

190-8

Suppose $\{E_k\}$ are measurable sets in \mathbb{R}^n with finite measure, and

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |\chi_{E_k} - f| \, dm = 0.$$

Prove that there exists a measurable set E , such that $f(x) = \chi_E(x)$ a.e. $x \in \mathbb{R}^n$.

Proof: Consider the measures $\mu_1(E) = \int_E |f| \, dm$ and $\mu_2(E) = \int_E |f - 1| \, dm$, then $\lim_{k \rightarrow \infty} \mu_1(E_k^C) + \mu_2(E_k) = 0$, so $\mu_1(E_k^C) \rightarrow 0$ and $\mu_2(E_k) \rightarrow 0$. We can assume $\mu_1(E_k^C), \mu_2(E_k) < 2^{-k}$, otherwise consider a sub-sequence E_{n_k} . Let $E = \bigcap_{N \geq 1} \bigcup_{k \geq N} E_k$, then $\mu_2(E) \leq \mu_2(\bigcup_{k \geq N} E_k) = 2^{1-N} \rightarrow 0$ so $\mu_2(E) = 0$, and $E^C = \bigcup_{N \geq 1} \bigcap_{k \geq N} E_k^C$ where $\mu_1(\bigcap_{k \geq N} E_k^C) = 0$ so $\mu_1(E^C) = 0$. Hence $f = \chi_E$ a.e. $x \in \mathbb{R}^n$.

190-9

Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is increasing. Prove that for $E \subset [0, 1]$ such that $m(E) = t$,

$$\int_{[0,t]} f \, dm \leq \int_E f \, dm.$$

Proof: Let $\mu_f(E) = \int_E f \, dm$, we show that $\mu_f(E) \geq \mu_f(I)$ where $I = [0, t]$. Let $A = E \setminus I, B = I \setminus E$, then $m(A) = m(B)$, and $\inf A \geq t \geq \sup B$, hence

$$\int_A f \, dm \geq f(t)m(A) = f(t)m(B) \geq \int_B f \, dm.$$

Therefore $\mu_f(E) = \mu_f(A) + \mu_f(E \cap I) \geq \mu_f(B) + \mu_f(E \cap I) = \mu_f(I)$.

191-15

Suppose $f \in \mathcal{L}^+([0, 1])$. If there exists a constant c such that

$$\int_{[0,1]} f^n \, dm = c, \, n = 1, 2, \dots$$

Prove that there exists $E \subset (0, 1)$ measurable, such that $f = \chi_E$ a.e. What if f is not non-negative?

Proof: For $t \geq 0$, let $A_t = f^{-1}(\{t\})$, then A_t is measurable. If $m(A_t) > 0$ for some $t > 1$, then $\int_{[0,1]} f^n \, dm \geq \int_{A_t} f^n \, dm = t^n m(A_t) \rightarrow \infty$, leading to contradiction. Hence $f \leq 1$ a.e., so $g_n = f - f^n \in \mathcal{L}^+([0, 1])$, and $\int_{[0,1]} g_n \, dm = 0 \, \forall n \geq 1$. Therefore $f = f^2$ a.e., so for $E = A_1$, $f = \chi_E$ a.e. Even if f may not be non-negative, this result holds: apply the argument above we get $E \subset (0, 1)$ measurable such that $f^2 = \chi_E$ a.e., and $\int_{[0,1]} f \, dm = \int_{[0,1]} f^2 \, dm$, hence $f = \chi_E$ a.e.

192-23

Suppose $f_k, f \in \mathcal{L}(\mathbb{R}^n)$, and for any measurable set $E \subset \mathbb{R}^n$,

$$\int_E f_k \, dm \leq \int_E f_{k+1} \, dm, \, k = 1, 2, \dots$$

and $\lim_{k \rightarrow \infty} \int_E f_k \, dm = \int_E f \, dm$.

Prove that $\lim_{k \rightarrow \infty} f_k = f$ a.e. $x \in \mathbb{R}^n$.

Proof: $\mu_{f_k}(E) \leq \mu_{f_{k+1}}(E)$ for any measurable E implies $f_k \leq f_{k+1}$ a.e. (otherwise take $E = \{x : f_k(x) - f_{k+1}(x) > 1/n\}$ such that $m(E) > 0$). Suppose $g = \lim_{k \rightarrow \infty} f_k$, then by monotone convergence theorem,

$$\int_E g \, dm = \lim_{k \rightarrow \infty} \int_E f_k \, dm = \int_E f \, dm$$

for every measurable E . Therefore $f = g$ a.e.

193-26

Suppose f is bounded on \mathbb{R} . If for every $x \in \mathbb{R}$, the limit $\lim_{h \rightarrow 0} f(x+h)$ exists, prove that $f(x)$ is Riemann integrable on any interval $[a, b]$.

Proof: Let $g(x) = \lim_{h \rightarrow 0} f(x+h)$, we show that $g \in C(\mathbb{R})$: For any $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists δ such that $|x - y| < \delta \implies |f(y) - g(x)| < \varepsilon/2$. For any such y , $\lim_{h \rightarrow 0} f(y+h) = g(y)$ implies there exists $z \in B(x, \delta)$ such that $|f(z) - g(y)| < \varepsilon/2$, hence $|g(x) - g(y)| \leq |g(x) - f(z)| + |f(z) - g(y)| < \varepsilon$. Therefore g is continuous.

The discontinuous points of f are $A = \{x \in [a, b] : f(x) \neq g(x)\}$, we show that

$A_n = \{x \in [a, b] : |f(x) - g(x)| > 1/n\}$ are finite, hence A is countable.

Otherwise if A_n is infinite, consider a limit point $L = \lim_{k \rightarrow \infty} x_k$ (since A_n is bounded). By definition, $\lim_{k \rightarrow \infty} g(x_k) = g(L) = \lim_{k \rightarrow \infty} f(x_k)$, so $\lim_{k \rightarrow \infty} |g(x_k) - f(x_k)| = 0$ leading to contradiction.

Therefore A is countable so f is Riemann integrable on any interval $[a, b]$.

193-28

Suppose $f \in \mathcal{R}([0, 1])$, prove that $f(x^2) \in \mathcal{R}([0, 1])$.

Proof: Otherwise there exists n such that $A = \{x \in [0, 1] : \omega_g(x) > 1/n\}$ has positive measure, where $g(x) = f(x^2)$. $\omega_g(x) = \lim_{h \rightarrow 0} \sup_{y \in (x-h, x+h)} |f(y^2) - f(x^2)|$, so $\{x^2 : x \in A\} \subset B = \{x \in [0, 1] : \omega_f(x) > 1/n\}$, leading to contradiction.

193-32

Suppose $f \in \mathcal{L}(\mathbb{R})$, and xf is Lebesgue integrable. Let

$$F(x) = \int_{-\infty}^x f \, dm.$$

If $\int_{\mathbb{R}} f \, dm = 0$ prove that $F \in \mathcal{L}(\mathbb{R})$.

Proof: For $x \in (-\infty, 0)$, by Tonelli theorem,

$$\int_{-\infty}^0 |F(x)| \, dx \leq \int_{-\infty}^0 \int_{-\infty}^x |f(t)| \, dt \, dx = \int_{-\infty}^0 \int_t^0 |f(t)| \, dx \, dt = \int_{-\infty}^0 |tf(t)| \, dt < \infty.$$

For $x > 0$, note that $F(x) = -\int_x^{\infty} f \, dm$, so likewise

$$\int_0^{\infty} |F(x)| \, dx \leq \int_0^{\infty} \int_x^{\infty} |f(t)| \, dt \, dx = \int_0^{\infty} |tf(t)| \, dt.$$

Therefore $F \in \mathcal{L}(\mathbb{R})$.

193-34

Suppose $f \in \mathcal{L}((0, a))$, $g(x) = \int_{[x, a]} \frac{f(t)}{t} \, dm$. Prove that $g \in \mathcal{L}((0, a))$ and

$$\int_0^a g \, dm = \int_0^a f \, dm.$$

Proof: Note that by Tonelli theorem,

$$\int_0^a |g| \, dx \leq \int_0^a \int_x^a \left| \frac{f(t)}{t} \right| \, dt \, dx = \int_0^a \int_0^t \left| \frac{f(t)}{t} \right| \, dx \, dt = \int_0^a |f(t)| \, dt$$

hence $g \in \mathcal{L}((0, a))$. Apply Fubini theorem we have

$$\int_0^a g \, dm = \int_0^a \int_x^a f(t)/t \, dt \, dx = \int_0^a \int_0^t f(t)/t \, dx \, dt = \int_0^a f \, dx.$$