2025/11/20

周民强《实变函数论（第三版》P189-192 习题4: 8, 9, 15, 23, 26, 28, 32, 34

190-8

Suppose $\{E_k\}$ are measurable sets in $\mathbb{R}^n$ with finite measure, and

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}|\chi_{E_k}-f|\,\mathrm{d}m=0.$$

Prove that there exists a measurable set E, such that $f(x)=\chi_E(x)$ a.e. $x\in\mathbb{R}^n$.
Proof: Consider the measures $\mu_1(E)=\int_E|f|\,\mathrm{d}m$ and $\mu_2(E)=\int_E|f-1|\,\mathrm{d}m$, then $\lim_{k\to\infty}\mu_1(E_k^C)+\mu_2(E_k)=0$, so $\mu_1(E_k^C)\to 0$ and $\mu_2(E_k)\to 0$. We can assume $\mu_1(E_k^C),\mu_2(E_k)<2^{-k}$, otherwise consider a sub-sequence E_{n_k}. Let $E=\bigcap_{N\geqslant 1}\bigcup_{k\geqslant N}E_k$, then $\mu_2(E)\leqslant\mu_2\left(\bigcup_{k\geqslant N}E_k\right)=2^{1-N}\to 0$ so $\mu_2(E)=0$, and $E^C=\bigcup_{N\geqslant 1}\bigcap_{k\geqslant N}E_k^C$ where $\mu_1\left(\bigcap_{k\geqslant N}E_k^C\right)=0$ so $\mu_1(E^C)=0$. Hence $f=\chi_E$ a.e. $x\in\mathbb{R}^n$.

190-9

Suppose $f:[0,1]\to\mathbb{R}$ is increasing. Prove that for $E\subset[0,1]$ such that $m(E)=t$,

$$\int_{[0,t]}f\,\mathrm{d}m\leqslant\int_E f\,\mathrm{d}m.$$

Proof: Let $\mu_f(E)=\int_E f\,\mathrm{d}m$, we show that $\mu_f(E)\geqslant\mu_f(I)$ where $I=[0,t]$. Let $A=E\backslash I, B=I\backslash E$, then $m(A)=m(B)$, and $\inf A\geqslant t\geqslant\sup B$, hence

$$\int_A f\,\mathrm{d}m\geqslant f(t)m(A)=f(t)m(B)\geqslant\int_B f\,\mathrm{d}m.$$

Therefore $\mu_f(E)=\mu_f(A)+\mu_f(E\cap I)\geqslant\mu_f(B)+\mu_f(E\cap I)=\mu_f(I)$.

191-15

Suppose $f\in\mathcal{L}^+([0,1])$. If there exists a constant c such that

$$\int_{[0,1]}f^n\,\mathrm{d}m=c,\ n=1,2,\cdots$$

Prove that there exists $E\subset(0,1)$ measurable, such that $f=\chi_E$ a.e. What if f is not non-negative?
Proof: For $t\geqslant 0$, let $A_t=f^{-1}([t,\infty))$, then A_t is measurable. If $m(A_t)>0$ for some $t>1$, then $\int_{[0,1]}f^n\,\mathrm{d}m\geqslant\int_{A_t}f^n\,\mathrm{d}m\geqslant t^n m(A_t)\to\infty$, leading to contradiction. Hence $f\leqslant 1$ a.e., so $g_n=f-f^n\in\mathcal{L}^+([0,1])$, and $\int_{[0,1]}g_n\,\mathrm{d}m=0\forall n\geqslant 1$. Therefore $f=f^2$ a.e., so for $E=A_1$, $f=\chi_E$ a.e.
Even if f may not be non-negative, this result holds: apply the argument above we get $E\subset(0,1)$ measurable such that $f^2=\chi_E$ a.e., and $\int_{[0,1]}f\,\mathrm{d}m=\int_{[0,1]}f^2\,\mathrm{d}m$, hence $f=\chi_E$ a.e.

192-23

Suppose $f_k,f\in\mathcal{L}(\mathbb{R}^n)$, and for any measurable set $E\subset\mathbb{R}^n$,

$$\int_E f_k\,\mathrm{d}m\leqslant\int_E f_{k+1}\,\mathrm{d}m,\ k=1,2,\cdots$$

and $\lim_{k\to\infty}\int_E f_k \,\mathrm{d}m = \int_E f \,\mathrm{d}m$.
Prove that $\lim_{k\to\infty} f_k = f$ a.e. $x \in \mathbb{R}^n$.
Proof: $\mu_{f_k}(E) \leqslant \mu_{f_{k+1}}(E)$ for any measurable E implies $f_k \leqslant f_{k+1}$ a.e. (otherwise take $E = \{x : f_k(x) - f_{k+1}(x) > 1/n\}$ such that $m(E) > 0$). Suppose $g = \lim_{k\to\infty} f_k$, then by monotone convergence theorem,

$$\int_E g \,\mathrm{d}m = \lim_{k\to\infty}\int_E f_k \,\mathrm{d}m = \int_E f \,\mathrm{d}m$$

for every measurable E. Therefore $f = g$ a.e.

193-26

Suppose f is bounded on $\mathbb{R}$. If for every $x \in \mathbb{R}$, the limit $\lim_{h\to 0} f(x+h)$ exists, prove that $f(x)$ is Riemann integrable on any interval $[a,b]$.
Proof: Let $g(x) = \lim_{h\to 0} f(x+h)$, we show that $g \in C(\mathbb{R})$: For any $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists δ such that $|x-y| < \delta \implies |f(y) - g(x)| < \varepsilon/2$. For any such y, $\lim_{h\to 0} f(y+h) = g(y)$ implies there exists $z \in B(x,\delta)$ such that $|f(z) - g(y)| < \varepsilon/2$, hence $|g(x) - g(y)| \leqslant |g(x) - f(z)| + |f(z) - g(y)| < \varepsilon$.
Therefore g is continuous.
The discontinuous points of f are $A = \{x \in [a,b] : f(x) \neq g(x)\}$, we show that $A_n = \{x \in [a,b] : |f(x) - g(x)| > 1/n\}$ are finite, hence A is countable.
Otherwise if A_n is infinite, consider a limit point $L = \lim_{k\to\infty} x_k$ (since A_n is bounded). By definition, $\lim_{k\to\infty} g(x_k) = g(L) = \lim_{k\to\infty} f(x_k)$, so $\lim_{k\to\infty}|g(x_k) - f(x_k)| = 0$ leading to contradiction.
Therefore A is countable so f is Riemann integrable on any interval $[a,b]$.

193-28

Suppose $f \in \mathcal{R}([0,1])$, prove that $f(x^2) \in \mathcal{R}([0,1])$.
Proof: Otherwise there exists n such that $A = \{x \in [0,1] : \omega_g(x) > 1/n\}$ has positive measure, where $g(x) = f(x^2)$. $\omega_g(x) = \lim_{h\to 0}\sup_{y\in(x-h,x+h)}|f(y^2) - f(x^2)|$, so $\{x^2 : x \in A\} \subset B = \{x \in [0,1] : \omega_f(x) > 1/n\}$, leading to contradiction. ($A' = \{x^2 : x \in A\}$ is also a null set, since if the intervals $\{[l_i, r_i]\}$ cover A, then $[l_i^2, r_i^2]$ cover A', and $\sum r_i^2 - l_i^2 \leqslant 2\sum r_i - l_i$).

193-32

Suppose $f \in \mathcal{L}(\mathbb{R})$, and xf is Lebesgue integrable. Let

$$F(x) = \int_{-\infty}^{x} f \,\mathrm{d}m.$$

If $\int_{\mathbb{R}} f \,\mathrm{d}m = 0$ prove that $F \in \mathcal{L}(\mathbb{R})$.
Proof: For $x \in (-\infty, 0)$, by Tonelli theorem,

$$\int_{-\infty}^{0}|F(x)| \,\mathrm{d}x \leqslant \int_{-\infty}^{0}\int_{-\infty}^{x}|f(t)| \,\mathrm{d}t \,\mathrm{d}x = \int_{-\infty}^{0}\int_{t}^{0}|f(t)| \,\mathrm{d}x \,\mathrm{d}t = \int_{-\infty}^{0}|tf(t)| \,\mathrm{d}t < \infty.$$

For $x > 0$, note that $F(x) = -\int_x^\infty f \,\mathrm{d}m$, so likewise

$$\int_{0}^{\infty}|F(x)| \,\mathrm{d}x \leqslant \int_{0}^{\infty}\int_{x}^{\infty}|f(t)| \,\mathrm{d}t \,\mathrm{d}x = \int_{0}^{\infty}|tf(t)| \,\mathrm{d}t.$$

Therefore $F \in \mathcal{L}(\mathbb{R})$.

193-34

Suppose $f \in \mathcal{L}((0,a))$, $g(x) = \int_{[x,a]} \frac{f(t)}{t}\,\mathrm{d}m$. Prove that $g \in \mathcal{L}((0,a))$ and

$$\int_0^a g\,\mathrm{d}m = \int_0^a f\,\mathrm{d}m.$$

Proof: Note that by Tonelli theorem,

$$\int_0^a |g|\,\mathrm{d}x \leqslant \int_0^a \int_x^a \left|\frac{f(t)}{t}\right| \mathrm{d}t\,\mathrm{d}x = \int_0^a \int_0^t \left|\frac{f(t)}{t}\right| \mathrm{d}x\,\mathrm{d}t = \int_0^a |f(t)|\,\mathrm{d}t$$

hence $g \in \mathcal{L}((0,a))$. Apply Fubini theorem we have

$$\int_0^a g\,\mathrm{d}m = \int_0^a \int_x^a f(t)/t\,\mathrm{d}t\,\mathrm{d}x = \int_0^a \int_0^t f(t)/t\,\mathrm{d}x\,\mathrm{d}t = \int_0^a f\,\mathrm{d}x.$$