周民强《实变函数论（第三版）》P218 思考题:9; P231-232 思考题: 3, 4; P241-244 习题5: 7, 8, 9, 10, 11

218-9

Suppose $f \in \mathrm{BV}([a,b])$. If f has a primitive on $[a,b]$, is f continuous on $[a,b]$?
Proof: Suppose $g:[a,b] \to \mathbb{R}$ satisfy $g'(x)=f(x)$. If f is not continuous at x_0, take $y_n \to x_0$ such that $|f(x_0)-f(y_n)|>\varepsilon_0$, and suppose $y_n<x_0$. By Darboux's theorem, for any n there is a $z_n \in (y_n, x_0)$ such that $|f(x_0)-f(z_n)|<\frac{\varepsilon_0}{2}$. We can suppose $y_n<z_n<y_{n+1}$, by defining it inductively. Then we obtain disjoint intervals $(y_n, z_n) \subset [a,b]$, such that $|f(y_n)-f(z_n)|>\frac{\varepsilon_0}{2}$, contradicting with $f \in \mathrm{BV}([a,b])$.

232-3

Suppose f_n is a sequence of increasing & absolutely continuous functions on $[a,b]$. If $\sum_{n=1}^{\infty} f_n(x)$ converges on $[a,b]$, then the sum is absolutely continuous on $[a,b]$.
Proof: Note that f_n' exists a.e. and $f_n' \geqslant 0, f_n' \in \mathcal{L}([a,b])$. Let $F(x)=\sum_{n=1}^{\infty} f_n(x)$, then by Monotone Convergence Theorem,

$$F(x)=\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty} f_n(a)+\int_a^x f_n' \,\mathrm{d}m=F(a)+\sum_{n=1}^{\infty}\int_a^x f_n' \,\mathrm{d}m=F(a)+\int_a^x \sum_{n=1}^{\infty} f_n' \,\mathrm{d}m.$$

Since $F(b), F(a)$ are finite and $f_n' \geqslant 0$, $G=\sum_{n=1}^{\infty} f_n' \in \mathcal{L}([a,b])$ and $F(x)=F(a)+\int_a^x G \,\mathrm{d}m$. Therefore $F \in \mathrm{AC}([a,b])$.

232-4

Suppose $f \in \mathrm{BV}([0,1])$. If $f \in \mathrm{AC}([\varepsilon,1])$ for any $\varepsilon>0$, and f is continuous at $x=0$, then $f \in \mathrm{AC}([0,1])$.
Proof: $f \in \mathrm{AC}([\varepsilon,1])$ implies f' exists a.e. on $[\varepsilon,1]$ and $f' \in \mathcal{L}([\varepsilon,1])$ and $f(x)=f(b)-\int_x^b f' \,\mathrm{d}m \forall x \in [\varepsilon,1]$. Hence f' exists a.e. on $[0,1]=\bigcup_{n\geqslant 1}\left[\frac{1}{n},1\right]$, so by $f \in \mathrm{BV}([0,1])$ we know that $f' \in \mathcal{L}([0,1])$, using $\int_0^1 |f'| \,\mathrm{d}m \leqslant \bigvee_0^1(f)<\infty$. Also $f(x)=f(b)-\int_x^b f' \,\mathrm{d}m \forall x \in (0,1]$. Since f is continuous at $x=0$, $f(0)=\lim_{x\to 0} f(x)=f(b)-\int_0^b f' \,\mathrm{d}m$, so $f(x)=f(0)+\int_0^x f' \,\mathrm{d}m$ and $f \in \mathrm{AC}([0,1])$.

242-7

Suppose $f:[a,b] \to [c,d]$ is continuous, and for any $y \in [c,d]$, $f^{-1}(\{y\})$ has at most 10 points. Prove that

$$\bigvee_a^b(f) \leqslant 10(d-c).$$

Proof: For any $\Delta=\{a=x_0<x_1<\cdots<x_n=b\}$, we show that $\sum_{k=1}^n |f(x_k)-f(x_{k-1})| \leqslant 10(d-c)$: Let $I_k=[f(x_k), f(x_{k-1})]$ (or $[f(x_k), f(x_{k-1})]$), then

$$\sum_{k=1}^n |f(x_k)-f(x_{k-1})|=\sum_{k=1}^n \int_c^d \chi_{I_k} \,\mathrm{d}m=\int_c^d \sum_{k=1}^n \chi_{I_k} \,\mathrm{d}m.$$

By intermediate value property, every $y \in [c,d]$ falls in at most 10 intervals I_k, so $\sum_{k=1}^n \chi_{I_k} \leqslant 10$, and $\sum_{k=1}^n |f(x_k)-f(x_{k-1})| \leqslant 10(d-c)$.

242-8

Suppose $f \in \mathcal{L}([0,1])$, $g:[0,1] \to \mathbb{R}$ is increasing. If for any $[a,b] \subset [0,1]$,

$$\left|\int_a^b f \,\mathrm{d}m\right|^2 \leqslant (g(b)-g(a))(b-a),$$

prove that $f \in \mathcal{L}^2([0,1])$.
Proof: $\left|\frac{1}{b-a}\int_a^b f \,\mathrm{d}m\right|^2 \leqslant \frac{g(b)-g(a)}{b-a}$ for any $[a,b] \subset [0,1]$. By Lebesgue Differentiation Theorem, let $b = a+h \to a$, then $|f|^2 \leqslant g', a.e.$ so $f \in \mathcal{L}^2([0,1])$.

242-9

Suppose $f \in \mathrm{AC}([a,b])$ is non-negative, prove that $f^p \in \mathrm{AC}([a,b])$ for $p \geqslant 1$.
Proof: $f \in \mathrm{AC}([a,b])$ implies $f \in C([a,b]) \cap \mathrm{BV}([a,b])$ & $m(f(Z)) = 0 \forall m(Z) = 0$. Clearly $f^p \in C([a,b])$.
$f \in C([a,b])$ implies $|f| \leqslant M$ on $[a,b]$ for some $M > 0$. Then x^p is Lipschitz (hence AC) on $[-M,M]$, so $f^p \in \mathrm{BV}([a,b])$ and $\forall m(Z) = 0, m(f(Z)) = 0$ so $m(f^p(Z)) = 0$. Therefore by Banach-Zaretsky Theorem, $f^p \in \mathrm{AC}([a,b])$.

242-10

Suppose f is increasing on $[a,b]$, and $\int_a^b f' \,\mathrm{d}m = f(b) - f(a)$. Prove that $f \in \mathrm{AC}([a,b])$.
Proof: By Lebesgue Differentiation Theorem, $\forall x \in [a,b]$, $f(x) - f(a) \geqslant \int_a^x f' \,\mathrm{d}m$ and $f(b) - f(x) \geqslant \int_x^b f' \,\mathrm{d}m$, so $f(x) = f(a) + \int_a^x f' \,\mathrm{d}m \forall x \in [a,b]$ and hence $f \in \mathrm{AC}([a,b])$.

242-11

Suppose $f \in \mathrm{BV}([a,b])$. If $\int_a^b |f'| \,\mathrm{d}m = \bigvee_a^b(f)$, prove that $f \in \mathrm{AC}([a,b])$.
Proof: First we show that $\int_a^b |f'| \,\mathrm{d}m \leqslant \bigvee_a^b(f)$ for $f \in \mathrm{BV}([a,b])$. By Jordan decomposition theorem, take $g(x) = \bigvee_a^{x+}(f) + f(a)$ and $h(x) = g(x) - f(x)$, then g, h are increasing and $f = g - h$. By Lebesgue Differentiation Theorem,

$$\int_a^b |f'| \,\mathrm{d}m \leqslant \int_a^b |g'| + |h'| \,\mathrm{d}m \leqslant g(b) + h(b) - h(a) - g(a) = \bigvee_a^b(f).$$

Now if $\int_a^b |f'| \,\mathrm{d}m = \bigvee_a^b(f)$, we obtain $\forall [c,d] \subset [a,b]$, $\int_c^d |f'| \,\mathrm{d}m = \bigvee_c^d(f)$.
If $f \notin \mathrm{AC}([a,b])$ then there exists $\varepsilon_0 > 0$ such that $\forall \delta > 0$ there exists disjoint (x_i, y_i) such that $\sum |x_i - y_i| < \delta$ but $\sum |f(x_i) - f(y_i)| > \varepsilon_0$.
Then

$$\mu_{|f'|}\left(\bigcup (x_i, y_i)\right) = \sum_{k=1}^N \int_{x_i}^{y_i} |f'| \,\mathrm{d}m = \sum_{k=1}^N \bigvee_{x_i}^{y_i}(f) \geqslant \sum_{k=1}^N |f(y_i) - f(x_i)| > \varepsilon_0,$$

while $m\left(\bigcup (x_i, y_i)\right) < \delta$. This contradicts the absolute continuity of integrals.