作业一 **1.1**：14、15、20、23；**1.2**： 6，7

1.1.14

Let $\mathscr{A}=\{A_1,A_2,\cdots,A_n\}$, $\mathscr{B}=\{B_1,B_2,\cdots,B_n\}$ be subsets of the space. Prove the for any bijection $f:\mathscr{A}\to\mathscr{B}$, the vector

$$\overrightarrow{A_1f(A_1)}+\overrightarrow{A_2f(A_2)}+\cdots+\overrightarrow{A_nf(A_n)}$$

is the same.
Proof: Let $O=A_1$, then

$$\sum_{i=1}^{n}\overrightarrow{A_if(A_i)}=\sum_{i=1}^{n}\overrightarrow{Of(A_i)}-\overrightarrow{OA_i}=\sum_{i=1}^{n}\overrightarrow{OB_i}-\overrightarrow{OA_i}$$

is independent of f.

1.1.15

Prove that A,B,C are collinear iff there exists λ,μ,ν not identically zero, such that $\lambda+\mu+\nu=0$ and

$$\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\nu\overrightarrow{OC}=\mathbf{0},$$

where O is an arbitrary point.
Proof: Note that for $\lambda+\mu+\nu=0$,

$$\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\nu\overrightarrow{OC}=\mathbf{0}\iff\mu\overrightarrow{AB}+\nu\overrightarrow{AC}=0.$$

Hence A,B,C are collinear iff $\exists\mu,\nu$ such that $\mu\overrightarrow{AB}+\nu\overrightarrow{AC}=\mathbf{0}$, iff $\exists\mu,\nu$ and $\lambda=-\mu-\nu$ such that $\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\nu\overrightarrow{OC}=\mathbf{0}$.

1.1.20

Suppose D,E,F are on the edges BC,CA,AB respectively, such that AD,BE,CE intersect at O, and

$$(A,B,F)=1/3,(C,F,O)=2,$$

determine the values of $(A,D,O),(B,C,D),(C,A,E),(B,E,O)$.
Solution: Let $\alpha=\overrightarrow{AB},\beta=\overrightarrow{AC}$, then $\overrightarrow{AF}=\frac{1}{4}\alpha$, $\overrightarrow{AO}=\frac{2}{3}\overrightarrow{AF}+\frac{1}{3}\overrightarrow{AC}=\frac{1}{6}\alpha+\frac{1}{3}\beta$.

$$\overrightarrow{AD}=\lambda\alpha+(1-\lambda)\beta=2\overrightarrow{AO}=\frac{1}{3}\alpha+\frac{2}{3}\beta.$$
$$\overrightarrow{AE}=\mu\beta=\nu\alpha+(1-\nu)\overrightarrow{AO}=\frac{2}{5}\beta.$$

Hence

$$(A,D,O)=1,\ (B,C,D)=2,\ (C,A,E)=3/2,\ (B,E,O)=5.$$

1.1.23

Suppose A, B, C are not collinear, and P, Q, R are on the lines AB, BC, CA respectively. Denote

$$\lambda = (A, B, P),\ \mu = (B, C, Q), \nu = (C, A, R).$$

Prove that P, Q, R are collinear iff $\lambda\mu\nu = -1$.
Let $\alpha = \overrightarrow{AB}$ and $\beta = \overrightarrow{AC}$, then $\overrightarrow{AP} = \frac{\lambda}{1+\lambda}\alpha$, $\overrightarrow{AR} = \frac{1}{1+\nu}\beta$, and $\overrightarrow{AQ} = \frac{1}{1+\mu}\alpha + \frac{\mu}{1+\mu}\beta$.
Hence

$$\overrightarrow{AQ} = \frac{1+\lambda}{\lambda(1+\mu)}\overrightarrow{AP} + \frac{(1+\nu)\mu}{1+\mu}\overrightarrow{AR}.$$

P, Q, R are collinear iff

$$\frac{1+\lambda}{\lambda(1+\mu)} + \frac{(1+\nu)\mu}{1+\mu} = 1 \iff 1 + \lambda + (1+\nu)\mu\lambda = \lambda(1+\mu) \iff \lambda\mu\nu = -1.$$

1.2.6

Given collinear point A, B, C such that $(A, B, C) = 5/2$, and suppose the coordinates of A, C are $(3, 7, 3), (8, 2, 3)$, determine the coordinates of B.
Solution:

$$\overrightarrow{OC} = \frac{5}{7}\overrightarrow{OB} + \frac{2}{7}\overrightarrow{OA} \implies \overrightarrow{OB} = \frac{7}{5}\overrightarrow{OC} - \frac{2}{5}\overrightarrow{OA} = (10, 0, 3).$$

1.2.7

Suppose the coordinates of vectors α, β, γ are respectively $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$. Prove that if α, β, γ are coplanar, than

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Proof: α, β, γ are coplanar implies $(x_1, y_1, z_1), (x_2, y_2, z_2), (z_3, y_3, z_3)$ are linearly dependent, hence

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$