周明强《实变函数论》P211思考题：1，3；P241习题5:2，3

211-1

Suppose $f:[a,b]\to\mathbb{R}$ is non-negative. If $f\in L([a,b])$, must f have a primitive on $[a,b]$?
Solution: Not necessarily, since by Darboux's theorem, functions like $f=\chi_{[0,1]}:[-1,1]\to\mathbb{R}$ does not have a primitive.

211-3

Prove that the result of Vitali covering theorem can be changed to: there exists countable $\{I_j\}$ such that $m^*\left(E\backslash\bigcup_{j\geqslant 1}I_j\right)=0$.
Proof: Just take all the intervals I_n in the original proof:
WLOG we assume I_α are closed intervals. $m^*(E)<\infty$, then we can find G open, $E\subset G$ & $m(G)<\infty$. We can assume $I_\alpha\subset G\forall\alpha\in J$ (throw away the others). Let $\delta_1=\sup\{|I|:I\in\Gamma\}$, we take $I_1\in\Gamma$ such that $|I_1|\geqslant\delta_1/2$. Likewise consider $\delta_n=\sup\{|I|:I\in\Gamma,I\cap I_j=\emptyset\}$, and $I_n\cap I_j=\emptyset$ and $|I_n|\geqslant\delta_n/2$.
Clearly $\bigcup I_j\subset G$, so $m(G)\geqslant m\left(\bigcup I_j\right)=\sum|I_j|$, hence $\delta_n\to 0$. For any $\varepsilon>0$, take N such that $\sum_{n\geqslant N}|I_n|<\varepsilon/5$. Let $S_N=E\backslash\bigcup_{j=1}^N I_j$, then for any $x\in S_N$, find I such that $x\in I$, and $I\cap\bigcup I_j\neq\emptyset$. We claim $I\cap I_j\neq\emptyset$ for j large enough, otherwise $\delta_n\geqslant|I|$ implies $|I|=0$.
Consider I_{n_0} to be the first to intersect with I, then $I\cap I_j=\emptyset\forall x<n_0$ and $I\cap I_{n_0}\neq\emptyset$. Then by definition, $\delta_{n_0}\geqslant|I|$ and $|I_n|\geqslant\frac{1}{2}\delta_{n_0}$, so $|I|\leqslant 2|I_{n_0}|$. We can prove $I\subset 5I_{n_0}$ (keep center and enlarge radius).
Hence $S_N\subset\bigcup_{j\geqslant N+1}5I_j$. So $m^*(S_N)\leqslant\sum_{j\geqslant N+1}m^*(5I_j)\leqslant\varepsilon$. $E\backslash\bigcup_{j\geqslant 1}I_j\subset S_N$ so it is a null set.

241-2

Suppose $\{x_n\}\subset[a,b]$, give an increasing function on $[a,b]$, such that its discontinuities are exactly $\{x_n\}$.
Proof: Consider $f(x)=\sum_{n=1}^\infty 2^{-n}\chi_{[0,\infty)}(x-x_n)$.

241-3

Suppose $f:(a,b)\to\mathbb{R}$ is increasing, $E\subset(a,b)$. If for any $\varepsilon>0$, there exists $(a_i,b_i)\subset(a,b)$ such that

$$E\subset\bigcup_i(a_i,b_i),\ \sum_i f(b_i)-f(a_i)<\varepsilon.$$

Prove that $f'(x)=0, a.e. x\in E$.
Proof: By Lebesgue monotone differentiation theorem, f' exists and is finite a.e. Clearly $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\geqslant 0$. We show that for any $k\geqslant 1$, $A_k=\{x\in E:f'(x)>k^{-1}\}$ is null.
Otherwise if $m(A_k)>0$, then for any cover $E\subset\bigcup_i(a_i,b_i)$,

$$\sum_i f(b_i)-f(a_i)\geqslant\sum_i\int_{(a_i,b_i)}f'\,\mathrm{d}m\geqslant\int_E f'\,\mathrm{d}m\geqslant\int_{A_k}f'\,\mathrm{d}m>\frac{m(A_k)}{k}.$$

Leading to contradiction. Hence $m(A_k)=0$ so $f'=0, a.e. x\in E$.

Extra 1

If $F\in C([a,b])$, prove that the Dini derivatives are measurable.
Proof: Consider $D^+F(x)=\limsup_{h\to 0^+}\frac{F(x+h)-F(x)}{h}=\inf_{h>0}\sup_{k\in(0,h)}\frac{F(x+k)-F(x)}{k}$.
Since $F\in C([a,b])$, the $\inf$ and $\sup$ can be taken on rationals, so D^+F is measurable.

Extra 2

If $F \in C([a, b])$, and $D^+ F(x) \geqslant 0, \forall x \in [a, b]$, prove that F is monotonically increasing.
Proof: For any $\varepsilon > 0$, consider $g_\varepsilon(x) = F(x) + \varepsilon x$, then $D^+ g_\varepsilon(x) \geqslant \varepsilon$. For any $x < y$, consider $A = \{t \in [x, y] : g_\varepsilon(t) \geqslant g_\varepsilon(x)\}$, then $x \in A$ so $A \neq \emptyset$. Take $t = \sup A$, then $t \in A$ since $g_\varepsilon \in C([a, b])$. If $t < y$, then for any $h \in (0, y - t)$, $g_\varepsilon(t + h) < g_\varepsilon(x) \leqslant g_\varepsilon(t)$, so $D^+ g_\varepsilon(t) \leqslant 0$, leading to contradiction. Hence $t = y$ and $g_\varepsilon(y) \geqslant g_\varepsilon(x)$ for any $x < y$. Let $\varepsilon \to 0$ we obtain $F(x) \leqslant F(y)$ for any $x \leqslant y$.

Extra 3

If $f \in C([a, b])$, prove that

$$\sup_{x_1, x_2 \in [a,b]} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \sup_{x \in [a,b]} D^+ f(x) = \sup_{x \in [a,b]} D^- f(x) = \sup_{x \in [a,b]} D_- f(x) = \sup_{x \in [a,b]} D_+ f(x).$$

(The statement holds when they are ∞)
Proof: Note that for $t = \sup_{x_1, x_2 \in [a,b]} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$,

$$\sup_{x \in [a,b]} D^+ f(x) = \sup_x \inf_{r > 0} \sup_{h < r} \frac{f(x+h) - f(x)}{h} \leqslant \sup_x \sup_{h > 0} \frac{f(x+h) - f(x)}{h} = t.$$

For any $\varepsilon > 0$, take $x_1 < x_2$ such that $f(x_2) - f(x_1) > (t - \varepsilon)(x_2 - x_1)$, so $g(x) = f(x) - (t - \varepsilon)x$ satisfy $g(x_2) > g(x_1)$. By the previous problem, there exists $x \in [x_1, x_2]$ such that $D^+ g(x) > 0$, so $D^+ f(x) > t - \varepsilon$. Let $\varepsilon \to 0$ we obtain $\sup_{x \in [a,b]} D^+ f(x) \geqslant t$, so $\sup_{x \in [a,b]} D^+ f(x) = t$.
By considering $f'(x) = f(a + b - x)$ and $-f$, we obtain the results for D^-, D_-, D_+.

Extra 4

As a corollary, consider the following result:
$F \in C([a, b])$. If one of the four Dini derivatives is continuous at $x_0 \in (a, b)$, prove that the other three are also continuous at x_0, and the four derivatives are equal. (Hence f is differentiable at x_0)
Proof: For any $\varepsilon > 0$, there is a neighborhood $U(x_0)$ such that $D^+ f(x_0) - \varepsilon \leqslant D^+ f(x) \leqslant D^+ f(x_0) + \varepsilon$, then $\sup_{x \in U(x_0)} D^+ f(x) \leqslant D^+ f(x_0) + \varepsilon$. By the previous problem, for any Dini derivative D^*, $\sup_{x \in U(x_0)} D^* f(x) \leqslant D^+ f(x_0) + \varepsilon$, and likewise $\inf_{x \in U(x_0)} D^* f(x) \geqslant D^+ f(x_0) - \varepsilon$, so $D^* f$ is continuous at x_0. Hence $|D^* f(x_0) - D^+ f(x_0)| \leqslant \varepsilon$. Let $\varepsilon \to 0$ we obtain $D^* f(x_0) = D^+ f(x_0)$, so all four Dini derivatives are equal.