

5.2.1

Is $\{f_1 \otimes f_2 : f_1, f_2 \in V^*\}$ a subspace of $\mathcal{T}^2(V)$?

Solution: If $\dim V \geq 2$, the answer is negative: Consider $f_1(x) = x_1, f_2(y) = y_1$ and $g_1(x) = x_2, g_2(y) = y_2$, then $(f_1 \otimes f_2 + g_1 \otimes g_2)(x, y) = x_1y_1 + x_2y_2$. If $f \otimes g(x, y) = f(x)g(y) = x_1y_1 + x_2y_2$, then $2 = f(e_1 + e_2)g(e_1 + e_2) = f(e_1)g(e_1) + f(e_2)g(e_1) + f(e_2)g(e_2) + f(e_1)g(e_2)$ so $f(e_2)g(e_1) + f(e_1)g(e_2) = 0$, while $f(e_1)g(e_1) = f(e_2)g(e_2) = 1$, leading to contradiction. If $\dim V = 1$, then $\dim \mathcal{T}^2(V) = 1$ so $\mathcal{T}^2(V) = \{f_1 \otimes f_2 : f_1, f_2 \in V^*\}$, and it is a subspace of $\mathcal{T}^2(V)$.

5.2.2

Suppose $\dim V = n$, prove that $\dim \mathcal{T}^2(V) = n^2$.

Proof: We show that $\dim \mathcal{T}^r(V) = n^r$. Take a basis e_1, \dots, e_n of V , and the dual basis f_1, \dots, f_n of V^* . We show that $\mathcal{B} = \{f_{j_1} \otimes \dots \otimes f_{j_r} : j_1, \dots, j_r \in \{1, \dots, n\}\}$ form a basis of $\mathcal{T}^r(V)$: Note that for any $L \in \mathcal{T}^r(V)$,

$$L = \sum_{1 \leq j_1, \dots, j_r \leq n} L(e_{j_1}, \dots, e_{j_r}) f_{j_1} \otimes \dots \otimes f_{j_r}.$$

Since $f_{j_1} \otimes \dots \otimes f_{j_r}(e_{i_1}, \dots, e_{i_r}) = \delta_{j_1, \dots, j_r}^{i_1, \dots, i_r}$.

Also, if $F = \sum_{1 \leq j_1, \dots, j_r \leq n} c_{j_1, \dots, j_r} f_{j_1} \otimes \dots \otimes f_{j_r} = 0$, then $c_{j_1, \dots, j_r} = F(e_{j_1}, \dots, e_{j_r}) = 0$, so $f_{j_1} \otimes \dots \otimes f_{j_r}$ are linearly independent, therefore they form a basis of $\mathcal{T}^r(V)$, so $\dim \mathcal{T}^r(V) = n^r$.

5.2.9

Suppose $\dim V = n$,

(1) Prove that $\dim \Lambda^2(V) = \frac{1}{2}n(n-1)$.

(2) Suppose $r > n$, prove that $\Lambda^r(V) = \{0\}$.

Proof: We show that $\dim \Lambda^r(V) = \binom{n}{r}$. Take a basis e_1, \dots, e_n of V , and the dual basis $f_1, \dots, f_n \in V^*$. We call an index set $J = (j_1, \dots, j_r)$ increasing if $j_1 < \dots < j_r$ and denote $f^J = f_{j_1} \wedge \dots \wedge f_{j_r}$ where $\alpha \wedge \beta = A \left(\frac{\alpha}{s!} \otimes \frac{\beta}{t!} \right)$. Let $\mathcal{B} = \{f^J : J \text{ increasing}\}$ then $|\mathcal{B}| = \binom{n}{r}$.

For $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$, $f^I(e_{j_1}, \dots, e_{j_r}) = A(f_{i_1} \otimes \dots \otimes f_{i_r})(e_{j_1}, \dots, e_{j_r}) = \varepsilon_J^I$ where $\varepsilon_J^I = \text{sign}(\sigma)$ if $J = \sigma(I)$, and otherwise $\varepsilon_J^I = 0$. Therefore \mathcal{B} is linearly independent, otherwise

$\sum_{I \in \mathcal{B}} c_I f^I = 0$ implies $c_J = \sum_{I \in \mathcal{B}} c_I f^I(u_{j_1}, \dots, u_{j_r}) = 0$.

For any $\alpha \in \Lambda^r(V)$, suppose

$$\alpha = \sum_{1 \leq j_1, \dots, j_r \leq n} c_{j_1, \dots, j_r} f_{j_1} \otimes \dots \otimes f_{j_r}.$$

Then

$$\begin{aligned} r! \alpha &= A \alpha = A \sum c_{j_1, \dots, j_r} f_{j_1} \otimes \dots \otimes f_{j_r} = \sum_{1 \leq j_1, \dots, j_r \leq n} c_{j_1, \dots, j_r} A(f_{j_1} \otimes \dots \otimes f_{j_r}) \\ &= \sum_{J \in \mathcal{B}} \left(\sum_{\sigma \in S_r} \text{sign}(\sigma) c_{j_{\sigma(1)}, \dots, j_{\sigma(r)}} \right) A f^J = \sum_{J \in \mathcal{B}} d_J A f^J. \end{aligned}$$

Hence \mathcal{B} forms a basis of $\Lambda^r(V)$, so $\dim \Lambda^r(V) = \binom{n}{r}$.

5.3.1

Suppose $B \in F^{n \times n}$. Consider $T_B \in \mathcal{L}(F^{n \times n})$, $A \mapsto AB - BA$. Prove that $\det T_B = 0$.

Proof: If $B = 0$ then $T_B = 0$ so $\det T_B = 0$. Otherwise $T_B(B) = 0$ so T_B is not injective, hence $\det T_B = 0$.

5.3.2

Suppose V is a n -dimensional real vector space, $n \geq 1$. We call a subset $\mathcal{O} \subset \Lambda^n(V) \setminus \{0\}$ a connected component, if for any $L \in \mathcal{O}$, $\mathcal{O} = \{cL : c > 0\}$.

(1) Prove that $\Lambda^n(V) \setminus \{0\}$ has exactly two connected components, each called an orientation of V .

(2) Given an orientation \mathcal{O}^+ of V , denote the other orientation as \mathcal{O}^- . If $T \in \mathcal{L}(V)$ is invertible, and $T^{(n)}(\mathcal{O}^+) = \mathcal{O}^+$, then we call T orientation-preserving; if $T^{(n)}(\mathcal{O}^+) = \mathcal{O}^-$, we call T orientation-reversing. Prove that T is orientation-preserving iff $\det T > 0$ and orientation-reversing iff $\det T < 0$.

Proof: (1) $\dim \Lambda^n(V) = 1$, so take any $L \in \Lambda^n(V)$, let $\mathcal{O}^+ = \{cL : c > 0\}$ and $\mathcal{O}^- = \{c(-L) : c > 0\}$, then $\Lambda^n(V) \setminus \{0\} = \mathcal{O}^+ \cup \mathcal{O}^-$. For any $T \in \mathcal{O}^+$, suppose $T = tL$, then $cL = \frac{c}{t}T$ so $\mathcal{O}^+ = \{cT : c > 0\}$, and likewise for any $T \in \mathcal{O}^-$, $\mathcal{O}^- = \{cT : c > 0\}$. Hence $\Lambda^n(V) \setminus \{0\}$ has two connected components \mathcal{O}^+ and \mathcal{O}^- .

(2) Note that $T^{(n)}(L) = \det(T)L$, so T is orientation-preserving iff $\det T > 0$, and orientation-reversing iff $\det T < 0$.