

6.1: 2, 3-8 >= 5道; 6.2: 1的两小题, 2, 3, 4-9 >= 3道;

6.3: 1-5, 7-9 每题 >= 2 小题. 其余选 >= 6 道.

6.4: >= 3 题. 6.5: 1-9 >= 8 题, 10, 11

6.1

6.1.2

尺规作图画出 $f(x)$ 在 $(t, f(t))$ 处的切线:

(1) $f(x) = x^k$ (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (3) $f(x) = \sin x$ (4) $f(x) = \tan x$ (5) $f(x) = e^x$ (6) $f(x) = \log x$.

引理: 尺规作图可以做加法、数乘有理数, 且有单位长度时, 可以做乘法、倒数、开根. 且做切线只需作出斜率的长度.

证明: 加法: 拼接两条线段即可. 数乘正整数: 若干次加法.

除正整数: 对于线段 AB , 过 A 作射线 AC_1 , 并在射线上取 C_2, \dots, C_n 使得 $AC_k = kAC_1$. 过 C_1 做 BC_n 的平行线交 AB 于 D 则 $AD = AB/n$. 故可以数乘任意有理数.

下面设已有单位长度:

倒数: 作以 $x, 1$ 为直角边的直角三角形 ABC , $AC = 1, BC = x$, 作 $AD \perp AB$ 交 BC 于 D 则 $CD = 1/x$.

乘法: 做 $OA = a, OB = b$, 在 OB 上取 E 使得 $OE = 1$, 过 B 作 AE 平行线交 OA 于 D , 则 $OD = ab$.

开根: 设射线 OA 上 $OA = a, OB = 1$, 以 OA 为直径作圆, 过 B 作 OA 垂线交圆于 C , 则 $BC = \sqrt{a}$.

若要作切线, $l: y = f'(t)(x - t) + f(t)$, 只需画出 $f'(t)$, 则会有 $(t, f(t))$ 和 $(0, -tf'(t) + f(t))$, 连接可得到 l .

原题:

(1) $f'(x) = kx^{k-1} = k \cdot x \cdots x, -tf'(t) + f(t) = (1 - k)t^k$ 可以作出.

(2) 用 Pascal 定理.

(3) 找到正半轴首个根 π , 取中点得到 $\pi/2$, 再做坐标轴垂线得到单位长度. $f'(t) = \cos t = \sqrt{1 - \sin^2 t}$ 可以作出.

(4) 同样通过根找到 π , 再用 $\tan \frac{\pi}{4} = 1$ 找到单位长度. $f'(t) = \sec^2 t = 1 + \tan^2 t$ 可以作出.

(5) $f(0) = 1$ 是单位长度, $f'(t) = e^t$ 可以作出.

(6) 唯一一个根是单位长度 1, $f'(x) = 1/x$ 可以作出.

6.1.3

Suppose f is differentiable at x_0 , calculate $\lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0 - h))/h$.

Solution: Note that

$$\frac{f(x_0 + h) - f(x_0 - h)}{h} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f(x_0) - f(x_0 - h)}{h}$$

hence the limit is $2f'(x_0)$.

6.1.4

Suppose f is continuous at 0 and $\lim_{x \rightarrow 0} (f(2x) - f(x))/x = k$, prove that $f'(0) = k$.

Proof: Let $f(2x) - f(x) = xk + xh(x)$, then $h(x) \rightarrow 0$, and

$$f(x) = f(2^{-n}x) + \sum_{k=0}^{n-1} 2^{-k-1} x h(2^{-k-1}x)$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $x < \delta$ implies $-\varepsilon < h(x), f(x) < \varepsilon$, then

$$\left| \frac{f(x) - f(0)}{x} - k \right| \leq \left| \frac{f(2^{-n}x) - f(0)}{x} - k \right| + \sum_{k=0}^{n-1} 2^{-k-1} |h(2^{-k-1}x)| \leq \left| \frac{f(2^{-n}x) - f(0)}{x} - k \right| + 4\varepsilon.$$

Let $n \rightarrow \infty$ then $|(f(x) - f(0))/x - k| < 4\varepsilon$. Hence $f'(0) = k$.

6.1.5

Prove that if $f(x)$ is differentiable and even/odd/periodic, then $f'(x)$ is odd/even/periodic.

Proof: If $f(x) = -f(-x)$, then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} = f'(x).$$

If $f(x) = f(-x)$, then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} -\frac{f(x-h) - f(x)}{h} = -f'(x).$$

If $f(x) = f(x+T)$ then

$$f'(x+T) = \lim_{h \rightarrow 0} \frac{f(x+T+h) - f(x+T)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

6.1.7

Suppose $g(0) = g'(0) = 0$, and

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine the value of $f'(0)$.

Solution:

$$f'(0) = \lim_{h \rightarrow 0} \frac{g(h)}{h} \sin \frac{1}{h} = 0,$$

since $g(h)/h \rightarrow 0$ and $\sin \frac{1}{h}$ is bounded.

6.1.8

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$, $\forall x, y \in \mathbb{R}$. If $f'(0) = 1$, prove that $f'(x) = f(x)$ for any $x \in \mathbb{R}$.

Proof: $f'(0) = 1$ so f is not constant. Hence let $x = 0$ we have $f(y) = f(0)f(y)$ so $f(0) = 1$. Note that

$f(x) = f(x/2)^2 \geq 0$, so we can let $g(x) = \log f(x)$, then $g'(x) = \frac{f'(x)}{f(x)}$ and $g'(0) = 1$, $g(0) = 0$,

$g(x+y) = g(x) + g(y)$. Since g is continuous at 0, $g(x) = cx$ so $g(x) = x \forall x \in \mathbb{R}$, hence $g'(x) = 1$ and $f'(x) = f(x)$.

6.2

6.2.1

(1) $f(x) = x^{1/3}$, $f'_+(0)$:

$$f'_0(x) = \lim_{h \rightarrow 0^+} \frac{h^{1/3}}{h} = \infty.$$

(2) $f(x) = |x-2|^3$, $f'_-(2)$:

$$f'_-(2) = \lim_{h \rightarrow 0^+} \frac{-h^3}{h} = 0.$$

6.2.2

Determine the values of a, b such that

$$f(x) = \begin{cases} x^2, & x \geq 1, \\ ax + b, & x < 1 \end{cases}$$

is differentiable at $x = 1$.

Solution: Clearly $f'_+(1) = 2$, while

$$f'_-(1) = \lim_{h \rightarrow 0^+} \frac{f(1) - f(1-h)}{h} = \lim_{h \rightarrow 0^+} \frac{1 - a(1-h) - b}{h} = 2$$

Hence $a = 2$ and $1 - a - b = 0$, so $(a, b) = (2, -1)$.

6.2.3

Suppose f is continuous at 0. Prove that $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x}$ exists if $\lim_{x \rightarrow 0} \frac{f(x) - f(x-x^3)}{x^3}$ exists.

Proof: Let $f(x) - f(0) = xg(x)$, and $f(x) - f(x-x^3) = x^3h(x)$.

If $\lim_{x \rightarrow 0} \frac{f(x) - f(x-x^3)}{x^3} = L$ exists, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $|h(x) - L| < \varepsilon$. Consider $x = x_0 \in (0, \delta)$, and $x_{k+1} = x_k - x_k^3$, then clearly $\lim_{n \rightarrow \infty} x_n = 0$. Note that

$$xg(x) = \sum_{k=0}^{\infty} f(x_k) - f(x_{k+1}) = \sum_{k=0}^{\infty} x_k^3 S(x_k) < (L + \varepsilon) \sum_{k=0}^{\infty} x_k - x_{k+1} = (L + \varepsilon)x$$

and likewise $xg(x) > (L - \varepsilon)x$, hence $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \rightarrow 0} g(x) = L$.

(The reverse is incorrect, for example $x^2 \sin x^{-1}$).

6.2.4

Calculate the Dini derivatives of f at $x = 0$:

$$(1) f(x) = x^2 \sin x^{-1}; (2) f(x) = |x|^{3/2}; (3) f(x) = e^{-|x|^2}.$$

Solution: (1) $|f(x)| \leq |x|^2$, so all four Dini derivatives are 0.

(2)(3) Likewise all four Dini derivatives are 0.

6.2.5

Calculate the Dini derivatives of f :

$$(1) f(x) = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0 \end{cases} \text{ at } x = 0;$$

(2) $f(x) = \sin x \cos x$ at $x = \pi$:

(3) $f(x) = x^2 \log|x|$ at $x = 0$:

Solution:

(1) $|f(x)/x| \leq |x^2|$ so all Dini derivatives are 0.

(2) $f'(\pi) = \cos 2\pi = 1$ so all Dini derivatives are 1.

(3) $|f(x)/x| = |x| \log|x| \rightarrow 0$ so all Dini derivatives are 0.

6.2.6

Calculate the symmetric derivatives of f :

(1) $f(x) = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0 \end{cases}$ at $x = 0$:

(2) $f(x) = \sin x^2$ at $x = \pi$:

(3) $f(x) = \log|x^2 - 1|$ at $x = 1$:

(4) $f(x) = x^2 e^{-|x|}$ at $x = 0$:

Solution: (1) $f(x) - f(-x) = 0$ so the derivative is 0.

(2) f is differentiable so the symmetric derivative is $f'(\pi) = 2x \cos x^2|_{\pi} = 2\pi \cos \pi^2$.

(3) $f(1+t) - f(1-t) = \log|2t + t^2| - \log|2t - t^2| = \log(2+t)/(2-t)$, and

$$\lim_{h \rightarrow 0} \frac{\log|2+h| - \log|2-h|}{2h} = \frac{1}{2}.$$

So the derivative is $1/2$.

(4) f is even so the derivative is clearly 0.

6.3

Calculate derivatives:

6.3.1

If $f = \prod_{k=1}^n g_k$, then

$$\frac{f'}{f} = \sum_{k=1}^n \frac{g'_k}{g_k}.$$

(1) $f(x) = (x+1)(x+2)^2(x+3)^3$

$$f'(x) = f(x) \left(\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} \right).$$

(2) $f(x) = (1+nx^m)(1+mx^n)$

$$f'(x) = f(x)nm \left(\frac{x^{m-1}}{1+nx^m} + \frac{x^{n-1}}{1+mx^n} \right).$$

(3) $f(x) = (ax^m + b)^n(cx^n + d)^m$

$$f'(x) = nmf(x) \left(\frac{ax^{m-1}}{ax^m + b} + \frac{cx^{n-1}}{cx^n + d} \right).$$

6.3.2

(1) $f(x) = \sqrt{\sin^2 x}$

$f'(x) = \text{sign}(\sin x) \cos x$ when $x \neq k\pi$.

(2) $f(x) = \frac{x}{(1-x^2)(1+x)}$.

$$f'(x) = \frac{1+x^2+2x^3}{(1-x)^2(1+x)^4}.$$

(3) $f(x) = x^p(1-x)^q/(1+x)$

$$f'(x) = -\frac{x^{p-1}(1-x)^{q-1}((p+q-1)x^2 + (q+1)x - p)}{(x+1)^2}.$$

$$(4) f(x) = \frac{x \sin x + \cos x}{x \sin x - \cos x}.$$

$$f'(x) = -\frac{2x + \sin 2x}{(x \sin x - \cos x)^2}.$$

6.3.3

$$(1) f(x) = \frac{x}{\sqrt{a^2+x^2}}: f'(x) = a^2(a^2+x^2)^{-3/2}.$$

$$(2) f(x) = \arcsin(\cos^2 x): f'(x) = -\sin 2x / \sqrt{1 - \cos^4 x}.$$

$$(3) f(x) = \arcsin x \sqrt{1-x^2}: f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}\sqrt{x^4-x^2+1}}.$$

$$(4) f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}: f'(x) = \frac{1+2\sqrt{x}+4\sqrt{x+\sqrt{x}}\sqrt{x}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}.$$

$$(5) f(x) = e^{ax} \sin bx: f'(x) = e^{ax}(a \sin bx + b \cos bx).$$

6.3.4

$$(1) f(x) = (x \sin x)^x: f'(x) = (x \sin x)^x(x \cot x + \log(x \sin x) + 1).$$

$$(2) f(x) = x \sqrt{(1-x)/(1+x)}: f'(x) = \frac{1-x-x^2}{(1+x)^{3/2}\sqrt{1-x}}.$$

$$(3) f(x) = (x)^{a^x}: f'(x) = a^x x^{a^x-1}(x \log a \log x + 1).$$

$$(4) f(x) = a^{x^n}: f'(a^{x^n} n x^{n-1} \log a).$$

$$(5) f(x) = (\cos x)^{\sin x} + (\sin x)^{\cos x}:$$

$$f'(x) = (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x) + (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x).$$

$$(6) f(x) = (x + \sqrt{1+x^2})^n: f'(x) = \frac{n(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}}.$$

6.3.5

Suppose $f(x) = \cos(2 \arctan(\sin(\arccot \sqrt{(1-x)/x})))$, prove that $\frac{f'(x)}{f(x)^2} = \pm \frac{2}{(1 \pm x)^2}$.

Proof: $f(x) = \frac{1-x}{1+x}$ so $\frac{f'}{f^2} = -\frac{2}{(1-x)^2}$.

6.3.6

Suppose $y = \frac{1}{4\sqrt{2}} \log \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} - \frac{1}{2\sqrt{2}} \arctan \frac{\sqrt{2}x}{x^2 - 1}$, calculate $y'(x)$.

Solution:

$$y'(x) = \frac{1}{x^4 + 1}.$$

6.3.7

Determine dy/dx :

$$(1) \begin{cases} x = \cos^4 t, \\ y = \sin^4 t \end{cases}, \text{ at } t = \pi/3;$$

$$dx/dt = 4 \cos^3 t (-\sin t) = -\frac{\sqrt{3}}{4}, dy/dt = 4 \sin^3 t \cos t, \text{ so } dy/dx = -3.$$

$$(2) \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \text{ at } t = \pi/2, \pi;$$

$$\begin{aligned}\frac{dx}{dt} &= a(1 - \cos t), \frac{dy}{dt} = a \sin t. \\ t = \frac{\pi}{2} : \frac{dy}{dx} &= 1. t = \pi. \frac{dy}{dx} = 0.\end{aligned}$$

6.3.8

Suppose f is differentiable, calculate y' :

- (1) $y = f(x \sin x)$: $y' = (\sin x + x \cos x)f'(x \sin x)$.
- (2) $y = f(e^x)e^{f(x)}$: $y' = f'(e^x)e^{x+f(x)} + f(e^x)e^{f(x)}f'(x)$.

6.3.9

Suppose u, v differentiable, calculate y' :

$$\begin{aligned}(1) y &= \sqrt{u^2 + v^2}: y' = \frac{uu' + vv'}{\sqrt{u^2 + v^2}}. \\ (5) y &= 1/\sqrt{u^2 + v^2}: y' = -\frac{u'u + v'v}{(u^2 + v^2)^{3/2}}.\end{aligned}$$

6.3.10

Suppose $a_{ij}(x)$ are differentiable, determine the derivative of $\det(a_{ij}(x))_{n \times n}$.

Solution: Let $\varphi_i(x) = (a_{i1}(x), \dots, a_{in}(x))$, then $\det(\varphi_1, \dots, \varphi_n)$ is a multi-linear form, so

$$\frac{d}{dx} \Big|_{x=x_0} \det(\varphi_1, \dots, \varphi_n) = \sum_{k=1}^n \det(\varphi_1, \dots, \varphi'_k, \dots, \varphi_n).$$

(If $A = (a_{ij})$ is invertible, then suppose $A' = CA$, we have $\varphi'_k = \sum_j C_{k,j} \varphi_j$. Since \det is alternating, $\det(\varphi_1, \dots, \varphi'_k, \dots, \varphi_n) = C_{k,k} \det A$, hence $(\det A)' = \det A \sum C_{k,k} = \det A \cdot \text{tr}(A'A^{-1})$.)

6.3.14

Suppose x_1, \dots, x_n are distinct, and $y(x) = \prod_{k=1}^n (x - x_k)$. Calculate $\sum_{k=1}^n 1/y'(x_k)$. What if x_k are not distinct?

Solution: Note that

$$\frac{y'}{y} = \sum \frac{1}{x - x_k},$$

hence $y'(x_k) = \prod_{j \neq k} (x_k - x_j)$.

By Lagrange interpolation,

$$1 = \sum_{k=1}^n \prod_{j \neq k} \frac{x - x_j}{x_k - x_j},$$

so by considering the coefficient of x^{n-1} , we obtain

$$\sum_{k=1}^n 1/y'(x_k) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

If $f(x) = \prod_{k=1}^n (x - x_k)^{\alpha_k}$, then $f'(x_k) = \alpha_k \prod_{j \neq k} (x_k - x_j)^{\alpha_j}$, hence

$$\sum \frac{1}{f'(x_k)} = \sum_{k=1}^n \frac{\alpha_k}{f'(x_k)} = \sum_{k=1}^n \frac{1}{\prod_{j \neq k} (x_k - x_j)^{\alpha_j}} = \begin{cases} 1, & \deg f = 1, \\ 0, & \deg f > 1. \end{cases}$$

6.3.19

Suppose $f(x_0) = g(x_0)$, f, g are differentiable at x_0 , and $f'(x_0) = g'(x_0) = k$. Prove that for any h such that $\min\{f, g\} \leq h \leq \max\{f, g\}$, h is differentiable at x_0 and $h'(x_0) = k$.

Proof: Clearly $h(x_0) = f(x_0)$ and we can assume it to be 0. For any t ,

$$\min \left\{ \frac{f(x_0 + t)}{t}, \frac{g(x_0 + t)}{t} \right\} \leq \frac{h(x_0 + t) - h(x_0)}{t} = \frac{h(x_0 + t)}{t} \leq \max \left\{ \frac{g(x_0 + t)}{t}, \frac{f(x_0 + t)}{t} \right\}.$$

Since $\lim_{t \rightarrow 0} f(x_0 + t)/t = \lim_{t \rightarrow 0} g(x_0 + t)/t$, the two sides of the inequality both tend to k , therefore $\lim_{t \rightarrow 0} (h(x_0 + t) - h(x_0))/t = k$. Hence $h'(x_0) = k$.

6.3.22

Prove that there is no $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable such that $f \circ f(x) = -x^3 + x^2 + 1$.

Proof: $f'(f(x))f'(x) = -3x^2 + 2x$, and $f(f(x)) = x \iff x = 1$. If $a = f(1)$ then

$f(f(a)) = f(f(f(1))) = f(1) = a$, so by uniqueness $a = 1$. However, $-1 = f'(f(1))f'(1) = (f'(1))^2$, leading to contradiction.

6.3.23

Prove that there is no $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable such that $f \circ f(x) = x^2 - 3x + 3$.

Proof: Suppose $f(f(x)) = x^2 - 3x + 3$, then $f'(f(x))f'(x) = 2x - 3$, but $f(f(x)) = x \iff x = 1, 3$.

Suppose $a = f(1)$, then $f(a) = 1$, so $-1 = f'(f(1))f'(1) = f'(a)f'(1)$, and

$2a - 3 = f'(f(a))f'(a) = f'(a)f'(1)$, leading to contradiction.

(Note: $x^2 - 3x + 3$ is not the iteration of any mapping f and any order r , so the differentiable condition is not needed.)

6.4

6.4.1

If $x^y = y^x$, prove that $x(y \log x - x)dy = y(x \log y - y)dx$.

Proof: $x \log y = y \log x$, so $y' \log x + y/x = \log y + xy'/y$. Hence $dy(\log x - x/y) = dx(\log y - y/x)$.

Therefore $x(y \log x - x)dy = y(x \log y - y)dx$.

6.4.2

If $y = \frac{x-a}{1+ax}$, then $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$.

Proof: Note that $y = \tan(\arctan x - \arctan a)$, so $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ since $d\arctan y = d\arctan x$.

($ds = \frac{dx}{1+x^2}$ is the angular distance in spherical geometry, and clearly rotation on the sphere is an isometry. We can do likewise for hyperbolas and the Blaschke factors.)

6.4.3-6.3.6

If $xy = -1$ prove that $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$.

Proof: Clearly $\frac{dy}{dx} = x^{-2}$, so $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$.

($ds = \frac{dx}{\sqrt{1+x^4}}$ is the parametrization of the hyperbolic lemniscate, and $y = -1/x$ is the inversion which is an isometry).

Note: For the next 3 problems, they are all about preserving the arc length of the lemniscate (the ∞ shaped

curve). (The hyperbolic lemniscate $\frac{dx}{\sqrt{1+x^4}}$ is just the lemniscate $\frac{dx}{\sqrt{1-x^4}}$ rotated $\frac{\pi}{4}$.)
6.4.4 comes from $(1 + \text{sl}^2(u))(1 + \text{cl}^2(u)) = 2$, since $x^2 + y^2 + x^2y^2 = 1$ implies
 $x = \text{sl}(u)$, $y = \pm \text{cl}(u) = \pm \text{sl}(\varpi/2 - u)$, and $\frac{dx}{\sqrt{1-x^4}} = du = \pm \frac{dy}{\sqrt{1-y^4}}$.
6.4.5 comes from the rotation $\text{sl}(u+v) = \frac{\text{sl}(u)\text{cl}(v)+\text{cl}(u)\text{sl}(v)}{1+\text{sl}^2(u)\text{sl}^2(v)}$, since the equation implies
 $x = \text{sl}(u)$, $c = \text{sl}(v)$, $y = \text{sl}(u+v)$.
6.4.6 comes from duplication $\text{sl}(2u) = \frac{2\text{sl}(u)\text{cl}(u)}{1+\text{sl}^4(u)}$, since $(y^4 - 2y^2 - 1)x + y^4 + 2y^2 - 1 = 0$ implies
 $x = \text{sl}(2u)$, $y = \text{sl}(u)$.

We can unify the problems above using elliptic integrals:

Consider the elliptic metric $\omega_J = \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, and the isometry $x = \text{sn}(u) \mapsto y = \text{sn}(u+v)$, where
 $t = \int_0^{\text{sn}(t)} \omega$ is the Jacobi elliptic integral (of the first kind). Since it is an isometry, we have

$$(x^2 + y^2 + a^2) - 2(x^2y^2 + y^2a^2 + a^2x^2) + 4k^2x^2y^2a^2 - k^4(x^2y^2a^2)^2 = 0$$

implies $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$.

For $k = 0, 1, i$ we obtain all the results mentioned above (spherical, hyperbolic, lemniscate).

6.5

6.5.1

- (1) $f(x) = e^x \cos x$, $f^{(5)}(x) =$
 $f(x) = \text{Re}(e^{(1+i)x})$, then $f^{(5)}(x) = \text{Re}((-4 - 4i)e^{(1+i)x}) = 4e^x(\sin x - \cos x)$.
(2) $f(x) = x^2 \log x + x \log^2 x$, $f'' =$
 $f''(x) = 2 \log x + 3 + 2 \frac{\log x + 1}{x}$.
(3) $f(x) = x^2 e^x$, $f^{(10)} =$
 $f^{(10)}(x) = \sum_{k=0}^{10} \binom{10}{k} (x^2)^{(k)} (e^x)^{(10-k)} = e^x(x^2 + 20x + 90)$.
(4) $f(x) = x^5 \cos x$, $f^{(50)} =$
Likewise $f^{(50)}(x) = (-x^5 + 20 \binom{50}{2} x^3 - 120 \binom{50}{4}) \cos x + (-250x^4 + 60 \binom{50}{3} x^2 - 120 \binom{50}{5}) \sin x$.

6.5.2

Prove that

$$(1) \sum_{k=0}^n \binom{n}{k} k = n2^{n-1} \quad (2) \sum_{k=0}^n \binom{n}{k} k^2 = n(n+1)2^{n-2}.$$

Proof: $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$, hence

$$\sum_{k=0}^n \binom{n}{k} kx^k = xn(1+x)^{n-1}, \sum_{k=0}^n \binom{n}{k} k^2 x^{k-1} = n(1+x)^{n-1} + xn(n-1)(1+x)^{n-2}.$$

Let $x = 1$ we obtain the desired identities.

6.5.3

Prove that $y = \cos(n \arccos x)$ satisfy

$$\frac{ndx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}}, (1-x^2)y'' - xy' + n^2y = 0.$$

Proof: $y = P_n$ is the n^{th} Chebyshev polynomial.

Note that $\arccos y = n \arccos x$, hence $\frac{dy}{\sqrt{1-y^2}} = d \arccos y = n \frac{dx}{\sqrt{1-x^2}}$.

Let $x = \cos \theta$, and $y = \cos(n\theta)$, then $y' = -\sin(n\theta)n\theta' = n \sin(n\theta)/\sin \theta$. Differentiate $y' \sin \theta = n \sin n\theta$, we obtain

$$y'' \sin \theta + y'\theta' \cos \theta = n^2 \theta' \cos(n\theta) \implies y'' \sqrt{1-x^2} - \frac{y'x}{\sqrt{1-x^2}} = -\frac{n^2 y}{\sqrt{1-x^2}}.$$

Hence $(1-x^2)y'' - xy' + n^2 y = 0$.

(We can use the fact the P_n are orthogonal under the weight $w = \frac{1}{\sqrt{1-x^2}}$, and solve the Pearson differential

equation $\frac{d}{dx}(\sigma w) = \tau w$ to get $\sigma = 1-x^2$, $\tau = -x$ and directly obtain this differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0.)$$

6.5.4

Prove that: For points (x, y) on the circle $(x-a)^2 + (y-b)^2 = c^2$, $y \neq b$,

$$(1) y''/(1+(y')^2)^{3/2} = \pm 1/c; (2) y'''(1+(y')^2) = 3y'(y'')^2.$$

Proof: (1) $1+(y')^2 = 1 + \frac{(x-a)^2}{(y-b)^2} = \frac{c^2}{(y-b)^2}$, and $y'' = -\frac{c^2}{(y-b)^3}$, hence $y''/(1+(y')^2)^{3/2} = \pm 1/c$. Its absolute value is the curvature at (x, y) .

$$(2) y''' = 3c^2 y'/(y-b)^4$$
, so $y'''(1+(y')^2) = 3c^4 y'/(y-b)^6 = 3y'(y'')^2$.

6.5.5

Given the polar coordinate form $r^2 = \cos 2\theta$ of $y = f(x)$, calculate $\frac{d^2y}{dx^2}$.

Solution: $r^2 = \cos 2\theta \implies x^2 + y^2 = \frac{1-y^2/x^2}{1+y^2/x^2}$. Hence $(x^2 + y^2)^2 = x^2 - y^2$. Therefore $\frac{d^2y}{dx^2} = -\frac{3r}{\sin^3(3\theta)}$.

6.5.6

Suppose $f(x) = x \log(2^{1/x} + 3^{1/x})$, $\forall x > 0$ prove that $f'(x), f''(x) > 0$.

Proof: Let $g(x) = \log(2^x + 3^x)/x$ and $h(x) = \log(2^x + 3^x)$. Note that by Cauchy-Schwarz,

$$h''(x) = \frac{(2^x \log^2 2 + 3^x \log^2 3)(2^x + 3^x) - (2^x \log 2 + 3^x \log 3)^2}{(2^x + 3^x)^2} > 0.$$

Since $f''(x) = x^{-3} h''(x^{-1})$, we infer $f''(x) > 0$.

Let $F(t) = f'(t^{-1}) = h(t) - th'(t)$, then $F'(t) = -th''(t) < 0$, hence F is strictly decreasing. Clearly $\lim_{t \rightarrow \infty} F(t) = 0$, so $f'(x) = F(x^{-1}) > 0$.

6.5.7

Suppose P is a polynomial of degree n with n distinct roots. Prove that $P'(x)^2 \geq P(x)P''(x)$.

Proof: Suppose $P(x) = \prod_{k=1}^n (x - x_k)$, and $P(t) \neq 0$. Then

$$\frac{PP'' - P'(t)^2}{P(t)^2} = \frac{d}{dt} \frac{P'(t)}{P(t)} = \frac{d}{dt} \sum_{k=1}^n \frac{1}{t - x_k} = \sum_{k=1}^n -(t - x_k)^{-2} < 0.$$

Therefore $P'(t)^2 \geq P(t)P''(t)$.

6.5.9

Suppose $f'(x) \neq 0$ and f^{-1} exists. Express $(f^{-1})^{(3)}$ in terms of f' , f'' , $f^{(3)}$.

Solution:

$$(f^{-1})^{(3)}(y) = \frac{3[f''(f^{-1}(y))]^2 - f'(f^{-1}(y))f^{(3)}(f^{-1}(y))}{[f'(f^{-1}(y))]^5}$$

6.5.10

$a, b \in \mathbb{R}$, $b > 0$. Consider $f : [-1, 1] \rightarrow \mathbb{R}$, where

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(If $b \leq 0$, then f n-times differentiable, $f \in C^n$, $f^{(n)}$ is bounded are all equivalent to $a - b \geq n$ which is very easy.)

Prove that

1. $f \in C([-1, 1])$ iff $a > 0$;

Proof: $f \in C([-1, 1])$ iff $\lim_{x \rightarrow 0} x^a \sin(x^{-b}) = 0$. If $a > 0$ then $|x^a \sin(x^{-b})| \leq |x|^a \rightarrow 0$. If $a < 0$ then let $x = ((2n + 1/2)\pi)^{-1/b}$, when $n \rightarrow \infty$, $x \rightarrow 0$ but $|x^a \sin(x^{-b})| \rightarrow \infty$. If $a = 0$, then let $x = ((2n + 1/2)\pi)^{-1/b}$, $|x^a \sin(x^{-b})| = 1$. Therefore $f \in C([-1, 1])$ iff $a > 0$.

2. f is differentiable at 0 iff $a > 1$;

Proof: f is differentiable at 0 iff $\lim_{x \rightarrow 0} x^{-a} \sin(x^{-b})$ exists. By 1 we know that $a > 1$. ($a = 1$ is invalid since $x = (2n\pi)^{-1/b}$ and $x = ((2n + 1/2)\pi)^{-1/b}$ converge to different values.)

3. f' is bounded on $[-1, 1]$ ($\iff f$ is Lipschitz) iff $a \geq 1 + b$;

Proof: $f'(x) = ax^{a-1} \sin(x^{-b}) + x^a \cos(x^{-b})(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geq 1 + b$.

4. $f \in C^1([-1, 1])$ iff $a > 1 + b$;

Proof: $f \in C^1([-1, 1])$ iff $f'(0) = 0 = \lim_{x \rightarrow 0} f'(x)$. By 1 we know it is equivalent to $a > 1 + b$.

5. f' is differentiable at 0 iff $a > 2 + b$;

6. f'' is bounded on $[-1, 1]$ iff $a \geq 2 + 2b$;

7. $f \in C^2([-1, 1])$ iff $a > 2 + 2b$.

Proof: 5,6,7 are exactly the same as 2,3,4.

Likewise $f \in C^n([-1, 1])$ iff $a > n(1 + b)$, $f^{(n-1)}$ differentiable at 0 iff $a > n + (n - 1)b$, and $f^{(n)}$ is bounded iff $a \geq n + nb$.

6.5.11

Suppose $f(x) = \prod_{k=1}^n (x - x_k)$ where x_k are distinct, and ξ_1, \dots, ξ_{n-1} are the roots of $f'(x)$. Calculate $\sum_{k=1}^{n-1} 1/f''(\xi_k)$.

Solution: Same as 6.3.14, use Lagrange interpolation, the sum for the k^{th} derivative is $\frac{\delta_{k,n}}{k!}$.