讲义习题5.6:1-6.

Denote ${}^{\sigma}L : (\alpha_1, \cdots, \alpha_r) \mapsto T(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(r)})$.

5.6.1

Verify that $\mathsf{A}L \in \Lambda^r(V) \forall L \in \mathcal{T}^r(V)$.
Proof: Suppose $\alpha_1 = \alpha_2$, then $\tau = (12)$ is odd, so

$$\mathsf{A}L = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)^{\sigma}L = \sum_{\sigma \in A_r} \operatorname{sgn}(\sigma)^{\sigma}L + \operatorname{sgn}(\sigma\tau)^{\sigma\tau}L.$$

Since $\alpha_1 = \alpha_2$, $\operatorname{sgn}(\sigma\tau)^{\sigma\tau}L(\alpha_1, \cdots, \alpha_n) = -\operatorname{sgn}(\sigma)^{\sigma}L$, so $\mathsf{A}L(\alpha_1, \cdots, \alpha_r) = 0$.

5.6.2

Prove that $\mathsf{A}L$ is skew-symmetric.
Proof: Suppose $\alpha_1 = \alpha_2$, then $\tau = (12)$ is odd, so

$$\begin{aligned}\mathsf{A}L(\alpha_1, \alpha_2, \cdots) &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)({}^{\sigma}L)(\alpha_1, \cdots, \alpha_r) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)({}^{\sigma}L)(\alpha_2, \alpha_1, \cdots, \alpha_r) \\ &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)({}^{\sigma\tau}L)(\alpha_1, \cdots, \alpha_r) = -\mathsf{A}L(\alpha_1, \cdots, \alpha_r).\end{aligned}$$

5.6.3

Prove that for $L \in \Lambda^r(V), M \in \Lambda^s(V), N \in \Lambda^t(V)$, then
$(s+t)!\mathsf{A}(\mathsf{A}(L \otimes M) \otimes N) = (r+s)!\mathsf{A}(L \otimes \mathsf{A}(M \otimes N))$.
Proof:

$$\begin{aligned}\mathsf{A}(\mathsf{A}(L \otimes M) \otimes N) &= \sum_{\sigma \in S_{r+s+t}} \operatorname{sgn}(\sigma)^{\sigma}(\mathsf{A}(L \otimes M) \otimes N) = \sum_{\sigma \in S_{r+s+t}} \sum_{\tau \in S_{r+s}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)^{\sigma}({}^{\tau}(L \otimes M) \otimes N) \\ &= \sum_{\tau \in S_{r+s}} \sum_{\sigma \in S_{r+s+t}} \operatorname{sgn}(\sigma\tilde{\tau})^{\sigma\tilde{\tau}}(L \otimes M \otimes N) \\ &= \sum_{\tau \in S_{r+s}} \sum_{\sigma \in S_{r+s+t}} \operatorname{sgn}(\sigma)^{\sigma}(L \otimes M \otimes N) = (r+s)! \sum_{\sigma \in S_{r+s+t}} \operatorname{sgn}(\sigma)^{\sigma}(L \otimes M \otimes N).\end{aligned}$$

Likewise we obtain $(s+t)!\mathsf{A}(\mathsf{A}(L \otimes M) \otimes N) = (r+s)!\mathsf{A}(L \otimes \mathsf{A}(M \otimes N))$.

5.6.4

Suppose $\operatorname{char} F = 0$. Verify that $L \wedge M = \frac{1}{(r+s)!}\mathsf{A}(L \otimes M)$ and $L \wedge_1 M = \frac{1}{r!s!}\mathsf{A}(L \otimes M)$ are both communicative.
Proof: By 5.6.3

$$\begin{aligned}(L \wedge M) \wedge N &= \left(\frac{1}{(r+s)!}\mathsf{A}(L \otimes M)\right) \wedge N = \frac{1}{(r+s)!(r+s+t)!}\mathsf{A}(\mathsf{A}(L \otimes M) \otimes N) \\ &= \frac{1}{(s+t)!(r+s+t)!}\mathsf{A}(L \otimes \mathsf{A}(M \otimes N)) = L \wedge (M \wedge N).\end{aligned}$$

And likewise $(L \wedge_1 M) \wedge_1 N = L \wedge_1 (M \wedge_1 N)$.

5.6.5

Suppose $\mathrm{char} F = 0$ verify that for $L \in \Lambda^r(V), M \in \Lambda^s(V)$,

$$\mathsf{A}\left(\frac{L}{r!} \otimes \frac{M}{s!}\right) = \sum_{\sigma \in Sh(r,s)} \mathrm{sgn}(\sigma)^{\sigma}(L \otimes M).$$

Proof:

$$\begin{aligned}\mathsf{A}(L \otimes M) &= \sum_{\sigma \in S_{r+s}} \mathrm{sgn}(\sigma)^{\sigma}(L \otimes M) = \sum_{\sigma \in Sh(r,s)} \sum_{\tau \in S_r, \rho \in S_s} \mathrm{sgn}(\sigma \circ (\tau \times \rho))^{\sigma}({}^{\tau}L \otimes^{\rho} M) \\ &= r!s! \sum_{\sigma \in Sh(r,s)} \mathrm{sgn}(\sigma)(L \otimes M)\end{aligned}$$

5.6.6

Suppose r is odd, $L \in \Lambda^r(V)$. Prove that $L \wedge L = 0$.
Proof: Take $\tau = (1, r+1)(2, r+2) \cdots (r, 2r)$, then $\mathrm{sgn}(\tau) = -1$ and ${}^{\tau}(L \otimes L) = L \otimes L$ so

$$L \wedge L = \frac{1}{(2r)!} \sum_{\sigma \in S_{2r}} \mathrm{sgn}(\sigma)^{\sigma}(L \otimes L) = \frac{1}{(2r)!} \sum_{\sigma \in A_{2r}} \mathrm{sgn}(\sigma)^{\sigma}(L \otimes L) + \mathrm{sgn}(\sigma\tau)^{\sigma\tau}(L \otimes L) = 0.$$