PSA

A1) Prove that e^x is uniformly continuous on $(-\infty, 0]$ but not on $\mathbb{R}$.

Proof: For $y < x \leqslant 0$ and $|x - y| < \varepsilon$,

$$e^x - e^y = e^y(e^{y-x} - 1) \leqslant e^{\varepsilon} - 1.$$

Hence e^x is uniformly continuous on $(-\infty, 0]$. But for any $\delta > 0$, there exists y and $x = y + \delta$ such that

$$e^x - e^y = e^y \cdot (e^{\delta} - 1) > 1.$$

Therefore e^x is not uniformly continuous on $\mathbb{R}$.

A2) Prove that the function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R},\ (x, \alpha) \mapsto x^{\alpha}$ is continuous on $\mathbb{R}_{>0} \times \mathbb{R}$.

Proof: For $(x, \alpha), (y, \beta)$,

$$|x^{\alpha} - y^{\beta}| \leqslant |x^{\alpha} - y^{\alpha}| + |y^{\alpha} - y^{\beta}|.$$

Since x^{α} and a^x are both continuous (as functions of x), so is $x^{\alpha} : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$.

A3) Prove that for any $x, y > 0$ and α, β, $(xy)^{\alpha} = x^{\alpha}y^{\alpha}$, $(x^{\alpha})^{\beta} = x^{\alpha\beta}$, $a^{\log_a x} = x$. If $x > 0, y > 0$, then $a^{x+y} = a^x a^y$, $\log_a(x \cdot y) = \log_a x + \log_a y$.

Proof: See PSE of HW2.

A4) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$ defined on $[0, 1]$, where $f_n(x) = x^n$. Prove that for any $a < 1$, $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0, a]$, but $\{f_n(x)\}_{n\geqslant 1}$ does not converge uniformly on $[0, 1)$.

Proof: For any $a < 1$, and any $\varepsilon > 0$, let $N = \log_a x$, then for any $n > N$, $f_n(x) < \varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0, a]$. Let $\varepsilon = 1/2$, then for any $N \in \mathbb{N}$, there exists $1 > x > 2^{-1/N}$ such that $f_N(x) > \varepsilon$. Hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent on $[0, 1)$.

A5) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$. Prove that $\{f_n(x)\}_{n\geqslant 1}$ converges point-wise to 0 on $\mathbb{R}$, but does not converge uniformly.

Proof: For any $x \in \mathbb{R}$, and any $\varepsilon > 0$, there exists $N = 1/(x\varepsilon)$ such that for any $n \geqslant N$,

$$\left|\frac{nx}{1 + n^2x^2}\right| \leqslant \frac{1}{|nx|} < \varepsilon.$$

Hence $f_n(x)$ converges to 0 for any $x \in \mathbb{R}$.
Let $\varepsilon = 1/2$, then for any $n \in \mathbb{N}$, there exists $x = 1/n$ such that $f_n(x) = \varepsilon$, so f is not uniformly continuous on $\mathbb{R}$.

A6) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where

$$f_n(x)=\begin{cases}\frac{nx^2}{1+nx}, & x>0;\\ \frac{nx^3}{1+nx^2}, & x\leqslant 0.\end{cases}$$

Determine the convergence of $\{f_n(x)\}_{n\geqslant 1}$ on $\mathbb{R}$ (both point-wise and uniformly).
Proof: For any $\varepsilon>0$, let $N=\max\{1/\varepsilon, 1/4\varepsilon^2\}$, then for any $x>0$ and $n>N$,

$$|f_n(x)-x|=\left|\frac{x}{1+nx}\right|<\frac{1}{n}<\varepsilon.$$

For any $x<0$,

$$|f_n(x)-x|=\left|\frac{x}{1+nx^2}\right|\leqslant\frac{1}{2\sqrt{n}}<\varepsilon.$$

Hence $\{f_n\}_{n\geqslant 1}$ converges uniformly to x.

A7) Given $\varphi:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ such that $\varphi(0)=0$, $\lim_{x\to\infty}\varphi(x)=0$, φ is continuous and not identically zero. Prove that the sequences $\{f_n(x)\}_{n\geqslant 1}$ and $\{g_n(x)\}_{n\geqslant 1}$ converge point-wise to 0, but uniformly, where $f_n(x)=\varphi(nx)$, $g_n(x)=\varphi(x/n)$.

Proof: Point-wise convergence is trivial. Let $\varepsilon=|\varphi(1)|>0$, then for any n there exists $x=1/n>0$ such that $|f_n(x)|=\varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent. Likewise $\{g_n(x)\}_{n\geqslant 1}$ is not uniformly continuous.

A8) $f\in C([a,b])$. For $n\geqslant 1$, let $a_k=a+(k-1)(b-a)/n$. Define

$$S_n=\sum_{k=1}^{n}\frac{b-a}{n}f(a_k).$$

Prove that $\{S_n\}_{n\geqslant 1}$ converges, and denote this limit by $\int_a^b f$. Further show that the mapping

$$\int_a^b : C([a,b])\to\mathbb{R},\ f\mapsto\int_a^b f$$

is linear and continuous with metric d_∞ on $C([a,b])$.
Proof: For any $n,m\in\mathbb{N}$, note that $|S_n-S_m|\leqslant|S_n-S_{nm}|+|S_{nm}-S_m|$, and

$$|S_n-S_{nm}|\leqslant\sum_{k=1}^{n}\frac{b-a}{n}\left|f(a_k^{(n)})-\frac{1}{m}\sum_{j=1}^{m}f(a_{n(k-1)+j}^{(nm)})\right|\leqslant(b-a)\sup_{|x-y|<1/n}|f(x)-f(y)|.$$

Since f is uniformly continuous on $[a,b]$, the sequence $\{S_n\}_{n\geqslant 1}$ is Cauchy.
Obviously $\int_a^b\cdot$ is linear, and for $f,g\in C([a,b])$,

$$\left|\int_a^b f-\int_a^b g\right|=\lim_{n\to\infty}|S_n(f)-S_n(g)|\leqslant(b-a)\|f-g\|_\infty.$$

Hence $\int_a^b\cdot$ is continuous on $C([a,b])$ with metric d_∞.

A9) For any $f:[a,\infty)\to\mathbb{R}$, suppose f is bounded on any closed interval $[a,b]$, then when the limits in RHS exist,

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}f(x+1)-f(x).$$
$$\lim_{x\to\infty}f(x)^{1/x}=\lim_{x\to\infty}\frac{f(x+1)}{f(x)},\text{if for any }x\in[a,\infty),f(x)\geqslant c>0.$$

Proof: Suppose $\lim_{x\to\infty}f(x+1)-f(x)=A$, then for any $\varepsilon>0$ there exists M such that for any $x>M$, $|f(x+1)-f(x)-A|<\varepsilon$, so for any $n\geqslant 1$, $|f(x+n)-f(x)-nA|<n\varepsilon$. Hence

$$\left|\frac{f(n+x)}{n+x}-A\right|\leqslant\left|\frac{f(n+x)-f(x)-nA}{n+x}\right|+\left|\frac{f(x)-xA}{n+x}\right|\leqslant\varepsilon A+\frac{|f(x)-xA|}{n}\to 0.$$

For any $x>M$. Therefore (since f is bounded on any closed interval) there exists N such that for any $x>N$, $|f(x)/x-A|<2\varepsilon A$, and hence

$$\lim_{x\to\infty}\frac{f(x)}{x}=A=\lim_{x\to\infty}f(x+1)-f(x).$$

Substitute f by $\log f$ and we obtain the second identity.

PSB: Uniform Continuity

Determine whether the following functions f are uniformly continuous on I:

B1) $f(x)=x^{1/3}$, $I=(0,\infty)$

For any $\varepsilon>0$ and $x-y\in(0,\varepsilon)$,

$$x^{1/3}-y^{1/3}=\frac{x-y}{x^{2/3}+x^{1/3}y^{1/3}+y^{2/3}}\leqslant\frac{\varepsilon}{\varepsilon^{2/3}}=\varepsilon^{1/3}.$$

Hence $f(x)$ is uniformly continuous on I.

B2) $f(x)=\log x$, $I=(0,1)$

For any $\varepsilon>0$ and $x-y\in(0,\varepsilon)$,

$$\log x-\log y=\log\left(1+\frac{x-y}{y}\right).$$

When $y\to 0$ and $x-y$ is constant, $\log x-\log y\to\infty$, hence $\log x$ is not uniformly continuous on I.

B3) $f(x)=\cos x^{-1}$, $I=(0,1)$

Note that for $x_n=1/(2n\pi)$ and $y_n=1/(2n\pi+\pi)$, $f(x_n)=1$ and $f(y_n)=-1$. Hence for $\varepsilon=1$ and any $\delta>0$, there exists n such that $|x_n-y_n|<\delta$ but $|f(x_n)-f(y_n)|=2>\varepsilon$. Therefore f is not uniformly continuous on I.

B4) $f(x)=x\cos x^{-1}$, $I=(0,\infty)$

For $x>y>1$ and $|x-y|<\varepsilon$,

$$\begin{aligned}|x\cos x^{-1}-y\cos y^{-1}|&\leqslant|x-y||\cos x^{-1}|+|y|\cdot|\cos x^{-1}-\cos y^{-1}|\\&\leqslant\varepsilon+2|y|\cdot|\sin\left(x^{-1}+y^{-1}\right)/2\sin\left(x^{-1}-y^{-1}\right)/2|\leqslant\varepsilon+\frac{y}{2}\left(\frac{1}{y^2}-\frac{1}{x^2}\right)\leqslant 2\varepsilon.\end{aligned}$$

For $1 > x > y$ and $|x-y| < \varepsilon$,

$$|x \cos x^{-1} - y \cos y^{-1}| \leqslant |x| + |y| < 2\varepsilon.$$

Hence f is uniformly continuous on I.

PSC: Existence of Limits

C1) $\alpha > 0$,

$$\lim_{x \to 1} \frac{\log x}{(x-1)^\alpha} = \lim_{t \to 0} \frac{\log(1+t)}{t^\alpha} = \lim_{t \to 0} t^{1-\alpha}$$

exists iff $\alpha \leqslant 1$.

C2) $\alpha > 0$,

$$\lim_{x \to 1} \frac{e^x - e}{(x-1)^\alpha} = e \lim_{t \to 0} \frac{e^t - 1}{t^\alpha} = \lim_{t \to 0} e t^{1-\alpha}.$$

exists iff $\alpha \leqslant 1$.

C3) $\alpha > 0$,

$$\lim_{x \to 1} \frac{x^x - 1}{(x-1)^\alpha} = \lim_{x \to 1} \frac{x^x(\log x + 1)}{\alpha (x-1)^{\alpha-1}}$$

exists iff $\alpha \leqslant 1$.

C4) $\alpha > 0$,

$$\lim_{x \to 1} \frac{\sqrt[3]{1-\sqrt{x}}}{(x-1)^\alpha}$$

exists iff $\alpha \leqslant 1/3$.

C5)

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{1 - \cos x} = 1.$$

C6)

$$\lim_{x \to 0} \frac{\sqrt{1+x^4} - 1}{1 - \cos^2 x} = 0.$$

C7) $\alpha > 0$,

$$\lim_{x \to 1} \frac{(x-1)^\alpha}{\sin(\pi x)}$$

exists iff $\alpha \geqslant 1$.

PSD: Problems on Uniform Continuity

D1) If f is continuous, monotonic and bounded on the open interval I, then f is uniformly continuous on I.

Proof: Otherwise if there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $|x-y| < \delta$ such that $|f(x) - f(y)| > \varepsilon$. We define x_n, y_n inductively as follows: Let $L = \min\{x_1, \cdots, x_{n-1}\}$, $R = \max\{y_1, \cdots, y_{n-1}\}$. Since f is uniformly continuous on $[L, R]$, there exists $\delta > 0$ such that for any $|s-t| < \delta$, $|f(s) - f(t)| < \varepsilon$. Hence there exists $x < y$ such that $x, y \notin [L, R]$, $|x-y| < \delta$ and $|f(x) - f(y)| > \varepsilon$. Let $x_n = x, y_n = y$, then (x_n, y_n) are disjoint intervals and $|f(x_n) - f(y_n)| > \varepsilon$. Which contradicts the fact that f is monotonic and bounded. Therefore f is uniformly continuous on I.

D2) I is an interval with finite length. Prove that the function f on I is uniformly continuous iff for any Cauchy sequence $\{x_n\}_{n\geqslant 1} \subset I$, $\{f(x_n)\}_{n\geqslant 1}$ is also a Cauchy sequence.

(f should be continuous, otherwise after changing the value of f at one point, $\{f(x_n)\}$ remains a Cauchy sequence.)
Proof: ==> If $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. There exists N such that for all $n, m > N$, $|a_n - a_m| < \delta$, hence $|f(a_n) - f(a_m)| < \varepsilon$, so $\{f(x_n)\}_{n\geqslant 1}$ is a Cauchy sequence.
<== If $I = (a, b)$ is open we can take $x_n \to a$ and define $f(a) = \lim_{n\to\infty} f(x_n)$, hence we can assume that I is closed. Therefore f is uniformly continuous.

D3) f is uniformly continuous on $\mathbb{R}$. Prove that there exists $a, b \in \mathbb{R}_{>0}$ such that for any $x \in \mathbb{R}$,

$$|f(x)| \leqslant a|x| + b.$$

Proof: For $\varepsilon = 1$, there exists $\delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < 1$. Hence let $C = \sup_{x\in[0,\delta]} |f(x)|$, then $|f(x)| \leqslant C + |x| \cdot (\frac{1}{\delta} + 1)$.

D4) Suppose f is uniformly continuous on $[0, \infty)$ and for any $x \in [0, 1]$, $\lim_{n\to\infty} f(x+n) = 0$. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

If we change the condition to f is continuous, will the statement still hold?
Proof: For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Let $N = \lfloor 1/\delta \rfloor + 1$, then for any $1 \leqslant n \leqslant N$, there exists M_n such that for all $m > M_n$, $|f(m + n/N)| < \varepsilon$. Let $M = \max\{M_1, \cdots, M_N\}$, then for all $x > M$, there exists $m \in \mathbb{Z}_{>M}$ and $1 \leqslant n \leqslant N$ such that $|x - m - n/N| < \delta$. Hence

$$|f(x)| \leqslant \varepsilon + |f(m + n/N)| < 2\varepsilon.$$

Therefore $\lim_{x\to\infty} f(x) = 0$.

D5) Suppose X is an interval, $f : X \to \mathbb{R}$ is continuous. If there is a constant $L > 0$ such that for any $x, y \in X$,

$$|f(x) - f(y)| \leqslant L|x-y|.$$

We say f satisfy the Lipschitz condition on X.

1. Prove that f satisfy the Lipschitz condition implies f is uniformly continuous.
   Proof: For any $\varepsilon > 0$, let $\delta = \varepsilon/L$, then for any $|x-y| < \delta$, $|f(x) - f(y)| \leqslant L|x-y| < \varepsilon$.

2. Determine whether the reversed statement holds.
   Consider the function $f(x) = x^{1/2}$, then f is uniformly continuous but $\frac{f(x)-f(y)}{x-y} = \frac{1}{\sqrt{x}+\sqrt{y}}$ is unbounded, hence does not satisfy the Lipschitz condition.
3. If f satisfy the Lipschitz condition on $[a, \infty)$, where $a > 0$, prove that $f(x)/x$ is uniformly continuous on $[a, \infty)$.
   Proof: Same as D3), there exists C such that $|f(x)| \leqslant C|x|$ for $x \in [a, \infty)$, then for $a < x < y$,

$$\begin{aligned}\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| &= \frac{|xf(y) - yf(x)|}{xy} \leqslant \frac{x|f(y) - f(x)| + |f(x)|(y - x)}{xy} \\ &\leqslant \frac{L + C}{y} \cdot |x - y|.\end{aligned}$$

Hence $f(x)/x$ satisfy the Lipschitz condition.

PSE:

Exactly the same as PSC in HW4?

PSF: Calculate Limits

F1)

$$\lim_{x \to \pi} \frac{\sin mx}{\sin nx} = \frac{m(-1)^m}{n(-1)^n}.$$

F2)

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x} \sqrt[3]{\cos 3x}}{x^2} = 3.$$

F3)

$$\lim_{x \to \infty} \sin \sqrt{1 + x} - \sin \sqrt{x} = 0.$$

Since the function $\sin x$ is uniformly continuous and $\lim_{x \to \infty} \sqrt{1 + x} - \sqrt{x} = 0$.

F4)

$$\lim_{x \to 0} \frac{\sqrt{1 + x \sin x} - 1}{e^{x^2} - 1} = \frac{1}{2}.$$

Since $\lim_{x \to 0} x^2/(e^{x^2} - 1) = 1$, $\lim_{x \to 0} x \sin x / x^2 = 1$ and $\lim_{x \to 0} 1/(1 + \sqrt{1 + x \sin x}) = 1/2$.

F5)

$$\lim_{n \to \infty} \sin^{(n)}(x) = 0.$$

Since the sequence $\{a_n = \sin^{(n)}(x)\}_{n \geqslant 1}$ is decreasing and bounded by 0, and its limit A satisfy $A = \sin A$. Therefore $\lim_{n \to \infty} \sin^{(n)}(x) = 0$.

Problem G

The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfy the following property: for any $\delta > 0$,

$$\lim_{n \to \infty} f(n\delta) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.
Proof: Consider any $\varepsilon > 0$. For any $N \in \mathbb{N}$,

$$A_N = \{\delta > 0 : \forall n \geqslant N, |f(n\delta)| < \varepsilon\}.$$

Then by the continuity of f, A_N is closed, and by $\lim_{n\to\infty} f(n\delta) = 0$ for any $\delta > 0$, $\bigcup_{N\geqslant 1} A_N = \mathbb{R}_{>0}$. Hence by Baire Category Theorem, there exists an $N > 0$ such that $(a, b) \subset A_N$ for some interval (a, b). Let $X = \{x \in \mathbb{R}_{>0} : |f(x)| < \varepsilon\}$, then since $(a, b) \subset A_N$, for any $n \geqslant N$, $(na, nb) \subset X$. Note that when $n > b/(b-a)$, $nb > (n+1)a$, hence there exists $M > 0$ such that $(M, \infty) \subset X$. Therefore $\lim_{x\to\infty} f(x) = 0$.

Problem H

The continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

1. $\lim_{x\to\infty}(\varphi(x) - x) = \infty$.
2. $\{x \in \mathbb{R} : \varphi(x) = x\}$ is a non-empty finite set.

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f \circ \varphi = f$, then f is constant.
(Probably need the condition $\lim_{x\to-\infty} \varphi(x) - x = -\infty$).
Proof: Suppose $\{x \in \mathbb{R} : \varphi(x) = x\} = \{a_1, \cdots, a_n\}$ where $a_1 < \cdots < a_n$. For any $x \in \mathbb{R}$, we will show that $f(x) \in \{f(a_1), \cdots, f(a_n)\}$ hence f is constant.
If $a_i < x < a_{i+1}$. Suppose $\varphi(x) > x$, then let $x_0 = x$, and inductively define x_k as a point in (a_k, x_{k-1}) such that $\varphi(a_i) = a_i < \varphi(x_k) = x_{k-1} < \varphi(x_{k-1})$. Since φ is continuous and $a_1, \cdots, a_n$ are all the roots of $\varphi(x) = x$, we know that $\varphi(x_k) > x_k$ for all $k \geqslant 0$. The sequence $\{x_k\}_{k\geqslant 0}$ is decreasing and bounded by a_i, hence has a limit A. From $\varphi(x_k) = x_{k-1}$ we know that $\varphi(A) = A$, so $A = a_i$. Note that $f(x_k) = f(\varphi(x_k)) = f(x_{k-1})$, hence $f(x) = f(x_k) = \lim_{k\to\infty} f(x_k) = f(a_i)$. The case $\varphi(x) < x$ is the same, by constructing a sequence which tends to a_{i+1}.
If $x > a_n$, then $\varphi(x) > x$, likewise we can construct a sequence x_k such that $x_{k-1} = \varphi(x_k)$ and $\lim_{k\to\infty} x_k = a_n$. The case $x < a_1$ is the same.
Hence for all $x \in \mathbb{R}$, $f(x) \in \{f(a_1), \cdots, f(a_n)\}$.

Problem I

The continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ satisfy $\lim_{x\to\infty} f(x)/x = 0$. Suppose $\{a_n\}_{n\geqslant 1}$ is a sequence of non-negative real numbers and the sequence $\{a_n/n\}_{n\geqslant 1}$ is bounded. Prove that $\lim_{n\to\infty} f(a_n)/n = 0$.
Proof: Suppose $\{a_n/n\}$ is bounded by M.
For any $\varepsilon > 0$, we need to find N such that $n \geqslant N \implies |f(a_n)| < \varepsilon n$. For $C > 0$, we can divide n into two parts: If $a_n \leqslant C$, then $|f(a_n)| \leqslant \sup_{x\in[0,C]} |f(x)|$, otherwise $a_n \geqslant C$, then $|f(a_n)| \leqslant \sup_{x\geqslant C} |f(x)/x| \cdot Mn$. Therefore, if we choose $C > 0$ such that $\sup_{x\geqslant C} |f(x)/x| < \varepsilon/M$, and choose N such that $N > \sup_{x\in[0,C]} |f(x)|/\varepsilon$, then for any $n \geqslant N$, $|f(a_n)| < \varepsilon n$, hence

$$\lim_{n\to\infty} \frac{f(a_n)}{n} = 0.$$

Ex: Proof of the infinity of primes using topology

Proof: Assume otherwise that the set $\mathcal{P}$ of primes is finite. Let $L_{a,b} = \{at + b : t \in \mathbb{Z}\}, \forall (a, b) \in I = \mathbb{Z}_{>0} \times \mathbb{Z}$. Then

$$\mathbb{Z} \subset \bigcup_{b\in\mathbb{Z}} L_{1,b} \subset \bigcup_{(a,b)\in I} L_{a,b} \subset \mathbb{Z} \implies \bigcup_{(a,b)\in I} L_{a,b} = \mathbb{Z}.$$

and for any $x \in \bigcap_{i=1}^n L_{a_i,b_i}$, let $a = \operatorname{lcm}(a_1, \cdots, a_n)$, then

$$x \in L_{a,x} \subset \bigcap_{i=1}^{n} L_{a_i,bi}.$$

Hence $L_{a,b}$ form a base.Consider the topology $\mathcal{T}$ on $\mathbb{Z}$ generated by the base $\{L_{a,b} : (a,b) \in I\}$. Note that

$$L_{a,b} = \mathbb{Z} \backslash \bigcup_{r=1}^{a-1} L_{a,b+r}$$

so $L_{a,b}$ is also closed. Since $\mathcal{P}$ is finite, the set

$$\bigcup_{p \in \mathcal{P}} L_{p,0} = \mathbb{Z} \backslash \{-1, 1\}$$

is closed, hence $\{-1, 1\}$ is open. However, an open set G is the union of $L_{a,b}$ which is infinite, so G is infinite, leading to contradiction.

> Quote:
> As for everything else, so for a mathematical theory: beauty can be perceived but not explained.
> ——A. Cayley