周民强《实变函数论（第三版》P235-236 思考题：3, 4; P241-244 习题5: 12,13,14,17,18,20

235-3

Suppose $g \in \mathcal{L}(\mathbb{R})$, then there exists C such that for $f \in C_c^2(\mathbb{R})$,

$$\left|\int_{\mathbb{R}} g f^2 \,\mathrm{d}m\right| \leqslant C \int_{\mathbb{R}} (f^2 + (f')^2)\,\mathrm{d}m.$$

Proof: Note that $\|gf^2\|_{L^1} \leqslant \|g\|_{L^1} \cdot \|f^2\|_{L^\infty}$, and for any $a < b$, $f \in C_c^2(\mathbb{R})$ implies $f^2 \in \mathrm{AC}([a, b])$ so

$$|f^2(a) - f^2(b)| = \left|\int_a^b 2ff' \,\mathrm{d}m\right| \leqslant \int_a^b |2ff'|\,\mathrm{d}m \leqslant \int_a^b f^2 + (f')^2 \,\mathrm{d}m.$$

Suppose $\operatorname{supp} f \subset [-N+1, N-1]$, then

$$\sup_{x \in [-N,N]} f^2(x) = \sup_{x,y \in [-N,N]} |f^2(x) - f^2(y)| \leqslant \int_{\mathbb{R}} f^2 + (f')^2 \,\mathrm{d}m.$$

Therefore $\left|\int_{\mathbb{R}} g f^2 \,\mathrm{d}m\right| \leqslant C \int_{\mathbb{R}} (f^2 + (f')^2)\,\mathrm{d}m$ where $C = \int_{\mathbb{R}} |g|\,\mathrm{d}m$.

236-4

Suppose $f \in \mathcal{L}([a, b])$, $F(x) = \int_a^x f(t)(x-t)^n \,\mathrm{d}t, \forall x \in [a, b]$, then F is n times differentiable and

$$F^{(n)} \in \mathrm{AC}([a, b]),\ F^{(n+1)}(x) = n! f(x), a.e.\, x \in [a, b].$$

Proof: Let $G_k(x) = \int_a^x f(t)(x-t)^k \,\mathrm{d}t$, then $G_k = f * x^k$ so $G_k' = kG_{k-1}$. $F = G_n$ so $F^{(n)} = n! \int_a^x f \,\mathrm{d}m \in \mathrm{AC}([a, b])$, and $F^{(n+1)}(x) = n! f(x), a.e.\, x \in [a, b]$.

243-12

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, increasing and differentiable on $\mathbb{R}$. Denote $A = \lim_{x \to -\infty} f(x) = A$, $B = \lim_{x \to \infty} f(x)$. Prove that

$$\int_{\mathbb{R}} f' \,\mathrm{d}m = B - A.$$

Proof: f is bounded and increasing so $f \in \mathrm{BV}(a, b)$. f differentiable every where and $f' \geqslant 0$ implies $f \in \mathrm{AC}([-N, N])$ for any $N > 0$ (using Banach-Zaretosky and Growth Lemma), hence $\int_{[-N,N]} f' \,\mathrm{d}m = f(N) - f(-N)$. Since f is increasing, $f' \geqslant 0$ hence

$$\int_{\mathbb{R}} f' \,\mathrm{d}m = \lim_{N \to \infty} \int_{[-N,N]} f' \,\mathrm{d}m = \lim_{N \to \infty} f(N) - f(-N) = B - A.$$

243-13

Suppose f is differentiable on $\mathbb{R}$, and $f, f' \in \mathcal{L}(\mathbb{R})$. Prove that

$$\int_{\mathbb{R}} f' \,\mathrm{d}m = 0.$$

Proof: By Banach-Zaretosky theorem and Growth Lemma we obtain $f \in \mathrm{AC}([-N, N])$ for every $N > 0$, so $\int_{[-N,N]} f' \,\mathrm{d}m = f(N) - f(-N)$. $f \in \mathcal{L}(\mathbb{R})$ implies $\liminf_{|x| \to \infty} |f(x)| = 0$, so $\liminf_{N \to \infty} \left|\int_{[-N,N]} f' \,\mathrm{d}m\right| \leqslant \liminf_{N \to \infty} |f(N)| + |f(-N)| = 0$.

243-14

Suppose $f:[a,b]\times[c,d]\to\mathbb{R}$, and exists $y_0\in(c,d)$ such that $f(\cdot,y_0)\in\mathcal{L}([a,b])$, and for every $x\in[a,b]$, $f(x,\cdot)$ is absolutely continuous on $[c,d]$. $\frac{\partial}{\partial y}f(x,y)\in\mathcal{L}([a,b]\times[c,d])$, prove that $F(y)=\int_a^b f(x,y)\,\mathrm{d}x$ is absolutely continuous on $[c,d]$, and $F'(y)=\int_a^b\frac{\partial}{\partial y}f(x,y)\,\mathrm{d}x, a.e.\, y\in[c,d]$.
Proof: For any $x\in[a,b]$, $f(x,y)=f(x,y_0)+\int_{y_0}^y\frac{\partial}{\partial y}f(x,t)\,\mathrm{d}t$. By Fubini's theorem,

$$F(y)=\int_a^b f(x,y)\,\mathrm{d}x=\int_a^b f(x,y_0)+\int_{y_0}^y\frac{\partial}{\partial y}f(x,t)\,\mathrm{d}t\,\mathrm{d}x=F(y_0)+\int_{y_0}^y\left(\int_a^b\frac{\partial}{\partial y}f(x,t)\,\mathrm{d}x\right)\mathrm{d}t.$$

Where $g(t)=\int_a^b\frac{\partial}{\partial y}f(x,t)\,\mathrm{d}x$ is integrable. Hence $F\in\mathrm{AC}([c,d])$ and $F'=g, a.e.\, y\in[c,d]$.

243-17

Suppose $\{g_k(x)\}$ are absolutely continuous functions on $[a,b]$, and $|g_k'(x)|\leqslant F(x), a.e.$ where $F\in\mathcal{L}([a,b])$. If $\lim_{k\to\infty}g_k(x)=g(x)\forall x\in[a,b]$ and $\lim_{k\to\infty}g_k'(x)=f(x), a.e.\, x\in[a,b]$, prove that $g'(x)=f(x), a.e.\, x\in[a,b]$.
Proof: Since $g_k\in\mathrm{AC}([a,b])$,

$$g_k(x)=g_k(a)+\int_a^x g_k'\,\mathrm{d}m, \forall x\in[a,b].$$

By Dominated Convergence Theorem, $\lim_{k\to\infty}\int_a^x g_k'\,\mathrm{d}m=\int_a^x f\,\mathrm{d}m\forall x\in[a,b]$, hence

$$g(x)=\lim_{k\to\infty}g_k(x)=g(a)+\int_a^x f\,\mathrm{d}m, \forall x\in[a,b].$$

Therefore $g\in\mathrm{AC}([a,b])$, so g' exists a.e., $g'\in\mathcal{L}([a,b])$ and $\int_a^x g'\,\mathrm{d}m=\int_a^x f\,\mathrm{d}m\forall x\in[a,b]$, which implies $g'=f, a.e.\, x\in[a,b]$.

243-18

Suppose f is absolutely continuous and strictly increasing on $[a,b]$, g is absolutely continuous on $[f(a),f(b)]$, prove that $g\circ f$ is absolutely continuous on $[a,b]$.
Proof: For any $\varepsilon>0$, take $\delta>0$ such that $\sum_{k=1}^N|y_k-x_k|<\delta\implies\sum_{k=1}^N|g(y_k)-g(x_k)|<\varepsilon$, and take $\iota>0$ such that $\sum_{k=1}^N|y_k-x_k|<\iota\implies\sum_{k=1}^N|f(y_k)-f(x_k)|<\delta$. Then For any $\sum_{k=1}^N|y_k-x_k|<\iota$ where (x_k,y_k) are disjoint, f is strictly increasing so $(f(x_k),f(y_k))$ are disjoint and $\sum_{k=1}^N|f(y_k)-f(x_k)|<\delta$. Hence $\sum_{k=1}^N|g(f(x_k))-g(f(y_k))|<\varepsilon$, so $g\circ f\in\mathrm{AC}([a,b])$.

244-20

Suppose f is differentiable on $[a,b]$. If $f'=0, a.e.\, x\in[a,b]$, prove that f is constant on $[a,b]$.
Proof: By Growth Lemma $m(f([a,b]))=0$. Since $f\in C([a,b])$, $f([a,b])$ is a compact, connected set, hence f is constant on $[a,b]$.