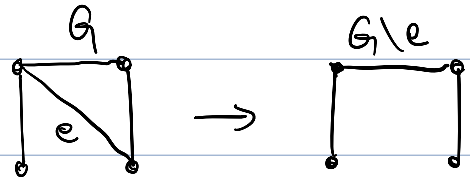
Lemma 2.5: Let G be a graph Let $H \subseteq G$ be non-null and connected. Then H is a connected componene of G iff for every $e \in E(G)$ with an end in $V(H)$ we have $e \in E(H)$

For $e \in E(G)$ a graph $G \backslash e$ obtained from G by deleting e is a graph $V(G\backslash e) = V(G)$ & $E(G) = E(G) - \{e\}$



For $v \in V(G)$ a graph $G \backslash v$ obtained from G by deleting v is a graph $V(G\backslash v) = V(G) - \{v\}$ & $E(G\backslash v)$ consisting of all edges of G not incident to v

Let comp(G) be the number of connected components of G comp$(G)=1 \iff G$ is connected

For which $e \in E(G)$ do we have comp$(G\backslash e)$ = comp(G)

Def: e is a <u>cut-edge</u> of G if e is not an edge of any cycle in G.

Lemma 2.6 Let $e = \{u, v\} \in E(G)$. Then exacely one of following holds.

- e is a cut-edge, $comp(G \backslash e) = comp(G) + 1$
  u, v belong to different components of $G \backslash e$
- e is not a cut edge, $comp(G \backslash e) = comp(G)$
  u & v are in the same component of $G \backslash e$

Proof: Suppose e is a cut-edge

Let $H_1, H_2, \cdots, H_k$ be the connected components of $G \backslash e$

If u and v belong to the same component H_i of $G \backslash e$, then there exists a path P in H_i with ends u & v and adding e to it, we get circle $\lightning$

So we may assume $u \in V(H_1)$, $v \in V(H_2)$.

Let $H_1' \subseteq G$ be obtained from $H_1 \cup H_2$ by adding e.

We claim that $H_1', H_3 \cdots H_k$ are all components of G which would imply $comp(G) = comp(G \backslash e) - 1$

By Lemma 2.5 $\Rightarrow$ H_1' is connected

3. Trees and Forests

Forest is a graph with no cycles

Tree is a non-null connected forest.

- for any $u, v \in V(T)$ there exists a unique path in T with ends u & v
- $|E(T)| = |V(T)| - 1$
- every tree with $|V(T)| \geq 2$ has a leaf vertex of degree 1

Lemma 3.1: Let F be a non-null forest then

$comp(F) = |V(F)| - |E(F)|$ (If T is a tree, $comp(T) = 1$

i.e. $|E(T)| = |V(T)| - 1$)

Proof: Induction on $|E(F)|$

Base Case ($|E(F)| = 0$): $comp(F) = |V(F)|$

every vertex is a component of F

Induction Step ($|E(F)| \geq 1$): Let $e \in E(F)$ then e is a cut-edge so by 2.6 $comp(F \backslash e) = comp(F) + 1$

$$LHS = |V(F\backslash e)| - |E(F\backslash e)| = |V(F)| - |E(F)| + 1$$

$$\Rightarrow \quad comp(F) = |V(F)| - |E(F)|$$

A <u>leaf</u> in a graph G is a vertex with degree one.

Lemma 3.2. Let T be a tree with $|V(T)| \geq 2$ Let X be the set of leaves of T, Let Y be the set of all vertices of T of degree ≥ 2. Then $|X| \geq |Y| + 2$

Proof:

By 1.1 $\sum_{v \in V(T)} \deg(v) = 2|E(T)| \overset{3.1}{=} \underbrace{2(|V(T)|-1) = 2|V(T)|-2}$

$-2 = \sum_{v \in V(T)} (\deg(v) - 2) = \underbrace{\sum_{v \in X} (\deg(v)-2)}_{= -|X|} + \underbrace{\sum_{v \in Y} (\deg(v)-2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T)-X-Y} (\deg(v)-2)}_{= 0} \geq |Y| - |X|$

$\Rightarrow \quad |X| \geq |Y| + 2$

Lemma 3.3: Let T be a tree with exactly two leaves u & v. Then T is a path with ends u & v

Proof: Let $P \subseteq T$ be a path with ends u & v. By 3.2 $\deg_T(w) = 2$ for every $w \in V(P) \setminus \{u, v\}$. So $\deg_T(w) = \deg_P(w)$ so no vertex $V(P)$ is incident to an edge $E(T) \setminus E(P)$. So 2.5 P is a connected component of T, but T is connected, so $T = P$

Lemma 3.4: Let T be a tree, $v \in V(T)$ be a leaf. Then $T \setminus v$ is a tree.

Proof: T had a path between two vertices in $V(T) - v$

$\rightarrow$ $T \setminus v$ is connected

v has neighbor $\rightarrow$ non-null

no cycles $\rightarrow$ obvious

Lemma 3.5: Let G be graph $v \in V(G)$ be a leaf. If $G \setminus v$ is tree, so is G

Lemma 3.6: Let T be tree, $u, v \in V(T)$ Then T contains a unique path with ends u & v

Proof: Induction on $|V(T)|$

Base Case $|V(T)|=1$ ✓

Induction Step

4. Spanning Tree

Let G be a graph. A subgraph $T \subseteq G$ is a spanning tree of G if T is a tree and $V(T) = V(G)$

Lemma 4.1: Let G be a connected non-null graph. Let $H \subseteq G$ chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.

proof: <u>no cycle</u>: If H has a cycle C

Let $e \in E(C)$. Then $H \setminus e$ is connected by 2.6 & minimal

Lemma 4.2: Let G be the connected non-null graph. Let $H \subseteq G$ be chosen maximal such that H contains no cycles. Then H is a spanning tree of G

Proof: Show that $V(H) = V(G)$, H is connected.

If $\exists$ $v \in V(G) \setminus V(H)$ adding v to H $\lightning$ maximality

Suppose H is not connected. Then by 2.2 there exists a partition (X, Y) of $V(H) = V(G)$ s.t. $X, Y \neq \emptyset$ and no edge of H has one end in X and another in Y. But as G is connected there exists an edge $\{u, v\} \in E(G)$ s.t. $u \in X, v \in Y$. Adding the edge to H, produces $H \subseteq G$ which contradicts maximality of H as $\{u, v\}$ belongs to no cycle in H, because H has no path from u to v. $\square$

Let T be a spanning tree of G

Let f be $E(G) \setminus E(T)$

Then the subgraph $T + F$ of G obtained by adding f to T, contains exactly one cycle by 3.6

C called fundamental cycle of f with respect to T, denoted $FC(T, f)$

Lemma 4.3. Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$, let $C = FC(T, f)$, let $e \in E(C)$. Then $(T+f) \setminus \{e\}$ is a spanning tree of G.

Proof: Let $T' = (T+f) \setminus \{e\}$. $T+f$ is connected, e is not a cut edge of $T+f$ so T' is connected. C is a unique cycle in $T+f$ so T' contains no cycle.

Let G be a graph, let $w: E(G) \to \mathbb{R}_+$

For a subgraph $H \subseteq C$ let

$$w(H) = \sum_{e \in E(H)} w(e)$$

The minimum spanning tree of (G, w)

$MST(G, w)$ is a spanning tree T of G s.t. $w(T)$ is min.

THM 4.4. Let G be a connected non-null graph,

$w: E(G) \to \mathbb{R}_+$

Let $T = MST(G, W)$, Let $E(T) = \{e_1, \cdots e_k\}$

$w(e_1) \leq w(e_2) \leq \cdots \leq w(e_k)$

Then for every $1 \leq i \leq k$ the edge e_i is an edge of minimum weight subject to

- $e_i \notin \{e_1, \cdots e_{i-1}\}$ and
- $\{e_1, \cdots, e_i\}$ does **not** contain an edge set of a cycle in G.

Kruskal's algorithm

Input: connected non-null graph G, $w: E(G) \to \mathbb{R}$

Steps: For each $1 \le i \le |V(G)|-1$

Let $e_i \in E(G)$ be chosen with $w(e_i)$ minimum subject to $e_i \notin \{e_1, e_2, \cdots e_{i-1}\}$ & $\{e_1, \cdots e_{i-1}, e_i\}$ does not contain edges set of circles.

Output: A graph T s.t. $V(T) = V(G)$, $E(T) = \{e_1, \cdots e_{|V(G)|-1}\}$

THM 4.5 Kruskal's Alg outputs $MST(G, w)$

Proof: If all edges of G have pair-wise different weights then 4.4. implies the $MST(G, w)$ is unique and outputed by Krustal's algorithm.

How many spanning trees are there in K_n? (complete graph of n vertices)

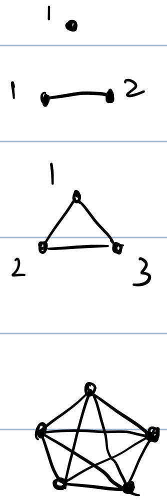
$\boxed{n^{n-2}}$

How many spanning tree of K_n are paths

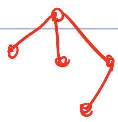
$\frac{n!}{2}$

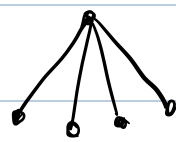
How many spanning tree of K_n are stars (have a vertex of degree of $n-1$)

n



n	
1	1
2	1
3	3
4	$\frac{4!}{2} + 4 = 16$
5	$\frac{5!}{2} + 5 + 5\cdot4\cdot3 = 125$





Theorem 4.6 (Cayley's Theorem) The complete graph K_n has n^{n-2} spanning trees

Proof: We will count spanning rooted forests in K_n with k component

in every component one vertex is designated as a root.

Let T_k be such set of rooted forest with k components in K_n. ($T_1 = n^{n-1} = n^{n-2} \cdot n$ each vertex can be a root)

We will orient all the edges of our forests s.t. each root has no edges directed towards it, and every other vertex has exactly one.

Claim 1. There is a unique way of directing edges of a rooted forest in such way.

Proof: For a tree, one can prove it by induction by deleting a non-root leaf in induction step.

Claim 2. $|T_n| = 1$

Claim 3 $n(k-1)|T_k| = (n-k+1)|T_{k-1}|$

Proof: A forest F with $k-1$ components is a parent of a forest F' with k components, if $F' = F \setminus e$ for some $e \in E(F)$ and in the component of F' containing no root of F' the end of e is chosen as a root.

F' is a child of F

We will count the num of (parent, child) pairs.

$F \in T_{k-1}$ has $|E(F)|$ children

$$|E(F)| = |V(F)| - comp(F) = n - (k-1) = n-k+1$$

$\Rightarrow$ $(n-k+1)|T_{k-1}|$ pairs

For $F' \in T_k$ we can obtain a parent by adding an edge from any edge from a vertex to a root of any of $(k-1)$ components not containing it

Any child has $n(k-1)$ parent. $|T_k| \cdot n \cdot (k-1)$

Claim 4. $|T_k| = \binom{n}{k} k n^{n-1-k}$

Proof

Induction on $n-k$

Base case $(n-k=0)$ Claim 2 ✓

Induction Step $(k \to k-1)$

$|T_{k-1}| \overset{\text{Claim 3}}{=} \frac{n(k-1)}{n-k+1} |T_k| \overset{\text{ind hyp}}{=} \frac{n(k-1)}{n-k+1} \binom{n}{k} k n^{n-1-k}$

$= \binom{n}{k-1} (k-1) n^{n-1-(k-1)}$

5. Euler's Theorem & Hamiltonian cycles

A walk in a graph G is a sequence $(v_0, v_1, \cdots, v_k)$ of vertices of G (perhaps with repetitions)

s.t. $v_i v_{i+1} \in E(G)$ for all $i = 0, \cdots, k-1$

A walk uses an edge e if $e = v_i v_{i+1}$ for some $i = 0, \cdots, k-1$

A trail is a walk that uses every edge at most once,

so $\{v_0, \cdots, v_k\}$ is a trail if $\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\}$ for $i \neq j$

An Eulerian trail is a trail which uses every edge exactly once

An Euler tour in G is a closed Eulerian trail

What dose a graph have an Euler Tour?

- All degrees must be even
- All edges have to belong to the same component.

Goal: show that it is sufficient.

lemma 5.1. Let G be a graph with $E(G) \neq \emptyset$. If G has no leaves then G has a cycle.

Proof Suppose not. Then G is a forest. Let C be a component with $E(C) \neq \emptyset$ then C is tree with $|V(C)| \geq 2$

Then C has leaves ↯

Lemma 5.2 Let G be a graph with all degrees even. Then there exist cycles $C_1, C_2, \cdots, C_k$ in G s.t. $(E(C_1), E(C_2), \cdots E(C_k))$ is a partition of $E(G)$

Proof: By induction on $|E(G)|$.

Base case $|E(G)| = 0$

Induction Step $(|E(G)| \geq 1)$ By 5.1 there exists a cycle C_1 in G. Let $G' = G \setminus E(C_1)$ Then all degree in G' are even so by induction hypothesis there exist $C_2 \cdots C_k$ s.t.

$(E(G_2), \cdots E(G_k))$ are partitions $E(G') = E(G) \setminus E(C_1)$. So

$C_1, C_2 \cdots C_k$ satisfies Lemma.

THM 5.3:

(Euler Theorem)

Proof: Let $W = (v_0, v_1, \cdots, v_k)$ be a closed trail in G of maximum length. If we uses all edges of G, then we're done.

Otherwise, Let $H \subseteq G$: $e \in E(H)$ if $e \in E(G)$ and $e \notin E(W)$

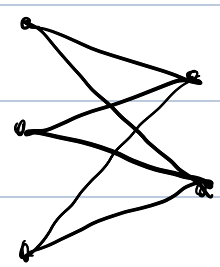
Then all degrees in H are even.

Let H' be a subgraph of G consisting of edges and vertices used by W

Then H' is not a component of G, so by 2.5 there exists $e \in E(G) - E(H')$ with an end $w \in V(H')$. By 5.2 there exists $C \subseteq H$ a cycle s.t. $e \in E(C)$, adding vertices of C in order at the occurence of w in W (a closed trail longer than W)

Corollary 5.4. Let G' be a connected graph with at most 2 vertices of odd degree. Then G contains a Euler trail.

Hamiltonian cycle in G is a cycle $C \subseteq G$ s.t. $V(C) = V(G)$



Lemma 5.5. Let G be a graph, $X \subseteq V(G)$ $X \neq \emptyset$. If $|X| <$ $\mathrm{comp}(G \setminus X)$ then G has no Hamiltonian cycle.