PSA

A1) Construct continuous functions $f_n, f \in C([0,1])$, such that for every $x \in [0,1]$, when $n \to \infty$, $f_n(x) \to f(x)$, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Solution: Let $f_n(x) = nxe^{-nx^2}$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2} \int_0^1 e^{-nx^2}\,d(x^2) = \lim_{n\to\infty} \frac{n}{2}\left(\frac{1}{n} - \frac{1}{ne^n}\right) = \frac{1}{2}$$

Hence $\lim_{n\to\infty} \int_0^1 f_n = 1/2 \neq 0 = \int_0^1 f$.

A2) $\alpha \in \mathbb{R}_{\geqslant 0}$. Prove that $\int_{100}^{\infty} \frac{dx}{x\log^\alpha(x)}$ converges iff $\alpha > 1$.

Proof: Substitute $y = \log x$, then

$$\int_{100}^{\infty} \frac{\mathrm{d}x}{x\log^\alpha(x)} = \int_{\log 100}^{\infty} \frac{\mathrm{d}y}{y^\alpha}$$

which converges iff $\alpha > 1$.

A3) f, F are defined on I, and for every bounded closed interval $J \subset I$, f, F are both Riemann integrable on J. Assume for all $x \in I$, $|f(x)| \leqslant F(x)$. Then if the improper integer of F on I converges, so does f.

Proof: This is because

$$\int_I f(x)\,\mathrm{d}x \text{ converges } \iff \forall\varepsilon > 0 \exists N \forall u, v \in I, N < u < v, |\int_u^v f(x)\,\mathrm{d}x| < \varepsilon.$$

A4) Prove the integrals below converge:

(1) $\int_0^\infty e^{-x^2}\,\mathrm{d}x$ (2) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}}$ (3) $\int_1^\infty \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x$

(1):

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x \leqslant 1 + \int_1^\infty e^{-x}\,\mathrm{d}x \leqslant 1 + \frac{1}{e}.$$

(2):

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}} \leqslant \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} = 2.$$

(3):

$$\int_1^\infty \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x \leqslant 5000 + \int_{100}^\infty \frac{1}{x(\log x)^3}\,\mathrm{d}x, \text{ which converges by A2.}$$

A5) Prove the series below converge:

(1) $\sum_{n=1}^{\infty} e^{-n}(n^2+\log n)$ (2) $\sum_{n=1}^{\infty} \frac{\log n}{1+n(\log n)^3}$

(1):

$$\sum_{n=1}^{\infty} e^{-n}(n^2+\log n) \leqslant \sum_{n=1}^{\infty} \frac{2n^2}{e^n} \leqslant 2\int_0^{\infty} x^2 e^{-x}\,\mathrm{d}x = 4.$$

(2):

$$\sum_{n=1}^{\infty} \frac{\log n}{1+n(\log n)^3} \leqslant \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \leqslant \frac{1}{2(\log 2)^2} + \int_2^{\infty} \frac{1}{x(\log x)^2}\,\mathrm{d}x \leqslant 3.$$

A6) Calculate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{n^{\alpha+1}}, \alpha > -1.$$

Solution:

$$\sum_{k=1}^{n} k^\alpha \leqslant \int_1^{n+1} x^\alpha\,\mathrm{d}x = \frac{1}{\alpha+1}((n+1)^{\alpha+1}-1).$$
$$\sum_{k=1}^{n} k^\alpha \geqslant 1 + \int_1^{n} x^\alpha\,\mathrm{d}x = 1 + \frac{1}{\alpha+1} n^{\alpha+1}.$$

Therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{n^{\alpha+1}} = \frac{1}{\alpha+1}.$$

A7) Calculate $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x$, to show that $\pi = 3.14\cdots$.

Solution:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x = \int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\,\mathrm{d}x = \frac{22}{7} - \pi.$$
$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \leqslant \int_0^1 \frac{x^3(1-x)^4}{2}\,\mathrm{d}x = \frac{1}{560} < 0.02, \frac{22}{7} > 3.1428.$$

A8) Assume $a, b, n \in \mathbb{Z}$, let

$$f_{a,b;n} = \frac{x^n(a-bx)^n}{n!}.$$

- Prove that for $k = 0, 1, \cdots, 2n$, $f_{a,b;n}^{(k)}(x) \in \mathbb{Z}$ when $x = 0, \frac{a}{b}$.
  See B10)
- If $\pi = \frac{a}{b} \in \mathbb{Q}$, then for every $n \in \mathbb{N}$,

$$\int_0^\pi f_{a,b;n}(x) \sin x\,\mathrm{d}x$$

is an integer.
Proof: By Darboux's formula of integration of parts

$$\int_0^\pi f_{a,b;n}(x) \sin x\,\mathrm{d}x = \sum_{k=0}^{2n} f_{a,b;n}^{(k)}(x) \sin\left(x - \frac{(k+1)\pi}{2}\right)\Big|_0^\pi \in \mathbb{Z}.$$

- Prove that $\pi \notin \mathbb{Q}$.
  Proof: Let $n = 2a^4 + 10$, then $\forall 0 \leqslant x \leqslant a/b$,

$$f_{a,b;n} \leqslant \frac{a^{2n}}{n!} < \frac{1}{2}\frac{(a^4)^{n/2}}{n \cdot (n-1) \cdots \left(\frac{n}{2}\right)} < \frac{1}{2}.$$

Hence

$$0 < \int_0^\pi f_{a,b;n}(x) \sin x \, \mathrm{d}x < \frac{1}{2}\int_0^\pi \sin x \, \mathrm{d}x = 1,$$

leading to contradiction.

A9) Let $I_n = \int_0^{\pi/2} \sin^n x \, \mathrm{d}x$, prove that $I_n \sim \sqrt{\frac{\pi}{2n}}$.

Proof: Since $I_n = \frac{n-1}{n} I_{n-2}$,

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ is even}, \\ \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is odd}. \end{cases}$$

Combined with $I_{2n+1} < I_{2n} < I_{2n-1}$, we get

$$\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n},$$

where

$$0 < -\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} + \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n(2n+1)} < \frac{\pi}{4n}.$$

Therefore

$$\lim_{n\to\infty} \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \frac{\pi}{2}.$$

Hence $I_n \sim \sqrt{\frac{\pi}{2n}}$.

A10) Assume $f : [0,1] \to [0,1]$ is monotonously increasing, $g = f^{-1} : [0,1] \to [0,1]$ is its inverse, and f, g are both continuously differentiable, then

$$\int_0^1 f(x) \, \mathrm{d}x + \int_0^1 g(x) \, \mathrm{d}x = 1.$$

Proof: We show that

$$\int_0^x f(t) \, \mathrm{d}t + \int_0^{f(x)} g(t) \, \mathrm{d}t = x f(x), \forall 0 \leqslant x \leqslant 1. (1)$$

$x = 0$ is trivial, hence it suffices to show that the derivatives of the two sides match.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t) \, \mathrm{d}t = f(x), \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{f(x)} g(t) \, \mathrm{d}t = f'(x) \cdot g(f(x)) = x f'(x).$$

Hence (1) holds.

A11) Prove that

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

Proof: By Dirichlet's test, $\sum_{k=0}^{\infty}(-1)^k(1-\varepsilon)^{2k+1}/(2k+1)$ converges uniformly.
Hence for any $\delta>0$, there exists an $N\in\mathbb{Z}$ such that

$$|\sum_{k=N}^{\infty}\frac{(-1)^k x^{2k}}{2k+1}|<\delta, \forall x\in[0,1].$$

Then $\forall\varepsilon<\frac{\delta}{N}$,

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}-\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\right|\\ \leqslant&\sum_{k=0}^{N-1}\frac{|(1-\varepsilon)^{2k+1}-1|}{2k+1}+\left|\sum_{k=N}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}\right|+\left|\sum_{k=N}^{\infty}\frac{(-1)^k}{2k+1}\right|<3\delta.\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

A12) For any continuous function $f:[a,b]\times[c,d]\to\mathbb{R},(x,y)\mapsto f(x,y)$, where $a,b,c,d\in\mathbb{R}$, show that f is uniformly continuous on $[a,b]\times[c,d]$.

Proof: $K=[a,b]\times[c,d]$ is a compact set. Consider an arbitrary $\varepsilon>0$.
For any $x\in K$, there is an open ball $B(x,2r_x)$ with center x such that
$\forall y\in B(x,2r_x), |f(x)-f(y)|<\varepsilon/2$. Let $O_x=B(x,r_x)$. Note that $\bigcup_{x\in K}O_x=K$ and K is compact, hence we can find $x_1,\cdots,x_n$ such that $\bigcup_{k\leqslant n}O_{x_k}=K$.
Let $\delta=\min\{r_{x_k}:k\leqslant n\}$, then $\forall|u-v|<\delta$, suppose $u\in O_{x_1}$, then

$$|v-x_1|\leqslant|v-u|+|u-x_1|<2r_{x_1}\implies v\in B(x_1,2r_{x_1}).$$

Hence

$$|f(u)-f(v)|\leqslant|f(u)-f(x_1)|+|f(v)-f(x_1)|<\varepsilon.$$

Therefore f is uniformly continuous on K.

PSB: On $\zeta(2)$

Part 1: The sequence $\{\sum_{k=1}^{n}1/k^p\}$

Define the sequence $S_n(p)=\sum_{k=1}^{n}1/k^p$ where $p\in\mathbb{Z}_{\geqslant 1}$.

B1) Prove that for any $k\in\mathbb{Z}_{\geqslant 1}$, we have

$$\frac{1}{(k+1)^p}\leqslant\int_k^{k+1}\frac{1}{x^p}\,\mathrm{d}x\leqslant\frac{1}{k^p}.$$

Proof: $\frac{1}{(k+1)^p}\leqslant\frac{1}{x^p}\leqslant\frac{1}{k^p},\forall k\leqslant x\leqslant k+1.$

B2) Prove that for any $n \in \mathbb{Z}_{\geqslant 2}$, we have

$$S_n(p) - 1 \leqslant \int_1^n \frac{1}{x^p}\,\mathrm{d}x \leqslant S_{n-1}(p).$$

Proof:

$$S_n(p) - 1 = \sum_{k=1}^{n-1} \frac{1}{(k+1)^p} \leqslant \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x^p}\,\mathrm{d}x = \int_1^n \frac{1}{x^p}\,\mathrm{d}x.$$

Likewise we have $\int_1^n \frac{1}{x^p}\,\mathrm{d}x \leqslant S_{n-1}(p)$.

B3) Let $p \in \mathbb{Z}_{\geqslant 1}$. Prove that $x \mapsto \frac{1}{x^p}$ is integrable on $[1, \infty)$ iff $p \geqslant 2$.

Proof: For $p \geqslant 2$,

$$\lim_{n\to\infty} \int_1^n \frac{1}{x^p}\,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^n = \frac{1}{1-p}.$$

If $p = 1$, $\lim_{n\to\infty} \int_1^n \frac{1}{x}\,\mathrm{d}x = \lim_{n\to\infty} \log x\big|_1^n = \infty$.

B4) Prove that $\{S_n(p)\}_{n\geqslant 1}$ converges iff $p \geqslant 2$. For $p \geqslant 2$ let

$$\zeta(p) = \lim_{n\to\infty} S_n(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Proof: If $p = 1$, $S_n(p) \geqslant \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \to \infty$.
For $p \geqslant 2$, $S_n(p) \leqslant S_{n+1}(p)$, and $S_n(p) \leqslant 1 + \int_1^n \frac{1}{x^p}\,\mathrm{d}x \leqslant 1 + \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$. Hence $\lim_{n\to\infty} S_n(p)$ exists.

Part 2: Calculate $\zeta(2)$

(We can also use Bernolli numbers and the Taylor expansion of $\tan x$).

Let $h(t) = \frac{t^2}{2\pi} - t$, $\varphi : [0, \pi] \to \mathbb{R}$:

$$\varphi(x) = \begin{cases} -1, & x = 0; \\ \frac{h(x)}{2\sin\left(\frac{x}{2}\right)}, & 0 < x \leqslant \pi. \end{cases}$$

B5) Prove that $\varphi \in C^1([0, \pi])$.

Proof:

$$\lim_{x\to 0} \frac{h(x)}{2\sin\left(\frac{x}{2}\right)} = \lim_{x\to 0} \frac{-x + o(x)}{2\sin\left(\frac{x}{2}\right)} = -1 = \varphi(0).$$

Hence $\varphi \in C^1([0, \pi])$.

B6) For all $k \geqslant 1$, calculate

$$\int_0^\pi h(x)\cos(kx)\,\mathrm{d}x.$$

Solution:

$$
\begin{aligned}
\int_0^\pi \left(\frac{x^2}{2\pi}-x\right)\cos(kx)\,\mathrm{d}x &= \frac{1}{k}\int_0^\pi \left(\frac{x^2}{2\pi}-x\right)\mathrm{d}\sin(kx)\\
&= -\frac{1}{k}\int_0^\pi \sin(kx)\left(\frac{x}{\pi}-1\right)\mathrm{d}x\\
&= \frac{1}{k^2}\int_0^\pi \left(\frac{x}{\pi}-1\right)\mathrm{d}\cos(kx)\\
&= \frac{1}{k^2}-\frac{1}{\pi k^2}\int_0^\pi \cos(kx)\,\mathrm{d}x=\frac{1}{k^2}.
\end{aligned}
$$

B7) Prove that there is a constant λ, such that for any $x\in(0,\pi)$,

$$
\sum_{k=1}^n \cos(kx)=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}-\lambda.
$$

Proof: Note that $2\cos(kx)\sin\left(\frac{x}{2}\right)=\sin(k+1/2)x-\sin(k-1/2)x$, hence

$$
\sum_{k=1}^n \cos(kx)\cdot 2\sin\frac{x}{2}=\sin\left(n+\frac{1}{2}\right)x-\sin\frac{x}{2},\lambda=\frac{1}{2}.
$$

B8) Prove that for any $\psi\in C^1([0,\pi])$,

$$
\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x\,\mathrm{d}x=0.
$$

Proof: Since $\sin(n+1/2)x=c_1\sin nx+c_2\cos nx$, where c_1,c_2 are constant, it suffices to show that

$$
\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x=\lim_{n\to\infty}\int_0^\pi \psi(x)\cos(2nx)\,\mathrm{d}x=0.
$$

Note that

$$
\begin{aligned}
\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x &= \sum_{k=1}^n \int_{(k-1)\pi/n}^{k\pi/n}\psi(x)\sin(2nx)\,\mathrm{d}x\\
&= \sum_{k=1}^n \frac{1}{2n}\int_0^{2\pi}\psi\left(\frac{x}{2n}+\frac{(k-1)\pi}{n}\right)\sin x\,\mathrm{d}x\\
\left(t=\frac{(k-1)\pi}{n}\right) &\leqslant \sum_{k=1}^n \frac{\pi}{n}\sup_{x\leqslant\pi}\left|\psi\left(\frac{x+\pi}{2n}+t\right)-\psi\left(\frac{x}{2n}+t\right)\right|\\
&\leqslant \pi \sup_{0\leqslant x\leqslant \pi-\pi/2n}\left|\psi\left(x+\frac{\pi}{2n}\right)-\psi(x)\right|\to 0.
\end{aligned}
$$

since ψ is uniformly continuous on $[0,\pi]$.

B9) Prove that $\zeta(2)=\frac{\pi^2}{6}$.

Proof:

$$
\begin{aligned}
\zeta(2) &= \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2}=\lim_{n\to\infty}\sum_{k=1}^n\int_0^\pi h(x)\cos(kx)\,\mathrm{d}x\\
&= \lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x-\frac{1}{2}\left(\frac{x^2}{2\pi}-x\right)\mathrm{d}x\\
(\mathrm{B8}) &= \frac{1}{2}\int_0^\pi\left(x-\frac{x^2}{2\pi}\right)\mathrm{d}x=\frac{\pi^2}{6}.
\end{aligned}
$$

Part 3: $\zeta(2)$ is irrational

Otherwise assume $\pi^2 = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

B10) Define a sequence of polynomials $f_n(x) = \frac{x^n(1-x)^n}{n!}$, where $n \in \mathbb{Z}_{\geqslant 1}$. Prove that for any $k \in \mathbb{Z}$, $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

Proof: If $k \leqslant n-1$, then $f_n^{(k)}(0) = f_n^{(k)}(1) = 0$. If $k \geqslant n$, then

$$\text{if } x^n(1-x)^n = \sum_{k=n}^{2n} c_k x^k, \text{then } f_n^{(k)}(x) = \sum_{m=n}^{2n} c_k \binom{m}{k} x^{m-k} \in \mathbb{Z}[x].$$

Hence $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

B11) Define the sequence

$$F_n(x) = b^n(\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)).$$

Prove that $F_n(0), F_n(1) \in \mathbb{Z}$.
Proof: For $0 \leqslant k \leqslant n$, $b^n \pi^{2n-2k}, f_n^{(2k)}(x) \in Z,$ when $x \in \{0,1\}$.

B12) For $n \geqslant 1$, define $\{g_n\}_{n\geqslant 1}, \{A_n\}_{n \geqslant 1}$ as below:

$$g_n(x) = F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x),\ A_n = \pi \int_0^1 a^n f_n(x)\sin(\pi x)\,\mathrm{d}x.$$

Prove that $A_n \in \mathbb{Z}$ and $g_n' = \pi^2 a^n f_n(x)\sin(\pi x)$.
Proof: Note that

$$\begin{aligned} g_n'(x) &= b^n \pi^{2n} \sum_{k=0}^{n} \left(f_n^{(2k)}(x)\sin(\pi x) - \pi f_n^{(2k+1)}(x)\cos(\pi x)\right)'(-\pi^2)^k \\ &= b^n \pi^{2n+2} f_n(x)\sin(\pi x). \end{aligned}$$

And

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^1 \mathrm{d}g_n(x) = \frac{1}{\pi}(g_n(1) - g_n(0)) \\ &= F_n(0) + F_n(1) \in \mathbb{Z}. \end{aligned}$$

B13) Prove that there exists $n \in \mathbb{Z}$ such that for all $x \in [0,1]$, $a^n f_n(x) < 1/2$.

Proof:

$$f_n(x) = \frac{1}{n!}(x(1-x))^n \leqslant \frac{1}{n!4^n} \to 0.$$

B14) Prove that there exists $n \in \mathbb{Z}$ such that $A_n \in (0,1)$, leading to contradiction.

Proof: $f_n, \sin(\pi x) \geq 0$, when $x \in [0,1]$, hence $A_n > 0$.
Take n such that $a^n f_n < 1/2$ then $A_n < \frac{\pi}{2}\int_0^1 \sin(\pi x)\,\mathrm{d}x = 1$.
Therefore $A_n \in (0,1)$, contradicting with $A_n \in \mathbb{Z}$.

PSC: Calculation of Integrals

$a \neq 0, b \neq 0$

(1) $\int_0^\pi \sin^3 x \, \mathrm{d}x$

$$\int_0^\pi \sin^3(x)\,\mathrm{d}x = -2\int_0^{\pi/2} \sin^2(x)\,\mathrm{d}\cos(x) = 2\int_0^1 (1-x^2)\,\mathrm{d}x = \frac{4}{3}.$$

(2) $\int_{-\pi}^{\pi} x^2 \cos x \, \mathrm{d}x$

$$\int_{-\pi}^{\pi} x^2 \cos(x)\,\mathrm{d}x = (x^2-2)\sin(x) + 2x\cos(x)\Big|_{-\pi}^{\pi} = -4\pi.$$

(3) $\int_0^1 \frac{x}{1+\sqrt{1+x}} \, \mathrm{d}x$

$$\int_0^1 \frac{x}{1+\sqrt{1+x}}\,\mathrm{d}x = \int_0^1 \sqrt{1+x} - 1\,\mathrm{d}x = \frac{2}{3}(1+x)^{3/2} - x\Big|_0^1 = \frac{4\sqrt{2}-5}{3}.$$

(4) $\int_0^{\sqrt{3}} x \arctan x \, \mathrm{d}x$

$$\begin{aligned}\int_0^{\sqrt{3}} x\arctan x\,\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{3}} \arctan x\,\mathrm{d}x^2 \\ &= \frac{1}{2}x^2\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,\mathrm{d}x \\ &= \frac{3}{2}\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\arctan\sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.\end{aligned}$$

(5) $\int_{-1}^{0} (2x+1)\sqrt{1-x-x^2} \, \mathrm{d}x$

$$\int_{-1}^{0} (2x+1)\sqrt{1-x-x^2}\,\mathrm{d}x = \int_{-1}^{1} \frac{y}{4}\sqrt{5-y^2}\,\mathrm{d}y = 0$$

(6) $\int_{\frac{1}{e}}^{e} |\log x| \, \mathrm{d}x$

$$\begin{aligned}\int_{\frac{1}{e}}^{e} |\log x|\,\mathrm{d}x &= \int_1^e \log x\,\mathrm{d}x + \int_1^{1/e} \log x\,\mathrm{d}x \\ &= (x\log x - x)\Big|_1^e + (x\log x - x)\Big|_1^{1/e} = 2 - \frac{2}{e}\end{aligned}$$

(7) $\int_0^a x^2\sqrt{a^2-x^2} \, \mathrm{d}x$

$$\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x = a^4\int_0^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{a^4\pi}{16}$$

(8) $\int_0^{\log 2} \sqrt{e^x-1} \, \mathrm{d}x$

$$\begin{aligned}\int_0^{\log 2} \sqrt{e^x-1}\,\mathrm{d}x &= \int_1^2 \frac{\sqrt{y-1}}{y}\,\mathrm{d}y = \int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \\ &= \int_0^{\pi/4} 2\tan^2\theta\,\mathrm{d}\theta = 2 - \frac{\pi}{2}.\end{aligned}$$

(9) $\int_1^2 x^{100} \log x \, \mathrm{d}x$

$$\int_1^2 x^{100} \log x \, \mathrm{d}x = \int_1^2 \log x \, \mathrm{d}\frac{x^{101}}{101} = \frac{2^{101}\log 2}{101} - \int_1^2 \frac{x^{100}}{101}\,\mathrm{d}x$$

$$= \frac{2^{101}\log 2}{101} - \frac{2^{101}-1}{101^2}.$$

(10) $\int_0^a \log\left(x+\sqrt{x^2+a^2}\right)\mathrm{d}x$

$$\int_0^a \log\left(x+\sqrt{x^2+a^2}\right)\mathrm{d}x =$$

$$\begin{aligned}\int_0^a \log\left(x+\sqrt{x^2+a^2}\right)\mathrm{d}x &= a\int_0^1 \log a + \log\left(t+\sqrt{t^2+1}\right)\mathrm{d}t \\ &= a\log a + a\int_0^1 \log\left(t+\sqrt{t^2+1}\right)\mathrm{d}t \\ &= a\log a + (\log\left(1+\sqrt{2}\right) + \sqrt{2} - 1)a.\end{aligned}$$

$$\begin{aligned}\int_0^1 \log\left(x+\sqrt{x^2+1}\right)\mathrm{d}x &= x\log\left(x+\sqrt{x^2+1}\right)\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x \\ (x=\tan\theta) &= \log\left(1+\sqrt{2}\right) + \int_0^{\pi/4}\frac{1}{\cos^2\theta}\,\mathrm{d}\cos\theta \\ &= \log\left(1+\sqrt{2}\right) + \sqrt{2} - 1.\end{aligned}$$

(11) $\int_0^{\pi/2} \frac{\cos x \sin x}{a^2\sin^2 x + b^2\cos^2 x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x \sin x}{a^2\sin^2 x + b^2\cos^2 x} &= \int_0^{\pi/2}\frac{\sin 2x}{a^2+b^2+(b^2-a^2)\cos 2x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1}\frac{1}{(a^2+b^2)+(b^2-a^2)t}\,\mathrm{d}t \\ &= \frac{1}{2(a^2-b^2)}\log\left(\frac{a^2}{b^2}\right).\end{aligned}$$

(12) $\int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\,\mathrm{d}x &= \int_0^{\pi/2}\frac{\sin^2 x}{\cos x + \sin x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\pi/2}\frac{1}{\cos x + \sin x}\,\mathrm{d}x \\ &= \int_{\pi/4}^{\pi/2}\frac{1}{\sqrt{2}\sin x}\,\mathrm{d}x \\ &= -\frac{\log\tan\left(\frac{\pi}{8}\right)}{\sqrt{2}}.\end{aligned}$$

(13) $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x}\,\mathrm{d}x$

See (12)

(14) $\int_0^{\pi/4} \log\left(1+\tan x\right) \mathrm{d}x$

$$\begin{aligned}
\int_0^{\pi/4} \log\left(1+\tan x\right) \mathrm{d}x &= \int_0^{\pi/4} \log\frac{\sin x+\cos x}{\cos x}\,\mathrm{d}x \\
&= \int_0^{\pi/4} \log\frac{\sqrt{2}\sin\left(x+\pi/4\right)}{\cos x}\,\mathrm{d}x \\
&= \frac{\pi}{8}\log 2.
\end{aligned}$$

(15) $\int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\mathrm{d}x$

$$\begin{aligned}
\int_0^1 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{1-x}{5-2x}\,\mathrm{d}x = \frac{1}{2}-\frac{3}{4}\log\frac{5}{3}, \\
\int_1^2 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x}{3-2x}\,\mathrm{d}x = -\frac{1}{2}+\frac{3}{4}\log 3, \\
\int_2^3 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 (x+1)\,\mathrm{d}x = \frac{3}{2}, \\
\int_3^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x+2}{2x+1}\,\mathrm{d}x = \frac{1}{2}+\frac{3}{4}\log 3, \\
\implies \int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= 2+\frac{3}{4}\log\frac{27}{5}
\end{aligned}$$

(16) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$

$$\begin{aligned}
\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x &= \int_0^\pi x\,\mathrm{d}\arctan\cos x \\
&= -x\arctan\cos x\Big|_0^\pi + \int_0^\pi \arctan\cos x\,\mathrm{d}x \\
&= \frac{\pi^2}{4}+0 = \frac{\pi^2}{4}.
\end{aligned}$$

(17) $\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\sin t-\cos t}{\sin t}\,\mathrm{d}t \\
&= \frac{\pi}{8}-\frac{\log 2}{4}.
\end{aligned}$$

(18) $\int_0^{\pi/2} \frac{\sin 2019x}{\sin x}\,\mathrm{d}x$

$$\int_0^{\pi/2} \frac{\sin\left(2n+1\right)x}{\sin x}\,\mathrm{d}x = \int_0^{\pi/2} 1+\sum_{k=1}^n \cos\left(2kx\right)\mathrm{d}x = \frac{\pi}{2}.$$

(19) $\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x$

$$\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x = \int_{-1}^1 \frac{\log\sqrt{6+t}}{\log\sqrt{6+t}+\log\sqrt{6-t}}\,\mathrm{d}x = 1.$$

(20) $\int_0^1 \frac{1}{\sqrt{1+x^2}+\sqrt{1-x^2}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1+x^2}+\sqrt{1-x^2}} &= \int_0^1 \frac{1}{2}(\sqrt{1+x^2}-\sqrt{1-x^2})\,\mathrm{d}x \\ &= -\frac{\pi}{8}+\frac{\sqrt{2}}{4}+\frac{1}{8}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}.\end{aligned}$$

(21) $\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x &= \int_1^{1+\sqrt{2}} \sqrt{y}\,\mathrm{d}\frac{2y+1-\sqrt{4y+5}}{2} \\ &= \int_1^{1+\sqrt{2}} \sqrt{y}-\frac{\sqrt{y}}{\sqrt{4y+5}}\,\mathrm{d}y \\ \left(y=\frac{z^2-5}{4}\right) &= \frac{2}{3}y^{3/2}\Big|_1^{1+\sqrt{2}} - \int_3^{1+2\sqrt{2}} \frac{\sqrt{z^2-5}}{4}\,\mathrm{d}z \\ &= \frac{2}{3}((1+\sqrt{2})^{3/2}-1)-\frac{3\sqrt{2}-1}{8}+\frac{5}{32}\log\frac{3+\sqrt{2}}{5}.\end{aligned}$$

(22) $\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x$

$$\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x = 0.\text{(by symmetry)}$$