McGill University
Department of Mathematics and Statistics
MATH 254 Honours Analysis 1, Fall 2024
Assignment 4

You should carefully work out **all** problems. However, you only have to hand in solutions to **problem 5 and 7.**

This assignment is due **Thursday, October 3, at 8:00pm** and is to be submitted on Crowdmark. **Late assignments will not be accepted!**

1. Use the definition of the limit of a sequence to show that:

   (a) $\lim\left(\frac{n^2+n}{2n^2-3}\right)=\frac{1}{2}$ (b) $\lim\left(\sqrt{n^2+1}-n\right)=0$ (c) $\lim\left(\frac{\sqrt{n}+(-1)^n}{\sqrt{n}+1}\right)=1$

2. Let (x_n) be a convergent sequence with $\lim(x_n)=x>0$. Prove that there exists an $N\in\mathbb{N}$ such that $\frac{1}{2}x<x_n<2x$ for all $n\geq N$.

3. Prove that a sequence (a_n) of real numbers is a null sequence iff the sequence of absolute values $(|a_n|)$ is a null sequence.

4. Prove that if $x_n\geq 0$ for all $n\in\mathbb{N}$ and $\lim(x_n)=0$, then $\lim(\sqrt{x_n})=0$.

5. (a) Prove that $\lim\left(\frac{2^n}{n!}\right)=0$. Hint: Show that $0<\frac{2^n}{n!}\leq 2\left(\frac{2}{3}\right)^{n-2}$ for all $n\geq 2$.

   (b) Prove that $\lim\left(\frac{n!}{n^n}\right)=0$.

6. (a) Let $a>1$. Prove that $\lim\left(\frac{1}{a^n}\right)=0$. Hint: Use Bernoulli's inequality.

   (b) Let $0<a<1$. Prove that $\lim(a^n)=0$.

   (c) Let $-1<a<0$. Prove that $\lim(a^n)=0$.

7. Let $a>1$. Prove that $\lim\left(\frac{n}{a^n}\right)=0$.

   Remark: The proof is similar to the proof of question 6a, but requires an extra step. Hint: $a=(\sqrt{a})^2$.

8. Let $0<a<1$. Prove that $\lim(\sqrt[n]{a})=1$.