A. Young's Inequality

$f:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ is continuously differentiable. Assume $f(0)=0$, $\lim_{x\to\infty}f(x)=\infty$, and for any $x\geqslant 0$, $f'(x)>0$. Denote f^{-1} by $g(x)$.

A1) Prove that for any $a\geqslant 0$,

$$af(a)=\int_0^a f(x)\,\mathrm{d}x+\int_0^{f(a)}g(y)\,\mathrm{d}y.$$

Proof: $a=0$ is trivial. Take derivatives on both sides, then

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a f(x)\,\mathrm{d}x=f(x),\ \frac{\mathrm{d}}{\mathrm{d}a}\int_0^{f(a)}g(y)\,\mathrm{d}y=g(f(a))f'(a)=af'(a),$$

and $(af(a))'=f(a)+af'(a)$.

A2) Prove Young's inequality: for any $a,b\geqslant 0$,

$$ab\leqslant\int_0^a f(x)\,\mathrm{d}x+\int_0^b g(y)\,\mathrm{d}y.$$

Proof: Assume that $b\geqslant f(a)$, then

$$\begin{aligned}ab&\leqslant\int_0^a f(x)\,\mathrm{d}x+\int_0^{f(a)}g(y)\,\mathrm{d}y+a(b-f(a))\\&\leqslant\int_0^a f(x)\,\mathrm{d}x+\int_0^b g(y)\,\mathrm{d}y.\end{aligned}$$

A3) Suppose $f:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ is strictly monotonically increasing, and $f(0)=0$, $\lim_{x\to\infty}f(x)=\infty$. Prove the inequality in A2)

Proof: It suffices to prove the equality in A1). For any $a,b\geqslant 0$, f can be uniformly approximated by functions P such that $P(0)=0$, P is continuously differentiable and monotonically increasing. Then

$$aP(a)=\int_0^a P(x)\,\mathrm{d}x+\int_0^{P(a)}P^{-1}(y)\,\mathrm{d}y.$$

Note that $P(a)-f(a)\to 0$, $\int_0^a P(x)\,\mathrm{d}x-\int_0^a f(x)\,\mathrm{d}x\to 0$, $\int_0^b P^{-1}(y)-g(y)\,\mathrm{d}y\to 0$(since P is continuously differentiable), and $\int_b^{P(a)}P^{-1}(y)\,\mathrm{d}y\to 0$ since P^{-1} is bounded by a.
Hence the equality in A1) also holds for f.

A4) Assume $a,b\geqslant 0$, $p>1,q>1$, and $1/p+1/q=1$. Prove that

$$ab\leqslant\frac{a^p}{p}+\frac{b^q}{q}.$$

Proof: Consider $f(x)=x^{p-1}$, $g(x)=x^{1/(p-1)}=x^{q-1}$, then by A2),

$$ab\leqslant\int_0^a f(x)\,\mathrm{d}x+\int_0^b g(y)\,\mathrm{d}y=\frac{a^p}{p}+\frac{b^q}{q}.$$

where equality holds iff $b=f(a)=a^{p-1}$, i.e. $a^p=b^q$.

B. Sobolev's Inequality

Consider any compact interval $[a,b]$.

B1) (Cauchy-Schwarz Inequality) Assume $f, g \in \mathcal{R}([a,b])$, prove that

$$\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \leqslant \left(\int_a^b |f(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_a^b |g(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Proof: Consider the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x.$$

Then

$$0 \leqslant \langle f+tg, f+tg\rangle = \langle f, f\rangle + 2t\langle f, g\rangle + t^2\langle g, g\rangle.$$

Hence

$$\Delta = 4(\langle f, g\rangle^2 - \langle f, f\rangle\langle g, g\rangle) \leqslant 0.$$

B2) Prove that for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that for any $f \in C^1([a,b])$, and any $x \in [a,b]$,

$$|f(x)^2 - f(a)^2| \leqslant C_\varepsilon \int_a^b f(x)^2\,\mathrm{d}x + \varepsilon\int_a^b f'(x)^2\,\mathrm{d}x.$$

Proof: Let $C_\varepsilon = 4/\varepsilon$, then

$$\begin{aligned}|f(x)^2 - f(a)^2| &= \left|\int_a^x \mathrm{d}f^2\right| = \left|\int_a^x 2ff'\right| \\ &\leqslant \varepsilon\int_a^b f'(x)^2\,\mathrm{d}x + \frac{4}{\varepsilon}\int_a^b f^2.\end{aligned}$$

B3) Prove that for any $\varepsilon > 0$, there is a constant $D_\varepsilon > 0$ such that for any $f \in C^1([a,b])$,

$$\sup_{x\in[a,b]} |f(x)|^2 \leqslant D_\varepsilon \int_a^b f(x)^2\,\mathrm{d}x + \varepsilon\int_a^b f'(x)^2\,\mathrm{d}x.$$

Proof: Let $D_\varepsilon = \frac{1}{b-a} + C_\varepsilon$, then

$$\begin{aligned}\sup_{x\in[a,b]} |f(x)|^2 &\leqslant \inf_{x\in[a,b]} |f(x)|^2 + \sup_{x,y\in[a,b]} |f(x)^2 - f(y)^2| \\ &\leqslant \frac{1}{b-a}\int_a^b f^2 + C_\varepsilon\int_a^b f^2 + \varepsilon\int_a^b f'(x)^2\,\mathrm{d}x.\end{aligned}$$

C. Wirtinger's Inequality

Let $E = \{f \in C^1([0,1]) : f(0) = f(1) = 0\}$.

C1) For any $f \in E$, define the improper integral

$$\mathbf{I}_1 = \int_0^1 \frac{f(x)f'(x)}{\tan(\pi x)}\,\mathrm{d}x,\ \mathbf{I}_2 = \int_0^1 \frac{f(x)^2}{\tan^2(\pi x)}(1+\tan^2(\pi x))\,\mathrm{d}x.$$

Prove that they converge and determine the value I_1/I_2.
Proof:

$$\lim_{x\to 0^+} \frac{f(x)f'(x)}{\tan(\pi x)} = f'(0)\lim_{x\to 0^+}\frac{f'(x)}{\pi/\cos^2(\pi x)} = \frac{f'(0)^2}{\pi}.$$

Hence $ff'/\tan(\pi x) \in C([0,1])$, so the integral converges.

$$
\begin{aligned}
I_1 &= \int_0^1 \frac{\mathrm{d} f^2}{2\tan(\pi x)} = \frac{f^2}{2\tan \pi x}\Big|_0^1 + \int_0^1 f^2 \frac{\pi}{2} \sec^2(\pi x)\,\mathrm{d}x \\
&= \frac{\pi}{2} I_2.
\end{aligned}
$$

C2C3) (Wirtinger's Inequality) For any $f \in E$,

$$
\int_0^1 f^2 \leqslant \pi^{-2} \int_0^1 f'(x)^2 \,\mathrm{d}x.
$$

Proof:

$$
\begin{aligned}
\mathbf{I}_2 &= \int_0^1 f(x)^2 \,\mathrm{d}x + \int_0^1 \frac{f^2}{\tan^2(\pi x)} \,\mathrm{d}x = \frac{2}{\pi} \mathbf{I}_1 \\
&= \frac{2}{\pi} \int_0^1 \frac{f}{\tan(\pi x)} f' \,\mathrm{d}x \leqslant \frac{2}{\pi} \sqrt{\int_0^1 \frac{f^2}{\tan^2(\pi x)} \,\mathrm{d}x \cdot \int_0^1 (f')^2} \\
&\leqslant \int_0^1 \frac{f^2}{\tan^2(\pi x)} \,\mathrm{d}x + \pi^{-2} \int_0^1 (f')^2.
\end{aligned}
$$

Therefore

$$
\int_0^1 f(x)^2 \,\mathrm{d}x \leqslant \pi^{-2} \int_0^1 f'(x)^2 \,\mathrm{d}x.
$$

If equality holds, then $f' = \frac{\pi f}{\tan(\pi x)}$, hence $df/f = \frac{\mathrm{d}\pi x}{\tan(\pi x)}$, so $f = C\sin(\pi x)$. Combined with $f \in E$, we obtain $f = C\sin(\pi x), \forall C \in \mathbb{R}$.

C4) Assume $f \in \mathcal{R}([0, 2\pi])$, determine the value of $A \in \mathbb{R}$ which minimizes the integral

$$
\int_0^{2\pi} |f(x) - A|^2 \,\mathrm{d}x.
$$

Solution:

$$
\langle f - A, f - A \rangle = 2\pi A^2 + \langle f, f \rangle - 2A \int_0^{2\pi} f(x) \,\mathrm{d}x.
$$

Hence

$$
A = \frac{1}{2\pi} \int_0^{2\pi} f(x) \,\mathrm{d}x.
$$

(Also, since e^{inx} forms a base of the linear space $\mathcal{R}([0,1])$, A is the projection of f onto e^{i0x}, hence $A = \tilde{f}(0)$.)

C5) Another version of Wirtinger's inequality: for any $f \in C^1([0, 2\pi])$, if $f(0) = f(2\pi)$, and $\int_0^{2\pi} f(x) \,\mathrm{d}x = 0$, then

$$
\int_0^{2\pi} f(x)^2 \,\mathrm{d}x \leqslant \int_0^{2\pi} f'(x)^2 \,\mathrm{d}x.
$$

Proof: Consider $g(x) = f(x) - f(x + \pi)$, then $g(0) = -g(\pi)$, hence $g(x) = 0$ for some $x \in [0, \pi]$. Assume that $g(0) = 0$, i.e. $f(0) = f(\pi)$. Apply C2) to $f|_{[0,\pi]} - f(0)$ and $f|_{[\pi,2\pi]} - f(0)$, then by C4),

$$
\int_0^{2\pi} f(x)^2 \,\mathrm{d}x \leqslant \int_0^{2\pi} |f(x) - f(0)|^2 \,\mathrm{d}x \leqslant \int_0^{2\pi} f'(x)^2 \,\mathrm{d}x.
$$

Proof using Fourier series: $\tilde{f}(0) = 0$, hence

$$\int_0^{2\pi} f^2 = 2\pi \sum_n |\tilde{f}(n)|^2 \leqslant 2\pi \sum_n n^2 |\tilde{f}(n)|^2 = \int_0^{2\pi} |f'|^2.$$

C6) (Isoperimetric inequality) Assume $\gamma : [0, 2\pi] \to \mathbb{R}^2$ is a continuously differentiable parameterization of a closed non-intersecting curve. Let $\gamma(t) = (x(t), y(t))$,

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t,\ A = \int_0^{2\pi} x'(t)y(t)\,\mathrm{d}t.$$

Prove that $L^2 \geqslant 4\pi A$, and equality holds iff γ forms a circle.
Proof: We can choose the parameterization γ such that $|\gamma'(t)| = 1$ for any $t \in [0, 2\pi]$. Furthermore, we can let $\gamma(0) = \gamma(\pi) = 0$. Then $L = 2\pi$ and

$$\int_0^{\pi} x'y \leqslant \frac{1}{2} \int_0^{\pi} |x'|^2 + y^2 = \frac{1}{2} \int_0^{\pi} y^2 + 1 - |y'|^2 \leqslant \frac{\pi}{2}.$$

Where the last inequality comes from C2).
If inequality holds, then by C2), $y = C \sin t, x = C \cos t$, hence γ forms a circle.

D. Gauss-Legendre Quadrature

For any $n \geqslant 1$, define the Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n ((x^2 - 1)^n).$$

Assume $P_0 = 1$.

D1) Prove that for any $\varphi(x) \in C([-1, 1])$, and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $c_1, c_2, \cdots, c_N \in \mathbb{R}$ such that

$$\|\varphi(x) - \sum_{k=1}^{N} c_k P_k(x)\|_\infty < \varepsilon.$$

Proof: $\deg P_n = n$, hence P_n are linearly independent. By Stone-Weierstrass theorem, there exists such N and $c_1, c_2, \cdots, c_N$.

D2) Prove that for any $n \geqslant 1$, P_n satisfy the following differential equation:

$$(1 - x^2)f'' - 2xf' + n(n + 1)f = 0.$$

Proof: Let $g(x) = (x^2 - 1)^n$, then

$$(1 - x^2)g'' + 2ng' + 2g = 0.???$$

We can prove by induction that $g^{(k)}$ satisfy

$$(1 - x^2)f'' + 2(n - k)f' + k(k + 1)f = 0.$$

Let $k = n$ and we obtain the required equation.

D3) Prove that for any $n, m \geqslant 1$,

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \begin{cases} 0, & m \neq n; \\ \frac{2}{2n+1}, & m = n. \end{cases}$$

Proof: Note that for all $0 \leqslant m < n$, $\dfrac{\mathrm{d}^m}{\mathrm{d}x^m}((x^2-1)^n) = 0$ when $x \in \{-1, 1\}$. Hence by Darboux's integration by parts formula,

$\langle P_n 2^n n!, P_m 2^m m! \rangle = \sum_{k=0}^{m} (-1)^k \left((x^2-1)^n\right)^{(n-k-1)} \left((x^2-1)^m\right)^{m+k} \Big|_{x=-1}^{x=1} = 0$

Hence $\{P_n\}$ form an orthogonal base, and also

$\langle P_n, P_n \rangle = \dfrac{(2n)!}{2^n n!} \int_{-1}^{1} (1-x^2)^n \mathrm{d}x = \frac{2}{2n+1}$. (Wallis formula)

D4) Given $n \geqslant 1$, Prove that if $Q(x)$ is a polynomial with degree at most $n-1$, then

$$\int_{-1}^{1} Q(x) P_n(x) \, \mathrm{d}x = 0.$$

Proof: $Q \in \mathrm{Span}\langle P_0, P_1, \cdots, P_{n-1} \rangle$ hence $\langle Q, P_n \rangle = 0$.

D5) Prove that for any $n \geqslant 1$, $P_n(x)$ has exactly n roots on the interval $(-1, 1)$. Denote them by $x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)}$.

Proof: If P_n has less than n distinct roots on $(-1, 1)$, take all roots with odd multiplicity $r_1, \cdots, r_k, k < n$. Let $Q(x) = \prod_{i=1}^{k} (x - r_i)$ then $P_n(x)$ and $Q(x)$ always have the same sign on $(-1, 1)$. Therefore $\langle P_n, Q \rangle \neq 0$, a contradiction.

D6) Prove that for any $n \geqslant 1$, there exists $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_n^{(n)}$ such that for any polynomial Q with degree at most $2n-1$,

$$\int_{-1}^{1} Q(x) \, \mathrm{d}x = \sum_{i=1}^{n} \alpha_i^{(n)} Q(x_i^{(n)}).$$

Proof: Suppose $Q(x) = P_n(x) T(x) + R(x)$, where $\deg R < \deg P_n = n$. Since $\deg Q \leqslant 2n-1$, $\deg T \leqslant n-1$, so $\langle P_n, T \rangle = 0$, i.e. $Q(x_i^{(n)}) = R(x_i^{(n)})$ and

$$\int_{-1}^{1} Q(x) \, \mathrm{d}x = \int_{-1}^{1} R(x) \, \mathrm{d}x.$$

By Lagrange interpolation, let

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j^{(n)}}{x_i^{(n)} - x_j^{(n)}}$$

then $R(x) = \sum_{i=1}^{n} L_i(x) R(x_i^{(n)})$. Let $\alpha_i^{(n)} = \int_{-1}^{1} L_i(x) \, \mathrm{d}x$, then

$$\int_{-1}^{1} R(x) \, \mathrm{d}x = \sum_{i=1}^{n} \alpha_i^{(n)} R(x_i^{(n)}).$$

Hence

$$\int_{-1}^{1} Q(x) \, \mathrm{d}x = \sum_{i=1}^{n} \alpha_i^{(n)} Q(x_i^{(n)}).$$

D7) An approximation of integrals by Gauss: for any $\varphi \in C([-1, 1])$, let

$$G_n(\varphi) = \sum_{i=1}^{n} \alpha_i^{(n)} \varphi(x_i^{(n)}).$$

Prove that $\lim_{n \to \infty} G_n(\varphi) = \int_{-1}^{1} \varphi(x) \, \mathrm{d}x$.

Proof: We show that $\alpha_i^{(n)}$ are all non-negative, hence take a sequence of polynomials $P_n(x)$ uniformly convergent to φ where $\deg P_n = n$, then

$$\left|G_n(\varphi)-\int_{-1}^{1}\varphi(x)\,\mathrm{d}x\right|\leqslant\int_{-1}^{1}|P_n(x)-\varphi(x)|\,\mathrm{d}x+\sum_{i=1}^{n}\alpha_i^{(n)}|\varphi(x_i^{(n)})-P_n(x_i^{(n)})|\to 0.$$

since $\sum_{i=1}^{n}\alpha_i^{(n)}=\int_{-1}^{1}1\,\mathrm{d}x=2$.
Note that $\alpha_i=\int_{-1}^{1}L_i(x)\,\mathrm{d}x$ where

$$L_i(x)=\prod_{j\neq i}\frac{x-x_j^{(n)}}{x_i^{(n)}-x_j^{(n)}}.$$

Since $\deg L=n-1$,

$$\int_{-1}^{1}L_i^2(x)\,\mathrm{d}x=\sum_{j=1}^{n}\alpha_j^{(n)}L_j^2(x)=\alpha_i^{(n)},$$

Hence $\alpha_i^{(n)}\geqslant 0$.

E. Equidistribution (Weyl)

Given a sequence $\{x_k\}_{k\geqslant 1}\subset[0,1]$, for any $0\leqslant a<b\leqslant 1$, let

$$S_n([a,b])=|\{x_k:k\leqslant n,x_k\in[a,b]\}|.$$

If for any $0\leqslant a<b\leqslant 1$,

$$\lim_{n\to\infty}\frac{S_n([a,b])}{n}=b-a,$$

we say that $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$.

E1) Prove that if the sequence $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$, then $\{x_k\}_{k\geqslant 1}$ is dense on $[0,1]$.

Proof: For any $x\in(0,1)$ and $\min\{x,1-x\}>\varepsilon>0$, since $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$, $\lim_{n\to\infty}S_n([x-\varepsilon,x+\varepsilon])/n=2\varepsilon>0$, hence $\{x_k\}_{k\geqslant 1}\cap[x-\varepsilon,x+\varepsilon]\neq\emptyset$ so $\{x_k\}$ is dense.

E2) Construct a dense subset $\{x_k\}_{k\geqslant 1}$ of $[0,1]$ such that it is not equidistributed.

Solution: List all rational numbers in $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$ as $\{q_1,q_2,\cdots\}$ and $\{r_1,r_2,\cdots\}$. Let

$$x_n=\begin{cases}q_{3k+1}, & n=4k+1,\\ q_{3k+2}, & n=4k+2,\\ q_{3k+3}, & n=4k+3,\\ r_{k+1}, & n=4k+4.\end{cases}$$

Then $\{x_n\}$ includes all rational numbers in $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$ so it is dense in $[0,1]$, but

$$\lim_{n\to\infty}\frac{S_n\left(\left[0,\frac{1}{2}\right]\right)}{n}=\frac{3}{4}$$

so it is not equidistributed.

E3) For an arbitrary sequence $\{x_k\}_{k\geqslant 1}\subset[0,1]$, let

$$D_n=\sup_{0\leqslant a<b\leqslant 1}\left|\frac{S_n([a,b])}{n}-(b-a)\right|,\ D_n^*=\sup_{0<b<1}\left|\frac{S_n([0,b])}{n}-b\right|.$$

Prove that $D_n^*\leqslant D_n\leqslant 2D_n^*$.
Proof: $D_n\geqslant D_n^*$ is trivial. Note that

$$\frac{S_n([a,b])}{n}-(b-a)=\frac{S_n([0,b])}{n}-b-\left(\frac{S_n([0,a])}{n}-a\right).$$

Hence $D_n \leqslant 2D_n^*$.

E4) Prove that the sequence $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$ iff $\lim_{n\to\infty} D_n^* = 0$.

Proof: By E3) $\lim_{n\to\infty} D_n^* = 0 \iff \lim_{n\to\infty} D_n = 0$. If $\lim_{n\to\infty} D_n = 0$, then for any $0 \leqslant a < b \leqslant 1$
,

$$\lim_{n\to\infty} \left| \frac{S_n([a,b])}{n} - (b-a) \right| = 0.$$

Hence $\{x_k\}_{k\geqslant 1}$ is equidistributed.
Suppose $\{x_k\}_{k\geqslant 1}$ is equidistributed, then for any $0 < b < 1$, $\lim_{n\to\infty} |S_n([0,b])/n - b| = 0$.

E5) Prove that the sequence $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$ iff for any $f \in R([0,1])$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,\mathrm{d}x. \tag{1}$$

Proof: See E6).

E6) The sequence $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$ iff for any $p \in \mathbb{Z}_{\geqslant 1}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i p x_k} = 0. \tag{2}$$

Proof:
i. Equidistribution ==> (1): $\{x_k\}_{k\geqslant 1}$ is equidistributed on $[0,1]$ implies that for any $\chi = \mathbf{1}_{[a,b]}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(x_k) = \int_0^1 \chi(x)\,\mathrm{d}x.$$

(which implies (1) ==> equidistribution)
Hence (1) holds for any $\varphi = \sum_{k=1}^{n-1} c_k \mathbf{1}_{[x_k, x_{k+1}]}$. For any Riemann integrable function f and any $\varepsilon > 0$, take $\varphi_1 \leqslant f \leqslant \varphi_2$ where φ_1, φ_2 are step functions such that

$$\int_0^1 \varphi_2(x)\,\mathrm{d}x - \varepsilon \leqslant \int_0^1 f(x)\,\mathrm{d}x \leqslant \int_0^1 \varphi_1(x)\,\mathrm{d}x + \varepsilon$$

There exists N such that for any $n \geqslant N$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} \varphi_1(x_k) - \int_0^1 \varphi_1(x)\,\mathrm{d}x \right|, \left| \frac{1}{n} \sum_{k=1}^{n} \varphi_2(x_k) - \int_0^1 \varphi_2(x)\,\mathrm{d}x \right| < \varepsilon.$$

Hence

$$\int_0^1 f(x)\,\mathrm{d}x \leqslant \int_0^1 \varphi_1(x)\,\mathrm{d}x + \varepsilon < \frac{1}{n} \sum_{k=1}^{n} \varphi_1(x_k) + 2\varepsilon \leqslant \frac{1}{n} \sum_{k=1}^{n} f(x_k) + 2\varepsilon.$$

Likewise

$$\int_0^1 f(x)\,\mathrm{d}x \geqslant \frac{1}{n} \sum_{k=1}^{n} f(x_k) - 2\varepsilon.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,\mathrm{d}x$$

for any $f \in R([0,1])$.
ii. (1) ==> (2): Since $e^{2\pi ipx} \in R([0,1])$ and

$$\int_0^1 e^{2\pi ipx}\,\mathrm{d}x = 0.$$

iii. (2) ==> equidistribution: From (2) we know that (1) holds for trigonometric polynomials. By Stone-Weierstrass theorem, continuous functions can by uniformly approximated by trigonometric polynomials, hence (1) holds for continuous functions. Likewise, step functions can be uniformly approximated by continuous functions, hence we obtain $\{x_k\}$ is equidistributed.

Ex: (Van Der Corput) Suppose $\{\xi_n\}_{n\geqslant 1}$ is a sequence on $[0,1]$. If for any $h \geqslant 1$, the sequence $\{\xi_{n+h}-\xi_n\}_{n\geqslant 1}$ is equidistributed, then $\{\xi_n\}_{n\geqslant 1}$ is equidistributed.

Proof: From E5) and E6) we know that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ip(\xi_{n+h}-\xi_n)} = 0,\ \forall p \in \mathbb{Z}-\{0\}, h \geqslant 1.$$

We only need to prove the following lemma:
Suppose $\{u_n\}_{n\geqslant 1} \subset \mathbb{C}$ is bounded, and for any $h \geqslant 1$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_{n+h}\bar{u}_n = 0,$$

then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_n = 0.$$

Suppose $M = \sup_{n\in\mathbb{N}}|u_n|$. Note that for any $N > D > 0$,

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_n - \frac{1}{D}\frac{1}{N}\sum_{h=1}^{D}\sum_{n=1}^{N} u_{n+h}\right| \leqslant \frac{1}{N}\sum_{k=1}^{D}\frac{D+1-k}{D}(|u_k|+|u_{N+k}|) \leqslant \frac{(D+1)M}{N}.$$

For a constant D, and any $\varepsilon > 0$, there exists N_0 such that for any $n > N_0$, and any $d_1 \neq d_2 \in [1,D]$, $\left|\sum_{n=1}^{N} u_{n+d_1}\bar{u}_{n+d_2}/N\right| < \varepsilon^2/2$, then

$$\begin{aligned}\left|\frac{1}{ND}\sum_{h=1}^{D}\sum_{n=1}^{N} u_{n+h}\right|^2 &\leqslant \frac{1}{N}\sum_{n=1}^{N}\left|\frac{1}{D}\sum_{h=1}^{D} u_{n+h}\right|^2 = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{D^2}\sum_{h,k} u_{n+h}\bar{u}_{n+k}\\ &= \frac{1}{ND^2}\left(\sum_{h=1}^{D}\sum_{n=1}^{N}|u_{n+h}|^2 + \frac{D(D-1)}{2}\varepsilon^2\right) \leqslant \frac{M^2}{D} + \frac{D-1}{2D}\varepsilon^2.\end{aligned}$$

Hence for any $\varepsilon > 0$ and $D \geqslant 1$, there exists N_0 such that for any $n > N_0$,

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_n\right| \leqslant \frac{(D+1)M}{N} + \sqrt{\frac{M^2}{D} + \frac{D-1}{2D}\varepsilon^2} \to 0.$$

Therefore

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_n = 0.$$

This implies that any polynomial $\sum_{i=0}^{n} c_i x^i$ with $c_n \in \mathbb{Q}^C$ is equidistributed.

Ex: (Fejér)

Suppose $g(t)$ ($t \geqslant 1$) satisfy: (a) $g \in C^1$; (b) g is monotonically increasing and $\lim_{t\to\infty} g(t) = +\infty$; (c) g' is monotonically decreasing and $\lim_{t\to\infty} g'(t) = 0$; (d) $\lim_{t\to\infty} tg'(t) = +\infty$. Prove that $\{\langle g(n)\rangle\}$ is equidistributed.
Proof: Consider

$$\begin{aligned}\left|\sum_{n=1}^{N-1} e^{2\pi i g(n)} - \int_1^N e^{2\pi i g(x)}\,\mathrm{d}x\right| &\leqslant \sum_{n=1}^{N-1}\int_n^{n+1} |\cos(2\pi g(n)) - \cos(2\pi g(x))| + |\sin(2\pi g(n)) - \sin(2\pi g(x))|\,\mathrm{d}x \\ &\leqslant \sum_{n=1}^{N-1}\int_n^{n+1} 4\pi|g'(\xi_x)|\,\mathrm{d}x \leqslant 4\pi\sum_{n=1}^{N-1}\sup_{\xi\in[n,n+1]}|g'(\xi)|\end{aligned}$$

and

$$\int_1^N e^{2\pi i g(x)}\,\mathrm{d}x = \int_{g(1)}^{g(N)} \frac{e^{2\pi i t}}{g'(g^{-1}(t))}\,\mathrm{d}t = \frac{1}{g'(N)}\int_\gamma^{g(N)} e^{2\pi i t}\,\mathrm{d}t$$

(using mean value theorem for integrals).
Hence

$$\lim_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N-1} e^{2\pi i g(n)} - \int_1^N e^{2\pi i g(x)}\,\mathrm{d}x\right| \leqslant \lim_{N\to\infty}\frac{4\pi}{N}\sum_{n=1}^{N-1}\sup_{\xi\in[n,n+1]}|g'(\xi)| = 0 \text{ (Cesaro sum)}$$

and

$$\frac{1}{N}\left|\int_1^N e^{2\pi i g(x)}\,\mathrm{d}x\right| \leqslant \frac{1}{2Ng'(N)} \to 0.$$

E7) Suppose $\theta > 0$, then the sequence $\{\langle n\theta\rangle\}_{n\geqslant 1}$ is equidistributed on $[0,1]$ iff θ is irrational.

Proof: Note that

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i p n\theta} = \frac{e^{2\pi i p\theta}}{N}\frac{1 - e^{2\pi i p N\theta}}{1 - e^{2\pi i p\theta}} \to 0.$$

Hence by Weyl's Theorem $\{\langle n\theta\rangle\}_{n\geqslant 1}$ is equidistributed.

E8) Prove that the sequence $\{\xi_n = \langle\sqrt{n}\rangle\}_{n\geqslant 1}$ is equidistributed on $[0,1]$.

Proof: See E8).

E9) For an arbitrary $a \neq 0$, $\sigma \in (0,1)$, prove that the sequence $\{\xi_n = \langle a n^\sigma\rangle\}_{n\geqslant 1}$ is equidistributed on $[0,1]$.

Proof: Use Fejér's theorem above.
Note: Using Van Der Corput, we can prove the statement for any $\sigma \in \mathbb{R}_{>0} - \mathbb{Z}$, by considering $\Delta^k(an^\sigma)$.

E10) Prove that for any $a \in \mathbb{R}$, the sequence $\{\xi_n = \langle a\log n\rangle\}_{n\geqslant 1}$ is not equidistributed on $[0,1]$.

Proof: Let $f(x) = a\log x$, then $\lim_{x\to\infty} f'(x) = 0$. Consider

$$\left|\sum_{n=1}^{N-1} e^{2\pi i f(n)} - \int_1^N e^{2\pi i f(x)}\,\mathrm{d}x\right| \leqslant 4\pi\sum_{n=1}^{N-1}\sup_{\xi\in[n,n+1]}|g'(\xi)| = o(N)$$

and

$$\frac{1}{N}\left|\int_1^N e^{2\pi i f(x)}\,\mathrm{d}x\right| = \frac{1}{N}\left|\int_0^{\log N} e^{(2\pi i a+1)t}\,\mathrm{d}t\right| = \left|\frac{e^{2\pi i\log N}-N^{-1}}{2\pi i a+1}\right| \to \frac{1}{|2\pi i a+1|} \neq 0.$$

Hence $\{\xi_n\}$ is not equidistributed on $[0,1]$.

F: Winding Number of Closed Curve

Let $E = \{f : \mathbb{R}\to\mathbb{C}^\times : f\in C^1, f(x+2\pi)=f(x)\}$, where $\mathbb{C}^\times = \mathbb{C}-\{0\}$. For $f\in E$, $f|_{[0,2\pi]}$ represents a closed curve on $\mathbb{C}$ that does not contain 0.
For any $f\in E$, define

$$d(f) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f'(t)}{f(t)}\,\mathrm{d}t.$$

F1) Prove that $d(f)$ is well-defined and calculate $d(f_n)$ where

$$f_n : \mathbb{R}\to\mathbb{C}^\times,\ x\mapsto f_n(x) = e^{inx}.$$

Proof: Since $f_n'(x) = ine^{inx}$, then

$$d(f_n) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f_n'(t)}{f_n(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\int_0^{2\pi} in\,\mathrm{d}t = n.$$

F2) Represent $d(f)$ in polar coordinates.

Suppose $f(t)=\rho(t)e^{i\theta(t)}$ where $\rho(t),\theta(t)\in\mathbb{R}$, then $f'(t)=\rho'(t)e^{i\theta(t)}+i\rho(t)\theta'(t)e^{i\theta(t)}$, hence

$$\begin{aligned} d(f) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{\rho'(t)}{\rho(t)}\,\mathrm{d}t + \frac{1}{2\pi i}\int_0^{2\pi}\theta'(t)\,\mathrm{d}t = \frac{\log\rho(2\pi)-\log\rho(0)}{2\pi i} + \frac{\theta(2\pi)-\theta(0)}{2\pi} \\ &= \frac{\theta(2\pi)-\theta(0)}{2\pi}. \end{aligned}$$

Which is the number of times f circles around 0, counterclockwise.

F3) Use the function

$$\psi(t) = \exp\int_0^t\frac{f'(s)}{f(s)}\,\mathrm{d}s$$

to show that $d(f)\in\mathbb{Z}$.
Proof: Note that

$$\psi'(t) = \psi(t)\cdot\frac{f'(t)}{f(t)}.$$

Hence let $g(t)=\psi(t)/f(t)$, then $g'(t) = (\psi'(t)f(t)-\psi(t)f'(t))/f^2(t) = 0$, so $\psi(t)=Cf(t)$ for all $t\in[0,2\pi]$. Therefore

$$1 = \psi(0) = Cf(0) = Cf(2\pi) = \psi(2\pi) = e^{2\pi i d(f)},$$

i.e. $d(f)\in\mathbb{Z}$.

F4) Prove that for any $f\in E$, there exists $\varepsilon>0$ such that for any $g\in E$, if $\|f-g\|_\infty<\varepsilon$, then $d(g)=d(f)$.

Proof: Suppose $f=\rho_1e^{i\theta_1}$ and $g=\rho_2e^{i\theta_2}$. Note that for any $z=\rho e^{i\theta}$ and $w=re^{it}$, $|z-w|\geqslant\sup_{r>0}|z-re^{it}| = \rho|\sin(\theta-t)|$. Hence $\|f-g\|_\infty<\varepsilon$ implies $\|\sin(\theta_1-\theta_2)\|_\infty<\delta=\varepsilon/\inf\rho_1$. Let $\theta(t)=\theta_1(t)-\theta_2(t)$, then when $\delta<1$, $\theta(t)$ lies in $\bigcup(2k\pi-\alpha,2k\pi+\alpha)$ for some $\alpha<\pi$. Since $\theta(t)$ is continuous, it must stay in the same interval, hence $|\theta(0)-\theta(2\pi)|<2\pi$, i.e. $|d(f)-d(g)|<1$ so $d(f)=d(g)$. Hence $\varepsilon=\inf\min\rho_1/2$ is sufficient.

F5) Try to define $d(f)$ for $f \in E$ using F4) and Weierstrass-Stone for trigonometric polynomials.

Solution: For $f_n(x) = e^{inx}$, let $d(f) = n$. For a trigonometric polynomial $P(x) = \sum_{k=-n}^{n} c_k f_k(x)$, let $Q(x) = \sum_{k=0}^{2n} c_{k-n} x^k$ and $d(P)$ be the number of roots of Q on the unit circle.
For an arbitrary $f \in E$, take a sequence of trigonometric polynomials $\{P_n\}_{n \geqslant 1}$ uniformly convergent to f. The sequence $\{P_n\}_{n \geqslant 1}$ is Cauchy, so by F4) $d(P_n)$ is eventually constant, and let $d(f) = \lim_{n \to \infty} d(P_n)$. From F4) this definition is the same as the original one.

F6) Prove invariance under homotopy: Suppose $F(t, \tau) : \mathbb{R} \times [0, 1] \to \mathbb{C}^\times$ is continuous and for any $\tau \in [0, 1]$, $F(\cdot, \tau) \in E$, then

$$d(F(x, 0)) = d(F(x, 1)).$$

Proof: Let $S = \{\tau \in [0, 1] : d(F(x, 0)) = d(F(x, \tau))\}$, and $s = \sup S$. ($0 \in S$ so S is non-empty). Using F4), there is an $\varepsilon > 0$ such that for any $g \in E$, if $\|F(\cdot, s) - g\|_\infty < \varepsilon$, then $d(F(\cdot, s)) = d(g)$. Since $F(t, \tau)$ is continuous, there exists $\delta > 0$ such that $r \in (s - \delta, s + \delta)$ implies $\|F(\cdot, s) - F(\cdot, r)\|_\infty < \varepsilon$ so $d(F(\cdot, s)) = d(F(\cdot, r))$. Hence we can obtain $s \in S$ and $s = 1$, i.e.

$$d(F(x, 0)) = d(F(x, 1)).$$

F7) Suppose $P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a complex polynomial, and $P(0) \neq 0$. Prove that there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the function $f_\varepsilon(x) = P(\varepsilon e^{ix}) \in E$ and calculate $d(f_\varepsilon)$.

Proof: Note that $\|f_\varepsilon - c_0\|_\infty \to 0$ so by F4) such ε_0 exists and $d(f_\varepsilon) = 0$.

F8) Following F7), prove that there exists $R_0 > 0$ such that for any $R \in (R_0, \infty)$, $f_R(x) = P(Re^{ix}) \in E$ and calculate $d(f_R)$.

Proof: Let $g_R(x) = f_R(x)/R^n$, then $d(f_R) = d(g_R)$ and $\|g_R - e^{nix}\|_\infty \to 0$, hence $d(f_R) = d(g_R) = n$.

F9) Prove that every complex polynomial of degree n has at least one root.

Proof: Let $F(x, t) = P(Rte^{ix})$ where $F : \mathbb{R} \times [0, 1] \to \mathbb{C}^\times$ and R as in F7). If P has no roots, then $F(\cdot, t) \in E$ for any $t \in [0, 1]$, but for some $\varepsilon > 0$, $d(F(\cdot, \varepsilon)) = 0$ while $d(F(\cdot, 1)) = n$, leading to contradiction with F6).

G: Bolzano Curve

Define the sequence of functions on $[0, 1]$ $\{f_n\}_{n \geqslant 0}$ inductively, where $f_0(x) = x$, and for $n \geqslant 0$, $0 \leqslant k \leqslant 3^m$,

$$\begin{cases} f_{n+1}(k3^{-n}) & = f_n(k3^{-n}), \\ f_{n+1}(k3^{-n} + 3^{-n-1}) & = f_n(k3^{-n} + 2 \cdot 3^{-n-1}), \\ f_{n+1}(k3^{-n} + 2 \cdot 3^{-n-1}) & = f_n(k3^{-n} + 3^{-n-1}). \end{cases}$$

and f_{n+1} is linear on every interval $[k3^{-n-1}, (k+1)3^{-n-1}]$.

G1) Prove that for any $n \geqslant 0$, and any $0 \leqslant x, y \leqslant 1$,

$$|f_n(x) - f_n(y)| \leqslant 2^n |x - y|.$$

Proof: We can assume that $x = k \cdot 3^{-n}$ and $y = (k+1) \cdot 3^{-n}$, then $|x - y| = 3^{-n}$. Suppose $f_n(x) = f_{n-1}(v)$ and $f_n(y) = f_{n-1}(u)$, we obtain $|f_n(x) - f_n(y)| \leqslant 2^{n-1}|u - v| \leqslant 2^n \cdot 3^{-n} = 2^n |x - y|$.

G2) Prove that for any $n \geqslant 0$ and any $x \in [0,1]$,

$$|f_{n+1}(x) - f_n(x)| \leqslant \frac{2^n}{3^{n+1}}.$$

Proof: The function $f_{n+1}(x) - f_n(x)$ is linear on every interval $[k3^{-n-1}, (k+1)3^{-n-1}]$, so we can assume that $x = k \cdot 3^{-n} + r \cdot 3^{-n-1}$ for some $k \leqslant 3^m$ and $r \in \{0,1,2\}$. By G1), in all three cases $|f_{n+1}(x) - f_n(x)| \leqslant 2^n \cdot 3^{-n-1}$.

G3) Prove that $\{f_n\}_{n \geqslant 1}$ converges uniformly to some $f \in C([0,1])$.

Proof: For any N, any $x \in [0,1]$ and $n > m > N$,

$$|f_n(x) - f_m(x)| \leqslant \frac{1}{3} \sum_{n=N}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2^{N-1}}{3^N} \to 0.$$

Hence $\{f_n\}_{n \geqslant 1}$ converges uniformly to f, and $f \in C([0,1])$ since f_n are all continuous.

G4) Prove that for any $n \geqslant 1$, and any $0 \leqslant k \leqslant 3^n$, $f(k3^{-n}) = f_n(k3^{-n})$.

Proof: Trivial since for any $m > n$, $f_m(k \cdot 3^{-n}) = f_{m-1}(k \cdot 3^{-n}) = f_n(k \cdot 3^{-n})$.

G5) Prove that for any $n \geqslant 1$ and any $0 \leqslant k \leqslant 3^n$, f is not differentiable at $k3^{-n}$.

Proof: As in G1), we can show by induction that for any $m > k$,
$|f(k \cdot 3^{-n}) - f(k \cdot 3^{-n} + 3^{-m})| = 2^m 3^{-m}$ so $f'(k \cdot 3^{-n})$ does not exist. (Same for $f'(1)$).

G6) Prove that f is nowhere differentiable on $[0,1]$.

Proof: If f is differentiable at x, then for any $\{h_n\}$ and $\{k_n\}$ such that $h_n, k_n > 0$ and $\lim_{n\to\infty} h_n = \lim_{n\to\infty} k_n = 0$,

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x-k_n)}{h_n + k_n} = f'(x).$$

Define h_n and k_n such that $x + h_n$ and $x - k_n$ are in the form $\{(k+1)\cdot 3^{-n}, k \cdot 3^{-n}\}$, then we infer f is not differentiable at x.

Problem W: e is Transcendental

Suppose $P(x)$ is a real polynomial of degree n. Let

$$I(t) = \int_0^t e^{t-x} P(x)\,\mathrm{d}x.$$

W1) Prove that $I(t) = e^t \sum_{i=0}^n P^{(i)}(0) - \sum_{i=0}^n P^{(i)}(t)$.

Proof: Note that $(e^{t-x})^{(n)} = (-1)^n e^{t-x}$, then using integration by part we obtain

$$I(t) = (-1)^n \sum_{i=0}^n (-1)^i (e^{t-x})^{(n-i+1)} P^{(i)}(x)\Big|_0^t = e^t \sum_{i=0}^n P^{(i)}(0) - \sum_{i=0}^n P^{(i)}(t).$$

W2) Suppose there exist integers $a_0, a_1, \cdots, a_n$, where $a_0 \neq 0$ such that

$$a_0 + a_1 e + a_2 e^2 + .. + a_n e^n = 0.$$

For $p \in \mathbb{Z}_{\geqslant 0}$, let

$$P(x) = x^{p-1}(x-1)^p (x-2)^p \cdots (x-n)^p$$

and define

$$J = a_0 I(0) + a_1 I(1) + \cdots + a_n I(n).$$

Prove that $J \in \mathbb{Z}$ and $(p-1)!|J$.
Proof: Denote $C = \sum_{i=0}^{\infty} P^{(i)}(0)$, then from W1),

$$J = \sum_{k=0}^{n} a_k I(k) = \sum_{k=0}^{n} a_k \left(e^k C - \sum_{i=0}^{\infty} P^{(i)}(k) \right) = -\sum_{k=0}^{n} \sum_{i=0}^{\infty} a_k P^{(i)}(k) \in \mathbb{Z},$$

Denote $F(k) = \sum_{i=0}^{\infty} P^{(i)}(k)/(p-1)!$, we show that $F(k) \in \mathbb{Z}$ for any $0 \leqslant k \leqslant n$.
When $k = 0$, let $u(x) = x^{p-1}/(p-1)!$ and $v(x) = (x-1)^p \cdots (x-n)^p$, then

$$P^{(m)}(0) = \sum_{j=0}^{m} u^{(j)}(0) v^{(m-j)}(0) \binom{m}{j}$$

so

$$F(0) = \sum_{j=p-1}^{\infty} v^{(j-p+1)}(0) \binom{j}{p-1} \in \mathbb{Z}.$$

If p is a prime greater then n, then $F(0) = v(0) + p(\cdots) \in \mathbb{Z} - p\mathbb{Z}$.
When $1 \leqslant k \leqslant n$, let $u(x) = (x-k)^p/(p-1)!$ and $v(x) = P(x)/(x-k)^p$, then

$$P^{(m)}(k) = \sum_{j=0}^{m} u^{(j)}(k) v^{(m-j)}(0) \binom{m}{j}$$

so

$$F(k) = \sum_{j=p}^{\infty} p v^{(j-p)}(0) \binom{j}{p} \in p\mathbb{Z}.$$

Therefore $(p-1)!|J$.

W3) Prove that if p is a prime greater than n then $J \neq 0$ hence $|J| \geqslant (p-1)!$.

Proof: From W2) we know that $F(0) \not\equiv 0 \pmod{p}$, and $p|F(k)$ for any $1 \leqslant k \leqslant n$. hence $J \not\equiv 0 \pmod{p}$ so $J \neq 0$.

W4) Prove that there exists $C > 0$ such that for any $p \in \mathbb{Z}_{\geqslant 1}$, $|J| \leqslant C^p$.

Proof: For any $0 \leqslant k \leqslant n$,

$$|I(k)| \leqslant \int_0^k |e^{k-x} P(x)| \, \mathrm{d}x \leqslant e^n \cdot n^{(n+1)p},$$

hence

$$|J| \leqslant e^n \cdot n^{(n+1)p} \cdot \sum_{j=0}^{n} |a_n| < C^p.$$

where $C = n^{n+1} \cdot e \cdot (1 + \sum_{j=0}^{n} |a_n|)$.

W5) Prove that e is transcendental.

Proof: Otherwise by W3) and W4), there exists $C > 0$ such that for any prime $p > n$, $C^p \leqslant (p-1)!$ which contradicts with the infinity of primes.

Ex: π is also transcendental

Consider the identity $e^{i\pi}+1=0$. Suppose πi is algebraic (with degree n), then

$$0=\prod_{i=1}^{n}(1+e^{\gamma_i})=\sum_{\varepsilon_i\in\{0,1\}} e^{\sum_i \varepsilon_i\gamma_i}=a+\sum_{i=1}^{m} e^{\alpha_i}.$$

where γ_i are the conjugates of πi, $a=2^n-m\geqslant 1$ are the number of zero exponents in the first sum, and α_i are all the non-zero exponents. Note that

$$\phi(x)=\prod_{\varepsilon\in\{0,1\}}\left(x-\sum_{i=1}^{n}\varepsilon_i\gamma_i\right)\in\mathbb{Q}[x].$$

Let

$$\psi(x)=C\frac{\phi(x)}{x^a}=\sum_{i=0}^{m} b_i x^i\in\mathbb{Z}[x], b_m>0, b_0\neq 0,$$

whose roots are exactly α_i. Furthermore, $b_m\alpha_i$ are all algebraic integers.
Apply the identity W1) to the polynomial

$$f(x)=\frac{b_m^{(m-1)p}}{(p-1)!}x^{p-1}\psi^p(x)=\frac{b_m^{mp}}{(p-1)!}x^{p-1}\prod_{i=1}^{m}(x-\alpha_i)^p.$$

Plug in $x=\alpha_i$ and sum over i, we obtain

$$-aF(0)-\sum_{i=1}^{m}F(\alpha_i)=\sum_{i=1}^{m}e^{\alpha_i}\int_0^{\alpha_i}f(t)e^{-t}\,\mathrm{d}t. \tag{1}$$

Note that

$$F(0)=(-1)^{mp}b_m^{mp}\left(\prod_i\alpha_i\right)^p\in\mathbb{Z}-p\mathbb{Z}.$$

Also

$$\sum_{i=1}^{m}F(\alpha_i)=pb_m^{mp}\sum_i\alpha_i^{p-1}\prod_{j\neq i}(\alpha_i-\alpha_j)^p\in p\mathbb{Z}$$

for large p since it is symmetric in α_i and the denominator is cleared by b_m^{mp}. Therefore the LHS of (1) is a non-zero integer.
We estimate the integral:

$$\left|e^{\alpha_i}\int_0^{\alpha_i}f(t)e^{-t}\,\mathrm{d}t\right|\leqslant(|\alpha_i b_m^{m-1}\psi(\alpha_i)|)^p/(p-1)!\to 0$$

as $p\to\infty$, reaching contradiction.

Further Theorems:

1. Hilbert's 7th question: For any algebraic number $a\notin\{0,1\}$, and irrational number b, a^b is transcendental.
2. Hermite-Lindemann: e^α is transcendental for any $\alpha\in\bar{\mathbb{Q}}-\{0\}$.