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周民强《实变函数论（第三版）》:P78 思考题 5；P84 思考题 1；P86 思考题 2；P94-95 习题2： 1， 8， 9， 15，16

78-5

Suppose $E \subset \mathbb{R}$, and $0 < \alpha < m(E)$. Prove that there exists a compact subset $F \subset E$ such that $m(F) = \alpha$.
Proof: Take $\varepsilon \in (0, m(E) - \alpha)$, then there exists a closed set $F \subset E$ such that $m(E \backslash F) < \varepsilon$, so $m(F) > \alpha$. Let $f(x) = m(F \cap [-x, x])$ for all $x \geqslant 0$, then since $F \cap [-x, x]$ is compact, f is well-defined, monotonic, and

$$f(x+\varepsilon) - f(x) = m(F \cap [-x-\varepsilon, x+\varepsilon]) - m(F \cap [-x, x]) \leqslant 2\varepsilon$$

so f is continuous. Since $f(0) = 0$, $\lim_{x \to \infty} f(x) = m(F)$, and $\alpha \in (0, m(F))$, there exists $x \in \mathbb{R}_{>0}$ such that $m(F \cap [-x, x]) = \alpha$. Hence $F \cap [-x, x] \subset E$ is bounded and has measure α.

84-1

Suppose $E \subset \mathbb{R}^n$, and $m^*(E) < \infty$. If

$$m^*(E) = \sup\{m(F) : F \subset E \text{ is compact}\}.$$

Prove that E is measurable.
Proof: For any $\varepsilon > 0$ there exists $F \subset E$ compact such that $m^*(E) \leqslant m(F) + \varepsilon$, so $m^*(E \backslash F) \leqslant \varepsilon$, hence E is measurable.

86-2

Suppose $E \subset \mathbb{R}$ is measurable, $a \in \mathbb{R}, \delta > 0$. If for any $|x| < \delta$, either $a + x \in E$ or $a - x \in E$, prove that $m(E) \geqslant \delta$.
Proof: Let $E' = -E + 2a$, then $(a - \delta, a + \delta) \subset E \cup E'$, so
$2\delta = m((a-\delta, a+\delta)) \leqslant m(E \cup E') \leqslant 2m(E)$, hence $m(E) \geqslant \delta$.

94-1

Suppose $E \subset \mathbb{R}$, and there exists $q \in (0, 1)$ such that for any (a, b) there exists a sequence $\{I_n\}$ of open intervals,

$$E \cap (a, b) \subset \bigcup_{n=1}^{\infty} I_n,\ \sum_{n=1}^{\infty} m(I_n) < (b-a)q,$$

prove that $m(E) = 0$.
Proof: We can suppose E is bounded by considering $E \cap [-M, M]$. If $m^*(E) > 0$, take $0 < \varepsilon < \frac{m^*(E)}{q^{-1}-1}$, then there exists open intervals C_k such that $E \subset \bigcup_{k \geqslant 1} C_k$, and $\sum_{k=1}^{\infty} |C_k| \leqslant m^*(E) + \varepsilon$. For any $k \geqslant 1$, take open intervals $I_n^{(k)}$ such that $E \cap C_k \subset \bigcup_{n \geqslant 1} I_n^{(k)}$ and $\sum_{n \geqslant 1} |I_n^{(k)}| < q|C_k|$. Then

$$E \subset \bigcup_{k \geqslant 1} C_k \cap E \subset \bigcup_{k \geqslant 1} \bigcup_{n \geqslant 1} I_n^{(k)},$$

so $m^*(E) \leqslant \sum_{k \geqslant 1} \sum_{n \geqslant 1} |I_n^{(k)}| < q \sum_{k \geqslant 1} |C_k| < q(m^*(E) + \varepsilon) < m^*(E)$, leading to contradiction.

94-8

Suppose $\{E_k\}$ are measurable sets in $[0,1]$, and $m(E_k)=1$. Prove that

$$m\left(\bigcap_{k=1}^{\infty} E_k\right)=1.$$

Proof: Note that $m(E_k)<\infty$, so

$$m\left(\bigcap_{k=1}^{\infty} E_k\right)=\lim_{n\to\infty} m\left(\bigcap_{k=1}^{n} E_k\right).$$

We only need to show that $\bigcap_{k=1}^{n} E_k$ has measure 1. Clearly

$$m\left(\bigcap_{k=1}^{n} E_k\right)\geqslant m([0,1])-\sum_{k=1}^{n} m(E_k^C)=1.$$

So $m\left(\bigcap_{k=1}^{\infty} E_k\right)=1$. (Or use the fact that the countable union of null sets are still null sets.)

94-9

Suppose $E_1,\cdots,E_k$ are measurable sets in $[0,1]$, and

$$\sum_{i=1}^{k} m(E_i)>k-1.$$

Prove that $m\left(\bigcap_{i=1}^{k} E_i\right)>0$.
Proof: Suppose $F_i=E_i^C$, then $\sum_{i=1}^{k} m(F_i)=\sum_{i=1}^{k} 1-m(E_i)<1$, so $m\left(\bigcup_{i=1}^{k} F_i\right)\leqslant\sum_{i=1}^{k} m(F_i)<1$. Hence $m\left(\bigcap_{i=1}^{k} E_i\right)=m([0,1])-m\left(\bigcup_{i=1}^{k} F_i\right)>0$.

95-15

Suppose $E\subset[0,1]$ is measurable, and

$$m(E)\geqslant\varepsilon>0,\ x_i\in[0,1], i=1,2,\cdots,n.$$

where $n>2/\varepsilon$. Prove that E contains a pair of points whose distance is equal to the distance of a pair of points in $\{x_1,\cdots,x_n\}$.
Proof: Otherwise, the sets $E+x_1,E+x_2,\cdots,E+x_n$ are disjoint subsets of $[0,2]$, hence

$$m([0,2])\geqslant m\left(\bigcup_{k=1}^{n}(E+x_k)\right)=\sum_{k=1}^{n} m(E+x_k)=n\cdot m(E),$$

which contradicts with $m(E)\geqslant\varepsilon>2/n$.

95-16

Suppose W is an unmeasurable subset of $[0,1]$. Prove that there exists $\varepsilon\in(0,1)$ such that for any measurable set $E\subset[0,1]$ with $m(E)\geqslant\varepsilon$, $W\cap E$ is unmeasurable.
Proof: Otherwise suppose for any $\varepsilon=1-1/n$, there exists $E_n\subset[0,1]$ measurable such that $m(E_n)\geqslant\varepsilon$ and $W\cap E_n$ is measurable.

Let $E = \bigcup_{n \geqslant 1} E_n$, then $E \subset [0,1]$ is measurable and $m(E) \geqslant m(E_n) \geqslant 1 - 1/n$ so $m(E) = 1$. Note that $W \cap E = \bigcup_{n \geqslant 1} (W \cap E_n)$ is the countable union of measurable sets, so $W \cap E$ is measurable. Also $m^*(W \cap E^C) = 0$ since E^C is a null set, so $W = (W \cap E) \cup (W \cap E^C)$ is measurable, leading to contradiction.