33-5

Suppose $F \subset \mathbb{R}$ is compact, $f(x) : F \to \mathbb{R}$. If for any $x_0 \in F'$, $f(x) \to \infty$ ($x \in F$ and $x \to x_0$), prove that F is countable.
Proof: Let $F_n = \{x \in F : f(x) \leqslant n\}$ then $F = \bigcup_{n \geqslant 1} F_n$. If F_n is infinite for some n, then F_n is bounded so $F_n' \neq \emptyset$. Take $x_n \to x \in F_n'$ where $x_m \in F_n$, then $x \in F'$ so $f(x_m) \to \infty$, contradicting $f(x_m) \leqslant n$. Hence F_n is finite so F is countable.

45-4

(i) $\chi_{\mathbb{Q}}(x)$ is not the limit of a sequence of continuous functions.
Proof: We show that the point-wise limit of a sequence of continuous functions $\{f_n(x)\}$ is continuous on a dense set.
Let $f^+(x) = \inf_{\varepsilon > 0} \sup_{y \in (x-\varepsilon, x+\varepsilon)} f(y)$ and $f^-(x) = \sup_{\varepsilon > 0} \inf_{y \in (x-\varepsilon, x+\varepsilon)} f(y)$, then f is continuous at x iff $f^+(x) = f^-(x)$, and $f^-(x) \leqslant f^+(x)$. Let $G_n = \{x \in \mathbb{R} : f^+(x) - f^-(x) < 1/n\}$ then $\bigcap_{n \geqslant 1} G_n$ are all continuous points of f.
By Baire Category theorem, we only need to show that G_n is open and dense. Clearly $f^+, -f^-$ is upper semi-continuous so G_n is open. Now we prove that $G_n = \{x \in \mathbb{R} : \omega_f(x) < 1/n\}$ is dense.
Consider any open set $O \neq \emptyset$. Let $E_m = \{x \in \mathbb{R} : \forall s, t \geqslant m, |f_s(x) - f_t(x)| \leqslant 1/n\}$, then $E_m = \bigcap_{s,t \geqslant m} \{x \in \mathbb{R} : f_s(x) - f_t(x) \in [-1/n, 1/n]\}$ is closed, and $X = \bigcup_{m \geqslant 1} E_m$ since $\{f_m(x)\}_{m \geqslant 1}$ is Cauchy. So $O = \bigcap_{m \geqslant 1} (O \cap E_m)$ is a non-empty open set, and by Baire Category O is of second category, so there is an E_m and an open set $U \subset G \cap E_m$. For any $x \in U$ and $s, t \geqslant m$, $|f_s(x) - f_t(x)| \leqslant 1/n$ so $|f_s(x) - f(x)| \leqslant 1/n$. Take $U' \subset U$ open such that $\forall x, y \in U'$, $|f_m(x) - f_m(y)| \leqslant 1/n$, then $|f(x) - f(y)| \leqslant |f(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f(x)| \leqslant 3/n$. Therefore $U' \subset O \cap G_{n/3}$. Hence G_n is dense.

50-1

Assume $E \subset \mathbb{R}$ is a perfect set, prove that for any $x \in E$ there exists $y \in E$ such that $x - y \in \mathbb{Q}^C$.
Proof: Otherwise if there exists $x \in E$ such that $x - y \in \mathbb{Q}$ for all $y \in E$, then $E \subset x + \mathbb{Q}$ is countable.

54-7

Let $f : [0,1] \to \mathbb{R}$, and there is a constant M such that for any $n \in \mathbb{N}$ and $x_1, x_2, \cdots, x_n \in [0,1]$,

$$|f(x_1) + \cdots + f(x_n)| \leqslant M$$

Prove that $E = \{x \in [0,1] : f(x) \neq 0\}$ is countable.
Proof: Let $E_n = \{x \in [0,1] : |f(x)| > 1/n\}$, then $E = \bigcup_{n \geqslant 1} E_n$. Clearly, every E_n is finite, since $E_n^+ = \{x \in E_n : f(x) > 1/n\}$ and $E_n^- = \{x \in E_n : f(x) < -1/n\}$ are both finite, otherwise there are infinite x such that $|f(x)| > 1/n$ and $f(x)$ have the same sign. Take $x_1, \cdots, x_{2Mn}$ then $|f(x_1) + \cdots + f(x_{2Mn})| > M$, a contradiction. Hence E_n is finite so E is countable.

55-11

Let $\{f_\alpha(x)\}_{\alpha\in I}$ be real valued functions on $[a,b]$ where I is infinite. If there exists $M>0$ such that

$$|f_\alpha(x)|\leqslant M, x\in[a,b], \alpha\in I.$$

Prove that for any countable subset $E\subset[a,b]$, there exists $\{f_{\alpha_n}(x)\}$ such that the limit $\lim_{n\to\infty} f_{\alpha_n}(x)$ exists for any $x\in E$.
Proof: Let $E=\{x_1,x_2,\cdots\}$, $I_{m,k}=[Mk2^{-m}, M(k+1)2^{-m}]$ for $-2^m\leqslant k\leqslant 2^m-1$. Define α_n and $k_n^{(l)}, 1\leqslant l\leqslant n$ inductively such that $I_{n,k_n^{(l)}}\subset I_{n-1,k_{n-1}^{(l)}}$ for all $l\leqslant n-1$, and $\alpha_n\in\{\alpha\in I:\forall l\leqslant n, f_\alpha(x_l)\in I_{n,k_n^{(l)}}\}$ which is an infinite set (there are only finite choices of $\{k_n^{(l)}\}_{1\leqslant l\leqslant n}$ so we can choose one such that the set is infinite). Then for any $l\leqslant n$, $f_{\alpha_n}(x_l)$ and $f_{\alpha_{n+m}}(x_l)$ lie in the same interval $I_{n,k_n^{(l)}}$, which implies $|f_{\alpha_n}(x_l)-f_{\alpha_m}(x_l)|\leqslant M2^{-n}$, so for any $x_l\in E$, $\{f_{\alpha_n}(x_l)\}_{n\geqslant 1}$ is Cauchy hence has a limit.

55-19

Suppose for any $a<b$, and $y\in(f(a),f(b))$ (or $y\in(f(b),f(a))$), there exists $c\in(a,b)$ such that $f(c)=y$. If for any $r\in\mathbb{Q}$, $\{x\in\mathbb{R}: f(x)=r\}$ is closed, prove that $f\in C(\mathbb{R})$.
Proof: Otherwise if f is discontinuous at x_0, i.e. there exists $\varepsilon>0$ such that for any $\delta>0$, there exists $y\in(x_0-\delta,x_0+\delta)$ such that $|f(x_0)-f(y)|>\varepsilon$. Either $f(y)>f(x_0)+\varepsilon$ or $f(y)<f(x_0)-\varepsilon$ so we can assume that there is a sequence $x_n\to x_0$ such that $f(x_n)>f(x_0)+\varepsilon$. Take $r\in\mathbb{Q}\cap(f(x_0),f(x_0)+\varepsilon)$, then there is a sequence $y_m\to x_0$ such that $f(y_m)=r$. Since $f^{-1}(\{r\})$ is closed, $x_0\in f^{-1}(\{r\})$, leading to contradiction.

56-27

Let $\{F_\alpha\}$ are compact sets in $\mathbb{R}^n$. If for any $\alpha_1,\cdots,\alpha_n$,

$$\bigcap_{i=1}^m F_{\alpha_i}\neq\emptyset,$$

prove that $\bigcap_\alpha F_\alpha\neq\emptyset$.
Proof: Let $G_\alpha=F_\alpha^C$ which is an open set. Suppose $\bigcap_{\alpha\in I}F_\alpha=\emptyset$, then take $\beta\in I$, $F_\beta\subset\bigcup_{\alpha\in I\setminus\{\beta\}}G_\beta$. Since F_β is compact, there is a finite subset $J\subset I\setminus\{\beta\}$ such that $F_\beta\subset\bigcup_{\alpha\in J}G_\alpha$ i.e. $\bigcap_{\alpha\in J\cup\{\beta\}}F_\alpha=\emptyset$, a contradiction. Hence $\bigcap_{\alpha\in I}F_\alpha\neq\emptyset$.

57-35

Prove that there does not exist $f(x,y)$ such that
(i) $f\in C(\mathbb{R}^2)$; (ii) $\frac{\partial}{\partial x}f$, $\frac{\partial}{\partial y}f$ exists on $\mathbb{R}^2$; (iii) f is not differentiable on all of $\mathbb{R}^2$.
Proof: Note that $\frac{\partial}{\partial x}f(x,y)=\lim_{n\to\infty} n(f(x+1/n,y)-f(x,y))$ is the limit of continuous functions. In 45-4(i) we proved that the continuous points of $\frac{\partial}{\partial x}f$ is a dense G_δ set, and so is that of $\frac{\partial}{\partial y}f$. By Baire Category theorem, their intersection is also a dense set, so we can take $(x,y)\in\mathbb{R}^2$ such that $\frac{\partial}{\partial x}f$, $\frac{\partial}{\partial y}f$ both exist at (x,y). Hence f is differentiable at x, leading to contradiction.