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95-6

Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.
(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?
(b) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$ where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?
(c) Prove that for every real number c the operator $T - cI$ is invertible.
(d) Prove that if $\mathcal{B}$ is any ordered basis for $\mathbb{R}^2$ and $[T]_\mathcal{B} = A$, then $A_{12}A_{21} \neq 0$.
Solution: (a) $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
(b) $T\alpha_1 = (-2, 1) = -\frac{1}{3}(1, 2) - \frac{5}{3}(1, -1)$ and $T\alpha_2 = (1, 1) = \frac{2}{3}(1, 2) + \frac{1}{3}(1, -1)$ so
$[T]_\mathcal{B} = \begin{pmatrix} -1/3 & -5/3 \\ 2/3 & 1/3 \end{pmatrix}$.
(c) If T has an eigenvalue c and a corresponding eigenvector $v = (x, y)$, then $Tv = (-y, x) = c(x, y)$ so $y = -cx, x = cy = -c^2x$ leading to contradiction.
(d) Otherwise suppose $A_{12} = 0$, then $T - A_{22}I$ is invertible since its matrix under $\mathcal{B}$ has a zero column, leading to contradiction.

95-7

Let $T \in \mathcal{L}(\mathbb{R}^3)$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^3$?
(b) What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$?
(c) Prove that T is invertible and give a rule for T^{-1} like the one which defines T.
Solution: (a) $T = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}$.
(b) $T\alpha_1 = (4, -2, 3) = \frac{17}{4}\alpha_1 - \frac{3}{4}\alpha_2 - \frac{1}{2}\alpha_3$, $T\alpha_2 = (-2, 4, 9) = \frac{35}{4}\alpha_1 + \frac{15}{4}\alpha_2 - \frac{7}{2}\alpha_3$, $T\alpha_3 = (7, -3, 4) = \frac{11}{2}\alpha_1 - \frac{3}{2}\alpha_2$, so the matrix of T is

$$\begin{pmatrix} \frac{17}{4} & -\frac{3}{4} & -\frac{1}{2} \\ \frac{35}{4} & \frac{15}{4} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{3}{2} & 0 \end{pmatrix}$$

(c) $T^{-1} = \begin{pmatrix} 4/9 & 2/9 & -1/9 \\ 8/9 & 13/9 & -2/9 \\ -1/3 & 2/3 & -1/3 \end{pmatrix}$ so

$$T^{-1}(x_1, x_2, x_3) = \left(\frac{4}{9}x_1 + \frac{2}{9}x_2 - \frac{1}{9}x_3, \frac{8}{9}x_1 + \frac{13}{9}x_2 - \frac{2}{9}x_3, -\frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3\right).$$

96-8

Let $\theta \in \mathbb{R}$, prove that the following are similar over $\mathbb{C}$:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Proof: Consider $T \in \mathcal{L}(\mathbb{C}^2) : (z, w) \mapsto (z\cos\theta - w\sin\theta, z\sin\theta + w\cos\theta)$, then for $\alpha_1 = (1, -i)$ and $\alpha_2 = (1, i)$, $T\alpha_1 = e^{i\theta}\alpha_1$ and $T\alpha_2 = e^{-i\theta}\alpha_2$, and α_1, α_2 form a base of $\mathbb{C}^2$, so they are similar matrices.

96-9

Let V be a finite dimensional vector space over the field F and let $S, T \in \mathcal{L}(V)$. We ask: When do there exist ordered bases $\mathcal{B}, \mathcal{B}'$ for V such that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$? Prove that such bases exist iff there is an invertible linear operator $U \in \mathcal{L}(V)$ such that $T = USU^{-1}$.
Proof: $\exists\mathcal{B}, \mathcal{B}'$ such that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'} \iff \exists\mathcal{B}$ such that $[S]_{\mathcal{B}} = [T]_E$ where $E = \{e_1, \cdots, e_n\} \iff$ $\exists P \in GL(n, F)$ such that $P[S]_E P^{-1} = [T]_E \iff \exists U = L_P \in \mathcal{L}(V)$ such that $T = USU^{-1}$.

96-10

We have seen that $T \in \mathcal{L}(\mathbb{R}^2)$ defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This operator satisfies $T^2 = T$. Prove that if S is a linear operator on $\mathbb{R}^2$ such that $S^2 = S$, then $S = 0$ or $S = I$ or there is an ordered basis $\mathcal{B}$ such that $[S]_{\mathcal{B}} = A$.
Proof: If the minimal polynomial $P(x) = x$ then $S = 0$, if $P(x) = x - 1$ then $S = I$. Otherwise $P(x) = x^2 - x$. Then there exists $v \neq 0$ such that $(S - I)v = 0$, and $u \neq 0$ such that $Su = 0$. So $Sv = v$ and $Su = 0$. Clearly u, v are linearly independent, so $[S]_{\mathcal{B}} = A$ under the base $\mathcal{B} = \{u, v\}$.

96-12

Let V be a $n-$dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$ be an ordered basis for V .
(a) According to Theorem1, there is a unique $T \in \mathcal{L}(V)$ such that $T\alpha_j = \alpha_{j+1}, j = 1, \cdots, n-1, T\alpha_n = 0$. What is the matrix of T in the ordered basis $\mathcal{B}$?
(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.
(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix of S in the ordered basis $\mathcal{B}'$ is the matrix A of part (a).
(d) Prove that if $M, N \in F^{n\times n}$ such that $M^n = N^n = 0$ but $M^{n-1}, N^{n-1} \neq 0$, then $M \sim N$.
Proof: (a) $[T]_{\mathcal{B}} = (\delta_{i+1,j})_{1\leqslant i,j\leqslant n}$. (b) Note that for $k < n$, $T^k(x_1, \cdots, x_n) = (0, \cdots, 0, x_1, \cdots, x_{n-k})$ under the base $\mathcal{B}$, so $T^n = 0$ but $T^{n-1}(1, 0, \cdots, 0) = (0, \cdots, 0, 1) \neq 0$.
(c) Since $S^n = 0$ but $S^{n-1} \neq 0$, the minimal polynomial of S is $P(x) = x^n$, so take v such that $S^{n-1}v \neq 0$, then $v, Sv, \cdots, S^{n-1}v$ are linearly independent, forming a base of V (if $c_0 v + c_1 Sv + \cdots + c_{n-1}S^{n-1}v = 0$ then $c_0 S^{n-1}v = 0$ so $c_0 = 0$ etc). Under this basis, the matrix of S is A.
(d) Such M, N are the matrices of T under different bases, so they are similar.

97-13

Let V, W be finite dimensional vector spaces over the field F and let $T \in \mathcal{L}(V, W)$. If $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \cdots, \beta_m\}$ are ordered bases for V, W, define the linear transformations $E^{p,q}$ as in the proof of Theorem5: $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$. Then $E^{p,q}$ form a basis for $\mathcal{L}(V, W)$, and so

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}$$

for certain scalars A_{pq}. Show that the matrix A with entries $A(p,q) = A_{pq}$ is precisely the matrix $[T]_{\mathcal{B},\mathcal{B}'}$.
Solution: For any $v = \sum_{i=1}^{n} c_i \alpha_i$,

$$T(v) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \left(\sum_{i=1}^{n} c_i \alpha_i \right) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} c_q \beta_p = \sum_{p=1}^{m} \left(\sum_{q=1}^{n} A_{pq} c_q \right) \beta_p$$

Hence $A(p,q)$ is the matrix $[T]_{\mathcal{B},\mathcal{B}'}$.