讲义习题5.1:1,2; 讲义习题5.2:3-8.

5.1.1

List the elements of A_4.
Solution: $A_4 = \{(1),(12)(34),(13)(24),(14)(23),(123),(124),(134),(234),(132),(142),(143),(243)\}$.

5.1.2

Determine all n such that $\sigma = \begin{pmatrix} 1 & \cdots & n \\ n & \cdots & 1 \end{pmatrix} \in A_n$.
Solution: $\sigma \in A_n \iff 2 \mid \frac{n(n-1)}{2} \iff n \equiv 0,1 \pmod 4$.

5.2.3

Prove that all bilinear forms on $F^{n\times 1}$ are in the form $L(X,Y) = Y^tAX$, where $A \in F^{n\times n}$.
Proof: Let $f_1,\cdots,f_n \in (F^n)^*$ be the dual basis of $e_1,\cdots,e_n$. Clearly $\mathcal{T}^2(V)$ has a basis $\{f_i\otimes f_j : i,j\in\{1,\cdots,n\}\}$, so $\mathrm{dim}\mathcal{T}^2(V) = n^2$ (note that if $c_{i,j} = L(e_i,e_j)$, then $L = \sum_{i,j} c_{i,j} f_i \otimes f_j$). Let $L_A : (X,Y) \mapsto Y^tAX$, then $\varphi : F^{n\times n} \to \mathcal{T}^2(V), A \mapsto L_A$ is linear. Clearly $\mathrm{Ker}\varphi = \{0\}$, so φ is an isomorphism, and all bilinear forms are in the form $L(X,Y) = Y^tAX$.

5.2.4

$L \in \mathcal{T}^2(V)$ is called non-degenerate, if for any $\alpha \in V\backslash\{0\}$, there exists $\beta \in V$ such that $L(\alpha,\beta) \neq 0$. Prove that $L(X,Y) = Y^tAX$ is non-degenerate iff A is invertible.
Proof: L is degenerate iff $\exists X \neq 0 \forall Y, L(X,Y) = 0$, iff $\exists X \neq 0, AX = 0$ iff A is not invertible. Hence L is non-degenerate iff A is invertible.

5.2.5

Prove that the standard symplectic form on F^{2n} is non-degenerate.
Proof: If $\omega_0 = \sum_{i=1}^n f_i \wedge f_{n+i}$ is degenerate, i.e. there exists $\alpha = (a_1,\cdots,a_{2n})$ such that for any $\beta = (b_1,\cdots,b_{2n})$,

$$\omega_0(\alpha,\beta) = \sum_{i=1}^n (f_i \otimes f_{n+1} - f_{n+1} \otimes f_i)(\alpha,\beta) = \sum_{i=1}^n a_i b_{n+i} - a_{n+i} b_i = 0.$$

Hence take $\beta = (0,\cdots,0,1,0,\cdots,0)$ we obtain $\alpha = (0,0,\cdots,0)$.

5.2.6

Suppose $\mathrm{dim}V < \infty$, L is a non-degenerate bilinear form. For $W \subset V$, define $W^\perp = \{\alpha \in V : L(\alpha,\beta) = 0, \forall \beta \in W\}$. Prove that $W^\perp$ is a subspace of V, and $\mathrm{dim}W + \mathrm{dim}W^\perp = \mathrm{dim}V$.
Proof: Clearly $W^\perp$ is a subspace. Take a basis $\{\beta_1,\cdots,\beta_k\}$ of W and extend it to a basis $\{\beta_1,\cdots,\beta_n\}$ of V. Consider $T : V \mapsto F^k, \alpha \mapsto (L(\alpha,\beta_1),\cdots,L(\alpha,\beta_k))$ then $\mathrm{Ker}T = W^\perp$ and $\mathrm{Im}T = F^k$ since L is non-degenerate. Hence $\mathrm{dim}W^\perp = \mathrm{dim}V - \mathrm{dim}W$.

5.2.7

Prove that if $\mathrm{char} F \neq 2$, then $L \in \Lambda^r(V) \iff L$ is skew-symmetric.
Proof: If $L \in \Lambda^r(V)$ then $L(a_1 + a_2, a_1 + a_2, a_3, \cdots, a_n) = 0$ so $L(a_1, a_2, \cdots, a_n) = -L(a_2, a_1, a_3, \cdots, a_n)$. Likewise we obtain $L(^\sigma a) = \mathbf{sign}(\sigma) L(a)$ so L is skew-symmetric.
If L is skew-symmetric, then $L(a_1, a_1, a_3, \cdots, a_n) = -L(a_1, a_1, a_3, \cdots, a_n)$. Since $\mathrm{char} F \neq 2$, $L(a_1, a_1, a_3, \cdots, a_n) = 0$, and likewise we obtain $L \in \Lambda^r(V)$.

5.2.8

Suppose $\mathrm{char} F = 2$, show that there is a skew-symmetric but not alternating bilinear form on F^n.
Proof: For odd n, consider $L(x, y) = \langle x, y \rangle$ where $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ for $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k)$. Clearly $L(x, y) = L(y, x)$ so L is skew-symmetric. But for $x = (1, 0, \cdots, 0)$, $L(x, x) = 1 \neq 0$.