2025/10/16

21页4; 26-27页1, 6-10.

21-4

For

$$A=\begin{pmatrix}1&-1&1\\2&0&1\\3&0&1\end{pmatrix}$$

find elementary matrices $E_1,\cdots,E_k$ such that

$$E_k\cdots E_2E_1A=I.$$

Solution: Let $E_1=\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}$, $E_2=\begin{pmatrix}1&&\\&1&-2\\&&1\end{pmatrix}$, $E_3=\begin{pmatrix}1&&-1\\&1&\\&&1\end{pmatrix}$, $E_4=\begin{pmatrix}1&-1&\\&1&\\&&1\end{pmatrix}$,
$E_5=\begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix}$, $E_6=\begin{pmatrix}&&1\\&1&\\1&&\end{pmatrix}$, $E_7=\begin{pmatrix}1&&\\&&1\\&1&\end{pmatrix}$.

26-1

Let

$$A=\begin{pmatrix}1&2&1&0\\-1&0&3&5\\1&-2&1&1\end{pmatrix}$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R=PA$.

Solution:

$$R=\begin{pmatrix}1&0&0&-\frac{1}{8}\\0&1&0&-\frac{1}{4}\\0&0&1&\frac{11}{8}\end{pmatrix}$$

$$P=\begin{pmatrix}\frac{3}{8}&-\frac{1}{4}&\frac{3}{8}\\\frac{1}{4}&0&-\frac{1}{4}\\\frac{1}{8}&\frac{1}{4}&\frac{1}{8}\end{pmatrix}$$

27-6

Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C=AB$ is not invertible.

Proof: (Clearly $\mathrm{rank}C\leqslant\min\{\mathrm{rank}A,\mathrm{rank}B\}=1$ so it is not invertible). Suppose $A=(a,b)^T$ and $B=(c,d)$, then $AB=\begin{pmatrix}ac&ad\\bc&bd\end{pmatrix}$ and clearly the two rows are linearly dependent, hence C is not invertible.

27-7

Let A be an $n \times n$ matrix. Prove the following two statements:
(a) If A is invertible for some $B \in F^{n\times n}$ and $AB = 0$, then $B = 0$.
(b) If A is not invertible, then there exists $B \in F^{n\times n}$ such that $AB = 0$ but $B \neq 0$.
Proof: (a) $0 = A^{-1}(AB) = (A^{-1}A)B = B$.
(b) If A is not invertible, then take a base $v_1, \cdots, v_k$ of $\mathrm{Ker}A$ and add some vectors to form a base $v_1, \cdots, v_n$ of $F^{n\times 1}$. Let B be the linear transformation that maps $\sum_{i=1}^{n} c_i v_i$ to $\sum_{i=1}^{k} c_i v_i$, then clearly B is linear and $\mathrm{Im}B = \mathrm{Ker}A$ so $AB = 0$ while $B \neq 0$.

27-8

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove using elementary row operations, that A is invertible iff $(ad - bc) \neq 0$.
Proof: A is invertible iff A can be transformed into I using elementary row operations. If $ad - bc = 0$ then the two rows of A are linearly dependent, so A is not invertible. If $ad - bc \neq 0$, then suppose $a \neq 0$ (otherwise change the two rows)

$$A \to \begin{pmatrix} a & b \\ ac & ad \end{pmatrix} \to \begin{pmatrix} a & b \\ 0 & ad - bc \end{pmatrix} \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \to I.$$

27-9

An $n \times n$ matrix A is called upper-triangular if $A_{ij} = 0$ for $i > j$. Prove that an upper-triangular matrix is invertible iff every entry on its main diagonal is different from 0.
Proof: A is invertible iff the columns of A are linearly independent. If the j-th element of the diagonal is 0, let j be minimal then the $j-$th column is spanned by the first $j - 1$ columns, hence A is not invertible. If no element of the diagonal is 0, then clearly all columns are linearly independent, hence A is invertible.

27-10

Prove that if A is an $m \times n$ matrix, B is an $n \times m$ matrix and $n < m$, then AB is not invertible.
Proof: If AB is invertible, then for the base $e_1, \cdots, e_m$ of $F^{m\times 1}$, $AB(e_i)$ are linearly independent, but $B(e_i)$ are m vectors in $F^{n\times 1}$ so they must be linearly dependent, leading to contradiction.