2025/9/30
教材5页6; 16页3,10. 写出定理"对F^n的任意非零子空间W, 存在唯一的无零行的行简化阶梯矩阵, 其行向量构成W的基"证明的细节.

5-6

Prove that if two homogeneous systems of linear equations in two unknowns have the same solutions, then they are equivalent.
Proof: The solutions of any set of linear equations $a_i x + b_i y = 0$ is $V^\perp$ where V is the subspace spanned by (a_i, b_i). Hence if $V^\perp = V'^\perp$ then $V = V'$ so any (a'_i, b'_i) is a linear combination of (a_j, b_j). Hence the two systems of linear equations are equivalent.

16-3

Describe explicitly all 2×2 row-reduced echelon matrices.
Solution:

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

16-10

Suppose R and R' are 2×3 row-reduced echelon matrices and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.
Proof: There are only a few choices of R:

- $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ the solutions are $\mathbb{R}^3$.
- $R = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix}$ the solutions are $\mathrm{Span}\langle(-a, 1, 0), (-b, 0, 1)\rangle$.
- $R = \begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}$ the solutions are $\mathrm{Span}\langle(0, -b, 1), (1, 0, 0)\rangle$.
- $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ the solutions are $\{(x, y, 0) : x, y \in \mathbb{R}\}$.
- $R = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$ the solutions are $\mathrm{Span}\langle(-a, -b, 1)\rangle$.
- $R = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the solutions are $\mathrm{Span}\langle(-a, 1, 0)\rangle$
- $R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the solutions are $\{(x, 0, 0) : x \in \mathbb{R}\}$.
  All are distinct, hence $R = R'$.

对F^n的任意非零子空间W, 存在唯一的无零行的行简化阶梯矩阵, 其行向量构成W的基.
Proof: Take a base $\{u_1, \cdots, u_n\}$ of W.
In the $k-$th step, suppose u_k has the smallest non-zero index among $u_k, \cdots, u_n$, and i being the first non-zero coordinate of u_k. Let $v_k = c^{-1} u_k$ and $v_j = u_j - c_j \cdot u_k$ where c_j is the $i-$th coordinate of u_j. Replace $\{u_1, \cdots, u_n\}$ with $\{v_1, \cdots, v_n\}$.

After n steps, we obtain a set of venture $\{u_1, \cdots, u_n\}$ that form a base of W, and $\begin{pmatrix} u_1 \\ \cdots \\ u_n \end{pmatrix}$ is a row-reduced echelon matrix.