Distributed Economic Dispatch over Networks with Markovian Communication Losses

Junfeng Wu, Shengjun Zhang, Tao Yang, Ling Shi, Hong Wang

***Abstract*—The economic dispatch problem (EDP) is one of fundamental problems in power system, which is used for generation scheduling. The goal of a standard EDP is to minimize the total generation cost while meeting total demand and satisfying individual generator output limit. In this paper, we consider the economic dispatch problem (EDP), where a strictly convex cost function is assigned to each of buses, over lossy communication networks with packet dropping communication links. This paper proposes a distributed algorithm for solving the EDP over Markovian lossy communication networks. Under the assumption that the underlying communication network is strongly connected with a positive probability and the packet drops are Markovian, we show that the proposed algorithm is able to solve the EDP even in the presence of packet drops. Numerical simulation results are used to validate and illustrate the proposed algorithm.**

***Index Terms*—Distributed algorithms; economic dispatch problem; Markovian packet drops; power systems; smart grids.**

I. Introduction

The economic dispatch problem (EDP) is one of fundamental problems in power system, which is used for generation scheduling. It is essentially an optimization problem where the objective is to minimize the total generation cost while meeting total demand and satisfying individual generator output limits.

Traditionally, the EDP problem is solved by the centralized approaches, such as lambda-iteration method and the gradient search method [1]. In these methods, there is a single control center which gathers the entire networks's information, perform the computation, and provides control signals to the entire system. However, centralized approaches suffer from a few drawbacks, such as a single point failure, high communication and computation burden, and limited flexibility.

To overcome these limitations, recently distributed EDP algorithms have been proposed in the literature [2]–[9]. In these proposed algorithms, each agent (generator) maintains a local estimate of an optimal incremental cost, which is the consensus variable, and updates it by exchanging information with only a few neighboring agents. It then follows from the consensus theory [10], [11], if the communication network is connected, all these local estimates converge to an optimal increment cost. The distributed algorithms for the EDP are progressing with generalization of communication networks, from fixed undirected networks to fixed directed networks and time-varying networks and networks with packet-dropping links.

To solve the EDP over fixed communication networks, the authors of [2] proposed a leader-follower consensus-based algorithm where the leader collects the mismatch between demand and generation. A leaderless algorithm was later developed in [3], where in addition to the consensus part, an innovation term is introduced to ensure the balance between system generation and demand. For directed fixed communication networks, the authors of [4] proposed a distributed algorithm based on the ratio consensus algorithm, and a consensus-based algorithm, where agents collectively learn the system imbalance, was developed in [5]. Other distributed algorithms for solving the EDP over fixed networks include the distributed algorithm based on the consensus and bisection method [7] and a minimum-time consensus-based algorithm [8].

Since varying communication links and communication time delays are ubiquitous in communication networks, distributed EDP algorithms over communication networks with varying communication links and/or communication time delays have been proposed in [12], [13]. In particular, the distributed algorithm proposed in [13] is based on the push-sum and gradient method, and is applicable to networks with both time-varying topologies and communication time delays. In our earlier work [14], we have proposed a robustified extension of this algorithm and showed that the robustified algorithm is able to solve the EDP over over communication networks even in the presence of i.i.d. packet drops. In this paper, we consider the case where the packet drops are Markovian. The Markovian packet loss model is used to capture loss of connectivity that occurs with temporal correlations. The challenges brought into the distributed DER algorithm also exist in consensus problem [23], [24]. The main contribution of this paper is to analyze a robustified extension of the distributed algorithm proposed in our earlier work [14] and show that this robustified distributed algorithm is able to solve the EDP over communication networks even in the presence of packet drops.

The remainder of the paper is organized as follows: In Section II, we introduce some preliminaries on graph theory and the notations used throughout the paper. In Section III, we presents the problem formulation for the economic dispatch problem (EDP) and motivate our study. In Section IV, a

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distributed algorithm based on our previous algorithm and a robustified strategy is proposed to solve the EDP over unreliable communication networks with packet-dropping communication links. Case studies are presented in Section V to illustrate and validate the proposed algorithm. Finally, concluding remarks are offered in Section VI.

II. PRELIMINARY

In this paper, we assign each bus in the power system an agent (node). Information exchanges among the agents occur over a communication network, described by a directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ denotes the index set of the agents with N being the number of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of communication links in between some pairs of the agents. In particular, $(j, i) \in \mathcal{E}$ if there exists a directed communication link from agent i to agent j. For notational convenience, we assume that $(j, j) \notin \mathcal{E}$ for all $j \in \mathcal{V}$ although each agent has an access to its own information. A directed path from node i_1 to node i_k is a sequence of nodes $i_1, \ldots, i_k$ such that $(i_{j+1}, i_j) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. If there exists a directed path from node i to node j, then node j is said to be reachable from node i. A directed graph $\mathcal{G}$ is said to be strongly connected if every node is reachable from every other node. Let $\mathcal{N}_j^{\text{in}}$ and $\mathcal{N}_j^{\text{out}}$ denote the in- and out-neighbors of node j, respectively, i.e.,

$$\mathcal{N}_j^{\text{in}} = \{i \in \mathcal{V} \mid (j, i) \in \mathcal{E}\},$$
$$\mathcal{N}_j^{\text{out}} = \{\ell \in \mathcal{V} \mid (\ell, j) \in \mathcal{E}\},$$

and d_j^{out} denotes the out-degree of node j, i.e., $d_j^{\text{out}} = |\mathcal{N}_j^{\text{out}}|$.

III. PROBLEM FORMULATION AND MOTIVATION

In this section, we first presents the mathematical formulation for the economic dispatch problem (EDP) and review our previous results. We then motivates our study.

A. Problem Formulation

The goal of the EDP is to minimize the total generation cost while meeting total demand and satisfying individual generator output limits, formulated as:

$$\min_{x_i} \quad \sum_{i=1}^{N} C_i(x_i) \tag{1a}$$
$$\text{subject to} \quad \sum_{i=1}^{N} x_i = D, \tag{1b}$$
$$x_i \in X_i := [x_i^{\min}, x_i^{\max}],\ i = 1, \ldots, N, \tag{1c}$$

where x_i is the power generation of agent i, $C_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function of agent i, $x_i^{\min}$ and $x_i^{\max}$ are respectively the lower and upper bounds of the power generation of agent i, and D is the total demand satisfying $\sum_{i=1}^{N} x_i^{\min} \leq D \leq \sum_{i=1}^{N} x_i^{\max}$ in order to ensure the feasibility of problem (1).

Traditionally, this problem have been solved by centralized algorithms, such as lambda-iteration method and the gradient search method [1]. To overcome the limitations of centralized approaches, such as a single point failure, high communication and computation burden, and limited flexibility, recently various distributed EDP algorithms have been proposed in the literature, see., e.g., [2]–[8], [13].

B. Previous Results

Since i) each cost function $C_i(\cdot)$ is convex, ii) the constraint (1b) is affine, and iii) the set $X_1 \times \cdots \times X_N$ is a polyhedral set, if we dualize problem (1) with respect to the constraint (1b), there is zero duality gap. Moreover, the dual optimal set is nonempty [16]. Consequently, solutions of the EDP can be obtained by solving its dual problem:

$$\max_{\lambda \in \mathbb{R}_+} \sum_{i=1}^{N} \Phi_i(\lambda), \tag{2}$$

where

$$\Phi_i(\lambda) = \min_{x_i \in X_i} C_i(x_i) - \lambda(x_i - D_i), \tag{3}$$

and D_i is a virtual local demand at each bus such that $\sum_{i=1}^{N} D_i = D$. Note that there is no physical meaning to D_i's.

In our previous work [13], we have develop a distributed algorithm based on the push-sum gradient method [20] to solve the dual problem (2) (which in turn solves the EDP in primal problem (1) due to the zero-duality) under the following two assumptions:

Assumption 1. *For each $i \in \{1, \ldots, N\}$, the cost function $C_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly convex and continuously differentiable, where $\mathbb{R}_+$ is the set of non-negative real numbers.*

Assumption 2. *The graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is strongly connected. Each agent j knows its own out-degree d_j^{out}.*

In the proposed algorithm, each agent j maintains a scalar variables $v_j(t)$, $w_j(t)$, $y_j(t)$, $\lambda_j(t)$, $x_j(t)$, where $x_j(t)$ and $\lambda_j(t)$ are the estimates of the optimal generation and the optimal incremental cost, respectively. At each time step t, each agent $j \in \mathcal{V}$ updates its variables through information exchange with its neighbors according to

$$w_j(t+1) = \sum_{i \in \mathcal{N}_j^{\text{in}}(t) \cup \{j\}} \frac{v_i(t)}{d_i^{\text{out}} + 1}, \tag{4a}$$
$$y_j(t+1) = \sum_{i \in \mathcal{N}_j^{\text{in}}(t) \cup \{j\}} \frac{y_i(t)}{d_i^{\text{out}} + 1}, \tag{4b}$$
$$\lambda_j(t+1) = \frac{w_j(t+1)}{y_j(t+1)}, \tag{4c}$$
$$x_j(t+1) = \text{proj}_{X_j}\left(\nabla C_j^{-1}(\lambda)\right), \tag{4d}$$
$$v_j(t+1) = w_j(t+1) - \gamma(t+1)(x_j(t+1) - D_j). \tag{4e}$$

The algorithm (4) is initialized at time instant $t = 0$, with an arbitrary value to $v_j(0)$ at agent j and $y_j(0) = 1$ for all $j \in \mathcal{V}$. In order to ensure the convergence of distributed algorithm (4) under Assumptiona 1 and 2, the step-size $\gamma(t+1) > 0$ needs to satisfy the following assumption.

Assumption 3. *The sequence $(\gamma(t))_{t \in \mathbb{N}}$ satisfies the following conditions:*

$$\sum_{t=1}^{\infty} \gamma(t) = \infty, \quad \sum_{t=1}^{\infty} \gamma^2(t) < \infty, \text{ and}$$
$$0 < \gamma(t) \leq \gamma(s) \quad \text{for all } t > s \geq 0. \tag{5}$$

Algorithm 1 Distributed algorithm for the EDP over networks with packet-dropping communication links

1: **Input:** $\upsilon_j(0)$, $\sigma_j(0)=0$, $\rho_{ji}(0)=0$, $\forall i\in\mathcal{N}_j^{\text{in}}$,
$y_j(0)=1$, $\eta_j(0)=0$, $\upsilon_{ji}(0)=0$, $\forall i\in\mathcal{N}_j^{\text{in}}$.

2: **for** $t\geq 0$:

3: **Compute:**

$$\sigma_j(t+1)=\sigma_j(t)+\upsilon_j(t)/(1+d_j^{\text{out}}), \tag{6a}$$
$$\eta_j(t+1)=\eta_j(t)+y_j(t)/(1+d_j^{\text{out}}). \tag{6b}$$

4: **Broadcast:** $\sigma_j(t+1)$ and $\eta_j(t+1)$ to all $\ell\in\mathcal{N}_j^{\text{out}}$.

5: **Receive:** From each $i\in\mathcal{N}_j^{\text{in}}$ receive $\sigma_i(t+1)$ and $\eta_i(t+1)$ if $r_{ji}(t)=1$.

6: **Set:**

$$\rho_{ji}(t+1)=\begin{cases}\sigma_i(t+1), & \text{if } r_{ji}(t)=1 \text{ or } i=j,\\ \rho_{ji}(t), & \text{if } r_{ji}(t)=0.\end{cases} \tag{7a}$$
$$\upsilon_{ji}(t+1)=\begin{cases}\eta_i(t+1), & \text{if } r_{ji}(t)=1 \text{ or } i=j,\\ \upsilon_{ji}(t), & \text{if } r_{ji}(t)=0.\end{cases} \tag{7b}$$

7: **Compute:**

$$w_j(t+1)=\sum_{i\in\mathcal{N}_j^{\text{in}}\cup\{j\}}(\rho_{ji}(t+1)-\rho_{ji}(t)), \tag{8a}$$
$$y_j(t+1)=\sum_{i\in\mathcal{N}_j^{\text{in}}\cup\{j\}}(\upsilon_{ji}(t+1)-\upsilon_{ji}(t)), \tag{8b}$$
$$\lambda_j(t+1)=\frac{w_j(t+1)}{y_j(t+1)}, \tag{8c}$$
$$x_j(t+1)=\text{proj}_{X_j}\left(\nabla C_j^{-1}(\lambda_j(t+1))\right), \tag{8d}$$
$$\upsilon_j(t+1)=w_j(t+1)-\gamma(t+1)(x_j(t+1)-D_j). \tag{8e}$$

The proposed algorithm is applicable to the case where the communication network is perfect with reliable communication links. However, the communication network maybe unreliable with packet-dropping communication links. To propose a resilient distributed algorithm for solving the EDP over networks with packet-dropping communication links, we integrate the algorithm (4) with the running-sum method proposed in [18], [19] and propose the algorithm given in Algorithm 1 in our previous work [14].

Intuitively, compared to the distributed algorithm (4), in Algorithm 1, each agent j keeps track of certain additional variables, includes them in the message it broadcasts, and uses them in the update equations. In particular, besides variables $w_j(t+1)$, $y_j(t+1)$, $\lambda_j(t+1)$, $x_j(t+1)$ and $\upsilon_j(t+1)$, each agent j at time instant $t+1$ also maintains additional variables $\sigma_j(t+1)=\sum_{k=0}^{t}\frac{\upsilon_j(k)}{1+d_j^{\text{out}}}$ and $\eta_j(t+1)=\sum_{k=0}^{t}\frac{y_j(k)}{1+d_j^{\text{out}}}$, which are the running sums of υ_j and y_j respectively, and $\rho_{ji}(t+1)$ and $\upsilon_{ji}(t+1)$ for $i\in\mathcal{N}_j^{\text{in}}$ which keep track of the running sum of υ_j and y_j received at agent j from agent i. These variables are updated according to Algorithm 1. Notice that each agent j computes the running sums $\sigma_j(t+1)$ and $\eta_j(t+1)$ according to (6) and sends them to all its outgoing neighboring agents. The running sums are initialized to $\sigma_j(0)=0$ and $\eta_j(0)=0$ for all $j\in\mathcal{V}$. The variables ρ_{ji} and υ_{ji} remain unchanged until a transmission is successfully received on link $(j,i)\in\mathcal{E}$, i.e., $r_{ji}(t)=1$. It is clear that each agent knows the running sum of itself, i.e., $\rho_{jj}(k+1)=\sigma_j(t+1)$ and $\upsilon_{jj}(t+1)=\eta_j(t+1)$. Finally, agent j updates the values of w_j and y_j to be the sum of the differences between the two most recently received running sum values according to (8a) and (8b) while other update equations (8c) to (8e) are the same as (4c)-(4e).

C. Motivation

Note that although in [14], we have showed that Algorithm 1 solves the EDP almost surely for the case where the communication network is unreliable with packet-dropping communication links. The packet drops are assumed to be independent and identically distributed (i.i.d.). However, in practice the switching of the network is always with temporal correlations, for example, the communication links are intermittent due to Rayleigh fading. This motivate our study in this paper. More specifically, we will consider a more general and realistic case where the packet drops are Markovian, and show that the proposed robustified algorithm in Algorithm 1 also solves the EDP almost surely even when the packe-drops are Markovian.

For this purpose, we introduce the communication network model For a fixed strongly connected communication network $\mathcal{G}:=(\mathcal{V},\mathcal{E})$, due to packet-dropping communication, the existing communication link from agent i to agent j, $(j,i)\in\mathcal{E}$ may randomly fail with some nonzero probability. Let $(\Omega,\mathcal{F})$ denote the measurable space generated by the intermittent communication over $\mathcal{E}$. We use an indicator variable $r_{ji}(t;\omega):\Omega\to\{0,1\}$ to denote if the communication over $(j,i)\in\mathcal{E}$ is successful or not: let $r_{ji}(t;\omega)=1$ if the information from agent i is received by agent j at time t; otherwise let $r_{ji}(t;\omega)=0$. Notice that for each link $(j,i)\in\mathcal{E}$, $r_{ji}(t;\omega)$ can be defined accordingly at time t. We let $r(t;\omega)$ be a random vector containing all random variables of $\{r_{ji}(t;\omega):(j,i)\in\mathcal{E}\}$ in a fixed order and denote $p_{ji}(t):=\mathbf{E}[r_{ji}(t)]$. We make the following assumption on the sequence $(r(t))_{t\in\mathbb{N}}$, where $\mathbb{N}=\{0,1,2\ldots\}$.

Assumption 4. *The binary random vectors $r(0),r(1),\ldots$ has the following property:*

(i). *The random vector $r(0),r(1),\ldots$ are two-state Markov chain.*

(ii). *The Markov chains are ergodic.*

IV. Main Results

In this section, we present the convergence results for the proposed Algorithm 1 under Markovian packet-dropping assumption. We first show that, for each agent j, a subsequence of $(\lambda_j(t))_{t\in\mathbb{N}}$ almost surely (a.s.) converges to the same optimal incremental cost λ^*, which is an optimal incremental cost. By doing so, we then show that the proposed distributed Algorithm 1 is able to solve the EDP over networks with Markovian packet-dropping communication links. Here we focus on the almost sure convergence analysis (i.e., pointwise convergence on the sample space Ω). Our main result is obtained with the help of results from weak ergodicity theory and the supermartingale convergence theorem.

In order to present the main results, the following property from [19] is needed. The property holds under both i.i.d. and Markovian packet drops. The presentation of the property will be adopted to the context of this paper.

Lemma 1 ([19] Lemma 2)**.** *Assume that Assumptions 2 and 4 are satisfied. For Algorithm 1, we have* $\mathbf{P}(y_j(t) \geq C \text{ i.o.}) = 1$*, where* $C := \frac{1}{N^N}$ *and "i.o." is short for "infinitely often".*

By virtue of Lemma 1, we can define a sequence of time instants for each agent j, at which $y_j(t) \geq C$ is satisfied, as follows:

$$
\begin{aligned}
t_{j,1} &= \min\{t : y_j(t) \geq C\}, \\
t_{j,2} &= \min\{t : y_j(t) \geq C, t > t_{j,1}\}, \\
&\vdots \\
t_{j,k} &= \min\{t : y_j(t) \geq C, t > t_{j,k-1}\}.
\end{aligned}
$$

Lemma 1 implies that the sequence $\mathcal{T}_j := (t_{j,1}, \ldots, t_{j,k}, \ldots)$ has countably infinite elements a.s. for all $j \in \mathcal{V}$.

We now present our main result, which states that $\lambda_j(t_{j,k})$, where $t_{j,k} \in \mathcal{T}_j$, converges to λ^* almost surely.

Theorem 1 (Almost Sure Convergence)**.** *Assume that Assumptions 1, 2, 3, and 4 are satisfied. Then the sequence* $(\lambda_j(t_{j,k}))_{t_{j,k} \in \mathcal{T}_j}$ *for any* $j \in \mathcal{V}$ *converges to the same random optimal incremental cost* $\lambda^*(\omega)$ *almost surely, i.e.,* $\mathbf{P}\left(\lim_{k\to\infty} \|\lambda_j(t_{j,k};\omega) - \lambda^*(\omega)\| = 0\right) = 1$ *for all* $j \in \mathcal{V}$*.*

Proof. The proof of Theorem 1 is somewhat involved. Overall, the proof is similar to that of Lemma 2 [14]. In what follows, we will give a sketch.

As we do in [14], we will build our analysis by using the augmented graph idea. For each communication link $(j, i) \in \mathcal{E}$, we add a virtual buffer agent $b_{(j,i)}$ which stores the mass that may have otherwise been lost due to packet drops over the link (j, i).

We next introduce the variables $\tilde{w}_\ell$, $\tilde{y}_\ell$, and $\tilde{v}_\ell$ for the virtual agents $\ell = b_{(j,i)}$ in the augmented graph, with initial conditions 0. The updates of $\tilde{w}_\ell$, $\tilde{y}_\ell$, and $\tilde{v}_\ell$ for $\ell = b_{(j,i)}$ are as follows:

$$
\tilde{w}_l(t+1) = \begin{cases} \tilde{w}_l(t) + v_i(t)/(1 + d_j^{\text{out}}), & \text{if } r_{ji}(t) = 0, \\ 0, & \text{otherwise;} \end{cases}
$$

$$
\tilde{y}_l(t+1) = \begin{cases} \tilde{y}_l(t) + y_i(t)/(1 + d_j^{\text{out}}), & \text{if } r_{ji}(t) = 0, \\ 0, & \text{otherwise;} \end{cases}
$$

and $\tilde{v}_l(t+1) = \tilde{w}_l(t+1)$. Given an arbitrarily given order $l_1, l_2, \ldots, l_L$ for the elements of $\mathcal{E}$, we define $\tilde{w} = [w_1, \ldots, w_N, \tilde{w}_{b_{l_1}}, \ldots, \tilde{w}_{b_{l_L}}]^\top$, $\tilde{y} = [y_1, \ldots, y_N, \tilde{y}_{b_{l_1}}, \ldots, \tilde{y}_{b_{l_L}}]^\top$, $\tilde{v} = [v_1, \ldots, v_N, \tilde{v}_{b_{l_1}}, \ldots, \tilde{v}_{b_{l_L}}]^\top$. With these notations, Algorithm 1 can be rewritten into a matrix form as

$$
\tilde{w}(t+1) = M(t)\tilde{v}(t), \tag{9a}
$$

$$
\tilde{y}(t+1) = M(t)\tilde{y}(t), \tag{9b}
$$

$$
\lambda_j(t+1) = \frac{w_j(t+1)}{y_j(t+1)},\ j \in \mathcal{V}, \tag{9c}
$$

$$
x_j(t+1) = \text{proj}_{X_j}\left(\nabla C_j^{-1}(\lambda)\right), \tag{9d}
$$

$$
\tilde{v}(t+1) = \tilde{w}(t+1) - \gamma(t+1)[x^\top(t+1) - \tilde{D}^\top, \mathbf{0}]^\top, \tag{9e}
$$

where $\tilde{D} = [D_1, \ldots, D_N]^\top$. Some immediate observations from (9) are as follows: $M(t) \in \mathbb{R}^{\tilde{N}\times\tilde{N}}$ is a random matrix, depending on a set of random variables $\{r_{ji}(t) | (j,i) \in \mathcal{E}\}$ and is column stochastic.

Consider a sequence $(x(t))_{t\in\mathbb{N}}$, where $x(t)$ is defined as

$$
x(t) = M(t)^\top \cdot \cdots \cdot M(s)^\top x(s), \tag{10}
$$

where $t \geq s$. For a constant $B > 0$, consider a specific event for time t, which is defined as

$$
A_B(t) = \{\omega \in \Omega : r_{ij}(k;\omega) = 1,\ (j,i) \in \mathcal{E}, k \in [t, k+B]\}.
$$

It contains all situations, for which the links of $\mathcal{E}$ are all reliable. The outcome of $M(t) \cdot \cdots \cdot M(t+B)$ corresponding to the realizations in $A(t)$ is denoted as M. By the irrecuciblity of the Markov chain, there is a $B > 0$ such that the probability $A(t) > 0$.

When there is no packet drop over $\mathcal{G}$, there are paths from each virtual buffer agent to the actual agents and paths from any actual agents to the other actual agents. Also observe that in M the diagonal elements corresponding to the actual agents are all strictly positive. Therefore, if we look at N consecutive matrices M jointly, there are at least one path from any actual or virtual agent to any actual ones. A mathematical formulation of this property is that: letting $T = M^N$ for $i \in \mathcal{V}$, there holds $T(i, :) > (1/N)^N$.

Then, following results on coefficients of ergodicity [21], [22], we have that: if

$$
z' = T^\top z,
$$

then

$$
\max_{i\in\mathcal{V}} z' - \min_{i\in\mathcal{V}} z' \leq \left(1 - (1/N)^N\right)\left(\max_{i\in\mathcal{V}} z - \min_{i\in\mathcal{V}} z\right). \tag{11}
$$

The rest of the proof directly follows the proof of Lemma 2 in [14]. □

Theorem 1 together with the update equation for the generation in (8d) and the zero duality between the primal problem (1) and the dual problem (**??**) leads to the following theorem.

Theorem 2. *Assume that Assumptions 1, 2, 3, and 4 are satisfied. Then the distributed Algorithm 1 solves the EDP in the sense that* $\lambda_j(t_{j,k}) \to \lambda^*$ *and* $x_j(t_{j,k}) \to x_j^*$ *a.s. as* $k \to \infty$*, where* λ^* *and* x_j^* *are respectively an optimal incremental cost and the optimal generation for generator* j*.*

V. Case Studies

In this section, we first present an motivating example for this study. We consider the IEEE 5-bus system shown in Fig. 1, where each bus is connected with a generator. The parameters of generators are given in Table I. The communication network is not necessarily the same as the physical topology and is modeled by a fixed directed graph depicted in Fig. 2.

We consider the case where the network randomly switches following a Markov chain. Since the packet drops are random, the iteration results at each agent vary from one simulation to another. Nevertheless, the proposed Algorithm 1 always solves the EDP. The simulation results of a particular run are given

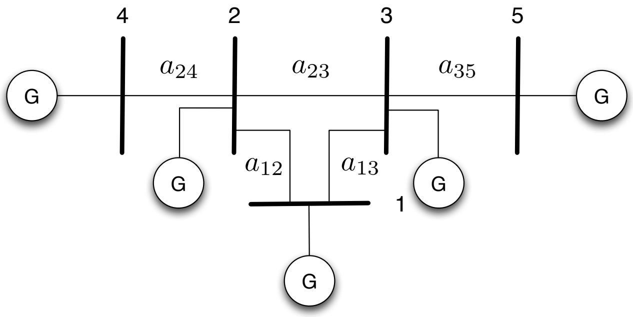


Fig. 1. IEEE five-bus power system.

TABLE I
GENERATOR PARAMETERS

Bus	a_i (kW^{2}h)	b_i (\$/kWh)	c_i (\$/h)	Range (kW)
1	0.00024	0.0267	0.38	[30,60]
2	0.00052	0.0152	0.65	[20,60]
3	0.00042	0.0185	0.4	[50,200]
4	0.00052	0.0152	0.65	[20,60]
5	0.00031	0.0297	0.3	[20,140]

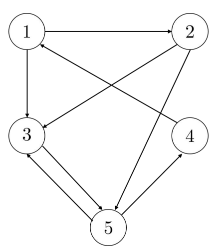
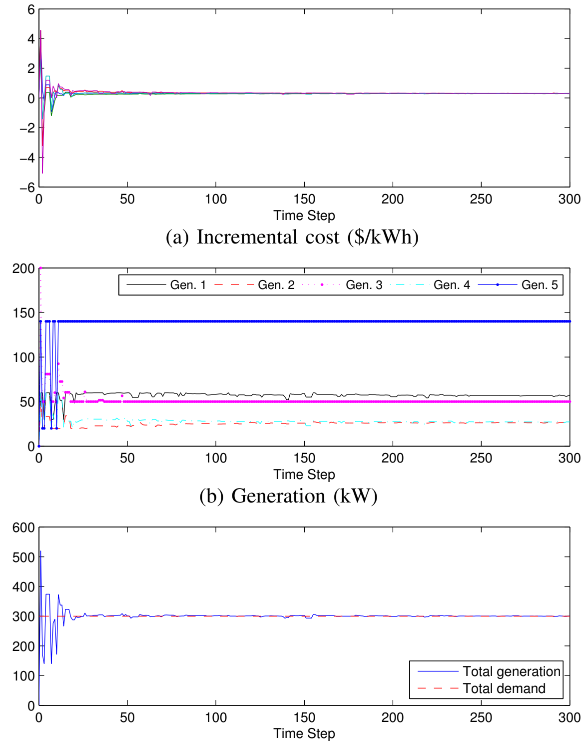


Fig. 2. Directed communication network.

in Fig. 3. As can been seen, even in the presence of Markovian packet-dropping communication links, each variable still converges to the optimal value as the case yet with a slower convergence rate.

VI. Conclusions

This paper considers the economic dispatch problem (EDP) over directed communication networks with Markovian packet-dropping links. We propose a robustified extension of the distributed algorithm and show that this robustified distributed algorithm is able to solve the EDP even in the presence of Makovian packet drops as long as the underlying communication network is strongly connected with a positive probability. One interesting direction is to explicitly characterize the convergence rate of the proposed algorithm. Another interesting direction is to extend the proposed distributed algorithm to accommodate additional physical models and constraints, such as transmission line loss, power flow, and transmission line flow constraints.



(a) Incremental cost (\$/kWh)

(b) Generation (kW)

(c) Generation (kW)

Fig. 3. Results for networks with packet-dropping communication links

References

[1] A. J. Wood and B. F. Wollenberg, *Power Generation, Operation, and Control*, 2nd ed. John Wiley & Sons, 1996.

[2] Z. Zhang and M.-Y. Chow, "Convergence analysis of the incremental cost consensus algorithm under different communication network topologies in a smart grid," *IEEE Trans. Power Syst.*, vol. 27, no. 4, pp. 1761–1768, Nov. 2012.

[3] S. Kar and G. Hug, "Distributed robust economic dispatch in power systems: A consensus+innovations approach," in *Proc. IEEE Power Energy Soc. Gene. Meet.*, Jul. 2012.

[4] A. Dominguez-Garcia, S. Cady, and C. N. Hadjicostis, "Decentralized optimal dispatch of distributed energy resources," in *Proc. 51st IEEE Conf. Decision and Control (CDC)*, Dec. 2012, pp. 3688–3693.

[5] S. Yang, S. Tan, and J.-X. Xu, "Consensus based approach for economic dispatch problem in a smart grid," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4416–4426, Nov. 2013.

[6] N. Rahbari-Asr, U. Ojha, Z. Zhang, and M.-Y. Chow, "Incremental welfare consensus algorithm for cooperative distributed generation/demand response in smart grid," *IEEE Trans. Smart Grid*, vol. 6, no. 5, pp. 2836–2845, Nov. 2014.

[7] H. Xing, Y. Mou, M. Fu, and Z. Lin, "Distributed bisection method for economic power dispatch in smart grid," *IEEE Trans. Power Syst.*, vol. 30, no. 6, pp. 3024–3035, Nov. 2015.

[8] T. Yang, D. Wu, Y. Sun, and J. Lian, "Minimum-time consensus based approach for power system applications," *IEEE Trans. Ind. Electron.*, vol. 63, no. 2, pp. 1318–1328, Feb. 2016.

[9] C. Zhao, J. He, P. Cheng, and J. Chen, "Consensus-based energy management in smart grid with transmission losses and directed communication," *IEEE Trans. Smart Grid*, 2016, to appear.

[10] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[11] W. Ren, R. W. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[12] Y. Xu, K. Cai, T. Han, and Z. Lin, "A fully distributed approach to resource allocation problem under directed and switching topologies," in *Proc. 10th Asian Control Conference (ASCC)*, May 2015.

[13] T. Yang, J. Lu, D. Wu, J. Wu, G. Shi, Z. Meng, and K. H. Johansson, "A distributed algorithm for economic dispatch over time-varying directed networks with delays," *IEEE Trans. Ind. Electron.*, vol. 64, no. 6, pp. 5095–5106, Jun. 2017.

[14] J. Wu, T. Yang, D. Wu, K. Kalsi, and K. H. Johansson, "Distributed

optimal dispatch of distributed energy resources over lossy communication networks," *IEEE Transactions on Smart Grid*, vol. 8, no. 6, pp. 3125–3137, Nov. 2017, to appear.

[15] C. Godsi and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 2001, vol. 207.

[16] D. P. Bertsekas, A. Nedić, and A. Ozdaglar, *Convex Analysis and Optimization*. Belmont, MA: Athena Scientific, 2003.

[17] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 1999.

[18] A. Dominguez-Garcia, C. N. Hadjicostis, and N. Vaidya, "Resilient networked control of distributed energy resources," *IEEE J. Select. Areas Commun.*, vol. 30, no. 6, pp. 1137–1148, Jul. 2012.

[19] C. N. Hadjicostis, N. Vaidya, and A. Dominguez-Garcia, "Robust distributed average consensus via exchange of running sums," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1492–1507, Jun. 2016.

[20] A. Nedić and A. Olshevsky, "Distributed optimization over time-varying directed graphs," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 601–615, Mar. 2015.

[21] E. Seneta, *Non-negative matrices and Markov chains*. Springer Science & Business Media, 2006.

[22] J. Hajnal and M. S. Bartlett, "Weak ergodicity in non-homogeneous markov chains," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 54, no. 02, pp. 233–246, 1958.

[23] M. Huang and S. Dey and G. N. Nair and J. H. Manton, "Stochastic consensus over noisy networks with Markovian and arbitrary switches," *Automatica*, vol. 46, no. 10, pp. 1571–1583, 2010.

[24] I. Matei and J. S. Baras and C. Somarakis "Convergence results for the linear consensus problem under markovian random graphs," *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 1574-1591 2013.