

162-3

An $n \times n$ matrix A over a field F is skew-symmetric if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det A = 0$.

Proof: Note that $\det A = \det A^t = \det(-A) = (-1)^n \det A = -\det A$, so $\det A = 0$.

162-4

An $n \times n$ matrix A over a field F is called orthogonal if $AA^t = I$. If A is orthogonal, show that $\det A = \pm 1$.

Give an example of an orthogonal matrix for which $\det A = -1$.

Proof: $A = \text{diag}(1, -1)$ is orthogonal, but $\det A = -1$.

If A is orthogonal, then $1 = \det I = \det AA^t = \det A \det A^t = (\det A)^2$, so $\det A = \pm 1$.

162-5

An $n \times n$ matrix A over \mathbb{C} is said to be unitary if $AA^* = I$. If A is unitary, show that $|\det A| = 1$.

Proof: Note that $|Av|^2 = \langle Av, Av \rangle = \langle v, A^*Av \rangle = \langle v, v \rangle = |v|^2$, so any eigenvalue λ of A satisfy $|\lambda| = 1$, so $|\det A| = 1$.

5.4.2

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26-1(2)

Calculate the determinant

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & 0 \\ & & \cdots & & \\ 0 & a_{n-1} & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

Solution: This matrix is $\text{diag}(a_1, \dots, a_n) \cdot R(\sigma)$ where $\sigma = \begin{pmatrix} 1 & \cdots & n \\ n & \cdots & 1 \end{pmatrix}$ so the determinant is $(-1)^{n(n-1)/2} a_1 \cdots a_n$.

35-1(3)

Calculate the determinant

$$\begin{vmatrix} 1 & 0 & -3 & 2 \\ -4 & -1 & 0 & -5 \\ 2 & 3 & -1 & -6 \\ 3 & 3 & -4 & 1 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 0 & -3 & 2 \\ -4 & -1 & 0 & -5 \\ 2 & 3 & -1 & -6 \\ 3 & 3 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 & 2 \\ 0 & 5 & -2 & -17 \\ 0 & 3 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \cdot (5 \cdot 5 + 2 \cdot 3) \cdot 5 = 155.$$

35-2(1)

Calculate

$$\begin{vmatrix} a & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & \cdots & 1 \\ & & & \cdots & \\ 1 & 1 & 1 & \cdots & a \end{vmatrix}$$

Solution: Note that

$$\begin{vmatrix} a & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & \cdots & 1 \\ & & & \cdots & \\ 1 & 1 & 1 & \cdots & a \end{vmatrix} = \begin{vmatrix} a + (n-1) & 1 & \cdots & 1 \\ a + (n-1) & a & \cdots & 1 \\ & & \cdots & \\ a + (n-1) & 1 & \cdots & a \end{vmatrix} = (a + n - 1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a - 1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & a - 1 \end{vmatrix}$$

so the determinant is $(a + n - 1)(a - 1)^{n-1}$.

35-3(1)

Prove that

$$\begin{vmatrix} a_1 - b_1 & b_1 - c_1 & c_1 - a_1 \\ a_2 - b_2 & b_2 - c_2 & c_2 - a_2 \\ a_3 - b_3 & b_3 - c_3 & c_3 - a_3 \end{vmatrix} = 0.$$

Proof: Note that $R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ satisfy $(x, y, z) = R(y, z, x)$ and 1 is an eigenvalue of R since

$R(1, 1, 1)^T = (1, 1, 1)^T$, so $\det(A - AR) = 0$.

35-4(2)

Calculate

$$\begin{vmatrix} a_1 + b_1 & a_1 + b_2 & \cdots & a_1 + b_n \\ a_2 + b_1 & a_2 + b_2 & \cdots & a_2 + b_n \\ & & \cdots & \\ a_n + b_1 & a_n + b_2 & \cdots & a_n + b_n \end{vmatrix}$$

Solution: For $n \geq 3$, the determinant is a polynomial of a_i with degree 1, and it vanishes when $a_i = a_j$, so it is zero.

When $n = 2$, it is $(a_1 + b_1)(a_2 + b_2) - (a_1 + b_2)(a_2 + b_1) = -(a_1 - a_2)(b_1 - b_2)$. When $n = 1$ it is $a_1 + b_1$.

5.4.4

Prove that $\text{sgn}(\sigma) = \det R(\sigma)$ where $R(\sigma) \in F^{n \times n}$ is the matrix where the i^{th} row is $e_{\sigma(j)}$.

Proof: Trivial since $\det I = 1$ and interchanging two rows reverses the sign of the determinant.

5.4.6

Suppose K is a subfield of F , such that F is finite dimensional viewed as a linear space of K . For $x \in F$, the determinant of the K -linear map $T_x : F \rightarrow F, y \mapsto xy$ is call the norm of x , denoted $N_{F/K}(x) = \det T_x$.

Try to give the expression of the norm:

(1) $K = \mathbb{R}, F = \mathbb{C}$.

(2) $K = \mathbb{Q}, F = \mathbb{Q}(\sqrt{2})$.

(3) $K = \mathbb{Q}, F = \mathbb{Q}(\sqrt[3]{2})$.

Proof:

(1) For $w = s + it \in \mathbb{C}, T_w : F \rightarrow F, z \mapsto wz$, then

$T_w(x + iy) = (s + it)(x + iy) = (sx - ty) + i(tx + sy)$, so the matrix of T_w is $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ and the

determinant is $s^2 + t^2 = |w|^2$, which is exactly the normal norm.

(2) For $x = a + b\sqrt{2} \in F, T_x : F \rightarrow F, y \mapsto xy$, then $T_x(u + v\sqrt{2}) = (a + b\sqrt{2})(u + v\sqrt{2})$, so the matrix of T is $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$, and the determinant is $a^2 - 2b^2$.

(3) For $x = a + b\omega + c\omega^2 \in F$,

$$T_x(u + v\omega + w\omega^2) = (au + 2bw + 2cv) + (av + bu + 2cw)\omega + (aw + bv + cu)\omega^2,$$

and the matrix of T_x is $\begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix}$ and its determinant is $a^3 + 2b^3 + 4c^3 - 6abc$.

5.4.7

Suppose V is a finite-dimensional complex linear space, $T \in \mathcal{L}(V)$. Applying the forgetful functor to T , we obtain $T_{\mathbb{R}} \in \mathcal{L}(V_{\mathbb{R}})$. Prove that $\det T_{\mathbb{R}} = |\det T|^2$.

Proof: For any $\omega \in \Lambda^n(V)$ where $n = \dim V_{\mathbb{C}}, T^*\omega = (\det T)\omega$. If e_1, \dots, e_n form a basis of $V_{\mathbb{C}}$, then $e_1, ie_1, \dots, e_n, ie_n$ form a basis of $V_{\mathbb{R}}$. Let z_1, \dots, z_n be the dual basis, then $z_k = x_k + iy_k$ so

$x_1, y_1, \dots, x_n, y_n$ is the dual basis of $V_{\mathbb{R}}^*$.

Consider $\bar{\omega}(x_1, \dots, x_n) = \overline{\omega(x_1, \dots, x_n)}$, notice that

$$T^*\bar{\omega}(x_1, \dots, x_n) = \bar{\omega}(Tx_1, \dots, Tx_n) = \overline{T^*\omega(x_1, \dots, x_n)} = \overline{(\det T)\omega(x_1, \dots, x_n)} \text{ so } T^*\bar{\omega} = \overline{(\det T)\omega}.$$

Let $\Omega = \omega \wedge \bar{\omega}$, note that $z_k \wedge \bar{z}_k = (x_k + iy_k) \wedge (x_k - iy_k) = -2ix_k \wedge y_k$, so $\Omega \in \Lambda^{2n}(V_{\mathbb{R}})$ and

$$T^*\Omega = \det T_{\mathbb{R}} \cdot \Omega. \text{ By } T^*\Omega = (T^*\omega) \wedge (T^*\bar{\omega}) = ((\det T)\omega) \wedge (\overline{(\det T)\omega}) = |\det T|^2 \Omega \text{ we obtain}$$

$$\det T_{\mathbb{R}} = |\det T|^2.$$