習題 7.1，任选不少于十道，题号之和不小于 120！（"！"不是阶乘【旺柴】）
习题 7.2，任选不少于十道题目，其中奇数和偶数题分别不得少于五个。
习题 7.3，同习题 7.2 的选择方式。
习题 7.5，题目选择方式同习题 7.1

7.1

7.1.8

Prove that the Laguerre polynomial $L_n(x) = \frac{e^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$ has n different real roots.
Proof: We know that the Laguerre polynomials are orthonormal on the space $L^2([0,\infty))$ with weight e^{-x}, by applying integration by parts n times, hence using the lemma below, it must have n distinct roots.
Or note that $f(x) = x^n e^{-x}$ has a root with multiplicity n at 0 and it vanishes at ∞, hence use Rolle's theorem and induction we can show that $f^{(k)}(x)$ has a root with multiplicity $n-k$ at 0 and k roots between 0 and ∞.
Lemma: If the class of polynomials P_n where $\deg P_n = n$ are orthogonal under the (real) inner product $\langle f, g \rangle = \int_X fgw\,\mathrm{d}x$ where $w \geqslant 0$, then P_n has all n distinct roots.
Proof: Otherwise if P_n only changes signs at $x_1, \cdots, x_k$ for $k \leqslant n - 1$, let $Q(x) = \pm(x - x_1)\cdots(x - x_k)$, then $P(x)Q(x) \geqslant 0$, and it cannot vanish on X (it has only finitely many roots), so $\langle P, Q \rangle > 0$. However, $\deg Q < n$ so it is the linear combination of $P_0, \cdots, P_{n-1}$, leading to contradiction.

7.1.9

Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$ has n different roots in the interval $(-1, 1)$.
Proof: We know that the polynomials $\sqrt{(2n+1)/2}P_n(x)$ form a set of orthonormal basis on the space $L^2([-1, 1])$, using integration by parts, hence it must have n different roots in the interval $(-1, 1)$.
Likewise the Hermite polynomial $H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$ has n different real roots.

7.1.10

Suppose $f \in C([a, b])$, differentiable on (a, b), $\xi \in (a, b)$. Must there exist $a_1, b_1 \in (a, b), a_1 < \xi < b_1$ such that $f'(\xi) = \frac{f(b_1) - f(a_1)}{b_1 - a_1}$?
Solution: No, for example, let $f(x) = x^2 \cdot \mathrm{sign}(x)$, then $f'(0) = 0$ but $f(a) - f(-b) = a^2 + b^2 > 0$.

7.1.11-12

Suppose f is differentiable on $[a, \infty)$, and $|f'(x)| \leqslant M|f(x)|$, $f(a) = 0$, $M > 0$. Prove that $f \equiv 0$.
Proof: (Application of Gronwell's inequality) $|f(x)| \leqslant \int_a^x |f'(t)|\,\mathrm{d}t \leqslant M\int_a^x |f(t)|\,\mathrm{d}t$. Let $g(x) = \left|f\left(\frac{x}{M} + a\right)\right|e^{-x}$, then

$$g(x)e^x \leqslant \int_0^x g(t)e^t\,\mathrm{d}t.$$

Suppose $g(x) = \sup_{x \in [0, N]} g(x)$, then

$$g(x)e^x \leqslant g(x)\int_0^x e^t\,\mathrm{d}t = g(x)(e^x - 1).$$

Hence $g(x) = 0$, so $f \equiv 0$.

7.1.13

Suppose $f(x) = \sum_{k=1}^n a_k \sin kx$, and $|f(x)| \leqslant |\sin x|$. Prove that $\left|\sum_{k=1}^n ka_k\right| \leqslant 1$.
Proof: $|f(x)| \leqslant |\sin x|$ implies $|f(x)/x| \leqslant |\sin x/x|$, so

$$\left|\sum_{k=1}^n ka_k\right| = \lim_{x\to 0}\left|\frac{f(x)}{x}\right| \leqslant \lim_{x\to 0}\left|\frac{\sin x}{x}\right| = 1.$$

7.1.14

Suppose $f \in C([0,1])$, and differentiable on $(0,1)$. If $f(0) = 0$, $f(1) = 1$, prove that there exists
$\xi_1 \neq \xi_2 \in (0,1)$ such that $f'(\xi_1)f'(\xi_2) = 1$.
Proof: If for $u, v \in (0,1)$ we have $f'(u) > 1$, $f'(v) < 1$, then by Darboux's theorem there exists such ξ_1, ξ_2
(since the range of f' contains $(f'(v), f'(u))$ and $t, t^{-1} \in (f'(v), f'(u))$ for $t \to 1$.). Otherwise suppose
$f'(\xi) \geqslant 1 \forall \xi \in (0,1)$, then $f' - 1 \in \mathcal{L}([0,1])$ and $0 = \int_0^1 f'(t) - 1\,dt$. Hence $f'(t) = 1$, a. e., so there
exists $\xi_1 \neq \xi_2$ such that $f'(\xi_1) = f'(\xi_2) = 1$.
Another proof: Take $f(t) = 1 - t$ (since $f(0) + 0 < 1 < f(1) + 1$), then there exists $\xi_1 \in (0,t)$ and
$\xi_2 \in (t,1)$ such that $f'(\xi_1) = \frac{f(t)-f(0)}{t-0} = \frac{1-t}{t}$, and $f'(\xi_2) = \frac{f(1)-f(t)}{1-t} = \frac{t}{1-t}$, so $f'(\xi_1)f'(\xi_2) = 1$.

7.1.17

Suppose $f, g, h \in C([a,b])$ are differentiable on (a,b). Prove that there exists $\xi \in (a,b)$ such that

$$\det\begin{pmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix} = 0.$$

Proof: Let $F(x) = (f(x), g(x), h(x))$, and $G(x) = \det(F(x), F(a), F(b))$, then $G : \mathbb{R} \to \mathbb{R}$ and
$G(a) = G(b) = 0$, hence there exists $\xi \in (a,b)$ such that $G'(\xi) = 0$. Therefore
$0 = G'(\xi) = \det(F'(\xi), F(a), F(b))$.

7.1.23

Suppose f is differentiable on $[a,b]$, twice differentiable on (a,b). Prove that there exists $\xi \in (a,b)$ such that
$f'(b) - f'(a) = f''(\xi)(b-a)$.
Proof: Let $A = \frac{f'(b)-f'(a)}{b-a}$ and $g(x) = f'(x) - A(x-a)$, then $g(a) = f'(a) = g(b)$. We find ξ such that
$g'(\xi) = 0$. Note that g is continuous on (a,b), so if $g'(\xi) \neq 0 \forall \xi \in (a,b)$, then we can suppose
$g'(x) > 0 \forall x \in (a,b)$. Hence g is monotonic on (a,b), and $\lim_{x\to a} g(x)$ exists. By Darboux's theorem for any
x there exists $\xi_x \in (a,x)$ such that $g(\xi_x) = \frac{1}{2}(g(a) + g(x))$, so $\lim_{x\to a} g(x) = \frac{1}{2}(g(a) + \lim_{x\to a} g(x))$,
which implies g is continuous at a. Likewise, g is continuous on $[a,b]$, so applying the mean value theorem we
obtain a contradiction.

7.1.25

Consider a degenerate case of Darboux's theorem: Suppose f is differentiable on $[a,b]$, and
$f'(a) = A = f'(b)$, must there exist $\xi \in (a,b)$ such that $f'(\xi) = A$?
Solution: No. Let $g(x) = A - \left|\frac{a-b}{2}\right| + \left|x - \frac{a+b}{2}\right|$, then $g(a) = g(b) = A$, and $g(x) \neq A \forall x \in (a,b)$. Let
$f(x) = \int_a^x g(t)\,dt$, then $f'(a) = f'(b) = A$ but $f'(\xi) \neq A \forall \xi \in (a,b)$.

7.1.26

Suppose f is differentiable on $[a, b]$, and $f'(a) = f'(b)$. Prove that there exists $\xi \in (a, b)$ such that

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

Proof: Let $g(x) = \frac{f(x) - f(a)}{x - a}$, then $g'(x) = \frac{f'(x)(x-a) - (f(x) - f(a))}{(x-a)^2}$ so $g'(\xi) = 0 \implies f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}$. Note that $g'(b) = -\frac{g(b) - g(a)}{b - a}$.

If $g'(\xi) \neq 0 \forall \xi \in (a, b)$, then by Darboux's theorem, suppose $g'(\xi) > 0 \forall \xi \in (a, b)$. Then $g(b) > g(a)$ so $g'(b) < 0$, hence by Darboux's theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$.

7.2

7.2.8

Suppose $f : (a, \infty) \to \mathbb{R}$ bounded on every bounded interval, and $\lim_{x \to \infty} \frac{f(x+1) - f(x)}{x^n} = l$. Calculate the limit $\lim_{x \to \infty} \frac{f(x)}{x^{n+1}}$.

Solution: Let $f(x + 1) - f(x) = (l + g(x))x^n$, then $g(x) \to 0$, and

$f(x + N) - f(x) = l \sum_{k=0}^{N-1} (x + k)^n + \sum_{k=0}^{N-1} g(x + k)(x + k)^n$. Suppose $a < 0$ and $M = \sup_{x \in [0,1]} |f(x)|$. Then for $x = a + N$ where $a \in (0, 1)$ and $N \in \mathbb{N}$,

$$\left| \frac{f(x)}{x^{n+1}} - \frac{l}{n + 1} \right| \leq \left| \frac{f(a)}{x^{n+1}} \right| + l \left| \frac{1}{n + 1} - \frac{\sum_{k=0}^{N-1} (a + k)^n}{(a + N)^{n+1}} \right| + \sum_{k=0}^{K} + \sum_{k=K+1}^{N-1} |g(x + k)| \cdot \left| \frac{(a + k)^n}{x^{n+1}} \right|.$$

Since $\left| \frac{f(a)}{x^{n+1}} \right| \leq \frac{M}{x^{n+1}} \to 0$, $\frac{\sum_{k=0}^{N-1} (a+k)^n}{(a+N)^{n+1}} \to \frac{1}{n+1}$, and for given K, $\frac{(a+K)^n}{x^{n+1}} \to 0$, we obtain

$\limsup_{x \to \infty} \left| \frac{f(x)}{x^{n+1}} - \frac{l}{n+1} \right| \leq (n + 1) \sup_{x \geq K} |g(x)| \to 0$, so $\lim_{x \to \infty} \frac{f(x)}{x^{n+1}} = \frac{l}{n+1}$.

7.2.9

Suppose f is a solution to

$$\begin{cases} f'' + 3f' + 4f = \frac{4x^2}{x^2 + x + 1}, & x \geq 0, \\ f(0) = 1, \quad f'(0) = 2 \end{cases}$$

Prove that $\lim_{x \to \infty} f''(x) = \lim_{x \to \infty} f'(x) = 0$.

Proof: The solution to this equation exists and is unique, since it can be viewed as $y_1' = y_2, y_2' = -4y_1 - 3y_2 + 4x^2/(x^2 + x + 1)$, and the function $f(x, Y) = (y_2, -4y_1 - 3y_2 + 4x^2/(x^2 + x + 1))$ is globally Lipschitz, so by Picard's theorem there is a unique solution locally. Since the equation is linear, it can be extended to the real line.

We first solve the homogeneous equation, the damped harmonic oscillator: $\varphi'' + 3\varphi' + 4\varphi = 0$, and $\varphi(0) = 0, \varphi'(0) = 1$. This is a second order linear equation, and by solving $\lambda^2 + 3\lambda + 4 = 0$ we obtain $\varphi(t) = \frac{2}{\sqrt{7}} e^{-3t/2} \sin\left(\frac{\sqrt{7}}{2} t\right)$.

Now, the solution f should be $f(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + \varphi * g$ where $g(x) = \frac{4x^2}{x^2 + x + 1}$. By $f(0) = 1$ and $f'(0) = 2$ we obtain $f_0(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} = e^{-3t/2}\left(\cos\left(\frac{\sqrt{7}}{2} t\right) + \sqrt{7} \sin\left(\frac{\sqrt{7}}{2} x\right)\right)$.

Hence $f'(t) = f_0'(t) + \varphi' * g$ and $f''(t) = g + \varphi'' * g$. Since $\lim_{x \to \infty} g(x) = 4$, we have $\lim_{x \to \infty} f'(t) = \lim_{x \to \infty} \varphi' * g = 4 \int_0^\infty \varphi' \, dt = 0$ and $\lim_{x \to \infty} f''(t) = 4 + 4 \int_0^\infty \varphi''(t) \, dt = 0$.

(Furthermore we have $\lim_{x \to \infty} f(x) = 1$).

Proof without solving the ODE: Let $y = f(x) - 1$ then the equation becomes $\ddot{y} + 3\dot{y} + 4y = g$ where $g(x) = O(x^{-1})$. This is the equation $m\ddot{x} = -k^2 x - r\dot{x} + f$.

Consider the total energy $E = \frac{1}{2}k^2 y^2 + \frac{1}{2}m\dot{y}^2 = 2y^2 + \frac{1}{2}\dot{y}^2$. Note that

$E' = 4\dot{y}y + \dot{y}\ddot{y} = \dot{y}(4y + \ddot{y}) = \dot{y}(g - 3\dot{y}) \leqslant g^2 - 2\dot{y}^2$. Hence $E(x) - E(0) + \int_0^x 2(y')^2\,dt \leqslant \int_0^x g^2\,dt$.

Since $g \in \mathcal{L}^2(\mathbb{R})$ and $E \geqslant 0$, we obtain $y' \in \mathcal{L}^2(\mathbb{R})$, and E is bounded, so y, y', g are bounded, hence y'' is bounded. Therefore y' is uniformly continuous, so combined with $y' \in \mathcal{L}^2(\mathbb{R})$ we have $\lim_{x\to\infty} y'(x) = 0$.

$E' = \dot{y}g - 3\dot{y}^2 \in \mathcal{L}(\mathbb{R})$, since both $\dot{y}g, \dot{y}^2 \in \mathcal{L}(\mathbb{R})$ ($\int \dot{y}g \leqslant \int \dot{y}^2 \int g^2$), so $\lim_{x\to\infty} E(x) = 2L$ exists. By $E = 2y^2 + \frac{1}{2}\dot{y}^2$ we obtain $|y| \to \sqrt{L}$, and since y is continuous we can assume $y \to \sqrt{L}$.

$\ddot{y} + 3\dot{y} + 4y = g \to 0$, so $\ddot{y} \to -\frac{\sqrt{L}}{4}$. Combined with $\lim_{x\to\infty} y' = 0$, we obtain $L = 0$ and $\lim_{x\to\infty} y'' = 0$.

7.2.11

Suppose $f \in C^1(\mathbb{R})$ and $\forall x \in \mathbb{R}$, $f(x+1) - f(x) = f'(x)$, $\lim_{x\to\infty} f'(x) = c$. Prove that $f' \equiv c$.

Proof: Note that for any $x \in \mathbb{R}$, there exists $\xi_x \in (x, x+1)$ such that $f'(\xi_x) = f(x+1) - f(x) = f'(x)$. If $f'(x_0) \neq c$, consider $A = \{x \in \mathbb{R} : f'(x) = f'(x_0)\}$ which is closed, then if $\sup A < \infty$, $t = \sup A \in A$ so $\xi_t \in A$, leading to contradiction. Hence A is unbounded, contradicting $\lim_{x\to\infty} f'(x) = c$. Therefore $f' \equiv c$.

7.2.12

Suppose f is differentiable on $\mathbb{R}$ and $|f'(x)| < 1$, and $x_0 \in \mathbb{R}$, $x_{n+1} = f(x_n)$. Show that $\{x_n\}$ may diverge.

Solution: Let $f(x) = \sqrt{x^2 + 1}$, then $f'(x) = \frac{x}{\sqrt{x^2+1}}$ so $|f'(x)| < 1$, but $x_n = \sqrt{n}$ diverges.

7.2.20

Suppose f is twice differentiable at x_0 and $f''(x_0) \neq 0$. For h small enough, there exists $\theta = \theta(h) \in (0,1)$ such that $f(x_0 + h) - f(x_0) = f'(x_0 + \theta h)h$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$.

Proof: Note that

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0 + \theta h) - f'(x_0)}{\theta h} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - hf'(x_0)}{\theta h^2} = f''(x_0)\lim_{h\to 0} \frac{1}{2\theta}.$$

So $\lim_{h\to 0} \theta = \frac{1}{2}$.

7.2.21

Suppose $a_i \in \mathbb{R}$, calculate the limit

$$\lim_{x\to 0} x^{-2}(1 - \cos(a_1 x) \cdots \cos(a_n x)).$$

Solution: Note that

$$1 - \prod_{k=1}^n \cos(a_k x) = 1 - \prod_{k=1}^n \left(1 - \frac{(a_k x)^2}{2} + O(x^4)\right) = \sum_{k=1}^n \frac{(a_k x)^2}{2} + O(x^4)$$

so

$$\lim_{x\to 0} x^{-2}\left(1 - \prod_{k=1}^n \cos(a_k x)\right) = \frac{1}{2}\sum_{k=1}^n a_k^2.$$

7.2.22

Suppose $f''(x_0)$ exists, $f'(x_0) \neq 0$. Calculate the limit

$$\lim_{x \to x_0} \frac{1}{f(x) - f(x_0)} - \frac{1}{f'(x_0)(x - x_0)}.$$

Solution: Suppose $g(x) = \frac{1}{(x-x_0)^2}(f(x) - f(x_0) - f'(x_0)(x - x_0))$, then

$$\lim_{x \to x_0} \frac{1}{f(x) - f(x_0)} - \frac{1}{f'(x_0)(x - x_0)} = \lim_{x \to x_0} -\frac{(x - x_0)^2 g(x)}{f'(x_0)(x - x_0)(f'(x_0)(x - x_0) + g(x)(x - x_0)^2)}.$$

Note that $g(x) = \frac{1}{2}f''(x_0) + o(1)$, so the limit equals

$$-\frac{f''(x_0)}{2(f'(x_0))^2}.$$

7.2.23

Suppose $a_1 \in (0, \pi), a_{n+1} = \sin a_n$, prove that $\lim_{n \to \infty} \sqrt{n} a_n = \sqrt{3}$.
Proof: Clearly $a_{n+1} < a_n$ so $\lim_{n \to \infty} a_n = 0$. By Stolz,

$$\lim_{n \to \infty} na_n^2 = \lim_{n \to \infty} \frac{1}{a_{n+1}^{-2} - a_n^{-2}} = \lim_{n \to \infty} \frac{a_n^2 \sin^2 a_n}{a_n^2 - \sin^2 a_n} = \lim_{x \to 0} \frac{x^2 \sin^2 x}{x^2 - \sin^2 x} = 3.$$

7.2.24

Suppose f is twice differentiable in a neighborhood O of 0, and $f(0) = 0$. Let $g(x) = f(x)/x$ and $g(0) = f'(0)$, prove that $g \in C^1(O)$.
Proof: Clearly g is continuously differentiable in $O \backslash \{0\}$. Note that

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} \frac{f(x)/x - f'(0)}{x} = \lim_{x \to 0} \frac{f(x) - xf'(0)}{x^2} = \frac{1}{2}f''(0).$$

And for any $x \neq 0$, $g'(x) = \frac{xf'(x) - f(x)}{x^2}$, where $f'(x) = f'(0) + xf''(0) + o(x)$ and $f(x) = xf'(0) + \frac{1}{2}f''(0)x^2 + o(x^2)$, so

$$\lim_{x \to 0} g'(x) = \frac{1}{2}f''(0) = g'(0).$$

Hence $g \in C^1(O)$.

7.2.25

Suppose f is differentiable on (a, ∞). If $\lim_{x \to \infty}(f(x) + xf'(x) \log x) = l$, prove that $\lim_{x \to \infty} f(x) = l$.
Proof: Let $F(x) = f(x) \log x$, then $F'(x) = \frac{f(x)}{x} + f'(x) \log x$, so $\lim_{x \to \infty} xF'(x) = l$. By l'Hôpital,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{F(x)}{\log x} = \lim_{x \to \infty} xF'(x) = l.$$

7.3

7.3.13

If $f(x) = \sum_{k=0}^{n+1} a_k x^k + o(x^{n+1})$, $x \to 0$, does $f'(x) = \sum_{k=0}^{n} (k+1)a_{k+1}x^k + o(x^n)$, $x \to 0$ hold? If not, add a condition to make it hold.
Solution: No, for example $f(x) = x + x^2 \sin x^{-1}$, but $f'(x) = 1 + 2x \sin x^{-1} - \sin x^{-1}$.
Fix: If $f^{(n+1)}$ exists, then we can apply the Taylor's theorem with Peano remainder to f' to get
$f'(x) = \sum_{k=0}^{n} (k+1)a_{k+1}x^k + o(x^n)$.

7.3.14

Find a function f such that $f((-\infty, 0]) = \{0\}$ and $f([1, \infty)) = \{1\}$, and $f \in C^k \backslash C^{k+1}$.
Solution: Consider $\varphi(x) = \max\left\{0, \frac{1}{2} - \left|x - \frac{1}{2}\right|\right\}$. Let $g_0 = \varphi$, $g_n(x) = \int_0^x g_{n-1}(t)\, \mathrm{d}t$. Then
$\varphi \in C(\mathbb{R}) \backslash C^1$, so $g_n \in C^n \backslash C^{n+1}$. Also it is clear that $g_n(x) = 0 \forall x \leqslant 0$ and $g_n(x) = g_n(1) \forall x \geqslant 1$. Hence we can take $f(x) = \frac{g_n(x)}{g_n(1)}$.

7.3.15&16

Suppose $n \geqslant 1$, $f^{(n+1)}$ exists, and $M_0, M_{n+1} < \infty$ where $M_m = \sup_{x \in \mathbb{R}} |f^{(m)}(x)|$. Prove that for $1 \leqslant m \leqslant n$, there exists a constant $C_m > 0$ independent of f, such that $M_m \leqslant C_m M_0^{1-m/(n+1)} M_{n+1}^{m/(n+1)}$.
Proof: Note that $f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi)$, so $|f'(x)| \leqslant \frac{1}{h}M_0 + hM_2$. Let $h = \sqrt{M_0/M_2}$, then $M_1 \leqslant 2\sqrt{M_0 M_2}$.
Apply this to $f^{(k-1)}$ and we obtain $M_k \leqslant 2\sqrt{M_{k-1}M_{k+1}}$. Let $a_k = \log M_k + k^2 \log 2$, then
$a_k \leqslant \frac{1}{2}(a_{k-1} + a_{k+1})$, so a_k is convex, and $a_m \leqslant \frac{n+1-m}{n+1}a_0 + \frac{m}{n+1}a_{n+1}$. Hence
$M_m \leqslant C_m M_0^{1-m/(n+1)} M_{n+1}^{m/(n+1)}$, where $C_m = 2^{m(n+1-m)}$.

7.3.19

Suppose $\{a_n\}$ is bounded, and $\lim_{n \to \infty}(a_{n+2} - 2a_{n+1} + a_n) = 0$, does $\lim_{n \to \infty}(a_{n+1} - a_n) = 0$ hold?
Solution: Suppose $|a_n| \leqslant M/2$, and let $d_n = a_{n+1} - a_n$, then $\forall u < v$, $\sum_{i=u}^{v} d_i \leqslant a_v - a_{u-1} \leqslant M$. For any $\varepsilon > 0$, take $t > 2M/\varepsilon$, then there exists N such that $n > N \implies |d_{n+1} - d_n| < \frac{\varepsilon}{t}$. If $d_n > \varepsilon$ for some $n > N$, then $d_{n+k} > \frac{\varepsilon(t-k)}{t}$, so $d_n + \cdots + d_{n+t} > \frac{\varepsilon}{t}(1 + \cdots + t) > \frac{t\varepsilon}{2} > M$, leading to contradiction.

7.3.20

$f : (-1, 1) \to \mathbb{R}$ is twice differentiable at 0 and $f(0) = 0$, prove that

$$\lim_{n \to \infty} \sum_{k=-n}^{n} f\left(\frac{k}{n^{3/2}}\right) = \frac{1}{3}f''(0).$$

Proof: Let $g(x) = f(x) + f(-x)$, then $g(0) = g'(0) = 0$, and $g''(0) = 2f''(0)$. For any x, since $g(x) = x^2 f''(0) + o(x^2)$, suppose $\varepsilon(x) = \sup_{|y| \leqslant x} |y^{-2}(g(y) - y^2 f''(0))|$, then

$$\lim_{n \to \infty} \sum_{k=-n}^{n} f\left(\frac{k}{n^{3/2}}\right) = \lim_{n \to \infty} \sum_{k=1}^{n} g\left(\frac{k}{n^{3/2}}\right) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^2}{n^3}f''(0) + \varepsilon(n^{-1/2})\frac{k^2}{n^3} = \frac{1}{3}f''(0).$$

7.3.21

Suppose $f : [0, \infty)$ is twice-differentiable, and $f(0) = f'(0) = 0$, $f''(x) + 3f'(x) + 2f(x) \geqslant 0$. Prove that $f(x) \geqslant 0$.

Proof: $L = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2 = \left(\frac{d}{dx} + 1\right)\left(\frac{d}{dx} + 2\right)$, so consider $g(x) = f'(x) + 2f(x)$, then $g' + g = f'' + 2f' + f' + 2f \geqslant 0$, so $(e^x g)' = e^x(g + g') \geqslant 0$. $f(0) = f'(0) = 0$ implies $g(0) = 0$, so $(e^x g)' \geqslant 0$ implies $g \geqslant 0$. Therefore $(e^{2x} f)' = e^{2x}(f' + 2f) \geqslant 0$, so $f \geqslant 0$.

7.3.22

Suppose $f \in C([-1, 1])$ and three times differentiable on $(-1, 1)$. Prove that there exists $\xi \in (-1, 1)$ such that $f^{(3)}(\xi) = 3(f(1) - f(-1) - 2f'(0))$.

Proof: Consider $g(x) = f(x) - f(-x) - 2xf'(0)$, then $g(x) + g(-x) = 0$ and $g(0) = g'(0) = g''(0) = 0$. Hence there exists $\eta \in (0, 1)$ such that $g(1) = \frac{1}{6}g^{(3)}(\eta)$. Note that $g^{(3)}(\eta) = f^{(3)}(\eta) + f^{(3)}(-\eta)$ so there exists $\xi \in (-1, 1)$ such that $g^{(3)}(\eta) = 2f^{(3)}(\xi)$, therefore $3(f(1) - f(-1) - 2f'(0)) = f^{(3)}(\xi)$.

7.3.23

Suppose f is bounded and differentiable on $\mathbb{R}$, and $|f'(x)| < 1$. Prove that there is a constant $M < 1$ such that $|f(x) - f(0)| \leqslant M|x|$.

Furthermore, does there exist $K < 1$ such that $|f(x) - f(y)| \leqslant K|x - y|$?

Proof: Suppose $|f| \leqslant N$, than for any $|x| > 4N$, $|f(x) - f(0)| \leqslant 2N \leqslant \frac{1}{2}|x|$. Suppose $M = \max\left\{\frac{1}{2}, \sup_{x \in [-4N, 4N]} |g(x)|\right\}$, where $g(x) = \frac{f(x) - f(0)}{x}$ and $g(0) = f'(0)$, then $g(x) < 1$ and is continuous, so $M < 1$ and for any $x \in [-4N, 4N]$, $|f(x) - f(0)| \leqslant M|x|$. Therefore $|f(x) - f(0)| \leqslant M|x|$ for any $x \in \mathbb{R}$.

However, consider

$$g(x) = \begin{cases} x - 4n, & x \in [4n - 1, 4n + 1) \\ 4n + 2 - x, & x \in [4n + 1, 4n + 3) \end{cases}$$

and $h(x) = (1 - (|n| + 1)^{-1})g(x)$, $x \in [4n, 4n + 4)$, then $h \in C(\mathbb{R})$, $|h| < 1$, and $\int_{4n}^{4n+4} h(t)\, dt = 0 \forall n \in \mathbb{Z}$. Hence let $f(x) = \int_0^x h(t)\, dt$, f is bounded by 4, differentiable, $|f'(x)| < 1$, but $\sup_{x \in \mathbb{R}} |f'(x)| = 1$, so there does not exist such K.

7.3.24

Suppose $f \in C^1(\mathbb{R})$ and $\forall x \in \mathbb{R}$, $f'(x) > f(f(x))$, prove that $\forall x \geqslant 0$, $f(f(f(x))) \leqslant 0$.

Proof: (IMC2012P4)

Step1: $\lim_{x \to \infty} f(x) = \infty$ cannot hold.

If $\lim_{x \to \infty} f(x) = \infty$, then $f(f(x)) \to \infty$ so $f'(x) \to \infty$.

For x large enough, $f'(x) > 2$, so $\exists N, \forall x > N$, $f(x) > x$. Then $f'(x) > f(f(x)) > f(x)$, so $f(x) > Ce^x$ for some constant C. $f'(x) > f(f(x)) > Ce^{f(x)}$, so $f'e^{-f} > C$. Integrate from N to x, we obtain $e^{-f(N)} - e^{-f(x)} > C(x - N)$, but the right side tends to ∞, while the left is bounded, leading to contradiction.

Step2: $f(t) < t$, $\forall t > 0$.

By Step1, there exists $t > 0$ such that $f(t) < t$, so if the statement is false then take $t_0 > 0$ such that $f(t_0) = t_0$. If $f(t) \geqslant t_0$, $\forall t > t_0$, then $f'(t) > f(f(t)) \geqslant t_0$ so $f \to \infty$, a contradiction. So take $T = \inf\{t \geqslant t_0 : f(t) < t_0\}$, then $f(T) = t_0$. $f'(T) > f(f(T)) = f(t_0) = t_0 > 0$, so $f(t) > t_0$ in some neighborhood $(T, T + \varepsilon)$, leading to contradiction.

Step3. If $f(f(x_0)) \geqslant 0$, then $f(x) > f(x_0)$, $\forall x > x_0$, $f(x) < f(x_0)$, $\forall x < x_0$.

Proof: If $f(x_1) \leqslant f(x_0)$ for some $x_1 > x_0$, let $T = \inf\{t > x_0 : f(t) \leqslant f(x_0)\}$, then $f(T) \leqslant f(x_0)$. Note that $f'(x_0) > f(f(x_0)) \geqslant 0$, so $f(x) > f(x_0)$ in a neighborhood $(x_0, x_0 + \varepsilon)$, then $T > x_0$. By continuity, $f(T) = f(x_0)$, so $f'(T) > f(f(T)) = f(f(x_0)) \geqslant 0$. Hence in a neighborhood $(T, T - \delta)$, $f(t) < f(T) \leqslant f(x_0)$, leading to contradiction.
The other side is the same.

Step4: $f(f(f(x))) \leqslant 0, \forall x \geqslant 0$.

Suppose $f(f(f(t_0))) > 0$ where $t_0 > 0$, then let $t_1 = f(t_0), t_2 = f(t_1), t_3 = f(t_2) > 0$. Note that $f(f(t_1)) = t_3 > 0$, and by Step1 $t_0 > 0$ so $t_0 > f(t_0) = t_1$ so by Step2,
$t_0 > t_1 \implies f(t_0) > f(t_1) \implies t_1 > t_2$. Then by Step2, $t_2 < t_1 \implies f(t_2) < f(t_1) \implies t_3 < t_2$, so $t_0 > t_1 > t_2 > t_3 > 0$.

Note that $f(f(t_0)) = t_2 > 0$, so by Step2, for all $t > t_0$, $f(t) > f(t_0) = t_1$. $f(f(t_1)) = t_3 > 0$ so by Step2, $f(t) > t_1 \implies f(f(t)) > f(t_1) = t_2$, so $f'(t) > f(f(t)) > t_2 > 0$. Hence $f(t) \to \infty$, leading to contradiction.

7.3.25

Suppose $c > e^{-1}$ prove that there does not exist $f : \mathbb{R} \to \mathbb{R}_+$ differentiable such that $f'(x) \geqslant f(x + c)$.
Find a solution for $c \in (0, e^{-1}]$.
Proof: Consider $f(x) = -x \log x, x \in (0, \infty)$, then $f'(x) = -1 - \log x$ so the range of x is $(0, e^{-1}]$. If $c \in (0, e^{-1}]$ take a such that $c = a^{-1} \log a$, then $f(x) = e^{ax}$ satisfy $f'(x) = ae^{ax} = e^{a(x+c)} = f(x + c)$.
If $c > e^{-1}$, from $f'(x) \geqslant f(x + c) > f(x)$ we obtain $(\log f)' \geqslant 1$ so $f(x + t) \geqslant f(x)e^t$.
Suppose $\lambda(t) = (\log f)'$ and $\lambda_0 = \inf_{x \in \mathbb{R}} \lambda(x)$, then $\lambda_0 \geqslant 1$.
$\log f(x + c) - \log f(x) = \int_x^{x+c} \lambda(t) \, dt \geqslant c\lambda_0$, so $f'(x) \geqslant f(x + c) \geqslant e^{c\lambda_0} f(x)$. Hence $e^{c\lambda_0} \leqslant \lambda_0$, leading to contradiction.

7.3.26

Suppose $f \in C^4[0, 1]$, $p(x)$ is a polynomial of degree 3, such that $p(0) = f(0), p'(0) = f'(0), p(1) = f(1), p'(1) = f'(1)$. Prove that $|f(x) - p(x)| \leqslant \frac{1}{384} \sup_{x \in [0,1]} |f^{(4)}(x)|$.
Proof: For any $x \in (0, 1)$, consider

$$F(t) = f(t) - p(t) - \lambda t^2 (1 - t)^2$$

where $\lambda = \frac{f(x) - p(x)}{x^2(1 - x^2)}$, then $F(0) = F(1) = F(x) = 0, F'(0) = F'(1) = 0$. Hence there exists $\xi_1 \in (0, x), \xi_2 \in (x, 1)$ such that $F'(\xi_1) = F'(\xi_2) = 0$, so F' has four roots. So $F^{(4)}$ has one root ξ, which means $0 = F^4(\xi) = f^{(4)}(\xi) - 24\lambda$. We obtain $\xi \in (0, 1)$ such that $\lambda = \frac{1}{24} f^{(4)}(\xi)$. Therefore

$$|f(x) - p(x)| \leqslant \frac{1}{16} \lambda \leqslant \frac{1}{384} \|f^{(4)}\|_\infty.$$

7.3.27

Suppose $f \in C^\infty(\mathbb{R})$, and $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, f^{(k)}(0) \neq 0$. Prove that $g(x) = f(x)x^{-k} \in C^\infty$.
Proof: Note that

$$f(x) = \int_0^x \frac{(x - t)^{k-1}}{(k - 1)!} f^{(k)}(t) \, dt = x^k \int_0^1 \frac{(1 - u)^{k-1}}{(k - 1)!} f^{(k)}(ux) \, du,$$

hence

$$g(x) = \int_0^1 \frac{(1-u)^{k-1}}{(k-1)!} f^{(k)}(ux)\, du$$

is smooth.

7.5

7.5.9

Prove that convex functions on bounded intervals are bounded.

Proof: If $f : [a, b] \to \mathbb{R}$ is convex, then $f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda)f(b) \leqslant \max\{f(a), f(b)\}$, so f is bounded from above.

Let $c = \frac{a+b}{2}$, then for $x \in (a, b)$, $2f(c) \leqslant f(x) + f(2c - x) \leqslant M + f(x)$, so $f(x) \geqslant 2f(c) - M$ is bounded from below. Hence f is bounded.

7.5.10

Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, and differentiable on $(0, \infty)$. If $f(0) = 0$, $f'(x)$ is strictly increasing, prove that $\frac{f(x)}{x}$ is strictly increasing on $(0, \infty)$.

Proof: Let $g(x) = \frac{f(x)}{x}$, then $g'(x) = \frac{xf'(x)-f(x)}{x}$, so we need to show that $f'(x) > \frac{f(x)}{x}$. By mean value theorem, there exists $\xi \in (0, x)$ such that $f'(\xi) = \frac{f(x)-f(0)}{x-0}$, so $f'(x) > f'(\xi) \geqslant \frac{f(x)}{x}$.

7.5.15

Discuss the convexity of $f(x) = -\log x$, and prove the inequality

$$\sqrt[n]{a_1 \cdots a_n} \leqslant \frac{1}{n} \sum_{k=1}^n a_k$$

for $a_i > 0, i = 1, \cdots, n$.

Proof: $f''(x) = \frac{1}{x^2} > 0$ so f is convex, hence by Jensen's inequality,

$$\sum_{k=1}^n f(a_k) \geqslant nf\left(\frac{1}{n} \sum_{k=1}^n a_k\right),$$

therefore

$$\frac{1}{n} \sum_{k=1}^n a_k \geqslant \sqrt[n]{a_1 \cdots a_n}.$$

7.5.16

Suppose $a_i > 0, i = 1, 2, \cdots$, prove that

$$\frac{n}{\sum_{k=1}^n \frac{1}{a_k}} \leqslant \sqrt[n]{a_1 \cdots a_n}.$$

Proof: Apply 7.5.15 to $\frac{1}{a_i}$.

7.5.17

Suppose $a_i > 0$ and $\sum_{i=1}^{n} a_i = 1$, prove that

$$\sum_{i=1}^{n} a_i x_i \geqslant \prod_{k=1}^{n} x_k^{a_k}, \ \forall x_i \geqslant 0.$$

And determine when equality holds.

Solution: Note that $f(x) = -\log x$ is strictly convex, so by Jensen's inequality,

$$\sum_{k=1}^{n} a_k f(x_k) \geqslant f\left(\sum_{k=1}^{n} a_k x_k\right).$$

Therefore $\sum_{i=1}^{n} a_i x_i \geqslant \prod_{k=1}^{n} x_k^{a_k}$, and equality holds only when x_i are all equal.

7.5.19

Prove that the non-trivial global solution to $y'' + a(x)y = 0, a(x) \geqslant c > 0$ must have infinitely many points such that $f''(x) = 0$.

Proof: It suffices to show that y has infinitely many roots. First, note that $f(x) = \sin\left(\sqrt{c}x\right)$ is a solution to $f'' + cf = 0$, and f has infinitely many roots $x_k = k\pi/\sqrt{c}$ (which are the only roots).

We show that y has a root in $[x_k, x_{k+1}]$: Otherwise suppose f, y are both positive on (x_k, x_{k+1}). Let $g = f'y - y'f$, then $g'(t) = f''(t)y(t) - y''(t)f(t) = (a - c)yf \geqslant 0$, so $g(x_{k+1}) \geqslant g(x_k) = f'(x_k)y(x_k) > 0$, but $g(x_{k+1}) = f'(x_{k+1})y(x_{k+1}) \leqslant 0$, leading to contradiction. Therefore f has infinitely many roots.

(Note: This is a special case of the Sturm Comparison Theorem in Sturm-Liouville theory: If $y'' + ay = 0, z'' + bz = 0$ and $a(t) \geqslant b(t)$, then y oscillates faster than z, i.e. given $a, b \in C^1$, between neighboring roots of z there is a root of y. If $a(t) \to M$ monotonically, then an upper bound for the number of roots is $Z(t)/t \to \sqrt{M}/\pi$.)

7.5.22

Suppose $P_0(a, b) \in \mathbb{R}^2$, prove that $f(x) = d((x, 0), (a, b))$ is convex.

Proof: $f(x) = \sqrt{(a - x)^2 + b^2}$, so by Cauchy's inequality,

$$\lambda f(x) + (1 - \lambda)f(y) = \lambda\sqrt{(a - x)^2 + b^2} + (1 - \lambda)\sqrt{(a - y)^2 + b^2}$$
$$\geqslant \sqrt{(\lambda(a - x) + (1 - \lambda)(a - y))^2 + b^2} = f(\lambda x + (1 - \lambda)y).$$

Hence f is convex.

7.5.23

Suppose $f \in C(\mathbb{R})$. If for any $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = 0.$$

Prove that f is linear.

Proof: For any $a < b$, let $g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$, then $g(a) = f(a), g(b) = f(b)$, and $g(x + h) + g(x - h) - 2g(x) = 0$. Denote $D\varphi(x) = \lim_{h \to 0} \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{h^2}$, then $Df = Dg \equiv 0$. We show that $f \leqslant g$ on the interval $[a, b]$: for any $\varepsilon > 0$, consider $\phi(x) = f(x) - g(x) + \varepsilon x^2$, then $D\phi > 0$,

so by lemma, $\phi(x) \leqslant \max\{\phi(a), \phi(b)\} \leqslant \varepsilon \max\{a^2, b^2\} \to 0$. Hence $f \leqslant g$.

Likewise $f \geqslant g$, so $f = g$ is linear.

Lemma: If $Df > 0$ on $[a, b]$ and $f \in C([a, b])$ then $f(x) \leqslant \max\{f(a), f(b)\}$.

Proof: If f has a local maxima at x, then $Df(x) < 0$. Since $f \in C([a, b])$, f reaches its maximum, so $f(x) \leqslant \max\{f(a), f(b)\}$.

7.5.24

Determine the approximating lines of $x^3 + y^3 = 3xy$.

Solution: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, so asymptote is $x + y = t$. Substitute $y = t - x$ we obtain $t^3 - 3t^2 x + 3tx^2 = 3x(t - x) = 3xt - 3x^2$, so $3t = -3, t = -1$ and the asymptote is $x + y + 1 = 0$.

7.5.26

Does there exists $f : \mathbb{R} \to \mathbb{R}$ convex, such that $f(0) < 0$ (or $f(0) > 0$) such that

$$\lim_{|x| \to \infty} (f(x) - |x|) = 0.$$

Solution: For $f(0) > 0$, the answer is positive. For example $f(x) = \sqrt{x^2 + 1}$.

For $f(0) < 0$, the answer is negative. Let $g(x) = f(x) - x \to 0$, then $f(0) \geqslant 2f(x) - f(2x) = 2g(x) - g(2x) \to 0$, hence $f(0) \geqslant 0$.