

2025/10/9

教材10-11页2,7,8; 16页7,9. 求 F_p^n 中二维子空间的个数.

10-2

If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

find all solutions of $AX = 0$ by row-reducing A .

Solution:

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 8 & 2 \\ 0 & 7 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

hence the three rows are linearly independent, and the only solution is $(0, 0, 0)$.

11-7

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: To interchange rows i, j , denote the k -th row by r_k . Let $r_i \rightarrow r_i + r_j$, $r_j \rightarrow r_j - r_i$, $r_j \rightarrow -r_j$, $r_i \rightarrow r_i - r_j$, then these operations interchange r_i, r_j .

11-8

Consider the system of equations $AX = 0$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F . Prove the following

(a) If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.

(b) If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.

(c) If $ad - bc = 0$ and some entry of A is not 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution iff there is a scalar y such that $x_1 = yx_1^0, x_2 = yx_2^0$.

Proof: (a) is trivial. (b): If $ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$, then

$0 = c(ax_1 + bx_2) - a(cx_1 + dx_2) = -(ad - bc)x_2$ so $x_2 = 0$. Likewise $x_1 = 0$ so the only solution is $(0, 0)$.

(c) Proof: Let $(x_1^0, x_2^0) = (b, -a)$ which is a solution of $AX = 0$. For any solution (x_1, x_2) , let $y = x_1/b$ (or $-x_2/a$ if $b = 0$), then $(x_1, x_2) = y(x_1^0, x_2^0)$.

16-7

Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_6 &= -2 \\ 2x_1 + 0x_2 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

Solution: Apply elementary row operations to the matrix:

$$\begin{pmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence all solutions are:

$$\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 1 + 2x_3 - x_4, x_2 = 2 - x_3 + x_4, x_5 = 1\}.$$

16-9

Let

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

Proof: Since A map each base $(0, \dots, 1, 0, \dots, 0)$ to its corresponding column, the answer is

$\text{Span}\langle (3, -2, 0, 1), (-6, 4, 0, 2), (2, 1, 1, 1), (-1, 3, 1, 0) \rangle$ which is $\text{Span}\langle (3, -2, 0, 1), (-1, 3, 1, 0) \rangle$.

Ex1

求 F_p^n 中二维子空间的个数.

Solution: Any subspace $W \subset F_p^n$ such that $\dim W = 2$ corresponds to a $2 \times n$ non-zero row-reduced echelon matrix. Hence there are

$$\begin{aligned} \sum_{1 \leq i < j \leq n} p^{n-i-1} \cdot p^{n-j} &= \sum_{i=1}^{n-1} p^{n-i-1} \sum_{j=i+1}^n p^{n-j} = \sum_{i=1}^{n-1} p^{n-i-1} \cdot \frac{p^{n-i} - 1}{p - 1} = \frac{1}{p - 1} \left(\sum_{i=1}^{n-1} p^{2(n-i)-1} - p^{n-i-1} \right) \\ &= \frac{1}{p - 1} \left(p \frac{p^{2(n-1)} - 1}{p^2 - 1} - \frac{p^{n-1} - 1}{p - 1} \right) = \frac{(p^n - 1)(p^{n-1} - 1)}{(p - 1)(p^2 - 1)} \end{aligned}$$

distinct subspaces.