周明强《实变函数论（第三版）》
P184思考题2, P189-192习题4: 29,33, P222思考题1,2

184-2

Suppose $A, B \subset \mathbb{R}^n$ are measurable, prove that

$$\int_{\mathbb{R}^n} m((A-x)\cap B)\,\mathrm{d}x = m(A)m(B).$$

Proof: Note that $m((A-x)\cap B) = \int_{\mathbb{R}^n} \chi_{(A-x)\cap B}\,\mathrm{d}y = \int_{\mathbb{R}^n} \chi_A(y+x)\chi_B(y)\,\mathrm{d}y$. Hence by Tonelli's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} m((A-x)\cap B)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_A(y+x)\chi_B(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \chi_B(y)\int_{\mathbb{R}^n} \chi_A(x+y)\,\mathrm{d}x\,\mathrm{d}y = m(A)m(B).\end{aligned}$$

193-29

Suppose $f, g : E \to \mathbb{R}$ is measurable and $m(E) < \infty$. If $f(x) + g(y)$ is integrable on $E \times E$, prove that f, g are integrable on E.
Proof: Otherwise suppose f is not integrable on E, then $\int_E |f|\,\mathrm{d}m = \infty$. Take N such that $A = \{y \in E : |g(y)| \leqslant N\}$ has positive measure, then

$$\int_{E\times A} |f(x) + g(y)|\,\mathrm{d}m < \infty.$$

However, $|f(x) + g(y)| \geqslant |f(x)| - N$, so

$$\int_{E\times A} |f(x)|\,\mathrm{d}m \leqslant Nm(E)m(A) + \int_{E\times A} |f(x) + g(y)|\,\mathrm{d}m < \infty.$$

By Tonelli's theorem, $\int_{E\times A} |f(x)|\,\mathrm{d}m = m(A)\int_E |f(x)|\,\mathrm{d}x = \infty$, leading to contradiction.

193-33

Calculate the value of

$$\lim_{n\to\infty} \int_0^{\pi/2} \cos x \arctan(nx)\,\mathrm{d}x.$$

Solution: Note that $|\cos x \arctan(nx)| \leqslant \frac{\pi}{2}$ which is integrable on $[0, \pi/2]$, and $\arctan nx \to \frac{\pi}{2}\chi_{(0,\pi/2]}$, so by Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^{\pi/2} \cos x \arctan nx\,\mathrm{d}x = \int_0^{\pi/2} \frac{\pi}{2}\chi_{(0,\pi/2]}(x)\cos x\,\mathrm{d}x = \frac{\pi}{2}.$$

222-1

Suppose $E \subset [0, 1]$. If there exists $l \in (0, 1)$, such that for any sub interval $[a, b] \subset [0, 1]$, $m(E \cap [a, b]) \geqslant l(b - a)$. Prove that $m(E) = 1$.
Proof: If $m(E) < 1$, then take $F = [0, 1]\backslash E, m(F) = 1 - m(E) > 0$. Take $\lambda > 1 - l$, then there exists an interval $I \subset [0, 1]$ such that $m(F \cap I) > \lambda m(I)$. Hence

$m(E \cap I) = m(I) - m(F \cap I) \leqslant (1-\lambda)m(I) < lm(I)$, leading to contradiction.

222-2

For $\chi_{\mathbb{Q}} : [0,1] \to \mathbb{R}$, what are its Lebesgue points on $[0,1]$?
Solution: The Lebesgue points are $\mathbb{Q}^C$:
If $x_0 \in \mathbb{Q}$, then $f(x_0) = 1$ but $f = 0, a.e.$ so $\int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| \,\mathrm{d}y = 2r$ and the limit

$$\lim_{r \to 0} \frac{1}{2r} \int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| \,\mathrm{d}y = 1.$$

Hence $x_0 \in \mathbb{Q}$ is not a Lebesgue point.
If $x_0 \in \mathbb{Q}^C$ then $f(x_0) = 0$ and $f = 0, a.e.$ so $\int_{x_0-r}^{x_0+r} |f(y) - f(x_0)| \,\mathrm{d}y = 0$ and the limit is 0.

Extra 1

If $f \in \mathcal{L}(\mathbb{R}^n)$, f is not a.e. zero, prove that there exists constants $C, R > 0$ such that for $|x| > R$, we have $Mf(x) \geqslant C|x|^{-n}$, which implies Mf is not integrable.
Proof: Since f is not a.e. zero and $f \in \mathcal{L}(\mathbb{R}^n)$, $I = \int_{\mathbb{R}^n} |f| \,\mathrm{d}m > 0$. Take R such that $\int_{B(0,R)} |f| \,\mathrm{d}m > I/2$. Then for $|x| > R$, take $r = 2|x|$ then $B(0,R) \subset B(x,r)$, so

$$Mf(x) \geqslant \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| \,\mathrm{d}m \geqslant \frac{1}{c(2|x|)^n} \int_{B(0,R)} |f| \,\mathrm{d}m > \frac{I}{c2^{n+1}} |x|^{-n}.$$

Therefore

$$\int_{\mathbb{R}^n} |Mf| \,\mathrm{d}m \geqslant \int_{|x|>R} C|x|^{-n} \,\mathrm{d}x = \infty,$$

so Mf is not integrable.

Extra 2

$f \in \mathcal{L}(\mathbb{R}^n)$, we define

$$M^*f(x) := \sup \left\{ \frac{1}{m(B)} \int_B |f(y)| \,\mathrm{d}y : B \text{ an open ball containing } x \right\}$$

Prove that $Mf \leqslant M^*f \leqslant 2^n Mf$.
Proof: Denote $F(B,f) = \frac{1}{m(B)} \int_B |f| \,\mathrm{d}m$, then $Mf = \sup\{F(B,f) : B = B(x,r)\}$, while $M^*f = \sup\{F(B,f) : x \in B\}$, hence $Mf \leqslant M^*f$. For any $x \in B$, suppose $B = B(y,R)$, then $B \subset B(x,2R)$, so

$$F(B(x,2R),f) = \frac{1}{m(B(x,2R))} \int_{B(x,2R)} |f| \,\mathrm{d}m \geqslant 2^{-n} \frac{1}{m(B)} \int_B |f| \,\mathrm{d}m = 2^{-n} F(B,f).$$

Therefore $M^*f \leqslant 2^n Mf$.

Extra 3

$E \subset \mathbb{R}^n$ is a Borel set. Define the “density” of E at x (if it exists):

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))}.$$

(a) Prove that $D_E(x) = 1, a.e.\, x \in E$; $D_E(x) = 0, a.e.\, x \in E^C$.
(b) Given $\alpha \in (0,1)$, give an example of E, x such that $D_E(x) = \alpha$, and give an example such that $D_E(x)$ does not exist.
Proof: (a) Consider the function $\chi_E \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, then

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} \chi_E \,\mathrm{d}m.$$

By Lebesgue differentiation theorem, $D_E(x) = 1, a.e.\, x \in E$. Likewise $D_E(x) = 0, a.e.\, x \in E^C$ is proved by applying the theorem to χ_{E^C}.
(b) For $\alpha \in (0,1)$, consider the cone $E_\lambda = \{y \in \mathbb{R}^n : |\langle y, e_1 \rangle| \leqslant \lambda |y_1|\}$ where $\lambda \in [0,1]$, then for any $r > 0$,

$$D_{E_\lambda}(0) = \frac{m(E_\lambda \cap B(0,r))}{m(B(0,r))} = \frac{m(E_\lambda \cap B(0,1))}{m(B(0,1))},$$

and $D_{E_0}(0) = 0, D_{E_1}(0) = 1$. Note that $D_{E_\lambda}(0)$ is strictly increasing and continuous, hence for any $\alpha \in (0,1)$ there exists $\lambda \in (0,1)$ such that $D_{E_\lambda}(0) = \alpha$.

Consider $E = \left\{y \in \mathbb{R}^n : |y| \in \bigcup_{n \geqslant 1} (2^{-2n}, 2^{-2n+1})\right\}$, then for $r = 2^{-2n}$,

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} = \frac{\sum_{k=n+1}^{\infty} (2^{-2k+1})^n - (2^{-2k})^n}{(2^{-2n})^n} = \sum_{k=1}^{\infty} (2^n)^{-k} (-1)^{k-1} = \frac{2^{-n}}{1 + 2^{-n}} = \frac{1}{1 + 2^n}.$$

And for $r = 2^{-2n+1}$,

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} = \frac{\sum_{k=n}^{\infty} (2^n)^{-2k+1} - (2^n)^{-2k}}{(2^n)^{-2n+1}} = \sum_{k=0}^{\infty} (2^n)^{-k} (-1)^k = \frac{2^n}{1 + 2^n}.$$

Therefore $\lim_{r \to 0} \frac{m(E \cap B(0,r))}{m(B(0,r))}$ does not exist.