2025/9/22

1.

If a/b is rational, let $n=\lfloor a/b \rfloor, b=p^s\cdot c$ and $d=\mathrm{ord}_p(c), m=(p^d-1)/c$, then

$$\frac{a}{b}=n+p^{-s}\cdot\frac{(a-nb)m}{p^d-1}=n+p^{-s}\sum_{j=1}^{\infty}(a-nb)mp^{-jd}.$$

Conversely, if

$$x=n+\sum_{j=1}^{N}c_jp^{-j}+p^{-N}\sum_{j=1}^{\infty}mp^{-jd}$$

where $n\in\mathbb{Z}, c_j\in\{0,1,\cdots,p-1\}, m\in\{0,1,\cdots,p^d-1\}$, then

$$x=n+\sum_{j=1}^{N}c_jp^{-j}+p^{-N}m\frac{1}{p^d-1}$$

is clearly rational.

3.

Determine all $(a,b)\in\mathbb{Q}^2$ such that for any

$$q=p+\frac{2-p^2}{ap+b},$$

(1) $p>0, p^2<2$ implies $q>p$ and $q^2<2$,
and (2) $p>0, p^2>2$ implies $0<q<p$ and $q^2>2$.
Solution: Since $(q-p)/(2-p^2)>0$ for any $p>0$, $a,b\geqslant 0$. Note that

$$q^2-2=(q+\sqrt{2})(p-\sqrt{2})\left(1-\frac{p+\sqrt{2}}{ap+b}\right),$$

so for any $p>0$, $ap+b>p+\sqrt{2}$, i.e. $a\geqslant 1$ and $b^2\geqslant 2$. Clearly such (a,b) satisfy (1)(2).

5.

Prove that for $a>0, n\in N, \min\{a,1\}\leqslant a^{1/n}\leqslant\max\{a,1\}$.
Proof: If $a\leqslant 1$, then a^x is monotonically decreasing, so $a^1\leqslant a^{1/n}\leqslant a^0$. Otherwise a^x is monotonically increasing, so $a^0\leqslant a^{1/n}\leqslant a^1$. In both cases $\min\{a,1\}\leqslant a^{1/n}\leqslant\max\{a,1\}$.

6.

Let $n\geqslant 2$, $\alpha_1,\cdots,\alpha_n\geqslant 0$, such that $\alpha_1+\cdots+\alpha_n=1$. Prove that for $x_1,\cdots,x_n>0$,
$x_1^{\alpha_1}\cdots x_n^{\alpha_n}\leqslant\alpha_1x_1+\cdots+\alpha_nx_n$.
Proof: Let $G=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, then

$$\sum_{i=1}^{n}\alpha_ix_i-G=\sum_{i=1}^{n}\alpha_i\int_G^{x_i}1-\frac{G}{x}\,\mathrm{d}x\geqslant 0.$$

(Since each $\int_G^{x_i}(1-G/x)\,\mathrm{d}x\geqslant 0$).

8.

Prove that in $\mathbb{R} \cup \{-\infty, \infty\}$,
(1) $\inf_{\alpha\in I}(-x_\alpha) = -\sup_{\alpha\in I} x_\alpha$,
Let $M = \sup_{\alpha\in I} x_\alpha$, then $-M$ is a lower bound of $\{-x_\alpha\}$. For any lower bound m' of $\{-x_\alpha\}$, $-m'$ is an upper bound of $\{x_\alpha\}$, so $-m' \geqslant M$ i.e. $m' \leqslant -M$. Hence $-M = \inf_{\alpha\in I} x_\alpha$.
(2) Prove that

$$\begin{aligned}\inf_{\alpha\in I} x_\alpha + \inf_{\alpha\in I} y_\alpha \leqslant \inf_{\alpha\in I}(x_\alpha + y_\alpha) \leqslant \sup_{\alpha\in I} x_\alpha + \inf_{\alpha\in I} y_\alpha \\ \leqslant \sup_{\alpha\in I}(x_\alpha + y_\alpha) \leqslant \sup_{\alpha\in I} x_\alpha + \sup_{\alpha\in I} y_\alpha.\end{aligned}$$

Proof: Note that $\inf_{\alpha\in I} x_\alpha + \inf_{\alpha\in I} y_\alpha = \inf_{\alpha\in I}(y_\alpha + \inf_{\beta\in I} x_\beta) \leqslant \inf_{\alpha\in I}(x_\alpha + y_\alpha)$. Likewise $\inf_{\alpha\in I}(x_\alpha + y_\alpha) \leqslant \inf_{\alpha\in I}(\sup_{\beta\in I} x_\beta + y_\alpha) \leqslant \sup_{\alpha\in I} x_\alpha + \inf_{\alpha\in I} y_\alpha$, and the other two inequalities are the same.
(3) If $x_\alpha > 0$, then

$$\sup_{\alpha\in I}\frac{1}{x_\alpha} = \frac{1}{\inf_{\alpha\in I} x_\alpha}.$$

Proof: Let $m = \inf_{\alpha\in I} x_\alpha$, then $1/x_\alpha \leqslant 1/m$ for any $\alpha \in I$. If $m = 0$, then for any $n \in \mathbb{N}$, there exists $\alpha \in I$ such that $x_\alpha < 1/n$, then $1/x_\alpha > n$ so $\sup_{\alpha\in I} 1/x_\alpha = \infty$. Otherwise $m > 0$, so for any $\varepsilon > 0$, there exists $\alpha \in I$ such that $x_\alpha < m + \varepsilon$, so

$$\frac{1}{x_\alpha} > \frac{1}{m+\varepsilon} = \frac{1}{m} - \frac{\varepsilon}{m(m+\varepsilon)} > \frac{1}{m} - \frac{\varepsilon}{m^2}.$$

Hence $\sup_{\alpha\in I} 1/x_\alpha = 1/m$.
(4) Suppose $x_\alpha, y_\alpha > 0$, then

$$\inf_{\alpha\in I} x_\alpha \inf_{\alpha\in I} y_\alpha \leqslant \inf_{\alpha\in I}(x_\alpha y_\alpha) \leqslant \sup_{\alpha\in I} x_\alpha \inf_{\alpha\in I} y_\alpha \leqslant \sup_{\alpha\in I}(x_\alpha y_\alpha) \leqslant \sup_{\alpha\in I} x_\alpha \sup_{\alpha\in I} y_\alpha.$$

Same as (2).

10.

For $x \in \mathbb{R}$, let $B(x) = \{b^t : t \in \mathbb{Q}, t \leqslant x\}$, and define $b^x = \sup B(x)$. $B(x)$ is clearly non-empty and bounded by $b^{\lfloor x\rfloor+1}$, so it is well-defined. If $r \in \mathbb{Q}$, then

$$b^r = \sup B(r),\ \forall r \in \mathbb{Q}.$$

Proof: $b^r \in B(r)$ and since b^t is monotonically increasing, $b^r \geqslant \sup B(r)$, hence $b^r = \sup B(r)$.

11.

Prove that for any $x, y > 0$, $(a^x)^y = a^{xy}$, and $a^x b^x = (ab)^x$.
Proof:

$$a^{xy} = \sup\{a^t : t \leqslant xy\} = \sup\{a^{uv} : u \leqslant x, v \leqslant y\} = (a^u)^v$$

$$(ab)^x = \sup\{(ab)^t : t \leqslant x\} = \sup\{a^t b^t : t \leqslant x\} = a^x b^x.$$

15.

Let $a, x, y > 0$, $a \neq 1$. Prove that $\log_a(xy) = \log_a x + \log_a y$.
Proof: Note that

$$a^{\log_a(xy)} = xy = a^{\log_a(x)} a^{\log_a(y)} = a^{\log_a(x)+\log_a(y)}.$$

Since a^x is strictly monotonic, $\log_a(xy) = \log_a x + \log_a y$.