

211-1

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is non-negative. If $f \in L([a, b])$, must f have a primitive on $[a, b]$?

Solution: Not necessarily, since by Darboux's theorem, functions like $f = \chi_{[0,1]} : [-1, 1] \rightarrow \mathbb{R}$ does not have a primitive.

211-3

Prove that the result of Vitali covering theorem can be changed to: there exists countable $\{I_j\}$ such that $m^*(E \setminus \bigcup_{j \geq 1} I_j) = 0$.

Proof: Just take all the intervals I_n in the original proof:

WLOG we assume I_α are closed intervals. $m^*(E) < \infty$, then we can find G open, $E \subset G$ & $m(G) < \infty$. We can assume $I_\alpha \subset G \forall \alpha \in J$ (throw away the others). Let $\delta_1 = \sup\{|I| : I \in \Gamma\}$, we take $I_1 \in \Gamma$ such that $|I_1| \geq \delta_1/2$. Likewise consider $\delta_n = \sup\{|I| : I \in \Gamma, I \cap I_j = \emptyset\}$, and $I_n \cap I_j = \emptyset$ and $|I_n| \geq \delta_n/2$.

Clearly $\bigcup I_j \subset G$, so $m(G) \geq m(\bigcup I_j) = \sum |I_j|$, hence $\delta_n \rightarrow 0$. For any $\varepsilon > 0$, take N such that $\sum_{n \geq N} |I_n| < \varepsilon/5$. Let $S_N = E \setminus \bigcup_{j=1}^N I_j$, then for any $x \in S_N$, find I such that $x \in I$, and $I \cap \bigcup I_j \neq \emptyset$. We claim $I \cap I_j \neq \emptyset$ for j large enough, otherwise $\delta_n \geq |I|$ implies $|I| = 0$.

Consider I_{n_0} to be the first to intersect with I , then $I \cap I_j = \emptyset \forall x < n_0$ and $I \cap I_{n_0} \neq \emptyset$. Then by definition, $\delta_{n_0} \geq |I|$ and $|I_{n_0}| \geq \frac{1}{2}\delta_{n_0}$, so $|I| \leq 2|I_{n_0}|$. We can prove $I \subset 5I_{n_0}$ (keep center and enlarge radius).

Hence $S_N \subset \bigcup_{j \geq N+1} 5I_j$. So $m^*(S_N) \leq \sum_{j \geq N+1} m^*(5I_j) \leq \varepsilon$. $E \setminus \bigcup_{j \geq 1} I_j \subset S_N$ so it is a null set.

241-2

Suppose $\{x_n\} \subset [a, b]$, give an increasing function on $[a, b]$, such that its discontinuities are exactly $\{x_n\}$.

Proof: Consider $f(x) = \sum_{n=1}^{\infty} 2^{-n} \chi_{[0, \infty)}(x - x_n)$.

241-3

Suppose $f : (a, b) \rightarrow \mathbb{R}$ is increasing, $E \subset (a, b)$. If for any $\varepsilon > 0$, there exists $(a_i, b_i) \subset (a, b)$ such that

$$E \subset \bigcup_i (a_i, b_i), \sum_i f(b_i) - f(a_i) < \varepsilon.$$

Prove that $f'(x) = 0$, a. e. $x \in E$.

Proof: By Lebesgue monotone differentiation theorem, f' exists and is finite a.e. Clearly

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \geq 0$. We show that for any $k \geq 1$, $A_k = \{x \in E : f'(x) > k^{-1}\}$ is null.

Otherwise if $m(A_k) > 0$, then for any cover $E \subset \bigcup_i (a_i, b_i)$,

$$\sum_i f(b_i) - f(a_i) \geq \sum_i \int_{(a_i, b_i)} f' dm \geq \int_E f' dm \geq \int_{A_k} f' dm > \frac{m(A_k)}{k}.$$

Leading to contradiction. Hence $m(A_k) = 0$ so $f' = 0$, a. e. $x \in E$.

Extra 1

If $F \in C([a, b])$, prove that the Dini derivatives are measurable.

Proof: Consider $D^+ F(x) = \limsup_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h} = \inf_{h > 0} \sup_{k \in (0, h)} \frac{F(x+k) - F(x)}{k}$.

Since $F \in C([a, b])$, the \inf and \sup can be taken on rationals, so $D^+ F$ is measurable.

Extra 2

If $F \in C([a, b])$, and $D^+F(x) \geq 0, \forall x \in [a, b]$, prove that F is monotonically increasing.

Proof: For any $\varepsilon > 0$, consider $g_\varepsilon(x) = F(x) + \varepsilon x$, then $D^+g_\varepsilon(x) \geq \varepsilon$. For any $x < y$, consider $A = \{t \in [x, y] : g_\varepsilon(t) \geq g_\varepsilon(x)\}$, then $x \in A$ so $A \neq \emptyset$. Take $t = \sup A$, then $t \in A$ since $g_\varepsilon \in C([a, b])$. If $t < y$, then for any $h \in (0, y - t)$, $g_\varepsilon(t + h) < g_\varepsilon(x) \leq g_\varepsilon(t)$, so $D^+g_\varepsilon(t) \leq 0$, leading to contradiction. Hence $t = y$ and $g_\varepsilon(y) \geq g_\varepsilon(x)$ for any $x < y$. Let $\varepsilon \rightarrow 0$ we obtain $F(x) \leq F(y)$ for any $x \leq y$.

Extra 3

If $f \in C([a, b])$, prove that

$$\sup_{x_1, x_2 \in [a, b]} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \sup_{x \in [a, b]} D^+f(x) = \sup_{x \in [a, b]} D^-f(x) = \sup_{x \in [a, b]} D_-f(x) = \sup_{x \in [a, b]} D_+f(x).$$

(The statement holds when they are ∞)

Proof: Note that for $t = \sup_{x_1, x_2 \in [a, b]} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$,

$$\sup_{x \in [a, b]} D^+f(x) = \sup_x \inf_{r > 0} \sup_{h < r} \frac{f(x + h) - f(x)}{h} \leq \sup_x \sup_{h > 0} \frac{f(x + h) - f(x)}{h} = t.$$

For any $\varepsilon > 0$, take $x_1 < x_2$ such that $f(x_2) - f(x_1) > (t - \varepsilon)(x_2 - x_1)$, so $g(x) = f(x) - (t - \varepsilon)x$ satisfy $g(x_2) > g(x_1)$. By the previous problem, there exists $x \in [x_1, x_2]$ such that $D^+g(x) > 0$, so $D^+f(x) > t - \varepsilon$. Let $\varepsilon \rightarrow 0$ we obtain $\sup_{x \in [a, b]} D^+f(x) \geq t$, so $\sup_{x \in [a, b]} D^+f(x) = t$.

By considering $f'(x) = f(a + b - x)$ and $-f$, we obtain the results for D^- , D_- , D_+ .

Extra 4

As a corollary, consider the following result:

$F \in C([a, b])$. If one of the four Dini derivatives is continuous at $x_0 \in (a, b)$, prove that the other three are also continuous at x_0 , and the four derivatives are equal. (Hence f is differentiable at x_0)

Proof: For any $\varepsilon > 0$, there is a neighborhood $U(x_0)$ such that $D^+f(x_0) - \varepsilon \leq D^+f(x) \leq D^+f(x_0) + \varepsilon$, then $\sup_{x \in U(x_0)} D^+f(x) \leq D^+f(x_0) + \varepsilon$. By the previous problem, for any Dini derivative D^* ,

$\sup_{x \in U(x_0)} D^*f(x) \leq D^+f(x_0) + \varepsilon$, and likewise $\inf_{x \in U(x_0)} D^*f(x) \geq D^+f(x_0) - \varepsilon$, so D^*f is continuous at x_0 . Hence $|D^*f(x_0) - D^+f(x_0)| \leq \varepsilon$. Let $\varepsilon \rightarrow 0$ we obtain $D^*f(x_0) = D^+f(x_0)$, so all four Dini derivatives are equal.