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1.

Prove that for any $a < b$, the interval (a, b) contains a rational number and an irrational number.
Proof: Let $q > 2/(b-a)$ and $p = \lfloor qa \rfloor + 1$, then $bq - aq > 2$ so $p \in (aq, bq)$ i.e. $p/q \in (a, b)$. Hence (a, b) contains a rational number. Apply this to the interval $(a/\sqrt{2}, b/\sqrt{2})$, then (a, b) contains an irrational number of the form $\sqrt{2}p/q$.

2.

Given an ellipsis Γ with foci F_1, F_2.
(1) Suppose a light ray is emitted from point A_1 on Γ, that intersects F_1F_2 and then Γ at A_2. After reflecting on Γ, it again intersects Γ at points $A_3, A_4, \cdots$, prove that A_nA_{n+1} always intersect F_1F_2.
Proof: It suffices to show that A_2A_3 intersects F_1F_2. Note that the line A_2A_2 bisects both angles $\angle F_1A_2F_2$ and $\angle A_1A_2A_3$, hence A_2A_3 intersects F_1F_2 iff A_1A_2 intersects F_1F_2.
(2) is similar

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1.

Prove that the Cantor set is uncountable but has measure 0.
Proof: The Cantor set

$$\mathcal{C} = \left\{ \sum_{i=1}^{\infty} a_n 3^{-n} : a_n \in \{0, 2\} \right\}$$

is uncountable since it can be mapped bijectively to $(0, 1)$ by $\sum_{i=1}^{\infty} a_n 3^{-n} \mapsto \sum_{i=1}^{\infty} b_n 2^{-n}$ where $b_n = a_n/2$. By the construction of $\mathcal{C}$ we know $m(\mathcal{C}) = 0$, where m is the Lebesgue measure.

2.

Prove that any infinite set contains a countable subset.
Proof: For any infinite set X, let x_1 be an arbitrary element of X, and define x_n inductively by $x_n \in X \backslash \{x_1, x_2, \cdots, x_{n-1}\}$. x_n always exist since X is infinite, so X has a countable subset $\{x_1, x_2, \cdots\}$.

3.

Prove that the union of a countable set and an uncountable set is uncountable.
Proof: Let A be countable and B uncountable. If there is an injective map $f : A \cup B \to \mathbb{N}$, then let $g : B \to \mathbb{N}$, $b \mapsto f(b)$ is injective, leading to contradiction.

4.

Suppose S is countable, and $f : S \to X$ is surjective, then X is countable.
Proof: Consider the injective mapping $g : S \to \mathbb{N}$, then $\varphi : X \to \mathbb{N}$, $x \mapsto \min g(f^{-1}(\{x\}))$ is injective, so X is countable.

5.

Prove that any interval of positive length on $\mathbb{R}$ is uncountable.
Proof: An unbounded interval contains a bounded interval. Any bounded open interval (a, b) can be mapped bijectively to $(0, 1)$ by $x \mapsto (x - a)/(b - a)$, hence it is uncountable. By problem3, any interval is uncountable.

6.

Construct metrics with radius r and diameter d such that $d = r$, and $d < r$.
6.7(i): Consider the discrete metric $d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ on $\mathbb{Z}$.
6.7(ii): Consider the metric $d(x, y) = \begin{cases} 1, & xy < 0 \\ 0, & xy > 0 \end{cases}$ on $\mathbb{Z} \backslash \{0\}$.