6.1: 2, 3-8 >= 5道; 6.2: 1的两小题,2,3,4-9 >= 3道;
6.3: 1-5,7-9每题>=2小题. 其余选>=6道.
6.4: >=3题. 6.5: 1-9>=8题,10,11

6.1

6.1.2

尺规作图画出 $f(x)$ 在 $(t, f(t))$ 处的切线:
(1) $f(x)=x^k$ (2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (3) $f(x)=\sin x$ (4) $f(x)=\tan x$ (5) $f(x)=e^x$ (6) $f(x)=\log x$.
引理：尺规作图可以做加法、数乘有理数，且有单位长度时，可以做乘法、倒数、开根. 且做切线只需作出斜率的长度.
证明：加法：拼接两条线段即可。数乘正整数：若干次加法。
除正整数：对于线段 AB, 过 A 作射线 AC_1, 并在射线上取 $C_2, \cdots, C_n$ 使得 $AC_k = kAC_1$. 过 C_1 做 BC_n 的平行线交 AB 于 D 则 $AD = AB/n$. 故可以数乘任意有理数。
下面设已有单位长度:
倒数：作以 $x, 1$ 为直角边的直角三角形 $ABC, AC=1, BC=x$, 作 $AD \perp AB$ 交 BC 于 D 则 $CD = 1/x$.
乘法：做 $OA=a, OB=b$, 在 OB 上取 E 使得 $OE=1$, 过 B 作 AE 平行线交 OA 于 D, 则 $OD=ab$.
开根：设射线 OA 上 $OA=a, OB=1$, 以 OA 为直径作圆，过 B 作 OA 垂线交圆于 C, 则 $BC=\sqrt{a}$.
若要作切线, $l: y = f'(t)(x-t)+f(t)$, 只需画出 $f'(t)$, 则会有 $(t, f(t))$ 和 $(0, -tf'(t)+f(t))$, 连接可得到 l.
原题:
(1) $f'(x) = kx^{k-1} = k \cdot x \cdots x, -tf'(t)+f(t) = (1-k)t^k$ 可以作出.
(2) 用Pascal定理.
(3) 找到正半轴首个根 π, 取中点得到 $\pi/2$, 再做坐标轴垂线得到单位长度. $f'(t) = \cos t = \sqrt{1-\sin^2 t}$ 可以作出.
(4) 同样通过根找到 π, 再用 $\tan\frac{\pi}{4}=1$ 找到单位长度. $f'(t) = \sec^2 t = 1+\tan^2 t$ 可以作出.
(5) $f(0)=1$ 是单位长度, $f'(t)=e^t$ 可以作出.
(6) 唯一一个根是单位长度 1, $f'(x)=1/x$ 可以作出.

6.1.3

Suppose f is differentiable at x_0, calculate $\lim_{h\to 0}(f(x_0+h)-f(x_0-h))/h$.
Solution: Note that

$$\frac{f(x_0+h)-f(x_0-h)}{h} = \frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0)-f(x_0-h)}{h}$$

hence the limit is $2f'(x_0)$.

6.1.4

Suppose f is continuous at 0 and $\lim_{x\to 0}(f(2x)-f(x))/x = k$, prove that $f'(0)=k$.
Proof: Let $f(2x)-f(x) = xk + xh(x)$, then $h(x)\to 0$, and

$$f(x) = f(2^{-n}x) + \sum_{k=0}^{n-1} 2^{-k-1}xh(2^{-k-1}x)$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $x < \delta$ implies $-\varepsilon < h(x), f(x) < \varepsilon$, then

$$\left|\frac{f(x)-f(0)}{x}-k\right| \leqslant \left|\frac{f(2^{-n}x)-f(0)}{x}-k\right| + \sum_{k=0}^{n-1} 2^{-k-1}|h(2^{-k-1}x)| \leqslant \left|\frac{f(2^{-n}x)-f(0)}{x}-k\right| + 4\varepsilon.$$

Let $n \to \infty$ then $|(f(x) - f(0))/x - k| < 4\varepsilon$. Hence $f'(0) = k$.

6.1.5

Prove that if $f(x)$ is differentiable and even/odd/periodic, then $f'(x)$ is odd/even/periodic.
Proof: If $f(x) = -f(-x)$, then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x).$$

If $f(x) = f(-x)$, then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} -\frac{f(x) - f(x-h)}{h} = -f'(x).$$

If $f(x) = f(x+T)$ then

$$f'(x+T) = \lim_{h \to 0} \frac{f(x+T+h) - f(x+T)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

6.1.7

Suppose $g(0) = g'(0) = 0$, and

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine the value of $f'(0)$.
Solution:

$$f'(0) = \lim_{h \to 0} \frac{g(h)}{h} \sin \frac{1}{h} = 0,$$

since $g(h)/h \to 0$ and $\sin \frac{1}{h}$ is bounded.

6.1.8

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y), \forall x, y \in \mathbb{R}$. If $f'(0) = 1$, prove that $f'(x) = f(x)$ for any $x \in \mathbb{R}$.
Proof: $f'(0) = 1$ so f is not constant. Hence let $x = 0$ we have $f(y) = f(0)f(y)$ so $f(0) = 1$. Note that $f(x) = f(x/2)^2 \geqslant 0$, so we can let $g(x) = \log f(x)$, then $g'(x) = \frac{f'(x)}{f(x)}$ and $g'(0) = 1, g(0) = 0$, $g(x+y) = g(x) + g(y)$. Since g is continuous at 0, $g(x) = cx$ so $g(x) = x \forall x \in \mathbb{R}$, hence $g'(x) = 1$ and $f'(x) = f(x)$.

6.2

6.2.1

(1) $f(x) = x^{1/3}$, $f'_+(0)$:

$$f'_0(x) = \lim_{h \to 0^+} \frac{h^{1/3}}{h} = \infty.$$

(2) $f(x) = |x-2|^3$, $f'_-(2)$:

$$f'_-(2) = \lim_{h\to 0^+} \frac{-h^3}{h} = 0.$$

6.2.2

Determine the values of a, b such that

$$f(x) = \begin{cases} x^2, & x \geqslant 1, \\ ax+b, & x < 1 \end{cases}$$

is differentiable at $x = 1$.
Solution: Clearly $f'_+(1) = 2$, while

$$f'_-(1) = \lim_{h\to 0^+} \frac{f(1) - f(1-h)}{h} = \lim_{h\to 0^+} \frac{1 - a(1-h) - b}{h} = 2$$

Hence $a = 2$ and $1 - a - b = 0$, so $(a, b) = (2, -1)$.

6.2.3

Suppose f is continuous at 0. Prove that $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ exists if $\lim_{x\to 0} \frac{f(x)-f(x-x^3)}{x^3}$ exists.
Proof: Let $f(x) - f(0) = xg(x)$, and $f(x) - f(x - x^3) = x^3 h(x)$.
If $\lim_{x\to 0} \frac{f(x)-f(x-x^3)}{x^3} = L$ exists, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $|h(x) - L| < \varepsilon$. Consider $x = x_0 \in (0, \delta)$, and $x_{k+1} = x_k - x_k^3$, then clearly $\lim_{n\to\infty} x_n = 0$. Note that

$$xg(x) = \sum_{k=0}^{\infty} f(x_k) - f(x_{k+1}) = \sum_{k=0}^{\infty} x_k^3 S(x_k) < (L+\varepsilon) \sum_{k=0}^{\infty} x_k - x_{k+1} = (L+\varepsilon)x$$

and likewise $xg(x) > (L - \varepsilon)x$, hence $|g(x) - L| < \varepsilon$. Therefore $\lim_{x\to 0} g(x) = L$.
(The reverse is incorrect, for example $x^2 \sin x^{-1}$).

6.2.4

Calculate the Dini derivatives of f at $x = 0$:
(1) $f(x) = x^2 \sin x^{-1}$; (2) $f(x) = |x|^{3/2}$; (3) $f(x) = e^{-|x|^2}$.
Solution: (1) $|f(x)| \leqslant |x|^2$, so all four Dini derivatives are 0.
(2)(3) Likewise all four Dini derivatives are 0.

6.2.5

Calculate the Dini derivatives of f:
(1) $f(x) = \begin{cases} x^3, & x \geqslant 0, \\ -x^3, & x < 0 \end{cases}$ at $x = 0$:
(2) $f(x) = \sin x \cos x$ at $x = \pi$:
(3) $f(x) = x^2 \log|x|$ at $x = 0$:
Solution:
(1) $|f(x)/x| \leqslant |x^2|$ so all Dini derivatives are 0.
(2) $f'(\pi) = \cos 2\pi = 1$ so all Dini derivatives are 1.
(3) $|f(x)/x| = |x| \log|x| \to 0$ so all Dini derivatives are 0.

6.2.6

Calculate the symmetric derivatives of f:
(1) $f(x)=\begin{cases}x^3, & x\geqslant 0,\\ -x^3, & x<0\end{cases}$ at $x=0$:
(2) $f(x)=\sin x^2$ at $x=\pi$:
(3) $f(x)=\log|x^2-1|$ at $x=1$:
(4) $f(x)=x^2e^{-|x|}$ at $x=0$:
Solution: (1) $f(x)-f(-x)=0$ so the derivative is 0.
(2) f is differentiable so the symmetric derivative is $f'(\pi)=2x\cos x^2|_\pi=2\pi\cos\pi^2$.
(3) $f(1+t)-f(1-t)=\log|2t+t^2|-\log|2t-t^2|=\log(2+t)/(2-t)$, and

$$\lim_{h\to 0}\frac{\log|2+h|-\log|2-h|}{2h}=\frac{1}{2}.$$

So the derivative is $1/2$.
(4) f is even so the derivative is clearly 0.

6.3

Calculate derivatives:

6.3.1

If $f=\prod_{k=1}^n g_k$, then

$$\frac{f'}{f}=\sum_{k=1}^n\frac{g_k'}{g_k}.$$

(1) $f(x)=(x+1)(x+2)^2(x+3)^2$

$$f'(x)=f(x)\left(\frac{1}{x+1}+\frac{2}{x+2}+\frac{2}{x+3}\right).$$

(2) $f(x)=(1+nx^m)(1+mx^n)$

$$f'(x)=f(x)nm\left(\frac{x^{m-1}}{1+nx^m}+\frac{x^{n-1}}{1+mx^n}\right).$$

(3) $f(x)=(ax^m+b)^n(cx^n+d)^m$

$$f'(x)=nmf(x)\left(\frac{ax^{m-1}}{ax^m+b}+\frac{cx^{n-1}}{cx^n+d}\right).$$

6.3.2

(1) $f(x)=\sqrt{\sin^2 x}$
$f'(x)=\operatorname{sign}(\sin x)\cos x$ when $x\neq k\pi$.
(2) $f(x)=\frac{x}{(1-x^2)(1+x)}$.

$$f'(x)=\frac{2x^2-x+1}{(1-x)^2(1+x)^3}.$$

(3) $f(x)=x^p(1-x)^q/(1+x)$

$$f'(x)=-\frac{x^{p-1}(1-x)^{q-1}((p+q-1)x^2+(q+1)x-p)}{(x+1)^2}.$$

(4) $f(x)=\frac{x\sin x+\cos x}{x\sin x-\cos x}$.

$$f'(x)=-\frac{2x+\sin 2x}{(x\sin x-\cos x)^2}.$$

6.3.3

(1) $f(x)=\frac{x}{\sqrt{a^2+x^2}}$: $f'(x)=a^2(a^2+x^2)^{-3/2}$.
(2) $f(x)=\arcsin\left(\cos^2 x\right)$: $f'(x)=-\sin 2x/\sqrt{1-\cos^4 x}$.
(3) $f(x)=\arcsin x\sqrt{1-x^2}$: $f'(x)=\frac{1-2x^2}{\sqrt{1-x^2}\sqrt{x^4-x^2+1}}$.
(4) $f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}$: $f'(x)=\frac{1+2\sqrt{x}+4\sqrt{x+\sqrt{x}}\sqrt{x}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$.
(5) $f(x)=e^{ax}\sin bx$: $f'(x)=e^{ax}(a\sin bx+b\cos bx)$.

6.3.4

(1) $f(x)=(x\sin x)^x$: $f'(x)=(x\sin x)^x(x\cot x+\log(x\sin x)+1)$.
(2) $f(x)=x\sqrt{(1-x)/(1+x)}$: $f'(x)=\frac{1-x-x^2}{(1+x)^{3/2}\sqrt{1-x}}$.
(3) $f(x)=(x)^{a^x}$: $f'(x)=a^x x^{a^x-1}(x\log a\log x+1)$.
(4) $f(x)=a^{x^n}$: $f'(a^{x^n}nx^{n-1}\log a)$.
(5) $f(x)=(\cos x)^{\sin x}+(\sin x)^{\cos x}$:

$$f'(x)=(\cos x)^{\sin x}(\cos x\log\cos x-\sin x\tan x)+(\sin x)^{\cos x}(\cos x\cot x-\sin x\log\sin x).$$

(6) $f(x)=(x+\sqrt{1+x^2})^n$: $f'(x)=\frac{n(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}}$.

6.3.5

Suppose $f(x)=\cos\left(2\arctan\left(\sin\left(\operatorname{arccot}\sqrt{(1-x)/x}\right)\right)\right)$, prove that $\frac{f'(x)}{f(x)^2}=\pm\frac{2}{(1\pm x)^2}$.
Proof: $f(x)=\frac{1-x}{1+x}$ so $\frac{f'}{f^2}=-\frac{2}{(1-x)^2}$.

6.3.6

Suppose $y=\frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}-\frac{1}{2\sqrt{2}}\arctan\frac{\sqrt{2}x}{x^2-1}$, calculate $y'(x)$.
Solution:

$$y'(x)=\frac{1}{x^4+1}.$$

6.3.7

Determine dy/dx:
(1) $\begin{cases}x=\cos^4 t,\\ y=\sin^4 t\end{cases}$, at $t=\pi/3$:
$dx/dt=4\cos^3 t(-\sin t)=-\frac{\sqrt{3}}{4}$, $dy/dt=4\sin^3 t\cos t$, so $dy/dx=-3$.
(2) $\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$, at $t=\pi/2,\pi$:

$\frac{dx}{dt} = a(1-\cos t)$, $\frac{dy}{dt} = a\sin t$.
$t = \frac{\pi}{2}$: $\frac{dy}{dx} = 1$. $t = \pi$: $\frac{dy}{dx} = 0$.

6.3.8

Suppose f is differentiable, calculate y':
(1) $y = f(x\sin x)$: $y' = (\sin x + x\cos x)f'(x\sin x)$.
(2) $y = f(e^x)e^{f(x)}$: $y' = f'(e^x)e^{x+f(x)} + f(e^x)e^{f(x)}f'(x)$.

6.3.9

Suppose u, v differentiable, calculate y':
(1) $y = \sqrt{u^2+v^2}$: $y' = \frac{uu'+vv'}{\sqrt{u^2+v^2}}$.
(5) $y = 1/\sqrt{u^2+v^2}$: $y' = -\frac{u'u+v'v}{(u^2+v^2)^{3/2}}$.

6.3.10

Suppose $a_{ij}(x)$ are differentiable, determine the derivative of $\det\left(a_{ij}(x)\right)_{n\times n}$.
Solution: Let $\varphi_i(x) = (a_{i1}(x), \cdots, a_{in}(x))$, then $\det(\varphi_1, \cdots, \varphi_n)$ is a multi-linear form, so

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=x_0} \det(\varphi_1, \cdots, \varphi_n) = \sum_{k=1}^{n} \det\left(\varphi_1, \cdots, \varphi_k', \cdots, \varphi_n\right).$$

6.3.14

Suppose $x_1, \cdots, x_n$ are distinct, and $y(x) = \prod_{k=1}^{n}(x-x_k)$. Calculate $\sum_{k=1}^{n} 1/y'(x_k)$. What if x_k are not distinct?
Solution: Note that

$$\frac{y'}{y} = \sum \frac{1}{x-x_k},$$

hence $y'(x_k) = \prod_{j\neq k}(x_k - x_j)$.
By Lagrange interpolation,

$$1 = \sum_{k=1}^{n}\prod_{j\neq k}\frac{x-x_j}{x_k-x_j},$$

so by considering the coefficient of x^{n-1}, we obtain

$$\sum_{k=1}^{n} 1/y'(x_k) = \begin{cases} 1, & n=1, \\ 0, & n>1. \end{cases}$$

If $f(x) = \prod_{k=1}^{n}(x-x_k)^{\alpha_k}$, then $f'(x_k) = \alpha_k\prod_{j\neq k}(x_k-x_j)^{\alpha_j}$, hence

$$\sum \frac{1}{f'(z_k)} = \sum_{k=1}^{n}\frac{\alpha_k}{f'(x_k)} = \sum_{k=1}^{n}\frac{1}{\prod_{j\neq k}(x_k-x_j)^{\alpha_j}} = \begin{cases} 1, & \deg f = 1, \\ 0, & \deg f > 1. \end{cases}$$

6.3.19

Suppose $f(x_0)=g(x_0)$, f,g are differentiable at x_0, and $f'(x_0)=g'(x_0)=k$. Prove that for any h such that $\min\{f,g\}\leqslant h\leqslant\max\{f,g\}$, h is differentiable at x_0 and $h'(x_0)=k$.
Proof: Clearly $h(x_0)=f(x_0)$ and we can assume it to be 0. For any t,

$$\min\left\{\frac{f(x_0+t)}{t},\frac{g(x_0+t)}{t}\right\}\leqslant\frac{h(x_0+t)-h(x_0)}{t}=\frac{h(x_0+t)}{t}\leqslant\max\left\{\frac{g(x_0+t)}{t},\frac{f(x_0+t)}{t}\right\}.$$

Since $\lim_{t\to 0}f(x_0+t)/t=\lim_{t\to 0}g(x_0+t)/t$, the two sides of the inequality both tend to k, therefore $\lim_{t\to 0}(h(x_0+t)-h(x_0))/t=k$. Hence $h'(x_0)=k$.

6.3.22

Prove that there is no $f:\mathbb{R}\to\mathbb{R}$ differentiable such that $f\circ f(x)=-x^3+x^2+1$.
Proof: $f'(f(x))f'(x)=-3x^2+2x$, and $f(f(x))=x\iff x=1$. If $a=f(1)$ then $f(f(a))=f(f(f(1)))=f(1)=a$, so by uniqueness $a=1$. However, $-1=f'(f(1))f'(1)=(f'(1))^2$, leading to contradiction.

6.3.23

Prove that there is no $f:\mathbb{R}\to\mathbb{R}$ differentiable such that $f\circ f(x)=x^2-3x+3$.
Proof: Suppose $f(f(x))=x^2-3x+3$, then $f'(f(x))f'(x)=2x-3$, but $f(f(x))=x\iff x=1,3$. Suppose $a=f(1)$, then $f(a)=1$, so $-1=f'(f(1))f'(1)=f'(a)f'(1)$, and $2a-3=f'(f(a))f'(a)=f'(a)f'(1)$, leading to contradiction.

6.4

6.4.1

If $x^y=y^x$, prove that $x(y\log x-x)dy=y(x\log y-y)dx$.
Proof: $x\log y=y\log x$, so $y'\log x+y/x=\log y+xy'/y$. Hence $dy(\log x-x/y)=dx(\log y-y/x)$. Therefore $x(y\log x-x)dy=y(x\log y-y)dx$.

6.4.2

If $y=\frac{x-a}{1+ax}$, then $\frac{dx}{1+x^2}=\frac{dy}{1+y^2}$.
Proof: Note that $y=\tan(\arctan x-\arctan a)$, so $\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$ since $d\arctan y=d\arctan x$.
($ds=\frac{dx}{1+x^2}$ is the angular distance in spherical geometry, and clearly rotation on the sphere is an isometry. We can do likewise for hyperbolas and the Blaschke factors.)

6.4.3

If $xy=-1$ prove that $\frac{dy}{\sqrt{1+y^4}}=\frac{dx}{\sqrt{1+x^4}}$.
Proof: Clearly $\frac{dy}{dx}=x^{-2}$, so $\frac{dy}{\sqrt{1+y^4}}=\frac{dx}{\sqrt{1+x^4}}$.
($ds=\frac{dx}{\sqrt{1+x^4}}$ is the parametrization of the lemniscate, and $y=-1/x$ is the inversion which is an isometry).
Note: For the next 3 problems, they are all about preserving the arc length of the lemniscate.
6.4.4 comes from $(1+\mathrm{sl}^2(u))(1+\mathrm{cl}^2(u))=2$;

6.4.5 comes from the rotation $\mathrm{sl}(u+v)=\frac{\mathrm{sl}(u)\mathrm{cl}(v)+\mathrm{cl}(u)\mathrm{sl}(v)}{1-\mathrm{sl}(u)\mathrm{sl}(v)\mathrm{cl}(u)\mathrm{cl}(v)}$;
6.4.6 comes from duplication $\mathrm{sl}(2u)=\frac{2\mathrm{sl}(u)\mathrm{cl}(u)}{1+\mathrm{sl}^4(u)}$.

6.5

6.5.1

(1) $f(x)=e^x\cos x$, $f^{(5)}(x)=$
$f(x)=\mathrm{Re}(e^{(1+i)x})$, then $f^{(5)}(x)=\mathrm{Re}((-4-4i)e^{(1+i)x})=4e^x(\sin x-\cos x)$.
(2) $f(x)=x^2\log x+x\log^2 x$, $f''=$
$f''(x)=2\log x+3+2\frac{\log x+1}{x}$.
(3) $f(x)=x^2e^x$, $f^{(10)}=$
$f^{(10)}(x)=\sum_{k=0}^{10}\binom{10}{k}(x^2)^{(k)}(e^x)^{(10-k)}=e^x(x^2+20x+90)$.
(4) $f(x)=x^5\cos x$, $f^{(50)}=$
Likewise $f^{(50)}(x)=(-x^5+20\binom{50}{2}x^3-120\binom{50}{4})\cos x+(-250x^4+60\binom{50}{3}x^2-120\binom{50}{5})\sin x$.

6.5.2

Prove that
(1) $\sum_{k=0}^n\binom{n}{k}k=n2^{n-1}$ (2) $\sum_{k=0}^n\binom{n}{k}k^2=n(n+1)2^{n-2}$.
Proof: $(1+x)^n=\sum_{k=0}^n\binom{n}{k}x^k$, hence

$$\sum_{k=0}^n\binom{n}{k}kx^k=xn(1+x)^{n-1},\sum_{k=0}^n\binom{n}{k}k^2x^{k-1}=n(1+x)^{n-1}+xn(n-1)(1+x)^{n-2}.$$

Let $x=1$ we obtain the desired identities.

6.5.3

Prove that $y=\cos(n\arccos x)$ satisfy

$$\frac{n\mathrm{d}x}{\sqrt{1-x^2}}=\frac{\mathrm{d}y}{\sqrt{1-y^2}},(1-x^2)y''-xy'+n^2y=0.$$

Proof: $y=P_n$ is the n^{th} Chebyshev polynomial.
Note that $\arccos y=n\arccos x$, hence $\frac{\mathrm{d}y}{\sqrt{1-y^2}}=\mathrm{d}\arccos y=n\frac{\mathrm{d}x}{\sqrt{1-x^2}}$.
Let $x=\cos\theta$, and $y=\cos(n\theta)$, then $y'=-\sin(n\theta)n\theta'=n\sin(n\theta)/\sin\theta$. Differentiate $y'\sin\theta=n\sin n\theta$, we obtain

$$y''\sin\theta+y'\theta'\cos\theta=n^2\theta'\cos(n\theta)\implies y''\sqrt{1-x^2}-\frac{y'x}{\sqrt{1-x^2}}=-\frac{n^2y}{\sqrt{1-x^2}}.$$

Hence $(1-x^2)y''-xy'+n^2y=0$.
(We can use the fact the P_n are orthogonal under the weight $w=\frac{1}{\sqrt{1-x^2}}$, and solve the Pearson differential equation $\frac{d}{dx}(\sigma w)=\tau w$ to get $\sigma=1-x^2,\tau=-x$ and directly obtain this differential equation $(1-x^2)y''-xy'+n^2y=0$.)

6.5.4

Prove that: For points (x, y) on the circle $(x-a)^2+(y-b)^2=c^2$, $y \neq b$,
(1) $y''/(1+(y')^2)^{3/2}=\pm 1/c$; (2) $y'''(1+(y')^2)=3y'(y'')^2$.
Proof: (1) $1+(y')^2=1+\frac{(x-a)^2}{(y-b)^2}=\frac{c^2}{(y-b)^2}$., and $y''=-\frac{c^2}{(y-b)^3}$, hence $y''/(1+(y')^2)^{3/2}=\pm 1/c$. Its absolute value is the curvature at (x, y).
(2) $y'''=3c^2y'/(y-b)^4$, so $y'''(1+(y')^2)=3c^4y'/(y-b)^6=3y'(y'')^2$.

6.5.5

Given the polar coordinate form $r^2=\cos 2\theta$ of $y=f(x)$, calculate $\frac{d^2y}{dx^2}$.
Solution: $r^2=\cos 2\theta \implies x^2+y^2=\frac{1-y^2/x^2}{1+y^2/x^2}$. Hence $(x^2+y^2)^2=x^2-y^2$. Therefore $\frac{d^2y}{dx^2}=-\frac{3r}{\sin^3(3\theta)}$.

6.5.6

Suppose $f(x)=x\log\left(2^{1/x}+3^{1/x}\right), \forall x>0$ prove that $f'(x), f''(x)>0$.
Proof: Let $g(x)=\log(2^x+3^x)/x$ and $h(x)=\log(2^x+3^x)$. Note that by Cauchy-Schwarz,

$$h''(x)=\frac{(2^x\log^2 2+3^x\log^2 3)(2^x+3^x)-(2^x\log 2+3^x\log 3)^2}{(2^x+3^x)^2}>0.$$

Since $f''(x)=x^{-3}h''(x^{-1})$, we infer $f''(x)>0$.
Let $F(t)=f'(t^{-1})=h(t)-th'(t)$, then $F'(t)=-th''(t)<0$, hence F is strictly decreasing. Clearly $\lim_{t\to\infty}F(t)=0$, so $f'(x)=F(x^{-1})>0$.

6.5.7

Suppose P is a polynomial of degree n with n distinct roots. Prove that $P'(x)^2 \geqslant P(x)P''(x)$.
Proof: Suppose $P(x)=\prod_{k=1}^n(x-x_k)$, and $P(t)\neq 0$. Then

$$\frac{PP''-P'(t)^2}{P(t)^2}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{P'(t)}{P(t)}=\frac{d}{dt}\sum_{k=1}^n\frac{1}{t-x_j}=\sum_{k=1}^n-(t-x_j)^{-2}<0.$$

Therefore $P'(t)^2\geqslant P(t)P''(t)$.

6.5.9

Suppose $f'(x)\neq 0$ and f^{-1} exists. Express $(f^{-1})^{(3)}$ in terms of $f', f'', f^{(3)}$.
Solution:

$$(f^{-1})^{(3)}(y)=\frac{3[f''(f^{-1}(y))]^2-f'(f^{-1}(y))f^{(3)}(f^{-1}(y))}{[f'(f^{-1}(y))]^5}$$

6.5.10

$a, b\in\mathbb{R}$, $a>0$. Consider $f:[-1,1]\to\mathbb{R}$, where

$$f(x)=\begin{cases}x^a\sin\left(x^{-b}\right), & x\neq 0,\\ 0, & x=0.\end{cases}$$

Prove that

1. $f \in C([-1,1])$ iff $a > 0$;
   Proof: $f \in C([-1,1])$ iff $\lim_{x\to 0} x^a \sin\left(x^{-b}\right) = 0$. If $a > 0$ then $|x^a \sin\left(x^{-b}\right)| \leqslant |x|^a \to 0$. If $a < 0$ then let $x = ((2n+1/2)\pi)^{-1/b}$, when $n \to \infty$, $x \to 0$ but $|x^a \sin\left(x^{-b}\right)| \to \infty$. If $a = 0$, then let $x = ((2n+1/2)\pi)^{-1/b}$, $|x^a \sin\left(x^{-b}\right)| = 1$. Therefore $f \in C([-1,1])$ iff $a > 0$.
2. f is differentiable at 0 iff $a > 1$;
   Proof: f is differentiable at 0 iff $\lim_{x\to 0} x^{-a} \sin\left(x^{-b}\right)$ exists. By 1 we know that $a > 1$. ($a = 1$ is invalid since $x = (2n\pi)^{-1/b}$ and $x = ((2n+1/2)\pi)^{-1/b}$ converge to different values.)
3. f' is bounded on $[-1,1]$ ($\iff$ f is Lipschitz) iff $a \geqslant 1+b$;
   Proof: $f'(x) = ax^{a-1}\sin\left(x^{-b}\right) + x^a \cos\left(x^{-b}\right)(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geqslant 1+b$.
4. $f \in C^1([-1,1])$ iff $a > 1+b$;
   Proof: $f \in C^1([-1,1])$ iff $f'(0) = 0 = \lim_{x\to 0} f'(x)$. By 1 we know it is equivalent to $a > 1+b$.
5. f' is differentiable at 0 iff $a > 2+b$;
6. f'' is bounded on $[-1,1]$ iff $a \geqslant 2+2b$;
7. $f \in C^2([-1,1])$ iff $a > 2+2b$.
   Proof: 5,6,7 are exactly the same as 2,3,4.
   Likewise $f \in C^n([-1,1])$ iff $a > n(1+b)$, $f^{(n-1)}$ differentiable at 0 iff $a > n+(n-1)b$, and $f^{(n)}$ is bounded iff $a \geqslant n+nb$.

6.5.11

Suppose $f(x) = \prod_{k=1}^{n}(x - x_k)$ where x_k are distinct, and $\xi_1, \cdots, \xi_{n-1}$ are the roots of $f'(x)$.
Calculate $\sum_{k=1}^{n-1} 1/f''(\xi_k)$.
Solution: Same as 6.3.14, use Lagrange interpolation, the sum for the k^{th} derivative is $\frac{\delta_{k,n}}{k!}$.