111页2; 115-116页1,3,4,5,7,8.

111-2

Use Theorem 20 to prove the following. If W is a subspace of a finite-dimensional vector space V and if $\{g_1,\cdots,g_r\}$ is any basis for W^0, then

$$W=\bigcap_{i=1}^{r}\mathrm{Ker}g_i.$$

Proof: Since $h\in\mathrm{Span}\{g_1,\cdots,g_n\}\iff\bigcap_{i=1}^{r}\mathrm{Ker}g_i\subset\mathrm{Ker}h$, so $W^0=\left\{h:f\left(\bigcap_{i=1}^{r}\mathrm{Ker}g_i\right)=\{0\}\right\}=\left(\bigcap_{i=1}^{r}\mathrm{Ker}g_i\right)^0$, hence $W=\bigcap_{i=1}^{r}\mathrm{Ker}g_i$.

115-1

Let F be a field and let f be the linear functional on F^2 defined by $f(x_1,x_2)=ax_1+bx_2$. For each of the following linear operators T, let $g=T^tf$ and find $g(x_1,x_2)$.
(a) $T(x_1,x_2)=(x_1,0)$; (b) $T(x_1,x_2)=(-x_2,x_1)$; (c) $T(x_1,x_2)=(x_1-x_2,x_1+x_2)$.
Proof: (a) $T^tf(x_1,x_2)=f\circ T(x_1,x_2)=f(x_1,0)=ax_1$.
(b) $T^tf(x_1,x_2)=f\circ T(x_1,x_2)=f(-x_2,x_1)=-ax_2+bx_1$.
(c) $T^tf(x_1,x_2)=f\circ T(x_1,x_2)=f(x_1-x_2,x_1+x_2)=(a+b)x_1+(b-a)x_2$.

115-3

Let V be the space of all $n\times n$ matrices over a field F and let B be a fixed $n\times n$ matrix. If T is the linear operator on V defined by $T(A)=AB-BA$, and if f is the trace function, what is T^tf?
Solution: $T^tf=f\circ T$, and $T^tf(A)=\mathrm{tr}(AB-BA)=0$, hence $T^tf=0$.

115-4

& V \ {\mathrm{Im}T} && {F } \ & {V/\mathrm{Ker}T} \arrow["T"', from=1-2, to=2-1] \arrow["f", from=1-2, to=2-3] \arrow["\pi", two heads, from=1-2, to=3-2] \arrow["{g_0=f^{\prime}∘ (T^{\prime})^{-1}}"', from=2-1, to=2-3] \arrow["{T^{\prime}}", hook', two heads, from=3-2, to=2-1] \arrow["{f^{\prime}}"', from=3-2, to=2-3] tikz

116-5

Let A be an $m \times n$ matrix with real entries. Prove that $A = 0$ iff $\mathrm{tr}(A^t A) = 0$.
Proof: Note that $(A^t A)_{i,i} = \sum_{j=1}^m A^t_{i,j} A_{j,i} = \sum_{j=1}^n A^2_{j,i} \geqslant 0$, hence $\mathrm{tr}(A^t A) = 0$ implies $(A^t A)_{i,i} = 0 \forall 1 \leqslant i \leqslant n$, hence $A_{j,i} = 0 \forall i, j$ so $A = 0$.
If $A = 0$ clearly $\mathrm{tr}(A^t A) = \mathrm{tr}(0) = 0$.

116-7

Let V be a finite-dimensional vector space over the field F. Show that $\varphi : \mathcal{L}(V, V) \to \mathcal{L}(V^*, V^*),\ T \to T^t$ is an isomorphism.
Proof: Clearly φ is linear, $\mathrm{Ker}\varphi = \{0\}$, and $\mathrm{dim}\mathcal{L}(V, V) = (\mathrm{dim}V)^2 = (\mathrm{dim}V^*)^2 = \mathrm{dim}\mathcal{L}(V^*, V^*)$, so φ is an isomorphism.

116-8

Let V be the vector space of $n \times n$ matrices over the field F.
(a) If B is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(A) = \mathrm{tr}(B^t A)$. Show that f_B is a linear functional on V.
(b) Show that every linear functional on V is of the above form, i.e., is f_B for some B.
(c) Show that $B \to f_B$ is an isomorphism of $V \to V^*$.
Proof: (a) $\varphi : A \mapsto B^t A \in \mathcal{L}(V, V)$, and $\mathrm{tr} \in V^*$, so $f_B = \mathrm{tr} \circ \varphi \in V^*$.
(b) (We can use Riesz representation theorem on the Euclidean inner product.)
Suppose $f(E_{i,j}) = c_{i,j}$ where $E_{i,j}$ is the matrix where only (i, j) is 1 and the other entries are 0. Then for $B = (c_{i,j})_{i,j \leqslant n}$, $f = f_B$.
(c) $\Phi : V \to V^*, B \mapsto f(B)$ is an epimorphism, and $\mathrm{dim}V = \mathrm{dim}V^*$, so Φ is an isomorphism.