2025/9/17

作业二：【尤】**1.3**：5、13；**1.4**：7；**1.5**：1、4、7

1.3.5

$\alpha=(1,0,2)$, $\beta=(1,1,1)$, $\gamma=(1,0,-2)$. Try to write α in the form $\alpha_1+\alpha_2$ where $\alpha_1\in\langle\beta,\gamma\rangle$, and $\alpha_2\in\langle\beta,\gamma\rangle^\perp$.
Solution: Note that $\langle\beta,\gamma\rangle^\perp=\langle\beta\times\gamma\rangle$ since β,γ are linearly independent. $\beta\times\gamma=(-2,3,-1)$, so $\alpha_2=\langle\alpha,\beta\times\gamma\rangle(\beta\times\gamma)/|\beta\times\gamma|=(8,-12,4)/\sqrt{14}$, and $\alpha_1=\alpha-\alpha_2$.

1.3.13

For $A,B,C,D\in\mathbb{R}^3$, prove that
(1) $\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{AD}+\overrightarrow{CA}\cdot\overrightarrow{BD}=0$;
Proof: Let $\alpha=\overrightarrow{AB}$, $\beta=\overrightarrow{AC}$, $\gamma=\overrightarrow{AD}$, then the left hand side becomes

$$\alpha\cdot(\gamma-\beta)+(\beta-\alpha)\cdot\gamma+(-\beta)\cdot(\gamma-\alpha)=0,$$

which is trivial.
(2) $\overrightarrow{AB^2}+\overrightarrow{CD^2}=\overrightarrow{AC^2}+\overrightarrow{BD^2}\iff\overrightarrow{AD}\cdot\overrightarrow{BC}=0$.
Likewise the first identity becomes

$$\alpha^2+(\beta-\gamma)^2=\beta^2+(\alpha-\gamma)^2\iff\beta\gamma=\alpha\gamma,$$

and the second one becomes $\gamma(\alpha-\beta)=0$.

1.4.7

Suppose $A,B,C\in\mathbb{R}^2$ with coordinates $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ respectively. Prove that the area of $\triangle ABC$ is

$$S=\left|\frac{1}{2}\begin{vmatrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{vmatrix}\right|.$$

Proof: Note that

$$\begin{vmatrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{vmatrix}=\begin{vmatrix}x_1&y_1&1\\x_2-x_1&y_2-y_1&0\\x_3-x_1&y_3-y_1&0\end{vmatrix}=\begin{vmatrix}x_2-x_1&y_2-y_1\\x_3-x_1&y_3-y_1\end{vmatrix}=\overrightarrow{AB}\times\overrightarrow{AC}$$

Hence $S=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$, implying the givin identity.

1.5.1

Prove that for any $\alpha,\beta,\gamma\in\mathbb{R}^3$,

$$\alpha\times(\beta\times\gamma)+\beta\times(\gamma\times\alpha)+\gamma\times(\alpha\times\beta)=\mathbf{0}.$$

Proof: We know that $\alpha\times(\beta\times\gamma)=(\alpha\cdot\gamma)\beta-(\alpha\cdot\beta)\gamma$, so adding up all three terms we obtain this identity.

1.5.4

Given points $A, B, C, D \in \mathbb{R}^3$ with coordinates $(1,0,1), (-1,1,5), (-1,-3,-3), (0,3,4)$ respectively. Determine this tetrahedral's volume.

Solution: Note that $\alpha = \overrightarrow{AB} = (-2,1,4)$, $\beta = \overrightarrow{AC} = (-2,-3,-4)$, $\gamma = \overrightarrow{AD} = (-1,3,3)$.
The volume should be

$$V = \frac{1}{3}|\alpha \cdot (\beta \times \gamma)| = \frac{1}{3}|(-2,1,4) \cdot (3,10,-9)| = \frac{1}{3} \cdot |-6+10-36| = \frac{32}{3}.$$

1.5.7

Prove that

$$\alpha \times (\beta \times (\gamma \times \delta)) = (\beta \cdot \delta)(\alpha \times \gamma) - (\beta \cdot \gamma)(\alpha \times \delta).$$

Proof:

$$\alpha \times (\beta \times (\gamma \times \delta)) = \alpha \times ((\beta \cdot \delta)\gamma - (\beta \cdot \gamma)\delta) = (\beta \cdot \delta)(\alpha \times \gamma) - (\beta \cdot \gamma)(\alpha \times \delta),$$

where the first equality is Lagrange's identity and the second from linearity of $\times$.