

149-5

Suppose $f \in \mathcal{L}([0, 1])$. If $\exp \int_{[0,1]} f \, dm = \int_{[0,1]} e^f \, dm$, then $f = C$ a.e. $x \in [0, 1]$.

Proof: $(e^x)'' = e^x > 0$ so it is strictly convex, i.e. $e^a > e^x + (a - x)e^x$ for any $a \neq x$. Let $C = \int_{[0,1]} f \, dm$, then $e^{f(x)} \geq e^C + (f(x) - C)e^C$, so

$$\int_{[0,1]} |e^f - e^C - (f - C)e^C| \, dm = \int_{[0,1]} e^f - e^C - (f - C)e^C \, dm = 0.$$

Hence $e^{f(x)} = e^C + (f(x) - C)e^C$ a.e., so $f = C$ a.e.

162-8

Suppose $f_n, f \in \mathcal{L}(\mathbb{R})$, and

$$\int_{\mathbb{R}} |f_n - f| \, dm \leq \frac{1}{n^2}, \forall n = 1, 2, \dots$$

then $f_n \rightarrow f$ a.e. $x \in \mathbb{R}$.

Proof: Denote $B_{n,k} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > 1/k\}$, then by Chebyshev's inequality, $m(B_{n,k}) \leq k \int_{\mathbb{R}} |f_n - f| \, dm \leq \frac{k}{n^2}$. Note that $\sum_{n=1}^{\infty} m(B_{n,k}) < \infty$, so by Borel-Cantelli lemma, $A_k = \limsup_{n \rightarrow \infty} B_{n,k} = \bigcap_{N \geq 1} \bigcup_{n \geq N} B_{n,k}$ is a null set. Hence $\bigcup_{k \geq 1} A_k$ is null, so $f_n \rightarrow f$ a.e. $x \in \mathbb{R}$.

190-7

For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if for any $\varepsilon > 0$, there exists $g, h \in \mathcal{L}(\mathbb{R}^n)$ such that $g \leq f \leq h$, and $\int_{\mathbb{R}^n} h - g \, dm < \varepsilon$. Prove that $f \in \mathcal{L}(\mathbb{R}^n)$.

Proof: For $\varepsilon = 1/k > 0$, take g_k, h_k such that $g_k \leq f \leq h_k$ and $\int_{\mathbb{R}^n} h_k - g_k \, dm < 1/k$. We can assume that $g_k \leq g_{k+1}$ and $h_k \geq h_{k+1}$, otherwise consider $\tilde{g}_k = \max\{g_1, \dots, g_k\}$ and $\tilde{h}_k = \min\{h_1, \dots, h_k\}$, we still have $\tilde{g}_k \leq f \leq \tilde{h}_k$ and $\int_{\mathbb{R}^n} \tilde{h}_k - \tilde{g}_k \, dm < \varepsilon$. Then let $g = \lim_{n \rightarrow \infty} g_n$ and $h = \lim_{n \rightarrow \infty} h_n$, we have $g \leq f \leq h$, and $0 \leq \int_{\mathbb{R}^n} h - g \, dm \leq \int_{\mathbb{R}^n} h_n - g_n \, dm < 1/n$, so $\int_{\mathbb{R}^n} h - g \, dm = 0$. Hence $g = h$ a.e., so $f = \lim_{n \rightarrow \infty} g_n$ a.e., therefore f is measurable. Since $g_1 \leq g \leq h \leq h_1$, $g, h \in \mathcal{L}(\mathbb{R}^n)$ so $f \in \mathcal{L}(\mathbb{R}^n)$.

191-12

Suppose $f \in \mathcal{L}(\mathbb{R})$, $a > 0$, prove that the series

$$S(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{x}{a} + n\right)$$

converges absolutely almost everywhere, and $S(x)$ has period a , $S \in \mathcal{L}([0, a])$.

Proof: $f \in \mathcal{L}(\mathbb{R}) \implies |f| \in \mathcal{L}(\mathbb{R})$ so we can assume $f \geq 0$ (note that $|S(x)| \leq \sum_{n=-\infty}^{\infty} |f(x/a + n)| \in \mathcal{L}([0, a])$ implies $S \in \mathcal{L}([0, a])$). Assume $a = 1$, then

$$\int_{\mathbb{R}} f \, dm = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_{[0,1]} f(x + n) \, dm = \int_{[0,1]} S \, dm < \infty.$$

where the second equality comes from monotone convergence theorem. Hence S is a.e. finite, so the series converges absolutely a.e., and $S \in \mathcal{L}([0, a])$.

191-18

Suppose $f \in \mathcal{L}(E)$, and $f(x) > 0, \forall x \in E$, prove that

$$\lim_{k \rightarrow \infty} \int_E f(x)^{1/k} dm = m(E).$$

Proof: Denote $\mu_f(A) = \int_A f dm$, and $A_N = \{x \in E : |f(x)| \leq N\}$. Then for $N \geq 1$,

$$\int_E f^{1/k} dm = \int_{A_N} f^{1/k} + \int_{E \setminus A_N} f^{1/k} \leq \int_{A_N} N^{1/k} + \int_{E \setminus A_N} f \leq N^{1/k} m(A_N) + \mu_f(E \setminus A_N).$$

Hence

$$\limsup_{k \rightarrow \infty} \int_E f^{1/k} dm \leq m(A_N) + \mu_f(E \setminus A_N),$$

since μ_f is absolutely continuous with respect to m ($\mu_f \ll m$) and $E \setminus \bigcup_{N \geq 1} A_N = f^{-1}(\infty)$ is null, we obtain $\lim_{N \rightarrow \infty} m(A_N) + \mu_f(E \setminus A_N) = m(E)$, so $\limsup_{k \rightarrow \infty} \int_E f^{1/k} \leq m(E)$. Likewise consider $B_N = \{x \in E : |f(x)| > 1/N\}$, we have $\liminf_{k \rightarrow \infty} \int_E f^{1/k} \geq m(E)$, hence $\lim_{k \rightarrow \infty} \int_E f^{1/k} = m(E)$.

191-19

Suppose $\{f_n\} \in \mathcal{L}^+([0, 1])$, and $f_n \rightarrow f$ in measure. If

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n dm = \int_{[0,1]} f dm,$$

prove that for any measurable set $E \subset [0, 1]$,

$$\lim_{n \rightarrow \infty} \int_E f_n dm = \int_E f dm.$$

Proof: Note that $||f_n| - |f| - |f - f_n|| \leq |f| + ||f_n| - |f - f_n|| \leq 2|f|$, and $|f_n| - |f| - |f - f_n| \rightarrow 0$ in measure, so by Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{[0,1]} |f_n| - |f| - |f - f_n| dm = 0.$$

Since $f_n, f \in \mathcal{L}^+([0, 1])$, we obtain $\lim_{n \rightarrow \infty} \int_{[0,1]} |f - f_n| dm = 0$, so for any measurable set $E \subset [0, 1]$, $\lim_{n \rightarrow \infty} \int_E f_n dm = \int_E f dm$.

192-20

Suppose $\{f_k\} \in \mathcal{L}^+(E)$, and $f_k \rightarrow 0$ a.e. on E . If

$$\int_E \max\{f_1, \dots, f_k\} dm \leq M, k = 1, 2, \dots$$

prove that

$$\lim_{k \rightarrow \infty} \int_E f_k dm = 0.$$

Proof: Let $g_k = \max\{f_1, \dots, f_k\}$, then $0 \leq g_k \leq g_{k+1}$, and $\int_E g_k \, dm \leq M$. Let $g = \lim_{k \rightarrow \infty} g_k$, then by monotone convergence theorem, $g \in \mathcal{L}^+(E)$. Hence $0 \leq f_k \leq g$ and g is integrable, so we can apply dominated convergence theorem to obtain $\lim_{k \rightarrow \infty} \int_E f_k \, dm = 0$.

192-21

Suppose $\{f_k\} \in \mathcal{L}^+(E)$ and $f_k \rightarrow f$ in measure, prove that

$$\int_E f \, dm \leq \liminf_{k \rightarrow \infty} \int_E f_k \, dm.$$

Proof: Take a sub-sequence $\lim_{k \rightarrow \infty} \int_E f_{n_k} \, dm = L = \liminf_{k \rightarrow \infty} \int_E f_k \, dm$, $f_{n_k} \rightarrow f$ in measure, so by Riesz lemma there is a sub-sequence $f_{m_t} \rightarrow f$ a.e. Apply Fatou's lemma, we have

$$\int_E f \, dm \leq \liminf_{t \rightarrow \infty} \int_E f_{m_t} \, dm, \text{ while } \liminf_{t \rightarrow \infty} \int_E f_{m_t} \, dm = L. \text{ Hence}$$

$$\int_E f \, dm \leq \liminf_{k \rightarrow \infty} \int_E f_k \, dm.$$