2025/11/5

周民强《实变函数论（第三版）》P126-127 习题3:9,10,12,13; P143 思考题 8; P189-192 习题4:3,4,5

# 126-9

Suppose $m(E) < \infty$, $f, f_1, \cdots, f_k$ are a.e. finite & measurable functions on $E$, prove that $\{f_k(x)\}$ converges to $f$ in measure iff

$$\lim_{k\to\infty} \inf_{\alpha>0}\{\alpha + m(\{x \in E : |f_k(x) - f(x)| > \alpha\})\} = 0. \tag{$\star$}$$

Proof: If $f_k \to f$ in measure, then for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \inf_{\alpha>0}\{\alpha + m(\{x \in E : |f_k(x) - f(x)| > \alpha\})\} \leqslant \lim_{k\to\infty} \varepsilon + m(\{x \in E : |f_k(x) - f(x)| > \varepsilon\}) = \varepsilon$$

let $\varepsilon \to 0$ we obtain $(\star)$.

If $(\star)$ holds, then for any $n > 0$ there exists $N(n)$ such that $k > N$ implies there exists $\alpha_{k,n}$ such that $\alpha_{k,n} + m(\{x \in E : |f_k(x) - f(x)| > \alpha_{k,n}\}) < 1/n$.

For any $\varepsilon > 0$, and any $n > 1/\varepsilon, k > N(n)$, $\alpha_{k,n} < 1/n < \varepsilon$ so

$$m(\{x \in E : |f_k(x) - f(x)| > \varepsilon\}) \leqslant m(\{x \in E : |f_k(x) - f(x)| > \alpha_{k,n}\}) < 1/n.$$

Hence $m(\{x \in E : |f_k(x) - f(x)| > \varepsilon\}) < (\sup\{n > \varepsilon^{-1} : k > N(n)\})^{-1}$ so it tends to $0$. Therefore $f_k \to f$ in measure.

# 127-10

Suppose $f_n(x)$ is monotonically increasing on $[0,1]$, and $\{f_n(x)\}$ converges to $f$ in measure. Prove that if $f$ is continuous at $x_0$, then $f_n(x_0) \to f(x_0)$ as $n \to \infty$.

Proof: For any $\varepsilon > 0$ and $x_0 < 1$ ($x_0 = 1$ automatically holds) we find $N$ such that $n > N \implies f_n(x_0) < f(x_0) + \varepsilon$. Take $\delta > 0$ such that $(x_0, x_0 + \delta) \subset [0,1]$ and $y \in (x_0 - \delta, x_0 + \delta) \implies |f(y) - f(x_0)| < \varepsilon/2$. Since $f_n \to f$ in measure, there exists $N$ such that $n > N$ implies $m(\{x \in [0,1] : |f_n(x) - f(x)| > \varepsilon/2\}) < \delta/2$. Note that $m((x_0, x_0 + \delta/2)) = \delta/2$ so there exists $y_0 \in (x_0, x_0 + \delta/2)$ such that $|f_n(y_0) - f(y_0)| \leqslant \varepsilon/2$. Combine this with $|f(y_0) - f(x_0)| < \varepsilon/2$ we obtain $|f_n(y_0) - f(x_0)| < \varepsilon$ so $f_n(y_0) < f(x_0) + \varepsilon$. Since $f_n$ is monotonically increasing, $f_n(x_0) \leqslant f_n(y_0) < f(x_0) + \varepsilon$.

The other side $f_n(x_0) > f(x_0) - \varepsilon$ is exactly the same, therefore $\lim_{n\to\infty} f_n(x_0) = f(x_0)$.

# 127-12

Suppose $\{f_k(x)\}$ and $\{g_k(x)\}$ both converge to $0$ in measure on $E$. Prove that $f_k \cdot g_k \to 0$ in measure on $E$.

Proof: For any $\varepsilon > 0$, $\lim_{n\to\infty} m(\{x \in E : |f_k(x)| > \varepsilon\}) = 0$ and $\lim_{n\to\infty} m(\{x \in E : |g_k(x)| > \varepsilon\}) = 0$. Then any $|f_k(x)g_k(x)| > \varepsilon$, either $|f_k(x)| > \sqrt{\varepsilon}$ or $|g_k(x)| > \sqrt{\varepsilon}$, so

$$\{x \in E : |f_k(x)g_k(x)| > \varepsilon\} \subset \{x \in E : |f_k(x)| > \sqrt{\varepsilon}\} \cup \{x \in E : |g_k(x)| > \sqrt{\varepsilon}\}$$

hence $\lim_{n\to\infty} m(\{x \in E : |f_k(x)g_k(x)| > \varepsilon\}) = 0$ so $f_k g_k \to 0$ in measure.

## 127-13

Suppose $f_k \to f$ in measure on $[a,b]$, $g \in C(\mathbb{R})$, prove that $g(f_k) \to g(f)$ in measure on $[a,b]$. Does the statement still hold when $[a,b]$ is changed to $[0,\infty)$?
Proof: For any $\varepsilon > 0$, we show that

$$\lim_{n\to\infty} m(\{x \in [a,b] : |g(f_k(x)) - g(f(x))| > \varepsilon\}) = 0.$$

For any $\eta > 0$, there exists $M > 0$ such that $m(\{x \in [a,b] : |f(x)| > M\}) < \eta/2$ (since $m([a,b]) < \infty$ so $\lim_{M\to\infty} m(\{x \in [a,b] : |f(x)| > M\}) = 0$).
Note that $g$ is uniformly continuous on $[-M-10, M+10]$, so there exists $\delta < 1$ such that $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Note that $\lim_{n\to\infty} m(\{x \in [a,b] : |f_k(x) - f(x)| > \delta\}) = 0$ so there exists $N$ such that $k > N$ implies $m(\{x \in [a,b] : |f_k(x) - f(x)| > \delta\}) < \varepsilon/2$. Then if $|g(f_k(x)) - g(f(x))| > \varepsilon$, either $|f(x)| > M$ or $|f_k(x) - f(x)| > \delta$, so

$$\begin{aligned} & m(\{x \in [a,b] : |g(f_k(x)) - g(f(x))| > \varepsilon\}) \\ \leqslant & m(\{x \in [a,b] : |f_k(x) - f(x)| > \delta\}) + m(\{x \in [a,b] : |f(x) > M|\}) < \eta. \end{aligned}$$

Therefore $\lim_{n\to\infty} m(\{x \in [a,b] : |g(f_k(x)) - g(f(x))| > \varepsilon\}) = 0$.
Counterexample:
Let $g(x) = x^2$ which is not uniformly continuous, and let

$$f_k(x) = \begin{cases} x + \frac{1}{k}, & x > k^2, \\ x, & x \in [0, k^2]. \end{cases}$$

Then for $k > 1/\varepsilon$, $\{x > 0 : |f(x) - f_k(x)| > \varepsilon\} = \emptyset$ so $f_k \to f$ in measure. However, for $x > k^2$, $|f_k(x)^2 - f(x)^2| = \frac{2x}{k} + \frac{1}{k^2} > k$, so for $\varepsilon = 1$, and any $k \geqslant 1$, $m(\{x > 0 : |f_k^2(x) - f^2(x)| > \varepsilon\}) = \infty$, hence $f_k$ does not converge to $f$ in measure.

## 143-8

Suppose $f$ is non-negative and measurable on $[a,b]$, then $f^3 \in \mathcal{L}([a,b])$ iff

$$\sum_{n=1}^{\infty} n^2 m(\{x \in [a,b] : f(x) \geqslant n\}) < \infty.$$

Proof: Let $A_n = \{x \in [a,b] : f(x) \in [n, n+1)\}$ and $B_n = \{x \in [a,b] : f(x) \geqslant n\}$, then $B_n = \bigcup_{k\geqslant n} A_k$ and $A_n$ are disjoint, so $m(B_n) = \sum_{k\geqslant n} m(A_k)$ and

$$\sum_{n=1}^{\infty} n^2 m(B_n) = \sum_{n=1}^{\infty} n^2 \sum_{k\geqslant n} m(A_k) = \sum_{k=1}^{\infty} m(A_k) \sum_{n\leqslant k} n^2$$

has the same convergence with $\sum_{k=1}^{\infty} k^3 m(A_k)$.
If $f^3 \in \mathcal{L}([a,b])$, then let $g_N = \sum_{n=1}^{N} n^3 \mathbf{1}_{A_n}$ where $A_n = \{x \in [a,b] : f(x) \in [n, n+1)\}$, then $g_N$ are simple measurable non-negative functions, so $\int_{[a,b]} g_N \,\mathrm{d}m \leqslant \int_{[a,b]} f^3 \,\mathrm{d}m < \infty$ and $\sum_{n=1}^{N} n^3 m(A_n) < \infty$, which implies $\sum_{k=1}^{\infty} k^3 m(A_k) < \infty$.
If $\sum_{k=1}^{\infty} k^3 m(A_k) < \infty$, then $\sum_{k=1}^{\infty} (k+1)^3 m(A_k) < \infty$. Note that

$$\int_{\bigcup_{n=1}^{N} A_n} f^3 \,\mathrm{d}m = \sum_{n=1}^{N} \int_{A_n} f^3 \,\mathrm{d}m \leqslant \sum_{n=1}^{N} (n+1)^3 m(A_n) \leqslant \sum_{n=1}^{\infty} (n+1)^3 m(A_n).$$

Use Beppo-Levi theorem and $[a,b] = \bigcup_{n=1}^{\infty} A_n$, we obtain

$$\int_{[a,b]} f^3 \,\mathrm{d}m = \lim_{N\to\infty} \int_{\bigcup_{n=1}^N A_n} f^3 \,\mathrm{d}m \leqslant \sum_{n=1}^{\infty} (n+1)^3 m(A_n) < \infty$$

so $f^3$ is integrable.

## 189-3

Suppose $f$ is non-negative and measurable on $E \subset \mathbb{R}^n$. If there exists $E_k \subset E$, $m(E\backslash E_k) < 1/k$ for $k = 1, 2, \cdots$, such that the limit

$$\lim_{k\to\infty} \int_{E_k} f(x) \,\mathrm{d}m$$

exists, prove that $f$ is integrable on $E$.
Proof: Let $f_N = \min\{f, N\}$, then $0 \leqslant f_N \leqslant f_{N+1} \leqslant f$ and $\lim_{N\to\infty} f_N = f$, so by Beppo-Levi theorem, $\int_E f \,\mathrm{d}m = \lim_{N\to\infty} \int_E f_N \,\mathrm{d}m$. Note that

$$\int_E f_N \,\mathrm{d}m = \int_{E_k} f_N \,\mathrm{d}m + \int_{E\backslash E_k} f_N \,\mathrm{d}m \leqslant \int_{E_K} f \,\mathrm{d}m + Nm(E\backslash E_k).$$

So for any $N \geqslant 1$,

$$\int_E f_N \,\mathrm{d}m \leqslant \lim_{k\to\infty} \int_{E_k} f \,\mathrm{d}m$$

is bounded, therefore $\int_E f \,\mathrm{d}m$ is bounded.

## 189-4

Suppose $f \in \mathcal{L}(\mathbb{R})$ non-negative, let

$$F(x) = \int_{(-\infty,x]} f \,\mathrm{d}m,\ \forall x \in \mathbb{R}.$$

If $F \in \mathcal{L}(\mathbb{R})$ and $\int_{\mathbb{R}} F \,\mathrm{d}m < \infty$, prove that $\int_{\mathbb{R}} f \,\mathrm{d}m = 0$.
Proof: Let $E = \{x \in \mathbb{R} : f(x) > 0\}$ and $E_n = \{x \in \mathbb{R} : f(x) > 1/n\}$, then $E = \bigcup_{n\geqslant 1} E_n$ and $E_n$ are measurable. If $m(E_n) > 0$ for some $n \geqslant 1$, then $E_n = \bigcup_{M\geqslant 1}(E_n \cap [-M, M])$ so there exists $M > 0$ such that $m(E_n \cap [-M, M]) > 0$. Let $A = E_n \cap [-M, M]$ be a measurable set, then for any $x > M$,

$$F(x) = \int_{(-\infty,x]} f \,\mathrm{d}m \geqslant \int_A f \,\mathrm{d}m \geqslant \frac{m(A)}{n},$$

so $\int_{\mathbb{R}} F \,\mathrm{d}m \geqslant \int_{[M+1,\infty)} F \,\mathrm{d}m \geqslant \frac{m(A)}{n} m([M+1,\infty)) = \infty$, leading to contradiction. Hence $m(E_n) = 0$ so $m(E) = 0$ and $f = 0$ a.e., so $\int_{\mathbb{R}} f \,\mathrm{d}m = 0$.

## 190-5

Suppose $f_k \in \mathcal{L}(\mathbb{R}^n)$ are non-negative. If for any measurable set $E \subset \mathbb{R}^n$,

$$\int_E f_k \,\mathrm{d}m \leqslant \int_E f_{k+1} \,\mathrm{d}m,$$

prove that

$$\lim_{k\to\infty}\int_E f_k\,\mathrm{d}m=\int_E \lim_{k\to\infty} f_k(x)\,\mathrm{d}m.$$

Proof: Note that for any $E\subset\mathbb{R}^n$ measurable, $\int_E (f_{k+1}-f_k)\,\mathrm{d}m\geqslant 0$. Let $A=\{x\in\mathbb{R}^n : f_{k+1}(x)<f_k(x)\}$, and $A_n=\{x\in\mathbb{R}^n : f_{k+1}(x)<f_k(x)-1/n\}$, then $A=\bigcup_{n\geqslant 1} A_n$ and $A_n$ are measurable. If $m(A_n)>0$ for some $n\geqslant 1$, then

$$\int_{A_n}(f_k-f_{k+1})\,\mathrm{d}m\geqslant\int_{A_n}\frac{1}{n}\,\mathrm{d}m=\frac{m(A_n)}{n}>0,$$

leading to contradiction. Hence $A_n$ are null sets, and so is $A$.
We obtain $f_k\leqslant f_{k+1}$ a.e., so by Beppo-Levi monotone convergence theorem,

$$\lim_{k\to\infty}\int_E f_k\,\mathrm{d}m=\int_E\lim_{k\to\infty} f_k\,\mathrm{d}m,$$

for any measurable set $E\subset\mathbb{R}^n$.