspach, Goddyn, and Zhang [1994], and for cographic matroids, results on the cut cone mentioned in Section 75.7.

The proof of Theorem 81.5 is by decomposing any matroid satisfying (81.24)(ii) into graphic matroids without $M(\mathbf{P}_{10})$ minor (to which Theorem 29.4 applies), and copies of F_7 and $M^*(V_8)$ (cf. Figure 3.2).

Goddyn [1993] conjectured (more strongly than Theorem 81.5) that for each matroid without $\mathbf{P}_{10}$ minor, the circuits form a Hilbert base. However, Laurent [1996b] showed that this is not true for $M^*(K_6)$.

A survey on this type of problems was given by Goddyn [1993].

81.10. Nowhere-zero flows and circuit double covers in matroids

Let $M = (E, \mathcal{I})$ be a binary matroid. A *flow over* GF(4) is a function $f : E \to \mathrm{GF}(4)$ with $f(D) = 0$ for each cocircuit D of M. The flow is *nowhere-zero* if $f(e) \neq 0$ for each $e \in E$. By linear algebra, each flow over GF(4) can be decomposed as a sum of vectors $\alpha \cdot \chi^C$, where $\alpha \in \mathrm{GF}(4)$ and C is a circuit.

Seymour [1981c] proved that the 4-flow conjecture of Tutte [1966] ('each bridgeless graph without a Petersen graph minor has a nowhere-zero 4-flow' — see Section 28.4) is equivalent to the following stronger conjecture, also given by Tutte [1966]:

(81.25) (?) each bridgeless matroid without F_7^*, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor has a nowhere-zero flow over GF(4). (?)

For graphic matroids, this clearly includes the 4-flow conjecture. For cographic matroids, the existence of a nowhere-zero flow over GF(4) is equivalent to the 4-vertex-colourability of the underlying graph G. By the four-colour theorem and Wagner's theorem (cf. Section 64.3b), any graph without K_5 minor is 4-vertex-colourable — so conjecture (81.25) includes this.

The existence of a nowhere-zero 4-flow is equivalent to the existence of three cycles (= disjoint unions of circuits) that cover each $e \in E$ precisely twice. Indeed, for each nonzero $z \in \mathrm{GF}(4)$, let $C_z := \{e \in E \mid f(e) \neq z\}$. Then the C_z are cycles as required, and the construction can be reversed.

Weaker is the concept of a *circuit double cover* in a binary matroid, which is a family of circuits covering each element precisely twice. Trivially, each bridgeless cographic matroid has a circuit double cover (just take all stars in the corresponding (loopless) graph). The *circuit double cover conjecture* (cf. Sections 29.8 and 38.8) asserts that also each bridgeless graphic matroid has a circuit double cover. Jamshy and Tarsi [1989] proved that this conjecture is equivalent to a generalization to matroids:

(81.26) (?) each bridgeless binary matroid without F_7^* minor has a circuit double cover. (?)

The property of having a circuit double cover need not be closed under taking deletions. So (81.26) gives no necessary and sufficient conditions. One may not relax the condition in (81.26) to requiring that M is binary and $\mathbf{2}$ belongs both to the circuit lattice and to the circuit cone, as is shown by the matroid whose circuits are the even-size cuts of K_{12} (M. Laurent (cf. Goddyn [1993])). This matroid M has an F_7^* minor, and hence does not contradict (81.26).

What has been proved by Jamshy and Tarsi [1989] is:

(81.27) each bridgeless binary matroid without F_7^* minor has a family of circuits covering each element precisely four times.

This extends the corresponding result for graphic matroids of Bermond, Jackson, and Jaeger [1983].

More on nowhere-zero flows and circuit covers in matroids can be found in Tarsi [1985,1986], Jamshy, Raspaud, and Tarsi [1987], and Jamshy and Tarsi [1989].

Chapter 82

Covering and antiblocking in hypergraphs

In this chapter we study the notions of stable set and edge cover in hypergraphs. These concepts are dual to those of matching and vertex cover, by taking the dual hypergraph. Yet, the way we study them is not dual: the classes of hypergraphs considered are closed under operations performed on the *vertex set* (like contraction), while when dualizing the results obtained above, would lead to operations on the *edge set*.
So, although several of the concepts considered in this chapter are just the duals of concepts considered before, we do not dualize the way we studied them above.
As it will turn out, the antiblocking analogues corresponding to the blocking concepts of ideal and Mengerian hypergraphs, all boil down to perfect graph theory.

82.1. Elementary concepts

Let $H = (V, \mathcal{E})$ be a hypergraph. A subset S of V is called *stable* if $|F \cap S| \leq 1$ for each $F \in \mathcal{E}$. An *edge cover* is a collection of edges covering V. So a stable set of H can be considered as a matching of the dual hypergraph H^*, and an edge cover of H as a vertex cover of H^*.

For any hypergraph $H = (V, \mathcal{E})$, define

(82.1) $\alpha(H) :=$ the maximum size of a stable set in H,
$\rho(H) :=$ the minimum size of an edge cover in H.

Determining these numbers is NP-complete, since finding a maximum-size stable set or a minimum-size vertex cover in a graph can be easily reduced to it.

There is the following straightforward inequality:

(82.2) $\alpha(H) \leq \rho(H)$.

For any hypergraph $H = (V, \mathcal{E})$, define

(82.3) $H^{\max} := (V, \{F \in \mathcal{E} \mid \text{there is no } E \in \mathcal{E} \text{ with } E \supset F\})$ and
$H^{\downarrow} := (V, \{F \mid \text{there is an } E \in \mathcal{E} \text{ with } E \supseteq F\})$

So for any hypergraph, $H^{\max}$ is a clutter. Moreover, we have $\alpha(H) = \alpha(H^{\max}) = \alpha(H^{\downarrow})$ and $\rho(H) = \rho(H^{\max}) = \rho(H^{\downarrow})$.

82.2. Fractional edge covers and stable sets

Let $H = (V, \mathcal{E})$ be a hypergraph. A *fractional stable set* is a function $x : V \to \mathbb{R}_+$ satisfying

(82.4) $$\sum_{v \in F} x_v \leq 1 \text{ for each } F \in \mathcal{F}.$$

A *fractional edge cover* is a function $y : \mathcal{E} \to \mathbb{R}_+$ satisfying

(82.5) $$\sum_{F \ni v} y_F \geq 1 \text{ for each } v \in V.$$

(Here and below, F ranges over the edges of H.) Let $\alpha^*(H)$ denote the maximum size of a fractional stable set and let $\rho^*(H)$ denote the minimum size of a fractional edge cover (where the *size* of a vector is the sum of its components).

So $\rho^*(H)$ can be described as

(82.6) $$\rho^*(H) = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{E}}, y^{\mathsf{T}}M \geq \mathbf{1}^{\mathsf{T}}\},$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. Similarly,

(82.7) $$\alpha^*(H) = \max\{\mathbf{1}^{\mathsf{T}}x \mid x \in \mathbb{R}_+^V, Mx \leq \mathbf{1}\}.$$

As these represent dual linear programs, this gives:

(82.8) $$\rho^*(H) = \alpha^*(H).$$

82.3. k-edge covers and k-stable sets

Like in the blocking case, there is an alternative interpretation of the parameters $\rho^*(H)$ and $\alpha^*(H)$. A *k-stable set* is a function $x : V \to \mathbb{Z}_+$ such that

(82.9) $$\sum_{v \in F} x_v \leq k \text{ for each } F \in \mathcal{F}.$$

Let $\alpha_k(H)$ denote the maximum size of a k-stable set. As 1-stable sets are precisely the incidence vectors of the stable sets, $\alpha_1(H) = \alpha(H)$.

A *k-edge cover* is a function $y : \mathcal{E} \to \mathbb{Z}_+$ such that

(82.10) $$\sum_{F \ni v} y_F \geq k \text{ for each } v \in V.$$

Let $\rho_k(H)$ denote the minimum size of a k-edge cover in H. The minimal 1-edge covers are precisely the incidence vectors of the edge covers, and hence $\rho_1(H) = \rho(H)$.

One easily checks that, for any $k \in \mathbb{Z}_+$:

(82.11) $\alpha_k(H) \leq \rho_k(H)$.

In fact, for each $k \geq 1$:

(82.12) $\alpha(H) \leq \dfrac{\alpha_k(H)}{k} \leq \alpha^*(H) = \rho^*(H) \leq \dfrac{\rho_k(H)}{k} \leq \rho(H)$.

Also one has (Lovász [1974]):

(82.13) $\rho^*(H) = \min\limits_k \dfrac{\rho_k(H)}{k} = \lim\limits_{k\to\infty} \dfrac{\rho_k(H)}{k}$.

Here the left-hand side equality holds as the minimum in (82.6) is attained by a rational optimum solution y. The right-hand side equality follows from Fekete's lemma (Theorem 2.2), using the fact that for all $k, l \geq 1$:

(82.14) $\rho_{k+l}(H) \leq \rho_k(H) + \rho_l(H)$,

since the sum of a k-edge cover and an l-edge cover is a $k + l$-edge cover. Similarly we have:

(82.15) $\alpha^*(H) = \max\limits_k \dfrac{\alpha_k(H)}{k} = \lim\limits_{k\to\infty} \dfrac{\alpha_k(H)}{k}$,

using (82.7) and the fact that for all $k, l \geq 1$:

(82.16) $\alpha_{k+l}(H) \geq \alpha_k(H) + \alpha_l(H)$.

82.4. The antiblocker and conformality

For any hypergraph $H = (V, \mathcal{E})$, the *antiblocking hypergraph*, or the *antiblocker*, of H is the hypergraph $a(H)$ with vertex set V and edges all inclusionwise maximal stable sets of H. So $a(H)$ is a clutter, and $\alpha(H) = r_{\max}(a(H))$ (=the maximum edge-size of $a(H)$).

In Section 77.6 we saw that for any clutter H we have $b(b(H)) = H$. A similar duality phenomenon does not hold for antiblockers. For instance, for the hypergraph $H = K_3$ (with $V := \{1,2,3\}$ and $\mathcal{E} := \{\{1,2\},\{1,3\},\{2,3\}\}$) one has $a(H) = (V, \{\{1\},\{2\},\{3\}\})$, and hence $a(a(H)) = (V, \{\{1,2,3\}\}) \neq H$.

However, by adding a further condition, we can restore this duality relation for the antiblocking operation. Call a hypergraph $H = (V, \mathcal{E})$ *conformal* if for each $U \subseteq V$:

(82.17) if each pair in U is contained in some edge of H, then U is contained in some edge of H.

So H is conformal $\iff$ $H^{\max}$ is conformal $\iff$ $H^{\downarrow}$ is conformal. Moreover:

(82.18) H is conformal $\iff$ there exists a graph G on V such that $H^{\max}$ consists of the inclusionwise maximal cliques of G.

One may check that for each hypergraph H, the hypergraph $a(H)$ is conformal. Also:

Theorem 82.1. *A hypergraph H is conformal if and only if $a(a(H)) = H^{\max}$. In particular, if H is a conformal clutter, then $a(a(H)) = H$.*

Proof. If H is conformal, there is a graph G on V such that $H^{\max}$ is the collection of inclusionwise maximal cliques of G. Then $a(H)$ is the collection of inclusionwise maximal stable sets of G. Hence $a(a(H))$ is the collection of inclusionwise maximal cliques of G. So $a(a(H)) = H^{\max}$. ∎

82.4a. Gilmore's characterization of conformality

Conformality of hypergraphs has been characterized by Gilmore [1962] as follows:

Theorem 82.2. *A hypergraph $H = (V, \mathcal{E})$ is conformal if and only if $V = \cup\mathcal{E}$ and for all $E_1, E_2, E_3 \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with*

(82.19) $E \supseteq (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$.

Proof. Necessity follows from the definition of conformality, since any two vertices in $(E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$ are contained in some E_i.

To see sufficiency, suppose that the condition is satisfied, but that H is not conformal. Let U be a minimal set such that any pair of vertices in U is contained in some edge of H, but U is contained in no edge of H. So $|U| \geq 3$. Choose distinct $u_1, u_2, u_3 \in U$ and let $F_i := U \setminus \{u_i\}$ for $i = 1, 2, 3$. By the minimality of U, each F_i is contained in some edge, E_i say, of H. Now $U = (F_1 \cap F_2) \cup (F_1 \cap F_3) \cup (F_2 \cap F_3) \subseteq (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$. By the condition, the latter set is contained in an edge of H, and hence also U is contained in an edge of H. This contradicts our assumption. ∎

As was noted by M. Conforti, Theorem 82.2 implies a polynomial-time test of conformality of a hypergraph, if all maximal edges are given.

82.5. Perfect hypergraphs

We now define the antiblocking analogue of the blocking concept of ideal hypergraph. A hypergraph $H = (V, \mathcal{E})$ is called *perfect*, if $\bigcup \mathcal{E} = V$ and each vertex of the polyhedron Q_H in $\mathbb{R}^V$ determined by:

(82.20)
(i) $x_v \geq 0$ for $v \in V$,
(ii) $x(F) \leq 1$ for $F \in \mathcal{E}$

is integer. (Lovász [1972c] called a hypergraph H *normal* if its dual H^* is perfect.)

We first observe:

Theorem 82.3. *A perfect hypergraph is conformal.*

Proof. Suppose that $H = (V, \mathcal{E})$ is perfect but not conformal. Let U be a minimal subset of V such that any two vertices are contained in an edge of H, but U is contained in no edge of H. So $|U| \geq 3$ and $U \setminus \{u\}$ is contained in an edge of H, for each $u \in U$. Define $z : V \to \mathbb{R}_+$ by:

(82.21) $$z := \frac{1}{|U|-1}\chi^U.$$

Then z belongs to Q_H, and hence z is a convex combination of integer vectors in Q_H. However, each integer vector x satisfies $x(U) \leq 1$ (since $x(U \setminus \{u\}) \leq 1$ for each $u \in U$ and since $|U| \geq 3$). As $z(U) = |U|/(|U|-1) > 1$, this is a contradiction. ∎

Note that each integer vector in Q_H is a 0,1 vector, and hence is the incidence vector of a stable set of H. So H is perfect if and only if Q_H is the convex hull of the incidence vectors of stable sets of H. By the theory of antiblocking polyhedra, this implies that if H is perfect, then each vertex of the polytope $Q_{a(H)}$, by definition determined by

(82.22) $$\begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(S) \leq 1 & \text{for } S \in a(H), \end{array}$$

is integer — hence $a(H)$ is perfect.

We cannot simply reverse this implication: if H is the complete graph K_3, then H is not perfect (as $\frac{1}{2} \cdot \mathbf{1}$ is a noninteger vertex of Q_H), but $a(H)$ is perfect: its edges are all singleton vertices of K_3.

However, if we require H to be conformal, the duality is restored (Fulkerson [1971a,1972a]):

Corollary 82.3a. *A hypergraph H is perfect $\iff$ H is conformal and its antiblocker $a(H)$ is perfect.*

Proof. If H is perfect, then H is conformal by Theorem 82.3. Moreover, $a(H)$ is perfect, by the theory of antiblocking polyhedra.

Conversely, if $a(H)$ is perfect, then $a(a(H))$ is perfect. As H is conformal, $H = a(a(H))$, and hence H is perfect. ∎

The following theorem implies that most of hypergraph theory related to antiblocking boils down to the theory of perfect graphs (the 'only if' part is due to Fulkerson [1972a] and the 'if' part to Lovász [1972c]):

Corollary 82.3b. *A hypergraph $H = (V, \mathcal{E})$ is perfect if and only if $H^{\max}$ consists of the maximal cliques of some perfect graph $G = (V, E)$.*

Proof. To see necessity, as H is perfect, it is conformal, and hence $H^{\max}$ consists of the maximal cliques of some graph $G = (V, E)$. Then G is a perfect graph, by Corollary 65.2e.

To see sufficiency, if $H^{\max}$ consists of the maximal cliques of a perfect graph, then (82.20) has integer vertices (again by Corollary 65.2e) and hence H is perfect. ∎

Perfect hypergraphs can be characterized by a weaker, and also by stronger, conditions than the definition. In the following corollary we collect some of them (Fulkerson [1972a]: (iii)⇔(iv)⇔(v), Lovász [1972c]: (i)⇔(ii)⇔(iv)⇔(vi), Lovász [1972a]: (i)⇔(viii), Berge [1973a]: (i)⇔(vii)).

Theorem 82.4. *For any hypergraph $H = (V, \mathcal{E})$ with $\bigcup \mathcal{E} = V$ the following are equivalent, where M denotes the $\mathcal{E} \times V$ incidence matrix of H:*

(82.23)
(i) *$H^{\max}$ consists of the maximal cliques of some perfect graph;*
(ii) *$\alpha(H') = \rho(H')$ for each contraction H' of H;*
(iii) *H is perfect, that is, $\{x \geq \mathbf{0} \mid Mx \leq \mathbf{1}\}$ is an integer polytope;*
(iv) *the system $x \geq \mathbf{0}, Mx \leq \mathbf{1}$ is totally dual integral;*
(v) *$a(H)$ is perfect;*
(vi) *$\alpha^*(H')$ is an integer for each contraction H' of H;*
(vii) *$\rho_2(H') = 2\rho(H')$ for each contraction H' of H;*
(viii) *$\alpha(H')r_{\max}(H') \geq |VH'|$ for each contraction H' of H.*

Proof. The equivalence of (i) and (iii) is Corollary 82.3b. The equivalence of (i), (iii), and (v) then follows from the perfect graph theorem (Corollary 65.2a). The implication (i)⇒(iv) follows from Corollary 65.2f. Since contractions of H correspond to taking induced subgraphs of G, the implication (i)⇒(ii) is the definition of perfect graph. The implication (ii)⇒(viii) is direct, as $\rho(H')r_{\max}(H') \geq |VH'|$ for any hypergraph H'. The implications (ii)⇒(vi) and (ii)⇒(vii) follow from (82.12). The implication (iv)⇒(iii) is general polyhedral theory (Theorem 5.22).

So it suffices to show that each of (vi), (vii), and (viii) implies (i). We first show that each of (vi), (vii), and (viii) implies that H is conformal.

Suppose that H is not conformal. Then there is a minimal subset U of V such that each pair in U is covered by an edge of H, but U is not covered by any edge of H. So $|U| \geq 3$. Let H' be obtained from H by contracting $V \setminus U$.

Then H' has a 2-edge cover of size 3 (taking $U \setminus \{u\}$ for three vertices $u \in U$), while $\rho(H') \geq 2$, contradicting (vii). Moreover, $\alpha(H') = 1$ and $r_{\max}(H') = |U| - 1 < |U| = |VH'|$, contradicting (viii).

As U is contained in no edge of H, we know that $\alpha^*(H') \geq |U|/(|U|-1)$, since $(|U|-1)^{-1} \cdot \mathbf{1}$ is a fractional stable set of H'. Also, $\alpha^*(H') \leq |U|/(|U|-1)$, since for any fractional stable set x of H' we have $x(U \setminus \{u\}) \leq 1$ for each $u \in U$ (as $U \setminus \{u\}$ is contained in an edge of H), and hence

$$x(U) = \frac{1}{|U|-1} \sum_{u \in U} x(U \setminus \{u\}) \leq \frac{|U|}{|U|-1}. \tag{82.24}$$

So $\alpha^*(H')$ is not an integer, contradicting (vi).

So each of (vi), (vii), and (viii) implies that H is conformal. Knowing that H is conformal, let $H^{\max}$ consist of the maximal cliques of a graph $G = (V, E)$. To show that H is perfect, it suffices to show that G is perfect if (vi), (vii), or (viii) holds. This follows from Theorems 65.10, 65.11, and 65.2, respectively (using that G is perfect if $\overline{G}$ is perfect, and that $\alpha^*(H) = \chi^*(\overline{G})$). ∎

By definition, 'perfect hypergraph' is the antiblocking analogue of 'ideal hypergraph'. By Theorem 82.4, we know that the antiblocking analogue of 'Mengerian hypergraph' coincides with 'perfect hypergraph' (since (82.23)(iii) and (iv) are equivalent). So perfect hypergraph theory reduces to perfect graph theory, and minimally imperfect hypergraphs can be characterized with the strong perfect graph theorem. We will not expand further on this but refer to the chapters in Part VI on perfect graphs.

82.6. Further notes

82.6a. Some equivalences for the k-parameters

Let $H = (V, \mathcal{E})$ be a hypergraph and let $v \in V$. Adding a *serial* vertex to v means extending V by a new vertex v' and replacing $\mathcal{E}$ by

$$\{E \mid v \notin E \in \mathcal{E}\} \cup \{E \cup \{v'\} \mid v \in E \in \mathcal{E}\}. \tag{82.25}$$

A hypergraph obtained from H by a sequence of contractions of vertices and adding serial vertices, is called a *serialization* of H. If $w : V \to \mathbb{Z}_+$ indicates the size of the final series classes of the vertices, we denote the serialization by H_w. So contractions are special cases of serializations and correspond to functions $w : V \to \{0, 1\}$. In a certain sense, also restrictions are special cases of parallelizations and correspond to functions $w : V \to \{1, \infty\}$.

Theorem 82.5. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(82.26) (i) *$k \cdot \alpha^*(H') = \alpha_k(H')$ for each serialization H' of H;*
(ii) *$k \cdot \alpha^*(H')$ is an integer for each serialization H' of H.*

Proof. Similar to the proof of Theorem 78.3. ∎

This is used in proving:

Theorem 82.6. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(82.27) (i) $\rho_k(H') = \alpha_k(H')$ *for each serialization H' of H;*
(ii) $k \cdot \rho^*(H') = \rho_k(H')$ *for each serialization H' of H;*
(iii) $\rho_{2k}(H') = 2\rho_k(H')$ *for each serialization H' of H.*

Proof. Similar to the proof of Theorem 79.2. ∎

Are Theorems 82.5 and 82.6 maintained if serializations are replaced by just contractions? As we will see, this is the case for $k = 2$ and $k = 3$ but not for general k.

As for $k = 2$, Lovász [1977b] showed:

Theorem 82.7. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.28) (i) $\alpha^*(H') = \frac{1}{2}\alpha_2(H')$ *for each contraction H' of H;*
(ii) $\alpha^*(H') \in \frac{1}{2}\mathbb{Z}$ *for each contraction H' of H.*

Proof. The implication (i)⇒(ii) is trivial. To see the reverse implication, we can assume that (ii) holds and that $\alpha^*(H') = \frac{1}{2}\alpha_2(H')$ for each contraction $H' \neq H$ of H, while $\alpha^*(H) > \frac{1}{2}\alpha_2(H)$.

Since $\alpha^*(H) > \frac{1}{2}\alpha_2(H)$ and $\alpha^*(H) \in \frac{1}{2}\mathbb{Z}$, we know $\alpha^*(H) \geq \frac{1}{2}\alpha_2(H) + \frac{1}{2}$. Let x be a fractional stable set of H of size $\alpha^*(H)$. Then for each $v \in V$, $x|V \setminus \{v\}$ is a fractional stable set of H/v, and so

(82.29) $$x(V \setminus \{v\}) \leq \alpha^*(H/v) = \tfrac{1}{2}\alpha_2(H/v) \leq \tfrac{1}{2}\alpha_2(H) \leq \alpha^*(H) - \tfrac{1}{2} = x(V) - \tfrac{1}{2}.$$

So $x_v \geq \frac{1}{2}$ for each $v \in V$. Hence $|F| \leq 2$ for each $F \in \mathcal{E}$. So H is (essentially) a graph, and hence $\alpha_2(H) = \rho_2(H)$ (by Corollary 30.9a). This implies $\alpha_2(H) = \frac{1}{2}\alpha^*(H)$. ∎

As a consequence one has (Lovász [1975a]: (i)⇔(ii)):

Corollary 82.7a. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.30) (i) $\alpha_2(H') = \rho_2(H')$ *for each contraction H' of H;*
(ii) $2\alpha^*(H') = \rho_2(H')$ *for each contraction H' of H;*
(iii) $\rho_6(H') = 3\rho_2(H')$ *for each contraction H' of H.*

Proof. The equivalence of (i) and (ii) follows directly from Theorem 82.7. Also the implication (i)⇒(iii) is direct, since $\alpha_2(H) \leq \frac{1}{3}\rho_6(H) \leq \rho_2(H)$ for any hypergraph H.

To see (iii)⇒(i), let $H = (V, \mathcal{E})$ be a counterexample with $|V|$ minimal. So $\rho_2(H') = \alpha_2(H')$ for each contraction $H' \neq H$ of H, and $\rho_6(H) = 3\rho_2(H)$. If each edge of H has size at most 2, then $\rho_2(H) = \alpha_2(H)$, by Corollary 30.9a. So H has an edge F of size at least 3. Choose distinct $v_1, v_2, v_3 \in F$. Then for each $i = 1, 2, 3$ we have;

(82.31) $$\rho_2(H/v_i) = \alpha_2(H/v_i) \leq \alpha_2(H) < \rho_2(H).$$

Hence $\rho_2(H/v_i) \leq \rho_2(H) - 1$.

For $i = 1, 2, 3$, let y_i be a 2-edge cover of H/v_i of size $\rho_2(H/v_i)$. Then $y_1 + y_2 + y_3 + 2\chi^{\{F\}}$ is a 6-edge cover of H of size

(82.32) $$\rho_2(H/v_1) + \rho_2(H/v_2) + \rho(H/v_3) + 2 \leq 3(\rho_2(H) - 1) + 2 < 3\rho_2(H).$$

This contradicts the fact that $\rho_6(H) = 3\rho_2(H)$. ∎

Lovász [1977b] showed that Theorem 82.7 also holds if we replace 2 by 3:

Theorem 82.8. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.33)
(i) $\alpha^*(H') = \frac{1}{3}\alpha_3(H')$ *for each contraction H' of H;*
(ii) $\alpha^*(H') \in \frac{1}{3}\mathbb{Z}$ *for each contraction H' of H.*

Proof. The implication (i)⇒(ii) being direct, we prove (ii)⇒(i). Let $H = (V, \mathcal{E})$ be a counterexample with $|V|$ minimal. So $\alpha^*(H) \in \frac{1}{3}\mathbb{Z}$, $\alpha^*(H) > \frac{1}{3}\alpha_3(H)$, and $\alpha^*(H') = \frac{1}{3}\alpha_3(H')$ for each contraction $H' \neq H$ of H. So $\alpha^*(H) \geq \frac{1}{3}\alpha_3(H) + \frac{1}{3}$.

Let x be a fractional stable set of H with $x(V) = \alpha^*(H)$. Then for each $v \in V$, $x|V \setminus \{v\}$ is a fractional stable set of H/v, and hence:

(82.34) $$\begin{aligned} x(V \setminus \{v\}) &\leq \alpha^*(H/v) = \tfrac{1}{3}\alpha_3(H/v) \leq \tfrac{1}{3}\alpha_3(H) \leq \alpha^*(H) - \tfrac{1}{3} \\ &= x(V) - \tfrac{1}{3}. \end{aligned}$$

So $x_v \geq \frac{1}{3}$ for each $v \in V$. Therefore, $|F| \leq 3$ for each $F \in \mathcal{E}$. Let U be the union of the edges of H of size 3. Then $x_v = \frac{1}{3}$ for each $v \in U$.

Let $W := V \setminus U$. Then the edges of H contained in W form a bipartite graph. Otherwise, it contains an odd circuit C, and then $H' := H/(V \setminus VC)$ satisfies $\alpha^*(H') = \frac{1}{2}|VC|$. So $\alpha^*(H')$ does not belong to $\frac{1}{3}\mathbb{Z}$, a contradiction.

Let N be the set of vertices w in W for which there is a $u \in U$ with $\{u, w\} \in \mathcal{E}$. Since x is a maximum-size fractional stable set of H and since $x_v = \frac{1}{3}$ for each $v \in U$, we know that $x|W$ attains the maximum in the linear program of maximizing $z(W)$ over $z \in \mathbb{R}^W$ satisfying

(82.35) $$\begin{array}{ll} 0 \leq z(v) \leq 1 & \text{for each } v \in V, \\ z(v) \leq \frac{2}{3} & \text{for each } v \in N, \\ z(u) + z(v) \leq 1 & \text{for each edge } \{u, v\} \subseteq W \text{ of } H. \end{array}$$

Since the constraint matrix of this LP-problem is totally unimodular and since the right-hand side is $\frac{1}{3}$-integer, there is a $\frac{1}{3}$-integer optimum solution z. We can assume that $x|W = z$. So $x \in \frac{1}{3}\mathbb{Z}^V$, implying that $3x$ is a 3-stable set. Hence $\alpha_3(H) \geq 3\alpha^*(H)$, contradicting our assumption. ∎

This implies (Lovász [1977b]):

Corollary 82.8a. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.36)
(i) $\alpha_3(H') = \rho_3(H')$ *for each contraction H' of H;*
(ii) $3\alpha^*(H') = \rho_3(H')$ *for each contraction H' of H.*

Proof. Directly from Theorem 82.8. ∎

Lovász [1977b] raised the question if in these results 3 can be replaced by any arbitrary integer k. However, Schrijver and Seymour [1979] gave the following example of a hypergraph $H = (V, \mathcal{E})$ satisfying $\alpha_{60}(H) < \rho_{60}(H)$ while $60\alpha^*(H') = \rho_{60}(H')$ for each contraction H' of H:

(82.37) $V := \{1,2,3,4,5,6,7\}$, $\mathcal{E} := \{V \setminus \{1,2\}, V \setminus \{1,3\}, V \setminus \{1,4\}, V \setminus \{2,3\}, V \setminus \{2,4\}, V \setminus \{3,4\}, V \setminus \{5\}, V \setminus \{6\}, V \setminus \{7\}\}$.

To see that $\rho_{60}(H') = 60\alpha^*(H')$ for each contraction H' of H, observe that if we contract two of the vertices $1,2,3,4$ or one of the vertices $5,6,7$, there is an edge covering all vertices, and $\alpha = \rho$ follows. So by symmetry it suffices to show that $\rho_{60}(H') = 60\alpha^*(H')$ for $H' := H$ and for $H' := H/1$.

The fractional stable set x of $H/1$ defined by $x := \frac{1}{5} \cdot \mathbf{1}$ shows that $\alpha^*(H/1) \geq \frac{6}{5}$. Then the 5-edge cover y of $H/1$ defined by: $y(V \setminus \{1,i\}) := 1$ for $i = 2, \ldots, 7$, and $y(E) := 0$ for any other edge E of $H/1$, shows that $\rho_5(H/1) \leq 6$. Hence $\rho_{60}(H/1) \leq 12\rho_5(H/1) \leq 72 \leq 60\alpha^*(H/1)$.

Finally we consider H. Let x be the fractional stable set defined by:

(82.38) $x(1) := x(2) := x(3) := x(4) := \frac{1}{8}$, $x(5) := x(6) := x(7) := \frac{1}{4}$,

and let y be the fractional edge cover defined by:

(82.39) $y(V \setminus \{i,j\}) := \frac{1}{12}$ for all $1 \leq i < j \leq 4$ and $y(V \setminus \{i\}) := \frac{1}{4}$ for $i = 5,6,7$.

So $x(V) = \frac{5}{4} = y(\mathcal{E})$. Hence $\alpha^*(H) = \frac{5}{4}$. However, x is the only fractional stable set of size $\frac{5}{4}$. Indeed, for any fractional stable set x of size $\frac{5}{4}$ one has $x(\{i,j\}) \geq \frac{1}{4}$ for all $1 \leq i < j \leq 4$ and $x(\{i\}) \geq \frac{1}{4}$ for all $5 \leq i \leq 7$. So $x(\{5,6,7\}) \geq \frac{3}{4}$, hence $x(\{1,2,3,4\}) \leq \frac{1}{2}$. Therefore, $x(\{i\}) = \frac{1}{4}$ for all $5 \leq i \leq 7$ and $x(\{i,j\}) = \frac{1}{4}$ for all $1 \leq i < j \leq 4$. This gives $x(\{i\}) = \frac{1}{8}$ for each $1 \leq i \leq 4$.

As $60x \notin \mathbb{Z}$, this shows that $\alpha_{60}(H) < 60\alpha^*(H)$.

82.6b. Further notes

The complete graphs show that $\rho(H)$ cannot be bounded in terms of $\alpha(H)$. Ding, Seymour, and Winkler [1994] showed that for each fixed k, $\rho(H)$ is bounded by a polynomial in $\alpha(H)$ if we restrict H to hypergraphs not having the complete graph on k vertices as partial subhypergraph. Here, a*partial subhypergraph* arises by deleting edges and contracting vertices.

A $\{0, \pm 1\}$ matrix M is *perfect* if the polytope

(82.40) $\{x \mid \mathbf{0} \leq x \leq \mathbf{1}, Mx \leq \mathbf{1} - b\}$

is integer, where b is the vector with b_i equal to the number of -1's in the ith row of M. These matrices generalize the incidence matrices of perfect hypergraphs and were studied by Conforti, Cornuéjols, and de Francesco [1997] (who gave a characterization in terms of perfect graphs), Boros and Čepek [1997], Guenin [1998b], and Tamura [2000].

An extension of the equivalence of (iii) and (iv) in Theorem 82.4 was proved by Korach [1982]: Let M_1 and M_2 be integer matrices such that each row of M_2 is a nonnegative linear combination of rows of M_1. Consider the system

(82.41) $M_1 x \geq \mathbf{0}, M_2 x \leq \mathbf{1}$.

Then (82.41) is TDI if and only if $M_1 x \geq \mathbf{0}$ is TDI and (82.41) determines an integer polyhedron.

The intersection of the polyhedra made by perfect and ideal hypergraphs was investigated by Sebő [1998]. Related results were given by Shepherd [1994a] and Gasparyan [1998]. Monma and Trotter [1979] gave an alternative proof of the relation between perfect graphs and perfect hypergraphs.

Determining the stable set number $\alpha(H)$ of a hypergraph H is equivalent to the vertex packing problem (equivalently, the set packing problem). In Section 64.9e we gave further references for this problem. Determining the edge cover number $\rho(H)$ of H amounts to the *set covering problem*. This NP-complete problem is studied by Lawler [1966], Roth [1969], Lemke, Salkin, and Spielberg [1971], Thiriez [1971], Balas and Padberg [1972,1975a], Garfinkel and Nemhauser [1972b] (survey), Even [1973], Guha [1973], Salkin and Koncal [1973], Christofides and Korman [1974], Fulkerson, Nemhauser, and Trotter [1974], Johnson [1974a], Gondran and Laurière [1975], Lovász [1975c], Etcheberry [1977], Chvátal [1979], Padberg [1979], Avis [1980a], Balas [1980], Balas and Ho [1980], Baker [1981], Bar-Yehuda and Even [1981], Ho [1982], Hochbaum [1982,1983b], Lifschitz and Pittel [1983], Vasko and Wilson [1984a,1984b], Beasley [1987,1990], Bertolazzi and Sassano [1987, 1988], Balas and Ng [1989a,1989b], Cornuéjols and Sassano [1989], Feo and Resende [1989], Nobili and Sassano [1989,1992], Sassano [1989], Fisher and Kedia [1990], Karmarkar, Resende, and Ramakrishnan [1991], El-Darzi and Mitra [1992], Goldschmidt, Hochbaum, and Yu [1993], Khuller, Vishkin, and Young [1993,1994], Lorena and Lopes [1994], Mannino and Sassano [1995], Halldórsson [1995,1996], Caprara, Fischetti, and Toth [1996,1999], Feige [1996,1998], Duh and Fürer [1997], Bar-Yehuda [2000], Halperin [2000,2002], and Holmerin [2002].

The related *set partitioning problem* was investigated by Garfinkel and Nemhauser [1969], Michaud [1972], Marsten [1973], Nemhauser, Trotter, and Nauss [1973], Gondran and Laurière [1974], Balas and Padberg [1975a,1975b,1976], Balas [1977], Nemhauser and Weber [1979], Johnson [1980], Hwang, Sun, and Yao [1985], John [1988], Fisher and Kedia [1990], El-Darzi and Mitra [1992], and Sherali and Lee [1996].

Chapter 83

Balanced and unimodular hypergraphs

In the preceding chapters we investigated conditions under which $\tau(H) = \nu(H)$ or $\alpha(H) = \rho(H)$ holds for all hypergraphs H obtained by deleting or multiplying vertices of some hypergraph. Although these parameters transfer to each other by taking the dual hypergraph, the study was unsymmetric as we considered only deleting or multiplying of vertices, not of edges. In the applications, generally the number of edges is exponentially large in the number of vertices.
In the present chapter we study hypergraphs for which these equalities hold in a symmetric fashion. This leads to the classes of balanced and unimodular matrices.

83.1. Balanced hypergraphs

A 0,1 matrix M is called *balanced* if M has no submatrix which is the incidence matrix of an odd circuit. A hypergraph H is *balanced* if its incidence matrix is balanced.

Another way of characterizing balancedness of a hypergraph $H = (V, \mathcal{E})$ is by the associated bipartite graph G with colour classes V and $\mathcal{E}$, and $v \in V$ and $F \in \mathcal{E}$ adjacent if and only if $v \in F$:

(83.1) H is balanced $\iff$ the length of each chordless circuit in G is a multiple of 4.

The class of balanced hypergraphs is closed under taking 'partial subhypergraphs'. A *partial hypergraph* of a hypergraph H is a hypergraph $(V, \mathcal{E}')$ with $\mathcal{E}' \subseteq \mathcal{E}$. A *partial subhypergraph* of H is a contraction of a partial hypergraph of H. So the incidence matrices of partial subhypergraphs of H arise by deleting rows and columns of the incidence matrix of H. In this terminology,

(83.2) a hypergraph H is balanced $\iff$ H has no odd circuit as partial subhypergraph.

Trivially, the dual of a balanced hypergraph is again balanced. Also, the class of balanced hypergraphs is closed under contractions and restrictions. More generally, it is closed under parallelization and serialization. Hence also the class of blockers of balanced hypergraphs is closed under parallelization and serialization.

Note that for graphs (that is, hypergraphs with each edge of size 2), balancedness coincides with bipartiteness.

In a deep theorem, Conforti, Cornuéjols, and Rao [1999] showed that balancedness of a hypergraph can be tested in polynomial time. The method is based on decomposition of balanced matrices into totally unimodular matrices. An outline of the method was given by Conforti and Cornuéjols [1990]. Related work is reported in Conforti, Cornuéjols, and Rao [1995].

83.2. Characterizations of balanced hypergraphs

Balanced hypergraphs can be characterized in several ways in terms of polyhedra and optimization, as in the following theorem. As before, the hypergraphs $b(H)$ and $a(H)$ denote the blocker and antiblocker of H, respectively. (Berge and Las Vergnas [1970] proved (i)⇔(ii)⇔(iii) and Berge [1972] proved (i)⇔(iv). Given the equivalence of (i) and (iii), the pluperfect graph theorem of Fulkerson [1971a] implies the equivalence of (i), (iii), and (v) (conjectured by Berge [1969]), since balancedness is closed under parallelization.)

Theorem 83.1. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(83.3)
(i) *H is balanced;*
(ii) *$\nu(H') = \tau(H')$ for each partial subhypergraph H' of H;*
(iii) *$\alpha(H') = \rho(H')$ for each partial subhypergraph H' of H;*
(iv) *$\nu(b(H')) = r_{\min}(H')$ for each partial subhypergraph H' of H;*
(v) *$\rho(a(H')) = r_{\max}(H')$ for each partial subhypergraph H' of H.*

Proof. Each of (ii), (iii), (iv), (v) implies (i), since if H is not balanced, it has a partial subhypergraph that is an odd circuit. It is easy to see that none of (ii)-(v) hold for any odd circuit. To show the reverse implications, it suffices to derive from (i) that each of the equalities holds for $H' = H$, since the class of balanced matrices is closed under taking partial subhypergraphs.

We first show (i)⇒(ii). Since the class of balanced hypergraphs is closed under parallelization, by Theorem 79.2 it suffices to show that $\nu_2(H) = 2\nu(H)$. Let $y : \mathcal{E} \to \mathbb{Z}_+$ be a 2-matching of size $\nu_2(H)$. Let $\mathcal{M} := \{E \in \mathcal{E} \mid y(E) = 2\}$ and $\mathcal{F} := \{E \in \mathcal{E} \mid y(E) = 1\}$. The dual of the hypergraph $(V, \mathcal{F})$ is a graph G, added with some edges of size ≤ 1. Since H is balanced, G is bipartite. Let $\mathcal{N}$ be the largest of the two colour classes of G. Then $|\mathcal{N}| \geq \frac{1}{2}|\mathcal{F}|$, and hence $\mathcal{M} \cup \mathcal{N}$ is a matching of size $\geq \frac{1}{2}\nu_2(H)$.

This shows (i)⇒(ii). By taking the dual of H, we see (i)⇒(iii). By Theorem 82.4, (iii) implies that the maximal edges of H are the maximal cliques of some perfect graph G on V. Then $\chi(G) = \omega(G)$ implies $\rho(a(H)) = r_{\max}(H)$.

We finally show (i)⇒(iv). We first show that

(83.4) the vertex set V of a balanced hypergraph $H = (V, \mathcal{E})$ can be partitioned into two sets, each intersecting each edge of size ≥ 2.

The proof is by induction on $|V|$. Let E be the collection of pairs in $\mathcal{E}$. Then the graph $G = (V, E)$ contains a vertex u such that any two neighbours of u belong to the same component of $G - u$. (This is true for any graph. To see it, we can assume that G is connected. Then choose an arbitrary vertex v and let u be a vertex at maximum distance from v.)

By induction, we can partition $V \setminus \{u\}$ into two sets V_1, V_2 each intersecting each edge F of H with $|F \setminus \{u\}| \geq 2$. Now any two neighbours of u in G are connected by a path in $G - u$ of even length, since G is bipartite (as H is balanced). Hence the neighbours belong either all to V_1 or all to V_2. By symmetry, we can assume that they all belong to V_1. Then $V_1, V_2 \cup \{u\}$ is a partition as required. This shows (83.4).

To show (i)⇒(iv), we prove $\nu(b(H)) = r_{\min}(H)$, that is, the maximum number of disjoint vertex covers of H is equal to the minimum edge size r. This is shown by induction on $|\mathcal{E}|$. Choose $F \in \mathcal{E}$ and define $\mathcal{E}' := \mathcal{E} \setminus \{F\}$. Then, by induction, the hypergraph $(V, \mathcal{E}')$ has r disjoint vertex covers $B_1, \ldots, B_r$. We can assume that they partition V. Choose $B_1, \ldots, B_r$ such that a maximum number of the B_i intersect F.

If each B_i intersects F we are done, so we may assume that $B_1 \cap F = \emptyset$. As $|F| \geq r$, we can assume that $|B_2 \cap F| \geq 2$. Now apply (83.4) to the contraction H' of H to $B_1 \cup B_2$. Then $r_{\min}(H') \geq 2$. So, by (83.4), $B_1 \cup B_2$ can be partitioned into two vertex covers of H', hence of H. Replacing B_1, B_2 by B_1', B_2' gives a partition of V into vertex covers of H' thereby increasing the number of them intersecting F. This contradicts our assumption. ∎

Since the incidence matrix of a bipartite graph is balanced, Theorem 83.1 generalizes several theorems of Kőnig, like Kőnig's matching theorem (Theorem 16.2), the Kőnig-Rado edge cover theorem (Theorem 19.4), and Kőnig's edge-colouring theorem (Theorem 20.1).

Theorem 83.1 implies some more extensive characterizations (cf. Fulkerson, Hoffman, and Oppenheim [1974], Berge [1980]):

Corollary 83.1a. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(83.5)
(i) *H is balanced;*
(ii) *$\tau^*(H') \in \mathbb{Z}$ for each partial subhypergraph H' of H;*
(iii) *each partial hypergraph of H is ideal;*
(iv) *each partial hypergraph of H is Mengerian;*
(v) *the blocker of each partial hypergraph of H is Mengerian;*
(vi) *$\alpha^*(H') \in \mathbb{Z}$ for each partial subhypergraph H' of H;*
(vii) *each partial hypergraph of H is perfect.*

Proof. We know the implications (iv)⇒(iii) (Section 79.1), (iii)⇒(ii) (Corollary 78.4b), (vii)⇒(vi) (Theorem 82.4), and (v)⇒(iii) (Theorem 78.1). Since the class of balanced hypergraphs is closed under parallelization, (i)⇒(ii) and (i)⇒(iii) in Theorem 83.1 give (i)⇒(iv) and (i)⇒(vii) in (83.5). Also the class of blockers of balanced hypergraphs is closed under parallelization (as the class of balanced matrices is closed under duplicating columns); so (i)⇒(iv) in Theorem 83.1 gives (i)⇒(v) in (83.5).

So it suffices to show (ii)⇒(i) and (vi)⇒(i). Suppose that H is not balanced. Let $U \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$ induce a partial subhypergraph that is an odd circuit. We can assume that $U = V$ and $\mathcal{E}' = \mathcal{E}$. Then $\frac{1}{2} \cdot \mathbf{1}$ is a minimum-size vertex cover and a maximum-size stable set of H, and hence $\tau^*(H)$ and $\alpha^*(H)$ are noninteger. ▮

These characterizations imply that certain linear programs have integer optimum solutions (taking $\infty \cdot 0 = 0$):

Corollary 83.1b. *For any $\{0,1\}$-valued $m \times n$ matrix M, the following are equivalent:*

(83.6)

(i) *M is balanced;*

(ii) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \{0,1\}^n : \min\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \geq w^\mathsf{T}\}$ *has an integer optimum solution y;*

(iii) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \geq w^\mathsf{T}\}$ *has an integer optimum solution y;*

(iv) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{0,1\}^n : \min\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \geq w^\mathsf{T}\}$ *has an integer optimum solution y;*

(v) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \{0,1\}^n : \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*

(vi) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*

(vii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{0,1\}^n : \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*

(viii) $\forall b \in \{0,1\}^m \;\; \forall w \in \{1,\infty\}^n : \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x;*

(ix) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{1,\infty\}^n : \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x.*

(x) $\forall b \in \{0,1\}^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x;*

(xi) $\forall b \in \{0,1\}^m \;\; \forall w \in \{1,\infty\}^n : \max\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \leq w^\mathsf{T}\}$ *has an integer optimum solution y;*

(xii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{1,\infty\}^n : \max\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \leq w^\mathsf{T}\}$ *has an integer optimum solution y;*

(xiii) $\forall b \in \{0,1\}^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \leq w^\mathsf{T}\}$ *has an integer optimum solution y.*

Proof. Observe that each of (ii)-(vii) is equivalent to each of (viii)-(xiii), respectively, after replacing M by M^{T}. The implications (iii)⇒(ii)⇒(i), (iv)⇒(ii), (vi)⇒(v)⇒(i), and (vii)⇒(v) are direct. Here we use that (ii) and (v) are closed under taking submatrices, and that the incidence matrix of an odd circuit does not satisfy (ii) and (v) for $b = \mathbf{1}$ and $w = \mathbf{1}$.

Finally, (i)⇒(iii) and (i)⇒(vi) follow from (83.5)(i)⇒(vii), (i)⇒(x) (hence (i)⇒(iv)) follows from (83.5)(i)⇒(iii), and (i)⇒(xiii) (hence (i)⇒(vii)) follows from (83.5)(i)⇒(iv). ∎

Berge [1970] gave the following further characterization:

(83.7) a hypergraph is balanced $\iff$ each partial subhypergraph is bicolourable,

where a hypergraph is *bicolourable* if its vertex set can be coloured with two colours such that each edge of size at least 2 gets both colours. While ⇐ in (83.7) is easy, ⇒ can be shown with the proof of (83.4).

More generally, Theorem 83.1 gives the following generalization of Theorem 20.6 for bipartite graphs (Berge [1973b]):

Corollary 83.1c. *Let $H = (V, \mathcal{E})$ be a balanced hypergraph and let $k \in \mathbb{Z}_+$. Then V can be partitioned into $V_1, \ldots, V_k$ such that each $E \in \mathcal{E}$ is intersected by $\min\{k, |E|\}$ of the V_i.*

Proof. Choose $F \in \mathcal{E}$. By induction on $|\mathcal{E}|$, there is a partition $V_1, \ldots, V_k$ of V such that each $E \in \mathcal{E}$ with $E \neq F$ is intersected by $\min\{k, |E|\}$ of the V_i. Choose the partition such that F is intersected by a maximum number of the V_i. If F is not intersected by $\min\{k, |F|\}$ of the V_i, there exist V_i, V_j with $V_i \cap F = \emptyset$ and $|V_j \cap F| \geq 2$. The hypergraph H' obtained from H by contracting $V \setminus (V_i \cup V_j)$ and after that deleting all edges of size ≤ 1, has $r_{\min}(H') \geq 2$. Hence by Theorem 83.1, $\nu(b(H')) \geq 2$, that is $V_i \cup V_j$ can be partitioned into two vertex covers V_i' and V_j' of H'. Then replacing V_i, V_j by V_i', V_j' increases the number of intersections with F, a contradiction. ∎

Another consequence was given by Conforti, Cornuéjols, Kapoor, and Vušković [1996]. Call a matching $\mathcal{M}$ in a hypergraph $H = (V, \mathcal{E})$ *perfect* if $\mathcal{M}$ covers all vertices — that is, if $\mathcal{M}$ is a partition of V.

Corollary 83.1d. *Let $H = (V, \mathcal{E})$ be a balanced hypergraph. Then H has a perfect matching if and only if there are no disjoint vertex sets B, R with $|B| > |R|$ and $|B \cap E| \leq |R \cap E|$ for each $E \in \mathcal{E}$.*

Proof. Necessity is easy, since if $\mathcal{M}$ is a perfect matching, then

(83.8) $$|B| = \sum_{E \in \mathcal{M}} |B \cap E| \leq \sum_{E \in \mathcal{M}} |R \cap E| = |R|.$$

To see sufficiency, let M be the $\mathcal{E} \times V$ incidence matrix of H. Suppose that H has no perfect matching. Since $\{y \geq \mathbf{0} \mid y^{\mathsf{T}} M \leq \mathbf{1}^{\mathsf{T}}\}$ is an integer polytope (by (83.6)(xii)), it implies that there is no vector $y \geq \mathbf{0}$ with $y^{\mathsf{T}} M = \mathbf{1}^{\mathsf{T}}$. Hence, by Farkas' lemma, there is an x with $Mx \geq \mathbf{0}$ and $\mathbf{1}^{\mathsf{T}} x < 0$. We can assume $-\mathbf{1} \leq x \leq \mathbf{1}$. Set $z := \mathbf{1} - x$. Then $\mathbf{0} \leq z \leq \mathbf{2}$, $Mz \leq M\mathbf{1}$, and $\mathbf{1}^{\mathsf{T}} z > \mathbf{1}^{\mathsf{T}} \mathbf{1}$. By (83.6)(vii), applied to the balanced matrix

$$(83.9) \qquad \begin{pmatrix} I \\ M \end{pmatrix},$$

we can assume that z is integer. Hence we can assume that x is integer and $-\mathbf{1} \leq x \leq \mathbf{1}$. Then $B := \{v \in V \mid x_v = -1\}$ and $R := \{v \in V \mid x_v = +1\}$ contradict the condition of the corollary. ∎

A combinatorial proof of this theorem was given by Huck and Triesch [2002].

83.2a. Totally balanced matrices

A 0,1 matrix is called *totally balanced* if it has no submatrix that is the incidence matrix of a circuit of length at least 3. Obviously, each totally balanced matrix is balanced.

Totally balanced matrices have several nice properties so as to apply 'perfect elimination' and 'greedy' methods when solving optimization problems. They might be considered as the bipartite analogue of chordal graphs.

Call a bipartite graph *totally balanced* (or *chordal bipartite*) if it has no chordless circuit of length at least 6. So a 0,1 matrix is totally balanced if and only if the associated bipartite graph is totally balanced. (The bipartite graph *associated* to an $m \times n$ matrix M is the bipartite graph with colour classes $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, where u_i and v_j are adjacent if and only if $M_{i,j} \neq 0$.)

The first important property of totally balanced matrices was found by Golumbic and Goss [1978]. Call an entry M_{i_0,j_0} of a $\{0,1\}$-valued $m \times n$ matrix M *simplicial* if

(83.10) (i) $M_{i_0,j_0} = 1$,
(ii) for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$: if $M_{i_0,j} = M_{i,j_0} = 1$, then $M_{i,j} = 1$.

Theorem 83.2. *Each nonzero totally balanced matrix M has a simplicial entry.*

Proof. Let $G = (V, E)$ be the bipartite graph associated to M, with colour classes U and W. To prove that M has a simplicial entry, we must show that G has an edge uw such that each vertex in $N(u)$ is adjacent to each vertex in $N(w)$.

We can assume that G is not a complete bipartite graph, since otherwise M trivially has a simplicial entry. Choose an inclusionwise maximal nonempty set $X \subseteq V$ such that

(83.11) the subgraph $G[X]$ of G induced by X is connected and G has an edge disjoint from $X \cup N(X)$.

Such a set X exists, since $X := \{u\}$ satisfies (83.11) for any vertex u that is isolated or (if no isolated vertices exist) any vertex $u \in U$ nonadjacent to at least one vertex in W.

Define $Z := V \setminus (X \cup N(X))$. The maximality of X gives:

(83.12) each vertex y in $N(X)$ is adjacent to one of the ends of any edge contained in Z,

since otherwise we can add y to X without violating (83.11), contradicting the maximality of X.

Also we have:

(83.13) each vertex in $N(X) \cap U$ is adjacent to each vertex in $N(X) \cap W$.

For choose $y \in N(X) \cap U$ and $z \in N(X) \cap W$. Let uw be an edge in Z, with $u \in U$ and $w \in W$. As $G[X]$ is connected, there is a path P in $G[X]$ connecting $N(y)$ and $N(z)$. Choose P shortest. Then y, P, z, u, w, y is a circuit of length at least 6 in G. Hence it has a chord. It cannot connect $\{u, w\}$ and P, since $u, w \notin N(X)$. So it is a chord of the path y, P, z. Since P is shortest, it follows that y and z are adjacent. This proves (83.13).

Now by induction we know that Z contains an edge uw such that $N(\{u, w\}) \cap Z$ induces a complete bipartite graph. Then (83.12) and (83.13) imply that $N(\{u, w\})$ induces a complete bipartite graph. ∎

Most of the properties of totally balanced matrices (including that described in the theorem above, which however is used in the proof) follow from the next theorem, saying that the rows and columns of a totally balanced matrix can be permuted such that it has no submatrix

(83.14) $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(in this order). Following Lubiw [1982], we call such a matrix *Γ-free*. In other words, M is Γ-free if for all row indices $i < i'$ and column indices $j < j'$ one has

(83.15) if $M_{i,j} = M_{i',j} = M_{i,j'} = 1$, then $M_{i',j'} = 1$.

The following was shown by Hoffman, Kolen, and Sakarovitch [1985] and Lubiw [1982]:

Theorem 83.3. *The rows and columns of a totally balanced matrix M can be permuted such that the matrix becomes Γ-free.*

Proof. We apply induction on the number of nonzero entries of M. If M is all-zero, the theorem is trivial. So we can assume that M has at least one nonzero entry. By Theorem 83.2, M has a simplicial entry M_{i_0,j_0}.

Reset M_{i_0,j_0} to 0, to obtain matrix $\widetilde{M}$. Then $\widetilde{M}$ is again totally balanced. For suppose that $\widetilde{M}$ has a submatrix C that is the incidence matrix of a circuit of length ≥ 3. Since M is totally balanced, C contains the entry $\widetilde{M}_{i_0,j_0}$. Row i_0 has two 1's in C and column j_0 has two 1's in C. Hence, by (83.10), C has a row with three 1's, a contradiction.

So $\widetilde{M}$ is totally balanced again. By induction, we can permute the rows and columns of $\widetilde{M}$ such that it becomes Γ-free. We can assume that entry $\widetilde{M}_{i_0,j_0}$ of $\widetilde{M}$

has moved to position i_0, j_0. We can also assume that among all valid permutations, we have chosen one which minimizes $i_0 + j_0$. Then

(83.16) $\quad M_{i,j_0} = 0$ for each $i < i_0$ and $M_{i_0,j} = 0$ for each $j < j_0$.

For suppose that $M_{i,j_0} = 1$ for some $i < i_0$. By the minimality of $i_0 + j_0$, we cannot exchange rows i_0 and $i_0 - 1$ of $\widetilde{M}$ without violating Γ-freeness. Hence there exist j, j' with $j < j'$ with $M_{i_0,j} = M_{i_0,j'} = 1$ and $M_{i_0-1,j} = 1$, $M_{i_0-1,j'} = 0$. Since $M_{i_0,j} = M_{i_0,j'} = 1$ and $M_{i,j_0} = 1$ we know by (83.10) that $M_{i,j} = M_{i,j'} = 1$.

So $i \neq i_0 - 1$ and hence $i < i_0 - 1$. But then $M_{i,j} = M_{i,j'} = M_{i_0-1,j} = 1$ while $M_{i_0-1,j'} = 0$, contradicting the Γ-freeness of $\widetilde{M}$.

This proves (83.16). Then resetting the (i_0, j_0)th entry to its original value 1, the matrix remains Γ-free (by (83.10) and (83.16)). ∎

Call a hypergraph $H = (V, \mathcal{E})$ *totally balanced* if its incidence matrix is totally balanced. Call two sets X and Y *comparable* if $X \subseteq Y$ or $Y \subseteq X$. Then (Brouwer and Kolen [1980], Anstee and Farber [1984]):

Corollary 83.3a. *Each totally balanced hypergraph $H = (V, \mathcal{E})$ with $V \neq \emptyset$, has a vertex v such that any two edges containing v are comparable.*

Proof. By Theorem 83.3, we can assume that the incidence matrix M of H is Γ-free. Then the vertex of H corresponding to the first column of M is as required. ∎

Other consequences of Theorem 83.3 are algorithmic. It gives a good characterization of total balancedness. In fact, the method gives a polynomial-time algorithm to test total balancedness: we iteratively find a simplicial entry and set it to 0. If we succeed in this until the matrix is all-zero, the original matrix is totally balanced, and otherwise not.

Lubiw [1982] gave the following simple algorithm to permute the rows and columns of a totally balanced matrix such that it becomes Γ-free. Iteratively, choose a column j such that the supports of the rows with a 1 in column j form a chain, and remove column j. The order in which we remove the columns, gives the permutation of the columns. Next order the rows lexicographically (reading from right to left). The final matrix is Γ-free. (Hoffman, Kolen, and Sakarovitch [1985] gave an $O(nm^2)$-time algorithm to transform a totally balanced $m \times n$ matrix to a Γ-free matrix, speeded up by Paige and Tarjan [1987] and Spinrad [1993].)

Also, if A is a nonsingular matrix whose support is totally balanced, then we can solve a system $Ax = b$ of linear equalities with Gaussian elimination, by repeatedly choosing a simplicial entry and pivoting on it. If we create no 0's 'by accident', then we can keep pivoting on simplicial entries throughout the process (since then we never change any zero entry to nonzero). So if the initial matrix is sparse, it remains sparse during the Gaussian elimination.

Lubiw [1982], Farber [1984], and Hoffman, Kolen, and Sakarovitch [1985] gave polynomial-time algorithms for optimization problems over a totally balanced matrix.

Call a hypergraph $H = (V, \mathcal{E})$ a *tree-hypergraph* if V is the vertex set of a tree T and each edge $E \in \mathcal{E}$ of H induces a subtree of T. Lehel [1985] showed that a hypergraph H is totally balanced if and only if each contraction of H is a tree-hypergraph.

Lubiw [1982] showed that for any totally balanced hypergraph $H = (V, \mathcal{E})$, its intersection matrix (the $\{0,1\}$-valued $\mathcal{E} \times \mathcal{E}$ matrix N with $N_{E,E'} = 1$ if and only if $E \cap E' \neq \emptyset$) is totally balanced.

Chvátal [1993] pointed out that a bipartite graph is totally balanced ($\equiv$ chordal bipartite) if and only if its complementary graph is perfectly orderable (cf. Hoàng [1996a]). More results on totally balanced matrices are reported by Golumbic [1980], Lubiw [1982,1987], Anstee and Farber [1984], and Dragan and Voloshin [1996], and applications by Tamir [1987].

83.2b. Examples of balanced hypergraphs

A graph $G = (V, E)$ is balanced if and only if it is bipartite. This follows directly from the definition of balancedness.

A second example was given by Frank [1977]. Let $D = (V, A)$ be a rooted tree. Let $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_n$ be directed paths in D. Define the $m \times n$ matrix M by:

(83.17) $$M_{i,j} := \begin{cases} 1 & \text{if } VP_i \cap VQ_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then M is a balanced matrix, as one easily checks. M need not be totally unimodular, as example (83.22) below shows. As Lubiw [1982] observed, these matrices are even totally balanced. The fact that for the corresponding hypergraphs $\alpha(H) = \rho(H)$ and (equivalently) $\nu(H) = \tau(H)$ hold was shown by Meir and Moon [1975]. Related results can be found in Slater [1977].

A third example was given by Giles [1978a]. Let $G = (V, E)$ be an (undirected) tree. For each $a : V \to \mathbb{Z}_+$, define

(83.18) $$U_v := \{u \in V \mid \mathrm{dist}_G(v, u) \leq a_v\}.$$

Then $(V, \{U_v \mid v \in V\})$ is a balanced hypergraph. Lubiw [1982] showed that these hypergraphs are in fact totally balanced.

83.2c. Balanced $0, \pm 1$ matrices

Truemper [1982] extended the concept of balancedness to $0, \pm 1$ matrices: A $0, \pm 1$ matrix is *balanced* if in each square submatrix with precisely two nonzeros in each row and in each column, the sum of the entries is a multiple of 4.

Most of the results described above for balanced $0, 1$ matrices, can be extended to $0, \pm 1$ matrices. Conforti and Cornuéjols [1995b] showed that for any balanced $0, \pm 1$ matrix M the following systems are TDI, and hence determine an integer polytope:

(83.19) $$\mathbf{0} \leq x \leq \mathbf{1}, Mx \leq \mathbf{1} - b,$$

and

(83.20) $$\mathbf{0} \leq x \leq \mathbf{1}, Mx \geq \mathbf{1} - b,$$

where b is the vector with b_i equal to the number of negative entries in the ith row of M. So balanced matrices are both perfect and ideal. By requiring this for each submatrix, each of this characterizes balancedness.

Conforti and Cornuéjols [1995b] also proved a bicolouring theorem extending Corollary 83.1c:

(83.21) the columns of a balanced $0, \pm 1$ matrix M can be split into two sets such that each row of M with at least two nonzeros, has nonzero entries of the same sign in both sets, or of opposite signs in one of the two sets.

Again, by requiring this for each submatrix, this characterizes balancedness.

Finally, the decomposition results and algorithms for balanced $0, 1$ matrices were extended to $0, \pm 1$ matrices by Conforti, Cornuéjols, Kapoor, and Vušković [1994,2001a,2001b]. For surveys, see Conforti, Cornuéjols, Kapoor, Vušković, and Rao [1994], Conforti and Cornuéjols [2001], and Cornuéjols [2001].

83.3. Unimodular hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *unimodular* if its incidence matrix M is totally unimodular; that is, each square submatrix of M has determinant 0, $+1$, or -1.

Since the incidence matrix of an odd circuit has determinant ± 2, each unimodular hypergraph is balanced. Not every balanced hypergraph is unimodular, as is shown by the hypergraph with incidence matrix

(83.22)
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Trivially, the dual of a unimodular hypergraph is again unimodular. Also, contracting vertices and deleting edges maintain unimodularity of a hypergraph.

For graphs (that is, hypergraphs with each edge of size 2), the concept of unimodular coincides with bipartite.

Characterizations of totally unimodular matrices imply corresponding characterizations of unimodular hypergraphs. We describe some of them in the following theorem. (The equivalence of (i)-(vii) is due to Hoffman and Kruskal [1956], characterization (viii) to Ghouila-Houri [1962b], characterization (ix) to Camion [1963,1965], and characterization (x) to R.E. Gomory (cf. Camion [1965]).)

For the proof we refer to Chapter 19 of Schrijver [1986b].

Theorem 83.4. *Let $H = (V, \mathcal{E})$ be a hypergraph, with incidence matrix M. Then the following are equivalent:*

(83.23) (i) *H is unimodular, that is, each square submatrix of M has determinant in $\{0, \pm 1\}$;*
(ii) *for each $b \in \mathbb{Z}_+^{\mathcal{E}}$, the polyhedron $\{x \geq \mathbf{0} \mid Mx \leq b\}$ is integer;*
(iii) *for each $b \in \mathbb{R}_+^{\mathcal{E}}$, the system $x \geq \mathbf{0}$, $Mx \leq b$ is totally dual integral;*
(iv) *for each $b \in \mathbb{Z}_+^{\mathcal{E}}$, the polyhedron $\{x \geq \mathbf{0} \mid Mx \geq b\}$ is integer;*
(v) *for each $b \in \mathbb{R}_+^{\mathcal{E}}$, the system $x \geq \mathbf{0}$, $Mx \geq b$ is totally dual integral;*
(vi) *for all $a, b \in \mathbb{Z}^{\mathcal{E}}$ and $c, d \in \mathbb{Z}^V$, the polyhedron $\{x \mid c \leq x \leq d, a \leq Mx \leq b\}$ is integer;*
(vii) *for all $a, b \in \mathbb{R}^{\mathcal{E}}$ and $c, d \in \mathbb{R}^V$, the system $c \leq x \leq d, a \leq Mx \leq b$ is totally dual integral;*
(viii) *each $U \subseteq V$ can be partitioned into sets U_1 and U_2 such that each $E \in \mathcal{E}$ satisfies $\big||E \cap U_1| - |E \cap U_2|\big| \leq 1$;*
(ix) *the sum of the entries in any square submatrix of M with even row and column sums, is divisible by 4;*
(x) *no square submatrix of M has determinant ± 2.*

Proof. See Chapter 19 of Schrijver [1986b]. ∎

This implies a characterization similar to Corollary 83.1b:

Corollary 83.4a. *For any $\{0,1\}$-valued $m \times n$ matrix M, the following are equivalent:*

(83.24) (i) *M is totally unimodular;*
(ii) *$\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \geq w^\mathsf{T}\}$ has an integer optimum solution y;*
(iii) *$\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \leq b\}$ has an integer optimum solution x;*
(iv) *$\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \leq w^\mathsf{T}\}$ has an integer optimum solution y;*
(v) *$\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \geq b\}$ has an integer optimum solution x.*

Proof. From Theorem 83.4. ∎

Unimodular hypergraphs have the following property stronger than was shown for balanced hypergraphs in Corollary 83.1c:

Theorem 83.5. *Let $H = (V, \mathcal{E})$ be a unimodular matrix and let $k \in \mathbb{Z}_+$ with $k \geq 1$. Then V can be partitioned into sets $V_1, \ldots, V_k$ such that*

(83.25) $$\left\lfloor \frac{|E|}{k} \right\rfloor \leq |E \cap V_i| \leq \left\lceil \frac{|E|}{k} \right\rceil$$

for each $E \in \mathcal{E}$ and each $i = 1, \ldots, k$.

Proof. Choose $F \in \mathcal{E}$. By induction, there is a partition $V_1, \ldots, V_k$ as required for the hypergraph $H' := (V, \mathcal{E} \setminus \{F\})$. Choose the partition with

$$(83.26) \qquad \sum_{i=1}^{k} |F \cap V_i|^2$$

as small as possible. Suppose that (83.25) does not hold for $E := F$. Then there exist i and j such that

$$(83.27) \qquad |F \cap V_i| \geq |F \cap V_j| + 2.$$

Consider the contraction of H to $V_i \cup V_j$. By (83.23)(viii), we can split $V_i \cup V_j$ into V_i' and V_j' such that $\big||E \cap V_i'| - |E \cap V_j'|\big| \leq 1$ for each $E \in \mathcal{E}$. So replacing V_i, V_j by V_i', V_j' gives again a valid partition, but decreases the sum (83.26), a contradiction. ∎

A basic theorem of Seymour [1980a] states that each totally unimodular matrix can be decomposed into network matrices, their transposes, and two special 5×5 matrices. As J. Edmonds noted, it yields a polynomial-time test of total unimodularity of matrices, and hence of unimodularity of hypergraphs — Bixby [1982], Schrijver [1986b], and Truemper [1990,1992] described implementations.

83.3a. Further notes

Truemper and Chandrasekaran [1978] proved the following characterization, that includes the polyhedral characterizations of both the balanced and the totally unimodular matrices. For any pair of an $\{0, 1\}$-valued $m \times n$ matrix A and a vector $b \in \mathbb{Z}_+^n$, the following are equivalent:

(83.28)
- (i) the polyhedron $\{x \geq \mathbf{0} \mid A'x \leq d'\}$ is integer for each row submatrix A' of A and each integer vector d' with $\mathbf{0} \leq d' \leq b'$, where b' is the part of b corresponding to A';
- (ii) A has no square submatrix M with the following properties: $\det M = \pm 2$, each entry of M^{-1} is $\pm\frac{1}{2}$, and $M\mathbf{1} \leq 2b'$, where b' is the part of b corresponding to M.

For $b = \mathbf{1}$ this characterizes balanced matrices. For b sufficiently large, it characterizes total unimodularity. Related results can be found in Conforti, Cornuéjols, and Truemper [1994] and Conforti, Cornuéjols, and Zambelli [2002a].

Conforti and Rao [1992c] reduced testing if a hypergraph is balanced, to testing if some derived hypergraphs are perfect. Conforti and Rao [1993] gave a polynomial-time algorithm to test if a given hypergraph H is balanced, provided that any two edges of H intersect in at most one vertex. Related results can be found in Lubiw [1988] and Conforti and Rao [1989,1992d].

Berge and Hoffman [1978] gave a formula for the minimum number of stable vertex covers needed to cover the vertex set of a unimodular hypergraph. Dahlhaus, Kratochvil, Manuel, and Miller [1997] described a polynomial-time algorithm to find a maximum number of disjoint vertex covers of a balanced hypergraph.

Conforti and Cornuéjols [1995a] applied balanced matrices to logic problems. Conforti, Cornuéjols, and Vušković [1999] gave a linear-time algorithm to find a chordless circuit in a bipartite graph of length $\equiv 0 \pmod 4$.

Survey of Problems, Questions, and Conjectures

We here collect unsolved problems, questions, and conjectures mentioned in this book. For terminology and background, we refer to the pages indicated.

1 (page 41). Is NP$\neq$P?

2 (page 42). Is P=NP$\cap$co-NP?

3 (page 65). The *Hirsch conjecture*: A d-dimensional polytope with m facets has diameter at most $m - d$.

4 (page 161). Is there an $O(nm)$-time algorithm for finding a maximum flow?

5 (page 232). Berge [1982b] posed the following conjecture generalizing the Gallai-Milgram theorem. Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then for each path collection $\mathcal{P}$ partitioning V and minimizing

$$(1) \qquad \sum_{P \in \mathcal{P}} \min\{|VP|, k\},$$

there exist disjoint stable sets $C_1, \ldots, C_k$ in D such that each $P \in \mathcal{P}$ intersects $\min\{|VP|, k\}$ of them. This was proved by Saks [1986] for acyclic graphs.

6 (page 403). The following open problem was mentioned by Fulkerson [1971b]: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S and let $w \in \mathbb{Z}_+^S$. What is the maximum number k of common transversals $T_1, \ldots, T_k$ of $\mathcal{A}$ and $\mathcal{B}$ such that

$$(2) \qquad \chi^{T_1} + \cdots + \chi^{T_k} \leq w?$$

7 (page 459). Can the weighted matching problem be formulated as a linear programming problem of size bounded by a polynomial in the size of the graph, by extending the set of variables? That is, is the matching polytope of a graph $G = (V, E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with A and b having size bounded by a polynomial in $|V| + |E|$?

8 (pages 472,646). The 5-*flow conjecture* of Tutte [1954a]:

(3) (?) each bridgeless graph has a nowhere-zero 5-flow. (?)

(A *nowhere-zero k-flow* is a flow over $\mathbb{Z}_k$ in some orientation of the graph, taking value 0 nowhere.)

9 (pages 472,498,645,1426). The 4-*flow conjecture* of Tutte [1966]:

(4) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow. (?)

This implies the four-colour theorem. For cubic graphs, (4) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Seymour [1981c] showed that the 4-flow conjecture is equivalent to the following more general conjecture, also due to Tutte [1966]:

(5) (?) each bridgeless matroid without F_7^*, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor has a nowhere-zero flow over GF(4). (?)

Here $\mathbf{P}_{10}$ denotes the Petersen graph.

10 (page 472). The 3-*flow conjecture* (W.T. Tutte, 1972 (cf. Bondy and Murty [1976], Unsolved problem 48)):

(6) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

11 (page 473). The *weak* 3-*flow conjecture* of Jaeger [1988]:

(7) (?) there exists a number k such that each k-edge-connected graph has a nowhere-zero 3-flow. (?)

12 (page 473). The following *circular flow conjecture* of Jaeger [1984] generalizes both the 3-flow and the 5-flow conjecture:

(8) (?) for each $k \geq 1$, any $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k+1$. (?)

13 (pages 475,645). The *generalized Fulkerson conjecture* of Seymour [1979a]:

(9) (?) $\lceil \chi'^*(G) \rceil = \lceil \frac{1}{2}\chi'(G_2) \rceil$ (?)

for each graph G. (Here $\chi'^*(G)$ denotes the fractional edge-colouring number of G, and G_2 the graph obtained from G by replacing each edge by two parallel edges.) This is equivalent to the conjecture that

(10) (?) for each k-graph G there exists a family of $2k$ perfect matchings, covering each edge precisely twice. (?)

(A *k-graph* is a k-regular graph $G = (V, E)$ with $|\delta(U)| \geq k$ for each odd-size subset U of V.)

14 (pages 476,645). Fulkerson [1971a] asked if in each bridgeless cubic graph there exist 6 perfect matchings, covering each edge precisely twice (the *Fulkerson conjecture*). It is a special case of Seymour's generalized Fulkerson conjecture.

15 (page 476). Berge [1979a] conjectures that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. (This would follow from the Fulkerson conjecture.)

16 (page 476). Gol'dberg [1973] and Seymour [1979a] conjecture that for each (not necessarily simple) graph G one has

(11) $$(?)\ \chi'(G) \leq \max\{\Delta(G)+1, \lceil \chi'^{*}(G)\rceil\}.\ (?)$$

An equivalent conjecture was stated by Andersen [1977].

17 (page 476). Seymour [1981c] conjectures the following generalization of the four-colour theorem:

(12) (?) each planar k-graph is k-edge-colourable. (?)

For $k = 3$, this is equivalent to the four-colour theorem. For $k = 4$ and $k = 5$, it was derived from the case $k = 3$ by Guenin [2002b].

18 (pages 476,644). Lovász [1987] conjectures more generally:

(13) (?) each k-graph without Petersen graph minor is k-edge-colourable. (?)

This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base.

19 (page 481). The following question was asked by Vizing [1968]: Is there a simple planar graph of maximum degree 6 and with edge-colouring number 7?

20 (page 481). Vizing [1965a] asked if a minimum edge-colouring of a graph can be obtained from an arbitrary edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

21 (page 482). Vizing [1976] conjectures that the list-edge-colouring number of any graph is equal to its edge-colouring number.

(The *list-edge-colouring number* $\chi^{l}(G)$ of a graph $G = (V, E)$ is the minimum number k such that for each choice of sets L_e for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges e, f one has $l_e \neq l_f$.)

22 (page 482). Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph G is at most $\Delta(G)+2$. (The *total colouring* *number* of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint.)

23 (page 482). More generally, Vizing [1968] conjectures that the total colouring number of a graph G is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum edge multiplicity of G.

24 (pages 497,645). Seymour [1979b] conjectures that each even integer vector in the circuit cone of a graph is a nonnegative integer combination of incidence vectors of circuits.

25 (pages 497,645,1427). A special case of this is the *circuit double cover conjecture* (asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them.

Jamshy and Tarsi [1989] proved that the circuit double cover conjecture is equivalent to a generalization to matroids:

(14) (?) each bridgeless binary matroid without F_7^* minor has a circuit double cover. (?)

26 (page 509). Is the system of T-join constraints totally dual quarter-integral?

27 (page 517). L. Lovász asked for the complexity of the following problem: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$ such that each circuit has nonnegative length, find a shortest odd $s - t$ path.

28 (page 545). What is the complexity of deciding if a given graph has a 2-factor without circuits of length at most 4?

29 (page 545). What is the complexity of finding a maximum-weight 2-factor without circuits of length at most 3?

30 (page 646). Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

(15) (?) in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice. (?)

31 (page 657). Is the dual of any algebraic matroid again algebraic?

32 (page 892). A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

(16) (?) Let $G = (V, E)$ be an undirected graph and let $s \in V$. Suppose that for each vertex $t \neq s$, there exist k internally vertex-disjoint $s - t$ paths. Then G has k spanning trees such that for each vertex

$t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint. (?)

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (16) was proved by Itai and Rodeh [1984, 1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

33 (page 962). Can a maximum number of disjoint directed cut covers in a directed graph be found in polynomial time?

34 (page 962). Woodall [1978a,1978b] conjectures (*Woodall's conjecture*):

(17) (?) In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers. (?)

35 (page 985). Let $G = (V, E)$ be a complete undirected graph, and consider the system

(18)
$$\begin{aligned} &0 \leq x_e \leq 1 \quad \text{for each edge } e, \\ &x(\delta(v)) = 2 \quad \text{for each vertex } v, \\ &x(\delta(U)) \geq 2 \text{ for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{aligned}$$

Let $l : E \to \mathbb{R}_+$ be a length function. Is the minimum length of a Hamiltonian circuit at most $\frac{4}{3}$ times the minimum value of $l^{\mathsf{T}}x$ over (18)?

36 (page 990). Padberg and Grötschel [1985] conjecture that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 2.

37 (page 1076). Frank [1994a] conjectures:

(19) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a k-vertex-connector for D is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$. (?)

(A *k-vertex-connector* for D is a set of (new) arcs whose addition to D makes it k-vertex-connected.)

38 (page 1087). *Hadwiger's conjecture* (Hadwiger [1943]): If $\chi(G) \geq k$, then G contains K_k as a minor.

Hadwiger's conjecture is trivial for $k = 1, 2, 3$, was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]), is equivalent to the four-colour theorem for $k = 5$ (by a theorem of Wagner [1937a]), and was derived from the four-colour theorem for $k = 6$ by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, the conjecture is unsettled.

39 (page 1099). Chvátal [1973a] asked if for each fixed t, the stable set problem for graphs for which the stable set polytope arises from $P(G)$ by at most

t rounds of cutting planes, is polynomial-time solvable. Here $P(G)$ is the polytope determined by the nonnegativity and clique inequalities.

40 (page 1099). Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph G with n vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from the system defining $Q(G)$ by adding at most $p(n)$ cutting planes. Here $Q(G)$ is the polytope determined by the nonnegativity and edge inequalities. (This conjecture would be implied by NP$\neq$co-NP.)

41 (page 1105). Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\chi(G) \leq g(\omega(G))$ for each graph G without odd holes.

42 (page 1107). Can perfection of a graph be tested in polynomial time?

43 (page 1131). Berge [1982a] conjectures the following. A directed graph $D = (V, A)$ is called *α-diperfect* if for every induced subgraph $D' = (V', A')$ and each maximum-size stable set S in D' there is a partition of V' into directed paths each intersecting S in exactly one vertex. Then for each directed graph D:

(20) (?) D is α-diperfect if and only if D has no induced subgraph C whose underlying undirected graph is a chordless odd circuit of length ≥ 5, say with vertices $v_1, \ldots, v_{2k+1}$ (in order) such that each of $v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k}$ is a source or a sink. (?)

44 (page 1170). Is $\vartheta(C_n) = \Theta(C_n)$ for each odd n?

45 (page 1170). Can Haemers' bound $\eta(G)$ on the Shannon capacity of a graph G be computed in polynomial time?

46 (page 1187). Is every t-perfect graph strongly t-perfect?

Here a graph is *t-perfect* if its stable set polytope is determined by the nonnegativity, edge, and odd circuit constraints. It is *strongly t-perfect* if this system is totally dual integral.

47 (page 1195). T-perfection is closed under taking induced subgraphs and under contracting all edges in $\delta(v)$ where v is a vertex not contained in a triangle. What are the minimally non-t-perfect graphs under this operation?

48 (page 1242). For any k, let $f(k)$ be the smallest number such that in any $f(k)$-connected undirected graph, for any choice of distinct vertices $s_1, t_1, \ldots, s_k, t_k$ there exist vertex-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

49 (page 1242). For any k, let $g(k)$ be the smallest number such that in any $g(k)$-edge-connected undirected graph, for any choice of vertices $s_1, t_1, \ldots, s_k, t_k$ there exist edge-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $g(k) = k$ if k is odd and $g(k) = k + 1$ if k is even.

50 (page 1243). What is the complexity of the k arc-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$? This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

51 (page 1274). Karzanov [1991] conjectures that if the nets in a multiflow problem form two disjoint triangles and if the capacities and demands are integer and satisfy the Euler condition, then the existence of a fractional multiflow implies the existence of a half-integer multiflow.

52 (page 1274). The previous conjecture implies that for each graph $H = (T, R)$ without three disjoint edges, there is an integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and any $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{k}$-integer multiflow.

53 (page 1276). Okamura [1998] conjectures the following. Let $G = (V, E)$ be an l-edge-connected graph (for some l). Let $H = (T, R)$ be a 'demand' graph, with $T \subseteq V$, such that $d_R(U) \leq l$ for each $U \subseteq V$. Then the edge-disjoint paths problem has a half-integer solution.

54 (page 1293). Is each Mader matroid a gammoid?

55 (page 1294). Is each Mader matroid linear?

56 (page 1299). Is the undirected edge-disjoint paths problem for planar graphs polynomial-time solvable if all terminals are on the outer boundary? Is it NP-complete?

57 (page 1310). Is the integer multiflow problem polynomial-time solvable if the graph and the nets form a planar graph such that the nets are spanned by a fixed number of faces?

58 (page 1310). Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if the graph $G + H$ (the union of the supply graph and the demand graph) is embeddable in the torus and there exists a quarter-integer solution.

59 (page 1320). Let $G = (V, E)$ be a planar bipartite graph and let q be a vertex on the outer boundary. Do there exist disjoint cuts $C_1, \ldots, C_p$ such that any pair s, t of vertices with s and t on the outer boundary, or with $s = q$, is separated by $\text{dist}_G(s, t)$ cuts?

60 (page 1345). Fu and Goddyn [1999] asked: Is the class of graphs for which the incidence vectors of cuts form a Hilbert base, closed under taking minors?

61 (page 1382). Füredi, Kahn, and Seymour [1993] conjecture that for each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

$$\sum_{F\in\mathcal{M}} \big(|F|-1+\frac{1}{|F|}\big)w(F) \geq \nu_w^*(H), \tag{21}$$

where $\nu_w^*(H)$ is the maximum weight $w^\mathsf{T}y$ of a fractional matching $y:\mathcal{E}\to\mathbb{R}_+$.

62 (pages 1387,1408). Seymour [1981a] conjectures:

(22) (?) a binary hypergraph is ideal if and only if it has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, or F_7 minor. (?)

63 (page 1392). Seymour [1990b] asked the following. Suppose that $H = (V,\mathcal{E})$ is a hypergraph without J_n minor ($n \geq 3$). Let $l, w : V \to \mathbb{Z}_+$ be such that

$$\tau(H^w)\cdot\tau(b(H)^l) > l^\mathsf{T}w. \tag{23}$$

Is there a minor H' of H and $l', w' : VH' \to \{0,1\}$ such that

$$\tau((H')^{w'})\cdot\tau(b(H')^{l'}) > l'^\mathsf{T}w' \tag{24}$$

and such that $\tau((H')^{w'}) \leq \tau(H^w)$ and $\tau(b(H')^{l'}) \leq \tau(b(H)^l)$?

Here, for each $n\geq 3$: $J_n :=$ the hypergraph with vertex set $\{1,\ldots,n\}$ and edges $\{2,\ldots,n\}$, $\{1,2\},\ldots,\{1,n\}$.

64 (page 1392). Seymour [1990b] also asked the following. Let $H = (V,\mathcal{E})$ be a nonideal hypergraph. Is the minimum of $\tau(H')$ over all parallelizations and minors H' of H with $\tau^*(H') < \tau(H')$ attained by a minor of H?

65 (page 1395). Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs H with $r_{\min}(H) > 2$ and $\tau(H) > 2$.

66 (page 1396). Ding [1993] asked whether there exists a number t such that each minimally nonideal hypergraph H satisfies $r_{\min}(H) \leq t$ or $\tau(H) \leq t$.

(The above conjecture of Cornuéjols and Novick [1994] implies a positive answer to this question.)

67 (page 1396). Ding [1993] conjectures that for each fixed $k \geq 2$, each minor-minimal hypergraph H with $\tau_k(H) < k\cdot\tau(H)$, contains some J_n minor ($n\geq 3$) or satisfies the regularity conditions of Lehman's theorems (Theorem 78.4 and 78.5).

68 (page 1401). Conforti and Cornuéjols [1993] conjecture:

(25) (?) a hypergraph is Mengerian if and only if it is packing. (?)

69 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture:

(26) (?) each minimally nonideal hypergraph H with $r_{\min}(H)\tau(H) = |VH|+1$ is minimally nonpacking. (?)

70 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture that $\tau(H) = 2$ for each ideal minimally nonpacking hypergraph H.

71 (page 1404). Seymour [1981a] conjectures that T_{30} is the unique minor-minimal binary ideal hypergraph H with the property $\nu_2(H) < 2\tau(H)$.

Here the hypergraph T_{30} arises as follows. Replace each edge of the Petersen graph by a path of length 2, making the graph G. Let $T := VG \setminus \{v\}$, where v is an arbitrary vertex of v of degree 3. Let $\mathcal{E}$ be the collection of T-joins. Then $T_{30} := (EG, \mathcal{E})$.

72 (page 1405). P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph H there exists an integer k such that $\nu_k(H) = k \cdot \tau(H)$ and such that $k = 2^i$ for some i. He also asks if $k = 4$ would do in all cases.

73 (page 1405). Seymour [1979a] conjectures that for each ideal hypergraph H, the g.c.d. of those k with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

74 (page 1409). Is the following true for binary hypergraphs H:

(27) (?) $\nu(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$ with $w(B)$ even for all $B \in b(H) \iff \frac{1}{2}\nu_2(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+ \iff$ H has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, F_7, or T_{15} minor. (?)

Here T_{15} is the hypergraph of $V\mathbf{P}_{10}$-joins in the Petersen graph $\mathbf{P}_{10}$.

75 (page 1421). Seymour [1981a] conjectures that for any binary matroid M:

(28) (?) M is 1-cycling $\iff$ M is 1-flowing $\iff$ M has no AG(3,2), T_{11}, or T_{11}^* minor. (?)

Here T_{11} is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, AG(3,2) is the matroid with 8 elements obtained from the 3-dimensional affine geometry over GF(2).

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Name Index

Subject Index

Bold numbers refer to pages where items are introduced.

Greek graph and hypergraph functions

This book offers an in-depth overview of polyhedral methods and efficient algorithms in combinatorial optimization. These techniques form a powerful, coherent, and unifying kernel in combinatorial optimization, with strong links to discrete mathematics, mathematical programming, and computer science.
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Volume A, Chapters 1–38 pp. 1–648

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