PSA: Stieltjes Integral

Let μ be a monotonic function on $I=[a,b]$.

A1) For any pair of partitions $\sigma,\sigma'\in\mathcal{S}(I)$,

$$\underline{S}_\mu(f;\sigma)\leqslant\overline{S}_\mu(f;\sigma').$$

Proof: Suppose $\mathcal{C}=\sigma\cup\sigma'$, then

$$\underline{S}_\mu(f;\sigma)\leqslant\underline{S}_\mu(f;\mathcal{C})\leqslant\overline{S}_\mu(f;\mathcal{C})\leqslant\overline{S}_\mu(f;\sigma').$$

A2) For any $\rho\in C([a,b]),\rho\geqslant 0$, $\mu(x)=\int_a^x\rho(t)\,\mathrm{d}t$. Prove that for any $f\in\mathcal{R}([a,b])$, $f\in\mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu=\int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

Proof: Consider any $\mathcal{C}=\{x_0,x_1,\cdots,x_n\}$, then if we denote $m_i=\inf_{x\in[x_{i-1},x_i]}f(x),u_i=\inf_{x\in[x_{i-1},x_i]}\rho(x),v_i=\inf_{x\in[x_{i-1},x_i]}f(x)\rho(x),M=\sup_{x\in[a,b]}f(x)$, $v_i-m_iu_i\leqslant f(t)\rho(t)-f(t)u_i\leqslant M\omega_\rho(x_{i-1},x_i)$. Hence for any $\varepsilon>0$ there exists a $\delta>0$, for any $\max\{x_i-x_{i-1}\}<\delta$, $\sup_{x,y\in[x_{i-1},x_i]}|\rho(x)-\rho(y)|<\varepsilon$. Then

$$\begin{aligned}\underline{S}(f\rho;\mathcal{C})&=\sum_{k=1}^n v_k(x_k-x_{k-1})\leqslant\sum_{k=1}^n u_im_i(x_k-x_{k-1})+M\varepsilon(b-a)\\&\leqslant M\varepsilon(b-a)+\sum_{k=1}^n m_i\int_{x_{k-1}}^{x_k}\rho(t)\,\mathrm{d}t=M\varepsilon(b-a)+\underline{S}_\mu(f;\mathcal{C}).\end{aligned}$$

The other side is similar, hence $\sup\{\underline{S}_\mu(f;\mathcal{C})\}=\inf\{\overline{S}_\mu(f;\mathcal{C})\}$ so $f\in\mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu=\int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

A3) Prove that $\mathcal{R}(I;\mu)$ is a linear space on $\mathbb{R}$ and the integration operator

$$\int_a^b\cdot\,\mathrm{d}\mu:\mathcal{R}(I;\mu)\to\mathbb{R},f\mapsto\int_a^b f\,\mathrm{d}\mu.$$

is linear.
Proof: Since $\underline{S}_\mu(\cdot;\mathcal{C})$ and $\overline{S}_\mu(\cdot;\mathcal{C})$ is linear for any $\mathcal{C}$, $\mathcal{R}(I;\mu)$ is clearly a linear space on $\mathbb{R}$, and $\int_a^b\cdot\,\mathrm{d}\mu$ is a linear operator.

A4) Suppose $f_1,f_2\in\mathcal{R}(I;\mu)$. If the any $x\in I$, $f_1(x)\leqslant f_2(x)$, then

$$\int_a^b f_1\,\mathrm{d}\mu\leqslant\int_a^b f_2\,\mathrm{d}\mu.$$

Proof: By A3), we can assume $f_1=0$. Then for any $\mathcal{C}$, $\underline{S}_\mu(f;\mathcal{C})\geqslant 0$ since $f\geqslant 0$, hence $\int_a^b f\,\mathrm{d}\mu=\sup\{\underline{S}_\mu(f;\mathcal{C})\}\geqslant 0$.

A5) If $f \in \mathcal{R}([a,b];\mu)$, then for any $c \in [a,b]$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Stieltjes integrable and

$$\int_a^b f \,\mathrm{d}\mu = \int_a^c f \,\mathrm{d}\mu + \int_c^b f \,\mathrm{d}\mu.$$

Proof: For any partition σ, let $\sigma' = \sigma \cup \{c\}$, then σ' can be split into two partitions of the intervals $[a,c]$ and $[c,b]$: $\sigma' = \sigma_1 \cup \sigma_2$, such that $\underline{S}_\mu(f;\sigma') = \underline{S}_\mu(f;\sigma_1) + \underline{S}_\mu(f;\sigma_2)$ and $\overline{S}_\mu(f;\sigma') = \overline{S}_\mu(f;\sigma_1) + \overline{S}_\mu(f;\sigma_2)$. Hence

$$\inf \underline{S}_\mu(f;\sigma_1) + \inf \underline{S}_\mu(f;\sigma_2) \leqslant \inf \underline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma_1) + \sup \overline{S}_\mu(f;\sigma_2).$$

Therefore

$$\int_a^b f \,\mathrm{d}\mu = \int_a^c f \,\mathrm{d}\mu + \int_c^b f \,\mathrm{d}\mu.$$

A6) If $f, g \in \mathcal{R}([a,b];\mu)$, then $f \cdot g \in \mathcal{R}([a,b];\mu)$.

Proof: Same as in the case of the Riemann integral.

A7) Define Stieltjes integral on the interval $[0,\infty)$: Suppose $f \in C([0,\infty))$ is continuous and bounded, define

$$\int_0^\infty f \,\mathrm{d}\mu = \lim_{M\to\infty} \int_0^M f \,\mathrm{d}\mu.$$

Suppose $\{\alpha_n\}_{n\geqslant 1}$ is a sequence of positive real numbers and $\sum_{n=1}^\infty \alpha_n$ converges, define the monotonic function $\mu = \sum_{n=1}^\infty \alpha_n \mathbf{1}_{\geqslant n}$, then

$$\int_1^\infty f \,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

Proof: Note that

$$\mu(x+0) - \mu(x-0) = \begin{cases} 0, & x \notin \mathbb{Z}, \\ \alpha_x, & x \in \mathbb{Z}. \end{cases}$$

Hence

$$\int_0^N f \,\mathrm{d}\mu = \sum_{n=1}^{N-1} f(n)\alpha_n.$$

By definition,

$$\int_0^\infty f \,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

A8) $f, g \in \mathcal{R}([a,b];\mu)$ are real-valued Riemann integrable functions. Suppose for any $x \in [a,b]$, $g(x) \geqslant 0$. Let

$$m = \inf_{x\in I} f(x),\ M = \sup_{x\in I} f(x).$$

Then there exists $\ell \in [m, M]$ such that

$$\int_a^b fg \,\mathrm{d}\mu = \ell \int_a^b g \,\mathrm{d}\mu.$$

Proof: Note that $mg \leqslant fg \leqslant Mg$, and use A4).

A9) Construct a Stieltjes integral to show that Abel summation method is a special case of integration by parts.

Proof:
The Abel summation formula states that

$$\sum_{i=1}^{n} T_i(S_i - S_{i-1}) = T_n S_n - T_1 S_0 - \sum_{i=1}^{n-1} S_i(T_{i+1} - T_i).$$

Consider the monotonically increasing function $\mu : [0, n] \to \mathbb{R}, x \mapsto T_{\lceil x \rceil}, \mu(0) = T_1$, and f be a polynomial such that $f(k) = S_k$ for $k = 0, 1, \cdots, n$. Then

$$\int_0^n f'\mu \,\mathrm{d}x = \sum_{k=1}^{n} \int_{k-1}^{k} f'\mu = \sum_{k=1}^{n} \int_{k-1}^{k} f'(x) T_k \,\mathrm{d}x = \sum_{k=1}^{n} T_k(S_k - S_{k-1}).$$

While

$$\int_0^n f \,\mathrm{d}\mu = \sum_{k=1}^{n-1} f(k)(\mu(k+0) - \mu(k)) = \sum_{k=1}^{n-1} S_k(T_{k+1} - T_k).$$

and

$$f\mu\Big|_0^n = T_n S_n - T_1 S_0.$$

Hence the formula is a special case of integration by parts.

PSB: Convergence of Improper Integrals

b can be ∞.

B1) Assume $f : [a, b) \to \mathbb{R}$, and for any $b^- < b$, f is integrable on $[a, b^-]$. Prove that the integral $\int_a^b f(x) \,\mathrm{d}x$ exists iff: for any $\varepsilon > 0$, $\exists b(\varepsilon) \in (a, b)$ such that for any $b', b'' > b(\varepsilon)$, $\left|\int_{b'}^{b''} f(x) \,\mathrm{d}x\right| < \varepsilon$.

Proof: Let

$$F(t) = \int_a^t f(x) \,\mathrm{d}x, \forall t \in [a, b).$$

Then $\int_a^b f(x) \,\mathrm{d}x$ exists iff $\lim_{t \to b^-} F(t)$ exists, which is equivalent to

$$\forall \varepsilon > 0, \exists b(\varepsilon) \in (a, b), \forall b', b'' > b(\varepsilon), \left|\int_{b'}^{b''} f(x) \,\mathrm{d}x\right| = |F(b'') - F(b')| < \varepsilon.$$

B2) If $|f(x)| \leqslant F(x), x \in [a, b)$ and $\int_a^b F(x) \,\mathrm{d}x$ converges, then $\int_a^b f(x) \,\mathrm{d}x$ converges.

Proof: Use B1) and

$$\left|\int_u^v f(x) \,\mathrm{d}x\right| \leqslant \int_u^v F(x) \,\mathrm{d}x.$$

B3) Prove the Dirichlet test for convergence: if $f, g : [a, \infty) \to \mathbb{R}$ satisfy

- f is continuous and there exists $A > 0$, such that for any $M \geqslant a$,

$$\left|\int_a^M f(x)\,\mathrm{d}x\right| \leqslant A.$$

- g is monotonic and $\lim_{x\to\infty} g(x) = 0$.
  Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

Lemma: The Second Integral Mean Value Theorem

If f is integrable and g is monotonic and non-negative(or non-positive) on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^c f(x)\,\mathrm{d}x + g(b)\int_c^b f(x)\,\mathrm{d}x.$$

Proof: Assume that g is non-negative and monotonically decreasing. It is easy to see that there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x.$$

Apply the above formula to $f(x)$ and $g(x) - g(b)$ and we get

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x.$$

Proof of B3): Since $|\int_u^v f(x)\,\mathrm{d}x| \leqslant 2A$, by lemma

$$\left|\int_u^v f(x)g(x)\,\mathrm{d}x\right| \leqslant 2A(|g(u)| + |g(v)|).$$

By B1), the integral converges.

B4) Prove the Abel test of convergence:

If $f, g : [a, \infty) \to \mathbb{R}$ satisfy:

- $\int_a^\infty f(x)\,\mathrm{d}x$ exists.
- g is monotonic and g is bounded.
  Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.
  Proof: Suppose g is monotonically increasing, then

$$\left|\int_u^v f(x)(g(x) - g(a))\,\mathrm{d}x\right| \leqslant 2M\left(\left|\int_u^\xi f(x)\,\mathrm{d}x\right| + \left|\int_\xi^v f(x)\,\mathrm{d}x\right|\right) \to 0$$

since $\int_a^\infty f(x)\,\mathrm{d}x$ converges. Therefore both $\int_a^\infty f(x)(g(x) - g(a))\,\mathrm{d}x$ and $\int_a^\infty f(x)g(a)\,\mathrm{d}x$ converges, hence $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

B5) Determine whether the following integrals converges:

(1)

$$\int_0^\infty \frac{\log(1+x)}{x^p}\,\mathrm{d}x$$

(absolutely) convergent when $1 < p < 2$, diverges when $p \leqslant 1$ or $p \geqslant 2$.

(2)

$$\int_1^\infty \frac{\sin x}{x^p}\,\mathrm{d}x$$

Absolutely convergent when $p > 1$, conditionally convergent when $0 < p \leqslant 1$, diverges when $p \geqslant 0$.

(3)

$$\int_1^\infty \sin x^2\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \frac{\sin y}{y^{1/2}}\,\mathrm{d}y$$

is conditionally convergent.

(4)

$$\int_0^\infty \frac{\sin^2 x}{x}\,\mathrm{d}x$$

diverges

(5) $p, q > 0$,

$$\int_0^{2\pi} \sin^{-p} x \cos^{-q} x\,\mathrm{d}x$$

Absolutely convergent when $p, q < 1$, diverges when $p \geqslant 1$ or $q \geqslant 1$.

(6)

$$\int_0^\infty x^p \sin\left(x^q\right)\mathrm{d}x$$

If $q = 0$ the integral diverges. Assume $q \neq 0$ below.

$$\int_0^\infty x^p \sin\left(x^q\right)\mathrm{d}x = \frac{1}{q}\int_0^\infty y^{(p+1)/q-1}\sin y\,\mathrm{d}y.$$

Let $\alpha = \frac{p+1}{q} - 1$, then the integral

- diverges if $\alpha \leqslant -2$ or $\alpha \geqslant 0$,
- converges absolutely if $-2 < \alpha < -1$.
- converges conditionally if $-1 \leqslant \alpha < 0$.
  (7) $q \geqslant 0$,

$$\int_0^\infty \frac{x^p \sin x}{1+x^q}\,\mathrm{d}x$$

If $p \leqslant -2$, then the integral diverges near 0, since $x^p \sin x \sim x^{p+1}$. The integral converges (absolutely) near 0 otherwise. Assume $p > -2$ below.

If $p - q < -1$ then the integral converges absolutely when it tends to infinity, since $\frac{x^p}{1+x^q} \sim x^{p-q}$.

If $-1 \leqslant p - q < 0$ then the integral converges conditionally, since the integral of $\left(x^{p-q}\right)'$ converges.

(8)

$$\begin{aligned}\int_0^{\pi/2} \frac{\log \sin x}{\sqrt{x}}\,\mathrm{d}x &= 2\int_0^{\pi/2} \log \sin x\,\mathrm{d}\sqrt{x}\\ &= 2\sqrt{x}\log \sin x\Big|_0^{\pi/2} - 2\int_0^{\pi/2} \sqrt{x}\cot x\,\mathrm{d}x\\ &= -2\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\cos x\,\mathrm{d}x\end{aligned}$$

converges, since $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges.

(9)

$$\int_e^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x = \int_1^\infty \frac{\log y}{y}e^y \sin e^y\,\mathrm{d}y.$$

It is easy to see the integral does not converge absolutely.
Meanwhile

$$f'(x) = \left(\frac{\log\log x}{\log x}\right)' = \frac{1-\log\log x}{(\log x)^2 x},$$

and

$$\int_e^\infty \frac{\log\log x - 1}{(\log x)^2 x}\,\mathrm{d}x = \int_1^\infty \frac{\log y - 1}{y^2}\,\mathrm{d}y = \int_0^\infty \frac{t-1}{e^t}\,\mathrm{d}t.$$

converges.
By Lagrange mean value theorem,

$$\begin{aligned}\int_{2\pi}^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x &= \sum_{n=1}^\infty \int_{2n\pi}^{(2n+1)\pi} (f(x) - f(x+\pi))\sin x\,\mathrm{d}x \\ &\leqslant \sum_{n=1}^\infty -2\pi f'(2n\pi) \leqslant 2\pi \int_e^\infty -f'(x)\,\mathrm{d}x\end{aligned}$$

converges.

PSC: Oscillatory Integral

$F(t), G(t) : [1,\infty) \to \mathbb{R}$, $\lim_{t\to\infty} G(t) = 0$. Assume that for any $t \geqslant 1$, $G(t) \neq 0$. If

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = 1.$$

Then we say F, G have the same order, and $F \sim G$.

Part 1

C1) $d > 0$ is a given real number. Assume $g \in C^1([0,d])$. Prove that there is a constant C, such that

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant \frac{C}{t}.$$

Proof: Let $C = \sup_{x\in[0,d]} |g(x)|$, then

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant C\int_0^d e^{-tx}\,\mathrm{d}x = \frac{C}{t}(1-e^{-td}) \leqslant \frac{C}{t}.$$

C2) Assume $d > 0$, $g \in C([0,d])$ and $g(0) \neq 0$. Prove that

$$\int_0^d e^{-tx} g(x)\,\mathrm{d}x \sim \frac{g(0)}{t}.$$

Proof: Let $M = \sup_{x\in[0,d]} |g(x)|$, then

$$\left|\int_0^d e^{-tx} t \frac{g(x)}{g(0)}\,\mathrm{d}x - 1\right| = \left|\int_0^{td} e^{-u} \frac{g(u/t)}{g(0)}\,\mathrm{d}u - \int_0^{\infty} e^{-u}\,\mathrm{d}u\right|$$
$$\leqslant \int_{td}^{\infty} e^{-u}\,\mathrm{d}u + \int_0^N e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u + \int_N^{td} e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u$$
$$\leqslant e^{-td} + \sup_{0\leqslant x\leqslant N/t}\left|\frac{g(x)}{g(0)} - 1\right| + \left(\frac{M}{|g(0)|} + 1\right)\int_N^{td} e^{-u}\,\mathrm{d}u \to 0.$$

(let $t \to \infty$ then let $N \to \infty$).

C3) $d > 0, g \in C([0,d]), g(0) \neq 0$. Prove that

$$\int_0^d e^{-tx^2} g(x)\,\mathrm{d}x \sim \frac{\sqrt{\pi}\cdot g(0)}{2\sqrt{t}}.$$

Proof: Same as C2), let $M = \sup_{x\in[0,d]} |g(x)/g(0)|$, then

$$\left|\int_0^d e^{-tx^2} \sqrt{t}\frac{g(x)}{g(0)}\,\mathrm{d}x - \frac{\sqrt{\pi}}{2}\right| = \left|\int_0^{d\sqrt{t}} e^{-u^2} \frac{g(u/\sqrt{t})}{g(0)}\,\mathrm{d}u - \int_0^{\infty} e^{-u^2}\,\mathrm{d}u\right|$$
$$\leqslant \int_{d\sqrt{t}}^{\infty} e^{-u^2}\,\mathrm{d}u + \int_0^N e^{-u^2}\left|\frac{g(u/\sqrt{t})}{g(0)} - 1\right|\mathrm{d}x + \int_N^{d\sqrt{t}} e^{-u^2}(M+1)\,\mathrm{d}u.$$

which tends to 0, same as C2).

For $t \geqslant 1$, $f, \varphi \in C([a,b])$, define the function

$$F(t) = \int_a^b e^{-t\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $F(t)$ when $t \to \infty$.

C4) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Further assume that $\varphi'(x) > 0$. Let $d = \varphi(b) - \varphi(a)$. Prove that

$$\Psi : [a,b] \to [0,d],\ x \mapsto \varphi(x) - \varphi(a),$$

is a continuously differentiable bijection on $[a,b]$.
Proof: φ is monotonic by $\varphi'(x) > 0$, hence Ψ is a bijection and $\Psi' = \psi'$.

C5) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Prove that if $f(a) \neq 0$, then when $t \to \infty$,

$$F(t) \sim \frac{f(a)}{\varphi'(a)}\frac{e^{-t\varphi(a)}}{t}.$$

Proof: Let $g(x) = f(x)/\Psi'(x)$, and $h = (t\Psi)^{-1}$ then

$$\left|F(t)\frac{t}{e^{-t\varphi(a)}}-\frac{f(a)}{\varphi'(a)}\right|=\left|\int_a^b e^{-t\Psi(x)}tf(x)\,\mathrm{d}x-\frac{f(a)}{\Psi'(a)}\right|=\left|\int_a^b e^{-t\Psi(x)}g(x)\,\mathrm{d}t\Psi(x)-g(a)\right|$$
$$=\left|\int_0^{t\Psi(b)} e^{-u}g(h(u))\,\mathrm{d}u-g(h(0))\int_0^\infty e^{-u}\,\mathrm{d}u\right|$$
$$=|g(h(0))|\int_{t\Psi(b)}^\infty e^{-u}\,\mathrm{d}u+\int_0^{N\Psi(b)} e^{-u}|g(h(u))-g(h(0))|\,\mathrm{d}u$$
$$+\int_{N\Psi(b)}^{t\Psi(b)} e^{-u}|g(h(u))-g(h(0))|\,\mathrm{d}u$$
$$\leqslant |g(a)|e^{-t\Psi(b)}+\sup_{x\in[a,\Psi^{-1}(N\Psi(b)/t)]}|g(x)-g(a)|+\int_{N\Psi(b)}^{t\Psi(b)} e^{-u}2M\,\mathrm{d}u.$$

which tends to 0 since g is continuous. ($M=\sup_{x\in[a,b]}|g(x)|$).

C6) Assume that $\varphi\in C^2([a,b]),\varphi'(a)=0,\varphi''(x)>0$ and for any $x\in(a,b],\varphi'(x)>0$. Let $d=\sqrt{\varphi(b)-\varphi(a)}$. Prove that

$$\Psi:[a,b]\to[0,d],\ x\mapsto\sqrt{\varphi(x)-\varphi(a)}.$$

is a continuously differentiable bijection on $[a,,b]$, and calculate $\Psi'(a)$.
Proof: Trivial. $\Psi'=\frac{\varphi'}{2\Psi}$, hence

$$\Psi'(a)=\lim_{x\to a^+}\frac{\varphi'(x)}{2\sqrt{\varphi(x)-\varphi(a)}}=\lim_{x\to a^+}\frac{\varphi''(x)}{\varphi'(x)/\sqrt{\varphi(x)-\varphi(a)}}=\sqrt{\frac{\varphi''(a)}{2}}.$$

C7) Assume $\varphi\in C^2([a,b]),\varphi'(a)=0,\varphi''(a)>0$. Prove that if $f(a)\neq 0$, when $t\to\infty$,

$$F(t)\sim\frac{\sqrt{\pi}f(a)}{\sqrt{2\varphi''(a)}}\frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: Let $g=f/\Psi', h=(\sqrt{t}\Psi)^{-1}$, then

$$F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}}=\int_a^b e^{-t\Psi^2(x)}f(x)\sqrt{t}\,\mathrm{d}x=\int_a^b e^{-t\Psi^2(x)}g(x)\,\mathrm{d}\sqrt{t}\Psi(x)=\int_0^{\sqrt{t}\Psi(b)} e^{-u^2}g(h(u))\,\mathrm{d}u.$$

Hence

$$\left|F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}}-\frac{\sqrt{\pi}}{2}g(a)\right|=\left|\int_0^{\sqrt{t}\Psi(b)} e^{-u^2}g(h(u))\,\mathrm{d}u-\int_0^\infty e^{-u^2}g(h(0))\,\mathrm{d}u\right|$$
$$\leqslant g(a)\int_{\sqrt{t}\Psi(b)}^\infty e^{-u^2}\,\mathrm{d}u+\int_0^{N\Psi(b)} e^{-u^2}|g(h(u))-g(h(0))|\,\mathrm{d}u$$
$$+\int_{N\Psi(b)}^{\sqrt{t}\Psi(b)} e^{-u^2}2M\,\mathrm{d}u$$
$$\leqslant g(a)e^{-\sqrt{t}\Psi(b)}+\sqrt{\pi}\sup_{x\in[a,\Psi^{-1}(N\Psi(b)/\sqrt{t})]}|g(x)-g(a)|+2Me^{-N\Psi(b)}.$$

which tends to 0 as $t\to\infty$ and $N\to\infty$, since g is continuous.
(A much simpler solution can be given using the Laplace method)

C8) Given $f \in C((0,\infty)), \varphi \in C^2((0,\infty))$. Assume that

- exists a unique $a \in (0,\infty)$ such that $\varphi'(a) = 0$;
- $\varphi''(a) > 0, f(a) \neq 0$;
- $\int_0^\infty e^{-\varphi(x)}|f(x)|\,\mathrm{d}x$ converges.
  Prove that when $t \to \infty$, the function $G(t) = \int_0^\infty e^{-t\varphi(x)} f(x)\,\mathrm{d}x$ satisfy

$$G(t) \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: (Simple application of the Laplace method)
Apply C7) to the intervals $[a/2, a]$ and $[a, 2a]$, then

$$\int_{a/2}^{2a} e^{-t\varphi(x)} f(x)\,\mathrm{d}x \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \cdot \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

While the integral on the intervals $(0, a/2), (2a, \infty)$ converges rapidly. Hence

$$G(t) \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

C9) $\Gamma(n) = (n-1)!$.

Proof:

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,\mathrm{d}t = -\int_0^\infty t^n\,\mathrm{d}e^{-t} = n\int_0^\infty t^{n-1}e^{-t}\,\mathrm{d}t = n\Gamma(n).$$

C10) Prove Stirling's approximation

$$n! \sim \sqrt{2\pi}\frac{n^{n+1/2}}{e^n}.$$

Proof: Note that, by substituting $t = ns$,

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t}t^n\,\mathrm{d}t = n^{n+1}\int_0^\infty e^{-n(s-\log s)}\,\mathrm{d}s.$$

Hence

$$\frac{\Gamma(t+1)}{t^{t+1}} \sim \sqrt{2\pi}\frac{e^{-t}}{\sqrt{t}}.$$

Part 2

For $\lambda \geqslant 1$, $f, \varphi \in C^\infty([a,b])$, define the function

$$I(\lambda) = \int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $I(\lambda)$ when $\lambda \to \infty$.

C11) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Define the maps

$$L : C^\infty([a,b]) \to C^\infty([a,b]), h \mapsto \frac{1}{i\lambda\varphi'(x)} h'(x),$$
$$M : C^\infty([a,b]) \to C^\infty([a,b]), h \mapsto -\left(\frac{h}{i\varphi'}\right)'(x).$$

Assume that $f, g \in C^\infty([a,b])$. Prove that if there exists $c > 0$ such that for any $x \in [a, a+c] \cup [b-c, b]$, $h(x) = 0$, then M/λ is the adjoint of L:

$$\int_a^b h \cdot Lg = \frac{1}{\lambda} \int_a^b g \cdot Mh.$$

Proof: By integration of parts,

$$\int_a^b h \cdot Lg = \int_a^b \frac{h}{i\lambda\varphi'} \, \mathrm{d}g = -\int_a^b g \, \mathrm{d}\left(\frac{h}{i\lambda\varphi'}\right) = \frac{1}{\lambda}\int_a^b g \cdot Mh.$$

C12) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$ and f vanishes near a and b. prove that for any $n \in \mathbb{Z}_{\geqslant 1}$, there is a constant c_n independent of λ such that

$$|I(\lambda)| \leqslant \frac{c_n}{\lambda^n}.$$

Proof: Let $g = e^{i\lambda}\varphi$, then $Lg = g$. $f \in C^\infty([a,b])$ vanishes near a, b hence $M^n f$ vanishes near a, b for any $n \in \mathbb{Z}_{\geqslant 0}$. Therefore

$$|I(\lambda)| = \left|\int_a^b fg\right| = \frac{1}{\lambda}\left|\int_a^b g \cdot Mf\right| = \cdots = \frac{1}{\lambda^n}\left|\int_a^b g \cdot M^n f\right|.$$

so $c_n = \left|\int_a^b g \cdot M^n f\right|$ is valid.

C13) If there exists $\delta > 0$, such that for any $x \in [a,b]$, $|\varphi'(x)| \geqslant \delta$ and $\varphi'(x)$ is monotonic on $[a,b]$. Prove that there is a constant C_1 independent of λ, φ, a, b such that

$$\left|\int_a^b e^{i\lambda\varphi(x)} \, \mathrm{d}x\right| \leqslant \frac{C_1}{\lambda\delta}.$$

Proof: Let $C_1 = 4$ then (since φ' maintains the same sign)

$$\begin{aligned}
\left|\int_a^b e^{i\lambda\varphi(x)} \, \mathrm{d}x\right| &= \left|\int_a^b \frac{\mathrm{d}e^{i\lambda\varphi}}{\lambda\varphi'}\right| \leqslant \left|\frac{e^{i\lambda\varphi}}{\lambda\varphi'}\Big|_a^b\right| + \frac{1}{\lambda}\left|\int_a^b e^{i\lambda\varphi}\frac{\varphi''}{(\varphi')^2} \, \mathrm{d}x\right| \\
&\leqslant \frac{2}{\lambda\delta} + \frac{1}{\lambda}\int_a^b \left|\frac{\varphi''}{(\varphi')^2}\right| \\
&= \frac{2}{\lambda\delta} + \frac{1}{\lambda}\int_a^b \mathrm{d}\frac{1}{\varphi'} \leqslant \frac{4}{\lambda\delta}.
\end{aligned}$$

C14) Suppose for any $x \in [a,b]$, $|\varphi''(x)| \geqslant 1$. Prove that there is a unique $c \in [a,b]$ such that

$$|\varphi'(c)| = \inf_{x \in [a,b]} |\varphi'(x)|.$$

Further prove that for any $x \in [a,b]$,

$$|\varphi'(x)| \geqslant |x - c|.$$

Proof: Since $\varphi \in C^\infty([a,b])$ and $|\varphi''| \geqslant 1$, φ'' maintains the same sign. Assume that $\forall x \in [a,b], \varphi''(x) \geqslant 1$, then φ' is monotonically increasing. Therefore, if $\varphi' \neq 0$, then $c \in \{a, b\}$, otherwise, c is the unique null point of φ'.
Either $\varphi'(c) = 0$ or $c = a$, when φ' maintains the same sign, so we always have $|\varphi'(x)| \geqslant |\varphi'(x) - \varphi'(c)|$, and

$$\forall x \in [a,b], \exists \xi \in [x,c],\ |\varphi'(x) - \varphi'(c)| \geqslant |x-c| \cdot \varphi'(\xi) \geqslant |x-c|.$$

Therefore $|\varphi'(x)| \geqslant |x-c|$.

!C15) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of λ, φ, a, b, such that

$$\left| \int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x \right| \leqslant \frac{C_2}{\sqrt{\lambda}}.$$

Proof: Since $\varphi \in C^\infty([a,b])$, we can assume $\varphi''(x) \geqslant 1$. For an arbitrary $\delta > 0$, divide the interval $[a,b]$ into two parts:
$E_1 = \{x : |\varphi'(x)| \leqslant \delta\}$ and $E_2 = \{x : |\varphi'(x)| > \delta\}$.
Note that $\forall x, y \in E_1$, $|\varphi'(x) - \varphi'(y)| \leqslant 2\delta$, but $|\varphi'(x) - \varphi'(y)| \geqslant \left|\int_x^y \varphi''(t)\,\mathrm{d}t\right| \geqslant |x-y|$.
Therefore E_1 is an interval of length at most 2δ, so

$$\left| \int_{E_1} e^{i\lambda\varphi(x)}\,\mathrm{d}x \right| \leqslant 2\delta.$$

Now consider the integral on E_2, which is the union of one or two intervals. By C13),

$$\left| \int_{E_2} e^{i\lambda\varphi(x)}\,\mathrm{d}x \right| \leqslant 2 \cdot \frac{4}{\lambda\delta}.$$

Therefore

$$\left| \int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x \right| \leqslant 2\delta + \frac{8}{\lambda\delta} = \frac{8}{\sqrt{\lambda}}.$$

(if we let $\delta = \sqrt{4/\lambda}$.)

C16) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of $\lambda, \varphi, f, a, b$ such that

$$\left| \int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x \right| \leqslant \frac{C_2}{\sqrt{\lambda}} \left(|f(a)| + \int_a^b |f'(x)|\,\mathrm{d}x \right).$$

Proof: By C15),

$$\begin{aligned}
\left| \int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x \right| &\leqslant \left| \int_a^b e^{i\lambda\varphi(x)} f(a)\,\mathrm{d}x \right| + \left| \int_a^b e^{i\lambda\varphi(x)} \int_a^x f'(t)\,\mathrm{d}t\,\mathrm{d}x \right| \\
&\leqslant |f(a)| \frac{C_2}{\sqrt{\lambda}} + \left| \int_a^b f'(t) \int_t^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leqslant \frac{C_2}{\sqrt{\lambda}} \left(|f(a)| + \int_a^b |f'(x)|\,\mathrm{d}x \right).
\end{aligned}$$