111页2; 115-116页1,3,4,5,7,8.

111-2

Use Theorem 20 to prove the following. If W is a subspace of a finite-dimensional vector space V and if $\{g_1, \cdots, g_r\}$ is any basis for W^0, then

$$W = \bigcap_{i=1}^{r} \mathrm{Ker} g_i.$$

Proof: Since $h \in \mathrm{Span}\{g_1, \cdots, g_n\} \iff \bigcap_{i=1}^{r} \mathrm{Ker} g_i \subset \mathrm{Ker} h$, so $W^0 = \left\{h : f\left(\bigcap_{i=1}^{r} \mathrm{Ker} g_i\right) = \{0\}\right\} = \left(\bigcap_{i=1}^{r} \mathrm{Ker} g_i\right)^0$, hence $W = \bigcap_{i=1}^{r} \mathrm{Ker} g_i$.

115-1

Let F be a field and let f be the linear functional on F^2 defined by $f(x_1, x_2) = ax_1 + bx_2$. For each of the following linear operators T, let $g = T^t f$ and find $g(x_1, x_2)$.
(a) $T(x_1, x_2) = (x_1, 0)$; (b) $T(x_1, x_2) = (-x_2, x_1)$; (c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$.
Proof: (a) $T^t f(x_1, x_2) = f \circ T(x_1, x_2) = f(x_1, 0) = ax_1$.
(b) $T^t f(x_1, x_2) = f \circ T(x_1, x_2) = f(-x_2, x_1) = -ax_2 + bx_1$.
(c) $T^t f(x_1, x_2) = f \circ T(x_1, x_2) = f(x_1 - x_2, x_1 + x_2) = (a + b)x_1 + (b - a)x_2$.

115-3

Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is the linear operator on V defined by $T(A) = AB - BA$, and if f is the trace function, what is $T^t f$?
Solution: $T^t f = f \circ T$, and $T^t f(A) = \mathrm{tr}(AB - BA) = 0$, hence $T^t f = 0$.

115-4

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Let c be a scalar and suppose there is a non-zero vector $\alpha \in V$ such that $T\alpha = c\alpha$. Prove that there is a non-zero linear functional $f \in V^*$ such that $T^t f = cf$.
Proof: $T - cI$ is not invertible, and $(T - cI)^t = T^t - cI$. Since $\mathrm{rank}(T^t) = \mathrm{rank}(T)$, $T^t - cI$ is also not invertible, hence c is an eigenvalue and there exists $f \in V^* \backslash \{0\}$ such that $T^t f = cf$.

116-5

Let A be an $m \times n$ matrix with real entries. Prove that $A = 0$ iff $\mathrm{tr}(A^t A) = 0$.
Proof: Note that $(A^t A)_{i,i} = \sum_{j=1}^{m} A^t_{i,j} A_{j,i} = \sum_{j=1}^{n} A^2_{j,i} \geqslant 0$, hence $\mathrm{tr}(A^t A) = 0$ implies $(A^t A)_{i,i} = 0 \forall 1 \leqslant i \leqslant n$, hence $A_{j,i} = 0 \forall i, j$ so $A = 0$.
If $A = 0$ clearly $\mathrm{tr}(A^t A) = \mathrm{tr}(0) = 0$.

116-7

Let V be a finite-dimensional vector space over the field F. Show that $\varphi : \mathcal{L}(V, V) \to \mathcal{L}(V^*, V^*)$, $T \to T^t$ is an isomorphism.
Proof: Clearly φ is linear, $\mathrm{Ker}\varphi = \{0\}$, and $\mathrm{dim}\mathcal{L}(V, V) = (\mathrm{dim}V)^2 = (\mathrm{dim}V^*)^2 = \mathrm{dim}\mathcal{L}(V^*, V^*)$, so φ is an isomorphism.

116-8

Let V be the vector space of $n \times n$ matrices over the field F.
(a) If B is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(A) = \mathrm{tr}(B^t A)$. Show that f_B is a linear functional on V.
(b) Show that every linear functional on V is of the above form, i.e., is f_B for some B.
(c) Show that $B \to f_B$ is an isomorphism of $V \to V^*$.
Proof: (a) $\varphi : A \mapsto B^t A \in \mathcal{L}(V, V)$, and $\mathrm{tr} \in V^*$, so $f_B = \mathrm{tr} \circ \varphi \in V^*$.
(b) (We can use Riesz representation theorem on the Euclidean inner product.)
Suppose $f(E_{i,j}) = c_{i,j}$ where $E_{i,j}$ is the matrix where only (i, j) is 1 and the other entries are 0. Then for $B = (c_{i,j})_{i,j \leqslant n}$, $f = f_B$.
(c) $\Phi : V \to V^*, B \mapsto f(B)$ is an epimorphism, and $\mathrm{dim} V = \mathrm{dim} V^*$, so Φ is an isomorphism.