Contents

1 Homework 1: Schröder-Bernstein Theorem

1.1 PSA

A1) Suppose a non-empty set $X \subset \mathbb{R}$ has an upper bound, and M is an upper bound of X. The following two propositions are equivalent:

- $M = \sup X$.
- For any $\varepsilon > 0$, there exists an $x \in X$ such that $x > M - \varepsilon$.

Proof:

$$M = \sup X \iff \forall M' < M, \exists x \in X, x > M' \iff \forall \varepsilon = M - M' > 0, \exists x \in X, x > M - \varepsilon.$$

A2) Prove that every non-empty open interval contains infinitely many rational numbers.

Proof: We only need to find one rational number q in the interval (a, b), then we can apply the process to (a, q) and so on.
By the Archimedean rule, there is a positive integer N such that $N(b - a) > 2$, hence there exists an integer p such that $p = \lfloor bN \rfloor \in (aN, bN)$, and $q = \frac{p}{N} \in (a, b) \cap \mathbb{Q}$.

A3) Let (X, d) be a metric space, $Y \subset X$. We define the distance function on Y:

$d_Y : Y \times Y \to \mathbb{R}, (y_1, y_2) \mapsto d_Y(y_1, y_2) = d(y_1, y_2)$.
Prove that d_Y is a distance function, and (Y, d_Y) is a metric space. We call d_Y the induced metric on Y, and (Y, d_Y) is called a subspace.
Proof: Trivial, since $d_Y(y_1, y_2) = d(y_1, y_2)$.

A4) Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, \dots, x_n \in \mathbb{R}\}$, for any $x, y \in \mathbb{R}^n$, we define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n).$$

Prove that $(\mathbb{R}^n, d)$ is a metric space.

Proof:

1. $d(x, y) = 0 \iff x_i = y_i, \forall 1 \leqslant i \leqslant n \iff x = y$.
2. $d(x, y) = d(y, x)$ is trivial.
3. $d(x, y) + d(y, z) \geqslant d(x, z)$ is the Minkowski inequality:

$$\left(\sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}\right)^2 = \sum_{i=1}^n a_i^2 + b_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2} \geqslant \sum_{i=1}^n a_i^2 + b_i^2 + 2a_i b_i = \sum_{i=1}^n (a_i + b_i)^2.$$

A5) Given a metric space (X, d), and $Y \subset X$. If for any $x \in X$ and $\varepsilon > 0$, there exists $y \in Y$ such that $d(y, x) < \varepsilon$, then we say Y is dense in X. Prove that the set of rational numbers is dense in $\mathbb{R}$.

Proof: For any $x \in \mathbb{R}$, let $N = \lfloor x \rfloor$, then for any $\varepsilon > 0$, let $q > 1/\varepsilon$. Then for $p \in [Nq, (N+1)q] \cap \mathbb{Z}$, choose p such that $|x - p/q|$ is minimal. Suppose $p/q < x$, then

$$2\left|x - \frac{p}{q}\right| < \left|x - \frac{p}{q}\right| + \left|x - \frac{p+1}{q}\right| = \frac{1}{q} < \varepsilon.$$

Hence $d(x, p/q) < \varepsilon$.

A6) For $(x, y) \in \mathbb{R}^2$, if its coordinates x and y are rational numbers, then we call this point a rational point. Prove that $(\mathbb{R}^2, d)$ (refer to question A4) the set of rational points in $\mathbb{R}^2$ is dense.

Proof: By A5), $\overline{\mathbb{Q}} = \mathbb{R}$. Hence for any $(x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$, there exists $(a, b) \in \mathbb{Q}^2$ such that $|a - x|, |b - y| < \varepsilon/2$. Then

$$d((x, y), (a, b)) = \sqrt{(a - x)^2 + (b - y)^2} < \varepsilon.$$

Hence $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$

A7) Prove that the axiom (F) and (O), and the boundedness principle imply the Archimedean axiom (A).

Proof: Otherwise assume that $\mathbb{N}$ has an upper bound. Then $M = \sup \mathbb{N}$ exists. Let $\varepsilon = 1/2$ then there is an $n \in \mathbb{N}$ such that $n > M - \varepsilon$. Hence $n + 1 > M$, leading to contradiction.

A8) (Existence of irrational numbers) Let $X = \left\{x \in \mathbb{Q} \mid x^2 < 2\right\}$ be a bounded set, and $\sqrt{2} = \sup X$. Prove that $\sqrt{2}$ is an irrational number.

Proof: If $\sqrt{2} = s = p/q$ is rational, then $p^2 \geqslant 2q^2$, otherwise let $x = s(2 - s^2)/4 + s$, then $s < x$ and $x^2 < 2$, a contradiction. If $s^2 > 2$, then $x = s(2 - s^2)/4 < s$ and $x^2 > 2$, hence x is an upper bound of X, leading to contradiction. Therefore $s^2 = 2$ which is impossible.

A9) Prove that every open interval contains infinitely many irrational numbers.

Proof: Otherwise the open interval will be a countable set.

1.2 PSB: Countable and Uncountable Sets

Let $\mathbb{N}$ denote the set of natural numbers (including 0). X is a set, if there is an injective map $f : X \to \mathbb{N}$, then we say X is countable; if X is not countable, then we say X is uncountable.

B1) Prove that finite sets are countable.

Proof: For any finite set $X = \{a_1, \cdots, a_n\}$, the map $f : a_k \mapsto k$ is an injective, hence X is countable.

B2) Prove that subsets of countable sets are countable.

Proof: If X is countable and $Y \subset X$, then there is an injective map $f : X \to N$, so $f|_Y : Y \to N$ is an injective map, hence Y is countable.

B3) Prove that if X is a countable set, then we can always write $X = \{x_1, x_2, x_3, \ldots\}$ (that is, the elements of X can be indexed by natural numbers).

Proof: Let $I = \{n \in \mathbb{N} : f^{-1}(n) \neq \emptyset\}$, $x_k = f^{-1}(\min I \backslash \{f(x_1), \cdots, (x_{k-1})\})$. Then $x_x \in X$, and for any $x \in X$, $f(x) \in I$ hence $x \in \{x_1, \cdots, x_{f(x)}\}$. Therefore $X = \{x_1, \cdots, x_n, \cdots\}$.

B4) Prove that the set of rational numbers $\mathbb{Q}$ is countable.

Proof: List every positive rational number as below:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \cdots$$

such that p/q is before m/n if $p+q < m+n$ or $p+q = m+n$ and $p < m$, then every number in $\mathbb{Q}_{>0}$ is listed at least once. Hence $\mathbb{Q}_{>0}$ is countable and so is $\mathbb{Q}$.

B5) Prove that the countable union of countable sets is countable, that is, if $X_1, X_2, \ldots, X_n, \ldots$ are all countable sets, then their union $\bigcup_{n=1}^{\infty} X_n$ is also a countable set.

Proof: Assume X_n are disjoint. Since X_n are countable, we can write

$$X_n = \{a_1^{(n)}, a_2^{(n)}, \cdots, a_m^{(n)}, \cdots\}.$$

Then

$$\bigcup_{n=1}^{\infty} X_n = \{a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_3^{(1)}, \cdots\}$$

where the order is the same as in B4). Hence $\bigcup_{n\geqslant 1} X_n$ is countable.

B6) If X is countable, and the map $f : X \to Y$ is surjective, then Y is countable.

Proof: Since X is countable, there is an injective map $g : X \to \mathbb{N}$. Let

$$h : Y \to \mathbb{N},\ y \mapsto \min g(f^{-1}(\{y\})).$$

then g is injective, hence Y is countable.

B7) Prove the following using proof by contradiction: $\mathbb{R}$ is uncountable.

B7-1) Suppose $J \subset \mathbb{R}$ is a closed interval and its length $|J| > 0$. For any $x \in \mathbb{R}$, there always exists an interval $I \subset J$ such that $|I| > 0$ and $x \notin I$.
Proof: Any closed interval $J = [a, b]$ can be written in the form $J = A \cup B \cup C$, where $A = \left[a, \frac{2a+b}{3}\right], B = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], C = \left[\frac{a+2b}{3}, b\right]$, and x can only be in at most 2 of these sets. Hence we can choose a set I in A, B, C.
B7-2) Prove that if $\{x_1, x_2, \ldots\}$ is a countable subset of $\mathbb{R}$, then there exists a nested interval sequence $I_1 \supset I_2 \supset \cdots$ such that for any n, $x_n \notin I_n$.
Proof: Simple application of B7-1)
B7-3) Prove that $\mathbb{R}$ is uncountable.
Proof: If $\mathbb{R}$ is countable, write $\mathbb{R} = \{r_1, r_2, \cdots\}$, then set $I_0 = [0, 1]$. By B7-2) we can obtain a sequence $I_0 \supset I_1 \supset \cdots$ such that $x_n \notin I_n$ for any n. Hence

$$\bigcap_{n=0}^{\infty} I_n = \emptyset,$$

leading to contradiction.

B8) Prove that if X is an uncountable set, and A is a countable subset of X, then $X - A$ is uncountable.

Proof: Otherwise suppose that both A and $X - A$ is countable, then there exist injective mappings $f : A \to \mathbb{N}$ and $g : X - A \to \mathbb{N}$. Define

$$h : X \to \mathbb{N},\, x \mapsto \begin{cases} 2f(x), & x \in A, \\ 2g(x) + 1, & x \notin A. \end{cases}$$

Then h is injective, hence X is countable.

B9) Prove that any interval of non-zero length (open or closed) is uncountable.

Proof: Same as B7).
Or use the fact that $\mathbb{R}$ is the countable union of intervals of the same length, and the countable union of countable sets is still countable.

B10) Prove that the set of complex numbers $\mathbb{C}$ is uncountable.

Proof: $\mathbb{C}$ has an uncountable subset $\mathbb{R}$.

B11) Suppose $\mathcal{I}$ is a collection of non-overlapping closed intervals, satisfying the following property: for any $I, J \in \mathcal{I}$, if $I \neq J$, then their intersection is empty, i.e., $I \cap J = \emptyset$. Prove that $\mathcal{I}$ is countable.

Proof: For any $I \in \mathcal{I}$, there exists a rational number $r_I \in I$. Consider $f : \mathcal{I} \to \mathbb{Q}$, $I \mapsto r_I$, then f is injective. Since $\mathbb{Q}$ is countable, so is $\mathcal{I}$.

1.3 PSC: Schröder-Bernstein Theorem

Suppose X and Y are two sets, and mappings $f : X \to Y$ and $g : Y \to X$ are both injective. Let $X' = X - g(Y)$.

C1) If X is a finite set, prove that there exists a bijection $\varphi : X \to Y$.

Proof: $g : Y \to X$ is injective and X is finite, $\implies$ Y is finite. Hence $|X| \leqslant |Y|$, and $|X| \geqslant |Y|$, so $|X| = |Y|$. Therefore we can write $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, and obtain

$$\varphi : X \to Y,\, x_k \mapsto y_k.$$

C2) If X is countable, prove that there exists a bijection $\varphi : X \to Y$.

Proof: Assume X is infinite, then Y is countable (by g) and infinite (by f). Hence we can list $X = \{x_1, x_2, \cdots\}$ and $Y = \{y_1, y_2, , ..\}$ and define

$$\varphi : X \to Y,\, x_k \mapsto y_k.$$

From now on, we impose no restrictions on X. Let $h : X \to X$ be the composite map $h = g \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow g \\ X & \leftarrow & \end{array}$$

C3) Consider the set family $\mathcal{F} = \{A \subset X \mid X' \cup h(A) \subset A\}$. Prove that $\mathcal{F}$ is non-empty.

Proof: $X \in \mathcal{F}$.

C4) Prove that if $A \in \mathcal{F}$, then $X' \cup h(A) \in \mathcal{F}$.

Proof: If $A \in \mathcal{F}$ then $X' \cup h(A) \subset A$, hence (let B denote $X' \cup h(A)$)

$$X' \cup h(B) \subset X' \cup h(A) = B.$$

C5) We define

$$A_0 = \bigcap_{A \in \mathcal{F}} A = \{x \in X \mid \text{for any } A \in \mathcal{F}, \text{ we have } x \in A\}.$$

Prove that $A_0 \in \mathcal{F}$.
Proof:

$$X' \cup h(A_0) \subset X' \cup (\bigcap_{A \in \mathcal{F}} h(A)) = \bigcap_{A \in \mathcal{F}} X' \cup h(A) \subset \bigcap_{A \in \mathcal{F}} A = A_0.$$

Hence $A_0 \in \mathcal{F}$.

C6) Prove that $X' \cup h(A_0) = A_0$.

Proof:

$$A_0 \in \mathcal{F} \implies X' \cup h(A_0) \in \mathcal{F} \implies A_0 \subset X' \cup h(A_0).$$

The other side is proved in C5).

C7) Let $B_0 = X - A_0$. Prove that $f(A_0) \cap g^{-1}(B_0) = \emptyset$ and $f(A_0) \cup g^{-1}(B_0) = Y$.

Proof: If $f(A_0) \cap g^{-1}(B_0) \neq \emptyset$, then there exist $a \in A_0, b \in B_0$ such that $f(a) = g^{-1}(b)$, i.e. $b = h(a)$. Since $a \in A_0$, for any $A \in \mathcal{F}$, $a \in A$, hence $b = h(a) \in X' \cup h(A) \subset A$. Therefore $b \in A_0$, a contradiction.
Otherwise if there exists $y \in Y$ such that $y \notin f(A_0) \cup g^{-1}(B_0)$, then $g(y) \notin B_0 \implies g(y) \in A_0$. Let $z = g(y) \in A_0 \cap g(Y)$, then $z \notin X'$ so $z \in h(A_0)$ by C6). Let $z = h(t)$ then $y = f(t) \in f(A_0)$ since g is injective, leading to contradiction.

C8) We define the map $\varphi : X \to Y$. For $x \in X$, we require

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in A_0; \\ g^{-1}(x), & \text{if } x \in B_0. \end{cases}$$

Prove that this is a bijection.
Proof:

1. φ is injective: for any $x, y \in A_0, x \neq y$, $\varphi(x) \neq \varphi(y)$ since f is injective. For any $x, y \in B_0$, $x \neq y$, $\varphi(x) \neq \varphi(y)$ since g is a mapping. For any $x \in A_0, y \in B_0$, $\varphi(x) \neq \varphi(y)$ since $f(A_0) \cap g^{-1}(B_0) = \emptyset$.
2. φ is subjective: $\varphi(X) = \varphi(A_0 \cup B_0) = f(A_0) \cup g^{-1}(B_0) = Y$.

Based on the above, we have proved:
Theorem (Schroeder-Bernstein). If there exist injective maps $f : X \to Y$ and $g : Y \to X$, then there exists a bijection $\varphi : X \to Y$ between the two sets.

1.4 PSD: Details of Dedekind Cut

The goal of this part of the exercise is to complete the part of the Dedekind cut construction method taught in class, thereby providing a complete proof for the construction of real numbers.

D1) Prove that if X and Y are both Dedekind cuts, then the product $X \cdot Y$ as defined in the lecture is also a Dedekind cut, i.e.,

$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (X, Y) \mapsto X \cdot Y,$
is well-defined. (Hint: You only need to prove the case where $X > 0, Y > 0$.)
Proof: The set $X \cdot Y$ is define as $Z = \bar{0} \cup \{x \cdot y : x, y \geqslant 0, x \in X, y \in Y\}$. Let $Z' = \mathbb{Q} - Z$, then

1. $Z \neq \emptyset, Z' \neq \emptyset$, since for any $x \in X', y \in Y'$, $x \cdot y \notin Z$.
2. For any $z \in Z, z' \in Z'$, if $z' < z$ then $z > 0$. So assume $z = x \cdot y, x \in X, y \in Y, x, y \geqslant 0$, then $z' = x \cdot (yz'/z) \in Z$, a contradiction.
3. If Z has a maximal element $z = x \cdot y, x, y \geqslant 0, x \in X, y \in Y$, then since x, y are both not maximal, there exists $x' \in X, y' \in Y$, such that $x < x', y < y'$ so $z < z' = x' \cdot y' \in Z$, a contradiction.

D2) Prove that $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$. ($\implies$ (F5))

Proof: We only need to verify the case where $X, Y, Z > 0$. Then both $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$ are the set

$$\bar{0} \cup \{x \cdot y \cdot z : x, y, z \geqslant 0, x \in X, y \in Y, z \in Z\}.$$

D3) Prove that $X \cdot Y = Y \cdot X$. ($\implies$ (F6))

Proof: Same as D2).

D4) Prove that $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$. ($\implies$ (F9))

Proof: We can assume that $X, Y, Z > 0$, then

$$X \cdot (Y + Z) = \{xy + xz : x \in X, y \in Y, z \in Z\}$$

while

$$X \cdot Y + X \cdot Z = \{xy + x'z : x, x' \in X, y \in Y, Z \in Z\}.$$

Hence $X \cdot (Y + Z) \subset X \cdot Y + X \cdot Z$.
For any $xy + x'z \in X \cdot Y + X \cdot Z$, suppose $x \geqslant x'$, then
$xy + xz \in X \cdot (Y + Z)$ and $xy + x'z \leqslant xy + xz$, so $xy + x'z \in X \cdot Y + X \cdot Z$, therefore $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$.

D5) Prove that $\bar{1} \cdot X = X$ and $\bar{1} \neq \bar{0}$. ($\implies$ (F7))

Proof: Assume that $X > 0$, then $\bar{1} \cdot X = \{u \cdot v : u < 1, v \in X\}$. Foy any $u < 1, v \in X$, $u \cdot v < v$ hence $u \cdot v \in X$. For any $x \in X$, there exists $x' \in X, x' > x$, then $x = x' \cdot (x/x') \in \bar{1} \cdot X$. Therefore $\bar{1} \cdot X = X$ and $1/2 \in \bar{1} \backslash \bar{0}$, so $\bar{1} \neq \bar{0}$.

D6) Prove that if $X \cdot Y = \overline{0}$, then $X = \overline{0}$ or $Y = \overline{0}$; conversely, if $X \geq \overline{0}, Y \geq \overline{0}$, then $X \cdot Y \geq \overline{0}$. ($\implies$ (O5))

Proof: Otherwise there exists $x, x' \in X, y, y' \in Y$, such that $x, y > 0, x', y' < 0$. Hence $xy, x'y \in X \cdot Y$, where $xy > 0 > x'y$, so $X \cdot Y \neq \overline{0}$.
Suppose $X, Y > 0$, then there exists $x \in X, y \in Y$ such that $x, y > 0$, hence $0 < xy \in X \cdot Y$, so $X \cdot Y > \overline{0}$.

D7) X is a positive Dedekind cut. Prove that for any integer n, there exist $x \in X, x' \in X'$ such that

$1 < \frac{x'}{x} < 1 + \frac{1}{n}$.
Proof: Let $l_0 = x \in X, r_0 = x' \in X'$. Define l_n, r_n as follows: If $(l_{n-1} + r_{n-1})/2 \in X$, then $l_n = (l_{n-1} + r_{n-1})/2, r_n = r_{n-1}$, otherwise $l_n = l_{n-1}, r_n = (l_{n-1} + r_{n-1})/2$. Then

$$0 < \frac{r_n - l_n}{l_n} \leqslant \frac{1}{2} \frac{r_{n-1} - l_{n-1}}{l_{n-1}}.$$

Hence there exist such x, x'.

D8) Prove that for any Dedekind cuts X and Y, if $Y \neq \overline{0}$, there exists a unique Dedekind cut Z such that

$Y \cdot Z = X$.
We denote Z as $\frac{X}{Y}$. When $X = \overline{1}$, we also denote it as Y^{-1}. ($\implies$ (F8))
Proof: By D6), Z is unique. By D2) we can assume that $X = \overline{1}$, and $Y > 0$. Let

$$Z = \left\{ \frac{1}{y} : y \in Y' \right\} \cup \overline{0} \cup \{0\}.$$

Then by D7), $Y \cdot Z = \overline{1}$.

2 Homework 2: Cesàro sum

2.1 PSA

A1) $\{x_n\}_{n \geqslant 1}$ is a bounded real sequence. Prove that there is a subsequence $\{x_{n_i}\}_{i \geqslant 1}$ such that $\lim_{i \to \infty} x_{n_i}$ exists and

$$\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n.$$

Proof: Let $M = \limsup_{n \to \infty} x_n < \infty$, then for any $\varepsilon = 1/i > 0$ there exists $N \geqslant n_{i-1}$ such that $M \leqslant \sup_{k \geqslant N} x_k < M + \varepsilon$. Hence there exists $n_i \geqslant N$ such that $x_{n_i} \in (M - \varepsilon, M + \varepsilon)$. Take the sequence $\{x_{n_i}\}_{i \geqslant 1}$ then $\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n$.

A2) $\{x_n\}_{n \geqslant 1}$ is a real sequence. Prove that $\{x_n\}_{n \geqslant 1}$ converges iff $\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n$.

Proof: Since a sub-sequence of a Cauchy sequence converge to the same value as the original sequence, $\implies$ is trivial by A1).
$\impliedby$ $\lim_{n \to \infty} \sup_{k \geqslant n} x_k - \inf_{k \geqslant n} x_k = 0$ implies x_n is Cauchy, hence convergent.

A3) $\{x^{(k)}\}_{k\geqslant 1} \subset \mathbb{R}^n$**, where** $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$**. Then** $\{x^{(k)}\}_{k\geqslant 1}$ **converges in** $\mathbb{R}^n$ **iff for any** $i = 1, 2, \cdots, n$**,** $\{x_i^{(k)}\}_{k\geqslant 1}$ **converges.**

Proof: Use Cauchy sequences and the fact that for $x = (x_1, x_2, \cdots, x_n)$,

$$\max\{|x_k| : 1 \leqslant k \leqslant n\} \leqslant \|x\| \leqslant \sum_{k=1}^{n} |x_k|.$$

A4) Suppose $\{z_n\}_{n\geqslant 1}, \{w_n\}_{n\geqslant 1}$ **are two convergent complex sequences. Prove that if** $\lim_{n\to\infty} w_n \neq 0$**, then the sequence** $\{z_n/w_n\}_{n\geqslant 1}$ **converges.**

Proof: Suppose $z = \lim_{n\to\infty} z_n$ and $w = \lim_{n\to\infty} w_n$, then

$$\left|\frac{z_n}{w_n} - \frac{z}{w}\right| \leqslant \frac{|w| \cdot |z_n - z|}{|w \cdot w_n|} + \frac{|z| \cdot |w_n - w|}{|w \cdot w_n|}.$$

Hence $\left|\frac{z_n}{w_n} - \frac{z}{w}\right| \to 0$, so $\lim_{n\to\infty} z_n/w_n = z/w$.

A5) Suppose $\{a_n\}_{n\geqslant 1}$ **is a monotonically decreasing sequence of positive reals, and** $\lim_{n\to\infty} a_n = 0$**. Prove that the series**

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

converges.

Proof: Suppose $a_n = a_1 - \sum_{k=1}^{n} b_k$, then $b_k \geqslant 0$ and $\sum_{k=1}^{\infty} b_k = a_1$. The series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} b_{2n} < a_1$$

clearly converges.

A6) $\{a_n\}_{n\geqslant 1} \subset \mathbb{C}$**. Prove that if** $\sum_{k=1}^{\infty} |a_k|$ **converges, then** $\sum_{k=1}^{\infty} a_k$ **converges.**

Proof: $\sum_{k=1}^{\infty} |a_k|$ converges implies for any $\varepsilon > 0$, there exists N such that for any $n \geqslant N$, $p \geqslant 0$, $\sum_{k=n}^{n+p} |a_k| < \varepsilon$. Note that $\left|\sum_{k=n}^{n+p} a_k\right| \leqslant \sum_{k=n}^{n+p} |a_k|$, so $\sum_{k=1}^{\infty} a_k$ converges.

A7) Prove that we can define the exponential function on $\mathbb{C}$**:**

$$\exp : \mathbb{C} \to \mathbb{C},\ z \mapsto \exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proof: Use A6).

A8) $\{a_n\} \subset \mathbb{C}$**. Suppose for any** $n \in \mathbb{N}$**,** $a_n \neq 0$**. Let** $P_n = a_1 \cdot a_2 \cdots a_n$**. If** $\lim_{n\to\infty} P_n$ **exists and is not** 0**, we call** $\prod_{n=1}^{\infty} a_n$ **convergent and let** $\prod_{n=1}^{\infty} a_n = \lim_{n\to\infty} P_n$**. Prove that** $\prod_{n=1}^{\infty} a_n$ **converges iff for any** $\varepsilon > 0$**, there exists** N **such that for any** $n \geqslant N$**,** $p \geqslant 0$**,**

$$|a_n \cdot a_{n+1} \cdots a_{n+p} - 1| < \varepsilon.$$

Proof: If $\lim_{n\to\infty} P_n = P$ exists and is non-zero, then for any $\varepsilon > 0$, there exists N such that for any $n \geqslant N$, $|P_n - P| < \varepsilon P/4$ and $|P_n| > P/2$. Then for any $n \geqslant N$, $p \geqslant 0$, $|P_{n+p}/P_n - 1| < \varepsilon$.
If for any $\varepsilon > 0$, there exists N such that for any $n \geqslant N$, $p \geqslant 0$, $|P_{n+p} - P_n| < \varepsilon |P_n|$, then let $\varepsilon = 1$ we infer that P_n is bounded by some constant M. Hence the sequence $\{P_n\}$ is Cauchy, and $P = \lim_{n\to\infty} P_n$ cannot be zero, otherwise there is no such N for $\varepsilon = 1/2$.

A9) Prove that $\exp(x)$ is monotonically increasing on $\mathbb{R}$.

Proof: For $x, y \in \mathbb{R}$,

$$\exp(x) \cdot \exp(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{k=0}^{\infty} \sum_{n+m=k} \frac{x^n y^n \binom{k}{n}}{k!} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = \exp(x+y).$$

$\exp(x) \cdot \exp(-x) = \exp(0) = 1$ implies $\exp(x) > 0$ for all $x \in \mathbb{R}$, so if $x > y$, $\exp(x)/\exp(y) = \exp(x-y) > 1 \implies \exp(x) > \exp(y)$.

A10) Suppose $P(x)$ and $Q(x)$ are polynomials of degree n, m, where $m > n$. Prove that

$$\lim_{n\to\infty} \frac{Q(n)}{P(n)} = 0, \lim_{n\to\infty} \frac{Q(n)}{e^n} = 0.$$

Proof: Suppose $P(x) = \sum_{k=0}^{n} a_k x^k$ and $Q(x) = \sum_{k=0}^{m} b_k x^k$, then there exists N such that for any $x \geqslant N$, $|P(x)| > |a_n| x^n/2$, $|Q(x)| \leqslant \sum_{k=0}^{m} |b_k| \cdot x^m$, and $e^x \geqslant x^{m+1}/(m+1)!$, hence

$$\left|\frac{Q(x)}{P(x)}\right| \leqslant \frac{2\sum_{k=0}^{m} |b_k|}{|a_n|} \cdot x^{m-n} \to 0, \left|\frac{Q(x)}{e^x}\right| \leqslant (m+1)! \sum_{k=0}^{m} |b_k| \cdot x^{-1} \to 0.$$

2.2 PSB: Calculation of Limits

B1)

$$\lim_{n\to\infty} \frac{n+10}{2n-1} = \frac{1}{2}.$$

B2)

$$\lim_{n\to\infty} \frac{\sqrt{n}+10}{2\sqrt{n}-1} = \frac{1}{2}.$$

B3)

$$\lim_{n\to\infty} 0.\underbrace{99\cdots9}_{n \text{ times}} = 1.$$

B4)

$$\lim_{n\to\infty} \frac{1}{n(n+3)} = 0.$$

B5)

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

B6)

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

B7)

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

B8)

$$\lim_{n\to\infty} \sqrt{n+10} - \sqrt{n+1} = 0.$$

B9)

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}.$$

B10)

$$\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{3}.$$

B11) $a > 0$

$$\lim_{n\to\infty} a^{1/n} = 1.$$

B12) $a > 1$

$$\lim_{n\to\infty} \frac{n^{10000}}{a^n} = 0.$$

B13)

$$\lim_{n\to\infty} \frac{2^n+n}{3^n+n^2} = 0.$$

B14)

$$\lim_{n\to\infty} \frac{3^n+2^n}{3^n+n^2} = 1.$$

B15)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{1}{2}.$$

B16) same as B12)

B17)

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1}.$$

B18)

$$\lim_{n\to\infty} \left(1-\frac{1}{5n}\right)^{n+2019} = e^{-1/5}.$$

B19)

$$\lim_{n\to\infty} (n^3+n^2+9n+1)^{1/n} = 1.$$

B20)

$$\lim_{n\to\infty} (2018^n+2019^n)^{1/n} = 2019.$$

2.3 PSC: Riemann Rearrangement Theorem

Suppose $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, we will prove that for and $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$, we can rearrange the sequence such that the new series sums to α. Suppose $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ is a bijection, let $b_k = a_{\varphi(k)}$, then the sequence $\{b_k\}_{k\geqslant 1}$ is called a rearrangement of $\{a_n\}_{n\geqslant 1}$.
Let all non-negative terms of $\{a_n\}_{n\geqslant 1}$, listed in the same order as in $\{a_n\}$ be $c_1, c_2, \cdots$, and the negative terms be $d_1, d_2, \cdots$.

C1) Prove that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$**.**

Proof: Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, c_n, d_n both have infinite terms and $\lim_{n\to\infty} a_n = 0$. Therefore $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$.

C2) Prove that $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} b_n = \infty$**.**

Proof: Since $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent, the two series can not be both convergent. If one converges and the other doesn't, then $\sum_{n=1}^{\infty} a_n$ will diverge. Hence they both diverge.

C3) Prove that for any $\alpha \in \mathbb{R}$, there exists a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} b_k = \alpha$.

Proof: Suppose $\alpha \geqslant 0$. Inductively define the indices u_i and v_i as follows ($u_0 = v_0 = 0$): For $i \geqslant 1$, let u_i be the least index such that $u_i > u_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant \alpha,$$

and v_i be the least index such that $v_i > v_{i-1}$ and

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j \leqslant \alpha.$$

Let φ be the permutation such that

$$b_1 = c_1, b_2 = c_2, \cdots, b_{u_1} = c_{u_1}, b_{u_1+1} = -d_1, \cdots, b_{u_1+v_1} = -d_{u_1}, \cdots$$

Since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} d_n = \infty$, u_i and v_i all exists, so φ is indeed a bijection. By definition we know that

$$\left| \sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j - \alpha \right| \leqslant c_{u_i - 1},$$

and

$$\left| \sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j - \alpha \right| \leqslant d_{v_i - 1}.$$

Since $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$, the two values above both tend to 0. Note that the series $\sum_{n=1}^{\infty} b_n$ is monotonic between these indices, hence $\sum_{n=1}^{\infty} b_n = \alpha$.

C4) Prove that there exists a rearrangement $\{x_k\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} x_k = \infty$.

Proof: Define u_i and v_i as in C3), such that

$$\sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_{i-1}} d_j \geqslant i \geqslant \sum_{j=1}^{u_i} c_j - \sum_{j=1}^{v_i} d_j.$$

Same as C3) define the sequence x_k and clearly $\sum_{n=1}^{\infty} x_k = \infty$.

2.4 PSD: Cesàro Sum

For a real sequence $\{a_n\}_{n \geqslant 1}$, let $\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$.

D1) Suppose $\lim_{n\to\infty} a_n = a$, prove that $\lim_{n\to\infty} \sigma_n = a$.

Proof: For any $n > 0$,

$$|\sigma_n - a| \leqslant \sum_{i=1}^{N} \frac{|a_i - a|}{n} + \sum_{i=N+1}^{n} \frac{|a_i - a|}{n} \leqslant \frac{MN}{n} + \varepsilon(N),$$

where $M = |a| + \sup_{i \leqslant N} |a_i|$, and $\varepsilon(N) = \sup_{i > N} |a_i - a|$. By $\lim_{n\to\infty} a_n = a$ we know $\varepsilon(N) \to 0$, hence $\lim_{n\to\infty} \sigma_n = a$.

D2) Construct a divergent sequence $\{a_n\}$ such that $\lim_{n\to\infty}\sigma_n = 0$.

Solution: $a_n = (-1)^{n-1}$, $\sigma_n \in [0, 1/n]$.

D3) Determine whether there exists $\{a_n\}_{n\geqslant 1}$ such that for any $n \geqslant 1$, $a_n > 0$ and $\limsup_{n\to\infty} a_n = \infty$ but $\lim_{n\to\infty}\sigma_n = 0$.

Solution: Let

$$a_n = \begin{cases} 2^{-n}, & n \neq 2^k, \\ k, & n = 2^k. \end{cases}$$

Then $\limsup_{n\to\infty} a_n = \infty$ and $a_n > 0$, but for any n, suppose $n \in [2^{k-1}, 2^k]$, then

$$\sigma_n \leqslant \frac{1}{n}\cdot\left(1 + \frac{k(k+1)}{2}\right) \leqslant \frac{k(k+1)}{2^{k-1}}.$$

Hence $\lim_{n\to\infty}\sigma_n = 0$.

D4) For $k \geqslant 1$, denote $b_k = a_{k+1} - a_k$. Prove that for any $n \geqslant 2$, $a_n - \sigma_n = \sum_{k=1}^{n-1} kb_k/n$.

Proof:

$$\sum_{k=1}^{n-1} kb_k = \sum_{k=1}^{n-1} k(a_{k+1} - a_k) = (n-1)a_n - \sum_{k=1}^{n-1} a_k = n(a_n - \sigma_n).$$

D5) Suppose $\lim_{k\to\infty} kb_k = 0$ and $\{\sigma_n\}_{n\geqslant 1}$ converges. Prove that $\{a_n\}_{n\geqslant 1}$ also converges.

Proof: By D1), $\lim_{k\to\infty} kb_k = 0$ implies

$$\lim_{n\to\infty} a_n - \sigma_n = \lim_{n\to\infty}\frac{\sum_{k=1}^{n-1} kb_k}{n} = \lim_{k\to\infty} kb_k = 0.$$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sigma_n$ exists.

D6) Suppose $\{kb_k\}_{k\geqslant 1}$ is bounded, i.e. $b_k = O(k^{-1})$, and $\lim_{n\to\infty}\sigma_n = \sigma$. Prove that $\lim_{n\to\infty} a_n = \sigma$.

Proof: Note that for $m < n$,

$$a_n - \sigma_n = \frac{m}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n} a_n - a_k.$$

Therefore since σ_n is a Cauchy sequence, and $|a_n - a_k| \leqslant M(n-k)/k$, we can choose n, m to show that $\lim_{n\to\infty} a_n - \sigma_n = 0$.

2.5 PSE: Definition of $\sqrt[n]{x}$ and b^x

E1) Given $n \in \mathbb{N}$ and $x > 0$, prove that if $y_1, y_2 > 0$ satisfy $y_1^n = x = y_2^n$, then $y_1 = y_2$.

Proof: Note that $y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1} > 0$, and

$$0 = y_1^n - y_2^n = (y_1 - y_2)\cdot(y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2^{n-1}).$$

Hence $y_1 = y_2$.

E2) Prove that if $x > 0$, then the set $E(x) = \{t \in \mathbb{R} : t^n < x\}$ is non-empty and has an upper-bound.

Proof: Note that $0 \in E(x)$ and $E(x)$ has the upper-bound $\max\{1, x\}$.

E3) Prove that $y = \sup E(x)$ satisfy $y^n = x$ and $y > 0$.

Proof: $y = \sup E(x) \implies y^n = x$ since t^n is continuous on $\mathbb{R}$, and $y^n = x$ and $0 \in E(x)$ implies $y > 0$.

E4) Prove that the mapping $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto \sqrt[n]{x} = y$ is well-defined. Denote $\sqrt[n]{x}$ as $x^{1/n}$.

Proof: Use E3).

E5) Prove the $\sqrt[n]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a bijection.

Proof: By E1) it is injective, and $\sqrt[n]{y^n} = y$ implies it is surjective. Hence it is a bijection.

E6) $a, b > 0$, $n \in \mathbb{N}$, prove that $(ab)^{1/n} = a^{1/n}b^{1/n}$.

Proof: Use E5) and $(xy)^n = x^n y^n$.

E7) Suppose $b > 1$, $m, n, p, q \in \mathbb{Z}$ where $n, q > 0$. Let $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Proof: Use $(b^m)^q = (b^p)^n$ and E5).

E8) Prove that for any $r \in \mathbb{Q}$, $r \mapsto b^r$ is well-defined.

Proof: For $r = p/q$, where $q > 0, \gcd(p, q) = 1$, let $b^r = (b^p)^{1/q}$, then for any $r = m/n$, $b^r = (b^m)^{1/n}$.

E9) Prove that for $r, s \in \mathbb{Q}$, $b^{r+s} = b^r b^s$.

Proof: Suppose $r = p/q, s = m/n$, where $n, q > 0$, then

$$b^{r+s} = b^{(mq+np)/nq} = (b^{mq} \cdot b^{np})^{1/nq} = (b^m)^{1/n} \cdot (b^p)^{1/q} = b^r b^s.$$

E10) For $x \in \mathbb{R}$, let $B(x) = \{b^t : t \in \mathbb{Q}, t \leqslant x\}$. Prove that $B(x)$ is non-empty and has an upper-bound. Define $b^x = \sup B(x)$.

Proof: $B(x)$ is clearly non-empty and bounded by $b^{\lfloor x \rfloor + 1}$.

E11) Prove that if $r \in \mathbb{Q}$, then

$$b^r = \sup B(r), \forall r \in \mathbb{Q}.$$

Proof: $b^r \in B(r)$ and since b^t is monotonically increasing, $b^r \geqslant \sup B(r)$, hence $b^r = \sup B(r)$.

E12) Prove that for any $x, y \in \mathbb{R}$, $b^{x+y} = b^x b^y$.

Proof: For any $b^t \in B(x)$, $b^s \in B(y)$, $t \leqslant x$ and $s \leqslant y$, so $t + s \leqslant x + y$ and $b^{t+s} \in B(x+y)$, hence $b^{x+y} \geqslant b^x b^y$. For any $b^t \in B(x+y)$, t can be written in the form $t = u+v$ where $b^u \in B(x)$, $b^v \in B(y)$, so $b^{x+y} \leqslant b^x b^y$.

E13*) Prove that when $b = e$, the function derived from E10) (denoted as e^x) is the same as $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Proof: From $\exp(1) = e$, $\exp(0) = 1$ and $\exp(x + y) = \exp(x) \cdot \exp(y)$ we know that for $n \in \mathbb{Z}$, $\exp(n) = e^n$. For $r = p/q \in \mathbb{Q}$,

$$(e^r)^q = e^p = \exp(p) = \exp(r)^q,$$

so by E5) $e^r = \exp(r)$. Since exp is continuous, for any $x \in \mathbb{R}$, $e^x = \exp(x)$.

2.6 PSF

Given $\alpha > 0$ and $x_1 > \sqrt{\alpha}$, we define inductively $\{x_n\}_{n\geqslant 1}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right), n \geqslant 1.$$

F1) Prove that $\{x_n\}$ is monotonically decreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$ (which is defined in E).

Proof: Note that

$$x_{n+1} - x_n = \frac{\alpha - x_n^2}{2x_n}.$$

Hence we can prove by induction that $x_n > \sqrt{\alpha}$ and $x_n > x_{n+1}$. x_n is decreasing and bounded, so $\lim_{n\to\infty} x_n = A$ exists, and $A = (A + \alpha/A)/2$. Therefore $\lim_{n\to\infty} x_n = A = \sqrt{\alpha}$.

F2) Let $\varepsilon_n = x_n - \sqrt{\alpha}$. Prove that $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$.

Proof:

$$\frac{\varepsilon_n^2}{2x_n} = \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = x_{n+1} - \sqrt{\alpha} = \varepsilon_{n+1}.$$

F3) Prove that if $\beta = 2\sqrt{\alpha}$, then $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

Proof: $\varepsilon_{n+1}/\beta < (\varepsilon_n/\beta)^2$, hence $\varepsilon_{n+1} < \beta(\varepsilon_1/\beta)^{2^n}$.

F4) Let $\alpha = 3, x_1 = 2$. Verify that $\varepsilon_1/\beta < 0.1$, $\varepsilon_5 < 4 \cdot 10^{-16}$, $\varepsilon_6 < 4 \cdot 10^{-32}$.

Now we consider $\alpha > 1$ and $y_1 > \sqrt{\alpha}$, and define

$$y_{n+1} = \frac{\alpha + y_n}{1 + y_n} = y_n + \frac{\alpha - y_n^2}{1 + y_n}, n \geqslant 1$$

F6) Prove that $\{y_{2k-1}\}$ is monotonically decreasing.

Proof: Note that

$$y_{n+2} = \frac{\alpha + y_{n+1}}{1 + y_{n+1}} = \frac{\alpha + \frac{\alpha+y_n}{1+y_n}}{1 + \frac{\alpha+y_n}{1+y_n}} = \frac{2\alpha + (\alpha+1)y_n}{(\alpha+1) + 2y_n}$$

hence

$$y_{n+2} - y_n = \frac{2(\alpha - y_n^2)}{(\alpha+1) + 2y_n},\ y_{n+2} - \sqrt{\alpha} = \frac{(\sqrt{\alpha}-1)^2}{(\alpha+1)+2y_n}(y_n - \sqrt{\alpha}).$$

Therefore $y_1 > \sqrt{\alpha}$ implies $\sqrt{\alpha} < y_{2n+1} < y_{2n-1}$.

F7) Prove that $\{y_{2k}\}$ is monotonically increasing.

Proof: $y_2 = (\alpha + y_1)/(1 + y_1) < \sqrt{\alpha}$, so same as F6), $y_{2k} > y_{2k-2}$ and $y_{2k} < \sqrt{\alpha}$.

F8) Prove that $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

Proof: $\{y_{2n-1}\}$ is decreasing and bounded by $\sqrt{\alpha}$, so $\lim_{n\to\infty} y_{2n-1} = A$ exists and $A = (2\alpha + (\alpha + 1)A)/((\alpha+1) + 2A)$, so $A = \sqrt{\alpha}$. Likewise $\lim_{n\to\infty} y_{2n} = \sqrt{\alpha}$, hence $\lim_{n\to\infty} y_n = \sqrt{\alpha}$.

F9) Compare the rates of convergence between x_n and y_n.

Solution: Let $\delta_n = |y_n - \sqrt{\alpha}|$, then $\delta_n \sim c^n \delta_1$, hence x_n converges faster then y_n.

2.7 PSG: Banach-Mazur Game

Alice and Bob are playing a game: Alice selects a closed interval W_1 first, then Bob choose a subinterval L_1 of W_1, such that the length of L_1 is less than half of the length of W_1; they take turns choosing intervals W_n and L_n, such that $L_n \subset W_n \subset L_{n-1}$ and $|L_n| < |W_n|/2 < |L_{n-1}|/4$, obtaining

$$W_1 \supset L_1 \supset W_2 \supset L_2 \supset \cdots \supset W_n \supset L_n \supset \cdots$$

Alice and Bob find that

$$\bigcap_{n\geqslant 1} W_n = \bigcap_{n\geqslant 1} L_n = \{x\}$$

is a real number. If $x \in \mathbb{Q}$ then Alice wins, otherwise Bob wins. Who has a winning strategy?
Solution: Bob will win. We show that if $\mathbb{Q}$ is replaced with any set M that is of first category, Bob can still win.
M can be written as the union of a countable number of nowhere dense sets. Then in every move of Bob, he can choose L_n such that it does not intersect the nth such nowhere dense set. Hence the final number x is not in M.

2.8 Problem H

Consider the set $\mathcal{P} = \{\{p_n\}_{n\geqslant 1} : p_n \in \mathbb{Z}, p_1 \geqslant 2, p_{n+1} \geqslant p_n^2\}$.

H1) For any $p = \{p_n\}_{n\geqslant 1} \in \mathcal{P}$, define the sequence

$$a_n = \prod_{k=1}^{n} \left(1 + \frac{1}{p_k}\right).$$

Prove that $f(p) = \lim_{n\to\infty} a_n$ exists and $f(p) \in (1, 2]$.
Proof: Note that $p_n \geqslant p_1^{2^{n-1}}$, then

$$a_n \leqslant \prod_{k=1}^{n} \left(1 + \frac{1}{p_1^{2^{k-1}}}\right) = \frac{1 - p_1^{-2^n}}{1 - p_1^{-1}} < \frac{1}{1 - p_1^{-1}}.$$

So the sequence $\{a_n\}$ is monotonic and bounded, hence $f(p) = \lim_{n\to\infty} a_n$ exists. Since $a_n \in (1 + 1/p_1, \frac{1}{1-p_1^{-1}})$, we obtain $f(p) \in [1 + 1/p_1, \frac{1}{1-p_1^{-1}}] \subset (1, 2]$.

H2) Prove that $f : \mathcal{P} \to (1, 2]$ is a bijection.

Proof: For any $p = \{p_n\}, q = \{q_n\} \in \mathcal{P}$, if $p \neq q$, take the least k such that $p_k \neq q_k$ and suppose $q_k \geqslant p_k + 1$, then for any $n > k$,

$$a_n = \prod_{t=1}^{n} \left(1 + \frac{1}{p_t}\right) \geqslant \prod_{t=1}^{k} \left(1 + \frac{1}{p_t}\right) \cdot \left(1 + \frac{1}{p_{k+1}}\right)$$
$$b_n = \prod_{t=1}^{n} \left(1 + \frac{1}{q_t}\right) \leqslant \prod_{i=1}^{k-1} \left(1 + \frac{1}{p_t}\right) \cdot \frac{1 - q_k^{-2^{n-k}}}{1 - q_k^{-1}}$$

Therefore

$$b_n \leqslant \prod_{t=1}^{k} \left(1 + \frac{1}{p_t}\right) \leqslant (1 + C)a_n$$

for all $n > k$ where $C = p_{k+1}^{-1} > 0$, hence $f(q) \leqslant (1 + C)f(p) < f(p)$, hence f is injective.
For any $x \in (1, 2]$, inductively define $p = \{p_n\} \in \mathcal{P}$ as follows: For any $n \geqslant 1$, Let t be the least integer such that $a_n \leqslant x$ and $t \geqslant p_{n-1}^2$ (clearly such t exists). If $a_n = x$, then let $p_n = t - 1$, $p_m = p_n^{2^{m-n}}$ for all $m > n$, then $f(p) = x$. Otherwise let $p_n = t$. Note that for any n such that $p_n > p_{n-1}^2$,

$$|x - a_n| \leqslant 2^{-2^n},$$

therefore $f(p) = x$, and f is surjective.

H3) Prove that $\mathcal{P}$ is uncountable.

Proof: By H2) and the fact that $(1, 2]$ is uncountable.

2.9 Problem I: Binary Expansion

Consider the set $\mathcal{S} = \{\{s_n\}_{n\geqslant 0} : s_n \in \{-1, 1\}\}$.

I1) For any $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, define the sequence

$$c_n = \sum_{k=0}^{n} \frac{s_0 s_1 \cdots s_k}{2^k}.$$

Prove that $h(s) = \lim_{n\to\infty} c_n$ exists and $h(s) \in [-2, 2]$.
Proof: $h(s)$ exists since c_n is clearly a Cauchy sequence, and $c_n \in [-2, 2]$ hence $h(s) \in [-2, 2]$.

I2) Prove that $h : \mathcal{S} \to [-2, 2]$ is surjective. Determine whether is is injective.

Proof: Consider any $x \in [-2, 2]$, we can choose s_n such that $|c_n - x| \leqslant 2^{-n}$. Hence there exists $s = \{s_n\} \in \mathcal{S}$ such that $h(s) = \lim_{n\to\infty} c_n = x$, so h is surjective.
Consider $s = \{1, -1, 1, 1, 1, \cdots\} \in \mathcal{S}$ and $s' = \{-1, -1, 1, 1, \cdots\}$, then $h(s) = h(s') = 0$, hence h is not injective.

I3) For $s = \{s_n\}_{n\geqslant 0} \in \mathcal{S}$, prove that

$$2\sin\left(\frac{\pi}{4}c_n\right) = s_0\sqrt{2 + s_1\sqrt{2 + \cdots + s_n\sqrt{2}}}.$$

Proof: We prove by induction on n. The base $n = 0$ is trivial. If the statement holds for $n - 1$, then let $s' = \{s_{n+1}\}_{n\geqslant 0} \in \mathcal{S}$, we have

$$2\sin\left(\frac{\pi}{4}c_n\right) = 2\sin s_0\left(\frac{\pi}{4} + \frac{1}{2}\cdot\frac{\pi}{4}c'_{n-1}\right) = s_0\sqrt{2 + \sin\left(\frac{\pi}{4}c'_{n-1}\right)}.$$

By the induction hypothesis, the statement also holds for n.

I4) Calculate the limit

$$\lim_{n\to\infty}\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}$$

Solution: Consider $s = \{s_n = 1\}_{n\geqslant 0} \in \mathcal{S}$, then $c_n = 2 - 2^n$ hence $\lim_{n\to\infty} 2\sin(\pi c_n/4) = 2$.

2.10 Problem J

Problem: $k \geqslant 2$ is a given integer. Define the sequence $\{a_n\}$ as follows:

$$a_0 > 0 \text{ already given}, a_{n+1} = a_n + a_n^{-1/k}, n \geqslant 0.$$

Calculate $\lim_{n\to\infty} a_n^{k+1}/n^k$.
Solution: It is easy to see that $a_n \to \infty$, hence

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n^{\frac{k+1}{k}}}{n} &= \lim_{n\to\infty} a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}} = \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1 + \frac{a_{n+1} - a_n}{a_n}\right)^{\frac{k+1}{k}} - 1\right) \\ &= \lim_{n\to\infty} a_n^{\frac{k+1}{k}}\left(\left(1 + a_n^{-\frac{k+1}{k}}\right)^{\frac{k+1}{k}} - 1\right) = \frac{k+1}{k}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{a_n^{k+1}}{n^k} = \left(1 + \frac{1}{k}\right)^k.$$

3 Homework 3: Basel Problem

3.1 PSA

A1) Given $f : (a, x_0) \cup (x_0, b) \to \mathbb{R}$, then $\lim_{x\to x_0} f(x)$ exists iff for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in (x_0 - \delta, x_0 + \delta)$, $|f(x_1) - f(x_2)| < \varepsilon$.

Proof: $\Longleftarrow$ Let $x_n = x_0 + 1/n$, then $\{f(x_n)\}$ form a Cauchy sequence, hence $f(x_0) = \lim_{n\to\infty} f(x_n)$ exists. For any $\varepsilon > 0$, there exists $N, \delta > 0$ such that for any $x, y \in (x_0 - \delta, x_0 + \delta), |f(x) - f(y)| < \varepsilon$

and for any $n > N$, $|f(x_n) - f(x_0)| < \varepsilon$, hence let $\delta' = \min\{\delta, 1/N\}$, then for any $x \in (x_0 - \delta', x_0 + \delta')$, $|f(x) - f(x_0)| \leqslant |f(x) - f(x_N)| + |f(x_N) - f(x_0)| < 2\varepsilon$.
Hence $\lim_{x\to x_0} f(x) = f(x_0)$ exists.
$\Longrightarrow$ For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(x_0)| < \varepsilon$, hence for any $x, y \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(y)| < 2\varepsilon$.

A2) Suppose I is an interval (not a point), prove that the linear space $C(I)$ on $\mathbb{R}$ is of infinite dimension.

Proof: $C(I)$ contains the subspace of all polynomials, hence is of infinite dimension.

A3) Suppose $f : X \to Y$ and $g : Y \to Z$ are both continuous, prove that $g \circ f : X \to Z$ is also continuous.

Proof: For any open set $U \in Z$, $g^{-1}(U) \subset Y$ is an open set, and $f^{-1}(g^{-1}(U)) \subset X$ is an open set, hence $(g \circ f)^{-1}(U)$ is an open set in X and therefore $g \circ f$ is continuous on X.

A4) Suppose (X, d_X) and (Y, d_Y) are metric spaces, $f : X \to Y$ is continuous. If d'_X and d_X are equivalent metrics, and so are d'_Y and d_Y, then in the spaces (X, d'_X) and (Y, d'_Y), f is also continuous.

Proof: The topology generated by equivalent metrics are the same.

A5) The mapping $f : X \to \mathbb{R}^n$ can be written in the form

$$f : X \to \mathbb{R}^n,\ x \mapsto f(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

Prove that f is continuous iff f_i is continuous for every $i = 1, 2, \cdots, n$.
Proof: Since f is continuous iff $\forall x_n \to x, f(x_n) \to f(x)$, and $\{x_k = (x_k^{(1)}, \cdots, x_k^{(n)})\}_{k\geqslant 1}$ converges iff every $\{x_k^{(i)}\}_{k\geqslant 1}$ converges, f is continuous iff every f_i is continuous.

A6) Suppose (X, d_X) is a metric space, $(V, \|\cdot\|)$ is a normed linear space. $f : X \to V$ and $g : X \to V$ are continuous mappings. Prove that $f \pm g : X \to V$ is continuous. If $V = \mathbb{C}$ then $f \cdot g : X \to \mathbb{C}$ is continuous. If $V = \mathbb{C}$ and for any $x \in X$, $g(x) \neq 0$, then $f/g : X \to \mathbb{C}$ is continuous.

(Choose one statement to prove.)
Proof: Since for $\{x_n\}, \{y_n\} \subset \mathbb{C}$, $\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$ and if $y_n \neq 0$, then

$$\lim_{n\to\infty} x_n/y_n = \lim_{n\to\infty} x_n / \lim_{n\to\infty} y_n.$$

Hence $f \cdot g$, f/g are both continuous.
For $\{x_n\}, \{y_n\} \subset V$, if $A = \lim_{n\to\infty} x_n$ and $B = \lim_{n\to\infty} y_n$ then

$$\|x_n + y_n - A - B\| \leqslant \|x_n - A\| + \|y_n - B\| \to 0.$$

Hence $f \pm g$ is continuous.

A7) Find all discontinuities of the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 1/q, & \text{if } x = p/q \in \mathbb{Q}, \text{where } q \geqslant 1, (p, q) = 1. \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Solution: For any $x \in \mathbb{Q}$, $f(x) \neq 0$ but for any $\delta > 0$ there exists $y \in (x-\delta, x+\delta)$ such that $y \notin \mathbb{Q}$. Hence $|f(x) - f(y)| = f(x)$, so f is not continuous at x.
For any $x \notin \mathbb{Q}$, and any $\varepsilon > 0$, let $N = \lfloor 1/\varepsilon \rfloor + 1$ and $\delta = \inf_{n \leqslant N} \|xn\|/n$, then for any $y \in (x-\delta, y+\delta)$, if $y \notin \mathbb{Q}$ then $f(x) = f(y) = 0$, if $y = p/q \in \mathbb{Q}$ then $q > N > 1/\varepsilon$, hence $|f(x) - f(y)| = f(y) = 1/q < \varepsilon$. Therefore f is continuous at x iff $x \notin \mathbb{Q}$.

A8) Calculate

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1.$$

A9) Calculate

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{n\to\infty}(1+1/n)^n = e$ and $(1+1/x)^x$ is monotonic on $[100, \infty)$.

A10) Calculate

$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Since $\lim_{x\to\infty}(1-1/x)^x = \lim_{x\to\infty}(1-1/x)^{x-1} = e$.

3.2 PSB

B1) Calculate the following series:

1.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1.$$

2.
$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1} = \frac{1}{2}.$$

3.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{1}{4}.$$

4.
$$\sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1} = \sum_{n=1}^{\infty} \arctan \frac{1}{n} - \arctan \frac{1}{n+1} = \frac{\pi}{4}.$$

5.
$$\sum_{n=0}^{\infty} \frac{(-1)^n + 2}{3^n} = \frac{1}{1+1/3} + \frac{2}{1-1/3} = \frac{3}{4} + 3 = \frac{15}{4}.$$

6.
$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}.$$

7.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \frac{1}{1+1/2} = \frac{2}{3}.$$

8.
$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = 3.$$

9.
$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1.$$

10.
$$\sum_{n=1}^{\infty} \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = 1 - \sqrt{2}.$$

11.
$$\sum_{n=1}^{\infty} \log\left(\frac{n(2n+1)}{(n+1)(2n-1)}\right) = \lim_{n\to\infty} \log\left(\frac{2n+1}{n+1}\right) = \log 2.$$

12.
$$\sum_{n=1}^{\infty} \frac{1}{n(n+m)} = \frac{1}{m}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+m} = \frac{1}{m}\sum_{n=1}^{m} \frac{1}{n}.$$

B2) Determine whether the following series converge:

1.
$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \sqrt{n+1} - 1 = \infty.$$

2.
$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

converges.

3.
$$\sum_{n=2}^{\infty} (\sqrt[n]{n} - 1)^n$$

converges, since $\limsup_{n\to\infty} \sqrt[n]{(\sqrt[n]{n} - 1)^n} = 0 < 1$.

4.
$$\sum_{n=1}^{\infty} \frac{1}{1+x^n}$$

converges if $|x| > 1$ and diverges if $|x| \leqslant 1$.

5.
$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

converges

6.
$$\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+1}\right)^n \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} < 1.$$

converges.
7.
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \geqslant \sum_{n=1}^{\infty} \frac{1}{2n}$$
diverges.
8.
$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\log \log n}} \leqslant C + \sum_{n=100}^{\infty} \frac{1}{n^2}$$
converges.
9.
$$\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n}$$
diverges, since
$$\lim_{n\to\infty} \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n} = \exp \lim_{n\to\infty} \left(\frac{\log n}{n} - n \log\left(1 + \frac{1}{n^2}\right)\right) = 1.$$
10.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$$
converges (conditionally), since the partial sum of $(-1)^{n-1}$ is bounded and $\frac{\sqrt{n}}{n+1}$ monotonically tends to 0.
11.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[n]{n}}$$
diverges since $(-1)^{n-1}n^{-1/n}$ does not tend to 0.
12.
Let $H_n = 1 + 1/2 + \cdots + 1/n$.
$$\sum_{n=1}^{\infty} \frac{H_n \sin nx}{n}$$
converges since the partial sum of $\sin nx$ is bounded and $\frac{H_n}{n}$ monotonically tends to 0.

B3) Determine whether the following series converge (absolutely):

1.
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$
converges since the partial sum of $(-1)^n$ is bounded and $\frac{1}{n\log n}$ monotonically tends to 0, but only conditionally by C3).
2.
$$\sum_{n=2}^{\infty} \frac{\sin(n\pi/4)}{\log n}$$
converges since the partial sum of $\sin(n\pi/4)$ is bounded and $\frac{1}{\log n}$ monotonically tends to 0, but only conditionally since $\sum_{n=2}^{\infty} \frac{1}{\log(4n+2)}$ tends to infinity.
3.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{\sqrt[3]{n}}$$

converges since $\frac{n-1}{(n+1)\sqrt[3]{n}}$ monotonically tends to 0, but only conditionally since $\sum_{n=1}^{\infty} n^{-1/3}$ diverges.
4.
$a > 1$.

$$\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \frac{n^{10}}{a^n}$$

converges absolutely since there exists $C > 0$ such that for $n > C$, $n^{10}a^{-n} \leqslant a^{-n/2}$, and $\sum_{n=1}^{\infty} a^{-n/2}$ converges.

3.3 PSC

Suppose the integer $b \geqslant 2$, $f : [1, \infty) \to \mathbb{R}_{>0}$ is monotonically decreasing.

C1) Prove that

$$(b-1)b^{k-1} f(b^k) \leqslant \sum_{j=b^{k-1}}^{b^k - 1} f(j) \leqslant (b-1)b^{k-1} f(b^{k-1}).$$

Proof: There are $(b-1)b^{k-1}$ integers in $[b^{k-1}, b^k - 1]$, and since f is monotonically decreasing, for any $j \in [b^{k-1}, b^k - 1]$, $f(j) \in [f(b^k), f(b^{k-1})]$.

C2) Prove that the series

$$\sum_{n=1}^{\infty} f(n) \text{ and } \sum_{n=1}^{\infty} b^n f(b^n)$$

converge or diverge simultaneously.
Proof: From C1),

$$\sum_{k=1}^{\infty} (b-1)b^{k-1} f(b^k) \leqslant \sum_{n=1}^{\infty} f(n) = \sum_{k=1}^{\infty} \sum_{j=b^{k-1}}^{b^k - 1} f(j) \leqslant \sum_{k=1}^{\infty} (b-1)b^{k-1} f(b^{k-1}).$$

Therefore the two series converge or diverge simultaneously.

C3) Prove that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

Proof: Consider $f(x) = \frac{1}{x \log x}$ which is monotonically decreasing. Note that

$$\sum_{n=2}^{\infty} 2^n f(2^n) = \sum_{n=2}^{\infty} \frac{1}{n \log 2} = \infty.$$

From C2) we know that $\sum_{n=2}^{\infty} f(n)$ diverges.

C4) Prove that $\sum_{n=100}^{\infty} \frac{1}{n \log n \log\log n}$ diverges.

Proof: Consider $f(x) = \frac{1}{x \log x \log\log x}$ which is monotonically decreasing. From C3),

$$\sum_{n=100}^{\infty} 2^n f(2^n) = \sum_{n=100}^{\infty} \frac{1}{n \log 2 \cdot \log(n \log 2)}$$

diverges. Hence from C2) we know that $\sum_{n=100}^{\infty} f(n)$ diverges.

C5) Prove that $\sum_{n=1}^{\infty} n^{-s}$ converges iff $s > 1$.

Proof: Consider $f(x) = x^{-s}$ which is monotonically decreasing. Note that

$$\sum_{n=1}^{\infty} 2^n f(2^n) = \sum_{n=1}^{\infty} 2^{-n(s-1)} = \frac{2^{1-s}}{1-2^{1-s}}.$$

C6) Suppose $s > 1$, prove that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$ and $\sum_{n=10}^{\infty} \frac{1}{n \log n (\log \log n)^s}$ converges.

Proof: Same as C3) and C4).

3.4 PSD

For $\{a_n\}_{n \geqslant 1} \subset \mathbb{R}$,

- $\alpha \in \mathbb{R}$, if for any $\varepsilon > 0$, there are infinitely many n such that $a_n \in (\alpha - \varepsilon, \alpha + \varepsilon)$, then we call α a limit point of $\{a_n\}_{n \geqslant 1}$.
- Likewise define limit points for $\alpha = \pm\infty$.

D1) Prove that $\alpha \in \mathbb{R}$ is a limit point of $\{a_n\}_{n \geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k \geqslant 1}$ which converges to α.

Proof: $\Longleftarrow$ is trivial. $\Longrightarrow$ Let $\varepsilon = 1/k$ then there exists a_{n_k} such that $|a_{n_k} - \alpha| < \varepsilon$. Hence $\lim_{k \to \infty} a_{n_k} = \alpha$.

D2) Prove that $+\infty$ is a limit point of $\{a_n\}_{n \geqslant 1}$ iff there is a sub-sequence $\{a_{n_k}\}_{k \geqslant 1}$ such that $\lim_{k \to \infty} a_{n_k} = \infty$.

Proof: Same as D1).

D3) Let $E = \{\alpha \in \mathbb{R} \cup \{\pm\infty\} : \alpha$ is a limit point of $\{a_n\}\}$. Prove that $E \neq \emptyset$.

Proof: If $\{a_n\}$ is unbounded, then by D2) $E \cap \{\pm\infty\} \neq 0$. If $\{a_n\}$ is bounded, then by Bolzano-Weierstrass theorem, $E \neq \emptyset$.

D4) Prove that $E \subset \mathbb{R}$ iff $\{a_n\}$ is bounded.

Proof: Use D2)

D5) Suppose $\{a_n\}_{n \geqslant 1}$ is bounded. Prove that $\sup E = \limsup_{n \to \infty} a_n$, $\inf E = \liminf_{n \to \infty} a_n$.

Proof: Let $M = \limsup_{n \to \infty} a_n$, then for any $\varepsilon > 0$, there exists n such that $M \leqslant \sup_{k \geqslant n} a_k < M + \varepsilon$, hence there exists $k \geqslant n$ such that $|a_k - M| < \varepsilon$, so $M \in E$.
For any $\alpha \in E$, there is a sub-sequence $\{a_{n_k}\} \to \alpha$, hence

$$\alpha = \lim_{k \to \infty} a_{n_k} \leqslant \lim_{k \to \infty} \sup_{m \geqslant n_k} a_{n_k} = \limsup_{n \to \infty} a_n = M.$$

Therefore $M = \sup E$. Substitute $a_n \to -a_n$ and we obtain $\inf E = \liminf_{n \to \infty} a_n$.

D6) Suppose $\{a_n\}_{n\geqslant 1}$ is bounded. Let $a^* = \limsup_{n\to\infty} a_n$. Prove that

i) $a^* \in E$, i.e. $\sup E \in E$.
Proof: See the proof of D5).
ii) For any $x > a^*$, there exists $N \in \mathbb{Z}_{\geqslant 1}$ such that for any $n > N$, $a_n < x$.
Proof: If there is an infinite sub-sequence $\{a_{n_k}\}_{k\geqslant 1}$ such that $a_{n_k} \geqslant x$, then $\{a_{n_k}\}$ has a limit point $a' > x > a^*$, contradicting $a^* = \sup E$.

D7) Construct an example of $\{a_n\}_{n\geqslant 1}$ such that $E \cap \mathbb{R} \neq \emptyset$ and $E \not\subset \mathbb{R}$.

Solution: Since $\mathbb{Q}$ is countable, let $\{a_n\}_{n\geqslant 1}$ iterate every element of $\mathbb{Q}$, then $E = \mathbb{R} \cup \{\pm\infty\}$ is an infinite set.

D8) Construct $\{a_n\}_{n\geqslant 1}$ such that E is an infinite set.

Solution: Same as D7).

3.5 PSE: Reciprocal Sum of Primes

Define the ζ-function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

We have proved the formula:

$$\zeta(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}.$$

Prove that the series

$$\sum_{p\in\mathcal{P}} p^{-s}$$

converges when $s > 1$, and diverges when $0 < s \leqslant 1$.
Proof: We know that for $|a_n| < 1$, $\prod_{n=1}^{\infty}(1-a_n)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges. Hence by $\zeta(s)^{-1} = \prod_{p\in\mathcal{P}}(1-p^{-s})$, we obtain $\sum_{p\in\mathcal{P}} p^{-s}$ converges iff $s > 1$.

3.6 PSF: Euler's "Proof" of the Basel Problem

For any $\theta \in \mathbb{R}, n \in \mathbb{Z}$, prove the identity

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta} = \prod_{k=1}^{n}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

Further prove that for any $x \in \mathbb{R}$,

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

Proof: (1) By induction there is a polynomial $P_n(x)$ such that $P_n(\sin\theta) = \sin(2n+1)\theta$ for any $\theta \in \mathbb{R}$ and $\deg P_n = 2n+1$. For any $k = 1, 2, \cdots, n$, and $\theta = \pm k\pi/(2n+1)$, $\sin((2n+1)\theta) = 0$, hence P_n has roots 0 and $\pm\sin(k\pi/(2n+1))$ for $k = 1, 2, \cdots, n$. Since $\deg P_n = 2n+1$,

$$P_n(x) = Cx\prod_{k=1}^{n}\left(1 - \frac{x^2}{\sin^2(k\pi/(2n+1))}\right)$$

for some $C \in \mathbb{R}$. Let $x = \sin\theta$ and consider the derivatives on both sides when $\theta = 0$, then we obtain $C = 2n+1$, therefore

$$\frac{\sin((2n+1)\theta)}{(2n+1)\sin\theta} = \prod_{k=1}^{n}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/(2n+1))}\right).$$

(2) Let $m = 2n+1$. From (1) we know that for any $x \in \mathbb{C}$ and $k < n$, $\sin x = U_k^{(n)} \cdot V_k^{(n)}$, where

$$\begin{aligned} U_k^{(n)} &= m\sin\frac{x}{m}\prod_{j=1}^{k}\left(1 - \frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right), \\ V_k^{(n)} &= \prod_{j=k+1}^{n}\left(1 - \frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right). \end{aligned}$$

Clearly, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} U_k^{(n)} = U_k = x\prod_{j=1}^{k}\left(1 - \frac{x^2}{j^2\pi^2}\right).$$

and for any $x \in \mathbb{C}$ and $j \in \mathbb{N}$,

$$\left|\frac{\sin^2(x/m)}{\sin^2(j\pi/m)}\right| \leqslant \frac{x^2}{4j^2} \cdot K(|x|/m)^2,$$

where $K(x) = \sum_{n=0}^{\infty} |x|^n/(2n+1)!$ is monotonic on $[0,\infty)$ and $K(0) = 1$.
Note that for $\alpha_i \in \mathbb{C}$,

$$\left|1 - \prod_{j=1}^{n}(1-\alpha_n)\right| \leqslant \sum_{j=1}^{n}\left(\sum_{k=1}^{n}|\alpha_k|\right)^j.$$

Hence for any $x \in \mathbb{C}$ and $\varepsilon > 0$, there exists N such that for any $k \geqslant N$, and any $n > k$, $|V_k^{(n)} - 1| < \varepsilon$, since

$$|V_k^{(n)} - 1| \leqslant \sum_{j=1}^{\infty}\left(\sum_{l=k+1}^{\infty}\frac{x^2}{4l^2}K(|x|/m)^2\right)^j \leqslant \sum_{j=1}^{\infty}\left(K(|x|/(2k+1))^2 \cdot \frac{x^2}{k}\right)^j \to 0.$$

i.e. for any $x \in \mathbb{C}$

$$\lim_{k\to\infty}\sup_{n>k}|V_k^{(n)} - 1| = 0.$$

And likewise we know that there is a constant M such that for any $n > k$, $|x| < k$, $|U_k^{(n)}| \leqslant M$.
Therefore for any $x \in \mathbb{C}$,

$$\sin x = x\lim_{n\to\infty}\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2\pi^2}\right) = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Note:
From the formula above, we can formally deduce that

$$\sin(\pi x) = \pi x(1 - \zeta(2)x^2 + \zeta(4)x^4 + \cdots).$$

Compare it to $\sin z = x - x^3/6 + \cdots$, and we get $\zeta(2) = \pi^2/6$.

4 Homework 4: Topology

4.1 PSA: Topology on Metric Spaces

A1) Suppose (X, d_x) and (Y, d_Y) are metric spaces, $f : X \to Y$ is a mapping. Prove that the two following definitions of continuity is equivalent:

- Suppose $x_0 \in X$, if for any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$, then we say f is continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
- Suppose $x_0 \in X$, $y_0 = f(x_0) \in Y$. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $d_X(x, x_0) < \delta$, $x \in X$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, we call f continuous at x_0. If f is continuous at every point $x \in X$, then f is a continuous mapping.
  Proof: 1=>2: If there exists $\varepsilon > 0$ such that for any $n \geqslant 1$, there exists x_n such that $d_X(x_0, x_n) < 1/n$ but $d_Y(f(x_n), f(x_0)) > \varepsilon$, then $\lim_{n\to\infty} x_n = x_0$ but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$, a contradiction.
  2=>1: For any $\{x_n\}_{n\geqslant 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, and any $\varepsilon > 0$, take the corresponding δ and N such that $n > N \implies d(x_n, x_0) < \delta$. Then for any $n > N$, $d(x_n, x_0) < \delta$ so $d(f(x_n), f(x_0)) < \varepsilon$, hence $\lim_{n\to\infty} f(x_n) = f(x_0)$.

A2) (X, d) is a metric space. For any $x \in X$, $r > 0$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. Proved that for any $x \in X$, $r > 0$, if $x' \in B(x, r)$, then there exists $r' > 0$ such that $B(x', r') \subset B(x, r)$.

If $U = \bigcup_{\alpha\in\mathcal{A}} B(x_\alpha, r_\alpha)$, then we call U an open set. Prove that $U \subset X$ is open iff for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$.
Proof: If $x' \in B(x, r)$, let $r' = r - d(x, x')$, then for any $y \in B(x', r')$, $d(x, y) \leqslant d(x, x') + d(x', y) < d(x, x') + r' = r$, hence $y \in B(x, r)$ so $B(x', r') \subset B(x, r)$.
If for any $x \in U$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset U$, then $U = \bigcup_{x\in U} B(x, \delta_x)$ is open.
If U is open then for any $x \in U$, suppose $x \in B(x_\alpha, r_\alpha)$ for some $\alpha \in \mathcal{A}$, then there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset B(x_\alpha, r_\alpha) \subset U$.

A3) Let $\mathcal{T}$ denote all open sets on (X, d). Prove that $\mathcal{T}$ is a topology.

Proof: 1. $\emptyset \in \mathcal{T}$, $X = \bigcup_{x\in X} B(x, 1) \in \mathcal{T}$. 2. If $\{U_\alpha : \alpha \in J\} \subset \mathcal{T}$, where $U_\alpha = \bigcup_{x\in\mathcal{A}_\alpha} B(x, r_{\alpha,x})$ then let $\mathcal{A} = \bigcup_{\alpha\in J} \mathcal{A}_\alpha$,

$$\bigcup_{\alpha\in J} U_\alpha = \bigcup_{x\in\mathcal{A}} B(x, \sup_{\alpha, x\in\mathcal{A}_\alpha} r_{\alpha,x}) \in \mathcal{T}.$$

1. If $U_1, \cdots, U_n \in \mathcal{T}$, where $U_k = \bigcup_{x\in\mathcal{A}_k} B(x, r_{k,x})$, then let $\mathcal{A} = \bigcup_{k=1}^n \mathcal{A}_k$

$$\bigcap_{k=1}^n U_k = \bigcup_{x\in\mathcal{A}} B(x, \min_{x\in\mathcal{A}_\|} r_{k,x}) \in \mathcal{T}.$$

Therefore $\mathcal{T}$ is a topology on X.

A4) (X, d) is a metric space. If $F \subset X$ and F^C is open, then we call F a closed set. Prove that F is closed iff for any sequence $\{x_n\}_{n\geqslant 1} \in F$, if $\lim_{n\to\infty} x_n = x$ then $x \in F$.

Proof: Suppose F is closed, if a sequence $\{x_n\}_{n\geqslant 1}$ satisfy $\lim_{n\to\infty} x_n = x$ and $x \in F^C$, then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset F^C$. However $B(x, \varepsilon) \cap \{x_n\} \neq \emptyset$, leading to contradiction.
If for any sequence $\{x_n\}_{n\geqslant 1}$ such that $\lim_{n\to\infty} x_n = x$, there is $x \in F$, then for any $x \in F^C$, if for any $\varepsilon > 0$ $B(x, \varepsilon) \not\subset F^C$, then for any $n \geqslant 1$, take $x_n \in B(x, \varepsilon) \cap F$. The sequence $\{x_n\}$ has the limit $\lim_{n\to\infty} x_n = x$ but $x \in F^C$, a contradiction. Hence F is closed.

A5) Prove that

1. $\emptyset$ and X are closed sets.
2. Any intersection of closed sets are still closed.
3. Finite unions of closed sets are still closed.
   Proof: Use A3) and de Morgan's theorem.

A6) Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$, then the following statements are equivalent:

1. f is continuous.
2. For any $U \subset Y$ open, $f^{-1}(U)$ is an open set in X.
3. For any $F \subset Y$ closed, $f^{-1}(F)$ is a closed set in X.
   Proof: 1=>2: If f is continuous, then for any $U \subset Y$ open, consider any point $x \in f^{=1}(U)$. Let $y = f(x) \in U$, then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subset U$. Since f is continuous, there exists $\delta > 0$ such that for any $x' \in B(x, \delta)$, $f(x') \in B(y, \varepsilon) \subset U$, hence $B(x, \delta) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is an open set in X.
   2=>1: For any $x \in X$ and $\varepsilon > 0$, consider the open set $U = B(y, \varepsilon)$, where $y = f(x)$. Since $x \in f^{-1}(U)$ and $f^{-1}(U)$ is an open set, there exists $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(U)$, therefore f is continuous.
   2<=>3: Note that $f^{-1}(Y \backslash A) = X \backslash f^{-1}(A)$.

A7) Let A' be the set of limit points of A. Prove that $\bar{A} = A' \cup A$.

Proof: For any closed set $F \supset A$, by A4) we know $A' \subset F$, hence $A' \cup A \subset \bar{A}$. Consider a sequence $\{x_n\}_{n \geqslant 1} \subset A' \cup A$ such that $\lim_{n \to \infty} x_n = x$ exists, for any $n \geqslant 1$ we can find a $y_n \in A$ such that $d(x_n, y_n) \leqslant 2^{-n}$, hence $\lim_{n \to \infty} y_n = \lim_{n \to \infty} x_n = x$ so $x \in A' \cup A$. Therefore $A' \cup A$ is closed, and hence $\bar{A} = A' \cup A$.

A8) Suppose (Y, d_Y) and (Z, d_Z) are metric spaces, define the metric on $Y \times Z$:

$$d_{Y \times Z} : (Y \times Z)^2 \to \mathbb{R}_{\geqslant 0}, \ ((y_1, z_1), (y_2, z_2)) \to \sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2}.$$

Prove that this defines a metric and the projection mappings are continuous:

$$\pi_Y : Y \times Z \to Y, \ (y, z) \mapsto y; \ \pi_Z : Y \times Z \to Z, \ (y, z) \mapsto z.$$

Given a mapping $F : X \to Y \times Z$, then F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.
Proof: $d((y_1, z_1), (y_2, z_2)) = 0 \iff (y_1, z_1) = (y_2, z_2)$, $d((y_1, z_1), (y_2, z_2)) = d((y_2, z_2), (y_1, z_1))$, and $d((y_1, z_1), (y_2, z_2)) \leqslant d((y_1, z_1), (y_3, z_3)) + d((y_3, z_3), (y_2, z_2))$ (since $\sqrt{(x+y)^2 + (u+v)^2} \leqslant \sqrt{x^2 + u^2} + \sqrt{y^2 + v^2}$), hence $d_{Y \times Z}$ is a metric.
Note that $d((y_1, z_1), (y_2, z_2) \geqslant d(y_1, y_2)$, hence π_Y and π_Z are continuous.
$d((y_1, z_1), (y_2, z_2)) \leqslant d(y_1, y_2) + d(z_1, z_2)$, hence F is continuous iff $\pi_Y \circ F$ and $\pi_Z \circ F$ are both continuous.

A9) Prove that the operators $+$ and $\cdot$ on real numbers are continuous.

Proof: For any $(x, y), (u, v) \in \mathbb{R}^2$,

$$|(x + y) - (u + v)| \leqslant |x - u| + |y - v| \leqslant 2|(x, y) - (u, v)|.$$

Hence $+$ is uniformly continuous.

$$|x \cdot y - u \cdot v| \leqslant |x| \cdot |y - v| + |v| \cdot |x - u|.$$

Therefore $\cdot$ is continuous.

A10) Prove that the operators $+$ and $\cdot$ on $\mathbf{M}_n(\mathbb{R})$ are continuous.

Proof: The proof of A9) only uses the properties of norms, and the fact that $\|A \cdot B\| \leqslant \|A\| \cdot \|B\|$. This also holds for the norm $\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|$ on $\mathbf{M}_n(\mathbb{R})$, therefore $+$ and $\cdot$ are continuous on $\mathbf{M}_n(\mathbb{R})$.

A11) Prove that $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

Proof: The mapping $\det : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous, since view as $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ it is a multi-linear mapping. The set $\mathbf{GL}_n(\mathbb{R}) = \det^{-1}(\{x \in \mathbb{R} : x \neq 0\})$, where $\{x \in \mathbb{R} : x \neq 0\}$ is an open set on $\mathbb{R}$, therefore $\mathbf{GL}_n(\mathbb{R})$ is an open set on $\mathbf{M}_n(\mathbb{R})$.

A12) Prove that $\mathrm{Inv} : \mathbf{GL}_n(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{R}), A \mapsto A^{-1}$ is continuous.

Proof: Note that for any $A, B \in \mathbf{GL}_n(\mathbb{R})$,

$$\|A^{-1} - B^{-1}\| \leqslant \frac{\|A - B\|}{\|A\| \cdot \|B\|}.$$

Hence Inv is continuous.

4.2 PSB

Prove the following equalities:

B1) $\lambda > 0$, $\lim_{x \to \infty} \frac{x^n}{e^{\lambda x}} = 0$.

Proof: By definition, for $x > 0$, $e^{\lambda x} \geqslant (\lambda x)^{n+1}/(n+1)!$. Hence for any $\varepsilon > 0$, let $M = \frac{(n+1)!}{\lambda^{n+1}\varepsilon}$, then for any $x > M$,

$$\left|\frac{x^n}{e^{\lambda x}}\right| \leqslant \frac{(n+1)!}{\lambda^{n+1} x} < \varepsilon.$$

Therefore

$$\lim_{x \to \infty} \frac{x^n}{e^{\lambda x}} = 0.$$

B2) $\alpha > 0$, then

$$\lim_{x \to \infty} x^\alpha \log\left(1 + \frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

Proof: If $0 < \alpha < 1$, then for any $\varepsilon > 0$, there exists $\delta = \varepsilon^{1/(\alpha-1)}$ such that for any $x > \delta$,

$$\left|x^\alpha \log\left(1 + \frac{1}{x}\right)\right| \leqslant x^{\alpha - 1} < \varepsilon.$$

If $\alpha > 1$, then for any $\varepsilon > 0$, there exists $\delta = (2\varepsilon)^{1/\alpha-1}$ such that for any $x > \delta$,

$$\left|x^\alpha \log\left(1+\frac{1}{x}\right)\right| \geqslant \frac{x^\alpha}{x+1} \geqslant \frac{1}{2}x^{\alpha-1} > \varepsilon.$$

If $\alpha = 1$, then for any $\varepsilon > 0$, there exists $\delta = 1/\varepsilon$ such that for any $x > \delta$,

$$1-\varepsilon \leqslant \frac{x}{x+1} \leqslant x\log\left(1+\frac{1}{x}\right) \leqslant 1.$$

Therefore

$$\lim_{x\to\infty} x^\alpha \log\left(1+\frac{1}{x}\right) = \begin{cases} \infty, & \alpha > 1; \\ 1, & \alpha = 1; \\ 0, & 0 < \alpha < 1. \end{cases}$$

B3) $\lim\limits_{x\to 0^+} x^{-n}e^{-1/x^2} = 0$**.**

Proof: If $x < 1$, then $e^{-1/x^2} \leqslant e^{-1/x} \leqslant (n+1)!x^{n+1}$, hence for any $\varepsilon > 0$, let $\delta = \varepsilon/(n+1)!$, then for any $x \in (0,\delta)$, $x^{-n}e^{-1/x^2} \leqslant (n+1)!x \leqslant \varepsilon$. Therefore

$$\lim_{x\to 0^+} x^{-n}e^{-1/x^2} = 0.$$

B4) We know that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$**. Calculate**

$$\lim_{x\to 0}\frac{\cos x - 1}{x}, \text{and } \lim_{x\to 0}\frac{\cos x - 1}{x^2/2}.$$

Solution: For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|x| < \delta$, $\sin x \in ((1-\varepsilon)x, (1+\varepsilon)x)$. Hence

$$\left|\frac{\cos x - 1}{x}\right| \leqslant \left|\frac{\sqrt{1-\sin^2 x}-1}{x}\right| \leqslant \left|\frac{\sin^2 x}{x(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (1+\varepsilon)^2 x \leqslant \delta(1+\varepsilon)^2.$$

Therefore

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Likewise

$$\left|\frac{\cos x - 1}{x^2/2} + 1\right| \leqslant \left|\frac{\sin^2 x - x^2(1+\sqrt{1-\sin^2 x})/2}{x^2/2\cdot(\sqrt{1-\sin^2 x}+1)}\right| \leqslant (2\varepsilon + \sqrt{1-\sin^2 x} - 1).$$

Therefore

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2/2} = -1.$$

4.3 PSC: Root of Function:

C1) Prove that $x^3 + 2x - 1 = 0$ **has exactly one root which lies in** $(0,1)$**.**

Proof: Let $f(x) = x^3 + 2x - 1$, then $f(0) = -1$ and $f(1) = 2$, so $f(0) < 0 < f(1)$. Since f is continuous and monotonically increasing on $(0,1)$, there is exactly one root in $(0,1)$.

C2) Suppose $0 \leqslant \lambda < 1$, $b > 0$, determine whether the equation $x - \lambda \sin x = b$ has a solution.

Solution:

C3) Prove that $\sin x = 1/x$ has infinitely many roots.

Proof: For any $n \in \mathbb{N}$, let $x_n = (2n + 1/2)\pi$, $y_n = (2n + 3/2)\pi$, and $f(x) = \sin x - 1/x$, then $f(x_n) = 1 - 1/x_n > 0$, $f(y_n) = -1 - 1/y_n < 0$, therefore f has a root in (x_n, y_n), and hence f has infinitely many roots.

C4) Assume $f \in C([0,2])$ and $f(0) = f(2)$. Prove that $f(x) - f(x+1) = 0$ has a root in $[0,1]$.

Proof: Let $g(x) = f(x) - f(x+1)$, then $g(0) = f(0) - f(1) = -g(1)$ and $g \in C([0,1])$. Therefore g has a root in $[0,1]$.

C5) Prove that $x^3 + 3 = e^x$ has a solution in $\mathbb{R}$.

Proof: Let $f(x) = e^x - x^3 - 3$, then $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, therefore f has a root in $\mathbb{R}$.

C6) Suppose $f : [0,2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$ then there exists $x \in [1,2]$ such that $f(x) = f(x-1)$.

Exactly the same as C4)?

C7) $f : \mathbb{R} \to \mathbb{R}$, Prove that if for any $c \in \mathbb{R}$, $|f^{-1}(c)| = 2$, then f is not continuous.

Proof: If f is continuous on $\mathbb{R}$, suppose $f^{-1}(0) = \{a < b\}$, then $f|_{[a,b]}$ is bounded. Suppose $f\left(\frac{a+b}{2}\right) > 0$, then for any $t \in (a,b)$, $f(t) > 0$ (otherwise $|f^{-1}(0)| > 2$). Consider an arbitrary $M > y = \sup_{x\in[a,b]} f(x)$, and take $t \in f^{-1}(M)$. Assume $t < a$, then $f(t) = M > y/2 > f(a) = 0$, hence there exists $s \in (t,a)$ such that $f(s) = y/2$. However there are at least two elements of $f^{-1}(y/2)$ in (a,b), leading to contradiction.

C8) Suppose the continuous function $f : [a,b] \to \mathbb{R}$ is injective. If $f(a) < f(b)$, prove that f is monotonically increasing.

Proof: Otherwise suppose $f(u) > f(v)$ for some $u < v$. Note that for any $c \in (a,b)$, $f(a) < f(c) < f(b)$, otherwise $f(c) < f(a) \implies \exists d \in (c,b), f(d) = f(a)$, or $f(c) > f(b) \implies \exists d \in (a,c), f(d) = f(b)$. Hence $a < u < v < b$. Likewise consider $u < v < b$ we get $f(u) > f(v) > f(b)$, and by $a < u < v$ we get $f(a) > f(u) > f(v)$, therefore $f(a) > f(b)$, a contradiction.

4.4 PSD: Calculation of Limits

n, m are positive integers.

(1)

$$\lim_{x\to\infty} \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m} = \begin{cases} 0, & m > n, \\ \infty, & m < n, a_0 > 0, \\ -\infty, & m < n, a_0 < 0, \\ \frac{a_0}{b_0}, & m = n. \end{cases}$$

(2) $a > 1, b > 0$

$$\lim_{x\to\infty} \frac{x^b}{a^x} = 0.$$

(3) $a > 0$

$$\lim_{x\to\infty} \frac{\log x}{x^a} = 0.$$

(4) $a > 0$

$$\lim_{x\to 0^+} x^a \log x = 0.$$

(5)

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^{x} = e^3.$$

(6)

$$\lim_{x\to\infty} (x - \sqrt{x^2-a}) = \lim_{x\to\infty} \frac{a}{x+\sqrt{x^2-a}} = 0.$$

(7)

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x-1} = \lim_{x\to\infty} \frac{2}{\sqrt{x+1}-\sqrt{x-1}} = 0.$$

(8)

$$\lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x)-1}{x} = 1+2+3 = 6.$$

(9)

$$\lim_{x\to 1} \frac{x+x^2+\cdots+x^n-n}{x-1} = \frac{n(n+1)}{2}.$$

(10)

$$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \frac{49}{24}.$$

(11)

$$\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-n}{2}.$$

Proof: Note that

$$\begin{aligned}\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)&=\lim_{x\to 1}\frac{m(1+x+\cdots+x^{n-1})-n(1+x+\cdots+x^{m-1})}{(1+x+\cdots+x^{m-1})(1+x+\cdots+x^{n-1})(1-x)}\\&=\frac{1}{mn}\cdot\lim_{x\to 1}\frac{m(x-1+\cdots+x^{n-1}-1)-n(x-1+\cdots+x^{m-1}-1)}{1-x}\\&=\frac{1}{mn}\cdot(-m(1+2+\cdots+(n-1))+n(1+2+\cdots+(m-1)))\\&=\frac{m-n}{2}.\end{aligned}$$

(12)

$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a.$$

(diverges if $a=0$).

(13)

$$\lim_{x\to 1}\frac{x^a-1}{x^b-1}=\frac{a}{b}.$$

(14)

$$\lim_{x\to\infty}(\log x)^{1/x}=\lim_{x\to\infty}e^{(\log\log x)/x}=1.$$

(15) $a,b>0$

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}=\sqrt{ab}.$$

(16)

$$\lim_{x\to\infty}\sqrt[k]{(x+a_1)(x+a_2)\cdots(x+a_k)}-x$$

Proof: Let $y=(x+a_1)(x+a_2)\cdots(x+a_k)$ and $s=a_1+\cdots+a_k$, then

$$\frac{sx^{k-1}}{ky^{(k-1)/k}}\leqslant\sqrt[k]{y}-x=\frac{y-x^k}{y^{(k-1)/k}+\cdots+x^{k-1}}\leqslant\frac{sx^{k-1}+\prod_{i=1}^k(1+a_i)x^{k-2}}{kx^{k-1}}.$$

Therefore

$$\lim_{x\to\infty}\sqrt[k]{y}-x=s=\sum_{i=1}^k a_i.$$

(17)

$$\lim_{x\to 0}\frac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}=2n.$$

(18)

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1.$$

(19)

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x=e.$$

(20) $\alpha>0$**,**

$$\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha>\frac{1}{2},\\ 1, & \alpha=\frac{1}{2},\\ \infty, & \alpha<\frac{1}{2}.\end{cases}$$

(21) $\alpha>0$**,**

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

Proof: Note that for $x\in(0,1)$,

$$x^{1/8-\alpha}\leqslant\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}\leqslant 2x^{1/8-\alpha}.$$

And for any $\varepsilon>0$ there exists $\delta=(1+\varepsilon)\varepsilon$ such that for any $x<\delta$, $\sqrt{x+\sqrt{x+\sqrt{x}}}<\varepsilon x^{1/8}$. Therefore

$$\lim_{x\to 0^+}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^\alpha}=\begin{cases}0, & \alpha<\frac{1}{8},\\ 1, & \alpha=\frac{1}{8},\\ \infty, & \alpha>\frac{1}{8}.\end{cases}$$

4.5 Problem E

Prove that for any $A\subset\mathbb{R}$ that is countable, there exists a monotonic function $f:\mathbb{R}\to\mathbb{R}$, such that the set of discontinuities of f is exactly A.

Proof: Let $A=\{x_1,x_2,\cdots\}$ and $f(x)=\sup\{1-2^n : x_n<x\}$, (define $\sup\emptyset=0$) then f is monotonically increasing and the set of discontinuities is exactly A.

4.6 Problem F

$f : [0,1] \to [0,1]$ is monotonic. Prove that f has a fixed point.
Proof: Otherwise suppose that f has no fixed points. Let $S = \{t \in [0,1] : f(t) > t\}$ and $x = \sup S$. Note that $0 \in S$ so S is non-empty. If $x \in S$, then $f(x) > x$ so $f(f(x)) > f(x)$ (f is monotonic) then $x < f(x) \in S$ which leads to contradiction. If $x \notin S$, then $f(x) < x$. Take $y \in (f(x), x) \cap S$, (y exists since $x = \sup S$) then $f(x) > f(y) > y > f(x)$, a contradiction.

4.7 Problem G

Consider all self-homeomorphisms of $[0,1]$, i.e.

$$\mathrm{Homeo}([0,1]) = \{f : [0,1] \to [0,1] : f \text{ is a continuous bijective}\}$$

We know that for any $f \in \mathrm{Homeo}([0,1])$, $f^{-1} \in \mathrm{Homeo}([0,1])$. Suppose $f, g \in \mathrm{Homeo}([0,1])$ and the only fixed points of f, g are $0, 1$. Prove that there exists $h \in \mathrm{Homeo}([0,1])$, such that

$$h \circ f \circ h^{-1} = g.$$

Proof: Take $x_0 = 1/2$, and let $I_n = [f^n(x_0), f^{n+1}(x_0)]$, $J_n = [g^n(x_0), g^{n+1}(x_0)]$. Note that $(0,1) = \bigcup_{n\in\mathbb{Z}} I_n = \bigcup_{n\in\mathbb{Z}} J_n$. Define $h_0 : I_0 \to J_0$, $x \mapsto kx + b$ such that the line h_0 passes (x_0, x_0) and $(f(x_0), g(x_0))$, i.e. $x \mapsto \frac{g(x_0)-x_0}{f(x_0)-x_0}(x - x_0) + x_0$. Define $h_n : I_n \to J_n$, $x \mapsto g^n \circ f^{-n}(x)$, and $h : [0,1] \to [0,1]$ such that

$$h(x) = \begin{cases} x, & x \in \{0,1\}, \\ h_n(x), & x \in I_n. \end{cases}$$

Then for any $x \in I_n$, $f(x) \in I_{n+1}$ hence $h(f(x)) = g^{n+1} \circ f^{-n}(x) = g(h(x))$. Since h maps I_n to J_n bijectively, h is a bijection on $[0,1]$. For any $x \in I_n \cap I_{n+1}$ the value of h does not depend on which interval we choose, and h is continuous on any interval I_n, therefore h is a continuous bijection.

5 Homework 5: Infinity of Prime

5.1 PSA

A1) Prove that e^x is uniformly continuous on $(-\infty, 0]$ but not on $\mathbb{R}$.

Proof: For $y < x \leqslant 0$ and $|x - y| < \varepsilon$,

$$e^x - e^y = e^y(e^{y-x} - 1) \leqslant e^{\varepsilon} - 1.$$

Hence e^x is uniformly continuous on $(-\infty, 0]$. But for any $\delta > 0$, there exists y and $x = y + \delta$ such that

$$e^x - e^y = e^y \cdot (e^{\delta} - 1) > 1.$$

Therefore e^x is not uniformly continuous on $\mathbb{R}$.

A2) Prove that the function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$, $(x, \alpha) \mapsto x^\alpha$ is continuous on $\mathbb{R}_{>0} \times \mathbb{R}$.

Proof: For $(x, \alpha), (y, \beta)$,

$$|x^\alpha - y^\beta| \leqslant |x^\alpha - y^\alpha| + |y^\alpha - y^\beta|.$$

Since x^α and a^x are both continuous (as functions of x), so is $x^\alpha : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$.

A3) Prove that for any $x, y > 0$ and α, β, $(xy)^\alpha = x^\alpha y^\alpha$, $(x^\alpha)^\beta = x^{\alpha\beta}$, $a^{\log_a x} = x$. If $x > 0, y > 0$, then $a^{x+y} = a^x a^y$, $\log_a(x \cdot y) = \log_a x + \log_a y$.

Proof: See PSE of HW2.

A4) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$ defined on $[0,1]$, where $f_n(x) = x^n$. Prove that for any $a < 1$, $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0,a]$, but $\{f_n(x)\}_{n\geqslant 1}$ does not converge uniformly on $[0,1)$.

Proof: For any $a < 1$, and any $\varepsilon > 0$, let $N = \log_a x$, then for any $n > N$, $f_n(x) < \varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ converges uniformly to 0 on $[0,a]$. Let $\varepsilon = 1/2$, then for any $N \in \mathbb{N}$, there exists $1 > x > 2^{-1/N}$ such that $f_N(x) > \varepsilon$. Hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent on $[0,1)$.

A5) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$. Prove that $\{f_n(x)\}_{n\geqslant 1}$ converges point-wise to 0 on $\mathbb{R}$, but does not converge uniformly.

Proof: For any $x \in \mathbb{R}$, and any $\varepsilon > 0$, there exists $N = 1/(x\varepsilon)$ such that for any $n \geqslant N$,

$$\left|\frac{nx}{1+n^2x^2}\right| \leqslant \frac{1}{|nx|} < \varepsilon.$$

Hence $f_n(x)$ converges to 0 for any $x \in \mathbb{R}$.
Let $\varepsilon = 1/2$, then for any $n \in \mathbb{N}$, there exists $x = 1/n$ such that $f_n(x) = \varepsilon$, so f is not uniformly continuous on $\mathbb{R}$.

A6) Consider the sequence of functions $\{f_n(x)\}_{n\geqslant 1}$, where

$$f_n(x) = \begin{cases} \frac{nx^2}{1+nx}, & x > 0; \\ \frac{nx^3}{1+nx^2}, & x \leqslant 0. \end{cases}$$

Determine the convergence of $\{f_n(x)\}_{n\geqslant 1}$ on $\mathbb{R}$ (both point-wise and uniformly).
Proof: For any $\varepsilon > 0$, let $N = \max\{1/\varepsilon, 1/4\varepsilon^2\}$, then for any $x > 0$ and $n > N$,

$$|f_n(x) - x| = \left|\frac{x}{1+nx}\right| < \frac{1}{n} < \varepsilon.$$

For any $x < 0$,

$$|f_n(x) - x| = \left|\frac{x}{1+nx^2}\right| \leqslant \frac{1}{2\sqrt{n}} < \varepsilon.$$

Hence $\{f_n\}_{n\geqslant 1}$ converges uniformly to x.

A7) Given $\varphi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ such that $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = 0$, φ is continuous and not identically zero. Prove that the sequences $\{f_n(x)\}_{n\geqslant 1}$ and $\{g_n(x)\}_{n\geqslant 1}$ converge point-wise to 0, but uniformly, where $f_n(x) = \varphi(nx)$, $g_n(x) = \varphi(x/n)$.

Proof: Point-wise convergence is trivial. Let $\varepsilon = |\varphi(1)| > 0$, then for any n there exists $x = 1/n > 0$ such that $|f_n(x)| = \varepsilon$, hence $\{f_n(x)\}_{n\geqslant 1}$ is not uniformly convergent. Likewise $\{g_n(x)\}_{n\geqslant 1}$ is not uniformly continuous.

A8) $f \in C([a,b])$. For $n \geqslant 1$, let $a_k = a + (k-1)(b-a)/n$. Define

$$S_n = \sum_{k=1}^{n} \frac{b-a}{n} f(a_k).$$

Prove that $\{S_n\}_{n\geqslant 1}$ converges, and denote this limit by $\int_a^b f$. Further show that the mapping

$$\int_a^b : C([a,b]) \to \mathbb{R},\ f \mapsto \int_a^b f$$

is linear and continuous with metric d_∞ on $C([a,b])$.
Proof: For any $n, m \in \mathbb{N}$, note that $|S_n - S_m| \leqslant |S_n - S_{nm}| + |S_{nm} - S_m|$, and

$$|S_n - S_{nm}| \leqslant \sum_{k=1}^{n} \frac{b-a}{n} \left| f(a_k^{(n)}) - \frac{1}{m}\sum_{j=1}^{m} f(a_{n(k-1)+j}^{(nm)}) \right| \leqslant (b-a) \sup_{|x-y|<1/n} |f(x)-f(y)|.$$

Since f is uniformly continuous on $[a,b]$, the sequence $\{S_n\}_{n\geqslant 1}$ is Cauchy.
Obviously $\int_a^b \cdot$ is linear, and for $f, g \in C([a,b])$,

$$\left| \int_a^b f - \int_a^b g \right| = \lim_{n\to\infty} |S_n(f) - S_n(g)| \leqslant (b-a)\|f-g\|_\infty.$$

Hence $\int_a^b \cdot$ is continuous on $C([a,b])$ with metric d_∞.

A9) For any $f : [a,\infty) \to \mathbb{R}$, suppose f is bounded on any closed interval $[a,b]$, then when the limits in RHS exist,

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f(x+1) - f(x).$$
$$\lim_{x\to\infty} f(x)^{1/x} = \lim_{x\to\infty} \frac{f(x+1)}{f(x)}, \text{if for any } x \in [a,\infty), f(x) \geqslant c > 0.$$

Proof: Suppose $\lim_{x\to\infty} f(x+1) - f(x) = A$, then for any $\varepsilon > 0$ there exists M such that for any $x > M$, $|f(x+1) - f(x) - A| < \varepsilon$, so for any $n \geqslant 1$, $|f(x+n) - f(x) - nA| < n\varepsilon$. Hence

$$\left| \frac{f(n+x)}{n+x} - A \right| \leqslant \left| \frac{f(n+x) - f(x) - nA}{n+x} \right| + \left| \frac{f(x) - xA}{n+x} \right| \leqslant \varepsilon A + \frac{|f(x) - xA|}{n} \to 0.$$

For any $x > M$. Therefore (since f is bounded on any closed interval) there exists N such that for any $x > N$, $|f(x)/x - A| < 2\varepsilon A$, and hence

$$\lim_{x\to\infty} \frac{f(x)}{x} = A = \lim_{x\to\infty} f(x+1) - f(x).$$

Substitute f by $\log f$ and we obtain the second identity.

5.2 PSB: Uniform Continuity

Determine whether the following functions f are uniformly continuous on I:

B1) $f(x) = x^{1/3}$**,** $I = (0,\infty)$

For any $\varepsilon > 0$ and $x - y \in (0,\varepsilon)$,

$$x^{1/3} - y^{1/3} = \frac{x-y}{x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}} \leqslant \frac{\varepsilon}{\varepsilon^{2/3}} = \varepsilon^{1/3}.$$

Hence $f(x)$ is uniformly continuous on I.

B2) $f(x) = \log x$**,** $I = (0, 1)$

For any $\varepsilon > 0$ and $x - y \in (0, \varepsilon)$,

$$\log x - \log y = \log\left(1 + \frac{x-y}{y}\right).$$

When $y \to 0$ and $x - y$ is constant, $\log x - \log y \to \infty$, hence $\log x$ is not uniformly continuous on I.

B3) $f(x) = \cos x^{-1}$**,** $I = (0, 1)$

Note that for $x_n = 1/(2n\pi)$ and $y_n = 1/(2n\pi + \pi)$, $f(x_n) = 1$ and $f(y_n) = -1$. Hence for $\varepsilon = 1$ and any $\delta > 0$, there exists n such that $|x_n - y_n| < \delta$ but $|f(x_n) - f(y_n)| = 2 > \varepsilon$. Therefore f is not uniformly continuous on I.

B4) $f(x) = x \cos x^{-1}$**,** $I = (0, \infty)$

For $x > y > 1$ and $|x - y| < \varepsilon$,

$$\begin{aligned} |x\cos x^{-1} - y\cos y^{-1}| &\leqslant |x-y||\cos x^{-1}| + |y| \cdot |\cos x^{-1} - \cos y^{-1}| \\ &\leqslant \varepsilon + 2|y| \cdot |\sin(x^{-1} + y^{-1})/2 \sin(x^{-1} - y^{-1})/2| \leqslant \varepsilon + \frac{y}{2}\left(\frac{1}{y^2} - \frac{1}{x^2}\right) \leqslant 2\varepsilon. \end{aligned}$$

For $1 > x > y$ and $|x - y| < \varepsilon$,

$$|x\cos x^{-1} - y\cos y^{-1}| \leqslant |x| + |y| < 2\varepsilon.$$

Hence f is uniformly continuous on I.

5.3 PSC: Existence of Limits

C1) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{\log x}{(x-1)^\alpha} = \lim_{t\to 0} \frac{\log(1+t)}{t^\alpha} = \lim_{t\to 0} t^{1-\alpha}$$

exists iff $\alpha \leqslant 1$.

C2) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{e^x - e}{(x-1)^\alpha} = e \lim_{t\to 0} \frac{e^t - 1}{t^\alpha} = \lim_{t\to 0} et^{1-\alpha}.$$

exists iff $\alpha \leqslant 1$.

C3) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{x^x - 1}{(x-1)^\alpha} = \lim_{x\to 1} \frac{x^x(\log x + 1)}{\alpha(x-1)^{\alpha-1}}$$

exists iff $\alpha \leqslant 1$.

C4) $\alpha > 0$**,**

$$\lim_{x\to 1} \frac{\sqrt[3]{1-\sqrt{x}}}{(x-1)^\alpha}$$

exists iff $\alpha \leqslant 1/3$.

C5)

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x}=1.$$

C6)

$$\lim_{x\to 0}\frac{\sqrt{1+x^4}-1}{1-\cos^2 x}=0.$$

C7) $\alpha>0$**,**

$$\lim_{x\to 1}\frac{(x-1)^\alpha}{\sin(\pi x)}$$

exists iff $\alpha\geqslant 1$.

5.4 PSD: Problems on Uniform Continuity

D1) If f is continuous, monotonic and bounded on the open interval I, then f is uniformly continuous on I.

Proof: Otherwise if there exists $\varepsilon>0$ such that for any $\delta>0$ there exists $|x-y|<\delta$ such that $|f(x)-f(y)|>\varepsilon$. We define x_n,y_n inductively as follows: Let $L=\min\{x_1,\cdots,x_{n-1}\}$, $R=\max\{y_1,\cdots,y_{n-1}\}$. Since f is uniformly continuous on $[L,R]$, there exists $\delta>0$ such that for any $|s-t|<\delta$, $|f(s)-f(t)|<\varepsilon$. Hence there exists $x<y$ such that $x,y\notin[L,R]$, $|x-y|<\delta$ and $|f(x)-f(y)|>\varepsilon$. Let $x_n=x,y_n=y$, then (x_n,y_n) are disjoint intervals and $|f(x_n)-f(y_n)|>\varepsilon$. Which contradicts the fact that f is monotonic and bounded. Therefore f is uniformly continuous on I.

D2) I is an interval with finite length. Prove that the function f on I is uniformly continuous iff for any Cauchy sequence $\{x_n\}_{n\geqslant 1}\subset I$, $\{f(x_n)\}_{n\geqslant 1}$ is also a Cauchy sequence.

(f should be continuous, otherwise after changing the value of f at one point, $\{f(x_n)\}$ remains a Cauchy sequence.)
Proof: $\Longrightarrow$ If $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence, then for any $\varepsilon>0$ there exists $\delta>0$ such that for all $|x-y|<\delta$, $|f(x)-f(y)|<\varepsilon$. There exists N such that for all $n,m>N$, $|a_n-a_m|<\delta$, hence $|f(a_n)-f(a_m)|<\varepsilon$, so $\{f(x_n)\}_{n\geqslant 1}$ is a Cauchy sequence.
$\Longleftarrow$ If $I=(a,b)$ is open we can take $x_n\to a$ and define $f(a)=\lim_{n\to\infty}f(x_n)$, hence we can assume that I is closed. Therefore f is uniformly continuous.

D3) f is uniformly continuous on $\mathbb{R}$. Prove that there exists $a,b\in\mathbb{R}_{>0}$ such that for any $x\in\mathbb{R}$,

$$|f(x)|\leqslant a|x|+b.$$

Proof: For $\varepsilon=1$, there exists $\delta>0$ such that $|x-y|<\delta\implies|f(x)-f(y)|<1$. Hence let $C=\sup_{x\in[0,\delta]}|f(x)|$, then $|f(x)|\leqslant C+|x|\cdot(\frac{1}{\delta}+1)$.

D4) Suppose f is uniformly continuous on $[0,\infty)$ and for any $x\in[0,1]$, $\lim_{n\to\infty} f(x+n)=0$. Prove that

$$\lim_{x\to\infty} f(x)=0.$$

If we change the condition to f is continuous, will the statement still hold?
Proof: For any $\varepsilon>0$, there exists $\delta>0$ such that $|x-y|<\delta \implies |f(x)-f(y)|<\varepsilon$. Let $N=\lfloor 1/\delta\rfloor+1$, then for any $1\leqslant n\leqslant N$, there exists M_n such that for all $m>M_n$, $|f(m+n/N)|<\varepsilon$. Let $M=\max\{M_1,\cdots,M_N\}$, then for all $x>M$, there exists $m\in\mathbb{Z}_{>M}$ and $1\leqslant n\leqslant N$ such that $|x-m-n/N|<\delta$. Hence

$$|f(x)|\leqslant\varepsilon+|f(m+n/N)|<2\varepsilon.$$

Therefore $\lim_{x\to\infty} f(x)=0$.

D5) Suppose X is an interval, $f:X\to\mathbb{R}$ is continuous. If there is a constant $L>0$ such that for any $x,y\in X$,

$$|f(x)-f(y)|\leqslant L|x-y|.$$

We say f satisfy the Lipschitz condition on X.

1. Prove that f satisfy the Lipschitz condition implies f is uniformly continuous.
   Proof: For any $\varepsilon>0$, let $\delta=\varepsilon/L$, then for any $|x-y|<\delta$, $|f(x)-f(y)|\leqslant L|x-y|<\varepsilon$.

2. Determine whether the reversed statement holds.
   Consider the function $f(x)=x^{1/2}$, then f is uniformly continuous but $\frac{f(x)-f(y)}{x-y}=\frac{1}{\sqrt{x}+\sqrt{y}}$ is unbounded, hence does not satisfy the Lipschitz condition.

3. If f satisfy the Lipschitz condition on $[a,\infty)$, where $a>0$, prove that $f(x)/x$ is uniformly continuous on $[a,\infty)$.
   Proof: Same as D3), there exists C such that $|f(x)|\leqslant C|x|$ for $x\in[a,\infty)$, then for $a<x<y$,

$$\begin{aligned}\left|\frac{f(x)}{x}-\frac{f(y)}{y}\right| &= \frac{|xf(y)-yf(x)|}{xy}\leqslant\frac{x|f(y)-f(x)|+|f(x)|(y-x)}{xy}\\ &\leqslant\frac{L+C}{y}\cdot|x-y|.\end{aligned}$$

Hence $f(x)/x$ satisfy the Lipschitz condition.

5.5 PSE:

Exactly the same as PSC in HW4?

5.6 PSF: Calculate Limits

F1)

$$\lim_{x\to\pi}\frac{\sin mx}{\sin nx}=\frac{m(-1)^m}{n(-1)^n}.$$

F2)

$$\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}}{x^2}=3.$$

F3)

$$\lim_{x\to\infty} \sin\sqrt{1+x} - \sin\sqrt{x} = 0.$$

Since the function $\sin x$ is uniformly continuous and $\lim_{x\to\infty} \sqrt{1+x} - \sqrt{x} = 0$.

F4)

$$\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1} = \frac{1}{2}.$$

Since $\lim_{x\to 0} x^2/(e^{x^2}-1) = 1$, $\lim_{x\to 0} x\sin x/x^2 = 1$ and $\lim_{x\to 0} 1/(1+\sqrt{1+x\sin x}) = 1/2$.

F5)

$$\lim_{n\to\infty} \sin^{(n)}(x) = 0.$$

Since the sequence $\{a_n = \sin^{(n)}(x)\}_{n\geqslant 1}$ is decreasing and bounded by 0, and its limit A satisfy $A = \sin A$. Therefore $\lim_{n\to\infty} \sin^{(n)}(x) = 0$.

5.7 Problem G

The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfy the following property: for any $\delta > 0$,

$$\lim_{n\to\infty} f(n\delta) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.
Proof: Consider any $\varepsilon > 0$. For any $N \in \mathbb{N}$,

$$A_N = \{\delta > 0 : \forall n \geqslant N, |f(n\delta)| < \varepsilon\}.$$

Then by the continuity of f, A_N is closed, and by $\lim_{n\to\infty} f(n\delta) = 0$ for any $\delta > 0$, $\bigcup_{N\geqslant 1} A_N = \mathbb{R}_{>0}$. Hence by Baire Category Theorem, there exists an $N > 0$ such that $(a,b) \subset A_N$ for some interval (a,b). Let $X = \{x \in \mathbb{R}_{>0} : |f(x)| < \varepsilon\}$, then since $(a,b) \subset A_N$, for any $n \geqslant N$, $(na, nb) \subset X$. Note that when $n > b/(b-a)$, $nb > (n+1)a$, hence there exists $M > 0$ such that $(M,\infty) \subset X$. Therefore $\lim_{x\to\infty} f(x) = 0$.

5.8 Problem H

The continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

1. $\lim_{x\to\infty}(\varphi(x) - x) = \infty$.
2. $\{x \in \mathbb{R} : \varphi(x) = x\}$ is a non-empty finite set.

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f \circ \varphi = f$, then f is constant.
(Probably need the condition $\lim_{x\to -\infty} \varphi(x) - x = -\infty$).
Proof: Suppose $\{x \in \mathbb{R} : \varphi(x) = x\} = \{a_1, \cdots, a_n\}$ where $a_1 < \cdots < a_n$. For any $x \in \mathbb{R}$, we will show that $f(x) \in \{f(a_1), \cdots, f(a_n)\}$ hence f is constant.
If $a_i < x < a_{i+1}$. Suppose $\varphi(x) > x$, then let $x_0 = x$, and inductively define x_k as a point in (a_k, x_{k-1}) such that $\varphi(a_i) = a_i < \varphi(x_k) = x_{k-1} < \varphi(x_{k-1})$. Since φ is continuous and $a_1, \cdots, a_n$ are all the roots of $\varphi(x) = x$, we know that $\varphi(x_k) > x_k$ for all $k \geqslant 0$. The sequence $\{x_k\}_{k\geqslant 0}$ is decreasing and bounded by a_i, hence has a limit A. From $\varphi(x_k) = x_{k-1}$ we know that $\varphi(A) = A$, so $A = a_i$. Note that $f(x_k) = f(\varphi(x_k)) = f(x_{k-1})$, hence $f(x) = f(x_k) = \lim_{k\to\infty} f(x_k) = f(a_i)$. The case $\varphi(x) < x$ is the same, by constructing a sequence which tends to a_{i+1}.

If $x > a_n$, then $\varphi(x) > x$, likewise we can construct a sequence x_k such that $x_{k-1} = \varphi(x_k)$ and $\lim_{k\to\infty} x_k = a_n$. The case $x < a_1$ is the same.
Hence for all $x \in \mathbb{R}$, $f(x) \in \{f(a_1), \cdots, f(a_n)\}$.

5.9 Problem I

The continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ satisfy $\lim_{x\to\infty} f(x)/x = 0$. Suppose $\{a_n\}_{n\geqslant 1}$ is a sequence of non-negative real numbers and the sequence $\{a_n/n\}_{n\geqslant 1}$ is bounded. Prove that $\lim_{n\to\infty} f(a_n)/n = 0$.
Proof: Suppose $\{a_n/n\}$ is bounded by M.
For any $\varepsilon > 0$, we need to find N such that $n \geqslant N \implies |f(a_n)| < \varepsilon n$. For $C > 0$, we can divide n into two parts: If $a_n \leqslant C$, then $|f(a_n)| \leqslant \sup_{x\in[0,C]} |f(x)|$, otherwise $a_n \geqslant C$, then $|f(a_n)| \leqslant \sup_{x\geqslant C} |f(x)/x| \cdot Mn$. Therefore, if we choose $C > 0$ such that $\sup_{x\geqslant C} |f(x)/x| < \varepsilon/M$, and choose N such that $N > \sup_{x\in[0,C]} |f(x)|/\varepsilon$, then for any $n \geqslant N$, $|f(a_n)| < \varepsilon n$, hence

$$\lim_{n\to\infty} \frac{f(a_n)}{n} = 0.$$

5.10 Ex: Proof of the infinity of primes using topology

Proof: Assume otherwise that the set $\mathcal{P}$ of primes is finite. Let $L_{a,b} = \{at + b : t \in \mathbb{Z}\}, \forall (a, b) \in I = \mathbb{Z}_{>0} \times \mathbb{Z}$. Then

$$\mathbb{Z} \subset \bigcup_{b\in\mathbb{Z}} L_{1,b} \subset \bigcup_{(a,b)\in I} L_{a,b} \subset \mathbb{Z} \implies \bigcup_{(a,b)\in I} L_{a,b} = \mathbb{Z}.$$

and for any $x \in \bigcap_{i=1}^n L_{a_i,b_i}$, let $a = \operatorname{lcm}(a_1, \cdots, a_n)$, then

$$x \in L_{a,x} \subset \bigcap_{i=1}^{n} L_{a_i,bi}.$$

Hence $L_{a,b}$ form a base.Consider the topology $\mathcal{T}$ on $\mathbb{Z}$ generated by the base $\{L_{a,b} : (a, b) \in I\}$. Note that

$$L_{a,b} = \mathbb{Z} \backslash \bigcup_{r=1}^{a-1} L_{a,b+r}$$

so $L_{a,b}$ is also closed. Since $\mathcal{P}$ is finite, the set

$$\bigcup_{p\in\mathcal{P}} L_{p,0} = \mathbb{Z} \backslash \{-1, 1\}$$

is closed, hence $\{-1, 1\}$ is open. However, an open set G is the union of $L_{a,b}$ which is infinite, so G is infinite, leading to contradiction.

> Quote:
> As for everything else, so for a mathematical theory: beauty can be perceived but not explained.
> ——A. Cayley

6 Homework 6: Takagi Function

6.1 PSA: Calculating Derivatives

A1) Consider the function

$$f : \mathbb{R} \to \mathbb{R}^n,\ x \mapsto f(x) = (f_1(x), \cdots, f_n(x)).$$

Prove that f is differentiable at x_0 iff every f_k is differentiable at x_0 and

$$f'(x) = (f_1'(x), \cdots, f_n'(x)).$$

Proof: For any $h \in \mathbb{R}$,

$$\|\frac{f(x+h)-f(x)}{h} - (f_1'(x), \cdots, f_n'(x))\|_2 \leqslant n \max_{1\leqslant k\leqslant n} \left\{ \left| \frac{f_k(x+h)-f_k(x)}{h} - f_k'(x) \right| \right\} \to 0.$$

Therefore $f'(x) = (f_1'(x), \cdots, f_n'(x))$.

A2) Consider the function

$$f : \mathbb{R} \to \mathbb{C},\ x \mapsto e^{ix}.$$

Prove by definition, $f'(0) = i$ and $(e^{ix})' = ie^{ix}$.
Proof: For any $h \in \mathbb{R}$,

$$\left| \frac{f(h)-f(0)}{h} - i \right| = \left| \frac{e^{ih}-ih-1}{h} \right| \leqslant \sum_{n=2}^{\infty} \left| \frac{1}{h} \frac{(ih)^n}{n!} \right| \leqslant |h| \to 0.$$

Therefore $f'(0) = i$. For any $x \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} e^{ix} \frac{f(h)-f(0)}{h} = ie^{ix}.$$

Hence $(e^{ix})' = ie^{ix}$.

A3) Calculate the derivatives of $\sin x$ and $\cos x$.

Solution: $\sin x = (e^{ix} - e^{-ix})/2i$, so $(\sin x)' = (e^{ix} + e^{-ix})/2 = \cos x$. Likewise $(\cos x)' = -\sin x$.

A4) Prove Faà di Bruno's formula for $n = 3$.

Proof:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(f \circ g) = f'(g) \cdot g'. \\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f \circ g) = f'(g) \cdot g'' + f''(g) \cdot (g')^2. \\
&\frac{\mathrm{d}^3}{\mathrm{d}x^3}(f \circ g) = f'(g) \cdot g''' + f''(g) \cdot g'' \cdot g' + f'''(g) \cdot (g')^3 + f''(g) \cdot 2g'g''.
\end{aligned}$$

A5) Define the map

$$E : \mathbb{R} \to \mathbb{C} = \mathbb{R}^2,\ \theta \mapsto (\cos\theta, \sin\theta).$$

Prove that the points in $\mathbf{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ can be written in the form $(\sin\theta, \cos\theta)$, i.e. $E(\mathbb{R}) = \mathbf{S}^1$. Calculate $E'(\theta)$ and show that Rolle's mean-value theorem is invalid for E.
Proof: Obviously $E(\mathbb{R}) \subset \mathbf{S}^1$. Consider any $(x,y) \in \mathbf{S}^1$, then $x \in [-1,1]$. Note that $\cos 0 = 1, \cos\pi = -1$, hence there exists $\theta \in [0,\pi]$ such that $\cos\theta = x$, and $|\sin\theta| = |y|$. If $\sin\theta = y$ then $(x,y) = (\cos\theta, \sin\theta) \in E(\mathbb{R})$. Otherwise $(x,y) = (\cos(-\theta), \sin(-\theta)) \in E(\mathbb{R})$, therefore $E(\mathbb{R}) = \mathbf{S}^1$.
By A1) and A3), $E'(\theta) = (-\sin\theta, \cos\theta)$. Since $E'(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ and $E'(\theta) = E'(\theta + 2\pi)$, Rolle's mean-value theorem is invalid.

A6) Calculate the derivatives of the following functions:

(1) $f(x) = a^x$, $a > 0$.

$$f'(x) = (e^{x \log a})' = a^x \log a.$$

(2) $f(x) = \arcsin x$.
Let $y = \arcsin x$, then $x = \sin y$, so $1 = \cos y \cdot y'$, hence

$$f'(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

(3) $f(x) = \arctan x$.
Let $y = \arctan x$, then $x = \tan y$, so $1 = \sec^2 y \cdot y'$, hence

$$f'(x) = \cos^2 y = \frac{1}{1+x^2}.$$

(4) $f(x) = x^{x^x}$, $x > 0$.
Let $y = x^x, z = x^y$, then $\log y = x \log x$, so $y'/y = \log x + 1$, $y' = x^x(1 + \log x)$. $\log z = y \log x$, so $z'/z = y' \log x + y/x = x^x \log x(1 + \log x) + x^{x-1}$. Therefore

$$f'(x) = x^{x^x} \cdot x^x \cdot (\log x + \log^2 x + x^{-1}).$$

(5) $f(x) = \log(\log(\log x))$.

$$f'(x) = \frac{(\log\log x)'}{\log\log x} = \frac{1}{x \log x \log\log x}.$$

(6) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

$$\begin{aligned} f'(x) &= \frac{(x + \sqrt{x+\sqrt{x}})'}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \left(1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right) / 2\sqrt{x+\sqrt{x+\sqrt{x}}} \\ &= \frac{2\sqrt{x+\sqrt{x}} + 1 + 1/2\sqrt{x}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}. \end{aligned}$$

(7) $f(x) = |x|$.
If $x > 0$, $f'(x) = (x)' = 1$. If $x < 0$, $f'(x) = (-x)' = -1$. If $x = 0$, f is not differentiable at x.
(8) $f(x) = \log|x|$.
If $x > 0$, $f'(x) = \frac{1}{x}$. If $x < 0$, $f'(x) = -\frac{1}{x}$. If $x = 0$, f is not differentiable at x.
(9)

$$f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \quad n = 1, 2, \cdots$$

For $x \neq 0$, $f'(x) = nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}$. When $x = 0$,

$$f'(0) = \lim_{h \to 0} h^{n-1} \sin \frac{1}{h} = \begin{cases} 0, & n \geqslant 2; \\ \text{diverges}, & n = 1. \end{cases}$$

A7) Calculate $f^{(3)}(x)$:

(1) $f(x) = \log(x+1)$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\log(x+1) = \frac{2}{(x+1)^3}.$$

(2) $f(x) = x^{-1}\log x$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{\log x}{x} = \frac{11-6\log x}{x^4}.$$

(3) $f(x) = \frac{x^m}{1-x}$, $m \in \mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{x^m}{1-x} = \frac{(m-2)(m-1)mx^{m-3}}{1-x} + \frac{3(m-1)mx^{m-2}}{(1-x)^2} + \frac{6mx^{m-1}}{(1-x)^3} + \frac{6x^m}{(1-x)^4}.$$

(4) $f(x) = x^m e^x$, $m \in \mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^m e^x) = e^x x^{m-3}(m^3 + 3m^2(x-1) + m(3x^2-3x+2) + x^3).$$

(5) $f(x) = e^{ax}\sin(bx)$, $a, b \in \mathbb{R}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{ax}\sin(bx)) = e^{ax}((3a^2b-b^3)\cos(bx) + a(a^2-3b^2)\sin(bx)).$$

(6) $f(x) = e^{-x^2}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-x^2} = -4e^{-x^2}x(2x^2-3).$$

A8) $f'(x_0) > 0$ does not imply f is increasing in a neighborhood of x_0: consider

$$f(x) = \begin{cases} x + 2x^2\sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Prove that $f'(0) > 0$ but for any $\varepsilon > 0$, f is not monotonic on $(-\varepsilon, \varepsilon)$.
Proof:

$$f'(0) = \lim_{h\to 0} 1 + 2h\sin\frac{1}{h} = 1 > 0.$$

However, for any $n \in \mathbb{N}$, let $x_n = \frac{1}{(2n+1/2)\pi}$, $y_n = \frac{1}{(2n-1/2)\pi}$, then

$$f(x_n) = x_n + 2x_n^2,\ f(y_n) = y_n - 2y_n^2.$$

Note that $0 < x_n < y_n$, but

$$f(x_n) - f(y_n) = 2x_n^2 + 2y_n^2 - \pi x_n y_n > 0,$$

i.e. $f(x_n) > f(0), f(y_n)$, therefore f is not monotonic on any $(-\varepsilon, \varepsilon)$.

A9) $A \in \mathbf{M}_n(\mathbb{R})$, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} \det(\mathbf{I}_n + xA).$$

Solution: Let $\Phi(x) = I_n + xA$, then $\Phi(0) = I_n$. Denote $\Phi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$. Note that det is a multi-linear function for n rows, hence by Euler's formula:

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \Phi(t) = \det(\varphi_1'(t), \varphi_2(t), \cdots, \varphi_n(t)) + \cdots + \det(\varphi_1(t), \varphi_2(t), \cdots, \varphi_n'(t)).$$

When $t = 0$, the formula becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \det \Phi(t) = \varphi_{1,1}' + \cdots + \varphi_{n,n}' = \operatorname{tr} \Phi'(0) = \operatorname{tr} A.$$

A10) Prove that the derivation of odd functions are even, and that of even functions are odd.

Proof: If f is an odd function then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x),$$

so f' is even. If f is an even function then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = -\lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = -f'(x),$$

so f' is odd.

A11) Prove that

$$f(x) = \begin{cases} 1/q, & x = \frac{p}{q} \in \mathbb{Q}, q \geqslant 1, \gcd(p, q) = 1; \\ 0, & x \in \mathbb{Q}^C. \end{cases}$$

is nowhere differentiable on $\mathbb{R}$.
Proof: For any $x \in \mathbb{Q}$, $f(x) \neq 0$, but for any $\varepsilon > 0$, there exists $y \in (x - \varepsilon, x + \varepsilon) \cap \mathbb{Q}^C$, such that $f(y) = 0$. Therefore f is not continuous at x, and clearly not differentiable.
Consider any $x \in \mathbb{Q}^C$, there is a sequence of irrational numbers $\{y_n\}_{n \geqslant 1}$ that converges to x, then

$$\lim_{n \to \infty} \frac{f(x) - f(y_n)}{x - y_n} = 0.$$

Choose any sequence of rational numbers $\{r_n = p_n/q_n\}_{n \geqslant 1}$ that converges to x, then

$$\lim_{n \to \infty} \frac{f(x) - f(r_n)}{x - r_n} = \lim_{n \to \infty} \frac{1}{xq_n - p_n} = \infty.$$

Therefore f is nowhere differentiable on $\mathbb{R}$.

6.2 PSB

B1) Define the hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2},\ \tanh x = \frac{\sinh x}{\cosh x}.$$

1. Prove that
   (1) $\cosh^2 x - \sinh^2 x = 1$
   Proof:$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1.$
   (2) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
   Proof: $\sinh x \cosh y + \cosh x \sinh y = \frac{e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}}{4} = \sinh(x+y)$
   (3) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
   Proof: Same as (2).
   (4) $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.
   Proof: $\tanh(x+y) = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

2. Calculate $\sinh'(x)$, $\cosh'(x)$ and $\tanh'(x)$.
   Solution: $\sinh'(x) = \cosh x$, $\cosh'(x) = \sinh x$, $\tanh'(x) = \frac{1}{\cosh^2 x}$.

3. Prove that $\sinh : \mathbb{R} \to \mathbb{R}$ has an inverse $\operatorname{arcsinh} : \mathbb{R} \to \mathbb{R}$ and calculate $\operatorname{arcsinh}'(x)$.
   Proof: $\sinh'(x) = \cosh x > 0$, so sinh is monotonically increasing over $\mathbb{R}$. Also $\lim_{x\to\infty} \sinh x = \infty$, $\lim_{x\to-\infty} \sinh x = -\infty$, therefore $\sinh : \mathbb{R} \to \mathbb{R}$ is a bijection and hence has an inverse.
   $\operatorname{arcsinh}'(x) = \frac{1}{\sqrt{1+x^2}}$.

B2) $a, b \in \mathbb{R}$**,** $a > 0$**. Consider** $f : [-1,1] \to \mathbb{R}$**, where**

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

1. $f \in C([-1,1])$ iff $a > 0$;
   Proof: $f \in C([-1,1])$ iff $\lim_{x\to0} x^a \sin(x^{-b}) = 0$. If $a > 0$ then $|x^a \sin(x^{-b})| \leqslant |x|^a \to 0$. If $a < 0$ then let $x = ((2n+1/2)\pi)^{-1/b}$, when $n \to \infty$, $x \to 0$ but $|x^a \sin(x^{-b})| \to \infty$. If $a = 0$, then let $x = ((2n+1/2)\pi)^{-1/b}$, $|x^a \sin(x^{-b})| = 1$. Therefore $f \in C([-1,1])$ iff $a > 0$.

2. f is differentiable at 0 iff $a > 1$;
   Proof: f is differentiable at 0 iff $\lim_{x\to0} x^{-a} \sin(x^{-b})$ exists. By 1 we know that $a > 1$. ($a = 1$ is invalid since $x = (2n\pi)^{-1/b}$ and $x = ((2n+1/2)\pi)^{-1/b}$ converge to different values.)

3. f' is bounded on $[-1,1]$ iff $a \geqslant 1 + b$;
   Proof: $f'(x) = ax^{a-1} \sin(x^{-b}) + x^a \cos(x^{-b})(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geqslant 1 + b$.

4. $f \in C^1([-1,1])$ iff $a > 1 + b$;
   Proof: $f \in C^1([-1,1])$ iff $f'(0) = 0 = \lim_{x\to0} f'(x)$. By 1 we know it is equivalent to $a > 1 + b$.

5. f' is differentiable at 0 iff $a > 2 + b$;

6. f'' is bounded on $[-1,1]$ iff $a \geqslant 2 + 2b$;

7. $f \in C^2([-1,1])$ iff $a > 2+2b$.
   Proof: 5,6,7 are exactly the same as 2,3,4.

6.3 PSC

If f satisfy $\lim_{x\to x_0} f(x) = 0$ near x_0, we call f an infinitesimal when $x \to x_0$. Likewise when $\lim_{x\to x_0} f(x) = +\infty$ or $\lim_{x\to x_0} f(x) = -\infty$, we call f an infinite quantity when $x \to x_0$.
Suppose f, g are both infinitesimal when $x \to x_0$, and $g(x)$ does not vanish near x_0. We introduce the notations

- if $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$, we say f is an infinitesimal of higher order than g, and denote $f(x) = o(g(x))$, $x \to x_0$;
- If $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \neq 0$, we say f and g are of the same order;
- If $=1$, denote $f \sim g$, $x \to x_0$;
- If $\limsup_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| < +\infty$, denote $f(x) = O(g(x))$, $x \to x_0$.

C1) Suppose $a(x) = o(1)$ when $x \to x_0$, prove that:

(1) $o(a) + o(a) = o(a)$
Proof: If $f, g = o(a)$, then

$$\lim_{x\to x_0} \frac{f(x)+g(x)}{a(x)} = \lim_{x\to x_0} \frac{f(x)}{a(x)} + \lim_{x\to x_0} \frac{g(x)}{a(x)} = 0,$$

hence $f + g = o(a)$.
(2) $co(a) = o(ca)$, $c \in \mathbb{R}$
Proof: If $f = o(a)$, then

$$\lim_{x\to x_0} \frac{cf(x)}{a(x)} = c \lim_{x\to x_0} \frac{f(x)}{a(x)} = 0,$$

hence $cf = o(a) = o(ca)$.
(3) $o(a)^k = o(a^k)$
Proof: If $f = o(a)$ then

$$\lim_{x\to x_0} \frac{f(x)^k}{a(x)^k} = \left(\lim_{x\to x_0} \frac{f(x)}{a(x)}\right)^k = 0,$$

hence $f^k = o(a^k)$.
(4) $1/(1+a) = 1 - a + o(a)$
Proof:

$$\lim_{x\to x_0} \frac{1/(1+a) - 1 + a}{a(x)} = \lim_{x\to x_0} \frac{a(x)}{1+a(x)} = 0,$$

hence $1/(1+a) = 1 - a + o(a)$.

C2) Suppose f, g are infinitesimals when $x \to x_0$, then

1. Prove that $f \sim g \iff f(x) - g(x) = o(g(x))$, $x \to x_0$.
   Proof: $f \sim g \iff \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1 \iff \lim\limits_{x\to x_0} \dfrac{f(x)-g(x)}{g(x)} = 0$, i.e. $f(x) - g(x) = o(g(x))$.

2. If $f \sim cg^k$, we call cg^k the leading term of f. Find the leading terms of the following functions, compared to $x - x_0$ or x:
   (1) $1/\sin \pi x$, $x \to 1$.
   $\frac{1}{\sin \pi x} = -\frac{1}{\pi(x-1)} + o(1)$.
   (2) $\sqrt{1+x} - \sqrt{1-x}$, $x \to 0$.
   $\sqrt{1+x} - \sqrt{1-x} = x + o(x)$.
   (3) $\sin(\sqrt{1+\sqrt{1+\sqrt{x}}} - \sqrt{2})$, $x \to 0^+$.
   $= \dfrac{\sqrt{2}x^{1/2}}{8} + o(x^{1/2})$.
   (4) $\sqrt{1+\tan x} - \sqrt{1-\sin x}$, $x \to 0$.
   $= x + o(x)$.
   (5) $\sqrt{x + \sqrt{x + \sqrt{x}}}$, $x \to 0^+$.
   $= x^{1/8} + o(x^{1/8})$.
   (6) $\sqrt{x + \sqrt{x + \sqrt{x}}}$, $x \to \infty$.
   $=\sqrt{x} + o(\sqrt{x})$.

3. Suppose $f \sim cx^k$, $x \to 0$, i.e. $f(x) = cx^k + o(x^k)$. If $f(x) - c^k$ has a leading term $c'x^{k'}$, we denote $f(x) = cx^k + c'x^{k'} + o(x^{k'})$. Expand the following terms to $o(x^2)$:
   (1) $\sqrt{1+x} - 1$.
   $\sqrt{1+x} - 1 = \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$.
   (2) $(1+x)^{1/m} - 1$, $m \in \mathbb{Z}_{\geqslant 1}$.
   $(1+x)^{1/m} - 1 = \frac{1}{m}x - \frac{m-1}{2m^2}x^2 + o(x^2)$.

6.4 PST: Takagi Function

Define $\psi : [0,1] \to \mathbb{R}$ as

$$\psi(x) = \begin{cases} x, & 0 \leqslant x < \frac{1}{2}; \\ 1 - x, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

For $x \in R$, let $\psi(x) = \psi(\{x\})$, then $\psi \in C(\mathbb{R})$.
Define the Takagi function $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \psi(2^k x),$$

and the partial sum $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{2^k} \psi(2^k x)$.

T1) Prove that $T(x)$ is a well-defined bounded continuous function on $\mathbb{R}$.

Proof: Note that $\psi(x) \in [0, 1/2]$ so the series $\sum_{k=0}^{\infty} 2^{-k} \psi(2^k x)$ converges absolutely, and hence $T(x)$ is well-defined and bounded by $T(x) \in [0,1]$.

T2) For $x \in [0,1]$, suppose $x = \sum_{n=1}^{\infty} a_n 2^{-n}$ is the binary form of x. Let $v_n = \sum_{k=1}^{n} a_k$, and $\sigma_n(y) = a_n + (1 - 2a_n)y$, where $y \in \{0,1\}$. Prove that

$$\psi(2^m x) = \sum_{n=1}^{\infty} \frac{\sigma_{m+1}(a_{m+n})}{2^n}.$$

Proof:

$$\psi(2^m x) = \psi\left(\sum_{n=1}^{\infty} a_n 2^{m-n}\right) = \psi\left(\sum_{n=m+1}^{\infty} a_n 2^{m-n}\right) = \begin{cases} \sum_{n=1}^{\infty} a_{m+n} 2^{-n}, & a_{m+1} = 0; \\ 1 - \sum_{n=1}^{\infty} a_{m+n} 2^{-n}, & a_{m+1} = 1. \end{cases}$$

Therefore

$$\psi(2^m x) = \sigma_n\left(\sum_{n=1}^{\infty} a_{m+n} 2^{-n}\right) = \sum_{n=1}^{\infty} \sigma_{m+1}(a_{m+n}) 2^{-n}.$$

T3) $x = \sum_{n=1}^{\infty} a_n 2^{-n} \in [0,1]$, prove that

$$T(x) = \sum_{n=1}^{\infty} \frac{(1-a_n)v_n + a_n(n-v_n)}{2^n}.$$

Proof: By T2),

$$T(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sigma_{m+1}(a_{m+n}) 2^{-m-n} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \sigma_{m+1}(a_n) 2^{-n} = \sum_{n=1}^{\infty} \frac{(1-a_n)v_n + a_n(n-v_n)}{2^n}.$$

T4) Suppose $x_0 = k_0 2^{-m_0} \in [0,1]$, where $k_0 \in \mathbb{Z}_{\geqslant 1}$ is odd, $m_0 \in \mathbb{Z}_{\geqslant 0}$. Let $h_n = 2^{-n}$, where $n \in \mathbb{Z}_{\geqslant m_0}$. Prove that the sequence $\left\{\frac{T(x+h_n)-T(x)}{h_n}\right\}_{n \geqslant m_0}$ does not converge.

Proof: By T3),

$$\frac{T(x+h_n)-T(x)}{h_n} = \frac{1}{h_n}\left(\frac{n-v_n}{2^n} - \frac{v_n}{2^n}\right) = n - 2\sum_{k=1}^{n} a_k = n - 2 - 2S_2(k_0) \to \infty.$$

T5) $f : I \to \mathbb{R}$, where I is an open interval. If f is differentiable at a, prove that

$$\lim_{(h,h')\to(0,0),h,h'>0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

i.e. it converges for any sequence $(h_n, h'_n) \to (0,0), h_n, h'_n > 0$.
Proof: Consider any sequence $(h_n, h'_n) \to (0,0)$, then

$$\frac{f(a+h)-f(a-h')}{h+h'} = \frac{f(a+h)-f(a)}{h} \cdot \frac{h}{h+h'} + \frac{f(a)-f(a-h')}{h'} \cdot \frac{h'}{h+h'} \to f'(a).$$

T6) Same as T5), if $f \in C^1(I)$, $a \in I$, prove that

$$\lim_{(h,h')\to(0,0),h+h'\neq 0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

Proof: For any $h+h' \neq 0$, there exists $\xi \in [a, a+h]$ and $\eta \in [a-h', a]$ such that $f(a+h) = f(a) + hf'(\xi)$ and $f(a-h') = f(a) - h'f'(\eta)$, then

$$\left|\frac{f(a+h)-f(a-h')}{h+h'} - f'(a)\right| \leqslant \frac{h}{h+h'}|f'(\xi) - f'(a)| + \frac{h'}{h+h'}|f'(\eta) - f'(a)| \to 0.$$

Hence

$$\lim_{(h,h')\to(0,0),h+h'\neq 0} \frac{f(a+h)-f(a-h')}{h+h'} = f'(a).$$

T7) Suppose $x \in [0,1]$, such that for any $n \in \mathbb{N}$, $2^n x \notin \mathbb{Z}$. For every $n \in N$, define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as follows:

$$\lfloor 2^n x \rfloor = 2^n(x - h'_n),\ \lfloor 2^n x \rfloor + 1 = 2^n(x + h_n).$$

Prove that for an arbitrary n, $h_n + h'_n = 2^{-n}$ and for every integer $1 \leqslant \ \leqslant n-1$, the interval $(2(x - h'_n), 2(x + h_n))$ does not include integers or half-integers.
Proof: $1 = 2^n(x+h_n) - 2^n(x-h'_n) = 2^n(h_n + h'_n)$, hence $h_n + h'_n = 2^{-n}$. For any integer $1 \leqslant \ \leqslant n-1$, $2(x - h'_n) = \lfloor 2^n x \rfloor \cdot 2^{-n}$ and $2(x+h_n) = (\lfloor 2^n x \rfloor + 1)2^{-n}$. Since $n - \ \geqslant 1$, the interval does not include integers or half-integers.

T8) Follow the notations of T7), prove that the sequence $\left\{\frac{T(x+h_n)-T(x-h'_n)}{h_n+h'_n}\right\}_{n\geqslant 1}$ diverges.

Proof: Let $t = \lfloor 2^n x \rfloor$, then

$$a_n = \frac{T(x+h_n) - T(x-h'_n)}{h_n + h'_n} = \sum_{k=0}^{n-1} 2^{n-k}\left(\psi\left(\frac{t+1}{2^{n-k}}\right) - \psi\left(\frac{t}{2^{n-k}}\right)\right).$$

Since the interval $(2^{k-n}(t+1), 2^{k-n}t)$ does not contain any integers or half-integers, $2^{n-k}(\psi(2^{k-n}(t+1)) - \psi(2^{k-n}t)) \in \{-1, 1\}$, so $a_n \in \mathbb{Z}$ and n, a_n have the same parity. Therefore the sequence $\{a_n\}_{n\geqslant 1}$ diverges.

T9) Prove that $T(x)$ is continuous but nowhere differentiable on $\mathbb{R}$.

Proof: For any $x \in [0,1]$, if $x = k_0 \cdot 2^{-m_0}$ as in T4), by T4) the sequence $\left\{\dfrac{T(x+h_n) - T(x)}{h_n}\right\}$ diverges, hence T is not differentiable at x. Otherwise for any $n \in \mathbb{N}$, $2^n x \notin \mathbb{Z}$. Define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as in T7), then by T8), the sequence $\left\{\dfrac{T(x+h_n) - T(x-h'_n)}{h_n + h'_n}\right\}_{n\geqslant 1}$ diverges. Combined with T5) we know that T is not differentiable at x. Therefore T is nowhere differentiable on $\mathbb{R}$, since T is periodic.
For any x, y in $\mathbb{R}$,

$$|T(x) - T(y)| \leqslant \sum_{k=0}^{N} 2^{-k}|T(2^k x) - T(2^k y)| + \sum_{k=N+1}^{\infty} 2^{-k} \leqslant 2 \max_{0\leqslant k\leqslant N} |T(2^k x) - T(2^k y)| + 2^{-N}.$$

Hence for any $N > 0$, when $\varepsilon \to 0$, $|T(x) - T(x+\varepsilon)| \leqslant 2^{1-N} \to 0$, so T is (uniformly) continuous on $\mathbb{R}$.

T10) Prove that

$$T(x) = \begin{cases} 2x + \frac{T(4x)}{4}, & 0 \leqslant x < \frac{1}{4}; \\ \frac{1}{2} + \frac{T(4x-1)}{4}, & \frac{1}{4} \leqslant x < \frac{1}{2}; \\ \frac{1}{2} + \frac{T(4x-2)}{4}, & \frac{1}{2} \leqslant x < \frac{3}{4}; \\ 2 - 2x + \frac{T(4x-3)}{4}, & \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

Proof: If $0 \leqslant x < 1/4$, then

$$T(x) = \psi(x) + \psi(2x)/2 + \sum_{k=2}^{\infty} \psi(2^k x)2^{-k} = 2x + \frac{T(4x)}{4}.$$

The other cases are exactly the same.

T11) Let $\Gamma = \{(x, T(x)) : 0 \leqslant x \leqslant 1\} \subset \mathbb{R}^2$. Define $\Phi_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Phi_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \Phi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$
$$\Phi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \qquad \Phi_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix}.$$

Prove that Φ_i maps Γ to $\left\{(x, T(x)) : x \in \left[\frac{i}{4}, \frac{i+1}{4}\right]\right\}$.
Proof: Consider $(x, T(x)) \in \Gamma$, then by T10),

$$\Phi_0 \begin{pmatrix} x \\ T(x) \end{pmatrix} = \begin{pmatrix} x/4 \\ x/2 + T(x)/4 \end{pmatrix} = \begin{pmatrix} x/4 \\ T(x/4) \end{pmatrix}.$$

Hence $\Phi_0(\Gamma) = \{(x, T(x)) : x \in [0, 1/4]\}$. The cases $i = 1, 2, 3$ are similar.

T12) Let $S_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. For every $n \geqslant 0$, define $S_{n+1} = \bigcup_{k=0}^{3} \Phi_k(S_n)$. Prove that S_n is a sequence of monotonically decreasing compact sets and $\Gamma = \bigcap_{n \geqslant 0} S_n$.

Proof: Let $S_n(x) = \{y \in [0, 1] : (x, y) \in S_n\}$. We prove by induction that $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$ for any $x \in [0, 1]$. The base $n = 0$ is trivial. Suppose $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$, then consider S_{n+1}. Note that $\Phi_k(S_n)$ are disjoint, since for any $(x, y) \in \Phi_k(S_n)$, $x \in [k/4, (k+1)/4]$. Hence for any $x \in [0, 1/4]$, $S_{n+1}(x) = \{y : (x, y) = \Phi_0(4x, z), z \in S_n(4x)\} = \{2x + z/4 : z \in S_n(4x)\}$ is a closed interval containing $T(x) = 2x + T(4x)/4$. By the induction hypothesis $S_n(x) = \{2x + z/4 : z \in S_{n-1}(4x)\}$ and $S_n(4x) \subset S_{n-1}(4x)$ so $S_{n+1}(x) \subset S_n(x)$. The case $x \in [1/4, 1]$ is similar. Therefore $S_{n+1} \subset S_n$ and S_{n+1} is compact, so by induction $S_n \subset S_{n-1}$ for all $n > 0$ and S_n is compact.
Clearly $\Gamma \subset \bigcap_{n \geqslant 0} S_n$, so it suffices to show that $|S_n(x)| \to 0$ for all $x \in [0, 1]$. From the proof above we get $\sup_{x \in [0,1]} |S_n(x)| \leqslant \sup_{x \in [0,1]} |S_{n-1}(x)|/4$, hence $|S_n(x)| \to 0$, therefore

$$\Gamma = \bigcap_{n \geqslant 0} S_n.$$

T13) Prove that $\sup_{x \in \mathbb{R}} T(x) \geqslant \frac{2}{3}$.

Proof: For any $(x, y) \in \Gamma$, by T11) we know that $(x/4 + 1/4, y/4 + 1/2) \in \Gamma$, hence if $a = \sup_{x \in \mathbb{R}} T(x)$ then $a \geqslant a/4 + 1/2$, i.e. $a \geqslant 2/3$.

T14) Find a $c \in [0, 1]$ such that $T(c) = \frac{2}{3}$.

Solution: Consider $c = 1/3$, then by T10), $T(c) = T(c)/4 + 1/2$, hence $T(c) = \frac{2}{3}$.

T15) For $x \in [0, 1]$, suppose $x = \sum_{n=1}^{\infty} b_n 4^{-n}$, where $b_n \in \{0, 1, 2, 3\}$. Prove that

$$\left\{x \in [0, 1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0, 1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1, 2\}\right\}.$$

Proof: If $x = \sum_{n=1}^{\infty} b_n 4^{-n}$, where $b_n \in \{1, 2\}$, then by T10),

$$T(x) = \frac{1}{2} + \frac{1}{4} T\left(\sum_{n=1}^{\infty} b_{n+1} 4^{-n}\right) = \cdots = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) = \frac{2}{3}.$$

Otherwise take the least n such that $b_n \in \{0,3\}$, denote $y = \sum_{k=1}^{\infty} b_{n+k-1}4^{-n}$, then

$$T(x) = \frac{1}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-2}}\right) + \frac{\min\{2y, 2-2y\}}{4^{n-1}} + \frac{1}{4^n}T(4y - b_n) < \frac{2}{3},$$

since $T(4y - b_n) \leqslant 2/3$ and $\min\{2y, 2-2y\} < 1/2$. Therefore

$$\left\{x \in [0,1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0,1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1,2\}\right\}.$$

T16) As in T11), study the self-similarity of Φ_1, Φ_2 on $\left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, which is a cantor set of Hausdorff dimension $\frac{1}{2}$.

Solution: Same as T11), denote $\Gamma' = \left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, then

$$\Phi_1(\Gamma') = \left\{(x, T(x)) : x \in \left[0, \frac{1}{2}\right], T(x) = \frac{2}{3}\right\},\ \Phi_2(\Gamma') = \left\{(x, T(x)) : x \in \left[\frac{1}{2}, 1\right], T(x) = \frac{2}{3}\right\}.$$

7 Homework 7: Émile Borel Lemma

7.1 PSA

f is a function on the interval I.

A1) Suppose f is twice-differentiable at x, prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Proof: For any $h > 0$, consider the function $g(t) = f(t) - f(t-h)$, then there exists $\xi \in [0,h]$ such that $g(x+h) = g(x) + hg'(\xi)$, and there exists $\eta \in [\xi - h, \xi] \subset [-h, h]$ such that $f'(\xi) - f'(\xi - h) = hf''(\eta)$

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \frac{f'(\xi) - f'(\xi - h)}{h} = f''(\eta),$$

therefore

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

A2) Suppose $x_0 \in I$, and

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + o(|x - x_0|^n) \\ &= b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + o(|x - x_0|^n). \end{aligned}$$

when $x \to x_0$, then for any $i = 0, 1, \cdots, n$, $a_i = b_i$.
Proof: Otherwise let $c_i = a_i - b_i$ and take the least k such that $c_k \neq 0$, then

$$c_k(x-x_0)^k + \cdots + c_n(x-x_0)^n + o(|x-x_0|^n) = 0 \implies c_k = -c_{k+1}(x-x_0) - \cdots - c_n(x-x_0)^{n-k} + o(|x-x_0|^{n-k}),$$

which leads to contradiction when $x \to x_0$.

A3) Suppose f is n-times differentiable at 0. Prove that if f is an even (odd) function, then the Taylor expansion of f at 0 has only even (odd) terms.

Proof: Use the fact that if f is even (odd) then f' is odd (even).

A4) If f is differentiable on (a, b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ prove that exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof: Otherwise if $f'(x) \neq 0$ for all $x \in (a, b)$, by Darboux's theorem $f'(x)$ have the same sign over (a, b), hence f is monotonic and non-constant on (a, b), contradicting $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$.

A5) Suppose $f \in C([a, b])$ and is differentiable on (a, b). Prove that f is strictly increasing on $[a, b]$ iff for any $x \in (a, b)$, $f'(x) \geqslant 0$ and on any sub-interval $(c, d) \subset (a, b)$, $f'(x)$ does not vanish.

Proof: $\Longrightarrow$ For any $x \in (a, b)$, $(f(x + h) - f(x))/h \geqslant 0$ so

$$f'(x) = \lim_{h\to\infty} \frac{f(x + h) - f(x)}{h} \geqslant 0.$$

If $f'(x)$ vanish on some sub-interval (c, d) then $f(c) = f(d)$, a contradiction.
$\Longleftarrow$ For any $a \leqslant x < y \leqslant b$, there exists $\xi \in (a, b)$ such that $f(y) - f(x) = (y - x)f'(\xi)$, hence $f(y) \geqslant f(x)$ and f is increasing. If $f(x) = f(y)$ for some $x < y$ then $f(t)$ is constant on $[x, y]$ and hence f' vanish on (x, y), a contradiction.

7.2 PSB

Use L'Hôpital theorem to calculate limits:

B1) $a > 0$, then

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to\infty} \frac{x^{-1}}{ax^{a-1}} = 0.$$

B2) $a > 0, b > 1$ then

$$\lim_{x\to\infty} \frac{x^a}{b^x} = \lim_{x\to\infty} \frac{ax^{a-1}}{b^x \ln b} = \cdots = \lim_{x\to\infty} \frac{a(a-1)\cdots\{a\}}{b^x(\ln b)^{\lfloor a\rfloor} x^{1-\{a\}}} = 0.$$

B3)

$$\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx} = \lim_{x\to 0} \frac{ae^{ax} - be^{bx}}{a\cos ax - b\cos bx} = 1.$$

B4)

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x\to 0} \frac{1 + \cos x}{\cos^2 x} = 2.$$

B5)

$$\lim_{x\to0}\frac{1-\cos x^2}{x^2\sin x^2}=\lim_{x\to0}\frac{1-\cos x}{x\sin x}=\lim_{x\to0}\frac{\sin x}{\sin x+x\cos x}=\lim_{x\to0}\frac{\cos x}{2\cos x-x\sin x}=\frac{1}{2}.$$

B6)

$$\lim_{x\to1}\frac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{4/3}}=\lim_{x\to1}\frac{(2x-x^4)^{-1/2}(1-2x^3)-x^{-2/3}/3}{-\frac{4}{3}x^{1/3}}=1.$$

B7)

$$\lim_{x\to1^-}(\log x)(\log(1-x))=\lim_{x\to1^-}\frac{\log(1-x)}{1/\log x}=\lim_{x\to1^-}\frac{x\log^2 x}{1-x}=0.$$

B8)

$$\lim_{x\to0^+}\frac{\log\sin ax}{\log\sin bx}=\lim_{x\to0^+}\frac{\sin bx}{\sin ax}\cdot\frac{a\cos ax}{b\cos bx}=1.$$

B9)

$$\lim_{x\to0^+}x^x=\exp\lim_{x\to0^+}\frac{\log x}{x^{-1}}=\exp\lim_{x\to0^+}-x=1.$$

B10)

$$\lim_{x\to1}x^{1/(1-x)}=\exp\lim_{x\to1}\frac{\log x}{1-x}=e^{-1}.$$

B11)

$$\begin{aligned}\lim_{x\to1}\left(\frac{1}{\log x}-\frac{1}{x-1}\right)&=\lim_{x\to1}\frac{x-1-\log x}{(x-1)\log x}=\lim_{x\to1}\frac{1-x^{-1}}{1-x^{-1}+\log x}\\&=\lim_{x\to1}\frac{x-1}{x-1+x\log x}=\frac{1}{2}.\end{aligned}$$

B12)

$$\lim_{x\to0^+}(\sin x)^x=\exp\lim_{x\to0^+}\frac{\log\sin x}{x^{-1}}=\exp\lim_{x\to0^+}-\frac{x^2}{\tan x}=1.$$

B13)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/(1-\cos x)} &= \exp\lim_{x\to 0}\frac{\log\sin x-\log x}{1-\cos x} = \exp\lim_{x\to 0}\frac{\cot x - x^{-1}}{\sin x}\\ &= \exp\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x} = \exp\lim_{x\to 0}\frac{-x\sin x}{\sin^2 x + x\sin 2x}\\ &= e^{-1/3}.\end{aligned}$$

B14)

$$\lim_{x\to a}\frac{a^x - x^a}{x-a} = \lim_{x\to a}\frac{a^x\log a - ax^{a-1}}{1} = a^a(\log a - 1).$$

B15)

$$\begin{aligned}\lim_{x\to\infty}\frac{(1+1/x)^x-e}{1/x} &= \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}(1+x)^{1/x}\cdot\frac{x/(x+1)-\log(1+x)}{x^2}\\ &= e\lim_{x\to 0}\frac{(x+1)^{-2}-(x+1)^{-1}}{2x} = \frac{e}{2}.\end{aligned}$$

B16)

$$\lim_{x\to\infty}\frac{x^{\log x}}{(\log x)^x} = \exp\lim_{x\to\infty}(\log x)^2 - x\log\log x = 0.$$

B17)

$$\begin{aligned}&\lim_{x\to\infty}(x+a)^{1+1/x}-x^{1+1/(x+a)} = \lim_{x\to\infty}\frac{(x+a)^{1+1/x}x^{-1}-x^{1/(x+a)}}{x^{-1}}\\ =&a.\end{aligned}$$

B18)

$$\lim_{x\to\infty}\sqrt[3]{x^3+x^2+x+1}-\sqrt{x^2+x+1}\cdot\frac{\log(e^x+x)}{x} = -\frac{1}{6}.$$

(Using WolframAlpha)

7.3 PSC

Calculate the maximum and minimum values of the following functions:
1.$f(x) = x^4 - 2x^2 + 5$, $x \in [-2, 2]$.
$f(x) = (x^2 - 1)^2 + 4 \in [4, 13]$.
2.$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$
$1 - f(x) = (1 + x^2)^{-1}(x - 1)^2 \geqslant 0$, $f(x) + 1 = (1 + x^2)^{-1}(x + 1)^2 \geqslant 0$, therefore $f(x) \in [-1, 1]$.
3.$f(x) = \arctan x - \frac{1}{2}\log(1 + x^2)$, $x \in \mathbb{R}$.
$f'(x) = \frac{1-x}{x^2+1}$, hence $\sup_{x\in\mathbb{R}} f(x) = f(1) = \frac{\pi}{4} - \frac{\log 2}{2}$, and f has no minimum.
4.$f(x) = x\log x$, $x \in (0, \infty)$.
$f'(x) = \log x + 1$, hence $\inf_{x\in(0,\infty)} f(x) = f(e^{-1}) = -e^{-1}$, and f has no maximum.
5.$f(x) = \sqrt{x}\log x$,$x \in (0, \infty)$.
$f'(x) = x^{-1/2}\left(1 + \frac{\log x}{2}\right)$, hence $\inf_{x\in(0,\infty)} f(x) = f(e^2) = -2e^{-1}$.
6.$f(x) = 2\tan x - \tan^2 x$, $x \in [0, \pi/2)$.
$f(x) = 1 - (1 - \tan x)^2 \in (-\infty, 1]$.

7.4 PSD

f is differentiable on (a, b). Suppose $x_0 \in (a, b)$ and $f'(x_0) = 0$.

D1) Prove that $f(x_0)$ is a local maximum if there exists $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ such that

$$f'(x) = \begin{cases} > 0, & \forall x \in (x_0 - \delta, x_0), \\ < 0, & \forall x \in (x_0, x_0 + \delta). \end{cases}$$

Proof: Trivial by Lagrange mean-value theorem.

D2) Prove that if $f''(x_0)$ exists and $f''(x_0) < 0$ then $f(x_0)$ is a local maximum.

Proof: $f''(x_0) < 0$ and $f'(x_0) = 0$ implies for some $\delta > 0$, $f'(x) < 0$ for $x \in (x_0, x_0 + \delta)$ and $f'(x) > 0$ for $x \in (x_0 - \delta, x_0)$. Hence by D1), $f(x_0)$ is a local maximum.

D3) Suppose f is n-times differentiable at x_0, $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Determine the conditions that $f(x_0)$ is a local maximum.

Solution: n is even and $f^{(n)}(x_0) < 0$.

7.5 PSE: Roots of Polynomials

E1) Prove that if all the roots of the polynomial $P_n(x) \in \mathbb{R}[x]$ are real numbers, then so are the polynomials $P_n'(x), P_n''(x), \cdots, P_n^{n-1}(x)$, where $n = \deg P_n$.

Proof: We only need to prove that P_n' has $n - 1$ real roots. By Rolle's mean-value theorem, between any two roots of P_n there is a root of P_n' hence P_n' has $n - 1$ real roots.

E2) Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$ has n different roots in the interval $(-1, 1)$.

Proof: We know that the polynomials $\sqrt{(2n + 1)/2}P_n(x)$ form a set of orthogonal base on the space $L^2([-1, 1])$, hence it must have n different roots in the interval $(-1, 1)$.

E3) Prove that the Laguerre polynomial $L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$ has n different real roots.

Proof: We know that the Laguerre polynomials are orthogonal on the space $L^2([0,\infty))$ with weight e^{-x}, hence it must have n distinct roots.
Or note that $f(x) = x^n e^{-x}$ has a root with multiplicity n at 0 and it vanishes at ∞, hence use Rolle's theorem and induction we can show that $f^{(k)}(x)$ has a root with multiplicity $n-k$ at 0 and k roots between 0 and ∞.

E4) Prove that the Hermite polynomial $H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$ has n different real roots.

Proof: We know that the polynomials $H_n(x)/\sqrt{2^n n!\sqrt{n}}$ form a set of orthogonal base on the Hilbert space $L^2(\mu)$ where $\mu(\mathrm{d}x) = e^{-x^2}\mathrm{d}x$, hence it must have n distinct real roots.

7.6 PSF: Émile Borel's Lemma

Part 1:

F1) Define $\phi : \mathbb{R} \to \mathbb{R}$:

$$\phi(x) = \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Prove that $\phi \in C^\infty(\mathbb{R})$.
Proof: We prove by induction that for any $n \in \mathbb{Z}_{\geqslant 0}$, there is a polynomial $P_n \in \mathbb{R}[x]$ such that

$$\phi^{(n)}(x) = \begin{cases} P_n(1/x) \cdot e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

(Which implies $\phi^{(n)}$ is continuous.)
The case $n = 0$ is trivial. Suppose it holds for n, then for any $x > 0$,

$$\phi^{(n+1)}(x) = e^{-1/x^2}\left(P_n(1/x)\frac{2}{x^3} - P_n'(1/x)\frac{1}{x^2}\right),$$

for any $x < 0$, $\phi^{(n+1)}(x) = 0$, and for $x = 0$,

$$\phi_+^{(n+1)}(0) = \lim_{x\to 0^+} e^{-1/x^2} P_n(1/x)\frac{1}{x} = 0.$$

Hence the claim holds for $n+1$ too.
Therefore $\phi \in C^\infty(\mathbb{R})$.

F2) Define $\chi : \mathbb{R} \to \mathbb{R}$:

$$\chi(x) = \frac{\phi(2-|x|)}{\phi(2-|x|) + \phi(|x|-1)}.$$

Prove that $\chi(x) \in C^\infty(\mathbb{R})$ and $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $0 \leqslant \chi(x) \leqslant 1$ and χ is an even function.
Proof: $2-|x|$ and $|x|-1$ cannot be both negative, hence the denominator is always positive, so $\chi \in C^\infty(\mathbb{R})$. The fact that $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $\chi(x) \in [0,1]$ and χ is even is trivial.

F3) Prove that for any $0 < a < b$, there exists a smooth function $\rho(x) \in C^\infty(\mathbb{R})$ such that $\rho|_{[-a,a]} \equiv 1$, $\rho|_{(-\infty,-b]\cup[b,\infty)} \equiv 0$, and $0 \leqslant \rho(x) \leqslant 1$.

Proof: Same as F2), define

$$\rho(x) = \frac{\phi(b-|x|)}{\phi(b-|x|)+\phi(|x|-a)}.$$

F4) Prove that there exists an even function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi|_{\{x:|x|\leqslant 1\}} \equiv 1$, $\psi|_{\{x:|x|\geqslant 2\}} \equiv 0$, and $0 \leqslant \psi(x) \leqslant 1$.

Proof: (A special case of Urysohn's lemma)
Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(\mathbf{x}) = \phi(1-|\mathbf{x}|^2)$ and $g : \mathbb{R}^n \to \mathbb{R}$ as $g(\mathbf{x}) = \phi(|x^2|/4-1)$, then f vanishes on $B(0,1)^C$ and g vanishes on $\bar{B}(0,2)$. Therefore

$$\psi(\mathbf{x}) = \frac{f(\mathbf{x})}{f(\mathbf{x})+g(\mathbf{x})}$$

satisfy the requirements.

Part 2: Interchanging $\sum$ and $\frac{\mathrm{d}}{\mathrm{d}x}$

$I = [a,b]$ is a closed interval, $\{f_k\}_{k\geqslant 0}$ is a sequence of functions in $C^1(I)$. Assume $\sum_{k=0}^\infty f_k$ converges point-wise on I, and let $f(x) = \sum_{k=0}^\infty f_k(x)$.

F5) Assume the series $\sum_{k=0}^\infty f_k'(x)$ converges absolutely on I, i.e. $\sum_{k=0}^\infty \|f_k'\|_\infty$ converges. Prove that f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Note that

$$\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty \frac{f_k(x+h)-f_k(x)}{h} = \sum_{k=0}^\infty f_k'(x+\xi_k).$$

Hence

$$\left|\frac{f(x+h)-f(x)}{h} - \sum_{k=0}^\infty f_k'(x)\right| \leqslant \sum_{n=0}^N |f_k'(x+\xi_n) - f_k'(x)| + 2\sum_{n=N+1}^\infty \|f_k'\|$$

Note that f_k' is uniformly continuous, so

$$\lim_{h\to 0}\sum_{n=0}^N |f_k'(x+\xi_k) - f_k'(x)| = 0,\ \lim_{N\to\infty} 2\sum_{n=N+1}^\infty \|f_k'\| = 0.$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty f_k'(x).$$

F6) Assume $\sum_{k=0}^\infty f_k'(x)$ converges uniformly on I, then f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Let $g(x) = \sum_{k=0}^\infty f_k'(x)$, since the series converges uniformly, $g(x)$ is continuous on I. By Lebesgue's Dominated Convergence Theorem,

$$\int_{x_0}^x g(t)\,\mathrm{d}t = \sum_{k=0}^\infty f_k(t)\Big|_{x_0}^x = f(x) - f(x_0).$$

Hence $f'(x) = g(x) = \sum_{k=0}^\infty f_k'(x)$.

F7) Calculate the derivative of e^x using F6).

Solution: On any closed interval $[-M, M]$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

converges uniformly. Hence

$$(e^x)' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Part 3: Borel's Lemma

Given an arbitrary sequence $\{a_k\}_{k\geqslant 0}$.

F8) For any $t_k > 0$, let $f_k(x) = \frac{a_k}{k!}x^k\chi(t_kx)$, determine the derivatives of any order of f_k at $x = 0$.

Solution: Note that when $x = 0$, $\chi^{(m)}(t_kx) = 0$ for any $m \geqslant 1$ and $\chi(t_kx) = 1$. Hence

$$f_k^{(n)}(0) = \frac{a_k}{k!}\sum_{j=0}^{n}\binom{n}{j}(x^k)^{(j)}\chi^{(n-j)}(t_kx)\Big|_{x=0} = \frac{a_k}{k!}(x^k)^{(n)}\Big|_{x=0} = a_k\delta_{n,k}.$$

F9) Prove that when $k \geqslant 2n$,

$$f_k^{(n)}(x) = a_k\sum_{=0}^{n}\binom{n}{}\frac{t_k^{n-}}{(k-)!}x^{k-}\chi^{(n-)}(t_kx).$$

Proof: Leibniz's Formula.

F10) (Borel's lemma) Prove that for any sequence $\{a_k\}_{k\geqslant 0}$, there exists a smooth function f on $\mathbb{R}$, such that for any $k \geqslant 0$, $f^{(k)}(0) = a_k$.

Proof: Let $f_k(x) = \frac{a_k}{k!}x^k\chi(t_kx)$ where t_k is yet to be determined, and

$$f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty}\frac{a_kx^k}{k!}\chi(t_kx).$$

For any $n \geqslant 0$, we want to show that $\sum_{k=0}^{\infty} f_k^{(n)}(x)$ converges uniformly on $\mathbb{R}$.
Suppose $M_n = \sup_{x\in\mathbb{R}, m\leqslant n}|\chi^{(m)}(x)|$, and

$$C_k = \sup_{n<k/2}\sum_{=0}^{n}\frac{\binom{n}{}}{(k-)!},$$

then for any $x \in \mathbb{R}$,

$$|f_k^{(n)}(x)| \leqslant |a_k|C_kM_kt_k^{-k/2}.$$

Hence if we choose t_k such that

$$|a_k|C_kM_kt_k^{-k/2} < 2^{-k},$$

then the series

$$\sum_{k=0}^{\infty} f_k^{(n)}(x)$$

converges uniformly on $\mathbb{R}$. By F6) we know that $f(x) = \sum_{k=0}^{\infty} f_k(x)$ is smooth, and by F8) we obtain $f^{(n)}(0) = a_n$ for any $n \geqslant 0$,

Part 4: Peano's Proof

F11) $\{c_k\}$ and $\{b_k\}$ are two sequences, and $b_k > 0$. Prove that

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = \begin{cases} n!(-1)^j c_{n-2j} b_{n-2j}^j, & \text{if } k = n-2j, j \in \mathbb{Z}_{\geqslant 0}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: For $x \to 0$,

$$\frac{c_k x^k}{1+b_k x^2} = c_k \sum_{n=0}^{\infty} (-1)^n x^{2n+k} b_k^n.$$

Which converges absolutely on the interval $[-b_k^{-1/2}/2, b_k^{-1/2}/2]$, and so are its n-times derivations, hence by F5)

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = c_k \sum_{j=0}^{\infty} (-1)^j \frac{(2j+k)!}{(2j+k-n)!} x^{2j+k-n} b_k^j \Big|_{x=0} = \begin{cases} n!(-1)^j c_k b_k^j, & k = n-2j, \\ 0, & \text{otherwise.} \end{cases}$$

F12) Prove that there is a constant C such that for any $k \geqslant n+2$, and any x,

$$\left|\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(x)\right| \leqslant C(n+1)! \frac{|c_k| k!}{b_k} |x|^{k-n-2}.$$

Proof: Use du Bois-Reymond, we can let $C = 1$.

F13) Prove that for a given $\{c_k\}$, we can choose $\{b_k\}$ such that b_k depends only on the value of c_k, and the function

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$$

is infinitely differentiable.
Proof: Let $b_k = (k!)^2 c_k$, then by F12),

$$\left|\sum_{k \geqslant n+2} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}\right| \leqslant (n+1)! \sum_{k \geqslant n+2} \frac{|x|^{k-n-2}}{k!}$$

hence the series

$$\sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}$$

converges uniformly for any $n \geqslant 1$. By F6) the function $f(x)$ is infinitely differentiable, and

$$f^{(n)}(x) = \sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}.$$

F14) Prove that $f(0) = c_0, f'(0) = c_1$ and when $n \geqslant 2$,

$$\frac{f^{(n)}(0)}{n!} = c_n + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j.$$

Proof: Combine F11) and F13).

F15) Prove that by carefully choosing $\{c_k\}$ and $\{b_k\}$, we can prove Borel's lemma.

Proof: Let $b_k = (k!)^2 c_k$ and define c_k inductively such that

$$c_n = \frac{a_n}{n!} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j$$

Then let $f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$

7.7 PSG: Midterm Test Part B

Consider $f : \mathbb{R} \to \mathbb{R}$.

- Let $\mathcal{B}$ be all bounded function on $\mathbb{R}$.
- Let $\mathcal{L}$ be all Lipschitz functions on $\mathbb{R}$.
  Suppose $a, \lambda \in \mathbb{R}$, $f \in \mathcal{B} \cap \mathcal{L}$, the goal is to find a function $F \in \mathcal{L}$ to solve:

$$F(x) - \lambda F(x+a) = f(x),\ x \in \mathbb{R}. \tag{$\star$}$$

Part 1: Basic Properties of Lipschitz Functions

B1) Prove that if $f, g \in \mathcal{B} \cap \mathcal{L}$, then $fg \in \mathcal{L}$.

Proof: Suppose $|f(x) - f(y)|, |g(x) - g(y)| \leqslant A|x-y|$, and $|f(x)|, |g(x)| \leqslant C$, then for any $x, y \in \mathbb{R}$,

$$|f(x)g(x) - f(y)g(y)| \leqslant 2MA|x-y|.$$

Hence $fg \in \mathcal{L}$.

B2) Prove that if f is differentiable and $f \in \mathcal{L}$ then $f' \in \mathcal{B}$.

Proof: If $|f(x) - f(y)| \leqslant C|x-y|$ then for any $x \in \mathbb{R}$,

$$|f'(x)| = \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leqslant C.$$

Hence $f' \in \mathcal{B}$.

B3) Prove that if f is differentiable and $f' \in \mathcal{B}$ then $f \in \mathcal{L}$.

Proof: For any $x, y \in \mathbb{R}$, there exists $\xi \in (x, y)$ such that

$$|f(x) - f(y)| = |x-y| \cdot |f'(\xi)| \leqslant \sup_{t \in \mathbb{R}} |f'(t)| \cdot |x-y|.$$

Hence $f \in \mathcal{L}$.

B4) If $f \in \mathcal{B}$ and there exists $B > 0$ such that for any $x, y \in \mathbb{R}$, $|x - y| \leqslant 1$ implies $|f(x) - f(y)| \leqslant B|x - y|$. Prove that $f \in \mathcal{L}$.

Proof: Suppose $M = \sup_{x\in\mathbb{R}} |f(x)|$, then for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant \max\{B, 2M\}|x - y|.$$

Hence $f \in \mathcal{L}$.

Part 2: Solution of ($\star$) when $|\lambda| < 1$.

Suppose $f \in \mathcal{B} \cap \mathcal{L}$ and $|\lambda| < 1$.

B5) Suppose F satisfy ($\star$). Prove that for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geqslant 1}$,

$$F(x) = \lambda^n F(x + na) + \sum_{k=0}^{n-1} \lambda^k f(x + ka),$$
$$F(x) = \lambda^{-n} F(x - na) - \sum_{k=1}^{n} \lambda^{-k} f(x - ka).$$

Proof: Use induction and apply ($\star$).
(Let $n \to \infty$ and we can obtain F formally.)

B6) Prove that for any $x \in \mathbb{R}$, $\sum_{k\geqslant 0} \lambda^k f(x + ka)$ converges.

Proof: Since f is bounded,

$$\left|\sum_{k=n}^{n+p} \lambda^k f(x + ka)\right| \leqslant \frac{M\lambda^n}{1 - \lambda}.$$

Hence the series converges.

B7-8) Let $F(x) = \sum_{k\geqslant 0} \lambda^k f(x + ka)$. Prove that $F \in \mathcal{L}$ and solve ($\star$).

Proof: For any $x, y \in \mathbb{R}$,

$$|F(x) - F(y)| \leqslant \sum_{k=0}^{\infty} \lambda^k |f(x + ka) - f(y + ka)| \leqslant \sum_{k=0}^{\infty} \lambda^k C|x - y| = \frac{C}{1 - \lambda}|x - y|.$$

Hence $F \in \mathcal{L}$. For any $x \in \mathbb{R}$,

$$F(x) - \lambda F(x + a) = \sum_{k\geqslant 0} \lambda^k f(x + ka) - \sum_{k\geqslant 1} \lambda^k f(x + ka) = f(x).$$

Therefore F solves ($\star$).
If F' also solves ($\star$), let $G = F - F'$, then G is bounded and

$$G(x) = \lambda G(x + a),\ x \in \mathbb{R}.$$

Therefore for any $x \in \mathbb{R}$,

$$|G(x)| = \lambda^n |G(x + na)| \leqslant M\lambda^n \to 0.$$

Hence $G \equiv 0$ and $F \equiv F'$, so the solution to ($\star$) is F.

B9) Solve $(\star)$ when $f(x) \equiv 1$ and $f(x) = \cos x$.

Solution: When $f(x) \equiv 1$,

$$F(x) = \sum_{k=0}^{\infty} \lambda^k f(x+ka) = \frac{1}{1-\lambda}.$$

When $f(x) = \cos x$,

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} \lambda^k \cos(x+ka) = \sum_{k=0}^{\infty} \lambda^k \frac{e^{i(x+ka)} + e^{-i(x+ka)}}{2} = \frac{1}{2}\left(\frac{e^{ix}}{1-\lambda e^{ia}} + \frac{e^{-ix}}{1-\lambda e^{-ia}}\right) \\ &= \frac{\cos x - \lambda \cos(x-a)}{1 - 2\lambda \cos a + \lambda^2}. \end{aligned}$$

Part 3: Solution of $(\star)$ when $|\lambda| > 1$.

B10) Solve $(\star)$ as in Part 2.

Solution: By B5), the solution should be

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka).$$

$f \in \mathcal{B}$ implies the series converges. Same as B8) we can show that the solution to $(\star)$ is unique, and like B7) we can show that $F \in \mathcal{L}$ and F satisfy $(\star)$.

B11) Solve $(\star)$ for $f(x) \equiv 1$ and $f(x) = \cos x$.

Solution: When $f(x) \equiv 1$,

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka) = \frac{1}{1-\lambda}.$$

When $f(x) = \cos x$, same as B9) we have

$$F(x) = -\sum_{k=1}^{\infty} \lambda^{-k} f(x-ka) = \frac{\cos x - \lambda \cos(x-a)}{1 - 2\lambda \cos a + \lambda^2}.$$

Part 4: The Case when $|\lambda| = 1$.

B12) Suppose $\lambda = 1$. Prove that there exists $F \in \mathcal{L}$ not identically zero, such that for any x, $F(x) - F(x+a) = 0$.

Proof: Let $F(x) = |\{x/a\} - 1/2|$, then $F(x) = F(x+a)$, and $F \in \mathcal{L} \cap \mathcal{B}$.

B13) Let $f(x) = \cos x$ in $(\star)$. Prove that if $\cos a \neq 1$, then there exists $F \in \mathcal{L}$ that solves $(\star)$. Determine whether the solution is unique.

Proof: The equation $(\star)$ becomes $F(x) = F(x+a) + \cos x$. Let

$$F(x) = \{x/a\} - \sum_{k=0}^{\lfloor x/a \rfloor - 1} \cos(k + \{x/a\})a,$$

(if $x < 0$ the sum is viewed as from $\lfloor x/a \rfloor - 1$ to 0) then clearly $F(x) = F(x+a) + \cos x$, and F is bounded since $\cos a \neq 1$.
For any $x, y \in \mathbb{R}$, if $|x - y| < a/2$, then suppose $na \leqslant x < y < (n+1)a$,

$$|F(x) - F(y)| \leqslant \left| \left\{ \frac{x}{a} \right\} - \left\{ \frac{y}{a} \right\} \right| + 2 \left| \sin \frac{\{x/a\} - \{y/a\}}{2} a \right| \cdot \left| \sum_{k=0}^{n-1} \sin(k + (\{x/a\} + \{y/a\})/2)a \right|$$
$$\leqslant \frac{|x-y|}{a} + \frac{|x-y|}{|\sin a|}.$$

Hence $F \in \mathcal{L}$ by B4), so F solves $(\star)$.
The solution is clearly not unique since we can add any factor of the F in B12) to the solution.

B14) Following B13), if $a = 2\pi$, then $(\star)$ has no solution in $\mathcal{L}$.

Proof: If $a = 2\pi$ and F is a solution to $(\star)$, then for any $x, y \in \mathbb{R}$,

$$|F(x + 2\pi n) - F(y + 2\pi n)| = n|\cos x - \cos y| \to \infty.$$

Hence $F \notin \mathcal{L}$.

B15) Suppose $\lambda = -1$, Prove that there exists $F \in \mathcal{L}$ not identically zero, such that for any x, $F(x) + F(x+a) = 0$.

Proof: Let $F(x) = |2\{x/2a\} - 1| - 1/2$, then $F \in \mathcal{L}$ and $F(x) + F(x+a) = 0$.

B16) Suppose $\lambda = -1$, $a = 1$, $f \in \mathcal{L}$ is monotonically decreasing and $\lim_{x\to\infty} f(x) = 0$, f is differentiable and f' is increasing. Prove that there exists $F \in \mathcal{L}$ such that

$$F(x) + F(x+1) = f(x),\ x \in \mathbb{R}.$$

Further show that if we require $F \in \mathcal{L}$ and $\lim_{x\to\infty} F(x) = 0$, then the solution is unique.
Proof: Since f is monotonically decreasing, for any $x \in \mathbb{R}$, the series

$$F(x) = \sum_{n=0}^{\infty} (-1)^n f(x+n)$$

converges.
For any $x, y \in \mathbb{R}$, $|x-y| < 1$, there exists $\xi_n \in (x+n, y+n)$ such that $f(y+n) - f(x+n) = (y-x)f'(\xi_n)$, hence (by B3) f' is bounded)

$$|F(x) - F(y)| = |y - x| \cdot \left| \sum_{n=0}^{\infty} (-1)^n f'(\xi_n) \right| \leqslant \sup_{t\in\mathbb{R}} |f'(t)| \cdot |y - x|.$$

so $F \in \mathcal{L}$. Clearly $F(x) + F(x+1) = f(x)$, so F solves $(\star)$, and $0 < F(x) < f(x)$ so $\lim_{x\to\infty} F(x) = 0$. If $F' \in \mathcal{L}$ also satisfy $(\star)$ and $\lim_{x\to\infty} F(x) = 0$, let $G = F - F'$, then $G(x) + G(x+1) = 0$ and $\lim_{x\to\infty} G(x) = 0$. Hence $G(x) = \lim_{n\to\infty} (-1)^n G(x+n) = 0$ for any $x \in \mathbb{R}$, so $G \equiv 0$. Therefore F is the unique solution.

8 Homework 8; Strum-Liouville Theory

8.1 PSA: Convex functions

A1)

(1) $f(x) = |x|$, $I = \mathbb{R}$ is convex, since

$$|\lambda x + (1-\lambda)y| \leqslant \lambda|x| + (1-\lambda)|y|.$$

(2) $f(x) = x^p$, $p \in \mathbb{R}$, $I = \mathbb{R}_{>0}$
$f''(x) = p(p-1)x^{p-2}$ so f is concave if $p \in [0,1]$ and convex if $p \in (-\infty, 0] \cup [1, \infty)$.
(3) $f(x) = \sin x$, $I = [0, \pi]$ is concave since $f''(x) = -\sin x \leqslant 0$ when $x \in [0, \pi]$.
(4) $f(x) = x \log x$, $I = \mathbb{R}_{\geqslant 0}$ is (strictly) convex since $f''(x) = 1/x > 0$.
(5) $f(x) = \mathbf{1}_{\{0,1\}}$, $I = [0,1]$ is convex since

$$f(\lambda x + (1-\lambda)y) = 0 \leqslant \lambda f(x) + (1-\lambda)f(y).$$

A2) Prove the following properties:

1. If f, g are convex on I, then $f+g$ is convex on I.
   Proof: By definition,$(f+g)(\lambda x + (1-\lambda)y) \leqslant \lambda(f+g)(x) + (1-\lambda)(f+g)(y)$,so $f+g$ is convex.

2. If f, g are monotonically increasing, non-negative, convex functions on I, then fg is convex.
   Proof: Note that

$$f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y) \leqslant (\lambda f(x) + (1-\lambda)f(y)) \cdot (\lambda g(x) + (1-\lambda)g(y))$$

and

$$\begin{aligned}
&\lambda f(x)g(x) + (1-\lambda)f(y)g(y) - (\lambda f(x) + (1-\lambda)f(y))(\lambda g(x) + (1-\lambda)g(y)) \\
=&\lambda(1-\lambda)(f(x) - f(y))(g(x) - g(y)) \geqslant 0.
\end{aligned}$$

hence

$$(fg)(\lambda x + (1-\lambda)y) \leqslant \lambda(fg)(x) + (1-\lambda)(fg)(y).$$

1. If f is convex on I, g is a monotonically increasing convex function on $J \supset f(I)$, then $g \circ f$ is convex.
   Proof: Note that

$$g(f(\lambda x + (1-\lambda)y)) \leqslant g(\lambda f(x) + (1-\lambda)f(y)) \leqslant \lambda g(f(x)) + (1-\lambda)g(f(y)).$$

hence $g \circ f$ is convex.

1. If f, g are convex on I, then $h(x) = \max\{f(x), g(x)\}$ is convex.
   Proof: For any x, y, λ and $t = \lambda x + (1-\lambda)y$, suppose $h(t) = f(t)$, then

$$h(t) \leqslant \lambda f(x) + (1-\lambda)f(y) \leqslant \lambda h(x) + (1-\lambda)h(y)$$

hence h is convex.

A3) Suppose $f \in C((a,b))$. If for any $x, y \in (a,b)$, $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$, prove that f is convex.

Proof: For any $x, y \in (a,b)$ and $\lambda \in [0,1]$, we need to prove that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

Note that it holds for any dyadic number λ, since the cases $\lambda = 0, 1, 1/2$ is trivial, and for $\lambda = (2m+1)/2^t$, let $u = m/2^{t-1}, v = (m+1)/2^{t-1}$, then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \frac{f(ux + (1-u)y) + f(vx + (1-v)y)}{2} \\ &\leqslant \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

Now since $f \in C((a,b))$, for any $\lambda \in (0,1)$ there is a sequence of dyadic numbers λ_n such that $\lim_{n\to\infty} \lambda_n = \lambda$, hence

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \lim_{n\to\infty} f(\lambda_n x + (1-\lambda_n)y) \leqslant \lim_{n\to\infty} \lambda_n f(x) + (1-\lambda_n) f(y) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

A4) f is a convex function on $[a,b]$. Prove that if there exists $c \in (a,b)$ such that $f(c) \geqslant \max\{f(a), f(b)\}$ then f is constant.

Proof: For any $t \in (a,b)$, let $\lambda = (t-a)/(b-a)$ then

$$f(t) = f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda) f(b) \leqslant \max\{f(a), f(b)\}.$$

By $f(c) \geqslant \max\{f(a), f(b)\}$ we know that $f(a) = f(b)$. If for some $t \in (a,b)$, $f(t) \neq f(a)$, suppose $c \in (a,t)$, then

$$f(c) = \lambda f(a) + (1-\lambda) f(t) < f(a)$$

a contradiction. Hence $f(t) = f(a)$ for all $t \in [a,b]$.

A5) f is convex on $\mathbb{R}$. Prove that if f has an upper-bound, then f is constant.

Proof: Otherwise suppose that $f(a) < f(b)$, where $a < b$. (If $f(a) > f(b)$ let $g(x) = f(-x)$). Let $x_0 = a, x_1 = b, x_n = a + n(b-a)$, then

$$f(x_{n+1}) - f(x_n) \geqslant f(x_n) - f(x_{n-1}) \geqslant f(b) - f(a),$$

hence $f(x_n) \geqslant f(a) + n(f(b) - f(a)) \to \infty$, leading to contradiction.

A6) f is strictly convex on I. Suppose $f(x_0)$ is a local minimum of f, prove that x_0 is the unique global minimum point of f.

Proof: Suppose there is another $x_1 \neq x_0$ such that $f(x_1) \leqslant f(x_0)$, then let $x_n = x_0 + n(x_1 - x_0)$. Since f is strictly convex, $f(x_n) < \max\{f(x_1), f(x_0)\} = f(x_0)$, contradicting the fact that $f(x_0)$ is a local minimum.

A7) I is an open interval. Prove that f is convex on I, iff for any $x_0 \in I$, there exists $a \in \mathbb{R}$, such that for any $x \in I$, $f(x) \geqslant a(x-x_0)+f(x_0)$.

Proof: Suppose f is convex on I, then the any $x_0 \in I$, the function $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$ is monotonically increasing. Hence we can let $a = \sup_{x<x_0} g(x) < \infty$.
If for any $x_0 \in I$, and and $x \in I$, $f(x) \geqslant g(x_0)(x-x_0)+f(x_0)$, then for any $x, y \in I$ and $\lambda \in (0,1)$, let $t = \lambda x + (1-\lambda)y$,

$$\begin{aligned}\lambda f(x) + (1-\lambda) f(y) &\geqslant \lambda(f(t) + (x-t)g(t)) + (1-\lambda)(f(t) + (1-\lambda)(y-t)g(t)) \\ &= f(t) = f(\lambda x + (1-\lambda)y).\end{aligned}$$

Hence f is convex.

8.2 PSB

B1) Prove the following inequalities:

(1)

$$x - \frac{x^2}{2} < \log(1+x) < x,\ x > 0.$$

Proof: If $f(x) = \log(x+1) - x$, then $f'(x) = \frac{1}{x+1} - 1 < 0$ hence $f(x) < f(0) = 0$. Let $g(x) = \log(1+x) - x + x^2/2$, then $g'(x) = \frac{1}{x+1} + (x+1) - 2 \geqslant 0$, hence $g(x) > g(0) = 0$.
(2)

$$(x^\alpha + y^\alpha)^{1/\alpha} > (x^\beta + y^\beta)^{1/\beta},\ x, y > 0, \beta > \alpha > 0.$$

Proof: Assume that $x^\alpha + y^\alpha = 1$, then $0 < x, y < 1$, so

$$x^\beta + y^\beta < x^\alpha + y^\alpha < 1 \implies (x^\beta + y^\beta)^{1/\beta} < (x^\alpha + y^\alpha)^{1/\alpha}.$$

(3)

$$x - \frac{x^3}{6} < \sin x < x,\ x > 0.$$

Proof: Let $f(x) = \sin x - x$, then $f'(x) = \cos x - 1 \leqslant 0$, so $f(x) < f(0) = 0$. Let $g(x) = \sin x - x + x^3/6$, then $g'(x) = \cos x - 1 + x^2/2$, $g''(x) = x - \sin x > 0$, so $g'(x) > g(0) = 0$ and $g(x) > g(0) = 0$.
(4)

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leqslant \frac{1}{2}(1+x^p),\ p \in [2,\infty), x \in [0,1].$$

Proof: ???

B2) Find all $a > 0$ such that $a^x \geqslant x^a$ for any $x > 0$.

Solution: $f(x) = x^{1/x}$ then $f'(x) = x^{1/x}\frac{1-\log x}{x^2}$ hence f has a unique minimum at e.

B3) Prove that for any $x_i, t_i, i = 1, 2, \cdots, n$, $\sum_{i=1}^n t_i = 1$,

$$\left(\sum_{i=1}^n t_i x_i\right)^{\sum_{i=1}^n t_i x_i} \leqslant \prod_{i=1}^n x_i^{t_i x_i}.$$

Proof: Let $f(x) = x\log x$, then $f''(x) = 1/x > 0$, so f is convex. By Jensen's inequality,

$$\sum_{i=1}^n t_i f(x_i) \geqslant f\left(\sum_{i=1}^n t_i x_i\right)$$

hence

$$\left(\sum_{i=1}^{n} t_i x_i\right)^{\sum_{i=1}^{n} t_i x_i} \leqslant \prod_{i=1}^{n} x_i^{t_i x_i}.$$

and equality holds iff $x_i = x_1$.

B4) Prove that for any $a, b > 0$, $1/p + 1/q = 1$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1; \; ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1.$$

Proof: The function $-\log x$ is convex, so when $p > 1$, $q > 1$, then

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geqslant \frac{1}{p}\log a^p + \frac{1}{q}\log b^q$$

so $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.
When $p < 1$, then $pq < 0$, so likewise $ab \geqslant \frac{a^p}{p} + \frac{b^q}{q}$.

B5) Prove that if $x_i, y_i \geqslant 0, i = 1, 2, \cdots, n$, $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}, \text{ if } p > 1;$$

and the inequality reverses when $p < 1$.
Proof: Assume that $\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} y_i^q = 1$, then by B4), if $p > 1$,

$$\sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{n} \frac{x_i^p}{p} + \frac{y_i^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

The case $p < 1$ is similar.

8.3 PSC

C1) Suppose $f \in C([0,1])$, g is differentiable on $[0,1]$ and $g(0) = 0$. If there is a constant $\lambda \neq 0$, such that for any $x \in [0,1]$, $|g(x)f(x) + \lambda g'(x)| \leqslant |g(x)|$, prove that $g(x) \equiv 0$.

Proof: Otherwise assume that $\forall \varepsilon > 0 \exists t \in (0, \varepsilon)$, such that $g(t) \neq 0$. Let $C = (1 + \sup_{x\in[0,1]} |f(x)|)/\lambda$, then $|g'(x)| \leqslant C|g(x)|, \forall x \in [0,1]$. For any $t \in (0,1)$, there exists $\xi \in [0,t]$ such that $g(t) = g(0) + tg'(\xi)$, hence

$$\frac{|g(t)|}{t} = |g'(\xi)| \leqslant C \sup_{\xi\in[0,t]} |g(\xi)|.$$

For any $t > 0$ suppose $|g(s)| = \sup_{\xi\in[0,t]} |g(\xi)| > 0$, then $|g(s)|/s \leqslant C|g(s)|$ hence $t \geqslant s \geqslant \frac{1}{C}$, a contradiction.

C2) f is twice differentiable on $(-1,1)$, $f(0) = f'(0) = 0$. If for any $x \in (-1,1)$, $|f''(x)| \leqslant |f(x)| + |f'(x)|$, prove that $f(x) \equiv 0$.

Proof: We prove that $f''(x) \equiv 0$. Otherwise suppose $\forall \varepsilon > 0$, $\exists x \in [0, \varepsilon]$, $f''(x) \neq 0$. Note that

$$|f''(x)| \leqslant |f(x)| + |f'(x)| \leqslant \left(\frac{x^2}{2} + |x|\right) \sup_{y\in[0,x]} |f''(y)|.$$

Since $f''(0) = 0$, take $x \in [0, 1/2]$ such that $f''(x) \neq 0$, and suppose $|f''(t)| = \sup_{y\in[0,x]} |f''(y)|$, then $|f''(t)| \leqslant (t^2/2 + t)|f''(t)|$, a contradiction.

C3) f is n-times differentiable on $\mathbb{R}$, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. If there exists $C \in \mathbb{R}_{>0}$ and $\in \mathbb{Z}_{\geqslant 0}$ such that for any $x \in \mathbb{R}$, $|f^{(n)}(x)| \leqslant C|f^{()}(x)|$. Prove that $f(x) \equiv 0$.

Proof: We can assume that $=0$. Since $f^{(k)}(x) = 0, \forall 0 \leqslant k < n$, we have

$$|f^{(n)}(x)| \leqslant C|f(x)| \leqslant C\frac{x^n}{n!}\sup_{y\in[0,x]}|f^{(n)}(y)|.$$

Hence for any $x \in [0,\varepsilon]$, $\varepsilon = (n!/C)^{1/n}$, $f^{(n)}(x) = 0$, so $f^{(k)}(x) = 0$ for all $x \in [0,\varepsilon], 0 \leqslant k < n$. Likewise we get $f(x) \equiv 0$.

C4) $n \in \mathbb{Z}_{>0}$, prove that the polynomial $P(x) = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k(x-k)^n \equiv 0$.

Proof: We know the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^l = 0, \, \forall 0 \leqslant l \leqslant n-1.$$

Since $\Delta^n x^l \equiv 0$
Likewise by considering $f(t) = (x-t)^n$ we have $P(x) \equiv 0$.
(Or we can use C3)

C5) $f \in C^\infty(\mathbb{R})$. Assume there exists $C > 0$ such that for any $n \in \mathbb{Z}_{\geqslant 0}$ and $x \in \mathbb{R}$, $|f^{(n)}(x)| \leqslant C$.

i. Prove that given an arbitrary $x_0 \in \mathbb{R}$,

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \, \forall x \in \mathbb{R}.$$

Proof: The Lagrange remainder

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

tends to zero as $n \to \infty$, hence the Taylor series

$$f(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

ii. $E \subset \mathbb{R}$ is an infinite bounded set. Prove that if $f(E) = \{0\}$, then $f \equiv 0$.
Proof: Suppose $E \subset [-M,M]$, then by Bolzano-Weierstrass theorem, there exists a sequence $\{z_n\}_{n\geqslant 1} \subset E$ such that $z = \lim_{n\to\infty} z_n$ exists. Since $f \in C(\mathbb{R})$, $f(z) = \lim_{n\to\infty} f(z_n) = 0$, so

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^k.$$

If f does not vanish on $\mathbb{R}$, then take the least $m > 0$ such that $f^{(m)}(z) \neq 0$. When $z_n \to z$,

$$0 = \frac{f^{(m)}(z)}{m!} + \sum_{k=m+1}^{\infty}\frac{f^{(k)}(z)}{k!}(x-z)^{k-m}$$

which leads to contradiction. Hence f vanishes on $\mathbb{R}$.

C6) Assume $f \in C^2((0,1))$, $\lim_{x\to 1^-} f(x) = 0$. If there exists $C > 0$, such that for any $x \in (0,1)$, $(1-x)^2|f''(x)| \leqslant C$. Prove that $\lim_{x\to 1^-}(1-x)f'(x) = 0$.

Proof: For any $0 < x < y < 1$, there exists $\xi \in (x, y)$ such that

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2} f''(\xi).$$

For any $\lambda > 0$, let $y = (\lambda + x)/(\lambda + 1) \in (x, 1)$, then

$$|(y-x)f'(x)| \leqslant |f(y)| + |f(x)| + \frac{\lambda^2}{2}(1-y)^2|f''(\xi)| \leqslant |f(y)| + |f(x)| + \frac{C\lambda^2}{2}.$$

Therefore

$$|(1-x)f'(x)| \leqslant (|f(t)| + |f(x)|)\frac{\lambda+1}{\lambda} + \frac{1}{2}\lambda(\lambda+1)C$$

Hence for any $\lambda > 0$,

$$\lim_{x\to 1^-} |(1-x)f'(x)| \leqslant \frac{1}{2}\lambda(\lambda+1)C \to 0,$$

so $\lim_{x\to 1^-}(1-x)f'(x) = 0$.

8.4 PSD

Calculate $\sup_{x\in I} f(x)$ and $\inf_{x\in I} f(x)$:

D1) $f(x) = \frac{(\log x)^2}{x}$**,** $I = \mathbb{R}_{>0}$

Solution: Let $y = \log x \in \mathbb{R}$, then $f(x) = y^2 e^{-y}$.

$$\frac{\mathrm{d}}{\mathrm{d}y} y^2 e^{-y} = y e^{-y}(2-y).$$

Hence $\sup_{x\in I} f(x) = f(e^2) = 4e^{-2}$, $\inf_{x\in I} f(x) = \min\{f(0), f(\infty)\} = 0$.

D2) $f(x) = |x(x^2-1)|$**,** $I = \mathbb{R}$

Solution: $\sup = \infty, \inf = 0$.

D3)

$$f(x) = \frac{x(x^2+1)}{x^4 - x^2 + 1},\ I = \mathbb{R}.$$

Solution: Note that

$$2(x^4 - x^2 + 1) - x(x^2+1) = (x^2-1)^2 + (x-1)^2(x^2+x+1) \geqslant 0.$$

Therefore $f(x) \leqslant 2$ where equality holds at $x = 1$. Since $f(x) = f(-x)$, $\sup = 2, \inf = -2$.

D4)

$$f(x) = x^{1/3}(1-x)^{2/3},\ I = (0,1).$$

Solution: By AM-GM, $f(x) \leqslant \frac{2^{2/3}}{3}$ where equality holds at $x = 1/3$. Hence $\sup = \frac{2^{2/3}}{3}, \inf = 0$.

D5)

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) e^{-x},\ I = \mathbb{R}.$$

Solution: $f'(x) = -e^{-x}\frac{x^n}{n!}$, so if n is even, $\sup = \infty, \inf = 0$, and if n is odd, $\sup = 1, \inf = -\infty$.

D6) $f(x) = \sin^{2m} x \cos^{2n} x$**,** $I = \mathbb{R}$**.**

Solution: Let $t = \sin^2 x \in [0, 1]$, then $f(x) = t^m(1-t)^n \in [0, n^n m^n/(n+m)^{n+m}]$.

8.5 PSE

Compare the two functions (or real numbers).

E1) $f(x) = e^x$**,** $g(x) = 1 + xe^x$**,** $x > 0$**.**

Solution: The case $x \geqslant 1$ is trivial. If $x \in (0, 1)$, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \leqslant \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

hence $f(x) \leqslant g(x)$. Therefore $f(x) \leqslant g(x)$ for all $x > 0$.

E2) $f(x) = xe^{x/2}$**,** $g(x) = e^x - 1$**,** $x > 0$**.**

Solution: $(x/2 \leqslant \sinh(x/2))$ Consider $h(x) = e^{x/2} - e^{-x/2} - x$, then $h(0) = 0$ and

$$h'(x) = \frac{1}{2}(e^{x/2} + e^{-x/2} - 2) \geqslant 0.$$

Hence $h(x) \geqslant 0$, i.e. $g(x) \geqslant f(x)$ for all $x > 0$.

E3) $f(x) = \left(\frac{x+1}{2}\right)^{(x+1)}$**,** $g(x) = x^x$**,** $x > 0$**.**

Solution: Consider $h(x) = x \log x - (x+1) \log \frac{x+1}{2}$, then $h(1) = 0$ and

$$h'(x) = \log \frac{2x}{x+1} \geqslant 0 \iff x \geqslant 1.$$

Hence $f(x) \leqslant g(x)$ for all $x > 0$.

E4) $2^{\sqrt{2}}$ **and** e**.**

Solution: Note that

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{2^n n} \leqslant \frac{2}{3} + \sum_{n=4}^{\infty} \frac{1}{2^n \cdot 4} = \frac{2}{3} + \frac{1}{32} < \frac{2}{3} + \frac{1}{30} = 0.7 < \frac{1}{\sqrt{2}},$$

hence $2^{\sqrt{2}} < e$.

E5) $f(x) = \log(1+\sqrt{1+x^2})$**,** $g(x) = 1/x + \log x$**,** $x > 0$**.**

Solution: Consider $h(x) = \log x + 1/x - \log(1+\sqrt{1+x^2})$, then

$$h'(x) = \frac{1}{x} - \frac{1}{x^2} - \frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}} \leqslant 0.$$

$(\iff (x-1)(\sqrt{1+x^2}+1+x^2) - x^3 \leqslant 0 \iff (x1)\sqrt{1+x^2} \leqslant x^2)$
Therefore $h(x) \geqslant \lim_{x\to\infty} h(x) = 0$.

E6) $\log 8$ **and** 2**.**

Solution: Note that

$$\log 2 = \log \frac{1}{1-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{2^n n} \geqslant \sum_{n=1}^{3} \frac{1}{2^n n} = \frac{2}{3},$$

hence $\log 8 > 2$.

8.6 PSF

If f satisfy $f(x) = (x-x_0)^r g(x)$ in a neighborhood of x_0, where $r \in \mathbb{Z}_{\geqslant 0}$, g is continuous at x_0 and $g(x_0) \neq 0$, then we call x_0 an r-fold root of f.

F1) Suppose x_0 is an r-fold root of f where $r > 0$. Prove that if $g(x) = f(x)/(x-x_0)^r$ is continuous, then x_0 is an $(r-1)$-fold root of f'.

Proof: Suppose $f(x) = (x-x_0)^r g(x)$ in the neighborhood $O(x_0)$, then $f'(x) = (x-x_0)^r g'(x) + r(x-x_0)^{r-1} g(x)$ in $O(x_0)$. Therefore let $h(x) = (x-x_0)g'(x) + g(x)$, then $f'(x) = (x-x_0)^{r-1}h(x)$ and $h(x_0) = g(x_0) \neq 0$, so x_0 is an $(r-1)$-fold root of f'.

F2) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x) = 0$ has $n+1$ distinct real roots, then $f^{(n)}(x) = 0$ has at least one root.

Proof: Use induction and Rolle's mean-value theorem to prove that $f^{(n-k)}(x)$ has at least $k+1$ different real roots.

F3) f is differentiable on $\mathbb{R}$. Suppose $f(x) = 0$ has r roots (counting multiplicity), then $f'(x) = 0$ has at least $r-1$ roots (counting multiplicity).

Proof: Combine F1) and F2).

F4) Suppose f is n-times differentiable on $\mathbb{R}$. Prove that if $f(x) = 0$ has exactly $n+1$ roots counting multiplicity, then $f^{(n)}(x) = 0$ has at least one root.

Proof: Use F3) and induction.

8.7 PSG

Let $a \in \mathbb{R}$, $f : (a,\infty) \to \mathbb{R}$ twice differentiable on (a,∞), and

$$M_0 := \sup_{x\in(a,\infty)} |f(x)|,\ M_1 := \sup_{x\in(a,\infty)} |f'(x)|,\ M_2 := \sup_{x\in(a,\infty)} |f''(x)|,$$

are real numbers.

G1) Prove that $M_1^2 \leqslant 4M_0M_2$.

Proof: Let $h = \sqrt{M_0/M_2}$, then for any $x \in (a, \infty)$, there exists $\xi \in (x, x+2h)$ such that

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi) \implies f'(x) = hf''(\xi) + \frac{f(x+2h) - f(x)}{2h}.$$

Therefore $f'(x) \leqslant M_0/h + M_2 h = 2\sqrt{M_0M_2}$, hence $M_1^2 \leqslant 4M_0M_2$.

G2) Let $a = -1$, consider the function

$$f(x) = \begin{cases} 2x^2 - 1, & x \in (-1, 0), \\ \frac{x^2-1}{x^2+1}, & x \in [0, \infty), \end{cases}$$

verify that f is twice differentiable, and $M_0 = 1, M_1 = 4, M_2 = 4$.
Proof: Note that $\lim\limits_{x \to 0^-} f(x) = -1 = f(0)$ so f is continuous, and

$$f'(x) = \begin{cases} 4x, & x \in (-1, 0), \\ \frac{4x}{(x^2+1)^2}, & x \in [0, \infty). \end{cases}$$

f' is also continuous, so

$$f''(x) = \begin{cases} 4, & x \in (-1, 0), \\ 4\frac{1-3x^2}{(x^2+1)^2}, & x \in [0, \infty). \end{cases}$$

Therefore $f \in C^2((-1, \infty))$ and $M_0 = 1, M_1 = 4, M_2 = 4$.

G3) Suppose $\mathbf{f} : (a, \infty) \to \mathbb{R}^n$ is twice differentiable, Let M_0, M_1, M_2 be the least upper bounds of $|\mathbf{f}|, |\mathbf{f}'|, |\mathbf{f}''|$. Prove that $M_1^2 \leqslant 4M_0M_2$.

Proof: Use G1) and Cauchy-Schwarz inequality.

8.8 Problem S: Strum-Liouville Theory

Assume the following uniqueness theorem holds:

> Theorem:
> Suppose $a(t) \in C^1(\mathbb{R})$, $t_0 \in \mathbb{R}$. If $x(t), y(t) \in C^2(\mathbb{R})$ both satisfy the equation
>
> $$x''(t) + a(t)x(t) = 0, y''(t) + a(t)y(t) = 0,$$
>
> and $(x(t_0), x'(t_0)) = (y(t_0), y'(t_0))$, then $x(t) \equiv y(t)$.

(Can be proved using Exercise C3?)
For any $f : \mathbb{R} \to \mathbb{R}$, $t \geqslant 0$, denote
$Z_t(f) = |\{x \in [0, t] : f(x) = 0\}|$.

Part 1

Let $a(t), b(t) \in C^1(\mathbb{R})$ and for any $t \in \mathbb{R}$, $a(t) \leqslant b(t)$. Suppose $x(t), y(t) \in C^2(\mathbb{R})$ satisfy the following equation

$$x''(t) + a(t)x(t) = 0,\ y''(t) + b(t)y(t) = 0.$$

Further assume that $x(t), y(t)$ are not identically zero.

S1) Assume $x(t_1) = 0$, if there exists $t > t_1$, such that $x(t) = 0$. Prove that there exists $t_2 > t_1$ such that $x(t_2) = 0$ and x has no roots in (t_1, t_2). We call t_1, t_2 neighboring roots.

Proof: Consider the set $S = \{t > t_1 : x(t) = 0\}$, and let $t_2 = \inf S$. Note that $|x''(t)| \leqslant |a(t)| \cdot |x(t)|$, so by C3) $x'(t_1) \neq 0$. Assume $x'(t_1) > 0$, since $x \in C^2(\mathbb{R})$, there exists $\varepsilon > 0$ such that $x'(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$, hence $x(t) > 0$ for all $t \in (t_1, t_1 + \varepsilon)$. Therefore $t_2 > t_1$, so by $x \in C(\mathbb{R})$, $x(t_2) = 0$ and x has no roots in (t_1, t_2).

S2) If $t_2 > t_1$ are two neighboring roots of x, prove that y has a root in $(t_1, t_2]$.

Proof: Otherwise assume that x, y are positive on (t_1, t_2), and $y(t_2) \neq 0$. Consider the function $h(t) = x'y - xy'$, then $h'(t) = (b - a)xy \geqslant 0$, so $h(t_2) \geqslant h(t_1) = x'(t_1)y(t_1) \geqslant 0$, but $h(t_2) = x'(t_2)y(t_2) < 0$, a contradiction.

S3) Prove that for any $t \geqslant 0$, $Z_t(y) \geqslant Z_t(x) - 1$.

Proof: Use S2).

S4) Suppose $t_2 > t_1$ and $x(t_1) = x'(t_2) = 0$, prove that

- If $y(t_1) = 0$, then there exists $t_3 \in [t_1, t_2]$, such that $y'(t_3) = 0$.
  Proof: We can assume that $t_2 = \inf\{t > t_1 : x'(t) = 0\}$($t_2 > t_1$ since $x'(t_1) \neq 0$). If there is no such t_3, we can further assume that $x'(t), y'(t), x(t), y(t) > 0$ for all $t \in (t_1, t_2)$. Again consider $h(t) = x'y - xy'$, then $h(t_1) = 0$, $h(t_2) = -x(t_2)y'(t_2) < 0$, but $h'(t) = (b - a)xy \geqslant 0$, leading to contradiction.
- If $y'(t_2) = 0$, then there exists $t_4 \in [t_1, t_2]$ such that $y(t_4) = 0$.
  (The two theorems are similar.)

Part 2

In this section, $p(t) \in C^1(\mathbb{R})$ is a positive function. $x(t), y(t) \in C^2(\mathbb{R})$ are not identically zero and satisfy

$$x''(t) + p(t)x(t) = 0,\ y''(t) + p(t)y(t) = 0.$$

S5) Prove that for any $t \geqslant 0$, $|Z_t(x) - Z_t(y)| \leqslant 1$.

Proof: Use S3).

S6) Prove that

- If t_1, t_2 are neighboring roots of x, then there exists a unique $t_3 \in [t_1, t_2]$ such that $x'(t_3) = 0$.
  Proof: The existence of t_3 is given by Rolle's mean-value theorem. If there exists $t_3 < t_4 \in [t_1, t_2]$ such that $x'(t_3) = x'(t_4) = 0$, then $t_3, t_4 \neq t_1, t_2$ and there exists $t_5 \in [t_3, t_4]$ such that $x''(t_5) = 0$. Hence $x(t_5) = 0$, which contradicts the fact that t_1, t_2 are neighboring roots. Therefore t_3 is unique.
- If t_1', t_2' are neighboring roots of x', then there exists a unique $t_3' \in [t_1', t_2']$ such that $x(t_3') = 0$.
  Proof: Exactly the same.

S7) Prove that

- t_0 is a local maximum of $|x(t)|$ iff $x'(t_0) = 0$.
  Proof: Trivial?
- t_0' is a local maximum of $|x'(t)|$ iff $x(t_0') = 0$.

Part 3

In this section, $p(t) \in C^1(\mathbb{R})$ is monotonically decreasing and $\lim_{t\to\infty} p(t) > 0$. Denote

$$p(\infty) := \lim_{t\to\infty} p(t).$$

$x(t) \in C^2(\mathbb{R})$ is not identically zero and

$$x''(t) + p(t)x(t) = 0.$$

***S8) Calculate**

$$\lim_{t\to\infty} \frac{Z_t(x)}{t}.$$

Solution: By S5) we can ignore initial conditions. First consider the ODE $y''(t)+p(\infty)y(t) = 0$, where one solution is $y = \sin(t\sqrt{p(\infty)})$, so $\lim\limits_{t\to\infty} Z_t(y)/t = \sqrt{p(\infty)}/\pi$.
Since $p(t) \geqslant p(\infty)$, by S3) we know $\lim\limits_{t\to\infty} Z_t(x)/t \geqslant \lim\limits_{t\to\infty} Z_t(y)/t = \sqrt{p(\infty)}/\pi$. For any $\varepsilon > 0$, there exists $M > 0$ such that for any $t > M$, $p(t) < p(\infty) + \varepsilon$. By S3), $\lim_{t\to\infty} Z_t(x)/t \leqslant \sqrt{p(\infty)+\varepsilon}/\pi$. Therefore

$$\lim_{t\to\infty} \frac{Z_t(x)}{t} = \frac{\sqrt{p(\infty)}}{\pi}.$$

S9) Suppose $0 \leqslant t_1 < t_2 < t_3 < \cdots$ are all the roots of $x(t)$ on $[0,\infty)$, $0 \leqslant t_1' < t_2' < \cdots$ are all the roots of $x'(t)$ on $[0,\infty)$. Prove that the sequence $\{|x'(t_k)|\}_{k\geqslant 1}$ is monotonically decreasing and the sequence $\{|x(t_k')|\}_{k\geqslant 1}$ is monotonically increasing, and

$$\lim_{k\to\infty} |x'(t_k)| = \sqrt{p(\infty)} \lim_{k\to\infty} |x(t_k')|.$$

Proof: Consider the (energy) function $E(t) = p(t)x^2(t) + x'(t)^2$, then $E'(t) = p'x^2 \leqslant 0$ so E is monotonically decreasing. For $k \geqslant 1$, $E(t_k) = x'(t_k)^2$ is decreasing, so $\{|x'(t_k)|\}_{k\geqslant 1}$ is decreasing. Likewise, consider $F(t) = x(t)^2 + x'(t)^2/p(t)$, then $F'(t) = -p'(x'/p)^2 \geqslant 0$, so $F(t_k') = x(t_k')^2$ is increasing, and

$$\lim_{k\to\infty} |x'(t_k)| = \sqrt{\lim_{k\to\infty} E(t_k)} = \sqrt{p(\infty) \lim_{k\to\infty} F(t_k)} = \sqrt{p(\infty)} \lim_{k\to\infty} |x(t_k')|.$$

***S10) Suppose $0 \leqslant \tilde{t}_1 < \tilde{t}_2 < \cdots$ are all the roots of $x(t)x'(t)$ on $[0,\infty)$. Prove that the sequence $\{\tilde{t}_{k+1} - \tilde{t}_k\}_{k\geqslant 1}$ is monotonically increasing and calculate its limit.**

Proof: By S6), the roots of x and x' appear alternating in $\{\tilde{t}_k\}$. Since t is a root of x iff t is a root of x'', we only need to prove that if t_1, t_2 are neighboring roots of x, and $t_3 \in [t_1, t_2]$ satisfy $x'(t_3) = 0$, then $t_3 - t_1 \leqslant t_2 - t_3$.

Same as before we can prove that, for $p(t), q(t), x(t), y(t)$ such that $p(0) = q(0)$, $p(t) \leqslant q(t)$, $x'(0) = y'(0) = 0$, $x(0) = y(0)$ and

$$x''(t) + p(t)x(t) = 0,\, y''(t) + q(t)y(t) = 0,$$

then the first roots a, b of x, y satisfy $a \leqslant b$.
Since the sequence is increasing, by S8) we know that
$\lim_{k\to\infty} \tilde{t}_{k+1} - \tilde{t}_k = \frac{1}{2}\lim_{t\to\infty} Z_t(x)/t = \sqrt{p(\infty)}/2\pi$.

9 Homework 9: Stone-Weierstrass Theorem

9.1 PSA

Assume $I = [a, b] \subset \mathbb{R}$, V is a normed linear space.

A1) $\sigma_1, \sigma_2 \in \mathcal{S}$ are two partitions. Prove that for any $\varepsilon > 0$, there exists a partition σ such that $\sigma \prec \sigma_1, \sigma \prec \sigma_2$ and $|\sigma| < \varepsilon$.

Proof: Take $n > 1/\varepsilon$, and let

$$\sigma = \sigma_1 \cup \sigma_2 \cup \left\{ \frac{k}{n}a + \frac{n-k}{n}b : 0 \leqslant k \leqslant n \right\}.$$

A2) Consider the space of simple functions $\mathcal{E}(I)$ with range V. Prove that it is a linear space on $\mathbb{R}$, and the integration operator $\int_a^b : \mathcal{E}(I) \to V$ is well-defined and is linear. Use this to define Riemann integrable functions with range V.

Proof: For any simple function $f = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$ (where A_i are disjoint), let

$$\int_a^b f = \sum_{i=1}^n c_i \mu(A_i)$$

For any function $f : I \to V$, partition $\mathcal{C} = \{x_0, x_1, \cdots, x_n\}$ and $\xi_i \in [x_{i-1}, x_i]$, define

$$\mathcal{R}(f; \mathcal{C}, \xi) = \sum_{k=0}^n f(\xi_i)(x_i - x_{i-1}).$$

Then f is Riemann integrable iff $\lim_{|\mathcal{C}|\to 0} \mathcal{R}(f; \mathcal{C}, \xi)$ exists.

A3) Suppose $f : I \to \mathbb{R}^n$ and f_i be the components of f, then $f \in \mathcal{R}(I)$ iff for every i, $f_i \in \mathcal{R}(I)$.

Proof: Note that

$$\max\{|x_k|\} \leqslant |(x_1, \cdots, x_n)|_{\mathbb{R}^n} \leqslant |x_1| + \cdots + |x_n|.$$

Hence the limit $|\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0$ iff the components of f are all Riemenn integrable.

A4) Assume $a < c < b$, then for any $f \in \mathcal{R}(I)$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Riemann integrable, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: They are both obviously Riemann integrable, and for any partition σ, let $\sigma' = \sigma \cup \{c\} = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are partitions of $[a, c]$ and $[c, b]$, then

$$\underline{S}(f; \sigma) \leqslant \underline{S}(f; \sigma') = \underline{S}(f|_{[a,c]}; \sigma_1) + \underline{S}(f|_{[c,b]}; \sigma_2),$$

and the other side is the same. Hence

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A5) Prove that for any two partition σ and σ', $\underline{S}(f; \sigma) \leqslant \overline{S}(f; \sigma')$. Use this to prove that if $f \in \mathcal{R}(I)$, then $\lim_{|\sigma| \to 0} |\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0$.

Proof: Let $\sigma'' = \sigma \cup \sigma'$, then

$$\underline{S}(f; \sigma) \leqslant \underline{S}(f; \sigma'') \leqslant \overline{S}(f; \sigma'') \leqslant \overline{S}(f; \sigma').$$

If $f \in \mathcal{R}(I)$, then $\sup_\sigma \underline{S}(f; \sigma) = \inf_\sigma \overline{S}(f; \sigma)$ hence

$$\lim_{|\sigma| \to 0} |\underline{S}(f; \sigma) - \overline{S}(f; \sigma)| = 0.$$

A6) $f \in \mathcal{R}(I)$. Prove that if we change the value of f at a finite number of points to g, then g is Riemann integrable and $\int_I g = \int_I f$.

Proof: We can assume that f and g differ only at the point $c \in (a, b)$. Let $M = \sup_{x \in I} |f(x)|$. For any $\varepsilon > 0$, and any partition σ, let $\sigma' = \sigma \cup \{c - \varepsilon, c + \varepsilon\}$, then $|\underline{S}(f; \sigma') - \underline{S}(f; \sigma)| \leqslant 4\varepsilon M \to 0$.

A7) $f \in C([a, b])$. Assume for any $x \in I$, $f(x) \geqslant 0$ and there exists $x_0 \in I$ such that $f(x_0) > 0$. Prove that $\int_a^b f > 0$.

Proof: Since f is continuous and $f(x_0) > 0$, there is an $\varepsilon > 0$ such that for all $y \in (x_0 - \varepsilon, x_0 + \varepsilon)$, $f(y) > 0$. Hence for any partition $\sigma = \{x_0, x_1, \cdots, x_n\}$ such that $|\sigma| < \varepsilon/2$, there is a $k \in \{1, \cdots, n\}$ such that $(x_{k-1}, x_k) \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. Hence $\mathcal{R}(f; \sigma, \xi) > 0$ whenever $|\sigma| < \varepsilon/2$, so $\int_a^b f(x)\,\mathrm{d}x > 0$.

A8) Suppose $f, g \in C^1(I)$, then

$$\int f' \cdot g = f \cdot g - \int f \cdot g'.$$

Proof:

$$d(f \cdot g) = df \cdot g + f \cdot dg.$$

A9) Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is differentiable, f is a continuous function, then

$$\int (f \circ \Phi)\Phi' = \int f.$$

Proof:

$$(f(\Phi(x)))' = f'(\Phi(x))\Phi'(x).$$

9.2 PSB: Calculating Integrals

(1)

$$\begin{aligned}\int \frac{x^5}{1+x}\,\mathrm{d}x &= \int x^4 - x^3 + x^2 - x + 1 - \frac{1}{1+x}\,\mathrm{d}x \\ &= \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C.\end{aligned}$$

(2)

$$\int \sqrt{x\sqrt{x\sqrt{x}}}\,\mathrm{d}x = \int x^{7/8}\,\mathrm{d}x = \frac{8}{15}x^{15/8}.$$

(3)

$$\begin{aligned}\int \left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right)\,\mathrm{d}x &= \int \left(\frac{2}{1-x} + \frac{2}{1+x} - 2\right)\,\mathrm{d}x \\ &= -2x + 2\log\frac{1+x}{1-x} + C.\end{aligned}$$

(4)

$$\int \frac{e^{3x}+1}{1+e^x}\,\mathrm{d}x = \int 1 - e^x + e^{2x}\,\mathrm{d}x = x - e^x + \frac{e^{2x}}{2} + C.$$

(5)

$$\int \sqrt{1-\sin(2x)}\,\mathrm{d}x = \int \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)\,\mathrm{d}x = -\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + C.$$

(6)

$$\int \frac{\cos(2x)}{\cos x - \sin x}\,\mathrm{d}x = \int \cos x + \sin x\,\mathrm{d}x = \sin x - \cos x + C.$$

(7)

$$\int \tan^2 x \,\mathrm{d}x = -x + \tan x + C.$$

(8)

$$\int |x| \,\mathrm{d}x = \frac{x|x|}{2} + C.$$

(9)

$$\int e^{-|x|} \,\mathrm{d}x = -\mathrm{sgn}(x)e^{-|x|} + C.$$

(10)

$$\int \frac{x^2}{(1-x)^{2018}} \,\mathrm{d}x = \frac{1}{2017(1-x)^{2017}} - \frac{1}{1013(1-x)^{2016}} + \frac{1}{2015(1-x)^{2015}}.$$

(11)

$$\int |x-1| \,\mathrm{d}x = \frac{(x-1)|x-1|}{2} + C.$$

(12)

$$\int \frac{1}{\sqrt{b^2+x^2}} \,\mathrm{d}x = \frac{1}{b} \log \frac{1 + \tan \frac{\arctan \frac{x}{b}}{2}}{1 - \tan \frac{\arctan \frac{x}{b}}{2}} + C.$$

(13)

Let $x = t^2$, then

$$\int \frac{\mathrm{d}x}{\sqrt{x}(1+x)} = 2 \arctan \sqrt{x} + C.$$

(14)

$$\int \frac{x^4}{(1-x^5)^4} \,\mathrm{d}x = \frac{1}{5} \int \frac{\mathrm{d}x^5}{(1-x^5)^4} = \frac{1}{15(1-x^5)^3} + C.$$

(15)

$$\int \left(\frac{1}{\sqrt{3-x^2}} + \frac{1}{1-3x^2} \right) \mathrm{d}x = \arcsin \frac{x}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \log \frac{1+\sqrt{3}x}{1-\sqrt{3}x} + C.$$

(16)

$$\int \frac{2x-3}{x^2-3x+8}\,\mathrm{d}x = \log(x^2-3x+8)+C.$$

(17)

$$\int \frac{\mathrm{d}x}{\sin^2\left(2x+\frac{\pi}{4}\right)} = \frac{\tan(2x-\pi/4)}{2}+C.$$

(18)

$$\int \frac{\mathrm{d}x}{1+\cos x} = \tan\frac{x}{2}+C.$$

(19)

$$\int \frac{1}{x^2}\sin\frac{1}{x}\,\mathrm{d}x = \cos\frac{1}{x}+C.$$

(20)

$$\int \cos^5 x\,\mathrm{d}x = \frac{\sin^5 x}{5}-\frac{2\sin^3 x}{3}+\sin x+C.$$

(21)

$$\int \cos(ax)\sin(bx)\,\mathrm{d}x = \frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}+C.$$

(22)

$$\int \frac{\mathrm{d}x}{a\cos x+b\sin x} = \frac{2}{\sqrt{a^2+b^2}}\tanh^{-1}\frac{a\tan(x/2)-b}{\sqrt{a^2+b^2}}+C.$$

(23)

$$\int \frac{\sin(2x)}{a^2\cos^2 x+b^2\sin^2 x}\,\mathrm{d}x = \frac{\log((b^2-a^2)\sin^2 x+a^2)}{b^2-a^2}+C.$$

(24)

$$\int \frac{\mathrm{d}x}{2-\sin^2 x} = \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)+C.$$

(25)

$$\int \frac{\mathrm{d}x}{x \ln x \ln \ln x} = \ln \ln \ln x + C.$$

(26)

$$\int \frac{\log x}{x\sqrt{1+\log x}}\,\mathrm{d}x = \frac{2}{3}(1+\log x)^{3/2} - 2\sqrt{1+\log x} + C.$$

(27)

$$\int \frac{\cos x + \sin x}{(\sin x - \cos x)^{1/3}}\,\mathrm{d}x = \frac{3}{2}(\sin x - \cos x)^{2/3} + C.$$

(28)

$$\int e^{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

(29)

$$\int \frac{x^{n/2}}{1+x^{n+2}}\,\mathrm{d}x = \frac{2}{n+2}\arctan x^{n/2+1} + C.$$

(30)

$$\int \frac{\sqrt{x}}{1-x^{1/3}}\,\mathrm{d}x = 6\arctan x^{1/6} - \frac{6}{5}x^{5/6} - \frac{6}{7}x^{7/6} - 2x^{1/2} - 6x^{1/6} + C.$$

(31)

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$$

(32)

$$\int \frac{\mathrm{d}x}{\cos^4 x} = \frac{\sin x}{2\cos^3 x} + \frac{\sin(3x)}{6\cos^3 x} + C.$$

(33)

$$\int \arcsin^2 x\,\mathrm{d}x = x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C.$$

(34)

$$\int x \arcsin x \, \mathrm{d}x = \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4}\arcsin x(1-2x^2) + C.$$

(35)

$$\int x \arctan x = \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.$$

(36)

$$\int \frac{\arctan x}{x^2} = \log x - \frac{\arctan x}{x} - \frac{1}{2}\log(1+x^2) + C.$$

(37)

$$\int x^2 \sin x = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

(38)

$$\int \frac{x}{\cos^2 x} = x\tan x + \log\cos x + C.$$

(39)

$$\int \log(x+\sqrt{1+x^2}) = x\log(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

(40)

$$\int \sin\log x = \frac{x}{2}(\sin\log x - \cos\log x) + C.$$

(41)

$$\int \sqrt{x^2+a^2} = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{4}\log\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}-x^2} + C.$$

(42)

$$\int \frac{x^2}{\sqrt{x^2-a^2}} = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{a^2}{4}\log\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}} + C.$$

(43)

$$\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} = \sqrt{x^2+1}\log(x+\sqrt{1+x^2}) - x + C.$$

Let $u = \sqrt{x^2+1} + x$ then $\mathrm{d}u/\mathrm{d}x = 1 + x/\sqrt{1+x^2}$, so it becomes

$$\int \frac{(u^2-1)\log u}{2u^2}\,\mathrm{d}x = -\frac{u}{2} + \frac{1}{2u} + \frac{1}{2}u\log u + \frac{\log u}{2u} + C.$$

(44)

$$\int \frac{1}{\sqrt{x^2+a^2}} = \log\frac{\sin t + \cos t}{\sin t - \cos t} + C = \tanh^{-1}\frac{x}{\sqrt{x^2+a^2}} + C.$$

where $t = \frac{1}{2}\arctan(x/a)$.

(45)

$$\int \frac{xe^x}{(1+x)^2} = \frac{e^x}{1+x} + C.$$

(46)

$$\int \arctan(1+\sqrt{x}) = x\arctan(1+\sqrt{x}) - \sqrt{x} + \log(2+2\sqrt{x}+x) + C.$$

(47)

$$\int \left(1-\frac{2}{x}\right)^2 e^x = e^x - \frac{4e^x}{x} + C.$$

since $\int e^x/x^2\,\mathrm{d}x = -e^x/x + \int e^x/x\,\mathrm{d}x$.

(48)

$$\int \sqrt{2+\tan^2 x} = \theta + \log\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} + C.$$

where $\theta = \arcsin(\sin x/\sqrt{2})$.

(49)

$$\int \frac{1}{1+x^3} = -\frac{1}{6}\log(x^2-x+1) + \frac{1}{3}\log(x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

(50)

$$\int \frac{x^7}{x^4+2} = \frac{x^4}{4} - \frac{1}{2}\log(2+x^4).$$

(51)

$$\int \frac{2x^2+1}{(x+3)(x-1)(x-4)} = -\frac{1}{4}\log(1-x) + \frac{11}{7}\log(4-x) + \frac{19}{28}\log(x+3) + C.$$

(52)

$$\int \frac{1+x^2}{1+x^4} = \frac{1}{\sqrt{2}}(\arctan(\sqrt{2}x+1) - \arctan(1-\sqrt{2}x)) + C.$$

Note that

$$\frac{1+x^2}{1+x^4} = \frac{1}{2(x^2+\sqrt{2}x+1)} - \frac{1}{2(-x^2+\sqrt{2}x+1)}.$$

(53)

Let $x = y^6 - 1$ then

$$\begin{aligned} &\int \frac{x}{\sqrt{x+1}+(x+1)^{1/3}} = \int (y^3-1)(1-y+y^2)6y^3\mathrm{d}y \\ =&\frac{2x\sqrt{x+1}}{3} - \frac{3x(x+1)^{1/3}}{4} + \frac{6x(x+1)^{1/6}}{7} - x + \frac{6}{5}(x+1)^{5/6} \\ &- \frac{3}{2}(x+1)^{2/3} + \frac{2\sqrt{x+1}}{3} - \frac{3(x+1)^{1/3}}{4} + \frac{6(x+1)^{1/6}}{7} + C. \end{aligned}$$

(54)

Let $x = y^2$, then

$$\int \frac{1}{\sqrt{x+x^2}} = \int \frac{\mathrm{d}y}{\sqrt{y^2+1}} = \tanh^{-1}\left(\sqrt{\frac{x}{x+1}}\right) + C.$$

(55)

The Poisson kernel

$$\int \frac{1-r^2}{1-2r\cos x + r^2} = 2\arctan(\frac{1+r}{1-r}\tan\frac{x}{2}) + C.$$

(56)

Let $x = \tan\theta$ then

$$\int \frac{1}{x\sqrt{1+x^2}} = \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\arctan x}{2} + C.$$

(57)

Let $t = \tan x/2$ then

$$\int \frac{1}{5-3\cos x} = \frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C.$$

(58)

Let $t = \tan x$, then

$$\int \frac{1}{2+\sin^2 x} = \frac{1}{\sqrt{6}} \arctan(\sqrt{\frac{3}{2}} \tan x) + C.$$

(59)

$$\int \frac{\sin^3 x}{\cos^4 x} = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$$

(60)

Let $t = \cos x$ then

$$\int \frac{1}{\sin x \cos^4 x} = -\int t^{-4} + t^{-2} + \frac{1}{1-t^2} = \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \frac{1}{2} \log \frac{1+\cos x}{1-\cos x} + C.$$

9.3 Problem W: Stone-WeierstraSS Theorem

Part 1: Approximating $|x|$

W1) (Dini) Suppose $K \subset \mathbb{R}^n$ is compact, $f_n : K \to \mathbb{R}$ is a sequence of continuous functions, which converges point-wise to $f : K \to \mathbb{R}$. If f is continuous and $f_n \leqslant f_{n+1}$, then f_n converges uniformly to f.

Proof: For any $\varepsilon > 0$, and any $x \in K$, there is an integer $n_x > 0$ such that $|f_{n_x}(x) - f(x)| < \varepsilon/4$. There exists $\delta > 0$, such that $\forall y \in \mathrm{B}(x, \delta) \cap K$, $|f(x) - f(y)| < \varepsilon/4$ and $|f_{n_x}(x) - f_{n_x}(y)| < \varepsilon/4$, then $|f_{n_x}(y) - f(y)| < \varepsilon/4$. Note that $K \subset \bigcup_{x \in K} \mathrm{B}(x, \delta_x)$ hence we can choose a finite set of $x_1, x_2, \cdots, x_N$ such that $K \subset \bigcup_{i=1}^N \mathrm{B}(x_i, \delta_{x_i})$. Let $M = \max\{n_{x_i} : i = 1, 2, \cdots, N\}$ then for any $m \geqslant M$ and $x \in K$, $|f_m(x) - f(x)| < \varepsilon$. Hence f_n converges uniformly to f.

W2) Consider the interval $[-1, 1]$. Define inductively a sequence of polynomials:

$$P_0(x) = 0,\ P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x)).$$

Prove that for any n, x, $0 \leqslant P_n(x) \leqslant P_{n+1}(x) \leqslant |x|$.
Proof: Assume $x > 0$, we prove by induction. If $t = P_n(x) \in [0, x]$, then

$$P_{n+1}(x) = \frac{1}{2}x^2 - \frac{1}{2}(t-1)^2 + \frac{1}{2} \leqslant \frac{1}{2}(x^2 - (1-x)^2 + 1) = x,$$

and $P_{n+1}(x) \geqslant P_n(x) = t$, hence $P_{n+1}(x) \in [0, x]$.

W3) Prove that $|x|$ can be uniformly approximated by polynomials on the interval $[-1, 1]$, i.e. for any $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(x)$ such that $\sup_{x \in [-1,1]} ||x| - P_\varepsilon(x)| < \varepsilon$.

Proof: By W2), the sequence of polynomials $\{P_n\}$ converge point-wise to $|x|$, hence by W1) P_n converge uniformly to $|x|$.

Part 3: Bernstein Polynomial

Assume $I = [0,1]$, and n is an integer.

W4) For any $0 \leqslant k \leqslant n$, define $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Prove that

$$\sum_{0 \leqslant k \leqslant n} p_{n,k}(x) \left(x - \frac{k}{n} \right)^2 = \frac{x(1-x)}{n}.$$

Proof:
Note that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} x^2 = x^2,$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} = x,$$

$$\sum_{k=0}^{n} p_{n,k}(x) k(k-1) = n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1-x)^{n-k} = n(n-1)x^2.$$

Therefore

$$\sum_{k=0}^{n} p_{n,k}(x) \left(x - \frac{k}{n} \right)^2 = \frac{x(1-x)}{n}.$$

W5) For any $f \in C([0,1])$, define

$$B_{f,n} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

For $x \in [0,1]$, prove that

$$|f(x) - B_{f,n}(x)| \leqslant \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x).$$

Proof: Note that

$$\sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k} = f(x).$$

W6) For any $f \in C([0,1])$, prove that for any $\varepsilon > 0$, there exists n such that $\|f - B_{f,n}\|_\infty < \varepsilon$.

Proof:
Let

$$\mathrm{I} = \sum_{|m-nx| < n^{3/4}} \left(f(x) - f\left(\frac{m}{n}\right) \right) p_{n,m}(x),$$

$$\mathrm{II} = \sum_{|m-nx| > n^{3/4}} \left(f(x) - f\left(\frac{m}{n}\right) \right) p_{n,m}(x).$$

Then $|f - B_{f,n}| \leqslant |\mathrm{I}| + |\mathrm{II}|$.
For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall x \in [0,1]$, $n \geqslant N \implies |\mathrm{I}| < \varepsilon$, since

$$|\mathrm{I}| \leqslant \sup_{|x-m/n|<n^{-1/4}} |f(x) - f(m/n)| \to 0.$$

Suppose $M = \sup_{x\in[0,1]} |f(x)|$, then

$$|\mathrm{II}| \leqslant 2M \sum_{|m-nx|>n^{3/4}} p_{n,m}(x) \leqslant 2M\sqrt{n} \sum_{m=0}^{n} (x - m/n)^2 p_{n,m}(x) = \frac{2Mx(1-x)}{\sqrt{n}}.$$

Hence $\|f - B_{f,n}\|_\infty \to 0$.

Part 3: Stone-Weierstrass Theorem

W7-14):

Let X be a compact Hausdorff space, $\mathcal{A} \subset C(X,\mathbb{R})$ satisfy the following properties:
(a) $\forall c \in \mathbb{R}$, $c \cdot 1_X \in \mathcal{A}$, (b) $\forall f, g \in \mathcal{A}$, $f+g, f-g, fg \in \mathcal{A}$,
(c) $\mathcal{A}$ can separate any pair of points in X.
Then $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

Lemma 1

There is a list of polynomials $\{P_n(x)\}$ that converges uniformly to $|x|$ on $[-1,1]$.

Lemma 2

If $\mathcal{A}$ is a subspace of $C(X,\mathbb{R})$, such that (a) $\mathcal{A}$ is a lattice, (b) $1_X \in \mathcal{A}$, and (c) $\mathcal{A}$ can separate any pair of points, then $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

Proof of main theorem

Assume WLOG $\mathcal{A}$ is closed, then by Lemma 1, $\forall f \in \mathcal{A}$, $P_n(f) \in \mathcal{A}$, hence $|f| \in \mathcal{A}$.(Since X is compact, $|f|$ is bounded.) Note that

$$\max\{f,g\} = \frac{1}{2}(|f+g| + |f-g|), \min\{f,g\} = \frac{1}{2}(|f+g| - |f-g|).$$

Hence $\mathcal{A}$ is a lattice, by Lemma 2 $\mathcal{A} = C(X,\mathbb{R})$.

Proof of Lemma 1

Proof 1: Let

$$Q_n(x) = \int_0^x (1-t^2)^n\, dt \Big/ \int_0^1 (1-t^2)^n\, dt.$$
$$P_n(x) = \int_0^x Q_n(t)\, dt.$$

Note that

$$\int_\varepsilon^1 (1-t^2)^n\, dt \leqslant (1-\varepsilon^2)^n(1-\varepsilon) \to 0$$

Hence (combined with Wallis's formula), $P_n(x)$ converges uniformly to $|x|$ on $[a,b]$.
Proof 2: WLOG change the interval to $[-1/2,1/2]$. The series

$$(1-t)^{1/2} = 1 + \sum_{n=1}^{\infty} (-t)^n \binom{\frac{1}{2}}{n} = 1 - \sum_{n=1}^{\infty} c_n t^n.$$

converges when $|t|<1$. Hence $\forall \varepsilon > 0$, there exists $Q \in \mathbb{R}[x]$ such that $\sup_{|t|\leqslant 1/2} |Q(t) - (1-t)^{1/2}| < \varepsilon/2$.
Let $t = 1-x^2$, then $|Q(1-x^2) - |x|| < \varepsilon/2$, so $P(x) = Q(1-x^2) - Q(1)$ converges to $|x|$ uniformly on $[-1/2,1/2]$.

Proof of Lemme 2

Step 1: Take any $f \in C(X,\mathbb{R})$, and any $x, y \in X$, we can find $g_{xy} \in \mathcal{A}$, such that $g_{xy}(x) = f(x), g_{xy}(y) = f(y)$. Since there exists $u \in \mathcal{A}$ such that $u(x) \neq u(y)$,

$$\begin{pmatrix} u(x), 1 \\ u(y), 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} f(x) \\ f(y) \end{pmatrix}$$

has a solution. (If $x = y$ it is trivial.)
Step 2:
For all $\varepsilon > 0$, $x, y \in X$, there is an open neighborhood $O_{x,y}$ of y, such that $\forall z \in O_{x,y}$, $f(z) - g_{xy}(z) \leqslant \varepsilon$. Note that $\bigcup_{y\in X} O_{x,y} = X$, so by X is compact, there is a list $y_1, \cdots, y_N$ such that $\bigcup_{k\leqslant N} O_{x,y_k} = X$. Let $h_x = \max\{g_{xy_k} : k \leqslant N\}$, then $h_x(y) - f(y) \geqslant -\varepsilon$, and $f(x) = h_x(x)$.
Step 3:
For all $x \in X$, there is an open neighborhood G_x of x, such that $\forall z \in G_x$, $h_x(z) - f(z) \leqslant \varepsilon$. Note that $\bigcup_{x\in X} G_x = X$, so by X is compact, there is a list $x_1, \cdots, x_M$ such that $\bigcup_{k\leqslant M} G_x = X$. Let $F = \min\{h_{x_k} : k \leqslant M\}$, then $|F(x) - f(x)| \leqslant \varepsilon, \forall x \in X$.
Therefore $\bar{\mathcal{A}} = C(X,\mathbb{R})$.

For complex numbers, there is an additional requirement: for any $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$.

W15-16):

It is easy to see that polynomials and trigonometric polynomials both satisfy the requirements of the theorem.

10 Homework 10: Irrationality of π

10.1 PSA

A1) Construct continuous functions $f_n, f \in C([0,1])$, such that for every $x \in [0,1]$, when $n \to \infty$, $f_n(x) \to f(x)$, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Solution: Let $f_n(x) = nxe^{-nx^2}$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2} \int_0^1 e^{-nx^2}\,d(x^2) = \lim_{n\to\infty} \frac{n}{2}\left(\frac{1}{n} - \frac{1}{ne^n}\right) = \frac{1}{2}$$

Hence $\lim_{n\to\infty} \int_0^1 f_n = 1/2 \neq 0 = \int_0^1 f$.

A2) $\alpha \in \mathbb{R}_{\geqslant 0}$**. Prove that** $\int_{100}^{\infty} \frac{dx}{x \log^{\alpha}(x)}$ **converges iff** $\alpha > 1$**.**

Proof: Substitute $y = \log x$, then

$$\int_{100}^{\infty} \frac{\mathrm{d}x}{x \log^{\alpha}(x)} = \int_{\log 100}^{\infty} \frac{\mathrm{d}y}{y^{\alpha}}$$

which converges iff $\alpha > 1$.

A3) f, F **are defined on** I**, and for every bounded closed interval** $J \subset I$**,** f, F **are both Riemann integrable on** J**. Assume for all** $x \in I$**,** $|f(x)| \leqslant F(x)$**. Then if the improper integer of** F **on** I **converges, so does** f**.**

Proof: This is because

$$\int_I f(x)\,\mathrm{d}x \text{ converges } \iff \forall \varepsilon > 0 \exists N \forall u, v \in I, N < u < v, |\int_u^v f(x)\,\mathrm{d}x| < \varepsilon.$$

A4) Prove the integrals below converge:

(1) $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$ (2) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}}$ (3) $\int_1^{\infty} \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x$
(1):

$$\int_0^{\infty} e^{-x^2}\,\mathrm{d}x \leqslant 1 + \int_1^{\infty} e^{-x}\,\mathrm{d}x \leqslant 1 + \frac{1}{e}.$$

(2):

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^3}} \leqslant \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} = 2.$$

(3):

$$\int_1^{\infty} \frac{(\log x)^2}{1+x(\log x)^5}\,\mathrm{d}x \leqslant 5000 + \int_{100}^{\infty} \frac{1}{x(\log x)^3}\,\mathrm{d}x, \text{ which converges by A2.}$$

A5) Prove the series below converge:

(1) $\sum_{n=1}^{\infty} e^{-n}(n^2 + \log n)$ (2) $\sum_{n=1}^{\infty} \frac{\log n}{1+n(\log n)^3}$
(1):

$$\sum_{n=1}^{\infty} e^{-n}(n^2 + \log n) \leqslant \sum_{n=1}^{\infty} \frac{2n^2}{e^n} \leqslant 2\int_0^{\infty} x^2 e^{-x}\,\mathrm{d}x = 4.$$

(2):

$$\sum_{n=1}^{\infty} \frac{\log n}{1+n(\log n)^3} \leqslant \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \leqslant \frac{1}{2(\log 2)^2} + \int_2^{\infty} \frac{1}{x(\log x)^2}\,\mathrm{d}x \leqslant 3.$$

A6) Calculate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^{\alpha}}{n^{\alpha+1}}, \alpha > -1.$$

Solution:

$$\sum_{k=1}^{n} k^\alpha \leqslant \int_1^{n+1} x^\alpha \,\mathrm{d}x = \frac{1}{\alpha+1}((n+1)^{\alpha+1}-1).$$
$$\sum_{k=1}^{n} k^\alpha \geqslant 1+\int_1^{n} x^\alpha \,\mathrm{d}x = 1+\frac{1}{\alpha+1}n^{\alpha+1}.$$

Therefore

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^\alpha}{n^{\alpha+1}} = \frac{1}{\alpha+1}.$$

A7) Calculate $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x$, to show that $\pi = 3.14\cdots$.

Solution:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x = \int_0^1 x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\,\mathrm{d}x = \frac{22}{7}-\pi.$$
$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \leqslant \int_0^1 \frac{x^3(1-x)^4}{2}\,\mathrm{d}x = \frac{1}{560} < 0.02, \frac{22}{7} > 3.1428.$$

A8) Assume $a, b, n \in \mathbb{Z}$, let

$$f_{a,b;n} = \frac{x^n(a-bx)^n}{n!}.$$

- Prove that for $k = 0, 1, \cdots, 2n$, $f_{a,b;n}^{(k)}(x) \in \mathbb{Z}$ when $x = 0, \frac{a}{b}$. See B10)
- If $\pi = \frac{a}{b} \in \mathbb{Q}$, then for every $n \in \mathbb{N}$,

$$\int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x$$

is an integer.
Proof: By Darboux's formula of integration of parts

$$\int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x = \sum_{k=0}^{2n} f_{a,b;n}^{(k)}(x)\sin\left(x-\frac{(k+1)\pi}{2}\right)\Big|_0^\pi \in \mathbb{Z}.$$

- Prove that $\pi \notin \mathbb{Q}$.
  Proof: Let $n = 2a^4+10$, then $\forall 0 \leqslant x \leqslant a/b$,

$$f_{a,b;n} \leqslant \frac{a^{2n}}{n!} < \frac{1}{2}\frac{(a^4)^{n/2}}{n\cdot(n-1)\cdots\left(\frac{n}{2}\right)} < \frac{1}{2}.$$

Hence

$$0 < \int_0^\pi f_{a,b;n}(x)\sin x\,\mathrm{d}x < \frac{1}{2}\int_0^\pi \sin x\,\mathrm{d}x = 1,$$

leading to contradiction.

A9) Let $I_n = \int_0^{\pi/2} \sin^n x \,\mathrm{d}x$, prove that $I_n \sim \sqrt{\frac{\pi}{2n}}$.

Proof: Since $I_n = \frac{n-1}{n} I_{n-2}$,

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ is even,} \\ \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is odd.} \end{cases}$$

Combined with $I_{2n+1} < I_{2n} < I_{2n-1}$, we get

$$\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n},$$

where

$$0 < -\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} + \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n(2n+1)} < \frac{\pi}{4n}.$$

Therefore

$$\lim_{n\to\infty} \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n} = \frac{\pi}{2}.$$

Hence $I_n \sim \sqrt{\frac{\pi}{2n}}$.

A10) Assume $f : [0,1] \to [0,1]$ is monotonously increasing, $g = f^{-1} : [0,1] \to [0,1]$ is its inverse, and f, g are both continuously differentiable, then

$$\int_0^1 f(x)\,\mathrm{d}x + \int_0^1 g(x)\,\mathrm{d}x = 1.$$

Proof: We show that

$$\int_0^x f(t)\,\mathrm{d}t + \int_0^{f(x)} g(t)\,\mathrm{d}t = xf(x), \forall 0 \leqslant x \leqslant 1.(1)$$

$x = 0$ is trivial, hence it suffices to show that the derivatives of the two sides match.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\,\mathrm{d}t = f(x), \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{f(x)} g(t)\,\mathrm{d}t = f'(x)\cdot g(f(x)) = xf'(x).$$

Hence (1) holds.

A11) Prove that

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

Proof: By Dirichlet's test, $\sum_{k=0}^{\infty}(-1)^k(1-\varepsilon)^{2k+1}/(2k+1)$ converges uniformly.
Hence for any $\delta > 0$, there exists an $N \in \mathbb{Z}$ such that

$$|\sum_{k=N}^{\infty}\frac{(-1)^k x^{2k}}{2k+1}| < \delta, \forall x \in [0,1].$$

Then $\forall \varepsilon < \frac{\delta}{N}$,

$$\begin{aligned} &\left|\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1} - \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\right| \\ \leqslant &\sum_{k=0}^{N-1}\frac{|(1-\varepsilon)^{2k+1}-1|}{2k+1} + \left|\sum_{k=N}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}\right| + \left|\sum_{k=N}^{\infty}\frac{(-1)^k}{2k+1}\right| < 3\delta. \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k(1-\varepsilon)^{2k+1}}{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

A12) For any continuous function $f:[a,b]\times[c,d]\to\mathbb{R},(x,y)\mapsto f(x,y)$, where $a,b,c,d\in\mathbb{R}$, show that f is uniformly continuous on $[a,b]\times[c,d]$.

Proof: $K=[a,b]\times[c,d]$ is a compact set. Consider an arbitrary $\varepsilon>0$.
For any $x\in K$, there is an open ball $B(x,2r_x)$ with center x such that $\forall y\in B(x,2r_x),|f(x)-f(y)|<\varepsilon/2$. Let $O_x=B(x,r_x)$. Note that $\bigcup_{x\in K}O_x=K$ and K is compact, hence we can find $x_1,\cdots,x_n$ such that $\bigcup_{k\leqslant n}O_{x_k}=K$.
Let $\delta=\min\{r_{x_k}:k\leqslant n\}$, then $\forall|u-v|<\delta$, suppose $u\in O_{x_1}$, then

$$|v-x_1|\leqslant|v-u|+|u-x_1|<2r_{x_1}\implies v\in B(x_1,2r_{x_1}).$$

Hence

$$|f(u)-f(v)|\leqslant|f(u)-f(x_1)|+|f(v)-f(x_1)|<\varepsilon.$$

Therefore f is uniformly continuous on K.

10.2 PSB: On $\zeta(2)$

Part 1: The sequence $\{\sum_{k=1}^{n}1/k^p\}$

Define the sequence $S_n(p)=\sum_{k=1}^{n}1/k^p$ where $p\in\mathbb{Z}_{\geqslant 1}$.

B1) Prove that for any $k\in\mathbb{Z}_{\geqslant 1}$, we have

$$\frac{1}{(k+1)^p}\leqslant\int_k^{k+1}\frac{1}{x^p}\,\mathrm{d}x\leqslant\frac{1}{k^p}.$$

Proof: $\frac{1}{(k+1)^p}\leqslant\frac{1}{x^p}\leqslant\frac{1}{k^p},\forall k\leqslant x\leqslant k+1$.

B2) Prove that for any $n\in\mathbb{Z}_{\geqslant 2}$, we have

$$S_n(p)-1\leqslant\int_1^n\frac{1}{x^p}\,\mathrm{d}x\leqslant S_{n-1}(p).$$

Proof:

$$S_n(p)-1=\sum_{k=1}^{n-1}\frac{1}{(k+1)^p}\leqslant\sum_{k=1}^{n-1}\int_k^{k+1}\frac{1}{x^p}\,\mathrm{d}x=\int_1^n\frac{1}{x^p}\,\mathrm{d}x.$$

Likewise we have $\int_1^n\frac{1}{x^p}\,\mathrm{d}x\leqslant S_{n-1}(p)$.

B3) Let $p\in\mathbb{Z}_{\geqslant 1}$. Prove that $x\mapsto\frac{1}{x^p}$ is integrable on $[1,\infty)$ iff $p\geqslant 2$.

Proof: For $p\geqslant 2$,

$$\lim_{n\to\infty}\int_1^n\frac{1}{x^p}\,\mathrm{d}x=\lim_{n\to\infty}\frac{1}{1-p}x^{1-p}\Big|_1^n=\frac{1}{1-p}.$$

If $p=1$, $\lim_{n\to\infty}\int_1^n\frac{1}{x}\,\mathrm{d}x=\lim_{n\to\infty}\log x\big|_1^n=\infty$.

B4) Prove that $\{S_n(p)\}_{n\geqslant 1}$ converges iff $p \geqslant 2$. For $p \geqslant 2$ let

$$\zeta(p) = \lim_{n\to\infty} S_n(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Proof: If $p = 1$, $S_n(p) \geqslant \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \to \infty$.
For $p \geqslant 2$, $S_n(p) \leqslant S_{n+1}(p)$, and $S_n(p) \leqslant 1 + \int_1^n \frac{1}{x^p}\,\mathrm{d}x \leqslant 1 + \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$. Hence $\lim_{n\to\infty} S_n(p)$ exists.

Part 2: Calculate $\zeta(2)$

(We can also use Bernolli numbers and the Taylor expansion of $\tan x$).

Let $h(t) = \frac{t^2}{2\pi} - t$, $\varphi : [0,\pi] \to \mathbb{R}$:

$$\varphi(x) = \begin{cases} -1, & x = 0; \\ \frac{h(x)}{2\sin\left(\frac{x}{2}\right)}, & 0 < x \leqslant \pi. \end{cases}$$

B5) Prove that $\varphi \in C^1([0,\pi])$.

Proof:

$$\lim_{x\to 0} \frac{h(x)}{2\sin\left(\frac{x}{2}\right)} = \lim_{x\to 0} \frac{-x + o(x)}{2\sin\left(\frac{x}{2}\right)} = -1 = \varphi(0).$$

Hence $\varphi \in C^1([0,\pi])$.

B6) For all $k \geqslant 1$, calculate

$$\int_0^\pi h(x)\cos(kx)\,\mathrm{d}x.$$

Solution:

$$\begin{aligned}
\int_0^\pi \left(\frac{x^2}{2\pi} - x\right)\cos(kx)\,\mathrm{d}x &= \frac{1}{k}\int_0^\pi \left(\frac{x^2}{2\pi} - x\right)\mathrm{d}\sin(kx) \\
&= -\frac{1}{k}\int_0^\pi \sin(kx)\left(\frac{x}{\pi} - 1\right)\mathrm{d}x \\
&= \frac{1}{k^2}\int_0^\pi \left(\frac{x}{\pi} - 1\right)\mathrm{d}\cos(kx) \\
&= \frac{1}{k^2} - \frac{1}{\pi k^2}\int_0^\pi \cos(kx)\,\mathrm{d}x = \frac{1}{k^2}.
\end{aligned}$$

B7) Prove that there is a constant λ, such that for any $x \in (0,\pi)$,

$$\sum_{k=1}^{n} \cos(kx) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)} - \lambda.$$

Proof: Note that $2\cos(kx)\sin(\frac{x}{2}) = \sin(k+1/2)x - \sin(k-1/2)x$, hence

$$\sum_{k=1}^{n} \cos(kx)\cdot 2\sin\frac{x}{2} = \sin\left(n + \frac{1}{2}\right)x - \sin\frac{x}{2}, \lambda = \frac{1}{2}.$$

B8) Prove that for any $\psi \in C^1([0,\pi])$,

$$\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x\,\mathrm{d}x = 0.$$

Proof: Since $\sin(n+1/2)x = c_1\sin nx + c_2\cos nx$, where c_1, c_2 are constant, it suffices to show that

$$\lim_{n\to\infty}\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x = \lim_{n\to\infty}\int_0^\pi \psi(x)\cos(2nx)\,\mathrm{d}x = 0.$$

Note that

$$\begin{aligned}
\int_0^\pi \psi(x)\sin(2nx)\,\mathrm{d}x &= \sum_{k=1}^n \int_{(k-1)\pi/n}^{k\pi/n} \psi(x)\sin(2nx)\,\mathrm{d}x \\
&= \sum_{k=1}^n \frac{1}{2n}\int_0^{2\pi} \psi\left(\frac{x}{2n}+\frac{(k-1)\pi}{n}\right)\sin x\,\mathrm{d}x \\
\left(t=\frac{(k-1)\pi}{n}\right) &\leqslant \sum_{k=1}^n \frac{\pi}{n}\sup_{x\leqslant\pi}\left|\psi\left(\frac{x+\pi}{2n}+t\right)-\psi\left(\frac{x}{2n}+t\right)\right| \\
&\leqslant \pi \sup_{0\leqslant x\leqslant \pi-\pi/2n}\left|\psi\left(x+\frac{\pi}{2n}\right)-\psi(x)\right| \to 0.
\end{aligned}$$

since ψ is uniformly continuous on $[0,\pi]$.

B9) Prove that $\zeta(2) = \frac{\pi^2}{6}$.

Proof:

$$\begin{aligned}
\zeta(2) &= \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2} = \lim_{n\to\infty}\sum_{k=1}^n \int_0^\pi h(x)\cos(kx)\,\mathrm{d}x \\
&= \lim_{n\to\infty}\int_0^\pi \psi(x)\sin(n+1/2)x - \frac{1}{2}\left(\frac{x^2}{2\pi}-x\right)\,\mathrm{d}x \\
(\text{B8}) &= \frac{1}{2}\int_0^\pi \left(x-\frac{x^2}{2\pi}\right)\,\mathrm{d}x = \frac{\pi^2}{6}.
\end{aligned}$$

Part 3: $\zeta(2)$ is irrational

Otherwise assume $\pi^2 = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

B10) Define a sequence of polynomials $f_n(x) = \frac{x^n(1-x)^n}{n!}$, where $n \in \mathbb{Z}_{\geqslant 1}$. Prove that for any $k \in \mathbb{Z}$, $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

Proof: If $k \leqslant n-1$, then $f_n^{(k)}(0) = f_n^{(k)}(1) = 0$. If $k \geqslant n$, then

$$\text{if } x^n(1-x)^n = \sum_{k=n}^{2n} c_k x^k, \text{then } f_n^{(k)}(x) = \sum_{m=n}^{2n} c_k \binom{m}{k} x^{m-k} \in \mathbb{Z}[x].$$

Hence $f_n^{(k)}(0), f_n^{(k)}(1) \in \mathbb{Z}$.

B11) Define the sequence

$$F_n(x) = b^n(\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)).$$

Prove that $F_n(0), F_n(1) \in \mathbb{Z}$.
Proof: For $0 \leqslant k \leqslant n$, $b^n\pi^{2n-2k}, f_n^{(2k)}(x) \in Z$, when $x \in \{0,1\}$.

B12) For $n \geqslant 1$, define $\{g_n\}_{n\geqslant 1}, \{A_n\}_{n\geqslant 1}$ as below:

$$g_n(x) = F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x),\ A_n = \pi \int_0^1 a^n f_n(x) \sin(\pi x)\,\mathrm{d}x.$$

Prove that $A_n \in \mathbb{Z}$ and $g_n' = \pi^2 a^n f_n(x)\sin(\pi x)$.
Proof: Note that

$$\begin{aligned} g_n'(x) &= b^n \pi^{2n} \sum_{k=0}^{n} \left(f_n^{(2k)}(x)\sin(\pi x) - \pi f_n^{(2k+1)}(x)\cos(\pi x) \right)' (-\pi^2)^k \\ &= b^n \pi^{2n+2} f_n(x)\sin(\pi x). \end{aligned}$$

And

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^1 \mathrm{d}g_n(x) = \frac{1}{\pi}(g_n(1) - g_n(0)) \\ &= F_n(0) + F_n(1) \in \mathbb{Z}. \end{aligned}$$

B13) Prove that there exists $n \in \mathbb{Z}$ such that for all $x \in [0,1]$, $a^n f_n(x) < 1/2$.

Proof:

$$f_n(x) = \frac{1}{n!}(x(1-x))^n \leqslant \frac{1}{n!4^n} \to 0.$$

B14) Prove that there exists $n \in \mathbb{Z}$ such that $A_n \in (0,1)$, leading to contradiction.

Proof: $f_n, \sin(\pi x) \geq 0$, when $x \in [0,1]$, hence $A_n > 0$.
Take n such that $a^n f_n < 1/2$ then $A_n < \frac{\pi}{2}\int_0^1 \sin(\pi x)\,\mathrm{d}x = 1$.
Therefore $A_n \in (0,1)$, contradicting with $A_n \in \mathbb{Z}$.

10.3 PSC: Calculation of Integrals

$a \neq 0, b \neq 0$

(1) $\int_0^\pi \sin^3 x\,\mathrm{d}x$

$$\int_0^\pi \sin^3(x)\,\mathrm{d}x = -2\int_0^{\pi/2} \sin^2(x)\,\mathrm{d}\cos(x) = 2\int_0^1 (1-x^2)\,\mathrm{d}x = \frac{4}{3}.$$

(2) $\int_{-\pi}^\pi x^2 \cos x\,\mathrm{d}x$

$$\int_{-\pi}^\pi x^2\cos(x)\,\mathrm{d}x = (x^2-2)\sin(x) + 2x\cos(x)\Big|_{-\pi}^{\pi} = -4\pi.$$

(3) $\int_0^1 \frac{x}{1+\sqrt{1+x}}\,\mathrm{d}x$

$$\int_0^1 \frac{x}{1+\sqrt{1+x}}\,\mathrm{d}x = \int_0^1 \sqrt{1+x} - 1\,\mathrm{d}x = \frac{2}{3}(1+x)^{3/2} - x\Big|_0^1 = \frac{4\sqrt{2}-5}{3}.$$

(4) $\int_0^{\sqrt{3}} x\arctan x\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\sqrt{3}} x\arctan x\,\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{3}} \arctan x\,\mathrm{d}x^2 \\ &= \frac{1}{2}x^2\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,\mathrm{d}x \\ &= \frac{3}{2}\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\arctan\sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.\end{aligned}$$

(5) $\int_{-1}^0 (2x+1)\sqrt{1-x-x^2}\,\mathrm{d}x$

$$\int_{-1}^0 (2x+1)\sqrt{1-x-x^2}\,\mathrm{d}x = \int_{-1}^1 \frac{y}{4}\sqrt{5-y^2}\,\mathrm{d}y = 0$$

(6) $\int_{\frac{1}{e}}^e |\log x|\,\mathrm{d}x$

$$\begin{aligned}\int_{\frac{1}{e}}^e |\log x|\,\mathrm{d}x &= \int_1^e \log x\,\mathrm{d}x + \int_1^{1/e} \log x\,\mathrm{d}x \\ &= (x\log x - x)\Big|_1^e + (x\log x - x)\Big|_1^{1/e} = 2 - \frac{2}{e}\end{aligned}$$

(7) $\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x$

$$\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x = a^4\int_0^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{a^4\pi}{16}$$

(8) $\int_0^{\log 2} \sqrt{e^x-1}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\log 2} \sqrt{e^x-1}\,\mathrm{d}x &= \int_1^2 \frac{\sqrt{y-1}}{y}\,\mathrm{d}y = \int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \\ &= \int_0^{\pi/4} 2\tan^2\theta\,\mathrm{d}\theta = 2 - \frac{\pi}{2}.\end{aligned}$$

(9) $\int_1^2 x^{100}\log x\,\mathrm{d}x$

$$\int_1^2 x^{100}\log x\,\mathrm{d}x = \int_1^2 \log x\,\mathrm{d}\frac{x^{101}}{101} = \frac{2^{101}\log 2}{101} - \int_1^2 \frac{x^{100}}{101}\,\mathrm{d}x$$
$$= \frac{2^{101}\log 2}{101} - \frac{2^{101}-1}{101^2}.$$

(10) $\int_0^a \log(x+\sqrt{x^2+a^2})\,\mathrm{d}x$

$$\int_0^a \log(x+\sqrt{x^2+a^2})\,\mathrm{d}x =$$

$$\begin{aligned}\int_0^a \log(x+\sqrt{x^2+a^2})\,\mathrm{d}x &= a\int_0^1 \log a + \log(t+\sqrt{t^2+1})\,\mathrm{d}t \\ &= a\log a + a\int_0^1 \log(t+\sqrt{t^2+1})\,\mathrm{d}t \\ &= a\log a + (\log(1+\sqrt{2})+\sqrt{2}-1)a.\end{aligned}$$

$$\begin{aligned}\int_0^1 \log(x+\sqrt{x^2+1})\,\mathrm{d}x &= x\log(x+\sqrt{x^2+1})\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x \\ (x=\tan\theta) &= \log(1+\sqrt{2}) + \int_0^{\pi/4} \frac{1}{\cos^2\theta}\,\mathrm{d}\cos\theta \\ &= \log(1+\sqrt{2})+\sqrt{2}-1.\end{aligned}$$

(11) $\int_0^{\pi/2} \frac{\cos x\sin x}{a^2\sin^2 x+b^2\cos^2 x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x\sin x}{a^2\sin^2 x+b^2\cos^2 x} &= \int_0^{\pi/2} \frac{\sin 2x}{a^2+b^2+(b^2-a^2)\cos 2x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^1 \frac{1}{(a^2+b^2)+(b^2-a^2)t}\,\mathrm{d}t \\ &= \frac{1}{2(a^2-b^2)}\log\left(\frac{a^2}{b^2}\right).\end{aligned}$$

(12) $\int_0^{\pi/2} \frac{\cos^2 x}{\cos x+\sin x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos^2 x}{\cos x+\sin x}\,\mathrm{d}x &= \int_0^{\pi/2} \frac{\sin^2 x}{\cos x+\sin x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cos x+\sin x}\,\mathrm{d}x \\ &= \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{2}\sin x}\,\mathrm{d}x \\ &= -\frac{\log\tan\left(\frac{\pi}{8}\right)}{\sqrt{2}}.\end{aligned}$$

(13) $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x}\,\mathrm{d}x$

See (12)

(14) $\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x &= \int_0^{\pi/4} \log\frac{\sin x+\cos x}{\cos x}\,\mathrm{d}x \\ &= \int_0^{\pi/4} \log\frac{\sqrt{2}\sin(x+\pi/4)}{\cos x}\,\mathrm{d}x \\ &= \frac{\pi}{8}\log 2.\end{aligned}$$

(15) $\int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{1-x}{5-2x}\,\mathrm{d}x = \frac{1}{2}-\frac{3}{4}\log\frac{5}{3}, \\ \int_1^2 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x}{3-2x}\,\mathrm{d}x = -\frac{1}{2}+\frac{3}{4}\log 3, \\ \int_2^3 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 (x+1)\,\mathrm{d}x = \frac{3}{2}, \\ \int_3^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= \int_0^1 \frac{x+2}{2x+1}\,\mathrm{d}x = \frac{1}{2}+\frac{3}{4}\log 3, \\ \Longrightarrow \int_0^4 \frac{|x-1|}{|x-2|+|x-3|}\,\mathrm{d}x &= 2+\frac{3}{4}\log\frac{27}{5}\end{aligned}$$

(16) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x &= \int_0^{\pi} x\,\mathrm{d}\arctan\cos x \\ &= -x\arctan\cos x\Big|_0^{\pi} + \int_0^{\pi}\arctan\cos x\,\mathrm{d}x \\ &= \frac{\pi^2}{4}+0 = \frac{\pi^2}{4}.\end{aligned}$$

(17) $\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\sin t-\cos t}{\sin t}\,\mathrm{d}t \\ &= \frac{\pi}{8}-\frac{\log 2}{4}.\end{aligned}$$

(18) $\int_0^{\pi/2} \frac{\sin 2019x}{\sin x}\,\mathrm{d}x$

$$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,\mathrm{d}x = \int_0^{\pi/2} 1 + \sum_{k=1}^{n} \cos(2kx)\,\mathrm{d}x = \frac{\pi}{2}.$$

(19) $\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x$

$$\int_2^4 \frac{\log\sqrt{9-x}}{\log\sqrt{9-x}+\log\sqrt{x+3}}\,\mathrm{d}x = \int_{-1}^1 \frac{\log\sqrt{6+t}}{\log\sqrt{6+t}+\log\sqrt{6-t}}\,\mathrm{d}x = 1.$$

(20) $\int_0^1 \frac{1}{\sqrt{1+x^2}+\sqrt{1-x^2}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1+x^2}+\sqrt{1-x^2}} &= \int_0^1 \frac{1}{2}(\sqrt{1+x^2}-\sqrt{1-x^2})\,\mathrm{d}x \\ &= -\frac{\pi}{8} + \frac{\sqrt{2}}{4} + \frac{1}{8}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}.\end{aligned}$$

(21) $\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \sqrt{x+\sqrt{x+1}}\,\mathrm{d}x &= \int_1^{1+\sqrt{2}} \sqrt{y}\,\mathrm{d}\frac{2y+1-\sqrt{4y+5}}{2} \\ &= \int_1^{1+\sqrt{2}} \sqrt{y} - \frac{\sqrt{y}}{\sqrt{4y+5}}\,\mathrm{d}y \\ \left(y = \frac{z^2-5}{4}\right) &= \frac{2}{3}y^{3/2}\Big|_1^{1+\sqrt{2}} - \int_3^{1+2\sqrt{2}} \frac{\sqrt{z^2-5}}{4}\,\mathrm{d}z \\ &= \frac{2}{3}((1+\sqrt{2})^{3/2}-1) - \frac{3\sqrt{2}-1}{8} + \frac{5}{32}\log\frac{3+\sqrt{2}}{5}.\end{aligned}$$

(22) $\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x$

$$\int_{-1}^1 \frac{\sin\sin\sin x}{x^{800}+1}\,\mathrm{d}x = 0.\text{(by symmetry)}$$

11 Homework 11: Density of Sum of Squares

11.1 PSA: Riemann Integral

A1) $f \in C([a,b]), g \in \mathcal{R}([a,b])$**, where** g **is positive. Prove that there exists** $\xi \in (a,b)$**, such that**

$$\int_a^b fg = f(\xi)\int_a^b g.$$

Proof: Since g is positive on $[a,b]$,

$$\inf_{x\in[a,b]} f(x)\int_a^b g \leqslant \int_a^b fg \leqslant \sup_{x\in[a,b]} f(x)\int_a^b g.$$

By $f\in C([a,b])$, there exists such an $\xi\in(a,b)$.

A2) Prove without using Lebesgue theorem: if f is monotonously increasing on $[a,b]$, then $f\in\mathcal{R}([a,b])$.

Proof: For any $\varepsilon>0$ let $n=[1/\varepsilon]+1$, and

$$\mathcal{C}=\left\{x_k=a+(b-a)\frac{k}{n}:k=0,1,\cdots,n\right\}.$$

Then

$$g(x)=\max_{x_k\leqslant x}\{f(x_k)\}\leqslant f, h(x)=\min_{x_k\geqslant x}\{f(x_k)\}\geqslant f.$$

and both are monotonous simple functions.
Therefore

$$\overline{\int_a^b} f-\underline{\int_a^b} f\leqslant \overline{S}(f;\mathcal{C})-\underline{S}(f;\mathcal{C})=\frac{1}{n}(f(b)-f(a))\to 0.$$

Hence f is Riemann integrable.

A3) Prove that $1_{\mathbb{Q}}$ is not Riemann integrable on $[0,1]$.

Proof: Let $\varepsilon=\frac{1}{2}$. For any $\mathcal{C}=\{0=x_0<\cdots<x_n=1\}$, $\omega(x_{k-1},x_k)=1$, hence

$$\sum_{k=1}^n \omega(x_{k-1},x_k)(x_k-x_{k-1})=1>\varepsilon.$$

Therefore $1_{\mathbb{Q}}$ is not Riemann integrable.

A4) Prove that if $f\in\mathcal{R}([a,b])$, then $|f|^p\in\mathcal{R}([a,b])$, where $p\geqslant 0$.

Proof: Since $x\mapsto |x|^p$ is continuous, $|f|^p$ is continuous as x whenever f is continuous at x. Hence

$$f\in\mathcal{R}([a,b]) \implies |f|^p\in\mathcal{R}([a,b]).$$

A5) Prove Hölder's Inequality: if $f,g\in\mathcal{R}([a,b]),p,q>0,1/p+1/q=1$, then

$$\left|\int_a^b fg\right|\leqslant\left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}.$$

Proof: By A4) the functions are integrable. We can assume that

$$\int_a^b |f|^p=\int_a^b |g|^q=1.$$

Then by Young's inequality,

$$\left|\int_a^b fg\right|\leqslant\int_a^b |f|\cdot|g|\leqslant\int_a^b \frac{1}{p}|f|^p+\frac{1}{q}|g|^q=\frac{1}{p}+\frac{1}{q}=1.$$

A6) Prove Minkowski's inequality: if $f, g \in \mathcal{R}([a,b]), p \geqslant 1$, then

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

Proof: Note that if $1/p + 1/q = 1$, then

$$\begin{aligned}\int_a^b |f+g|^p &= \int_a^b |f| \cdot |f+g|^{1-p} + \int_a^b |g| \cdot |f+g|^{1-p} \\ &\leqslant \left(\left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}\right)\left(\int_a^b |f+g|^{(1-p)q}\right)^{1/q}\end{aligned}$$

Hence

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

The equality holds, when $|f|/|f+g|^{1-p}, |g|/|f+g|^{1-p}$ are both constant, which is equivalent to $|f|/|g|$ is constant.

11.2 PSB: Convex Functions

B1) Assume $f \in \mathcal{R}([a,b])$ and f is convex, prove that

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

Proof: Note that $f\left(\frac{a+b}{2}\right) \leqslant \frac{f(x)+f(a+b-x)}{2} \leqslant \frac{f(a)+f(b)}{2}$, and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \frac{f(x)+f(a+b-x)}{2}\,\mathrm{d}x.$$

Hence

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

B2) Assume f is twice differentiable on $[a,b]$ and for any $x, f''(x) > 0, f(x) \leqslant 0$. Prove that for any x,

$$f(x) \geqslant \frac{2}{b-a}\int_a^b f(y)\,\mathrm{d}y.$$

Proof: For any $x \leqslant y \leqslant b$,

$$f(y) \leqslant \frac{b-y}{b-x}f(x) + \frac{y-x}{b-x}f(b) \leqslant \frac{b-y}{b-x}f(x),$$

hence

$$\int_x^b f(y)\,\mathrm{d}y \leqslant f(x)\int_x^b \frac{b-y}{b-x}\,\mathrm{d}y = \frac{b-x}{2}f(x).$$

Likewise,
$$\int_a^x f(y)\,\mathrm{d}y \leqslant f(x)\int_a^x \frac{y-a}{x-a}\,\mathrm{d}y = \frac{x-a}{2}f(x).$$
Therefore
$$f(x) \geqslant \frac{2}{b-a}\int_a^b f(y)\,\mathrm{d}y.$$

B3) Assume f is twice differentiable on $\mathbb{R}$ and $f''(x) \geqslant 0$, $\varphi \in C([a,b])$. Prove that
$$\frac{1}{b-a}\int_a^b (f\circ\varphi)(t)\,\mathrm{d}t \geqslant f\left(\frac{1}{b-a}\int_a^b \varphi(t)\,\mathrm{d}t\right).$$
Proof: We prove the proposition for any convex function f and φ on the set X.
Let
$$\langle g\rangle = \frac{1}{\mu(X)}\int_X g\,\mathrm{d}\mu.$$
Then since f is convex, there is a constant K such that $f(y) - f(\langle\varphi\rangle) \geqslant K(y - \langle\varphi\rangle)$. Hence
$$\begin{aligned}\langle f(\varphi)\rangle &= \frac{1}{\mu(X)}\int_X f(\varphi(t))\,\mathrm{d}\mu\\ &\geqslant \frac{1}{\mu(x)}\int_X f(\langle\varphi\rangle)\,\mathrm{d}\mu + \frac{1}{\mu(X)}\int_X K(\varphi(t)-\langle\varphi\rangle)\,\mathrm{d}\mu\\ &= f(\langle\varphi\rangle).\end{aligned}$$

B4) Assume $f \in C([a,b])$ and for any x, $f(x) > 0$. Prove that
$$\log\left(\frac{1}{b-a}\int_a^b f\right) \geqslant \frac{1}{b-a}\int_a^b \log f.$$
Proof: Since $-\log x$ is convex, we can use B3).

B5) Prove that if f is convex on $\mathbb{R}$, $\varphi \in C([0,1])$, then
$$f\left(\int_0^1 \varphi\right) \leqslant \int_0^1 f\circ\varphi.$$
Proof: A special case of what we proved in B3).

11.3 PSC: Integrals and Derivatives

C1) Assume $f \in C^1([0,2])$, $|f'| \leqslant 1$, $f(0) = f(2) = 1$. Prove that
$$1 \leqslant \int_0^2 f \leqslant 3.$$
Proof: Note that for $0 \leqslant x \leqslant 1$,
$$|f(x)-1| = x|f'(\xi)| \leqslant x.$$
and for $1 \leqslant x \leqslant 2$,
$$|f(x)-1| = (2-x)|f'(\xi)| \leqslant 2-x.$$

Hence

$$\int_0^2 |f(x)-1|\,\mathrm{d}x \leqslant \int_0^1 x\,\mathrm{d}x + \int_1^2 (2-x)\,\mathrm{d}x = 1.$$

C2) Assume that $f \in C^2([0,1])$. Prove that $\exists \xi \in [0,1]$, such that

$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$

Proof: Let $g(x) = f(x) + f(1-x)$, then

$$\begin{aligned}
\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) &= \int_0^{1/2} g(x) - 2f\left(\frac{1}{2}\right)\,\mathrm{d}x \\
\text{(integration by parts)} &= -\int_0^{1/2} x g'(x)\,\mathrm{d}x = -\frac{1}{2}\int_0^{1/2} g'(x)\,\mathrm{d}x^2 \\
\text{(integration by parts)} &= \frac{1}{2}\int_0^{1/2} x^2 g''(x)\,\mathrm{d}x.
\end{aligned}$$

Note that $g'' \in C([0,1])$ hence by A1), $\exists \eta \in (0, \frac{1}{2})$,

$$\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) = g''(\eta)\frac{1}{2}\int_0^{1/2} x^2\,\mathrm{d}x = \frac{1}{48} g''(\eta).$$

Since $f'' \in C([0,1])$, there exists $\xi \in (\eta, 1-\eta)$, such that

$$f''(\xi) = \frac{f''(\eta) + f''(1-\eta)}{2} = \frac{g''(\eta)}{2}.$$

Therefore

$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$

C3) Assume $f \in C^1([0,1])$. Prove that

$$\max_{x\in[a,b]} |f(x)| \leqslant \frac{1}{b-a}\left|\int_a^b f(x)\,\mathrm{d}x\right| + \int_a^b |f'(x)|\,\mathrm{d}x.$$

Proof: For any $t \in [a,b]$,

$$(b-a)|f(t)| \leqslant \left|\int_a^b f(x)\,\mathrm{d}x\right| + \left|\int_a^b f(x) - f(t)\,\mathrm{d}x\right|$$

where

$$\begin{aligned}
\left|\int_a^b f(x) - f(t)\,\mathrm{d}x\right| &= \left|\int_a^b \left(\int_t^x f'(u)\,\mathrm{d}u\right)\,\mathrm{d}x\right| \\
&\leqslant \int_a^b \int_t^x |f'(u)|\,\mathrm{d}u\,\mathrm{d}x \\
&\leqslant (b-a)\int_a^b |f'(u)|\,\mathrm{d}u.
\end{aligned}$$

C4) Suppose $f \in C([0,1])$ and for any $g \in C([0,1]), g(0) = g(1) = 0,$ we have

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise assume $f(t) > 0$ for some $t \in (0,1)$, then there exists an $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon) \subset [0,1]$ and $\forall x \in (t-\varepsilon, t+\varepsilon), f(x) > f(t)/2$.
Let

$$g(x) = \begin{cases} 0, & x \notin (t-\varepsilon, t+\varepsilon), \\ 1 - \frac{|x-t|}{\varepsilon}, & x \in (t-\varepsilon, t+\varepsilon). \end{cases}$$

Then

$$\int_0^1 f(x)g(x)\,\mathrm{d}x > \int_{t-\varepsilon}^{t+\varepsilon} \frac{f(t)}{2} g(x)\,\mathrm{d}x > 0,$$

leading to contradiction. Hence $f(x) \equiv 0$.

C5) Suppose $f \in C([0,1])$ and for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$\int_0^1 f(x)x^n\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise, $\int_0^1 f^2 > 0$. By Stone-Weierstrass theorem, for any $\varepsilon > 0$, there is a polynomial P such that $\sup_{x\in[0,1]} |f(x) - P(x)| < \varepsilon$. Hence

$$0 = \int_0^1 f(x)P(x)\,\mathrm{d}x = \int_0^1 f^2 - \int_0^1 f(x)(f(x) - P(x))\,\mathrm{d}x \geqslant \int_0^1 f^2 - \sup_{x\in[0,1]} |f(x)|\varepsilon > 0$$

when $\varepsilon \to 0$, leading to contradiction.

C6) (Gronwall's Inequality) Suppose $\varphi \in C([0,T])$ and for any $t \in [0,T]$, $|\varphi(t)| \leqslant M + k\int_0^t |\varphi(s)|\,\mathrm{d}s$, where M, k are positive real numbers. Prove that $\forall t \in [0,T]$, $|\varphi(t)| \leqslant Me^{kt}$.

Proof: Let
$f : \left[0, \frac{T}{k}\right] \to \mathbb{R}, t \mapsto \frac{e^{-t}|\varphi(t/k)|}{M}$,
then for any $t \in [0, T/k]$,

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(s)e^s\,\mathrm{d}s.$$

Let $f(t) = \sup_{s\in[0,T/k]}\{f(s)\}$ then

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(t)e^s\,\mathrm{d}x = e^{-t} + f(t)(1 - e^{-t}).$$

Hence $f(s) \leqslant f(t) \leqslant 1, \implies |\varphi(t)| \leqslant Me^{kt}$.

C7) Assume $a, b > 0$, $f \in C([-a,b])$. If for any $x \in (-a,b)$, $f(x) > 0$ and $\int_{-a}^b xf(x)\,\mathrm{d}x = 0$. Prove that

$$\int_{-a}^b x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^b f(x)\,\mathrm{d}x.$$

Proof: Note that

$$\int_{-a}^{b}(x+a)(x-b)f(x)\,\mathrm{d}x \leqslant 0.$$

Combined with $\int_{-a}^{b} xf(x)\,\mathrm{d}x = 0$ we get

$$\int_{-a}^{b} x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^{b} f(x)\,\mathrm{d}x.$$

C8) Assume $f \in C([-1,1])$. Prove that

$$\lim_{\lambda\to 0^+}\int_{-1}^{1}\frac{\lambda}{\lambda^2+x^2}f(x)\,\mathrm{d}x = \pi f(0).$$

Proof: Let $M = \sup_{|x|\leqslant 1}|f(x)|$ and

$$g(x) = \frac{\lambda}{\lambda^2+x^2},$$

then (g is sort of a good kernel)

$$\int_{-1}^{1} g(x)\,\mathrm{d}x = 2\arctan\frac{1}{\lambda}.$$

Hence

$$\begin{aligned}
&\left|\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x - \pi f(0)\right| \\
\leqslant &\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \int_{-\varepsilon}^{\varepsilon}|f(x)-f(0)|g(x)\,\mathrm{d}x + \int_{\varepsilon\leqslant x\leqslant 1} Mg(x)\,\mathrm{d}x \\
\leqslant &\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \sup_{|x|\leqslant\varepsilon}|f(x)-f(0)|\pi + 2M\left|\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda}\right| \\
&\to 0
\end{aligned}$$

since

$$\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda} = \arctan\frac{\lambda(1-\varepsilon)}{\lambda^2+\varepsilon} \to 0, \text{when } \lambda \to 0^+.$$

and $\sup_{|x|\leqslant\varepsilon}|f(x)-f(0)| \to 0$ when $\varepsilon \to 0$.

C9) Assume f is differentiable on $[1,\infty)$ and both $\int_1^\infty f(x)\,\mathrm{d}x$ and $\int_1^\infty f'(x)\,\mathrm{d}x$ converges. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

Proof: For any $\varepsilon > 0$, there exists $N > 1$, such that $\forall u, v > N$,

$$\left|\int_u^v f'(x)\,\mathrm{d}x\right| < \varepsilon, \text{ i.e. } |f(u)-f(v)| < \varepsilon$$

Hence for any $u > N$, if $|f(u)| > \varepsilon$,

$$\left|\int_u^M f(x)\,\mathrm{d}x\right| \geqslant (M-u)(|f(u)-\varepsilon|) \to \infty, \text{ as } M \to \infty,$$

which contradicts the fact that $\int_1^\infty f(x)\,\mathrm{d}x$ converges. Therefore $|f(u)| < \varepsilon$ for any $u > N$, which implies $\lim_{x\to\infty} f(x) = 0$.

C10) Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x, \int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

Proof:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = -\int_0^\infty \sin^2 x\,\mathrm{d}\frac{1}{x} = \int_0^\infty \frac{\sin 2x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$
$$\int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{1}{1+x}\,\mathrm{d}\sin x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

11.4 PSD: Calculation of improper integrals

D1)

$$\int_0^1 \log x\,\mathrm{d}x = (x\log x - x)\Big|_0^1 = -1.$$

D2)

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x\Big|_{-\infty}^\infty = \pi.$$

D3)

Calculating residues, we get

$$\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^4+1} = 2\pi i\cdot(Res(f;e^{i\pi/4}) + Res(f;e^{3i\pi/4})) = \frac{\pi}{\sqrt{2}}.$$

Hence

$$\int_0^\infty \frac{\mathrm{d}x}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

D4)

Same as D3)

$$\int_{-\infty}^\infty \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \sqrt{2}\pi.$$

Hence

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

D5)

$$\int_{-\infty}^0 xe^x\,\mathrm{d}x = \int_{-\infty}^0 x\,\mathrm{d}e^x = -\int_{-\infty}^0 e^x\,\mathrm{d}x = -1.$$

D6)

$$\int_0^\infty e^{-\sqrt{x}}\,\mathrm{d}x = 2\int_0^\infty ye^{-y}\,\mathrm{d}y = 2\int_0^\infty e^{-y}\,\mathrm{d}y = 2.$$

D7)

$$\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)^{3/2}} = \frac{1}{a^2}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^{3/2}} = \frac{1}{2a^2}B\left(\frac{1}{2},1\right) = \frac{1}{a^2}.$$

(We can also substitute $x = a\tan\theta$).

D8)

$$\int_2^\infty \frac{\mathrm{d}x}{x^2+x-2} = \frac{1}{3}\log\frac{x-1}{x+2}\Big|_2^\infty = \frac{\log 3}{3}.$$

D9)

$$\begin{aligned}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+x+1)^2} &= \frac{8}{3\sqrt{3}}\int_{-\infty}^\infty \frac{\mathrm{d}u}{(1+u^2)^2}\\ (u=\tan\theta) &= \frac{8}{3\sqrt{3}}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \frac{4\sqrt{3}\pi}{9}.\end{aligned}$$

D10)

$$\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} 1\,\mathrm{d}\theta = \pi.$$

D11)

$$\int_{-1}^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2}\theta\,\mathrm{d}\theta = 0.$$

D12)

Let γ be the unit circle, then

$$\begin{aligned}\int_{-1}^1 \frac{\mathrm{d}x}{(2-x)^2\sqrt{1-x^2}} &= \int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}\theta}{(2-\sin\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{(2-\sin\theta)^2}\\ &= \frac{1}{2}\int_\gamma -\frac{4}{i}\frac{z\mathrm{d}z}{(z^2-4iz-1)^2}\\ &= -4\pi\mathrm{Res}\left(\frac{z}{(z^2-4iz-1)^2};(2-\sqrt{3})i\right)\\ &= \frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

D13)

$$\int_0^1 \frac{\arcsin\sqrt{x}}{x(1-x)}\,\mathrm{d}x > \int_{1/4}^1 \frac{\pi}{6}\frac{1}{1-x}\,\mathrm{d}x \text{ which diverges.}$$

D14)

$$\int_0^1 (1-x)^n x^{1/2-1}\,\mathrm{d}x = B\left(n+1,\frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{n!2^{n+1}}{(2n+1)!!}.$$

D15)

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x = \begin{cases} \frac{(n-1)!!}{n!!}, n \text{ is odd,} \\ \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, n \text{ is even.} \end{cases}$$

D16)

Using integration by parts, and substitute $x = e^{-y}$,

$$\begin{aligned} \int_0^1 x^m(\log x)^n\,\mathrm{d}x &= (-1)^n \int_0^\infty e^{-(m+1)y}y^n\,\mathrm{d}y \\ &= (-1)^n\frac{n!}{(m+1)^n}\int_0^\infty e^{-(m+1)y}\,\mathrm{d}y = \frac{(-1)^n n!}{(m+1)^{n+1}}. \end{aligned}$$

D17)

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p} = \int_{\log 2}^\infty \frac{\mathrm{d}y}{y^p} = \frac{(\log 2)^{1-p}}{p-1}.$$

D18)

Substitute $x = ay$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\log a}{2a} + \frac{1}{a}\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = \frac{\pi\log a}{2a}.$$

since by substituting $y = 1/z$,

$$\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = -\int_0^\infty \frac{\log z}{1+z^2}\,\mathrm{d}z = 0.$$

D19)

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \Gamma(n) = (n-1)!.$$

D20)

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(ax^2+2bx+c)^n} = \frac{1}{d^n}\sqrt{\frac{d}{a}}\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1+u^2)^n} = \frac{1}{d^n}\sqrt{\frac{d}{a}}\pi\frac{(2n-3)!!}{(2n-2)!!}.$$

where $d = \frac{ac-b^2}{a}$

D21)

$$\int_0^{\infty} x^{2n-1}e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\infty} y^{n-1}e^{-y}\,\mathrm{d}y = \frac{(n-1)!}{2}.$$

D22)

The Poisson kernel

$$\begin{aligned}\frac{1-r^2}{1-2r\cos x+r^2} &= \frac{1-r^2}{(1-re^{ix})(1-re^{-ix})}\\ &= (1-r^2)\sum_{n=0}^{\infty} r^n e^{inx}\sum_{m=0}^{\infty} r^m e^{-imx}\\ &= \sum_{k=-\infty}^{\infty} r^{|k|}e^{ikx}.\end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos x+r^2}\,\mathrm{d}x = 2\pi.$$

D23)

$$\begin{aligned}\int_0^{\infty} e^{-ax}\cos bx\,\mathrm{d}x &= \frac{1}{b}\int_0^{\infty} e^{-ax}\,\mathrm{d}\sin bx = \frac{a}{b}\int_0^{\infty} e^{-ax}\sin bx\,\mathrm{d}x\\ &= -\frac{a}{b^2}\int_0^{\infty} e^{-ax}\,\mathrm{d}\cos bx = \frac{a}{b^2} - \frac{a^2}{b^2}\int_0^{\infty} e^{-ax}\cos bx\,\mathrm{d}x\\ &= \frac{a}{a^2+b^2}.\end{aligned}$$

D24)

Same as (23),

$$\int_0^{\infty} e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2}.$$

D25)

$$\begin{aligned}&\int_1^\infty \frac{\mathrm{d}x}{x(x+1)\cdots(x+n)} = \lim_{N\to\infty}\int_0^N \sum_{k=0}^n \frac{(-1)^k}{x+k}\binom{n}{k}\,\mathrm{d}x\\
&= \lim_{N\to\infty}\sum_{k=0}^n (-1)^k \binom{n}{k}\log\left(\frac{N+k}{(k+1)}\right)\\
&= -\sum_{k=0}^n (-1)^k\binom{n}{k}\log(k+1) + \lim_{N\to\infty}\sum_{k=0}^n (-1)^k\binom{n}{k}\log\left(1+\frac{k}{N}\right)\\
&= -\sum_{k=0}^n (-1)^k\binom{n}{k}\log(k+1).\end{aligned}$$

D26)

$$\begin{aligned}&\int_0^\pi \log\sin x\,\mathrm{d}x = 2\int_0^{\pi/2}\log\sin x\,\mathrm{d}x = 2\int_0^{\pi/2}\log\cos x\,\mathrm{d}x\\
&= \int_0^{\pi/2}\log\sin 2x - \log 2\,\mathrm{d}x = \frac{1}{2}\int_0^\pi \log\sin x\,\mathrm{d}x - \frac{\pi}{2}\log 2\\
&= -\pi\log 2.\end{aligned}$$

D27)

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Note that

$$\max\{0, 1-x^2\} < e^{-x^2} < \frac{1}{1+x^2}.$$

Hence

$$\frac{(2n)!!}{(2n+1)!!} < \int_0^\infty e^{-nx^2}\,\mathrm{d}x < \frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

Therefore

$$\sqrt{n}\frac{(2n)!!}{(2n+1)!!} < \int_0^\infty e^{-x^2}\,\mathrm{d}x < \sqrt{n}\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

By Wallis's formula,

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

11.5 PSE: Density of sum of squares

Let $I = (0, \infty)$.

Part 1

E1) Prove that $e^{-u}/\sqrt{u}$ is integrable on I, and let $K = \int_0^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u$.

Proof:

$$\int_1^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u < \int_1^\infty e^{-u}\,\mathrm{d}u = \frac{1}{e}.$$
$$\int_0^1 e^{-u}/\sqrt{u}\,\mathrm{d}u < \int_0^1 u^{-1/2}\,\mathrm{d}u = \frac{1}{2}.$$

E2) Prove that for any $x \in I$,

$F(x) = \int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u$
is well-defined.
Proof:

$$F(x) < \int_0^\infty \frac{e^{-u}}{x\sqrt{u}}\,\mathrm{d}u \text{ converges.}$$

E3) Prove that $F \in C^1(I)$ and calculate $F'(x)$.

Solution: Let $f(x,u) = \frac{e^{-u}}{\sqrt{u}(u+x)}$, then f is uniformly continuous on any closed subinterval of I, and so is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x,u) = -\frac{e^{-u}}{\sqrt{u}(u+x)^2}.$$

Also,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} f(x,u)\,\mathrm{d}u$$

converges uniformly.
Hence F is continuously differentiable and

$$F'(x) = -\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u.$$

E4) Prove that for any $x \in I$,

$$xF'(x) - \left(x - \frac{1}{2}\right) F(x) = -K.$$

Proof: We show that

$$x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u + \left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u = \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u.$$

Note that, by substituting $u \to ux$,

$$x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u = \frac{1}{\sqrt{x}}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u,$$
$$\left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u = \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+x)}\,\mathrm{d}u,$$
$$\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u = \sqrt{x}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.$$

Hence it is equivalent to
$$x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u = \int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u + \left(x-\frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u.$$
Note that $\mathrm{d}e^{-ux}\sqrt{u} = -e^{-ux}\left(x\sqrt{u}-\frac{1}{2\sqrt{u}}\right)\mathrm{d}u$, hence
$$\begin{aligned}
&x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u - \left(x+\frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u\\
&=\int_0^\infty e^{-ux}\left(x\sqrt{u}-\frac{1}{2\sqrt{u}}\right)\frac{\mathrm{d}u}{1+u}\\
&=-\int_0^\infty \frac{\mathrm{d}e^{-ux}\sqrt{u}}{1+u} = -\int_0^\infty e^{-ux}\sqrt{u}\,\frac{\mathrm{d}u}{(1+u)^2}\\
&=\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{(1+u)^2} - \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{1+u}.
\end{aligned}$$
($\sqrt{u} = \frac{1}{\sqrt{u}}((1+u)-1)$)

E5) Define $G: I\to\mathbb{R}, x\mapsto \sqrt{x}e^{-x}F(x)$. Prove that $\exists C\in\mathbb{R}$ such that
$$G(x) = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$
Proof: By B4)
$$G'(x) = \sqrt{x}e^{-x}F'(x) + \left(\frac{1}{2\sqrt{x}}-\sqrt{x}\right)e^{-x}F(x) = -K\frac{e^{-x}}{\sqrt{x}}.$$
Hence let $C = G(0)$, then
$$G(x) = C + \int_0^x G'(x)\,\mathrm{d}x = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$

E6) Calculate the value of K.

Solution: Note that when $x\to\infty$, $F(x)\to 0$ hence $G(x)\to 0$. Therefore
$$0 = \lim_{x\to\infty} G(x) = G(0) - K\int_0^\infty \frac{e^{-t}}{t}\,\mathrm{d}t = G(0) - K^2.$$
Where
$$\begin{aligned}
G(0) &= \lim_{x\to 0^+}\frac{\sqrt{x}}{e^x}\int_0^\infty \frac{e^{-u}}{\sqrt{u}(x+u)}\,\mathrm{d}u = \lim_{x\to 0}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u\\
&= \int_0^\infty \frac{1}{\sqrt{u}(1+u)}\,\mathrm{d}u = \int_0^\infty \frac{2\mathrm{d}t}{1+t^2} = \pi.
\end{aligned}$$

Hence $K = \sqrt{\pi}$.

Part 2

Define
$$f(x) = \sum_{n=1}^\infty \frac{e^{-nx}}{\sqrt{n}}, g(x) = \sum_{n=0}^\infty \sqrt{n}e^{-nx}.$$

E7) Prove that f, g are well-defined on I and are both continuous on I.

Proof: Let $C = \sup_{x \geqslant 0} x^3 e^{-x}$, then

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{\sqrt{n}} < \sum_{n=0}^{\infty} \sqrt{n} e^{-nx} \leqslant \sum_{n=1}^{\infty} \frac{C}{(nx)^2 \sqrt{x}} \text{ converges.}$$

On any closed sub-interval of I, the two series both converge uniformly, and e^{-nx} is continuous, hence f, g are both continuous on I.

E8) Prove that $\forall x \in I$,

$$\int_1^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

Proof: The function $e^{-ux}/\sqrt{u}$ is monotonously decreasing by u, hence

$$\int_1^{N} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant \sum_{n=1}^{N-1} \frac{e^{-nx}}{\sqrt{n}} \leqslant f(x).$$

$$\sum_{n=1}^{N} \frac{e^{-nx}}{\sqrt{n}} \leqslant \int_0^{N} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

Therefore

$$\int_1^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u \leqslant f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{u}} \,\mathrm{d}u.$$

E9) Prove that $\exists C_0$ such that

$$\lim_{x \to 0^+} \sqrt{x} f(x) = C_0.$$

Proof: By E8)

$$\sqrt{x} f(x) \leqslant \int_0^{\infty} \frac{e^{-ux}}{\sqrt{ux}} \,\mathrm{d}ux = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} \,\mathrm{d}t = \sqrt{\pi}.$$

$$\sqrt{x} f(x) \geqslant \int_1^{\infty} \frac{e^{-ux}}{\sqrt{ux}} \,\mathrm{d}ux = \int_x^{\infty} \frac{e^{-t}}{\sqrt{t}} \,\mathrm{d}t \to \sqrt{\pi}.$$

Hence

$$\lim_{x \to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E10) Define the sequence $\{a_n\}_{n \geqslant 1}$ as follows:

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right) - 2\sqrt{n}.$$

Prove that $\{a_n\}$ converges.
Proof: By Euler-Maclaurin formula, for $f(x) = 1/\sqrt{x}$,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &= \frac{f(1)+f(n)}{2} + \int_1^n \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x \\ &= 2\sqrt{n} - \frac{3}{2} + \frac{1}{2\sqrt{n}} + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x\end{aligned}$$

Hence

$$\lim_{n\to\infty} a_n = -\int_1^\infty \frac{\widetilde{B}_1(x)}{2x^{3/2}}\,\mathrm{d}x - \frac{3}{2}.$$

E11) Prove that for any $x \in I$, the function

$$h(x) = \sum_{n\geqslant 1}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) e^{-nx}$$

is well-defined.
Proof: By E10), $|a_n|$ is bounded, hence

$$h(x) = \sum_{n\geqslant 1} 2\sqrt{n} e^{-nx} + a_n e^{-nx} = 2g(x) + \sum_{n\geqslant 1} a_n e^{-nx} \leqslant 2g(x) + \sup_n |a_n| \cdot \frac{1}{e^x - 1}.$$

E12) Express $h(x)$ using $f(x)$ and find a constant C_1 such that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} h(x) = C_1.$$

Proof: Since $e^{-nx}/k > 0$, we can interchange the sums

$$h(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{n=k}^{\infty} e^{-nx} = \sum_{k=1}^{\infty} \frac{e^{-kx}}{\sqrt{k}} \frac{1}{1-e^{-x}} = \frac{1}{1-e^{-x}} f(x).$$

Therefore

$$\lim_{x\to 0^+} x^{3/2} h(x) = \lim_{x\to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E13) Prove that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} g(x) = \frac{\sqrt{\pi}}{2}.$$

Proof:

$$\lim_{x\to 0^+} x^{3/2} |h(x) - 2g(x)| \leqslant \lim_{x\to 0^+} \sup_n |a_n| \cdot \frac{x^{3/2}}{e^x - 1} = 0.$$

Hence

$$\lim_{x\to 0^+} x^{3/2} g(x) = \frac{1}{2} \lim_{x\to 0^+} x^{3/2} h(x) = \frac{\sqrt{\pi}}{2}.$$

Part 3

Given $A \subset \mathbb{Z}_{\geqslant 0}$, we can define a sequence $\{a_n\}_{n\geqslant 0}$:

$$a_n = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$$

Define the set $I_A \subset \mathbb{R}_{\geqslant 0}$ as follows:

$$I_A = \left\{ x \geqslant 0 : \text{the series } \sum_{n\geqslant 0} a_n e^{-nx} \text{ converges} \right\}.$$

Define the function $f_A : I_A \to \mathbb{R}$ as follows:

$$f_A(x) = \sum_{n\geqslant 0} a_n e^{-nx}.$$

Let $\Phi(A) = \lim_{x\to 0} x f_A(x)$ (if the limit exists) and let

$$\mathcal{S} = \{A \subset \mathbb{Z}_{\geqslant 0} : \lim_{x\to 0^+} x f_A(x) \text{ exists}\}.$$

For example, let

$$A_1 = \{n^2 : n \in \mathbb{Z}_{\geqslant 1}\}, A_2 = \{p^2 + q^2 : p, q \in \mathbb{Z}_{\geqslant 1}\}.$$

E14) Determine the set I_A.

Solution: If A is finite, then $I_A = \mathbb{R}_{\geqslant 0}$. Otherwise $I_A = \mathbb{R}_{>0} = I$.

E15) Given $A \subset \mathbb{Z}_{\geqslant 0}$, for any $n \geqslant 0$, define the set $A_{\leqslant n}$:

$$A_{\leqslant n} = \{ \in A : \leqslant n\}.$$

Prove that for any $x > 0$, the series

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx}$$

converges, and satisfy

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \frac{f_A(x)}{1 - e^{-x}}.$$

Proof: $|A_{\leqslant n}| \leqslant n + 1$, hence

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} \text{ converges.}$$

Therefore

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k \cdot e^{-nx} = \sum_{k=0}^{\infty} a_k \cdot \frac{e^{-kx}}{1 - e^{-x}} = \frac{f_A(x)}{1 - e^{-x}}.$$

E16) Prove that for any $x > 0$

$$\frac{f_{A_1}(x)}{1-e^{-x}} = \sum_{n=0}^{\infty} \lfloor \sqrt{n} \rfloor e^{-nx}.$$

Proof: By E15),

$$|A_{1\,\leqslant n}| = \sum_{k=0}^{n} [\sqrt{k} \in \mathbb{Z}_{\geqslant 1}] = \lfloor \sqrt{n} \rfloor.$$

E17) Prove that

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x)$$

exists and calculate the value of $\Phi(A_1)$.
Proof:

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x) = \lim_{x\to 0^+} \sqrt{x}(1-e^{-x}) \left(g(x) - \sum_{n=0}^{\infty} \{\sqrt{n}\} e^{-nx} \right).$$

Since $1-e^{-x} \sim x$, $g(x) \sim \frac{\sqrt{\pi}}{2} x^{-3/2}$, and

$$\left| \sum_{n=0}^{\infty} \left\{ \sqrt{n} \right\} e^{-nx} \right| \leqslant \frac{1}{1-e^{-x}}.$$

Hence

$$\lim_{x\to 0^+} \sqrt{x} f_{A_1}(x) = \frac{\sqrt{\pi}}{2}.$$

and

$$\Phi(A_1) = \lim_{x\to 0^+} x f_{A_1}(x) = 0.$$

E18) Let $v(n) = \#\{(p,q) \in \mathbb{Z}^2_{\geqslant 1} : p^2+q^2 = n\}$. Prove that for any $x > 0$, the series

$$\sum_{n\geqslant 1} v(n) e^{-nx}$$

converges and

$$\sum_{n\geqslant 1} v(n) e^{-nx} = (f_{A_1}(x))^2.$$

Proof: Since $v(n) \leqslant n$, $\sum_{n\geqslant 1} v(n)e^{-nx}$ converges.

$$\sum_{n\geqslant 1} v(n)e^{-nx} = \sum_{n\geqslant 1}\sum_{k=0}^{n} a_k a_{n-k} e^{-nx} = \sum_{n\geqslant 1}\sum_{k=0}^{n} a_k e^{-kx} \cdot a_{n-k} e^{-(n-k)x} = (f_{A_1}(x))^2.$$

E19) Prove that for any $x > 0$

$$f_{A_2}(x) \leqslant (f_{A_1}(x))^2$$

and give an upper-bound of $\Phi(A_2)$ (assuming it exists).
Proof:

$$f_{A_2}(x) = \sum_{n\geqslant 1} [v(n) \geqslant 1] \cdot e^{-nx} \leqslant \sum_{n\geqslant 1} v(n)e^{-nx} = (f_{A_1}(x))^2.$$

Hence

$$\Phi(A_2) = \lim_{x\to 0^+} xf_{A_2}(x) \leqslant \lim_{x\to 0^+} (\sqrt{x}f_{A_1}(x))^2 = \frac{\pi}{4}.$$

Part 4

Assume $\{a_n\}_{n\geqslant 0}$ is a sequence of non-negative numbers, such that for any $x > 0$ the series

$$S(x) = \sum_{n\geqslant 0} a_n e^{-nx}$$

converges. Moreover, assume that the limit below exists:

$$\lim_{x\to 0^+} xS(x) = \lim_{x\to 0^+} x \sum_{n\geqslant 0} a_n e^{-nx} = \in [0, +\infty).$$

Let $F = \{f : [0, 1] \to \mathbb{R}\}$, $E_0 = C([0, 1])$. Let E be the space of piecewise continuous functions, and define the norm on E:

$$\|\psi\|_\infty = \sup_{x\in[0,1]} |\psi(x)|.$$

E20) Define $L : E \to F$ as follows:

$$(L(\psi))(x) = \sum_{n=0}^{\infty} a_n e^{-nx}\psi(e^{-nx}),\ \psi \in E.$$

Prove that L is well-defined and is linear. Moreover, if for any $x \in [0, 1]$, $\psi_1(x) \leqslant \psi_2(x)$, the for any $x \in [0, 1]$,

$$(L(\psi_1))(x) \leqslant (L(\psi_2))(x).$$

Proof: Since $\psi \in E$, ψ is bounded, hence L is well-defined and is clearly linear. The inequality holds since a_n are non-negative.

E21) Define the subspace of E

$$E_1 = \{\psi \in E : \lim_{x\to 0^+} x(L(\psi))(x) \text{ exists}\}.$$

Define the linear map $\Delta : E_1 \to \mathbb{R}$ as follows:

$$\Delta(\psi) = \lim_{x\to 0^+} x(L(\psi))(x),\ \psi \in E_1.$$

Prove that E_1 is a subspace of E and there is a constant $M > 0$ such that for any $\psi \in E_1$,

$$|\Delta(\psi)| \leqslant M\|\psi\|_\infty.$$

Proof: Since L is linear, so is Δ, thus E_1 is clearly a subspace of E.

$$|\Delta(\psi)| = \left|\lim_{x\to 0^+} x \sum_{n=0}^{\infty} a_n e^{-nx}\psi(e^{-nx})\right| \leqslant \|\psi\|_\infty \cdot |\lim_{x\to 0^+} xS(x)| = \|\psi\|_\infty.$$

E22) For the polynomial $P_n(x) = x^n$, prove that $P_n \in E_1$ and calculate $\Delta(P_n)$.

Proof:

$$\Delta(P_n) = \lim_{x \to 0^+} x \sum_{k=0}^{\infty} a_k e^{-kx} e^{-nkx} = \frac{1}{n+1}.$$

E23) Prove that $E_0 \subset E_1$ and for every $\psi \in E_0$ calculate $\Delta(\psi)$.

Proof: Since Δ is linear, by E22) we know that for any polynomial P,

$$\Delta(P) = \int_0^1 P(x) \, \mathrm{d}x.$$

By Stone-WeierstraSS theorem, any continuous function on $[0,1]$ can be uniformly approximated with polynomials, hence (same as E24)

$$\Delta(\psi) = \int_0^1 \psi(x) \, \mathrm{d}x, \, \forall \psi \in E_0.$$

E24) For $a \in (0,1)$, $\varepsilon \in (0, \min(a, 1-a))$, define the functions

$$g_-(x) = \begin{cases} 1, & x \in [0, a-\varepsilon]; \\ \frac{a-x}{\varepsilon}, & x \in (a-\varepsilon, a) \\ 0, & x \in [a, 1] \end{cases}, g_+(x) = \begin{cases} 1, & x \in [0, a]; \\ \frac{a+\varepsilon-x}{\varepsilon}, & x \in (a, a+\varepsilon) \\ 0, & x \in [a+\varepsilon, 1] \end{cases}.$$

Prove that $g_\pm \in E_0$ and calculate $\Delta(g_\pm)$. Further prove that $\mathbf{1}_{[0,a]} \in E_1$ and calculate $\Delta(\mathbf{1}_{[0,a]})$.
Proof: $g_\pm \in E_0$ is trivial, and $\Delta(g_\pm) = \int_0^1 g_\pm = (a \pm \varepsilon/2)$. Since $g_- \leqslant \mathbf{1}_{[0,a]} \leqslant g_+$,

$$x(L(g_-))(x) \leqslant x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x)$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < x < \delta$,

$$|x(L(g_-))(x) - \Delta(g_-)|, |x(L(g_+))(x) - \Delta(g_+)| < \frac{\varepsilon}{2}.$$

Hence for any $0 < x < \delta$,

$$x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x) \leqslant \Delta(g_+) + \frac{\varepsilon}{2} = a + \varepsilon.$$
$$x(L(\mathbf{1}_{[0,a]}))(x) \geqslant x(L(g_-))(x) \geqslant \Delta(g_-) - \frac{\varepsilon}{2} = a - \varepsilon.$$

Therefore

$$\Delta(\mathbf{1}_{[0,a]}) = \lim_{x \to 0^+} x(L(\mathbf{1}_{[0,a]}))(x) = a.$$

E25) Prove that $E_1 = E$ and for $\psi \in E$ determine the formula of $\Delta(\psi)$.

Proof: Use the same method as E24) applied to Darboux's sum. Hence

$$E_1 = E, \text{ and } \Delta(\psi) = \int_0^1 \psi(x) \, \mathrm{d}x.$$

E26) Define the function

$$\psi(x) = \begin{cases} 0, & x \in [0, e^{-1}); \\ \frac{1}{x}, & x \in [e^{-1}, 1]. \end{cases}$$

Prove the following equation by calculating $L(\psi)\left(\frac{1}{N}\right)$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} a_k = .$$

Proof:

$$(L(\psi))\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} a_n e^{-n/N} \psi(e^{-n/N}) = \sum_{n=0}^{N} a_n.$$

Hence by E25),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} a_n = \Delta(\psi) = \int_0^1 \psi(x)\,\mathrm{d}x = .$$

E27) Consider $A \in \mathcal{S}$, and calculate

$$\lim_{n\to\infty} \frac{|A_{\leqslant n}|}{n}.$$

which is called the asymptomatic density of A on $\mathbb{Z}_{\geqslant 0}$.
Solution:

$$\lim_{n\to\infty} \frac{|A_{\leqslant n}|}{n} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} a_n = \lim_{x\to 0^+} x \sum_{n=0}^{\infty} a_n e^{-nx} = \Phi(A).$$

E28) Calculate

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} v(k)}{n},$$

and give an upper-bound of the asymptomatic density of A_2.
Solution:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} v(k)}{n} = \lim_{x\to 0^+} x \sum_{n=0}^{\infty} v(n) e^{-nx} = \lim_{x\to 0^+} x (f_{A_1}(x))^2 = \frac{\pi}{4}.$$

From E19) $\Phi(A_2) \leqslant \frac{\pi}{4}$.

Quote:
God does not care about our mathematical difficulties. He integrates empirically.
——Albert Einstein

12 Homework 12: Oscillatory Intergral

12.1 PSA: Stieltjes Integral

Let μ be a monotonic function on $I = [a, b]$.

A1) For any pair of partitions $\sigma, \sigma' \in \mathcal{S}(I)$,

$$\underline{S}_\mu(f;\sigma) \leqslant \overline{S}_\mu(f;\sigma').$$

Proof: Suppose $\mathcal{C} = \sigma \cup \sigma'$, then

$$\underline{S}_\mu(f;\sigma) \leqslant \underline{S}_\mu(f;\mathcal{C}) \leqslant \overline{S}_\mu(f;\mathcal{C}) \leqslant \overline{S}_\mu(f;\sigma').$$

A2) For any $\rho \in C([a,b]), \rho \geqslant 0$, $\mu(x) = \int_a^x \rho(t)\,\mathrm{d}t$. Prove that for any $f \in \mathcal{R}([a,b])$, $f \in \mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu = \int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

Proof: Consider any $\mathcal{C} = \{x_0, x_1, \cdots, x_n\}$, then if we denote $m_i = \inf_{x\in[x_{i-1},x_i]} f(x), u_i = \inf_{x\in[x_{i-1},x_i]} \rho(x), v_i = \inf_{x\in[x_{i-1},x_i]} f(x)\rho(x), M = \sup_{x\in[a,b]} f(x)$, $v_i - m_i u_i \leqslant f(t)\rho(t) - f(t)u_i \leqslant M\omega_\rho(x_{i-1}, x_i)$. Hence for any $\varepsilon > 0$ there exists a $\delta > 0$, for any $\max\{x_i - x_{i-1}\} < \delta$, $\sup_{x,y\in[x_{i-1},x_i]} |\rho(x) - \rho(y)| < \varepsilon$. Then

$$\begin{aligned}\underline{S}(f\rho;\mathcal{C}) &= \sum_{k=1}^n v_k(x_k - x_{k-1}) \leqslant \sum_{k=1}^n u_i m_i (x_k - x_{k-1}) + M\varepsilon(b-a)\\ &\leqslant M\varepsilon(b-a) + \sum_{k=1}^n m_i \int_{x_{k-1}}^{x_k} \rho(t)\,\mathrm{d}t = M\varepsilon(b-a) + \underline{S}_\mu(f;\mathcal{C}).\end{aligned}$$

The other side is similar, hence $\sup\{\underline{S}_\mu(f;\mathcal{C})\} = \inf\{\overline{S}_\mu(f;\mathcal{C})\}$ so $f \in \mathcal{R}([a,b];\mu)$ and

$$\int_a^b f\,\mathrm{d}\mu = \int_a^b f(x)\rho(x)\,\mathrm{d}x.$$

A3) Prove that $\mathcal{R}(I;\mu)$ is a linear space on $\mathbb{R}$ and the integration operator

$$\int_a^b \cdot\,\mathrm{d}\mu : \mathcal{R}(I;\mu) \to \mathbb{R}, f \mapsto \int_a^b f\,\mathrm{d}\mu.$$

is linear.

Proof: Since $\underline{S}_\mu(\cdot;\mathcal{C})$ and $\overline{S}_\mu(\cdot;\mathcal{C})$ is linear for any $\mathcal{C}$, $\mathcal{R}(I;\mu)$ is clearly a linear space on $\mathbb{R}$, and $\int_a^b \cdot\,\mathrm{d}\mu$ is a linear operator.

A4) Suppose $f_1, f_2 \in \mathcal{R}(I;\mu)$. If the any $x \in I$, $f_1(x) \leqslant f_2(x)$, then

$$\int_a^b f_1\,\mathrm{d}\mu \leqslant \int_a^b f_2\,\mathrm{d}\mu.$$

Proof: By A3), we can assume $f_1 = 0$. Then for any $\mathcal{C}$, $\underline{S}_\mu(f;\mathcal{C}) \geqslant 0$ since $f \geqslant 0$, hence $\int_a^b f\,\mathrm{d}\mu = \sup\{\underline{S}_\mu(f;\mathcal{C})\} \geqslant 0$.

A5) If $f \in \mathcal{R}([a,b];\mu)$, then for any $c \in [a,b]$, $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Stieltjes integrable and

$$\int_a^b f\,\mathrm{d}\mu = \int_a^c f\,\mathrm{d}\mu + \int_c^b f\,\mathrm{d}\mu.$$

Proof: For any partition σ, let $\sigma' = \sigma \cup \{c\}$, then σ' can be split into two partitions of the intervals $[a,c]$ and $[c,b]$: $\sigma' = \sigma_1 \cup \sigma_2$, such that $\underline{S}_\mu(f;\sigma') = \underline{S}_\mu(f;\sigma_1) + \underline{S}_\mu(f;\sigma_2)$ and $\overline{S}_\mu(f;\sigma') = \overline{S}_\mu(f;\sigma_1) + \overline{S}_\mu(f;\sigma_2)$. Hence

$$\inf \underline{S}_\mu(f;\sigma_1) + \inf \underline{S}_\mu(f;\sigma_2) \leqslant \inf \underline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma') \leqslant \sup \overline{S}_\mu(f;\sigma_1) + \sup \overline{S}_\mu(f;\sigma_2).$$

Therefore

$$\int_a^b f\,\mathrm{d}\mu = \int_a^c f\,\mathrm{d}\mu + \int_c^b f\,\mathrm{d}\mu.$$

A6) If $f, g \in \mathcal{R}([a,b];\mu)$, then $f \cdot g \in \mathcal{R}([a,b];\mu)$.

Proof: Same as in the case of the Riemann integral.

A7) Define Stieltjes integral on the interval $[0,\infty)$: Suppose $f \in C([0,\infty))$ is continuous and bounded, define

$$\int_0^\infty f\,\mathrm{d}\mu = \lim_{M\to\infty} \int_0^M f\,\mathrm{d}\mu.$$

Suppose $\{\alpha_n\}_{n\geqslant 1}$ is a sequence of positive real numbers and $\sum_{n=1}^\infty \alpha_n$ converges, define the monotonic function $\mu = \sum_{n=1}^\infty \alpha_n \mathbf{1}_{\geqslant n}$, then

$$\int_1^\infty f\,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

Proof: Note that

$$\mu(x+0) - \mu(x-0) = \begin{cases} 0, & x \notin \mathbb{Z}, \\ \alpha_x, & x \in \mathbb{Z}. \end{cases}$$

Hence

$$\int_0^N f\,\mathrm{d}\mu = \sum_{n=1}^{N-1} f(n)\alpha_n.$$

By definition,

$$\int_0^\infty f\,\mathrm{d}\mu = \sum_{n=1}^\infty \alpha_n f(n).$$

A8) $f, g \in \mathcal{R}([a,b];\mu)$ **are real-valued Riemann integrable functions. Suppose for any $x \in [a,b]$, $g(x) \geqslant 0$. Let**

$$m = \inf_{x\in I} f(x),\ M = \sup_{x\in I} f(x).$$

Then there exists $\in [m,M]$ such that

$$\int_a^b fg\,\mathrm{d}\mu = \int_a^b g\,\mathrm{d}\mu.$$

Proof: Note that $mg \leqslant fg \leqslant Mg$, and use A4).

A9) Construct a Stieltjes integral to show that Abel summation method is a special case of integration by parts.

Proof:
The Abel summation formula states that

$$\sum_{i=1}^{n} T_i(S_i - S_{i-1}) = T_n S_n - T_1 S_0 - \sum_{i=1}^{n-1} S_i(T_{i+1} - T_i).$$

Consider the monotonically increasing function $\mu : [0, n] \to \mathbb{R}, x \mapsto T_{\lceil x \rceil}, \mu(0) = T_1$, and f be a polynomial such that $f(k) = S_k$ for $k = 0, 1, \cdots, n$. Then

$$\int_0^n f'\mu \,\mathrm{d}x = \sum_{k=1}^{n} \int_{k-1}^{k} f'\mu = \sum_{k=1}^{n} \int_{k-1}^{k} f'(x) T_k \,\mathrm{d}x = \sum_{k=1}^{n} T_k(S_k - S_{k-1}).$$

While

$$\int_0^n f \,\mathrm{d}\mu = \sum_{k=1}^{n-1} f(k)(\mu(k+0) - \mu(k)) = \sum_{k=1}^{n-1} S_k(T_{k+1} - T_k).$$

and

$$f\mu\Big|_0^n = T_n S_n - T_1 S_0.$$

Hence the formula is a special case of integration by parts.

12.2 PSB: Convergence of Improper Integrals

b can be ∞.

B1) Assume $f : [a, b) \to \mathbb{R}$, and for any $b^- < b$, f is integrable on $[a, b^-]$. Prove that the integral $\int_a^b f(x) \,\mathrm{d}x$ exists iff: for any $\varepsilon > 0$, $\exists b(\varepsilon) \in (a, b)$ such that for any $b', b'' > b(\varepsilon)$, $\left|\int_{b'}^{b''} f(x) \,\mathrm{d}x\right| < \varepsilon$.

Proof: Let

$$F(t) = \int_a^t f(x) \,\mathrm{d}x, \forall t \in [a, b).$$

Then $\int_a^b f(x) \,\mathrm{d}x$ exists iff $\lim_{t \to b^-} F(t)$ exists, which is equivalent to

$$\forall \varepsilon > 0, \exists b(\varepsilon) \in (a, b), \forall b', b'' > b(\varepsilon), \left|\int_{b'}^{b''} f(x) \,\mathrm{d}x\right| = |F(b'') - F(b')| < \varepsilon.$$

B2) If $|f(x)| \leqslant F(x), x \in [a, b)$ and $\int_a^b F(x) \,\mathrm{d}x$ converges, then $\int_a^b f(x) \,\mathrm{d}x$ converges.

Proof: Use B1) and

$$\left|\int_u^v f(x) \,\mathrm{d}x\right| \leqslant \int_u^v F(x) \,\mathrm{d}x.$$

B3) Prove the Dirichlet test for convergence: if $f, g : [a, \infty) \to \mathbb{R}$ satisfy

- f is continuous and there exists $A > 0$, such that for any $M \geqslant a$,
$$\left| \int_a^M f(x)\,\mathrm{d}x \right| \leqslant A.$$
- g is monotonic and $\lim_{x\to\infty} g(x) = 0$.
Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

Lemma: The Second Integral Mean Value Theorem

If f is integrable and g is monotonic and non-negative(or non-positive) on $[a, b]$, then there exists $c \in (a, b)$ such that
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^c f(x)\,\mathrm{d}x + g(b)\int_c^b f(x)\,\mathrm{d}x.$$
Proof: Assume that g is non-negative and monotonically decreasing. It is easy to see that there exists $\xi \in (a, b)$ such that
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x.$$
Apply the above formula to $f(x)$ and $g(x) - g(b)$ and we get
$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x.$$
Proof of B3): Since $|\int_u^v f(x)\,\mathrm{d}x| \leqslant 2A$, by lemma
$$\left|\int_u^v f(x)g(x)\,\mathrm{d}x\right| \leqslant 2A(|g(u)| + |g(v)|).$$
By B1), the integral converges.

B4) Prove the Abel test of convergence:

If $f, g : [a, \infty) \to \mathbb{R}$ satisfy:

- $\int_a^\infty f(x)\,\mathrm{d}x$ exists.
- g is monotonic and g is bounded.
Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.
Proof: Suppose g is monotonically increasing, then
$$\left|\int_u^v f(x)(g(x) - g(a))\,\mathrm{d}x\right| \leqslant 2M\left(\left|\int_u^\xi f(x)\,\mathrm{d}x\right| + \left|\int_\xi^v f(x)\,\mathrm{d}x\right|\right) \to 0$$

since $\int_a^\infty f(x)\,\mathrm{d}x$ converges. Therefore both $\int_a^\infty f(x)(g(x) - g(a))\,\mathrm{d}x$ and $\int_a^\infty f(x)g(a)\,\mathrm{d}x$ converges, hence $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

B5) Determine whether the following integrals converges:

(1)
$$\int_0^\infty \frac{\log(1+x)}{x^p}\,\mathrm{d}x$$

(absolutely) convergent when $1 < p < 2$, diverges when $p \leqslant 1$ or $p \geqslant 2$.
(2)

$$\int_1^\infty \frac{\sin x}{x^p}\,\mathrm{d}x$$

Absolutely convergent when $p > 1$, conditionally convergent when $0 < p \leqslant 1$, diverges when $p \geqslant 0$.
(3)

$$\int_1^\infty \sin x^2\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \frac{\sin y}{y^{1/2}}\,\mathrm{d}y$$

is conditionally convergent.
(4)

$$\int_0^\infty \frac{\sin^2 x}{x}\,\mathrm{d}x$$

diverges
(5) $p, q > 0$,

$$\int_0^{2\pi} \sin^{-p} x \cos^{-q} x\,\mathrm{d}x$$

Absolutely convergent when $p, q < 1$, diverges when $p \geqslant 1$ or $q \geqslant 1$.
(6)

$$\int_0^\infty x^p \sin(x^q)\,\mathrm{d}x$$

If $q = 0$ the integral diverges. Assume $q \neq 0$ below.

$$\int_0^\infty x^p \sin(x^q)\,\mathrm{d}x = \frac{1}{q}\int_0^\infty y^{(p+1)/q-1}\sin y\,\mathrm{d}y.$$

Let $\alpha = \frac{p+1}{q} - 1$, then the integral

- diverges if $\alpha \leqslant -2$ or $\alpha \geqslant 0$,
- converges absolutely if $-2 < \alpha < -1$.
- converges conditionally if $-1 \leqslant \alpha < 0$.
  (7) $q \geqslant 0$,

$$\int_0^\infty \frac{x^p \sin x}{1 + x^q}\,\mathrm{d}x$$

If $p \leqslant -2$, then the integral diverges near 0, since $x^p \sin x \sim x^{p+1}$. The integral converges (absolutely) near 0 otherwise. Assume $p > -2$ below.
If $p - q < -1$ then the integral converges absolutely when it tends to infinity, since $\frac{x^p}{1+x^q} \sim x^{p-q}$.
If $-1 \leqslant p - q < 0$ then the integral converges conditionally, since the integral of $(x^{p-q})'$ converges.
(8)

$$\begin{aligned}
\int_0^{\pi/2} \frac{\log \sin x}{\sqrt{x}}\,\mathrm{d}x &= 2\int_0^{\pi/2} \log \sin x\,\mathrm{d}\sqrt{x} \\
&= 2\sqrt{x}\log \sin x\Big|_0^{\pi/2} - 2\int_0^{\pi/2} \sqrt{x}\cot x\,\mathrm{d}x \\
&= -2\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\cos x\,\mathrm{d}x
\end{aligned}$$

converges, since $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges.
(9)

$$\int_e^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x = \int_1^\infty \frac{\log y}{y}e^y \sin e^y\,\mathrm{d}y.$$

It is easy to see the integral does not converge absolutely.
Meanwhile

$$f'(x) = \left(\frac{\log\log x}{\log x}\right)' = \frac{1-\log\log x}{(\log x)^2 x},$$

and

$$\int_e^\infty \frac{\log\log x - 1}{(\log x)^2 x}\,\mathrm{d}x = \int_1^\infty \frac{\log y - 1}{y^2}\,\mathrm{d}y = \int_0^\infty \frac{t-1}{e^t}\,\mathrm{d}t.$$

converges.
By Lagrange mean value theorem,

$$\begin{aligned}\int_{2\pi}^\infty \frac{\log\log x}{\log x}\sin x\,\mathrm{d}x &= \sum_{n=1}^\infty \int_{2n\pi}^{(2n+1)\pi} (f(x)-f(x+\pi))\sin x\,\mathrm{d}x \\ &\leqslant \sum_{n=1}^\infty -2\pi f'(2n\pi) \leqslant 2\pi\int_e^\infty -f'(x)\,\mathrm{d}x\end{aligned}$$

converges.

12.3 PSC: Oscillatory Integral

$F(t), G(t) : [1,\infty) \to \mathbb{R}$, $\lim_{t\to\infty} G(t) = 0$. Assume that for any $t \geqslant 1$, $G(t) \neq 0$. If

$$\lim_{t\to\infty}\frac{F(t)}{G(t)} = 1.$$

Then we say F, G have the same order, and $F \sim G$.

Part 1

C1) $d > 0$ is a given real number. Assume $g \in C^1([0,d])$. Prove that there is a constant C, such that

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant \frac{C}{t}.$$

Proof: Let $C = \sup_{x\in[0,d]} |g(x)|$, then

$$\left|\int_0^d e^{-tx} g(x)\,\mathrm{d}x\right| \leqslant C\int_0^d e^{-tx}\,\mathrm{d}x = \frac{C}{t}(1-e^{-td}) \leqslant \frac{C}{t}.$$

C2) Assume $d > 0$, $g \in C([0,d])$ and $g(0) \neq 0$. Prove that

$$\int_0^d e^{-tx} g(x)\,\mathrm{d}x \sim \frac{g(0)}{t}.$$

Proof: Let $M = \sup_{x\in[0,d]} |g(x)|$, then

$$\begin{aligned}
\left|\int_0^d e^{-tx} t\frac{g(x)}{g(0)}\,\mathrm{d}x - 1\right| &= \left|\int_0^{td} e^{-u}\frac{g(u/t)}{g(0)}\,\mathrm{d}u - \int_0^\infty e^{-u}\,\mathrm{d}u\right| \\
&\leqslant \int_{td}^\infty e^{-u}\,\mathrm{d}u + \int_0^N e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u + \int_N^{td} e^{-u}\left|\frac{g(u/t)}{g(0)} - 1\right|\mathrm{d}u \\
&\leqslant e^{-td} + \sup_{0\leqslant x\leqslant N/t}\left|\frac{g(x)}{g(0)} - 1\right| + \left(\frac{M}{|g(0)|} + 1\right)\int_N^{td} e^{-u}\,\mathrm{d}u \to 0.
\end{aligned}$$

(let $t \to \infty$ then let $N \to \infty$).

C3) $d > 0, g \in C([0,d]), g(0) \neq 0$. **Prove that**

$$\int_0^d e^{-tx^2} g(x)\,\mathrm{d}x \sim \frac{\sqrt{\pi}\cdot g(0)}{2\sqrt{t}}.$$

Proof: Same as C2), let $M = \sup_{x\in[0,d]} |g(x)/g(0)|$, then

$$\begin{aligned}
\left|\int_0^d e^{-tx^2}\sqrt{t}\frac{g(x)}{g(0)}\,\mathrm{d}x - \frac{\sqrt{\pi}}{2}\right| &= \left|\int_0^{d\sqrt{t}} e^{-u^2}\frac{g(u/\sqrt{t})}{g(0)}\,\mathrm{d}u - \int_0^\infty e^{-u^2}\,\mathrm{d}u\right| \\
&\leqslant \int_{d\sqrt{t}}^\infty e^{-u^2}\,\mathrm{d}u + \int_0^N e^{-u^2}\left|\frac{g(u/\sqrt{t})}{g(0)} - 1\right|\mathrm{d}x + \int_N^{d\sqrt{t}} e^{-u^2}(M+1)\,\mathrm{d}u.
\end{aligned}$$

which tends to 0, same as C2).

For $t \geqslant 1$, $f, \varphi \in C([a,b])$, define the function

$$F(t) = \int_a^b e^{-t\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $F(t)$ when $t \to \infty$.

C4) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Further assume that $\varphi'(x) > 0$. Let $d = \varphi(b) - \varphi(a)$. Prove that

$$\Psi : [a,b] \to [0,d],\, x \mapsto \varphi(x) - \varphi(a),$$

is a continuously differentiable bijection on $[a,b]$.
Proof: φ is monotonic by $\varphi'(x) > 0$, hence Ψ is a bijection and $\Psi' = \psi'$.

C5) Assume $\varphi \in C^1([a,b])$, and for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Prove that if $f(a) \neq 0$, then when $t \to \infty$,

$$F(t) \sim \frac{f(a)}{\varphi'(a)}\frac{e^{-t\varphi(a)}}{t}.$$

Proof: Let $g(x) = f(x)/\Psi'(x)$, and $h = (t\Psi)^{-1}$ then

$$
\begin{aligned}
\left|F(t)\frac{t}{e^{-t\varphi(a)}} - \frac{f(a)}{\varphi'(a)}\right| &= \left|\int_a^b e^{-t\Psi(x)} t f(x)\,\mathrm{d}x - \frac{f(a)}{\Psi'(a)}\right| = \left|\int_a^b e^{-t\Psi(x)} g(x)\,\mathrm{d}t\Psi(x) - g(a)\right| \\
&= \left|\int_0^{t\Psi(b)} e^{-u} g(h(u))\,\mathrm{d}u - g(h(0))\int_0^\infty e^{-u}\,\mathrm{d}u\right| \\
&= |g(h(0))|\int_{t\Psi(b)}^\infty e^{-u}\,\mathrm{d}u + \int_0^{N\Psi(b)} e^{-u}|g(h(u)) - g(h(0))|\,\mathrm{d}u \\
&+ \int_{N\Psi(b)}^{t\Psi(b)} e^{-u}|g(h(u)) - g(h(0))|\,\mathrm{d}u \\
&\leqslant |g(a)|e^{-t\Psi(b)} + \sup_{x\in[a,\Psi^{-1}(N\Psi(b)/t)]}|g(x) - g(a)| + \int_{N\Psi(b)}^{t\Psi(b)} e^{-u}2M\,\mathrm{d}u.
\end{aligned}
$$

which tends to 0 since g is continuous. ($M = \sup_{x\in[a,b]}|g(x)|$).

C6) Assume that $\varphi \in C^2([a,b]), \varphi'(a) = 0, \varphi''(x) > 0$ and for any $x \in (a,b], \varphi'(x) > 0$. Let $d = \sqrt{\varphi(b) - \varphi(a)}$. Prove that

$$\Psi : [a,b] \to [0,d],\ x \mapsto \sqrt{\varphi(x) - \varphi(a)}.$$

is a continuously differentiable bijection on $[a,,b]$, and calculate $\Psi'(a)$.
Proof: Trivial. $\Psi' = \frac{\varphi'}{2\Psi}$, hence

$$\Psi'(a) = \lim_{x\to a^+}\frac{\varphi'(x)}{2\sqrt{\varphi(x) - \varphi(a)}} = \lim_{x\to a^+}\frac{\varphi''(x)}{\varphi'(x)/\sqrt{\varphi(x) - \varphi(a)}} = \sqrt{\frac{\varphi''(a)}{2}}.$$

C7) Assume $\varphi \in C^2([a,b]), \varphi'(a) = 0, \varphi''(a) > 0$. Prove that if $f(a) \neq 0$, when $t \to \infty$,

$$F(t) \sim \frac{\sqrt{\pi}f(a)}{\sqrt{2\varphi''(a)}}\frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: Let $g = f/\Psi'$, $h = (\sqrt{t}\Psi)^{-1}$, then

$$F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}} = \int_a^b e^{-t\Psi^2(x)} f(x)\sqrt{t}\,\mathrm{d}x = \int_a^b e^{-t\Psi^2(x)} g(x)\,\mathrm{d}\sqrt{t}\Psi(x) = \int_0^{\sqrt{t}\Psi(b)} e^{-u^2} g(h(u))\,\mathrm{d}u.$$

Hence

$$
\begin{aligned}
\left|F(t)\frac{\sqrt{t}}{e^{-t\varphi(a)}} - \frac{\sqrt{\pi}}{2}g(a)\right| &= \left|\int_0^{\sqrt{t}\Psi(b)} e^{-u^2} g(h(u))\,\mathrm{d}u - \int_0^\infty e^{-u^2} g(h(0))\,\mathrm{d}u\right| \\
&\leqslant g(a)\int_{\sqrt{t}\Psi(b)}^\infty e^{-u^2}\,\mathrm{d}u + \int_0^{N\Psi(b)} e^{-u^2}|g(h(u)) - g(h(0))|\,\mathrm{d}u \\
&+ \int_{N\Psi(b)}^{\sqrt{t}\Psi(b)} e^{-u^2} 2M\,\mathrm{d}u \\
&\leqslant g(a)e^{-\sqrt{t}\Psi(b)} + \sqrt{\pi}\sup_{x\in[a,\Psi^{-1}(N\Psi(b)/\sqrt{t})]}|g(x) - g(a)| + 2Me^{-N\Psi(b)}.
\end{aligned}
$$

which tends to 0 as $t \to \infty$ and $N \to \infty$, since g is continuous.
(A much simpler solution can be given using the Laplace method)

C8) Given $f \in C((0,\infty)), \varphi \in C^2((0,\infty))$. Assume that

- exists a unique $a \in (0,\infty)$ such that $\varphi'(a) = 0$;
- $\varphi''(a) > 0, f(a) \neq 0$;
- $\int_0^\infty e^{-\varphi(x)}|f(x)|\,\mathrm{d}x$ converges.
  Prove that when $t \to \infty$, the function $G(t) = \int_0^\infty e^{-t\varphi(x)} f(x)\,\mathrm{d}x$ satisfy

$$G(t) \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

Proof: (Simple application of the Laplace method)
Apply C7) to the intervals $[a/2, a]$ and $[a, 2a]$, then

$$\int_{a/2}^{2a} e^{-t\varphi(x)} f(x)\,\mathrm{d}x \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \cdot \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

While the integral on the intervals $(0, a/2), (2a, \infty)$ converges rapidly. Hence

$$G(t) \sim \frac{\sqrt{2\pi} f(a)}{\sqrt{\varphi''(a)}} \frac{e^{-t\varphi(a)}}{\sqrt{t}}.$$

C9) $\Gamma(n) = (n-1)!$**.**

Proof:

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,\mathrm{d}t = -\int_0^\infty t^n\,\mathrm{d}e^{-t} = n\int_0^\infty t^{n-1}e^{-t}\,\mathrm{d}t = n\Gamma(n).$$

C10) Prove Stirling‘s approximation

$$n! \sim \sqrt{2\pi}\frac{n^{n+1/2}}{e^n}.$$

Proof: Note that, by substituting $t = ns$,

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t}t^n\,\mathrm{d}t = n^{n+1}\int_0^\infty e^{-n(s-\log s)}\,\mathrm{d}s.$$

Hence

$$\frac{\Gamma(t+1)}{t^{t+1}} \sim \sqrt{2\pi}\frac{e^{-t}}{\sqrt{t}}.$$

Part 2

For $\lambda \geqslant 1,\ f, \varphi \in C^\infty([a,b])$, define the function

$$I(\lambda) = \int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x.$$

Our goal is to study $I(\lambda)$ when $\lambda \to \infty$.

C11) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$. Define the maps

$$\begin{aligned} L &: C^\infty([a,b]) \to C^\infty([a,b]), h \mapsto \frac{1}{i\lambda\varphi'(x)} h'(x), \\ M &: C^\infty([a,b]) \to C^\infty([a,b]), h \mapsto -\left(\frac{h}{i\varphi'}\right)'(x). \end{aligned}$$

Assume that $f, g \in C^\infty([a,b])$. Prove that if there exists $c > 0$ such that for any $x \in [a, a+c] \cup [b-c, b]$, $h(x) = 0$, then M/λ is the adjoint of L:

$$\int_a^b h \cdot Lg = \frac{1}{\lambda} \int_a^b g \cdot Mh.$$

Proof: By integration of parts,

$$\int_a^b h \cdot Lg = \int_a^b \frac{h}{i\lambda\varphi'} \,\mathrm{d}g = -\int_a^b g \,\mathrm{d}\left(\frac{h}{i\lambda\varphi'}\right) = \frac{1}{\lambda} \int_a^b g \cdot Mh.$$

C12) Assume that for any $x \in [a,b]$, $\varphi'(x) \neq 0$ and f vanishes near a and b. prove that for any $n \in \mathbb{Z}_{\geqslant 1}$, there is a constant c_n independent of λ such that

$$|I(\lambda)| \leqslant \frac{c_n}{\lambda^n}.$$

Proof: Let $g = e^{i\lambda}\varphi$, then $Lg = g$. $f \in C^\infty([a,b])$ vanishes near a, b hence $M^n f$ vanishes near a, b for any $n \in \mathbb{Z}_{\geqslant 0}$. Therefore

$$|I(\lambda)| = \left| \int_a^b fg \right| = \frac{1}{\lambda} \left| \int_a^b g \cdot Mf \right| = \cdots = \frac{1}{\lambda^n} \left| \int_a^b g \cdot M^n f \right|.$$

so $c_n = \left| \int_a^b g \cdot M^n f \right|$ is valid.

C13) If there exists $\delta > 0$, such that for any $x \in [a,b]$, $|\varphi'(x)| \geqslant \delta$ and $\varphi'(x)$ is monotonic on $[a,b]$. Prove that there is a constant C_1 independent of λ, φ, a, b such that

$$\left| \int_a^b e^{i\lambda\varphi(x)} \,\mathrm{d}x \right| \leqslant \frac{C_1}{\lambda\delta}.$$

Proof: Let $C_1 = 4$ then (since φ' maintains the same sign)

$$\begin{aligned} \left| \int_a^b e^{i\lambda\varphi(x)} \,\mathrm{d}x \right| = \left| \int_a^b \frac{\mathrm{d}e^{i\lambda\varphi}}{\lambda\varphi'} \right| &\leqslant \left| \frac{e^{i\lambda\varphi}}{\lambda\varphi'} \Big|_a^b \right| + \frac{1}{\lambda} \left| \int_a^b e^{i\lambda\varphi} \frac{\varphi''}{(\varphi')^2} \,\mathrm{d}x \right| \\ &\leqslant \frac{2}{\lambda\delta} + \frac{1}{\lambda} \int_a^b \left| \frac{\varphi''}{(\varphi')^2} \right| \\ &= \frac{2}{\lambda\delta} + \frac{1}{\lambda} \int_a^b \mathrm{d}\frac{1}{\varphi'} \leqslant \frac{4}{\lambda\delta}. \end{aligned}$$

C14) Suppose for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a unique $c \in [a,b]$ such that

$$|\varphi'(c)| = \inf_{x\in[a,b]} |\varphi'(x)|.$$

Further prove that for any $x \in [a,b]$,

$$|\varphi'(x)| \geqslant |x-c|.$$

Proof: Since $\varphi \in C^\infty([a,b])$ and $|\varphi''| \geqslant 1$, φ'' maintains the same sign. Assume that $\forall x \in [a,b], \varphi''(x) \geqslant 1$, then φ' is monotonically increasing. Therefore, if $\varphi' \neq 0$, then $c \in \{a,b\}$, otherwise, c is the unique null point of φ'.
Either $\varphi'(c) = 0$ or $c = a$, when φ' maintains the same sign, so we always have $|\varphi'(x)| \geqslant |\varphi'(x) - \varphi'(c)|$, and

$$\forall x \in [a,b], \exists \xi \in [x,c],\ |\varphi'(x) - \varphi'(c)| \geqslant |x-c| \cdot \varphi'(\xi) \geqslant |x-c|.$$

Therefore $|\varphi'(x)| \geqslant |x-c|$.

!C15) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of λ, φ, a, b, such that

$$\left| \int_a^b e^{i\lambda\varphi(x)}\, \mathrm{d}x \right| \leqslant \frac{C_2}{\sqrt{\lambda}}.$$

Proof: Since $\varphi \in C^\infty([a,b])$, we can assume $\varphi''(x) \geqslant 1$. For an arbitrary $\delta > 0$, divide the interval $[a,b]$ into two parts:
$E_1 = \{x : |\varphi'(x)| \leqslant \delta\}$ and $E_2 = \{x : |\varphi'(x)| > \delta\}$.
Note that $\forall x, y \in E_1$, $|\varphi'(x) - \varphi'(y)| \leqslant 2\delta$, but $|\varphi'(x) - \varphi'(y)| \geqslant \left|\int_x^y \varphi''(t)\, \mathrm{d}t\right| \geqslant |x-y|$. Therefore E_1 is an interval of length at most 2δ, so

$$\left| \int_{E_1} e^{i\lambda\varphi(x)}\, \mathrm{d}x \right| \leqslant 2\delta.$$

Now consider the integral on E_2, which is the union of one or two intervals. By C13),

$$\left| \int_{E_2} e^{i\lambda\varphi(x)}\, \mathrm{d}x \right| \leqslant 2 \cdot \frac{4}{\lambda\delta}.$$

Therefore

$$\left| \int_a^b e^{i\lambda\varphi(x)}\, \mathrm{d}x \right| \leqslant 2\delta + \frac{8}{\lambda\delta} = \frac{8}{\sqrt{\lambda}}.$$

(if we let $\delta = \sqrt{4/\lambda}$.)

C16) Assume that for any $x \in [a,b], |\varphi''(x)| \geqslant 1$. Prove that there is a constant C_2 independent of $\lambda, \varphi, f, a, b$ such that

$$\left| \int_a^b e^{i\lambda\varphi(x)} f(x)\, \mathrm{d}x \right| \leqslant \frac{C_2}{\sqrt{\lambda}} \left(|f(a)| + \int_a^b |f'(x)|\, \mathrm{d}x \right).$$

Proof: By C15),

$$\begin{aligned}\left|\int_a^b e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x\right| &\leqslant \left|\int_a^b e^{i\lambda\varphi(x)} f(a)\,\mathrm{d}x\right| + \left|\int_a^b e^{i\lambda\varphi(x)} \int_a^x f'(t)\,\mathrm{d}t\,\mathrm{d}x\right| \\ &\leqslant |f(a)|\frac{C_2}{\sqrt{\lambda}} + \left|\int_a^b f'(t)\int_t^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant \frac{C_2}{\sqrt{\lambda}}\left(|f(a)| + \int_a^b |f'(x)|\,\mathrm{d}x\right).\end{aligned}$$

13 Midterm Exam

13.1 Problem A

A1) Prove using $\varepsilon - N$:

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

Proof: For any $\varepsilon > 0$, there exists $N = \lfloor 1/\varepsilon \rfloor + 10$ such that for any $n > N$,

$$0 < \frac{n}{n^2+1} < \frac{1}{n} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 0$.

A2) Prove using $\varepsilon - \delta$ that $f(x) = \begin{cases} \sin(1/x), & x \neq 0; \\ 0, & x = 0 \end{cases}$ is not continuous at x.
Proof: Let $\varepsilon = 1$, for any $\delta > 0$, there exists $x_1 = (2\pi n)^{-1}$ and $x_2 = (2\pi n + \pi/2)^{-1}$ where $n = \lfloor 1/\delta \rfloor + 10$ such that $0 < x_2 < x_1 < \delta$ and $|\sin x_1 - \sin x_2| = 1$, so f is not continuous at x.

A3) Calculate

$$\lim_{n\to\infty} \left(\sum_{k=1}^n k^2\right)^{1/n}.$$

Solution: Clearly $\sum_{k=1}^n k^2 < \sum_{k=1}^n n^2 = n^3$ so $0 < \left(\sum_{k=1}^n k^2\right)^{1/n} < (n^{1/n})^3$. Since $\lim_{n\to\infty} n^{1/n} = 0$,

$$\lim_{n\to\infty} \left(\sum_{k=1}^n k^2\right)^{1/n} = 0.$$

A6) Prove that the series $\sum_{n=1}^\infty \frac{2n+1}{n^2(n+100)^2}$ converges.
Proof: Note that

$$\sum_{n=1}^\infty \frac{2n+1}{n^2(n+100)^2} < \sum_{n=1}^\infty \frac{2n+1}{n^2(n+1)^2} = 1.$$

Hence the series converges.

A5) Given a real sequence $\{a_n\}$, if $\sum_{n=1}^{\infty} a_n$ converges, prove that for any $x \in (-1, 1)$, $\sum_{n=1}^{\infty} a_n x^n$ converges.
Proof: $\sum_{n=1}^{\infty} a_n$ converges implies $\lim_{n\to\infty} a_n = 0$. For any $n, m \geqslant 0$,

$$\left|\sum_{k=n}^{n+m} a_k x^k\right| \leqslant \sum_{k=n}^{\infty} |a_k| x^k \leqslant \frac{x^n}{1-x} \sup_{k\geqslant n} |a_k|.$$

Since $x^n \to 0$ and $\sup_{k\geqslant n} |a_k| \to 0$, the series is Cauchy hence it converges.

A6) Suppose $f \in C((a, b))$, prove that for any $x_1, x_2 \in (a, b)$, there exists $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{1}{2}(f(x_1) + f(x_2)).$$

Proof: Let $g(x) = f(x) - \frac{1}{2}(f(x_1) + f(x_2))$, then $g(x_1)g(x_2) \leqslant 0$ so g has a solution $x_0 \in (a, b)$.

A7) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with period 1, prove that f is bounded and can reach $\sup f(x)$.
Proof: $\sup_{x\in\mathbb{R}} f(x) = \sup_{x\in[0,1]} f(x) < \infty$ and there exists $x \in [0, 1]$ such that $f(x) = \sup_{y\in[0,1]} f(y)$.

A8) Prove that $f(x) = \sqrt{x+1}$ is uniformly continuous on $\mathbb{R}_{\geqslant 0}$.
Proof: Note that

$$|f(x) - f(y)| = |\sqrt{x+1} - \sqrt{y+1}| = \frac{|x-y|}{\sqrt{x+1} + \sqrt{y+1}} \leqslant |x - y|$$

Hence f is uniformly continuous.

13.2 Problem B

Given a sequence $\{a_k\}$ of complex numbers, let

$$a_n^* = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k.$$

Part 1

Suppose $a_k = z^k$ for any $k \geqslant 0$, where $z \in \mathbb{C}$.

B1) Prove that if $|z| < 1$, then $\sum_{k=0}^{\infty} a_k$ converges. Denote the limit by $A(z)$.
Proof: Note that

$$\sum_{k=0}^{n} a_k = \frac{1 - z^{n+1}}{1 - z},$$

and $|z^{n+1}| = |z|^{n+1} \to 0$ so $\sum_{k=0}^{\infty} a_k$ converges and $A(z) = 1/(1 - z)$.

B2) Prove that if $|z| < 1$, then $\sum_{k=0}^{\infty} a_k^*$ converges, denote it by $A^*(z)$.
Proof: Since $a_k^* = \left(\frac{1+z}{2}\right)^k$, and $\left|\frac{1+z}{2}\right| \leqslant \frac{1+|z|}{2} < 1$, $\sum_{k=0}^{\infty} a_k^*$ converges and $A^*(z) = 1/\left(1 - \frac{1+z}{2}\right) = \frac{2}{1-z}$.

B3) Prove that if $|z| \geqslant 1$, then $\sum_{k=0}^{\infty} a_k$ does not converge.
Proof: $|a_k| = 1$ does not tend to 0, so $\sum_{k=0}^{\infty} a_k$ does not converge.

B4) Find $z \in \mathbb{C}$ such that $|z| > 1$ and $\sum_{k=0}^{\infty} a_k^*$ converges.
Solution: Let $z = i\sqrt{2}$, then $\left|\frac{1+z}{2}\right| = \frac{\sqrt{3}}{2} < 1$ so $\sum_{k=0}^{\infty} a_k^*$ converges.

B5) Prove that if $|z| = 1$ and $z \neq \pm 1$, then $\sum_{k=0}^{\infty} a_k^*$ converges.
Proof: Likewise $\left|\frac{1+z}{2}\right| \leqslant 1$ and equality holds iff $z = 1$, so $\sum_{k=0}^{\infty} a_k^*$ converges.

Part 2

Suppose $\{a_k\}_{k \geqslant 0}$ is a sequence of real numbers.

B6) Prove that if $k \in \mathbb{Z}_{\geqslant 0}$ is fixed,

$$\lim_{n\to\infty} \binom{n}{k} / \frac{n^k}{k!} = 1, \ \lim_{n\to\infty} \binom{n}{k} / 2^n = 0.$$

Proof: Note that

$$\binom{n}{k} / \frac{n^k}{k!} = \prod_{j=0}^{k-1} \frac{n-j}{n}$$

hence $\lim_{n\to\infty} \binom{n}{k} / \frac{n^k}{k!} = 1$.
For $n > 2k$, $\binom{n}{k} < \binom{n}{k+1} < \cdots < \binom{n}{\lfloor n/2 \rfloor}$ so

$$\binom{n}{k} / 2^n < \frac{1}{\lfloor n/2 \rfloor - k} \to 0.$$

B7) Given any non-negative integer $n > q$, we define

$$a_{n,q}^* = \frac{1}{2^n} \sum_{k=0}^{q} \binom{n}{k} a_k.$$

For any fixed q, calculate $\lim_{n\to\infty} a_{n,q}^*$.
Solution: By B6) $\lim_{n\to\infty} \binom{n}{k}/2^n = 0$ uniformly for any $0 \leqslant k \leqslant q$ so $\lim_{n\to\infty} a_{n,q}^* = 0$.

B8) If $\lim_{n\to\infty} a_n = 0$, prove that $\lim_{n\to\infty} a_n^* = 0$.
Proof: Note that

$$|a_n^*| \leqslant |a_{n,q}^*| + \frac{1}{2^n} \sum_{k=q+1}^{n} |a_k| \leqslant |a_{n,q}^*| + \sup_{k>q} |a_k|.$$

Since $\lim_{n\to\infty} a_{n,q}^* = 0$, and $\sup_{k>q} |a_k| \to 0$ as $q \to 0$, $\lim_{n\to\infty} a_n^* = 0$.

B9) If $\lim_{n\to\infty} a_n$ exists, prove that $\lim_{n\to\infty} a_n^*$ exists and is exactly $\lim_{n\to\infty} a_n$.
Proof: Suppose $a = \lim_{n\to\infty} a_n$, let $b_n = a_n - a$ and $b_n^* = a_n^* - a$, then $b_n^* = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k$ and $\lim_{n\to\infty} b_n = 0$. Hence $\lim_{n\to\infty} b_n^* = 0$ so $\lim_{n\to\infty} a_n^* = \lim_{n\to\infty} a_n$.

B10) If $\lim_{n\to\infty} a_n^*$ exists, does $\lim_{n\to\infty} a_n$ exist?
Solution: Not necessarily. If $a_n = (-1)^n$ then $a_n^* = 0$ so $\lim_{n\to\infty} a_n^* = 0$ but $\lim_{n\to\infty} a_n$ does not exist.

B11) For $n \geqslant 0$ define

$$S_n = \sum_{k=0}^{n} a_k,\ S_n^* = \sum_{k=0}^{n} a_k^*,\ U_n = 2^n S_n^*.$$

Prove that for any $n \geqslant 0$,

$$U_n = \sum_{k=0}^{n} \binom{n+1}{k+1} S_k.$$

Proof: Note that

$$U_n = 2^n \sum_{k=0}^{n} a_k^* = 2^n \sum_{k=0}^{n} 2^{-k} \sum_{j=0}^{k} \binom{k}{j} a_j = \sum_{j=0}^{n} a_j \sum_{k=0}^{n} 2^{n-k} \binom{k}{j}$$

while

$$\sum_{k=0}^{n} \binom{n+1}{k+1} S_k = \sum_{k=0}^{n} \binom{n+1}{k+1} \sum_{j=0}^{k} a_j = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} \binom{n+1}{k+1}.$$

It suffices to show that for any j and $n \geqslant j$,

$$\sum_{k=j}^{n} 2^{n-k} \binom{k}{j} = \sum_{k=j}^{n} \binom{n+1}{k+1}.$$

Note that

$$\sum_{k=j}^{n} 2^{n-k} \binom{k}{j} - \sum_{k=j}^{n-1} 2^{n-1-k} \binom{k}{j} = \binom{n}{j} + \sum_{k=j}^{n-1} 2^{n-1-k} \binom{k}{j}$$

and

$$\sum_{k=j}^{n} \binom{n+1}{k+1} - \sum_{k=j}^{n-1} \binom{n}{k+1} = 1 + \sum_{k=j}^{n-1} \binom{n}{k} = \binom{n}{j} + \sum_{k=j}^{n-1} \binom{n}{j+1}.$$

Hence we can prove this by induction.

B12) Prove that if $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} a_k^*$ converges.
Proof: Note that $S_n^* = 2 \cdot \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n+1}{k+1} S_k$ so S_n converges implies S_n^* converges, and $\sum_{k=0}^{\infty} a_k^* = 2 \sum_{k=0}^{\infty} a_k$.

13.3 Problem C: Maximal Ideal of $C([a,b])$

Suppose $a < b$ are real numbers, we study the properties of the ring $C([a,b])$.

C1) For any subset $A \subset [a,b]$, let $I(A) = \{f \in C([a,b]) : f(A) = \{0\}\}$. Prove that $I(A)$ is an ideal of $C([a,b])$. What is $I([a,b])$? Prove that if $A \subset B$ then $I(B) \subset I(A)$. Does there exists $A \subsetneq [a,b]$ such that $I(A) = \{0\}$?
Proof: For any $f, g \in I(A)$ and any $x \in A$, $(f+g)(x) = 0$ and for $f \in I(A)$, $g \in C([a,b])$, $(fg)(x) = 0$ so $I(A)$ is an ideal. If $A \subset B$ and $f \in I(B)$ then $f(A) \subset f(B) = \{0\}$ so $f \in I(A)$. Let $A = (a,b]$, then $f(A) = 0$ implies $f(a) = \lim_{x \to a^+} f(a) = 0$ so $I(A) = \{0\}$.

C2) Prove that $I \subset C([a,b])$ is an ideal, then $1 \notin I$. Further prove that for any $f \in I$, $f(x) = 0$ has a root in $[a,b]$.
Proof: If $f \neq 0$ on $[a,b]$, then $f^{-1} \in C([a,b])$, so $1 = f \cdot f^{-1} \in I$. Hence for all $g \in C([a,b])$, $g = 1 \cdot g \in I$, implying $I = C([a,b])$, a contradiction.

C3) For $f \in C([a,b])$, the set $V(f) = f^{-1}(\{0\})$ is closed. Prove that for $I \subset C([a,b])$, $V(I) = \{x \in [a,b] : \forall f \in I, f(x) = 0\}$ is closed. If $V(I) = [a,b]$, can I be determined?
Proof: $V(I) = \bigcap_{f \in I} f^{-1}(\{0\})$ is the union of closed sets, hence $V(I)$ is closed. If $V(I) = [a,b]$, then for all $f \in I$ and $x \in [a,b]$, $f(x) = 0$ so $I = \{0\}$.

C4) For any $x \in [a,b]$, let $A = \{x\}$ and $\mathfrak{m}_x = I(A) = I(\{x\})$. Prove that $\mathfrak{m}_x$ is a maximal ideal.
Proof: $\mathfrak{m}_x = \{f \in C([a,b]) : f(x) = 0\}$. If there is a larger ideal $\mathfrak{m}_k \subset \mathfrak{m}$, then there exists $g \in \mathfrak{m} \backslash \mathfrak{m}_k$ so $g(x) \neq 0$. For any $h \in C([a,b])$, note that $h = \frac{h(x)}{g(x)} g + \left(h - \frac{h(x)}{g(x)} g\right)$, where $\frac{h(x)}{g(x)} \cdot g \in \mathfrak{m}$ and $h - \frac{h(x)}{g(x)} g$ vanish at x so is in $\mathfrak{m}$. Hence $h \in \mathfrak{m}$ and $\mathfrak{m} = C([a,b])$, a contradiction.

C5) Prove that if $\mathfrak{m}$ is a maximal ideal of $C([a,b])$, then there exists $x \in [a,b]$ such that $\mathfrak{m} = \mathfrak{m}_x$.
Proof: If for some $x \in [a,b]$, $f(x) = 0$ for all $f \in \mathfrak{m}$, then $\mathfrak{m} \subset \mathfrak{m}_x$. So we suppose $V(\mathfrak{m}) = \emptyset$. For every $x \in [a,b]$ take $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$.
Note $f_x(x) \neq 0$ implies there exists $\varepsilon_x > 0$ such that for any $y \in (x - \varepsilon_x, x + \varepsilon_x) \cap [a,b]$, $f_x(y) \neq 0$. Since $[a,b] \subset \bigcup_{x \in [a,b]} (x - \varepsilon_x, x + \varepsilon_x)$ and $[a,b]$ is compact, there exists $y_1, \cdots, y_n \in [a,b]$ such that $[a,b] \subset \bigcup_{i=1}^n (y_i - \varepsilon_{y_i}, y_i + \varepsilon_{y_i})$.
For ever $1 \leqslant i \leqslant n$, take the continuous function $g_i(x) = \begin{cases} 0, & |x - y_i| \geqslant \varepsilon_i, \\ \varepsilon_i - |x - y_i|, & \text{otherwise} \end{cases}$ such that $\mathrm{supp} g_i = [y_i - \varepsilon_{y_i}, y_i + \varepsilon_{y_i}]$, then $h_i = g_i f_{y_i} \in \mathfrak{m}$ and $\mathrm{supp} h_i = [y_i - \varepsilon_i, y_i + \varepsilon_i]$. Suppose $h_i \geqslant 0$, otherwise replace it with $-h_i$ (since $h_i \neq 0$ on $(y_i - \varepsilon_i, y_i + \varepsilon_i)$, it does not change signs). Then let $f = \sum_{i=1}^n h_i$, we have $f(x) \neq 0$, $\forall x \in [a,b]$ since $[a,b] \subset \bigcup_{i=1}^n (y_i - \varepsilon_i, y_i + \varepsilon_i)$.

C6)** Suppose A is closed, prove that $V(I(A)) = A$.
Proof: Clearly $A \subset V(I(A))$, consider any $x \notin A$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap A = \emptyset$. Take $f(y) = \begin{cases} \varepsilon - |y - \varepsilon|, & |y - x| < \varepsilon \\ 0, & |y - x| \geqslant \varepsilon \end{cases}$, then $\mathrm{supp} f = [x - \varepsilon, x + \varepsilon]$, so $f \in I(A)$ but $f(x) \neq 0$ so $x \notin V(I(A))$. Therefore $A = V(I(A))$.

Note
This result can be applied to compact metric spaces: Suppose (X,d) is a compact metric space, $C(X)$ are all complex valued continuous functions on X, then we have the following bijection:

$$X \longrightarrow \{\text{maximal ideal of } C(x)\}, x \mapsto \mathfrak{m}_x,$$

where $\mathfrak{m}_x = \{f \in C(x) : f(x) = 0\}$.

14 Final Exam

14.1 PSA (60pts)

A1) (5pts) Compute the maximum value of $f(x) = x^3 - 12x + 1$ on the interval $[0,1]$.
Solution: For any $x \in [0,1]$,

$$f(x) = 1 + x(x^2 - 12) \leqslant 1,$$

and equality holds when $x = 0$. Hence the maximum value is 1.

A2) (5pts) Prove that the polynomial $f(x) = x^3 + ax^2 + bx + c$ cannot be convex. Proof: $f''(x) = 3x + a$ cannot be positive on $\mathbb{R}$.

A3) (5pts) Solution:

$$\sqrt{1+x} = 1 + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + o(x^3) \tag{1}$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3). \tag{2}$$

A4) (5pts) Consider the function on $[-1, 1]$**:**

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \in [-1,0) \cup (0,1]; \\ 0, & x = 0. \end{cases}$$

Prove that $f \in C([-1,1])$ but is not differentiable at 0.
Proof: Since $\sin(1/x) \in [-1, 1]$,

$$\lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0.$$

so $f \in C([-1,1])$. However,

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \sin\frac{1}{h},$$

so f is not differentiable at 0.

A5) (5pts) Calculate the integral

$$\int_0^1 \frac{x^2+4}{x^2+3x+2}\,\mathrm{d}x.$$

Solution:

$$\int_0^1 \frac{x^2+4}{x^2+3x+2}\,\mathrm{d}x = \int_0^1 1 - \frac{8}{x+2} + \frac{5}{x+1}\,\mathrm{d}x = 1 + 13\log 2 - 8\log 3.$$

A6) (5pts) Calculate the integral

$$\int_0^1 x^2 e^x\,\mathrm{d}x.$$

Solution:

$$\int_0^1 x^2 e^x\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}e^x = x^2e^x\Big|_0^1 - \int_0^1 2xe^x\,\mathrm{d}x \tag{3}$$
$$= e - 2\int_0^1 x\,\mathrm{d}e^x = e - 2xe^x\Big|_0^1 + 2\int_0^1 e^x\,\mathrm{d}x \tag{4}$$
$$= e - 2e + 2e^x\Big|_0^1 = e - 2. \tag{5}$$

A7) (5pts) Calculate the integral

$$\int_0^\pi \sin^3(x) + \sin(2x)\,\mathrm{d}x.$$

Solution:

$$\int_0^\pi \sin(2x)\,\mathrm{d}x = -\frac{\cos 2x}{2}\Big|_0^\pi = 0.$$

while

$$\int_0^\pi \sin^3 x\,\mathrm{d}x = -2\int_0^{\pi/2} \sin^2 x\,\mathrm{d}\cos x \tag{6}$$

$$= 2\int_0^1 (1-t^2)\,\mathrm{d}t = \frac{4}{3}. \tag{7}$$

A8) (5pts) Calculate the integral

$$\int_{-\pi}^\pi \sin(x^3+2x)\,\mathrm{d}x.$$

Solution: Note that $\sin(x^3+2x)$ is an odd function, so the integral is 0.

A9) (5pts) Calculate the limit

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{k^{2019}}{n^{2020}}.$$

Solution:

$$\int_0^1 x^{2019}\,\mathrm{d}x \leqslant \sum_{k=1}^n \left(\frac{k}{n}\right)^{2019}\frac{1}{n} \leqslant \int_{1/n}^{(n+1)/n} x^{2019}\,\mathrm{d}x$$

Hence

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{k^{2019}}{n^{2020}} = \frac{1}{2020}.$$

A10) (5pts) Determine whether the following improper integral converges:

$$\int_0^1 \frac{\log x}{x^{1/2}}\,\mathrm{d}x.$$

Solution: Substitute $x = y^2$, then

$$\int_0^1 \frac{\log x}{x^{1/2}}\,\mathrm{d}x = \int_0^1 4\log y\,\mathrm{d}y = -4.$$

A11) (5pts) Prove that for any $n \geqslant 1$,

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \geqslant \log n.$$

Proof: Note that

$$\frac{1}{n} \geqslant \int_n^{n+1} \frac{1}{x}\,\mathrm{d}x.$$

Hence

$$\sum_{k=1}^n \frac{1}{k} \geqslant \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x = \log(n+1) \geqslant \log n.$$

A12) (5pts) Prove that the following improper integral converges:

$$\int_2^\infty \frac{\sin x}{\log x}\,\mathrm{d}x.$$

Proof:

$$\int_{2\pi}^\infty \frac{\sin x}{\log x}\,\mathrm{d}x = \sum_{n=1}^\infty \int_{2n\pi}^{(2n+1)\pi} \left(\frac{1}{\log x} - \frac{1}{\log(x+\pi)}\right)\sin x\,\mathrm{d}x.$$

Note that

$$\left(\frac{1}{\log x}\right)' = -\frac{1}{x\log^2 x}.$$

Hence by Lagrange's mean value theorem,

$$\int_{2\pi}^\infty \frac{\sin x}{\log x}\,\mathrm{d}x \leqslant \sum_{n=1}^\infty 2\pi^2 \frac{1}{2n\pi \log^2(2n\pi)}$$

converges. Or use integration by parts

$$\int_2^\infty \frac{\sin x}{\log x}\,\mathrm{d}x = C + \int_2^\infty \frac{\cos x}{x\log^2 x}\,\mathrm{d}x.$$

14.2 PSB (21pts) The Arithmetic-Geometric Mean

B1) (2pts) Suppose $a, b > 0$. Prove that the following improper integral converges:

$$I(a,b) = \int_0^\infty \frac{1}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}\,\mathrm{d}x.$$

Proof:

$$I(a,b) \leqslant \frac{1}{ab} + \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x \leqslant 1 + \frac{1}{ab}.$$

B2) (2pts) Prove that

$$I(a,b) = \int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\,\mathrm{d}\theta.$$

Proof: Let $x = b\tan\theta$, then

$$I(a,b) = \int_0^{\pi/2} \frac{1}{\sqrt{a^2+b^2\tan^2\theta}\cdot b\sec\theta}\,\mathrm{d}b\tan\theta = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

B3) (2pts) Prove that the map defined by $I(a,b)$

$$I:(0,\infty)\times(0,\infty)\to\mathbb{R},\ (a,b)\mapsto I(a,b)$$

is a continuous function.
Proof: For any $\theta\in\mathbb{R}$, $a^2\cos^2\theta+b^2\sin^2\theta\geqslant\min\{a^2,b^2\}$. Hence for $\varepsilon,\delta\to 0$,

$$\left|\frac{1}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}-\frac{1}{\sqrt{(a-\varepsilon)^2\cos^2\theta+(b-\delta)^2\sin^2\theta}}\right| \tag{8}$$

$$\leqslant\frac{1}{2\min\{a^2/2,b^2/2\}^{3/2}}(|a^2-(a-\varepsilon)^2|+|b^2-(b-\delta)^2|)\to 0. \tag{9}$$

B4) (2pts) Prove that for any $\lambda>0$,

$$I(a,b)=\int_0^\infty\frac{\lambda}{ab}\frac{1}{\sqrt{x^2+\left(\frac{\lambda}{a}\right)^2}\sqrt{x^2+\left(\frac{\lambda}{b}\right)^2}}\,\mathrm{d}x.$$

Proof: Let $y=\lambda/x$, then

$$I(a,b)=\int_0^\infty\frac{1}{\sqrt{(\lambda/y)^2+a^2}\sqrt{(\lambda/y)^2+b^2}}\,\mathrm{d}\frac{\lambda}{y}=\int_0^\infty\frac{\lambda}{ab}\frac{1}{\sqrt{y^2+\left(\frac{\lambda}{a}\right)^2}\sqrt{y^2+\left(\frac{\lambda}{b}\right)^2}}\,\mathrm{d}y$$

B5) (2pts) Prove that

$$I(a,b)=2\int_{\sqrt{ab}}^\infty\frac{1}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}\,\mathrm{d}x.$$

Proof: By substituting $y=ab/x$,

$$\int_0^{\sqrt{ab}}\frac{\mathrm{d}x}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}=\int_{\sqrt{ab}}^\infty\frac{ab}{y^2}\frac{\mathrm{d}y}{\sqrt{(ab/y)^2+a^2}\sqrt{(ab/y)^2+b^2}} \tag{10}$$

$$=\int_{\sqrt{ab}}^\infty\frac{\mathrm{d}y}{\sqrt{y^2+a^2}\sqrt{y^2+b^2}}. \tag{11}$$

B6) (2pts) Prove that the map

$$\varphi:(\sqrt{ab},\infty)\to(0,\infty),\ x\mapsto\varphi(x)=\frac{1}{2}\left(x-\frac{ab}{x}\right)$$

is a continuously differentiable bijection.
Proof: $\varphi\in C^1$ is trivial. Since $\varphi'(x)=\frac{1}{2}\left(1+\frac{ab}{x^2}\right)>0$, φ is injective. $\lim_{x\to\sqrt{ab}}\varphi(x)=0,\lim_{x\to\infty}\varphi(x)=\infty$, hence φ is bijective.

B7) (3pts) Use the substitution $y=\frac{1}{2}\left(x-\frac{ab}{x}\right)$ to prove that

$$I(a,b)=\int_0^\infty\frac{1}{\sqrt{y^2+\left(\frac{a+b}{2}\right)^2}\sqrt{y^2+(\sqrt{ab})^2}}\,\mathrm{d}y.$$

Hence

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

Proof: Note that

$$I\left(\frac{a+b}{2}, \sqrt{ab}\right) = \int_0^\infty \frac{1}{\sqrt{y^2 + \left(\frac{a+b}{2}\right)^2}\sqrt{y^2+ab}}\,\mathrm{d}y = \frac{1}{2}\int_{\sqrt{ab}}^\infty \frac{\mathrm{d}x/(1+ab/x^2)}{\sqrt{y^2 + \left(\frac{a+b}{2}\right)^2}\sqrt{y^2+ab}} \tag{12}$$

$$= 2\int_{\sqrt{ab}}^\infty \frac{\mathrm{d}x}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}} = I(a,b). \tag{13}$$

B8) (2pts) Inductively define the sequence: $a_1 = a, b_1 = b$**; for** $n \geqslant 1$**, define** $a_{n+1} = \frac{a_n+b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$**. Prove that the limits** $\lim_{n\to\infty} a_n$ **and** $\lim_{n\to\infty} b_n$ **exist.** Proof: Clearly $\min\{a,b\} \leqslant a_n, b_n \leqslant \max\{a,b\}$, and $a_n \geqslant b_n$ for $n \geqslant 2$ so $a_{n+1} \leqslant a_n, b_{n+1} \geqslant b_n$. Hence $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

B9) (2pts) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$**. Denote this value by** $M(a,b)$ **and call it the arithmetic-geometric mean of** a **and** b**.** Proof: Use $2a_{n+1} = a_n + b_n$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

B10) (2pts) Prove that for any $a, b > 0$**, their arithmetic-geometric mean** $M(a,b)$ **and the elliptic integral** $I(a,b)$ **have the following formula:**

$$2M(a,b)I(a,b) = \pi.$$

Proof: Let $F(a,b) = 2M(a,b)I(a,b)$, then $F(a,b) = F\left(\frac{a+b}{2}, \sqrt{ab}\right)$ and

$$F(a,a) = 2a\int_0^\infty \frac{\mathrm{d}x}{x^2+a^2} = \pi.$$

Hence it suffices to show that F is continuous on $(0,\infty)^2$, then $F(a,b) = \lim_{n\to\infty} F(a_n, b_n) = F(M(a,b), M(a,b)) = \pi$.
Consider any (a,b), (a,b') and the deriving sequences $\{a_n\}, \{b_n\}, \{a'_n\}, \{b'_n\}$. Suppose $b < b'$, then $a_n \leqslant a'_n$ and $b_n \leqslant b'_n$. Let $d_n = a'_n - a_n$ and $\lambda_n = b'_n/b_n$, then $d_{n+1} \leqslant \max\{d_n, b_n(\lambda_n - 1)\}$ and $\lambda_{n+1} \leqslant \max\{\lambda_n, d_n/a_n + 1\}$. Hence $d_{n+1} \leqslant \max\{d_n, b_n d_{n-1}/a_{n-1}\} \leqslant \max\{d_n, d_{n-1}\}$ since $a_{n-1} \geqslant a_n \geqslant b_n$, and likewise $\lambda_{n+1} \leqslant \max\{\lambda_n, \lambda_{n-1}\}$. Therefore $M(a,b') - M(a,b) \leqslant |b - b'|$. Likewise we obtain $|M(a,b) - M(a',b')| \leqslant |a - a'| + |b - b'|$.

14.3 PSC (24pts) The Calculation of $\int_0^\infty \frac{1}{1+x^\alpha}\,\mathrm{d}x$

Assume $\alpha > 1, \beta \in (0,1), \alpha\beta = 1$.

C1) (2pts) Prove that the following improper integral converges:

$$I(\alpha) = \int_0^\infty \frac{1}{1+x^\alpha}\,\mathrm{d}x.$$

Proof:

$$I(\alpha) \leqslant 1 + \int_1^\infty \frac{1}{x^\alpha}\,\mathrm{d}x = 1 + \frac{1}{\alpha - 1}.$$

C2) (2pts) Prove that the following two improper integrals converge:

$$J_1(\beta) = \int_0^1 \frac{x^{\beta-1}}{1+x}\,\mathrm{d}x,\ J_2(\beta) = \int_0^1 \frac{x^{-\beta}}{1+x}\,\mathrm{d}x.$$

Proof:

$$J_1(\beta) \leqslant \int_0^1 x^{\beta-1}\,\mathrm{d}x = \frac{1}{\beta},\ J_2(\beta) \leqslant \int_0^1 x^{-\beta}\,\mathrm{d}x = \frac{1}{1-\beta}.$$

C3) (2pts) Prove that

$$\int_0^1 \frac{1}{1+x^\alpha}\,\mathrm{d}x = \beta J_1(\beta).$$

Proof:

$$\beta J_1(\beta) = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x^\beta = \int_0^1 \frac{1}{1+y^\alpha}\,\mathrm{d}y.$$

C4) Prove that

$$\int_1^\infty \frac{1}{1+x^\alpha}\,\mathrm{d}x = \beta J_2(\beta).$$

Proof: By substituting $x = 1/y$,

$$\beta J_2(\beta) = \beta \int_1^\infty \frac{y^{\beta-1}}{1+y}\,\mathrm{d}y = \int_1^\infty \frac{1}{1+y}\,\mathrm{d}y^\beta = \int_1^\infty \frac{1}{1+t^\alpha}\,\mathrm{d}t.$$

C5) For integers $n \geqslant 1$, define

$$h_n(x) = \sum_{k=0}^n (-1)^k x^k.$$

Prove that for any $x \in [0,1]$,

$$\left| h_n(x) - \frac{1}{1+x} \right| \leqslant x^n.$$

Proof: Note that

$$h_n(x) = \frac{1-(-x)^{n+1}}{1+x}$$

hence

$$\left| h_n(x) - \frac{1}{1+x} \right| = \frac{x^{n+1}}{1+x} \leqslant x^n.$$

C6) (2pts) Let

$$J_{1,n}(\beta) = \int_0^1 x^{\beta-1} h_n(x)\,\mathrm{d}x,\ J_{2,n}(\beta) = \int_0^1 x^{-\beta} h_n(x)\,\mathrm{d}x$$

Prove that

$$\lim_{n\to\infty} J_{1,n}(\beta) = J_1(\beta),\ \lim_{n\to\infty} J_{2,n}(\beta) = J_2(\beta).$$

Proof:

$$|J_{1,n}(\beta) - J_1(\beta)| \leqslant \int_0^1 x^{\beta-1} \left| h_n(x) - \frac{1}{1+x} \right| \mathrm{d}x \leqslant \int_0^1 x^{n+\beta-1}\,\mathrm{d}x = \frac{1}{n+\beta} \to 0.$$

The other equation can be proved in the same way.

C7) (2pts) Define the function g on $[0,\pi]$:

$$g(x)=\begin{cases}\frac{\cos(x/\alpha)-1}{\sin(x/2)}, & x\in(0,\pi];\\ 0, & x=0.\end{cases}$$

Prove that $g\in C^1([0,\pi])$.
Proof: Since $\cos(x/\alpha)-1=O(x^2), \sin(x/2)=O(x)$, $\lim_{x\to 0^+} g(x)=0=g(0)$, so g is continuous. The limit

$$\lim_{h\to 0^+}\frac{g(h)}{h}=\lim_{h\to 0^+}\frac{-h^2/2\alpha^2+o(h^2)}{h^2/2+o(h^2)}=-\frac{1}{\alpha^2}$$

exists, and

$$\lim_{x\to 0^+} g'(x)=\lim_{x\to 0^+}\frac{\sin(x/2)(-\sin(x/\alpha)/\alpha)+(1-\cos(x/\alpha))\cos(x/2)/2}{\sin^2(x/2)}=-\frac{1}{\alpha^2}.$$

Hence $g\in C^1([0,\pi])$.

C8) (3pts) For any $n\geqslant 1$, let

$$a_n=\int_0^\pi g(x)\sin\left((2n+1)\frac{x}{2}\right)\,\mathrm{d}x.$$

Prove that there is a constant C such that for any n,

$$|a_n|\leqslant\frac{C}{2n+1}.$$

Proof: Consider

$$a_n=\frac{2}{2n+1}\int_0^{(2n+1)\pi/2} g\left(\frac{2y}{2n+1}\right)\sin y\,\mathrm{d}y.$$

Note that

$$\left|\int_a^{a+2\pi} g\left(\frac{2y}{2n+1}\right)\sin y\,\mathrm{d}y\right|=\left|\int_a^{a+\pi}\left(g\left(\frac{2y}{2n+1}\right)-g\left(\frac{2(y+\pi)}{2n+1}\right)\right)\sin y\,\mathrm{d}y\right| \tag{14}$$

$$\leqslant\int_a^{a+\pi} g'(\xi)\frac{2\pi}{2n+1}\,\mathrm{d}x\leqslant M\frac{2\pi^2}{2n+1}. \tag{15}$$

Hence $a_n=O(1/n)$.

C9) (2pts) For any $n\geqslant 1$, let

$$\varphi_n(x)=\cos x+\cos 2x+\cdots+\cos nx.$$

Define the integral

$$A_n=\int_0^\pi \varphi_n(x)\cos\left(\frac{x}{\alpha}\right)\,\mathrm{d}x.$$

Prove that

$$A_n=\frac{\alpha}{2}\sin\left(\frac{\pi}{\alpha}\right)\sum_{k=1}^n(-1)^k\left(\frac{1}{1+\alpha k}+\frac{1}{1-\alpha k}\right).$$

Proof: Trivial, since

$$\int_0^\pi\cos kx\cos\left(\frac{x}{\alpha}\right)\,\mathrm{d}x=\frac{\alpha}{2}\sin\frac{\pi}{\alpha}(-1)^k\left(\frac{1}{1+\alpha k}+\frac{1}{1-\alpha k}\right).$$

C10) (3pts) Prove that

$$\varphi_n(x) = -\frac{1}{2} + \frac{1}{2}\frac{\sin((2n+1)x/2)}{\sin(x/2)}.$$

And use this to prove that

$$\lim_{n\to\infty} A_n = -\frac{\alpha}{2}\sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{2}.$$

Proof: $2\sin(x/2)\varphi_n(x) = \sum_{k=1}^n 2\sin(x/2)\cos kx = \sin((2n+1)x/2) - \sin(x/2)$. So

$$A_n = \frac{1}{2}\int_0^\pi \frac{\sin((2n+1)x/2)}{\sin(x/2)} - \cos(x/\alpha)\,\mathrm{d}x + \frac{1}{2}\int_0^\pi g(x)\sin((2n+1)x/2)\,\mathrm{d}x.$$

Clearly

$$\int_0^\pi \frac{\sin((2n+1)x/2)}{\sin(x/2)}\,\mathrm{d}x = \pi + 2\int_0^\pi \varphi_n(x)\,\mathrm{d}x = \pi$$

hence

$$\lim_{n\to\infty} A_n = \frac{\pi}{2} - \frac{\alpha}{2}\sin\left(\frac{\pi}{\alpha}\right).$$

C11) (2pts) Prove that

$$I(\alpha) = \frac{\pi}{\alpha\sin\left(\frac{\pi}{\alpha}\right)}.$$

Proof: We proved that $I(\alpha) = \beta(J_1(\beta) + J_2(\beta))$, and $J_i(\beta) = \lim_{n\to\infty} J_{i,n}(\beta)$. Note that

$$\frac{\pi}{2} - \frac{\alpha}{2}\sin\left(\frac{\pi}{\alpha}\right) = \lim_{n\to\infty} A_n = \sum_{n=1}^{\infty}\frac{\alpha}{2}\sin\left(\frac{\pi}{\alpha}\right)(-1)^k\left(\frac{1}{1+\alpha k} + \frac{1}{1-\alpha k}\right)$$

so

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n+\beta} = \frac{\pi}{\sin\pi\beta}.$$

Clearly

$$J_{1,n}(\beta) + J_{2,n}(\beta) = \sum_{k=-n-1}^{n}\frac{(-1)^k}{k+\beta} \to \frac{\pi}{\sin\pi\beta}.$$

Therefore $I(\alpha) = \frac{\pi}{\alpha\sin(\pi/\alpha)}$.

Extra Problems:

14.4 PSD Stone-Weierstrass Theorem (10pts)

D1) (2pts) Suppose $f \in C^1([0,1])$ and f is monotonically increasing, then there exists a sequence of real polynomials $\{P_n\}_{n\geqslant 1} \subset \mathbb{R}[X]$ such that

- for any n, P_n is monotonically increasing on $[0,1]$.
- P_n converges uniformly to f.
  Proof: By Stone-Weierstrass theorem, there exists a sequence of real polynomials $\{Q_n\}_{n\geqslant 1}$ such that $Q_n(x) \geqslant 0$ for $x \in [0,1]$ and Q_n converges uniformly to $g = f'$ (e.g., taking the Bernstein polynomials). Let $P_n(x) = f(0) + \int_0^x Q_n(x)\,\mathrm{d}x$, then $|P_n(x) - f(x)| \leqslant x\sup_{t\in[0,1]}|g(t) - Q_n(t)|$ so $\|f - P_n\| \leqslant \|g - Q_n\| \to 0$, and P_n is monotonically increasing.

D2) (1pts) Suppose $f \in C([0,1])$ and is monotonically increasing. For any $x \in (1,2]$, let $f(x) = f(1)$. Define the function $f_n : [0,1] \to \mathbb{R}$, such that

$$f_n(x) = n \int_x^{x+1/n} f(y) \, \mathrm{d}y.$$

Prove that sequence $\{f_n\}_{n \geqslant 1}$ satisfy:

- for any n, $f_n \in C^1([0,1])$;
- for any n, f_n is monotonically increasing on $[0,1]$.
- f_n converges uniformly to f.
  Proof: Clearly $f_n'(x) = n(f(x+1/n) - f(x))$ is continuous on $[0,1]$. For any $x > x'$, $f_n(x) - f_n(x') = n\int_0^{1/n} f(t+x) - f(t+x') \, \mathrm{d}t \geqslant 0$ so f_n is monotonically increasing. For any $x \in [0,1]$,

$$|f(x) - f_n(x)| \leqslant \int_0^1 |f(x+t/n) - f(x)| \, \mathrm{d}x \leqslant \sup_{|u-v| \leqslant 1/n} |f(u) - f(v)|$$

hence f_n converges uniformly to f.

D3) (1pts) Suppose $f \in C([0,1])$ and is monotonically increasing, then there exists a sequence of real polynomials $\{P_n\}_{n \geqslant 1} \subset \mathbb{R}[X]$ such that

- for any n, P_n is monotonically increasing on $[0,1]$.
- P_n converges uniformly to f.
  Proof: Take f_n such that $f_n \in C^1([0,1])$, f_n is monotonically increasing, and $\|f - f_n\|_\infty < 2^{-n}$. Take polynomials $P_{n,k}$ such that $P_{n,k}$ is monotonically increasing and $\|P_{n,k} - f_n\|_\infty < 2^{-k}$, then $\|f - P_{n,n}\|_\infty < 2^{1-n}$ so $P_{n,n}$ converges uniformly to f where $P_{n,n}$ are monotonically increasing polynomials.

D4) (2pts) (Walsh) Suppose $f \in C([0,1])$, and $x_1, x_2, \cdots, x_m$ are m given points on the interval $[0,1]$, then there exists a sequence of real polynomials $\{P_n\}_{n \geqslant 1} \subset \mathbb{R}[X]$ such that

- for any $n \geqslant 1$ and any $1 \leqslant i \leqslant m$, $P_n(x_i) = f(x_i)$;
- P_n converges uniformly to f.
  Proof: Let Q be the polynomial of degree $m-1$ such that $Q(x_i) = f(x_i)$. Now we find a sequence of polynomials P_n such that $P_n(x_i) = 0$ and P_n converges uniformly to $g = f - Q$. By Stone-Weierstrass theorem, we can take a sequence of polynomials Q_n that converges uniformly to g on $[0,1]$. Consider

$$R_n(x) = Q_n(x) - \sum_{i=1}^m Q_n(x_i) L_i(x), \text{ where } L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

then $R_n(x_i) = 0$ for any $1 \leqslant i \leqslant m$. Let $M = \sup\{|L_i(x)| : x \in [0,1], 1 \leqslant i \leqslant m\}$, then for any $x \in [0,1]$, $|R_n(x) - g(x)| \leqslant |Q_n(x) - g(x)| + M \sum_{i=1}^m |Q_n(x_i)| \leqslant (1 + Mm)\|Q_n - g\|_\infty$ since $g(x_i) = 0$. Therefore R_n converges uniformly to g and R_n has roots $x_1, \cdots, x_m$, and we obtain a sequence $P_n = R_n + Q$ that converges uniformly to f and $P_n(x_i) = f(x_i)$.

D5) (1pts) Suppose $I=[a,b]\subset(0,1)$, the polynomial $p(x)=2x(1-x)$, let $Q_n=p\circ p\circ\cdots\circ p$, prove that $\{Q_n\}$ converges uniformly to the constant function $1/2$ on I. Proof: Note that $|p(x)-1/2|=2|x-1/2|^2\leqslant\lambda|x-1/2|$ where $\lambda=2\max\{a,1-b\}<1$. Hence $\|Q_n-1/2\|_\infty\leqslant\lambda^n\to 0$ so Q_n converges uniformly to $1/2$ on I.

D6) (1pts) Suppose $I=[a,b]\subset(0,1)$, for any $k\in\mathbb{Z}$, prove that there exists a sequence of polynomials $\{P_n\}_{n\geqslant 1}\subset\mathbb{Z}[X]$, such that $\{P_n\}_{n\geqslant 1}$ converges uniformly to the constant function 2^k on I. Proof: The case $k\geqslant 0$ is trivial since $2^k\in\mathbb{Z}[X]$. If $k=-m\leqslant 0$, let $P_n=Q_n^m$, then $|P_n(x)-2^{-m}|=|Q_n-1/2|\cdot|Q_n^{m-1}+\cdots+2^{-(m-1)}|$. For n large enough, $|Q_n|<1$, so $|P_n(x)-2^{-m}|\leqslant m|Q_n-2^{-1}|$ therefore P_n converges uniformly to 2^k.

D7) (2pts) (Chudnovsky) Suppose $f\in C(I)$, where $I=[a,b]\subset(0,1)$. Prove that there exists a sequence of polynomials $\{P_n\}_{n\geqslant 1}\subset\mathbb{Z}[X]$, such that $\{P_n\}_{n\geqslant 1}$ converges uniformly to f on I. [[!!]]Proof: By D6) and binary representation we know the case $f=C\cdot\chi_{[a,b]}$ holds. Also if $P_n\to f$ uniformly then $xP_n\to xf$ uniformly. Hence all polynomials with real coefficients can be approximated. By Stone-Weierstrass theorem, all continuous functions can be uniformly approximated.
Note
For further results on the approximation of continuous functions by polynomials with integer coefficients, see the article by Hervé Pépin and Nicolas Tosel: *Approximation par des polynômes à coefficients dans* $\mathbb{Z}$, RMS,114ème année, 2003-2004.

14.5 PSE

Find all functions $f\in C(\mathbb{R})$ such that for any $x,y\in\mathbb{R}$,

$$f(x)f(y)=\int_{x-y}^{x+y}f(t)\,\mathrm{d}t.$$

Solution: $f\equiv 0$ is a trivial solution, now suppose otherwise. Let $F(x)=\int_0^x f(t)\,\mathrm{d}t$, then $f(x)f(y)=F(x+y)-F(x-y)$ where F is differentiable, so f is also differentiable (take y such that $f(y)\neq 0$), and $f'(x)f(y)=f(x+y)-f(x-y)$.
Take y such that $f(y)\neq 0$, then $f'(x)=\frac{f(x+y)-f(x-y)}{f(y)}$ is differentiable on $\mathbb{R}$. Hence $f''(x)f(y)=f'(x+y)-f'(x-y)$, and $f(x)f'(y)=f(x+y)+f(x-y)$ so $f(x)f''(y)=f'(x+y)-f'(x-y)$. Therefore $f''(x)f(y)=f''(y)f(x)$ for all $x,y\in\mathbb{R}$. Since $f(y)\neq 0$ for some y, there exists c such that $f''(x)=cf(x)$.
If $c=0$ then $f(x)=kx$ so $f(x)=2x$.
If $c\neq 0$ then $f(x)=Ae^{\sqrt{c}x}+Be^{-\sqrt{c}x}$ (let $\sqrt{-t}=i\sqrt{t}$), hence $f(x)=a\sin bx$ or $f(x)=a\sinh bx$ where $ab=2$.

14.6 PSF

Given an arbitrary set of distinct real numbers $\{\alpha_1,\alpha_2,\cdots,\alpha_{2020}\}$, and non-zero real numbers $a_1,a_2,\cdots,a_{2020}$, consider the function defined on $(0,\infty)$:

$$f(x)=a_1x^{\alpha_1}+a_2x^{\alpha_2}+\cdots+a_{2020}x^{\alpha_{2020}}.$$

Prove that $f(x)$ has at most 2019 roots on $(0,\infty)$.
Proof: We prove by induction that for $n\geqslant 1$ and distinct non-zero real numbers $a_1,\cdots,a_n$, the function $f(x)=\sum_{k=1}^n a_kx^{\alpha_k}$ has at most $n-1$ roots on $(0,\infty)$. The base $n=1$ is trivial.
Suppose $\alpha_1<\cdots<\alpha_n$, and f has n roots $r_1<\cdots<r_n$. Then they are also roots of $\tilde{f}(x)=a_1+\sum_{k=2}^n a_kx^{\alpha_k-\alpha_1}$. By Rolle's theorem, the function $g=\tilde{f}'=\sum_{k=2}^n a_k(\alpha_k-\alpha_1)x^{\alpha_k-\alpha_1-1}$ has $n-1$ distinct roots, leading to contradiction.