2025/10/21
26-27页3,5; 54-55页1,3,4,7; 讲义习题1.6:1-3.

26-3

For each of the two matrices

$$\begin{pmatrix}2 & 5 & -1\\ 4 & -1 & 2\\ 6 & 4 & 1\end{pmatrix}, \begin{pmatrix}1 & -1 & 2\\ 3 & 2 & 4\\ 0 & 1 & -2\end{pmatrix}$$

use elementary row operations to determine whether it is invertible, and to find the inverse in case it is.
Solution:

$$\begin{pmatrix}2 & 5 & -1 & 1 & 0 & 0\\ 4 & -1 & 2 & 0 & 1 & 0\\ 6 & 4 & 1 & 0 & 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}2 & 5 & -1 & 1 & 0 & 0\\ 0 & -11 & 4 & -2 & 1 & 0\\ 0 & -11 & 4 & -3 & 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}2 & 5 & -1 & 1 & 0 & 0\\ 0 & -11 & 4 & -2 & 1 & 0\\ 0 & 0 & 0 & -1 & -1 & 1\end{pmatrix}$$

so it is not invertible.

$$\begin{pmatrix}1 & -1 & 2 & 1 & 0 & 0\\ 3 & 2 & 4 & 0 & 1 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}1 & -1 & 2 & 1 & 0 & 0\\ 0 & 5 & -2 & -3 & 1 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 8 & -3 & 1 & -5\\ 0 & 1 & -2 & 0 & 0 & 1\end{pmatrix}$$
$$\rightarrow \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4}.\end{pmatrix}$$

Hence it is invertible and its inverse is

$$\begin{pmatrix}1 & 0 & 1\\ -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{pmatrix}$$

27-5

Determine whether

$$A = \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 2 & 3 & 4\\ 0 & 0 & 3 & 4\\ 0 & 0 & 0 & 4\end{pmatrix}$$

is invertible, and find A^{-1} if it exists.
Solution: It is clearly invertible since the rows are linearly independent, and

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 1 & & & \\
 & 2 & 3 & 4 & & 1 & & \\
 & & 3 & 4 & & & 1 & \\
 & & & 4 & & & & 1
\end{pmatrix}
\to
\begin{pmatrix}
1 & 2 & 3 & 0 & 1 & & & -1 \\
 & 2 & 3 & 0 & & 1 & & -1 \\
 & & 3 & 0 & & & 1 & -1 \\
 & & & 4 & & & & 1
\end{pmatrix}
$$

$$
\to
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 & & -1 & \\
 & 2 & 0 & 0 & & 1 & -1 & \\
 & & 3 & 0 & & & 1 & -1 \\
 & & & 4 & & & & 1
\end{pmatrix}
\to
\begin{pmatrix}
1 & & & & 1 & -1 & & \\
 & 2 & & & & 1 & -1 & \\
 & & 3 & & & & 1 & -1 \\
 & & & 4 & & & & 1
\end{pmatrix}
$$

So

$$
A^{-1} = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.
$$

54-1

Show that the vectors $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (0,0,1,1)$, $\alpha_3 = (1,0,0,4)$, $\alpha_4 = (0,0,0,2)$ form a basis for $\mathbb{R}^3$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.
Solution:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 4 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 1
\end{pmatrix}
\to
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
\to
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
$$

So the coordinates are $e_1 = \alpha_3 - 2\alpha_4$, $e_2 = \alpha_1 - \alpha_3 + 2\alpha_4$, $e_3 = \alpha_2 - \frac{1}{2}\alpha_4$, $e_4 = \frac{1}{2}\alpha_4$.

55-3

Let $\mathscr{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,1,1)$, $\alpha_3 = (1,0,0)$. What are the coordinates of the vector (a,b,c) in the ordered bases $\mathscr{B}$?
Solution: Note that $(a,b,c) = b(1,1,1) + (b-c)(1,0,-1) + (a+c-2b)(1,0,0)$, so its coordinates are $(b-c, b, a+c-2b)$.

55-4

Let W be the subspace of $\mathbb{C}^3$ spanned by $\alpha_1 = (1,0,i)$ and $\alpha_2 = (1+i, 1, -1)$.
(a) Show that α_1, α_2 form a basis of W.
(b) Show that the vectors $\beta_1 = (1,1,0)$ and $\beta_2 = (1, i, 1+i)$ are in W and form another bases for W.
(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?
Proof: (a) $\lambda\alpha_1 + \mu\alpha_2 = (\lambda + \mu + i\mu, \mu, \lambda i - \mu) = (0,0,0)$ implies $\mu = 0$ so $\lambda = 0$, hence they are linearly independent, and form a basis of W.
(b) Note that $\beta_1 = \alpha_2 - i\alpha_1 \in W$, $\beta_2 = i\alpha_2 + (2+i)\alpha_1 \in W$, and likewise $\lambda\beta_1 + \mu\beta_2 = (0,0,0)$ implies $\mu = 0$ so $\lambda = 0$. Hence they are linearly independent and $\dim W = 2$ so they form another basis for W.
(c) $\alpha_1 = \frac{1}{1+i}\beta_1 + \frac{1}{1-i}\beta_2$, $\alpha_2 = \frac{1+2i}{1+i}\beta_1 - \frac{1}{1+i}\beta_2$.

55-7

Let V be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less. Let t be a fixed real number and define $g_1(x) = 1$, $g_2(x) = x + t$, $g_3(x) = (x + t)^2$. Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V . If $f(x) = c_0 + c_1x + c_2x^2$, what are the coordinates of f in this ordered basis $\mathcal{B}$?
Proof: Clearly $\dim V = 3$, and for any $f(x) = c_0 + c_1x + c_2x^2$,

$$\begin{aligned} f(x) &= c_2(x+t)^2 + c_0 + c_1x - 2c_2tx - c_2t^2 \\ &= c_2(x+t)^2 + (c_1 - 2c_2t)(x+t) + c_0 - c_2t^2 - t(c_1 - 2c_2t) \\ &= c_2(x+t)^2 + (c_1 - 2c_2t)(x+t) + c_0 - tc_1 + t^2c_2. \end{aligned}$$

So $\mathrm{Span}\langle g_1, g_2, g_3\rangle = V$ hence they form a basis, and the coordinates of $f = c_0 + c_1x + c_2x^2$ are $(c_0 - tc_1 + t^2c_2, c_1 - 2c_2t, c_2)$.

1.6.1

Suppose $A \in F^{n\times n}$, and there exists $I, J \subset \{1, \cdots, n\}$ such that $|I| + |J| > n$ and for any $i \in I$ and $j \in J$, $A_{ij} = 0$. Prove that A is not invertible.
Proof: Let $|I| = k$, by elementary row operations we can assume $I = \{1, 2, \cdots, k\}$. Then the first k rows of A are elements of the $n - |J|$ dimensional subspace $\{v : v_j = 0 \forall j \in J\}$. Hence they are linearly dependent, so A is not invertible.

1.6.2

Given a basis $\{\alpha_1, \cdots, \alpha_n\}$ of F^n. Prove that $A \in F^{n\times n}$ is invertible iff $\{\alpha_1A, \cdots, \alpha_nA\}$ is a basis of F^n.
Proof: There exists an invertible matrix P such that $(\alpha_1, \cdots, \alpha_n) = (e_1, \cdots, e_n)P$. Hence $\{\alpha_1A, \cdots, \alpha_nA\}$ is a basis of F^n iff $(\alpha_1, \cdots, \alpha_n)A(c_1, \cdots, c_n)^T = 0 \iff c_i = 0$, which is equivalent to $(e_1, \cdots, e_n)(PA)(c_1, \cdots, c_n)^T = 0 \iff c_i = 0$. Therefore is is equivalent to A is invertible.

1.6.3

Prove that the subset $\{f_1, \cdots, f_n\}$ of F^F is linearly independent iff there exists $x_1, \cdots, x_n \in F$ such that

$$\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ & \cdots & \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix}$$

is invertible.
Proof: $f_1, \cdots, f_n$ are linearly independent iff $\sum_{i=1}^n c_if_i = 0 \iff c_i = 0$.
If there exists such $x_1, \cdots, x_n$, then $\sum_{i=1}^n c_if_i = 0$ implies

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ & \cdots & \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} = \left(\sum_{k=1}^n c_kf_k(x_i) \right)_{i=1,\cdots,n} = 0$$

leading to contradiction, so $f_1, \cdots, f_n$ are linearly independent.
If $f_1, \cdots, f_n$ are linearly independent, then we prove by induction that such $x_1, \cdots, x_n$ exists. The base $n = 1$ is trivial. Suppose it holds for $n - 1$.
Then there exists $x_1, \cdots, x_{n-1}$ such that

$$P = \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_{n-1}) \\ & \cdots & \\ f_{n-1}(x_1) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}$$

is invertible. If such x_n does not exist, then for any $x \in F$, there exists $c_1, \cdots, c_{n-1}$ such that $(f_1(x), \cdots, f_n(x)) = \sum_{i=1}^{n-1} c_i(f_1(x_i), \cdots, f_n(x_i))$. Note that $f_n(x) = \sum_{i=1}^{n-1} c_i f_n(x_i)$, where c_i satisfy $(c_1, \cdots, c_{n-1})P = (f_1(x), \cdots, f_{n-1}(x))$ so $(c_1, \cdots, c_{n-1}) = (f_1(x), \cdots, f_{n-1}(x))P^{-1}$. Hence $c_i(x) \in \mathrm{Span}\langle f_1, \cdots, f_{n-1}\rangle$ so $f_n \in \mathrm{Span}\langle f_1, \cdots, f_{n-1}\rangle$, leading to contradiction.