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73-74页7,10,13; 83-84页2,4,5,11,12; 86页2,7.

73-7

Let F a subfield of $\mathbb{C}$ and let $T \in \mathcal{L}(F^3)$ defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.
(a) Verify that T is linear.
(b) If $(a, b, c) \in F^3$, what are the conditions that $(a, b, c) \in \mathrm{Im}T$? Calculate $\dim\mathrm{Im}T$.
(c) What are the conditions that $(a, b, c) \in \mathrm{Ker}T$? Calculate $\dim\mathrm{Ker}T$.
Proof: (a) $T(cx_1 + y_1, cx_2 + y_2, cx_3 + y_3) = cT(x_1, x_2, x_3) + T(y_1, y_2, y_3)$ so T is linear.
(b) $T(1, 0, 0) = (1, 2, -1)$, $T(0, 1, 0) = (-1, 1, -2)$, $T(0, 0, 1) = (2, 0, 2)$ then
$T(1, 0, 0) \in \mathrm{Span}\langle T(0, 1, 0), T(0, 0, 1)\rangle$ so $\dim\mathrm{Im}T = 2$ and
$\mathrm{Im}T = \mathrm{Span}\langle (0, 1, -1), (1, 0, 1)\rangle = \{(x, y, z) : z = x - y\}$.
(c) $\mathrm{Ker}T = \left\{(x, y, z) : y = -2x, z = -\frac{3}{2}x\right\} = \mathrm{Span}\langle(-2, 4, 3)\rangle$, $\dim\mathrm{Ker}T = 1$.

73-10

Let V be $\mathbb{C}$ over the field $\mathbb{R}$. Find $T \in \mathcal{L}(V)$ but not complex linear.
Solution: Consider $T : \mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$, then $T(rz + w) = \bar{r}\bar{z} + \bar{w} = \bar{r}T(z) + T(w)$ so T is linear over the field $\mathbb{R}$ but not over $\mathbb{C}$.

74-13

For $T \in \mathcal{L}(V)$. Prove that the following are equivalent: (a) $\{0\} = \mathrm{Ker}T \cap \mathrm{Im}T$; (b)
$T^2(\alpha) = 0 \implies T\alpha = 0$.
Proof: (a)=>(b): If $T(T(\alpha)) = 0$ then $T(\alpha) \in \mathrm{Ker}T$. Clearly $T(\alpha) \in \mathrm{Im}T$ so $T(\alpha) = 0$.
(b)=>(a): If $0 \neq a \in \mathrm{Ker}T \cap \mathrm{Im}T$, then take $a = T\alpha$, we have $T\alpha \neq 0$ but $T^2\alpha = Ta = 0$.

83-2

Let $T \in \mathcal{L}(\mathbb{C}^3)$ such that $Te_1 = (1, 0, i), Te_2 = (0, 1, 1), Te_3 = (i, 1, 0)$. Is T invertible?
Solution: Te_j are linearly dependent, so T is not invertible.

83-4

For the linear operator $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$, prove that $(T^2 - I)(T - 3I) = 0$.
Proof: Note that $Te_1 = 3e_1$, $T(0, 0, 1) = (0, 0, 1)$, $T(0, -2, 1) = -(0, -2, 1)$, so T has three different eigenvalues $3, 1, -1$. Hence $(T - I)(T + I)(T - 3I) = 0$, since we can take the matrix A of T under the base $(1, 0, 0), (0, 0, 1), (0, -2, 1)$.

83-5

Let $B = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}$ and $T \in \mathcal{L}(\mathbb{C}^{2\times 2})$ be $T : A \mapsto BA$. What is the rank of T? Can you describe T?
Solution: For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $BA = \begin{pmatrix} a - c & -4(a - c) \\ b - d & -4(b - d) \end{pmatrix}$. So $\dim\mathrm{Im}T = 2$.

84-11

Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\dim \mathrm{Im} T^2 = \dim \mathrm{Im} T$. Prove that the range and null of T are disjoint.
Proof: $\dim \mathrm{Im} T^2 = \dim \mathrm{Im} T|_{\mathrm{Im} T} = \mathrm{Im} T - \dim \mathrm{Ker} T|\mathrm{Im} T$, so $\mathrm{Ker} T \cap \mathrm{Im} T = \{0\}$.

84-12

Let $V = F^{m\times n}, W = F^{p\times n}$, $B \in F^{p\times m}$ and $T \in \mathcal{L}(V, W) : A \mapsto BA$. Prove that T is invertible iff $p = m$ and $B \in GL(m, F)$.
Proof: <== is trivial: $T^{-1} : A \mapsto B^{-1}A$.
==> If T is invertible, then consider $C = T^{-1}(I)$, $I_V = T^{-1}T(I_V) = CBI_V$, and $I_W = TT^{-1}(I_W) = BCI_W$, so $BC = I_W$ and $CB = I_V$. Hence $p = m$ and $B \in GL(m, F)$.

86-2

Let V be a vector space over the field $\mathbb{C}$, and suppose there is an isomorphism $T : V \to \mathbb{C}^3$ Let $\alpha_1, \alpha_2, \alpha_3, \alpha_3 \in V$ such that $T\alpha_1 = (1, 0, i)$, $T\alpha_2 = (-2, 1 + i, 0)$, $T\alpha_3 = (-1, 1, 1)$, $T\alpha_4 = (\sqrt{2}, i, 3)$.
(a) Is α_1 in the subspace spanned by α_2 and α_3?
(b) Let $W_1 = \mathrm{Span}\langle\alpha_1, \alpha_2\rangle$ and $W_2 = \mathrm{Span}\langle\alpha_3, \alpha_4\rangle$. What is $W_1 \cap W_2$?
(c) Find a basis for the subspace of V spanned by the four vectors α_j.
Solution: (a) Note that $-iT\alpha_1 + \frac{1-i}{2}T\alpha_2 = T\alpha_3$, since T is an isomorphism, $\alpha_1 \in \mathrm{Span}\langle\alpha_2, \alpha_3\rangle$.
(b) $TW_1 = \mathrm{Span}\langle T\alpha_1, T\alpha_2\rangle$ and $TW_2 = \mathrm{Span}\langle T\alpha_3, T\alpha_4\rangle$, so $TW_1 \cap TW_2 = \mathrm{Span}\langle T\alpha_3\rangle$ hence $W_1 \cap W_2 = \mathrm{Span}\langle\alpha_3\rangle$.
(c) $\{\alpha_1, \alpha_2, \alpha_4\}$.

86-7

For an isomorphism $U \in \mathcal{L}(V, W)$, prove that $\varphi : T \mapsto UTU^{-1}$ is an isomorphism of $\mathcal{L}(V, V) \to \mathcal{L}(W, W)$.
Proof: Clearly φ is linear. Consider $\psi : \mathcal{L}(W, W) \to \mathcal{L}(V, V), P \mapsto U^{-1}PU$, then $\varphi\psi = 1_{\mathcal{L}(W,W)}$ and $\psi\varphi = 1_{\mathcal{L}(V,V)}$ so φ is an isomorphism.