PSA: Calculating Derivatives

A1) Consider the function

$$f:\mathbb{R}\to\mathbb{R}^n,\ x\mapsto f(x)=(f_1(x),\cdots,f_n(x)).$$

Prove that f is differentiable at x_0 iff every f_k is differentiable at x_0 and

$$f'(x)=(f_1'(x),\cdots,f_n'(x)).$$

Proof: For any $h\in\mathbb{R}$,

$$\|\frac{f(x+h)-f(x)}{h}-(f_1'(x),\cdots,f_n'(x))\|_2\leqslant n\max_{1\leqslant k\leqslant n}\left\{\left|\frac{f_k(x+h)-f_k(x)}{h}-f_k'(x)\right|\right\}\to 0.$$

Therefore $f'(x)=(f_1'(x),\cdots,f_n'(x))$.

A2) Consider the function

$$f:\mathbb{R}\to\mathbb{C},\ x\mapsto e^{ix}.$$

Prove by definition, $f'(0)=i$ and $(e^{ix})'=ie^{ix}$.
Proof: For any $h\in\mathbb{R}$,

$$\left|\frac{f(h)-f(0)}{h}-i\right|=\left|\frac{e^{ih}-ih-1}{h}\right|\leqslant\sum_{n=2}^{\infty}\left|\frac{1}{h}\frac{(ih)^n}{n!}\right|\leqslant|h|\to 0.$$

Therefore $f'(0)=i$. For any $x\in\mathbb{R}$,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}e^{ix}\frac{f(h)-f(0)}{h}=ie^{ix}.$$

Hence $(e^{ix})'=ie^{ix}$.

A3) Calculate the derivatives of $\sin x$ and $\cos x$.

Solution: $\sin x=(e^{ix}-e^{-ix})/2i$, so $(\sin x)'=(e^{ix}+e^{-ix})/2=\cos x$. Likewise $(\cos x)'=-\sin x$.

A4) Prove Faà di Bruno's formula for $n=3$.

Proof:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}(f\circ g)=f'(g)\cdot g'.\\&\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f\circ g)=f'(g)\cdot g''+f''(g)\cdot(g')^2.\\&\frac{\mathrm{d}^3}{\mathrm{d}x^3}(f\circ g)=f'(g)\cdot g'''+f''(g)\cdot g''\cdot g'+f'''(g)\cdot(g')^3+f''(g)\cdot 2g'g''.\end{aligned}$$

A5) Define the map

$$E:\mathbb{R}\to\mathbb{C}=\mathbb{R}^2,\ \theta\mapsto(\cos\theta,\sin\theta).$$

Prove that the points in $\mathbf{S}^1=\{(x,y)\in\mathbb{R}^2:x^2+y^2=1\}$ can be written in the form $(\sin\theta,\cos\theta)$, i.e. $E(\mathbb{R})=\mathbf{S}^1$. Calculate $E'(\theta)$ and show that Rolle's mean-value theorem is invalid for E.
Proof: Obviously $E(\mathbb{R})\subset\mathbf{S}^1$. Consider any $(x,y)\in\mathbf{S}^1$, then $x\in[-1,1]$. Note that $\cos 0=1,\cos\pi=-1$, hence there exists $\theta\in[0,\pi]$ such that $\cos\theta=x$, and $|\sin\theta|=|y|$. If $\sin\theta=y$ then $(x,y)=(\cos\theta,\sin\theta)\in E(\mathbb{R})$. Otherwise $(x,y)=(\cos(-\theta),\sin(-\theta))\in E(\mathbb{R})$, therefore $E(\mathbb{R})=\mathbf{S}^1$.
By A1) and A3), $E'(\theta)=(-\sin\theta,\cos\theta)$. Since $E'(\theta)\neq 0$ for all $\theta\in\mathbb{R}$ and $E'(\theta)=E'(\theta+2\pi)$, Rolle's mean-value theorem is invalid.

A6) Calculate the derivatives of the following functions:

(1) $f(x) = a^x$, $a > 0$.

$$f'(x) = (e^{x \log a})' = a^x \log a.$$

(2) $f(x) = \arcsin x$.
Let $y = \arcsin x$, then $x = \sin y$, so $1 = \cos y \cdot y'$, hence

$$f'(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

(3) $f(x) = \arctan x$.
Let $y = \arctan x$, then $x = \tan y$, so $1 = \sec^2 y \cdot y'$, hence

$$f'(x) = \cos^2 y = \frac{1}{1+x^2}.$$

(4) $f(x) = x^{x^x}$, $x > 0$.
Let $y = x^x$, $z = x^y$, then $\log y = x \log x$, so $y'/y = \log x + 1$, $y' = x^x(1+\log x)$. $\log z = y \log x$, so $z'/z = y' \log x + y/x = x^x \log x(1+\log x) + x^{x-1}$. Therefore

$$f'(x) = x^{x^x} \cdot x^x \cdot (\log x + \log^2 x + x^{-1}).$$

(5) $f(x) = \log(\log(\log x))$.

$$f'(x) = \frac{(\log\log x)'}{\log\log x} = \frac{1}{x \log x \log\log x}.$$

(6) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$.

$$\begin{aligned} f'(x) &= \frac{(x+\sqrt{x+\sqrt{x}})'}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \left(1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)/2\sqrt{x+\sqrt{x+\sqrt{x}}} \\ &= \frac{2\sqrt{x+\sqrt{x}}+1+1/2\sqrt{x}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}. \end{aligned}$$

(7) $f(x) = |x|$.
If $x > 0$, $f'(x) = (x)' = 1$. If $x < 0$, $f'(x) = (-x)' = -1$. If $x = 0$, f is not differentiable at x.
(8) $f(x) = \log|x|$.
If $x > 0$, $f'(x) = \frac{1}{x}$. If $x < 0$, $f'(x) = -\frac{1}{x}$. If $x = 0$, f is not differentiable at x.
(9)

$$f(x) = \begin{cases} x^n \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \quad n = 1, 2, \cdots$$

For $x \neq 0$, $f'(x) = nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{x}$. When $x = 0$,

$$f'(0) = \lim_{h\to 0} h^{n-1}\sin\frac{1}{h} = \begin{cases} 0, & n \geqslant 2; \\ \text{diverges}, & n = 1. \end{cases}$$

A7) Calculate $f^{(3)}(x)$:

(1) $f(x) = \log(x+1)$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\log(x+1) = \frac{2}{(x+1)^3}.$$

(2) $f(x) = x^{-1}\log x$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{\log x}{x} = \frac{11-6\log x}{x^4}.$$

(3) $f(x) = \frac{x^m}{1-x}$, $m \in \mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{x^m}{1-x}=\frac{(m-2)(m-1)mx^{m-3}}{1-x}+\frac{3(m-1)mx^{m-2}}{(1-x)^2}+\frac{6mx^{m-1}}{(1-x)^3}+\frac{6x^m}{(1-x)^4}.$$

(4) $f(x)=x^m e^x$, $m\in\mathbb{Z}_{\geqslant 0}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^m e^x)=e^x x^{m-3}(m^3+3m^2(x-1)+m(3x^2-3x+2)+x^3).$$

(5) $f(x)=e^{ax}\sin{(bx)}$, $a,b\in\mathbb{R}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{ax}\sin{(bx)})=e^{ax}((3a^2b-b^3)\cos{(bx)}+a(a^2-3b^2)\sin{(bx)}).$$

(6) $f(x)=e^{-x^2}$.

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-x^2}=-4e^{-x^2}x(2x^2-3).$$

A8) $f'(x_0)>0$ does not imply f is increasing in a neighborhood of x_0: consider

$$f(x)=\begin{cases}x+2x^2\sin\frac{1}{x}, & x\neq 0;\\ 0, & x=0.\end{cases}$$

Prove that $f'(0)>0$ but for any $\varepsilon>0$, f is not monotonic on $(-\varepsilon,\varepsilon)$.
Proof:

$$f'(0)=\lim_{h\to 0}1+2h\sin\frac{1}{h}=1>0.$$

However, for any $n\in\mathbb{N}$, let $x_n=\frac{1}{(2n+1/2)\pi}$, $y_n=\frac{1}{(2n-1/2)\pi}$, then

$$f(x_n)=x_n+2x_n^2,\ f(y_n)=y_n-2y_n^2.$$

Note that $0<x_n<y_n$, but

$$f(x_n)-f(y_n)=2x_n^2+2y_n^2-\pi x_n y_n>0,$$

i.e. $f(x_n)>f(0),f(y_n)$, therefore f is not monotonic on any $(-\varepsilon,\varepsilon)$.

A9) $A\in\mathbf{M}_n(\mathbb{R})$, calculate

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\det{(\mathbf{I}_n+xA)}.$$

Solution: Let $\Phi(x)=I_n+xA$, then $\Phi(0)=I_n$. Denote $\Phi(t)=(\varphi_1(t),\cdots,\varphi_n(t))$. Note that det is a multi-linear function for n rows, hence by Euler's formula:

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi(t)=\det\left(\varphi_1'(t),\varphi_2(t),\cdots,\varphi_n(t)\right)+\cdots+\det\left(\varphi_1(t),\varphi_2(t),\cdots,\varphi_n'(t)\right).$$

When $t=0$, the formula becomes

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\det\Phi(t)=\varphi_{1,1}'+\cdots+\varphi_{n,n}'=\mathrm{tr}\,\Phi'(0)=\mathrm{tr}A.$$

A10) Prove that the derivation of odd functions are even, and that of even functions are odd.

Proof: If f is an odd function then

$$f'(-x)=\lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}=\lim_{h\to 0}\frac{f(x)-f(x-h)}{h}=f'(x),$$

so f' is even. If f is an even function then

$$f'(-x)=\lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}=-\lim_{h\to 0}\frac{f(x)-f(x-h)}{h}=-f'(x),$$

so f' is odd.

A11) Prove that

$$f(x)=\begin{cases}1/q, & x=\frac{p}{q}\in\mathbb{Q}, q\geqslant 1, \gcd(p,q)=1;\\ 0, & x\in\mathbb{Q}^C.\end{cases}$$

is nowhere differentiable on $\mathbb{R}$.
Proof: For any $x\in\mathbb{Q}$, $f(x)\neq 0$, but for any $\varepsilon>0$, there exists $y\in(x-\varepsilon,x+\varepsilon)\cap\mathbb{Q}^C$, such that $f(y)=0$. Therefore f is not continuous at x, and clearly not differentiable.
Consider any $x\in\mathbb{Q}^C$, there is a sequence of irrational numbers $\{y_n\}_{n\geqslant 1}$ that converges to x, then

$$\lim_{n\to\infty}\frac{f(x)-f(y_n)}{x-y_n}=0.$$

Choose any sequence of rational numbers $\{r_n=p_n/q_n\}_{n\geqslant 1}$ that converges to x, then

$$\lim_{n\to\infty}\frac{f(x)-f(r_n)}{x-r_n}=\lim_{n\to\infty}\frac{1}{xq_n-p_n}=\infty.$$

Therefore f is nowhere differentiable on $\mathbb{R}$.

PSB

B1) Define the hyperbolic functions:

$$\sinh x=\frac{e^x-e^{-x}}{2},\ \cosh x=\frac{e^x+e^{-x}}{2},\ \tanh x=\frac{\sinh x}{\cosh x}.$$

1. Prove that
   (1) $\cosh^2 x-\sinh^2 x=1$
   Proof:$\cosh^2 x-\sinh^2 x=\frac{e^{2x}+e^{-2x}+2}{4}-\frac{e^{2x}+e^{-2x}-2}{4}=1.$
   (2) $\sinh{(x+y)}=\sinh x\cosh y+\cosh x\sinh y$.
   Proof:
   $$\sinh x\cosh y+\cosh x\sinh y=\frac{e^{x+y}-e^{y-x}+e^{x-y}-e^{-x-y}}{4}+\frac{e^{x+y}-e^{x-y}+e^{y-x}-e^{-x-y}}{4}=\sinh{(x+y)}$$
   (3) $\cosh{(x+y)}=\cosh x\cosh y+\sinh x\sinh y$.
   Proof: Same as (2).
   (4) $\tanh{(x+y)}=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}$.
   Proof: $\tanh{(x+y)}=\frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}$.
2. Calculate $\sinh'(x)$, $\cosh'(x)$ and $\tanh'(x)$.
   Solution: $\sinh'(x)=\cosh x, \cosh'(x)=\sinh x, \tanh'(x)=\frac{1}{\cosh^2 x}$.
3. Prove that $\sinh:\mathbb{R}\to\mathbb{R}$ has an inverse $\operatorname{arcsinh}:\mathbb{R}\to\mathbb{R}$ and calculate $\operatorname{arcsinh}'(x)$.
   Proof: $\sinh'(x)=\cosh x>0$, so $\sinh$ is monotonically increasing over $\mathbb{R}$. Also $\lim_{x\to\infty}\sinh x=\infty$, $\lim_{x\to-\infty}\sinh x=-\infty$, therefore $\sinh:\mathbb{R}\to\mathbb{R}$ is a bijection and hence has an inverse.
   $\operatorname{arcsinh}'(x)=\frac{1}{\sqrt{1+x^2}}$.

B2) $a,b\in\mathbb{R}$, $a>0$. Consider $f:[-1,1]\to\mathbb{R}$, where

$$f(x)=\begin{cases}x^a\sin{(x^{-b})}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Prove that

1. $f\in C([-1,1])$ iff $a>0$;
   Proof: $f\in C([-1,1])$ iff $\lim_{x\to 0}x^a\sin{(x^{-b})}=0$. If $a>0$ then $|x^a\sin{(x^{-b})}|\leqslant|x|^a\to 0$. If $a<0$ then let $x=((2n+1/2)\pi)^{-1/b}$, when $n\to\infty$, $x\to 0$ but $|x^a\sin{(x^{-b})}|\to\infty$. If $a=0$, then let $x=((2n+1/2)\pi)^{-1/b}$, $|x^a\sin{(x^{-b})}|=1$. Therefore $f\in C([-1,1])$ iff $a>0$.
2. f is differentiable at 0 iff $a>1$;
   Proof: f is differentiable at 0 iff $\lim_{x\to 0}x^{-a}\sin{(x^{-b})}$ exists. By 1 we know that $a>1$. ($a=1$ is invalid since $x=(2n\pi)^{-1/b}$ and $x=((2n+1/2)\pi)^{-1/b}$ converge to different values.)

3. f' is bounded on $[-1,1]$ iff $a \geqslant 1+b$;
   Proof: $f'(x) = ax^{a-1}\sin\left(x^{-b}\right) + x^a\cos\left(x^{-b}\right)(-b)x^{-b-1}$ is bounded iff x^{a-1} and x^{a-b-1} are bounded, i.e. $a \geqslant 1+b$.
4. $f \in C^1([-1,1])$ iff $a > 1+b$;
   Proof: $f \in C^1([-1,1])$ iff $f'(0) = 0 = \lim_{x\to 0} f'(x)$. By 1 we know it is equivalent to $a > 1+b$.
5. f' is differentiable at 0 iff $a > 2+b$;
6. f'' is bounded on $[-1,1]$ iff $a \geqslant 2+2b$;
7. $f \in C^2([-1,1])$ iff $a > 2+2b$.
   Proof: 5,6,7 are exactly the same as 2,3,4.

PSC

If f satisfy $\lim_{x\to x_0} f(x) = 0$ near x_0, we call f an infinitesimal when $x \to x_0$. Likewise when $\lim_{x\to x_0} f(x) = +\infty$ or $\lim_{x\to x_0} f(x) = -\infty$, we call f an infinite quantity when $x \to x_0$.
Suppose f, g are both infinitesimal when $x \to x_0$, and $g(x)$ does not vanish near x_0. We introduce the notations

- if $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$, we say f is an infinitesimal of higher order than g, and denote $f(x) = o(g(x))$, $x \to x_0$;
- If $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \ell \neq 0$, we say f and g are of the same order;
- If $\ell = 1$, denote $f \sim g$, $x \to x_0$;
- If $\limsup_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| < +\infty$, denote $f(x) = O(g(x))$, $x \to x_0$.

C1) Suppose $a(x) = o(1)$ when $x \to x_0$, prove that:

(1) $o(a) + o(a) = o(a)$
Proof: If $f, g = o(a)$, then

$$\lim_{x\to x_0} \frac{f(x)+g(x)}{a(x)} = \lim_{x\to x_0} \frac{f(x)}{a(x)} + \lim_{x\to x_0} \frac{g(x)}{a(x)} = 0,$$

hence $f + g = o(a)$.
(2) $co(a) = o(ca)$, $c \in \mathbb{R}$
Proof: If $f = o(a)$, then

$$\lim_{x\to x_0} \frac{cf(x)}{a(x)} = c\lim_{x\to x_0} \frac{f(x)}{a(x)} = 0,$$

hence $cf = o(a) = o(ca)$.
(3) $o(a)^k = o(a^k)$
Proof: If $f = o(a)$ then

$$\lim_{x\to x_0} \frac{f(x)^k}{a(x)^k} = \left(\lim_{x\to x_0} \frac{f(x)}{a(x)}\right)^k = 0,$$

hence $f^k = o(a^k)$.
(4) $1/(1+a) = 1 - a + o(a)$
Proof:

$$\lim_{x\to x_0} \frac{1/(1+a) - 1 + a}{a(x)} = \lim_{x\to x_0} \frac{a(x)}{1+a(x)} = 0,$$

hence $1/(1+a) = 1 - a + o(a)$.

C2) Suppose f, g are infinitesimals when $x \to x_0$, then

1. Prove that $f \sim g \iff f(x) - g(x) = o(g(x))$, $x \to x_0$.
   Proof: $f \sim g \iff \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1 \iff \lim\limits_{x\to x_0} \dfrac{f(x)-g(x)}{g(x)} = 0$, i.e. $f(x) - g(x) = o(g(x))$.
2. If $f \sim cg^k$, we call cg^k the leading term of f. Find the leading terms of the following functions, compared to $x - x_0$ or x:

(1) $1/\sin\pi x$, $x\to 1$.
$\frac{1}{\sin\pi x} = -\frac{1}{\pi(x-1)} + o(1)$.
(2) $\sqrt{1+x}-\sqrt{1-x}$, $x\to 0$.
$\sqrt{1+x}-\sqrt{1-x} = x+o(x)$.
(3) $\sin\left(\sqrt{1+\sqrt{1+\sqrt{x}}}-\sqrt{2}\right)$, $x\to 0^+$.
$= \dfrac{\sqrt{2}x^{1/2}}{8} + o(x^{1/2})$.
(4) $\sqrt{1+\tan x}-\sqrt{1-\sin x}$, $x\to 0$.
$= x+o(x)$.
(5) $\sqrt{x+\sqrt{x+\sqrt{x}}}$, $x\to 0^+$.
$= x^{1/8}+o(x^{1/8})$.
(6) $\sqrt{x+\sqrt{x+\sqrt{x}}}$, $x\to\infty$.
=$\sqrt{x}+o(\sqrt{x})$.

3. Suppose $f\sim cx^k$, $x\to 0$, i.e. $f(x)=cx^k+o(x^k)$. If $f(x)-c^k$ has a leading term $c'x^{k'}$, we denote $f(x)=cx^k+c'x^{k'}+o(x^{k'})$. Expand the following terms to $o(x^2)$:
   (1) $\sqrt{1+x}-1$.
   $\sqrt{1+x}-1=\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$.
   (2) $(1+x)^{1/m}-1$, $m\in\mathbb{Z}_{\geqslant 1}$.
   $(1+x)^{1/m}-1=\frac{1}{m}x-\frac{m-1}{2m^2}x^2+o(x^2)$.

PST: Takagi Function

Define $\psi:[0,1]\to\mathbb{R}$ as

$$\psi(x)=\begin{cases} x, & 0\leqslant x<\frac{1}{2};\\ 1-x, & \frac{1}{2}\leqslant x\leqslant 1.\end{cases}$$

For $x\in R$, let $\psi(x)=\psi(\{x\})$, then $\psi\in C(\mathbb{R})$.
Define the Takagi function $T:\mathbb{R}\to\mathbb{R}$ as follows:

$$T(x)=\sum_{k=0}^{\infty}\frac{1}{2^k}\psi(2^kx),$$

and the partial sum $T_n(x)=\displaystyle\sum_{k=0}^{n}\frac{1}{2^k}\psi(2^kx)$.

T1) Prove that $T(x)$ is a well-defined bounded continuous function on $\mathbb{R}$.

Proof: Note that $\psi(x)\in[0,1/2]$ so the series $\sum_{k=0}^{\infty}2^{-k}\psi(2^kx)$ converges absolutely, and hence $T(x)$ is well-defined and bounded by $T(x)\in[0,1]$.

T2) For $x\in[0,1]$, suppose $x=\sum_{n=1}^{\infty}a_n2^{-n}$ is the binary form of x. Let $v_n=\sum_{k=1}^{n}a_k$, and $\sigma_n(y)=a_n+(1-2a_n)y$, where $y\in\{0,1\}$. Prove that

$$\psi(2^mx)=\sum_{n=1}^{\infty}\frac{\sigma_{m+1}(a_{m+n})}{2^n}.$$

Proof:

$$\psi(2^mx)=\psi\left(\sum_{n=1}^{\infty}a_n2^{m-n}\right)=\psi\left(\sum_{n=m+1}^{\infty}a_n2^{m-n}\right)=\begin{cases}\sum_{n=1}^{\infty}a_{m+n}2^{-n}, & a_{m+1}=0;\\ 1-\sum_{n=1}^{\infty}a_{m+n}2^{-n}, & a_{m+1}=1.\end{cases}$$

Therefore

$$\psi(2^mx)=\sigma_n\left(\sum_{n=1}^{\infty}a_{m+n}2^{-n}\right)=\sum_{n=1}^{\infty}\sigma_{m+1}(a_{m+n})2^{-n}.$$

T3) $x=\sum_{n=1}^{\infty} a_n 2^{-n} \in [0,1]$, prove that

$$T(x)=\sum_{n=1}^{\infty} \frac{(1-a_n)v_n + a_n(n-v_n)}{2^n}.$$

Proof: By T2),

$$T(x)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sigma_{m+1}(a_{m+n})2^{-m-n}=\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sigma_{m+1}(a_n)2^{-n}=\sum_{n=1}^{\infty}\frac{(1-a_n)v_n+a_n(n-v_n)}{2^n}.$$

T4) Suppose $x_0=k_0 2^{-m_0}\in[0,1]$, where $k_0\in\mathbb{Z}_{\geqslant 1}$ is odd, $m_0\in\mathbb{Z}_{\geqslant 0}$. Let $h_n=2^{-n}$, where $n\in\mathbb{Z}_{\geqslant m_0}$. Prove that the sequence $\left\{\frac{T(x+h_n)-T(x)}{h_n}\right\}_{n\geqslant m_0}$ does not converge.

Proof: By T3),

$$\frac{T(x+h_n)-T(x)}{h_n}=\frac{1}{h_n}\left(\frac{n-v_n}{2^n}-\frac{v_n}{2^n}\right)=n-2\sum_{k=1}^{n}a_k=n-2-2S_2(k_0)\to\infty.$$

T5) $f:I\to\mathbb{R}$, where I is an open interval. If f is differentiable at a, prove that

$$\lim_{(h,h')\to(0,0),h,h'>0}\frac{f(a+h)-f(a-h')}{h+h'}=f'(a).$$

i.e. it converges for any sequence $(h_n,h'_n)\to(0,0), h_n,h'_n>0$.
Proof: Consider any sequence $(h_n,h'_n)\to(0,0)$, then

$$\frac{f(a+h)-f(a-h')}{h+h'}=\frac{f(a+h)-f(a)}{h}\cdot\frac{h}{h+h'}+\frac{f(a)-f(a-h')}{h'}\cdot\frac{h'}{h+h'}\to f'(a).$$

T6) Same as T5), if $f\in C^1(I)$, $a\in I$, prove that

$$\lim_{(h,h')\to(0,0),h+h'\neq 0}\frac{f(a+h)-f(a-h')}{h+h'}=f'(a).$$

Proof: For any $h+h'\neq 0$, there exists $\xi\in[a,a+h]$ and $\eta\in[a-h',a]$ such that $f(a+h)=f(a)+hf'(\xi)$ and $f(a-h')=f(a)-h'f'(\eta)$, then

$$\left|\frac{f(a+h)-f(a-h')}{h+h'}-f'(a)\right|\leqslant\frac{h}{h+h'}|f'(\xi)-f'(a)|+\frac{h'}{h+h'}|f'(\eta)-f'(a)|\to 0.$$

Hence

$$\lim_{(h,h')\to(0,0),h+h'\neq 0}\frac{f(a+h)-f(a-h')}{h+h'}=f'(a).$$

T7) Suppose $x\in[0,1]$, such that for any $n\in\mathbb{N}$, $2^n x\notin\mathbb{Z}$. For every $n\in N$, define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as follows:

$$\lfloor 2^n x\rfloor=2^n(x-h'_n),\ \lfloor 2^n x\rfloor+1=2^n(x+h_n).$$

Prove that for an arbitrary n, $h_n+h'_n=2^{-n}$ and for every integer $1\leqslant\ell\leqslant n-1$, the interval $(2^\ell(x-h'_n),2^\ell(x+h_n))$ does not include integers or half-integers.
Proof: $1=2^n(x+h_n)-2^n(x-h'_n)=2^n(h_n+h'_n)$, hence $h_n+h'_n=2^{-n}$. For any integer $1\leqslant\ell\leqslant n-1$, $2^\ell(x-h'_n)=\lfloor 2^n x\rfloor\cdot 2^{\ell-n}$ and $2^\ell(x+h_n)=(\lfloor 2^n x\rfloor+1)2^{\ell-n}$. Since $n-\ell\geqslant 1$, the interval does not include integers or half-integers.

T8) Follow the notations of T7), prove that the sequence $\left\{\frac{T(x+h_n)-T(x-h'_n)}{h_n+h'_n}\right\}_{n\geqslant 1}$ diverges.

Proof: Let $t=\lfloor 2^n x\rfloor$, then

$$a_n=\frac{T(x+h_n)-T(x-h'_n)}{h_n+h'_n}=\sum_{k=0}^{n-1}2^{n-k}\left(\psi\left(\frac{t+1}{2^{n-k}}\right)-\psi\left(\frac{t}{2^{n-k}}\right)\right).$$

Since the interval $(2^{k-n}(t+1),2^{k-n}t)$ does not contain any integers or half-integers, $2^{n-k}(\psi(2^{k-n}(t+1))-\psi(2^{k-n}t))\in\{-1,1\}$, so $a_n\in\mathbb{Z}$ and n,a_n have the same parity. Therefore the sequence $\{a_n\}_{n\geqslant 1}$ diverges.

T9) Prove that $T(x)$ is continuous but nowhere differentiable on $\mathbb{R}$.

Proof: For any $x\in[0,1]$, if $x=k_0\cdot 2^{-m_0}$ as in T4), by T4) the sequence $\left\{\dfrac{T(x+h_n)-T(x)}{h_n}\right\}$ diverges, hence T is not differentiable at x. Otherwise for any $n\in\mathbb{N}$, $2^n x\notin\mathbb{Z}$. Define $\{h_n\}_{n\geqslant 1}$ and $\{h'_n\}_{n\geqslant 1}$ as in T7), then by T8), the sequence $\left\{\dfrac{T(x+h_n)-T(x-h'_n)}{h_n+h'_n}\right\}_{n\geqslant 1}$ diverges. Combined with T5) we know that T is not differentiable at x. Therefore T is nowhere differentiable on $\mathbb{R}$, since T is periodic.
For any x,y in $\mathbb{R}$,

$$|T(x)-T(y)|\leqslant\sum_{k=0}^{N}2^{-k}|T(2^kx)-T(2^ky)|+\sum_{k=N+1}^{\infty}2^{-k}\leqslant 2\max_{0\leqslant k\leqslant N}|T(2^kx)-T(2^ky)|+2^{-N}.$$

Hence for any $N>0$, when $\varepsilon\to 0$, $|T(x)-T(x+\varepsilon)|\leqslant 2^{1-N}\to 0$, so T is (uniformly) continuous on $\mathbb{R}$.

T10) Prove that

$$T(x)=\begin{cases}2x+\frac{T(4x)}{4}, & 0\leqslant x<\frac14;\\ \frac12+\frac{T(4x-1)}{4}, & \frac14\leqslant x<\frac12;\\ \frac12+\frac{T(4x-2)}{4}, & \frac12\leqslant x<\frac34;\\ 2-2x+\frac{T(4x-3)}{4}, & \frac34\leqslant x\leqslant 1.\end{cases}$$

Proof: If $0\leqslant x<1/4$, then

$$T(x)=\psi(x)+\psi(2x)/2+\sum_{k=2}^{\infty}\psi(2^kx)2^{-k}=2x+\frac{T(4x)}{4}.$$

The other cases are exactly the same.

T11) Let $\Gamma=\{(x,T(x)):0\leqslant x\leqslant 1\}\subset\mathbb{R}^2$. Define $\Phi_i:\mathbb{R}^2\to\mathbb{R}^2$

$$\Phi_0\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/4&0\\1/2&1/4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\qquad \Phi_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/4&0\\0&1/4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/4\\1/2\end{pmatrix},$$
$$\Phi_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/4&0\\0&1/4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\1/2\end{pmatrix},\qquad \Phi_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/4&0\\0&1/4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}3/4\\1/2\end{pmatrix}.$$

Prove that Φ_i maps Γ to $\left\{(x,T(x)):x\in\left[\frac{i}{4},\frac{i+1}{4}\right]\right\}$.
Proof: Consider $(x,T(x))\in\Gamma$, then by T10),

$$\Phi_0\begin{pmatrix}x\\T(x)\end{pmatrix}=\begin{pmatrix}x/4\\x/2+T(x)/4\end{pmatrix}=\begin{pmatrix}x/4\\T(x/4)\end{pmatrix}.$$

Hence $\Phi_0(\Gamma)=\{(x,T(x)):x\in[0,1/4]\}$. The cases $i=1,2,3$ are similar.

T12) Let $S_0 = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. For every $n \geqslant 0$, define $S_{n+1} = \bigcup_{k=0}^{3} \Phi_k(S_n)$. Prove that S_n is a sequence of monotonically decreasing compact sets and $\Gamma = \bigcap_{n\geqslant 0} S_n$.

Proof: Let $S_n(x) = \{y \in [0,1] : (x,y) \in S_n\}$. We prove by induction that $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$ for any $x \in [0,1]$. The base $n = 0$ is trivial. Suppose $S_n \subset S_{n-1}$ and $S_n(x)$ is a closed interval containing $T(x)$, then consider S_{n+1}. Note that $\Phi_k(S_n)$ are disjoint, since for any $(x,y) \in \Phi_k(S_n)$, $x \in [k/4, (k+1)/4]$. Hence for any $x \in [0, 1/4]$,
$S_{n+1}(x) = \{y : (x,y) = \Phi_0(4x, z), z \in S_n(4x)\} = \{2x + z/4 : z \in S_n(4x)\}$ is a closed interval containing $T(x) = 2x + T(4x)/4$. By the induction hypothesis $S_n(x) = \{2x + z/4 : z \in S_{n-1}(4x)\}$ and $S_n(4x) \subset S_{n-1}(4x)$ so $S_{n+1}(x) \subset S_n(x)$. The case $x \in [1/4, 1]$ is similar. Therefore $S_{n+1} \subset S_n$ and S_{n+1} is compact, so by induction $S_n \subset S_{n-1}$ for all $n > 0$ and S_n is compact.
Clearly $\Gamma \subset \bigcap_{n\geqslant 0} S_n$, so it suffices to show that $|S_n(x)| \to 0$ for all $x \in [0,1]$. From the proof above we get $\sup_{x\in[0,1]} |S_n(x)| \leqslant \sup_{x\in[0,1]} |S_{n-1}(x)|/4$, hence $|S_n(x)| \to 0$, therefore

$$\Gamma = \bigcap_{n\geqslant 0} S_n.$$

T13) Prove that $\sup_{x\in\mathbb{R}} T(x) \geqslant \frac{2}{3}$.

Proof: For any $(x,y) \in \Gamma$, by T11) we know that $(x/4 + 1/4, y/4 + 1/2) \in \Gamma$, hence if $a = \sup_{x\in\mathbb{R}} T(x)$ then $a \geqslant a/4 + 1/2$, i.e. $a \geqslant 2/3$.

T14) Find a $c \in [0,1]$ such that $T(c) = \frac{2}{3}$.

Solution: Consider $c = 1/3$, then by T10), $T(c) = T(c)/4 + 1/2$, hence $T(c) = \frac{2}{3}$.

T15) For $x \in [0,1]$, suppose $x = \sum_{n=1}^{\infty} b_n 4^{-n}$, where $b_n \in \{0,1,2,3\}$. Prove that

$$\left\{x \in [0,1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0,1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1,2\}\right\}.$$

Proof: If $x = \sum_{n=1}^{\infty} b_n 4^{-n}$, where $b_n \in \{1,2\}$, then by T10),

$$T(x) = \frac{1}{2} + \frac{1}{4} T\left(\sum_{n=1}^{\infty} b_{n+1} 4^{-n}\right) = \cdots = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) = \frac{2}{3}.$$

Otherwise take the least n such that $b_n \in \{0,3\}$, denote $y = \sum_{k=1}^{\infty} b_{n+k-1} 4^{-n}$, then

$$T(x) = \frac{1}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-2}}\right) + \frac{\min\{2y, 2-2y\}}{4^{n-1}} + \frac{1}{4^n} T(4y - b_n) < \frac{2}{3},$$

since $T(4y - b_n) \leqslant 2/3$ and $\min\{2y, 2-2y\} < 1/2$. Therefore

$$\left\{x \in [0,1] : T(x) = \frac{2}{3}\right\} = \left\{x \in [0,1] : x = \sum_{n=1}^{\infty} b_n 4^{-n}, b_n \in \{1,2\}\right\}.$$

T16) As in T11), study the self-similarity of Φ_1, Φ_2 on $\left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, which is a cantor set of Hausdorff dimension $\frac{1}{2}$.

Solution: Same as T11), denote $\Gamma' = \left\{(x, T(x)) : x \in [0,1], T(x) = \frac{2}{3}\right\}$, then

$$\Phi_1(\Gamma') = \left\{(x, T(x)) : x \in \left[0, \frac{1}{2}\right], T(x) = \frac{2}{3}\right\},\ \Phi_2(\Gamma') = \left\{(x, T(x)) : x \in \left[\frac{1}{2}, 1\right], T(x) = \frac{2}{3}\right\}.$$