6.1: 2, 3-8 >= 5道; 6.2: 1的两小题,2,3,4-9 >= 3道;
6.3: 1-5,7-9每题>=2小题. 其余选>=6道.
6.4: >=3题. 6.5: 1-9>=8题,10,11

6.1

6.1.2

尺规作图画出 $f(x)$ 在 $(t, f(t))$ 处的切线:
(1) $f(x)=x^k$ (2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (3) $f(x)=\sin x$ (4) $f(x)=\tan x$ (5) $f(x)=e^x$ (6) $f(x)=\log x$.
引理：尺规作图可以做加法、数乘有理数，且有单位长度时，可以做乘法、倒数、开根. 且做切线只需作出斜率的长度.
证明：加法：拼接两条线段即可。数乘正整数：若干次加法。
除正整数：对于线段 AB, 过 A 作射线 AC_1, 并在射线上取 $C_2,\cdots,C_n$ 使得 $AC_k=kAC_1$. 过 C_1 做 BC_n 的平行线交 AB 于 D 则 $AD=AB/n$. 故可以数乘任意有理数。
下面设已有单位长度:
倒数：作以 $x,1$ 为直角边的直角三角形 $ABC, AC=1, BC=x$, 作 $AD\perp AB$ 交 BC 于 D 则 $CD=1/x$.
乘法：做 $OA=a, OB=b$, 在 OB 上取 E 使得 $OE=1$, 过 B 作 AE 平行线交 OA 于 D, 则 $OD=ab$.
开根：设射线 OA 上 $OA=a, OB=1$, 以 OA 为直径作圆，过 B 作 OA 垂线交圆于 C, 则 $BC=\sqrt{a}$.
若要作切线, $l: y=f'(t)(x-t)+f(t)$, 只需画出 $f'(t)$, 则会有 $(t,f(t))$ 和 $(0,-tf'(t)+f(t))$, 连接可得到 l.
原题:
(1) $f'(x)=kx^{k-1}=k\cdot x\cdots x, -tf'(t)+f(t)=(1-k)t^k$ 可以作出.
(2) 用Pascal定理.
(3) 找到正半轴首个根 π, 取中点得到 $\pi/2$, 再做坐标轴垂线得到单位长度. $f'(t)=\cos t=\sqrt{1-\sin^2 t}$ 可以作出.
(4) 同样通过根找到 π, 再用 $\tan\frac{\pi}{4}=1$ 找到单位长度. $f'(t)=\sec^2 t=1+\tan^2 t$ 可以作出.
(5) $f(0)=1$ 是单位长度, $f'(t)=e^t$ 可以作出.
(6) 唯一一个根是单位长度 1, $f'(x)=1/x$ 可以作出.

6.1.3

Suppose f is differentiable at x_0, calculate $\lim_{h\to 0}(f(x_0+h)-f(x_0-h))/h$.
Solution: Note that

$$\frac{f(x_0+h)-f(x_0-h)}{h}=\frac{f(x_0+h)-f(x_0)}{h}+\frac{f(x_0)-f(x_0-h)}{h}$$

hence the limit is $2f'(x_0)$.

6.1.4

Suppose f is continuous at 0 and $\lim_{x\to 0}(f(2x)-f(x))/x=k$, prove that $f'(0)=k$.
Proof: Let $f(2x)-f(x)=xk+xh(x)$, then $h(x)\to 0$, and

$$f(x)=f(2^{-n}x)+\sum_{k=0}^{n-1}2^{-k-1}xh(2^{-k-1}x)$$

For any $\varepsilon>0$ there exists $\delta>0$ such that $x<\delta$ implies $-\varepsilon<h(x), f(x)<\varepsilon$, then

$$\left|\frac{f(x)-f(0)}{x}-k\right|\leqslant\left|\frac{f(2^{-n}x)-f(0)}{x}-k\right|+\sum_{k=0}^{n-1}2^{-k-1}|h(2^{-k-1}x)|\leqslant\left|\frac{f(2^{-n}x)-f(0)}{x}-k\right|+4\varepsilon.$$

Let $n \to \infty$ then $|(f(x) - f(0))/x - k| < 4\varepsilon$. Hence $f'(0) = k$.

6.1.5

Prove that if $f(x)$ is differentiable and even/odd/periodic, then $f'(x)$ is odd/even/periodic.
Proof: If $f(x) = -f(-x)$, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = f'(x).$$

If $f(x) = f(-x)$, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} -\frac{f(x) - f(x-h)}{h} = -f'(x).$$

If $f(x) = f(x+T)$ then

$$f'(x+T) = \lim_{h\to 0} \frac{f(x+T+h) - f(x+T)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

6.1.7

Suppose $g(0) = g'(0) = 0$, and

$$f(x) = \begin{cases} g(x)\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine the value of $f'(0)$.
Solution:

$$f'(0) = \lim_{h\to 0} \frac{g(h)}{h} \sin\frac{1}{h} = 0,$$

since $g(h)/h \to 0$ and $\sin\frac{1}{h}$ is bounded.

6.1.8

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y), \forall x, y \in \mathbb{R}$. If $f'(0) = 1$, prove that $f'(x) = f(x)$ for any $x \in \mathbb{R}$.
Proof: $f'(0) = 1$ so f is not constant. Hence let $x = 0$ we have $f(y) = f(0)f(y)$ so $f(0) = 1$. Note that $f(x) = f(x/2)^2 \geqslant 0$, so we can let $g(x) = \log f(x)$, then $g'(x) = \frac{f'(x)}{f(x)}$ and $g'(0) = 1, g(0) = 0$, $g(x+y) = g(x) + g(y)$. Since g is continuous at 0, $g(x) = cx$ so $g(x) = x \forall x \in \mathbb{R}$, hence $g'(x) = 1$ and $f'(x) = f(x)$.

6.2

6.2.1

(1) $f(x) = x^{1/3}$, $f'_+(0)$:

$$f'_0(x) = \lim_{h\to 0^+} \frac{h^{1/3}}{h} = \infty.$$

(2) $f(x) = |x-2|^3$, $f'_-(2)$:

$$f'_-(2) = \lim_{h \to 0^+} \frac{-h^3}{h} = 0.$$

6.2.2

Determine the values of a, b such that

$$f(x) = \begin{cases} x^2, & x \geqslant 1, \\ ax + b, & x < 1 \end{cases}$$

is differentiable at $x = 1$.
Solution: Clearly $f'_+(1) = 2$, while

$$f'_-(1) = \lim_{h \to 0^+} \frac{f(1) - f(1-h)}{h} = \lim_{h \to 0^+} \frac{1 - a(1-h) - b}{h} = 2$$

Hence $a = 2$ and $1 - a - b = 0$, so $(a, b) = (2, -1)$.

6.2.3

Suppose f is continuous at 0. Prove that $\lim_{x \to 0} \frac{f(x) - f(0)}{x}$ exists if $\lim_{x \to 0} \frac{f(x) - f(x - x^3)}{x^3}$ exists.
Proof: Let $f(x) - f(0) = xg(x)$, and $f(x) - f(x - x^3) = x^3 h(x)$.
If $\lim_{x \to 0} \frac{f(x) - f(x - x^3)}{x^3} = L$ exists, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $|h(x) - L| < \varepsilon$. Consider $x = x_0 \in (0, \delta)$, and $x_{k+1} = x_k - x_k^3$, then clearly $\lim_{n \to \infty} x_n = 0$. Note that

$$xg(x) = \sum_{k=0}^{\infty} f(x_k) - f(x_{k+1}) = \sum_{k=0}^{\infty} x_k^3 S(x_k) < (L + \varepsilon) \sum_{k=0}^{\infty} x_k - x_{k+1} = (L + \varepsilon) x$$

and likewise $xg(x) > (L - \varepsilon)x$, hence $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to 0} g(x) = L$.
(The reverse is incorrect, for example $x^2 \sin x^{-1}$).

6.2.4

Calculate the Dini derivatives of f at $x = 0$:
(1) $f(x) = x^2 \sin x^{-1}$; (2) $f(x) = |x|^{3/2}$; (3) $f(x) = e^{-|x|^2}$.
Solution: (1) $|f(x)| \leqslant |x|^2$, so all four Dini derivatives are 0.
(2)(3) Likewise all four Dini derivatives are 0.

6.2.5

Calculate the Dini derivatives of f:
(1) $f(x) = \begin{cases} x^3, & x \geqslant 0, \\ -x^3, & x < 0 \end{cases}$ at $x = 0$:
(2) $f(x) = \sin x \cos x$ at $x = \pi$:
(3) $f(x) = x^2 \log|x|$ at $x = 0$:
Solution:
(1) $|f(x)/x| \leqslant |x^2|$ so all Dini derivatives are 0.
(2) $f'(\pi) = \cos 2\pi = 1$ so all Dini derivatives are 1.
(3) $|f(x)/x| = |x| \log|x| \to 0$ so all Dini derivatives are 0.

6.2.6

Calculate the symmetric derivatives of f:
(1) $f(x)=\begin{cases}x^3, & x\geqslant 0,\\ -x^3, & x<0\end{cases}$ at $x=0$:
(2) $f(x)=\sin x^2$ at $x=\pi$:
(3) $f(x)=\log|x^2-1|$ at $x=1$:
(4) $f(x)=x^2e^{-|x|}$ at $x=0$:
Solution: (1) $f(x)-f(-x)=0$ so the derivative is 0.
(2) f is differentiable so the symmetric derivative is $f'(\pi)=2x\cos x^2|_\pi=2\pi\cos\pi^2$.
(3) $f(1+t)-f(1-t)=\log|2t+t^2|-\log|2t-t^2|=\log(2+t)/(2-t)$, and

$$\lim_{h\to 0}\frac{\log|2+h|-\log|2-h|}{2h}=\frac{1}{2}.$$

So the derivative is $1/2$.
(4) f is even so the derivative is clearly 0.

6.3

Calculate derivatives:

6.3.1

If $f=\prod_{k=1}^n g_k$, then

$$\frac{f'}{f}=\sum_{k=1}^n\frac{g_k'}{g_k}.$$

(1) $f(x)=(x+1)(x+2)^2(x+3)^2$

$$f'(x)=f(x)\left(\frac{1}{x+1}+\frac{2}{x+2}+\frac{2}{x+3}\right).$$

(2) $f(x)=(1+nx^m)(1+mx^n)$

$$f'(x)=f(x)nm\left(\frac{x^{m-1}}{1+nx^m}+\frac{x^{n-1}}{1+mx^n}\right).$$

(3) $f(x)=(ax^m+b)^n(cx^n+d)^m$

$$f'(x)=nmf(x)\left(\frac{ax^{m-1}}{ax^m+b}+\frac{cx^{n-1}}{cx^n+d}\right).$$

6.3.2

(1) $f(x)=\sqrt{\sin^2 x}$
$f'(x)=\operatorname{sign}(\sin x)\cos x$ when $x\neq k\pi$.
(2) $f(x)=\frac{x}{(1-x^2)(1+x)}$.

$$f'(x)=\frac{2x^2-x+1}{(1-x)^2(1+x)^3}.$$

(3) $f(x)=x^p(1-x)^q/(1+x)$

$$f'(x) = -\frac{x^{p-1}(1-x)^{q-1}((p+q-1)x^2+(q+1)x-p)}{(x+1)^2}.$$

(4) $f(x) = \frac{x\sin x+\cos x}{x\sin x-\cos x}$.

$$f'(x) = -\frac{2x+\sin 2x}{(x\sin x-\cos x)^2}.$$

6.3.3

(1) $f(x) = \frac{x}{\sqrt{a^2+x^2}}$: $f'(x) = a^2(a^2+x^2)^{-3/2}$.
(2) $f(x) = \arcsin\left(\cos^2 x\right)$: $f'(x) = -\sin 2x/\sqrt{1-\cos^4 x}$.
(3) $f(x) = \arcsin x\sqrt{1-x^2}$: $f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}\sqrt{x^4-x^2+1}}$.
(4) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$: $f'(x) = \frac{1+2\sqrt{x}+4\sqrt{x+\sqrt{x}}\sqrt{x}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$.
(5) $f(x) = e^{ax}\sin bx$: $f'(x) = e^{ax}(a\sin bx + b\cos bx)$.

6.3.4

(1) $f(x) = (x\sin x)^x$: $f'(x) = (x\sin x)^x(x\cot x + \log(x\sin x)+1)$.
(2) $f(x) = x\sqrt{(1-x)/(1+x)}$: $f'(x) = \frac{1-x-x^2}{(1+x)^{3/2}\sqrt{1-x}}$.
(3) $f(x) = (x)^{a^x}$: $f'(x) = a^x x^{a^x-1}(x\log a\log x+1)$.
(4) $f(x) = a^{x^n}$: $f'(a^{x^n}nx^{n-1}\log a)$.
(5) $f(x) = (\cos x)^{\sin x} + (\sin x)^{\cos x}$:

$$f'(x) = (\cos x)^{\sin x}(\cos x\log\cos x - \sin x\tan x) + (\sin x)^{\cos x}(\cos x\cot x - \sin x\log\sin x).$$

(6) $f(x) = (x+\sqrt{1+x^2})^n$: $f'(x) = \frac{n(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}}$.

6.3.5

Suppose $f(x) = \cos\left(2\arctan\left(\sin\left(\operatorname{arccot}\sqrt{(1-x)/x}\right)\right)\right)$, prove that $\frac{f'(x)}{f(x)^2} = \pm\frac{2}{(1\pm x)^2}$.
Proof: $f(x) = \frac{1-x}{1+x}$ so $\frac{f'}{f^2} = -\frac{2}{(1-x)^2}$.

6.3.6

Suppose $y = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} - \frac{1}{2\sqrt{2}}\arctan\frac{\sqrt{2}x}{x^2-1}$, calculate $y'(x)$.
Solution:

$$y'(x) = \frac{1}{x^4+1}.$$

6.3.7

Determine dy/dx:
(1) $\begin{cases} x = \cos^4 t, \\ y = \sin^4 t \end{cases}$, at $t = \pi/3$:
$dx/dt = 4\cos^3 t(-\sin t) = -\frac{\sqrt{3}}{4}$, $dy/dt = 4\sin^3 t\cos t$, so $dy/dx = -3$.
(2) $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$, at $t = \pi/2, \pi$:

$\frac{dx}{dt} = a(1-\cos t)$, $\frac{dy}{dt} = a\sin t$.
$t = \frac{\pi}{2}$: $\frac{dy}{dx} = 1$. $t = \pi$: $\frac{dy}{dx} = 0$.

6.3.8

Suppose f is differentiable, calculate y':
(1) $y = f(x\sin x)$: $y' = (\sin x + x\cos x) f'(x\sin x)$.
(2) $y = f(e^x)e^{f(x)}$: $y' = f'(e^x)e^{x+f(x)} + f(e^x)e^{f(x)}f'(x)$.

6.3.9

Suppose u, v differentiable, calculate y':
(1) $y = \sqrt{u^2+v^2}$: $y' = \frac{uu'+vv'}{\sqrt{u^2+v^2}}$.
(5) $y = 1/\sqrt{u^2+v^2}$: $y' = -\frac{u'u+v'v}{(u^2+v^2)^{3/2}}$.

6.3.10

Suppose $a_{ij}(x)$ are differentiable, determine the derivative of $\det\big(a_{ij}(x)\big)_{n\times n}$.
Solution: Let $\varphi_i(x) = (a_{i1}(x), \cdots, a_{in}(x))$, then $\det(\varphi_1, \cdots, \varphi_n)$ is a multi-linear form, so

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} \det(\varphi_1, \cdots, \varphi_n) = \sum_{k=1}^{n} \det\big(\varphi_1, \cdots, \varphi_k', \cdots, \varphi_n\big).$$

6.3.14

Suppose $x_1, \cdots, x_n$ are distinct, and $y(x) = \prod_{k=1}^{n}(x-x_k)$. Calculate $\sum_{k=1}^{n} 1/y'(x_k)$. What if x_k are not distinct?
Solution: Note that

$$\frac{y'}{y} = \sum \frac{1}{x-x_k},$$

hence $y'(x_k) = \prod_{j\neq k}(x_k - x_j)$.
By Lagrange interpolation,

$$1 = \sum_{k=1}^{n} \prod_{j\neq k} \frac{x-x_j}{x_k-x_j},$$

so by considering the coefficient of x^{n-1}, we obtain

$$\sum_{k=1}^{n} 1/y'(x_k) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

If $f(x) = \prod_{k=1}^{n}(x-x_k)^{\alpha_k}$, then $f'(x_k) = \alpha_k \prod_{j\neq k}(x_k-x_j)^{\alpha_j}$, hence

$$\sum \frac{1}{f'(z_k)} = \sum_{k=1}^{n} \frac{\alpha_k}{f'(x_k)} = \sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(x_k-x_j)^{\alpha_j}} = \begin{cases} 1, & \deg f = 1, \\ 0, & \deg f > 1. \end{cases}$$

6.3.19

Suppose $f(x_0) = g(x_0)$, f, g are differentiable at x_0, and $f'(x_0) = g'(x_0) = k$. Prove that for any h such that $\min\{f, g\} \leqslant h \leqslant \max\{f, g\}$, h is differentiable at x_0 and $h'(x_0) = k$.
Proof: Clearly $h(x_0) = f(x_0)$ and we can assume it to be 0. For any t,

$$\min\left\{\frac{f(x_0+t)}{t}, \frac{g(x_0+t)}{t}\right\} \leqslant \frac{h(x_0+t)-h(x_0)}{t} = \frac{h(x_0+t)}{t} \leqslant \max\left\{\frac{g(x_0+t)}{t}, \frac{f(x_0+t)}{t}\right\}.$$

Since $\lim_{t\to 0} f(x_0+t)/t = \lim_{t\to 0} g(x_0+t)/t$, the two sides of the inequality both tend to k, therefore $\lim_{t\to 0}(h(x_0+t) - h(x_0))/t = k$. Hence $h'(x_0) = k$.

6.3.22

Prove that there is no $f : \mathbb{R} \to \mathbb{R}$ differentiable such that $f \circ f(x) = -x^3 + x^2 + 1$.
Proof: $f'(f(x))f'(x) = -3x^2 + 2x$, and $f(f(x)) = x \iff x = 1$. If $a = f(1)$ then $f(f(a)) = f(f(f(1))) = f(1) = a$, so by uniqueness $a = 1$. However, $-1 = f'(f(1))f'(1) = (f'(1))^2$, leading to contradiction.

6.3.23

Prove that there is no $f : \mathbb{R} \to \mathbb{R}$ differentiable such that $f \circ f(x) = x^2 - 3x + 3$.
Proof: Suppose $f(f(x)) = x^2 - 3x + 3$, then $f'(f(x))f'(x) = 2x - 3$, but $f(f(x)) = x \iff x = 1, 3$.
Suppose $a = f(1)$, then $f(a) = 1$, so $-1 = f'(f(1))f'(1) = f'(a)f'(1)$, and
$2a - 3 = f'(f(a))f'(a) = f'(a)f'(1)$, leading to contradiction.
(Note: $x^2 - 3x + 3$ is not the iteration of any mapping f and any order r, so the differentiable condition is not needed.)

6.4

6.4.1

If $x^y = y^x$, prove that $x(y \log x - x)dy = y(x \log y - y)dx$.
Proof: $x \log y = y \log x$, so $y' \log x + y/x = \log y + xy'/y$. Hence $dy(\log x - x/y) = dx(\log y - y/x)$.
Therefore $x(y \log x - x)dy = y(x \log y - y)dx$.

6.4.2

If $y = \frac{x-a}{1+ax}$, then $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$.
Proof: Note that $y = \tan(\arctan x - \arctan a)$, so $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ since $d \arctan y = d \arctan x$.
($ds = \frac{dx}{1+x^2}$ is the angular distance in spherical geometry, and clearly rotation on the sphere is an isometry.
We can do likewise for hyperbolas and the Blaschke factors.)

6.4.3-6.3.6

If $xy = -1$ prove that $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$.
Proof: Clearly $\frac{dy}{dx} = x^{-2}$, so $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$.
($ds = \frac{dx}{\sqrt{1+x^4}}$ is the parametrization of the hyperbolic lemniscate, and $y = -1/x$ is the inversion which is an isometry).
Note: For the next 3 problems, they are all about preserving the arc length of the lemniscate (the ∞ shaped

curve). (The hyperbolic lemniscate $\frac{dx}{\sqrt{1+x^4}}$ is just the lemniscate $\frac{dx}{\sqrt{1-x^4}}$ rotated $\frac{\pi}{4}$.)
6.4.4 comes from $(1+\mathrm{sl}^2(u))(1+\mathrm{cl}^2(u))=2$, since $x^2+y^2+x^2y^2=1$ implies $x=\mathrm{sl}(u), y=\pm\mathrm{cl}(u)=\pm\mathrm{sl}(\varpi/2-u)$, and $\frac{dx}{\sqrt{1-x^4}}=du=\pm\frac{dy}{\sqrt{1-y^4}}$.
6.4.5 comes from the rotation $\mathrm{sl}(u+v)=\frac{\mathrm{sl}(u)\mathrm{cl}(v)+\mathrm{cl}(u)\mathrm{sl}(v)}{1+\mathrm{sl}^2(u)\mathrm{sl}^2(v)}$, since the equation implies $x=\mathrm{sl}(u), c=\mathrm{sl}(v), y=\mathrm{sl}(u+v)$.
6.4.6 comes from duplication $\mathrm{sl}(2u)=\frac{2\mathrm{sl}(u)\mathrm{cl}(u)}{1+\mathrm{sl}^4(u)}$, since $(y^4-2y^2-1)x+y^4+2y^2-1=0$ implies $x=\mathrm{sl}(2u), y=\mathrm{sl}(u)$.

We can unify the problems above using elliptic integrals:
Consider the elliptic metric $\omega_J=\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, and the isometry $x=\mathrm{sn}(u)\mapsto y=\mathrm{sn}(u+v)$, where $t=\int_0^{\mathrm{sn}(t)}\omega$ is the Jacobi elliptic integral (of the first kind). Since it is an isometry, we have

$$(x^2+y^2+a^2)-2(x^2y^2+y^2a^2+a^2x^2)+4k^2x^2y^2a^2-k^4(x^2y^2a^2)^2=0$$

implies $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$.
For $k=0,1,i$ we obtain all the results mentioned above (spherical, hyperbolic, lemniscate).

6.5

6.5.1

(1) $f(x)=e^x\cos x$, $f^{(5)}(x)=$
$f(x)=\mathrm{Re}(e^{(1+i)x})$, then $f^{(5)}(x)=\mathrm{Re}((-4-4i)e^{(1+i)x})=4e^x(\sin x-\cos x)$.
(2) $f(x)=x^2\log x+x\log^2 x$, $f''=$
$f''(x)=2\log x+3+2\frac{\log x+1}{x}$.
(3) $f(x)=x^2e^x$, $f^{(10)}=$
$f^{(10)}(x)=\sum_{k=0}^{10}\binom{10}{k}(x^2)^{(k)}(e^x)^{(10-k)}=e^x(x^2+20x+90)$.
(4) $f(x)=x^5\cos x$, $f^{(50)}=$
Likewise $f^{(50)}(x)=(-x^5+20\binom{50}{2}x^3-120\binom{50}{4})\cos x+(-250x^4+60\binom{50}{3}x^2-120\binom{50}{5})\sin x$.

6.5.2

Prove that
(1) $\sum_{k=0}^n\binom{n}{k}k=n2^{n-1}$ (2) $\sum_{k=0}^n\binom{n}{k}k^2=n(n+1)2^{n-2}$.
Proof: $(1+x)^n=\sum_{k=0}^n\binom{n}{k}x^k$, hence

$$\sum_{k=0}^n\binom{n}{k}kx^k=xn(1+x)^{n-1},\sum_{k=0}^n\binom{n}{k}k^2x^{k-1}=n(1+x)^{n-1}+xn(n-1)(1+x)^{n-2}.$$

Let $x=1$ we obtain the desired identities.

6.5.3

Prove that $y=\cos(n\arccos x)$ satisfy

$$\frac{n\mathrm{d}x}{\sqrt{1-x^2}}=\frac{\mathrm{d}y}{\sqrt{1-y^2}},(1-x^2)y''-xy'+n^2y=0.$$

Proof: $y = P_n$ is the n^{th} Chebyshev polynomial.
Note that $\arccos y = n \arccos x$, hence $\frac{\mathrm{d}y}{\sqrt{1-y^2}} = \mathrm{d}\arccos y = n\frac{\mathrm{d}x}{\sqrt{1-x^2}}$.
Let $x = \cos\theta$, and $y = \cos(n\theta)$, then $y' = -\sin(n\theta)n\theta' = n\sin(n\theta)/\sin\theta$. Differentiate $y'\sin\theta = n\sin n\theta$, we obtain

$$y''\sin\theta + y'\theta'\cos\theta = n^2\theta'\cos(n\theta) \implies y''\sqrt{1-x^2} - \frac{y'x}{\sqrt{1-x^2}} = -\frac{n^2y}{\sqrt{1-x^2}}.$$

Hence $(1-x^2)y'' - xy' + n^2y = 0$.
(We can use the fact the P_n are orthogonal under the weight $w = \frac{1}{\sqrt{1-x^2}}$, and solve the Pearson differential equation $\frac{d}{dx}(\sigma w) = \tau w$ to get $\sigma = 1-x^2, \tau = -x$ and directly obtain this differential equation $(1-x^2)y'' - xy' + n^2y = 0$.)

6.5.4

Prove that: For points (x, y) on the circle $(x-a)^2 + (y-b)^2 = c^2$, $y \neq b$,
(1) $y''/(1+(y')^2)^{3/2} = \pm 1/c$; (2) $y'''(1+(y')^2) = 3y'(y'')^2$.
Proof: (1) $1+(y')^2 = 1 + \frac{(x-a)^2}{(y-b)^2} = \frac{c^2}{(y-b)^2}$., and $y'' = -\frac{c^2}{(y-b)^3}$, hence $y''/(1+(y')^2)^{3/2} = \pm 1/c$. Its absolute value is the curvature at (x, y).
(2) $y''' = 3c^2y'/(y-b)^4$, so $y'''(1+(y')^2) = 3c^4y'/(y-b)^6 = 3y'(y'')^2$.

6.5.5

Given the polar coordinate form $r^2 = \cos 2\theta$ of $y = f(x)$, calculate $\frac{d^2y}{dx^2}$.
Solution: $r^2 = \cos 2\theta \implies x^2 + y^2 = \frac{1-y^2/x^2}{1+y^2/x^2}$. Hence $(x^2+y^2)^2 = x^2 - y^2$. Therefore $\frac{d^2y}{dx^2} = -\frac{3r}{\sin^3(3\theta)}$.

6.5.6

Suppose $f(x) = x\log(2^{1/x} + 3^{1/x}), \forall x > 0$ prove that $f'(x), f''(x) > 0$.
Proof: Let $g(x) = \log(2^x + 3^x)/x$ and $h(x) = \log(2^x + 3^x)$. Note that by Cauchy-Schwarz,

$$h''(x) = \frac{(2^x\log^2 2 + 3^x\log^2 3)(2^x + 3^x) - (2^x\log 2 + 3^x\log 3)^2}{(2^x+3^x)^2} > 0.$$

Since $f''(x) = x^{-3}h''(x^{-1})$, we infer $f''(x) > 0$.
Let $F(t) = f'(t^{-1}) = h(t) - th'(t)$, then $F'(t) = -th''(t) < 0$, hence F is strictly decreasing. Clearly $\lim_{t\to\infty} F(t) = 0$, so $f'(x) = F(x^{-1}) > 0$.

6.5.7

Suppose P is a polynomial of degree n with n distinct roots. Prove that $P'(x)^2 \geqslant P(x)P''(x)$.
Proof: Suppose $P(x) = \prod_{k=1}^n (x - x_k)$, and $P(t) \neq 0$. Then

$$\frac{PP'' - P'(t)^2}{P(t)^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{P'(t)}{P(t)} = \frac{d}{dt}\sum_{k=1}^n \frac{1}{t-x_j} = \sum_{k=1}^n -(t-x_j)^{-2} < 0.$$

Therefore $P'(t)^2 \geqslant P(t)P''(t)$.

6.5.9

Suppose $f'(x) \neq 0$ and f^{-1} exists. Express $(f^{-1})^{(3)}$ in terms of $f', f'', f^{(3)}$.
Solution:

$$(f^{-1})^{(3)}(y) = \frac{3[f''(f^{-1}(y))]^2 - f'(f^{-1}(y))f^{(3)}(f^{-1}(y))}{[f'(f^{-1}(y))]^5}$$

6.5.10

$a, b \in \mathbb{R}, a > 0$. Consider $f : [-1, 1] \to \mathbb{R}$, where

$$f(x) = \begin{cases} x^a \sin\left(x^{-b}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

1. $f \in C([-1, 1])$ iff $a > 0$;
   Proof: $f \in C([-1, 1])$ iff $\lim_{x\to 0} x^a \sin\left(x^{-b}\right) = 0$. If $a > 0$ then $\left|x^a \sin\left(x^{-b}\right)\right| \leqslant |x|^a \to 0$. If $a < 0$ then let $x = ((2n + 1/2)\pi)^{-1/b}$, when $n \to \infty$, $x \to 0$ but $\left|x^a \sin\left(x^{-b}\right)\right| \to \infty$. If $a = 0$, then let $x = ((2n + 1/2)\pi)^{-1/b}$, $\left|x^a \sin\left(x^{-b}\right)\right| = 1$. Therefore $f \in C([-1, 1])$ iff $a > 0$.
2. f is differentiable at 0 iff $a > 1$;
   Proof: f is differentiable at 0 iff $\lim_{x\to 0} x^{-a} \sin\left(x^{-b}\right)$ exists. By 1 we know that $a > 1$. ($a = 1$ is invalid since $x = (2n\pi)^{-1/b}$ and $x = ((2n + 1/2)\pi)^{-1/b}$ converge to different values.)
3. f' is bounded on $[-1, 1]$ ($\iff$ f is Lipschitz) iff $a \geqslant 1 + b$;
   Proof: $f'(x) = ax^{a-1} \sin\left(x^{-b}\right) + x^a \cos\left(x^{-b}\right)(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geqslant 1 + b$.
4. $f \in C^1([-1, 1])$ iff $a > 1 + b$;
   Proof: $f \in C^1([-1, 1])$ iff $f'(0) = 0 = \lim_{x\to 0} f'(x)$. By 1 we know it is equivalent to $a > 1 + b$.
5. f' is differentiable at 0 iff $a > 2 + b$;
6. f'' is bounded on $[-1, 1]$ iff $a \geqslant 2 + 2b$;
7. $f \in C^2([-1, 1])$ iff $a > 2 + 2b$.
   Proof: 5,6,7 are exactly the same as 2,3,4.
   Likewise $f \in C^n([-1, 1])$ iff $a > n(1 + b)$, $f^{(n-1)}$ differentiable at 0 iff $a > n + (n - 1)b$, and $f^{(n)}$ is bounded iff $a \geqslant n + nb$.

6.5.11

Suppose $f(x) = \prod_{k=1}^{n} (x - x_k)$ where x_k are distinct, and $\xi_1, \cdots, \xi_{n-1}$ are the roots of $f'(x)$.
Calculate $\sum_{k=1}^{n-1} 1/f''(\xi_k)$.
Solution: Same as 6.3.14, use Lagrange interpolation, the sum for the k^{th} derivative is $\frac{\delta_{k,n}}{k!}$.