2025/10/14

序列极限一章的作业：

习题3.1，3.2，3.4 每一节后面的 n 道题，可自选不少于 n-3 道题，

习题3.3 自选不少于 20道

3.1

3.1.1

Prove by definition

$$\lim_{n\to\infty}\frac{3n^3+4n^2-100}{2n^3-9n-11}=\frac{3}{2}.$$

Proof: Note that

$$\frac{3n^3+4n^2-100}{2n^3-9n-11}-\frac{3}{2}=\frac{8n^2+27n-167}{2(2n^3-9n-11)}.$$

Hence for $n>100$,

$$\left|\frac{3n^3+4n^2-100}{2n^3-9n-11}-\frac{3}{2}\right|\leqslant\frac{9n^2}{n^3}=\frac{9}{n}$$

so the limit equals $\frac{3}{2}$.

3.1.2

Prove by definition

$$\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)=0.$$

Proof: Note that $e=\lim_{n\to\infty}(1+1/n)^n$ so $1\leqslant(1+1/n)^n<e$, then $0<\log(1+1/n)<1/n$. Therefore $\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)=0$.

3.1.4

Calculate

$$\lim_{n\to\infty}\frac{2\sqrt[n]{n}+\sqrt[n]{100}}{3\sqrt[n]{n}-1}.$$

Solution: Trivial. The answer is $\frac{2}{3}$.

3.1.5

Calculate

$$\lim_{n\to\infty}\left(\frac{3\log^3 n+2\log n+1}{2\log^2 n+\log n}-\frac{6\log^3 n+5\log^2 n+3\log n+1}{4\log^2 n-\log n+2}\right).$$

Solution: The answer is $-\frac{19}{8}$.

3.1.6

Calculate

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n+\left(2+\frac{3}{n}\right)^n+\left(3+\frac{5}{n}\right)^n\right)^{1/n}.$$

Solution: The answer is 3.

3.1.7

Let $0\leqslant k\leqslant p-1$, calculate

$$\lim_{n\to\infty}\frac{\binom{kn}{p}+\binom{kn}{p+k}+\cdots+\binom{kn}{p+(n-1)k}}{2^{kn}}.$$

Solution: Let $w=e^{2\pi i/kn}$, then

$$\sum_{i=0}^{n-1}\binom{kn}{p+ik}=\frac{1}{k}\sum_{j=0}^{k-1}(1+w^j)^{kn}w^{-jp}.$$

Note that $|1+w^j|\leqslant 2$ and equality holds iff $j=0$, hence the answer is $\frac{1}{k}$.

3.2

3.2.1

Calculate

$$\lim_{n\to\infty}\frac{n^{-1}-(n+1)^{-1}}{n^{-2}}.$$

Solution:

$$\lim_{n\to\infty}\frac{n^{-1}-(n+1)^{-1}}{n^{-2}}=\lim_{n\to\infty}n-\frac{n^2}{n+1}=\lim_{n\to\infty}\frac{n}{n+1}=1.$$

3.2.2

Prove that $\lim_{n\to\infty}\sin n$ does not exists.
Proof: We know that $\left\{\frac{n}{2\pi}\right\}$ is equidistributed in $[0,1]$, hence $1,-1$ are both limit points of $\{\sin n : n\geqslant 1\}$.

3.2.4 & 3.2.5

For $\alpha>0$ and $\alpha\in(-1,0)$ calculate

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n k^\alpha}{n^{\alpha+1}}.$$

Solution: For $\alpha>0$.

$$\frac{n^{\alpha+1}}{\alpha+1}=\int_0^n x^\alpha\,\mathrm{d}x\leqslant\sum_{k=1}^n k^\alpha\leqslant\int_1^{n+1}x^\alpha\,\mathrm{d}x=\frac{(n+1)^{\alpha+1}-1}{\alpha+1}.$$

Hence the answer is $1/(\alpha+1)$. It the same with $\alpha\in(-1,0)$, by simply reversing the inequalities.

3.2.6

For $\alpha < -1$ determine the value of

$$\lim_{n\to\infty} \frac{\sum_{k=n+1}^{2n} k^\alpha}{n^{\alpha+1}}.$$

Solution: Likewise

$$\lim_{n\to\infty} \frac{\sum_{k=n+1}^{2n} k^\alpha}{n^{\alpha+1}} = \lim_{n\to\infty} \frac{(2n)^{\alpha+1} - n^{\alpha+1}}{(\alpha+1)n^{\alpha+1}} = \frac{2^{\alpha+1}-1}{\alpha+1}.$$

3.2.8

Let $x_1 = a, x_2 = b, x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$, prove that $\lim_{n\to\infty} x_n$ exists and determine its value.
Solution: $x_n = \alpha + \left(-\frac{1}{2}\right)^{n-1}\beta$ where $a = \alpha + \beta$ and $b = \alpha - \frac{\beta}{2}$. Hence $\lim_{n\to\infty} x_n = \alpha = \frac{a+2b}{3}$.

3.2.9

Suppose $\{x_n\}$ satisfy $\lim_{n\to\infty} x_n \sum_{k=1}^n x_k^2 = 1$, prove that $\lim_{n\to\infty} \sqrt[3]{3n}x_n = 1$.
Proof: Let $S_n = \sum_{k=1}^n x_n^2$, then $\lim_{n\to\infty} S_n\sqrt{S_n - S_{n-1}} = 1$. Clearly $x_n \to 0$ so $S_n/S_{n-1} \to 1$.
Note that

$$\begin{aligned}\lim_{n\to\infty} nx_n^3 &= \lim_{n\to\infty} \frac{n}{S_n^3} = \lim_{n\to\infty} \frac{1}{S_n^3 - S_{n-1}^3} = \lim_{n\to\infty} \frac{1}{(S_n - S_{n-1})(S_n^2 + S_nS_{n-1} + S_{n-1}^2)} \\ &= \lim_{n\to\infty} \frac{1}{x_n^2(S_n^2 + S_nS_{n-1} + S_{n-1}^2)} = \lim_{n\to\infty} \frac{S_n^2}{S_n^2 + S_nS_{n-1} + S_{n-1}^2} = \frac{1}{3}.\end{aligned}$$

3.2.10

Given positive numbers a, d, let A_n, G_n denote respectively the arithmetic and geometric means of the sequence $a, a+d, \cdots, a+(n-1)d$. Calculate

$$\lim_{n\to\infty} \frac{G_n}{A_n}.$$

Solution: $A_n = a + \frac{n-1}{2}d \sim \frac{d}{2}n$.
$\log G_n = \frac{1}{n}\sum_{k=0}^{n-1} \log(a+kd) = \log d + \frac{\log(n-1)!}{n} + \frac{1}{n}\sum_{k=0}^{n-1} \log\left(1 + \frac{a}{kd}\right)$.
Clearly $\lim_{n\to\infty} \log\left(1 + \frac{a}{nd}\right) = 0$ so $\lim_{n\to\infty} \frac{G_n}{d(n!)^{1/n}} = 1$. We know that $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$ so

$$\lim_{n\to\infty} \frac{G_n}{A_n} = \frac{2}{e}.$$

3.2.11

Let $a_{n,k} = \frac{k}{n-k+1}$, prove that
(1) for any given k, $a_{n,k} = o(1)$.
(2) $A_n = \prod_{k=1}^n a_{n,k}$ is not $o(1)$.
Proof: (1) is trivial. (2) Note that

$$A_n = \prod_{k=1}^n \frac{k}{n-k+1} = 1$$

is constant.

3.2.12

Suppose $E \subset \mathbb{R}$ is nonempty. Prove that there exists $\{x_k\} \subset E$ such that $\lim_{n\to\infty} x_n = \sup E$ ($\sup E$ can be ∞).
Proof: If $\sup E$ is finite, for any $n \geqslant 1$, there exists $x_n \in E$ such that $x_n > \sup E - 1/n$. Hence $\lim_{n\to\infty} x_n = \sup E$. The case $\sup E = \infty$ is trivial.

3.2.13

Suppose $\{t_{n,m}\}_{n\geqslant m}$ satisfy:
(i) $\lim_{n\to\infty} t_{n,m} = 0$ for every m;
(ii) $\sum_{k=1}^{n} |t_{n,k}| < K$.
Let $y = \sum_{k=1}^{n} x_k t_{n,k}$, prove that
(1) If $\lim_{n\to\infty} x_n = 0$ then $\lim_{n\to\infty} y_n = 0$.
(2) Let $T_n = \sum_{k=1}^{n} t_{n,k}$. If $\lim_{n\to\infty} T_n = 1$, and $\lim_{n\to\infty} x_n = a$ is finite, then $\lim_{n\to\infty} y_n = a$.
Proof: (1) Clearly

$$|y_n| \leqslant \sum_{k=1}^{N} |t_{n,k}| \cdot |x_k| + \sum_{k=N+1}^{n} |t_{n,k}| \cdot |x_k| \leqslant MN\varepsilon(n,N) + K \cdot \delta(N).$$

where $M = \sup_{n\geqslant 0} |x_n|$, $\varepsilon(n,N) = \sup_{k\geqslant n, j\leqslant N} |t_{k,j}|$ and $\delta(N) = \sup_{k\geqslant N} |x_k|$. Let $n \to \infty$ then $\varepsilon(n,N) \to 0$, then let $N \to \infty$ then we obtain $\lim_{n\to\infty} y_n = 0$.
(2) Let $x'_n = x_n - a$ and $y'_n = \sum_{k=1}^{n} x'_k t_{n,k}$, then from (1) $\lim_{n\to\infty} y'_n = 0$. Note that $y_n - y'_n = aT_n$, hence $\lim_{n\to\infty} y_n = a$.

3.3

3.3.2

If E is closed, prove that $\sup E, \inf E \in E$.
If $\sup E \in E^C$, then there exists $\varepsilon > 0$ such that $B(\sup E, \varepsilon) \subset E^C$, leading to contradiction. Likewise $\sup E, \inf E \in E$.

3.3.3

Prove that $\overline{\mathbb{Q}^C} = \mathbb{R}$.
Proof: $\overline{\mathbb{Q}^C} \supset \overline{\sqrt{2}\mathbb{Q}} = \mathbb{R}$.

3.3.4

Let $x_0 \in (0,2)$, $x_{n+1} = x_n(2 - x_n)$, prove that $\lim_{n\to\infty} x_n = 1$.
(And likewise for $y_0 \in (0, c^{-1})$, $y_{n+1} = y_n(2 - cy_n)$, $\lim_{n\to\infty} y_n = c^{-1}$.)
Proof: Note that $|x_{n+1} - 1| = |x_n - 1|^2$ so $|x_n - 1| = |1 - x_0|^{2^n} \to 0$.

3.3.5

Let $a_0, b_0 > 0$, $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = \frac{2a_nb_n}{a_n+b_n}$. Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ both exists.
Proof: Note that $a_{n+1}b_{n+1} = a_nb_n$, and $a_n \geqslant b_n$ so $a_{n+1} \leqslant a_n$ and $b_{n+1} \geqslant b_n$. Also, $\min\{a_0, b_0\} \leqslant a_n, b_n \leqslant \max\{a_0, b_0\}$, hence $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist. Since $2a_{n+1} - a_n = b_n$, they are equal, and both are $\sqrt{ab}$.

3.3.6

Suppose $c \geqslant -3$, and $x_1 = c/2$, $x_{n+1} = \frac{c}{2} + \frac{x_n^2}{2}$. When does x_n converge and calculate that limit.
Solution: Let $y_n = x_n/2$, then $y_{n+1} = y_n^2 + y_0$. If y_n converges, then the limit should be $a = \frac{1+\sqrt{1-4y_0}}{2}$ or $b = \frac{1-\sqrt{1-4y_0}}{2}$, hence $y_0 \leqslant 1/4$.
When $y_0 \in [-3/4, 1/4]$, $|y_{n+1} - b| = |y_n - b| \cdot |y_n + b|$. If $y_0 \in [0, 1/4]$, then we can show that $x_n \leqslant x_{n+1} \leqslant b$, hence $\lim_{n\to\infty} y_n = b$. If $y_0 \in [-3/4, 0)$, then we can prove that $-1 < x_n < a$, so $|y_n + b| \leqslant |b - 1| < 1$. Therefore $\lim_{n\to\infty} y_n = b$.

3.3.8

Suppose $E \subset \mathbb{R}$. Prove that there exists a countable set $F \subset E$ such that F is dense in E.
Proof: Actually, subsets of a separable metric space is still separable. If $\mathcal{T} = (X, d)$ is a separable metric space with a countable dense set D, then $\mathcal{T}$ is second-countable (having a countable base $\mathcal{B} = \{B(x, r) : r \in \mathbb{Q}, x \in D\}$). Consider the relative topology $\mathcal{T}_A = \{G \cap A : G \in \mathcal{T}\}$, then clearly $\mathcal{T}_A$ is second-countable (with base $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$), so $\mathcal{T}_A$ is separable.

3.3.9

Prove that $[a, b]$ is connected (cannot be represented as the disjoint union of two closed sets).
Proof: If $[a, b] = U \cup V$ where U, V are disjoint closed sets, then suppose $a \in U$, let $c = \sup\{c \in [a, b] : [a, c] \subset U\}$ (c exists since $[a, a] \subset U$). Clearly there exists $c_n \to c$ such that $[a, c_n] \subset U$ so $c \in U$. For any $n > 0$, there exists $d_n \in (c, c + 1/n)$ such that $d_n \in V$ hence $c = \lim_{n\to\infty} d_n \in V$, leading to contradiction.

3.3.10

Prove that $\mathbb{R}$ is connected.
Proof: If $\mathbb{R} = U \cup V$ where U, V are closed and non-empty, then take $[a, b]$ intersecting U, V, $[a, b] = (U \cap [a, b]) \cup (V \cap [a, b])$ which are two disjoint closed sets, contradicting with 3.3.9.

3.3.11

Suppose $\{a_{k_n}\}$ and $\{a_{m_n}\}$ are two subsequences of $\{a_n\}$ with the same limit, and $\{k_n\} \cup \{m_n\} = \mathbb{N}_+$. Prove that $\{a_n\}$ converges.
Proof: Let $a = \lim_{n\to\infty} a_{k_n} = \lim_{n\to\infty} a_{m_n}$. For any $\varepsilon > 0$, there exists N_1, N_2 such that for any $n > N_1$, $|a - a_{k_n}| < \varepsilon$ and for any $n > N_2$, $|a - a_{m_n}| < \varepsilon$. Then let $N = \max\{k_{N_1}, m_{N_2}\}$, for any $n > N$, $|a - a_n| < \varepsilon$ hence $\lim_{n\to\infty} a_n = a$.

3.3.12

Suppose E is closed in $\mathbb{R}$. Prove that there exists a set F such that $F' = E$.
Proof: Let $A = E \backslash E'$ be all isolated points of E. For every $x \in A$, let r_x be such that $B(x, r_x) \cap A = \{x\}$, and $F = A \cup \left\{x + \frac{r_x}{n} : x \in A, n > 10\right\}$, we show that $F' = E$. Clearly $E \subset F'$. Consider every $y = \lim_{n \to \infty} y_n \in F'$ where $y_n \in F$. If there are infinitely $y_n \in E$, then $y \in E' \subset E$. Otherwise suppose $y_m = x_m + \frac{r_m}{n_m}$. If $\liminf_{m \to \infty} r_m = 0$ then $y \in E'$. Otherwise, let $s = \liminf_{m \to \infty} r_m$ then for m large enough, $|x_m - y| < \frac{r_m}{10} + \frac{s}{10}$ so $|x_n - x_m| < \max\{r_n, r_m\}$. By the definition of r_n, we obtain $x_n = x_m$ for all n, m large enough, hence $y = x_m \in E$.

3.3.14

Try to construct a sequence of non-empty sets such that $E_{k+1} = E_k' \subsetneq E_k$.
Solution: Consider the Cantor set $\mathcal{C}$.
Let $A_0 = \mathcal{C}$, from exercise 3.3.12 we know that there exists A_{n+1} such that $A_{n+1}' = A_n$ (we can let A_{n+1} be closed, otherwise use $\bar{A}_{n+1}$), then $A_n = A_{n+1}' \subsetneq A_{n+1}$.
Let $E_0 = \bigcup_{n \geqslant 1} (2n + A_n)$, then $E_m = \bigcup_{n \geqslant 1} (2n + A_n^{(m)})$, so $E_m \subsetneq E_{m-1}$ since $2n + A_n^{(n)} = 2n + \mathcal{C} \subsetneq 2n + A_1 = 2n + A_{n+1}^{(n)}$.

3.3.20

Prove that

$$\lim_{n \to \infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots n\sqrt{1 + (n+1)}}}} = 3.$$

Proof: Note that $\sqrt{1 + n(n+2)} = n + 1$, hence

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots + (n-1)\sqrt{1 + n \cdot (n+2)}}}}$$

is greater than the left side of the identity. Also

$$\sqrt{1 + 2\sqrt{1 + \cdots + n\sqrt{n+2}}} \geqslant \frac{1}{(n+2)^{2^{-n}}} \cdot \sqrt{1 + 2\sqrt{1 + \cdots + n(n+2)}} = \frac{3}{(n+2)^{2^{-n}}}.$$

Hence the identity holds.

3.3.22

Let $x_0 > 0$, $x_{n+1} = \sqrt{2 + x_n}$. Determine the value of $\lim_{n \to \infty} 4^n (2 - x_n)$.
Solution: If $x_0 < 2$, then let $x_n = 2\cos\theta_n$ where $\theta_n \in (0, \pi/2)$, we have $4\cos^2\theta_{n+1} = 2(1 + \cos\theta_n)$ so $\theta_{n+1} = \theta_n / 2$.

$$\lim_{n \to \infty} 4^n (2 - x_n) = \lim_{n \to \infty} 4^n \left(2 - 2\cos\frac{\theta_0}{2^n}\right) = \theta_0^2.$$

Likewise if $x_0 > 2$, let $x_n = 2\cosh\theta_n$ where $\theta_n > 0$, then $\cosh\theta_n = 2\cosh^2\theta_{n+1} - 1$ so $\theta_n = \theta_0 2^{-n}$.

$$\lim_{n \to \infty} 4^n (2 - x_n) = -\lim_{n \to \infty} 4^n \cdot (e^{\theta_0 2^{-(n+1)}} - e^{-\theta_0 2^{-(n+1)}})^2 = -\theta_0^2.$$

Hence

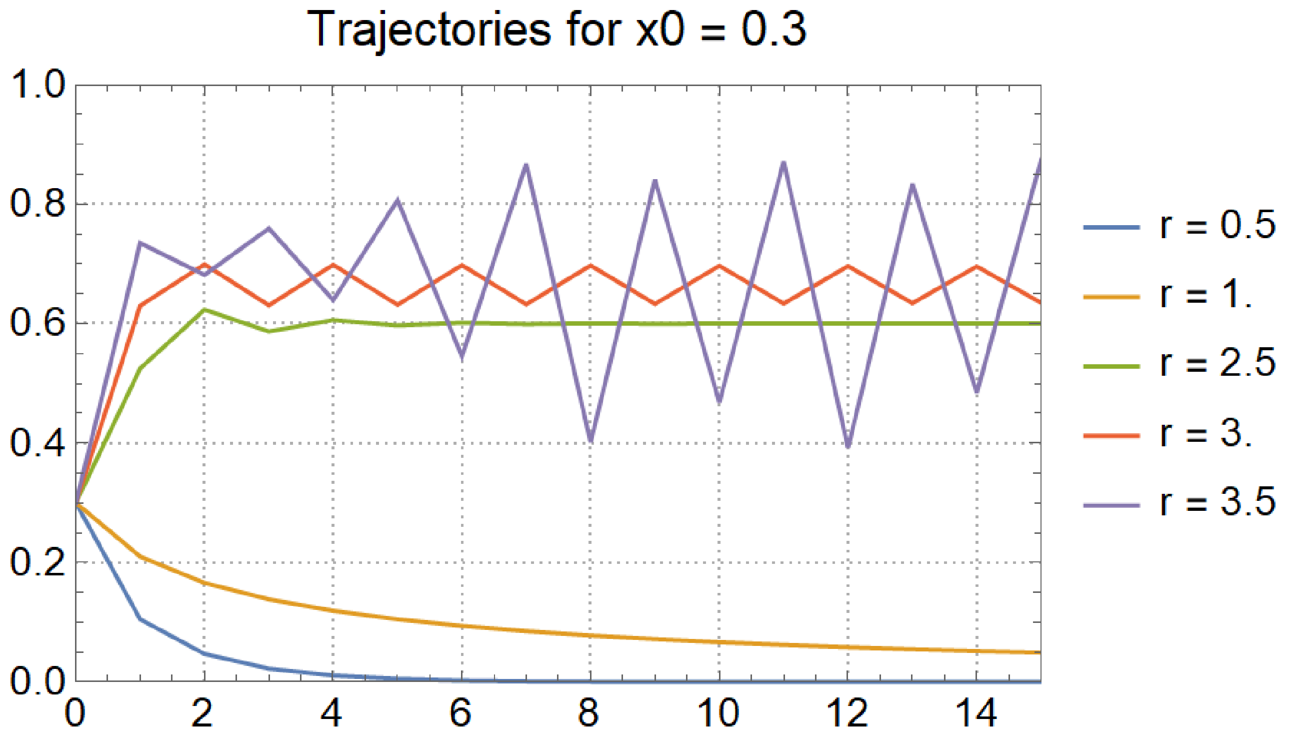
$$\lim_{n\to\infty} 4^n(2-x_n) = \arccos^2\frac{x_0}{2}.$$

(since $\cosh z = \cos iz$).

3.3.24

For $r \in (0,4)$, take an arbitrary $x_0 \in (0,1)$ and let $x_{n+1} = rx_n(1-x_n)$. For what values r, the sequence $\{x_n\}$ converges independent of the choice of x_0.
Try writing a computer program to compute how x_n converges with different values of r.
Solution: If $\{x_n\}$ has a limit a, then $a = ra(1-a)$ so $a = 1 - r^{-1}$ or $a = 0$.
Case 1: $r \in (0,1]$, then $x_{n+1} \leqslant x_n(1-x_n)$ so $\lim_{n\to\infty} x_n = 0$ converges.
Case 2: $r \in (1,3]$, then for $a = 1 - r^{-1}$, $x_{n+1} - a = (x_n - a)(1 - rx_n)$. Note that either $x_1 < \frac{2}{r}$ or $x_2 < \frac{2}{r}$, and if $x_n > \frac{2}{r}$ then $x_{n+1} < \frac{2}{r}$. Hence x_n converges to a.
Case 3: $r \in (3,4)$, then $\{x_n\}$ converges for some x_0 (e.g., $x_0 = 1 - r^{-1}$) but not for others (e.g., $x_0 = \frac{r+1-\sqrt{(r-3)(r+1)}}{2r}$).
Code: (Wolfram Language)

```
func[r_] :=
 r*#*(1 - #) &; rs = {0.5, 1.0, 2.5, 3.0, 3.5}; x0 = 0.3; dep = 15;
ListLinePlot[
 MapThread[
  Tooltip[#1, "r = " <> ToString[#2]] &, {Table[
    Module[{x = x0, trajectory = {{0, x0}}},
     Do[x = func[r][x]; AppendTo[trajectory, {n, x}];, {n, 1, dep}];
     trajectory], {r, rs}], rs}],
 PlotLegends -> (StringForm["r = ``", #] & /@ rs),
 PlotStyle -> Table[ColorData[97, i], {i, Length[rs]}],
 PlotLabel -> StringForm["Trajectories for x0 = ``", x0],
 ImageSize -> Medium, LabelStyle -> Directive[Black, 14],
 GridLines -> Automatic, PlotRange -> {{0, dep}, {0, 1}},
 PlotTheme -> "Detailed", BaseStyle -> {FontSize -> 10}]
```



3.3.28

For $E \subset \mathbb{R}$, what conditions must E satisfy, such that every closed covering of E contains a finite covering of E.
Solution: Since sets of a single element are closed, E must be finite, and this is clearly sufficient.

3.3.29

Prove that any open set $U \subset \mathbb{R}$ can be written as the disjoint countable union of open intervals.
Proof: Every connected set of $\mathbb{R}$ is an interval, hence every connected part of U is an open interval, and there are only countably many parts.

3.3.30

A closed interval $[a, b]$ cannot be written as the disjoint countable union of closed intervals.
Proof: Suppose $[a, b] = \bigcup_{\alpha \in I}[l_\alpha, r_\alpha]$, let $J = \{l_\alpha, r_\alpha\} \backslash \{a, b\}$. Clearly J is a perfect set, hence I is uncountable.

3.3.31

$\mathbb{R}$ is not the countable union of nowhere dense sets.
Proof: $\mathbb{R}$ is a complete metric space, so by Baire Category theorem $\mathbb{R}$ is of second category, hence not the countable union of nowhere dense sets.

3.3.32

$\{V_k\}_{k \geqslant 1}$ are open dense sets in $\mathbb{R}$, prove that $\bigcap_{k \geqslant 1} V_k$ is dense in $\mathbb{R}$.
Baire Category theorem: If X is a complete metric space, $G_n \subset X$ are all open, dense sets, then $\bigcap_{n \geqslant 1} G_n$ is dense.
(Or if F_n are closed and has no interior, then $\bigcup_{n \geqslant 1} F_n$ has no interior)

Proof: For any $x \in X$ and $\varepsilon > 0$, we show that $\bigcap_{n \geqslant 1} G_n \cap B(x, \varepsilon) \neq \emptyset$. Let $x_0 = x, \varepsilon_0 = \varepsilon$.
Take $x_1 \in G_1 \cap B(x_0, \varepsilon_0/2)$, then since G_1 is open we can take $\varepsilon_1 < \varepsilon_0/2$ such that $B(x_1, \varepsilon_1) \subset G_1 \cap B(x_0, \varepsilon_0/2)$. Likewise take $x_n \in G_n \cap B(x_{n-1}, \varepsilon_{n-1}/2)$, and $\varepsilon_n < \varepsilon_{n-1}/2$ such that $B(x_n, \varepsilon_n) \subset G_n \cap B(x_{n-1}, \varepsilon_{n-1}/2)$. Note that $d(x_n, x_{n-1}) < \varepsilon_{n-1}/2$, and $\varepsilon_n \leqslant \varepsilon_0 2^{-n}$, so $d(x_n, x_{n+m}) < \varepsilon_0 2^{-n}$ and $\{x_n\}$ is Cauchy. Let $x^* = \lim_{n\to\infty} x_n$, then for any n, $d(x_n, x^*) \leqslant d(x_n, x_{n+m}) + d(x_{n+m}, x^*) < \varepsilon_n/2 + d(x_{n+m}, x^*)$. Hence $d(x_n, x) \leqslant \varepsilon_n/2$ so $x^* \in G_n$, and likewise $d(x_0, x^*) < \varepsilon_0$. Therefore $x^* \in \bigcap_{n \geqslant 1} G_n \cap B(x, \varepsilon)$.

3.3.33

Prove that ρ is a metric iff $\rho(x, y) = 0 \iff x = y$ and $\forall x, y, z \in X$, $\rho(x, z) \leqslant \rho(y, x) + \rho(y, z)$.
Proof: $\implies$ is trivial. $\impliedby$: let $y = z$, then $\rho(x, y) \leqslant \rho(y, x)$ for any $x, y \in X$. Interchange x, y then $\rho(x, y) \geqslant \rho(y, x)$ so $\rho(x, y) = \rho(y, x)$. Hence $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

3.4

3.4.1

If $x_n > 0$ and $\overline{\lim}_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} x_n^{-1} = 1$, prove that $\{x_n\}$ converges.
Proof: Clearly $\limsup_{n\to\infty} x_n^{-1} = (\liminf_{n\to\infty} x_n)^{-1}$, hence $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ so $\{x_n\}$ converges.

3.4.2 & 3.4.3

Suppose $\{x_n\}$ satisfy $0 \leqslant x_{m+n} \leqslant x_m \cdot x_n$, prove that $\{\sqrt[n]{x_n}\}$ converges, and give an example where $\{\sqrt[n]{x_n}\}$ is not monotonic.
Proof: Let $L = \inf \sqrt[n]{x_n}$, we show that $L = \lim_{n\to\infty} \sqrt[n]{x_n}$. Denote $y_n = \sqrt[n]{x_n}$.
If $L = 0$, then there is a subsequence $\lim_{n\to\infty} y_{k_n} = 0$. For any $m \geqslant 1$, and any $q = k_n$, suppose $m = tq + r$ where $r \in \{0, 1, \cdots, q-1\}$, then $y_m^m \leqslant y_q^{tq} \cdot y_r^r$ so $\lim_{m\to\infty} y_m \leqslant y_q$. Let $q \to \infty$ then $\lim_{m\to\infty} y_m = 0$.
The case $L > 0$ is similar.
Example: $x_n = \begin{cases} 1, & x \text{ even} \\ 2, & x \text{ odd} \end{cases}$, then $x_{m+n} = 2$ implies one of m, n is odd so $x_{m+n} \leqslant x_m \cdot x_n$.

3.4.4

Suppose $\{a_n\}$ satisfy $a_m + a_n - 1 \leqslant a_{m+n} \leqslant a_m + a_n + 1$. Prove that
(1) $\lim_{n\to\infty} a_n/n$ exists.
(2) Suppose $\lim_{n\to\infty} a_n/n = q$, then $nq - 1 \leqslant a_n \leqslant nq + 1$.
Proof: Let $b_n = a_n/n$, take $L = \inf b_n$, and $\lim_{n\to\infty} b_{k_n} = L$. For any $m \geqslant 1$ and $q = k_n$, let $m = tq + r$ where $0 \leqslant r < q$, then

$$ta_q + a_r - t \leqslant a_m \leqslant ta_q + a_r + t$$

so $\frac{tq}{m} b_q + \frac{r}{m} b_r - \frac{t}{m} \leqslant b_m \leqslant \frac{tq}{m} b_q + \frac{r}{m} b_r + \frac{t}{m}$. Let $m \to \infty$, then $\lim_{m\to\infty} b_m = b_q + 1/q$ so $\lim_{m\to\infty} b_m = L$.
(2) Note that $ma_n - (m-1) \leqslant a_{mn} \leqslant ma_n + (m-1)$, hence let $m \to \infty$ we obtain $a_n \in [nq - 1, nq + 1]$.

3.4.5

Suppose $r \in (0,1)$, $x_0 = 0, x_{n+1} = r(1-x_n^2)$. Determine whether $\{x_n\}$ converges.
Solution: Let $f(x) = r(1-x^2)$, $g(x) = f(f(x))$, then note that $g(x)$ is monotonically increasing. Since $x_0 = 0$ and $x_1 = r$, we know that x_{2n} is increasing while x_{2n+1} is decreasing. Let $a = \lim_{n\to\infty} x_{2n}$ and $b = \lim_{n\to\infty} x_{2n+1}$, then $\{x_n\}$ converges iff $a = b$.
If $r \in (0, \sqrt{3}/2]$, then $g(x)$ has only one fixed point in $(0,1)$, hence $\{x_n\}$ converges to $\frac{\sqrt{1+4r^2}-1}{2r}$. If $r \in (\sqrt{3}/2, 1)$, then $a = \frac{1-\sqrt{4r^2-3}}{2r}$ and $b = \frac{1+\sqrt{4r^2-3}}{2r}$ so x_n does not converge.

3.4.6 & 3.4.7

Suppose $0 < q < 1$, $\{a_n\}, \{b_n\}$ satisfy $a_n = b_n - qa_{n+1}$, and a_n, b_n are bounded. Prove that $\lim_{n\to\infty} b_n$ exists iff $\lim_{n\to\infty} a_n$ exists. What if $q \notin (0,1)$?
Proof: If $\lim_{n\to\infty} b_n$ exists, let $\lim_{n\to\infty} a_n + qa_{n+1} = \lim_{n\to\infty} b_n = a$. Let $c_n = a_n - a/(1+q)$, $d_n = c_n + qc_{n+1}$ then $\lim_{n\to\infty} d_n = 0$ and c_n is bounded.
Note that

$$c_n + (-q)^m c_{n+m} = \sum_{j=0}^{m-1} d_{n+j}(-q)^j$$

hence if $M = \sup_{n\geqslant 1} |c_n|$ and $\varepsilon(n) = \sup_{k\geqslant n} |d_k|$, then

$$|c_n| \leqslant q^m M + \frac{\varepsilon(n)}{1-q}.$$

Let $m \to \infty$, we obtain $|c_n| \leqslant \varepsilon(n)/(1-q)$ hence $c_n \to 0$.
The reverse is trivial.
The proof works for $q \in (-1,0)$ too, since we can let $a_n' = (-1)^n a_n$. $q = 0$ is trivial.
If $q = 1$, let $a_n = (-1)^n$ and $b_n = 0$, then $\lim_{n\to\infty} b_n = 0$ but $\lim_{n\to\infty} a_n$ doesn't exist.
If $q = -1$, then $a_n = a_1 - \sum_{j=1}^n b_j$. Let $a_0 = 0$, $b_{2^n+k} = \frac{1}{2} - \frac{k}{2^n}$ for any $1 \leqslant k \leqslant 2^n - 1$, and $b_{2^n} = 0$, then $\lim_{n\to\infty} b_n = 0$ but $a_{2^n} = 0$ and $a_{2^n+2^{n-1}} = 1$ hence $\lim_{n\to\infty} a_n$ doesn't exist.
If $|q| > 1$, likewise define c_n and d_n, then

$$|c_{n+m}| = \left| -c_n(-q)^{-m} + \sum_{j=0}^{m-1} d_{n+j}(-q)^{m-j} \right| \leqslant \frac{|c_n|}{|q|^m} + \frac{\varepsilon(n)}{1-|q|^{-1}}.$$

So likewise $c_n \to 0$.

3.4.8

Suppose $a_n \geqslant 0$ and $a_{n+1} \leqslant a_n + \frac{1}{n^2}$. Prove that $\{a_n\}$ converges.
Proof: Let L be a limit point of $\{a_n\}$. Suppose $L = \lim_{n\to\infty} a_{k_n}$. For any $m \geqslant 1$, assume $k_n \leqslant m < k_{n+1}$, then $a_m \leqslant a_{k_n} + \sum_{j=k_n}^{m-1} j^{-2} \leqslant a_{k_n} + 1/k_n \leqslant L + \varepsilon(n) + 1/k_n$, and $a_m \geqslant a_{k_{n+1}} - \sum_{j=m}^{k_{n+1}-1} j^{-2} \geqslant a_{k_{n+1}} - 1/m \geqslant L - \varepsilon(n) - 1/m$, where $\varepsilon(n) \to 0$ since $a_{k_n} \to L$. Hence $\lim_{n\to\infty} a_n = L$.

3.4.12

Suppose $x_n > 0$, prove that

$$\limsup_{n\to\infty} n\left(\frac{1+x_{n+1}}{x_n}-1\right) \geqslant 1.$$

Proof: If $\sup_{k\geqslant n} k\left(\frac{1+x_{k+1}}{x_k}-1\right)=\lambda<1$ for some n, then $k(x_{k+1}-x_k+1)\leqslant \lambda x_k$ so $x_{k+1}\leqslant(1+\lambda/k)x_k-1$. Let $y_k=x_k/k$, then $y_{k+1}\leqslant\frac{k+\lambda}{k+1}y_k-\frac{1}{k+1}<y_k-\frac{1}{k+1}$. Since $\sum_{n=1}^{\infty}n^{-1}=\infty$, this contradicts with $y_k>0$.

3.4.13

(Improved Banach fixed point theorem) Suppose $f:\mathbb{R}\to\mathbb{R}$ has period 1, and $\forall x,y\in\mathbb{R}$, $x\neq y$ implies $|f(x)-f(y)|<|x-y|$. For any $x_0\in\mathbb{R}$, let $x_{n+1}=f(x_n)$. Prove that $\{x_n\}$ converges and the limit is independent of x_0.

Proof: Take any limit point L of $\{x_n\}$, and suppose $x_{k_n}\to L$, then $f(L)=\lim_{n\to\infty}f(x_{k_n})=\lim_{n\to\infty}x_{k_n+1}$ is another limit point.

Note that $|x_{k_n}-x_{k_n+1}|\geqslant|x_{k_n+1}-x_{k_n+2}|\geqslant\cdots\geqslant|x_{k_{n+1}}-x_{k_{n+1}+1}|$. Let $n\to\infty$ we obtain $|L-f(L)|\geqslant|f(L)-f^2(L)|\geqslant|L-f(L)|$ so $|L-f(L)|=|f(L)-f^2(L)|$, which implies $f(L)=L$. Hence the only limit point is the unique fixed point a, which does not depend on x_0.