教材163页9,12; 丘老师书上册51页4-7, 56页3.

193-9

Let A be an $n \times n$ matrix over a field, $A \neq 0$. If $r \in \{1, \cdots, n\}$, an $r \times r$ sub-matrix is any $r \times r$ matrix obtained by deleting $(n-r)$ rows and columns of A. The determinant rank of A is the largest positive integer r such that some $r \times r$ sub-matrix of A has non-zero determinant. Prove that the determinant rank of A is equal to the row rank of A.
Proof: If the determinant rank of A is r, then there is an $r \times r$ sub-matrix of A that has non-zero determinant, then the r rows of this sub-matrix are linearly independent, so the r rows in A are linearly independent, hence $\operatorname{rank}(A) \geqslant r$.
If $\operatorname{rank}(A) = r$, suppose the first r rows of A, $R_1, \cdots, R_r$ are linearly independent. We extend these vectors to a basis $\{R_1, \cdots, R_n\}$ of $F^{n \times n}$, and suppose $B = (R_1, \cdots, R_n)$. Then $\det B \neq 0$, so by Laplace theorem, there must be an $n \times n$ sub-matrix of the first r rows of A, that has non-zero determinant.

193-12

If $V = F^{n \times n}$ and B is a fixed $n \times n$ matrix over F, let L_B and R_B be the linear operators on V defined by $L_B(A) = BA$ and $R_B(A) = AB$. Show that
(a) $\det L_B = (\det B)^n$; (b) $\det R_B = (\det B)^n$.
Proof: (a) Denote $L : F^{n \times 1} \to F^{n \times 1}, x \mapsto Bx$. Consider any $\omega \in \Lambda^n(F^{n \times 1})$, then $L^* \omega = \det B \cdot \omega$.
Let $\pi_j : F^{n \times n} \to F^{n \times 1}$ be taking the j^{th} column of a matrix, and $\omega_j = \pi_j^* \omega \in \Lambda^n(F^{n \times n})$ (where f^* is the pullback of f). Note that $L_B(A) = (L\pi_1 A, \cdots, L\pi_n A)$ so $\pi_j L_B = L\pi_j$.
Let $\Omega = \omega_1 \wedge \cdots \wedge \omega_n \in \Lambda^{n \times n}(F^{n \times n})$, then denote $E_{i,j}$ be the matrix where only (i,j) is 1, and $\mathcal{C}_j = (E_{1,j}, \cdots, E_{n,j})$, then $\omega_k(\mathcal{C}_j) = 0 \forall k \neq j$, so $\Omega(\mathcal{C}_1, \cdots, \mathcal{C}_n) = \omega_1(\mathcal{C}_1) \cdots \omega_n(\mathcal{C}_n) \neq 0$, so $\Omega \neq 0$.
Then $\det L_B \cdot \Omega = L_B^* \Omega = \bigwedge_{j=1}^n L_B^* \omega_j$ note that $\omega_j = \pi_j^* \omega$, so
$L_B^* \omega_j = L_B^* \pi_j^* \omega = \pi_j^* L^* \omega = \pi_j^* (\det B) \omega = (\det B) \pi_j^* \omega = \det B \cdot \omega_j$. Therefore
$L_B^* \Omega = \bigwedge_{j=1}^n L_B^* \omega_j = \bigwedge_{j=1}^n \det B \cdot \omega_j = (\det B)^n \Omega$, so $\det L_B = (\det B)^n$.
(b) is exactly the same.

51-4

For what values a, b does the following linear equation has non-trivial solutions:

$$\begin{cases} ax_1 + x_2 + x_3 = 0, \\ x_1 + bx_2 + x_3 = 0, \\ x_1 + 2bx_2 + x_3 = 0. \end{cases}$$

Solution: It is equivalent to $\det A = 0$ where $A = \begin{pmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 2b & 1 \end{pmatrix}$.
Note that $\det A = a(b - 2b) + 1(2b - b) = -ab + b = b(1 - a)$, so it is equivalent to $b = 0$ or $a = 1$.

51-5

For what values a, b does the following linear equation have exactly one solution:

$$\begin{cases} ax_1 + x_2 + x_3 = 2, \\ x_1 + bx_2 + x_3 = 1, \\ x_1 + 2bx_2 + x_3 = 2. \end{cases}$$

Solution: Following 51-4, it is equivalent to $A : F^n \to F^n$ is bijective or $\det A = b(1 - a) \neq 0$.

51-6

For exercise51-5, for what values a, b does the equation has no solution/infinitely many solutions?
Solution: If $b = 0$, then $\mathrm{Im} A = \mathrm{Span}\{(a,1,1),(1,0,1)\}$, so $(2,1,2) \in \mathrm{Im} A$ iff $a = 1$. Hence it has infinitely many solutions iff $(a,b) = \left(1, \frac{1}{2}\right)$ and otherwise no solutions.

51-7

Discuss when the following equation has a unique solution/no solution/infinitely many solutions?

$$\begin{cases} ax_1 + x_2 + x_3 = 2, \\ x_1 + bx_2 + x_3 = 1, \\ x_1 + 2bx_2 + x_3 = 1. \end{cases}$$

Solution: Likewise, a unique solution $\iff b(a-1) \neq 0$. Infinitely many solutions iff $b = 0$ and otherwise no solutions.

56-3

Calculate the following determinant:

$$\begin{vmatrix} 0 & \cdots & 0 & a_{11} & \cdots & a_{1k} \\ & \cdots & & & & \\ 0 & \cdots & 0 & a_{k1} & \cdots & a_{kk} \\ b_{11} & \cdots & b_{1r} & c_{11} & \cdots & c_{1k} \\ & \cdots & & & & \\ b_{r1} & \cdots & b_{rr} & c_{r1} & \cdots & c_{rk} \end{vmatrix}$$

Solution: The determinant is

$$\det \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} = (-1)^{kr} \det \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = (-1)^{kr} \det A \cdot \det B$$