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周民强《实变函数论(第三版)》:P149 思考题 5; P162-163 思考题 8; P189-192 习题4： 7,12,18,19,20,21

149-5

Suppose $f \in \mathcal{L}([0,1])$. If $\exp\int_{[0,1]} f\,\mathrm{d}m = \int_{[0,1]} e^f\,\mathrm{d}m$, then $f = C$ a.e. $x \in [0,1]$.
Proof: $(e^x)'' = e^x > 0$ so it is strictly convex, i.e. $e^a > e^x + (a-x)e^x$ for any $a \neq x$. Let $C = \int_{[0,1]} f\,\mathrm{d}m$, then $e^{f(x)} \geqslant e^C + (f(x)-C)e^C$, so

$$\int_{[0,1]} |e^f - e^C - (f-C)e^C|\,\mathrm{d}m = \int_{[0,1]} e^f - e^C - (f-C)e^C\,\mathrm{d}m = 0.$$

Hence $e^{f(x)} = e^C + (f(x)-C)e^C$ a.e., so $f = C$ a.e.

162-8

Suppose $f_n, f \in \mathcal{L}(\mathbb{R})$, and

$$\int_{\mathbb{R}} |f_n - f|\,\mathrm{d}m \leqslant \frac{1}{n^2}, \forall n = 1, 2, \cdots$$

then $f_n \to f$ a.e. $x \in \mathbb{R}$.
Proof: Denote $B_{n,k} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > 1/k\}$, then by Chebyshev's inequality, $m(B_{n,k}) \leqslant k\int_{\mathbb{R}} |f_n - f|\,\mathrm{d}m \leqslant \frac{k}{n^2}$. Note that $\sum_{n=1}^{\infty} m(B_{n,k}) < \infty$, so by Borel-Cantelli lemma, $A_k = \limsup_{n\to\infty} B_{n,k} = \bigcap_{N\geqslant 1}\bigcup_{n\geqslant N} B_{n,k}$ is a null set. Hence $\bigcup_{k\geqslant 1} A_k$ is null, so $f_n \to f$ a.e. $x \in \mathbb{R}$.

190-7

For $f : \mathbb{R}^n \to \mathbb{R}$, if for any $\varepsilon > 0$, there exists $g, h \in \mathcal{L}(\mathbb{R}^n)$ such that $g \leqslant f \leqslant h$, and $\int_{\mathbb{R}^n} h - g\,\mathrm{d}m < \varepsilon$. Prove that $f \in \mathcal{L}(\mathbb{R}^n)$.
Proof: For $\varepsilon = 1/k > 0$, take g_k, h_k such that $g_k \leqslant f \leqslant h_k$ and $\int_{\mathbb{R}^n} h_k - g_k\,\mathrm{d}m < 1/k$. We can assume that $g_k \leqslant g_{k+1}$ and $h_k \geqslant h_{k+1}$, otherwise consider $\tilde{g}_k = \max\{g_1, \cdots, g_k\}$ and $\tilde{h}_k = \min\{h_1, \cdots, h_k\}$, we still have $\tilde{g}_k \leqslant f \leqslant \tilde{h}_k$ and $\int_{\mathbb{R}^n} \tilde{h}_k - \tilde{g}_k\,\mathrm{d}m < \varepsilon$. Then let $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$, we have $g \leqslant f \leqslant h$, and $0 \leqslant \int_{\mathbb{R}^n} h - g\,\mathrm{d}m \leqslant \int_{\mathbb{R}^n} h_n - g_n\,\mathrm{d}m < 1/n$, so $\int_{\mathbb{R}^n} h - g\,\mathrm{d}m = 0$. Hence $g = h$ a.e., so $f = \lim_{n\to\infty} g_n$ a.e., therefore f is measurable. Since $g_1 \leqslant g \leqslant h \leqslant h_1$, $g, h \in \mathcal{L}(\mathbb{R}^n)$ so $f \in \mathcal{L}(\mathbb{R}^n)$.

191-12

Suppose $f \in \mathcal{L}(\mathbb{R})$, $a > 0$, prove that the series

$$S(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{x}{a} + n\right)$$

converges absolutely almost everywhere, and $S(x)$ has period a, $S \in \mathcal{L}([0,a])$.
Proof: $f \in \mathcal{L}(\mathbb{R}) \implies |f| \in \mathcal{L}(\mathbb{R})$ so we can assume $f \geqslant 0$ (note that $|S(x)| \leqslant \sum_{n=-\infty}^{\infty} |f(x/a+n)| \in \mathcal{L}([0,a])$ implies $S \in \mathcal{L}([0,a])$). Assume $a = 1$, then

$$\int_{\mathbb{R}} f\,\mathrm{d}m = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{[0,1]} f(x+n)\,\mathrm{d}m = \int_{[0,1]} S\,\mathrm{d}m < \infty.$$

where the second equality comes from monotone convergence theorem. Hence S is a.e. finite, so the series converges absolutely a.e., and $S \in \mathcal{L}([0,a])$.

191-18

Suppose $f \in \mathcal{L}(E)$, and $f(x) > 0, \forall x \in E$, prove that

$$\lim_{k\to\infty} \int_E f(x)^{1/k} \,\mathrm{d}m = m(E).$$

Proof: Denote $\mu_f(A) = \int_A f \,\mathrm{d}m$, and $A_N = \{x \in E : |f(x)| \leqslant N\}$. Then for $N \geqslant 1$,

$$\int_E f^{1/k} \,\mathrm{d}m = \int_{A_N} f^{1/k} + \int_{E\backslash A_N} f^{1/k} \leqslant \int_{A_N} N^{1/k} + \int_{E\backslash A_N} f \leqslant N^{1/k} m(A_N) + \mu_f(E\backslash A_N).$$

Hence

$$\limsup_{k\to\infty} \int_E f^{1/k} \,\mathrm{d}m \leqslant m(A_N) + \mu_f(E\backslash A_N),$$

since μ_f is absolutely continuous with respect to m ($\mu_f \ll m$) and $E\backslash \bigcup_{N\geqslant 1} A_N = f^{-1}(\infty)$ is null, we obtain $\lim_{N\to\infty} m(A_N) + \mu_f(E\backslash A_N) = m(E)$, so $\limsup_{k\to\infty} \int_E f^{1/k} \leqslant m(E)$. Likewise consider $B_N = \{x \in E : |f(x)| > 1/N\}$, we have $\liminf_{k\to\infty} \int_E f^{1/k} \geqslant m(E)$, hence $\lim_{k\to\infty} \int_E f^{1/k} = m(E)$.

191-19

Suppose $\{f_n\} \in \mathcal{L}^+([0,1])$, and $f_n \to f$ in measure. If

$$\lim_{n\to\infty} \int_{[0,1]} f_n \,\mathrm{d}m = \int_{[0,1]} f \,\mathrm{d}m,$$

prove that for any measurable set $E \subset [0,1]$,

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}m = \int_E f \,\mathrm{d}m.$$

Proof: Note that $||f_n| - |f| - |f - f_n|| \leqslant |f| + ||f_n| - |f - f_n|| \leqslant 2|f|$, and $|f_n| - |f| - |f - f_n| \to 0$ in measure, so by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{[0,1]} |f_n| - |f| - |f - f_n| \,\mathrm{d}m = 0.$$

Since $f_n, f \in \mathcal{L}^+([0,1])$, we obtain $\lim_{n\to\infty} \int_{[0,1]} |f - f_n| \,\mathrm{d}m = 0$, so for any measurable set $E \subset [0,1]$, $\lim_{n\to\infty} \int_E f_n \,\mathrm{d}m = \int_E f \,\mathrm{d}m$.

192-20

Suppose $\{f_k\} \in \mathcal{L}^+(E)$, and $f_k \to 0$ a.e. on E. If

$$\int_E \max\{f_1, \cdots, f_k\} \,\mathrm{d}m \leqslant M, k = 1, 2, \cdots$$

prove that

$$\lim_{k\to\infty} \int_E f_k \,\mathrm{d}m = 0.$$

Proof: Let $g_k = \max\{f_1, \cdots, f_k\}$, then $0 \leqslant g_k \leqslant g_{k+1}$, and $\int_E g_k \, \mathrm{d}m \leqslant M$. Let $g = \lim_{n\to\infty} g_n$, then by monotone convergence theorem, $g \in \mathcal{L}^+(E)$. Hence $0 \leqslant f_k \leqslant g$ and g is integrable, so we can apply dominated convergence theorem to obtain $\lim_{k\to\infty} \int_E f_k \, \mathrm{d}m = 0$.

192-21

Suppose $\{f_k\} \in \mathcal{L}^+(E)$ and $f_k \to f$ in measure, prove that

$$\int_E f \, \mathrm{d}m \leqslant \liminf_{k\to\infty} \int_E f_k \, \mathrm{d}m.$$

Proof: Take a sub-sequence $\lim_{k\to\infty} \int_E f_{n_k} \, \mathrm{d}m = L = \liminf_{k\to\infty} \int_E f_k \, \mathrm{d}m$, $f_{n_k} \to f$ in measure, so by Riesz lemma there is a sub-sequence $f_{m_t} \to f$ a.e. Apply Fatou's lemma, we have $\int_E f \, \mathrm{d}m \leqslant \liminf_{t\to\infty} \int_E f_{m_t} \, \mathrm{d}m$, while $\liminf_{t\to\infty} \int_E f_{m_t} \, \mathrm{d}m = L$. Hence $\int_E f \, \mathrm{d}m \leqslant \liminf_{k\to\infty} \int_E f_k \, \mathrm{d}m$.