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周民强《实变函数论（第三版》

P11 思考题1, 2；P13 思考题 1,2；P23 思考题 7,8;P25 思考题 11，14

11-1

Suppose $\{f_n(x)\}$ and $f(x)$ are real valued functions on $\mathbb{R}$, and

$$\lim_{n\to\infty} f_n(x) = f(x),\ x \in \mathbb{R},$$

then for $t \in R$,

$$\{x \in \mathbb{R} : f(x) \leqslant t\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \mathbb{R} : f_n(x) < t + 1/k\}.$$

Proof: Let $B_{n,k} = \{x \in \mathbb{R} : f_n(x) < t + 1/k\}$, then

$$\{x \in \mathbb{R} : f(x) \leqslant t\} = \bigcap_{k\geqslant 1} \{x \in \mathbb{R} : f(x) < t + 1/k\},$$

For any $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x) < t + 1/2k$ then there exists $m \geqslant 1$ such that for any $n \geqslant m$, $f_n(x) < t + 1/k$, hence

$$\{x \in \mathbb{R} : f(x) < t + 1/2k\} \subset \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} B_{n,k} \subset \{x \in \mathbb{R} : f(x) \leqslant t + 1/k\}.$$

Therefore

$$\{x \in \mathbb{R} : f(x) \leqslant t\} = \bigcap_{k\geqslant 1} \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} B_{n,k}.$$

11-2

Suppose $a_n \to a$ as $n \to \infty$, then

$$\bigcap_{k\geqslant 1} \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} \left(a_n - \frac{1}{k}, a_n + \frac{1}{k}\right) = \{a\}.$$

Proof: $\lim_{n\to\infty} a_n = a$ iff for any $k \geqslant 1$ there exists $m \geqslant 1$ such that for any $n \geqslant m$, $a \in B_{n,k} = (a_n - 1/k, a_n + 1/k)$, i.e.

$$a \in \bigcap_{k\geqslant 1} \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} B_{n,k}.$$

Since the limit is unique,

$$\bigcap_{k\geqslant 1} \bigcup_{m\geqslant 1} \bigcap_{n\geqslant m} B_{n,k} = \{a\}$$

13-1

For $f : \mathbb{R} \to \mathbb{R}$, let $f_1(x) = f(x)$, $f_n(x) = f(f_{n-1}(x))$. If there exists n_0 such that $f_{n_0}(x) = x$, then f is injective.

Proof: If $f(x) = f(y)$, then $x = f_{n_0}(x) = f_{n_0-1}(f(x)) = f_{n_0-1}(f(y)) = y$, hence f is injective.

13-2

Prove that there does not exists a continuous function f on $\mathbb{R}$, such that it is a bijection on $\mathbb{Q}^C$, but not on $\mathbb{Q}$.
Proof: Suppose f is bijective on $\mathbb{Q}^C$, $f(\mathbb{R})$ should be connected, hence an interval. Since $\mathbb{Q}^C \subset f(\mathbb{R})$ which is dense in $\mathbb{R}$, $f(\mathbb{R}) = \mathbb{R}$, therefore f is surjective on $\mathbb{R}$.
If $a < b$ and $f(a) = f(b)$, then take $c \in (a, b)$ such that $f(c) \neq f(a)$. Assume $f(c) > f(a)$, then for any $y \in I = (f(a), f(c)) \cap \mathbb{Q}^C$, there exists $u \in (a, c)$ and $v \in (c, b)$ such that $f(u) = f(v) = y \notin \mathbb{Q}$, where at least one of u, v is rational. Hence for each $y \in I$, there exists $u \in (a, b) \cap \mathbb{Q}$ such that $f(u) = y$, but I is uncountable while $\mathbb{Q}$ is countable, a contradiction. Therefore f is bijective on $\mathbb{Q}$.

23-7

Determine whether there is a function $f \in C(\mathbb{R})$ such that

$$f(x) \begin{cases} \in \mathbb{Q}^C, & x \in \mathbb{Q}, \\ \in \mathbb{Q}, & x \in \mathbb{Q}^C. \end{cases}$$

Solution: The answer is no. Since $f(\mathbb{R}) \subset \mathbb{Q} \cup f(\mathbb{Q})$ is countable, and $f(I)$ is connected for any interval I, f is constant, leading to contradiction.

23-8

Suppose $E \subset (0, 1)$ is an infinite set. If for any sequence $\{a_n\} \subset E$, $\sum_{n=1}^{\infty} a_n$ converges, then E is countable.
Proof: Let $E_n = E \cap (1/n, 1)$ then for any $n \geqslant 1$, E_n is finite, and

$$E = \bigcup_{n \geqslant 1} E_n.$$

Hence E is countable.

25-11

Suppose $E \subset \mathbb{R}$, and $|E| < |\mathbb{R}|$. Prove that there exists $a \in \mathbb{R}$ such that $E + a \subset \mathbb{R} \backslash \mathbb{Q}$.
Proof: $E + a \cap \mathbb{Q} \neq \infty \iff a \in A = \{q - e : (q, e) \in \mathbb{Q} \times E\}$. Since $|\mathbb{Q} \times E| = \max\{|\mathbb{Q}|, |E|\}$, $\mathbb{R} \backslash A$ is nonempty, and we can take $a \in A^C$.

25-14

Prove that the cardinal of transcendental numbers is $|\mathbb{R}|$.
Proof: It suffices to show that algebraic numbers are countable. The set of polynomials with integer coefficients $\mathbb{Q}[x]$ is countable, since

$$\mathbb{Q}[x] = \bigcup_{n \geqslant 0} \mathbb{Q}_n[x]$$

and $\mathbb{Q}_n[x] \sim \mathbb{Z}^{n+1}$.
For any algebraic number α, map it to (P, k) where P is its minimal polynomial, and α is the k-th root of P, ($\alpha = re^{i\theta}$ ordered by first r then $\theta \pmod{2\pi}$). We obtain an injection from all algebraic numbers to $\mathbb{Q}[x] \times \mathbb{N}$, hence it is countable.