PSA: Convex functions

A1)

(1) $f(x) = |x|,\ I = \mathbb{R}$ is convex, since

$$|\lambda x + (1-\lambda)y| \leqslant \lambda|x| + (1-\lambda)|y|.$$

(2) $f(x) = x^p,\ p \in \mathbb{R},\ I = \mathbb{R}_{>0}$
$f''(x) = p(p-1)x^{p-2}$ so f is concave if $p \in [0,1]$ and convex if $p \in (-\infty, 0] \cup [1, \infty)$.
(3) $f(x) = \sin x,\ I = [0, \pi]$ is concave since $f''(x) = -\sin x \leqslant 0$ when $x \in [0, \pi]$.
(4) $f(x) = x \log x,\ I = \mathbb{R}_{\geqslant 0}$ is (strictly) convex since $f''(x) = 1/x > 0$.
(5) $f(x) = \mathbf{1}_{\{0,1\}},\ I = [0,1]$ is convex since

$$f(\lambda x + (1-\lambda)y) = 0 \leqslant \lambda f(x) + (1-\lambda)f(y).$$

A2) Prove the following properties:

1. If f, g are convex on I, then $f + g$ is convex on I.
   Proof: By definition, $(f+g)(\lambda x + (1-\lambda)y) \leqslant \lambda(f+g)(x) + (1-\lambda)(f+g)(y)$, so $f + g$ is convex.
2. If f, g are monotonically increasing, non-negative, convex functions on I, then fg is convex.
   Proof: Note that

$$f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y) \leqslant (\lambda f(x) + (1-\lambda)f(y)) \cdot (\lambda g(x) + (1-\lambda)g(y))$$

and

$$\begin{aligned} &\lambda f(x)g(x) + (1-\lambda)f(y)g(y) - (\lambda f(x) + (1-\lambda)f(y))(\lambda g(x) + (1-\lambda)g(y)) \\ =&\lambda(1-\lambda)(f(x) - f(y))(g(x) - g(y)) \geqslant 0. \end{aligned}$$

hence

$$(fg)(\lambda x + (1-\lambda)y) \leqslant \lambda(fg)(x) + (1-\lambda)(fg)(y).$$

3. If f is convex on I, g is a monotonically increasing convex function on $J \supset f(I)$, then $g \circ f$ is convex.
   Proof: Note that

$$g(f(\lambda x + (1-\lambda)y)) \leqslant g(\lambda f(x) + (1-\lambda)f(y)) \leqslant \lambda g(f(x)) + (1-\lambda)g(f(y)).$$

hence $g \circ f$ is convex.

4. If f, g are convex on I, then $h(x) = \max\{f(x), g(x)\}$ is convex.
   Proof: For any x, y, λ and $t = \lambda x + (1-\lambda)y$, suppose $h(t) = f(t)$, then

$$h(t) \leqslant \lambda f(x) + (1-\lambda)f(y) \leqslant \lambda h(x) + (1-\lambda)h(y)$$

hence h is convex.

A3) Suppose $f \in C((a,b))$. If for any $x, y \in (a,b)$, $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$, prove that f is convex.

Proof: For any $x, y \in (a,b)$ and $\lambda \in [0,1]$, we need to prove that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y).$$

Note that it holds for any dyadic number λ, since the cases $\lambda = 0, 1, 1/2$ is trivial, and for $\lambda = (2m+1)/2^t$, let $u = m/2^{t-1}, v = (m+1)/2^{t-1}$, then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \frac{f(ux + (1-u)y) + f(vx + (1-v)y)}{2} \\ &\leqslant \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Now since $f \in C((a,b))$, for any $\lambda \in (0,1)$ there is a sequence of dyadic numbers λ_n such that $\lim_{n\to\infty} \lambda_n = \lambda$, hence

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \lim_{n\to\infty} f(\lambda_n x + (1-\lambda_n)y) \leqslant \lim_{n\to\infty} \lambda_n f(x) + (1-\lambda_n)f(y) \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

A4) f is a convex function on $[a,b]$. Prove that if there exists $c \in (a,b)$ such that $f(c) \geqslant \max\{f(a), f(b)\}$ then f is constant.

Proof: For any $t \in (a,b)$, let $\lambda = (t-a)/(b-a)$ then

$$f(t) = f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda)f(b) \leqslant \max\{f(a), f(b)\}.$$

By $f(c) \geqslant \max\{f(a), f(b)\}$ we know that $f(a) = f(b)$. If for some $t \in (a,b)$, $f(t) \neq f(a)$, suppose $c \in (a,t)$, then

$$f(c) = \lambda f(a) + (1-\lambda)f(t) < f(a)$$

a contradiction. Hence $f(t) = f(a)$ for all $t \in [a,b]$.

A5) f is convex on $\mathbb{R}$. Prove that if f has an upper-bound, then f is constant.

Proof: Otherwise suppose that $f(a) < f(b)$, where $a < b$. (If $f(a) > f(b)$ let $g(x) = f(-x)$). Let $x_0 = a, x_1 = b, x_n = a + n(b-a)$, then

$$f(x_{n+1}) - f(x_n) \geqslant f(x_n) - f(x_{n-1}) \geqslant f(b) - f(a),$$

hence $f(x_n) \geqslant f(a) + n(f(b) - f(a)) \to \infty$, leading to contradiction.

A6) f is strictly convex on I. Suppose $f(x_0)$ is a local minimum of f, prove that x_0 is the unique global minimum point of f.

Proof: Suppose there is another $x_1 \neq x_0$ such that $f(x_1) \leqslant f(x_0)$, then let $x_n = x_0 + n(x_1 - x_0)$. Since f is strictly convex, $f(x_n) < \max\{f(x_1), f(x_0)\} = f(x_0)$, contradicting the fact that $f(x_0)$ is a local minimum.

A7) I is an open interval. Prove that f is convex on I, iff for any $x_0 \in I$, there exists $a \in \mathbb{R}$, such that for any $x \in I$, $f(x) \geqslant a(x - x_0) + f(x_0)$.

Proof: Suppose f is convex on I, then the any $x_0 \in I$, the function $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$ is monotonically increasing. Hence we can let $a = \sup_{x<x_0} g(x) < \infty$.
If for any $x_0 \in I$, and and $x \in I$, $f(x) \geqslant g(x_0)(x - x_0) + f(x_0)$, then for any $x, y \in I$ and $\lambda \in (0, 1)$, let $t = \lambda x + (1 - \lambda)y$,

$$\begin{aligned} \lambda f(x) + (1-\lambda)f(y) &\geqslant \lambda(f(t) + (x-t)g(t)) + (1-\lambda)(f(t) + (1-\lambda)(y-t)g(t)) \\ &= f(t) = f(\lambda x + (1-\lambda)y). \end{aligned}$$

Hence f is convex.

PSB

B1) Prove the following inequalities:

(1)

$$x - \frac{x^2}{2} < \log(1+x) < x,\ x > 0.$$

Proof: If $f(x) = \log(x+1) - x$, then $f'(x) = \frac{1}{x+1} - 1 < 0$ hence $f(x) < f(0) = 0$. Let $g(x) = \log(1+x) - x + x^2/2$, then $g'(x) = \frac{1}{x+1} + (x+1) - 2 \geqslant 0$, hence $g(x) > g(0) = 0$.
(2)

$$(x^\alpha + y^\alpha)^{1/\alpha} > (x^\beta + y^\beta)^{1/\beta},\ x, y > 0, \beta > \alpha > 0.$$

Proof: Assume that $x^\alpha + y^\alpha = 1$, then $0 < x, y < 1$, so

$$x^\beta + y^\beta < x^\alpha + y^\alpha < 1 \implies (x^\beta + y^\beta)^{1/\beta} < (x^\alpha + y^\alpha)^{1/\alpha}.$$

(3)

$$x - \frac{x^3}{6} < \sin x < x,\ x > 0.$$

Proof: Let $f(x) = \sin x - x$, then $f'(x) = \cos x - 1 \leqslant 0$, so $f(x) < f(0) = 0$. Let $g(x) = \sin x - x + x^3/6$, then $g'(x) = \cos x - 1 + x^2/2$, $g''(x) = x - \sin x > 0$, so $g'(x) > g(0) = 0$ and $g(x) > g(0) = 0$.
(4)

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leqslant \frac{1}{2}(1 + x^p),\ p \in [2, \infty), x \in [0, 1].$$

Proof: ???

B2) Find all $a > 0$ such that $a^x \geqslant x^a$ for any $x > 0$.

Solution: $f(x) = x^{1/x}$ then $f'(x) = x^{1/x}\frac{1-\log x}{x^2}$ hence f has a unique minimum at e.

B3) Prove that for any $x_i, t_i, i = 1, 2, \cdots, n$, $\sum_{i=1}^n t_i = 1$,

$$\left(\sum_{i=1}^n t_i x_i\right)^{\sum_{i=1}^n t_i x_i} \leqslant \prod_{i=1}^n x_i^{t_i x_i}.$$

Proof: Let $f(x) = x \log x$, then $f''(x) = 1/x > 0$, so f is convex. By Jensen's inequality,

$$\sum_{i=1}^{n} t_i f(x_i) \geqslant f\left(\sum_{i=1}^{n} t_i x_i\right)$$

hence

$$\left(\sum_{i=1}^{n} t_i x_i\right)^{\sum_{i=1}^{n} t_i x_i} \leqslant \prod_{i=1}^{n} x_i^{t_i x_i}.$$

and equality holds iff $x_i = x_1$.

B4) Prove that for any $a, b > 0$, $1/p + 1/q = 1$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1;\ ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1.$$

Proof: The function $-\log x$ is convex, so when $p > 1$, $q > 1$, then

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geqslant \frac{1}{p}\log a^p + \frac{1}{q}\log b^q$$

so $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.
When $p < 1$, then $pq < 0$, so likewise $ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}$.

B5) Prove that if $x_i, y_i \geqslant 0, i = 1, 2, \cdots, n$, $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}, \text{ if } p > 1;$$

and the inequality reverses when $p < 1$.
Proof: Assume that $\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} y_i^q = 1$, then by B4), if $p > 1$,

$$\sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{n} \frac{x_i^p}{p} + \frac{y_i^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

The case $p < 1$ is similar.

PSC

C1) Suppose $f \in C([0,1])$, g is differentiable on $[0,1]$ and $g(0) = 0$. If there is a constant $\lambda \neq 0$, such that for any $x \in [0,1]$, $|g(x)f(x) + \lambda g'(x)| \leqslant |g(x)|$, prove that $g(x) \equiv 0$.

Proof: Otherwise assume that $\forall \varepsilon > 0 \exists t \in (0, \varepsilon)$, such that $g(t) \neq 0$. Let $C = (1 + \sup_{x\in[0,1]} |f(x)|)/\lambda$, then $|g'(x)| \leqslant C|g(x)|, \forall x \in [0,1]$. For any $t \in (0,1)$, there exists $\xi \in [0,t]$ such that $g(t) = g(0) + tg'(\xi)$, hence

$$\frac{|g(t)|}{t} = |g'(\xi)| \leqslant C \sup_{\xi\in[0,t]} |g(\xi)|.$$

For any $t > 0$ suppose $|g(s)| = \sup_{\xi\in[0,t]} |g(\xi)| > 0$, then $|g(s)|/s \leqslant C|g(s)|$ hence $t \geqslant s \geqslant \frac{1}{C}$, a contradiction.

C2) f is twice differentiable on $(-1,1)$, $f(0)=f'(0)=0$. If for any $x\in(-1,1)$, $|f''(x)|\leqslant|f(x)|+|f'(x)|$, prove that $f(x)\equiv 0$.

Proof: We prove that $f''(x)\equiv 0$. Otherwise suppose $\forall\varepsilon>0$, $\exists x\in[0,\varepsilon]$, $f''(x)\neq 0$. Note that

$$|f''(x)|\leqslant|f(x)|+|f'(x)|\leqslant\left(\frac{x^2}{2}+|x|\right)\sup_{y\in[0,x]}|f''(y)|.$$

Since $f''(0)=0$, take $x\in[0,1/2]$ such that $f''(x)\neq 0$, and suppose $|f''(t)|=\sup_{y\in[0,x]}|f''(y)|$, then $|f''(t)|\leqslant(t^2/2+t)|f''(t)|$, a contradiction.

C3) f is n-times differentiable on $\mathbb{R}$, $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$. If there exists $C\in\mathbb{R}_{>0}$ and $\ell\in\mathbb{Z}_{\geqslant 0}$ such that for any $x\in\mathbb{R}$, $|f^{(n)}(x)|\leqslant C|f^{(\ell)}(x)|$. Prove that $f(x)\equiv 0$.

Proof: We can assume that $\ell=0$. Since $f^{(k)}(x)=0,\forall 0\leqslant k<n$, we have

$$|f^{(n)}(x)|\leqslant C|f(x)|\leqslant C\frac{x^n}{n!}\sup_{y\in[0,x]}|f^{(n)}(y)|.$$

Hence for any $x\in[0,\varepsilon]$, $\varepsilon=(n!/C)^{1/n}$, $f^{(n)}(x)=0$, so $f^{(k)}(x)=0$ for all $x\in[0,\varepsilon],0\leqslant k<n$. Likewise we get $f(x)\equiv 0$.

C4) $n\in\mathbb{Z}_{>0}$, prove that the polynomial $P(x)=\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k(x-k)^n\equiv 0$.

Proof: We know the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^l=0,\ \forall 0\leqslant l\leqslant n-1.$$

Since $\Delta^n x^l\equiv 0$
Likewise by considering $f(t)=(x-t)^n$ we have $P(x)\equiv 0$.
(Or we can use C3)

C5) $f\in C^\infty(\mathbb{R})$. Assume there exists $C>0$ such that for any $n\in\mathbb{Z}_{\geqslant 0}$ and $x\in\mathbb{R}$, $|f^{(n)}(x)|\leqslant C$.

i. Prove that given an arbitrary $x_0\in\mathbb{R}$,

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,\ \forall x\in\mathbb{R}.$$

Proof: The Lagrange remainder

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

tends to zero as $n\to\infty$, hence the Taylor series

$$f(x)=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+R_n(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

ii. $E\subset\mathbb{R}$ is an infinite bounded set. Prove that if $f(E)=\{0\}$, then $f\equiv 0$.
Proof: Suppose $E\subset[-M,M]$, then by Bolzano-Weierstrass theorem, there exists a sequence $\{z_n\}_{n\geqslant 1}\subset E$ such that $z=\lim_{n\to\infty}z_n$ exists. Since $f\in C(\mathbb{R})$, $f(z)=\lim_{n\to\infty}f(z_n)=0$, so

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^k.$$

If f does not vanish on $\mathbb{R}$, then take the least $m>0$ such that $f^{(m)}(z)\neq 0$. When $z_n\to z$,

$$0=\frac{f^{(m)}(z)}{m!}+\sum_{k=m+1}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^{k-m}$$

which leads to contradiction. Hence f vanishes on $\mathbb{R}$.

C6) Assume $f\in C^2((0,1))$, $\lim_{x\to1^-}f(x)=0$. If there exists $C>0$, such that for any $x\in(0,1)$, $(1-x)^2|f''(x)|\leqslant C$. Prove that $\lim_{x\to1^-}(1-x)f'(x)=0$.

Proof: For any $0<x<y<1$, there exists $\xi\in(x,y)$ such that

$$f(y)=f(x)+(y-x)f'(x)+\frac{(y-x)^2}{2}f''(\xi).$$

For any $\lambda>0$, let $y=(\lambda+x)/(\lambda+1)\in(x,1)$, then

$$|(y-x)f'(x)|\leqslant|f(y)|+|f(x)|+\frac{\lambda^2}{2}(1-y)^2|f''(\xi)|\leqslant|f(y)|+|f(x)|+\frac{C\lambda^2}{2}.$$

Therefore

$$|(1-x)f'(x)|\leqslant(|f(t)|+|f(x)|)\frac{\lambda+1}{\lambda}+\frac{1}{2}\lambda(\lambda+1)C$$

Hence for any $\lambda>0$,

$$\varlimsup_{x\to1^-}|(1-x)f'(x)|\leqslant\frac{1}{2}\lambda(\lambda+1)C\to0,$$

so $\lim_{x\to1^-}(1-x)f'(x)=0$.

PSD

Calculate $\sup_{x\in I}f(x)$ and $\inf_{x\in I}f(x)$:

D1) $f(x)=\frac{(\log x)^2}{x}$, $I=\mathbb{R}_{>0}$

Solution: Let $y=\log x\in\mathbb{R}$, then $f(x)=y^2e^{-y}$.

$$\frac{\mathrm{d}}{\mathrm{d}y}y^2e^{-y}=ye^{-y}(2-y).$$

Hence $\sup_{x\in I}f(x)=f(e^2)=4e^{-2}$, $\inf_{x\in I}f(x)=\min\{f(0),f(\infty)\}=0$.

D2) $f(x)=|x(x^2-1)|$, $I=\mathbb{R}$

Solution: $\sup=\infty,\inf=0$.

D3)

$$f(x)=\frac{x(x^2+1)}{x^4-x^2+1},\ I=\mathbb{R}.$$

Solution: Note that

$$2(x^4-x^2+1)-x(x^2+1)=(x^2-1)^2+(x-1)^2(x^2+x+1)\geqslant 0.$$

Therefore $f(x)\leqslant 2$ where equality holds at $x=1$. Since $f(x)=f(-x)$, $\sup=2, \inf=-2$.

D4)

$$f(x)=x^{1/3}(1-x)^{2/3},\ I=(0,1).$$

Solution: By AM-GM, $f(x)\leqslant \frac{2^{2/3}}{3}$ where equality holds at $x=1/3$. Hence $\sup=\frac{2^{2/3}}{3}, \inf=0$.

D5)

$$f(x)=\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)e^{-x},\ I=\mathbb{R}.$$

Solution: $f'(x)=-e^{-x}\frac{x^n}{n!}$, so if n is even, $\sup=\infty, \inf=0$, and if n is odd, $\sup=1, \inf=-\infty$.

D6) $f(x)=\sin^{2m}x\cos^{2n}x$, $I=\mathbb{R}$.

Solution: Let $t=\sin^2 x\in[0,1]$, then $f(x)=t^m(1-t)^n\in[0, n^n m^n/(n+m)^{n+m}]$.

PSE

Compare the two functions (or real numbers).

E1) $f(x)=e^x$, $g(x)=1+xe^x$, $x>0$.

Solution: The case $x\geqslant 1$ is trivial. If $x\in(0,1)$, then

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\leqslant\sum_{n=0}^{\infty}x^n=\frac{1}{1-x},$$

hence $f(x)\leqslant g(x)$. Therefore $f(x)\leqslant g(x)$ for all $x>0$.

E2) $f(x)=xe^{x/2}$, $g(x)=e^x-1$, $x>0$.

Solution: ($x/2\leqslant\sinh(x/2)$) Consider $h(x)=e^{x/2}-e^{-x/2}-x$, then $h(0)=0$ and

$$h'(x)=\frac{1}{2}(e^{x/2}+e^{-x/2}-2)\geqslant 0.$$

Hence $h(x)\geqslant 0$, i.e. $g(x)\geqslant f(x)$ for all $x>0$.

E3) $f(x)=\left(\frac{x+1}{2}\right)^{(x+1)}$, $g(x)=x^x$, $x>0$.

Solution: Consider $h(x)=x\log x-(x+1)\log\frac{x+1}{2}$, then $h(1)=0$ and

$$h'(x)=\log\frac{2x}{x+1}\geqslant 0\iff x\geqslant 1.$$

Hence $f(x)\leqslant g(x)$ for all $x>0$.

E4) $2^{\sqrt{2}}$ and e.

Solution: Note that

$$\log 2=\sum_{n=1}^{\infty}\frac{1}{2^n n}\leqslant\frac{2}{3}+\sum_{n=4}^{\infty}\frac{1}{2^n\cdot 4}=\frac{2}{3}+\frac{1}{32}<\frac{2}{3}+\frac{1}{30}=0.7<\frac{1}{\sqrt{2}},$$

hence $2^{\sqrt{2}} < e$.

E5) $f(x) = \log\left(1+\sqrt{1+x^2}\right)$, $g(x) = 1/x + \log x$, $x > 0$.

Solution: Consider $h(x) = \log x + 1/x - \log\left(1+\sqrt{1+x^2}\right)$, then

$$h'(x) = \frac{1}{x} - \frac{1}{x^2} - \frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}} \leqslant 0.$$

$(\iff (x-1)(\sqrt{1+x^2}+1+x^2) - x^3 \leqslant 0 \iff (x—1)\sqrt{1+x^2} \leqslant x^2)$
Therefore $h(x) \geqslant \lim_{x\to\infty} h(x) = 0$.

E6) $\log 8$ and 2.

Solution: Note that

$$\log 2 = \log \frac{1}{1-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{2^n n} \geqslant \sum_{n=1}^{3} \frac{1}{2^n n} = \frac{2}{3},$$

hence $\log 8 > 2$.

PSF

If f satisfy $f(x) = (x-x_0)^r g(x)$ in a neighborhood of x_0, where $r \in \mathbb{Z}_{\geqslant 0}$, g is continuous at x_0 and $g(x_0) \neq 0$, then we call x_0 an r-fold root of f.

F1) Suppose x_0 is an r-fold root of f where $r > 0$. Prove that if $g(x) = f(x)/(x-x_0)^r$ is continuous, then x_0 is an $(r-1)$-fold root of f'.

Proof: Suppose $f(x) = (x-x_0)^r g(x)$ in the neighborhood $O(x_0)$, then $f'(x) = (x-x_0)^r g'(x) + r(x-x_0)^{r-1} g(x)$ in $O(x_0)$. Therefore let $h(x) = (x-x_0)g'(x) + g(x)$, then $f'(x) = (x-x_0)^{r-1} h(x)$ and $h(x_0) = g(x_0) \neq 0$, so x_0 is an $(r-1)$-fold root of f'.

F2) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x) = 0$ has $n+1$ distinct real roots, then $f^{(n)}(x) = 0$ has at least one root.

Proof: Use induction and Rolle's mean-value theorem to prove that $f^{(n-k)}(x)$ has at least $k+1$ different real roots.

F3) f is differentiable on $\mathbb{R}$. Suppose $f(x) = 0$ has r roots (counting multiplicity), then $f'(x) = 0$ has at least $r-1$ roots (counting multiplicity).

Proof: Combine F1) and F2).

F4) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x)=0$ has exactly $n+1$ roots counting multiplicity, then $f^{(n)}(x)=0$ has at least one root.

Proof: Use F3) and induction.

PSG

Let $a \in \mathbb{R}$, $f:(a,\infty)\to\mathbb{R}$ twice differentiable on (a,∞), and

$$M_0 := \sup_{x\in(a,\infty)} |f(x)|,\ M_1 := \sup_{x\in(a,\infty)} |f'(x)|,\ M_2 := \sup_{x\in(a,\infty)} |f''(x)|,$$

are real numbers.

G1) Prove that $M_1^2 \leqslant 4M_0M_2$.

Proof: Let $h=\sqrt{M_0/M_2}$, then for any $x\in(a,\infty)$, there exists $\xi\in(x,x+2h)$ such that

$$f(x+2h)=f(x)+2hf'(x)+2h^2f''(\xi) \implies f'(x)=hf''(\xi)+\frac{f(x+2h)-f(x)}{2h}.$$

Therefore $f'(x)\leqslant M_0/h+M_2h=2\sqrt{M_0M_2}$, hence $M_1^2\leqslant 4M_0M_2$.

G2) Let $a=-1$, consider the function

$$f(x)=\begin{cases}2x^2-1, & x\in(-1,0),\\ \frac{x^2-1}{x^2+1}, & x\in[0,\infty),\end{cases}$$

verify that f is twice differentiable, and $M_0=1, M_1=4, M_2=4$.
Proof: Note that $\lim\limits_{x\to0^-} f(x)=-1=f(0)$ so f is continuous, and

$$f'(x)=\begin{cases}4x, & x\in(-1,0),\\ \frac{4x}{(x^2+1)^2}, & x\in[0,\infty).\end{cases}$$

f' is also continuous, so

$$f''(x)=\begin{cases}4, & x\in(-1,0),\\ 4\frac{1-3x^2}{(x^2+1)^2}, & x\in[0,\infty).\end{cases}$$

Therefore $f\in C^2((-1,\infty))$ and $M_0=1, M_1=4, M_2=4$.

G3) Suppose $\mathbf{f}:(a,\infty)\to\mathbb{R}^n$ is twice differentiable, Let M_0,M_1,M_2 be the least upper bounds of $|\mathbf{f}|,|\mathbf{f}'|,|\mathbf{f}''|$. Prove that $M_1^2\leqslant 4M_0M_2$.

Proof: Use G1) and Cauchy-Schwarz inequality.

Problem S: Strum-Liouville Theory

Assume the following uniqueness theorem holds:

> [!note] Theorem
> Suppose $a(t)\in C^1(\mathbb{R})$, $t_0\in\mathbb{R}$. If $x(t),y(t)\in C^2(\mathbb{R})$ both satisfy the equation
>
> $$x''(t)+a(t)x(t)=0, y''(t)+a(t)y(t)=0,$$
>
> and $(x(t_0),x'(t_0))=(y(t_0),y'(t_0))$, then $x(t)\equiv y(t)$.

(Can be proved using Exercise C3?)
For any $f : \mathbb{R} \to \mathbb{R}$, $t \geqslant 0$, denote
$Z_t(f) = |\{x \in [0,t] : f(x) = 0\}|$.

Part 1

Let $a(t), b(t) \in C^1(\mathbb{R})$ and for any $t \in \mathbb{R}$, $a(t) \leqslant b(t)$. Suppose $x(t), y(t) \in C^2(\mathbb{R})$ satisfy the following equation

$$x''(t) + a(t)x(t) = 0,\ y''(t) + b(t)y(t) = 0.$$

Further assume that $x(t), y(t)$ are not identically zero.

S1) Assume $x(t_1) = 0$, if there exists $t > t_1$, such that $x(t) = 0$. Prove that there exists $t_2 > t_1$ such that $x(t_2) = 0$ and x has no roots in (t_1, t_2). We call t_1, t_2 neighboring roots.

Proof: Consider the set $S = \{t > t_1 : x(t) = 0\}$, and let $t_2 = \inf S$. Note that $|x''(t)| \leqslant |a(t)| \cdot |x(t)|$, so by C3) $x'(t_1) \neq 0$. Assume $x'(t_1) > 0$, since $x \in C^2(\mathbb{R})$, there exists $\varepsilon > 0$ such that $x'(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$, hence $x(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$. Therefore $t_2 > t_1$, so by $x \in C(\mathbb{R})$, $x(t_2) = 0$ and x has no roots in (t_1, t_2).

S2) If $t_2 > t_1$ are two neighboring roots of x, prove that y has a root in $(t_1, t_2]$.

Proof: Otherwise assume that x, y are positive on (t_1, t_2), and $y(t_2) \neq 0$. Consider the function $h(t) = x'y - xy'$, then $h'(t) = (b - a)xy \geqslant 0$, so $h(t_2) \geqslant h(t_1) = x'(t_1)y(t_1) \geqslant 0$, but $h(t_2) = x'(t_2)y(t_2) < 0$, a contradiction.

S3) Prove that for any $t \geqslant 0$, $Z_t(y) \geqslant Z_t(x) - 1$.

Proof: Use S2).

S4) Suppose $t_2 > t_1$ and $x(t_1) = x'(t_2) = 0$, prove that

- If $y(t_1) = 0$, then there exists $t_3 \in [t_1, t_2]$, such that $y'(t_3) = 0$.
  Proof: We can assume that $t_2 = \inf \{t > t_1 : x'(t) = 0\}$($t_2 > t_1$ since $x'(t_1) \neq 0$). If there is no such t_3, we can further assume that $x'(t), y'(t), x(t), y(t) > 0$ for all $t \in (t_1, t_2)$.
  Again consider $h(t) = x'y - xy'$, then $h(t_1) = 0, h(t_2) = -x(t_2)y'(t_2) < 0$, but $h'(t) = (b - a)xy \geqslant 0$, leading to contradiction.
- If $y'(t_2) = 0$, then there exists $t_4 \in [t_1, t_2]$ such that $y(t_4) = 0$.
  (The two theorems are similar.)

Part 2

In this section, $p(t) \in C^1(\mathbb{R})$ is a positive function. $x(t), y(t) \in C^2(\mathbb{R})$ are not identically zero and satisfy

$$x''(t) + p(t)x(t) = 0,\ y''(t) + p(t)y(t) = 0.$$

S5) Prove that for any $t \geqslant 0$, $|Z_t(x) - Z_t(y)| \leqslant 1$.

Proof: Use S3).

S6) Prove that

- If t_1, t_2 are neighboring roots of x, then there exists a unique $t_3 \in [t_1, t_2]$ such that $x'(t_3) = 0$.
  Proof: The existence of t_3 is given by Rolle's mean-value theorem. If there exists $t_3 < t_4 \in [t_1, t_2]$ such that $x'(t_3) = x'(t_4) = 0$, then $t_3, t_4 \neq t_1, t_2$ and there exists $t_5 \in [t_3, t_4]$ such that $x''(t_5) = 0$. Hence $x(t_5) = 0$, which contradicts the fact that t_1, t_2 are neighboring roots. Therefore t_3 is unique.
- If t_1', t_2' are neighboring roots of x', then there exists a unique $t_3' \in [t_1', t_2']$ such that $x(t_3') = 0$.
  Proof: Exactly the same.

S7) Prove that

- t_0 is a local maximum of $|x(t)|$ iff $x'(t_0) = 0$.
  Proof: Trivial?
- t_0' is a local maximum of $|x'(t)|$ iff $x(t_0') = 0$.

Part 3

In this section, $p(t) \in C^1(\mathbb{R})$ is monotonically decreasing and $\lim_{t\to\infty} p(t) > 0$. Denote

$$p(\infty) := \lim_{t\to\infty} p(t).$$

$x(t) \in C^2(\mathbb{R})$ is not identically zero and

$$x''(t) + p(t)x(t) = 0.$$

*S8) Calculate

$$\lim_{t\to\infty} \frac{Z_t(x)}{t}.$$

Solution: By S5) we can ignore initial conditions. First consider the ODE $y''(t) + p(\infty)y(t) = 0$, where one solution is $y = \sin\left(t\sqrt{p(\infty)}\right)$, so $\lim\limits_{t\to\infty} Z_t(y)/t = \sqrt{p(\infty)}/\pi$.
Since $p(t) \geqslant p(\infty)$, by S3) we know $\lim\limits_{t\to\infty} Z_t(x)/t \geqslant \lim\limits_{t\to\infty} Z_t(y)/t = \sqrt{p(\infty)}/\pi$. For any $\varepsilon > 0$, there exists $M > 0$ such that for any $t > M$, $p(t) < p(\infty) + \varepsilon$. By S3), $\lim_{t\to\infty} Z_t(x)/t \leqslant \sqrt{p(\infty) + \varepsilon}/\pi$. Therefore

$$\lim_{t\to\infty} \frac{Z_t(x)}{t} = \frac{\sqrt{p(\infty)}}{\pi}.$$

S9) Suppose $0 \leqslant t_1 < t_2 < t_3 < \cdots$ are all the roots of $x(t)$ on $[0, \infty)$, $0 \leqslant t_1' < t_2' < \cdots$ are all the roots of $x'(t)$ on $[0, \infty)$. Prove that the sequence $\{|x'(t_k)|\}_{k\geqslant 1}$ is monotonically decreasing and the sequence $\{|x(t_k')|\}_{k\geqslant 1}$ is monotonically increasing, and

$$\lim_{k\to\infty} |x'(t_k)| = \sqrt{p(\infty)} \lim_{k\to\infty} |x(t_k')|.$$

Proof: Consider the (energy) function $E(t) = p(t)x^2(t) + x'(t)^2$, then $E'(t) = p'x^2 \leqslant 0$ so E is monotonically decreasing. For $k \geqslant 1$, $E(t_k) = x'(t_k)^2$ is decreasing, so $\{|x'(t_k)|\}_{k\geqslant 1}$ is decreasing. Likewise, consider $F(t) = x(t)^2 + x'(t)^2/p(t)$, then $F'(t) = -p'(x'/p)^2 \geqslant 0$, so $F(t_k') = x(t_k')^2$ is increasing, and

$$\lim_{k\to\infty} |x'(t_k)| = \sqrt{\lim_{k\to\infty} E(t_k)} = \sqrt{p(\infty) \lim_{k\to\infty} F(t_k)} = \sqrt{p(\infty)} \lim_{k\to\infty} |x(t'_k)|.$$

***S10) Suppose $0 \leqslant \tilde{t}_1 < \tilde{t}_2 < \cdots$ are all the roots of $x(t)x'(t)$ on $[0, \infty)$. Prove that the sequence $\{\tilde{t}_{k+1} - \tilde{t}_k\}_{k\geqslant 1}$ is monotonically increasing and calculate its limit.**

Proof: By S6), the roots of x and x' appear alternating in $\{\tilde{t}_k\}$. Since t is a root of x iff t is a root of x'', we only need to prove that if t_1, t_2 are neighboring roots of x, and $t_3 \in [t_1, t_2]$ satisfy $x'(t_3) = 0$, then $t_3 - t_1 \leqslant t_2 - t_3$.
Same as before we can prove that, for $p(t), q(t), x(t), y(t)$ such that $p(0) = q(0)$, $p(t) \leqslant q(t)$, $x'(0) = y'(0) = 0$, $x(0) = y(0)$ and

$$x''(t) + p(t)x(t) = 0,\ y''(t) + q(t)y(t) = 0,$$

then the first roots a, b of x, y satisfy $a \leqslant b$.
Since the sequence is increasing, by S8) we know that $\lim_{k\to\infty} \tilde{t}_{k+1} - \tilde{t}_k = \frac{1}{2} \lim_{t\to\infty} Z_t(x)/t = \sqrt{p(\infty)}/2\pi$.