习题 7.1，任选不少于十道，题号之和不小于 120！（“！”不是阶乘【旺柴】）
习题 7.2，任选不少于十道题目，其中奇数和偶数题分别不得少于五个。
习题 7.3，同习题 7.2 的选择方式。
习题 7.5，题目选择方式同习题 7.1

7.1

7.1.8

Prove that the Laguerre polynomial $L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$ has n different real roots.
Proof: We know that the Laguerre polynomials are orthonormal on the space $L^2([0,\infty))$ with weight e^{-x}, by applying integration by parts n times, hence using the lemma below, it must have n distinct roots.
Or note that $f(x) = x^n e^{-x}$ has a root with multiplicity n at 0 and it vanishes at ∞, hence use Rolle's theorem and induction we can show that $f^{(k)}(x)$ has a root with multiplicity $n-k$ at 0 and k roots between 0 and ∞.
Lemma: If the class of polynomials P_n where $\deg P_n = n$ are orthogonal under the (real) inner product $\langle f, g\rangle = \int_X fgw\,\mathrm{d}x$ where $w \geqslant 0$, then P_n has all n distinct roots.
Proof: Otherwise if P_n only changes signs at $x_1, \cdots, x_k$ for $k \leqslant n-1$, let $Q(x) = \pm(x-x_1)\cdots(x-x_k)$, then $P(x)Q(x) \geqslant 0$, and it cannot vanish on X (it has only finitely many roots), so $\langle P, Q\rangle > 0$. However, $\deg Q < n$ so it is the linear combination of $P_0, \cdots, P_{n-1}$, leading to contradiction.

7.1.9

Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$ has n different roots in the interval $(-1,1)$.
Proof: We know that the polynomials $\sqrt{(2n+1)/2P_n(x)}$ form a set of orthonormal basis on the space $L^2([-1,1])$, using integration by parts, hence it must have n different roots in the interval $(-1,1)$.
Likewise the Hermite polynomial $H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$ has n different real roots.

7.1.10

Suppose $f \in C([a,b])$, differentiable on (a,b), $\xi \in (a,b)$. Must there exist $a_1, b_1 \in (a,b), a_1 < \xi < b_1$ such that $f'(\xi) = \frac{f(b_1)-f(a_1)}{b_1-a_1}$?
Solution: No, for example, let $f(x) = x^2 \cdot \mathrm{sign}(x)$, then $f'(0) = 0$ but $f(a) - f(-b) = a^2 + b^2 > 0$.

7.1.11-12

Suppose f is differentiable on $[a,\infty)$, and $|f'(x)| \leqslant M|f(x)|$, $f(a) = 0$, $M > 0$. Prove that $f \equiv 0$.
Proof: If $f(a) = 0$, then suppose $|f(t)| = \sup_{t\in[a,a+\varepsilon]}|f(t)|$ where $\varepsilon = 1/2M$, then $|f(t)| = |f(t) - f(a)| = (t-a)|f'(\xi)| \leqslant \varepsilon M|f(t)|$. Since $\varepsilon M < 1$, $|f(t)| = 0$, hence $f(t) = 0 \forall t \in [a, a+\varepsilon]$. Likewise we obtain $f \equiv 0$.

7.1.13

Suppose $f(x) = \sum_{k=1}^n a_k \sin kx$, and $|f(x)| \leqslant |\sin x|$. Prove that $\left|\sum_{k=1}^n ka_k\right| \leqslant 1$.
Proof: $|f(x)| \leqslant |\sin x|$ implies $|f(x)/x| \leqslant |\sin x/x|$, so

$$\left|\sum_{k=1}^n ka_k\right| = \lim_{x\to 0}\left|\frac{f(x)}{x}\right| \leqslant \lim_{x\to 0}\left|\frac{\sin x}{x}\right| = 1.$$

7.1.14

Suppose $f \in C([0,1])$, and differentiable on $(0,1)$. If $f(0)=0, f(1)=1$, prove that there exists $\xi_1 \neq \xi_2 \in (0,1)$ such that $f'(\xi_1)f'(\xi_2)=1$.
Proof: Take $f(t)=1-t$ (since $f(0)+0<1<f(1)+1$), then there exists $\xi_1 \in (0,t)$ and $\xi_2 \in (t,1)$ such that $f'(\xi_1)=\frac{f(t)-f(0)}{t-0}=\frac{1-t}{t}$, and $f'(\xi_2)=\frac{f(1)-f(t)}{1-t}=\frac{t}{1-t}$, so $f'(\xi_1)f'(\xi_2)=1$.

7.1.17

Suppose $f,g,h \in C([a,b])$ are differentiable on (a,b). Prove that there exists $\xi \in (a,b)$ such that

$$\det \begin{pmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix} = 0.$$

Proof: Let $F(x)=(f(x),g(x),h(x))$, and $G(x)=\det(F(x),F(a),F(b))$, then $G:\mathbb{R}\to\mathbb{R}$ and $G(a)=G(b)=0$, hence there exists $\xi \in (a,b)$ such that $G'(\xi)=0$. Therefore $0=G'(\xi)=\det(F'(\xi),F(a),F(b))$.

7.1.23

Suppose f is differentiable on $[a,b]$, twice differentiable on (a,b). Prove that there exists $\xi \in (a,b)$ such that $f'(b)-f'(a)=f''(\xi)(b-a)$.
Proof: Let $A=\frac{f'(b)-f'(a)}{b-a}$ and $g(x)=f'(x)-A(x-a)$, then $g(a)=f'(a)=g(b)$. We find ξ such that $g'(\xi)=0$. Note that g is continuous on (a,b), so if $g'(\xi)\neq 0 \forall \xi \in (a,b)$, then we can suppose $g'(x)>0 \forall x \in (a,b)$. Hence g is monotonic on (a,b), and $\lim_{x\to a} g(x)$ exists. By Darboux's theorem for any x there exists $\xi_x \in (a,x)$ such that $g(\xi_x)=\frac{1}{2}(g(a)+g(x))$, so $\lim_{x\to a} g(x)=\frac{1}{2}(g(a)+\lim_{x\to a} g(x))$, which implies g is continuous at a. Likewise, g is continuous on $[a,b]$, so applying the mean value theorem we obtain a contradiction.

7.1.25

Consider a degenerate case of Darboux's theorem: Suppose f is differentiable on $[a,b]$, and $f'(a)=A=f'(b)$, must there exist $\xi \in (a,b)$ such that $f'(\xi)=A$?
Solution: No. Let $g(x)=A-\left|\frac{a-b}{2}\right|+\left|x-\frac{a+b}{2}\right|$, then $g(a)=g(b)=A$, and $g(x)\neq A \forall x \in (a,b)$. Let $f(x)=\int_a^x g(t)\,\mathrm{d}t$, then $f'(a)=f'(b)=A$ but $f'(\xi)\neq A \forall \xi \in (a,b)$.

7.1.26

Suppose f is differentiable on $[a,b]$, and $f'(a)=f'(b)$. Prove that there exists $\xi \in (a,b)$ such that

$$\frac{f(\xi)-f(a)}{\xi-a}=f'(\xi).$$

Proof: Let $g(x)=\frac{f(x)-f(a)}{x-a}$, then $g'(x)=\frac{f'(x)(x-a)-(f(x)-f(a))}{(x-a)^2}$ so $g'(\xi)=0 \implies f'(\xi)=\frac{f(\xi)-f(a)}{\xi-a}$. Note that $g'(b)=-\frac{g(b)-g(a)}{b-a}$.
If $g'(\xi)\neq 0 \forall \xi \in (a,b)$, then by Darboux's theorem, suppose $g'(\xi)>0 \forall \xi \in (a,b)$. Then $g(b)>g(a)$ so $g'(b)<0$, hence by Darboux's theorem, there exists $\xi \in (a,b)$ such that $g'(\xi)=0$.

7.2

7.2.8

Suppose $f:(a,\infty)\to\mathbb{R}$ bounded on every bounded interval, and $\lim_{x\to\infty}\frac{f(x+1)-f(x)}{x^n}=l$. Calculate the limit $\lim_{x\to\infty}\frac{f(x)}{x^{n+1}}$.
Solution: Let $f(x+1)-f(x)=(l+g(x))x^n$, then $g(x)\to 0$, and
$f(x+N)-f(x)=l\sum_{k=0}^{N-1}(x+k)^n+\sum_{k=0}^{N-1}g(x+k)(x+k)^n$. Suppose $a<0$ and
$M=\sup_{x\in[0,1]}|f(x)|$. Then for $x=a+N$ where $a\in(0,1)$ and $N\in\mathbb{N}$,

$$\left|\frac{f(x)}{x^{n+1}}-\frac{l}{n+1}\right|\leqslant\left|\frac{f(a)}{x^{n+1}}\right|+l\left|\frac{1}{n+1}-\frac{\sum_{k=0}^{N-1}(a+k)^n}{(a+N)^{n+1}}\right|+\sum_{k=0}^{K}+\sum_{k=K+1}^{N-1}|g(x+k)|\cdot\left|\frac{(a+k)^n}{x^{n+1}}\right|.$$

Since $\left|\frac{f(a)}{x^{n+1}}\right|\leqslant\frac{M}{x^{n+1}}\to 0$, $\frac{\sum_{k=0}^{N-1}(a+k)^n}{(a+N)^{n+1}}\to\frac{1}{n+1}$, and for given K, $\frac{(a+K)^n}{x^{n+1}}\to 0$, we obtain
$\limsup_{x\to\infty}\left|\frac{f(x)}{x^{n+1}}-\frac{l}{n+1}\right|\leqslant(n+1)\sup_{x\geqslant K}|g(x)|\to 0$, so $\lim_{x\to\infty}\frac{f(x)}{x^{n+1}}=\frac{l}{n+1}$.

7.2.9

Suppose f is a solution to

$$\begin{cases}f''+3f'+4f=\frac{4x^2}{x^2+x+1}, & x\geqslant 0,\\ f(0)=1, \quad f'(0)=2\end{cases}$$

Prove that $\lim_{x\to\infty}f''(x)=\lim_{x\to\infty}f'(x)=0$.
Proof: The solution to this equation exists and is unique, since it can be viewed as
$y_1'=y_2, y_2'=-4y_1-3y_2+4x^2/(x^2+x+1)$, and the function
$f(x,Y)=(y_2,-4y_1-3y_2+4x^2/(x^2+x+1))$ is globally Lipschitz, so by Picard's theorem there is a unique solution locally. Since the equation is linear, it can be extended to the real line.
We first solve the homogeneous equation, the damped harmonic oscillator: $\varphi''+3\varphi'+4\varphi=0$, and
$\varphi(0)=0,\varphi'(0)=1$. This is a second order linear equation, and by solving $\lambda^2+3\lambda+4=0$ we obtain
$\varphi(t)=\frac{2}{\sqrt{7}}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)$.
Now, the solution f should be $f(t)=Ae^{\lambda_1 t}+Be^{\lambda_2 t}+\varphi*g$ where $g(x)=\frac{4x^2}{x^2+x+1}$. By $f(0)=1$ and
$f'(0)=2$ we obtain $f_0(t)=Ae^{\lambda_1 t}+Be^{\lambda_2 t}=e^{-3t/2}\left(\cos\left(\frac{\sqrt{7}}{2}t\right)+\sqrt{7}\sin\left(\frac{\sqrt{7}}{2}x\right)\right)$.
Hence $f'(t)=f_0'(t)+\varphi'*g$ and $f''(t)=g+\varphi''*g$. Since $\lim_{x\to\infty}g(x)=4$, we have
$\lim_{x\to\infty}f'(t)=\lim_{x\to\infty}\varphi'*g=4\int_0^\infty\varphi'\,\mathrm{d}t=0$ and $\lim_{x\to\infty}f''(t)=4+4\int_0^\infty\varphi''(t)\,\mathrm{d}t=0$.
(Furthermore we have $\lim_{x\to\infty}f(x)=1$).

Proof without solving the ODE: Let $y=f(x)-1$ then the equation becomes $\ddot{y}+3\dot{y}+4y=g$ where
$g(x)=O(x^{-1})$. This is the equation $m\ddot{x}=-k^2x-r\dot{x}+f$.
Consider the total energy $E=\frac{1}{2}k^2y^2+\frac{1}{2}m\dot{y}^2=2y^2+\frac{1}{2}\dot{y}^2$. Note that
$E'=4\dot{y}y+\dot{y}\ddot{y}=\dot{y}(4y+\ddot{y})=\dot{y}(g-3\dot{y})\leqslant g^2-2\dot{y}^2$. Hence $E(x)-E(0)+\int_0^x 2(y')^2\,\mathrm{d}t\leqslant\int_0^x g^2\,\mathrm{d}t$.
Since $g\in\mathcal{L}^2(\mathbb{R})$ and $E\geqslant 0$, we obtain $y'\in\mathcal{L}^2(\mathbb{R})$, and E is bounded, so y,y',g are bounded, hence y'' is bounded. Therefore y' is uniformly continuous, so combined with $y'\in\mathcal{L}^2(\mathbb{R})$ we have $\lim_{x\to\infty}y'(x)=0$.
$E'=\dot{y}g-3\dot{y}^2\in\mathcal{L}(\mathbb{R})$, since both $\dot{y}g,\dot{y}^2\in\mathcal{L}(\mathbb{R})$ ($\int\dot{y}g\leqslant\int\dot{y}^2\int g^2$), so $\lim_{x\to\infty}E(x)=2L$ exists. By
$E=2y^2+\frac{1}{2}\dot{y}^2$ we obtain $|y|\to\sqrt{L}$, and since y is continuous we can assume $y\to\sqrt{L}$.
$\ddot{y}+3\dot{y}+4y=g\to 0$, so $\ddot{y}\to-\frac{\sqrt{L}}{4}$. Combined with $\lim_{x\to\infty}y'=0$, we obtain $L=0$ and $\lim_{x\to\infty}y''=0$.

7.2.11

Suppose $f \in C^1(\mathbb{R})$ and $\forall x \in \mathbb{R}, f(x+1) - f(x) = f'(x), \lim_{x\to\infty} f'(x) = c$. Prove that $f' \equiv c$.
Proof: Note that for any $x \in \mathbb{R}$, there exists $\xi_x \in (x, x+1)$ such that $f'(\xi_x) = f(x+1) - f(x) = f'(x)$. If $f'(x_0) \neq c$, consider $A = \{x \in \mathbb{R} : f'(x) = f'(x_0)\}$ which is closed, then if $\sup A < \infty, t = \sup A \in A$ so $\xi_t \in A$, leading to contradiction. Hence A is unbounded, contradicting $\lim_{x\to\infty} f'(x) = c$. Therefore $f' \equiv c$.

7.2.12

Suppose f is differentiable on $\mathbb{R}$ and $|f'(x)| < 1$, and $x_0 \in \mathbb{R}$, $x_{n+1} = f(x_n)$. Show that $\{x_n\}$ may diverge.
Solution: Let $f(x) = \sqrt{x^2+1}$, then $f'(x) = \frac{x}{\sqrt{x^2+1}}$ so $|f'(x)| < 1$, but $x_n = \sqrt{n}$ diverges.

7.2.20

Suppose f is twice differentiable at x_0 and $f''(x_0) \neq 0$. For h small enough, there exists $\theta = \theta(h) \in (0,1)$ such that $f(x_0+h) - f(x_0) = f'(x_0+\theta h)h$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$.
Proof: Note that

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+\theta h) - f'(x_0)}{\theta h} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - hf'(x_0)}{\theta h^2} = f''(x_0) \lim_{h\to 0} \frac{1}{2\theta}.$$

So $\lim_{h\to 0} \theta = \frac{1}{2}$.

7.2.21

Suppose $a_i \in \mathbb{R}$, calculate the limit

$$\lim_{x\to 0} x^{-2}(1 - \cos(a_1 x) \cdots \cos(a_n x)).$$

Solution: Note that

$$1 - \prod_{k=1}^{n} \cos(a_k x) = 1 - \prod_{k=1}^{n} \left(1 - \frac{(a_k x)^2}{2} + O(x^4)\right) = \sum_{k=1}^{n} \frac{(a_k x)^2}{2} + O(x^4)$$

so

$$\lim_{x\to 0} x^{-2} \left(1 - \prod_{k=1}^{n} \cos(a_k x)\right) = \frac{1}{2} \sum_{k=1}^{n} a_k^2.$$

7.2.22

Suppose $f''(x_0)$ exists, $f'(x_0) \neq 0$. Calculate the limit

$$\lim_{x\to x_0} \frac{1}{f(x) - f(x_0)} - \frac{1}{f'(x_0)(x - x_0)}.$$

Solution: Suppose $g(x) = \frac{1}{(x-x_0)^2}(f(x) - f(x_0) - f'(x_0)(x - x_0))$, then

$$\lim_{x\to x_0} \frac{1}{f(x) - f(x_0)} - \frac{1}{f'(x_0)(x - x_0)} = \lim_{x\to x_0} -\frac{(x - x_0)^2 g(x)}{f'(x_0)(x - x_0)(f'(x_0)(x - x_0) + g(x)(x - x_0)^2)}.$$

Note that $g(x) = \frac{1}{2} f''(x_0) + o(1)$, so the limit equals

$$-\frac{f''(x_0)}{2(f'(x_0))^2}.$$

7.2.23

Suppose $a_1 \in (0, \pi)$, $a_{n+1} = \sin a_n$, prove that $\lim_{n\to\infty} \sqrt{n} a_n = \sqrt{3}$.
Proof: Clearly $a_{n+1} < a_n$ so $\lim_{n\to\infty} a_n = 0$. By Stolz,

$$\lim_{n\to\infty} n a_n^2 = \lim_{n\to\infty} \frac{1}{a_{n+1}^{-2} - a_n^{-2}} = \lim_{n\to\infty} \frac{a_n^2 \sin^2 a_n}{a_n^2 - \sin^2 a_n} = \lim_{x\to 0} \frac{x^2 \sin^2 x}{x^2 - \sin^2 x} = 3.$$

7.2.24

Suppose f is twice differentiable in a neighborhood O of 0, and $f(0) = 0$. Let $g(x) = f(x)/x$ and $g(0) = f'(0)$, prove that $g \in C^1(O)$.
Proof: Clearly g is continuously differentiable in $O \backslash \{0\}$. Note that

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} \frac{f(x)/x - f'(0)}{x} = \lim_{x\to 0} \frac{f(x) - xf'(0)}{x^2} = \frac{1}{2} f''(0).$$

And for any $x \neq 0$, $g'(x) = \frac{xf'(x) - f(x)}{x^2}$, where $f'(x) = f'(0) + xf''(0) + o(x)$ and $f(x) = xf'(0) + \frac{1}{2} f''(0)x^2 + o(x^2)$, so

$$\lim_{x\to 0} g'(x) = \frac{1}{2} f''(0) = g'(0).$$

Hence $g \in C^1(O)$.

7.2.25

Suppose f is differentiable on (a, ∞). If $\lim_{x\to\infty}(f(x) + xf'(x)\log x) = l$, prove that $\lim_{x\to\infty} f(x) = l$.
Proof: Let $F(x) = f(x)\log x$, then $F'(x) = \frac{f(x)}{x} + f'(x)\log x$, so $\lim_{x\to\infty} xF'(x) = l$. By l'Hôpital,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{F(x)}{\log x} = \lim_{x\to\infty} xF'(x) = l.$$

7.3

7.3.13

If $f(x) = \sum_{k=0}^{n+1} a_k x^k + o(x^{n+1}), x \to 0$, does $f'(x) = \sum_{k=0}^{n} (k+1)a_{k+1}x^k + o(x^n), x \to 0$ hold? If not, add a condition to make it hold.
Solution: No, for example $f(x) = x + x^2 \sin x^{-1}$, but $f'(x) = 1 + 2x \sin x^{-1} - \sin x^{-1}$.
Fix: If $f^{(n+1)}$ exists, then we can apply the Taylor's theorem with Peano remainder to f' to get $f'(x) = \sum_{k=0}^{n} (k+1)a_{k+1}x^k + o(x^n)$.

7.3.14

Find a function f such that $f((-\infty, 0]) = \{0\}$ and $f([1, \infty)) = \{1\}$, and $f \in C^k \backslash C^{k+1}$.
Solution: First we can find $\varphi \in C^\infty(\mathbb{R})$ such that $f((-\infty, 0]) = \{0\}$ and $f([1, \infty)) = \{1\}$, for example $\varphi(x) = \frac{\psi(x)}{\psi(x) + \psi(1-x)}$ where $\psi(x) = \begin{cases} e^{-x^2}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$.
For $k = 0$, we have $f_0 = \max\{0, \min\{1, x\}\} \in C \backslash C^1$. If we already have such f_k, define

$g(x)=\int_0^x(f_k-\varphi)(t)\,\mathrm{d}t+\lambda\varphi(x)$ such that $g(1)\neq 0$, and $f_{k+1}(x)=\frac{g(x)}{g(1)}$, then $f_{k+1}\in C^{k+1}\backslash C^{k+2}$ and f_{k+1} satisfy the requirements.

7.3.15&16

Suppose $n\geqslant 1$, $f^{(n+1)}$ exists, and $M_0, M_{n+1}<\infty$ where $M_m=\sup_{x\in\mathbb{R}}|f^{(m)}(x)|$. Prove that for $1\leqslant m\leqslant n$, there exists a constant $C_m>0$ independent of f, such that $M_m\leqslant C_m M_0^{1-m/(n+1)}M_{n+1}^{m/(n+1)}$.
Proof: Note that $f(x+2h)=f(x)+2hf'(x)+2h^2f''(\xi)$, so $|f'(x)|\leqslant\frac{1}{h}M_0+hM_2$. Let $h=\sqrt{M_0/M_2}$, then $M_1\leqslant 2\sqrt{M_0M_2}$.
Apply this to $f^{(k-1)}$ and we obtain $M_k\leqslant 2\sqrt{M_{k-1}M_{k+1}}$. Let $a_k=\log M_k+k^2\log 2$, then $a_k\leqslant\frac{1}{2}(a_{k-1}+a_{k+1})$, so a_k is convex, and $a_m\leqslant\frac{n+1-m}{n+1}a_0+\frac{m}{n+1}a_{n+1}$. Hence $M_m\leqslant C_mM_0^{1-m/(n+1)}M_{n+1}^{m/(n+1)}$, where $C_m=2^{m(n+1-m)}$.

7.3.19

Suppose $\{a_n\}$ is bounded, and $\lim_{n\to\infty}(a_{n+2}-2a_{n+1}+a_n)=0$, does $\lim_{n\to\infty}(a_{n+1}-a_n)=0$ hold?
Solution: Suppose $|a_n|\leqslant M/2$, and let $d_n=a_{n+1}-a_n$, then $\forall u<v, \sum_{i=u}^{v}d_i\leqslant a_v-a_{u-1}\leqslant M$. For any $\varepsilon>0$, take $t>2M/\varepsilon$, then there exists N such that $n>N\implies|d_{n+1}-d_n|<\frac{\varepsilon}{t}$. If $d_n>\varepsilon$ for some $n>N$, then $d_{n+k}>\frac{\varepsilon(t-k)}{t}$, so $d_n+\cdots+d_{n+t}>\frac{\varepsilon}{t}(1+\cdots+t)>\frac{t\varepsilon}{2}>M$, leading to contradiction.

7.3.20

$f:(-1,1)\to\mathbb{R}$ is twice differentiable at 0 and $f(0)=0$, prove that

$$\lim_{n\to\infty}\sum_{k=-n}^{n}f\left(\frac{k}{n^{3/2}}\right)=\frac{1}{3}f''(0).$$

Proof: Let $g(x)=f(x)+f(-x)$, then $g(0)=g'(0)=0$, and $g''(0)=2f''(0)$. For any x, since $g(x)=x^2f''(0)+o(x^2)$, suppose $\varepsilon(x)=\sup_{|y|\leqslant x}|y^{-2}(g(y)-y^2f''(0))|$, then

$$\lim_{n\to\infty}\sum_{k=-n}^{n}f\left(\frac{k}{n^{3/2}}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}g\left(\frac{k}{n^{3/2}}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}f''(0)+\varepsilon(n^{-1/2})\frac{k^2}{n^3}=\frac{1}{3}f''(0).$$

7.3.21

Suppose $f:[0,\infty)$ is twice-differentiable, and $f(0)=f'(0)=0$, $f''(x)+3f'(x)+2f(x)\geqslant 0$. Prove that $f(x)\geqslant 0$.
Proof: $L=\frac{d^2}{dx^2}+3\frac{d}{dx}+2=\left(\frac{d}{dx}+1\right)\left(\frac{d}{dx}+2\right)$, so consider $g(x)=f'(x)+2f(x)$, then $g'+g=f''+2f'+f'+2f\geqslant 0$, so $(e^xg)'=e^x(g+g')\geqslant 0$. $f(0)=f'(0)=0$ implies $g(0)=0$, so $(e^xg)'\geqslant 0$ implies $g\geqslant 0$. Therefore $(e^{2x}f)'=e^{2x}(f'+2f)\geqslant 0$, so $f\geqslant 0$.

7.3.22

Suppose $f\in C([-1,1])$ and three times differentiable on $(-1,1)$. Prove that there exists $\xi\in(-1,1)$ such that $f^{(3)}(\xi)=3(f(1)-f(-1)-2f'(0))$.
Proof: Consider $g(x)=f(x)-f(-x)-2xf'(0)$, then $g(x)+g(-x)=0$ and $g(0)=g'(0)=g''(0)=0$. Hence there exists $\eta\in(0,1)$ such that $g(1)=\frac{1}{6}g^{(3)}(\eta)$. Note that $g^{(3)}(\eta)=f^{(3)}(\eta)+f^{(3)}(-\eta)$ so there exists $\xi\in(-1,1)$ such that $g^{(3)}(\eta)=2f^{(3)}(\xi)$, therefore $3(f(1)-f(-1)-2f'(0))=f^{(3)}(\xi)$.

7.3.23

Suppose f is bounded and differentiable on $\mathbb{R}$, and $|f'(x)| < 1$. Prove that there is a constant $M < 1$ such that $|f(x) - f(0)| \leqslant M|x|$.
Furthermore, does there exist $K < 1$ such that $|f(x) - f(y)| \leqslant K|x - y|$?
Proof: Suppose $|f| \leqslant N$, than for any $|x| > 4N$, $|f(x) - f(0)| \leqslant 2N \leqslant \frac{1}{2}|x|$. Suppose $M = \max\left\{\frac{1}{2}, \sup_{x\in[-4N,4N]}|g(x)|\right\}$, where $g(x) = \frac{f(x)-f(0)}{x}$ and $g(0) = f'(0)$, then $g(x) < 1$ and is continuous, so $M < 1$ and for any $x \in [-4N, 4N]$, $|f(x) - f(0)| \leqslant M|x|$. Therefore $|f(x) - f(0)| \leqslant M|x|$ for any $x \in \mathbb{R}$.
However, consider

$$g(x) = \begin{cases} x - 4n, & x \in [4n-1, 4n+1) \\ 4n + 2 - x, & x \in [4n+1, 4n+3) \end{cases}$$

and $h(x) = (1 - (|n| + 1)^{-1})g(x), x \in [4n, 4n + 4)$, then $h \in C(\mathbb{R})$, $|h| < 1$, and $\int_{4n}^{4n+4} h(t)\,\mathrm{d}t = 0 \forall n \in \mathbb{Z}$. Hence let $f(x) = \int_0^x h(t)\,\mathrm{d}t$, f is bounded by 4, differentiable, $|f'(x)| < 1$, but $\sup_{x\in\mathbb{R}} |f'(x)| = 1$, so there does not exist such K.

7.3.24

Suppose $f \in C^1(\mathbb{R})$ and $\forall x \in \mathbb{R}$, $f'(x) > f(f(x))$, prove that $\forall x \geqslant 0, f(f(f(x))) \leqslant 0$.
Proof: (IMC2012P4)
Step1: $\lim_{x\to\infty} f(x) = \infty$ cannot hold.
If $\lim_{x\to\infty} f(x) = \infty$, then $f(f(x)) \to \infty$ so $f'(x) \to \infty$.
For x large enough, $f'(x) > 2$, so $\exists N, \forall x > N, f(x) > x$. Then $f'(x) > f(f(x)) > f(x)$, so $f(x) > Ce^x$ for some constant C. $f'(x) > f(f(x)) > Ce^{f(x)}$, so $f'e^{-f} > C$. Integrate from N to x, we obtain $e^{-f(N)} - e^{-f(x)} > C(x - N)$, but the right side tends to ∞, while the left is bounded, leading to contradiction.
Step2: $f(t) < t$, $\forall t > 0$.
By Step1, there exists $t > 0$ such that $f(t) < t$, so if the statement is false then take $t_0 > 0$ such that $f(t_0) = t_0$. If $f(t) \geqslant t_0$, $\forall t > t_0$, then $f'(t) > f(f(t)) \geqslant t_0$ so $f \to \infty$, a contradiction. So take $T = \inf\{t \geqslant t_0 : f(t) < t_0\}$, then $f(T) = t_0$. $f'(T) > f(f(T)) = f(t_0) = t_0 > 0$, so $f(t) > t_0$ in some neighborhood $(T, T + \varepsilon)$, leading to contradiction.
Step3. If $f(f(x_0)) \geqslant 0$, then $f(x) > f(x_0)$, $\forall x > x_0$, $f(x) < f(x_0)$, $\forall x < x_0$.
Proof: If $f(x_1) \leqslant f(x_0)$ for some $x_1 > x_0$, let $T = \inf\{t > x_0 : f(t) \leqslant f(x_0)\}$, then $f(T) \leqslant f(x_0)$. Note that $f'(x_0) > f(f(x_0)) \geqslant 0$, so $f(x) > f(x_0)$ in a neighborhood $(x_0, x_0 + \varepsilon)$, then $T > x_0$. By continuity, $f(T) = f(x_0)$, so $f'(T) > f(f(T)) = f(f(x_0)) \geqslant 0$. Hence in a neighborhood $(T, T - \delta)$, $f(t) < f(T) \leqslant f(x_0)$, leading to contradiction.
The other side is the same.
Step4: $f(f(f(x))) \leqslant 0$, $\forall x \geqslant 0$.
Suppose $f(f(f(t_0))) > 0$ where $t_0 > 0$, then let $t_1 = f(t_0), t_2 = f(t_1), t_3 = f(t_2) > 0$. Note that $f(f(t_1)) = t_3 > 0$, and by Step1 $t_0 > 0$ so $t_0 > f(t_0) = t_1$ so by Step2, $t_0 > t_1 \implies f(t_0) > f(t_1) \implies t_1 > t_2$. Then by Step2, $t_2 < t_1 \implies f(t_2) < f(t_1) \implies t_3 < t_2$, so $t_0 > t_1 > t_2 > t_3 > 0$.
Note that $f(f(t_0)) = t_2 > 0$, so by Step2, for all $t > t_0$, $f(t) > f(t_0) = t_1$. $f(f(t_1)) = t_3 > 0$ so by Step2, $f(t) > t_1 \implies f(f(t)) > f(t_1) = t_2$, so $f'(t) > f(f(t)) > t_2 > 0$. Hence $f(t) \to \infty$, leading to contradiction.

7.3.25

Suppose $c > e^{-1}$ prove that there does not exist $f: \mathbb{R} \to \mathbb{R}_+$ differentiable such that $f'(x) \geqslant f(x+c)$. Find a solution for $c \in (0, e^{-1}]$.
Proof: Consider $f(x) = -x \log x, x \in (0, \infty)$, then $f'(x) = -1 - \log x$ so the range of x is $(0, e^{-1}]$. If $c \in (0, e^{-1}]$ take a such that $c = a^{-1} \log a$, then $f(x) = e^{ax}$ satisfy $f'(x) = ae^{ax} = e^{a(x+c)} = f(x+c)$.
If $c > e^{-1}$, from $f'(x) \geqslant f(x+c) > f(x)$ we obtain $(\log f)' \geqslant 1$ so $f(x+t) \geqslant f(x)e^t$.
Suppose $\lambda(t) = (\log f)'$ and $\lambda_0 = \inf_{x \in \mathbb{R}} \lambda(x)$, then $\lambda_0 \geqslant 1$.
$\log f(x+c) - \log f(x) = \int_x^{x+c} \lambda(t)\,\mathrm{d}t \geqslant c\lambda_0$, so $f'(x) \geqslant f(x+c) \geqslant e^{c\lambda_0} f(x)$. Hence $e^{c\lambda_0} \leqslant \lambda_0$, leading to contradiction.

7.3.26

Suppose $f \in C^4[0,1]$, $p(x)$ is a polynomial of degree 3, such that $p(0) = f(0)$, $p'(0) = f'(0)$, $p(1) = f(1)$, $p'(1) = f'(1)$. Prove that $|f(x) - p(x)| \leqslant \frac{1}{384} \sup_{x \in [0,1]} |f^{(4)}(x)|$.
Proof: For any $x \in (0,1)$, consider

$$F(t) = f(t) - p(t) - \lambda t^2 (1-t)^2$$

where $\lambda = \frac{f(x) - p(x)}{x^2(1-x^2)}$, then $F(0) = F(1) = F(x) = 0$, $F'(0) = F'(1) = 0$. Hence there exists $\xi_1 \in (0, x), \xi_2 \in (x, 1)$ such that $F'(\xi_1) = F'(\xi_2) = 0$, so F' has four roots. So $F^{(4)}$ has one root ξ, which means $0 = F^4(\xi) = f^{(4)}(\xi) - 24\lambda$. We obtain $\xi \in (0,1)$ such that $\lambda = \frac{1}{24} f^{(4)}(\xi)$. Therefore

$$|f(x) - p(x)| \leqslant \frac{1}{16}\lambda \leqslant \frac{1}{384} \|f^{(4)}\|_\infty.$$

7.3.27

Suppose $f \in C^\infty(\mathbb{R})$, and $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$, $f^{(k)}(0) \neq 0$. Prove that $g(x) = f(x) x^{-k} \in C^\infty$.
Proof: Note that

$$f(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,\mathrm{d}t = x^k \int_0^1 \frac{(1-u)^{k-1}}{(k-1)!} f^{(k)}(ux)\,\mathrm{d}u,$$

hence

$$g(x) = \int_0^1 \frac{(1-u)^{k-1}}{(k-1)!} f^{(k)}(ux)\,\mathrm{d}u$$

is smooth.

7.5

7.5.9

Prove that convex functions on bounded intervals are bounded.
Proof: If $f: [a,b] \to \mathbb{R}$ is convex, then $f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda) f(b) \leqslant \max\{f(a), f(b)\}$, so f is bounded from above.
Let $c = \frac{a+b}{2}$, then for $x \in (a,b)$, $2f(c) \leqslant f(x) + f(2c - x) \leqslant M + f(x)$, so $f(x) \geqslant 2f(c) - M$ is bounded from below. Hence f is bounded.

7.5.10

Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, and differentiable on $(0, \infty)$. If $f(0) = 0$, $f'(x)$ is strictly increasing, prove that $\frac{f(x)}{x}$ is strictly increasing on $(0, \infty)$.
Proof: Let $g(x) = \frac{f(x)}{x}$, then $g'(x) = \frac{xf'(x) - f(x)}{x}$, so we need to show that $f'(x) > \frac{f(x)}{x}$. By mean value theorem, there exists $\xi \in (0, x)$ such that $f'(\xi) = \frac{f(x)-f(0)}{x-0}$, so $f'(x) > f'(\xi) \geqslant \frac{f(x)}{x}$.

7.5.15

Discuss the convexity of $f(x) = -\log x$, and prove the inequality

$$\sqrt[n]{a_1 \cdots a_n} \leqslant \frac{1}{n} \sum_{k=1}^{n} a_k$$

for $a_i > 0, i = 1, \cdots, n$.
Proof: $f''(x) = \frac{1}{x^2} > 0$ so f is convex, hence by Jensen's inequality,

$$\sum_{k=1}^{n} f(a_k) \geqslant nf\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right),$$

therefore

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geqslant \sqrt[n]{a_1 \cdots a_n}.$$

7.5.16

Suppose $a_i > 0, i = 1, 2, \cdots$, prove that

$$\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \leqslant \sqrt[n]{a_1 \cdots a_n}.$$

Proof: Apply 7.5.15 to $\frac{1}{a_i}$.

7.5.17

Suppose $a_i > 0$ and $\sum_{i=1}^{n} a_i = 1$, prove that

$$\sum_{i=1}^{n} a_i x_i \geqslant \prod_{k=1}^{n} x_k^{a_k},\ \forall x_i \geqslant 0.$$

And determine when equality holds.
Solution: Note that $f(x) = -\log x$ is strictly convex, so by Jensen's inequality,

$$\sum_{k=1}^{n} a_k f(x_k) \geqslant f\left(\sum_{k=1}^{n} a_k x_k\right).$$

Therefore $\sum_{i=1}^{n} a_i x_i \geqslant \prod_{k=1}^{n} x_k^{a_k}$, and equality holds only when x_i are all equal.

7.5.19

Prove that the non-trivial global solution to $y''+a(x)y=0, a(x)\geqslant c>0$ must have infinitely many points such that $f''(x)=0$.
Proof: It suffices to show that y has infinitely many roots. First, note that $f(x)=\sin\left(\sqrt{c}x\right)$ is a solution to $f''+cf=0$, and f has infinitely many roots $x_k=k\pi/\sqrt{c}$ (which are the only roots).
We show that y has a root in $[x_k,x_{k+1})$: Otherwise suppose f,y are both positive on (x_k,x_{k+1}). Let $g=f'y-y'f$, then $g'(t)=f''(t)y(t)-y''(t)f(t)=(a-c)yf\geqslant 0$, so $g(x_{k+1})\geqslant g(x_k)=f'(x_k)y(x_k)>0$, but $g(x_{k+1})=f'(x_{k+1})y(x_{k+1})\leqslant 0$, leading to contradiction.
Therefore f has infinitely many roots.
(Note: This is a special case of the Sturm Comparison Theorem in Sturm-Liouville theory: If $y''+ay=0, z''+bz=0$ and $a(t)\geqslant b(t)$, then y oscillates faster than z, i.e. given $a,b\in C^1$, between neighboring roots of z there is a root of y. If $a(t)\to M$ monotonically, then an upper bound for the number of roots is $Z(t)/t\to\sqrt{M}/\pi$.)

7.5.22

Suppose $P_0(a,b)\in\mathbb{R}^2$, prove that $f(x)=d((x,0),(a,b))$ is convex.
Proof: $f(x)=\sqrt{(a-x)^2+b^2}$, so by Cauchy's inequality,

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y)&=\lambda\sqrt{(a-x)^2+b^2}+(1-\lambda)\sqrt{(a-y)^2+b^2}\\&\geqslant\sqrt{(\lambda(a-x)+(1-\lambda)(a-y))^2+b^2}=f(\lambda x+(1-\lambda)y).\end{aligned}$$

Hence f is convex.

7.5.23

Suppose $f\in C(\mathbb{R})$. If for any $x\in\mathbb{R}$,

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=0.$$

Prove that f is linear.
Proof: For any $a<b$, let $g(x)=\frac{f(b)-f(a)}{b-a}(x-a)+f(a)$, then $g(a)=f(a), g(b)=f(b)$, and $g(x+h)+g(x-h)-2g(x)=0$. Denote $D\varphi(x)=\lim_{h\to 0}\frac{\varphi(x+h)+\varphi(x-h)-2\varphi(x)}{h^2}$, then $Df=Dg\equiv 0$.
We show that $f\leqslant g$ on the interval $[a,b]$: for any $\varepsilon>0$, consider $\phi(x)=f(x)-g(x)+\varepsilon x^2$, then $D\phi>0$, so by lemma, $\phi(x)\leqslant\max\{\phi(a),\phi(b)\}\leqslant\varepsilon\max\{a^2,b^2\}\to 0$. Hence $f\leqslant g$.
Likewise $f\geqslant g$, so $f=g$ is linear.
Lemma: If $Df>0$ on $[a,b]$ and $f\in C([a,b])$ then $f(x)\leqslant\max\{f(a),f(b)\}$.
Proof: If f has a local maxima at x, then $Df(x)<0$. Since $f\in C([a,b])$, f reaches its maximum, so $f(x)\leqslant\max\{f(a),f(b)\}$.

7.5.24

Determine the approximating lines of $x^3+y^3=3xy$.
Solution: $x^3+y^3=(x+y)(x^2-xy+y^2)$, so asymptote is $x+y=t$. Substitute $y=t-x$ we obtain $t^3-3t^2x+3tx^2=3x(t-x)=3xt-3x^2$, so $3t=-3, t=-1$ and the asymptote is $x+y+1=0$.

7.5.26

Does there exists $f : \mathbb{R} \to \mathbb{R}$ convex, such that $f(0) < 0$ (or $f(0) > 0$) such that

$$\lim_{|x| \to \infty} (f(x) - |x|) = 0.$$

Solution: For $f(0) > 0$, the answer is positive. For example $f(x) = \sqrt{x^2 + 1}$.
For $f(0) < 0$, the answer is negative. Let $g(x) = f(x) - x \to 0$, then
$f(0) \geqslant 2f(x) - f(2x) = 2g(x) - g(2x) \to 0$, hence $f(0) \geqslant 0$.