Lemma 5.5 See in week 3&4

Proof: Suppose for contradiction that C is a Hamiltonian Cycle in G.

$$\text{comp}(C\setminus X) \geq \text{comp}(G\setminus X) > |X|$$

$$\text{comp}(C\setminus X) = |V(C\setminus X)| - |E(C\setminus X)| \leq |V(C)| - |X| - (|E(C)| - 2|X|) = |X|$$

as $C\setminus X$ is a forest by 3.1

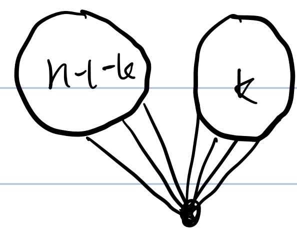
A contradiction! $\square$

What is the minimum m s.t. if $|E(G)| \geq m$ then G has a Hamiltonian Cycle

$$m = \binom{n}{2} - n + 2 \text{ is not enough!}$$

What is the minimum d s.t. if $\deg(v) \geq d \ \forall v \in V(G)$ then G has a Hamiltonian Cycle.

$d = \left\lfloor \frac{n-1}{2} \right\rfloor$ is not enough



(Dirac-Pósa)

THM 5.6. Let G be a graph on $n \geq 3$ vertices. If $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u & v in G then G has Hamiltonian Circle.

Proof By induction on $\binom{n}{2} - |E(G)|$

Base case $|E(G)| = \binom{n}{2}$

Induction Step:

Let $u, v \in V(G)$ that are not adjacent. Let G' be obtained by adding a edge uv. By IH, $\exists$ Hamiltonian Cycle in G'

If $uv \notin E(C)$ then C is a HC in G and we're done. So we may assume $uv \in E(C)$. Let $u = u_1, u_2, \dots, v = u_n$ be vertices of C in order.

Let $A = \{i : u_1 u_i \in E(G)\}$ $\quad |A| = \deg(u)$

$B = \{i : v u_{i-1} \in E(G)\}$ $\quad |B| = \deg(v)$

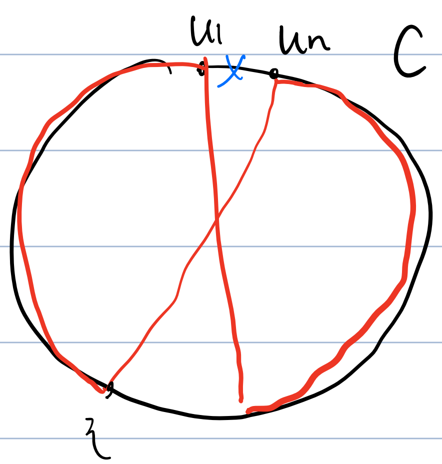
By condition $\quad |A| + |B| \geq n$

$A \subseteq \{2, \dots, n-1\} \quad B \subseteq \{3, \dots, n\}$

$A \cup B \subseteq \{2, \dots, n\}$ $\quad$ so $|A \cup B| \leq n-1$

So $|A \cap B| = |A| + |B| - |A \cup B| \geq 1$

so $\exists i : u u_i \in E(G) \quad v u_{i-1} \in E(G)$



Then $u_1 \cdots u_{i-1} \cdots u_n \cdots u_i \cdots u_1$ is HC.

Corallary 5.7. Let G be a graph on $n \geq 3$ vertices.

(a) if $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$, or

(b) if $|E(G)| \geq \binom{n}{2} - n + 3$

Then G contains a HC.

Proof.

By 5.6. it suffices to check that $\deg(u) + \deg(v) \geq n$ $\forall u, v \in V(G)$ u, v non-adjacent

Suppose (b) holds, Let $\bar{G}$ be the complement of G.

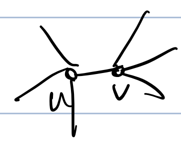
$V(\bar{G}) = V(G)$ and $\{u,v\} \in E(\bar{G})$ iff $\{u,v\} \notin E(G)$

$$|E(\bar{G})| = \binom{n}{2} - |E(G)| \leq n-3$$

$$\deg_G(u) + \deg_G(v) = (n-1-\deg_{\bar{G}}(u)) + (n-1-\deg_{\bar{G}}(v))$$

$$\geq 2n-2-(|E(\bar{G})|+1) \geq 2n-2-(n-3+1) = n \Rightarrow (a)$$

$\uparrow$



6. Bipartite Graph

A Bipartition of a graph G is a partition (A, B) of $V(G)$ s.t. every edge of G has one end in A and another in B.

A graph G is <u>bipartite</u> if it admits a bipartition.

What are minimally non-bipartite graphs?

Lemma 6.1 Trees are bipartite

Proof: By Induction on $|V(T)|$

Base Case $|V(T)|=1$ trivial

Induction step: Let v be a leaf of T with neighbor u. By IH $G\backslash v$ is bipartite. Let (A, B) be bipartition of $G\backslash v$, assume without losing generality $u \in A$

Then $(A, B \cup \{v\})$ is a bipartition.

THM 6.2. Let G be a graph. Then the following condition are equivalent.

(1) G is bipartite

(2) G contains no closed walk of odd length

(3) G contains no odd cycle

Proof:

$1 \Rightarrow 2$: <u>Observation</u>: If (A, B) is a bipartition of G and $v_0, v_1, v_2, \cdots, v_k$ is a walk in G s.t. $v_0 \in A$ then $v_i \in A$ iff i is even.

(Easy by induction)

$2 \Rightarrow 3$: Clear

$3 \Rightarrow 1$: As bipartitions of components can be combined into bipartition of G, we can assume that G is connected and non-null.

Let T be a spanning tree of G. By 6.1 there exists a bipartition (A, B) of T. We will show that (A, B) is a bipartition of G.

Let $f \in E(G) - E(T)$, Let $v_0, v_1, \cdots, v_k$ be a path in T between ends of f. Assume $v_0 \in A$ without losing generality. As the fund. cycle of f w.r.t. T is even, k is odd and so $v_k \in B$ by observation, so f has one end in A another in B.

7. Matchings in bipartite graphs.

A <u>matching</u> M in G is a set of edges of G s.t. no vertex is incident to more than one edge in M.

Let $\nu(G)$ denote the matching number of G the maximum number "nu" of edges in a matching in G.

$$\nu(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$$

A vertex cover is a set $X \subseteq V(G)$ s.t. every edge of G has at least one end in X.

$$|M| \leq |X|$$

Let $\tau(G)$ be the minimum size of a vertex cover in G.

so $\nu(G) \leq \tau(G)$

	$\nu(G)$	$\tau(G)$
P_n	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$
C_n	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$
K_n	$\lfloor \frac{n}{2} \rfloor$	$n-1$

Goal: Show that $\nu(G) = \tau(G)$ in bipa graph.

Lemma 7.1: For any graph G

$$\nu(G) \le \tau(G) \le 2\nu(G)$$

Proof: It remains to show $\tau(G) \le 2\nu(G)$

Let M be max matching G $|M| = \nu(G)$

We want to find a vertex cover s.t. $|X| \le 2|M|$

Let X be the set of all ends of edges in M. Then $|X| = 2|M|$, and X is a vertex cover C, if not $e \in E(G)$ with no end in X then $M \cup \{e\}$ is a matching contradicting maximality of M.

Let M be a matching in G. A path in G is <u>M-alternating</u> if edges of P alternate between edges of M and $E(G)-M$, i.e. every internal vertex of P is incident to an edge in $E(P) \cap M$.

An M-alternating P is <u>M-augmenting</u> if $|V(P)| \ge 2$ and ends of P are not incident to edges of M

<u>Observation</u>: If G contains an M-augmenting path then M is not maximum

Theorem 7.2: For any bipartite G, we have $\nu(G) = \tau(G)$ (König)

<u>Proof</u>: It suffices to show that $\tau(G) \leq \nu(G)$ i.e.

Let (A, B) be a bipartition of G

Let A' and B' be sets of all vertices not incident to edges of M in A and B, respectively

Let Z be the set of all vertices $v \in V(G)$ s.t. there is an M-alternating path in G with one end v and another in A'.

Then ① $A' \subseteq Z$

② $Z \cap B' = \emptyset$ (no M-augmenting)

③ every edge of M with one end in Z has both ends in Z

④ every edges with an end in $Z \cap A$ has second end in $Z \cap B$

Let $X = (Z \cap B) \cup (A - Z)$

Then $|X| = |M|$, because every vertex of X is incident to an edge of M and every edge of M has exactly one end in X by ③

And X is a vertex cover by ④.