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1 Homework 1: Schröder-Bernstein Theorem

1.1 PSA

A1) Suppose a non-empty set $X \subset \mathbb{R}$ has an upper bound, and M is an upper bound of X. The following two propositions are equivalent:

- $M = \sup X$.
- For any $\varepsilon > 0$, there exists an $x \in X$ such that $x > M - \varepsilon$.

Proof:

$$M = \sup X \iff \forall M' < M, \exists x \in X, x > M' \iff \forall \varepsilon = M - M' > 0, \exists x \in X, x > M - \varepsilon.$$

A2) Prove that every non-empty open interval contains infinitely many rational numbers.

Proof: We only need to find one rational number q in the interval (a, b), then we can apply the process to (a, q) and so on.
By the Archimedean rule, there is a positive integer N such that $N(b - a) > 2$, hence there exists an integer p such that $p = \lfloor bN \rfloor \in (aN, bN)$, and $q = \frac{p}{N} \in (a, b) \cap \mathbb{Q}$.

A3) Let (X, d) be a metric space, $Y \subset X$. We define the distance function on Y:

$d_Y : Y \times Y \to \mathbb{R}, (y_1, y_2) \mapsto d_Y(y_1, y_2) = d(y_1, y_2)$.
Prove that d_Y is a distance function, and (Y, d_Y) is a metric space. We call d_Y the induced metric on Y, and (Y, d_Y) is called a subspace.
Proof: Trivial, since $d_Y(y_1, y_2) = d(y_1, y_2)$.

A4) Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$, for any $x, y \in \mathbb{R}^n$, we define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n).$$

Prove that $(\mathbb{R}^n, d)$ is a metric space.

Proof:

1. $d(x, y) = 0 \iff x_i = y_i, \forall 1 \leqslant i \leqslant n \iff x = y$.
2. $d(x, y) = d(y, x)$ is trivial.
3. $d(x, y) + d(y, z) \geqslant d(x, z)$ is the Minkowski inequality:

$$\left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2 = \sum_{i=1}^{n} a_i^2 + b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2} \geqslant \sum_{i=1}^{n} a_i^2 + b_i^2 + 2a_i b_i = \sum_{i=1}^{n} (a_i + b_i)^2.$$

A5) Given a metric space (X, d), and $Y \subset X$. If for any $x \in X$ and $\varepsilon > 0$, there exists $y \in Y$ such that $d(y, x) < \varepsilon$, then we say Y is dense in X. Prove that the set of rational numbers is dense in $\mathbb{R}$.

Proof: For any $x \in \mathbb{R}$, let $N = \lfloor x \rfloor$, then for any $\varepsilon > 0$, let $q > 1/\varepsilon$. Then for $p \in [Nq, (N+1)q] \cap \mathbb{Z}$, choose p such that $|x - p/q|$ is minimal. Suppose $p/q < x$, then

$$2\left|x - \frac{p}{q}\right| < \left|x - \frac{p}{q}\right| + \left|x - \frac{p+1}{q}\right| = \frac{1}{q} < \varepsilon.$$

Hence $d(x, p/q) < \varepsilon$.

A6) For $(x, y) \in \mathbb{R}^2$, if its coordinates x and y are rational numbers, then we call this point a rational point. Prove that $(\mathbb{R}^2, d)$ (refer to question A4) the set of rational points in $\mathbb{R}^2$ is dense.

Proof: By A5), $\overline{\mathbb{Q}} = \mathbb{R}$. Hence for any $(x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$, there exists $(a, b) \in \mathbb{Q}^2$ such that $|a - x|, |b - y| < \varepsilon/2$. Then

$$d((x, y), (a, b)) = \sqrt{(a-x)^2 + (b-y)^2} < \varepsilon.$$

Hence $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$

A7) Prove that the axiom (F) and (O), and the boundedness principle imply the Archimedean axiom (A).

Proof: Otherwise assume that $\mathbb{N}$ has an upper bound. Then $M = \sup \mathbb{N}$ exists. Let $\varepsilon = 1/2$ then there is an $n \in \mathbb{N}$ such that $n > M - \varepsilon$. Hence $n + 1 > M$, leading to contradiction.

A8) (Existence of irrational numbers) Let $X = \left\{x \in \mathbb{Q} \mid x^2 < 2\right\}$ be a bounded set, and $\sqrt{2} = \sup X$. Prove that $\sqrt{2}$ is an irrational number.

Proof: If $\sqrt{2} = s = p/q$ is rational, then $p^2 \geqslant 2q^2$, otherwise let $x = s(2 - s^2)/4 + s$, then $s < x$ and $x^2 < 2$, a contradiction. If $s^2 > 2$, then $x = s(2 - s^2)/4 < s$ and $x^2 > 2$, hence x is an upper bound of X, leading to contradiction. Therefore $s^2 = 2$ which is impossible.

A9) Prove that every open interval contains infinitely many irrational numbers.

Proof: Otherwise the open interval will be a countable set.

1.2 PSB: Countable and Uncountable Sets

Let $\mathbb{N}$ denote the set of natural numbers (including 0). X is a set, if there is an injective map $f : X \to \mathbb{N}$, then we say X is countable; if X is not countable, then we say X is uncountable.

B1) Prove that finite sets are countable.

Proof: For any finite set $X = \{a_1, \cdots, a_n\}$, the map $f : a_k \mapsto k$ is an injective, hence X is countable.

B2) Prove that subsets of countable sets are countable.

Proof: If X is countable and $Y \subset X$, then there is an injective map $f : X \to N$, so $f|_Y : Y \to N$ is an injective map, hence Y is countable.

B3) Prove that if X is a countable set, then we can always write $X = \{x_1, x_2, x_3, \ldots\}$ (that is, the elements of X can be indexed by natural numbers).

Proof: Let $I = \{n \in \mathbb{N} : f^{-1}(n) \neq \emptyset\}$, $x_k = f^{-1}(\min I \backslash \{f(x_1), \cdots, (x_{k-1})\})$. Then $x_x \in X$, and for any $x \in X$, $f(x) \in I$ hence $x \in \{x_1, \cdots, x_{f(x)}\}$. Therefore $X = \{x_1, \cdots, x_n, \cdots\}$.

B4) Prove that the set of rational numbers $\mathbb{Q}$ is countable.

Proof: List every positive rational number as below:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \cdots$$

such that p/q is before m/n if $p+q < m+n$ or $p+q = m+n$ and $p < m$, then every number in $\mathbb{Q}_{>0}$ is listed at least once. Hence $\mathbb{Q}_{>0}$ is countable and so is $\mathbb{Q}$.

B5) Prove that the countable union of countable sets is countable, that is, if $X_1, X_2, \ldots, X_n, \ldots$ are all countable sets, then their union $\bigcup_{n=1}^{\infty} X_n$ is also a countable set.

Proof: Assume X_n are disjoint. Since X_n are countable, we can write

$$X_n = \{a_1^{(n)}, a_2^{(n)}, \cdots, a_m^{(n)}, \cdots\}.$$

Then

$$\bigcup_{n=1}^{\infty} X_n = \{a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_3^{(1)}, \cdots\}$$

where the order is the same as in B4). Hence $\bigcup_{n\geqslant 1} X_n$ is countable.

B6) If X is countable, and the map $f : X \to Y$ is surjective, then Y is countable.

Proof: Since X is countable, there is an injective map $g : X \to \mathbb{N}$. Let

$$h : Y \to \mathbb{N},\ y \mapsto \min g(f^{-1}(\{y\})).$$

then g is injective, hence Y is countable.

B7) Prove the following using proof by contradiction: $\mathbb{R}$ is uncountable.

B7-1) Suppose $J \subset \mathbb{R}$ is a closed interval and its length $|J| > 0$. For any $x \in \mathbb{R}$, there always exists an interval $I \subset J$ such that $|I| > 0$ and $x \notin I$.
Proof: Any closed interval $J = [a, b]$ can be written in the form $J = A \cup B \cup C$, where $A = \left[a, \frac{2a+b}{3}\right], B = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], C = \left[\frac{a+2b}{3}, b\right]$, and x can only be in at most 2 of these sets. Hence we can choose a set I in A, B, C.
B7-2) Prove that if $\{x_1, x_2, \ldots\}$ is a countable subset of $\mathbb{R}$, then there exists a nested interval sequence $I_1 \supset I_2 \supset \cdots$ such that for any n, $x_n \notin I_n$.
Proof: Simple application of B7-1)
B7-3) Prove that $\mathbb{R}$ is uncountable.
Proof: If $\mathbb{R}$ is countable, write $\mathbb{R} = \{r_1, r_2, \cdots\}$, then set $I_0 = [0, 1]$. By B7-2) we can obtain a sequence $I_0 \supset I_1 \supset \cdots$ such that $x_n \notin I_n$ for any n. Hence

$$\bigcap_{n=0}^{\infty} I_n = \emptyset,$$

leading to contradiction.

B8) Prove that if X is an uncountable set, and A is a countable subset of X, then $X - A$ is uncountable.

Proof: Otherwise suppose that both A and $X - A$ is countable, then there exist injective mappings $f : A \to \mathbb{N}$ and $g : X - A \to \mathbb{N}$. Define

$$h : X \to \mathbb{N},\ x \mapsto \begin{cases} 2f(x), & x \in A, \\ 2g(x) + 1, & x \notin A. \end{cases}$$

Then h is injective, hence X is countable.

B9) Prove that any interval of non-zero length (open or closed) is uncountable.

Proof: Same as B7).
Or use the fact that $\mathbb{R}$ is the countable union of intervals of the same length, and the countable union of countable sets is still countable.

B10) Prove that the set of complex numbers $\mathbb{C}$ is uncountable.

Proof: $\mathbb{C}$ has an uncountable subset $\mathbb{R}$.

B11) Suppose $\mathcal{I}$ is a collection of non-overlapping closed intervals, satisfying the following property: for any $I, J \in \mathcal{I}$, if $I \neq J$, then their intersection is empty, i.e., $I \cap J = \emptyset$. Prove that $\mathcal{I}$ is countable.

Proof: For any $I \in \mathcal{I}$, there exists a rational number $r_I \in I$. Consider $f : \mathcal{I} \to \mathbb{Q}$, $I \mapsto r_I$, then f is injective. Since $\mathbb{Q}$ is countable, so is $\mathcal{I}$.

1.3 PSC: Schröder-Bernstein Theorem

Suppose X and Y are two sets, and mappings $f : X \to Y$ and $g : Y \to X$ are both injective. Let $X' = X - g(Y)$.

C1) If X is a finite set, prove that there exists a bijection $\varphi : X \to Y$.

Proof: $g : Y \to X$ is injective and X is finite, $\implies$ Y is finite. Hence $|X| \leqslant |Y|$, and $|X| \geqslant |Y|$, so $|X| = |Y|$. Therefore we can write $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, and obtain

$$\varphi : X \to Y,\ x_k \mapsto y_k.$$

C2) If X is countable, prove that there exists a bijection $\varphi : X \to Y$.

Proof: Assume X is infinite, then Y is countable (by g) and infinite (by f). Hence we can list $X = \{x_1, x_2, \cdots\}$ and $Y = \{y_1, y_2, , ..\}$ and define

$$\varphi : X \to Y,\ x_k \mapsto y_k.$$

From now on, we impose no restrictions on X. Let $h : X \to X$ be the composite map $h = g \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow g \\ X & \leftarrow & \end{array}$$

C3) Consider the set family $\mathcal{F} = \{A \subset X \mid X' \cup h(A) \subset A\}$. Prove that $\mathcal{F}$ is non-empty.

Proof: $X \in \mathcal{F}$.

C4) Prove that if $A \in \mathcal{F}$, then $X' \cup h(A) \in \mathcal{F}$.

Proof: If $A \in \mathcal{F}$ then $X' \cup h(A) \subset A$, hence (let B denote $X' \cup h(A)$)

$$X' \cup h(B) \subset X' \cup h(A) = B.$$

C5) We define

$$A_0 = \bigcap_{A \in \mathcal{F}} A = \{x \in X \mid \text{for any } A \in \mathcal{F}, \text{ we have } x \in A\}.$$

Prove that $A_0 \in \mathcal{F}$.
Proof:

$$X' \cup h(A_0) \subset X' \cup (\bigcap_{A \in \mathcal{F}} h(A)) = \bigcap_{A \in \mathcal{F}} X' \cup h(A) \subset \bigcap_{A \in \mathcal{F}} A = A_0.$$

Hence $A_0 \in \mathcal{F}$.

C6) Prove that $X' \cup h(A_0) = A_0$.

Proof:

$$A_0 \in \mathcal{F} \implies X' \cup h(A_0) \in \mathcal{F} \implies A_0 \subset X' \cup h(A_0).$$

The other side is proved in C5).

C7) Let $B_0 = X - A_0$. Prove that $f(A_0) \cap g^{-1}(B_0) = \emptyset$ and $f(A_0) \cup g^{-1}(B_0) = Y$.

Proof: If $f(A_0) \cap g^{-1}(B_0) \neq \emptyset$, then there exist $a \in A_0, b \in B_0$ such that $f(a) = g^{-1}(b)$, i.e. $b = h(a)$. Since $a \in A_0$, for any $A \in \mathcal{F}$, $a \in A$, hence $b = h(a) \in X' \cup h(A) \subset A$. Therefore $b \in A_0$, a contradiction.
Otherwise if there exists $y \in Y$ such that $y \notin f(A_0) \cup g^{-1}(B_0)$, then $g(y) \notin B_0 \implies g(y) \in A_0$. Let $z = g(y) \in A_0 \cap g(Y)$, then $z \notin X'$ so $z \in h(A_0)$ by C6). Let $z = h(t)$ then $y = f(t) \in f(A_0)$ since g is injective, leading to contradiction.

C8) We define the map $\varphi : X \to Y$. For $x \in X$, we require

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in A_0; \\ g^{-1}(x), & \text{if } x \in B_0. \end{cases}$$

Prove that this is a bijection.
Proof:

1. φ is injective: for any $x, y \in A_0, x \neq y$, $\varphi(x) \neq \varphi(y)$ since f is injective. For any $x, y \in B_0$, $x \neq y$, $\varphi(x) \neq \varphi(y)$ since g is a mapping. For any $x \in A_0, y \in B_0$, $\varphi(x) \neq \varphi(y)$ since $f(A_0) \cap g^{-1}(B_0) = \emptyset$.
2. φ is subjective: $\varphi(X) = \varphi(A_0 \cup B_0) = f(A_0) \cup g^{-1}(B_0) = Y$.

Based on the above, we have proved:
Theorem (Schroeder-Bernstein). If there exist injective maps $f : X \to Y$ and $g : Y \to X$, then there exists a bijection $\varphi : X \to Y$ between the two sets.

1.4 PSD: Details of Dedekind Cut

The goal of this part of the exercise is to complete the part of the Dedekind cut construction method taught in class, thereby providing a complete proof for the construction of real numbers.

D1) Prove that if X and Y are both Dedekind cuts, then the product $X \cdot Y$ as defined in the lecture is also a Dedekind cut, i.e.,

$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (X, Y) \mapsto X \cdot Y$,
is well-defined. (Hint: You only need to prove the case where $X > 0, Y > 0$.)
Proof: The set $X \cdot Y$ is define as $Z = \bar{0} \cup \{x \cdot y : x, y \geqslant 0, x \in X, y \in Y\}$. Let $Z' = \mathbb{Q} - Z$, then

1. $Z \neq \emptyset, Z' \neq \emptyset$, since for any $x \in X', y \in Y'$, $x \cdot y \notin Z$.
2. For any $z \in Z, z' \in Z'$, if $z' < z$ then $z > 0$. So assume $z = x \cdot y, x \in X, y \in Y, x, y \geqslant 0$, then $z' = x \cdot (yz'/z) \in Z$, a contradiction.
3. If Z has a maximal element $z = x \cdot y, x, y \geqslant 0, x \in X, y \in Y$, then since x, y are both not maximal, there exists $x' \in X, y' \in Y$, such that $x < x', y < y'$ so $z < z' = x' \cdot y' \in Z$, a contradiction.

D2) Prove that $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$. ($\Longrightarrow$ (F5))

Proof: We only need to verify the case where $X, Y, Z > 0$. Then both $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$ are the set

$$\bar{0} \cup \{x \cdot y \cdot z : x, y, z \geqslant 0, x \in X, y \in Y, z \in Z\}.$$

D3) Prove that $X \cdot Y = Y \cdot X$. ($\Longrightarrow$ (F6))

Proof: Same as D2).

D4) Prove that $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$. ($\Longrightarrow$ (F9))

Proof: We can assume that $X, Y, Z > 0$, then

$$X \cdot (Y + Z) = \{xy + xz : x \in X, y \in Y, z \in Z\}$$

while

$$X \cdot Y + X \cdot Z = \{xy + x'z : x, x' \in X, y \in Y, Z \in Z\}.$$

Hence $X \cdot (Y + Z) \subset X \cdot Y + X \cdot Z$.
For any $xy + x'z \in X \cdot Y + X \cdot Z$, suppose $x \geqslant x'$, then
$xy + xz \in X \cdot (Y + Z)$ and $xy + x'z \leqslant xy + xz$, so $xy + x'z \in X \cdot Y + X \cdot Z$, therefore $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$.

D5) Prove that $\bar{1} \cdot X = X$ and $\bar{1} \neq \bar{0}$. ($\Longrightarrow$ (F7))

Proof: Assume that $X > 0$, then $\bar{1} \cdot X = \{u \cdot v : u < 1, v \in X\}$. Foy any $u < 1, v \in X$, $u \cdot v < v$ hence $u \cdot v \in X$. For any $x \in X$, there exists $x' \in X, x' > x$, then $x = x' \cdot (x/x') \in \bar{1} \cdot X$. Therefore $\bar{1} \cdot X = X$ and $1/2 \in \bar{1} \backslash \bar{0}$, so $\bar{1} \neq \bar{0}$.

D6) Prove that if $X \cdot Y = \overline{0}$, then $X = \overline{0}$ or $Y = \overline{0}$; conversely, if $X \geq \overline{0}, Y \geq \overline{0}$, then $X \cdot Y \geq \overline{0}$. ($\Longrightarrow$ (O5))

Proof: Otherwise there exists $x, x' \in X, y, y' \in Y$, such that $x, y > 0, x', y' < 0$. Hence $xy, x'y \in X \cdot Y$, where $xy > 0 > x'y$, so $X \cdot Y \neq \overline{0}$.
Suppose $X, Y > 0$, then there exists $x \in X, y \in Y$ such that $x, y > 0$, hence $0 < xy \in X \cdot Y$, so $X \cdot Y > \overline{0}$.

D7) X is a positive Dedekind cut. Prove that for any integer n, there exist $x \in X, x' \in X'$ such that

$1 < \frac{x'}{x} < 1 + \frac{1}{n}$.
Proof: Let $l_0 = x \in X, r_0 = x' \in X'$. Define l_n, r_n as follows: If $(l_{n-1} + r_{n-1})/2 \in X$, then $l_n = (l_{n-1} + r_{n-1})/2, r_n = r_{n-1}$, otherwise $l_n = l_{n-1}, r_n = (l_{n-1} + r_{n-1})/2$. Then

$$0 < \frac{r_n - l_n}{l_n} \leqslant \frac{1}{2} \frac{r_{n-1} - l_{n-1}}{l_{n-1}}.$$

Hence there exist such x, x'.

D8) Prove that for any Dedekind cuts X and Y, if $Y \neq \overline{0}$, there exists a unique Dedekind cut Z such that

$Y \cdot Z = X$.
We denote Z as $\frac{X}{Y}$. When $X = \overline{1}$, we also denote it as Y^{-1}. ($\Longrightarrow$ (F8))
Proof: By D6), Z is unique. By D2) we can assume that $X = \overline{1}$, and $Y > 0$. Let

$$Z = \left\{ \frac{1}{y} : y \in Y' \right\} \cup \overline{0} \cup \{0\}.$$

Then by D7), $Y \cdot Z = \overline{1}$.

2 Homework 2: Cesàro sum

2.1 PSA

A1) $\{x_n\}_{n \geqslant 1}$ is a bounded real sequence. Prove that there is a subsequence $\{x_{n_i}\}_{i \geqslant 1}$ such that $\lim_{i \to \infty} x_{n_i}$ exists and

$$\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n.$$

Proof: Let $M = \limsup_{n \to \infty} x_n < \infty$, then for any $\varepsilon = 1/i > 0$ there exists $N \geqslant n_{i-1}$ such that $M \leqslant \sup_{k \geqslant N} x_k < M + \varepsilon$. Hence there exists $n_i \geqslant N$ such that $x_{n_i} \in (M - \varepsilon, M + \varepsilon)$. Take the sequence $\{x_{n_i}\}_{i \geqslant 1}$ then $\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n$.

A2) $\{x_n\}_{n \geqslant 1}$ is a real sequence. Prove that $\{x_n\}_{n \geqslant 1}$ converges iff $\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n$.

Proof: Since a sub-sequence of a Cauchy sequence converge to the same value as the original sequence, $\Longrightarrow$ is trivial by A1).
$\Longleftarrow$ $\lim_{n \to \infty} \sup_{k \geqslant n} x_k - \inf_{k \geqslant n} x_k = 0$ implies x_n is Cauchy, hence convergent.

A3) $\{x^{(k)}\}_{k\geqslant 1}\subset\mathbb{R}^n$, where $x^{(k)}=(x_1^{(k)},x_2^{(k)},\cdots,x_n^{(k)})$. Then $\{x^{(k)}\}_{k\geqslant 1}$ converges in $\mathbb{R}^n$ iff for any $i=1,2,\cdots,n$, $\{x_i^{(k)}\}_{k\geqslant 1}$ converges.

Proof: Use Cauchy sequences and the fact that for $x=(x_1,x_2,\cdots,x_n)$,

$$\max\{|x_k|:1\leqslant k\leqslant n\}\leqslant\|x\|\leqslant\sum_{k=1}^{n}|x_k|.$$

A4) Suppose $\{z_n\}_{n\geqslant 1},\{w_n\}_{n\geqslant 1}$ are two convergent complex sequences. Prove that if $\lim_{n\to\infty}w_n\neq 0$, then the sequence $\{z_n/w_n\}_{n\geqslant 1}$ converges.

Proof: Suppose $z=\lim_{n\to\infty}z_n$ and $w=\lim_{n\to\infty}w_n$, then

$$\left|\frac{z_n}{w_n}-\frac{z}{w}\right|\leqslant\frac{|w|\cdot|z_n-z|}{|w\cdot w_n|}+\frac{|z|\cdot|w_n-w|}{|w\cdot w_n|}.$$

Hence $\left|\frac{z_n}{w_n}-\frac{z}{w}\right|\to 0$, so $\lim_{n\to\infty}z_n/w_n=z/w$.

A5) Suppose $\{a_n\}_{n\geqslant 1}$ is a monotonically decreasing sequence of positive reals, and $\lim_{n\to\infty}a_n=0$. Prove that the series

$$a_1-a_2+a_3-a_4+\cdots+(-1)^{n-1}a_n+\cdots$$

converges.

Proof: Suppose $a_n=a_1-\sum_{k=1}^{n}b_k$, then $b_k\geqslant 0$ and $\sum_{k=1}^{\infty}b_k=a_1$. The series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}b_{2n}<a_1$$

clearly converges.

A6) $\{a_n\}_{n\geqslant 1}\subset\mathbb{C}$. Prove that if $\sum_{k=1}^{\infty}|a_k|$ converges, then $\sum_{k=1}^{\infty}a_k$ converges.

Proof: $\sum_{k=1}^{\infty}|a_k|$ converges implies for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$, $\sum_{k=n}^{n+p}|a_k|<\varepsilon$. Note that $\left|\sum_{k=n}^{n+p}a_k\right|\leqslant\sum_{k=n}^{n+p}|a_k|$, so $\sum_{k=1}^{\infty}a_k$ converges.

A7) Prove that we can define the exponential function on $\mathbb{C}$:

$$\exp:\mathbb{C}\to\mathbb{C},\ z\mapsto\exp(z)=e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Proof: Use A6).

A8) $\{a_n\}\subset\mathbb{C}$. Suppose for any $n\in\mathbb{N}$, $a_n\neq 0$. Let $P_n=a_1\cdot a_2\cdots a_n$. If $\lim_{n\to\infty}P_n$ exists and is not 0, we call $\prod_{n=1}^{\infty}a_n$ convergent and let $\prod_{n=1}^{\infty}a_n=\lim_{n\to\infty}P_n$. Prove that $\prod_{n=1}^{\infty}a_n$ converges iff for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$,

$$|a_n\cdot a_{n+1}\cdots a_{n+p}-1|<\varepsilon.$$

Proof: If $\lim_{n\to\infty}P_n=P$ exists and is non-zero, then for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $|P_n-P|<\varepsilon P/4$ and $|P_n|>P/2$. Then for any $n\geqslant N$, $p\geqslant 0$, $|P_{n+p}/P_n-1|<\varepsilon$.
If for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$, $|P_{n+p}-P_n|<\varepsilon|P_n|$, then let $\varepsilon=1$ we infer that P_n is bounded by some constant M. Hence the sequence $\{P_n\}$ is Cauchy, and $P=\lim_{n\to\infty}P_n$ cannot be zero, otherwise there is no such N for $\varepsilon=1/2$.

A9) Prove that $\exp(x)$ is monotonically increasing on $\mathbb{R}$.

Proof: For $x, y \in \mathbb{R}$,

$$\exp(x) \cdot \exp(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{k=0}^{\infty} \sum_{n+m=k} \frac{x^n y^n \binom{k}{n}}{k!} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = \exp(x+y).$$

$\exp(x) \cdot \exp(-x) = \exp(0) = 1$ implies $\exp(x) > 0$ for all $x \in \mathbb{R}$, so if $x > y$, $\exp(x)/\exp(y) = \exp(x-y) > 1 \implies \exp(x) > \exp(y)$.

A10) Suppose $P(x)$ and $Q(x)$ are polynomials of degree n, m, where $m > n$. Prove that

$$\lim_{n\to\infty} \frac{Q(n)}{P(n)} = 0, \lim_{n\to\infty} \frac{Q(n)}{e^n} = 0.$$

Proof: Suppose $P(x) = \sum_{k=0}^{n} a_k x^k$ and $Q(x) = \sum_{k=0}^{m} b_k x^k$, then there exists N such that for any $x \geqslant N$, $|P(x)| > |a_n| x^n/2$, $|Q(x)| \leqslant \sum_{k=0}^{m} |b_k| \cdot x^m$, and $e^x \geqslant x^{m+1}/(m+1)!$, hence

$$\left|\frac{Q(x)}{P(x)}\right| \leqslant \frac{2\sum_{k=0}^{m} |b_k|}{|a_n|} \cdot x^{m-n} \to 0, \left|\frac{Q(x)}{e^x}\right| \leqslant (m+1)! \sum_{k=0}^{m} |b_k| \cdot x^{-1} \to 0.$$

2.2 PSB: Calculation of Limits

B1)

$$\lim_{n\to\infty} \frac{n+10}{2n-1} = \frac{1}{2}.$$

B2)

$$\lim_{n\to\infty} \frac{\sqrt{n}+10}{2\sqrt{n}-1} = \frac{1}{2}.$$

B3)

$$\lim_{n\to\infty} 0.\underbrace{99\cdots9}_{n \text{ times}} = 1.$$

B4)

$$\lim_{n\to\infty} \frac{1}{n(n+3)} = 0.$$

B5)

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

B6)

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

B7)

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

B8)

$$\lim_{n\to\infty} \sqrt{n+10} - \sqrt{n+1} = 0.$$

B9)

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}.$$

B10)

$$\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{3}.$$

B11) $a > 0$

$$\lim_{n\to\infty} a^{1/n} = 1.$$

B12) $a > 1$

$$\lim_{n\to\infty} \frac{n^{10000}}{a^n} = 0.$$

B13)

$$\lim_{n\to\infty} \frac{2^n+n}{3^n+n^2} = 0.$$

B14)

$$\lim_{n\to\infty} \frac{3^n+2^n}{3^n+n^2} = 1.$$

B15)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{1}{2}.$$

B16) same as B12)

B17)

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1}.$$

B18)

$$\lim_{n\to\infty} \left(1-\frac{1}{5n}\right)^{n+2019} = e^{-1/5}.$$

B19)

$$\lim_{n\to\infty} (n^3+n^2+9n+1)^{1/n} = 1.$$

B20)

$$\lim_{n\to\infty} (2018^n+2019^n)^{1/n} = 2019.$$

2.3 PSC: Riemann Rearrangement Theorem

Suppose $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, we will prove that for and $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$, we can rearrange the sequence such that the new series sums to α. Suppose $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ is a bijection, let $b_k = a_{\varphi(k)}$, then the sequence $\{b_k\}_{k\geqslant 1}$ is called a rearrangement of $\{a_n\}_{n\geqslant 1}$.
Let all non-negative terms of $\{a_n\}_{n\geqslant 1}$, listed in the same order as in $\{a_n\}$ be $c_1, c_2, \cdots$, and the negative terms be $d_1, d_2, \cdots$.

C1) Prove that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$**.**

Proof: Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, c_n, d_n both have infinite terms and $\lim_{n\to\infty} a_n = 0$. Therefore $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$.

C2) Prove that $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} b_n = \infty$**.**

Proof: Since $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent, the two series can not be both convergent. If one converges and the other doesn't, then $\sum_{n=1}^{\infty} a_n$ will diverge. Hence they both diverge.

C3) Prove that for any $\alpha \in \mathbb{R}$, there exists a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} b_k = \alpha$.

Proof: Suppose $\alpha \geqslant 0$. Inductively define the indices u_i and v_i as follows ($u_0 = v_0 = 0$): For $i \geqslant 1$, let u_i be the least index such that $u_i > u_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant \alpha,$$

and v_i be the least index such that $v_i > v_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j \leqslant \alpha.$$

Let φ be the permutation such that

$$b_1 = c_1, b_2 = c_2, \cdots, b_{u_1} = c_{u_1}, b_{u_1+1} = -d_1, \cdots, b_{u_1+v_1} = -d_{u_1}, \cdots$$

Since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} d_n = \infty$, u_i and v_i all exists, so φ is indeed a bijection. By definition we know that

$$\left|\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j - \alpha\right| \leqslant c_{u_i-1},$$

and

$$\left|\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j - \alpha\right| \leqslant d_{v_i-1}.$$

Since $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$, the two values above both tend to 0. Note that the series $\sum_{n=1}^{\infty} b_n$ is monotonic between these indices, hence $\sum_{n=1}^{\infty} b_n = \alpha$.

C4) Prove that there exists a rearrangement $\{x_k\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} x_k = \infty$.

Proof: Define u_i and v_i as in C3), such that

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant i \geqslant \sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j.$$

Same as C3) define the sequence x_k and clearly $\sum_{n=1}^{\infty} x_k = \infty$.

2.4 PSD: Cesàro Sum

For a real sequence $\{a_n\}_{n\geqslant 1}$, let $\sigma_n = \frac{a_1+a_2+\cdots+a_n}{n}$.

D1) Suppose $\lim_{n\to\infty} a_n = a$, prove that $\lim_{n\to\infty} \sigma_n = a$.

Proof: For any $n > 0$,

$$|\sigma_n - a| \leqslant \sum_{i=1}^{N} \frac{|a_i - a|}{n} + \sum_{i=N+1}^{n} \frac{|a_i - a|}{n} \leqslant \frac{MN}{n} + \varepsilon(N),$$

where $M = |a| + \sup_{i\leqslant N} |a_i|$, and $\varepsilon(N) = \sup_{i>N} |a_i - a|$. By $\lim_{n\to\infty} a_n = a$ we know $\varepsilon(N) \to 0$, hence $\lim_{n\to\infty} \sigma_n = a$.

D2) Construct a divergent sequence $\{a_n\}$ such that $\lim_{n\to\infty} \sigma_n = 0$.

Solution: $a_n = (-1)^{n-1}$, $\sigma_n \in [0, 1/n]$.

D3) Determine whether there exists $\{a_n\}_{n\geqslant 1}$ such that for any $n \geqslant 1$, $a_n > 0$ and $\limsup_{n\to\infty} a_n = \infty$ but $\lim_{n\to\infty} \sigma_n = 0$.

Solution: Let

$$a_n = \begin{cases} 2^{-n}, & n \neq 2^k, \\ k, & n = 2^k. \end{cases}$$

Then $\limsup_{n\to\infty} a_n = \infty$ and $a_n > 0$, but for any n, suppose $n \in [2^{k-1}, 2^k]$, then

$$\sigma_n \leqslant \frac{1}{n} \cdot \left(1 + \frac{k(k+1)}{2}\right) \leqslant \frac{k(k+1)}{2^{k-1}}.$$

Hence $\lim_{n\to\infty} \sigma_n = 0$.

D4) For $k \geqslant 1$, denote $b_k = a_{k+1} - a_k$. Prove that for any $n \geqslant 2$, $a_n - \sigma_n = \sum_{k=1}^{n-1} kb_k/n$.

Proof:

$$\sum_{k=1}^{n-1} kb_k = \sum_{k=1}^{n-1} k(a_{k+1} - a_k) = (n-1)a_n - \sum_{k=1}^{n-1} a_k = n(a_n - \sigma_n).$$

D5) Suppose $\lim_{k\to\infty} kb_k = 0$ and $\{\sigma_n\}_{n\geqslant 1}$ converges. Prove that $\{a_n\}_{n\geqslant 1}$ also converges.

Proof: By D1), $\lim_{k\to\infty} kb_k = 0$ implies

$$\lim_{n\to\infty} a_n - \sigma_n = \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} kb_k}{n} = \lim_{k\to\infty} kb_k = 0.$$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n$ exists.

D6) Suppose $\{kb_k\}_{k\geqslant 1}$ is bounded, i.e. $b_k = O(k^{-1})$, and $\lim_{n\to\infty} \sigma_n = \sigma$. Prove that $\lim_{n\to\infty} a_n = \sigma$.

Proof: Note that for $m < n$,

$$a_n - \sigma_n = \frac{m}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^{n} a_n - a_k.$$

Therefore since σ_n is a Cauchy sequence, and $|a_n - a_k| \leqslant M(n-k)/k$, we can choose n, m to show that $\lim_{n\to\infty} a_n - \sigma_n = 0$.

2.5 PSE: Definition of $\sqrt[n]{x}$ and b^x

E1) Given $n \in \mathbb{N}$ and $x > 0$, prove that if $y_1, y_2 > 0$ satisfy $y_1^n = x = y_2^n$, then $y_1 = y_2$.

Proof: Note that $y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1} > 0$, and

$$0 = y_1^n - y_2^n = (y_1 - y_2) \cdot (y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1}).$$

Hence $y_1 = y_2$.

E2) Prove that if $x > 0$, then the set $E(x) = \{t \in \mathbb{R} : t^n < x\}$ is non-empty and has an upper-bound.

Proof: Note that $0 \in E(x)$ and $E(x)$ has the upper-bound $\max\{1, x\}$.

E3) Prove that $y = \sup E(x)$ satisfy $y^n = x$ and $y > 0$.

Proof: $y = \sup E(x) \implies y^n = x$ since t^n is continuous on $\mathbb{R}$, and $y^n = x$ and $0 \in E(x)$ implies $y > 0$.

E4) Prove that the mapping $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto \sqrt[n]{x} = y$ is well-defined. Denote $\sqrt[n]{x}$ as $x^{1/n}$.

Proof: Use E3).

E5) Prove the $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a bijection.

Proof: By E1) it is injective, and $\sqrt[n]{y^n} = y$ implies it is surjective. Hence it is a bijection.

E6) $a, b > 0$, $n \in \mathbb{N}$, prove that $(ab)^{1/n} = a^{1/n}b^{1/n}$.

Proof: Use E5) and $(xy)^n = x^n y^n$.

E7) Suppose $b > 1$, $m, n, p, q \in \mathbb{Z}$ where $n, q > 0$. Let $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Proof: Use $(b^m)^q = (b^p)^n$ and E5).

E8) Prove that for any $r \in \mathbb{Q}$, $r \mapsto b^r$ is well-defined.

Proof: For $r = p/q$, where $q > 0, \gcd(p, q) = 1$, let $b^r = (b^p)^{1/q}$, then for any $r = m/n$, $b^r = (b^m)^{1/n}$.

E9) Prove that for $r, s \in \mathbb{Q}$, $b^{r+s} = b^r b^s$.

Proof: Suppose $r = p/q, s = m/n$, where $n, q > 0$, then

$$b^{r+s} = b^{(mq+np)/nq} = (b^{mq} \cdot b^{np})^{1/nq} = (b^m)^{1/n} \cdot (b^p)^{1/q} = b^r b^s.$$

E10) For $x \in \mathbb{R}$, let $B(x) = \{b^t : t \in \mathbb{Q}, t \leqslant x\}$. Prove that $B(x)$ is non-empty and has an upper-bound. Define $b^x = \sup B(x)$.

Proof: $B(x)$ is clearly non-empty and bounded by $b^{\lfloor x \rfloor + 1}$.

E11) Prove that if $r \in \mathbb{Q}$, then

$$b^r = \sup B(r), \forall r \in \mathbb{Q}.$$

Proof: $b^r \in B(r)$ and since b^t is monotonically increasing, $b^r \geqslant \sup B(r)$, hence $b^r = \sup B(r)$.

E12) Prove that for any $x, y \in \mathbb{R}$, $b^{x+y} = b^x b^y$.

Proof: For any $b^t \in B(x)$, $b^s \in B(y)$, $t \leqslant x$ and $s \leqslant y$, so $t + s \leqslant x + y$ and $b^{t+s} \in B(x + y)$, hence $b^{x+y} \geqslant b^x b^y$. For any $b^t \in B(x+y)$, t can be written in the form $t = u+v$ where $b^u \in B(x)$, $b^v \in B(y)$, so $b^{x+y} \leqslant b^x b^y$.

E13*) Prove that when $b = e$, the function derived from E10) (denoted as e^x) is the same as $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Proof: From $\exp(1) = e$, $\exp(0) = 1$ and $\exp(x + y) = \exp(x) \cdot \exp(y)$ we know that for $n \in \mathbb{Z}$, $\exp(n) = e^n$. For $r = p/q \in \mathbb{Q}$,

$$(e^r)^q = e^p = \exp(p) = \exp(r)^q,$$

so by E5) $e^r = \exp(r)$. Since exp is continuous, for any $x \in \mathbb{R}$, $e^x = \exp(x)$.

2.6 PSF

Given $\alpha > 0$ and $x_1 > \sqrt{\alpha}$, we define inductively $\{x_n\}_{n\geqslant 1}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right), n \geqslant 1.$$

F1) Prove that $\{x_n\}$ is monotonically decreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$ (which is defined in E).

Proof: Note that

$$x_{n+1} - x_n = \frac{\alpha - x_n^2}{2x_n}.$$

Hence we can prove by induction that $x_n > \sqrt{\alpha}$ and $x_n > x_{n+1}$. x_n is decreasing and bounded, so $\lim_{n\to\infty} x_n = A$ exists, and $A = (A + \alpha/A)/2$. Therefore $\lim_{n\to\infty} x_n = A = \sqrt{\alpha}$.

F2) Let $\varepsilon_n = x_n - \sqrt{\alpha}$. Prove that $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$.

Proof:

$$\frac{\varepsilon_n^2}{2x_n} = \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = x_{n+1} - \sqrt{\alpha} = \varepsilon_{n+1}.$$

F3) Prove that if $\beta = 2\sqrt{\alpha}$, then $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

Proof: $\varepsilon_{n+1}/\beta < (\varepsilon_n/\beta)^2$, hence $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

F4) Let $\alpha = 3, x_1 = 2$. Verify that $\varepsilon_1/\beta < 0.1$, $\varepsilon_5 < 4 \cdot 10^{-16}$, $\varepsilon_6 < 4 \cdot 10^{-32}$.

Now we consider $\alpha > 1$ and $y_1 > \sqrt{\alpha}$, and define

$$y_{n+1} = \frac{\alpha + y_n}{1 + y_n} = y_n + \frac{\alpha - y_n^2}{1 + y_n}, n \geqslant 1$$

F6) Prove that $\{y_{2k-1}\}$ is monotonically decreasing.

Proof: Note that

$$y_{n+2} = \frac{\alpha + y_{n+1}}{1 + y_{n+1}} = \frac{\alpha + \frac{\alpha + y_n}{1+y_n}}{1 + \frac{\alpha + y_n}{1+y_n}} = \frac{2\alpha + (\alpha+1)y_n}{(\alpha+1) + 2y_n}$$

hence

$$y_{n+2} - y_n = \frac{2(\alpha - y_n^2)}{(\alpha+1)+2y_n},\ y_{n+2} - \sqrt{\alpha} = \frac{(\sqrt{\alpha}-1)^2}{(\alpha+1)+2y_n}(y_n - \sqrt{\alpha}).$$

Therefore $y_1 > \sqrt{\alpha}$ implies $\sqrt{\alpha} < y_{2n+1} < y_{2n-1}$.

F7) Prove that $\{y_{2k}\}$ is monotonically increasing.

Proof: $y_2 = (\alpha + y_1)/(1 + y_1) < \sqrt{\alpha}$, so same as F6), $y_{2k} > y_{2k-2}$ and $y_{2k} < \sqrt{\alpha}$.

F8) Prove that $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

Proof: $\{y_{2n-1}\}$ is decreasing and bounded by $\sqrt{\alpha}$, so $\lim_{n\to\infty} y_{2n-1} = A$ exists and $A = (2\alpha + (\alpha + 1)A)/((\alpha+1)+2A)$, so $A = \sqrt{\alpha}$. Likewise $\lim_{n\to\infty} y_{2n} = \sqrt{\alpha}$, hence $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

F9) Compare the rates of convergence between x_n and y_n.

Solution: Let $\delta_n = |y_n - \sqrt{\alpha}|$, then $\delta_n \sim c^n \delta_1$, hence x_n converges faster then y_n.

2.7 PSG: Banach-Mazur Game

Alice and Bob are playing a game: Alice selects a closed interval W_1 first, then Bob choose a subinterval L_1 of W_1, such that the length of L_1 is less than half of the length of W_1; they take turns choosing intervals W_n and L_n, such that $L_n \subset W_n \subset L_{n-1}$ and $|L_n| < |W_n|/2 < |L_{n-1}|/4$, obtaining

$$W_1 \supset L_1 \supset W_2 \supset L_2 \supset \cdots \supset W_n \supset L_n \supset \cdots$$

Alice and Bob find that

$$\bigcap_{n\geqslant 1} W_n = \bigcap_{n\geqslant 1} L_n = \{x\}$$

is a real number. If $x \in \mathbb{Q}$ then Alice wins, otherwise Bob wins. Who has a winning strategy?
Solution: Bob will win. We show that if $\mathbb{Q}$ is replaced with any set M that is of first category, Bob can still win.
M can be written as the union of a countable number of nowhere dense sets. Then in every move of Bob, he can choose L_n such that it does not intersect the nth such nowhere dense set. Hence the final number x is not in M.

2.8 Problem H

Consider the set $\mathcal{P} = \{\{p_n\}_{n\geqslant 1} : p_n \in \mathbb{Z}, p_1 \geqslant 2, p_{n+1} \geqslant p_n^2\}$.

H1) For any $p = \{p_n\}_{n\geqslant 1} \in \mathcal{P}$, define the sequence

$$a_n = \prod_{k=1}^{n} \left(1 + \frac{1}{p_k}\right).$$

Prove that $f(p) = \lim_{n\to\infty} a_n$ exists and $f(p) \in (1, 2]$.
Proof: Note that $p_n \geqslant p_1^{2^{n-1}}$, then

$$a_n \leqslant \prod_{k=1}^{n}\left(1+\frac{1}{p_1^{2^{k-1}}}\right) = \frac{1-p_1^{-2^n}}{1-p_1^{-1}} < \frac{1}{1-p_1^{-1}}.$$

So the sequence $\{a_n\}$ is monotonic and bounded, hence $f(p) = \lim_{n\to\infty} a_n$ exists. Since $a_n \in (1 + 1/p_1, \frac{1}{1-p_1^{-1}})$, we obtain $f(p) \in [1 + 1/p_1, \frac{1}{1-p_1^{-1}}] \subset (1, 2]$.

H2) Prove that $f : \mathcal{P} \to (1, 2]$ is a bijection.

Proof: For any $p = \{p_n\}, q = \{q_n\} \in \mathcal{P}$, if $p \neq q$, take the least k such that $p_k \neq q_k$ and suppose $q_k \geqslant p_k + 1$, then for any $n > k$,

$$a_n = \prod_{t=1}^{n}\left(1+\frac{1}{p_t}\right) \geqslant \prod_{t=1}^{k}\left(1+\frac{1}{p_t}\right)\cdot\left(1+\frac{1}{p_{k+1}}\right)$$
$$b_n = \prod_{t=1}^{n}\left(1+\frac{1}{q_t}\right) \leqslant \prod_{i=1}^{k-1}\left(1+\frac{1}{p_t}\right)\cdot\frac{1-q_k^{-2^{n-k}}}{1-q_k^{-1}}$$

Therefore

$$b_n \leqslant \prod_{t=1}^{k}\left(1+\frac{1}{p_t}\right) \leqslant (1+C)a_n$$

for all $n > k$ where $C = p_{k+1}^{-1} > 0$, hence $f(q) \leqslant (1 + C)f(p) < f(p)$, hence f is injective.
For any $x \in (1, 2]$, inductively define $p = \{p_n\} \in \mathcal{P}$ as follows: For any $n \geqslant 1$, Let t be the least integer such that $a_n \leqslant x$ and $t \geqslant p_{n-1}^2$ (clearly such t exists). If $a_n = x$, then let $p_n = t - 1$, $p_m = p_n^{2^{m-n}}$ for all $m > n$, then $f(p) = x$. Otherwise let $p_n = t$. Note that for any n such that $p_n > p_{n-1}^2$,

$$|x - a_n| \leqslant 2^{-2^n},$$

therefore $f(p) = x$, and f is surjective.

H3) Prove that $\mathcal{P}$ is uncountable.

Proof: By H2) and the fact that $(1, 2]$ is uncountable.

2.9 Problem I: Binary Expansion

Consider the set $\mathcal{S} = \{\{s_n\}_{n\geqslant 0} : s_n \in \{-1, 1\}\}$.

I1) For any $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, define the sequence

$$c_n = \sum_{k=0}^{n}\frac{s_0 s_1 \cdots s_k}{2^k}.$$

Prove that $h(s) = \lim_{n\to\infty} c_n$ exists and $h(s) \in [-2, 2]$.
Proof: $h(s)$ exists since c_n is clearly a Cauchy sequence, and $c_n \in [-2, 2]$ hence $h(s) \in [-2, 2]$.

I2) Prove that $h : \mathcal{S} \to [-2,2]$ is surjective. Determine whether is is injective.

Proof: Consider any $x \in [-2,2]$, we can choose s_n such that $|c_n - x| \leqslant 2^{-n}$. Hence there exists $s = \{s_n\} \in \mathcal{S}$ such that $h(s) = \lim_{n\to\infty} c_n = x$, so h is surjective.
Consider $s = \{1,-1,1,1,1,\cdots\} \in \mathcal{S}$ and $s' = \{-1,-1,1,1,\cdots\}$, then $h(s) = h(s') = 0$, hence h is not injective.

I3) For $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, prove that

$$2\sin\left(\frac{\pi}{4}c_n\right) = s_0\sqrt{2 + s_1\sqrt{2 + \cdots + s_n\sqrt{2}}}.$$

Proof: We prove by induction on n. The base $n = 0$ is trivial. If the statement holds for $n-1$, then let $s' = \{s_{n+1}\}_{n\geqslant 0} \in \mathcal{S}$, we have

$$2\sin\left(\frac{\pi}{4}c_n\right) = 2\sin s_0\left(\frac{\pi}{4} + \frac{1}{2}\cdot\frac{\pi}{4}c'_{n-1}\right) = s_0\sqrt{2 + \sin\left(\frac{\pi}{4}c'_{n-1}\right)}.$$

By the induction hypothesis, the statement also holds for n.

I4) Calculate the limit

$$\lim_{n\to\infty}\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}$$

Solution: Consider $s = \{s_n = 1\}_{n\geqslant 0} \in \mathcal{S}$, then $c_n = 2 - 2^n$ hence $\lim_{n\to\infty} 2\sin(\pi c_n/4) = 2$.

2.10 Problem J

Problem: $k \geqslant 2$ is a given integer. Define the sequence $\{a_n\}$ as follows:

$$a_0 > 0 \text{ already given}, a_{n+1} = a_n + a_n^{-1/k}, n \geqslant 0.$$

Calculate $\lim_{n\to\infty} a_n^{k+1}/n^k$.
Solution: It is easy to see that $a_n \to \infty$, hence

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n^{\frac{k+1}{k}}}{n} &= \lim_{n\to\infty} a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}} = \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1 + \frac{a_{n+1} - a_n}{a_n}\right)^{\frac{k+1}{k}} - 1\right) \\ &= \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1 + a_n^{-\frac{k+1}{k}}\right)^{\frac{k+1}{k}} - 1\right) = \frac{k+1}{k}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{a_n^{k+1}}{n^k} = \left(1 + \frac{1}{k}\right)^k.$$

3 Homework 3: Basel Problem

3.1 PSA

A1) Given $f : (a,x_0) \cup (x_0,b) \to \mathbb{R}$, then $\lim_{x\to x_0} f(x)$ exists iff for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in (x_0 - \delta, x_0 + \delta)$, $|f(x_1) - f(x_2)| < \varepsilon$.

Proof: $\Longleftarrow$ Let $x_n = x_0 + 1/n$, then $\{f(x_n)\}$ form a Cauchy sequence, hence $f(x_0) = \lim_{n\to\infty} f(x_n)$ exists. For any $\varepsilon > 0$, there exists $N, \delta > 0$ such that for any $x, y \in (x_0 - \delta, x_0 + \delta), |f(x) - f(y)| < \varepsilon$

and for any $n > N$, $|f(x_n) - f(x_0)| < \varepsilon$, hence let $\delta' = \min\{\delta, 1/N\}$, then for any $x \in (x_0 - \delta', x_0 + \delta')$, $|f(x) - f(x_0)| \leqslant |f(x) - f(x_N)| + |f(x_N) - f(x_0)| < 2\varepsilon$.
Hence $\lim_{x \to x_0} f(x) = f(x_0)$ exists.
$\Longrightarrow$ For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(x_0)| < \varepsilon$, hence for any $x, y \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(y)| < 2\varepsilon$.

A2) Suppose I is an interval (not a point), prove that the linear space $C(I)$ on $\mathbb{R}$ is of infinite dimension.

Proof: $C(I)$ contains the subspace of all polynomials, hence is of infinite dimension.

A3) Suppose $f : X \to Y$ and $g : Y \to Z$ are both continuous, prove that $g \circ f : X \to Z$ is also continuous.

Proof: For any open set $U \in Z$, $g^{-1}(U) \subset Y$ is an open set, and $f^{-1}(g^{-1}(U)) \subset X$ is an open set, hence $(g \circ f)^{-1}(U)$ is an open set in X and therefore $g \circ f$ is continuous on X.

A4) Suppose (X, d_X) and (Y, d_Y) are metric spaces, $f : X \to Y$ is continuous. If d'_X and d_X are equivalent metrics, and so are d'_Y and d_Y, then in the spaces (X, d'_X) and (Y, d'_Y), f is also continuous.

Proof: The topology generated by equivalent metrics are the same.

A5) The mapping $f : X \to \mathbb{R}^n$ can be written in the form

$$f : X \to \mathbb{R}^n,\ x \mapsto f(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

Prove that f is continuous iff f_i is continuous for every $i = 1, 2, \cdots, n$.
Proof: Since f is continuous iff $\forall x_n \to x, f(x_n) \to f(x)$, and $\{x_k = (x_k^{(1)}, \cdots, x_k^{(n)})\}_{k \geqslant 1}$ converges iff every $\{x_k^{(i)}\}_{k \geqslant 1}$ converges, f is continuous iff every f_i is continuous.

A6) Suppose (X, d_X) is a metric space, $(V, \|\cdot\|)$ is a normed linear space. $f : X \to V$ and $g : X \to V$ are continuous mappings. Prove that $f \pm g : X \to V$ is continuous. If $V = \mathbb{C}$ then $f \cdot g : X \to \mathbb{C}$ is continuous. If $V = \mathbb{C}$ and for any $x \in X$, $g(x) \neq 0$, then $f/g : X \to \mathbb{C}$ is continuous.

(Choose one statement to prove.)
Proof: Since for $\{x_n\}, \{y_n\} \subset \mathbb{C}$, $\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$ and if $y_n \neq 0$, then

$$\lim_{n\to\infty} x_n/y_n = \lim_{n\to\infty} x_n / \lim_{n\to\infty} y_n.$$

Hence $f \cdot g$, f/g are both continuous.
For $\{x_n\}, \{y_n\} \subset V$, if $A = \lim_{n\to\infty} x_n$ and $B = \lim_{n\to\infty} y_n$ then

$$\|x_n + y_n - A - B\| \leqslant \|x_n - A\| + \|y_n - B\| \to 0.$$

Hence $f \pm g$ is continuous.

A7) Find all discontinuities of the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 1/q, & \text{if } x = p/q \in \mathbb{Q}, \text{where } q \geqslant 1, (p, q) = 1. \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Solution: For any $x \in \mathbb{Q}$, $f(x) \neq 0$ but for any $\delta > 0$ there exists $y \in (x-\delta, x+\delta)$ such that $y \notin \mathbb{Q}$. Hence $|f(x) - f(y)| = f(x)$, so f is not continuous at x.
For any $x \notin \mathbb{Q}$, and any $\varepsilon > 0$, let $N = \lfloor 1/\varepsilon \rfloor + 1$ and $\delta = \inf_{n \leqslant N} \|xn\|/n$, then for any $y \in (x-\delta, y+\delta)$, if $y \notin \mathbb{Q}$ then $f(x) = f(y) = 0$, if $y = p/q \in \mathbb{Q}$ then $q > N > 1/\varepsilon$, hence $|f(x) - f(y)| = f(y) = 1/q < \varepsilon$. Therefore f is continuous at x iff $x \notin \mathbb{Q}$.

A8) Calculate

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1.$$

A9) Calculate

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{n \to \infty} (1 + 1/n)^n = e$ and $(1 + 1/x)^x$ is monotonic on $[100, \infty)$.

A10) Calculate

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{x \to \infty} (1 - 1/x)^x = \lim_{x \to \infty} (1 - 1/x)^{x-1} = e$.

3.2 PSB

B1) Calculate the following series:

1.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1.$$

2.
$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1} = \frac{1}{2}.$$

3.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{1}{4}.$$

4.
$$\sum_{n=1}^{\infty} \arctan \frac{1}{n^2 + n + 1} = \sum_{n=1}^{\infty} \arctan \frac{1}{n} - \arctan \frac{1}{n+1} = \frac{\pi}{4}.$$

5.
$$\sum_{n=0}^{\infty} \frac{(-1)^n + 2}{3^n} = \frac{1}{1 + 1/3} + \frac{2}{1 - 1/3} = \frac{3}{4} + 3 = \frac{15}{4}.$$

6.
$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}.$$

7.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \frac{1}{1+1/2} = \frac{2}{3}.$$

8.
$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = 3.$$

9.
$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1.$$

10.
$$\sum_{n=1}^{\infty} \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = 1 - \sqrt{2}.$$

11.
$$\sum_{n=1}^{\infty} \log\left(\frac{n(2n+1)}{(n+1)(2n-1)}\right) = \lim_{n\to\infty} \log\left(\frac{2n+1}{n+1}\right) = \log 2.$$

12.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+m)} = \frac{1}{m}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+m} = \frac{1}{m}\sum_{n=1}^{m} \frac{1}{n}.$$

B2) Determine whether the following series converge:

1.
$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \sqrt{n+1} - 1 = \infty.$$

2.
$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

converges.

3.
$$\sum_{n=2}^{\infty} (\sqrt[n]{n} - 1)^n$$

converges, since $\limsup_{n\to\infty} \sqrt[n]{(\sqrt[n]{n}-1)^n} = 0 < 1$.

4.
$$\sum_{n=1}^{\infty} \frac{1}{1+x^n}$$

converges if $|x| > 1$ and diverges if $|x| \leqslant 1$.

5.
$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

converges

6.
$$\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+1}\right)^n \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} < 1.$$

converges.
7.
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \geqslant \sum_{n=1}^{\infty} \frac{1}{2n}$$
diverges.
8.
$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\log \log n}} \leqslant C + \sum_{n=100}^{\infty} \frac{1}{n^2}$$
converges.
9.
$$\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n}$$
diverges, since
$$\lim_{n\to\infty} \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n} = \exp \lim_{n\to\infty} \left(\frac{\log n}{n} - n \log\left(1+\frac{1}{n^2}\right)\right) = 1.$$
10.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$$
converges (conditionally), since the partial sum of $(-1)^{n-1}$ is bounded and $\frac{\sqrt{n}}{n+1}$ monotonically tends to 0.
11.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[n]{n}}$$
diverges since $(-1)^{n-1}n^{-1/n}$ does not tend to 0.
12.
Let $H_n = 1 + 1/2 + \cdots + 1/n$.
$$\sum_{n=1}^{\infty} \frac{H_n \sin nx}{n}$$
converges since the partial sum of $\sin nx$ is bounded and $\frac{H_n}{n}$ monotonically tends to 0.

B3) Determine whether the following series converge (absolutely):

1.
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$
converges since the partial sum of $(-1)^n$ is bounded and $\frac{1}{n \log n}$ monotonically tends to 0, but only conditionally by C3).
2.
$$\sum_{n=2}^{\infty} \frac{\sin(n\pi/4)}{\log n}$$
converges since the partial sum of $\sin(n\pi/4)$ is bounded and $\frac{1}{\log n}$ monotonically tends to 0, but only conditionally since $\sum_{n=2}^{\infty} \frac{1}{\log(4n+2)}$ tends to infinity.
3.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{\sqrt[3]{n}}$$

converges since $\frac{n-1}{(n+1)\sqrt[3]{n}}$ monotonically tends to 0, but only conditionally since $\sum_{n=1}^{\infty} n^{-1/3}$ diverges.
4.
$a > 1$.

$$\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \frac{n^{10}}{a^n}$$

converges absolutely since there exists $C > 0$ such that for $n > C$, $n^{10}a^{-n} \leqslant a^{-n/2}$, and $\sum_{n=1}^{\infty} a^{-n/2}$ converges.

3.3 PSC

Suppose the integer $b \geqslant 2$, $f : [1, \infty) \to \mathbb{R}_{>0}$ is monotonically decreasing.

C1) Prove that

$$(b-1)b^{k-1}f(b^k) \leqslant \sum_{j=b^{k-1}}^{b^k-1} f(j) \leqslant (b-1)b^{k-1}f(b^{k-1}).$$

Proof: There are $(b-1)b^{k-1}$ integers in $[b^{k-1}, b^k - 1]$, and since f is monotonically decreasing, for any $j \in [b^{k-1}, b^k - 1]$, $f(j) \in [f(b^k), f(b^{k-1})]$.

C2) Prove that the series

$$\sum_{n=1}^{\infty} f(n) \text{ and } \sum_{n=1}^{\infty} b^n f(b^n)$$

converge or diverge simultaneously.
Proof: From C1),

$$\sum_{k=1}^{\infty} (b-1)b^{k-1}f(b^k) \leqslant \sum_{n=1}^{\infty} f(n) = \sum_{k=1}^{\infty} \sum_{j=b^{k-1}}^{b^k-1} f(j) \leqslant \sum_{k=1}^{\infty} (b-1)b^{k-1}f(b^{k-1}).$$

Therefore the two series converge or diverge simultaneously.

C3) Prove that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

Proof: Consider $f(x) = \frac{1}{x \log x}$ which is monotonically decreasing. Note that

$$\sum_{n=2}^{\infty} 2^n f(2^n) = \sum_{n=2}^{\infty} \frac{1}{n \log 2} = \infty.$$

From C2) we know that $\sum_{n=2}^{\infty} f(n)$ diverges.

C4) Prove that $\sum_{n=100}^{\infty} \frac{1}{n \log n \log \log n}$ diverges.

Proof: Consider $f(x) = \frac{1}{x \log x \log \log x}$ which is monotonically decreasing. From C3),

$$\sum_{n=100}^{\infty} 2^n f(2^n) = \sum_{n=100}^{\infty} \frac{1}{n \log 2 \cdot \log(n \log 2)}$$

diverges. Hence from C2) we know that $\sum_{n=100}^{\infty} f(n)$ diverges.

C5) Prove that $\sum_{n=1}^{\infty} n^{-s}$ converges iff $s>1$.

Proof: Consider $f(x)=x^{-s}$ which is monotonically decreasing. Note that

$$\sum_{n=1}^{\infty} 2^n f(2^n) = \sum_{n=1}^{\infty} 2^{-n(s-1)} = \frac{2^{1-s}}{1-2^{1-s}}.$$

C6) Suppose $s>1$, prove that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$ and $\sum_{n=10}^{\infty} \frac{1}{n\log n(\log\log n)^s}$ converges.

Proof: Same as C3) and C4).

3.4 PSD

For $\{a_n\}_{n\geqslant 1}\subset\mathbb{R}$,

- $\alpha\in\mathbb{R}$, if for any $\varepsilon>0$, there are infinitely many n such that $a_n\in(\alpha-\varepsilon,\alpha+\varepsilon)$, then we call α a limit point of $\{a_n\}_{n\geqslant 1}$.
- Likewise define limit points for $\alpha=\pm\infty$.

D1) Prove that $\alpha\in\mathbb{R}$ is a limit point of $\{a_n\}_{n\geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ which converges to α.

Proof: $\Longleftarrow$ is trivial. $\Longrightarrow$ Let $\varepsilon=1/k$ then there exists a_{n_k} such that $|a_{n_k}-\alpha|<\varepsilon$. Hence $\lim_{k\to\infty} a_{n_k}=\alpha$.

D2) Prove that $+\infty$ is a limit point of $\{a_n\}_{n\geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ such that $\lim_{k\to\infty} a_{n_k}=\infty$.

Proof: Same as D1).

D3) Let $E=\{\alpha\in\mathbb{R}\cup\{\pm\infty\}:\alpha$ is a limit point of $\{a_n\}\}$. Prove that $E\neq\emptyset$.

Proof: If $\{a_n\}$ is unbounded, then by D2) $E\cap\{\pm\infty\}\neq 0$. If $\{a_n\}$ is bounded, then by Bolzano-Weierstrass theorem, $E\neq\emptyset$.

D4) Prove that $E\subset\mathbb{R}$ iff $\{a_n\}$ is bounded.

Proof: Use D2)

D5) Suppose $\{a_n\}_{n\geqslant 1}$ is bounded. Prove that $\sup E=\limsup_{n\to\infty} a_n$, $\inf E=\liminf_{n\to\infty} a_n$.

Proof: Let $M=\limsup_{n\to\infty} a_n$, then for any $\varepsilon>0$, there exists n such that $M\leqslant\sup_{k\geqslant n} a_k<M+\varepsilon$, hence there exists $k\geqslant n$ such that $|a_k-M|<\varepsilon$, so $M\in E$.
For any $\alpha\in E$, there is a sub-sequence $\{a_{n_k}\}\to\alpha$, hence

$$\alpha=\lim_{k\to\infty} a_{n_k}\leqslant\lim_{k\to\infty}\sup_{m\geqslant n_k} a_{n_k}=\limsup_{n\to\infty} a_n=M.$$

Therefore $M=\sup E$. Substitute $a_n\to -a_n$ and we obtain $\inf E=\liminf_{n\to\infty} a_n$.

D6) Suppose $\{a_n\}_{n\geqslant 1}$ is bounded. Let $a^* = \limsup_{n\to\infty} a_n$. Prove that

i) $a^* \in E$, i.e. $\sup E \in E$.
Proof: See the proof of D5).
ii) For any $x > a^*$, there exists $N \in \mathbb{Z}_{\geqslant 1}$ such that for any $n > N$, $a_n < x$.
Proof: If there is an infinite sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ such that $a_{n_k} \geqslant x$, then $\{a_{n_k}\}$ has a limit point $a' > x > a^*$, contradicting $a^* = \sup E$.

D7) Construct an example of $\{a_n\}_{n\geqslant 1}$ such that $E \cap \mathbb{R} \neq \emptyset$ and $E \not\subset \mathbb{R}$.

Solution: Since $\mathbb{Q}$ is countable, let $\{a_n\}_{n\geqslant 1}$ iterate every element of $\mathbb{Q}$, then $E = \mathbb{R} \cup \{\pm\infty\}$ is an infinite set.

D8) Construct $\{a_n\}_{n\geqslant 1}$ such that E is an infinite set.

Solution: Same as D7).

3.5 PSE: Reciprocal Sum of Primes

Define the ζ-function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

We have proved the formula:

$$\zeta(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}.$$

Prove that the series

$$\sum_{p\in\mathcal{P}} p^{-s}$$

converges when $s > 1$, and diverges when $0 < s \leqslant 1$.
Proof: We know that for $|a_n| < 1$, $\prod_{n=1}^{\infty}(1-a_n)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges. Hence by $\zeta(s)^{-1} = \prod_{p\in\mathcal{P}}(1-p^{-s})$, we obtain $\sum_{p\in\mathcal{P}} p^{-s}$ converges iff $s > 1$.

3.6 PSF: Euler's "Proof" of the Basel Problem

For any $\theta \in \mathbb{R}, n \in \mathbb{Z}$, prove the identity

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta} = \prod_{k=1}^{n}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

Further prove that for any $x \in \mathbb{R}$,

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

Proof: (1) By induction there is a polynomial $P_n(x)$ such that $P_n(\sin\theta) = \sin(2n+1)\theta$ for any $\theta \in \mathbb{R}$ and $\deg P_n = 2n+1$. For any $k = 1, 2, \cdots, n$, and $\theta = \pm k\pi/(2n+1)$, $\sin((2n+1)\theta) = 0$, hence P_n has roots 0 and $\pm\sin(k\pi/(2n+1))$ for $k = 1, 2, \cdots, n$. Since $\deg P_n = 2n+1$,

$$P_n(x) = Cx\prod_{k=1}^{n}\left(1 - \frac{x^2}{\sin^2(k\pi/(2n+1))}\right)$$

for some $C \in \mathbb{R}$. Let $x = \sin\theta$ and consider the derivatives on both sides when $\theta = 0$, then we obtain $C = 2n + 1$, therefore

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta} = \prod_{k=1}^{n}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

(2) Let $m = 2n + 1$. From (1) we know that for any $x \in \mathbb{C}$ and $k < n$, $\sin x = U_k^{(n)} \cdot V_k^{(n)}$, where

$$U_k^{(n)} = m \sin\frac{x}{m}\prod_{j=1}^{k}\left(1 - \frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right),$$
$$V_k^{(n)} = \prod_{j=k+1}^{n}\left(1 - \frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right).$$

Clearly, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} U_k^{(n)} = U_k = x\prod_{j=1}^{k}\left(1 - \frac{x^2}{j^2\pi^2}\right).$$

and for any $x \in \mathbb{C}$ and $j \in \mathbb{N}$,

$$\left|\frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right| \leqslant \frac{x^2}{4j^2} \cdot K(|x|/m)^2,$$

where $K(x) = \sum_{n=0}^{\infty} |x|^n/(2n+1)!$ is monotonic on $[0, \infty)$ and $K(0) = 1$.
Note that for $\alpha_i \in \mathbb{C}$,

$$\left|1 - \prod_{j=1}^{n}(1 - \alpha_n)\right| \leqslant \sum_{j=1}^{n}\left(\sum_{k=1}^{n}|\alpha_k|\right)^j.$$

Hence for any $x \in \mathbb{C}$ and $\varepsilon > 0$, there exists N such that for any $k \geqslant N$, and any $n > k$, $|V_k^{(n)} - 1| < \varepsilon$, since

$$|V_k^{(n)} - 1| \leqslant \sum_{j=1}^{\infty}\left(\sum_{l=k+1}^{\infty}\frac{x^2}{4l^2}K(|x|/m)^2\right)^j \leqslant \sum_{j=1}^{\infty}\left(K(|x|/(2k+1))^2 \cdot \frac{x^2}{k}\right)^j \to 0.$$

i.e. for any $x \in \mathbb{C}$

$$\lim_{k\to\infty}\sup_{n>k}|V_k^{(n)} - 1| = 0.$$

And likewise we know that there is a constant M such that for any $n > k$, $|x| < k$, $|U_k^{(n)}| \leqslant M$.
Therefore for any $x \in \mathbb{C}$,

$$\sin x = x\lim_{n\to\infty}\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2\pi^2}\right) = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Note:
From the formula above, we can formally deduce that

$$\sin(\pi x) = \pi x(1 - \zeta(2)x^2 + \zeta(4)x^4 + \cdots).$$

Compare it to $\sin z = x - x^3/6 + \cdots$, and we get $\zeta(2) = \pi^2/6$.

4 Homework 4: Topology

4.1 PSA: Topology on Metric Spaces

A1) Suppose (X, d_x) and (Y, d_Y) are metric spaces, $f : X \to Y$ is a mapping. Prove that the two following definitions of continuity is equivalent:

- Suppose $x_0 \in X$, if for any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$, then we say f is continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
- Suppose $x_0 \in X$, $y_0 = f(x_0) \in Y$. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $d_X(x, x_0) < \delta$, $x \in X$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, we call f continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
  Proof: 1=>2: If there exists $\varepsilon > 0$ such that for any $n \geqslant 1$, there exists x_n such that $d_X(x_0, x_n) < 1/n$ but $d_Y(f(x_n), f(x_0)) > \varepsilon$, then $\lim_{n\to\infty} x_n = x_0$ but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$, a contradiction.
  2=>1: For any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, and any $\varepsilon > 0$, take the corresponding δ and N such that $n > N \implies d(x_n, x_0) < \delta$. Then for any $n > N$, $d(x_n, x_0) < \delta$ so $d(f(x_n), f(x_0)) < \varepsilon$, hence $\lim_{n\to\infty} f(x_n) = f(x_0)$.

A2) (X, d) is a metric space. For any $x \in X$, $r > 0$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. Proved that for any $x \in X$, $r > 0$, if $x' \in B(x, r)$, then there exists $r' > 0$ such that $B(x', r') \subset B(x, r)$.

If $U = \bigcup_{\alpha\in\mathcal{A}} B(x_\alpha, r_\alpha)$, then we call U an open set. Prove that $U \subset X$ is open iff for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$.
Proof: If $x' \in B(x, r)$, let $r' = r - d(x, x')$, then for any $y \in B(x', r')$, $d(x, y) \leqslant d(x, x') + d(x', y) < d(x, x') + r' = r$, hence $y \in B(x, r)$ so $B(x', r') \subset B(x, r)$.
If for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$, then $U = \bigcup_{x\in U} B(x, \delta_x)$ is open.
If U is open then for any $x \in U$, suppose $x \in B(x_\alpha, r_\alpha)$ for some $\alpha \in \mathcal{A}$, then there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset B(x_\alpha, r_\alpha) \subset U$.

A3) Let $\mathcal{T}$ denote all open sets on (X, d). Prove that $\mathcal{T}$ is a topology.

Proof: 1. $\emptyset \in \mathcal{T}$, $X = \bigcup_{x\in X} B(x, 1) \in \mathcal{T}$. 2. If $\{U_\alpha : \alpha \in J\} \subset \mathcal{T}$, where $U_\alpha = \bigcup_{x\in\mathcal{A}_\alpha} B(x, r_{\alpha,x})$ then let $\mathcal{A} = \bigcup_{\alpha\in J} \mathcal{A}_\alpha$,

$$\bigcup_{\alpha\in J} U_\alpha = \bigcup_{x\in\mathcal{A}} B(x, \sup_{\alpha, x\in\mathcal{A}_\alpha} r_{\alpha,x}) \in \mathcal{T}.$$

1. If $U_1, \cdots, U_n \in \mathcal{T}$, where $U_k = \bigcup_{x\in\mathcal{A}_k} B(x, r_{k,x})$, then let $\mathcal{A} = \bigcup_{k=1}^n \mathcal{A}_k$

$$\bigcap_{k=1}^n U_k = \bigcup_{x\in\mathcal{A}} B(x, \min_{x\in\mathcal{A}_{\|}} r_{k,x}) \in \mathcal{T}.$$

Therefore $\mathcal{T}$ is a topology on X.

A4) (X, d) is a metric space. If $F \subset X$ and F^C is open, then we call F a closed set. Prove that F is closed iff for any sequence $\{x_n\}_{n\geqslant 1} \in F$, if $\lim_{n\to\infty} x_n = x$ then $x \in F$.

Proof: Suppose F is closed, if a sequence $\{x_n\}_{n\geqslant 1}$ satisfy $\lim_{n\to\infty} x_n = x$ and $x \in F^C$, then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset F^C$. However $B(x, \varepsilon) \cap \{x_n\} \neq \emptyset$, leading to contradiction.
If for any sequence $\{x_n\}_{n\geqslant 1}$ such that $\lim_{n\to\infty} x_n = x$, there is $x \in F$, then for any $x \in F^C$, if for any $\varepsilon > 0$ $B(x, \varepsilon) \not\subset F^C$, then for any $n \geqslant 1$, take $x_n \in B(x, \varepsilon) \cap F$. The sequence $\{x_n\}$ has the limit $\lim_{n\to\infty} x_n = x$ but $x \in F^C$, a contradiction. Hence F is closed.

A5) Prove that

1. $\emptyset$ and X are closed sets.
2. Any intersection of closed sets are still closed.
3. Finite unions of closed sets are still closed.
   Proof: Use A3) and de Morgan's theorem.

A6) Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$, then the following statements are equivalent:

1. f is continuous.
2. For any $U \subset Y$ open, $f^{-1}(U)$ is an open set in X.
3. For any $F \subset Y$ closed, $f^{-1}(F)$ is a closed set in X.
   Proof: 1=>2: If f is continuous, then for any $U \subset Y$ open, consider any point $x \in f^{=1}(U)$. Let $y = f(x) \in U$, then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subset U$. Since f is continuous, there exists $\delta > 0$ such that for any $x' \in B(x, \delta)$, $f(x') \in B(y, \varepsilon) \subset U$, hence $B(x, \delta) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is an open set in X.
   2=>1: For any $x \in X$ and $\varepsilon > 0$, consider the open set $U = B(y, \varepsilon)$, where $y = f(x)$. Since $x \in f^{-1}(U)$ and $f^{-1}(U)$ is an open set, there exists $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(U)$, therefore f is continuous.
   2<=>3: Note that $f^{-1}(Y \backslash A) = X \backslash f^{-1}(A)$.

A7) Let A' be the set of limit points of A. Prove that $\bar{A} = A' \cup A$.

Proof: For any closed set $F \supset A$, by A4) we know $A' \subset F$, hence $A' \cup A \subset \bar{A}$. Consider a sequence $\{x_n\}_{n \geqslant 1} \subset A' \cup A$ such that $\lim_{n \to \infty} x_n = x$ exists, for any $n \geqslant 1$ we can find a $y_n \in A$ such that $d(x_n, y_n) \leqslant 2^{-n}$, hence $\lim_{n \to \infty} y_n = \lim_{n \to \infty} x_n = x$ so $x \in A' \cup A$. Therefore $A' \cup A$ is closed, and hence $\bar{A} = A' \cup A$.

A8) Suppose (Y, d_Y) and (Z, d_Z) are metric spaces, define the metric on $Y \times Z$:

$$d_{Y \times Z} : (Y \times Z)^2 \to \mathbb{R}_{\geqslant 0},\ ((y_1, z_1), (y_2, z_2)) \to \sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2}.$$

Prove that this defines a metric and the projection mappings are continuous:

$$\pi_Y : Y \times Z \to Y,\ (y, z) \mapsto y;\ \pi_Z : Y \times Z \to Z,\ (y, z) \mapsto z.$$

Given a mapping $F : X \to Y \times Z$, then F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.
Proof: $d((y_1, z_1), (y_2, z_2)) = 0 \iff (y_1, z_1) = (y_2, z_2)$, $d((y_1, z_1), (y_2, z_2)) = d((y_2, z_2), (y_1, z_1))$, and $d((y_1, z_1), (y_2, z_2)) \leqslant d((y_1, z_1), (y_3, z_3)) + d((y_3, z_3), (y_2, z_2))$ (since $\sqrt{(x+y)^2 + (u+v)^2} \leqslant \sqrt{x^2 + u^2} + \sqrt{y^2 + v^2}$), hence $d_{Y \times Z}$ is a metric.
Note that $d((y_1, z_1), (y_2, z_2) \geqslant d(y_1, y_2)$, hence π_Y and π_Z are continuous.
$d((y_1, z_1), (y_2, z_2)) \leqslant d(y_1, y_2) + d(z_1, z_2)$, hence F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.

A9) Prove that the operators + and $\cdot$ on real numbers are continuous.

Proof: For any $(x, y), (u, v) \in \mathbb{R}^2$,

$$|(x + y) - (u + v)| \leqslant |x - u| + |y - v| \leqslant 2|(x, y) - (u, v)|.$$

Hence $+$ is uniformly continuous.

$$|x \cdot y - u \cdot v| \leqslant |x| \cdot |y - v| + |v| \cdot |x - u|.$$

Therefore $\cdot$ is continuous.

A10) Prove that the operators $+$ and $\cdot$ on $\mathbf{M}_n(\mathbb{R})$ are continuous.

Proof: The proof of A9) only uses the properties of norms, and the fact that $\|A \cdot B\| \leqslant \|A\| \cdot \|B\|$. This also holds for the norm $\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|$ on $\mathbf{M}_n(\mathbb{R})$, therefore $+$ and $\cdot$ are continuous on $\mathbf{M}_n(\mathbb{R})$.

A11) Prove that $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

Proof: The mapping $\det : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous, since view as $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ it is a multi-linear mapping. The set $\mathbf{GL}_n(\mathbb{R}) = \det^{-1}(\{x \in \mathbb{R} : x \neq 0\})$, where $\{x \in \mathbb{R} : x \neq 0\}$ is an open set on $\mathbb{R}$, therefore $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

A12) Prove that $\mathrm{Inv} : \mathbf{GL}_n(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{R}), A \mapsto A^{-1}$ is continuous.

Proof: Note that for any $A, B \in \mathbf{GL}_n(\mathbb{R})$,

$$\|A^{-1} - B^{-1}\| \leqslant \frac{\|A - B\|}{\|A\| \cdot \|B\|}.$$

Hence Inv is continuous.

4.2 PSB

Prove the following equalities:

B1) $\lambda > 0$, $\lim_{x \to \infty} \frac{x^n}{e^{\lambda x}} = 0$.

Proof: By definition, for $x > 0$, $e^{\lambda x} \geqslant (\lambda x)^{n+1}/(n+1)!$. Hence for any $\varepsilon > 0$, let $M = \frac{(n+1)!}{\lambda^{n+1}\varepsilon}$, then for any $x > M$,

$$\left| \frac{x^n}{e^{\lambda x}} \right| \leqslant \frac{(n+1)!}{\lambda^{n+1} x} < \varepsilon.$$

Therefore

$$\lim_{x \to \infty} \frac{x^n}{e^{\lambda x}} = 0.$$

B2) $\alpha > 0$, then

$$\lim_{x \to \infty} x^\alpha \log\left(1 + \frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

Proof: If $0 < \alpha < 1$, then for any $\varepsilon > 0$, there exists $\delta = \varepsilon^{1/(\alpha - 1)}$ such that for any $x > \delta$,

$$\left| x^\alpha \log\left(1 + \frac{1}{x}\right) \right| \leqslant x^{\alpha - 1} < \varepsilon.$$

If $\alpha > 1$, then for any $\varepsilon > 0$, there exists $\delta = (2\varepsilon)^{1/\alpha-1}$ such that for any $x > \delta$,

$$\left| x^\alpha \log\left(1+\frac{1}{x}\right)\right| \geqslant \frac{x^\alpha}{x+1} \geqslant \frac{1}{2}x^{\alpha-1} > \varepsilon.$$

If $\alpha = 1$, then for any $\varepsilon > 0$, there exists $\delta = 1/\varepsilon$ such that for any $x > \delta$,

$$1-\varepsilon \leqslant \frac{x}{x+1} \leqslant x\log\left(1+\frac{1}{x}\right) \leqslant 1.$$

Therefore

$$\lim_{x\to\infty} x^\alpha \log\left(1+\frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

B3) $\lim\limits_{x\to 0^+} x^{-n}e^{-1/x^2} = 0$**.**

Proof: If $x < 1$, then $e^{-1/x^2} \leqslant e^{-1/x} \leqslant (n+1)!x^{n+1}$, hence for any $\varepsilon > 0$, let $\delta = \varepsilon/(n+1)!$, then for any $x \in (0,\delta)$, $x^{-n}e^{-1/x^2} \leqslant (n+1)!x \leqslant \varepsilon$. Therefore

$$\lim_{x\to 0^+} x^{-n}e^{-1/x^2} = 0.$$

B4) We know that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$**. Calculate**

$$\lim_{x\to 0}\frac{\cos x - 1}{x}, \text{and } \lim_{x\to 0}\frac{\cos x-1}{x^2/2}.$$

Solution: For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|x| < \delta$, $\sin x \in ((1-\varepsilon)x, (1+\varepsilon)x)$. Hence

$$\left|\frac{\cos x-1}{x}\right| \leqslant \left|\frac{\sqrt{1-\sin^2 x}-1}{x}\right| \leqslant \left|\frac{\sin^2 x}{x(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (1+\varepsilon)^2 x \leqslant \delta(1+\varepsilon)^2.$$

Therefore

$$\lim_{x\to 0}\frac{\cos x-1}{x} = 0.$$

Likewise

$$\left|\frac{\cos x-1}{x^2/2}+1\right| \leqslant \left|\frac{\sin^2 x - x^2(1+\sqrt{1-\sin^2 x})/2}{x^2/2\cdot(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (2\varepsilon + \sqrt{1-\sin^2 x} - 1).$$

Therefore

$$\lim_{x\to 0}\frac{\cos x-1}{x^2/2} = -1.$$

4.3 PSC: Root of Function:

C1) Prove that $x^3+2x-1=0$ **has exactly one root which lies in** $(0,1)$**.**

Proof: Let $f(x) = x^3+2x-1$, then $f(0) = -1$ and $f(1) = 2$, so $f(0) < 0 < f(1)$. Since f is continuous and monotonically increasing on $(0,1)$, there is exactly one root in $(0,1)$.

C2) Suppose $0 \leqslant \lambda < 1$, $b > 0$, determine whether the equation $x - \lambda \sin x = b$ has a solution.

Solution:

C3) Prove that $\sin x = 1/x$ has infinitely many roots.

Proof: For any $n \in \mathbb{N}$, let $x_n = (2n + 1/2)\pi$, $y_n = (2n + 3/2)\pi$, and $f(x) = \sin x - 1/x$, then $f(x_n) = 1 - 1/x_n > 0$, $f(y_n) = -1 - 1/y_n < 0$, therefore f has a root in (x_n, y_n), and hence f has infinitely many roots.

C4) Assume $f \in C([0, 2])$ and $f(0) = f(2)$. Prove that $f(x) - f(x + 1) = 0$ has a root in $[0, 1]$.

Proof: Let $g(x) = f(x) - f(x + 1)$, then $g(0) = f(0) - f(1) = -g(1)$ and $g \in C([0, 1])$. Therefore g has a root in $[0, 1]$.

C5) Prove that $x^3 + 3 = e^x$ has a solution in $\mathbb{R}$.

Proof: Let $f(x) = e^x - x^3 - 3$, then $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, therefore f has a root in $\mathbb{R}$.

C6) Suppose $f : [0, 2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$ then there exists $x \in [1, 2]$ such that $f(x) = f(x - 1)$.

Exactly the same as C4)?

C7) $f : \mathbb{R} \to \mathbb{R}$, Prove that if for any $c \in \mathbb{R}$, $|f^{-1}(c)| = 2$, then f is not continuous.

Proof: If f is continuous on $\mathbb{R}$, suppose $f^{-1}(0) = \{a < b\}$, then $f|_{[a,b]}$ is bounded. Suppose $f\left(\frac{a+b}{2}\right) > 0$, then for any $t \in (a, b)$, $f(t) > 0$ (otherwise $|f^{-1}(0)| > 2$). Consider an arbitrary $M > y = \sup_{x\in[a,b]} f(x)$, and take $t \in f^{-1}(M)$. Assume $t < a$, then $f(t) = M > y/2 > f(a) = 0$, hence there exists $s \in (t, a)$ such that $f(s) = y/2$. However there are at least two elements of $f^{-1}(y/2)$ in (a, b), leading to contradiction.

C8) Suppose the continuous function $f : [a, b] \to \mathbb{R}$ is injective. If $f(a) < f(b)$, prove that f is monotonically increasing.

Proof: Otherwise suppose $f(u) > f(v)$ for some $u < v$. Note that for any $c \in (a, b)$, $f(a) < f(c) < f(b)$, otherwise $f(c) < f(a) \implies \exists d \in (c, b), f(d) = f(a)$, or $f(c) > f(b) \implies \exists d \in (a, c), f(d) = f(b)$. Hence $a < u < v < b$. Likewise consider $u < v < b$ we get $f(u) > f(v) > f(b)$, and by $a < u < v$ we get $f(a) > f(u) > f(v)$, therefore $f(a) > f(b)$, a contradiction.

4.4 PSD: Calculation of Limits

n, m are positive integers.

(1)

$$\lim_{x\to\infty} \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m} = \begin{cases} 0, & m > n, \\ \infty, & m < n, a_0 > 0, \\ -\infty, & m < n, a_0 < 0, \\ \frac{a_0}{b_0}, & m = n. \end{cases}$$

(2) $a > 1, b > 0$

$$\lim_{x\to\infty} \frac{x^b}{a^x} = 0.$$

(3) $a > 0$

$$\lim_{x\to\infty} \frac{\log x}{x^a} = 0.$$

(4) $a > 0$

$$\lim_{x\to 0^+} x^a \log x = 0.$$

(5)

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^{x} = e^3.$$

(6)

$$\lim_{x\to\infty} (x - \sqrt{x^2-a}) = \lim_{x\to\infty} \frac{a}{x+\sqrt{x^2-a}} = 0.$$

(7)

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x-1} = \lim_{x\to\infty} \frac{2}{\sqrt{x+1} - \sqrt{x-1}} = 0.$$

(8)

$$\lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x)-1}{x} = 1+2+3 = 6.$$

(9)

$$\lim_{x\to 1} \frac{x+x^2+\cdots+x^n-n}{x-1} = \frac{n(n+1)}{2}.$$

(10)

$$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \frac{49}{24}.$$

(11)

$$\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-n}{2}.$$

Proof: Note that

$$\begin{aligned}\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)&=\lim_{x\to 1}\frac{m(1+x+\cdots+x^{n-1})-n(1+x+\cdots+x^{m-1})}{(1+x+\cdots+x^{m-1})(1+x+\cdots+x^{n-1})(1-x)}\\&=\frac{1}{mn}\cdot\lim_{x\to 1}\frac{m(x-1+\cdots+x^{n-1}-1)-n(x-1+\cdots+x^{m-1}-1)}{1-x}\\&=\frac{1}{mn}\cdot(-m(1+2+\cdots+(n-1))+n(1+2+\cdots+(m-1)))\\&=\frac{m-n}{2}.\end{aligned}$$

(12)

$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a.$$

(diverges if $a=0$).

(13)

$$\lim_{x\to 1}\frac{x^a-1}{x^b-1}=\frac{a}{b}.$$

(14)

$$\lim_{x\to\infty}(\log x)^{1/x}=\lim_{x\to\infty}e^{(\log\log x)/x}=1.$$

(15) $a,b>0$

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}=\sqrt{ab}.$$

(16)

$$\lim_{x\to\infty}\sqrt[k]{(x+a_1)(x+a_2)\cdots(x+a_k)}-x$$

Proof: Let $y=(x+a_1)(x+a_2)\cdots(x+a_k)$ and $s=a_1+\cdots+a_k$, then

$$\frac{sx^{k-1}}{ky^{(k-1)/k}}\leqslant\sqrt[k]{y}-x=\frac{y-x^k}{y^{(k-1)/k}+\cdots+x^{k-1}}\leqslant\frac{sx^{k-1}+\prod_{i=1}^k(1+a_i)x^{k-2}}{kx^{k-1}}.$$

Therefore

$$\lim_{x\to\infty}\sqrt[k]{y}-x=s=\sum_{i=1}^k a_i.$$

(17)

$$\lim_{x\to 0}\frac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}=2n.$$

(18)

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1.$$

(19)

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x=e.$$

(20) $\alpha>0$**,**

$$\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha>\frac{1}{2},\\ 1, & \alpha=\frac{1}{2},\\ \infty, & \alpha<\frac{1}{2}.\end{cases}$$

(21) $\alpha>0$**,**

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

Proof: Note that for $x\in(0,1)$,

$$x^{1/8-\alpha}\leqslant\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}\leqslant 2x^{1/8-\alpha}.$$

And for any $\varepsilon>0$ there exists $\delta=(1+\varepsilon)\varepsilon$ such that for any $x<\delta$, $\sqrt{x+\sqrt{x+\sqrt{x}}}<\varepsilon x^{1/8}$. Therefore

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

4.5 Problem E

Prove that for any $A\subset\mathbb{R}$ that is countable, there exists a monotonic function $f:\mathbb{R}\to\mathbb{R}$, such that the set of discontinuities of f is exactly A.
Proof: Let $A=\{x_1,x_2,\cdots\}$ and $f(x)=\sup\{1-2^n : x_n<x\}$, (define $\sup\emptyset=0$) then f is monotonically increasing and the set of discontinuities is exactly A.

4.6 Problem F

$f : [0,1] \to [0,1]$ is monotonic. Prove that f has a fixed point.
Proof: Otherwise suppose that f has no fixed points. Let $S = \{t \in [0,1] : f(t) > t\}$ and $x = \sup S$. Note that $0 \in S$ so S is non-empty. If $x \in S$, then $f(x) > x$ so $f(f(x)) > f(x)$ (f is monotonic) then $x < f(x) \in S$ which leads to contradiction. If $x \notin S$, then $f(x) < x$. Take $y \in (f(x), x) \cap S$, (y exists since $x = \sup S$) then $f(x) > f(y) > y > f(x)$, a contradiction.

4.7 Problem G

Consider all self-homeomorphisms of $[0,1]$, i.e.

$$\mathrm{Homeo}([0,1]) = \{f : [0,1] \to [0,1] : f \text{ is a continuous bijective}\}$$

We know that for any $f \in \mathrm{Homeo}([0,1])$, $f^{-1} \in \mathrm{Homeo}([0,1])$. Suppose $f, g \in \mathrm{Homeo}([0,1])$ and the only fixed points of f, g are $0, 1$. Prove that there exists $h \in \mathrm{Homeo}([0,1])$, such that

$$h \circ f \circ h^{-1} = g.$$

Proof: Take $x_0 = 1/2$, and let $I_n = [f^n(x_0), f^{n+1}(x_0)]$, $J_n = [g^n(x_0), g^{n+1}(x_0)]$. Note that $(0,1) = \bigcup_{n\in\mathbb{Z}} I_n = \bigcup_{n\in\mathbb{Z}} J_n$. Define $h_0 : I_0 \to J_0$, $x \mapsto kx + b$ such that the line h_0 passes (x_0, x_0) and $(f(x_0), g(x_0))$, i.e. $x \mapsto \frac{g(x_0)-x_0}{f(x_0)-x_0}(x - x_0) + x_0$. Define $h_n : I_n \to J_n$, $x \mapsto g^n \circ f^{-n}(x)$, and $h : [0,1] \to [0,1]$ such that

$$h(x) = \begin{cases} x, & x \in \{0,1\}, \\ h_n(x), & x \in I_n. \end{cases}$$

Then for any $x \in I_n$, $f(x) \in I_{n+1}$ hence $h(f(x)) = g^{n+1} \circ f^{-n}(x) = g(h(x))$. Since h maps I_n to J_n bijectively, h is a bijection on $[0,1]$. For any $x \in I_n \cap I_{n+1}$ the value of h does not depend on which interval we choose, and h is continuous on any interval I_n, therefore h is a continuous bijection.

5 Homework 5: Infinity of Prime

5.1 PSA

A1) Prove that e^x is uniformly continuous on $(-\infty, 0]$ but not on $\mathbb{R}$.

Proof: For $y < x \leqslant 0$ and $|x - y| < \varepsilon$,

$$e^x - e^y = e^y(e^{y-x} - 1) \leqslant e^{\varepsilon} - 1.$$

Hence e^x is uniformly continuous on $(-\infty, 0]$. But for any $\delta > 0$, there exists y and $x = y + \delta$ such that

$$e^x - e^y = e^y \cdot (e^{\delta} - 1) > 1.$$

Therefore e^x is not uniformly continuous on $\mathbb{R}$.

A2) Prove that the function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$, $(x, \alpha) \mapsto x^{\alpha}$ is continuous on $\mathbb{R}_{>0} \times \mathbb{R}$.

Proof: For $(x, \alpha), (y, \beta)$,

$$|x^{\alpha} - y^{\beta}| \leqslant |x^{\alpha} - y^{\alpha}| + |y^{\alpha} - y^{\beta}|.$$

Since x^{α} and a^x are both continuous (as functions of x), so is $x^{\alpha} : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$.

A3) Prove that for any $x, y > 0$ and α, β, $(xy)^\alpha = x^\alpha y^\alpha$, $(x^\alpha)^\beta = x^{\alpha\beta}$, $a^{\log_a x} = x$. If $x > 0, y > 0$, then $a^{x+y} = a^x a^y$, $\log_a(x \cdot y) = \log_a x + \log_a y$.

Proof: See PSE of HW2.

A4) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$ defined on $[0,1]$, where $f_n(x) = x^n$. Prove that for any $a < 1$, $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0,a]$, but $\{f_n(x)\}_{n\geqslant 1}$ does not converge uniformly on $[0,1)$.

Proof: For any $a < 1$, and any $\varepsilon > 0$, let $N = \log_a x$, then for any $n > N$, $f_n(x) < \varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0,a]$. Let $\varepsilon = 1/2$, then for any $N \in \mathbb{N}$, there exists $1 > x > 2^{-1/N}$ such that $f_N(x) > \varepsilon$. Hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent on $[0,1)$.

A5) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$. Prove that $\{f_n(x)\}_{n\geqslant 1}$ converges point-wise to 0 on $\mathbb{R}$, but does not converge uniformly.

Proof: For any $x \in \mathbb{R}$, and any $\varepsilon > 0$, there exists $N = 1/(x\varepsilon)$ such that for any $n \geqslant N$,

$$\left|\frac{nx}{1+n^2x^2}\right| \leqslant \frac{1}{|nx|} < \varepsilon.$$

Hence $f_n(x)$ converges to 0 for any $x \in \mathbb{R}$.
Let $\varepsilon = 1/2$, then for any $n \in \mathbb{N}$, there exists $x = 1/n$ such that $f_n(x) = \varepsilon$, so f is not uniformly continuous on $\mathbb{R}$.

A6) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where

$$f_n(x) = \begin{cases} \frac{nx^2}{1+nx}, & x > 0; \\ \frac{nx^3}{1+nx^2}, & x \leqslant 0. \end{cases}$$

Determine the convergence of $\{f_n(x)\}_{n\geqslant 1}$ on $\mathbb{R}$ (both point-wise and uniformly).
Proof: For any $\varepsilon > 0$, let $N = \max\{1/\varepsilon, 1/4\varepsilon^2\}$, then for any $x > 0$ and $n > N$,

$$|f_n(x) - x| = \left|\frac{x}{1+nx}\right| < \frac{1}{n} < \varepsilon.$$

For any $x < 0$,

$$|f_n(x) - x| = \left|\frac{x}{1+nx^2}\right| \leqslant \frac{1}{2\sqrt{n}} < \varepsilon.$$

Hence $\{f_n\}_{n\geqslant 1}$ converges uniformly to x.

A7) Given $\varphi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ such that $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = 0$, φ is continuous and not identically zero. Prove that the sequences $\{f_n(x)\}_{n\geqslant 1}$ and $\{g_n(x)\}_{n\geqslant 1}$ converge point-wise to 0, but uniformly, where $f_n(x) = \varphi(nx)$, $g_n(x) = \varphi(x/n)$.

Proof: Point-wise convergence is trivial. Let $\varepsilon = |\varphi(1)| > 0$, then for any n there exists $x = 1/n > 0$ such that $|f_n(x)| = \varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent. Likewise $\{g_n(x)\}_{n\geqslant 1}$ is not uniformly continuous.

A8) $f \in C([a,b])$. For $n \geqslant 1$, let $a_k = a + (k-1)(b-a)/n$. Define

$$S_n = \sum_{k=1}^{n} \frac{b-a}{n} f(a_k).$$

Prove that $\{S_n\}_{n\geqslant 1}$ converges, and denote this limit by $\int_a^b f$. Further show that the mapping

$$\int_a^b : C([a,b]) \to \mathbb{R},\ f \mapsto \int_a^b f$$

is linear and continuous with metric d_∞ on $C([a,b])$.
Proof: For any $n, m \in \mathbb{N}$, note that $|S_n - S_m| \leqslant |S_n - S_{nm}| + |S_{nm} - S_m|$, and

$$|S_n - S_{nm}| \leqslant \sum_{k=1}^{n} \frac{b-a}{n} \left| f(a_k^{(n)}) - \frac{1}{m}\sum_{j=1}^{m} f(a_{n(k-1)+j}^{(nm)}) \right| \leqslant (b-a) \sup_{|x-y|<1/n} |f(x) - f(y)|.$$

Since f is uniformly continuous on $[a,b]$, the sequence $\{S_n\}_{n\geqslant 1}$ is Cauchy.
Obviously $\int_a^b \cdot$ is linear, and for $f, g \in C([a,b])$,

$$\left| \int_a^b f - \int_a^b g \right| = \lim_{n\to\infty} |S_n(f) - S_n(g)| \leqslant (b-a)\|f-g\|_\infty.$$

Hence $\int_a^b \cdot$ is continuous on $C([a,b])$ with metric d_∞.

A9) For any $f : [a,\infty) \to \mathbb{R}$, suppose f is bounded on any closed interval $[a,b]$, then when the limits in RHS exist,

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f(x+1) - f(x).$$
$$\lim_{x\to\infty} f(x)^{1/x} = \lim_{x\to\infty} \frac{f(x+1)}{f(x)}, \text{if for any } x \in [a,\infty), f(x) \geqslant c > 0.$$

Proof: Suppose $\lim_{x\to\infty} f(x+1) - f(x) = A$, then for any $\varepsilon > 0$ there exists M such that for any $x > M$, $|f(x+1) - f(x) - A| < \varepsilon$, so for any $n \geqslant 1$, $|f(x+n) - f(x) - nA| < n\varepsilon$. Hence

$$\left| \frac{f(n+x)}{n+x} - A \right| \leqslant \left| \frac{f(n+x) - f(x) - nA}{n+x} \right| + \left| \frac{f(x) - xA}{n+x} \right| \leqslant \varepsilon A + \frac{|f(x) - xA|}{n} \to 0.$$

For any $x > M$. Therefore (since f is bounded on any closed interval) there exists N such that for any $x > N$, $|f(x)/x - A| < 2\varepsilon A$, and hence

$$\lim_{x\to\infty} \frac{f(x)}{x} = A = \lim_{x\to\infty} f(x+1) - f(x).$$

Substitute f by $\log f$ and we obtain the second identity.

5.2 PSB: Uniform Continuity

Determine whether the following functions f are uniformly continuous on I:

B1) $f(x) = x^{1/3}$**,** $I = (0,\infty)$

For any $\varepsilon > 0$ and $x - y \in (0,\varepsilon)$,

$$x^{1/3} - y^{1/3} = \frac{x-y}{x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}} \leqslant \frac{\varepsilon}{\varepsilon^{2/3}} = \varepsilon^{1/3}.$$

Hence $f(x)$ is uniformly continuous on I.

B2) $f(x) = \log x$**,** $I = (0, 1)$

For any $\varepsilon > 0$ and $x - y \in (0, \varepsilon)$,

$$\log x - \log y = \log\left(1 + \frac{x - y}{y}\right).$$

When $y \to 0$ and $x - y$ is constant, $\log x - \log y \to \infty$, hence $\log x$ is not uniformly continuous on I.

B3) $f(x) = \cos x^{-1}$**,** $I = (0, 1)$

Note that for $x_n = 1/(2n\pi)$ and $y_n = 1/(2n\pi + \pi)$, $f(x_n) = 1$ and $f(y_n) = -1$. Hence for $\varepsilon = 1$ and any $\delta > 0$, there exists n such that $|x_n - y_n| < \delta$ but $|f(x_n) - f(y_n)| = 2 > \varepsilon$. Therefore f is not uniformly continuous on I.

B4) $f(x) = x \cos x^{-1}$**,** $I = (0, \infty)$

For $x > y > 1$ and $|x - y| < \varepsilon$,

$$\begin{aligned} |x\cos x^{-1} - y\cos y^{-1}| &\leqslant |x - y||\cos x^{-1}| + |y| \cdot |\cos x^{-1} - \cos y^{-1}| \\ &\leqslant \varepsilon + 2|y| \cdot |\sin(x^{-1} + y^{-1})/2 \sin(x^{-1} - y^{-1})/2| \leqslant \varepsilon + \frac{y}{2}\left(\frac{1}{y^2} - \frac{1}{x^2}\right) \leqslant 2\varepsilon. \end{aligned}$$

For $1 > x > y$ and $|x - y| < \varepsilon$,

$$|x\cos x^{-1} - y\cos y^{-1}| \leqslant |x| + |y| < 2\varepsilon.$$

Hence f is uniformly continuous on I.

5.3 PSC: Existence of Limits

C1) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{\log x}{(x-1)^\alpha} = \lim_{t\to 0} \frac{\log(1+t)}{t^\alpha} = \lim_{t\to 0} t^{1-\alpha}$$

exists iff $\alpha \leqslant 1$.

C2) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{e^x - e}{(x-1)^\alpha} = e \lim_{t\to 0} \frac{e^t - 1}{t^\alpha} = \lim_{t\to 0} e t^{1-\alpha}.$$

exists iff $\alpha \leqslant 1$.

C3) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{x^x - 1}{(x-1)^\alpha} = \lim_{x\to 1} \frac{x^x(\log x + 1)}{\alpha(x-1)^{\alpha-1}}$$

exists iff $\alpha \leqslant 1$.

C4) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{\sqrt[3]{1-\sqrt{x}}}{(x-1)^\alpha}$$

exists iff $\alpha \leqslant 1/3$.

C5)

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{1-\cos x} = 1.$$

C6)

$$\lim_{x\to 0} \frac{\sqrt{1+x^4}-1}{1-\cos^2 x} = 0.$$

C7) $\alpha > 0$,

$$\lim_{x\to 1} \frac{(x-1)^\alpha}{\sin(\pi x)}$$

exists iff $\alpha \geqslant 1$.

5.4 PSD: Problems on Uniform Continuity

D1) If f is continuous, monotonic and bounded on the open interval I, then f is uniformly continuous on I.

Proof: Otherwise if there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $|x-y| < \delta$ such that $|f(x) - f(y)| > \varepsilon$. We define x_n, y_n inductively as follows: Let $L = \min\{x_1, \cdots, x_{n-1}\}$, $R = \max\{y_1, \cdots, y_{n-1}\}$. Since f is uniformly continuous on $[L, R]$, there exists $\delta > 0$ such that for any $|s-t| < \delta$, $|f(s) - f(t)| < \varepsilon$. Hence there exists $x < y$ such that $x, y \notin [L, R]$, $|x-y| < \delta$ and $|f(x) - f(y)| > \varepsilon$. Let $x_n = x, y_n = y$, then (x_n, y_n) are disjoint intervals and $|f(x_n) - f(y_n)| > \varepsilon$. Which contradicts the fact that f is monotonic and bounded. Therefore f is uniformly continuous on I.

D2) I is an interval with finite length. Prove that the function f on I is uniformly continuous iff for any Cauchy sequence $\{x_n\}_{n\geqslant 1} \subset I$, $\{f(x_n)\}_{n\geqslant 1}$ is also a Cauchy sequence.

(f should be continuous, otherwise after changing the value of f at one point, $\{f(x_n)\}$ remains a Cauchy sequence.)
Proof: $\Longrightarrow$ If $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. There exists N such that for all $n, m > N$, $|a_n - a_m| < \delta$, hence $|f(a_n) - f(a_m)| < \varepsilon$, so $\{f(x_n)\}_{n\geqslant 1}$ is a Cauchy sequence.
$\Longleftarrow$ If $I = (a, b)$ is open we can take $x_n \to a$ and define $f(a) = \lim_{n\to\infty} f(x_n)$, hence we can assume that I is closed. Therefore f is uniformly continuous.

D3) f is uniformly continuous on $\mathbb{R}$. Prove that there exists $a, b \in \mathbb{R}_{>0}$ such that for any $x \in \mathbb{R}$,

$$|f(x)| \leqslant a|x| + b.$$

Proof: For $\varepsilon = 1$, there exists $\delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < 1$. Hence let $C = \sup_{x\in[0,\delta]} |f(x)|$, then $|f(x)| \leqslant C + |x| \cdot (\frac{1}{\delta} + 1)$.

D4) Suppose f is uniformly continuous on $[0,\infty)$ and for any $x \in [0,1]$, $\lim_{n\to\infty} f(x+n) = 0$. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

If we change the condition to f is continuous, will the statement still hold?
Proof: For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < \varepsilon$. Let $N = \lfloor 1/\delta \rfloor + 1$, then for any $1 \leqslant n \leqslant N$, there exists M_n such that for all $m > M_n$, $|f(m+n/N)| < \varepsilon$. Let $M = \max\{M_1, \cdots, M_N\}$, then for all $x > M$, there exists $m \in \mathbb{Z}_{>M}$ and $1 \leqslant n \leqslant N$ such that $|x-m-n/N| < \delta$. Hence

$$|f(x)| \leqslant \varepsilon + |f(m+n/N)| < 2\varepsilon.$$

Therefore $\lim_{x\to\infty} f(x) = 0$.

D5) Suppose X is an interval, $f : X \to \mathbb{R}$ is continuous. If there is a constant $L > 0$ such that for any $x, y \in X$,

$$|f(x) - f(y)| \leqslant L|x-y|.$$

We say f satisfy the Lipschitz condition on X.

1. Prove that f satisfy the Lipschitz condition implies f is uniformly continuous.
   Proof: For any $\varepsilon > 0$, let $\delta = \varepsilon/L$, then for any $|x-y| < \delta$, $|f(x)-f(y)| \leqslant L|x-y| < \varepsilon$.
2. Determine whether the reversed statement holds.
   Consider the function $f(x) = x^{1/2}$, then f is uniformly continuous but $\frac{f(x)-f(y)}{x-y} = \frac{1}{\sqrt{x}+\sqrt{y}}$ is unbounded, hence does not satisfy the Lipschitz condition.
3. If f satisfy the Lipschitz condition on $[a,\infty)$, where $a > 0$, prove that $f(x)/x$ is uniformly continuous on $[a,\infty)$.
   Proof: Same as D3), there exists C such that $|f(x)| \leqslant C|x|$ for $x \in [a,\infty)$, then for $a < x < y$,

$$\begin{aligned}\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| &= \frac{|xf(y) - yf(x)|}{xy} \leqslant \frac{x|f(y)-f(x)| + |f(x)|(y-x)}{xy} \\ &\leqslant \frac{L+C}{y} \cdot |x-y|.\end{aligned}$$

Hence $f(x)/x$ satisfy the Lipschitz condition.

5.5 PSE:

Exactly the same as PSC in HW4?

5.6 PSF: Calculate Limits

F1)

$$\lim_{x\to\pi} \frac{\sin mx}{\sin nx} = \frac{m(-1)^m}{n(-1)^n}.$$

F2)

$$\lim_{x\to 0} \frac{1 - \cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2} = 3.$$

F3)

$$\lim_{x\to\infty} \sin\sqrt{1+x} - \sin\sqrt{x} = 0.$$

Since the function $\sin x$ is uniformly continuous and $\lim_{x\to\infty} \sqrt{1+x} - \sqrt{x} = 0$.

F4)

$$\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1} = \frac{1}{2}.$$

Since $\lim_{x\to 0} x^2/(e^{x^2}-1) = 1$, $\lim_{x\to 0} x\sin x/x^2 = 1$ and $\lim_{x\to 0} 1/(1+\sqrt{1+x\sin x}) = 1/2$.

F5)

$$\lim_{n\to\infty} \sin^{(n)}(x) = 0.$$

Since the sequence $\{a_n = \sin^{(n)}(x)\}_{n\geqslant 1}$ is decreasing and bounded by 0, and its limit A satisfy $A = \sin A$. Therefore $\lim_{n\to\infty} \sin^{(n)}(x) = 0$.

5.7 Problem G

The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfy the following property: for any $\delta > 0$,

$$\lim_{n\to\infty} f(n\delta) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.
Proof: Consider any $\varepsilon > 0$. For any $N \in \mathbb{N}$,

$$A_N = \{\delta > 0 : \forall n \geqslant N, |f(n\delta)| < \varepsilon\}.$$

Then by the continuity of f, A_N is closed, and by $\lim_{n\to\infty} f(n\delta) = 0$ for any $\delta > 0$, $\bigcup_{N\geqslant 1} A_N = \mathbb{R}_{>0}$. Hence by Baire Category Theorem, there exists an $N > 0$ such that $(a,b) \subset A_N$ for some interval (a,b). Let $X = \{x \in \mathbb{R}_{>0} : |f(x)| < \varepsilon\}$, then since $(a,b) \subset A_N$, for any $n \geqslant N$, $(na, nb) \subset X$. Note that when $n > b/(b-a)$, $nb > (n+1)a$, hence there exists $M > 0$ such that $(M,\infty) \subset X$. Therefore $\lim_{x\to\infty} f(x) = 0$.

5.8 Problem H

The continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

1. $\lim_{x\to\infty}(\varphi(x) - x) = \infty$.
2. $\{x \in \mathbb{R} : \varphi(x) = x\}$ is a non-empty finite set.

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f \circ \varphi = f$, then f is constant.
(Probably need the condition $\lim_{x\to-\infty} \varphi(x) - x = -\infty$).
Proof: Suppose $\{x \in \mathbb{R} : \varphi(x) = x\} = \{a_1, \cdots, a_n\}$ where $a_1 < \cdots < a_n$. For any $x \in \mathbb{R}$, we will show that $f(x) \in \{f(a_1), \cdots, f(a_n)\}$ hence f is constant.
If $a_i < x < a_{i+1}$. Suppose $\varphi(x) > x$, then let $x_0 = x$, and inductively define x_k as a point in (a_k, x_{k-1}) such that $\varphi(a_i) = a_i < \varphi(x_k) = x_{k-1} < \varphi(x_{k-1})$. Since φ is continuous and $a_1, \cdots, a_n$ are all the roots of $\varphi(x) = x$, we know that $\varphi(x_k) > x_k$ for all $k \geqslant 0$. The sequence $\{x_k\}_{k\geqslant 0}$ is decreasing and bounded by a_i, hence has a limit A. From $\varphi(x_k) = x_{k-1}$ we know that $\varphi(A) = A$, so $A = a_i$. Note that $f(x_k) = f(\varphi(x_k)) = f(x_{k-1})$, hence $f(x) = f(x_k) = \lim_{k\to\infty} f(x_k) = f(a_i)$. The case $\varphi(x) < x$ is the same, by constructing a sequence which tends to a_{i+1}.

If $x > a_n$, then $\varphi(x) > x$, likewise we can construct a sequence x_k such that $x_{k-1} = \varphi(x_k)$ and $\lim_{k\to\infty} x_k = a_n$. The case $x < a_1$ is the same.
Hence for all $x \in \mathbb{R}$, $f(x) \in \{f(a_1), \cdots, f(a_n)\}$.

5.9 Problem I

The continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ satisfy $\lim_{x\to\infty} f(x)/x = 0$. Suppose $\{a_n\}_{n\geqslant 1}$ is a sequence of non-negative real numbers and the sequence $\{a_n/n\}_{n\geqslant 1}$ is bounded. Prove that $\lim_{n\to\infty} f(a_n)/n = 0$.
Proof: Suppose $\{a_n/n\}$ is bounded by M.
For any $\varepsilon > 0$, we need to find N such that $n \geqslant N \implies |f(a_n)| < \varepsilon n$. For $C > 0$, we can divide n into two parts: If $a_n \leqslant C$, then $|f(a_n)| \leqslant \sup_{x\in[0,C]} |f(x)|$, otherwise $a_n \geqslant C$, then $|f(a_n)| \leqslant \sup_{x\geqslant C} |f(x)/x| \cdot Mn$. Therefore, if we choose $C > 0$ such that $\sup_{x\geqslant C} |f(x)/x| < \varepsilon/M$, and choose N such that $N > \sup_{x\in[0,C]} |f(x)|/\varepsilon$, then for any $n \geqslant N$, $|f(a_n)| < \varepsilon n$, hence

$$\lim_{n\to\infty} \frac{f(a_n)}{n} = 0.$$

5.10 Ex: Proof of the infinity of primes using topology

Proof: Assume otherwise that the set $\mathcal{P}$ of primes is finite. Let $L_{a,b} = \{at + b : t \in \mathbb{Z}\}, \forall (a, b) \in I = \mathbb{Z}_{>0} \times \mathbb{Z}$. Then

$$\mathbb{Z} \subset \bigcup_{b\in\mathbb{Z}} L_{1,b} \subset \bigcup_{(a,b)\in I} L_{a,b} \subset \mathbb{Z} \implies \bigcup_{(a,b)\in I} L_{a,b} = \mathbb{Z}.$$

and for any $x \in \bigcap_{i=1}^n L_{a_i,b_i}$, let $a = \operatorname{lcm}(a_1, \cdots, a_n)$, then

$$x \in L_{a,x} \subset \bigcap_{i=1}^{n} L_{a_i,bi}.$$

Hence $L_{a,b}$ form a base.Consider the topology $\mathcal{T}$ on $\mathbb{Z}$ generated by the base $\{L_{a,b} : (a, b) \in I\}$. Note that

$$L_{a,b} = \mathbb{Z} \backslash \bigcup_{r=1}^{a-1} L_{a,b+r}$$

so $L_{a,b}$ is also closed. Since $\mathcal{P}$ is finite, the set

$$\bigcup_{p\in\mathcal{P}} L_{p,0} = \mathbb{Z}\backslash\{-1, 1\}$$

is closed, hence $\{-1, 1\}$ is open. However, an open set G is the union of $L_{a,b}$ which is infinite, so G is infinite, leading to contradiction.

> Quote:
> As for everything else, so for a mathematical theory: beauty can be perceived but not explained.
> ——A. Cayley

6 Homework 6: Takagi Function

6.1 PSA: Calculating Derivatives

A1) Consider the function

$$f : \mathbb{R} \to \mathbb{R}^n,\ x \mapsto f(x) = (f_1(x), \cdots, f_n(x)).$$

Prove that f is differentiable at x_0 iff every f_k is differentiable at x_0 and

$$f'(x) = (f_1'(x), \cdots, f_n'(x)).$$

Proof: For any $h \in \mathbb{R}$,

$$\|\frac{f(x+h)-f(x)}{h} - (f_1'(x), \cdots, f_n'(x))\|_2 \leqslant n \max_{1 \leqslant k \leqslant n} \left\{ \left| \frac{f_k(x+h)-f_k(x)}{h} - f_k'(x) \right| \right\} \to 0.$$

Therefore $f'(x) = (f_1'(x), \cdots, f_n'(x))$.

A2) Consider the function

$$f : \mathbb{R} \to \mathbb{C},\ x \mapsto e^{ix}.$$

Prove by definition, $f'(0) = i$ and $(e^{ix})' = ie^{ix}$.
Proof: For any $h \in \mathbb{R}$,

$$\left| \frac{f(h)-f(0)}{h} - i \right| = \left| \frac{e^{ih} - ih - 1}{h} \right| \leqslant \sum_{n=2}^{\infty} \left| \frac{1}{h} \frac{(ih)^n}{n!} \right| \leqslant |h| \to 0.$$

Therefore $f'(0) = i$. For any $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} e^{ix} \frac{f(h)-f(0)}{h} = ie^{ix}.$$

Hence $(e^{ix})' = ie^{ix}$.

A3) Calculate the derivatives of $\sin x$ and $\cos x$.

Solution: $\sin x = (e^{ix} - e^{-ix})/2i$, so $(\sin x)' = (e^{ix} + e^{-ix})/2 = \cos x$. Likewise $(\cos x)' = -\sin x$.

A4) Prove Faà di Bruno's formula for $n = 3$.

Proof:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(f \circ g) = f'(g) \cdot g'.\\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f \circ g) = f'(g) \cdot g'' + f''(g) \cdot (g')^2.\\
&\frac{\mathrm{d}^3}{\mathrm{d}x^3}(f \circ g) = f'(g) \cdot g''' + f''(g) \cdot g'' \cdot g' + f'''(g) \cdot (g')^3 + f''(g) \cdot 2g'g''.
\end{aligned}$$

A5) Define the map

$$E : \mathbb{R} \to \mathbb{C} = \mathbb{R}^2,\ \theta \mapsto (\cos\theta, \sin\theta).$$

Prove that the points in $\mathbf{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ can be written in the form $(\sin\theta, \cos\theta)$, i.e. $E(\mathbb{R}) = \mathbf{S}^1$. Calculate $E'(\theta)$ and show that Rolle's mean-value theorem is invalid for E.
Proof: Obviously $E(\mathbb{R}) \subset \mathbf{S}^1$. Consider any $(x, y) \in \mathbf{S}^1$, then $x \in [-1, 1]$. Note that $\cos 0 = 1, \cos\pi = -1$, hence there exists $\theta \in [0, \pi]$ such that $\cos\theta = x$, and $|\sin\theta| = |y|$. If $\sin\theta = y$ then $(x, y) = (\cos\theta, \sin\theta) \in E(\mathbb{R})$. Otherwise $(x, y) = (\cos(-\theta), \sin(-\theta)) \in E(\mathbb{R})$, therefore $E(\mathbb{R}) = \mathbf{S}^1$.
By A1) and A3), $E'(\theta) = (-\sin\theta, \cos\theta)$. Since $E'(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ and $E'(\theta) = E'(\theta + 2\pi)$, Rolle's mean-value theorem is invalid.

A6) Calculate the derivatives of the following functions:

(1) $f(x) = a^x$, $a > 0$.

$$f'(x) = (e^{x \log a})' = a^x \log a.$$

(2) $f(x) = \arcsin x$.
Let $y = \arcsin x$, then $x = \sin y$, so $1 = \cos y \cdot y'$, hence

$$f'(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

(3) $f(x) = \arctan x$.
Let $y = \arctan x$, then $x = \tan y$, so $1 = \sec^2 y \cdot y'$, hence

$$f'(x) = \cos^2 y = \frac{1}{1+x^2}.$$

(4) $f(x) = x^{x^x}$, $x > 0$.
Let $y = x^x, z = x^y$, then $\log y = x \log x$, so $y'/y = \log x + 1$, $y' = x^x(1 + \log x)$. $\log z = y \log x$, so $z'/z = y' \log x + y/x = x^x \log x(1 + \log x) + x^{x-1}$. Therefore

$$f'(x) = x^{x^x} \cdot x^x \cdot (\log x + \log^2 x + x^{-1}).$$

(5) $f(x) = \log(\log(\log x))$.

$$f'(x) = \frac{(\log\log x)'}{\log\log x} = \frac{1}{x \log x \log\log x}.$$

(6) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

$$\begin{aligned} f'(x) &= \frac{(x + \sqrt{x+\sqrt{x}})'}{2\sqrt{x + \sqrt{x+\sqrt{x}}}} = \left(1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right) /2\sqrt{x + \sqrt{x+\sqrt{x}}} \\ &= \frac{2\sqrt{x+\sqrt{x}} + 1 + 1/2\sqrt{x}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}. \end{aligned}$$

(7) $f(x) = |x|$.
If $x > 0$, $f'(x) = (x)' = 1$. If $x < 0$, $f'(x) = (-x)' = -1$. If $x = 0$, f is not differentiable at x.
(8) $f(x) = \log|x|$.
If $x > 0$, $f'(x) = \frac{1}{x}$. If $x < 0$, $f'(x) = -\frac{1}{x}$. If $x = 0$, f is not differentiable at x.
(9)

$$f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \quad n = 1, 2, \cdots$$

For $x \neq 0$, $f'(x) = nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}$. When $x = 0$,

$$f'(0) = \lim_{h \to 0} h^{n-1} \sin \frac{1}{h} = \begin{cases} 0, & n \geqslant 2; \\ \text{diverges}, & n = 1. \end{cases}$$

A7) Calculate $f^{(3)}(x)$:

(1) $f(x) = \log(x+1)$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\log(x+1) = \frac{2}{(x+1)^3}.$$

(2) $f(x) = x^{-1}\log x$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{\log x}{x} = \frac{11 - 6\log x}{x^4}.$$

(3) $f(x) = \frac{x^m}{1-x}$, $m \in \mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{x^m}{1-x} = \frac{(m-2)(m-1)mx^{m-3}}{1-x} + \frac{3(m-1)mx^{m-2}}{(1-x)^2} + \frac{6mx^{m-1}}{(1-x)^3} + \frac{6x^m}{(1-x)^4}.$$

(4) $f(x) = x^m e^x$, $m \in \mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^m e^x) = e^x x^{m-3}(m^3 + 3m^2(x-1) + m(3x^2 - 3x + 2) + x^3).$$

(5) $f(x) = e^{ax}\sin(bx)$, $a, b \in \mathbb{R}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{ax}\sin(bx)) = e^{ax}((3a^2b - b^3)\cos(bx) + a(a^2 - 3b^2)\sin(bx)).$$

(6) $f(x) = e^{-x^2}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-x^2} = -4e^{-x^2}x(2x^2 - 3).$$

A8) $f'(x_0) > 0$ does not imply f is increasing in a neighborhood of x_0: consider

$$f(x) = \begin{cases} x + 2x^2\sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Prove that $f'(0) > 0$ but for any $\varepsilon > 0$, f is not monotonic on $(-\varepsilon, \varepsilon)$.
Proof:

$$f'(0) = \lim_{h\to 0} 1 + 2h\sin\frac{1}{h} = 1 > 0.$$

However, for any $n \in \mathbb{N}$, let $x_n = \frac{1}{(2n+1/2)\pi}$, $y_n = \frac{1}{(2n-1/2)\pi}$, then

$$f(x_n) = x_n + 2x_n^2,\ f(y_n) = y_n - 2y_n^2.$$

Note that $0 < x_n < y_n$, but

$$f(x_n) - f(y_n) = 2x_n^2 + 2y_n^2 - \pi x_n y_n > 0,$$

i.e. $f(x_n) > f(0), f(y_n)$, therefore f is not monotonic on any $(-\varepsilon, \varepsilon)$.

A9) $A \in \mathbf{M}_n(\mathbb{R})$, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=0} \det(\mathbf{I}_n + xA).$$

Solution: Let $\Phi(x) = I_n + xA$, then $\Phi(0) = I_n$. Denote $\Phi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$. Note that det is a multi-linear function for n rows, hence by Euler's formula:

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = \det(\varphi_1'(t), \varphi_2(t), \cdots, \varphi_n(t)) + \cdots + \det(\varphi_1(t), \varphi_2(t), \cdots, \varphi_n'(t)).$$

When $t = 0$, the formula becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det \Phi(t) = \varphi_{1,1}' + \cdots + \varphi_{n,n}' = \mathrm{tr}\, \Phi'(0) = \mathrm{tr} A.$$

A10) Prove that the derivation of odd functions are even, and that of even functions are odd.

Proof: If f is an odd function then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x),$$

so f' is even. If f is an even function then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = -\lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = -f'(x),$$

so f' is odd.

A11) Prove that

$$f(x) = \begin{cases} 1/q, & x = \frac{p}{q} \in \mathbb{Q}, q \geqslant 1, \gcd(p,q) = 1; \\ 0, & x \in \mathbb{Q}^C. \end{cases}$$

is nowhere differentiable on $\mathbb{R}$.
Proof: For any $x \in \mathbb{Q}$, $f(x) \neq 0$, but for any $\varepsilon > 0$, there exists $y \in (x - \varepsilon, x + \varepsilon) \cap \mathbb{Q}^C$, such that $f(y) = 0$. Therefore f is not continuous at x, and clearly not differentiable.
Consider any $x \in \mathbb{Q}^C$, there is a sequence of irrational numbers $\{y_n\}_{n \geqslant 1}$ that converges to x, then

$$\lim_{n \to \infty} \frac{f(x) - f(y_n)}{x - y_n} = 0.$$

Choose any sequence of rational numbers $\{r_n = p_n/q_n\}_{n \geqslant 1}$ that converges to x, then

$$\lim_{n \to \infty} \frac{f(x) - f(r_n)}{x - r_n} = \lim_{n \to \infty} \frac{1}{xq_n - p_n} = \infty.$$

Therefore f is nowhere differentiable on $\mathbb{R}$.

6.2 PSB

B1) Define the hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2},\ \tanh x = \frac{\sinh x}{\cosh x}.$$

1. Prove that
   (1) $\cosh^2 x - \sinh^2 x = 1$
   Proof:$\cosh^2 x - \sinh^2 x = \dfrac{e^{2x} + e^{-2x} + 2}{4} - \dfrac{e^{2x} + e^{-2x} - 2}{4} = 1.$
   (2) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
   Proof: $\sinh x \cosh y + \cosh x \sinh y = \dfrac{e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}}{4} + \dfrac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}}{4} = \sinh(x + y)$
   (3) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
   Proof: Same as (2).
   (4) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.
   Proof: $\tanh(x + y) = \dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

2. Calculate $\sinh'(x)$, $\cosh'(x)$ and $\tanh'(x)$.
   Solution: $\sinh'(x) = \cosh x$, $\cosh'(x) = \sinh x$, $\tanh'(x) = \dfrac{1}{\cosh^2 x}$.

3. Prove that $\sinh : \mathbb{R} \to \mathbb{R}$ has an inverse $\operatorname{arcsinh} : \mathbb{R} \to \mathbb{R}$ and calculate $\operatorname{arcsinh}'(x)$.
   Proof: $\sinh'(x) = \cosh x > 0$, so sinh is monotonically increasing over $\mathbb{R}$. Also $\lim_{x\to\infty} \sinh x = \infty$, $\lim_{x\to-\infty} \sinh x = -\infty$, therefore $\sinh : \mathbb{R} \to \mathbb{R}$ is a bijection and hence has an inverse.
   $\operatorname{arcsinh}'(x) = \frac{1}{\sqrt{1+x^2}}$.

B2) $a, b \in \mathbb{R}$**,** $a > 0$**. Consider** $f : [-1, 1] \to \mathbb{R}$**, where**

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

1. $f \in C([-1, 1])$ iff $a > 0$;
   Proof: $f \in C([-1, 1])$ iff $\lim_{x\to 0} x^a \sin(x^{-b}) = 0$. If $a > 0$ then $|x^a \sin(x^{-b})| \leqslant |x|^a \to 0$. If $a < 0$ then let $x = ((2n + 1/2)\pi)^{-1/b}$, when $n \to \infty$, $x \to 0$ but $|x^a \sin(x^{-b})| \to \infty$. If $a = 0$, then let $x = ((2n + 1/2)\pi)^{-1/b}$, $|x^a \sin(x^{-b})| = 1$. Therefore $f \in C([-1, 1])$ iff $a > 0$.

2. f is differentiable at 0 iff $a > 1$;
   Proof: f is differentiable at 0 iff $\lim_{x\to 0} x^{-a} \sin(x^{-b})$ exists. By 1 we know that $a > 1$. ($a = 1$ is invalid since $x = (2n\pi)^{-1/b}$ and $x = ((2n + 1/2)\pi)^{-1/b}$ converge to different values.)

3. f' is bounded on $[-1, 1]$ iff $a \geqslant 1 + b$;
   Proof: $f'(x) = ax^{a-1} \sin(x^{-b}) + x^a \cos(x^{-b})(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geqslant 1 + b$.

4. $f \in C^1([-1, 1])$ iff $a > 1 + b$;
   Proof: $f \in C^1([-1, 1])$ iff $f'(0) = 0 = \lim_{x\to 0} f'(x)$. By 1 we know it is equivalent to $a > 1 + b$.

5. f' is differentiable at 0 iff $a > 2 + b$;

6. f'' is bounded on $[-1, 1]$ iff $a \geqslant 2 + 2b$;

7. $f \in C^2([-1,1])$ iff $a > 2 + 2b$.
   Proof: 5,6,7 are exactly the same as 2,3,4.

6.3 PSC

If f satisfy $\lim_{x\to x_0} f(x) = 0$ near x_0, we call f an infinitesimal when $x \to x_0$. Likewise when $\lim_{x\to x_0} f(x) = +\infty$ or $\lim_{x\to x_0} f(x) = -\infty$, we call f an infinite quantity when $x \to x_0$.
Suppose f, g are both infinitesimal when $x \to x_0$, and $g(x)$ does not vanish near x_0. We introduce the notations

- if $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$, we say f is an infinitesimal of higher order than g, and denote $f(x) = o(g(x))$, $x \to x_0$;
- If $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \neq 0$, we say f and g are of the same order;
- If $= 1$, denote $f \sim g$, $x \to x_0$;
- If $\limsup_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| < +\infty$, denote $f(x) = O(g(x))$, $x \to x_0$.

C1) Suppose $a(x) = o(1)$ when $x \to x_0$, prove that:

(1) $o(a) + o(a) = o(a)$
Proof: If $f, g = o(a)$, then

$$\lim_{x\to x_0} \frac{f(x) + g(x)}{a(x)} = \lim_{x\to x_0} \frac{f(x)}{a(x)} + \lim_{x\to x_0} \frac{g(x)}{a(x)} = 0,$$

hence $f + g = o(a)$.
(2) $co(a) = o(ca)$, $c \in \mathbb{R}$
Proof: If $f = o(a)$, then

$$\lim_{x\to x_0} \frac{cf(x)}{a(x)} = c \lim_{x\to x_0} \frac{f(x)}{a(x)} = 0,$$

hence $cf = o(a) = o(ca)$.
(3) $o(a)^k = o(a^k)$
Proof: If $f = o(a)$ then

$$\lim_{x\to x_0} \frac{f(x)^k}{a(x)^k} = \left(\lim_{x\to x_0} \frac{f(x)}{a(x)}\right)^k = 0,$$

hence $f^k = o(a^k)$.
(4) $1/(1+a) = 1 - a + o(a)$
Proof:

$$\lim_{x\to x_0} \frac{1/(1+a) - 1 + a}{a(x)} = \lim_{x\to x_0} \frac{a(x)}{1 + a(x)} = 0,$$

hence $1/(1+a) = 1 - a + o(a)$.

C2) Suppose f, g are infinitesimals when $x \to x_0$, then

1. Prove that $f \sim g \iff f(x) - g(x) = o(g(x))$, $x \to x_0$.
   Proof: $f \sim g \iff \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1 \iff \lim\limits_{x\to x_0} \dfrac{f(x) - g(x)}{g(x)} = 0$, i.e. $f(x) - g(x) = o(g(x))$.

2. If $f \sim cg^k$, we call cg^k the leading term of f. Find the leading terms of the following functions, compared to $x - x_0$ or x:
   (1) $1/\sin \pi x$, $x \to 1$.
   $\frac{1}{\sin \pi x} = -\frac{1}{\pi(x-1)} + o(1)$.
   (2) $\sqrt{1+x} - \sqrt{1-x}$, $x \to 0$.
   $\sqrt{1+x} - \sqrt{1-x} = x + o(x)$.
   (3) $\sin(\sqrt{1+\sqrt{1+\sqrt{x}}} - \sqrt{2})$, $x \to 0^+$.
   $= \dfrac{\sqrt{2}x^{1/2}}{8} + o(x^{1/2})$.
   (4) $\sqrt{1+\tan x} - \sqrt{1-\sin x}$, $x \to 0$.
   $= x + o(x)$.
   (5) $\sqrt{x + \sqrt{x + \sqrt{x}}}$, $x \to 0^+$.
   $= x^{1/8} + o(x^{1/8})$.
   (6) $\sqrt{x + \sqrt{x + \sqrt{x}}}$, $x \to \infty$.
   $=\sqrt{x} + o(\sqrt{x})$.

3. Suppose $f \sim cx^k$, $x \to 0$, i.e. $f(x) = cx^k + o(x^k)$. If $f(x) - c^k$ has a leading term $c'x^{k'}$, we denote $f(x) = cx^k + c'x^{k'} + o(x^{k'})$. Expand the following terms to $o(x^2)$:
   (1) $\sqrt{1+x} - 1$.
   $\sqrt{1+x} - 1 = \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$.
   (2) $(1+x)^{1/m} - 1$, $m \in \mathbb{Z}_{\geqslant 1}$.
   $(1+x)^{1/m} - 1 = \frac{1}{m}x - \frac{m-1}{2m^2}x^2 + o(x^2)$.

6.4 PST: Takagi Function

Define $\psi : [0,1] \to \mathbb{R}$ as

$$\psi(x) = \begin{cases} x, & 0 \leqslant x < \frac{1}{2}; \\ 1-x, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

For $x \in R$, let $\psi(x) = \psi(\{x\})$, then $\psi \in C(\mathbb{R})$.
Define the Takagi function $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k}\psi(2^k x),$$

and the partial sum $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{2^k}\psi(2^k x)$.

T1) Prove that $T(x)$ is a well-defined bounded continuous function on $\mathbb{R}$.

Proof: Note that $\psi(x) \in [0, 1/2]$ so the series $\sum_{k=0}^{\infty} 2^{-k}\psi(2^k x)$ converges absolutely, and hence $T(x)$ is well-defined and bounded by $T(x) \in [0,1]$.

T2) For $x \in [0,1]$, suppose $x = \sum_{n=1}^{\infty} a_n 2^{-n}$ is the binary form of x. Let $v_n = \sum_{k=1}^{n} a_k$, and $\sigma_n(y) = a_n + (1-2a_n)y$, where $y \in \{0,1\}$. Prove that

$$\psi(2^m x) = \sum_{n=1}^{\infty} \frac{\sigma_{m+1}(a_{m+n})}{2^n}.$$

Proof:

$$\psi(2^m x) = \psi\left(\sum_{n=1}^{\infty} a_n 2^{m-n}\right) = \psi\left(\sum_{n=m+1}^{\infty} a_n 2^{m-n}\right) = \begin{cases} \sum_{n=1}^{\infty} a_{m+n}2^{-n}, & a_{m+1}=0; \\ 1-\sum_{n=1}^{\infty} a_{m+n}2^{-n}, & a_{m+1}=1. \end{cases}$$

Therefore

$$\psi(2^m x) = \sigma_n\left(\sum_{n=1}^{\infty} a_{m+n}2^{-n}\right) = \sum_{n=1}^{\infty} \sigma_{m+1}(a_{m+n})2^{-n}.$$

T3) $x = \sum_{n=1}^{\infty} a_n 2^{-n} \in [0,1]$**, prove that**

$$T(x) = \sum_{n=1}^{\infty} \frac{(1-a_n)v_n + a_n(n-v_n)}{2^n}.$$

Proof: By T2),

$$T(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sigma_{m+1}(a_{m+n})2^{-m-n} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \sigma_{m+1}(a_n)2^{-n} = \sum_{n=1}^{\infty} \frac{(1-a_n)v_n + a_n(n-v_n)}{2^n}.$$

T4) Suppose $x_0 = k_0 2^{-m_0} \in [0,1]$**, where** $k_0 \in \mathbb{Z}_{\geqslant 1}$ **is odd,** $m_0 \in \mathbb{Z}_{\geqslant 0}$**. Let** $h_n = 2^{-n}$**, where** $n \in \mathbb{Z}_{\geqslant m_0}$**. Prove that the sequence** $\left\{\frac{T(x+h_n)-T(x)}{h_n}\right\}_{n\geqslant m_0}$ **does not converge.**

Proof: By T3),

$$\frac{T(x+h_n)-T(x)}{h_n} = \frac{1}{h_n}\left(\frac{n-v_n}{2^n} - \frac{v_n}{2^n}\right) = n - 2\sum_{k=1}^{n} a_k = n-2-2S_2(k_0) \to \infty.$$

T5) $f: I \to \mathbb{R}$**, where** I **is an open interval. If** f **is differentiable at** a**, prove that**

$$\lim_{(h,h')\to(0,0),h,h'>0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

i.e. it converges for any sequence $(h_n, h_n') \to (0,0), h_n, h_n' > 0$.
Proof: Consider any sequence $(h_n, h_n') \to (0,0)$, then

$$\frac{f(a+h)-f(a-h')}{h+h'} = \frac{f(a+h)-f(a)}{h}\cdot\frac{h}{h+h'} + \frac{f(a)-f(a-h')}{h'}\cdot\frac{h'}{h+h'} \to f'(a).$$

T6) Same as T5), if $f \in C^1(I)$**,** $a \in I$**, prove that**

$$\lim_{(h,h')\to(0,0),h+h'\neq 0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

Proof: For any $h+h' \neq 0$, there exists $\xi \in [a, a+h]$ and $\eta \in [a-h', a]$ such that $f(a+h) = f(a)+hf'(\xi)$ and $f(a-h') = f(a) - h'f'(\eta)$, then

$$\left|\frac{f(a+h)-f(a-h')}{h+h'} - f'(a)\right| \leqslant \frac{h}{h+h'}|f'(\xi)-f'(a)| + \frac{h'}{h+h'}|f'(\eta)-f'(a)| \to 0.$$

Hence

$$\lim_{(h,h')\to(0,0),h+h'\neq 0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

T7) Suppose $x \in [0,1]$, such that for any $n \in \mathbb{N}$, $2^n x \notin \mathbb{Z}$. For every $n \in N$, define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as follows:

$$\lfloor 2^n x \rfloor = 2^n(x - h'_n),\ \lfloor 2^n x \rfloor + 1 = 2^n(x + h_n).$$

Prove that for an arbitrary n, $h_n + h'_n = 2^{-n}$ and for every integer $1 \leqslant \ \leqslant n-1$, the interval $(2(x-h'_n), 2(x+h_n))$ does not include integers or half-integers.
Proof: $1 = 2^n(x+h_n) - 2^n(x-h'_n) = 2^n(h_n + h'_n)$, hence $h_n + h'_n = 2^{-n}$. For any integer $1 \leqslant \ \leqslant n-1$, $2(x-h'_n) = \lfloor 2^n x \rfloor \cdot 2^{-n}$ and $2(x+h_n) = (\lfloor 2^n x \rfloor + 1)2^{-n}$. Since $n - \ \geqslant 1$, the interval does not include integers or half-integers.

T8) Follow the notations of T7), prove that the sequence $\left\{\frac{T(x+h_n)-T(x-h'_n)}{h_n+h'_n}\right\}_{n\geqslant 1}$ diverges.

Proof: Let $t = \lfloor 2^n x \rfloor$, then

$$a_n = \frac{T(x+h_n) - T(x-h'_n)}{h_n + h'_n} = \sum_{k=0}^{n-1} 2^{n-k}\left(\psi\left(\frac{t+1}{2^{n-k}}\right) - \psi\left(\frac{t}{2^{n-k}}\right)\right).$$

Since the interval $(2^{k-n}(t+1), 2^{k-n}t)$ does not contain any integers or half-integers, $2^{n-k}(\psi(2^{k-n}(t+1)) - \psi(2^{k-n}t)) \in \{-1, 1\}$, so $a_n \in \mathbb{Z}$ and n, a_n have the same parity. Therefore the sequence $\{a_n\}_{n\geqslant 1}$ diverges.

T9) Prove that $T(x)$ is continuous but nowhere differentiable on $\mathbb{R}$.

Proof: For any $x \in [0,1]$, if $x = k_0 \cdot 2^{-m_0}$ as in T4), by T4) the sequence $\left\{\dfrac{T(x+h_n) - T(x)}{h_n}\right\}$ diverges, hence T is not differentiable at x. Otherwise for any $n \in \mathbb{N}$, $2^n x \notin \mathbb{Z}$. Define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as in T7), then by T8), the sequence $\left\{\dfrac{T(x+h_n) - T(x-h'_n)}{h_n + h'_n}\right\}_{n\geqslant 1}$ diverges. Combined with T5) we know that T is not differentiable at x. Therefore T is nowhere differentiable on $\mathbb{R}$, since T is periodic.
For any x, y in $\mathbb{R}$,

$$|T(x) - T(y)| \leqslant \sum_{k=0}^{N} 2^{-k}|T(2^k x) - T(2^k y)| + \sum_{k=N+1}^{\infty} 2^{-k} \leqslant 2 \max_{0\leqslant k\leqslant N} |T(2^k x) - T(2^k y)| + 2^{-N}.$$

Hence for any $N > 0$, when $\varepsilon \to 0$, $|T(x) - T(x+\varepsilon)| \leqslant 2^{1-N} \to 0$, so T is (uniformly) continuous on $\mathbb{R}$.

T10) Prove that

$$T(x) = \begin{cases} 2x + \frac{T(4x)}{4}, & 0 \leqslant x < \frac{1}{4}; \\ \frac{1}{2} + \frac{T(4x-1)}{4}, & \frac{1}{4} \leqslant x < \frac{1}{2}; \\ \frac{1}{2} + \frac{T(4x-2)}{4}, & \frac{1}{2} \leqslant x < \frac{3}{4}; \\ 2 - 2x + \frac{T(4x-3)}{4}, & \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

Proof: If $0 \leqslant x < 1/4$, then

$$T(x) = \psi(x) + \psi(2x)/2 + \sum_{k=2}^{\infty} \psi(2^k x)2^{-k} = 2x + \frac{T(4x)}{4}.$$

The other cases are exactly the same.

T11) Let $\Gamma = \{(x, T(x)) : 0 \leqslant x \leqslant 1\} \subset \mathbb{R}^2$**. Define** $\Phi_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Phi_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \Phi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$
$$\Phi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \qquad \Phi_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix}.$$

Prove that Φ_i maps Γ to $\left\{(x, T(x)) : x \in \left[\frac{i}{4}, \frac{i+1}{4}\right]\right\}$.
Proof: Consider $(x, T(x)) \in \Gamma$, then by T10),

$$\Phi_0 \begin{pmatrix} x \\ T(x) \end{pmatrix} = \begin{pmatrix} x/4 \\ x/2 + T(x)/4 \end{pmatrix} = \begin{pmatrix} x/4 \\ T(x/4) \end{pmatrix}.$$

Hence $\Phi_0(\Gamma) = \{(x, T(x)) : x \in [0, 1/4]\}$. The cases $i = 1, 2, 3$ are similar.

T12) Let $S_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$**. For every** $n \geqslant 0$**, define** $S_{n+1} = \bigcup_{k=0}^{3} \Phi_k(S_n)$**. Prove that** S_n **is a sequence of monotonically decreasing compact sets and** $\Gamma = \bigcap_{n \geqslant 0} S_n$**.**

Proof: Let $S_n(x) = \{y \in [0, 1] : (x, y) \in S_n\}$. We prove by induction that $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$ for any $x \in [0, 1]$. The base $n = 0$ is trivial. Suppose $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$, then consider S_{n+1}. Note that $\Phi_k(S_n)$ are disjoint, since for any $(x, y) \in \Phi_k(S_n)$, $x \in [k/4, (k+1)/4]$. Hence for any $x \in [0, 1/4]$, $S_{n+1}(x) = \{y : (x, y) = \Phi_0(4x, z), z \in S_n(4x)\} = \{2x + z/4 : z \in S_n(4x)\}$ is a closed interval containing $T(x) = 2x + T(4x)/4$. By the induction hypothesis $S_n(x) = \{2x + z/4 : z \in S_{n-1}(4x)\}$ and $S_n(4x) \subset S_{n-1}(4x)$ so $S_{n+1}(x) \subset S_n(x)$. The case $x \in [1/4, 1]$ is similar. Therefore $S_{n+1} \subset S_n$ and S_{n+1} is compact, so by induction $S_n \subset S_{n-1}$ for all $n > 0$ and S_n is compact.
Clearly $\Gamma \subset \bigcap_{n \geqslant 0} S_n$, so it suffices to show that $|S_n(x)| \to 0$ for all $x \in [0, 1]$. From the proof above we get $\sup_{x \in [0,1]} |S_n(x)| \leqslant \sup_{x \in [0,1]} |S_{n-1}(x)|/4$, hence $|S_n(x)| \to 0$, therefore

$$\Gamma = \bigcap_{n \geqslant 0} S_n.$$

T13) Prove that $\sup_{x \in \mathbb{R}} T(x) \geqslant \frac{2}{3}$**.**

Proof: For any $(x, y) \in \Gamma$, by T11) we know that $(x/4 + 1/4, y/4 + 1/2) \in \Gamma$, hence if $a = \sup_{x \in \mathbb{R}} T(x)$ then $a \geqslant a/4 + 1/2$, i.e. $a \geqslant 2/3$.

T14) Find a $c \in [0, 1]$ **such that** $T(c) = \frac{2}{3}$**.**

Solution: Consider $c = 1/3$, then by T10), $T(c) = T(c)/4 + 1/2$, hence $T(c) = \frac{2}{3}$.

T15) For $x \in [0, 1]$**, suppose** $x = \sum_{n=1}^{\infty} b_n 4^{-n}$**, where** $b_n \in \{0, 1, 2, 3\}$**. Prove that**

$$\left\{x \in [0, 1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0, 1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1, 2\}\right\}.$$

Proof: If $x = \sum_{n=1}^{\infty} b_n 4^{-n}$, where $b_n \in \{1, 2\}$, then by T10),

$$T(x) = \frac{1}{2} + \frac{1}{4} T\left(\sum_{n=1}^{\infty} b_{n+1} 4^{-n}\right) = \cdots = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) = \frac{2}{3}.$$

Otherwise take the least n such that $b_n \in \{0,3\}$, denote $y = \sum_{k=1}^{\infty} b_{n+k-1}4^{-n}$, then

$$T(x) = \frac{1}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-2}}\right) + \frac{\min\{2y, 2-2y\}}{4^{n-1}} + \frac{1}{4^n}T(4y - b_n) < \frac{2}{3},$$

since $T(4y - b_n) \leqslant 2/3$ and $\min\{2y, 2-2y\} < 1/2$. Therefore

$$\left\{x \in [0,1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0,1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1,2\}\right\}.$$

T16) As in T11), study the self-similarity of Φ_1, Φ_2 on $\left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, which is a cantor set of Hausdorff dimension $\frac{1}{2}$.

Solution: Same as T11), denote $\Gamma' = \left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, then

$$\Phi_1(\Gamma') = \left\{(x, T(x)) : x \in \left[0, \frac{1}{2}\right], T(x) = \frac{2}{3}\right\},\ \Phi_2(\Gamma') = \left\{(x, T(x)) : x \in \left[\frac{1}{2}, 1\right], T(x) = \frac{2}{3}\right\}.$$

7 Homework 7: Émile Borel Lemma

7.1 PSA

f is a function on the interval I.

A1) Suppose f is twice-differentiable at x, prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Proof: For any $h > 0$, consider the function $g(t) = f(t) - f(t-h)$, then there exists $\xi \in [0,h]$ such that $g(x+h) = g(x) + hg'(\xi)$, and there exists $\eta \in [\xi - h, \xi] \subset [-h, h]$ such that $f'(\xi) - f'(\xi - h) = hf''(\eta)$

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \frac{f'(\xi) - f'(\xi - h)}{h} = f''(\eta),$$

therefore

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

A2) Suppose $x_0 \in I$, and

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + \cdots + a_n(x-x_0)^n + o(|x-x_0|^n) \\ &= b_0 + b_1(x - x_0) + \cdots + b_n(x-x_0)^n + o(|x-x_0|^n). \end{aligned}$$

when $x \to x_0$, then for any $i = 0, 1, \cdots, n$, $a_i = b_i$.
Proof: Otherwise let $c_i = a_i - b_i$ and take the least k such that $c_k \neq 0$, then

$$c_k(x-x_0)^k + \cdots + c_n(x-x_0)^n + o(|x-x_0|^n) = 0 \implies c_k = -c_{k+1}(x-x_0) - \cdots - c_n(x-x_0)^{n-k} + o(|x-x_0|^{n-k}),$$

which leads to contradiction when $x \to x_0$.

A3) Suppose f is n-times differentiable at 0. Prove that if f is an even (odd) function, then the Taylor expansion of f at 0 has only even (odd) terms.

Proof: Use the fact that if f is even (odd) then f' is odd (even).

A4) If f is differentiable on (a,b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ prove that exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$.

Proof: Otherwise if $f'(x) \neq 0$ for all $x \in (a,b)$, by Darboux's theorem $f'(x)$ have the same sign over (a,b), hence f is monotonic and non-constant on (a,b), contradicting $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$.

A5) Suppose $f \in C([a,b])$ and is differentiable on (a,b). Prove that f is strictly increasing on $[a,b]$ iff for any $x \in (a,b)$, $f'(x) \geqslant 0$ and on any sub-interval $(c,d) \subset (a,b)$, $f'(x)$ does not vanish.

Proof: $\Longrightarrow$ For any $x \in (a,b)$, $(f(x+h) - f(x))/h \geqslant 0$ so

$$f'(x) = \lim_{h\to\infty} \frac{f(x+h) - f(x)}{h} \geqslant 0.$$

If $f'(x)$ vanish on some sub-interval (c,d) then $f(c) = f(d)$, a contradiction.
$\Longleftarrow$ For any $a \leqslant x < y \leqslant b$, there exists $\xi \in (a,b)$ such that $f(y) - f(x) = (y-x)f'(\xi)$, hence $f(y) \geqslant f(x)$ and f is increasing. If $f(x) = f(y)$ for some $x < y$ then $f(t)$ is constant on $[x,y]$ and hence f' vanish on (x,y), a contradiction.

7.2 PSB

Use L'Hôpital theorem to calculate limits:

B1) $a > 0$, then

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to\infty} \frac{x^{-1}}{ax^{a-1}} = 0.$$

B2) $a > 0, b > 1$ then

$$\lim_{x\to\infty} \frac{x^a}{b^x} = \lim_{x\to\infty} \frac{ax^{a-1}}{b^x \ln b} = \cdots = \lim_{x\to\infty} \frac{a(a-1)\cdots\{a\}}{b^x (\ln b)^{\lfloor a \rfloor} x^{1-\{a\}}} = 0.$$

B3)

$$\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx} = \lim_{x\to 0} \frac{ae^{ax} - be^{bx}}{a\cos ax - b\cos bx} = 1.$$

B4)

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x\to 0} \frac{1 + \cos x}{\cos^2 x} = 2.$$

B5)

$$\lim_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2}=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{\sin x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\cos x}{2\cos x-x\sin x}=\frac{1}{2}.$$

B6)

$$\lim_{x\to 1}\frac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{4/3}}=\lim_{x\to 1}\frac{(2x-x^4)^{-1/2}(1-2x^3)-x^{-2/3}/3}{-\frac{4}{3}x^{1/3}}=1.$$

B7)

$$\lim_{x\to 1^-}(\log x)(\log(1-x))=\lim_{x\to 1^-}\frac{\log(1-x)}{1/\log x}=\lim_{x\to 1^-}\frac{x\log^2 x}{1-x}=0.$$

B8)

$$\lim_{x\to 0^+}\frac{\log\sin ax}{\log\sin bx}=\lim_{x\to 0^+}\frac{\sin bx}{\sin ax}\cdot\frac{a\cos ax}{b\cos bx}=1.$$

B9)

$$\lim_{x\to 0^+}x^x=\exp\lim_{x\to 0^+}\frac{\log x}{x^{-1}}=\exp\lim_{x\to 0^+}-x=1.$$

B10)

$$\lim_{x\to 1}x^{1/(1-x)}=\exp\lim_{x\to 1}\frac{\log x}{1-x}=e^{-1}.$$

B11)

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\log x}-\frac{1}{x-1}\right)&=\lim_{x\to 1}\frac{x-1-\log x}{(x-1)\log x}=\lim_{x\to 1}\frac{1-x^{-1}}{1-x^{-1}+\log x}\\&=\lim_{x\to 1}\frac{x-1}{x-1+x\log x}=\frac{1}{2}.\end{aligned}$$

B12)

$$\lim_{x\to 0^+}(\sin x)^x=\exp\lim_{x\to 0^+}\frac{\log\sin x}{x^{-1}}=\exp\lim_{x\to 0^+}-\frac{x^2}{\tan x}=1.$$

B13)

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/(1-\cos x)} &= \exp\lim_{x\to 0}\frac{\log\sin x-\log x}{1-\cos x} = \exp\lim_{x\to 0}\frac{\cot x - x^{-1}}{\sin x}\\
&= \exp\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x} = \exp\lim_{x\to 0}\frac{-x\sin x}{\sin^2 x + x\sin 2x}\\
&= e^{-1/3}.
\end{aligned}
$$

B14)

$$
\lim_{x\to a}\frac{a^x-x^a}{x-a} = \lim_{x\to a}\frac{a^x\log a - ax^{a-1}}{1} = a^a(\log a - 1).
$$

B15)

$$
\begin{aligned}
\lim_{x\to\infty}\frac{(1+1/x)^x-e}{1/x} = \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} &= \lim_{x\to 0}(1+x)^{1/x}\cdot\frac{x/(x+1)-\log(1+x)}{x^2}\\
= e\lim_{x\to 0}\frac{(x+1)^{-2}-(x+1)^{-1}}{2x} &= \frac{e}{2}.
\end{aligned}
$$

B16)

$$
\lim_{x\to\infty}\frac{x^{\log x}}{(\log x)^x} = \exp\lim_{x\to\infty}(\log x)^2 - x\log\log x = 0.
$$

B17)

$$
\begin{aligned}
&\lim_{x\to\infty}(x+a)^{1+1/x}-x^{1+1/(x+a)} = \lim_{x\to\infty}\frac{(x+a)^{1+1/x}x^{-1}-x^{1/(x+a)}}{x^{-1}}\\
=&a.
\end{aligned}
$$

B18)

$$
\lim_{x\to\infty}\sqrt[3]{x^3+x^2+x+1}-\sqrt{x^2+x+1}\cdot\frac{\log(e^x+x)}{x} = -\frac{1}{6}.
$$

(Using WolframAlpha)

7.3 PSC

Calculate the maximum and minimum values of the following functions:
1.$f(x) = x^4 - 2x^2 + 5$, $x \in [-2, 2]$.
$f(x) = (x^2 - 1)^2 + 4 \in [4, 13]$.
2.$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$
$1 - f(x) = (1 + x^2)^{-1}(x - 1)^2 \geqslant 0$, $f(x) + 1 = (1 + x^2)^{-1}(x + 1)^2 \geqslant 0$, therefore $f(x) \in [-1, 1]$.
3.$f(x) = \arctan x - \frac{1}{2}\log(1 + x^2)$, $x \in \mathbb{R}$.
$f'(x) = \frac{1-x}{x^2+1}$, hence $\sup_{x\in\mathbb{R}} f(x) = f(1) = \frac{\pi}{4} - \frac{\log 2}{2}$, and f has no minimum.
4.$f(x) = x \log x$, $x \in (0, \infty)$.
$f'(x) = \log x + 1$, hence $\inf_{x\in(0,\infty)} f(x) = f(e^{-1}) = -e^{-1}$, and f has no maximum.
5.$f(x) = \sqrt{x} \log x$,$x \in (0, \infty)$.
$f'(x) = x^{-1/2}\left(1 + \frac{\log x}{2}\right)$, hence $\inf_{x\in(0,\infty)} f(x) = f(e^2) = -2e^{-1}$.
6.$f(x) = 2\tan x - \tan^2 x$, $x \in [0, \pi/2)$.
$f(x) = 1 - (1 - \tan x)^2 \in (-\infty, 1]$.

7.4 PSD

f is differentiable on (a, b). Suppose $x_0 \in (a, b)$ and $f'(x_0) = 0$.

D1) Prove that $f(x_0)$ is a local maximum if there exists $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ such that

$$f'(x) = \begin{cases} > 0, & \forall x \in (x_0 - \delta, x_0), \\ < 0, & \forall x \in (x_0, x_0 + \delta). \end{cases}$$

Proof: Trivial by Lagrange mean-value theorem.

D2) Prove that if $f''(x_0)$ exists and $f''(x_0) < 0$ then $f(x_0)$ is a local maximum.

Proof: $f''(x_0) < 0$ and $f'(x_0) = 0$ implies for some $\delta > 0$, $f'(x) < 0$ for $x \in (x_0, x_0 + \delta)$ and $f'(x) > 0$ for $x \in (x_0 - \delta, x_0)$. Hence by D1), $f(x_0)$ is a local maximum.

D3) Suppose f is n-times differentiable at x_0, $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Determine the conditions that $f(x_0)$ is a local maximum.

Solution: n is even and $f^{(n)}(x_0) < 0$.

7.5 PSE: Roots of Polynomials

E1) Prove that if all the roots of the polynomial $P_n(x) \in \mathbb{R}[x]$ are real numbers, then so are the polynomials $P_n'(x), P_n''(x), \cdots, P_n^{n-1}(x)$, where $n = \deg P_n$.

Proof: We only need to prove that P_n' has $n - 1$ real roots. By Rolle's mean-value theorem, between any two roots of P_n there is a root of P_n' hence P_n' has $n - 1$ real roots.

E2) Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$ has n different roots in the interval $(-1, 1)$.

Proof: We know that the polynomials $\sqrt{(2n + 1)/2}P_n(x)$ form a set of orthogonal base on the space $L^2([-1, 1])$, hence it must have n different roots in the interval $(-1, 1)$.

E3) Prove that the Laguerre polynomial $L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$ has n different real roots.

Proof: We know that the Laguerre polynomials are orthogonal on the space $L^2([0,\infty))$ with weight e^{-x}, hence it must have n distinct roots.
Or note that $f(x) = x^n e^{-x}$ has a root with multiplicity n at 0 and it vanishes at ∞, hence use Rolle's theorem and induction we can show that $f^{(k)}(x)$ has a root with multiplicity $n-k$ at 0 and k roots between 0 and ∞.

E4) Prove that the Hermite polynomial $H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$ has n different real roots.

Proof: We know that the polynomials $H_n(x)/\sqrt{2^n n!\sqrt{n}}$ form a set of orthogonal base on the Hilbert space $L^2(\mu)$ where $\mu(\mathrm{d}x) = e^{-x^2}\mathrm{d}x$, hence it must have n distinct real roots.

7.6 PSF: Émile Borel's Lemma

Part 1:

F1) Define $\phi : \mathbb{R} \to \mathbb{R}$:

$$\phi(x) = \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Prove that $\phi \in C^\infty(\mathbb{R})$.
Proof: We prove by induction that for any $n \in \mathbb{Z}_{\geqslant 0}$, there is a polynomial $P_n \in \mathbb{R}[x]$ such that

$$\phi^{(n)}(x) = \begin{cases} P_n(1/x)\cdot e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

(Which implies $\phi^{(n)}$ is continuous.)
The case $n = 0$ is trivial. Suppose it holds for n, then for any $x > 0$,

$$\phi^{(n+1)}(x) = e^{-1/x^2}\left(P_n(1/x)\frac{2}{x^3} - P_n'(1/x)\frac{1}{x^2}\right),$$

for any $x < 0$, $\phi^{(n+1)}(x) = 0$, and for $x = 0$,

$$\phi_+^{(n+1)}(0) = \lim_{x\to 0^+} e^{-1/x^2}P_n(1/x)\frac{1}{x} = 0.$$

Hence the claim holds for $n+1$ too.
Therefore $\phi \in C^\infty(\mathbb{R})$.

F2) Define $\chi : \mathbb{R} \to \mathbb{R}$:

$$\chi(x) = \frac{\phi(2-|x|)}{\phi(2-|x|) + \phi(|x|-1)}.$$

Prove that $\chi(x) \in C^\infty(\mathbb{R})$ and $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $0 \leqslant \chi(x) \leqslant 1$ and χ is an even function.
Proof: $2-|x|$ and $|x|-1$ cannot be both negative, hence the denominator is always positive, so $\chi \in C^\infty(\mathbb{R})$. The fact that $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $\chi(x) \in [0,1]$ and χ is even is trivial.

F3) Prove that for any $0 < a < b$, there exists a smooth function $\rho(x) \in C^\infty(\mathbb{R})$ such that $\rho|_{[-a,a]} \equiv 1$, $\rho|_{(-\infty,-b]\cup[b,\infty)} \equiv 0$, and $0 \leqslant \rho(x) \leqslant 1$.

Proof: Same as F2), define

$$\rho(x) = \frac{\phi(b-|x|)}{\phi(b-|x|)+\phi(|x|-a)}.$$

F4) Prove that there exists an even function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi|_{\{x:|x|\leqslant 1\}} \equiv 1$, $\psi|_{\{x:|x|\geqslant 2\}} \equiv 0$, and $0 \leqslant \psi(x) \leqslant 1$.

Proof: (A special case of Urysohn's lemma)
Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(\mathbf{x}) = \phi(1-|\mathbf{x}|^2)$ and $g : \mathbb{R}^n \to \mathbb{R}$ as $g(\mathbf{x}) = \phi(|x^2|/4-1)$, then f vanishes on $B(0,1)^C$ and g vanishes on $\bar{B}(0,2)$. Therefore

$$\psi(\mathbf{x}) = \frac{f(\mathbf{x})}{f(\mathbf{x})+g(\mathbf{x})}$$

satisfy the requirements.

Part 2: Interchanging $\sum$ and $\frac{\mathrm{d}}{\mathrm{d}x}$

$I = [a,b]$ is a closed interval, $\{f_k\}_{k\geqslant 0}$ is a sequence of functions in $C^1(I)$. Assume $\sum_{k=0}^\infty f_k$ converges point-wise on I, and let $f(x) = \sum_{k=0}^\infty f_k(x)$.

F5) Assume the series $\sum_{k=0}^\infty f_k'(x)$ converges absolutely on I, i.e. $\sum_{k=0}^\infty \|f_k'\|_\infty$ converges. Prove that f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Note that

$$\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty \frac{f_k(x+h)-f_k(x)}{h} = \sum_{k=0}^\infty f_k'(x+\xi_k).$$

Hence

$$\left|\frac{f(x+h)-f(x)}{h} - \sum_{k=0}^\infty f_k'(x)\right| \leqslant \sum_{n=0}^N |f_k'(x+\xi_n) - f_k'(x)| + 2\sum_{n=N+1}^\infty \|f_k'\|$$

Note that f_k' is uniformly continuous, so

$$\lim_{h\to 0}\sum_{n=0}^N |f_k'(x+\xi_k) - f_k'(x)| = 0,\ \lim_{N\to\infty} 2\sum_{n=N+1}^\infty \|f_k'\| = 0.$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty f_k'(x).$$

F6) Assume $\sum_{k=0}^\infty f_k'(x)$ converges uniformly on I, then f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Let $g(x) = \sum_{k=0}^\infty f_k'(x)$, since the series converges uniformly, $g(x)$ is continuous on I. By Lebesgue's Dominated Convergence Theorem,

$$\int_{x_0}^x g(t)\,\mathrm{d}t = \sum_{k=0}^\infty f_k(t)\Big|_{x_0}^x = f(x) - f(x_0).$$

Hence $f'(x) = g(x) = \sum_{k=0}^\infty f_k'(x)$.

F7) Calculate the derivative of e^x using F6).

Solution: On any closed interval $[-M, M]$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

converges uniformly. Hence

$$(e^x)' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Part 3: Borel's Lemma

Given an arbitrary sequence $\{a_k\}_{k\geqslant 0}$.

F8) For any $t_k > 0$, let $f_k(x) = \frac{a_k}{k!} x^k \chi(t_k x)$, determine the derivatives of any order of f_k at $x = 0$.

Solution: Note that when $x = 0$, $\chi^{(m)}(t_k x) = 0$ for any $m \geqslant 1$ and $\chi(t_k x) = 1$. Hence

$$f_k^{(n)}(0) = \frac{a_k}{k!} \sum_{j=0}^{n} \binom{n}{j} (x^k)^{(j)} \chi^{(n-j)}(t_k x)\Big|_{x=0} = \frac{a_k}{k!} (x^k)^{(n)}\Big|_{x=0} = a_k \delta_{n,k}.$$

F9) Prove that when $k \geqslant 2n$,

$$f_k^{(n)}(x) = a_k \sum_{=0}^{n} \binom{n}{} \frac{t_k^{n-}}{(k-)!} x^{k-} \chi^{(n-)}(t_k x).$$

Proof: Leibniz's Formula.

F10) (Borel's lemma) Prove that for any sequence $\{a_k\}_{k\geqslant 0}$, there exists a smooth function f on $\mathbb{R}$, such that for any $k \geqslant 0$, $f^{(k)}(0) = a_k$.

Proof: Let $f_k(x) = \frac{a_k}{k!} x^k \chi(t_k x)$ where t_k is yet to be determined, and

$$f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \chi(t_k x).$$

For any $n \geqslant 0$, we want to show that $\sum_{k=0}^{\infty} f_k^{(n)}(x)$ converges uniformly on $\mathbb{R}$.
Suppose $M_n = \sup_{x\in\mathbb{R}, m\leqslant n} |\chi^{(m)}(x)|$, and

$$C_k = \sup_{n<k/2} \sum_{=0}^{n} \frac{\binom{n}{}}{(k-)!},$$

then for any $x \in \mathbb{R}$,

$$|f_k^{(n)}(x)| \leqslant |a_k| C_k M_k t_k^{-k/2}.$$

Hence if we choose t_k such that

$$|a_k| C_k M_k t_k^{-k/2} < 2^{-k},$$

then the series

$$\sum_{k=0}^{\infty} f_k^{(n)}(x)$$

converges uniformly on $\mathbb{R}$. By F6) we know that $f(x) = \sum_{k=0}^{\infty} f_k(x)$ is smooth, and by F8) we obtain $f^{(n)}(0) = a_n$ for any $n \geqslant 0$,

Part 4: Peano's Proof

F11) $\{c_k\}$ and $\{b_k\}$ are two sequences, and $b_k > 0$. Prove that

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = \begin{cases} n!(-1)^j c_{n-2j} b_{n-2j}^j, & \text{if } k = n-2j, j \in \mathbb{Z}_{\geqslant 0}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: For $x \to 0$,

$$\frac{c_k x^k}{1+b_k x^2} = c_k \sum_{n=0}^{\infty} (-1)^n x^{2n+k} b_k^n.$$

Which converges absolutely on the interval $[-b_k^{-1/2}/2, b_k^{-1/2}/2]$, and so are its n-times derivations, hence by F5)

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = c_k \sum_{j=0}^{\infty} (-1)^j \frac{(2j+k)!}{(2j+k-n)!} x^{2j+k-n} b_k^j \Big|_{x=0} = \begin{cases} n!(-1)^j c_k b_k^j, & k = n-2j, \\ 0, & \text{otherwise.} \end{cases}$$

F12) Prove that there is a constant C such that for any $k \geqslant n+2$, and any x,

$$\left|\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(x)\right| \leqslant C(n+1)! \frac{|c_k| k!}{b_k} |x|^{k-n-2}.$$

Proof: Use du Bois-Reymond, we can let $C = 1$.

F13) Prove that for a given $\{c_k\}$, we can choose $\{b_k\}$ such that b_k depends only on the value of c_k, and the function

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$$

is infinitely differentiable.

Proof: Let $b_k = (k!)^2 c_k$, then by F12),

$$\left|\sum_{k \geqslant n+2} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}\right| \leqslant (n+1)! \sum_{k \geqslant n+2} \frac{|x|^{k-n-2}}{k!}$$

hence the series

$$\sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}$$

converges uniformly for any $n \geqslant 1$. By F6) the function $f(x)$ is infinitely differentiable, and

$$f^{(n)}(x) = \sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}.$$

F14) Prove that $f(0) = c_0, f'(0) = c_1$ and when $n \geqslant 2$,

$$\frac{f^{(n)}(0)}{n!} = c_n + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j.$$

Proof: Combine F11) and F13).

F15) Prove that by carefully choosing $\{c_k\}$ and $\{b_k\}$, we can prove Borel's lemma.

Proof: Let $b_k = (k!)^2 c_k$ and define c_k inductively such that

$$c_n = \frac{a_n}{n!} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j$$

Then let $f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$

7.7 PSG: Midterm Test Part B

Consider $f : \mathbb{R} \to \mathbb{R}$.

- Let $\mathcal{B}$ be all bounded function on $\mathbb{R}$.
- Let $\mathcal{L}$ be all Lipschitz functions on $\mathbb{R}$.
  Suppose $a, \lambda \in \mathbb{R}$, $f \in \mathcal{B} \cap \mathcal{L}$, the goal is to find a function $F \in \mathcal{L}$ to solve:

$$F(x) - \lambda F(x+a) = f(x),\ x \in \mathbb{R}. \tag{$\star$}$$

Part 1: Basic Properties of Lipschitz Functions

B1) Prove that if $f, g \in \mathcal{B} \cap \mathcal{L}$, then $fg \in \mathcal{L}$.

Proof: Suppose $|f(x) - f(y)|, |g(x) - g(y)| \leqslant A|x-y|$, and $|f(x)|, |g(x)| \leqslant C$, then for any $x, y \in \mathbb{R}$,

$$|f(x)g(x) - f(y)g(y)| \leqslant 2MA|x-y|.$$

Hence $fg \in \mathcal{L}$.

B2) Prove that if f is differentiable and $f \in \mathcal{L}$ then $f' \in \mathcal{B}$.

Proof: If $|f(x) - f(y)| \leqslant C|x-y|$ then for any $x \in \mathbb{R}$,

$$|f'(x)| = \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leqslant C.$$

Hence $f' \in \mathcal{B}$.

B3) Prove that if f is differentiable and $f' \in \mathcal{B}$ then $f \in \mathcal{L}$.

Proof: For any $x, y \in \mathbb{R}$, there exists $\xi \in (x, y)$ such that

$$|f(x) - f(y)| = |x-y| \cdot |f'(\xi)| \leqslant \sup_{t \in \mathbb{R}} |f'(t)| \cdot |x-y|.$$

Hence $f \in \mathcal{L}$.

B4) If $f \in \mathcal{B}$ and there exists $B > 0$ such that for any $x, y \in \mathbb{R}$, $|x-y| \leqslant 1$ implies $|f(x) - f(y)| \leqslant B|x-y|$. Prove that $f \in \mathcal{L}$.

Proof: Suppose $M = \sup_{x\in\mathbb{R}} |f(x)|$, then for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant \max\{B, 2M\}|x-y|.$$

Hence $f \in \mathcal{L}$.

Part 2: Solution of ($\star$) when $|\lambda| < 1$.

Suppose $f \in \mathcal{B} \cap \mathcal{L}$ and $|\lambda| < 1$.

B5) Suppose F satisfy ($\star$). Prove that for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geqslant 1}$,

$$F(x) = \lambda^n F(x+na) + \sum_{k=0}^{n-1} \lambda^k f(x+ka),$$
$$F(x) = \lambda^{-n} F(x-na) - \sum_{k=1}^{n} \lambda^{-k} f(x-ka).$$

Proof: Use induction and apply ($\star$).
(Let $n \to \infty$ and we can obtain F formally.)

B6) Prove that for any $x \in \mathbb{R}$, $\sum_{k\geqslant 0} \lambda^k f(x+ka)$ converges.

Proof: Since f is bounded,

$$\left|\sum_{k=n}^{n+p} \lambda^k f(x+ka)\right| \leqslant \frac{M\lambda^n}{1-\lambda}.$$

Hence the series converges.

B7-8) Let $F(x) = \sum_{k\geqslant 0} \lambda^k f(x+ka)$. Prove that $F \in \mathcal{L}$ and solve ($\star$).

Proof: For any $x, y \in \mathbb{R}$,

$$|F(x) - F(y)| \leqslant \sum_{k=0}^{\infty} \lambda^k |f(x+ka) - f(y+ka)| \leqslant \sum_{k=0}^{\infty} \lambda^k C|x-y| = \frac{C}{1-\lambda}|x-y|.$$

Hence $F \in \mathcal{L}$. For any $x \in \mathbb{R}$,

$$F(x) - \lambda F(x+a) = \sum_{k\geqslant 0} \lambda^k f(x+ka) - \sum_{k\geqslant 1} \lambda^k f(x+ka) = f(x).$$

Therefore F solves ($\star$).
If F' also solves ($\star$), let $G = F - F'$, then G is bounded and

$$G(x) = \lambda G(x+a),\ x \in \mathbb{R}.$$

Therefore for any $x \in \mathbb{R}$,

$$|G(x)| = \lambda^n |G(x+na)| \leqslant M\lambda^n \to 0.$$

Hence $G \equiv 0$ and $F \equiv F'$, so the solution to ($\star$) is F.

B9) Solve $(\star)$ when $f(x) \equiv 1$ and $f(x) = \cos x$.

Solution: When $f(x) \equiv 1$,

$$F(x) = \sum_{k=0}^{\infty} \lambda^k f(x+ka) = \frac{1}{1-\lambda}.$$

When $f(x) = \cos x$,

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} \lambda^k \cos(x+ka) = \sum_{k=0}^{\infty} \lambda^k \frac{e^{i(x+ka)} + e^{-i(x+ka)}}{2} = \frac{1}{2}\left(\frac{e^{ix}}{1-\lambda e^{ia}} + \frac{e^{-ix}}{1-\lambda e^{-ia}}\right) \\ &= \frac{\cos x - \lambda \cos(x-a)}{1 - 2\lambda \cos a + \lambda^2}. \end{aligned}$$

Part 3: Solution of $(\star)$ when $|\lambda| > 1$.

B10) Solve $(\star)$ as in Part 2.

Solution: By B5), the solution should be

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka).$$

$f \in \mathcal{B}$ implies the series converges. Same as B8) we can show that the solution to $(\star)$ is unique, and like B7) we can show that $F \in \mathcal{L}$ and F satisfy $(\star)$.

B11) Solve $(\star)$ for $f(x) \equiv 1$ and $f(x) = \cos x$.

Solution: When $f(x) \equiv 1$,

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka) = \frac{1}{1-\lambda}.$$

When $f(x) = \cos x$, same as B9) we have

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka) = \frac{\cos x - \lambda \cos(x-a)}{1 - 2\lambda \cos a + \lambda^2}.$$

Part 4: The Case when $|\lambda| = 1$.

B12) Suppose $\lambda = 1$. Prove that there exists $F \in \mathcal{L}$ not identically zero, such that for any x, $F(x) - F(x+a) = 0$.

Proof: Let $F(x) = |\{x/a\} - 1/2|$, then $F(x) = F(x+a)$, and $F \in \mathcal{L} \cap \mathcal{B}$.

B13) Let $f(x) = \cos x$ in $(\star)$. Prove that if $\cos a \neq 1$, then there exists $F \in \mathcal{L}$ that solves $(\star)$. Determine whether the solution is unique.

Proof: The equation $(\star)$ becomes $F(x) = F(x+a) + \cos x$. Let

$$F(x) = \{x/a\} - \sum_{k=0}^{\lfloor x/a \rfloor - 1} \cos(k + \{x/a\})a,$$

(if $x<0$ the sum is viewed as from $\lfloor x/a\rfloor-1$ to 0) then clearly $F(x)=F(x+a)+\cos x$, and F is bounded since $\cos a\neq 1$.
For any $x,y\in\mathbb{R}$, if $|x-y|<a/2$, then suppose $na\leqslant x<y<(n+1)a$,

$$\begin{aligned}|F(x)-F(y)| &\leqslant \left|\left\{\frac{x}{a}\right\}-\left\{\frac{y}{a}\right\}\right|+2\left|\sin\frac{\{x/a\}-\{y/a\}}{2}a\right|\cdot\left|\sum_{k=0}^{n-1}\sin(k+(\{x/a\}+\{y/a\})/2)a\right| \\ &\leqslant \frac{|x-y|}{a}+\frac{|x-y|}{|\sin a|}.\end{aligned}$$

Hence $F\in\mathcal{L}$ by B4), so F solves $(\star)$.
The solution is clearly not unique since we can add any factor of the F in B12) to the solution.

B14) Following B13), if $a=2\pi$, then $(\star)$ has no solution in $\mathcal{L}$.

Proof: If $a=2\pi$ and F is a solution to $(\star)$, then for any $x,y\in\mathbb{R}$,

$$|F(x+2\pi n)-F(y+2\pi n)|=n|\cos x-\cos y|\to\infty.$$

Hence $F\notin\mathcal{L}$.

B15) Suppose $\lambda=-1$, Prove that there exists $F\in\mathcal{L}$ not identically zero, such that for any x, $F(x)+F(x+a)=0$.

Proof: Let $F(x)=|2\{x/2a\}-1|-1/2$, then $F\in\mathcal{L}$ and $F(x)+F(x+a)=0$.

B16) Suppose $\lambda=-1$, $a=1$, $f\in\mathcal{L}$ is monotonically decreasing and $\lim_{x\to\infty}f(x)=0$, f is differentiable and f' is increasing. Prove that there exists $F\in\mathcal{L}$ such that

$$F(x)+F(x+1)=f(x),\ x\in\mathbb{R}.$$

Further show that if we require $F\in\mathcal{L}$ and $\lim_{x\to\infty}F(x)=0$, then the solution is unique.
Proof: Since f is monotonically decreasing, for any $x\in\mathbb{R}$, the series

$$F(x)=\sum_{n=0}^{\infty}(-1)^n f(x+n)$$

converges.
For any $x,y\in\mathbb{R}$, $|x-y|<1$, there exists $\xi_n\in(x+n,y+n)$ such that $f(y+n)-f(x+n)=(y-x)f'(\xi_n)$, hence (by B3) f' is bounded)

$$|F(x)-F(y)|=|y-x|\cdot\left|\sum_{n=0}^{\infty}(-1)^n f'(\xi_n)\right|\leqslant\sup_{t\in\mathbb{R}}|f'(t)|\cdot|y-x|.$$

so $F\in\mathcal{L}$. Clearly $F(x)+F(x+1)=f(x)$, so F solves $(\star)$, and $0<F(x)<f(x)$ so $\lim_{x\to\infty}F(x)=0$. If $F'\in\mathcal{L}$ also satisfy $(\star)$ and $\lim_{x\to\infty}F(x)=0$, let $G=F-F'$, then $G(x)+G(x+1)=0$ and $\lim_{x\to\infty}G(x)=0$. Hence $G(x)=\lim_{n\to\infty}(-1)^nG(x+n)=0$ for any $x\in\mathbb{R}$, so $G\equiv 0$. Therefore F is the unique solution.

8 Homework 8; Strum-Liouville Theory

8.1 PSA: Convex functions

A1)

(1) $f(x) = |x|$, $I = \mathbb{R}$ is convex, since

$$|\lambda x + (1-\lambda)y| \leqslant \lambda|x| + (1-\lambda)|y|.$$

(2) $f(x) = x^p$, $p \in \mathbb{R}$, $I = \mathbb{R}_{>0}$
$f''(x) = p(p-1)x^{p-2}$ so f is concave if $p \in [0,1]$ and convex if $p \in (-\infty, 0] \cup [1, \infty)$.
(3) $f(x) = \sin x$, $I = [0, \pi]$ is concave since $f''(x) = -\sin x \leqslant 0$ when $x \in [0, \pi]$.
(4) $f(x) = x \log x$, $I = \mathbb{R}_{\geqslant 0}$ is (strictly) convex since $f''(x) = 1/x > 0$.
(5) $f(x) = \mathbf{1}_{\{0,1\}}$, $I = [0,1]$ is convex since

$$f(\lambda x + (1-\lambda)y) = 0 \leqslant \lambda f(x) + (1-\lambda)f(y).$$

A2) Prove the following properties:

1. If f, g are convex on I, then $f + g$ is convex on I.
   Proof: By definition,$(f+g)(\lambda x + (1-\lambda)y) \leqslant \lambda(f+g)(x) + (1-\lambda)(f+g)(y)$,so $f+g$ is convex.

2. If f, g are monotonically increasing, non-negative, convex functions on I, then fg is convex.
   Proof: Note that

$$f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y) \leqslant (\lambda f(x) + (1-\lambda)f(y)) \cdot (\lambda g(x) + (1-\lambda)g(y))$$

and

$$\begin{aligned} &\lambda f(x)g(x) + (1-\lambda)f(y)g(y) - (\lambda f(x) + (1-\lambda)f(y))(\lambda g(x) + (1-\lambda)g(y)) \\ =&\lambda(1-\lambda)(f(x) - f(y))(g(x) - g(y)) \geqslant 0. \end{aligned}$$

hence

$$(fg)(\lambda x + (1-\lambda)y) \leqslant \lambda(fg)(x) + (1-\lambda)(fg)(y).$$

1. If f is convex on I, g is a monotonically increasing convex function on $J \supset f(I)$, then $g \circ f$ is convex.
   Proof: Note that

$$g(f(\lambda x + (1-\lambda)y)) \leqslant g(\lambda f(x) + (1-\lambda)f(y)) \leqslant \lambda g(f(x)) + (1-\lambda)g(f(y)).$$

hence $g \circ f$ is convex.

1. If f, g are convex on I, then $h(x) = \max\{f(x), g(x)\}$ is convex.
   Proof: For any x, y, λ and $t = \lambda x + (1-\lambda)y$, suppose $h(t) = f(t)$, then

$$h(t) \leqslant \lambda f(x) + (1-\lambda)f(y) \leqslant \lambda h(x) + (1-\lambda)h(y)$$

hence h is convex.

A3) Suppose $f \in C((a,b))$. If for any $x, y \in (a,b)$, $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$, prove that f is convex.

Proof: For any $x, y \in (a,b)$ and $\lambda \in [0,1]$, we need to prove that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

Note that it holds for any dyadic number λ, since the cases $\lambda = 0, 1, 1/2$ is trivial, and for $\lambda = (2m+1)/2^t$, let $u = m/2^{t-1}, v = (m+1)/2^{t-1}$, then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \frac{f(ux + (1-u)y) + f(vx + (1-v)y)}{2} \\ &\leqslant \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

Now since $f \in C((a,b))$, for any $\lambda \in (0,1)$ there is a sequence of dyadic numbers λ_n such that $\lim_{n\to\infty} \lambda_n = \lambda$, hence

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \lim_{n\to\infty} f(\lambda_n x + (1-\lambda_n)y) \leqslant \lim_{n\to\infty} \lambda_n f(x) + (1-\lambda_n) f(y) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

A4) f is a convex function on $[a,b]$. Prove that if there exists $c \in (a,b)$ such that $f(c) \geqslant \max\{f(a), f(b)\}$ then f is constant.

Proof: For any $t \in (a,b)$, let $\lambda = (t-a)/(b-a)$ then

$$f(t) = f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda) f(b) \leqslant \max\{f(a), f(b)\}.$$

By $f(c) \geqslant \max\{f(a), f(b)\}$ we know that $f(a) = f(b)$. If for some $t \in (a,b)$, $f(t) \neq f(a)$, suppose $c \in (a,t)$, then

$$f(c) = \lambda f(a) + (1-\lambda) f(t) < f(a)$$

a contradiction. Hence $f(t) = f(a)$ for all $t \in [a,b]$.

A5) f is convex on $\mathbb{R}$. Prove that if f has an upper-bound, then f is constant.

Proof: Otherwise suppose that $f(a) < f(b)$, where $a < b$. (If $f(a) > f(b)$ let $g(x) = f(-x)$). Let $x_0 = a, x_1 = b, x_n = a + n(b-a)$, then

$$f(x_{n+1}) - f(x_n) \geqslant f(x_n) - f(x_{n-1}) \geqslant f(b) - f(a),$$

hence $f(x_n) \geqslant f(a) + n(f(b) - f(a)) \to \infty$, leading to contradiction.

A6) f is strictly convex on I. Suppose $f(x_0)$ is a local minimum of f, prove that x_0 is the unique global minimum point of f.

Proof: Suppose there is another $x_1 \neq x_0$ such that $f(x_1) \leqslant f(x_0)$, then let $x_n = x_0 + n(x_1 - x_0)$. Since f is strictly convex, $f(x_n) < \max\{f(x_1), f(x_0)\} = f(x_0)$, contradicting the fact that $f(x_0)$ is a local minimum.

A7) I is an open interval. Prove that f is convex on I, iff for any $x_0 \in I$, there exists $a \in \mathbb{R}$, such that for any $x \in I$, $f(x) \geqslant a(x - x_0) + f(x_0)$.

Proof: Suppose f is convex on I, then the any $x_0 \in I$, the function $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$ is monotonically increasing. Hence we can let $a = \sup_{x<x_0} g(x) < \infty$.
If for any $x_0 \in I$, and and $x \in I$, $f(x) \geqslant g(x_0)(x - x_0) + f(x_0)$, then for any $x, y \in I$ and $\lambda \in (0, 1)$, let $t = \lambda x + (1 - \lambda)y$,

$$\begin{aligned}\lambda f(x) + (1-\lambda)f(y) &\geqslant \lambda(f(t) + (x-t)g(t)) + (1-\lambda)(f(t) + (1-\lambda)(y-t)g(t)) \\ &= f(t) = f(\lambda x + (1-\lambda)y).\end{aligned}$$

Hence f is convex.

8.2 PSB

B1) Prove the following inequalities:

(1)

$$x - \frac{x^2}{2} < \log(1+x) < x,\ x > 0.$$

Proof: If $f(x) = \log(x + 1) - x$, then $f'(x) = \frac{1}{x+1} - 1 < 0$ hence $f(x) < f(0) = 0$. Let $g(x) = \log(1 + x) - x + x^2/2$, then $g'(x) = \frac{1}{x+1} + (x + 1) - 2 \geqslant 0$, hence $g(x) > g(0) = 0$.
(2)

$$(x^\alpha + y^\alpha)^{1/\alpha} > (x^\beta + y^\beta)^{1/\beta},\ x, y > 0, \beta > \alpha > 0.$$

Proof: Assume that $x^\alpha + y^\alpha = 1$, then $0 < x, y < 1$, so

$$x^\beta + y^\beta < x^\alpha + y^\alpha < 1 \implies (x^\beta + y^\beta)^{1/\beta} < (x^\alpha + y^\alpha)^{1/\alpha}.$$

(3)

$$x - \frac{x^3}{6} < \sin x < x,\ x > 0.$$

Proof: Let $f(x) = \sin x - x$, then $f'(x) = \cos x - 1 \leqslant 0$, so $f(x) < f(0) = 0$. Let $g(x) = \sin x - x + x^3/6$, then $g'(x) = \cos x - 1 + x^2/2$, $g''(x) = x - \sin x > 0$, so $g'(x) > g(0) = 0$ and $g(x) > g(0) = 0$.
(4)

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leqslant \frac{1}{2}(1 + x^p),\ p \in [2, \infty), x \in [0, 1].$$

Proof: ???

B2) Find all $a > 0$ such that $a^x \geqslant x^a$ for any $x > 0$.

Solution: $f(x) = x^{1/x}$ then $f'(x) = x^{1/x}\frac{1-\log x}{x^2}$ hence f has a unique minimum at e.

B3) Prove that for any $x_i, t_i, i = 1, 2, \cdots, n$, $\sum_{i=1}^n t_i = 1$,

$$\left(\sum_{i=1}^n t_i x_i\right)^{\sum_{i=1}^n t_i x_i} \leqslant \prod_{i=1}^n x_i^{t_i x_i}.$$

Proof: Let $f(x) = x \log x$, then $f''(x) = 1/x > 0$, so f is convex. By Jensen's inequality,

$$\sum_{i=1}^n t_i f(x_i) \geqslant f\left(\sum_{i=1}^n t_i x_i\right)$$

hence

$$\left(\sum_{i=1}^n t_i x_i\right)^{\sum_{i=1}^n t_i x_i} \leqslant \prod_{i=1}^n x_i^{t_i x_i}.$$

and equality holds iff $x_i = x_1$.

B4) Prove that for any $a, b > 0$, $1/p + 1/q = 1$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1;\ ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1.$$

Proof: The function $-\log x$ is convex, so when $p > 1$, $q > 1$, then

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geqslant \frac{1}{p}\log a^p + \frac{1}{q}\log b^q$$

so $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.
When $p < 1$, then $pq < 0$, so likewise $ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}$.

B5) Prove that if $x_i, y_i \geqslant 0, i = 1, 2, \cdots, n$, $1/p + 1/q = 1$, then

$$\sum_{i=1}^n x_i y_i \leqslant \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q}, \text{ if } p > 1;$$

and the inequality reverses when $p < 1$.
Proof: Assume that $\sum_{i=1}^n x_i^p = \sum_{i=1}^n y_i^q = 1$, then by B4), if $p > 1$,

$$\sum_{i=1}^n x_i y_i \leqslant \sum_{i=1}^n \frac{x_i^p}{p} + \frac{y_i^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

The case $p < 1$ is similar.

8.3 PSC

C1) Suppose $f \in C([0,1])$, g is differentiable on $[0,1]$ and $g(0) = 0$. If there is a constant $\lambda \neq 0$, such that for any $x \in [0,1]$, $|g(x)f(x) + \lambda g'(x)| \leqslant |g(x)|$, prove that $g(x) \equiv 0$.

Proof: Otherwise assume that $\forall \varepsilon > 0 \exists t \in (0, \varepsilon)$, such that $g(t) \neq 0$. Let $C = (1 + \sup_{x\in[0,1]} |f(x)|)/\lambda$, then $|g'(x)| \leqslant C|g(x)|, \forall x \in [0,1]$. For any $t \in (0,1)$, there exists $\xi \in [0,t]$ such that $g(t) = g(0) + tg'(\xi)$, hence

$$\frac{|g(t)|}{t} = |g'(\xi)| \leqslant C \sup_{\xi\in[0,t]} |g(\xi)|.$$

For any $t > 0$ suppose $|g(s)| = \sup_{\xi\in[0,t]} |g(\xi)| > 0$, then $|g(s)|/s \leqslant C|g(s)|$ hence $t \geqslant s \geqslant \frac{1}{C}$, a contradiction.

C2) f is twice differentiable on $(-1,1)$, $f(0) = f'(0) = 0$. If for any $x \in (-1,1)$, $|f''(x)| \leqslant |f(x)| + |f'(x)|$, prove that $f(x) \equiv 0$.

Proof: We prove that $f''(x) \equiv 0$. Otherwise suppose $\forall \varepsilon > 0$, $\exists x \in [0, \varepsilon]$, $f''(x) \neq 0$. Note that

$$|f''(x)| \leqslant |f(x)| + |f'(x)| \leqslant \left(\frac{x^2}{2} + |x|\right) \sup_{y\in[0,x]} |f''(y)|.$$

Since $f''(0) = 0$, take $x \in [0, 1/2]$ such that $f''(x) \neq 0$, and suppose $|f''(t)| = \sup_{y\in[0,x]} |f''(y)|$, then $|f''(t)| \leqslant (t^2/2 + t)|f''(t)|$, a contradiction.

C3) f is n-times differentiable on $\mathbb{R}$, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. If there exists $C \in \mathbb{R}_{>0}$ and $\in \mathbb{Z}_{\geqslant 0}$ such that for any $x \in \mathbb{R}$, $|f^{(n)}(x)| \leqslant C|f^{()}(x)|$. Prove that $f(x) \equiv 0$.

Proof: We can assume that $= 0$. Since $f^{(k)}(x) = 0, \forall 0 \leqslant k < n$, we have

$$|f^{(n)}(x)| \leqslant C|f(x)| \leqslant C\frac{x^n}{n!}\sup_{y\in[0,x]}|f^{(n)}(y)|.$$

Hence for any $x \in [0,\varepsilon]$, $\varepsilon = (n!/C)^{1/n}$, $f^{(n)}(x) = 0$, so $f^{(k)}(x) = 0$ for all $x \in [0,\varepsilon], 0 \leqslant k < n$. Likewise we get $f(x) \equiv 0$.

C4) $n \in \mathbb{Z}_{>0}$, prove that the polynomial $P(x) = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k(x-k)^n \equiv 0$.

Proof: We know the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^l = 0,\ \forall 0 \leqslant l \leqslant n-1.$$

Since $\Delta^n x^l \equiv 0$
Likewise by considering $f(t) = (x-t)^n$ we have $P(x) \equiv 0$.
(Or we can use C3)

C5) $f \in C^\infty(\mathbb{R})$. Assume there exists $C > 0$ such that for any $n \in \mathbb{Z}_{\geqslant 0}$ and $x \in \mathbb{R}$, $|f^{(n)}(x)| \leqslant C$.

i. Prove that given an arbitrary $x_0 \in \mathbb{R}$,

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,\ \forall x \in \mathbb{R}.$$

Proof: The Lagrange remainder

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

tends to zero as $n \to \infty$, hence the Taylor series

$$f(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

ii. $E \subset \mathbb{R}$ is an infinite bounded set. Prove that if $f(E) = \{0\}$, then $f \equiv 0$.
Proof: Suppose $E \subset [-M, M]$, then by Bolzano-Weierstrass theorem, there exists a sequence $\{z_n\}_{n\geqslant 1} \subset E$ such that $z = \lim_{n\to\infty} z_n$ exists. Since $f \in C(\mathbb{R})$, $f(z) = \lim_{n\to\infty} f(z_n) = 0$, so

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^k.$$

If f does not vanish on $\mathbb{R}$, then take the least $m > 0$ such that $f^{(m)}(z) \neq 0$. When $z_n \to z$,

$$0 = \frac{f^{(m)}(z)}{m!} + \sum_{k=m+1}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^{k-m}$$

which leads to contradiction. Hence f vanishes on $\mathbb{R}$.

C6) Assume $f \in C^2((0,1))$, $\lim_{x\to 1^-} f(x) = 0$. If there exists $C > 0$, such that for any $x \in (0,1)$, $(1-x)^2|f''(x)| \leqslant C$. Prove that $\lim_{x\to 1^-}(1-x)f'(x) = 0$.

Proof: For any $0 < x < y < 1$, there exists $\xi \in (x,y)$ such that

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2} f''(\xi).$$

For any $\lambda > 0$, let $y = (\lambda + x)/(\lambda + 1) \in (x, 1)$, then

$$|(y-x)f'(x)| \leqslant |f(y)| + |f(x)| + \frac{\lambda^2}{2}(1-y)^2|f''(\xi)| \leqslant |f(y)| + |f(x)| + \frac{C\lambda^2}{2}.$$

Therefore

$$|(1-x)f'(x)| \leqslant (|f(t)| + |f(x)|)\frac{\lambda+1}{\lambda} + \frac{1}{2}\lambda(\lambda+1)C$$

Hence for any $\lambda > 0$,

$$\lim_{x\to 1^-} |(1-x)f'(x)| \leqslant \frac{1}{2}\lambda(\lambda+1)C \to 0,$$

so $\lim_{x\to 1^-}(1-x)f'(x) = 0$.

8.4 PSD

Calculate $\sup_{x\in I} f(x)$ and $\inf_{x\in I} f(x)$:

D1) $f(x) = \frac{(\log x)^2}{x}$**,** $I = \mathbb{R}_{>0}$

Solution: Let $y = \log x \in \mathbb{R}$, then $f(x) = y^2 e^{-y}$.

$$\frac{\mathrm{d}}{\mathrm{d}y} y^2 e^{-y} = y e^{-y}(2-y).$$

Hence $\sup_{x\in I} f(x) = f(e^2) = 4e^{-2}$, $\inf_{x\in I} f(x) = \min\{f(0), f(\infty)\} = 0$.

D2) $f(x) = |x(x^2-1)|$**,** $I = \mathbb{R}$

Solution: $\sup = \infty, \inf = 0$.

D3)

$$f(x) = \frac{x(x^2+1)}{x^4 - x^2 + 1},\ I = \mathbb{R}.$$

Solution: Note that

$$2(x^4 - x^2 + 1) - x(x^2+1) = (x^2-1)^2 + (x-1)^2(x^2+x+1) \geqslant 0.$$

Therefore $f(x) \leqslant 2$ where equality holds at $x = 1$. Since $f(x) = f(-x)$, $\sup = 2, \inf = -2$.

D4)

$$f(x) = x^{1/3}(1-x)^{2/3},\ I = (0,1).$$

Solution: By AM-GM, $f(x) \leqslant \frac{2^{2/3}}{3}$ where equality holds at $x = 1/3$. Hence $\sup = \frac{2^{2/3}}{3}, \inf = 0$.

D5)

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) e^{-x},\ I = \mathbb{R}.$$

Solution: $f'(x) = -e^{-x}\frac{x^n}{n!}$, so if n is even, $\sup = \infty, \inf = 0$, and if n is odd, $\sup = 1, \inf = -\infty$.

D6) $f(x) = \sin^{2m} x \cos^{2n} x$**,** $I = \mathbb{R}$**.**

Solution: Let $t = \sin^2 x \in [0,1]$, then $f(x) = t^m(1-t)^n \in [0, n^n m^n/(n+m)^{n+m}]$.

8.5 PSE

Compare the two functions (or real numbers).

E1) $f(x) = e^x$**,** $g(x) = 1 + xe^x$**,** $x > 0$**.**

Solution: The case $x \geqslant 1$ is trivial. If $x \in (0,1)$, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \leqslant \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

hence $f(x) \leqslant g(x)$. Therefore $f(x) \leqslant g(x)$ for all $x > 0$.

E2) $f(x) = xe^{x/2}$**,** $g(x) = e^x - 1$**,** $x > 0$**.**

Solution: $(x/2 \leqslant \sinh(x/2))$ Consider $h(x) = e^{x/2} - e^{-x/2} - x$, then $h(0) = 0$ and

$$h'(x) = \frac{1}{2}(e^{x/2} + e^{-x/2} - 2) \geqslant 0.$$

Hence $h(x) \geqslant 0$, i.e. $g(x) \geqslant f(x)$ for all $x > 0$.

E3) $f(x) = \left(\frac{x+1}{2}\right)^{(x+1)}$**,** $g(x) = x^x$**,** $x > 0$**.**

Solution: Consider $h(x) = x \log x - (x+1)\log\frac{x+1}{2}$, then $h(1) = 0$ and

$$h'(x) = \log\frac{2x}{x+1} \geqslant 0 \iff x \geqslant 1.$$

Hence $f(x) \leqslant g(x)$ for all $x > 0$.

E4) $2^{\sqrt{2}}$ **and** e**.**

Solution: Note that

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{2^n n} \leqslant \frac{2}{3} + \sum_{n=4}^{\infty} \frac{1}{2^n \cdot 4} = \frac{2}{3} + \frac{1}{32} < \frac{2}{3} + \frac{1}{30} = 0.7 < \frac{1}{\sqrt{2}},$$

hence $2^{\sqrt{2}} < e$.

E5) $f(x) = \log(1+\sqrt{1+x^2})$**,** $g(x) = 1/x + \log x$**,** $x > 0$**.**

Solution: Consider $h(x) = \log x + 1/x - \log(1+\sqrt{1+x^2})$, then

$$h'(x) = \frac{1}{x} - \frac{1}{x^2} - \frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}} \leqslant 0.$$

$(\iff (x-1)(\sqrt{1+x^2}+1+x^2) - x^3 \leqslant 0 \iff (x1)\sqrt{1+x^2} \leqslant x^2)$
Therefore $h(x) \geqslant \lim_{x\to\infty} h(x) = 0$.

E6) $\log 8$ **and** 2**.**

Solution: Note that

$$\log 2 = \log \frac{1}{1-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{2^n n} \geqslant \sum_{n=1}^{3} \frac{1}{2^n n} = \frac{2}{3},$$

hence $\log 8 > 2$.

8.6 PSF

If f satisfy $f(x) = (x-x_0)^r g(x)$ in a neighborhood of x_0, where $r \in \mathbb{Z}_{\geqslant 0}$, g is continuous at x_0 and $g(x_0) \neq 0$, then we call x_0 an r-fold root of f.

F1) Suppose x_0 is an r-fold root of f where $r > 0$. Prove that if $g(x) = f(x)/(x-x_0)^r$ is continuous, then x_0 is an $(r-1)$-fold root of f'.

Proof: Suppose $f(x) = (x-x_0)^r g(x)$ in the neighborhood $O(x_0)$, then $f'(x) = (x-x_0)^r g'(x) + r(x-x_0)^{r-1} g(x)$ in $O(x_0)$. Therefore let $h(x) = (x-x_0)g'(x) + g(x)$, then $f'(x) = (x-x_0)^{r-1}h(x)$ and $h(x_0) = g(x_0) \neq 0$, so x_0 is an $(r-1)$-fold root of f'.

F2) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x) = 0$ has $n+1$ distinct real roots, then $f^{(n)}(x) = 0$ has at least one root.

Proof: Use induction and Rolle's mean-value theorem to prove that $f^{(n-k)}(x)$ has at least $k+1$ different real roots.

F3) f is differentiable on $\mathbb{R}$. Suppose $f(x) = 0$ has r roots (counting multiplicity), then $f'(x) = 0$ has at least $r-1$ roots (counting multiplicity).

Proof: Combine F1) and F2).

F4) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x) = 0$ has exactly $n+1$ roots counting multiplicity, then $f^{(n)}(x) = 0$ has at least one root.

Proof: Use F3) and induction.

8.7 PSG

Let $a \in \mathbb{R}$, $f : (a,\infty) \to \mathbb{R}$ twice differentiable on (a,∞), and

$$M_0 := \sup_{x\in(a,\infty)} |f(x)|,\ M_1 := \sup_{x\in(a,\infty)} |f'(x)|,\ M_2 := \sup_{x\in(a,\infty)} |f''(x)|,$$

are real numbers.

G1) Prove that $M_1^2 \leqslant 4M_0M_2$.

Proof: Let $h = \sqrt{M_0/M_2}$, then for any $x \in (a, \infty)$, there exists $\xi \in (x, x+2h)$ such that

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi) \implies f'(x) = hf''(\xi) + \frac{f(x+2h) - f(x)}{2h}.$$

Therefore $f'(x) \leqslant M_0/h + M_2 h = 2\sqrt{M_0M_2}$, hence $M_1^2 \leqslant 4M_0M_2$.

G2) Let $a = -1$, consider the function

$$f(x) = \begin{cases} 2x^2 - 1, & x \in (-1, 0), \\ \frac{x^2-1}{x^2+1}, & x \in [0, \infty), \end{cases}$$

verify that f is twice differentiable, and $M_0 = 1, M_1 = 4, M_2 = 4$.
Proof: Note that $\lim\limits_{x \to 0^-} f(x) = -1 = f(0)$ so f is continuous, and

$$f'(x) = \begin{cases} 4x, & x \in (-1, 0), \\ \frac{4x}{(x^2+1)^2}, & x \in [0, \infty). \end{cases}$$

f' is also continuous, so

$$f''(x) = \begin{cases} 4, & x \in (-1, 0), \\ 4\frac{1-3x^2}{(x^2+1)^2}, & x \in [0, \infty). \end{cases}$$

Therefore $f \in C^2((-1, \infty))$ and $M_0 = 1, M_1 = 4, M_2 = 4$.

G3) Suppose $\mathbf{f} : (a, \infty) \to \mathbb{R}^n$ is twice differentiable, Let M_0, M_1, M_2 be the least upper bounds of $|\mathbf{f}|, |\mathbf{f}'|, |\mathbf{f}''|$. Prove that $M_1^2 \leqslant 4M_0M_2$.

Proof: Use G1) and Cauchy-Schwarz inequality.

8.8 Problem S: Strum-Liouville Theory

Assume the following uniqueness theorem holds:

> Theorem:
> Suppose $a(t) \in C^1(\mathbb{R})$, $t_0 \in \mathbb{R}$. If $x(t), y(t) \in C^2(\mathbb{R})$ both satisfy the equation
>
> $$x''(t) + a(t)x(t) = 0, y''(t) + a(t)y(t) = 0,$$
>
> and $(x(t_0), x'(t_0)) = (y(t_0), y'(t_0))$, then $x(t) \equiv y(t)$.

(Can be proved using Exercise C3?)
For any $f : \mathbb{R} \to \mathbb{R}$, $t \geqslant 0$, denote
$Z_t(f) = |\{x \in [0, t] : f(x) = 0\}|$.

Part 1

Let $a(t), b(t) \in C^1(\mathbb{R})$ and for any $t \in \mathbb{R}$, $a(t) \leqslant b(t)$. Suppose $x(t), y(t) \in C^2(\mathbb{R})$ satisfy the following equation

$$x''(t) + a(t)x(t) = 0,\ y''(t) + b(t)y(t) = 0.$$

Further assume that $x(t), y(t)$ are not identically zero.

S1) Assume $x(t_1) = 0$, if there exists $t > t_1$, such that $x(t) = 0$. Prove that there exists $t_2 > t_1$ such that $x(t_2) = 0$ and x has no roots in (t_1, t_2). We call t_1, t_2 neighboring roots.

Proof: Consider the set $S = \{t > t_1 : x(t) = 0\}$, and let $t_2 = \inf S$. Note that $|x''(t)| \leqslant |a(t)| \cdot |x(t)|$, so by C3) $x'(t_1) \neq 0$. Assume $x'(t_1) > 0$, since $x \in C^2(\mathbb{R})$, there exists $\varepsilon > 0$ such that $x'(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$, hence $x(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$. Therefore $t_2 > t_1$, so by $x \in C(\mathbb{R})$, $x(t_2) = 0$ and x has no roots in (t_1, t_2).

S2) If $t_2 > t_1$ are two neighboring roots of x, prove that y has a root in $(t_1, t_2]$.

Proof: Otherwise assume that x, y are positive on (t_1, t_2), and $y(t_2) \neq 0$. Consider the function $h(t) = x'y - xy'$, then $h'(t) = (b - a)xy \geqslant 0$, so $h(t_2) \geqslant h(t_1) = x'(t_1)y(t_1) \geqslant 0$, but $h(t_2) = x'(t_2)y(t_2) < 0$, a contradiction.

S3) Prove that for any $t \geqslant 0$, $Z_t(y) \geqslant Z_t(x) - 1$.

Proof: Use S2).

S4) Suppose $t_2 > t_1$ and $x(t_1) = x'(t_2) = 0$, prove that

- If $y(t_1) = 0$, then there exists $t_3 \in [t_1, t_2]$, such that $y'(t_3) = 0$.
  Proof: We can assume that $t_2 = \inf \{t > t_1 : x'(t) = 0\}$($t_2 > t_1$ since $x'(t_1) \neq 0$). If there is no such t_3, we can further assume that $x'(t), y'(t), x(t), y(t) > 0$ for all $t \in (t_1, t_2)$. Again consider $h(t) = x'y - xy'$, then $h(t_1) = 0$, $h(t_2) = -x(t_2)y'(t_2) < 0$, but $h'(t) = (b - a)xy \geqslant 0$, leading to contradiction.
- If $y'(t_2) = 0$, then there exists $t_4 \in [t_1, t_2]$ such that $y(t_4) = 0$.
  (The two theorems are similar.)

Part 2

In this section, $p(t) \in C^1(\mathbb{R})$ is a positive function. $x(t), y(t) \in C^2(\mathbb{R})$ are not identically zero and satisfy

$$x''(t) + p(t)x(t) = 0,\ y''(t) + p(t)y(t) = 0.$$

S5) Prove that for any $t \geqslant 0$, $|Z_t(x) - Z_t(y)| \leqslant 1$.

Proof: Use S3).

S6) Prove that

- If t_1, t_2 are neighboring roots of x, then there exists a unique $t_3 \in [t_1, t_2]$ such that $x'(t_3) = 0$.
  Proof: The existence of t_3 is given by Rolle's mean-value theorem. If there exists $t_3 < t_4 \in [t_1, t_2]$ such that $x'(t_3) = x'(t_4) = 0$, then $t_3, t_4 \neq t_1, t_2$ and there exists $t_5 \in [t_3, t_4]$ such that $x''(t_5) = 0$. Hence $x(t_5) = 0$, which contradicts the fact that t_1, t_2 are neighboring roots. Therefore t_3 is unique.
- If t'_1, t'_2 are neighboring roots of x', then there exists a unique $t'_3 \in [t'_1, t'_2]$ such that $x(t'_3) = 0$.
  Proof: Exactly the same.

S7) Prove that

- t_0 is a local maximum of $|x(t)|$ iff $x'(t_0)=0$.
  Proof: Trivial?
- t_0' is a local maximum of $|x'(t)|$ iff $x(t_0')=0$.

Part 3

In this section, $p(t)\in C^1(\mathbb{R})$ is monotonically decreasing and $\lim_{t\to\infty}p(t)>0$. Denote

$$p(\infty):=\lim_{t\to\infty}p(t).$$

$x(t)\in C^2(\mathbb{R})$ is not identically zero and

$$x''(t)+p(t)x(t)=0.$$

***S8) Calculate**

$$\lim_{t\to\infty}\frac{Z_t(x)}{t}.$$

Solution: By S5) we can ignore initial conditions. First consider the ODE $y''(t)+p(\infty)y(t)=0$, where one solution is $y=\sin(t\sqrt{p(\infty)})$, so $\lim\limits_{t\to\infty}Z_t(y)/t=\sqrt{p(\infty)}/\pi$.
Since $p(t)\geqslant p(\infty)$, by S3) we know $\lim\limits_{t\to\infty}Z_t(x)/t\geqslant\lim\limits_{t\to\infty}Z_t(y)/t=\sqrt{p(\infty)}/\pi$. For any $\varepsilon>0$, there exists $M>0$ such that for any $t>M$, $p(t)<p(\infty)+\varepsilon$. By S3), $\lim_{t\to\infty}Z_t(x)/t\leqslant\sqrt{p(\infty)+\varepsilon}/\pi$. Therefore

$$\lim_{t\to\infty}\frac{Z_t(x)}{t}=\frac{\sqrt{p(\infty)}}{\pi}.$$

S9) Suppose $0\leqslant t_1<t_2<t_3<\cdots$ are all the roots of $x(t)$ on $[0,\infty)$, $0\leqslant t_1'<t_2'<\cdots$ are all the roots of $x'(t)$ on $[0,\infty)$. Prove that the sequence $\{|x'(t_k)|\}_{k\geqslant1}$ is monotonically decreasing and the sequence $\{|x(t_k')|\}_{k\geqslant1}$ is monotonically increasing, and

$$\lim_{k\to\infty}|x'(t_k)|=\sqrt{p(\infty)}\lim_{k\to\infty}|x(t_k')|.$$

Proof: Consider the (energy) function $E(t)=p(t)x^2(t)+x'(t)^2$, then $E'(t)=p'x^2\leqslant0$ so E is monotonically decreasing. For $k\geqslant1$, $E(t_k)=x'(t_k)^2$ is decreasing, so $\{|x'(t_k)|\}_{k\geqslant1}$ is decreasing. Likewise, consider $F(t)=x(t)^2+x'(t)^2/p(t)$, then $F'(t)=-p'(x'/p)^2\geqslant0$, so $F(t_k')=x(t_k')^2$ is increasing, and

$$\lim_{k\to\infty}|x'(t_k)|=\sqrt{\lim_{k\to\infty}E(t_k)}=\sqrt{p(\infty)\lim_{k\to\infty}F(t_k)}=\sqrt{p(\infty)}\lim_{k\to\infty}|x(t_k')|.$$

***S10) Suppose $0\leqslant\tilde t_1<\tilde t_2<\cdots$ are all the roots of $x(t)x'(t)$ on $[0,\infty)$. Prove that the sequence $\{\tilde t_{k+1}-\tilde t_k\}_{k\geqslant1}$ is monotonically increasing and calculate its limit.**

Proof: By S6), the roots of x and x' appear alternating in $\{\tilde t_k\}$. Since t is a root of x iff t is a root of x'', we only need to prove that if t_1,t_2 are neighboring roots of x, and $t_3\in[t_1,t_2]$ satisfy $x'(t_3)=0$, then $t_3-t_1\leqslant t_2-t_3$.

Same as before we can prove that, for $p(t), q(t), x(t), y(t)$ such that $p(0) = q(0)$, $p(t) \leqslant q(t)$, $x'(0) = y'(0) = 0$, $x(0) = y(0)$ and

$$x''(t) + p(t)x(t) = 0,\ y''(t) + q(t)y(t) = 0,$$

then the first roots a, b of x, y satisfy $a \leqslant b$.
Since the sequence is increasing, by S8) we know that
$\lim_{k\to\infty} \tilde{t}_{k+1} - \tilde{t}_k = \frac{1}{2} \lim_{t\to\infty} Z_t(x)/t = \sqrt{p(\infty)}/2\pi$.

9 Homework 9: Stone-Weierstrass Theorem

9.1 PSA

Assume $I = [a, b] \subset \mathbb{R}$, V is a normed linear space.

A1) $\sigma_1, \sigma_2 \in \mathcal{S}$ are two partitions. Prove that for any $\varepsilon > 0$, there exists a partition σ such that $\sigma \prec \sigma_1, \sigma \prec \sigma_2$ and $|\sigma| < \varepsilon$.

Proof: Take $n > 1/\varepsilon$, and let

$$\sigma = \sigma_1 \cup \sigma_2 \cup \left\{ \frac{k}{n}a + \frac{n-k}{n}b : 0 \leqslant k \leqslant n \right\}.$$

A2) Consider the space of simple functions $\mathcal{E}(I)$ with range V. Prove that it is a linear space on $\mathbb{R}$, and the integration operator $\int_a^b : \mathcal{E}(I) \to V$ is well-defined and is linear. Use this to define Riemann integrable functions with range V.

Proof: For any simple function $f = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$ (where A_i are disjoint), let

$$\int_a^b f = \sum_{i=1}^n c_i \mu(A_i)$$

For any function $f : I \to V$, partition $\mathcal{C} = \{x_0, x_1, \cdots, x_n\}$ and $\xi_i \in [x_{i-1}, x_i]$, define

$$\mathcal{R}(f; \mathcal{C}, \xi) = \sum_{k=0}^n f(\xi_i)(x_i - x_{i-1}).$$

Then f is Riemann integrable iff $\lim_{|\mathcal{C}|\to 0} \mathcal{R}(f; \mathcal{C}, \xi)$ exists.

A3) Suppose $f : I \to \mathbb{R}^n$ and f_i be the components of f, then $f \in \mathcal{R}(I)$ iff for every i, $f_i \in \mathcal{R}(I)$.

Proof: Note that

$$\max\{|x_k|\} \leqslant |(x_1, \cdots, x_n)|_{\mathbb{R}^n} \leqslant |x_1| + \cdots + |x_n|.$$

Hence the limit $|\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0$ iff the components of f are all Riemenn integrable.

A4) Assume $a < c < b$, then for any $f \in \mathcal{R}(I)$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Riemann integrable, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: They are both obviously Riemann integrable, and for any partition σ, let $\sigma' = \sigma \cup \{c\} = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are partitions of $[a, c]$ and $[c, b]$, then

$$\underline{S}(f; \sigma) \leqslant \underline{S}(f; \sigma') = \underline{S}(f|_{[a,c]}; \sigma_1) + \underline{S}(f|_{[c,b]}; \sigma_2),$$

and the other side is the same. Hence

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A5) Prove that for any two partition σ and σ', $\underline{S}(f; \sigma) \leqslant \overline{S}(f; \sigma')$. Use this to prove that if $f \in \mathcal{R}(I)$, then $\lim_{|\sigma| \to 0} |\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0$.

Proof: Let $\sigma'' = \sigma \cup \sigma'$, then

$$\underline{S}(f; \sigma) \leqslant \underline{S}(f; \sigma'') \leqslant \overline{S}(f; \sigma'') \leqslant \overline{S}(f; \sigma').$$

If $f \in \mathcal{R}(I)$, then $\sup_\sigma \underline{S}(f; \sigma) = \inf_\sigma \overline{S}(f; \sigma)$ hence

$$\lim_{|\sigma| \to 0} |\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0.$$

A6) $f \in \mathcal{R}(I)$. Prove that if we change the value of f at a finite number of points to g, then g is Riemann integrable and $\int_I g = \int_I f$.

Proof: We can assume that f and g differ only at the point $c \in (a, b)$. Let $M = \sup_{x \in I} |f(x)|$. For any $\varepsilon > 0$, and any partition σ, let $\sigma' = \sigma \cup \{c - \varepsilon, c + \varepsilon\}$, then $|\underline{S}(f; \sigma') - \underline{S}(f; \sigma)| \leqslant 4\varepsilon M \to 0$.

A7) $f \in C([a, b])$. Assume for any $x \in I$, $f(x) \geqslant 0$ and there exists $x_0 \in I$ such that $f(x_0) > 0$. Prove that $\int_a^b f > 0$.

Proof: Since f is continuous and $f(x_0) > 0$, there is an $\varepsilon > 0$ such that for all $y \in (x_0 - \varepsilon, x_0 + \varepsilon)$, $f(y) > 0$. Hence for any partition $\sigma = \{x_0, x_1, \cdots, x_n\}$ such that $|\sigma| < \varepsilon/2$, there is a $k \in \{1, \cdots, n\}$ such that $(x_{k-1}, x_k) \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. Hence $\mathcal{R}(f; \sigma, \xi) > 0$ whenever $|\sigma| < \varepsilon/2$, so $\int_a^b f(x)\,\mathrm{d}x > 0$.

A8) Suppose $f, g \in C^1(I)$, then

$$\int f' \cdot g = f \cdot g - \int f \cdot g'.$$

Proof:

$$d(f \cdot g) = df \cdot g + f \cdot dg.$$

A9) Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is differentiable, f is a continuous function, then

$$\int (f \circ \Phi)\Phi' = \int f.$$

Proof:

$$(f(\Phi(x)))' = f'(\Phi(x))\Phi'(x).$$

9.2 PSB: Calculating Integrals

(1)

$$\begin{aligned}\int \frac{x^5}{1+x}\,\mathrm{d}x &= \int x^4 - x^3 + x^2 - x + 1 - \frac{1}{1+x}\,\mathrm{d}x \\ &= \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C.\end{aligned}$$

(2)

$$\int \sqrt{x\sqrt{x\sqrt{x}}}\,\mathrm{d}x = \int x^{7/8}\,\mathrm{d}x = \frac{8}{15}x^{15/8}.$$

(3)

$$\begin{aligned}\int \left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right)\mathrm{d}x &= \int \left(\frac{2}{1-x} + \frac{2}{1+x} - 2\right)\mathrm{d}x \\ &= -2x + 2\log\frac{1+x}{1-x} + C.\end{aligned}$$

(4)

$$\int \frac{e^{3x}+1}{1+e^x}\,\mathrm{d}x = \int 1 - e^x + e^{2x}\,\mathrm{d}x = x - e^x + \frac{e^{2x}}{2} + C.$$

(5)

$$\int \sqrt{1-\sin(2x)}\,\mathrm{d}x = \int \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)\mathrm{d}x = -\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + C.$$

(6)

$$\int \frac{\cos(2x)}{\cos x - \sin x}\,\mathrm{d}x = \int \cos x + \sin x\,\mathrm{d}x = \sin x - \cos x + C.$$

(7)

$$\int \tan^2 x \, \mathrm{d}x = -x + \tan x + C.$$

(8)

$$\int |x| \, \mathrm{d}x = \frac{x|x|}{2} + C.$$

(9)

$$\int e^{-|x|} \, \mathrm{d}x = -\operatorname{sgn}(x) e^{-|x|} + C.$$

(10)

$$\int \frac{x^2}{(1-x)^{2018}} \, \mathrm{d}x = \frac{1}{2017(1-x)^{2017}} - \frac{1}{1013(1-x)^{2016}} + \frac{1}{2015(1-x)^{2015}}.$$

(11)

$$\int |x-1| \, \mathrm{d}x = \frac{(x-1)|x-1|}{2} + C.$$

(12)

$$\int \frac{1}{\sqrt{b^2+x^2}} \, \mathrm{d}x = \frac{1}{b} \log \frac{1 + \tan \frac{\arctan \frac{x}{b}}{2}}{1 - \tan \frac{\arctan \frac{x}{b}}{2}} + C.$$

(13)

Let $x = t^2$, then

$$\int \frac{\mathrm{d}x}{\sqrt{x}(1+x)} = 2 \arctan \sqrt{x} + C.$$

(14)

$$\int \frac{x^4}{(1-x^5)^4} \, \mathrm{d}x = \frac{1}{5} \int \frac{\mathrm{d}x^5}{(1-x^5)^4} = \frac{1}{15(1-x^5)^3} + C.$$

(15)

$$\int \left(\frac{1}{\sqrt{3-x^2}} + \frac{1}{1-3x^2} \right) \mathrm{d}x = \arcsin \frac{x}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \log \frac{1+\sqrt{3}x}{1-\sqrt{3}x} + C.$$

(16)

$$\int \frac{2x-3}{x^2-3x+8}\,\mathrm{d}x = \log(x^2-3x+8)+C.$$

(17)

$$\int \frac{\mathrm{d}x}{\sin^2\left(2x+\frac{\pi}{4}\right)} = \frac{\tan(2x-\pi/4)}{2}+C.$$

(18)

$$\int \frac{\mathrm{d}x}{1+\cos x} = \tan\frac{x}{2}+C.$$

(19)

$$\int \frac{1}{x^2}\sin\frac{1}{x}\,\mathrm{d}x = \cos\frac{1}{x}+C.$$

(20)

$$\int \cos^5 x\,\mathrm{d}x = \frac{\sin^5 x}{5}-\frac{2\sin^3 x}{3}+\sin x+C.$$

(21)

$$\int \cos(ax)\sin(bx)\,\mathrm{d}x = \frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}+C.$$

(22)

$$\int \frac{\mathrm{d}x}{a\cos x+b\sin x} = \frac{2}{\sqrt{a^2+b^2}}\tanh^{-1}\frac{a\tan(x/2)-b}{\sqrt{a^2+b^2}}+C.$$

(23)

$$\int \frac{\sin(2x)}{a^2\cos^2 x+b^2\sin^2 x}\,\mathrm{d}x = \frac{\log((b^2-a^2)\sin^2 x+a^2)}{b^2-a^2}+C.$$

(24)

$$\int \frac{\mathrm{d}x}{2-\sin^2 x} = \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)+C.$$

(25)

$$\int \frac{\mathrm{d}x}{x\ln x\ln\ln x} = \ln\ln\ln x + C.$$

(26)

$$\int \frac{\log x}{x\sqrt{1+\log x}}\,\mathrm{d}x = \frac{2}{3}(1+\log x)^{3/2} - 2\sqrt{1+\log x} + C.$$

(27)

$$\int \frac{\cos x + \sin x}{(\sin x - \cos x)^{1/3}}\,\mathrm{d}x = \frac{3}{2}(\sin x - \cos x)^{2/3} + C.$$

(28)

$$\int e^{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

(29)

$$\int \frac{x^{n/2}}{1+x^{n+2}}\,\mathrm{d}x = \frac{2}{n+2}\arctan x^{n/2+1} + C.$$

(30)

$$\int \frac{\sqrt{x}}{1-x^{1/3}}\,\mathrm{d}x = 6\arctan x^{1/6} - \frac{6}{5}x^{5/6} - \frac{6}{7}x^{7/6} - 2x^{1/2} - 6x^{1/6} + C.$$

(31)

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$$

(32)

$$\int \frac{\mathrm{d}x}{\cos^4 x} = \frac{\sin x}{2\cos^3 x} + \frac{\sin(3x)}{6\cos^3 x} + C.$$

(33)

$$\int \arcsin^2 x\,\mathrm{d}x = x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C.$$

(34)

$$\int x \arcsin x \, \mathrm{d}x = \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4} \arcsin x (1 - 2x^2) + C.$$

(35)

$$\int x \arctan x = \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}x + C.$$

(36)

$$\int \frac{\arctan x}{x^2} = \log x - \frac{\arctan x}{x} - \frac{1}{2} \log(1+x^2) + C.$$

(37)

$$\int x^2 \sin x = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

(38)

$$\int \frac{x}{\cos^2 x} = x \tan x + \log \cos x + C.$$

(39)

$$\int \log(x + \sqrt{1+x^2}) = x \log(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

(40)

$$\int \sin \log x = \frac{x}{2}(\sin \log x - \cos \log x) + C.$$

(41)

$$\int \sqrt{x^2+a^2} = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{4} \log \frac{x + \sqrt{a^2+x^2}}{\sqrt{a^2+x^2} - x^2} + C.$$

(42)

$$\int \frac{x^2}{\sqrt{x^2-a^2}} = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{a^2}{4} \log \frac{x + \sqrt{x^2-a^2}}{x - \sqrt{x^2-a^2}} + C.$$

(43)

$$\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} = \sqrt{x^2+1}\log(x+\sqrt{1+x^2}) - x + C.$$

Let $u = \sqrt{x^2+1} + x$ then $\mathrm{d}u/\mathrm{d}x = 1 + x/\sqrt{1+x^2}$, so it becomes

$$\int \frac{(u^2-1)\log u}{2u^2}\,\mathrm{d}x = -\frac{u}{2} + \frac{1}{2u} + \frac{1}{2}u\log u + \frac{\log u}{2u} + C.$$

(44)

$$\int \frac{1}{\sqrt{x^2+a^2}} = \log\frac{\sin t+\cos t}{\sin t-\cos t} + C = \tanh^{-1}\frac{x}{\sqrt{x^2+a^2}} + C.$$

where $t = \frac{1}{2}\arctan(x/a)$.

(45)

$$\int \frac{xe^x}{(1+x)^2} = \frac{e^x}{1+x} + C.$$

(46)

$$\int \arctan(1+\sqrt{x}) = x\arctan(1+\sqrt{x}) - \sqrt{x} + \log(2+2\sqrt{x}+x) + C.$$

(47)

$$\int \left(1-\frac{2}{x}\right)^2 e^x = e^x - \frac{4e^x}{x} + C.$$

since $\int e^x/x^2\,\mathrm{d}x = -e^x/x + \int e^x/x\,\mathrm{d}x$.

(48)

$$\int \sqrt{2+\tan^2 x} = \theta + \log\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta} + C.$$

where $\theta = \arcsin(\sin x/\sqrt{2})$.

(49)

$$\int \frac{1}{1+x^3} = -\frac{1}{6}\log(x^2-x+1) + \frac{1}{3}\log(x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

(50)

$$\int \frac{x^7}{x^4+2} = \frac{x^4}{4} - \frac{1}{2}\log(2+x^4).$$

(51)

$$\int \frac{2x^2+1}{(x+3)(x-1)(x-4)} = -\frac{1}{4}\log(1-x) + \frac{11}{7}\log(4-x) + \frac{19}{28}\log(x+3) + C.$$

(52)

$$\int \frac{1+x^2}{1+x^4} = \frac{1}{\sqrt{2}}(\arctan(\sqrt{2}x+1) - \arctan(1-\sqrt{2}x)) + C.$$

Note that

$$\frac{1+x^2}{1+x^4} = \frac{1}{2(x^2+\sqrt{2}x+1)} - \frac{1}{2(-x^2+\sqrt{2}x+1)}.$$

(53)

Let $x = y^6 - 1$ then

$$\begin{aligned}
&\int \frac{x}{\sqrt{x+1}+(x+1)^{1/3}} = \int (y^3-1)(1-y+y^2)6y^3 \mathrm{d}y \\
=&\frac{2x\sqrt{x+1}}{3} - \frac{3x(x+1)^{1/3}}{4} + \frac{6x(x+1)^{1/6}}{7} - x + \frac{6}{5}(x+1)^{5/6} \\
&- \frac{3}{2}(x+1)^{2/3} + \frac{2\sqrt{x+1}}{3} - \frac{3(x+1)^{1/3}}{4} + \frac{6(x+1)^{1/6}}{7} + C.
\end{aligned}$$

(54)

Let $x = y^2$, then

$$\int \frac{1}{\sqrt{x+x^2}} = \int \frac{\mathrm{d}y}{\sqrt{y^2+1}} = \tanh^{-1}\left(\sqrt{\frac{x}{x+1}}\right) + C.$$

(55)

The Poisson kernel

$$\int \frac{1-r^2}{1-2r\cos x + r^2} = 2\arctan(\frac{1+r}{1-r}\tan\frac{x}{2}) + C.$$

(56)

Let $x = \tan\theta$ then

$$\int \frac{1}{x\sqrt{1+x^2}} = \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\arctan x}{2} + C.$$

(57)

Let $t = \tan x/2$ then

$$\int \frac{1}{5-3\cos x} = \frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C.$$

(58)

Let $t = \tan x$, then

$$\int \frac{1}{2+\sin^2 x} = \frac{1}{\sqrt{6}} \arctan(\sqrt{\frac{3}{2}} \tan x) + C.$$

(59)

$$\int \frac{\sin^3 x}{\cos^4 x} = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$$

(60)

Let $t = \cos x$ then

$$\int \frac{1}{\sin x \cos^4 x} = -\int t^{-4} + t^{-2} + \frac{1}{1-t^2} = \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \frac{1}{2}\log\frac{1+\cos x}{1-\cos x} + C.$$

9.3 Problem W: Stone-WeierstraSS Theorem

Part 1: Approximating $|x|$

W1) (Dini) Suppose $K \subset \mathbb{R}^n$ is compact, $f_n : K \to \mathbb{R}$ is a sequence of continuous functions, which converges point-wise to $f : K \to \mathbb{R}$. If f is continuous and $f_n \leqslant f_{n+1}$, then f_n converges uniformly to f.

Proof: For any $\varepsilon > 0$, and any $x \in K$, there is an integer $n_x > 0$ such that $|f_{n_x}(x) - f(x)| < \varepsilon/4$. There exists $\delta > 0$, such that $\forall y \in \mathrm{B}(x,\delta) \cap K$, $|f(x) - f(y)| < \varepsilon/4$ and $|f_{n_x}(x) - f_{n_x}(y)| < \varepsilon/4$, then $|f_{n_x}(y) - f(y)| < \varepsilon/4$. Note that $K \subset \bigcup_{x\in K} \mathrm{B}(x,\delta_x)$ hence we can choose a finite set of $x_1, x_2, \cdots, x_N$ such that $K \subset \bigcup_{i=1}^N \mathrm{B}(x_i, \delta_{x_i})$. Let $M = \max\{n_{x_i} : i = 1, 2, \cdots, N\}$ then for any $m \geqslant M$ and $x \in K$, $|f_m(x) - f(x)| < \varepsilon$. Hence f_n converges uniformly to f.

W2) Consider the interval $[-1, 1]$. Define inductively a sequence of polynomials:

$$P_0(x) = 0,\ P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x)).$$

Prove that for any n, x, $0 \leqslant P_n(x) \leqslant P_{n+1}(x) \leqslant |x|$.
Proof: Assume $x > 0$, we prove by induction. If $t = P_n(x) \in [0, x]$, then

$$P_{n+1}(x) = \frac{1}{2}x^2 - \frac{1}{2}(t-1)^2 + \frac{1}{2} \leqslant \frac{1}{2}(x^2 - (1-x)^2 + 1) = x,$$

and $P_{n+1}(x) \geqslant P_n(x) = t$, hence $P_{n+1}(x) \in [0, x]$.

W3) Prove that $|x|$ can be uniformly approximated by polynomials on the interval $[-1, 1]$, i.e. for any $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(x)$ such that $\sup_{x\in[-1,1]} ||x| - P_\varepsilon(x)| < \varepsilon$.

Proof: By W2), the sequence of polynomials $\{P_n\}$ converge point-wise to $|x|$, hence by W1) P_n converge uniformly to $|x|$.

Part 3: Bernstein Polynomial

Assume $I=[0,1]$, and n is an integer.

W4) For any $0 \leqslant k \leqslant n$, define $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$. Prove that

$$\sum_{0\leqslant k\leqslant n} p_{n,k}(x)\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n}.$$

Proof:
Note that

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}x^2=x^2,$$

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{k}{n}=\sum_{k=1}^{n}\binom{n-1}{k-1}x^k(1-x)^{n-k}=x,$$

$$\sum_{k=0}^{n}p_{n,k}(x)k(k-1)=n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2}x^k(1-x)^{n-k}=n(n-1)x^2.$$

Therefore

$$\sum_{k=0}^{n}p_{n,k}(x)\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n}.$$

W5) For any $f\in C([0,1])$, define

$$B_{f,n}=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

For $x\in[0,1]$, prove that

$$|f(x)-B_{f,n}(x)|\leqslant\sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right|p_{n,k}(x).$$

Proof: Note that

$$\sum_{k=0}^{n}f(x)\binom{n}{k}x^k(1-x)^{n-k}=f(x).$$

W6) For any $f\in C([0,1])$, prove that for any $\varepsilon>0$, there exists n such that $\|f-B_{f,n}\|_\infty<\varepsilon$.

Proof:
Let

$$\mathrm{I}=\sum_{|m-nx|<n^{3/4}}\left(f(x)-f\left(\frac{m}{n}\right)\right)p_{n,m}(x),$$

$$\mathrm{II}=\sum_{|m-nx|>n^{3/4}}\left(f(x)-f\left(\frac{m}{n}\right)\right)p_{n,m}(x).$$

Then $|f - B_{f,n}| \leqslant |\mathrm{I}| + |\mathrm{II}|$.
For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall x \in [0,1]$, $n \geqslant N \implies |\mathrm{I}| < \varepsilon$, since

$$|\mathrm{I}| \leqslant \sup_{|x-m/n|<n^{-1/4}} |f(x) - f(m/n)| \to 0.$$

Suppose $M = \sup_{x\in[0,1]} |f(x)|$, then

$$|\mathrm{II}| \leqslant 2M \sum_{|m-nx|>n^{3/4}} p_{n,m}(x) \leqslant 2M\sqrt{n} \sum_{m=0}^{n} (x - m/n)^2 p_{n,m}(x) = \frac{2Mx(1-x)}{\sqrt{n}}.$$

Hence $\|f - B_{f,n}\|_\infty \to 0$.

Part 3: Stone-Weierstrass Theorem

W7-14):

Let X be a compact Hausdorff space, $\mathcal{A} \subset C(X,\mathbb{R})$ satisfy the following properties:
(a) $\forall c \in \mathbb{R}$, $c \cdot 1_X \in \mathcal{A}$, (b) $\forall f, g \in \mathcal{A}$, $f+g, f-g, fg \in \mathcal{A}$,
(c) $\mathcal{A}$ can separate any pair of points in X.
Then $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

Lemma 1

There is a list of polynomials $\{P_n(x)\}$ that converges uniformly to $|x|$ on $[-1,1]$.

Lemma 2

If $\mathcal{A}$ is a subspace of $C(X,\mathbb{R})$, such that (a) $\mathcal{A}$ is a lattice, (b) $1_X \in \mathcal{A}$, and (c) $\mathcal{A}$ can separate any pair of points, then $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

Proof of main theorem

Assume WLOG $\mathcal{A}$ is closed, then by Lemma 1, $\forall f \in \mathcal{A}$, $P_n(f) \in \mathcal{A}$, hence $|f| \in \mathcal{A}$.(Since X is compact, $|f|$ is bounded.) Note that

$$\max\{f,g\} = \frac{1}{2}(|f+g| + |f-g|), \min\{f,g\} = \frac{1}{2}(|f+g| - |f-g|).$$

Hence $\mathcal{A}$ is a lattice, by Lemma 2 $\mathcal{A} = C(X,\mathbb{R})$.

Proof of Lemma 1

Proof 1: Let

$$Q_n(x) = \int_0^x (1-t^2)^n\,dt \Big/ \int_0^1 (1-t^2)^n\,dt.$$
$$P_n(x) = \int_0^x Q_n(t)\,dt.$$

Note that

$$\int_\varepsilon^1 (1-t^2)^n\,dt \leqslant (1-\varepsilon^2)^n(1-\varepsilon) \to 0$$

Hence (combined with Wallis's formula), $P_n(x)$ converges uniformly to $|x|$ on $[a,b]$.
Proof 2: WLOG change the interval to $[-1/2,1/2]$. The series

$$(1-t)^{1/2} = 1 + \sum_{n=1}^{\infty} (-t)^n \binom{\frac{1}{2}}{n} = 1 - \sum_{n=1}^{\infty} c_n t^n.$$

converges when $|t|<1$. Hence $\forall \varepsilon > 0$, there exists $Q \in \mathbb{R}[x]$ such that $\sup_{|t|\leqslant 1/2} |Q(t)-(1-t)^{1/2}| < \varepsilon/2$.
Let $t = 1-x^2$, then $|Q(1-x^2)-|x|| < \varepsilon/2$, so $P(x) = Q(1-x^2)-Q(1)$ converges to $|x|$ uniformly on $[-1/2,1/2]$.

Proof of Lemme 2

Step 1: Take any $f \in C(X,\mathbb{R})$, and any $x,y \in X$, we can find $g_{xy} \in \mathcal{A}$, such that $g_{xy}(x) = f(x), g_{xy}(y) = f(y)$. Since there exists $u \in \mathcal{A}$ such that $u(x) \neq u(y)$,

$$\begin{pmatrix} u(x),1 \\ u(y),1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} f(x) \\ f(y) \end{pmatrix}$$

has a solution. (If $x = y$ it is trivial.)
Step 2:
For all $\varepsilon > 0$, $x,y \in X$, there is an open neighborhood $O_{x,y}$ of y, such that $\forall z \in O_{x,y}$, $f(z)-g_{xy}(z) \leqslant \varepsilon$. Note that $\bigcup_{y\in X} O_{x,y} = X$, so by X is compact, there is a list $y_1,\cdots,y_N$ such that $\bigcup_{k\leqslant N} O_{x,y_k} = X$. Let $h_x = \max\{g_{xy_k} : k \leqslant N\}$, then $h_x(y)-f(y) \geqslant -\varepsilon$, and $f(x) = h_x(x)$.
Step 3:
For all $x \in X$, there is an open neighborhood G_x of x, such that $\forall z \in G_x$, $h_x(z)-f(z) \leqslant \varepsilon$. Note that $\bigcup_{x\in X} G_x = X$, so by X is compact, there is a list $x_1,\cdots,x_M$ such that $\bigcup_{k\leqslant M} G_x = X$. Let $F = \min\{h_{x_k} : k \leqslant M\}$, then $|F(x)-f(x)| \leqslant \varepsilon, \forall x \in X$.
Therefore $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

For complex numbers, there is an additional requirement: for any $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$.

W15-16):

It is easy to see that polynomials and trigonometric polynomials both satisfy the requirements of the theorem.

10 Homework 10: Irrationality of π

10.1 PSA

A1) Construct continuous functions $f_n, f \in C([0,1])$, such that for every $x \in [0,1]$, when $n \to \infty$, $f_n(x) \to f(x)$, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Solution: Let $f_n(x) = nxe^{-nx^2}$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2} \int_0^1 e^{-nx^2}\,d(x^2) = \lim_{n\to\infty} \frac{n}{2} \left(\frac{1}{n} - \frac{1}{ne^n}\right) = \frac{1}{2}$$

Hence $\lim_{n\to\infty} \int_0^1 f_n = 1/2 \neq 0 = \int_0^1 f$.

A2) $\alpha \in \mathbb{R}_{\geqslant 0}$. **Prove that** $\int_{100}^{\infty} \frac{dx}{x \log^\alpha(x)}$ **converges iff** $\alpha > 1$.

Proof: Substitute $y = \log x$, then

$$\int_{100}^{\infty} \frac{\mathrm{d}x}{x \log^\alpha(x)} = \int_{\log 100}^{\infty} \frac{\mathrm{d}y}{y^\alpha}$$

which converges iff $\alpha > 1$.

A3) f, F **are defined on** I**, and for every bounded closed interval** $J \subset I$**,** f, F **are both Riemann integrable on** J**. Assume for all** $x \in I$**,** $|f(x)| \leqslant F(x)$**. Then if the improper integer of** F **on** I **converges, so does** f**.**

Proof: This is because

$$\int_I f(x)\,\mathrm{d}x \text{ converges } \iff \forall \varepsilon > 0 \exists N \forall u, v \in I, N < u < v, |\int_u^v f(x)\,\mathrm{d}x| < \varepsilon.$$

A4) Prove the integrals below converge:

(1) $\int_0^\infty e^{-x^2}\,\mathrm{d}x$ (2) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}}$ (3) $\int_1^\infty \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x$
(1):

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x \leqslant 1 + \int_1^\infty e^{-x}\,\mathrm{d}x \leqslant 1 + \frac{1}{e}.$$

(2):

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}} \leqslant \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} = 2.$$

(3):

$$\int_1^\infty \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x \leqslant 5000 + \int_{100}^\infty \frac{1}{x(\log x)^3}\,\mathrm{d}x, \text{ which converges by A2.}$$

A5) Prove the series below converge:

(1) $\sum_{n=1}^\infty e^{-n}(n^2 + \log n)$ (2) $\sum_{n=1}^\infty \frac{\log n}{1+n(\log n)^3}$
(1):

$$\sum_{n=1}^\infty e^{-n}(n^2+\log n) \leqslant \sum_{n=1}^\infty \frac{2n^2}{e^n} \leqslant 2\int_0^\infty x^2 e^{-x}\,\mathrm{d}x = 4.$$

(2):

$$\sum_{n=1}^\infty \frac{\log n}{1+n(\log n)^3} \leqslant \sum_{n=2}^\infty \frac{1}{n(\log n)^2} \leqslant \frac{1}{2(\log 2)^2} + \int_2^\infty \frac{1}{x(\log x)^2}\,\mathrm{d}x \leqslant 3.$$

A6) Calculate

$$\lim_{n\to\infty} \sum_{k=1}^n \frac{k^\alpha}{n^{\alpha+1}}, \alpha > -1.$$

Solution:

$$\sum_{k=1}^{n} k^\alpha \leqslant \int_1^{n+1} x^\alpha \,\mathrm{d}x = \frac{1}{\alpha+1}((n+1)^{\alpha+1}-1).$$
$$\sum_{k=1}^{n} k^\alpha \geqslant 1+\int_1^{n} x^\alpha \,\mathrm{d}x = 1+\frac{1}{\alpha+1}n^{\alpha+1}.$$

Therefore
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^\alpha}{n^{\alpha+1}} = \frac{1}{\alpha+1}.$$

A7) Calculate $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x$, to show that $\pi = 3.14\cdots$.

Solution:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x = \int_0^1 x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\,\mathrm{d}x = \frac{22}{7}-\pi.$$
$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \leqslant \int_0^1 \frac{x^3(1-x)^4}{2}\,\mathrm{d}x = \frac{1}{560} < 0.02, \frac{22}{7} > 3.1428.$$

A8) Assume $a, b, n \in \mathbb{Z}$, let

$$f_{a,b;n} = \frac{x^n(a-bx)^n}{n!}.$$

- Prove that for $k = 0, 1, \cdots, 2n$, $f^{(k)}_{a,b;n}(x) \in \mathbb{Z}$ when $x = 0, \frac{a}{b}$. See B10)
- If $\pi = \frac{a}{b} \in \mathbb{Q}$, then for every $n \in \mathbb{N}$,
$$\int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x$$

is an integer.
Proof: By Darboux's formula of integration of parts

$$\int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x = \sum_{k=0}^{2n} f^{(k)}_{a,b;n}(x)\sin\left(x-\frac{(k+1)\pi}{2}\right)\Big|_0^\pi \in \mathbb{Z}.$$

- Prove that $\pi \notin \mathbb{Q}$.
Proof: Let $n = 2a^4+10$, then $\forall 0 \leqslant x \leqslant a/b$,
$$f_{a,b;n} \leqslant \frac{a^{2n}}{n!} < \frac{1}{2}\frac{(a^4)^{n/2}}{n\cdot(n-1)\cdots\left(\frac{n}{2}\right)} < \frac{1}{2}.$$

Hence
$$0 < \int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x < \frac{1}{2}\int_0^\pi \sin x\,\mathrm{d}x = 1,$$

leading to contradiction.

A9) Let $I_n = \int_0^{\pi/2} \sin^n x \, \mathrm{d}x$, prove that $I_n \sim \sqrt{\frac{\pi}{2n}}$.

Proof: Since $I_n = \frac{n-1}{n} I_{n-2}$,

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ is even}, \\ \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is odd}. \end{cases}$$

Combined with $I_{2n+1} < I_{2n} < I_{2n-1}$, we get

$$\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n},$$

where

$$0 < -\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} + \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n(2n+1)} < \frac{\pi}{4n}.$$

Therefore

$$\lim_{n\to\infty} \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \frac{\pi}{2}.$$

Hence $I_n \sim \sqrt{\frac{\pi}{2n}}$.

A10) Assume $f : [0,1] \to [0,1]$ is monotonously increasing, $g = f^{-1} : [0,1] \to [0,1]$ is its inverse, and f, g are both continuously differentiable, then

$$\int_0^1 f(x) \, \mathrm{d}x + \int_0^1 g(x) \, \mathrm{d}x = 1.$$

Proof: We show that

$$\int_0^x f(t) \, \mathrm{d}t + \int_0^{f(x)} g(t) \, \mathrm{d}t = x f(x), \forall 0 \leqslant x \leqslant 1. (1)$$

$x = 0$ is trivial, hence it suffices to show that the derivatives of the two sides match.

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t) \, \mathrm{d}t = f(x), \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{f(x)} g(t) \, \mathrm{d}t = f'(x) \cdot g(f(x)) = x f'(x).$$

Hence (1) holds.

A11) Prove that

$$\lim_{\varepsilon\to 0} \sum_{k=0}^{\infty} \frac{(-1)^k (1-\varepsilon)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

Proof: By Dirichlet's test, $\sum_{k=0}^{\infty} (-1)^k (1-\varepsilon)^{2k+1}/(2k+1)$ converges uniformly.
Hence for any $\delta > 0$, there exists an $N \in \mathbb{Z}$ such that

$$|\sum_{k=N}^{\infty} \frac{(-1)^k x^{2k}}{2k+1}| < \delta, \forall x \in [0,1].$$

Then $\forall \varepsilon < \frac{\delta}{N}$,

$$\begin{aligned} &\left|\sum_{k=0}^{\infty} \frac{(-1)^k (1-\varepsilon)^{2k+1}}{2k+1} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\right| \\ \leqslant &\sum_{k=0}^{N-1} \frac{|(1-\varepsilon)^{2k+1} - 1|}{2k+1} + \left|\sum_{k=N}^{\infty} \frac{(-1)^k (1-\varepsilon)^{2k+1}}{2k+1}\right| + \left|\sum_{k=N}^{\infty} \frac{(-1)^k}{2k+1}\right| < 3\delta. \end{aligned}$$

Hence

$$\lim_{\varepsilon \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

A12) For any continuous function $f : [a,b] \times [c,d] \to \mathbb{R}, (x,y) \mapsto f(x,y)$, where $a,b,c,d \in \mathbb{R}$, show that f is uniformly continuous on $[a,b] \times [c,d]$.

Proof: $K = [a,b] \times [c,d]$ is a compact set. Consider an arbitrary $\varepsilon > 0$.
For any $x \in K$, there is an open ball $B(x, 2r_x)$ with center x such that $\forall y \in B(x, 2r_x), |f(x) - f(y)| < \varepsilon/2$. Let $O_x = B(x, r_x)$. Note that $\bigcup_{x \in K} O_x = K$ and K is compact, hence we can find $x_1, \cdots, x_n$ such that $\bigcup_{k \leqslant n} O_{x_k} = K$.
Let $\delta = \min\{r_{x_k} : k \leqslant n\}$, then $\forall |u - v| < \delta$, suppose $u \in O_{x_1}$, then

$$|v - x_1| \leqslant |v - u| + |u - x_1| < 2r_{x_1} \implies v \in B(x_1, 2r_{x_1}).$$

Hence

$$|f(u) - f(v)| \leqslant |f(u) - f(x_1)| + |f(v) - f(x_1)| < \varepsilon.$$

Therefore f is uniformly continuous on K.

10.2 PSB: On $\zeta(2)$

Part 1: The sequence $\{\sum_{k=1}^{n} 1/k^p\}$

Define the sequence $S_n(p) = \sum_{k=1}^{n} 1/k^p$ where $p \in \mathbb{Z}_{\geqslant 1}$.

B1) Prove that for any $k \in \mathbb{Z}_{\geqslant 1}$, we have

$$\frac{1}{(k+1)^p} \leqslant \int_k^{k+1} \frac{1}{x^p} \,\mathrm{d}x \leqslant \frac{1}{k^p}.$$

Proof: $\frac{1}{(k+1)^p} \leqslant \frac{1}{x^p} \leqslant \frac{1}{k^p}, \forall k \leqslant x \leqslant k+1$.

B2) Prove that for any $n \in \mathbb{Z}_{\geqslant 2}$, we have

$$S_n(p) - 1 \leqslant \int_1^n \frac{1}{x^p} \,\mathrm{d}x \leqslant S_{n-1}(p).$$

Proof:

$$S_n(p) - 1 = \sum_{k=1}^{n-1} \frac{1}{(k+1)^p} \leqslant \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x^p} \,\mathrm{d}x = \int_1^n \frac{1}{x^p} \,\mathrm{d}x.$$

Likewise we have $\int_1^n \frac{1}{x^p} \,\mathrm{d}x \leqslant S_{n-1}(p)$.

B3) Let $p \in \mathbb{Z}_{\geqslant 1}$. Prove that $x \mapsto \frac{1}{x^p}$ is integrable on $[1, \infty)$ iff $p \geqslant 2$.

Proof: For $p \geqslant 2$,

$$\lim_{n \to \infty} \int_1^n \frac{1}{x^p} \,\mathrm{d}x = \lim_{n \to \infty} \frac{1}{1-p} x^{1-p} \Big|_1^n = \frac{1}{1-p}.$$

If $p = 1$, $\lim_{n \to \infty} \int_1^n \frac{1}{x} \,\mathrm{d}x = \lim_{n \to \infty} \log x \big|_1^n = \infty$.

B4) Prove that $\{S_n(p)\}_{n\geqslant 1}$ converges iff $p \geqslant 2$. For $p \geqslant 2$ let

$$\zeta(p) = \lim_{n\to\infty} S_n(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Proof: If $p = 1$, $S_n(p) \geqslant \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \to \infty$.
For $p \geqslant 2$, $S_n(p) \leqslant S_{n+1}(p)$, and $S_n(p) \leqslant 1 + \int_1^n \frac{1}{x^p}\,\mathrm{d}x \leqslant 1 + \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$. Hence $\lim_{n\to\infty} S_n(p)$ exists.

Part 2: Calculate $\zeta(2)$

(We can also use Bernolli numbers and the Taylor expansion of $\tan x$).

Let $h(t) = \frac{t^2}{2\pi} - t$, $\varphi : [0,\pi] \to \mathbb{R}$:

$$\varphi(x) = \begin{cases} -1, & x = 0; \\ \frac{h(x)}{2\sin\left(\frac{x}{2}\right)}, & 0 < x \leqslant \pi. \end{cases}$$

B5) Prove that $\varphi \in C^1([0,\pi])$.

Proof:

$$\lim_{x\to 0} \frac{h(x)}{2\sin\left(\frac{x}{2}\right)} = \lim_{x\to 0} \frac{-x + o(x)}{2\sin\left(\frac{x}{2}\right)} = -1 = \varphi(0).$$

Hence $\varphi \in C^1([0,\pi])$.

B6) For all $k \geqslant 1$, calculate

$$\int_0^\pi h(x)\cos(kx)\,\mathrm{d}x.$$

Solution:

$$\begin{aligned} \int_0^\pi \left(\frac{x^2}{2\pi} - x\right)\cos(kx)\,\mathrm{d}x &= \frac{1}{k}\int_0^\pi \left(\frac{x^2}{2\pi} - x\right)\mathrm{d}\sin(kx) \\ &= -\frac{1}{k}\int_0^\pi \sin(kx)\left(\frac{x}{\pi} - 1\right)\mathrm{d}x \\ &= \frac{1}{k^2}\int_0^\pi \left(\frac{x}{\pi} - 1\right)\mathrm{d}\cos(kx) \\ &= \frac{1}{k^2} - \frac{1}{\pi k^2}\int_0^\pi \cos(kx)\,\mathrm{d}x = \frac{1}{k^2}. \end{aligned}$$

B7) Prove that there is a constant λ, such that for any $x \in (0,\pi)$,

$$\sum_{k=1}^{n} \cos(kx) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)} - \lambda.$$

Proof: Note that $2\cos(kx)\sin(\frac{x}{2}) = \sin(k+1/2)x - \sin(k-1/2)x$, hence

$$\sum_{k=1}^{n} \cos(kx)\cdot 2\sin\frac{x}{2} = \sin\left(n + \frac{1}{2}\right)x - \sin\frac{x}{2}, \lambda = \frac{1}{2}.$$

B8) Prove that for any $\psi \in C^1([0,\pi])$,

$$\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x\,\mathrm{d}x = 0.$$

Proof: Since $\sin(n+1/2)x = c_1\sin nx + c_2\cos nx$, where c_1, c_2 are constant, it suffices to show that

$$\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x = \lim_{n\to\infty}\int_0^\pi \psi(x)\cos(2nx)\,\mathrm{d}x = 0.$$

Note that

$$\begin{aligned}
\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x &= \sum_{k=1}^n \int_{(k-1)\pi/n}^{k\pi/n} \psi(x)\sin(2nx)\,\mathrm{d}x \\
&= \sum_{k=1}^n \frac{1}{2n}\int_0^{2\pi} \psi\left(\frac{x}{2n}+\frac{(k-1)\pi}{n}\right)\sin x\,\mathrm{d}x \\
\left(t=\frac{(k-1)\pi}{n}\right) &\leqslant \sum_{k=1}^n \frac{\pi}{n}\sup_{x\leqslant\pi}\left|\psi\left(\frac{x+\pi}{2n}+t\right)-\psi\left(\frac{x}{2n}+t\right)\right| \\
&\leqslant \pi \sup_{0\leqslant x\leqslant \pi-\pi/2n}\left|\psi\left(x+\frac{\pi}{2n}\right)-\psi(x)\right| \to 0.
\end{aligned}$$

since ψ is uniformly continuous on $[0,\pi]$.

B9) Prove that $\zeta(2) = \frac{\pi^2}{6}$.

Proof:

$$\begin{aligned}
\zeta(2) &= \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2} = \lim_{n\to\infty}\sum_{k=1}^n \int_0^\pi h(x)\cos(kx)\,\mathrm{d}x \\
&= \lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x - \frac{1}{2}\left(\frac{x^2}{2\pi}-x\right)\,\mathrm{d}x \\
(\text{B8}) &= \frac{1}{2}\int_0^\pi \left(x-\frac{x^2}{2\pi}\right)\,\mathrm{d}x = \frac{\pi^2}{6}.
\end{aligned}$$

Part 3: $\zeta(2)$ is irrational

Otherwise assume $\pi^2 = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

B10) Define a sequence of polynomials $f_n(x) = \frac{x^n(1-x)^n}{n!}$, where $n \in \mathbb{Z}_{\geqslant 1}$. Prove that for any $k \in \mathbb{Z}$, $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

Proof: If $k \leqslant n-1$, then $f_n^{(k)}(0) = f_n^{(k)}(1) = 0$. If $k \geqslant n$, then

$$\text{if } x^n(1-x)^n = \sum_{k=n}^{2n} c_k x^k, \text{then } f_n^{(k)}(x) = \sum_{m=n}^{2n} c_k\binom{m}{k}x^{m-k} \in \mathbb{Z}[x].$$

Hence $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

B11) Define the sequence

$$F_n(x) = b^n(\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)).$$

Prove that $F_n(0), F_n(1) \in \mathbb{Z}$.
Proof: For $0 \leqslant k \leqslant n$, $b^n\pi^{2n-2k}, f_n^{(2k)}(x) \in Z$, when $x \in \{0,1\}$.

B12) For $n \geqslant 1$, define $\{g_n\}_{n\geqslant 1}, \{A_n\}_{n\geqslant 1}$ as below:

$$g_n(x) = F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x),\ \ A_n = \pi \int_0^1 a^n f_n(x)\sin(\pi x)\,\mathrm{d}x.$$

Prove that $A_n \in \mathbb{Z}$ and $g_n' = \pi^2 a^n f_n(x)\sin(\pi x)$.
Proof: Note that

$$\begin{aligned} g_n'(x) &= b^n\pi^{2n}\sum_{k=0}^{n}\left(f_n^{(2k)}(x)\sin(\pi x) - \pi f_n^{(2k+1)}(x)\cos(\pi x)\right)'(-\pi^2)^k \\ &= b^n\pi^{2n+2} f_n(x)\sin(\pi x). \end{aligned}$$

And

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^1 \mathrm{d}g_n(x) = \frac{1}{\pi}(g_n(1) - g_n(0)) \\ &= F_n(0) + F_n(1) \in \mathbb{Z}. \end{aligned}$$

B13) Prove that there exists $n \in \mathbb{Z}$ such that for all $x \in [0,1]$, $a^n f_n(x) < 1/2$.

Proof:

$$f_n(x) = \frac{1}{n!}(x(1-x))^n \leqslant \frac{1}{n!4^n} \to 0.$$

B14) Prove that there exists $n \in \mathbb{Z}$ such that $A_n \in (0,1)$, leading to contradiction.

Proof: $f_n, \sin(\pi x) \geq 0$, when $x \in [0,1]$, hence $A_n > 0$.
Take n such that $a^n f_n < 1/2$ then $A_n < \frac{\pi}{2}\int_0^1 \sin(\pi x)\,\mathrm{d}x = 1$.
Therefore $A_n \in (0,1)$, contradicting with $A_n \in \mathbb{Z}$.

10.3 PSC: Calculation of Integrals

$a \neq 0, b \neq 0$

(1) $\int_0^\pi \sin^3 x\,\mathrm{d}x$

$$\int_0^\pi \sin^3(x)\,\mathrm{d}x = -2\int_0^{\pi/2} \sin^2(x)\,\mathrm{d}\cos(x) = 2\int_0^1 (1-x^2)\,\mathrm{d}x = \frac{4}{3}.$$

(2) $\int_{-\pi}^{\pi} x^2\cos x\,\mathrm{d}x$

$$\int_{-\pi}^{\pi} x^2\cos(x)\,\mathrm{d}x = (x^2-2)\sin(x) + 2x\cos(x)\Big|_{-\pi}^{\pi} = -4\pi.$$

(3) $\int_0^1 \frac{x}{1+\sqrt{1+x}}\,\mathrm{d}x$

$$\int_0^1 \frac{x}{1+\sqrt{1+x}}\,\mathrm{d}x = \int_0^1 \sqrt{1+x}-1\,\mathrm{d}x = \frac{2}{3}(1+x)^{3/2}-x\Big|_0^1 = \frac{4\sqrt{2}-5}{3}.$$

(4) $\int_0^{\sqrt{3}} x\arctan x\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\sqrt{3}} x\arctan x\,\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{3}} \arctan x\,\mathrm{d}x^2\\ &= \frac{1}{2}x^2\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,\mathrm{d}x\\ &= \frac{3}{2}\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\arctan\sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.\end{aligned}$$

(5) $\int_{-1}^0 (2x+1)\sqrt{1-x-x^2}\,\mathrm{d}x$

$$\int_{-1}^0 (2x+1)\sqrt{1-x-x^2}\,\mathrm{d}x = \int_{-1}^1 \frac{y}{4}\sqrt{5-y^2}\,\mathrm{d}y = 0$$

(6) $\int_{\frac{1}{e}}^e |\log x|\,\mathrm{d}x$

$$\begin{aligned}\int_{\frac{1}{e}}^e |\log x|\,\mathrm{d}x &= \int_1^e \log x\,\mathrm{d}x + \int_1^{1/e} \log x\,\mathrm{d}x\\ &= (x\log x - x)\Big|_1^e + (x\log x - x)\Big|_1^{1/e} = 2 - \frac{2}{e}\end{aligned}$$

(7) $\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x$

$$\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x = a^4\int_0^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{a^4\pi}{16}$$

(8) $\int_0^{\log 2} \sqrt{e^x-1}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\log 2} \sqrt{e^x-1}\,\mathrm{d}x &= \int_1^2 \frac{\sqrt{y-1}}{y}\,\mathrm{d}y = \int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x\\ &= \int_0^{\pi/4} 2\tan^2\theta\,\mathrm{d}\theta = 2 - \frac{\pi}{2}.\end{aligned}$$

(9) $\int_1^2 x^{100} \log x \, \mathrm{d}x$

$$\int_1^2 x^{100} \log x \, \mathrm{d}x = \int_1^2 \log x \, \mathrm{d}\frac{x^{101}}{101} = \frac{2^{101}\log 2}{101} - \int_1^2 \frac{x^{100}}{101} \, \mathrm{d}x$$
$$= \frac{2^{101}\log 2}{101} - \frac{2^{101}-1}{101^2}.$$

(10) $\int_0^a \log(x+\sqrt{x^2+a^2}) \, \mathrm{d}x$

$$\int_0^a \log(x+\sqrt{x^2+a^2}) \, \mathrm{d}x =$$

$$\begin{aligned}\int_0^a \log(x+\sqrt{x^2+a^2}) \, \mathrm{d}x &= a\int_0^1 \log a + \log(t+\sqrt{t^2+1}) \, \mathrm{d}t \\ &= a\log a + a\int_0^1 \log(t+\sqrt{t^2+1}) \, \mathrm{d}t \\ &= a\log a + (\log(1+\sqrt{2}) + \sqrt{2} - 1)a.\end{aligned}$$

$$\begin{aligned}\int_0^1 \log(x+\sqrt{x^2+1}) \, \mathrm{d}x &= x\log(x+\sqrt{x^2+1})\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}} \, \mathrm{d}x \\ (x=\tan\theta) &= \log(1+\sqrt{2}) + \int_0^{\pi/4} \frac{1}{\cos^2\theta} \, \mathrm{d}\cos\theta \\ &= \log(1+\sqrt{2}) + \sqrt{2} - 1.\end{aligned}$$

(11) $\int_0^{\pi/2} \frac{\cos x \sin x}{a^2\sin^2 x + b^2\cos^2 x} \, \mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x \sin x}{a^2\sin^2 x + b^2\cos^2 x} &= \int_0^{\pi/2} \frac{\sin 2x}{a^2+b^2+(b^2-a^2)\cos 2x} \, \mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^1 \frac{1}{(a^2+b^2)+(b^2-a^2)t} \, \mathrm{d}t \\ &= \frac{1}{2(a^2-b^2)} \log\left(\frac{a^2}{b^2}\right).\end{aligned}$$

(12) $\int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} \, \mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} \, \mathrm{d}x &= \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} \, \mathrm{d}x \\ &= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cos x + \sin x} \, \mathrm{d}x \\ &= \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{2}\sin x} \, \mathrm{d}x \\ &= -\frac{\log\tan\left(\frac{\pi}{8}\right)}{\sqrt{2}}.\end{aligned}$$

(13) $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x}\,\mathrm{d}x$

See (12)

(14) $\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x &= \int_0^{\pi/4} \log\frac{\sin x + \cos x}{\cos x}\,\mathrm{d}x\\ &= \int_0^{\pi/4} \log\frac{\sqrt{2}\sin(x+\pi/4)}{\cos x}\,\mathrm{d}x\\ &= \frac{\pi}{8}\log 2.\end{aligned}$$

(15) $\int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{1-x}{5-2x}\,\mathrm{d}x = \frac{1}{2}-\frac{3}{4}\log\frac{5}{3},\\ \int_1^2 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x}{3-2x}\,\mathrm{d}x = -\frac{1}{2}+\frac{3}{4}\log 3,\\ \int_2^3 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 (x+1)\,\mathrm{d}x = \frac{3}{2},\\ \int_3^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x+2}{2x+1}\,\mathrm{d}x = \frac{1}{2}+\frac{3}{4}\log 3,\\ \Longrightarrow \int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= 2+\frac{3}{4}\log\frac{27}{5}\end{aligned}$$

(16) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x &= \int_0^{\pi} x\,\mathrm{d}\arctan\cos x\\ &= -x\arctan\cos x\Big|_0^{\pi} + \int_0^{\pi}\arctan\cos x\,\mathrm{d}x\\ &= \frac{\pi^2}{4}+0 = \frac{\pi^2}{4}.\end{aligned}$$

(17) $\int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\sin t - \cos t}{\sin t}\,\mathrm{d}t\\ &= \frac{\pi}{8} - \frac{\log 2}{4}.\end{aligned}$$

(18) $\int_0^{\pi/2} \frac{\sin 2019x}{\sin x}\,\mathrm{d}x$

$$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,\mathrm{d}x = \int_0^{\pi/2} 1 + \sum_{k=1}^{n} \cos(2kx)\,\mathrm{d}x = \frac{\pi}{2}.$$

(19) $\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x$

$$\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x = \int_{-1}^1 \frac{\log\sqrt{6+t}}{\log\sqrt{6+t}+\log\sqrt{6-t}}\,\mathrm{d}x = 1.$$

(20) $\int_0^1 \frac{1}{\sqrt{1+x^2}+\sqrt{1-x^2}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1+x^2}+\sqrt{1-x^2}} &= \int_0^1 \frac{1}{2}(\sqrt{1+x^2}-\sqrt{1-x^2})\,\mathrm{d}x \\ &= -\frac{\pi}{8}+\frac{\sqrt{2}}{4}+\frac{1}{8}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}.\end{aligned}$$

(21) $\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x &= \int_1^{1+\sqrt{2}} \sqrt{y}\,\mathrm{d}\frac{2y+1-\sqrt{4y+5}}{2} \\ &= \int_1^{1+\sqrt{2}} \sqrt{y}-\frac{\sqrt{y}}{\sqrt{4y+5}}\,\mathrm{d}y \\ \left(y=\frac{z^2-5}{4}\right) &= \frac{2}{3}y^{3/2}\Big|_1^{1+\sqrt{2}} - \int_3^{1+2\sqrt{2}} \frac{\sqrt{z^2-5}}{4}\,\mathrm{d}z \\ &= \frac{2}{3}((1+\sqrt{2})^{3/2}-1)-\frac{3\sqrt{2}-1}{8}+\frac{5}{32}\log\frac{3+\sqrt{2}}{5}.\end{aligned}$$

(22) $\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x$

$$\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x = 0.\text{(by symmetry)}$$

11 Homework 11: Density of Sum of Squares

11.1 PSA: Riemann Integral

A1) $f \in C([a,b]), g \in \mathcal{R}([a,b])$**, where** g **is positive. Prove that there exists** $\xi \in (a,b)$**, such that**

$$\int_a^b fg = f(\xi)\int_a^b g.$$

Proof: Since g is positive on $[a,b]$,

$$\inf_{x\in[a,b]} f(x)\int_a^b g \leqslant \int_a^b fg \leqslant \sup_{x\in[a,b]} f(x)\int_a^b g.$$

By $f\in C([a,b])$, there exists such an $\xi\in(a,b)$.

A2) Prove without using Lebesgue theorem: if f is monotonously increasing on $[a,b]$, then $f\in\mathcal{R}([a,b])$.

Proof: For any $\varepsilon>0$ let $n=[1/\varepsilon]+1$, and

$$\mathcal{C}=\left\{x_k=a+(b-a)\frac{k}{n}:k=0,1,\cdots,n\right\}.$$

Then

$$g(x)=\max_{x_k\leqslant x}\{f(x_k)\}\leqslant f, h(x)=\min_{x_k\geqslant x}\{f(x_k)\}\geqslant f.$$

and both are monotonous simple functions.
Therefore

$$\overline{\int_a^b} f-\underline{\int_a^b} f\leqslant \overline{S}(f;\mathcal{C})-\underline{S}(f;\mathcal{C})=\frac{1}{n}(f(b)-f(a))\to 0.$$

Hence f is Riemann integrable.

A3) Prove that $1_{\mathbb{Q}}$ is not Riemann integrable on $[0,1]$.

Proof: Let $\varepsilon=\frac{1}{2}$. For any $\mathcal{C}=\{0=x_0<\cdots<x_n=1\}$, $\omega(x_{k-1},x_k)=1$, hence

$$\sum_{k=1}^n \omega(x_{k-1},x_k)(x_k-x_{k-1})=1>\varepsilon.$$

Therefore $1_{\mathbb{Q}}$ is not Riemann integrable.

A4) Prove that if $f\in\mathcal{R}([a,b])$, then $|f|^p\in\mathcal{R}([a,b])$, where $p\geqslant 0$.

Proof: Since $x\mapsto |x|^p$ is continuous, $|f|^p$ is continuous as x whenever f is continuous at x. Hence

$$f\in\mathcal{R}([a,b]) \implies |f|^p\in\mathcal{R}([a,b]).$$

A5) Prove Hölder's Inequality: if $f,g\in\mathcal{R}([a,b]),p,q>0,1/p+1/q=1$, then

$$\left|\int_a^b fg\right|\leqslant\left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}.$$

Proof: By A4) the functions are integrable. We can assume that

$$\int_a^b |f|^p=\int_a^b |g|^q=1.$$

Then by Young's inequality,

$$\left|\int_a^b fg\right|\leqslant\int_a^b |f|\cdot|g|\leqslant\int_a^b \frac{1}{p}|f|^p+\frac{1}{q}|g|^q=\frac{1}{p}+\frac{1}{q}=1.$$

A6) Prove Minkowski's inequality: if $f, g \in \mathcal{R}([a,b]), p \geqslant 1$, then

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

Proof: Note that if $1/p + 1/q = 1$, then

$$\begin{aligned}\int_a^b |f+g|^p &= \int_a^b |f| \cdot |f+g|^{1-p} + \int_a^b |g| \cdot |f+g|^{1-p} \\ &\leqslant \left(\left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}\right)\left(\int_a^b |f+g|^{(1-p)q}\right)^{1/q}\end{aligned}$$

Hence

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

The equality holds, when $|f|/|f+g|^{1-p}, |g|/|f+g|^{1-p}$ are both constant, which is equivalent to $|f|/|g|$ is constant.

11.2 PSB: Convex Functions

B1) Assume $f \in \mathcal{R}([a,b])$ and f is convex, prove that

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

Proof: Note that $f\left(\frac{a+b}{2}\right) \leqslant \frac{f(x)+f(a+b-x)}{2} \leqslant \frac{f(a)+f(b)}{2}$, and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \frac{f(x)+f(a+b-x)}{2}\,\mathrm{d}x.$$

Hence

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

B2) Assume f is twice differentiable on $[a,b]$ and for any $x, f''(x) > 0, f(x) \leqslant 0$. Prove that for any x,

$$f(x) \geqslant \frac{2}{b-a}\int_a^b f(y)\,\mathrm{d}y.$$

Proof: For any $x \leqslant y \leqslant b$,

$$f(y) \leqslant \frac{b-y}{b-x}f(x) + \frac{y-x}{b-x}f(b) \leqslant \frac{b-y}{b-x}f(x),$$

hence

$$\int_x^b f(y)\,\mathrm{d}y \leqslant f(x)\int_x^b \frac{b-y}{b-x}\,\mathrm{d}y = \frac{b-x}{2}f(x).$$

Likewise,

$$\int_a^x f(y)\,\mathrm{d}y \leqslant f(x)\int_a^x \frac{y-a}{x-a}\,\mathrm{d}y = \frac{x-a}{2}f(x).$$

Therefore

$$f(x) \geqslant \frac{2}{b-a}\int_a^b f(y)\,\mathrm{d}y.$$

B3) Assume f is twice differentiable on $\mathbb{R}$ and $f''(x) \geqslant 0$, $\varphi \in C([a,b])$. Prove that

$$\frac{1}{b-a}\int_a^b (f\circ\varphi)(t)\,\mathrm{d}t \geqslant f\left(\frac{1}{b-a}\int_a^b \varphi(t)\,\mathrm{d}t\right).$$

Proof: We prove the proposition for any convex function f and φ on the set X.
Let

$$\langle g\rangle = \frac{1}{\mu(X)}\int_X g\,\mathrm{d}\mu.$$

Then since f is convex, there is a constant K such that $f(y) - f(\langle\varphi\rangle) \geqslant K(y - \langle\varphi\rangle)$. Hence

$$\begin{aligned}\langle f(\varphi)\rangle &= \frac{1}{\mu(X)}\int_X f(\varphi(t))\,\mathrm{d}\mu \\ &\geqslant \frac{1}{\mu(x)}\int_X f(\langle\varphi\rangle)\,\mathrm{d}\mu + \frac{1}{\mu(X)}\int_X K(\varphi(t) - \langle\varphi\rangle)\,\mathrm{d}\mu \\ &= f(\langle\varphi\rangle).\end{aligned}$$

B4) Assume $f \in C([a,b])$ and for any x, $f(x) > 0$. Prove that

$$\log\left(\frac{1}{b-a}\int_a^b f\right) \geqslant \frac{1}{b-a}\int_a^b \log f.$$

Proof: Since $-\log x$ is convex, we can use B3).

B5) Prove that if f is convex on $\mathbb{R}$, $\varphi \in C([0,1])$, then

$$f\left(\int_0^1 \varphi\right) \leqslant \int_0^1 f\circ\varphi.$$

Proof: A special case of what we proved in B3).

11.3 PSC: Integrals and Derivatives

C1) Assume $f \in C^1([0,2])$, $|f'| \leqslant 1$, $f(0) = f(2) = 1$. Prove that

$$1 \leqslant \int_0^2 f \leqslant 3.$$

Proof: Note that for $0 \leqslant x \leqslant 1$,

$$|f(x) - 1| = x|f'(\xi)| \leqslant x.$$

and for $1 \leqslant x \leqslant 2$,

$$|f(x) - 1| = (2-x)|f'(\xi)| \leqslant 2 - x.$$

Hence
$$\int_0^2 |f(x)-1|\,\mathrm{d}x \leqslant \int_0^1 x\,\mathrm{d}x + \int_1^2 (2-x)\,\mathrm{d}x = 1.$$

C2) Assume that $f \in C^2([0,1])$. Prove that $\exists \xi \in [0,1]$, such that
$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$
Proof: Let $g(x) = f(x) + f(1-x)$, then
$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) &= \int_0^{1/2} g(x) - 2f\left(\frac{1}{2}\right)\,\mathrm{d}x \\ \text{(integration by parts)} &= -\int_0^{1/2} x g'(x)\,\mathrm{d}x = -\frac{1}{2}\int_0^{1/2} g'(x)\,\mathrm{d}x^2 \\ \text{(integration by parts)} &= \frac{1}{2}\int_0^{1/2} x^2 g''(x)\,\mathrm{d}x.\end{aligned}$$

Note that $g'' \in C([0,1])$ hence by A1), $\exists \eta \in (0, \frac{1}{2})$,
$$\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) = g''(\eta)\frac{1}{2}\int_0^{1/2} x^2\,\mathrm{d}x = \frac{1}{48} g''(\eta).$$
Since $f'' \in C([0,1])$, there exists $\xi \in (\eta, 1-\eta)$, such that
$$f''(\xi) = \frac{f''(\eta) + f''(1-\eta)}{2} = \frac{g''(\eta)}{2}.$$
Therefore
$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$

C3) Assume $f \in C^1([0,1])$. Prove that
$$\max_{x\in[a,b]} |f(x)| \leqslant \frac{1}{b-a}\left|\int_a^b f(x)\,\mathrm{d}x\right| + \int_a^b |f'(x)|\,\mathrm{d}x.$$
Proof: For any $t \in [a,b]$,
$$(b-a)|f(t)| \leqslant \left|\int_a^b f(x)\,\mathrm{d}x\right| + \left|\int_a^b f(x) - f(t)\,\mathrm{d}x\right|$$
where
$$\begin{aligned}\left|\int_a^b f(x) - f(t)\,\mathrm{d}x\right| &= \left|\int_a^b \left(\int_t^x f'(u)\,\mathrm{d}u\right)\,\mathrm{d}x\right| \\ &\leqslant \int_a^b \int_t^x |f'(u)|\,\mathrm{d}u\,\mathrm{d}x \\ &\leqslant (b-a)\int_a^b |f'(u)|\,\mathrm{d}u.\end{aligned}$$

C4) Suppose $f \in C([0,1])$ and for any $g \in C([0,1]), g(0) = g(1) = 0,$ we have

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise assume $f(t) > 0$ for some $t \in (0,1)$, then there exists an $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon) \subset [0,1]$ and $\forall x \in (t-\varepsilon, t+\varepsilon), f(x) > f(t)/2$.
Let

$$g(x) = \begin{cases} 0, & x \notin (t-\varepsilon, t+\varepsilon), \\ 1 - \frac{|x-t|}{\varepsilon}, & x \in (t-\varepsilon, t+\varepsilon). \end{cases}$$

Then

$$\int_0^1 f(x)g(x)\,\mathrm{d}x > \int_{t-\varepsilon}^{t+\varepsilon} \frac{f(t)}{2} g(x)\,\mathrm{d}x > 0,$$

leading to contradiction. Hence $f(x) \equiv 0$.

C5) Suppose $f \in C([0,1])$ and for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$\int_0^1 f(x)x^n\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise, $\int_0^1 f^2 > 0$. By Stone-Weierstrass theorem, for any $\varepsilon > 0$, there is a polynomial P such that $\sup_{x\in[0,1]} |f(x) - P(x)| < \varepsilon$. Hence

$$0 = \int_0^1 f(x)P(x)\,\mathrm{d}x = \int_0^1 f^2 - \int_0^1 f(x)(f(x) - P(x))\,\mathrm{d}x \geqslant \int_0^1 f^2 - \sup_{x\in[0,1]} |f(x)|\varepsilon > 0$$

when $\varepsilon \to 0$, leading to contradiction.

C6) (Gronwall's Inequality) Suppose $\varphi \in C([0,T])$ and for any $t \in [0,T]$, $|\varphi(t)| \leqslant M + k\int_0^t |\varphi(s)|\,\mathrm{d}s$, where M, k are positive real numbers. Prove that $\forall t \in [0,T]$, $|\varphi(t)| \leqslant Me^{kt}$.

Proof: Let
$f : \left[0, \frac{T}{k}\right] \to \mathbb{R}, t \mapsto \frac{e^{-t}|\varphi(t/k)|}{M}$,
then for any $t \in [0, T/k]$,

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(s)e^s\,\mathrm{d}s.$$

Let $f(t) = \sup_{s\in[0,T/k]}\{f(s)\}$ then

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(t)e^s\,\mathrm{d}x = e^{-t} + f(t)(1 - e^{-t}).$$

Hence $f(s) \leqslant f(t) \leqslant 1, \implies |\varphi(t)| \leqslant Me^{kt}$.

C7) Assume $a, b > 0$, $f \in C([-a,b])$. If for any $x \in (-a,b)$, $f(x) > 0$ and $\int_{-a}^b xf(x)\,\mathrm{d}x = 0$. Prove that

$$\int_{-a}^b x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^b f(x)\,\mathrm{d}x.$$

Proof: Note that

$$\int_{-a}^{b}(x+a)(x-b)f(x)\,\mathrm{d}x \leqslant 0.$$

Combined with $\int_{-a}^{b} xf(x)\,\mathrm{d}x = 0$ we get

$$\int_{-a}^{b} x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^{b} f(x)\,\mathrm{d}x.$$

C8) Assume $f \in C([-1,1])$. Prove that

$$\lim_{\lambda\to 0^+}\int_{-1}^{1}\frac{\lambda}{\lambda^2+x^2}f(x)\,\mathrm{d}x = \pi f(0).$$

Proof: Let $M = \sup_{|x|\leqslant 1}|f(x)|$ and

$$g(x) = \frac{\lambda}{\lambda^2+x^2},$$

then (g is sort of a good kernel)

$$\int_{-1}^{1} g(x)\,\mathrm{d}x = 2\arctan\frac{1}{\lambda}.$$

Hence

$$\begin{aligned}
&\left|\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x - \pi f(0)\right| \\
\leqslant &\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \int_{-\varepsilon}^{\varepsilon}|f(x)-f(0)|g(x)\,\mathrm{d}x + \int_{\varepsilon\leqslant x\leqslant 1} Mg(x)\,\mathrm{d}x \\
\leqslant &\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \sup_{|x|\leqslant\varepsilon}|f(x)-f(0)|\pi + 2M\left|\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda}\right| \\
&\to 0
\end{aligned}$$

since

$$\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda} = \arctan\frac{\lambda(1-\varepsilon)}{\lambda^2+\varepsilon} \to 0, \text{when } \lambda\to 0^+.$$

and $\sup_{|x|\leqslant\varepsilon}|f(x)-f(0)| \to 0$ when $\varepsilon\to 0$.

C9) Assume f is differentiable on $[1,\infty)$ and both $\int_1^\infty f(x)\,\mathrm{d}x$ and $\int_1^\infty f'(x)\,\mathrm{d}x$ converges. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

Proof: For any $\varepsilon > 0$, there exists $N > 1$, such that $\forall u, v > N$,

$$\left|\int_u^v f'(x)\,\mathrm{d}x\right| < \varepsilon, \text{ i.e. } |f(u)-f(v)| < \varepsilon$$

Hence for any $u > N$, if $|f(u)| > \varepsilon$,

$$\left|\int_u^M f(x)\,\mathrm{d}x\right| \geqslant (M-u)(|f(u)-\varepsilon|) \to \infty, \text{ as } M\to\infty,$$

which contradicts the fact that $\int_1^\infty f(x)\,\mathrm{d}x$ converges. Therefore $|f(u)| < \varepsilon$ for any $u > N$, which implies $\lim_{x\to\infty} f(x) = 0$.

C10) Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x, \int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

Proof:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = -\int_0^\infty \sin^2 x\,\mathrm{d}\frac{1}{x} = \int_0^\infty \frac{\sin 2x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$
$$\int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{1}{1+x}\,\mathrm{d}\sin x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

11.4 PSD: Calculation of improper integrals

D1)

$$\int_0^1 \log x\,\mathrm{d}x = (x\log x - x)\Big|_0^1 = -1.$$

D2)

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x\Big|_{-\infty}^\infty = \pi.$$

D3)

Calculating residues, we get

$$\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^4+1} = 2\pi i\cdot(Res(f;e^{i\pi/4}) + Res(f;e^{3i\pi/4})) = \frac{\pi}{\sqrt{2}}.$$

Hence

$$\int_0^\infty \frac{\mathrm{d}x}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

D4)

Same as D3)

$$\int_{-\infty}^\infty \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \sqrt{2}\pi.$$

Hence

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

D5)

$$\int_{-\infty}^0 xe^x\,\mathrm{d}x = \int_{-\infty}^0 x\,\mathrm{d}e^x = -\int_{-\infty}^0 e^x\,\mathrm{d}x = -1.$$

D6)

$$\int_0^\infty e^{-\sqrt{x}}\,\mathrm{d}x = 2\int_0^\infty ye^{-y}\,\mathrm{d}y = 2\int_0^\infty e^{-y}\,\mathrm{d}y = 2.$$

D7)

$$\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)^{3/2}} = \frac{1}{a^2}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^{3/2}} = \frac{1}{2a^2}B\left(\frac{1}{2},1\right) = \frac{1}{a^2}.$$

(We can also substitute $x = a\tan\theta$).

D8)

$$\int_2^\infty \frac{\mathrm{d}x}{x^2+x-2} = \frac{1}{3}\log\frac{x-1}{x+2}\Big|_2^\infty = \frac{\log 3}{3}.$$

D9)

$$\begin{aligned}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+x+1)^2} &= \frac{8}{3\sqrt{3}}\int_{-\infty}^\infty \frac{\mathrm{d}u}{(1+u^2)^2}\\ (u=\tan\theta) &= \frac{8}{3\sqrt{3}}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \frac{4\sqrt{3}\pi}{9}.\end{aligned}$$

D10)

$$\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} 1\,\mathrm{d}\theta = \pi.$$

D11)

$$\int_{-1}^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2}\theta\,\mathrm{d}\theta = 0.$$

D12)

Let γ be the unit circle, then

$$\begin{aligned}\int_{-1}^1 \frac{\mathrm{d}x}{(2-x)^2\sqrt{1-x^2}} &= \int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}\theta}{(2-\sin\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{(2-\sin\theta)^2}\\ &= \frac{1}{2}\int_\gamma -\frac{4}{i}\frac{z\mathrm{d}z}{(z^2-4iz-1)^2}\\ &= -4\pi\mathrm{Res}\left(\frac{z}{(z^2-4iz-1)^2};(2-\sqrt{3})i\right)\\ &= \frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

D13)

$$\int_0^1 \frac{\arcsin\sqrt{x}}{x(1-x)}\,\mathrm{d}x > \int_{1/4}^1 \frac{\pi}{6}\frac{1}{1-x}\,\mathrm{d}x \text{ which diverges.}$$

D14)

$$\int_0^1 (1-x)^n x^{1/2-1}\,\mathrm{d}x = B\left(n+1, \frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{n!2^{n+1}}{(2n+1)!!}.$$

D15)

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x = \begin{cases} \frac{(n-1)!!}{n!!}, n \text{ is odd,} \\ \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, n \text{ is even.} \end{cases}$$

D16)

Using integration by parts, and substitute $x = e^{-y}$,

$$\begin{aligned} \int_0^1 x^m (\log x)^n\,\mathrm{d}x &= (-1)^n \int_0^\infty e^{-(m+1)y} y^n\,\mathrm{d}y \\ &= (-1)^n \frac{n!}{(m+1)^n}\int_0^\infty e^{-(m+1)y}\,\mathrm{d}y = \frac{(-1)^n n!}{(m+1)^{n+1}}. \end{aligned}$$

D17)

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p} = \int_{\log 2}^\infty \frac{\mathrm{d}y}{y^p} = \frac{(\log 2)^{1-p}}{p-1}.$$

D18)

Substitute $x = ay$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\log a}{2a} + \frac{1}{a}\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = \frac{\pi \log a}{2a}.$$

since by substituting $y = 1/z$,

$$\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = -\int_0^\infty \frac{\log z}{1+z^2}\,\mathrm{d}z = 0.$$

D19)

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \Gamma(n) = (n-1)!.$$

D20)

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(ax^2+2bx+c)^n} = \frac{1}{d^n}\sqrt{\frac{d}{a}} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1+u^2)^n} = \frac{1}{d^n}\sqrt{\frac{d}{a}}\pi\frac{(2n-3)!!}{(2n-2)!!}.$$

where $d = \frac{ac-b^2}{a}$

D21)

$$\int_0^{\infty} x^{2n-1}e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\infty} y^{n-1}e^{-y}\,\mathrm{d}y = \frac{(n-1)!}{2}.$$

D22)

The Poisson kernel

$$\begin{aligned} \frac{1-r^2}{1-2r\cos x + r^2} &= \frac{1-r^2}{(1-re^{ix})(1-re^{-ix})} \\ &= (1-r^2)\sum_{n=0}^{\infty} r^n e^{inx} \sum_{m=0}^{\infty} r^m e^{-imx} \\ &= \sum_{k=-\infty}^{\infty} r^{|k|}e^{ikx}. \end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos x + r^2}\,\mathrm{d}x = 2\pi.$$

D23)

$$\begin{aligned} \int_0^{\infty} e^{-ax}\cos bx\,\mathrm{d}x &= \frac{1}{b}\int_0^{\infty} e^{-ax}\,\mathrm{d}\sin bx = \frac{a}{b}\int_0^{\infty} e^{-ax}\sin bx\,\mathrm{d}x \\ &= -\frac{a}{b^2}\int_0^{\infty} e^{-ax}\,\mathrm{d}\cos bx = \frac{a}{b^2} - \frac{a^2}{b^2}\int_0^{\infty} e^{-ax}\cos bx\,\mathrm{d}x \\ &= \frac{a}{a^2+b^2}. \end{aligned}$$

D24)

Same as (23),

$$\int_0^{\infty} e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2}.$$

D25)

$$\int_1^\infty \frac{\mathrm{d}x}{x(x+1)\cdots(x+n)} = \lim_{N\to\infty}\int_0^N \sum_{k=0}^n \frac{(-1)^k}{x+k}\binom{n}{k}\,\mathrm{d}x$$
$$= \lim_{N\to\infty}\sum_{k=0}^n (-1)^k\binom{n}{k}\log\left(\frac{N+k}{(k+1)}\right)$$
$$= -\sum_{k=0}^n (-1)^k\binom{n}{k}\log(k+1) + \lim_{N\to\infty}\sum_{k=0}^n (-1)^k\binom{n}{k}\log\left(1+\frac{k}{N}\right)$$
$$= -\sum_{k=0}^n (-1)^k\binom{n}{k}\log(k+1).$$

D26)

$$\int_0^\pi \log\sin x\,\mathrm{d}x = 2\int_0^{\pi/2}\log\sin x\,\mathrm{d}x = 2\int_0^{\pi/2}\log\cos x\,\mathrm{d}x$$
$$= \int_0^{\pi/2}\log\sin 2x - \log 2\,\mathrm{d}x = \frac{1}{2}\int_0^\pi \log\sin x\,\mathrm{d}x - \frac{\pi}{2}\log 2$$
$$= -\pi\log 2.$$

D27)

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Note that

$$\max\{0, 1-x^2\} < e^{-x^2} < \frac{1}{1+x^2}.$$

Hence

$$\frac{(2n)!!}{(2n+1)!!} < \int_0^\infty e^{-nx^2}\,\mathrm{d}x < \frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

Therefore

$$\sqrt{n}\frac{(2n)!!}{(2n+1)!!} < \int_0^\infty e^{-x^2}\,\mathrm{d}x < \sqrt{n}\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

By Wallis's formula,

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

11.5 PSE: Density of sum of squares

Let $I = (0,\infty)$.

Part 1

E1) Prove that $e^{-u}/\sqrt{u}$ is integrable on I, and let $K = \int_0^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u$.

Proof:

$$\int_1^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u < \int_1^\infty e^{-u}\,\mathrm{d}u = \frac{1}{e}.$$
$$\int_0^1 e^{-u}/\sqrt{u}\,\mathrm{d}u < \int_0^1 u^{-1/2}\,\mathrm{d}u = \frac{1}{2}.$$

E2) Prove that for any $x \in I$,

$F(x) = \int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u$
is well-defined.
Proof:

$$F(x) < \int_0^\infty \frac{e^{-u}}{x\sqrt{u}}\,\mathrm{d}u \text{ converges.}$$

E3) Prove that $F \in C^1(I)$ and calculate $F'(x)$.

Solution: Let $f(x,u) = \frac{e^{-u}}{\sqrt{u}(u+x)}$, then f is uniformly continuous on any closed subinterval of I, and so is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x,u) = -\frac{e^{-u}}{\sqrt{u}(u+x)^2}.$$

Also,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} f(x,u)\,\mathrm{d}u$$

converges uniformly.
Hence F is continuously differentiable and

$$F'(x) = -\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u.$$

E4) Prove that for any $x \in I$,

$$xF'(x) - \left(x - \frac{1}{2}\right)F(x) = -K.$$

Proof: We show that

$$x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u + \left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u = \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u.$$

Note that, by substituting $u \to ux$,

$$x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u = \frac{1}{\sqrt{x}}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u,$$
$$\left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u = \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+x)}\,\mathrm{d}u,$$
$$\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u = \sqrt{x}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.$$

Hence it is equivalent to

$$x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u = \int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u + \left(x-\frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u.$$

Note that $\mathrm{d}e^{-ux}\sqrt{u} = -e^{-ux}\left(x\sqrt{u}-\frac{1}{2\sqrt{u}}\right)\mathrm{d}u$, hence

$$\begin{aligned}
&x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u - \left(x+\frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u\\
&=\int_0^\infty e^{-ux}\left(x\sqrt{u}-\frac{1}{2\sqrt{u}}\right)\frac{\mathrm{d}u}{1+u}\\
&=-\int_0^\infty \frac{\mathrm{d}e^{-ux}\sqrt{u}}{1+u} = -\int_0^\infty e^{-ux}\sqrt{u}\,\frac{\mathrm{d}u}{(1+u)^2}\\
&=\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{(1+u)^2} - \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{1+u}.
\end{aligned}$$

($\sqrt{u} = \frac{1}{\sqrt{u}}((1+u)-1)$)

E5) Define $G: I\to\mathbb{R}, x\mapsto \sqrt{x}e^{-x}F(x)$. Prove that $\exists C\in\mathbb{R}$ such that

$$G(x) = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$

Proof: By B4)

$$G'(x) = \sqrt{x}e^{-x}F'(x) + \left(\frac{1}{2\sqrt{x}}-\sqrt{x}\right)e^{-x}F(x) = -K\frac{e^{-x}}{\sqrt{x}}.$$

Hence let $C = G(0)$, then

$$G(x) = C + \int_0^x G'(x)\,\mathrm{d}x = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$

E6) Calculate the value of K.

Solution: Note that when $x\to\infty$, $F(x)\to 0$ hence $G(x)\to 0$. Therefore

$$0 = \lim_{x\to\infty} G(x) = G(0) - K\int_0^\infty \frac{e^{-t}}{t}\,\mathrm{d}t = G(0) - K^2.$$

Where

$$\begin{aligned}
G(0) &= \lim_{x\to 0^+}\frac{\sqrt{x}}{e^x}\int_0^\infty \frac{e^{-u}}{\sqrt{u}(x+u)}\,\mathrm{d}u = \lim_{x\to 0}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u\\
&= \int_0^\infty \frac{1}{\sqrt{u}(1+u)}\,\mathrm{d}u = \int_0^\infty \frac{2\mathrm{d}t}{1+t^2} = \pi.
\end{aligned}$$

Hence $K = \sqrt{\pi}$.

Part 2

Define

$$f(x) = \sum_{n=1}^\infty \frac{e^{-nx}}{\sqrt{n}}, g(x) = \sum_{n=0}^\infty \sqrt{n}e^{-nx}.$$

E7) Prove that f, g are well-defined on I and are both continuous on I.

Proof: Let $C = \sup_{x \geqslant 0} x^3 e^{-x}$, then

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{\sqrt{n}} < \sum_{n=0}^{\infty} \sqrt{n} e^{-nx} \leqslant \sum_{n=1}^{\infty} \frac{C}{(nx)^2 \sqrt{x}} \text{ converges.}$$

On any closed sub-interval of I, the two series both converge uniformly, and e^{-nx} is continuous, hence f, g are both continuous on I.

E8) Prove that $\forall x \in I$,

$$\int_1^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

Proof: The function $e^{-ux}/\sqrt{u}$ is monotonously decreasing by u, hence

$$\int_1^{N} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant \sum_{n=1}^{N-1} \frac{e^{-nx}}{\sqrt{n}} \leqslant f(x).$$

$$\sum_{n=1}^{N} \frac{e^{-nx}}{\sqrt{n}} \leqslant \int_0^{N} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

Therefore

$$\int_1^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

E9) Prove that $\exists C_0$ such that

$$\lim_{x \to 0^+} \sqrt{x} f(x) = C_0.$$

Proof: By E8)

$$\sqrt{x} f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{ux}} \,\mathrm{d}ux = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} \,\mathrm{d}t = \sqrt{\pi}.$$

$$\sqrt{x} f(x) \geqslant \int_1^{\infty} \frac{e^{-ux}}{\sqrt{ux}} \,\mathrm{d}ux = \int_x^{\infty} \frac{e^{-t}}{\sqrt{t}} \,\mathrm{d}t \to \sqrt{\pi}.$$

Hence

$$\lim_{x \to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E10) Define the sequence $\{a_n\}_{n \geqslant 1}$ as follows:

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right) - 2\sqrt{n}.$$

Prove that $\{a_n\}$ converges.
Proof: By Euler-Maclaurin formula, for $f(x) = 1/\sqrt{x}$,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &= \frac{f(1)+f(n)}{2} + \int_1^n \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x \\ &= 2\sqrt{n} - \frac{3}{2} + \frac{1}{2\sqrt{n}} + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x\end{aligned}$$

Hence

$$\lim_{n\to\infty} a_n = -\int_1^\infty \frac{\widetilde{B}_1(x)}{2x^{3/2}}\,\mathrm{d}x - \frac{3}{2}.$$

E11) Prove that for any $x \in I$, the function

$$h(x) = \sum_{n\geqslant 1} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right) e^{-nx}$$

is well-defined.
Proof: By E10), $|a_n|$ is bounded, hence

$$h(x) = \sum_{n\geqslant 1} 2\sqrt{n}e^{-nx} + a_n e^{-nx} = 2g(x) + \sum_{n\geqslant 1} a_n e^{-nx} \leqslant 2g(x) + \sup_n |a_n| \cdot \frac{1}{e^x - 1}.$$

E12) Express $h(x)$ using $f(x)$ and find a constant C_1 such that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} h(x) = C_1.$$

Proof: Since $e^{-nx}/k > 0$, we can interchange the sums

$$h(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{n=k}^{\infty} e^{-nx} = \sum_{k=1}^{\infty} \frac{e^{-kx}}{\sqrt{k}} \frac{1}{1-e^{-x}} = \frac{1}{1-e^{-x}} f(x).$$

Therefore

$$\lim_{x\to 0^+} x^{3/2} h(x) = \lim_{x\to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E13) Prove that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} g(x) = \frac{\sqrt{\pi}}{2}.$$

Proof:

$$\lim_{x\to 0^+} x^{3/2} |h(x) - 2g(x)| \leqslant \lim_{x\to 0^+} \sup_n |a_n| \cdot \frac{x^{3/2}}{e^x - 1} = 0.$$

Hence

$$\lim_{x\to 0^+} x^{3/2} g(x) = \frac{1}{2} \lim_{x\to 0^+} x^{3/2} h(x) = \frac{\sqrt{\pi}}{2}.$$

Part 3

Given $A \subset \mathbb{Z}_{\geqslant 0}$, we can define a sequence $\{a_n\}_{n\geqslant 0}$:

$$a_n = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$$

Define the set $I_A \subset \mathbb{R}_{\geqslant 0}$ as follows:

$$I_A = \left\{ x \geqslant 0 : \text{the series } \sum_{n\geqslant 0} a_n e^{-nx} \text{ converges} \right\}.$$

Define the function $f_A : I_A \to \mathbb{R}$ as follows:

$$f_A(x) = \sum_{n\geqslant 0} a_n e^{-nx}.$$

Let $\Phi(A) = \lim_{x\to 0} x f_A(x)$ (if the limit exists) and let

$$\mathcal{S} = \{A \subset \mathbb{Z}_{\geqslant 0} : \lim_{x\to 0^+} x f_A(x) \text{ exists}\}.$$

For example, let

$$A_1 = \{n^2 : n \in \mathbb{Z}_{\geqslant 1}\}, A_2 = \{p^2 + q^2 : p, q \in \mathbb{Z}_{\geqslant 1}\}.$$

E14) Determine the set I_A.

Solution: If A is finite, then $I_A = \mathbb{R}_{\geqslant 0}$. Otherwise $I_A = \mathbb{R}_{>0} = I$.

E15) Given $A \subset \mathbb{Z}_{\geqslant 0}$, for any $n \geqslant 0$, define the set $A_{\leqslant n}$:

$$A_{\leqslant n} = \{ \in A : \leqslant n\}.$$

Prove that for any $x > 0$, the series

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx}$$

converges, and satisfy

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \frac{f_A(x)}{1 - e^{-x}}.$$

Proof: $|A_{\leqslant n}| \leqslant n + 1$, hence

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} \text{ converges.}$$

Therefore

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \cdot e^{-nx} = \sum_{k=0}^{\infty} a_k \cdot \frac{e^{-kx}}{1 - e^{-x}} = \frac{f_A(x)}{1 - e^{-x}}.$$

E16) Prove that for any $x > 0$

$$\frac{f_{A_1}(x)}{1-e^{-x}} = \sum_{n=0}^{\infty} \lfloor \sqrt{n} \rfloor e^{-nx}.$$

Proof: By E15),

$$|A_{1\,\leqslant n}| = \sum_{k=0}^{n} [\sqrt{k} \in \mathbb{Z}_{\geqslant 1}] = \lfloor \sqrt{n} \rfloor.$$

E17) Prove that

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x)$$

exists and calculate the value of $\Phi(A_1)$.
Proof:

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x) = \lim_{x\to 0^+} \sqrt{x}(1-e^{-x}) \left(g(x) - \sum_{n=0}^{\infty} \{\sqrt{n}\} e^{-nx} \right).$$

Since $1-e^{-x} \sim x$, $g(x) \sim \frac{\sqrt{\pi}}{2} x^{-3/2}$, and

$$\left| \sum_{n=0}^{\infty} \left\{ \sqrt{n} \right\} e^{-nx} \right| \leqslant \frac{1}{1-e^{-x}}.$$

Hence

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x) = \frac{\sqrt{\pi}}{2}.$$

and

$$\Phi(A_1) = \lim_{x\to 0^+} x f_{A_1}(x) = 0.$$

E18) Let $v(n) = \#\{(p,q) \in \mathbb{Z}^2_{\geqslant 1} : p^2+q^2 = n\}$. Prove that for any $x > 0$, the series

$$\sum_{n\geqslant 1} v(n) e^{-nx}$$

converges and

$$\sum_{n\geqslant 1} v(n) e^{-nx} = (f_{A_1}(x))^2.$$

Proof: Since $v(n) \leqslant n$, $\sum_{n\geqslant 1} v(n) e^{-nx}$ converges.

$$\sum_{n\geqslant 1} v(n) e^{-nx} = \sum_{n\geqslant 1} \sum_{k=0}^{n} a_k a_{n-k} e^{-nx} = \sum_{n\geqslant 1} \sum_{k=0}^{n} a_k e^{-kx} \cdot a_{n-k} e^{-(n-k)x} = (f_{A_1}(x))^2.$$

E19) Prove that for any $x > 0$

$$f_{A_2}(x) \leqslant (f_{A_1}(x))^2$$

and give an upper-bound of $\Phi(A_2)$ (assuming it exists).
Proof:

$$f_{A_2}(x) = \sum_{n \geqslant 1} [v(n) \geqslant 1] \cdot e^{-nx} \leqslant \sum_{n \geqslant 1} v(n) e^{-nx} = (f_{A_1}(x))^2.$$

Hence

$$\Phi(A_2) = \lim_{x \to 0^+} x f_{A_2}(x) \leqslant \lim_{x \to 0^+} (\sqrt{x} f_{A_1}(x))^2 = \frac{\pi}{4}.$$

Part 4

Assume $\{a_n\}_{n \geqslant 0}$ is a sequence of non-negative numbers, such that for any $x > 0$ the series

$$S(x) = \sum_{n \geqslant 0} a_n e^{-nx}$$

converges. Moreover, assume that the limit below exists:

$$\lim_{x \to 0^+} x S(x) = \lim_{x \to 0^+} x \sum_{n \geqslant 0} a_n e^{-nx} = \ \in [0, +\infty).$$

Let $F = \{f : [0,1] \to \mathbb{R}\}$, $E_0 = C([0,1])$. Let E be the space of piecewise continuous functions, and define the norm on E:

$$\|\psi\|_\infty = \sup_{x \in [0,1]} |\psi(x)|.$$

E20) Define $L : E \to F$ as follows:

$$(L(\psi))(x) = \sum_{n=0}^{\infty} a_n e^{-nx} \psi(e^{-nx}),\ \psi \in E.$$

Prove that L is well-defined and is linear. Moreover, if for any $x \in [0,1]$, $\psi_1(x) \leqslant \psi_2(x)$, the for any $x \in [0,1]$,

$$(L(\psi_1))(x) \leqslant (L(\psi_2))(x).$$

Proof: Since $\psi \in E$, ψ is bounded, hence L is well-defined and is clearly linear. The inequality holds since a_n are non-negative.

E21) Define the subspace of E

$$E_1 = \{\psi \in E : \lim_{x \to 0^+} x(L(\psi))(x) \text{ exists}\}.$$

Define the linear map $\Delta : E_1 \to \mathbb{R}$ as follows:

$$\Delta(\psi) = \lim_{x \to 0^+} x(L(\psi))(x),\ \psi \in E_1.$$

Prove that E_1 is a subspace of E and there is a constant $M > 0$ such that for any $\psi \in E_1$,

$$|\Delta(\psi)| \leqslant M \|\psi\|_\infty.$$

Proof: Since L is linear, so is Δ, thus E_1 is clearly a subspace of E.

$$|\Delta(\psi)| = \left| \lim_{x \to 0^+} x \sum_{n=0}^{\infty} a_n e^{-nx} \psi(e^{-nx}) \right| \leqslant \|\psi\|_\infty \cdot |\lim_{x \to 0^+} x S(x)| = \|\psi\|_\infty.$$

E22) For the polynomial $P_n(x) = x^n$, prove that $P_n \in E_1$ and calculate $\Delta(P_n)$.

Proof:

$$\Delta(P_n) = \lim_{x \to 0^+} x \sum_{k=0}^{\infty} a_k e^{-kx} e^{-nkx} = \frac{1}{n+1}.$$

E23) Prove that $E_0 \subset E_1$ and for every $\psi \in E_0$ calculate $\Delta(\psi)$.

Proof: Since Δ is linear, by E22) we know that for any polynomial P,

$$\Delta(P) = \int_0^1 P(x)\,\mathrm{d}x.$$

By Stone-WeierstraSS theorem, any continuous function on $[0,1]$ can be uniformly approximated with polynomials, hence (same as E24)

$$\Delta(\psi) = \int_0^1 \psi(x)\,\mathrm{d}x,\ \forall \psi \in E_0.$$

E24) For $a \in (0,1)$, $\varepsilon \in (0, \min(a, 1-a))$, define the functions

$$g_-(x) = \begin{cases} 1, & x \in [0, a-\varepsilon]; \\ \frac{a-x}{\varepsilon}, & x \in (a-\varepsilon, a) \\ 0, & x \in [a, 1] \end{cases}, g_+(x) = \begin{cases} 1, & x \in [0, a]; \\ \frac{a+\varepsilon-x}{\varepsilon}, & x \in (a, a+\varepsilon) \\ 0, & x \in [a+\varepsilon, 1] \end{cases}.$$

Prove that $g_\pm \in E_0$ and calculate $\Delta(g_\pm)$. Further prove that $\mathbf{1}_{[0,a]} \in E_1$ and calculate $\Delta(\mathbf{1}_{[0,a]})$.
Proof: $g_\pm \in E_0$ is trivial, and $\Delta(g_\pm) = \int_0^1 g_\pm = (a \pm \varepsilon/2)$. Since $g_- \leqslant \mathbf{1}_{[0,a]} \leqslant g_+$,

$$x(L(g_-))(x) \leqslant x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x)$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < x < \delta$,

$$|x(L(g_-))(x) - \Delta(g_-)|, |x(L(g_+))(x) - \Delta(g_+)| < \frac{\varepsilon}{2}.$$

Hence for any $0 < x < \delta$,

$$x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x) \leqslant \Delta(g_+) + \frac{\varepsilon}{2} = a + \varepsilon.$$
$$x(L(\mathbf{1}_{[0,a]}))(x) \geqslant x(L(g_-))(x) \geqslant \Delta(g_-) - \frac{\varepsilon}{2} = a - \varepsilon.$$

Therefore

$$\Delta(\mathbf{1}_{[0,a]}) = \lim_{x \to 0^+} x(L(\mathbf{1}_{[0,a]}))(x) = a.$$

E25) Prove that $E_1 = E$ and for $\psi \in E$ determine the formula of $\Delta(\psi)$.

Proof: Use the same method as E24) applied to Darboux's sum. Hence

$$E_1 = E, \text{ and } \Delta(\psi) = \int_0^1 \psi(x)\,\mathrm{d}x.$$

E26) Define the function

$$\psi(x)=\begin{cases}0, & x\in[0,e^{-1});\\ \frac{1}{x}, & x\in[e^{-1},1].\end{cases}$$

Prove the following equation by calculating $L(\psi)\left(\frac{1}{N}\right)$:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}a_k = .$$

Proof:

$$(L(\psi))\left(\frac{1}{N}\right)=\sum_{n=0}^{\infty}a_n e^{-n/N}\psi(e^{-n/N})=\sum_{n=0}^{N}a_n.$$

Hence by E25),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}a_n=\Delta(\psi)=\int_0^1\psi(x)\,\mathrm{d}x= .$$

E27) Consider $A\in\mathcal{S}$, and calculate

$$\lim_{n\to\infty}\frac{|A_{\leqslant n}|}{n}.$$

which is called the asymptomatic density of A on $\mathbb{Z}_{\geqslant 0}$.
Solution:

$$\lim_{n\to\infty}\frac{|A_{\leqslant n}|}{n}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}a_n=\lim_{x\to 0^+}x\sum_{n=0}^{\infty}a_n e^{-nx}=\Phi(A).$$

E28) Calculate

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}v(k)}{n},$$

and give an upper-bound of the asymptomatic density of A_2.
Solution:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}v(k)}{n}=\lim_{x\to 0^+}x\sum_{n=0}^{\infty}v(n)e^{-nx}=\lim_{x\to 0^+}x(f_{A_1}(x))^2=\frac{\pi}{4}.$$

From E19) $\Phi(A_2)\leqslant\frac{\pi}{4}$.

> Quote:
> God does not care about our mathematical difficulties. He integrates empirically.
> ——Albert Einstein

12 Homework 12: Oscillatory Intergral

12.1 PSA: Stieltjes Integral

Let μ be a monotonic function on $I=[a,b]$.

A1) For any pair of partitions $\sigma, \sigma' \in \mathcal{S}(I)$,

$$\underline{S}_\mu(f;\sigma) \leqslant \overline{S}_\mu(f;\sigma').$$

Proof: Suppose $\mathcal{C} = \sigma \cup \sigma'$, then

$$\underline{S}_\mu(f;\sigma) \leqslant \underline{S}_\mu(f;\mathcal{C}) \leqslant \overline{S}_\mu(f;\mathcal{C}) \leqslant \overline{S}_\mu(f;\sigma').$$

A2) For any $\rho \in C([a,b]), \rho \geqslant 0$, $\mu(x) = \int_a^x \rho(t)\,\mathrm{d}t$. Prove that for any $f \in \mathcal{R}([a,b])$, $f \in \mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu = \int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

Proof: Consider any $\mathcal{C} = \{x_0, x_1, \cdots, x_n\}$, then if we denote $m_i = \inf_{x\in[x_{i-1},x_i]} f(x), u_i = \inf_{x\in[x_{i-1},x_i]} \rho(x), v_i = \inf_{x\in[x_{i-1},x_i]} f(x)\rho(x), M = \sup_{x\in[a,b]} f(x)$, $v_i - m_i u_i \leqslant f(t)\rho(t) - f(t)u_i \leqslant M\omega_\rho(x_{i-1}, x_i)$. Hence for any $\varepsilon > 0$ there exists a $\delta > 0$, for any $\max\{x_i - x_{i-1}\} < \delta$, $\sup_{x,y\in[x_{i-1},x_i]} |\rho(x) - \rho(y)| < \varepsilon$. Then

$$\begin{aligned}\underline{S}(f\rho;\mathcal{C}) &= \sum_{k=1}^n v_k(x_k - x_{k-1}) \leqslant \sum_{k=1}^n u_i m_i (x_k - x_{k-1}) + M\varepsilon(b-a)\\ &\leqslant M\varepsilon(b-a) + \sum_{k=1}^n m_i \int_{x_{k-1}}^{x_k} \rho(t)\,\mathrm{d}t = M\varepsilon(b-a) + \underline{S}_\mu(f;\mathcal{C}).\end{aligned}$$

The other side is similar, hence $\sup\{\underline{S}_\mu(f;\mathcal{C})\} = \inf\{\overline{S}_\mu(f;\mathcal{C})\}$ so $f \in \mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu = \int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

A3) Prove that $\mathcal{R}(I;\mu)$ is a linear space on $\mathbb{R}$ and the integration operator

$$\int_a^b \cdot\,\mathrm{d}\mu : \mathcal{R}(I;\mu) \to \mathbb{R}, f \mapsto \int_a^b f\,\mathrm{d}\mu.$$

is linear.

Proof: Since $\underline{S}_\mu(\cdot;\mathcal{C})$ and $\overline{S}_\mu(\cdot;\mathcal{C})$ is linear for any $\mathcal{C}$, $\mathcal{R}(I;\mu)$ is clearly a linear space on $\mathbb{R}$, and $\int_a^b \cdot\,\mathrm{d}\mu$ is a linear operator.

A4) Suppose $f_1, f_2 \in \mathcal{R}(I;\mu)$. If the any $x \in I$, $f_1(x) \leqslant f_2(x)$, then

$$\int_a^b f_1\,\mathrm{d}\mu \leqslant \int_a^b f_2\,\mathrm{d}\mu.$$

Proof: By A3), we can assume $f_1 = 0$. Then for any $\mathcal{C}$, $\underline{S}_\mu(f;\mathcal{C}) \geqslant 0$ since $f \geqslant 0$, hence $\int_a^b f\,\mathrm{d}\mu = \sup\{\underline{S}_\mu(f;\mathcal{C})\} \geqslant 0$.

A5) If $f \in \mathcal{R}([a,b];\mu)$, then for any $c \in [a,b]$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Stieltjes integrable and

$$\int_a^b f \,\mathrm{d}\mu = \int_a^c f \,\mathrm{d}\mu + \int_c^b f \,\mathrm{d}\mu.$$

Proof: For any partition σ, let $\sigma' = \sigma \cup \{c\}$, then σ' can be split into two partitions of the intervals $[a,c]$ and $[c,b]$: $\sigma' = \sigma_1 \cup \sigma_2$, such that $\underline{S}_\mu(f;\sigma') = \underline{S}_\mu(f;\sigma_1) + \underline{S}_\mu(f;\sigma_2)$ and $\overline{S}_\mu(f;\sigma') = \overline{S}_\mu(f;\sigma_1) + \overline{S}_\mu(f;\sigma_2)$. Hence

$$\inf \underline{S}_\mu(f;\sigma_1) + \inf \underline{S}_\mu(f;\sigma_2) \leqslant \inf \underline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma_1) + \sup \overline{S}_\mu(f;\sigma_2).$$

Therefore

$$\int_a^b f \,\mathrm{d}\mu = \int_a^c f \,\mathrm{d}\mu + \int_c^b f \,\mathrm{d}\mu.$$

A6) If $f, g \in \mathcal{R}([a,b];\mu)$, then $f \cdot g \in \mathcal{R}([a,b];\mu)$.

Proof: Same as in the case of the Riemann integral.

A7) Define Stieltjes integral on the interval $[0,\infty)$: Suppose $f \in C([0,\infty))$ is continuous and bounded, define

$$\int_0^\infty f \,\mathrm{d}\mu = \lim_{M\to\infty} \int_0^M f \,\mathrm{d}\mu.$$

Suppose $\{\alpha_n\}_{n\geqslant 1}$ is a sequence of positive real numbers and $\sum_{n=1}^\infty \alpha_n$ converges, define the monotonic function $\mu = \sum_{n=1}^\infty \alpha_n \mathbf{1}_{\geqslant n}$, then

$$\int_1^\infty f \,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

Proof: Note that

$$\mu(x+0) - \mu(x-0) = \begin{cases} 0, & x \notin \mathbb{Z}, \\ \alpha_x, & x \in \mathbb{Z}. \end{cases}$$

Hence

$$\int_0^N f \,\mathrm{d}\mu = \sum_{n=1}^{N-1} f(n)\alpha_n.$$

By definition,

$$\int_0^\infty f \,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

A8) $f, g \in \mathcal{R}([a,b];\mu)$ are real-valued Riemann integrable functions. Suppose for any $x \in [a,b]$, $g(x) \geqslant 0$. Let

$$m = \inf_{x\in I} f(x),\ M = \sup_{x \in I} f(x).$$

Then there exists $\in [m,M]$ such that

$$\int_a^b fg \,\mathrm{d}\mu = \int_a^b g \,\mathrm{d}\mu.$$

Proof: Note that $mg \leqslant fg \leqslant Mg$, and use A4).

A9) Construct a Stieltjes integral to show that Abel summation method is a special case of integration by parts.

Proof:
The Abel summation formula states that

$$\sum_{i=1}^{n} T_i(S_i - S_{i-1}) = T_n S_n - T_1 S_0 - \sum_{i=1}^{n-1} S_i(T_{i+1} - T_i).$$

Consider the monotonically increasing function $\mu : [0, n] \to \mathbb{R}, x \mapsto T_{\lceil x \rceil}, \mu(0) = T_1$, and f be a polynomial such that $f(k) = S_k$ for $k = 0, 1, \cdots, n$. Then

$$\int_0^n f'\mu \,\mathrm{d}x = \sum_{k=1}^{n} \int_{k-1}^{k} f'\mu = \sum_{k=1}^{n} \int_{k-1}^{k} f'(x) T_k \,\mathrm{d}x = \sum_{k=1}^{n} T_k(S_k - S_{k-1}).$$

While

$$\int_0^n f \,\mathrm{d}\mu = \sum_{k=1}^{n-1} f(k)(\mu(k+0) - \mu(k)) = \sum_{k=1}^{n-1} S_k(T_{k+1} - T_k).$$

and

$$f\mu\Big|_0^n = T_n S_n - T_1 S_0.$$

Hence the formula is a special case of integration by parts.

12.2 PSB: Convergence of Improper Integrals

b can be ∞.

B1) Assume $f : [a, b) \to \mathbb{R}$, and for any $b^- < b$, f is integrable on $[a, b^-]$. Prove that the integral $\int_a^b f(x)\,\mathrm{d}x$ exists iff: for any $\varepsilon > 0$, $\exists b(\varepsilon) \in (a, b)$ such that for any $b', b'' > b(\varepsilon)$, $\left|\int_{b'}^{b''} f(x)\,\mathrm{d}x\right| < \varepsilon$.

Proof: Let

$$F(t) = \int_a^t f(x)\,\mathrm{d}x, \,\forall t \in [a, b).$$

Then $\int_a^b f(x)\,\mathrm{d}x$ exists iff $\lim_{t\to b^-} F(t)$ exists, which is equivalent to

$$\forall \varepsilon > 0, \exists b(\varepsilon) \in (a, b), \forall b', b'' > b(\varepsilon), \left|\int_{b'}^{b''} f(x)\,\mathrm{d}x\right| = |F(b'') - F(b')| < \varepsilon.$$

B2) If $|f(x)| \leqslant F(x), x \in [a, b)$ and $\int_a^b F(x)\,\mathrm{d}x$ converges, then $\int_a^b f(x)\,\mathrm{d}x$ converges.

Proof: Use B1) and

$$\left|\int_u^v f(x)\,\mathrm{d}x\right| \leqslant \int_u^v F(x)\,\mathrm{d}x.$$

B3) Prove the Dirichlet test for convergence: if $f, g : [a, \infty) \to \mathbb{R}$ satisfy

- f is continuous and there exists $A > 0$, such that for any $M \geqslant a$,
$$\left| \int_a^M f(x)\,\mathrm{d}x \right| \leqslant A.$$
- g is monotonic and $\lim_{x\to\infty} g(x) = 0$.
Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

Lemma: The Second Integral Mean Value Theorem

If f is integrable and g is monotonic and non-negative(or non-positive) on $[a, b]$, then there exists $c \in (a, b)$ such that
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^c f(x)\,\mathrm{d}x + g(b)\int_c^b f(x)\,\mathrm{d}x.$$
Proof: Assume that g is non-negative and monotonically decreasing. It is easy to see that there exists $\xi \in (a, b)$ such that
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x.$$
Apply the above formula to $f(x)$ and $g(x) - g(b)$ and we get
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x.$$
Proof of B3): Since $|\int_u^v f(x)\,\mathrm{d}x| \leqslant 2A$, by lemma
$$\left| \int_u^v f(x)g(x)\,\mathrm{d}x \right| \leqslant 2A(|g(u)| + |g(v)|).$$
By B1), the integral converges.

B4) Prove the Abel test of convergence:

If $f, g : [a, \infty) \to \mathbb{R}$ satisfy:

- $\int_a^\infty f(x)\,\mathrm{d}x$ exists.
- g is monotonic and g is bounded.
Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.
Proof: Suppose g is monotonically increasing, then
$$\left| \int_u^v f(x)(g(x) - g(a))\,\mathrm{d}x \right| \leqslant 2M\left(\left| \int_u^\xi f(x)\,\mathrm{d}x \right| + \left| \int_\xi^v f(x)\,\mathrm{d}x \right| \right) \to 0$$

since $\int_a^\infty f(x)\,\mathrm{d}x$ converges. Therefore both $\int_a^\infty f(x)(g(x) - g(a))\,\mathrm{d}x$ and $\int_a^\infty f(x)g(a)\,\mathrm{d}x$ converges, hence $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

B5) Determine whether the following integrals converges:

(1)
$$\int_0^\infty \frac{\log(1+x)}{x^p}\,\mathrm{d}x$$

(absolutely) convergent when $1 < p < 2$, diverges when $p \leqslant 1$ or $p \geqslant 2$.
(2)

$$\int_1^\infty \frac{\sin x}{x^p}\,\mathrm{d}x$$

Absolutely convergent when $p > 1$, conditionally convergent when $0 < p \leqslant 1$, diverges when $p \geqslant 0$.
(3)

$$\int_1^\infty \sin x^2\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \frac{\sin y}{y^{1/2}}\,\mathrm{d}y$$

is conditionally convergent.
(4)

$$\int_0^\infty \frac{\sin^2 x}{x}\,\mathrm{d}x$$

diverges
(5) $p, q > 0$,

$$\int_0^{2\pi} \sin^{-p} x \cos^{-q} x\,\mathrm{d}x$$

Absolutely convergent when $p, q < 1$, diverges when $p \geqslant 1$ or $q \geqslant 1$.
(6)

$$\int_0^\infty x^p \sin(x^q)\,\mathrm{d}x$$

If $q = 0$ the integral diverges. Assume $q \neq 0$ below.

$$\int_0^\infty x^p \sin(x^q)\,\mathrm{d}x = \frac{1}{q}\int_0^\infty y^{(p+1)/q-1}\sin y\,\mathrm{d}y.$$

Let $\alpha = \frac{p+1}{q} - 1$, then the integral

- diverges if $\alpha \leqslant -2$ or $\alpha \geqslant 0$,
- converges absolutely if $-2 < \alpha < -1$.
- converges conditionally if $-1 \leqslant \alpha < 0$.
  (7) $q \geqslant 0$,

$$\int_0^\infty \frac{x^p \sin x}{1 + x^q}\,\mathrm{d}x$$

If $p \leqslant -2$, then the integral diverges near 0, since $x^p \sin x \sim x^{p+1}$. The integral converges (absolutely) near 0 otherwise. Assume $p > -2$ below.
If $p - q < -1$ then the integral converges absolutely when it tends to infinity, since $\frac{x^p}{1+x^q} \sim x^{p-q}$.
If $-1 \leqslant p - q < 0$ then the integral converges conditionally, since the integral of $(x^{p-q})'$ converges.
(8)

$$\begin{aligned}
\int_0^{\pi/2} \frac{\log \sin x}{\sqrt{x}}\,\mathrm{d}x &= 2\int_0^{\pi/2} \log \sin x\,\mathrm{d}\sqrt{x} \\
&= 2\sqrt{x}\log \sin x\Big|_0^{\pi/2} - 2\int_0^{\pi/2} \sqrt{x}\cot x\,\mathrm{d}x \\
&= -2\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\cos x\,\mathrm{d}x
\end{aligned}$$

converges, since $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges.
(9)

$$\int_e^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x = \int_1^\infty \frac{\log y}{y}e^y \sin e^y\,\mathrm{d}y.$$

It is easy to see the integral does not converge absolutely.
Meanwhile

$$f'(x) = \left(\frac{\log\log x}{\log x}\right)' = \frac{1-\log\log x}{(\log x)^2 x},$$

and

$$\int_e^\infty \frac{\log\log x - 1}{(\log x)^2 x}\,\mathrm{d}x = \int_1^\infty \frac{\log y - 1}{y^2}\,\mathrm{d}y = \int_0^\infty \frac{t-1}{e^t}\,\mathrm{d}t.$$

converges.
By Lagrange mean value theorem,

$$\begin{aligned}\int_{2\pi}^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x &= \sum_{n=1}^\infty \int_{2n\pi}^{(2n+1)\pi} (f(x) - f(x+\pi))\sin x\,\mathrm{d}x \\ &\leqslant \sum_{n=1}^\infty -2\pi f'(2n\pi) \leqslant 2\pi \int_e^\infty -f'(x)\,\mathrm{d}x\end{aligned}$$

converges.

12.3 PSC: Oscillatory Integral

$F(t), G(t) : [1,\infty) \to \mathbb{R}$, $\lim_{t\to\infty} G(t) = 0$. Assume that for any $t \geqslant 1$, $G(t) \neq 0$. If

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = 1.$$

Then we say F, G have the same order, and $F \sim G$.

Part 1

C1) $d > 0$ is a given real number. Assume $g \in C^1([0,d])$. Prove that there is a constant C, such that

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant \frac{C}{t}.$$

Proof: Let $C = \sup_{x\in[0,d]} |g(x)|$, then

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant C\int_0^d e^{-tx}\,\mathrm{d}x = \frac{C}{t}(1-e^{-td}) \leqslant \frac{C}{t}.$$

C2) Assume $d > 0$, $g \in C([0,d])$ and $g(0) \neq 0$. Prove that

$$\int_0^d e^{-tx} g(x)\,\mathrm{d}x \sim \frac{g(0)}{t}.$$

Proof: Let $M = \sup_{x\in[0,d]} |g(x)|$, then

$$\begin{aligned}\left|\int_0^d e^{-tx} t\frac{g(x)}{g(0)}\,\mathrm{d}x - 1\right| &= \left|\int_0^{td} e^{-u}\frac{g(u/t)}{g(0)}\,\mathrm{d}u - \int_0^\infty e^{-u}\,\mathrm{d}u\right| \\ &\leqslant \int_{td}^\infty e^{-u}\,\mathrm{d}u + \int_0^N e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u + \int_N^{td} e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u \\ &\leqslant e^{-td} + \sup_{0\leqslant x\leqslant N/t}\left|\frac{g(x)}{g(0)} - 1\right| + \left(\frac{M}{|g(0)|} + 1\right)\int_N^{td} e^{-u}\,\mathrm{d}u \to 0.\end{aligned}$$

(let $t\to\infty$ then let $N\to\infty$).

C3) $d > 0, g \in C([0,d]), g(0) \neq 0$. **Prove that**

$$\int_0^d e^{-tx^2} g(x)\,\mathrm{d}x \sim \frac{\sqrt{\pi}\cdot g(0)}{2\sqrt{t}}.$$

Proof: Same as C2), let $M = \sup_{x\in[0,d]} |g(x)/g(0)|$, then

$$\begin{aligned}\left|\int_0^d e^{-tx^2}\sqrt{t}\frac{g(x)}{g(0)}\,\mathrm{d}x - \frac{\sqrt{\pi}}{2}\right| &= \left|\int_0^{d\sqrt{t}} e^{-u^2}\frac{g(u/\sqrt{t})}{g(0)}\,\mathrm{d}u - \int_0^\infty e^{-u^2}\,\mathrm{d}u\right| \\ &\leqslant \int_{d\sqrt{t}}^\infty e^{-u^2}\,\mathrm{d}u + \int_0^N e^{-u^2}\left|\frac{g(u/\sqrt{t})}{g(0)} - 1\right|\mathrm{d}x + \int_N^{d\sqrt{t}} e^{-u^2}(M+1)\,\mathrm{d}u.\end{aligned}$$

which tends to 0, same as C2).

For $t \geqslant 1$, $f, \varphi \in C([a,b])$, define the function

$$F(t) = \int_a^b e^{-t\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $F(t)$ when $t\to\infty$.

C4) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Further assume that $\varphi'(x) > 0$. Let $d = \varphi(b) - \varphi(a)$. Prove that

$$\Psi : [a,b] \to [0,d],\ x \mapsto \varphi(x) - \varphi(a),$$

is a continuously differentiable bijection on $[a,b]$.
Proof: φ is monotonic by $\varphi'(x) > 0$, hence Ψ is a bijection and $\Psi' = \psi'$.

C5) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Prove that if $f(a) \neq 0$, then when $t \to \infty$,

$$F(t) \sim \frac{f(a)}{\varphi'(a)} \frac{e^{-t\varphi(a)}}{t}.$$

Proof: Let $g(x) = f(x)/\Psi'(x)$, and $h = (t\Psi)^{-1}$ then

$$\begin{aligned}
\left| F(t)\frac{t}{e^{-t\varphi(a)}} - \frac{f(a)}{\varphi'(a)} \right| &= \left| \int_a^b e^{-t\Psi(x)} tf(x)\,\mathrm{d}x - \frac{f(a)}{\Psi'(a)} \right| = \left| \int_a^b e^{-t\Psi(x)} g(x)\,\mathrm{d}t\Psi(x) - g(a) \right| \\
&= \left| \int_0^{t\Psi(b)} e^{-u} g(h(u))\,\mathrm{d}u - g(h(0)) \int_0^\infty e^{-u}\,\mathrm{d}u \right| \\
&= |g(h(0))| \int_{t\Psi(b)}^\infty e^{-u}\,\mathrm{d}u + \int_0^{N\Psi(b)} e^{-u} |g(h(u)) - g(h(0))|\,\mathrm{d}u \\
&+ \int_{N\Psi(b)}^{t\Psi(b)} e^{-u} |g(h(u)) - g(h(0))|\,\mathrm{d}u \\
&\leqslant |g(a)| e^{-t\Psi(b)} + \sup_{x \in [a, \Psi^{-1}(N\Psi(b)/t)]} |g(x) - g(a)| + \int_{N\Psi(b)}^{t\Psi(b)} e^{-u} 2M\,\mathrm{d}u.
\end{aligned}$$

which tends to 0 since g is continuous. ($M = \sup_{x\in[a,b]} |g(x)|$).

C6) Assume that $\varphi \in C^2([a,b]), \varphi'(a) = 0, \varphi''(x) > 0$ and for any $x \in (a,b], \varphi'(x) > 0$. Let $d = \sqrt{\varphi(b) - \varphi(a)}$. Prove that

$$\Psi : [a,b] \to [0,d],\ x \mapsto \sqrt{\varphi(x) - \varphi(a)}.$$

is a continuously differentiable bijection on $[a,,b]$, and calculate $\Psi'(a)$.
Proof: Trivial. $\Psi' = \frac{\varphi'}{2\Psi}$, hence

$$\Psi'(a) = \lim_{x\to a^+} \frac{\varphi'(x)}{2\sqrt{\varphi(x) - \varphi(a)}} = \lim_{x\to a^+} \frac{\varphi''(x)}{\varphi'(x)/\sqrt{\varphi(x) - \varphi(a)}} = \sqrt{\frac{\varphi''(a)}{2}}.$$

C7) Assume $\varphi \in C^2([a,b]), \varphi'(a) = 0, \varphi''(a) > 0$. Prove that if $f(a) \neq 0$, when $t \to \infty$,

$$F(t) \sim \frac{\sqrt{\pi} f(a)}{\sqrt{2\varphi''(a)}} \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: Let $g = f/\Psi'$, $h = (\sqrt{t}\Psi)^{-1}$, then

$$F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}} = \int_a^b e^{-t\Psi^2(x)} f(x)\sqrt{t}\,\mathrm{d}x = \int_a^b e^{-t\Psi^2(x)} g(x)\,\mathrm{d}\sqrt{t}\Psi(x) = \int_0^{\sqrt{t}\Psi(b)} e^{-u^2} g(h(u))\,\mathrm{d}u.$$

Hence

$$\left|F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}}-\frac{\sqrt{\pi}}{2}g(a)\right| = \left|\int_0^{\sqrt{t}\Psi(b)} e^{-u^2}g(h(u))\,\mathrm{d}u - \int_0^\infty e^{-u^2}g(h(0))\,\mathrm{d}u\right|$$
$$\leqslant g(a)\int_{\sqrt{t}\Psi(b)}^\infty e^{-u^2}\,\mathrm{d}u + \int_0^{N\Psi(b)} e^{-u^2}|g(h(u))-g(h(0))|\,\mathrm{d}u$$
$$+\int_{N\Psi(b)}^{\sqrt{t}\Psi(b)} e^{-u^2}2M\,\mathrm{d}u$$
$$\leqslant g(a)e^{-\sqrt{t}\Psi(b)} + \sqrt{\pi}\sup_{x\in[a,\Psi^{-1}(N\Psi(b)/\sqrt{t})]}|g(x)-g(a)| + 2Me^{-N\Psi(b)}.$$

which tends to 0 as $t\to\infty$ and $N\to\infty$, since g is continuous.
(A much simpler solution can be given using the Laplace method)

C8) Given $f\in C((0,\infty)),\varphi\in C^2((0,\infty))$**. Assume that**

- exists a unique $a\in(0,\infty)$ such that $\varphi'(a)=0$;
- $\varphi''(a)>0, f(a)\neq 0$;
- $\int_0^\infty e^{-\varphi(x)}|f(x)|\,\mathrm{d}x$ converges.
  Prove that when $t\to\infty$, the function $G(t)=\int_0^\infty e^{-t\varphi(x)}f(x)\,\mathrm{d}x$ satisfy

$$G(t)\sim\frac{\sqrt{2\pi}f(a)}{\sqrt{\varphi''(a)}}\frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: (Simple application of the Laplace method)
Apply C7) to the intervals $[a/2,a]$ and $[a,2a]$, then

$$\int_{a/2}^{2a} e^{-t\varphi(x)}f(x)\,\mathrm{d}x\sim\frac{\sqrt{2\pi}f(a)}{\sqrt{\varphi''(a)}}\cdot\frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

While the integral on the intervals $(0,a/2),(2a,\infty)$ converges rapidly. Hence

$$G(t)\sim\frac{\sqrt{2\pi}f(a)}{\sqrt{\varphi''(a)}}\frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

C9) $\Gamma(n)=(n-1)!$**.**

Proof:

$$\Gamma(n+1)=\int_0^\infty t^n e^{-t}\,\mathrm{d}t=-\int_0^\infty t^n\,\mathrm{d}e^{-t}=n\int_0^\infty t^{n-1}e^{-t}\,\mathrm{d}t=n\Gamma(n).$$

C10) Prove Stirling‘s approximation

$$n!\sim\sqrt{2\pi}\frac{n^{n+1/2}}{e^n}.$$

Proof: Note that, by substituting $t=ns$,

$$n!=\Gamma(n+1)=\int_0^\infty e^{-t}t^n\,\mathrm{d}t=n^{n+1}\int_0^\infty e^{-n(s-\log s)}\,\mathrm{d}s.$$

Hence

$$\frac{\Gamma(t+1)}{t^{t+1}} \sim \sqrt{2\pi}\frac{e^{-t}}{\sqrt{t}}.$$

Part 2

For $\lambda \geqslant 1,\ f, \varphi \in C^{\infty}([a,b])$, define the function

$$I(\lambda) = \int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $I(\lambda)$ when $\lambda \to \infty$.

C11) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Define the maps

$$L : C^{\infty}([a,b]) \to C^{\infty}([a,b]), h \mapsto \frac{1}{i\lambda\varphi'(x)} h'(x),$$
$$M : C^{\infty}([a,b]) \to C^{\infty}([a,b]), h \mapsto -\left(\frac{h}{i\varphi'}\right)'(x).$$

Assume that $f, g \in C^{\infty}([a,b])$. Prove that if there exists $c > 0$ such that for any $x \in [a, a+c] \cup [b-c, b]$, $h(x) = 0$, then M/λ is the adjoint of L:

$$\int_a^b h \cdot Lg = \frac{1}{\lambda}\int_a^b g \cdot Mh.$$

Proof: By integration of parts,

$$\int_a^b h \cdot Lg = \int_a^b \frac{h}{i\lambda\varphi'}\,\mathrm{d}g = -\int_a^b g\,\mathrm{d}\left(\frac{h}{i\lambda\varphi'}\right) = \frac{1}{\lambda}\int_a^b g \cdot Mh.$$

C12) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$ and f vanishes near a and b. prove that for any $n \in \mathbb{Z}_{\geqslant 1}$, there is a constant c_n independent of λ such that

$$|I(\lambda)| \leqslant \frac{c_n}{\lambda^n}.$$

Proof: Let $g = e^{i\lambda}\varphi$, then $Lg = g$. $f \in C^{\infty}([a,b])$ vanishes near a, b hence $M^n f$ vanishes near a, b for any $n \in \mathbb{Z}_{\geqslant 0}$. Therefore

$$|I(\lambda)| = \left|\int_a^b fg\right| = \frac{1}{\lambda}\left|\int_a^b g \cdot Mf\right| = \cdots = \frac{1}{\lambda^n}\left|\int_a^b g \cdot M^n f\right|.$$

so $c_n = \left|\int_a^b g \cdot M^n f\right|$ is valid.

C13) If there exists $\delta > 0$, such that for any $x \in [a,b]$, $|\varphi'(x)| \geqslant \delta$ and $\varphi'(x)$ is monotonic on $[a,b]$. Prove that there is a constant C_1 independent of λ, φ, a, b such that

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant \frac{C_1}{\lambda\delta}.$$

Proof: Let $C_1 = 4$ then (since φ' maintains the same sign)

$$\begin{aligned}\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| = \left|\int_a^b \frac{\mathrm{d}e^{i\lambda\varphi}}{\lambda\varphi'}\right| &\leqslant \left|\frac{e^{i\lambda\varphi}}{\lambda\varphi'}\Big|_a^b\right| + \frac{1}{\lambda}\left|\int_a^b e^{i\lambda\varphi}\frac{\varphi''}{(\varphi')^2}\,\mathrm{d}x\right| \\ &\leqslant \frac{2}{\lambda\delta} + \frac{1}{\lambda}\int_a^b \left|\frac{\varphi''}{(\varphi')^2}\right| \\ &= \frac{2}{\lambda\delta} + \frac{1}{\lambda}\int_a^b \mathrm{d}\frac{1}{\varphi'} \leqslant \frac{4}{\lambda\delta}.\end{aligned}$$

C14) Suppose for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a unique $c \in [a,b]$ such that

$$|\varphi'(c)| = \inf_{x\in[a,b]} |\varphi'(x)|.$$

Further prove that for any $x \in [a,b]$,

$$|\varphi'(x)| \geqslant |x-c|.$$

Proof: Since $\varphi \in C^\infty([a,b])$ and $|\varphi''| \geqslant 1$, φ'' maintains the same sign. Assume that $\forall x \in [a,b], \varphi''(x) \geqslant 1$, then φ' is monotonically increasing. Therefore, if $\varphi' \neq 0$, then $c \in \{a,b\}$, otherwise, c is the unique null point of φ'.
Either $\varphi'(c) = 0$ or $c = a$, when φ' maintains the same sign, so we always have $|\varphi'(x)| \geqslant |\varphi'(x) - \varphi'(c)|$, and

$$\forall x \in [a,b], \exists \xi \in [x,c],\ |\varphi'(x) - \varphi'(c)| \geqslant |x-c| \cdot \varphi'(\xi) \geqslant |x-c|.$$

Therefore $|\varphi'(x)| \geqslant |x-c|$.

!C15) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of λ, φ, a, b, such that

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant \frac{C_2}{\sqrt{\lambda}}.$$

Proof: Since $\varphi \in C^\infty([a,b])$, we can assume $\varphi''(x) \geqslant 1$. For an arbitrary $\delta > 0$, divide the interval $[a,b]$ into two parts:
$E_1 = \{x : |\varphi'(x)| \leqslant \delta\}$ and $E_2 = \{x : |\varphi'(x)| > \delta\}$.
Note that $\forall x, y \in E_1$, $|\varphi'(x) - \varphi'(y)| \leqslant 2\delta$, but $|\varphi'(x) - \varphi'(y)| \geqslant \left|\int_x^y \varphi''(t)\,\mathrm{d}t\right| \geqslant |x-y|$. Therefore E_1 is an interval of length at most 2δ, so

$$\left|\int_{E_1} e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant 2\delta.$$

Now consider the integral on E_2, which is the union of one or two intervals. By C13),

$$\left|\int_{E_2} e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant 2 \cdot \frac{4}{\lambda\delta}.$$

Therefore

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant 2\delta + \frac{8}{\lambda\delta} = \frac{8}{\sqrt{\lambda}}.$$

(if we let $\delta = \sqrt{4/\lambda}$.)

C16) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of $\lambda, \varphi, f, a, b$ such that

$$\left|\int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x\right| \leqslant \frac{C_2}{\sqrt{\lambda}}\left(|f(a)| + \int_a^b |f'(x)|\,\mathrm{d}x\right).$$

Proof: By C15),

$$\begin{aligned}\left|\int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x\right| &\leqslant \left|\int_a^b e^{i\lambda\varphi(x)} f(a)\,\mathrm{d}x\right| + \left|\int_a^b e^{i\lambda\varphi(x)} \int_a^x f'(t)\,\mathrm{d}t\,\mathrm{d}x\right| \\ &\leqslant |f(a)|\frac{C_2}{\sqrt{\lambda}} + \left|\int_a^b f'(t)\int_t^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant \frac{C_2}{\sqrt{\lambda}}\left(|f(a)| + \int_a^b |f'(x)|\,\mathrm{d}x\right).\end{aligned}$$