PSA: Topology on Metric Spaces

A1) Suppose (X, d_x) and (Y, d_Y) are metric spaces, $f : X \to Y$ is a mapping. Prove that the two following definitions of continuity is equivalent:

- Suppose $x_0 \in X$, if for any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$, then we say f is continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
- Suppose $x_0 \in X$, $y_0 = f(x_0) \in Y$. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $d_X(x, x_0) < \delta$, $x \in X$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, we call f continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
  Proof: 1=>2: If there exists $\varepsilon > 0$ such that for any $n \geqslant 1$, there exists x_n such that $d_X(x_0, x_n) < 1/n$ but $d_Y(f(x_n), f(x_0)) > \varepsilon$, then $\lim_{n\to\infty} x_n = x_0$ but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$, a contradiction.
  2=>1: For any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, and any $\varepsilon > 0$, take the corresponding δ and N such that $n > N \implies d(x_n, x_0) < \delta$. Then for any $n > N$, $d(x_n, x_0) < \delta$ so $d(f(x_n), f(x_0)) < \varepsilon$, hence $\lim_{n\to\infty} f(x_n) = f(x_0)$.

A2) (X, d) is a metric space. For any $x \in X$, $r > 0$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. Proved that for any $x \in X$, $r > 0$, if $x' \in B(x, r)$, then there exists $r' > 0$ such that $B(x', r') \subset B(x, r)$.

If $U = \bigcup_{\alpha\in\mathcal{A}} B(x_\alpha, r_\alpha)$, then we call U an open set. Prove that $U \subset X$ is open iff for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$.
Proof: If $x' \in B(x, r)$, let $r' = r - d(x, x')$, then for any $y \in B(x', r')$, $d(x, y) \leqslant d(x, x') + d(x', y) < d(x, x') + r' = r$, hence $y \in B(x, r)$ so $B(x', r') \subset B(x, r)$.
If for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$, then $U = \bigcup_{x\in U} B(x, \delta_x)$ is open.
If U is open then for any $x \in U$, suppose $x \in B(x_\alpha, r_\alpha)$ for some $\alpha \in \mathcal{A}$, then there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset B(x_\alpha, r_\alpha) \subset U$.

A3) Let $\mathcal{T}$ denote all open sets on (X, d). Prove that $\mathcal{T}$ is a topology.

Proof: 1. $\emptyset \in \mathcal{T}$, $X = \bigcup_{x\in X} B(x, 1) \in \mathcal{T}$. 2. If $\{U_\alpha : \alpha \in J\} \subset \mathcal{T}$, where $U_\alpha = \bigcup_{x\in\mathcal{A}_\alpha} B(x, r_{\alpha,x})$ then let $\mathcal{A} = \bigcup_{\alpha\in J} \mathcal{A}_\alpha$,

$$\bigcup_{\alpha\in J} U_\alpha = \bigcup_{x\in\mathcal{A}} B(x, \sup_{\alpha, x\in\mathcal{A}_\alpha} r_{\alpha,x}) \in \mathcal{T}.$$

3. If $U_1, \cdots, U_n \in \mathcal{T}$, where $U_k = \bigcup_{x\in\mathcal{A}_k} B(x, r_{k,x})$, then let $\mathcal{A} = \bigcup_{k=1}^n \mathcal{A}_k$

$$\bigcap_{k=1}^n U_k = \bigcup_{x\in\mathcal{A}} B(x, \min_{x\in\mathcal{A}_k} r_{k,x}) \in \mathcal{T}.$$

Therefore $\mathcal{T}$ is a topology on X.

A4) (X, d) is a metric space. If $F \subset X$ and F^C is open, then we call F a closed set. Prove that F is closed iff for any sequence $\{x_n\}_{n\geqslant 1} \in F$, if $\lim_{n\to\infty} x_n = x$ then $x \in F$.

Proof: Suppose F is closed, if a sequence $\{x_n\}_{n\geqslant 1}$ satisfy $\lim_{n\to\infty} x_n = x$ and $x \in F^C$, then there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset F^C$. However $B(x,\varepsilon) \cap \{x_n\} \neq \emptyset$, leading to contradiction.
If for any sequence $\{x_n\}_{n\geqslant 1}$ such that $\lim_{n\to\infty} x_n = x$, there is $x \in F$, then for any $x \in F^C$, if for any $\varepsilon > 0$ $B(x,\varepsilon) \not\subset F^C$, then for any $n \geqslant 1$, take $x_n \in B(x,\varepsilon) \cap F$. The sequence $\{x_n\}$ has the limit $\lim_{n\to\infty} x_n = x$ but $x \in F^C$, a contradiction. Hence F is closed.

A5) Prove that

1. $\emptyset$ and X are closed sets.
2. Any intersection of closed sets are still closed.
3. Finite unions of closed sets are still closed.
   Proof: Use A3) and de Morgan's theorem.

A6) Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$, then the following statements are equivalent:

1. f is continuous.
2. For any $U \subset Y$ open, $f^{-1}(U)$ is an open set in X.
3. For any $F \subset Y$ closed, $f^{-1}(F)$ is a closed set in X.
   Proof: 1=>2: If f is continuous, then for any $U \subset Y$ open, consider any point $x \in f^{=1}(U)$. Let $y = f(x) \in U$, then there exists $\varepsilon > 0$ such that $B(y,\varepsilon) \subset U$. Since f is continuous, there exists $\delta > 0$ such that for any $x' \in B(x,\delta)$, $f(x') \in B(y,\varepsilon) \subset U$, hence $B(x,\delta) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is an open set in X.
   2=>1: For any $x \in X$ and $\varepsilon > 0$, consider the open set $U = B(y,\varepsilon)$, where $y = f(x)$. Since $x \in f^{-1}(U)$ and $f^{-1}(U)$ is an open set, there exists $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(U)$, therefore f is continuous.
   2<=>3: Note that $f^{-1}(Y\backslash A) = X\backslash f^{-1}(A)$.

A7) Let A' be the set of limit points of A. Prove that $\bar{A} = A' \cup A$.

Proof: For any closed set $F \supset A$, by A4) we know $A' \subset F$, hence $A' \cup A \subset \bar{A}$. Consider a sequence $\{x_n\}_{n\geqslant 1} \subset A' \cup A$ such that $\lim_{n\to\infty} x_n = x$ exists, for any $n \geqslant 1$ we can find a $y_n \in A$ such that $d(x_n, y_n) \leqslant 2^{-n}$, hence $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = x$ so $x \in A' \cup A$. Therefore $A' \cup A$ is closed, and hence $\bar{A} = A' \cup A$.

A8) Suppose (Y, d_Y) and (Z, d_Z) are metric spaces, define the metric on $Y \times Z$:

$$d_{Y\times Z} : (Y \times Z)^2 \to \mathbb{R}_{\geqslant 0},\ ((y_1,z_1),(y_2,z_2)) \to \sqrt{d_Y(y_1,y_2)^2 + d_Z(z_1,z_2)^2}.$$

Prove that this defines a metric and the projection mappings are continuous:

$$\pi_Y : Y \times Z \to Y,\ (y,z) \mapsto y;\ \pi_Z : Y \times Z \to Z,\ (y,z) \mapsto z.$$

Given a mapping $F : X \to Y \times Z$, then F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.
Proof: $d((y_1,z_1),(y_2,z_2)) = 0 \iff (y_1,z_1) = (y_2,z_2)$,
$d((y_1,z_1),(y_2,z_2)) = d((y_2,z_2),(y_1,z_1))$, and
$d((y_1,z_1),(y_2,z_2)) \leqslant d((y_1,z_1),(y_3,z_3)) + d((y_3,z_3),(y_2,z_2))$ (since

$\sqrt{(x+y)^2+(u+v)^2} \leqslant \sqrt{x^2+u^2}+\sqrt{y^2+v^2}$), hence $d_{Y\times Z}$ is a metric.
Note that $d((y_1,z_1),(y_2,z_2) \geqslant d(y_1,y_2)$, hence π_Y and π_Z are continuous.
$d((y_1,z_1),(y_2,z_2)) \leqslant d(y_1,y_2)+d(z_1,z_2)$, hence F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.

A9) Prove that the operators $+$ and $\cdot$ on real numbers are continuous.

Proof: For any $(x,y),(u,v) \in \mathbb{R}^2$,

$$|(x+y)-(u+v)| \leqslant |x-u|+|y-v| \leqslant 2|(x,y)-(u,v)|.$$

Hence $+$ is uniformly continuous.

$$|x\cdot y-u\cdot v| \leqslant |x|\cdot|y-v|+|v|\cdot|x-u|.$$

Therefore $\cdot$ is continuous.

A10) Prove that the operators $+$ and $\cdot$ on $\mathbf{M}_n(\mathbb{R})$ are continuous.

Proof: The proof of A9) only uses the properties of norms, and the fact that $\|A\cdot B\| \leqslant \|A\|\cdot\|B\|$. This also holds for the norm $\|A\| = \sup_{|\mathbf{x}|=1}|A\mathbf{x}|$ on $\mathbf{M}_n(\mathbb{R})$, therefore $+$ and $\cdot$ are continuous on $\mathbf{M}_n(\mathbb{R})$.

A11) Prove that $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

Proof: The mapping $\det : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous, since view as $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ it is a multi-linear mapping. The set $\mathbf{GL}_n(\mathbb{R}) = \det^{-1}(\{x \in \mathbb{R} : x \neq 0\})$, where $\{x \in \mathbb{R} : x \neq 0\}$ is an open set on $\mathbb{R}$, therefore $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

A12) Prove that $\mathrm{Inv} : \mathbf{GL}_n(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{R}), A \mapsto A^{-1}$ is continuous.

Proof: Note that for any $A, B \in \mathbf{GL}_n(\mathbb{R})$,

$$\|A^{-1}-B^{-1}\| \leqslant \frac{\|A-B\|}{\|A\|\cdot\|B\|}.$$

Hence Inv is continuous.

PSB

Prove the following equalities:

B1) $\lambda > 0$, $\lim\limits_{x\to\infty} \dfrac{x^n}{e^{\lambda x}} = 0$.

Proof: By definition, for $x>0$, $e^{\lambda x} \geqslant (\lambda x)^{n+1}/(n+1)!$. Hence for any $\varepsilon > 0$, let $M = \frac{(n+1)!}{\lambda^{n+1}\varepsilon}$, then for any $x > M$,

$$\left|\frac{x^n}{e^{\lambda x}}\right| \leqslant \frac{(n+1)!}{\lambda^{n+1}x} < \varepsilon.$$

Therefore

$$\lim_{x\to\infty}\frac{x^n}{e^{\lambda x}} = 0.$$

B2) $\alpha > 0$**, then**

$$\lim_{x\to\infty} x^\alpha \log\left(1+\frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

Proof: If $0 < \alpha < 1$, then for any $\varepsilon > 0$, there exists $\delta = \varepsilon^{1/(\alpha-1)}$ such that for any $x > \delta$,

$$\left|x^\alpha \log\left(1+\frac{1}{x}\right)\right| \leqslant x^{\alpha-1} < \varepsilon.$$

If $\alpha > 1$, then for any $\varepsilon > 0$, there exists $\delta = (2\varepsilon)^{1/\alpha-1}$ such that for any $x > \delta$,

$$\left|x^\alpha \log\left(1+\frac{1}{x}\right)\right| \geqslant \frac{x^\alpha}{x+1} \geqslant \frac{1}{2}x^{\alpha-1} > \varepsilon.$$

If $\alpha = 1$, then for any $\varepsilon > 0$, there exists $\delta = 1/\varepsilon$ such that for any $x > \delta$,

$$1-\varepsilon \leqslant \frac{x}{x+1} \leqslant x\log\left(1+\frac{1}{x}\right) \leqslant 1.$$

Therefore

$$\lim_{x\to\infty} x^\alpha \log\left(1+\frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

B3) $\lim\limits_{x\to 0^+} x^{-n}e^{-1/x^2} = 0$**.**

Proof: If $x < 1$, then $e^{-1/x^2} \leqslant e^{-1/x} \leqslant (n+1)!x^{n+1}$, hence for any $\varepsilon > 0$, let $\delta = \varepsilon/(n+1)!$, then for any $x \in (0,\delta)$, $x^{-n}e^{-1/x^2} \leqslant (n+1)!x \leqslant \varepsilon$. Therefore

$$\lim_{x\to 0^+} x^{-n}e^{-1/x^2} = 0.$$

B4) We know that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$**. Calculate**

$$\lim_{x\to 0}\frac{\cos x - 1}{x}, \text{and } \lim_{x\to 0}\frac{\cos x - 1}{x^2/2}.$$

Solution: For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|x| < \delta$, $\sin x \in ((1-\varepsilon)x, (1+\varepsilon)x)$. Hence

$$\left|\frac{\cos x - 1}{x}\right| \leqslant \left|\frac{\sqrt{1-\sin^2 x} - 1}{x}\right| \leqslant \left|\frac{\sin^2 x}{x(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (1+\varepsilon)^2 x \leqslant \delta(1+\varepsilon)^2.$$

Therefore

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Likewise

$$\left|\frac{\cos x - 1}{x^2/2} + 1\right| \leqslant \left|\frac{\sin^2 x - x^2(1+\sqrt{1-\sin^2 x})/2}{x^2/2\cdot(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (2\varepsilon + \sqrt{1-\sin^2 x} - 1).$$

Therefore

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2/2} = -1.$$

PSC: Root of Function:

C1) Prove that $x^3 + 2x - 1 = 0$ has exactly one root which lies in $(0, 1)$.

Proof: Let $f(x) = x^3 + 2x - 1$, then $f(0) = -1$ and $f(1) = 2$, so $f(0) < 0 < f(1)$. Since f is continuous and monotonically increasing on $(0, 1)$, there is exactly one root in $(0, 1)$.

C2) Suppose $0 \leqslant \lambda < 1, b > 0$, determine whether the equation $x - \lambda \sin x = b$ has a solution.

Solution:

C3) Prove that $\sin x = 1/x$ has infinitely many roots.

Proof: For any $n \in \mathbb{N}$, let $x_n = (2n + 1/2)\pi$, $y_n = (2n + 3/2)\pi$, and $f(x) = \sin x - 1/x$, then $f(x_n) = 1 - 1/x_n > 0$, $f(y_n) = -1 - 1/y_n < 0$, therefore f has a root in (x_n, y_n), and hence f has infinitely many roots.

C4) Assume $f \in C([0, 2])$ and $f(0) = f(2)$. Prove that $f(x) - f(x + 1) = 0$ has a root in $[0, 1]$.

Proof: Let $g(x) = f(x) - f(x + 1)$, then $g(0) = f(0) - f(1) = -g(1)$ and $g \in C([0, 1])$. Therefore g has a root in $[0, 1]$.

C5) Prove that $x^3 + 3 = e^x$ has a solution in $\mathbb{R}$.

Proof: Let $f(x) = e^x - x^3 - 3$, then $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$, therefore f has a root in $\mathbb{R}$.

C6) Suppose $f : [0, 2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$ then there exists $x \in [1, 2]$ such that $f(x) = f(x - 1)$.

Exactly the same as C4)?

C7) $f : \mathbb{R} \to \mathbb{R}$, Prove that if for any $c \in \mathbb{R}$, $|f^{-1}(c)| = 2$, then f is not continuous.

Proof: If f is continuous on $\mathbb{R}$, suppose $f^{-1}(0) = \{a < b\}$, then $f|_{[a,b]}$ is bounded. Suppose $f\left(\frac{a+b}{2}\right) > 0$, then for any $t \in (a, b)$, $f(t) > 0$ (otherwise $|f^{-1}(0)| > 2$). Consider an arbitrary $M > y = \sup_{x \in [a,b]} f(x)$, and take $t \in f^{-1}(M)$. Assume $t < a$, then $f(t) = M > y/2 > f(a) = 0$, hence there exists $s \in (t, a)$ such that $f(s) = y/2$. However there are at least two elements of $f^{-1}(y/2)$ in (a, b), leading to contradiction.

C8) Suppose the continuous function $f : [a, b] \to \mathbb{R}$ is injective. If $f(a) < f(b)$, prove that f is monotonically increasing.

Proof: Otherwise suppose $f(u) > f(v)$ for some $u < v$. Note that for any $c \in (a, b)$, $f(a) < f(c) < f(b)$, otherwise $f(c) < f(a) \implies \exists d \in (c, b), f(d) = f(a)$, or $f(c) > f(b) \implies \exists d \in (a, c), f(d) = f(b)$. Hence $a < u < v < b$. Likewise consider $u < v < b$ we get $f(u) > f(v) > f(b)$, and by $a < u < v$ we get $f(a) > f(u) > f(v)$, therefore $f(a) > f(b)$, a contradiction.

PSD: Calculation of Limits

n, m are positive integers.

(1)

$$\lim_{x\to\infty}\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m}=\begin{cases}0, & m>n,\\ \infty, & m<n, a_0>0,\\ -\infty, & m<n, a_0<0,\\ \frac{a_0}{b_0}, & m=n.\end{cases}$$

(2) $a>1, b>0$

$$\lim_{x\to\infty}\frac{x^b}{a^x}=0.$$

(3) $a>0$

$$\lim_{x\to\infty}\frac{\log x}{x^a}=0.$$

(4) $a>0$

$$\lim_{x\to 0^+}x^a\log x=0.$$

(5)

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2}=\lim_{x\to\infty}\left(\frac{x+1}{x-2}\right)^{x}=e^3.$$

(6)

$$\lim_{x\to\infty}(x-\sqrt{x^2-a})=\lim_{x\to\infty}\frac{a}{x+\sqrt{x^2-a}}=0.$$

(7)

$$\lim_{x\to\infty}\sqrt{x+1}-\sqrt{x-1}=\lim_{x\to\infty}\frac{2}{\sqrt{x+1}-\sqrt{x-1}}=0.$$

(8)

$$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=1+2+3=6.$$

(9)

$$\lim_{x\to 1}\frac{x+x^2+\cdots+x^n-n}{x-1}=\frac{n(n+1)}{2}.$$

(10)

$$\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1}=\frac{49}{24}.$$

(11)

$$\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-n}{2}.$$

Proof: Note that

$$
\begin{aligned}
\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right) &= \lim_{x\to 1}\frac{m(1+x+\cdots+x^{n-1})-n(1+x+\cdots+x^{m-1})}{(1+x+\cdots+x^{m-1})(1+x+\cdots+x^{n-1})(1-x)} \\
&= \frac{1}{mn}\cdot\lim_{x\to 1}\frac{m(x-1+\cdots+x^{n-1}-1)-n(x-1+\cdots+x^{m-1}-1)}{1-x} \\
&= \frac{1}{mn}\cdot(-m(1+2+\cdots+(n-1))+n(1+2+\cdots+(m-1))) \\
&= \frac{m-n}{2}.
\end{aligned}
$$

(12)

$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a.$$

(diverges if $a=0$).

(13)

$$\lim_{x\to 1}\frac{x^a-1}{x^b-1}=\frac{a}{b}.$$

(14)

$$\lim_{x\to\infty}(\log x)^{1/x}=\lim_{x\to\infty}e^{(\log\log x)/x}=1.$$

(15) $a,b>0$

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}=\sqrt{ab}.$$

(16)

$$\lim_{x\to\infty}\sqrt[k]{(x+a_1)(x+a_2)\cdots(x+a_k)}-x$$

Proof: Let $y=(x+a_1)(x+a_2)\cdots(x+a_k)$ and $s=a_1+\cdots+a_k$, then

$$\frac{sx^{k-1}}{ky^{(k-1)/k}}\leqslant\sqrt[k]{y}-x=\frac{y-x^k}{y^{(k-1)/k}+\cdots+x^{k-1}}\leqslant\frac{sx^{k-1}+\prod_{i=1}^k(1+a_i)x^{k-2}}{kx^{k-1}}.$$

Therefore

$$\lim_{x\to\infty}\sqrt[k]{y}-x=s=\sum_{i=1}^k a_i.$$

(17)

$$\lim_{x\to 0}\frac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}=2n.$$

(18)

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1.$$

(19)

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x=e.$$

(20) $\alpha>0$**,**

$$\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha>\frac{1}{2},\\ 1, & \alpha=\frac{1}{2},\\ \infty, & \alpha<\frac{1}{2}.\end{cases}$$

(21) $\alpha>0$**,**

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

Proof: Note that for $x\in(0,1)$,

$$x^{1/8-\alpha}\leqslant\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}\leqslant 2x^{1/8-\alpha}.$$

And for any $\varepsilon>0$ there exists $\delta=(1+\varepsilon)\varepsilon$ such that for any $x<\delta$, $\sqrt{x+\sqrt{x+\sqrt{x}}}<\varepsilon x^{1/8}$. Therefore

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

Problem E

Prove that for any $A\subset\mathbb{R}$ that is countable, there exists a monotonic function $f:\mathbb{R}\to\mathbb{R}$, such that the set of discontinuities of f is exactly A.
Proof: Let $A=\{x_1,x_2,\cdots\}$ and $f(x)=\sup\{1-2^n : x_n<x\}$, (define $\sup\emptyset=0$) then f is monotonically increasing and the set of discontinuities is exactly A.

Problem F

$f:[0,1]\to[0,1]$ is monotonic. Prove that f has a fixed point.
Proof: Otherwise suppose that f has no fixed points. Let $S=\{t\in[0,1]:f(t)>t\}$ and $x=\sup S$. Note that $0\in S$ so S is non-empty. If $x\in S$, then $f(x)>x$ so $f(f(x))>f(x)$ (f is monotonic) then $x<f(x)\in S$ which leads to contradiction. If $x\notin S$, then $f(x)<x$. Take $y\in(f(x),x)\cap S$, (y exists since $x=\sup S$) then $f(x)>f(y)>y>f(x)$, a contradiction.

Problem G

Consider all self-homeomorphisms of $[0,1]$, i.e.

$$\mathrm{Homeo}([0,1])=\{f:[0,1]\to[0,1]:f \text{ is a continuous bijective}\}$$

We know that for any $f\in\mathrm{Homeo}([0,1])$, $f^{-1}\in\mathrm{Homeo}([0,1])$. Suppose $f,g\in\mathrm{Homeo}([0,1])$ and the only fixed points of f,g are $0,1$. Prove that there exists $h\in\mathrm{Homeo}([0,1])$, such that

$$h\circ f\circ h^{-1}=g.$$

Proof: Take $x_0 = 1/2$, and let $I_n = [f^n(x_0), f^{n+1}(x_0)]$, $J_n = [g^n(x_0), g^{n+1}(x_0)]$. Note that $(0,1) = \bigcup_{n\in\mathbb{Z}} I_n = \bigcup_{n\in\mathbb{Z}} J_n$. Define $h_0 : I_0 \to J_0,\ x \mapsto kx + b$ such that the line h_0 passes (x_0, x_0) and $(f(x_0), g(x_0))$, i.e. $x \mapsto \frac{g(x_0)-x_0}{f(x_0)-x_0}(x - x_0) + x_0$. Define $h_n : I_n \to J_n,\ x \mapsto g^n \circ f^{-n}(x)$, and $h : [0,1] \to [0,1]$ such that

$$h(x) = \begin{cases} x, & x \in \{0,1\}, \\ h_n(x), & x \in I_n. \end{cases}$$

Then for any $x \in I_n$, $f(x) \in I_{n+1}$ hence $h(f(x)) = g^{n+1} \circ f^{-n}(x) = g(h(x))$. Since h maps I_n to J_n bijectively, h is a bijection on $[0,1]$. For any $x \in I_n \cap I_{n+1}$ the value of h does not depend on which interval we choose, and h is continuous on any interval I_n, therefore h is a continuous bijection.