

# A Branch-and-Bound Algorithm for Single-Machine Scheduling Problems with Candidate Tools and Release Time (Supplementary File)

Shengchao Li, Shixin Liu, *Senior Member, IEEE*, Ziyang Zhao, *Member, IEEE*, and Mengchu Zhou, *Fellow, IEEE*

## APPENDIX A EXAMPLES

We show two examples to understand  $1|K_j, r_j|C_{max}$  and its degenerate problem  $1|K_j|C_{max}$ , as well as the corresponding B&B and BrFS-B&B algorithms presented in this paper. Both examples use the data in Table A.1 where there are 6 jobs and 5 tools. Each job has its release time  $r_j$ , processing time  $p_j$ , and candidate tool set  $K_j$ . Each tool show its setup time  $s_k$ .

TABLE A.1  
AN INSTANCE OF  $1|K_j, r_j|C_{max}$

|       | $j_1$      | $j_2$      | $j_3$ | $j_4$           | $j_5$ | $j_6$      |
|-------|------------|------------|-------|-----------------|-------|------------|
| $r_j$ | 83         | 963        | 1359  | 1500            | 2432  | 3532       |
| $p_j$ | 600        | 598        | 681   | 482             | 695   | 599        |
| $K_j$ | $k_4, k_5$ | $k_4, k_5$ | $k_3$ | $k_1, k_2, k_5$ | $k_3$ | $k_4, k_5$ |
| tool  | $k_1$      | $k_2$      | $k_3$ | $k_4$           | $k_5$ | -          |
| $s_k$ | 26         | 27         | 32    | 33              | 33    | -          |

Note.  $r_j$ ,  $p_j$ , and  $s_k$  are measured in minutes.

### A. An Example of $1|K_j, r_j|C_{max}$

**Example 1.** Consider the problem  $1|K_j, r_j|C_{max}$  corresponding to Table A.1. An optimal solution is shown in Fig. A.1. The optimal sequence is  $\langle (j_1, k_4), (j_2, k_5), (j_4, k_5), (j_3, k_3), (j_5, k_3), (j_6, k_4) \rangle$  with a makespan of 4181. Each column in Fig. A.1 represents the tools that can be used for the current job, and each row indicates the jobs matched with the current tool. The white background area indicates that the current tool is selected by the job; the shaded area indicates the current tool is a candidate for the job but not used; and the gray background area indicates the installation or removal of the current tool. For instance, job  $j_1$  selects tool  $k_4$ , thus  $k_4$  is installed on the machine before  $j_1$  begins processing. Since  $j_2$  selects  $k_5$ ,  $k_4$  must be removed after  $j_1$  is completed. Note that if FCFS is applied, the sequence is  $\langle (j_1, k_4), (j_2, k_4), (j_3, k_3), (j_4, k_1), (j_5, k_3), (j_6, k_4) \rangle$  with a makespan of 4297 where the tool with minimum setup time for each job is selected because the candidate tool sets for two adjacent jobs do not intersect. The sequence is not optimal for  $1|K_j, r_j|C_{max}$ .

The branch-and-bound tree of Example 1 is shown in Fig. A.2. Job  $j_1$  choosing tool  $k_4$  is reduced and is not shown in the figure. There are 5 remaining jobs, so the created nodes

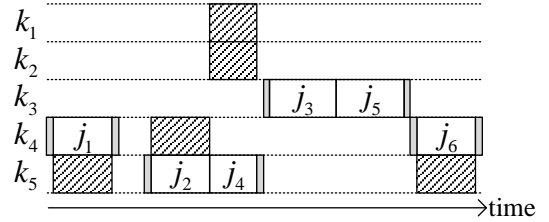


Fig. A.1. An optimal solution of Example 1.

15/20/24/31/32 at level 5 represent feasible solutions to the problem. The lower bound for each of these nodes is the upper bound for the current node and the problem. The initial upper bound  $UB_0$  of the problem obtained by Algorithm 3 is 4297. Notice that node 7 is dominated by node 16 according to Theorem 5. Nodes 25 and 10 are pruned at the 15th and 19th explorations, respectively, due to the lower bound pruning rule. Meanwhile, nodes 13/14/26/27 and 11/18/22/29 are pruned at the 15th and 19th explorations, respectively, because their lower bounds are not smaller than those of nodes 25 and 10. The Gantt chart of the optimal solution corresponding to node 31 is shown in Fig. A.1.

### B. An Example of $1|K_j|C_{max}$

**Example 2.** Consider the degenerate problem  $1|K_j|C_{max}$  corresponding to Table A.1, i.e., excluding the release time of jobs. The branch-and-bound tree is shown in Fig. A.3(a). Jobs  $j_3$  and  $j_5$  do not appear in the figure since they are reduced by the domain reduction of BrFS-B&B. The corresponding optimal sequence is  $\langle (j_3, k_3), (j_5, k_3), (j_1, k_5), (j_2, k_5), (j_4, k_5), (j_6, k_5) \rangle$  with a makespan of 3785, and the Gantt chart is shown in Fig. A.3(b).

## APPENDIX B PROOF OF THEOREM 2

*Proof.* Consider an arbitrary feasible sequence  $\langle (j_2, k^{j_2}), (j_3, k^{j_3}), \dots, (j_{|J|}, k^{j_{|J|}}) \rangle$  of  $J \setminus \{j_1\}$ , denoted as  $\mathcal{S}$ . Note that  $\tilde{T}_{j_2} = r_{j_2}$  must hold, because  $j_2$  is the first processed job in  $\mathcal{S}$  and satisfies inequality (31). The proof is divided into two cases below.

Case (a):  $j_1$  does not batch with any other jobs in  $\mathcal{S}$ . In this case, tool switching must occur before and after  $j_1$



removal of the tool selected by  $j_1$ . When assigning the tool for  $j_1$ , the following equation holds due to inequality (32).

$$\begin{aligned} \tilde{T}_{j_1} &= r_{j_2} - s_{k^{j_2}} + p_{j_1} + s_{k^{j_2}} - H(k^{j_1}, k^{j_2}) - p_{j_1} \\ &\geq r_{j_2} - \max_{k \in K_{j_1}, k' \in K_{j_2}} H(k, k') \geq r_{j_1}. \end{aligned}$$

Particularly, the equation still holds when  $j_1$  and  $j_2$  are in the same batch. The release time constraint of  $j_1$  is satisfied, so the above adjustment is feasible.

According to the above two cases, we have the inequality  $C_{max}(\langle j_1, j_2, j_3, \dots, j_{|J|} \rangle) \leq C_{max}(\langle j_2, j_3, \dots, j_1, \dots, j_{|J|} \rangle)$ . Due to the arbitrariness of  $\mathcal{S}$ , there must exist an optimal sequence where  $j_1$  is processed first.  $\square$

## APPENDIX C DATA GENERATION PROCESS

The complete data generation process is shown as Procedure 1. Firstly,  $|K|$  types of tools are generated, including the setup time of each tool type and the job types matched by each tool. Secondly, obtain the matched tool set for each type of job. Finally,  $|J|$  jobs are generated. Note that the job release time needs to be done after the processing time is generated. According to the current data generation parameters, it is guaranteed that  $KT_i$  ( $i \in \{J1, \dots, J7\}$ ) is non-empty (i.e., each instance is feasible) even if each tool type corresponds to only a minimum number of job types (i.e.,  $|JT_k| = 2, \forall k \in K$ ).

### Procedure 1: Data Generation

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**Output:**  $J, K$

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1  $JT := \{J1, J2, \dots, J7\};$ 
2 for  $k = 1, 2, \dots, |K|$  do
3   generate  $s_k$  and a random number  $num$  using
      $Ui(2, 3);$ 
4   if  $|JT| < num$  then
5      $JT_k \leftarrow JT; JT \leftarrow \{J1, J2, \dots, J7\};$ 
6   while True do
7      $JT'_k \leftarrow$  choose  $num - |JT_k|$  job types from
        $JT$  randomly;
8     if  $JT'_k \cap JT_k = \emptyset$  then
9        $JT \leftarrow JT \setminus JT'_k; JT_k \leftarrow JT_k \cup JT'_k;$ 
10    break while;
11 for  $i = J1, J2, \dots, J7$  and  $k = 1, 2, \dots, |K|$  do
12   if  $i \in JT_k$  then  $KT_i \leftarrow KT_i \cup \{k\};$ 
13 for  $j = 1, 2, \dots, |J|$  do
14    $j.type \leftarrow$  choose a job type using roulette;
15    $K_j \leftarrow KT_{j.type};$ 
16   generate  $p_j;$ 
17 for  $j = 1, 2, \dots, |J|$  do generate  $r_j;$ 
18 return  $J, K.$ 
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## APPENDIX D

### COMPUTATIONAL RESULTS AND ANALYSIS ON DATA SETS D2-D4

Cases under different data generation parameters are generated to illustrate the generalizability of the algorithm. The

setup time of tools in baseline data is much smaller compared to the processing time of jobs, so the range of data generated for setup time increases in this part. In addition, tightening and slackening the release time are considered to reflect the dense and sparse arrival of jobs, respectively.

#### A. Increase Setup Time (D2)

The results for the small-scale and large-scale cases are shown in Tables A.2 and A.3, respectively. The lower bounds obtained by MILP and CP are difficult to converge. In terms of the solving time for each solved instance, B&B significantly outperforms MILP and CP. Compared to data set D1, the solvable case scales of both MILP and B&B become smaller, since increasing setup time limits the application of some strategies, making the feasible solution space larger. This can be seen from the values of the columns *avg nodes* in Tables IV and A.3. Due to the unique constraint propagation technique, the solvable case scale for CP remains essentially constant.

The implementation details of B&B for all cases are shown in Fig. A.5(a) and Table A.4. Since  $r_j$  is only related to  $\sum_{j \in J} p_j$ , increasing the setup time makes job arrival dense. That makes the condition in line 15 of Algorithm 3 difficult to satisfy, and thus the percentage of job reduction decreases significantly. The percentage of the dominance rule pruning does not change significantly. However, it takes longer to determine whether the current node is dominated or not because of the increased number of explored nodes. In addition, the branching strategy for the root node is limited because the inequalities (31) and (32) are not easily satisfied. These all lead to limitations on solution performance.

#### B. Tighten Release Time (D3)

The results for the small-scale and large-scale cases are shown in Tables A.2 and A.3, respectively. Tightening the release time makes the feasible solution space larger, and thus the case scale that can be solved optimally by MILP and CP is smaller. In fact, the number of ready jobs in the scheduling process increases when the jobs arrive more centrally, which benefits the job batching (i.e., building the subtrees of lines 10 and 13 in Algorithm 4). In particular, when all jobs have the same release time, the problem degenerates to the easily solvable problem  $1|K_j|C_{max}$ . However, MILP and CP cannot recognize this property and performs poorly. On the contrary, B&B can utilize it to solve fairly large-scale cases.

The implementation details of B&B for all cases are shown in Fig. A.5(b) and Table A.4. As can be seen in the figure, the significantly large percentage of middle nodes suggests that Theorem 3 plays an important role. In contrast, the enumeration branching is hardly ever used. Although the percentage of job reduction decreases significantly and the branching strategy for the root node is limited, they have little effect on solving large-scale problems.

#### C. Slacken Release Time (D4)

The results for the small-scale and large-scale cases are shown in Tables A.2 and A.3, respectively. For MILP and

TABLE A.2  
COMPARISON OF SMALL-SCALE RESULTS BETWEEN MILP, CP, AND B&B UNDER DATA SETS D2-D4

| J                    | method | #opt | time/s |         |         | gap <sup>L</sup> /% |       |       | gap <sup>U</sup> /% |      |      |
|----------------------|--------|------|--------|---------|---------|---------------------|-------|-------|---------------------|------|------|
|                      |        |      | min    | avg     | max     | min                 | avg   | max   | min                 | avg  | max  |
| increase setup time  |        |      |        |         |         |                     |       |       |                     |      |      |
| 6                    | MILP   | 20   | 0.03   | 0.08    | 0.16    | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 0.05   | 0.23    | 0.48    | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | <0.01   | /                   | /     | /     | /                   | /    | /    |
| 9                    | MILP   | 20   | 0.72   | 6.87    | 65.13   | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 0.18   | 0.63    | 4.12    | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.02    | /                   | /     | /     | /                   | /    | /    |
| 12                   | MILP   | 14   | 2.66   | 406.38  | 2007.47 | 9.34                | 14.83 | 19.19 | 0.00                | 0.00 | 0.00 |
|                      | CP     | 20   | 0.23   | 3.39    | 15.34   | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.03    | /                   | /     | /     | /                   | /    | /    |
| 15                   | MILP   | 4    | 43.25  | 367.94  | 637.19  | 3.26                | 13.42 | 21.09 | 0.00                | 2.19 | 2.19 |
|                      | CP     | 19   | 0.47   | 119.93  | 1210.07 | 14.91               | 14.91 | 14.91 | 0.00                | 0.00 | 0.00 |
|                      | B&B    | 20   | <0.01  | 0.03    | 0.38    | /                   | /     | /     | /                   | /    | /    |
| tighten release time |        |      |        |         |         |                     |       |       |                     |      |      |
| 6                    | MILP   | 20   | 0.03   | 0.10    | 0.30    | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 0.05   | 0.40    | 0.75    | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.01    | /                   | /     | /     | /                   | /    | /    |
| 8                    | MILP   | 20   | 0.72   | 1.27    | 2.38    | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 0.11   | 1.52    | 2.70    | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.02    | /                   | /     | /     | /                   | /    | /    |
| 10                   | MILP   | 20   | 3.23   | 283.84  | 1749.91 | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 1.24   | 8.08    | 20.55   | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | <0.01   | /                   | /     | /     | /                   | /    | /    |
| 12                   | MILP   | 2    | 287.61 | 1037.79 | 1787.97 | 14.09               | 23.54 | 32.15 | 0.00                | 0.00 | 0.00 |
|                      | CP     | 16   | 6.02   | 147.35  | 796.47  | 45.34               | 48.02 | 51.08 | 0.00                | 0.00 | 0.00 |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.01    | /                   | /     | /     | /                   | /    | /    |
| slacken release time |        |      |        |         |         |                     |       |       |                     |      |      |
| 10                   | MILP   | 20   | 0.03   | 1.56    | 14.09   | /                   | /     | /     | /                   | /    | /    |
|                      | CP     | 20   | 0.03   | 0.73    | 2.84    | /                   | /     | /     | /                   | /    | /    |
|                      | B&B    | 20   | <0.01  | <0.01   | <0.01   | /                   | /     | /     | /                   | /    | /    |
| 15                   | MILP   | 17   | 0.03   | 31.06   | 198.39  | 1.51                | 3.72  | 5.86  | 0.00                | 0.14 | 0.14 |
|                      | CP     | 19   | 0.05   | 28.75   | 453.21  | 3.79                | 3.79  | 3.79  | 0.00                | 0.00 | 0.00 |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.02    | /                   | /     | /     | /                   | /    | /    |
| 20                   | MILP   | 7    | 0.06   | 175.46  | 1186.22 | 0.32                | 5.28  | 19.91 | 0.00                | 0.32 | 0.32 |
|                      | CP     | 16   | 0.05   | 127.08  | 1333.35 | 3.08                | 10.52 | 19.91 | 0.00                | 0.00 | 0.00 |
|                      | B&B    | 20   | <0.01  | <0.01   | 0.01    | /                   | /     | /     | /                   | /    | /    |

Note. Time limit is 3600s.

TABLE A.3  
RESULTS OF B&B FOR LARGE-SCALE CASES UNDER DATA SETS D2-D4

| $ J $                | #opt | time/s |         |         | gap/% |       |       | nodes |        |         |
|----------------------|------|--------|---------|---------|-------|-------|-------|-------|--------|---------|
|                      |      | min    | avg     | max     | min   | avg   | max   | min   | avg    | max     |
| increase setup time  |      |        |         |         |       |       |       |       |        |         |
| 20                   | 20   | <0.01  | 0.29    | 1.95    | /     | /     | /     | 313   | 4474   | 13461   |
| 30                   | 19   | <0.01  | 37.76   | 615.47  | 10.26 | 10.26 | 10.26 | 117   | 42670  | 327994  |
| 40                   | 18   | 0.02   | 569.12  | 3253.04 | 8.90  | 12.09 | 15.28 | 1843  | 144193 | 383911  |
| 50                   | 17   | 4.48   | 1258.69 | 3420.12 | 8.02  | 19.30 | 41.58 | 24963 | 248765 | 486979  |
| 60                   | 7    | 0.20   | 1054.46 | 2937.36 | 6.29  | 19.20 | 30.87 | 4513  | 340281 | 491620  |
| tighten release time |      |        |         |         |       |       |       |       |        |         |
| 100                  | 20   | <0.01  | 0.32    | 1.44    | /     | /     | /     | 232   | 5075   | 13886   |
| 300                  | 19   | 0.02   | 62.90   | 568.94  | 0.63  | 0.63  | 0.63  | 1122  | 87956  | 589234  |
| 500                  | 16   | 1.97   | 231.06  | 1292.66 | 0.16  | 0.33  | 0.41  | 6776  | 162203 | 521423  |
| 750                  | 12   | 2.09   | 766.12  | 3388.72 | 0.15  | 0.27  | 0.42  | 12570 | 215848 | 653107  |
| 1000                 | 9    | 0.09   | 675.06  | 3421.79 | 0.11  | 0.23  | 0.43  | 2027  | 160907 | 523065  |
| slacken release time |      |        |         |         |       |       |       |       |        |         |
| 100                  | 20   | <0.01  | 66.47   | 843.71  | /     | /     | /     | 1     | 44259  | 339892  |
| 200                  | 20   | <0.01  | 349.73  | 3320.65 | /     | /     | /     | 1     | 105751 | 872901  |
| 300                  | 17   | <0.01  | 213.36  | 1726.17 | 0.99  | 1.59  | 1.91  | 97    | 138366 | 551434  |
| 400                  | 16   | <0.01  | 0.03    | 0.30    | 1.18  | 2.52  | 4.10  | 1     | 100027 | 668720  |
| 500                  | 17   | <0.01  | 0.09    | 1.15    | 0.26  | 0.77  | 1.17  | 1     | 67643  | 463524  |
| 600                  | 14   | <0.01  | 101.57  | 1190.87 | 0.29  | 1.08  | 2.01  | 1     | 221473 | 1336733 |

Note. Time limit is 3600s.

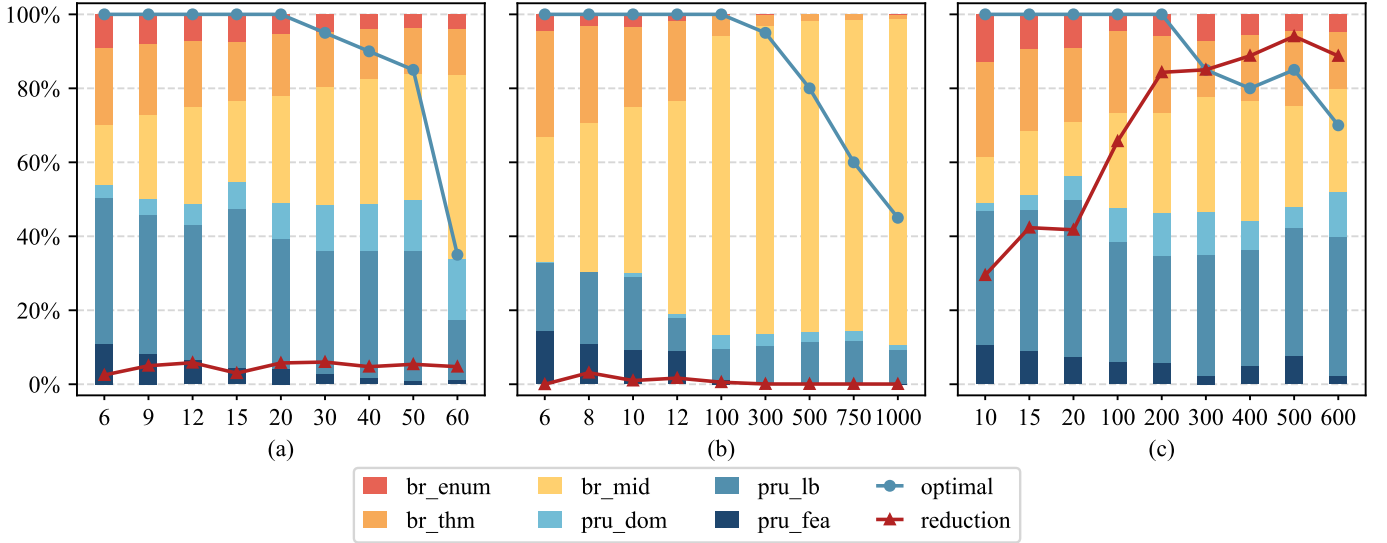


Fig. A.5. Implementation details of B&B for data sets D2-D4. Figs. (a)-(c) correspond to data sets D2-D4, respectively. See Fig. 1 for legend information.

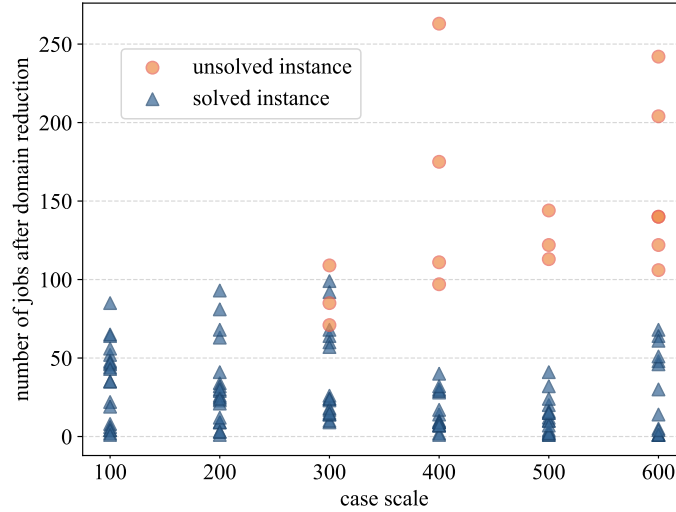


Fig. A.6. Number of jobs for each instance after domain reduction for data set D4. Only the results for large-scale cases are shown in the figure.

CP, the number of solved instances for each case increases slightly, but the case scale that can be solved optimally remains limited. In contrast, B&B can solve quite large-scale cases in a short time. Note that the solved instances consume quite short time to attain the optimal solutions when  $|J| = 400$  and  $|J| = 500$ . This is because, for the instances generated with the current parameters, the solving time of B&B is not positively related to the case scale, but basically depends on the number of jobs  $J'$  after domain reduction. By analyzing the experimental results, we find that: if  $|J'| < 50$ , B&B can solve the problem in a quite short time; if  $|J'| \in [50, 100]$ , most problems can be solved optimally in 3600s but need to consume some time; if  $|J'| > 100$ , the problem can hardly be solved optimally in 3600s. According to Table A.5 and Fig. A.6, the number of jobs after domain reduction is mostly either less than 50 or greater than 100 for the instances with case scales 400 and 500. Therefore, these instances are either solved optimally quite quickly or cannot be solved within 3600s.

The implementation details of B&B for all cases are shown in Fig. A.5(c) and Table A.4. As can be seen in the figure, the job reduction ratio increases with the case scale. But this does not mean that the problem is easier to solve. The difficulty of solving the problem for this data set mainly depends on the number of jobs after domain reduction. In addition, as can be seen in the table, the number of instances with optimal initial solutions increases. If the relaxation factor  $\alpha$  is larger, this number will be even larger, since the optimal sequence can basically be obtained by FCFS.

TABLE A.4  
IMPLEMENTATION DETAILS OF B&B FOR ALL DATA SETS

| $ J $       | $opt/\%$ | $\overline{DR}/\%$ | $\overline{UB}_0$ | $I^{sol}$ | $\overline{N}^{cr}$ | pruning/ $\%$ |       |       | branching/ $\%$ |       |        |
|-------------|----------|--------------------|-------------------|-----------|---------------------|---------------|-------|-------|-----------------|-------|--------|
|             |          |                    |                   |           |                     | $fea$         | $lb$  | $dom$ | $mid$           | $thm$ | $enum$ |
| data set D1 |          |                    |                   |           |                     |               |       |       |                 |       |        |
| 10          | 100      | 26.00              | 7651              | 6         | 79                  | 11.64         | 30.97 | 2.40  | 19.67           | 25.16 | 10.16  |
| 15          | 100      | 29.67              | 10865             | 3         | 439                 | 7.31          | 37.86 | 4.61  | 18.54           | 22.99 | 8.69   |
| 20          | 100      | 25.25              | 14344             | 3         | 2315                | 6.43          | 35.32 | 6.66  | 23.31           | 19.87 | 8.42   |
| 25          | 100      | 18.40              | 17724             | 1         | 3548                | 3.36          | 37.75 | 9.52  | 21.75           | 19.44 | 8.19   |
| 30          | 100      | 32.33              | 21107             | 2         | 7153                | 5.74          | 30.55 | 8.18  | 30.06           | 19.76 | 5.71   |
| 40          | 100      | 25.75              | 27793             | 0         | 42333               | 3.40          | 33.59 | 12.68 | 25.82           | 17.81 | 6.70   |
| 50          | 100      | 25.80              | 34381             | 0         | 87507               | 2.09          | 31.97 | 13.32 | 29.71           | 17.21 | 5.70   |
| 60          | 95       | 27.92              | 41618             | 0         | 254277              | 2.04          | 28.87 | 14.03 | 34.81           | 15.30 | 4.96   |
| 70          | 85       | 18.79              | 47270             | 0         | 451995              | 0.93          | 28.48 | 17.15 | 30.71           | 16.28 | 6.44   |
| 80          | 80       | 30.50              | 53396             | 1         | 551550              | 2.38          | 24.23 | 16.57 | 34.56           | 16.65 | 5.61   |
| 90          | 75       | 30.11              | 61061             | 0         | 697091              | 1.20          | 23.83 | 15.38 | 40.98           | 14.37 | 4.23   |
| 100         | 70       | 24.55              | 67167             | 0         | 867472              | 0.62          | 24.80 | 16.53 | 39.96           | 13.69 | 4.41   |
| data set D2 |          |                    |                   |           |                     |               |       |       |                 |       |        |
| 6           | 100      | 2.50               | 5596              | 9         | 109                 | 10.87         | 39.42 | 3.38  | 16.31           | 20.80 | 9.23   |
| 9           | 100      | 5.00               | 8225              | 4         | 394                 | 8.17          | 37.39 | 4.48  | 22.77           | 19.05 | 8.14   |
| 12          | 100      | 5.83               | 10478             | 2         | 1109                | 6.45          | 36.63 | 5.55  | 26.27           | 17.92 | 7.19   |
| 15          | 100      | 3.00               | 12562             | 1         | 3606                | 4.20          | 43.20 | 7.24  | 21.80           | 16.15 | 7.40   |
| 20          | 100      | 5.75               | 16345             | 1         | 13392               | 4.19          | 34.96 | 9.78  | 28.92           | 16.82 | 5.33   |
| 30          | 95       | 6.00               | 23657             | 0         | 117001              | 2.63          | 33.33 | 12.31 | 32.14           | 14.76 | 4.82   |
| 40          | 90       | 4.75               | 31152             | 0         | 428858              | 1.57          | 34.31 | 12.80 | 33.81           | 13.38 | 4.13   |
| 50          | 85       | 5.40               | 38290             | 0         | 776121              | 0.91          | 35.04 | 13.85 | 33.83           | 12.44 | 3.92   |
| 60          | 35       | 4.75               | 45859             | 0         | 1007978             | 0.99          | 16.42 | 16.34 | 49.81           | 12.48 | 3.97   |
| data set D3 |          |                    |                   |           |                     |               |       |       |                 |       |        |
| 6           | 100      | 0.00               | 4217              | 9         | 42                  | 14.28         | 18.41 | 0.26  | 33.81           | 28.78 | 4.46   |
| 8           | 100      | 3.13               | 5606              | 1         | 59                  | 10.76         | 19.45 | 0.21  | 40.06           | 26.41 | 3.10   |
| 10          | 100      | 1.00               | 6754              | 5         | 129                 | 9.16          | 19.80 | 0.95  | 45.12           | 21.51 | 3.47   |
| 12          | 100      | 1.67               | 8130              | 3         | 196                 | 8.92          | 8.89  | 1.03  | 57.71           | 21.58 | 1.87   |
| 100         | 100      | 0.55               | 63028             | 0         | 49277               | 1.16          | 8.18  | 3.97  | 80.67           | 5.70  | 0.32   |
| 300         | 95       | 0.07               | 187193            | 0         | 1500595             | 0.35          | 9.77  | 3.31  | 83.30           | 3.06  | 0.20   |
| 500         | 80       | 0.06               | 311108            | 0         | 3171258             | 0.22          | 11.08 | 2.86  | 84.04           | 1.72  | 0.08   |
| 750         | 60       | 0.06               | 465945            | 0         | 5406385             | 0.13          | 11.45 | 2.92  | 84.03           | 1.40  | 0.06   |
| 1000        | 45       | 0.05               | 621028            | 0         | 5735801             | 0.10          | 9.08  | 1.48  | 87.96           | 1.34  | 0.06   |
| data set D4 |          |                    |                   |           |                     |               |       |       |                 |       |        |
| 10          | 100      | 29.50              | 7941              | 12        | 74                  | 10.51         | 36.31 | 2.20  | 12.40           | 25.48 | 13.11  |
| 15          | 100      | 42.33              | 11346             | 6         | 251                 | 8.92          | 38.08 | 4.23  | 17.28           | 21.92 | 9.58   |
| 20          | 100      | 41.75              | 14993             | 6         | 714                 | 7.18          | 42.64 | 6.44  | 14.42           | 20.15 | 9.17   |
| 100         | 100      | 65.70              | 70836             | 5         | 126800              | 5.85          | 32.55 | 9.05  | 25.72           | 22.24 | 4.59   |
| 200         | 100      | 84.33              | 139442            | 4         | 230299              | 5.55          | 29.01 | 11.65 | 26.95           | 20.95 | 5.90   |
| 300         | 85       | 85.02              | 208587            | 1         | 366208              | 2.30          | 32.50 | 11.80 | 31.05           | 15.02 | 7.33   |
| 400         | 80       | 88.79              | 275700            | 4         | 241737              | 4.77          | 31.47 | 7.70  | 32.70           | 17.74 | 5.62   |
| 500         | 85       | 94.06              | 343459            | 4         | 212042              | 7.50          | 34.65 | 5.73  | 27.27           | 20.28 | 4.57   |
| 600         | 70       | 88.76              | 411401            | 6         | 571127              | 2.05          | 37.61 | 12.39 | 27.78           | 15.15 | 5.01   |

Note.  $opt$ : the percentage of solved instances;  $\overline{DR}$ : the average percentage of reduced jobs;  $\overline{UB}_0$ : the average initial upper bound for 20 instances;  $I^{sol}$ : the number of instances with an optimal initial solution;  $\overline{N}^{cr}$ : the average number of created nodes;  $fea$ ,  $lb$ ,  $dom$ : the average percentages of nodes in  $\overline{N}^{cr}$  pruned due to the feasibility, the lower bound, and the dominance rule, respectively;  $mid$ : the average percentage of nodes in  $\overline{N}^{cr}$  that are middle nodes in the branching process;  $thm$ ,  $enum$ : the average percentage of nodes in  $\overline{N}^{cr}$  that use theorem and branching, respectively.

TABLE A.5  
NUMBER OF JOBS FOR EACH INSTANCE AFTER DOMAIN REDUCTION FOR DATA SET D4

| $ J $ | 1          | 2  | 3  | 4  | 5          | 6          | 7          | 8  | 9          | 10 | 11 | 12         | 13         | 14 | 15         | 16 | 17 | 18         | 19         | 20        |
|-------|------------|----|----|----|------------|------------|------------|----|------------|----|----|------------|------------|----|------------|----|----|------------|------------|-----------|
| 100   | 64         | 8  | 65 | 22 | 4          | 52         | 2          | 6  | 35         | 4  | 56 | 46         | 44         | 48 | 35         | 47 | 1  | 85         | 43         | 19        |
| 200   | 63         | 29 | 93 | 9  | 1          | 23         | 25         | 68 | 34         | 3  | 26 | 21         | 9          | 41 | 32         | 30 | 81 | 12         | 24         | 3         |
| 300   | 64         | 9  | 10 | 24 | 23         | 99         | 60         | 14 | 26         | 68 | 17 | 57         | 14         | 18 | <u>71</u>  | 15 | 92 | <u>109</u> | 24         | <u>85</u> |
| 400   | 14         | 17 | 9  | 10 | <u>263</u> | 28         | 7          | 9  | <u>175</u> | 2  | 40 | 1          | <u>111</u> | 7  | <u>97</u>  | 30 | 29 | 9          | 7          | 32        |
| 500   | 10         | 20 | 12 | 1  | 2          | <u>144</u> | <u>122</u> | 24 | 3          | 4  | 1  | 10         | 2          | 41 | 7          | 15 | 16 | <u>113</u> | 32         | 15        |
| 600   | <u>204</u> | 5  | 1  | 68 | <u>140</u> | 64         | 61         | 48 | 46         | 1  | 1  | <u>242</u> | <u>140</u> | 30 | <u>122</u> | 51 | 14 | 1          | <u>106</u> | 4         |

Note. Only the results for large-scale cases are shown in the table. Columns 1-20 indicate the 20 instances of each case. The underlined data indicate that the optimal solutions for these instances cannot be found within 3600s using B&B, i.e., these instances are unsolved.