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1.

If a/b is rational, let $n = \lfloor a/b \rfloor, b = p^s \cdot c$ and $d = \mathrm{ord}_p(c), m = (p^d - 1)/c$, then

$$\frac{a}{b} = n + p^{-s} \cdot \frac{(a-nb)m}{p^d - 1} = n + p^{-s} \sum_{j=1}^{\infty} (a-nb)mp^{-jd}.$$

Conversely, if

$$x = n + \sum_{j=1}^{N} c_j p^{-j} + p^{-N} \sum_{j=1}^{\infty} mp^{-jd}$$

where $n \in \mathbb{Z}, c_j \in \{0, 1, \cdots, p-1\}, m \in \{0, 1, \cdots, p^d - 1\}$, then

$$x = n + \sum_{j=1}^{N} c_j p^{-j} + p^{-N} m \frac{1}{p^d - 1}$$

is clearly rational.

3.

Determine all $(a, b) \in \mathbb{Q}^2$ such that for any

$$q = p + \frac{2 - p^2}{ap + b},$$

(1) $p > 0, p^2 < 2$ implies $q > p$ and $q^2 < 2$,
and (2) $p > 0, p^2 > 2$ implies $0 < q < p$ and $q^2 > 2$.
Solution: Since $(q-p)/(2-p^2) > 0$ for any $p > 0$, $a, b \geqslant 0$. Note that

$$q^2 - 2 = (q + \sqrt{2})(p - \sqrt{2}) \left(1 - \frac{p + \sqrt{2}}{ap + b}\right),$$

so for any $p > 0$, $ap + b > p + \sqrt{2}$, i.e. $a \geqslant 1$ and $b^2 \geqslant 2$. Clearly such (a, b) satisfy (1)(2).

5.

Prove that for $a > 0, n \in N$, $\min\{a, 1\} \leqslant a^{1/n} \leqslant \max\{a, 1\}$.
Proof: If $a \leqslant 1$, then a^x is monotonically decreasing, so $a^1 \leqslant a^{1/n} \leqslant a^0$. Otherwise a^x is monotonically increasing, so $a^0 \leqslant a^{1/n} \leqslant a^1$. In both cases $\min\{a, 1\} \leqslant a^{1/n} \leqslant \max\{a, 1\}$.

6.

Let $n \geqslant 2$, $\alpha_1, \cdots, \alpha_n \geqslant 0$, such that $\alpha_1 + \cdots + \alpha_n = 1$. Prove that for $x_1, \cdots, x_n > 0$, $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leqslant \alpha_1 x_1 + \cdots + \alpha_n x_n$.
Proof: Let $G = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then

$$\sum_{i=1}^{n} \alpha_i x_i - G = \sum_{i=1}^{n} \alpha_i \int_G^{x_i} 1 - \frac{G}{x} \,\mathrm{d}x \geqslant 0.$$

(Since each $\int_G^{x_i} (1 - G/x) \,\mathrm{d}x \geqslant 0$).

8.

Prove that in $\mathbb{R}\cup\{-\infty,\infty\}$,
(1) $\inf_{\alpha\in I}(-x_\alpha)=-\sup_{\alpha\in I}x_\alpha$,
Let $M=\sup_{\alpha\in I}x_\alpha$, then $-M$ is a lower bound of $\{-x_\alpha\}$. For any lower bound m' of $\{-x_\alpha\}$, $-m'$ is an upper bound of $\{x_\alpha\}$, so $-m'\geqslant M$ i.e. $m'\leqslant -M$. Hence $-M=\inf_{\alpha\in I}x_\alpha$.
(2) Prove that

$$\begin{aligned}\inf_{\alpha\in I}x_\alpha+\inf_{\alpha\in I}y_\alpha\leqslant\inf_{\alpha\in I}(x_\alpha+y_\alpha)&\leqslant\sup_{\alpha\in I}x_\alpha+\inf_{\alpha\in I}y_\alpha\\&\leqslant\sup_{\alpha\in I}(x_\alpha+y_\alpha)\leqslant\sup_{\alpha\in I}x_\alpha+\sup_{\alpha\in I}y_\alpha.\end{aligned}$$

Proof: Note that $\inf_{\alpha\in I}x_\alpha+\inf_{\alpha\in I}y_\alpha=\inf_{\alpha\in I}(y_\alpha+\inf_{\beta\in I}x_\beta)\leqslant\inf_{\alpha\in I}(x_\alpha+y_\alpha)$. Likewise $\inf_{\alpha\in I}(x_\alpha+y_\alpha)\leqslant\inf_{\alpha\in I}(\sup_{\beta\in I}x_\beta+y_\alpha)\leqslant\sup_{\alpha\in I}x_\alpha+\inf_{\alpha\in I}y_\alpha$, and the other two inequalities are the same.
(3) If $x_\alpha>0$, then

$$\sup_{\alpha\in I}\frac{1}{x_\alpha}=\frac{1}{\inf_{\alpha\in I}x_\alpha}.$$

Proof: Let $m=\inf_{\alpha\in I}x_\alpha$, then $1/x_\alpha\leqslant 1/m$ for any $\alpha\in I$. If $m=0$, then for any $n\in\mathbb{N}$, there exists $\alpha\in I$ such that $x_\alpha<1/n$, then $1/x_\alpha>n$ so $\sup_{\alpha\in I}1/x_\alpha=\infty$. Otherwise $m>0$, so for any $\varepsilon>0$, there exists $\alpha\in I$ such that $x_\alpha<m+\varepsilon$, so

$$\frac{1}{x_\alpha}>\frac{1}{m+\varepsilon}=\frac{1}{m}-\frac{\varepsilon}{m(m+\varepsilon)}>\frac{1}{m}-\frac{\varepsilon}{m^2}.$$

Hence $\sup_{\alpha\in I}1/x_\alpha=1/m$.
(4) Suppose $x_\alpha,y_\alpha>0$, then

$$\inf_{\alpha\in I}x_\alpha\inf_{\alpha\in I}y_\alpha\leqslant\inf_{\alpha\in I}(x_\alpha y_\alpha)\leqslant\sup_{\alpha\in I}x_\alpha\inf_{\alpha\in I}y_\alpha\leqslant\sup_{\alpha\in I}(x_\alpha y_\alpha)\leqslant\sup_{\alpha\in I}x_\alpha\sup_{\alpha\in I}y_\alpha.$$

Same as (2).

10.

For $x\in\mathbb{R}$, let $B(x)=\{b^t:t\in\mathbb{Q},t\leqslant x\}$, and define $b^x=\sup B(x)$. $B(x)$ is clearly non-empty and bounded by $b^{\lfloor x\rfloor+1}$, so it is well-defined. If $r\in\mathbb{Q}$, then

$$b^r=\sup B(r),\ \forall r\in\mathbb{Q}.$$

Proof: $b^r\in B(r)$ and since b^t is monotonically increasing, $b^r\geqslant\sup B(r)$, hence $b^r=\sup B(r)$.

11.

Prove that for any $x,y>0$, $(a^x)^y=a^{xy}$, and $a^xb^x=(ab)^x$.
Proof:

$$a^{xy}=\sup\{a^t:t\leqslant xy\}=\sup\{a^{uv}:u\leqslant x,v\leqslant y\}=(a^u)^v$$
$$(ab)^x=\sup\{(ab)^t:t\leqslant x\}=\sup\{a^tb^t:t\leqslant x\}=a^xb^x.$$

15.

Let $a, x, y > 0$, $a \neq 1$. Prove that $\log_a(xy) = \log_a x + \log_a y$.
Proof: Note that

$$a^{\log_a(xy)} = xy = a^{\log_a(x)} a^{\log_a(y)} = a^{\log_a(x) + \log_a(y)}.$$

Since a^x is strictly monotonic, $\log_a(xy) = \log_a x + \log_a y$.