

5.1.3

Suppose $\{e_1, \dots, e_n\}$ is the standard basis of $F^{n \times 1}$. For $\sigma \in S_n$, denote $R(\sigma) \in F^{n \times n}$ as the (invertible) matrix where the j^{th} column is $e_{\sigma(j)}$. Prove that $R : S_n \rightarrow \text{GL}(n, F)$ is a homomorphism on groups.

Proof: Note that $R(\sigma)_{i,j} = \delta_{i,\sigma(j)}$, so $R(\sigma\tau)_{i,j} = \delta_{i,\sigma\tau(j)}$, and

$$(R(\sigma)R(\tau))_{i,j} = \sum_{k=1}^n R(\sigma)_{i,k} R(\tau)_{k,j} = \sum_{k=1}^n \delta_{i,\sigma(k)} \delta_{k,\tau(j)} = \delta_{i,\sigma(\tau(j))} = R(\sigma\tau)_{i,j}.$$

Hence $R(\sigma\tau^{-1}) = R(\sigma)R(\tau)^{-1}$.

5.1.4

Suppose $\{e_1, \dots, e_n\}$ is the standard basis of F^n . For $\sigma \in S_n$, denote $R'(\sigma)$ as the matrix where the i^{th} row is $e_{\sigma(i)}$. Does the identity $R'(\sigma\tau) = R'(\sigma)R'(\tau)$ hold?

Solution: Note that $R'(\sigma)_{i,j} = \delta_{\sigma(i),j}$, so

$$(R'(\sigma)R'(\tau))_{i,j} = \sum_{k=1}^n \delta_{\sigma(i),k} \delta_{\tau(k),j} = \delta_{\tau(\sigma(i)),j} = R'(\tau\sigma)_{i,j}$$

is not a homomorphism when $n \geq 3$ since S_n is not abelian, rather it is a contravariant one.

Or use $R'(\sigma\tau) = R(\sigma\tau)^T = (R(\sigma)R(\tau))^T = R(\tau)^T R(\sigma)^T = R'(\tau)R'(\sigma)$.