24Fall MATH350 Honours Discrete Mathmetics

Assignment #2: Trees

Shawn Cheng | September 24, 2024

Problem 1.

Proof:

"Only If":

$$\sum_{i=1}^{n} d_n = 2|E(T)|$$

By Lemma 3.1, $|E(T)| = |V(T)| - 1$. Then

$$\sum_{i=1}^{n} d_n = 2|E(T)| = 2|V(T)| - 2 = 2n - 2$$

"If":

$$\begin{cases} \sum_{i=1}^{n} d_n = 2|E(G)| = 2n-2 \\ 2|V(G)| - 2 = 2n - 2 \end{cases} \Rightarrow |V(G)| = |E(G)| + 1$$

Assume there does not exist a tree T, such that $|V(G)| = |E(G)| + 1$. Let T^* be such a tree. By Lemma 3,1,

$$T^* \text{is a tree} \Rightarrow |V(T^*)| = |E(T^*)| + 1$$

A contradiction. Therefore, there exist a tree, such that $|V(G)| = |E(G)| + 1$, i.e. $\sum_{i=1}^{n} d_n = 2n - 2$. $\square$

Problem 2.

Proof:

Since every pair of vertices $u, v \in V(G)$ there exists a path in G from u to v, G is connected. If G is a tree, then we are done. Otherwise, G contains circles.

Assume G only contains circle C of length larger than 2k+1. Now we arbitrarily pick a vertex u in C. Obviously, we are able to pick another vertex v in C such that there exist a path P from u to v of length k+1. By problem's setting, there also exists a P' from u to v of length n ($n \leq 2$).

As P and P' have the same end u, v and they are distinct, there exists a circle of length at most 2k+1. A contradiction to the assumption. Therefore G contains a circle of length at 2k+1 or G is a tree. $\square$

Problem 3.

Proof:

Base Case: $|V(C)| = 1$

Since $|V(T)| = 1$ only have one vertex v, we have $V(T_1 \cap \cdots \cap T_n) = v$

Induction Steps:

Assume $|V(T)| = k$, there exists connected subgraphs of T, $T_1, \ldots, T_n$, such that $V(T_i \cap T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq n$ that satisfies

$$V(T_1 \cap \cdots \cap T_n) \neq \emptyset$$

Now we consider $|V(T)| = k + 1$.

Arbitrarily pick a leaf v. Let $T_1, \ldots, T_i$ be subgraphs of T such that $\forall m \in \{1, \ldots, i\}$, $V(T_m) \subseteq V(T/v)$. Let T_j be an arbitrary subgraph of T such that $v \in V(T_j)$. Let $S = V(T_1 \cap \cdots \cap T_i)$. By induction hypothesis, $S \neq \emptyset$.

Assume that $S \cap V(T_j) = \emptyset$. Let $v_1 \in V(T_1 \cap T_j)$. By assumption, we are able to find a vertex v_2 such that $v_2 \in V(T_m \cap T_j),\ where\ m \in \{1, \ldots, i\}$ and $v_2 \notin V(T_1 \cap T_j)$

As $v_1, v_2 \in V(T_j)$ and T_j is connected, there exists a path P_1 from v_1 to v_2 and $P_1 \subseteq T_j$. Let $u \in S$, there exists another distinct path P_2 from v_1 to v_2 passing u. Then there exists a circle in T, which contradict to the setting that T is a tree. Therefore $S \cap V(T_j) \neq \emptyset$, i.e., $V(T_1 \cap \ldots T_i \cap T_j) \neq \emptyset$ for any T_j.

That is to say $|V(T)| = k + 1$,

$$V(T_1 \cap \cdots \cap T_n) \neq \emptyset \quad \square$$