PSA

Assume $I = [a, b] \subset \mathbb{R}$, V is a normed linear space.

A1) $\sigma_1, \sigma_2 \in \mathcal{S}$ are two partitions. Prove that for any $\varepsilon > 0$, there exists a partition σ such that $\sigma \prec \sigma_1, \sigma \prec \sigma_2$ and $|\sigma| < \varepsilon$.

Proof: Take $n > 1/\varepsilon$, and let

$$\sigma = \sigma_1 \cup \sigma_2 \cup \left\{ \frac{k}{n} a + \frac{n-k}{n} b : 0 \leqslant k \leqslant n \right\}.$$

A2) Consider the space of simple functions $\mathcal{E}(I)$ with range V. Prove that it is a linear space on $\mathbb{R}$, and the integration operator $\int_a^b : \mathcal{E}(I) \to V$ is well-defined and is linear. Use this to define Riemann integrable functions with range V.

Proof: For any simple function $f = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$ (where A_i are disjoint), let

$$\int_a^b f = \sum_{i=1}^n c_i \mu(A_i)$$

For any function $f : I \to V$, partition $\mathcal{C} = \{x_0, x_1, \cdots, x_n\}$ and $\xi_i \in [x_{i-1}, x_i]$, define

$$\mathcal{R}(f; \mathcal{C}, \xi) = \sum_{k=0}^n f(\xi_i)(x_i - x_{i-1}).$$

Then f is Riemann integrable iff $\lim_{|\mathcal{C}| \to 0} \mathcal{R}(f; \mathcal{C}, \xi)$ exists.

A3) Suppose $f : I \to \mathbb{R}^n$ and f_i be the components of f, then $f \in \mathcal{R}(I)$ iff for every i, $f_i \in \mathcal{R}(I)$.

Proof: Note that

$$\max\{|x_k|\} \leqslant |(x_1, \cdots, x_n)|_{\mathbb{R}^n} \leqslant |x_1| + \cdots + |x_n|.$$

Hence the limit $|\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0$ iff the components of f are all Riemenn integrable.

A4) Assume $a < c < b$, then for any $f \in \mathcal{R}(I)$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Riemann integrable, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: They are both obviously Riemann integrable, and for any partition σ, let $\sigma' = \sigma \cup \{c\} = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are partitions of $[a, c]$ and $[c, b]$, then

$$\underline{S}(f; \sigma) \leqslant \underline{S}(f; \sigma') = \underline{S}(f|_{[a,c]}; \sigma_1) + \underline{S}(f|_{[c,b]}; \sigma_2),$$

and the other side is the same. Hence

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A5) Prove that for any two partition σ and σ', $\underline{S}(f;\sigma) \leqslant \overline{S}(f;\sigma')$. Use this to prove that if $f \in \mathcal{R}(I)$, then $\lim_{|\sigma|\to 0}|\underline{S}(f;\sigma) - \overline{S}(f;\sigma)| = 0$.

Proof: Let $\sigma'' = \sigma \cup \sigma'$, then

$$\underline{S}(f;\sigma) \leqslant \underline{S}(f;\sigma'') \leqslant \overline{S}(f;\sigma'') \leqslant \overline{S}(f;\sigma').$$

If $f \in \mathcal{R}(I)$, then $\sup_\sigma \underline{S}(f;\sigma) = \inf_\sigma \overline{S}(f;\sigma)$ hence

$$\lim_{|\sigma|\to 0}|\underline{S}(f;\sigma) - \overline{S}(f;\sigma)| = 0.$$

A6) $f \in \mathcal{R}(I)$. Prove that if we change the value of f at a finite number of points to g, then g is Riemann integrable and $\int_I g = \int_I f$.

Proof: We can assume that f and g differ only at the point $c \in (a,b)$. Let $M = \sup_{x\in I}|f(x)|$. For any $\varepsilon > 0$, and any partition σ, let $\sigma' = \sigma \cup \{c-\varepsilon, c+\varepsilon\}$, then $|\underline{S}(f;\sigma') - \underline{S}(f;\sigma)| \leqslant 4\varepsilon M \to 0$.

A7) $f \in C([a,b])$. Assume for any $x \in I$, $f(x) \geqslant 0$ and there exists $x_0 \in I$ such that $f(x_0) > 0$. Prove that $\int_a^b f > 0$.

Proof: Since f is continuous and $f(x_0) > 0$, there is an $\varepsilon > 0$ such that for all $y \in (x_0 - \varepsilon, x_0 + \varepsilon)$, $f(y) > 0$. Hence for any partition $\sigma = \{x_0, x_1, \cdots, x_n\}$ such that $|\sigma| < \varepsilon/2$, there is a $k \in \{1, \cdots, n\}$ such that $(x_{k-1}, x_k) \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. Hence $\mathcal{R}(f;\sigma,\xi) > 0$ whenever $|\sigma| < \varepsilon/2$, so $\int_a^b f(x)\,\mathrm{d}x > 0$.

A8) Suppose $f, g \in C^1(I)$, then

$$\int f' \cdot g = f \cdot g - \int f \cdot g'.$$

Proof:

$$d(f \cdot g) = df \cdot g + f \cdot dg.$$

A9) Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is differentiable, f is a continuous function, then

$$\int (f \circ \Phi)\Phi' = \int f.$$

Proof:

$$(f(\Phi(x)))' = f'(\Phi(x))\Phi'(x).$$

PSB: Calculating Integrals

(1)

$$\begin{aligned}\int \frac{x^5}{1+x}\,\mathrm{d}x &= \int x^4 - x^3 + x^2 - x + 1 - \frac{1}{1+x}\,\mathrm{d}x \\ &= \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C.\end{aligned}$$

(2)

$$\int \sqrt{x\sqrt{x\sqrt{x}}}\,\mathrm{d}x = \int x^{7/8}\,\mathrm{d}x = \frac{8}{15}x^{15/8}.$$

(3)

$$\begin{aligned}\int \left(\frac{1+x}{1-x}+\frac{1-x}{1+x}\right)\mathrm{d}x &= \int \left(\frac{2}{1-x}+\frac{2}{1+x}-2\right)\mathrm{d}x \\ &= -2x+2\log\frac{1+x}{1-x}+C.\end{aligned}$$

(4)

$$\int \frac{e^{3x}+1}{1+e^x}\,\mathrm{d}x = \int 1-e^x+e^{2x}\,\mathrm{d}x = x-e^x+\frac{e^{2x}}{2}+C.$$

(5)

$$\int \sqrt{1-\sin{(2x)}}\,\mathrm{d}x = \int \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\mathrm{d}x = -\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+C.$$

(6)

$$\int \frac{\cos{(2x)}}{\cos x-\sin x}\,\mathrm{d}x = \int \cos x+\sin x\,\mathrm{d}x = \sin x-\cos x+C.$$

(7)

$$\int \tan^2 x\,\mathrm{d}x = -x+\tan x+C.$$

(8)

$$\int |x|\,\mathrm{d}x = \frac{x|x|}{2}+C.$$

(9)

$$\int e^{-|x|}\,\mathrm{d}x = -\mathrm{sgn}(x)e^{-|x|}+C.$$

(10)

$$\int \frac{x^2}{(1-x)^{2018}}\,\mathrm{d}x = \frac{1}{2017(1-x)^{2017}}-\frac{1}{1013(1-x)^{2016}}+\frac{1}{2015(1-x)^{2015}}.$$

(11)

$$\int |x-1|\,\mathrm{d}x = \frac{(x-1)|x-1|}{2}+C.$$

(12)

$$\int \frac{1}{\sqrt{b^2+x^2}}\,\mathrm{d}x = \frac{1}{b}\log\frac{1+\tan\frac{\arctan\frac{x}{b}}{2}}{1-\tan\frac{\arctan\frac{x}{b}}{2}}+C.$$

(13)

Let $x = t^2$, then

$$\int \frac{\mathrm{d}x}{\sqrt{x}(1+x)} = 2\arctan\sqrt{x} + C.$$

(14)

$$\int \frac{x^4}{(1-x^5)^4}\,\mathrm{d}x = \frac{1}{5}\int \frac{\mathrm{d}x^5}{(1-x^5)^4} = \frac{1}{15(1-x^5)^3} + C.$$

(15)

$$\int \left(\frac{1}{\sqrt{3-x^2}} + \frac{1}{1-3x^2}\right)\mathrm{d}x = \arcsin\frac{x}{\sqrt{3}} + \frac{1}{2\sqrt{3}}\log\frac{1+\sqrt{3}x}{1-\sqrt{3}x} + C.$$

(16)

$$\int \frac{2x-3}{x^2-3x+8}\,\mathrm{d}x = \log\left(x^2-3x+8\right) + C.$$

(17)

$$\int \frac{\mathrm{d}x}{\sin^2\left(2x+\frac{\pi}{4}\right)} = \frac{\tan\left(2x-\pi/4\right)}{2} + C.$$

(18)

$$\int \frac{\mathrm{d}x}{1+\cos x} = \tan\frac{x}{2} + C.$$

(19)

$$\int \frac{1}{x^2}\sin\frac{1}{x}\,\mathrm{d}x = \cos\frac{1}{x} + C.$$

(20)

$$\int \cos^5 x\,\mathrm{d}x = \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3} + \sin x + C.$$

(21)

$$\int \cos\left(ax\right)\sin\left(bx\right)\mathrm{d}x = \frac{\cos\left(a-b\right)x}{2(a-b)} - \frac{\cos\left(a+b\right)x}{2(a+b)} + C.$$

(22)

$$\int \frac{\mathrm{d}x}{a\cos x + b\sin x} = \frac{2}{\sqrt{a^2+b^2}}\tanh^{-1}\frac{a\tan\left(x/2\right)-b}{\sqrt{a^2+b^2}} + C.$$

(23)

$$\int \frac{\sin\left(2x\right)}{a^2\cos^2 x + b^2\sin^2 x}\,\mathrm{d}x = \frac{\log\left(\left(b^2-a^2\right)\sin^2 x + a^2\right)}{b^2-a^2} + C.$$

(24)

$$\int \frac{\mathrm{d}x}{2-\sin^2 x} = \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C.$$

(25)

$$\int \frac{\mathrm{d}x}{x\ln x\ln\ln x} = \ln\ln\ln x + C.$$

(26)

$$\int \frac{\log x}{x\sqrt{1+\log x}}\,\mathrm{d}x = \frac{2}{3}(1+\log x)^{3/2} - 2\sqrt{1+\log x} + C.$$

(27)

$$\int \frac{\cos x + \sin x}{(\sin x - \cos x)^{1/3}}\,\mathrm{d}x = \frac{3}{2}(\sin x - \cos x)^{2/3} + C.$$

(28)

$$\int e^{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

(29)

$$\int \frac{x^{n/2}}{1+x^{n+2}}\,\mathrm{d}x = \frac{2}{n+2}\arctan x^{n/2+1} + C.$$

(30)

$$\int \frac{\sqrt{x}}{1-x^{1/3}}\,\mathrm{d}x = 6\arctan x^{1/6} - \frac{6}{5}x^{5/6} - \frac{6}{7}x^{7/6} - 2x^{1/2} - 6x^{1/6} + C.$$

(31)

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$$

(32)

$$\int \frac{\mathrm{d}x}{\cos^4 x} = \frac{\sin x}{2\cos^3 x} + \frac{\sin(3x)}{6\cos^3 x} + C.$$

(33)

$$\int \arcsin^2 x\,\mathrm{d}x = x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C.$$

(34)

$$\int x\arcsin x\,\mathrm{d}x = \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4}\arcsin x(1-2x^2) + C.$$

(35)

$$\int x \arctan x = \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.$$

(36)

$$\int \frac{\arctan x}{x^2} = \log x - \frac{\arctan x}{x} - \frac{1}{2}\log\left(1+x^2\right) + C.$$

(37)

$$\int x^2 \sin x = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

(38)

$$\int \frac{x}{\cos^2 x} = x\tan x + \log\cos x + C.$$

(39)

$$\int \log\left(x+\sqrt{1+x^2}\right) = x\log\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C.$$

(40)

$$\int \sin\log x = \frac{x}{2}(\sin\log x - \cos\log x) + C.$$

(41)

$$\int \sqrt{x^2+a^2} = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{4}\log\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}-x^2} + C.$$

(42)

$$\int \frac{x^2}{\sqrt{x^2-a^2}} = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{a^2}{4}\log\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}} + C.$$

(43)

$$\int \frac{x\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = \sqrt{x^2+1}\log\left(x+\sqrt{1+x^2}\right) - x + C.$$

Let $u = \sqrt{x^2+1}+x$ then $\mathrm{d}u/\mathrm{d}x = 1 + x/\sqrt{1+x^2}$, so it becomes

$$\int \frac{(u^2-1)\log u}{2u^2}\mathrm{d}x = -\frac{u}{2} + \frac{1}{2u} + \frac{1}{2}u\log u + \frac{\log u}{2u} + C.$$

(44)

$$\int \frac{1}{\sqrt{x^2+a^2}} = \log\frac{\sin t+\cos t}{\sin t-\cos t} + C = \tanh^{-1}\frac{x}{\sqrt{x^2+a^2}} + C.$$

where $t = \frac{1}{2}\arctan(x/a)$.

(45)

$$\int \frac{xe^x}{(1+x)^2} = \frac{e^x}{1+x} + C.$$

(46)

$$\int \arctan\left(1+\sqrt{x}\right) = x\arctan\left(1+\sqrt{x}\right) - \sqrt{x} + \log\left(2+2\sqrt{x}+x\right) + C.$$

(47)

$$\int \left(1-\frac{2}{x}\right)^2 e^x = e^x - \frac{4e^x}{x} + C.$$

since $\int e^x/x^2 \, \mathrm{d}x = -e^x/x + \int e^x/x \, \mathrm{d}x$.

(48)

$$\int \sqrt{2+\tan^2 x} = \theta + \log\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta} + C.$$

where $\theta = \arcsin\left(\sin x/\sqrt{2}\right)$.

(49)

$$\int \frac{1}{1+x^3} = -\frac{1}{6}\log\left(x^2-x+1\right) + \frac{1}{3}\log\left(x+1\right) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

(50)

$$\int \frac{x^7}{x^4+2} = \frac{x^4}{4} - \frac{1}{2}\log\left(2+x^4\right).$$

(51)

$$\int \frac{2x^2+1}{(x+3)(x-1)(x-4)} = -\frac{1}{4}\log\left(1-x\right) + \frac{11}{7}\log\left(4-x\right) + \frac{19}{28}\log\left(x+3\right) + C.$$

(52)

$$\int \frac{1+x^2}{1+x^4} = \frac{1}{\sqrt{2}}\left(\arctan\left(\sqrt{2}x+1\right) - \arctan\left(1-\sqrt{2}x\right)\right) + C.$$

Note that

$$\frac{1+x^2}{1+x^4} = \frac{1}{2(x^2+\sqrt{2}x+1)} - \frac{1}{2(-x^2+\sqrt{2}x+1)}.$$

(53)

Let $x = y^6 - 1$ then

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}+(x+1)^{1/3}} &= \int (y^3-1)(1-y+y^2)6y^3 \mathrm{d}y \\ &= \frac{2x\sqrt{x+1}}{3} - \frac{3x(x+1)^{1/3}}{4} + \frac{6x(x+1)^{1/6}}{7} - x + \frac{6}{5}(x+1)^{5/6} \\ &\quad - \frac{3}{2}(x+1)^{2/3} + \frac{2\sqrt{x+1}}{3} - \frac{3(x+1)^{1/3}}{4} + \frac{6(x+1)^{1/6}}{7} + C.\end{aligned}$$

(54)

Let $x = y^2$, then

$$\int \frac{1}{\sqrt{x+x^2}} = \int \frac{\mathrm{d}y}{\sqrt{y^2+1}} = \tanh^{-1}\left(\sqrt{\frac{x}{x+1}}\right) + C.$$

(55)

The Poisson kernel

$$\int \frac{1-r^2}{1-2r\cos x + r^2} = 2\arctan\left(\frac{1+r}{1-r}\tan\frac{x}{2}\right) + C.$$

(56)

Let $x = \tan\theta$ then

$$\int \frac{1}{x\sqrt{1+x^2}} = \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\arctan x}{2} + C.$$

(57)

Let $t = \tan x/2$ then

$$\int \frac{1}{5-3\cos x} = \frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C.$$

(58)

Let $t = \tan x$, then

$$\int \frac{1}{2+\sin^2 x} = \frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{3}{2}}\tan x\right) + C.$$

(59)

$$\int \frac{\sin^3 x}{\cos^4 x} = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$$

(60)

Let $t = \cos x$ then

$$\int \frac{1}{\sin x\cos^4 x} = -\int t^{-4} + t^{-2} + \frac{1}{1-t^2} = \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \frac{1}{2}\log\frac{1+\cos x}{1-\cos x} + C.$$

Problem W: Stone-Weierstraß Theorem

Part 1: Approximating $|x|$

W1) (Dini) Suppose $K \subset \mathbb{R}^n$ is compact, $f_n : K \to \mathbb{R}$ is a sequence of continuous functions, which converges point-wise to $f : K \to \mathbb{R}$. If f is continuous and $f_n \leqslant f_{n+1}$, then f_n converges uniformly to f.

Proof: For any $\varepsilon > 0$, and any $x \in K$, there is an integer $n_x > 0$ such that $|f_{n_x}(x) - f(x)| < \varepsilon/4$. There exists $\delta > 0$, such that $\forall y \in \mathrm{B}(x, \delta) \cap K$, $|f(x) - f(y)| < \varepsilon/4$ and $|f_{n_x}(x) - f_{n_x}(y)| < \varepsilon/4$, then $|f_{n_x}(y) - f(y)| < \varepsilon/4$. Note that $K \subset \bigcup_{x \in K} \mathrm{B}(x, \delta_x)$ hence we can choose a finite set of $x_1, x_2, \cdots, x_N$ such that $K \subset \bigcup_{i=1}^N \mathrm{B}(x_i, \delta_{x_i})$. Let $M = \max\{n_{x_i} : i = 1, 2, \cdots, N\}$ then for any $m \geqslant M$ and $x \in K$, $|f_m(x) - f(x)| < \varepsilon$. Hence f_n converges uniformly to f.

W2) Consider the interval $[-1, 1]$. Define inductively a sequence of polynomials:

$$P_0(x) = 0,\ P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x)).$$

Prove that for any n, x, $0 \leqslant P_n(x) \leqslant P_{n+1}(x) \leqslant |x|$.
Proof: Assume $x > 0$, we prove by induction. If $t = P_n(x) \in [0, x]$, then

$$P_{n+1}(x) = \frac{1}{2}x^2 - \frac{1}{2}(t-1)^2 + \frac{1}{2} \leqslant \frac{1}{2}(x^2 - (1-x)^2 + 1) = x,$$

and $P_{n+1}(x) \geqslant P_n(x) = t$, hence $P_{n+1}(x) \in [0, x]$.

W3) Prove that $|x|$ can be uniformly approximated by polynomials on the interval $[-1, 1]$, i.e. for any $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(x)$ such that $\sup_{x \in [-1,1]} ||x| - P_\varepsilon(x)| < \varepsilon$.

Proof: By W2), the sequence of polynomials $\{P_n\}$ converge point-wise to $|x|$, hence by W1) P_n converge uniformly to $|x|$.

Part 3: Bernstein Polynomial

Assume $I = [0, 1]$, and n is an integer.

W4) For any $0 \leqslant k \leqslant n$, define $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Prove that

$$\sum_{0 \leqslant k \leqslant n} p_{n,k}(x) \left(x - \frac{k}{n} \right)^2 = \frac{x(1-x)}{n}.$$

Proof:
Note that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} x^2 = x^2,$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} = x,$$

$$\sum_{k=0}^{n} p_{n,k}(x) k(k-1) = n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1-x)^{n-k} = n(n-1)x^2.$$

Therefore

$$\sum_{k=0}^{n} p_{n,k}(x)\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n}.$$

W5) For any $f \in C([0,1])$, define

$$B_{f,n}=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

For $x \in [0,1]$, prove that

$$|f(x)-B_{f,n}(x)| \leqslant \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right| p_{n,k}(x).$$

Proof: Note that

$$\sum_{k=0}^{n} f(x)\binom{n}{k}x^k(1-x)^{n-k}=f(x).$$

W6) For any $f \in C([0,1])$, prove that for any $\varepsilon>0$, there exists n such that $\|f-B_{f,n}\|_\infty<\varepsilon$.

Proof:
Let

$$\begin{aligned}
\mathrm{I} &= \sum_{|m-nx|<n^{3/4}} \left(f(x)-f\left(\frac{m}{n}\right)\right) p_{n,m}(x), \\
\mathrm{II} &= \sum_{|m-nx|>n^{3/4}} \left(f(x)-f\left(\frac{m}{n}\right)\right) p_{n,m}(x).
\end{aligned}$$

Then $|f-B_{f,n}| \leqslant |\mathrm{I}|+|\mathrm{II}|$.
For any $\varepsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall x \in [0,1], n \geqslant N \implies |\mathrm{I}|<\varepsilon$, since

$$|\mathrm{I}| \leqslant \sup_{|x-m/n|<n^{-1/4}} |f(x)-f(m/n)| \to 0.$$

Suppose $M=\sup_{x\in[0,1]}|f(x)|$, then

$$|\mathrm{II}| \leqslant 2M \sum_{|m-nx|>n^{3/4}} p_{n,m}(x) \leqslant 2M\sqrt{n}\sum_{m=0}^{n}(x-m/n)^2 p_{n,m}(x)=\frac{2Mx(1-x)}{\sqrt{n}}.$$

Hence $\|f-B_{f,n}\|_\infty \to 0$.

Part 3: Stone-Weierstrass Theorem

W7-14):

Let X be a compact Hausdorff space, $\mathcal{A} \subset C(X,\mathbb{R})$ satisfy the following properties:
(a) $\forall c \in \mathbb{R}$, $c \cdot 1_X \in \mathcal{A}$, (b) $\forall f,g \in \mathcal{A}, f+g, f-g, fg \in \mathcal{A}$,
(c) $\mathcal{A}$ can separate any pair of points in X.
Then $\bar{\mathcal{A}}=C(X,\mathbb{R})$.

Lemma 1

There is a list of polynomials $\{P_n(x)\}$ that converges uniformly to $|x|$ on $[-1, 1]$.

Lemma 2

If $\mathcal{A}$ is a subspace of $C(X, \mathbb{R})$, such that (a) $\mathcal{A}$ is a lattice, (b) $1_X \in \mathcal{A}$, and (c) $\mathcal{A}$ can separate any pair of points, then $\bar{\mathcal{A}} = C(X, \mathbb{R})$.

Proof of main theorem

Assume WLOG $\mathcal{A}$ is closed, then by Lemma 1, $\forall f \in \mathcal{A}$, $P_n(f) \in \mathcal{A}$, hence $|f| \in \mathcal{A}$.(Since X is compact, $|f|$ is bounded.) Note that

$$\max\{f, g\} = \frac{1}{2}(|f+g| + |f-g|), \min\{f, g\} = \frac{1}{2}(|f+g| - |f-g|).$$

Hence $\mathcal{A}$ is a lattice, by Lemma 2 $\mathcal{A} = C(X, \mathbb{R})$.

Proof of Lemma 1

Proof 1: Let

$$Q_n(x) = \int_0^x (1-t^2)^n \, dt / \int_0^1 (1-t^2)^n \, dt.$$
$$P_n(x) = \int_0^x Q_n(t) \, dt.$$

Note that

$$\int_\varepsilon^1 (1-t^2)^n \, dt \leqslant (1-\varepsilon^2)^n (1-\varepsilon) \to 0$$

Hence (combined with Wallis's formula), $P_n(x)$ converges uniformly to $|x|$ on $[a, b]$.
Proof 2: WLOG change the interval to $[-1/2, 1/2]$. The series

$$(1-t)^{1/2} = 1 + \sum_{n=1}^{\infty} (-t)^n \binom{\frac{1}{2}}{n} = 1 - \sum_{n=1}^{\infty} c_n t^n.$$

converges when $|t| < 1$. Hence $\forall \varepsilon > 0$, there exists $Q \in \mathbb{R}[x]$ such that $\sup_{|t| \leqslant 1/2} |Q(t) - (1-t)^{1/2}| < \varepsilon/2$.
Let $t = 1 - x^2$, then $|Q(1-x^2) - |x|| < \varepsilon/2$, so $P(x) = Q(1-x^2) - Q(1)$ converges to $|x|$ uniformly on $[-1/2, 1/2]$.

Proof of Lemme 2

Step 1: Take any $f \in C(X, \mathbb{R})$, and any $x, y \in X$, we can find $g_{xy} \in \mathcal{A}$, such that $g_{xy}(x) = f(x), g_{xy}(y) = f(y)$. Since there exists $u \in \mathcal{A}$ such that $u(x) \neq u(y)$,

$$\begin{pmatrix} u(x), 1 \\ u(y), 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} f(x) \\ f(y) \end{pmatrix}$$

has a solution. (If $x = y$ it is trivial.)
Step 2:
For all $\varepsilon > 0$, $x, y \in X$, there is an open neighborhood $O_{x,y}$ of y, such that $\forall z \in O_{x,y}$, $f(z) - g_{xy}(z) \leqslant \varepsilon$. Note that $\bigcup_{y \in X} O_{x,y} = X$, so by X is compact, there is a list $y_1, \cdots, y_N$ such that $\bigcup_{k \leqslant N} O_{x,y_k} = X$. Let $h_x = \max\{g_{xy_k} : k \leqslant N\}$, then $h_x(y) - f(y) \geqslant -\varepsilon$, and $f(x) = h_x(x)$.

Step 3:
For all $x \in X$, there is an open neighborhood G_x of x, such that $\forall z \in G_x$, $h_x(z) - f(z) \leqslant \varepsilon$.
Note that $\bigcup_{x \in X} G_x = X$, so by X is compact, there is a list $x_1, \cdots, x_M$ such that
$\bigcup_{k \leqslant M} G_x = X$. Let $F = \min \{h_{x_k} : k \leqslant M\}$, then $|F(x) - f(x)| \leqslant \varepsilon, \forall x \in X$.
Therefore $\bar{\mathcal{A}} = C(X, \mathbb{R})$.

For complex numbers, there is an additional requirement: for any $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$.

W15-16):

It is easy to see that polynomials and trigonometric polynomials both satisfy the requirements of the theorem.