PSA

A1) Suppose a non-empty set $X \subset \mathbb{R}$ has an upper bound, and M is an upper bound of X. The following two propositions are equivalent:

- $M = \sup X$.
- For any $\varepsilon > 0$, there exists an $x \in X$ such that $x > M - \varepsilon$.

Proof:

$$M = \sup X \iff \forall M' < M, \exists x \in X, x > M' \iff \forall \varepsilon = M - M' > 0, \exists x \in X, x > M - \varepsilon.$$

A2) Prove that every non-empty open interval contains infinitely many rational numbers.

Proof: We only need to find one rational number q in the interval (a, b), then we can apply the process to (a, q) and so on.
By the Archimedean rule, there is a positive integer N such that $N(b-a) > 2$, hence there exists an integer p such that $p = \lfloor bN \rfloor \in (aN, bN)$, and $q = \frac{p}{N} \in (a, b) \cap \mathbb{Q}$.

A3) Let (X, d) be a metric space, $Y \subset X$. We define the distance function on Y:

$d_Y : Y \times Y \to \mathbb{R}, (y_1, y_2) \mapsto d_Y(y_1, y_2) = d(y_1, y_2)$.
Prove that d_Y is a distance function, and (Y, d_Y) is a metric space. We call d_Y the induced metric on Y, and (Y, d_Y) is called a subspace.
Proof: Trivial, since $d_Y(y_1, y_2) = d(y_1, y_2)$.

A4) Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$, for any $x, y \in \mathbb{R}^n$, we define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n).$$

Prove that $(\mathbb{R}^n, d)$ is a metric space.
Proof:

1. $d(x, y) = 0 \iff x_i = y_i, \forall 1 \leqslant i \leqslant n \iff x = y$.
2. $d(x, y) = d(y, x)$ is trivial.
3. $d(x, y) + d(y, z) \geqslant d(x, z)$ is the Minkowski inequality:

$$\left(\sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}\right)^2 = \sum_{i=1}^n a_i^2 + b_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2} \geqslant \sum_{i=1}^n a_i^2 + b_i^2 + 2a_i b_i = \sum_{i=1}^n (a_i + b_i)^2.$$

A5) Given a metric space (X, d), and $Y \subset X$. If for any $x \in X$ and $\varepsilon > 0$, there exists $y \in Y$ such that $d(y, x) < \varepsilon$, then we say Y is dense in X. Prove that the set of rational numbers is dense in $\mathbb{R}$.

Proof: For any $x \in \mathbb{R}$, let $N = \lfloor x \rfloor$, then for any $\varepsilon > 0$, let $q > 1/\varepsilon$. Then for $p \in [Nq, (N+1)q] \cap \mathbb{Z}$, choose p such that $|x - p/q|$ is minimal. Suppose $p/q < x$, then

$$2\left|x - \frac{p}{q}\right| < \left|x - \frac{p}{q}\right| + \left|x - \frac{p+1}{q}\right| = \frac{1}{q} < \varepsilon.$$

Hence $d(x, p/q) < \varepsilon$.

A6) For $(x, y) \in \mathbb{R}^2$, if its coordinates x and y are rational numbers, then we call this point a rational point. Prove that $(\mathbb{R}^2, d)$ (refer to question A4) the set of rational points in $\mathbb{R}^2$ is dense.

Proof: By A5), $\overline{\mathbb{Q}} = \mathbb{R}$. Hence for any $(x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$, there exists $(a, b) \in \mathbb{Q}^2$ such that $|a - x|, |b - y| < \varepsilon/2$. Then

$$d((x, y), (a, b)) = \sqrt{(a - x)^2 + (b - y)^2} < \varepsilon.$$

Hence $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$

A7) Prove that the axiom (F) and (O), and the boundedness principle imply the Archimedean axiom (A).

Proof: Otherwise assume that $\mathbb{N}$ has an upper bound. Then $M = \sup \mathbb{N}$ exists. Let $\varepsilon = 1/2$ then there is an $n \in \mathbb{N}$ such that $n > M - \varepsilon$. Hence $n + 1 > M$, leading to contradiction.

A8) (Existence of irrational numbers) Let $X = \left\{x \in \mathbb{Q} \mid x^2 < 2\right\}$ be a bounded set, and $\sqrt{2} = \sup X$. Prove that $\sqrt{2}$ is an irrational number.

Proof: If $\sqrt{2} = s = p/q$ is rational, then $p^2 \geqslant 2q^2$, otherwise let $x = s(2 - s^2)/4 + s$, then $s < x$ and $x^2 < 2$, a contradiction. If $s^2 > 2$, then $x = s(2 - s^2)/4 < s$ and $x^2 > 2$, hence x is an upper bound of X, leading to contradiction. Therefore $s^2 = 2$ which is impossible.

A9) Prove that every open interval contains infinitely many irrational numbers.

Proof: Otherwise the open interval will be a countable set.

PSB: Countable and Uncountable Sets

Let $\mathbb{N}$ denote the set of natural numbers (including 0). X is a set, if there is an injective map $f : X \to \mathbb{N}$, then we say X is countable; if X is not countable, then we say X is uncountable.

B1) Prove that finite sets are countable.

Proof: For any finite set $X = \{a_1, \cdots, a_n\}$, the map $f : a_k \mapsto k$ is an injective, hence X is countable.

B2) Prove that subsets of countable sets are countable.

Proof: If X is countable and $Y \subset X$, then there is an injective map $f : X \to N$, so $f|_Y : Y \to N$ is an injective map, hence Y is countable.

B3) Prove that if X is a countable set, then we can always write $X = \{x_1, x_2, x_3, \ldots\}$ (that is, the elements of X can be indexed by natural numbers).

Proof: Let $I = \{n \in \mathbb{N} : f^{-1}(n) \neq \emptyset\}$, $x_k = f^{-1}(\min I \backslash \{f(x_1), \cdots, (x_{k-1})\})$. Then $x_x \in X$, and for any $x \in X$, $f(x) \in I$ hence $x \in \{x_1, \cdots, x_{f(x)}\}$. Therefore $X = \{x_1, \cdots, x_n, \cdots\}$.

B4) Prove that the set of rational numbers $\mathbb{Q}$ is countable.

Proof: List every positive rational number as below:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \dots$$

such that p/q is before m/n if $p+q < m+n$ or $p+q = m+n$ and $p < m$, then every number in $\mathbb{Q}_{>0}$ is listed at least once. Hence $\mathbb{Q}_{>0}$ is countable and so is $\mathbb{Q}$.

B5) Prove that the countable union of countable sets is countable, that is, if $X_1, X_2, \dots, X_n, \dots$ are all countable sets, then their union $\bigcup_{n=1}^{\infty} X_n$ is also a countable set.

Proof: Assume X_n are disjoint. Since X_n are countable, we can write

$$X_n = \{a_1^{(n)}, a_2^{(n)}, \cdots, a_m^{(n)}, \cdots\}.$$

Then

$$\bigcup_{n=1}^{\infty} X_n = \{a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_3^{(1)}, \cdots\}$$

where the order is the same as in B4). Hence $\bigcup_{n \geqslant 1} X_n$ is countable.

B6) If X is countable, and the map $f : X \to Y$ is surjective, then Y is countable.

Proof: Since X is countable, there is an injective map $g : X \to \mathbb{N}$. Let

$$h : Y \to \mathbb{N},\ y \mapsto \min g(f^{-1}(\{y\})).$$

then g is injective, hence Y is countable.

B7) Prove the following using proof by contradiction: $\mathbb{R}$ is uncountable.

B7-1) Suppose $J \subset \mathbb{R}$ is a closed interval and its length $|J| > 0$. For any $x \in \mathbb{R}$, there always exists an interval $I \subset J$ such that $|I| > 0$ and $x \notin I$.
Proof: Any closed interval $J = [a, b]$ can be written in the form $J = A \cup B \cup C$, where $A = \left[a, \frac{2a+b}{3}\right], B = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], C = \left[\frac{a+2b}{3}, b\right]$, and x can only be in at most 2 of these sets. Hence we can choose a set I in A, B, C.
B7-2) Prove that if $\{x_1, x_2, \dots\}$ is a countable subset of $\mathbb{R}$, then there exists a nested interval sequence $I_1 \supset I_2 \supset \cdots$ such that for any n, $x_n \notin I_n$.
Proof: Simple application of B7-1)
B7-3) Prove that $\mathbb{R}$ is uncountable.
Proof: If $\mathbb{R}$ is countable, write $\mathbb{R} = \{r_1, r_2, \cdots\}$, then set $I_0 = [0, 1]$. By B7-2) we can obtain a sequence $I_0 \supset I_1 \supset \cdots$ such that $x_n \notin I_n$ for any n. Hence

$$\bigcap_{n=0}^{\infty} I_n = \emptyset,$$

leading to contradiction.

B8) Prove that if X is an uncountable set, and A is a countable subset of X, then $X - A$ is uncountable.

Proof: Otherwise suppose that both A and $X - A$ is countable, then there exist injective mappings $f : A \to \mathbb{N}$ and $g : X - A \to \mathbb{N}$. Define

$$h : X \to \mathbb{N},\ x \mapsto \begin{cases} 2f(x), & x \in A, \\ 2g(x) + 1, & x \notin A. \end{cases}$$

Then h is injective, hence X is countable.

B9) Prove that any interval of non-zero length (open or closed) is uncountable.

Proof: Same as B7).
Or use the fact that $\mathbb{R}$ is the countable union of intervals of the same length, and the countable union of countable sets is still countable.

B10) Prove that the set of complex numbers $\mathbb{C}$ is uncountable.

Proof: $\mathbb{C}$ has an uncountable subset $\mathbb{R}$.

B11) Suppose $\mathcal{I}$ is a collection of non-overlapping closed intervals, satisfying the following property: for any $I, J \in \mathcal{I}$, if $I \neq J$, then their intersection is empty, i.e., $I \cap J = \emptyset$. Prove that $\mathcal{I}$ is countable.

Proof: For any $I \in \mathcal{I}$, there exists a rational number $r_I \in I$. Consider $f : \mathcal{I} \to \mathbb{Q}$, $I \mapsto r_I$, then f is injective. Since $\mathbb{Q}$ is countable, so is $\mathcal{I}$.

PSC: Schröder-Bernstein Theorem

Suppose X and Y are two sets, and mappings $f : X \to Y$ and $g : Y \to X$ are both injective. Let $X' = X - g(Y)$.

C1) If X is a finite set, prove that there exists a bijection $\varphi : X \to Y$.

Proof: $g : Y \to X$ is injective and X is finite, $\implies$ Y is finite. Hence $|X| \leqslant |Y|$, and $|X| \geqslant |Y|$, so $|X| = |Y|$. Therefore we can write $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, and obtain

$$\varphi : X \to Y,\ x_k \mapsto y_k.$$

C2) If X is countable, prove that there exists a bijection $\varphi : X \to Y$.

Proof: Assume X is infinite, then Y is countable (by g) and infinite (by f). Hence we can list $X = \{x_1, x_2, \cdots\}$ and $Y = \{y_1, y_2, , ..\}$ and define

$$\varphi : X \to Y,\ x_k \mapsto y_k.$$

From now on, we impose no restrictions on X. Let $h : X \to X$ be the composite map $h = g \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow g \\ X & \leftarrow & \end{array}$$

C3) Consider the set family $\mathcal{F} = \{A \subset X \mid X' \cup h(A) \subset A\}$. Prove that $\mathcal{F}$ is non-empty.

Proof: $X \in \mathcal{F}$.

C4) Prove that if $A \in \mathcal{F}$, then $X' \cup h(A) \in \mathcal{F}$.

Proof: If $A \in \mathcal{F}$ then $X' \cup h(A) \subset A$, hence (let B denote $X' \cup h(A)$)

$$X' \cup h(B) \subset X' \cup h(A) = B.$$

C5) We define

$$A_0 = \bigcap_{A \in \mathcal{F}} A = \{x \in X \mid \text{for any } A \in \mathcal{F}, \text{ we have } x \in A\}.$$

Prove that $A_0 \in \mathcal{F}$.
Proof:

$$X' \cup h(A_0) \subset X' \cup (\bigcap_{A \in \mathcal{F}} h(A)) = \bigcap_{A \in \mathcal{F}} X' \cup h(A) \subset \bigcap_{A \in \mathcal{F}} A = A_0.$$

Hence $A_0 \in \mathcal{F}$.

C6) Prove that $X' \cup h(A_0) = A_0$.

Proof:

$$A_0 \in \mathcal{F} \implies X' \cup h(A_0) \in \mathcal{F} \implies A_0 \subset X' \cup h(A_0).$$

The other side is proved in C5).

C7) Let $B_0 = X - A_0$. Prove that $f(A_0) \cap g^{-1}(B_0) = \emptyset$ and $f(A_0) \cup g^{-1}(B_0) = Y$.

Proof: If $f(A_0) \cap g^{-1}(B_0) \neq \emptyset$, then there exist $a \in A_0, b \in B_0$ such that $f(a) = g^{-1}(b)$, i.e. $b = h(a)$. Since $a \in A_0$, for any $A \in \mathcal{F}$, $a \in A$, hence $b = h(a) \in X' \cup h(A) \subset A$. Therefore $b \in A_0$, a contradiction.
Otherwise if there exists $y \in Y$ such that $y \notin f(A_0) \cup g^{-1}(B_0)$, then $g(y) \notin B_0 \implies g(y) \in A_0$. Let $z = g(y) \in A_0 \cap g(Y)$, then $z \notin X'$ so $z \in h(A_0)$ by C6). Let $z = h(t)$ then $y = f(t) \in f(A_0)$ since g is injective, leading to contradiction.

C8) We define the map $\varphi : X \to Y$. For $x \in X$, we require

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in A_0; \\ g^{-1}(x), & \text{if } x \in B_0. \end{cases}$$

Prove that this is a bijection.
Proof:

1. φ is injective: for any $x, y \in A_0, x \neq y$, $\varphi(x) \neq \varphi(y)$ since f is injective. For any $x, y \in B_0$, $x \neq y$, $\varphi(x) \neq \varphi(y)$ since g is a mapping. For any $x \in A_0, y \in B_0$, $\varphi(x) \neq \varphi(y)$ since $f(A_0) \cap g^{-1}(B_0) = \emptyset$.
2. φ is subjective: $\varphi(X) = \varphi(A_0 \cup B_0) = f(A_0) \cup g^{-1}(B_0) = Y$.

Based on the above, we have proved:
Theorem (Schroeder-Bernstein). If there exist injective maps $f : X \to Y$ and $g : Y \to X$, then there exists a bijection $\varphi : X \to Y$ between the two sets.

PSD: Details of Dedekind Cut

The goal of this part of the exercise is to complete the part of the Dedekind cut construction method taught in class, thereby providing a complete proof for the construction of real numbers.

D1) Prove that if X and Y are both Dedekind cuts, then the product $X \cdot Y$ as defined in the lecture is also a Dedekind cut, i.e.,

$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (X, Y) \mapsto X \cdot Y,$
is well-defined. (Hint: You only need to prove the case where $X > 0, Y > 0$.)
Proof: The set $X \cdot Y$ is define as $Z = \bar{0} \cup \{x \cdot y : x, y \geqslant 0, x \in X, y \in Y\}$. Let $Z' = \mathbb{Q} - Z$, then

1. $Z \neq \emptyset, Z' \neq \emptyset$, since for any $x \in X', y \in Y', x \cdot y \notin Z$.
2. For any $z \in Z, z' \in Z'$, if $z' < z$ then $z > 0$. So assume $z = x \cdot y, x \in X, y \in Y, x, y \geqslant 0$, then $z' = x \cdot (yz'/z) \in Z$, a contradiction.
3. If Z has a maximal element $z = x \cdot y, x, y \geqslant 0, x \in X, y \in Y$, then since x, y are both not maximal, there exists $x' \in X, y' \in Y$, such that $x < x', y < y'$ so $z < z' = x' \cdot y' \in Z$, a contradiction.

D2) Prove that $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$. ($\implies$ (F5))

Proof: We only need to verify the case where $X, Y, Z > 0$. Then both $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$ are the set

$$\bar{0} \cup \{x \cdot y \cdot z : x, y, z \geqslant 0, x \in X, y \in Y, z \in Z\}.$$

D3) Prove that $X \cdot Y = Y \cdot X$. ($\implies$ (F6))

Proof: Same as D2).

D4) Prove that $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$. ($\implies$ (F9))

Proof: We can assume that $X, Y, Z > 0$, then

$$X \cdot (Y + Z) = \{xy + xz : x \in X, y \in Y, z \in Z\}$$

while

$$X \cdot Y + X \cdot Z = \{xy + x'z : x, x' \in X, y \in Y, Z \in Z\}.$$

Hence $X \cdot (Y + Z) \subset X \cdot Y + X \cdot Z$.
For any $xy + x'z \in X \cdot Y + X \cdot Z$, suppose $x \geqslant x'$, then $xy + xz \in X \cdot (Y + Z)$ and $xy + x'z \leqslant xy + xz$, so $xy + x'z \in X \cdot Y + X \cdot Z$, therefore $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$.

D5) Prove that $\bar{1} \cdot X = X$ and $\bar{1} \neq \bar{0}$. ($\implies$ (F7))

Proof: Assume that $X > 0$, then $\bar{1} \cdot X = \{u \cdot v : u < 1, v \in X\}$. Foy any $u < 1, v \in X$, $u \cdot v < v$ hence $u \cdot v \in X$. For any $x \in X$, there exists $x' \in X, x' > x$, then $x = x' \cdot (x/x') \in \bar{1} \cdot X$. Therefore $\bar{1} \cdot X = X$ and $1/2 \in \bar{1} \backslash \bar{0}$, so $\bar{1} \neq \bar{0}$.

D6) Prove that if $X \cdot Y = \overline{0}$, then $X = \overline{0}$ or $Y = \overline{0}$; conversely, if $X \geq \overline{0}, Y \geq \overline{0}$, then $X \cdot Y \geq \overline{0}$. ($\Longrightarrow$ (O5))

Proof: Otherwise there exists $x, x' \in X, y, y' \in Y$, such that $x, y > 0, x', y' < 0$. Hence $xy, x'y \in X \cdot Y$, where $xy > 0 > x'y$, so $X \cdot Y \neq \overline{0}$.
Suppose $X, Y > 0$, then there exists $x \in X, y \in Y$ such that $x, y > 0$, hence $0 < xy \in X \cdot Y$, so $X \cdot Y > \overline{0}$.

D7) X is a positive Dedekind cut. Prove that for any integer n, there exist $x \in X, x' \in X'$ such that

$1 < \frac{x'}{x} < 1 + \frac{1}{n}$.
Proof: Let $l_0 = x \in X, r_0 = x' \in X'$. Define l_n, r_n as follows: If $(l_{n-1} + r_{n-1})/2 \in X$, then $l_n = (l_{n-1} + r_{n-1})/2, r_n = r_{n-1}$, otherwise $l_n = l_{n-1}, r_n = (l_{n-1} + r_{n-1})/2$. Then

$$0 < \frac{r_n - l_n}{l_n} \leqslant \frac{1}{2} \frac{r_{n-1} - l_{n-1}}{l_{n-1}}.$$

Hence there exist such x, x'.

D8) Prove that for any Dedekind cuts X and Y, if $Y \neq \overline{0}$, there exists a unique Dedekind cut Z such that

$Y \cdot Z = X$.
We denote Z as $\frac{X}{Y}$. When $X = \overline{1}$, we also denote it as Y^{-1}. ($\Longrightarrow$ (F8))
Proof: By D6), Z is unique. By D2) we can assume that $X = \overline{1}$, and $Y > 0$. Let

$$Z = \left\{ \frac{1}{y} : y \in Y' \right\} \cup \overline{0} \cup \{0\}.$$

Then by D7), $Y \cdot Z = \overline{1}$.