

184-2

Suppose $A, B \subset \mathbb{R}^n$ are measurable, prove that

$$\int_{\mathbb{R}^n} m((A - x) \cap B) dx = m(A)m(B).$$

Proof: Note that $m((A - x) \cap B) = \int_{\mathbb{R}^n} \chi_{(A-x) \cap B} dy = \int_{\mathbb{R}^n} \chi_A(y+x)\chi_B(y) dy$. Hence by Tonelli's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} m((A - x) \cap B) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(y+x)\chi_B(y) dy dx \\ &= \int_{\mathbb{R}^n} \chi_B(y) \int_{\mathbb{R}^n} \chi_A(x+y) dx dy = m(A)m(B). \end{aligned}$$

193-29

Suppose $f, g : E \rightarrow \mathbb{R}$ is measurable and $m(E) < \infty$. If $f(x) + g(y)$ is integrable on $E \times E$, prove that f, g are integrable on E .

Proof: Otherwise suppose f is not integrable on E , then $\int_E |f| dm = \infty$. Take N such that $A = \{y \in E : |g(y)| \leq N\}$ has positive measure, then

$$\int_{E \times A} |f(x) + g(y)| dm < \infty.$$

However, $|f(x) + g(y)| \geq |f(x)| - N$, so

$$\int_{E \times A} |f(x)| dm \leq Nm(E)m(A) + \int_{E \times A} |f(x) + g(y)| dm < \infty.$$

By Tonelli's theorem, $\int_{E \times A} |f(x)| dm = m(A) \int_E |f(x)| dx = \infty$, leading to contradiction.

193-33

Calculate the value of

$$\lim_{n \rightarrow \infty} \int_0^{\pi/2} \cos x \arctan(nx) dx.$$

Solution: Note that $|\cos x \arctan(nx)| \leq \frac{\pi}{2}$ which is integrable on $[0, \pi/2]$, and $\arctan nx \rightarrow \frac{\pi}{2} \chi_{(0, \pi/2]}$, so by Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^{\pi/2} \cos x \arctan nx dx = \int_0^{\pi/2} \frac{\pi}{2} \chi_{(0, \pi/2]}(x) \cos x dx = \frac{\pi}{2}.$$

222-1

Suppose $E \subset [0, 1]$. If there exists $l \in (0, 1)$, such that for any sub interval $[a, b] \subset [0, 1]$, $m(E \cap [a, b]) \geq l(b - a)$. Prove that $m(E) = 1$.

Proof: If $m(E) < 1$, then take $F = [0, 1] \setminus E$, $m(F) = 1 - m(E) > 0$. Take $\lambda > 1 - l$, then there exists an interval $I \subset [0, 1]$ such that $m(F \cap I) > \lambda m(I)$. Hence

$m(E \cap I) = m(I) - m(F \cap I) \leq (1 - \lambda)m(I) < \lambda m(I)$, leading to contradiction.

222-2

For $\chi_{\mathbb{Q}} : [0, 1] \rightarrow \mathbb{R}$, what are its Lebesgue points on $[0, 1]$?

Solution: The Lebesgue points are \mathbb{Q}^C :

If $x_0 \in \mathbb{Q}$, then $f(x_0) = 1$ but $f = 0$, a. e. so $\int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| dy = 2r$ and the limit

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| dy = 1.$$

Hence $x_0 \in \mathbb{Q}$ is not a Lebesgue point.

If $x_0 \in \mathbb{Q}^C$ then $f(x_0) = 0$ and $f = 0$, a. e. so $\int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| dy = 0$ and the limit is 0.

Extra 1

If $f \in \mathcal{L}(\mathbb{R}^n)$, f is not a.e. zero, prove that there exists constants $C, R > 0$ such that for $|x| > R$, we have $Mf(x) \geq C|x|^{-n}$, which implies Mf is not integrable.

Proof: Since f is not a.e. zero and $f \in \mathcal{L}(\mathbb{R}^n)$, $I = \int_{\mathbb{R}^n} |f| dm > 0$. Take R such that $\int_{B(0,R)} |f| dm > I/2$. Then for $|x| > R$, take $r = 2|x|$ then $B(0, R) \subset B(x, r)$, so

$$Mf(x) \geq \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm \geq \frac{1}{c(2|x|)^n} \int_{B(0, R)} |f| dm > \frac{I}{c2^{n+1}} |x|^{-n}.$$

Therefore

$$\int_{\mathbb{R}^n} |Mf| dm \geq \int_{|x| > R} C|x|^{-n} dx = \infty,$$

so Mf is not integrable.

Extra 2

$f \in \mathcal{L}(\mathbb{R}^n)$, we define

$$M^*f(x) := \sup \left\{ \frac{1}{m(B)} \int_B |f(y)| dy : B \text{ an open ball containing } x \right\}$$

Prove that $Mf \leq M^*f \leq 2^n Mf$.

Proof: Denote $F(B, f) = \frac{1}{m(B)} \int_B |f| dm$, then $Mf = \sup\{F(B, f) : B = B(x, r)\}$, while $M^*f = \sup\{F(B, f) : x \in B\}$, hence $Mf \leq M^*f$. For any $x \in B$, suppose $B = B(y, R)$, then $B \subset B(x, 2R)$, so

$$F(B(x, 2R), f) = \frac{1}{m(B(x, 2R))} \int_{B(x, 2R)} |f| dm \geq 2^{-n} \frac{1}{m(B)} \int_B |f| dm = 2^{-n} F(B, f).$$

Therefore $M^*f \leq 2^n Mf$.

Extra 3

$E \subset \mathbb{R}^n$ is a Borel set. Define the "density" of E at x (if it exists):

$$D_E(x) = \lim_{r \rightarrow 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}.$$

(a) Prove that $D_E(x) = 1, a. e. x \in E$; $D_E(x) = 0, a. e. x \in E^C$.

(b) Given $\alpha \in (0, 1)$, give an example of E, x such that $D_E(x) = \alpha$, and give an example such that $D_E(x)$ does not exist.

Proof: (a) Consider the function $\chi_E \in \mathcal{L}_{loc}^1(\mathbb{R}^n)$, then

$$D_E(x) = \lim_{r \rightarrow 0} \frac{m(E \cap B(x, r))}{m(B(x, r))} = \lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} \chi_E \, dm.$$

By Lebesgue differentiation theorem, $D_E(x) = 1, a. e. x \in E$. Likewise $D_E(x) = 0, a. e. x \in E^C$ is proved by applying the theorem to χ_{E^C} .

(b) For $\alpha \in (0, 1)$, consider the cone $E_\lambda = \{y \in \mathbb{R}^n : |\langle y, e_1 \rangle| \leq \lambda |y_1|\}$ where $\lambda \in [0, 1]$, then for any $r > 0$

$$D_{E_\lambda}(0) = \frac{m(E_\lambda \cap B(0, r))}{m(B(0, r))} = \frac{m(E_\lambda \cap B(0, 1))}{m(B(0, 1))},$$

and $D_{E_0}(0) = 0, D_{E_1}(0) = 1$. Note that $D_{E_\lambda}(0)$ is strictly increasing and continuous, hence for any $\alpha \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that $D_{E_\lambda}(0) = \alpha$.

Consider $E = \{y \in \mathbb{R}^n : |y| \in \bigcup_{n \geq 1} (2^{-2n}, 2^{-2n+1})\}$, then for $r = 2^{-2n}$,

$$\frac{m(E \cap B(0, r))}{m(B(0, r))} = \frac{\sum_{k=n+1}^{\infty} (2^{-2k+1})^n - (2^{-2k})^n}{(2^{-2n})^n} = \sum_{k=1}^{\infty} (2^n)^{-k} (-1)^{k-1} = \frac{2^{-n}}{1 + 2^{-n}} = \frac{1}{1 + 2^n}.$$

And for $r = 2^{-2n+1}$,

$$\frac{m(E \cap B(0, r))}{m(B(0, r))} = \frac{\sum_{k=n}^{\infty} (2^n)^{-2k+1} - (2^n)^{-2k}}{(2^n)^{-2n+1}} = \sum_{k=0}^{\infty} (2^n)^{-k} (-1)^k = \frac{2^n}{1 + 2^n}.$$

Therefore $\lim_{r \rightarrow 0} \frac{m(E \cap B(0, r))}{m(B(0, r))}$ does not exist.