# PSA

### A1) Given $f:(a,x_0)\cup(x_0,b)\to\mathbb{R}$, then $\lim_{x\to x_0} f(x)$ exists iff for any $\varepsilon>0$, there exists $\delta>0$ such that for any $x_1,x_2\in(x_0-\delta,x_0+\delta)$, $|f(x_1)-f(x_2)|<\varepsilon$.

Proof: <== Let $x_n=x_0+1/n$, then $\{f(x_n)\}$ form a Cauchy sequence, hence $f(x_0)=\lim_{n\to\infty} f(x_n)$ exists. For any $\varepsilon>0$, there exists $N,\delta>0$ such that for any $x,y\in(x_0-\delta,x_0+\delta)$, $|f(x)-f(y)|<\varepsilon$ and for any $n>N$, $|f(x_n)-f(x_0)|<\varepsilon$, hence let $\delta'=\min\{\delta,1/N\}$, then for any $x\in(x_0-\delta',x_0+\delta')$,
$|f(x)-f(x_0)|\leqslant|f(x)-f(x_N)|+|f(x_N)-f(x_0)|<2\varepsilon$.
Hence $\lim_{x\to x_0} f(x)=f(x_0)$ exists.
==> For any $\varepsilon>0$ there exists $\delta>0$ such that for any $x\in(x_0-\delta,x_0+\delta)$, $|f(x)-f(x_0)|<\varepsilon$, hence for any $x,y\in(x_0-\delta,x_0+\delta)$, $|f(x)-f(y)|<2\varepsilon$.

### A2) Suppose $I$ is an interval (not a point), prove that the linear space $C(I)$ on $\mathbb{R}$ is of infinite dimension.

Proof: $C(I)$ contains the subspace of all polynomials, hence is of infinite dimension.

### A3) Suppose $f:X\to Y$ and $g:Y\to Z$ are both continuous, prove that $g\circ f:X\to Z$ is also continuous.

Proof: For any open set $U\in Z$, $g^{-1}(U)\subset Y$ is an open set, and $f^{-1}(g^{-1}(U))\subset X$ is an open set, hence $(g\circ f)^{-1}(U)$ is an open set in $X$ and therefore $g\circ f$ is continuous on $X$.

### A4) Suppose $(X,d_X)$ and $(Y,d_Y)$ are metric spaces, $f:X\to Y$ is continuous. If $d'_X$ and $d_X$ are equivalent metrics, and so are $d'_Y$ and $d_Y$, then in the spaces $(X,d'_X)$ and $(Y,d'_Y)$, $f$ is also continuous.

Proof: The topology generated by equivalent metrics are the same.

### A5) The mapping $f:X\to\mathbb{R}^n$ can be written in the form

$$f:X\to\mathbb{R}^n,\ x\mapsto f(x)=(f_1(x),f_2(x),\cdots,f_n(x)).$$

Prove that $f$ is continuous iff $f_i$ is continuous for every $i=1,2,\cdots,n$.
Proof: Since $f$ is continuous iff $\forall x_n\to x$, $f(x_n)\to f(x)$, and $\{x_k=(x_k^{(1)},\cdots,x_k^{(n)})\}_{k\geqslant1}$ converges iff every $\{x_k^{(i)}\}_{k\geqslant1}$ converges, $f$ is continuous iff every $f_i$ is continuous.

### A6) Suppose $(X,d_X)$ is a metric space, $(V,\|\cdot\|)$ is a normed linear space. $f:X\to V$ and $g:X\to V$ are continuous mappings. Prove that $f\pm g:X\to V$ is continuous. If $V=\mathbb{C}$ then $f\cdot g:X\to\mathbb{C}$ is continuous. If $V=\mathbb{C}$ and for any $x\in X$, $g(x)\neq0$, then $f/g:X\to\mathbb{C}$ is continuous.

(Choose one statement to prove.)
Proof: Since for $\{x_n\},\{y_n\}\subset\mathbb{C}$, $\lim_{n\to\infty}x_ny_n=\lim_{n\to\infty}x_n\cdot\lim_{n\to\infty}y_n$ and if $y_n\neq0$, then

$$\lim_{n\to\infty}x_n/y_n=\lim_{n\to\infty}x_n/\lim_{n\to\infty}y_n.$$

Hence $f \cdot g$, $f/g$ are both continuous.
For $\{x_n\}, \{y_n\} \subset V$, if $A = \lim_{n\to\infty} x_n$ and $B = \lim_{n\to\infty} y_n$ then

$$\|x_n + y_n - A - B\| \leqslant \|x_n - A\| + \|y_n - B\| \to 0.$$

Hence $f \pm g$ is continuous.

### A7) Find all discontinuities of the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 1/q, & \text{if } x = p/q \in \mathbb{Q}, \text{where } q \geqslant 1, (p,q) = 1. \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Solution: For any $x \in \mathbb{Q}$, $f(x) \neq 0$ but for any $\delta > 0$ there exists $y \in (x - \delta, x + \delta)$ such that $y \notin \mathbb{Q}$. Hence $|f(x) - f(y)| = f(x)$, so $f$ is not continuous at $x$.
For any $x \notin \mathbb{Q}$, and any $\varepsilon > 0$, let $N = \lfloor 1/\varepsilon \rfloor + 1$ and $\delta = \inf_{n \leqslant N} \|xn\|/n$, then for any $y \in (x - \delta, y + \delta)$, if $y \notin \mathbb{Q}$ then $f(x) = f(y) = 0$, if $y = p/q \in \mathbb{Q}$ then $q > N > 1/\varepsilon$, hence $|f(x) - f(y)| = f(y) = 1/q < \varepsilon$. Therefore $f$ is continuous at $x$ iff $x \notin \mathbb{Q}$.

### A8) Calculate

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1.$$

### A9) Calculate

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{n\to\infty}(1 + 1/n)^n = e$ and $(1 + 1/x)^x$ is monotonic on $[100, \infty)$.

### A10) Calculate

$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{x\to\infty}(1 - 1/x)^x = \lim_{x\to\infty}(1 - 1/x)^{x-1} = e$.

# PSB

### B1) Calculate the following series:

1.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1.$$

2.

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1} = \frac{1}{2}.$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{1}{4}.$$

4.

$$\sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1} = \sum_{n=1}^{\infty} \arctan \frac{1}{n} - \arctan \frac{1}{n+1} = \frac{\pi}{4}.$$

5.

$$\sum_{n=0}^{\infty} \frac{(-1)^n + 2}{3^n} = \frac{1}{1+1/3} + \frac{2}{1-1/3} = \frac{3}{4} + 3 = \frac{15}{4}.$$

6.

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}.$$

7.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \frac{1}{1+1/2} = \frac{2}{3}.$$

8.

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = 3.$$

9.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1.$$

10.

$$\sum_{n=1}^{\infty} \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = 1 - \sqrt{2}.$$

11.

$$\sum_{n=1}^{\infty} \log\left(\frac{n(2n+1)}{(n+1)(2n-1)}\right) = \lim_{n\to\infty} \log\left(\frac{2n+1}{n+1}\right) = \log 2.$$

12.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+m)} = \frac{1}{m}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+m} = \frac{1}{m}\sum_{n=1}^{m} \frac{1}{n}.$$

## B2) Determine whether the following series converge:

1.

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \sqrt{n+1} - 1 = \infty.$$

2.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

converges.

3.

$$\sum_{n=2}^{\infty}(\sqrt[n]{n}-1)^n$$

converges, since $\limsup_{n\to\infty}\sqrt[n]{(\sqrt[n]{n}-1)^n}=0<1$.

4.

$$\sum_{n=1}^{\infty}\frac{1}{1+x^n}$$

converges if $|x|>1$ and diverges if $|x|\leqslant 1$.

5.

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}\leqslant\sum_{n=1}^{\infty}\frac{1}{2^n}=1.$$

converges

6.

$$\sum_{n=1}^{\infty}\left(\frac{n^2}{3n^2+1}\right)^n\leqslant\sum_{n=1}^{\infty}\frac{1}{3^n}<1.$$

converges.

7.

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}\geqslant\sum_{n=1}^{\infty}\frac{1}{2n}$$

diverges.

8.

$$\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}=\sum_{n=2}^{\infty}\frac{1}{n^{\log\log n}}\leqslant C+\sum_{n=100}^{\infty}\frac{1}{n^2}$$

converges.

9.

$$\sum_{n=1}^{\infty}\frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n}$$

diverges, since

$$\lim_{n\to\infty}\frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n}=\exp\lim_{n\to\infty}\left(\frac{\log n}{n}-n\log\left(1+\frac{1}{n^2}\right)\right)=1.$$

10.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt{n}}{n+1}$$

converges (conditionally), since the partial sum of $(-1)^{n-1}$ is bounded and $\frac{\sqrt{n}}{n+1}$ monotonically tends to $0$.

11.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[n]{n}}$$

diverges since $(-1)^{n-1}n^{-1/n}$ does not tend to $0$.

12.

Let $H_n = 1 + 1/2 + \cdots + 1/n$.

$$\sum_{n=1}^{\infty} \frac{H_n \sin nx}{n}$$

converges since the partial sum of $\sin nx$ is bounded and $\frac{H_n}{n}$ monotonically tends to $0$.

### B3) Determine whether the following series converge (absolutely):

1.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$

converges since the partial sum of $(-1)^n$ is bounded and $\frac{1}{n \log n}$ monotonically tends to $0$, but only conditionally by C3).

2.

$$\sum_{n=2}^{\infty} \frac{\sin(n\pi/4)}{\log n}$$

converges since the partial sum of $\sin(n\pi/4)$ is bounded and $\frac{1}{\log n}$ monotonically tends to $0$, but only conditionally since $\sum_{n=2}^{\infty} \frac{1}{\log(4n+2)}$ tends to infinity.

3.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{\sqrt[3]{n}}$$

converges since $\frac{n-1}{(n+1)\sqrt[3]{n}}$ monotonically tends to $0$, but only conditionally since $\sum_{n=1}^{\infty} n^{-1/3}$ diverges.

4.

$a > 1$.

$$\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \frac{n^{10}}{a^n}$$

converges absolutely since there exists $C > 0$ such that for $n > C$, $n^{10}a^{-n} \leqslant a^{-n/2}$, and $\sum_{n=1}^{\infty} a^{-n/2}$ converges.

# PSC

Suppose the integer $b \geqslant 2$, $f : [1, \infty) \to \mathbb{R}_{>0}$ is monotonically decreasing.

### C1) Prove that

$$(b-1)b^{k-1}f(b^k) \leqslant \sum_{j=b^{k-1}}^{b^k-1} f(j) \leqslant (b-1)b^{k-1}f(b^{k-1}).$$

Proof: There are $(b-1)b^{k-1}$ integers in $[b^{k-1}, b^k - 1]$, and since $f$ is monotonically decreasing, for any $j \in [b^{k-1}, b^k - 1]$, $f(j) \in [f(b^k), f(b^{k-1})]$.

**C2) Prove that the series**

$$\sum_{n=1}^{\infty} f(n) \text{ and } \sum_{n=1}^{\infty} b^n f(b^n)$$

converge or diverge simultaneously.
Proof: From C1),

$$\sum_{k=1}^{\infty}(b-1)b^{k-1}f(b^k) \leqslant \sum_{n=1}^{\infty} f(n) = \sum_{k=1}^{\infty}\sum_{j=b^{k-1}}^{b^k-1} f(j) \leqslant \sum_{k=1}^{\infty}(b-1)b^{k-1}f(b^{k-1}).$$

Therefore the two series converge or diverge simultaneously.

**C3) Prove that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\log n}$ diverges.**

Proof: Consider $f(x) = \frac{1}{x\log x}$ which is monotonically decreasing. Note that

$$\sum_{n=2}^{\infty} 2^n f(2^n) = \sum_{n=2}^{\infty}\frac{1}{n\log 2} = \infty.$$

From C2) we know that $\sum_{n=2}^{\infty} f(n)$ diverges.

**C4) Prove that $\sum_{n=100}^{\infty}\frac{1}{n\log n\log\log n}$ diverges.**

Proof: Consider $f(x) = \frac{1}{x\log x\log\log x}$ which is monotonically decreasing. From C3),

$$\sum_{n=100}^{\infty} 2^n f(2^n) = \sum_{n=100}^{\infty}\frac{1}{n\log 2\cdot\log(n\log 2)}$$

diverges. Hence from C2) we know that $\sum_{n=100}^{\infty} f(n)$ diverges.

**C5) Prove that $\sum_{n=1}^{\infty} n^{-s}$ converges iff $s > 1$.**

Proof: Consider $f(x) = x^{-s}$ which is monotonically decreasing. Note that

$$\sum_{n=1}^{\infty} 2^n f(2^n) = \sum_{n=1}^{\infty} 2^{-n(s-1)} = \frac{2^{1-s}}{1-2^{1-s}}.$$

**C6) Suppose $s > 1$, prove that $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^s}$ and $\sum_{n=10}^{\infty}\frac{1}{n\log n(\log\log n)^s}$ converges.**

Proof: Same as C3) and C4).

# PSD

For $\{a_n\}_{n\geqslant 1}\subset\mathbb{R}$,

- $\alpha\in\mathbb{R}$, if for any $\varepsilon > 0$, there are infinitely many $n$ such that $a_n\in(\alpha-\varepsilon,\alpha+\varepsilon)$, then we call $\alpha$ a limit point of $\{a_n\}_{n\geqslant 1}$.
- Likewise define limit points for $\alpha = \pm\infty$.

### D1) Prove that $\alpha \in \mathbb{R}$ is a limit point of $\{a_n\}_{n\geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ which converges to $\alpha$.

Proof: <== is trivial. ==> Let $\varepsilon = 1/k$ then there exists $a_{n_k}$ such that $|a_{n_k} - \alpha| < \varepsilon$. Hence $\lim_{k\to\infty} a_{n_k} = \alpha$.

### D2) Prove that $+\infty$ is a limit point of $\{a_n\}_{n\geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ such that $\lim_{k\to\infty} a_{n_k} = \infty$.

Proof: Same as D1).

### D3) Let $E = \{\alpha \in \mathbb{R} \cup \{\pm\infty\} : \alpha \text{ is a limit point of } \{a_n\}\}$. Prove that $E \neq \emptyset$.

Proof: If $\{a_n\}$ is unbounded, then by D2) $E \cap \{\pm\infty\} \neq 0$. If $\{a_n\}$ is bounded, then by Bolzano-Weierstrass theorem, $E \neq \emptyset$.

### D4) Prove that $E \subset \mathbb{R}$ iff $\{a_n\}$ is bounded.

Proof: Use D2)

### D5) Suppose $\{a_n\}_{n\geqslant 1}$ is bounded. Prove that $\sup E = \limsup_{n\to\infty} a_n$, $\inf E = \liminf_{n\to\infty} a_n$.

Proof: Let $M = \limsup_{n\to\infty} a_n$, then for any $\varepsilon > 0$, there exists $n$ such that $M \leqslant \sup_{k\geqslant n} a_k < M + \varepsilon$, hence there exists $k \geqslant n$ such that $|a_k - M| < \varepsilon$, so $M \in E$. For any $\alpha \in E$, there is a sub-sequence $\{a_{n_k}\} \to \alpha$, hence

$$\alpha = \lim_{k\to\infty} a_{n_k} \leqslant \lim_{k\to\infty} \sup_{m\geqslant n_k} a_{n_k} = \limsup_{n\to\infty} a_n = M.$$

Therefore $M = \sup E$. Substitute $a_n \to -a_n$ and we obtain $\inf E = \liminf_{n\to\infty} a_n$.

### D6) Suppose $\{a_n\}_{n\geqslant 1}$ is bounded. Let $a^* = \limsup_{n\to\infty} a_n$. Prove that

i) $a^* \in E$, i.e. $\sup E \in E$.
Proof: See the proof of D5).
ii) For any $x > a^*$, there exists $N \in \mathbb{Z}_{\geqslant 1}$ such that for any $n > N$, $a_n < x$.
Proof: If there is an infinite sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ such that $a_{n_k} \geqslant x$, then $\{a_{n_k}\}$ has a limit point $a' > x > a^*$, contradicting $a^* = \sup E$.

### D7) Construct an example of $\{a_n\}_{n\geqslant 1}$ such that $E \cap \mathbb{R} \neq \emptyset$ and $E \not\subset \mathbb{R}$.

Solution: Since $\mathbb{Q}$ is countable, let $\{a_n\}_{n\geqslant 1}$ iterate every element of $\mathbb{Q}$, then $E = \mathbb{R} \cup \{\pm\infty\}$ is an infinite set.

### D8) Construct $\{a_n\}_{n\geqslant 1}$ such that $E$ is an infinite set.

Solution: Same as D7).

# PSE: Reciprocal Sum of Primes

Define the $\zeta$-function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

We have proved the formula:

$$\zeta(s)=\prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}}.$$

Prove that the series

$$\sum_{p\in\mathcal{P}}p^{-s}$$

converges when $s>1$, and diverges when $0<s\leqslant 1$.
Proof: We know that for $|a_n|<1$, $\prod_{n=1}^{\infty}(1-a_n)$ converges iff $\sum_{n=1}^{\infty}a_n$ converges. Hence by $\zeta(s)^{-1}=\prod_{p\in\mathcal{P}}(1-p^{-s})$, we obtain $\sum_{p\in\mathcal{P}}p^{-s}$ converges iff $s>1$.

# PSF: Euler's "Proof" of the Basel Problem

For any $\theta\in\mathbb{R}, n\in\mathbb{Z}$, prove the identity

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta}=\prod_{k=1}^{n}\left(1-\frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

Further prove that for any $x\in\mathbb{R}$,

$$\frac{\sin(\pi x)}{\pi x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right).$$

Proof: (1) By induction there is a polynomial $P_n(x)$ such that $P_n(\sin\theta)=\sin(2n+1)\theta$ for any $\theta\in\mathbb{R}$ and $\deg P_n=2n+1$. For any $k=1,2,\cdots,n$, and $\theta=\pm k\pi/(2n+1)$, $\sin((2n+1)\theta)=0$, hence $P_n$ has roots $0$ and $\pm\sin(k\pi/(2n+1))$ for $k=1,2,\cdots,n$. Since $\deg P_n=2n+1$,

$$P_n(x)=Cx\prod_{k=1}^{n}\left(1-\frac{x^2}{\sin^2(k\pi/(2n+1))}\right)$$

for some $C\in\mathbb{R}$. Let $x=\sin\theta$ and consider the derivatives on both sides when $\theta=0$, then we obtain $C=2n+1$, therefore

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta}=\prod_{k=1}^{n}\left(1-\frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

(2) Let $m=2n+1$. From (1) we know that for any $x\in\mathbb{C}$ and $k<n$, $\sin x=U_k^{(n)}\cdot V_k^{(n)}$, where

$$\begin{aligned}U_k^{(n)}&=m\sin\frac{x}{m}\prod_{j=1}^{k}\left(1-\frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right),\\ V_k^{(n)}&=\prod_{j=k+1}^{n}\left(1-\frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right).\end{aligned}$$

Clearly, for any $k\in\mathbb{N}$,

$$\lim_{n\to\infty}U_k^{(n)}=U_k=x\prod_{j=1}^{k}\left(1-\frac{x^2}{j^2\pi^2}\right).$$

and for any $x\in\mathbb{C}$ and $j\in\mathbb{N}$,

$$\left|\frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right|\leqslant\frac{x^2}{4j^2}\cdot K(|x|/m)^2,$$

where $K(x)=\sum_{n=0}^{\infty}|x|^n/(2n+1)!$ is monotonic on $[0,\infty)$ and $K(0)=1$.
Note that for $\alpha_i\in\mathbb{C}$,

$$\left|1-\prod_{j=1}^{n}(1-\alpha_n)\right|\leqslant\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|\alpha_k|\right)^j.$$

Hence for any $x\in\mathbb{C}$ and $\varepsilon>0$, there exists $N$ such that for any $k\geqslant N$, and any $n>k$, $|V_k^{(n)}-1|<\varepsilon$, since

$$|V_k^{(n)}-1|\leqslant\sum_{j=1}^{\infty}\left(\sum_{l=k+1}^{\infty}\frac{x^2}{4l^2}K(|x|/m)^2\right)^j\leqslant\sum_{j=1}^{\infty}\left(K(|x|/(2k+1))^2\cdot\frac{x^2}{k}\right)^j\to 0.$$

i.e. for any $x\in\mathbb{C}$

$$\lim_{k\to\infty}\sup_{n>k}|V_k^{(n)}-1|=0.$$

And likewise we know that there is a constant $M$ such that for any $n>k$, $|x|<k$, $|U_k^{(n)}|\leqslant M$.
Therefore for any $x\in\mathbb{C}$,

$$\sin x=x\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2\pi^2}\right)=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

> Note:
> From the formula above, we can formally deduce that
>
> $$\sin(\pi x)=\pi x(1-\zeta(2)x^2+\zeta(4)x^4+\cdots).$$
>
> Compare it to $\sin z=x-x^3/6+\cdots$, and we get $\zeta(2)=\pi^2/6$.