PSA

A1) $\{x_n\}_{n\geqslant 1}$ is a bounded real sequence. Prove that there is a subsequence $\{x_{n_i}\}_{i\geqslant 1}$ such that $\lim_{i\to\infty} x_{n_i}$ exists and

$$\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n.$$

Proof: Let $M = \limsup_{n\to\infty} x_n < \infty$, then for any $\varepsilon = 1/i > 0$ there exists $N \geqslant n_{i-1}$ such that $M \leqslant \sup_{k\geqslant N} x_k < M + \varepsilon$. Hence there exists $n_i \geqslant N$ such that $x_{n_i} \in (M-\varepsilon, M+\varepsilon)$. Take the sequence $\{x_{n_i}\}_{i\geqslant 1}$ then $\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$.

A2) $\{x_n\}_{n\geqslant 1}$ is a real sequence. Prove that $\{x_n\}_{n\geqslant 1}$ converges iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Proof: Since a sub-sequence of a Cauchy sequence converge to the same value as the original sequence, ==> is trivial by A1).
<== $\lim_{n\to\infty} \sup_{k\geqslant n} x_k - \inf_{k\geqslant n} x_k = 0$ implies x_n is Cauchy, hence convergent.

A3) $\{x^{(k)}\}_{k\geqslant 1} \subset \mathbb{R}^n$, where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$. Then $\{x^{(k)}\}_{k\geqslant 1}$ converges in $\mathbb{R}^n$ iff for any $i = 1, 2, \cdots, n$, $\{x_i^{(k)}\}_{k\geqslant 1}$ converges.

Proof: Use Cauchy sequences and the fact that for $x = (x_1, x_2, \cdots, x_n)$,

$$\max\{|x_k| : 1 \leqslant k \leqslant n\} \leqslant \|x\| \leqslant \sum_{k=1}^{n} |x_k|.$$

A4) Suppose $\{z_n\}_{n\geqslant 1}, \{w_n\}_{n\geqslant 1}$ are two convergent complex sequences. Prove that if $\lim_{n\to\infty} w_n \neq 0$, then the sequence $\{z_n/w_n\}_{n\geqslant 1}$ converges.

Proof: Suppose $z = \lim_{n\to\infty} z_n$ and $w = \lim_{n\to\infty} w_n$, then

$$\left|\frac{z_n}{w_n} - \frac{z}{w}\right| \leqslant \frac{|w|\cdot|z_n - z|}{|w\cdot w_n|} + \frac{|z|\cdot|w_n - w|}{|w\cdot w_n|}.$$

Hence $\left|\frac{z_n}{w_n} - \frac{z}{w}\right| \to 0$, so $\lim_{n\to\infty} z_n/w_n = z/w$.

A5) Suppose $\{a_n\}_{n\geqslant 1}$ is a monotonically decreasing sequence of positive reals, and $\lim_{n\to\infty} a_n = 0$. Prove that the series

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

converges.
Proof: Suppose $a_n = a_1 - \sum_{k=1}^{n} b_k$, then $b_k \geqslant 0$ and $\sum_{k=1}^{\infty} b_k = a_1$. The series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} b_{2n} < a_1$$

clearly converges.

A6) $\{a_n\}_{n\geqslant 1}\subset\mathbb{C}$. Prove that if $\sum_{k=1}^{\infty}|a_k|$ converges, then $\sum_{k=1}^{\infty}a_k$ converges.

Proof: $\sum_{k=1}^{\infty}|a_k|$ converges implies for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$, $\sum_{k=n}^{n+p}|a_k|<\varepsilon$. Note that $\left|\sum_{k=n}^{n+p}a_k\right|\leqslant\sum_{k=n}^{n+p}|a_k|$, so $\sum_{k=1}^{\infty}a_k$ converges.

A7) Prove that we can define the exponential function on $\mathbb{C}$:

$$\exp:\mathbb{C}\to\mathbb{C},\ z\mapsto\exp(z)=e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Proof: Use A6).

A8) $\{a_n\}\subset\mathbb{C}$. Suppose for any $n\in\mathbb{N}$, $a_n\neq 0$. Let $P_n=a_1\cdot a_2\cdots a_n$. If $\lim_{n\to\infty}P_n$ exists and is not 0, we call $\prod_{n=1}^{\infty}a_n$ convergent and let $\prod_{n=1}^{\infty}a_n=\lim_{n\to\infty}P_n$. Prove that $\prod_{n=1}^{\infty}a_n$ converges iff for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$,

$$|a_n\cdot a_{n+1}\cdots a_{n+p}-1|<\varepsilon.$$

Proof: If $\lim_{n\to\infty}P_n=P$ exists and is non-zero, then for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $|P_n-P|<\varepsilon P/4$ and $|P_n|>P/2$. Then for any $n\geqslant N$, $p\geqslant 0$, $|P_{n+p}/P_n-1|<\varepsilon$.
If for any $\varepsilon>0$, there exists N such that for any $n\geqslant N$, $p\geqslant 0$, $|P_{n+p}-P_n|<\varepsilon|P_n|$, then let $\varepsilon=1$ we infer that P_n is bounded by some constant M. Hence the sequence $\{P_n\}$ is Cauchy, and $P=\lim_{n\to\infty}P_n$ cannot be zero, otherwise there is no such N for $\varepsilon=1/2$.

A9) Prove that $\exp(x)$ is monotonically increasing on $\mathbb{R}$.

Proof: For $x,y\in\mathbb{R}$,

$$\exp(x)\cdot\exp(y)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot\sum_{m=0}^{\infty}\frac{y^m}{m!}=\sum_{k=0}^{\infty}\sum_{n+m=k}\frac{x^ny^n\binom{k}{n}}{k!}=\sum_{k=0}^{\infty}\frac{(x+y)^k}{k!}=\exp(x+y).$$

$\exp(x)\cdot\exp(-x)=\exp(0)=1$ implies $\exp(x)>0$ for all $x\in\mathbb{R}$, so if $x>y$, $\exp(x)/\exp(y)=\exp(x-y)>1\implies\exp(x)>\exp(y)$.

A10) Suppose $P(x)$ and $Q(x)$ are polynomials of degree n,m, where $m>n$. Prove that

$$\lim_{n\to\infty}\frac{Q(n)}{P(n)}=0,\ \lim_{n\to\infty}\frac{Q(n)}{e^n}=0.$$

Proof: Suppose $P(x)=\sum_{k=0}^{n}a_kx^k$ and $Q(x)=\sum_{k=0}^{m}b_kx^k$, then there exists N such that for any $x\geqslant N$, $|P(x)|>|a_n|x^n/2$, $|Q(x)|\leqslant\sum_{k=0}^{m}|b_k|\cdot x^m$, and $e^x\geqslant x^{m+1}/(m+1)!$, hence

$$\left|\frac{Q(x)}{P(x)}\right|\leqslant\frac{2\sum_{k=0}^{m}|b_k|}{|a_n|}\cdot x^{m-n}\to 0,\ \left|\frac{Q(x)}{e^x}\right|\leqslant(m+1)!\sum_{k=0}^{m}|b_k|\cdot x^{-1}\to 0.$$

PSB: Calculation of Limits

B1)

$$\lim_{n\to\infty} \frac{n+10}{2n-1} = \frac{1}{2}.$$

B2)

$$\lim_{n\to\infty} \frac{\sqrt{n}+10}{2\sqrt{n}-1} = \frac{1}{2}.$$

B3)

$$\lim_{n\to\infty} 0.\underbrace{99\cdots9}_{n \text{ times}} = 1.$$

B4)

$$\lim_{n\to\infty} \frac{1}{n(n+3)} = 0.$$

B5)

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

B6)

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

B7)

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

B8)

$$\lim_{n\to\infty} \sqrt{n+10} - \sqrt{n+1} = 0.$$

B9)

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}.$$

B10)

$$\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{3}.$$

B11) $a > 0$

$$\lim_{n\to\infty} a^{1/n} = 1.$$

B12) $a > 1$

$$\lim_{n\to\infty} \frac{n^{10000}}{a^n} = 0.$$

B13)

$$\lim_{n\to\infty} \frac{2^n+n}{3^n+n^2} = 0.$$

B14)

$$\lim_{n\to\infty} \frac{3^n+2^n}{3^n+n^2} = 1.$$

B15)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{1}{2}.$$

B16) same as B12)

B17)

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1}.$$

B18)

$$\lim_{n\to\infty} \left(1-\frac{1}{5n}\right)^{n+2019} = e^{-1/5}.$$

B19)

$$\lim_{n\to\infty} (n^3+n^2+9n+1)^{1/n} = 1.$$

B20)

$$\lim_{n\to\infty} (2018^n+2019^n)^{1/n} = 2019.$$

PSC: Riemann Rearrangement Theorem

Suppose $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, we will prove that for and $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$, we can rearrange the sequence such that the new series sums to α. Suppose $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ is a bijection, let $b_k = a_{\varphi(k)}$, then the sequence $\{b_k\}_{k\geqslant 1}$ is called a rearrangement of $\{a_n\}_{n\geqslant 1}$.
Let all non-negative terms of $\{a_n\}_{n\geqslant 1}$, listed in the same order as in $\{a_n\}$ be $c_1, c_2, \cdots$, and the negative terms be $d_1, d_2, \cdots$.

C1) Prove that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$.

Proof: Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, c_n, d_n both have infinite terms and $\lim_{n\to\infty} a_n = 0$. Therefore $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$.

C2) Prove that $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} b_n = \infty$.

Proof: Since $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent, the two series can not be both convergent. If one converges and the other doesn't, then $\sum_{n=1}^{\infty} a_n$ will diverge. Hence they both diverge.

C3) Prove that for any $\alpha \in \mathbb{R}$, there exists a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} b_k = \alpha$.

Proof: Suppose $\alpha \geqslant 0$. Inductively define the indices u_i and v_i as follows ($u_0 = v_0 = 0$): For $i \geqslant 1$, let u_i be the least index such that $u_i > u_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant \alpha,$$

and v_i be the least index such that $v_i > v_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j \leqslant \alpha.$$

Let φ be the permutation such that

$$b_1 = c_1, b_2 = c_2, \cdots, b_{u_1} = c_{u_1}, b_{u_1+1} = -d_1, \cdots, b_{u_1+v_1} = -d_{u_1}, \cdots$$

Since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} d_n = \infty$, u_i and v_i all exists, so φ is indeed a bijection. By definition we know that

$$\left|\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j - \alpha\right| \leqslant c_{u_i-1},$$

and

$$\left|\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j - \alpha\right| \leqslant d_{v_i-1}.$$

Since $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$, the two values above both tend to 0. Note that the series $\sum_{n=1}^{\infty} b_n$ is monotonic between these indices, hence $\sum_{n=1}^{\infty} b_n = \alpha$.

C4) Prove that there exists a rearrangement $\{x_k\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} x_k = \infty$.

Proof: Define u_i and v_i as in C3), such that

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant i \geqslant \sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j.$$

Same as C3) define the sequence x_k and clearly $\sum_{n=1}^{\infty} x_k = \infty$.

PSD: Cesàro Sum

For a real sequence $\{a_n\}_{n\geqslant 1}$, let $\sigma_n = \frac{a_1+a_2+\cdots+a_n}{n}$.

D1) Suppose $\lim_{n\to\infty} a_n = a$, prove that $\lim_{n\to\infty} \sigma_n = a$.

Proof: For any $n > 0$,

$$|\sigma_n - a| \leqslant \sum_{i=1}^{N} \frac{|a_i - a|}{n} + \sum_{i=N+1}^{n} \frac{|a_i - a|}{n} \leqslant \frac{MN}{n} + \varepsilon(N),$$

where $M = |a| + \sup_{i\leqslant N} |a_i|$, and $\varepsilon(N) = \sup_{i>N} |a_i - a|$. By $\lim_{n\to\infty} a_n = a$ we know $\varepsilon(N) \to 0$, hence $\lim_{n\to\infty} \sigma_n = a$.

D2) Construct a divergent sequence $\{a_n\}$ such that $\lim_{n\to\infty} \sigma_n = 0$.

Solution: $a_n = (-1)^{n-1}$, $\sigma_n \in [0, 1/n]$.

D3) Determine whether there exists $\{a_n\}_{n\geqslant 1}$ such that for any $n \geqslant 1$, $a_n > 0$ and $\limsup_{n\to\infty} a_n = \infty$ but $\lim_{n\to\infty} \sigma_n = 0$.

Solution: Let

$$a_n = \begin{cases} 2^{-n}, & n \neq 2^k, \\ k, & n = 2^k. \end{cases}$$

Then $\limsup_{n\to\infty} a_n = \infty$ and $a_n > 0$, but for any n, suppose $n \in [2^{k-1}, 2^k]$, then

$$\sigma_n \leqslant \frac{1}{n} \cdot \left(1 + \frac{k(k+1)}{2}\right) \leqslant \frac{k(k+1)}{2^{k-1}}.$$

Hence $\lim_{n\to\infty} \sigma_n = 0$.

D4) For $k \geqslant 1$, denote $b_k = a_{k+1} - a_k$. Prove that for any $n \geqslant 2$, $a_n - \sigma_n = \sum_{k=1}^{n-1} kb_k/n$.

Proof:

$$\sum_{k=1}^{n-1} kb_k = \sum_{k=1}^{n-1} k(a_{k+1} - a_k) = (n-1)a_n - \sum_{k=1}^{n-1} a_k = n(a_n - \sigma_n).$$

D5) Suppose $\lim_{k\to\infty} kb_k = 0$ and $\{\sigma_n\}_{n\geqslant 1}$ converges. Prove that $\{a_n\}_{n\geqslant 1}$ also converges.

Proof: By D1), $\lim_{k\to\infty} kb_k = 0$ implies

$$\lim_{n\to\infty} a_n - \sigma_n = \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} kb_k}{n} = \lim_{k\to\infty} kb_k = 0.$$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n$ exists.

D6) Suppose $\{kb_k\}_{k\geqslant 1}$ is bounded, i.e. $b_k = O(k^{-1})$, and $\lim_{n\to\infty} \sigma_n = \sigma$. Prove that $\lim_{n\to\infty} a_n = \sigma$.

Proof: Note that for $m < n$,

$$a_n - \sigma_n = \frac{m}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^{n} a_n - a_k.$$

Therefore since σ_n is a Cauchy sequence, and $|a_n - a_k| \leqslant M(n-k)/k$, we can choose n, m to show that $\lim_{n\to\infty} a_n - \sigma_n = 0$.

PSE: Definition of $\sqrt[n]{x}$ and b^x

E1) Given $n \in \mathbb{N}$ and $x > 0$, prove that if $y_1, y_2 > 0$ satisfy $y_1^n = x = y_2^n$, then $y_1 = y_2$.

Proof: Note that $y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1} > 0$, and

$$0 = y_1^n - y_2^n = (y_1 - y_2) \cdot (y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1}).$$

Hence $y_1 = y_2$.

E2) Prove that if $x > 0$, then the set $E(x) = \{t \in \mathbb{R} : t^n < x\}$ is non-empty and has an upper-bound.

Proof: Note that $0 \in E(x)$ and $E(x)$ has the upper-bound $\max\{1, x\}$.

E3) Prove that $y = \sup\, E(x)$ satisfy $y^n = x$ and $y > 0$.

Proof: $y = \sup\, E(x) \implies y^n = x$ since t^n is continuous on $\mathbb{R}$, and $y^n = x$ and $0 \in E(x)$ implies $y > 0$.

E4) Prove that the mapping $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto \sqrt[n]{x} = y$ is well-defined. Denote $\sqrt[n]{x}$ as $x^{1/n}$.

Proof: Use E3).

E5) Prove the $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a bijection.

Proof: By E1) it is injective, and $\sqrt[n]{y^n} = y$ implies it is surjective. Hence it is a bijection.

E6) $a, b > 0$, $n \in \mathbb{N}$, prove that $(ab)^{1/n} = a^{1/n}b^{1/n}$.

Proof: Use E5) and $(xy)^n = x^n y^n$.

E7) Suppose $b > 1$, $m, n, p, q \in \mathbb{Z}$ where $n, q > 0$. Let $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Proof: Use $(b^m)^q = (b^p)^n$ and E5).

E8) Prove that for any $r \in \mathbb{Q}$, $r \mapsto b^r$ is well-defined.

Proof: For $r = p/q$, where $q > 0, \gcd(p, q) = 1$, let $b^r = (b^p)^{1/q}$, then for any $r = m/n$, $b^r = (b^m)^{1/n}$.

E9) Prove that for r, $s \in \mathbb{Q}$, $b^{r+s} = b^r b^s$.

Proof: Suppose $r = p/q, s = m/n$, where $n, q > 0$, then

$$b^{r+s} = b^{(mq+np)/nq} = (b^{mq} \cdot b^{np})^{1/nq} = (b^m)^{1/n} \cdot (b^p)^{1/q} = b^r b^s.$$

E10) For $x \in \mathbb{R}$, let $B(x) = \{b^t : t \in \mathbb{Q}, t \leqslant x\}$. Prove that $B(x)$ is non-empty and has an upper-bound. Define $b^x = \sup\, B(x)$.

Proof: $B(x)$ is clearly non-empty and bounded by $b^{\lfloor x \rfloor + 1}$.

E11) Prove that if $r \in \mathbb{Q}$, then

$$b^r = \sup B(r),\ \forall r \in \mathbb{Q}.$$

Proof: $b^r \in B(r)$ and since b^t is monotonically increasing, $b^r \geqslant \sup\, B(r)$, hence $b^r = \sup\, B(r)$.

E12) Prove that for any $x, y \in \mathbb{R}$, $b^{x+y} = b^x b^y$.

Proof: For any $b^t \in B(x)$, $b^s \in B(y)$, $t \leqslant x$ and $s \leqslant y$, so $t + s \leqslant x + y$ and $b^{t+s} \in B(x + y)$, hence $b^{x+y} \geqslant b^x b^y$. For any $b^t \in B(x + y)$, t can be written in the form $t = u + v$ where $b^u \in B(x)$, $b^v \in B(y)$, so $b^{x+y} \leqslant b^x b^y$.

E13*) Prove that when $b = e$, the function derived from E10) (denoted as e^x) is the same as $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Proof: From $\exp(1) = e$, $\exp(0) = 1$ and $\exp(x+y) = \exp(x) \cdot \exp(y)$ we know that for $n \in \mathbb{Z}$, $\exp(n) = e^n$. For $r = p/q \in \mathbb{Q}$,

$$(e^r)^q = e^p = \exp(p) = \exp(r)^q,$$

so by E5) $e^r = \exp(r)$. Since $\exp$ is continuous, for any $x \in \mathbb{R}$, $e^x = \exp(x)$.

PSF

Given $\alpha > 0$ and $x_1 > \sqrt{\alpha}$, we define inductively $\{x_n\}_{n \geqslant 1}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right), n \geqslant 1.$$

F1) Prove that $\{x_n\}$ is monotonically decreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$ (which is defined in E).

Proof: Note that

$$x_{n+1} - x_n = \frac{\alpha - x_n^2}{2x_n}.$$

Hence we can prove by induction that $x_n > \sqrt{\alpha}$ and $x_n > x_{n+1}$. x_n is decreasing and bounded, so $\lim_{n\to\infty} x_n = A$ exists, and $A = (A + \alpha/A)/2$. Therefore $\lim_{n\to\infty} x_n = A = \sqrt{\alpha}$.

F2) Let $\varepsilon_n = x_n - \sqrt{\alpha}$. Prove that $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$.

Proof:

$$\frac{\varepsilon_n^2}{2x_n} = \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = x_{n+1} - \sqrt{\alpha} = \varepsilon_{n+1}.$$

F3) Prove that if $\beta = 2\sqrt{\alpha}$, then $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

Proof: $\varepsilon_{n+1}/\beta < (\varepsilon_n/\beta)^2$, hence $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

F4) Let $\alpha = 3$, $x_1 = 2$. Verify that $\varepsilon_1/\beta < 0.1$, $\varepsilon_5 < 4 \cdot 10^{-16}$, $\varepsilon_6 < 4 \cdot 10^{-32}$.

Now we consider $\alpha > 1$ and $y_1 > \sqrt{\alpha}$, and define

$$y_{n+1} = \frac{\alpha + y_n}{1 + y_n} = y_n + \frac{\alpha - y_n^2}{1 + y_n}, n \geqslant 1$$

F6) Prove that $\{y_{2k-1}\}$ is monotonically decreasing.

Proof: Note that

$$y_{n+2} = \frac{\alpha + y_{n+1}}{1 + y_{n+1}} = \frac{\alpha + \frac{\alpha + y_n}{1 + y_n}}{1 + \frac{\alpha + y_n}{1 + y_n}} = \frac{2\alpha + (\alpha+1)y_n}{(\alpha+1) + 2y_n},$$

hence

$$y_{n+2} - y_n = \frac{2(\alpha - y_n^2)}{(\alpha+1) + 2y_n}, \; y_{n+2} - \sqrt{\alpha} = \frac{(\sqrt{\alpha} - 1)^2}{(\alpha+1) + 2y_n}(y_n - \sqrt{\alpha}).$$

Therefore $y_1 > \sqrt{\alpha}$ implies $\sqrt{\alpha} < y_{2n+1} < y_{2n-1}$.

F7) Prove that $\{y_{2k}\}$ is monotonically increasing.

Proof: $y_2 = (\alpha + y_1)/(1+y_1) < \sqrt{\alpha}$, so same as F6), $y_{2k} > y_{2k-2}$ and $y_{2k} < \sqrt{\alpha}$.

F8) Prove that $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

Proof: $\{y_{2n-1}\}$ is decreasing and bounded by $\sqrt{\alpha}$, so $\lim_{n\to\infty} y_{2n-1} = A$ exists and $A = (2\alpha + (\alpha+1)A)/((\alpha+1)+2A)$, so $A = \sqrt{\alpha}$. Likewise $\lim_{n\to\infty} y_{2n} = \sqrt{\alpha}$, hence $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

F9) Compare the rates of convergence between x_n and y_n.

Solution: Let $\delta_n = |y_n - \sqrt{\alpha}|$, then $\delta_n \sim c^n \delta_1$, hence x_n converges faster then y_n.

PSG: Banach-Mazur Game

Alice and Bob are playing a game: Alice selects a closed interval W_1 first, then Bob choose a subinterval L_1 of W_1, such that the length of L_1 is less than half of the length of W_1; they take turns choosing intervals W_n and L_n, such that $L_n \subset W_n \subset L_{n-1}$ and $|L_n| < |W_n|/2 < |L_{n-1}|/4$, obtaining

$$W_1 \supset L_1 \supset W_2 \supset L_2 \supset \cdots \supset W_n \supset L_n \supset \cdots$$

Alice and Bob find that

$$\bigcap_{n\geqslant 1} W_n = \bigcap_{n\geqslant 1} L_n = \{x\}$$

is a real number. If $x \in \mathbb{Q}$ then Alice wins, otherwise Bob wins. Who has a winning strategy?
Solution: Bob will win. We show that if $\mathbb{Q}$ is replaced with any set M that is of first category, Bob can still win.
M can be written as the union of a countable number of nowhere dense sets. Then in every move of Bob, he can choose L_n such that it does not intersect the nth such nowhere dense set. Hence the final number x is not in M.

Problem H

Consider the set $\mathcal{P} = \{\{p_n\}_{n\geqslant 1} : p_n \in \mathbb{Z}, p_1 \geqslant 2, p_{n+1} \geqslant p_n^2\}$.

H1) For any $p = \{p_n\}_{n\geqslant 1} \in \mathcal{P}$, define the sequence

$$a_n = \prod_{k=1}^{n} \left(1 + \frac{1}{p_k}\right).$$

Prove that $f(p) = \lim_{n\to\infty} a_n$ exists and $f(p) \in (1, 2]$.
Proof: Note that $p_n \geqslant p_1^{2^{n-1}}$, then

$$a_n \leqslant \prod_{k=1}^{n} \left(1 + \frac{1}{p_1^{2^{k-1}}}\right) = \frac{1 - p_1^{-2^n}}{1 - p_1^{-1}} < \frac{1}{1 - p_1^{-1}}.$$

So the sequence $\{a_n\}$ is monotonic and bounded, hence $f(p) = \lim_{n\to\infty} a_n$ exists. Since $a_n \in (1 + 1/p_1, \frac{1}{1-p_1^{-1}})$, we obtain $f(p) \in [1 + 1/p_1, \frac{1}{1-p_1^{-1}}] \subset (1, 2]$.

H2) Prove that $f : \mathcal{P} \to (1, 2]$ is a bijection.

Proof: For any $p = \{p_n\}, q = \{q_n\} \in \mathcal{P}$, if $p \neq q$, take the least k such that $p_k \neq q_k$ and suppose $q_k \geqslant p_k + 1$, then for any $n > k$,

$$a_n = \prod_{t=1}^{n}\left(1+\frac{1}{p_t}\right) \geqslant \prod_{t=1}^{k}\left(1+\frac{1}{p_t}\right)\cdot\left(1+\frac{1}{p_{k+1}}\right)$$

$$b_n = \prod_{t=1}^{n}\left(1+\frac{1}{q_t}\right) \leqslant \prod_{i=1}^{k-1}\left(1+\frac{1}{p_t}\right)\cdot\frac{1-q_k^{-2^{n-k}}}{1-q_k^{-1}}$$

Therefore

$$b_n \leqslant \prod_{t=1}^{k}\left(1+\frac{1}{p_t}\right) \leqslant (1+C)a_n$$

for all $n > k$ where $C = p_{k+1}^{-1} > 0$, hence $f(q) \leqslant (1+C)f(p) < f(p)$, hence f is injective.
For any $x \in (1, 2]$, inductively define $p = \{p_n\} \in \mathcal{P}$ as follows: For any $n \geqslant 1$, Let t be the least integer such that $a_n \leqslant x$ and $t \geqslant p_{n-1}^2$ (clearly such t exists). If $a_n = x$, then let $p_n = t - 1$, $p_m = p_n^{2^{m-n}}$ for all $m > n$, then $f(p) = x$. Otherwise let $p_n = t$. Note that for any n such that $p_n > p_{n-1}^2$,

$$|x - a_n| \leqslant 2^{-2^n},$$

therefore $f(p) = x$, and f is surjective.

H3) Prove that $\mathcal{P}$ is uncountable.

Proof: By H2) and the fact that $(1, 2]$ is uncountable.

Problem I: Binary Expansion

Consider the set $\mathcal{S} = \{\{s_n\}_{n\geqslant 0} : s_n \in \{-1, 1\}\}$.

I1) For any $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, define the sequence

$$c_n = \sum_{k=0}^{n}\frac{s_0 s_1 \cdots s_k}{2^k}.$$

Prove that $h(s) = \lim_{n\to\infty} c_n$ exists and $h(s) \in [-2, 2]$.
Proof: $h(s)$ exists since c_n is clearly a Cauchy sequence, and $c_n \in [-2, 2]$ hence $h(s) \in [-2, 2]$.

I2) Prove that $h : \mathcal{S} \to [-2, 2]$ is surjective. Determine whether is is injective.

Proof: Consider any $x \in [-2, 2]$, we can choose s_n such that $|c_n - x| \leqslant 2^{-n}$. Hence there exists $s = \{s_n\} \in \mathcal{S}$ such that $h(s) = \lim_{n\to\infty} c_n = x$, so h is surjective.
Consider $s = \{1, -1, 1, 1, 1, \cdots\} \in \mathcal{S}$ and $s' = \{-1, -1, 1, 1, \cdots\}$, then $h(s) = h(s') = 0$, hence h is not injective.

I3) For $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, prove that

$$2\sin\left(\frac{\pi}{4}c_n\right) = s_0\sqrt{2 + s_1\sqrt{2 + \cdots + s_n\sqrt{2}}}.$$

Proof: We prove by induction on n. The base $n = 0$ is trivial. If the statement holds for $n - 1$, then let $s' = \{s_{n+1}\}_{n\geqslant 0} \in \mathcal{S}$, we have

$$2\sin\left(\frac{\pi}{4}c_n\right) = 2\sin s_0\left(\frac{\pi}{4}+\frac{1}{2}\cdot\frac{\pi}{4}c'_{n-1}\right) = s_0\sqrt{2+\sin\left(\frac{\pi}{4}c'_{n-1}\right)}.$$

By the induction hypothesis, the statement also holds for n.

I4) Calculate the limit

$$\lim_{n\to\infty}\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}$$

Solution: Consider $s = \{s_n = 1\}_{n\geqslant 0} \in \mathcal{S}$, then $c_n = 2 - 2^n$ hence $\lim_{n\to\infty} 2\sin\left(\pi c_n/4\right) = 2$.

Problem J

Problem: $k \geqslant 2$ is a given integer. Define the sequence $\{a_n\}$ as follows:

$$a_0 > 0 \text{ already given}, a_{n+1} = a_n + a_n^{-1/k}, n \geqslant 0.$$

Calculate $\lim_{n\to\infty} a_n^{k+1}/n^k$.
Solution: It is easy to see that $a_n \to \infty$, hence

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n^{\frac{k+1}{k}}}{n} &= \lim_{n\to\infty} a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}} = \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1+\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{k+1}{k}} - 1\right)\\ &= \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1+a_n^{-\frac{k+1}{k}}\right)^{\frac{k+1}{k}} - 1\right) = \frac{k+1}{k}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{a_n^{k+1}}{n^k} = \left(1+\frac{1}{k}\right)^k.$$