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1.

Suppose V is a $F-$linear space, $N \subset V$ is a subspace. Consider the map

$$\Phi : \{\text{subspace of } V \text{ containing } N\} \to \{\text{subspace of } V/N\}, \Phi(W) = W/N.$$

Prove that Φ is invertible, and its inverse is $W/N \mapsto Q^{-1}(W/N)$, where $Q : V \to V/N$ is the quotient map.
Proof: Note that $W = \bigcup_{A \in W/N} A$ so Φ is injective. For any subspace $\{x + N : x \in S\}$ of V/N, S is a subspace of V, so $W = S + N$ satisfy $\Phi(W) = W/N$, hence Φ is bijective.
For any $x \in W$, $Q(x) = x + N \in W/N$ so $W \subset Q^{-1}(W/N)$. If $Q(x) = x + N \in W/N$, then $x + N = w + N$ so $x - w \in N \subset W$, hence $x \in W$. Therefore $W = Q^{-1}(W/N)$, so the inverse of Φ is $W/N \mapsto Q^{-1}(W/N)$.

2.

Suppose N, W are subspaces of V, prove that $(W + N)/W \cong N/(W \cap N)$.
Proof: Consider $\varphi : N \to (W + N)/W, n \mapsto n + W$, then $\mathrm{Im}\varphi = (W + N)/W$, and $\mathrm{Ker}\varphi = W \cap N$, so $(W + N)/W \cong N/(W \cap N)$.

3.

Suppose W is a subspace of V and suppose $\dim V/W = m < \infty$. Prove that there exists $f_1, \cdots, f_m \in V^*$ such that $W = \bigcap_{i=1}^m \mathrm{Ker} f_i$.
Proof: Take a basis $\{\alpha_1 + W, \cdots, \alpha_m + W\}$ of V/W, and the dual basis $\{g_1, \cdots, g_n\}$ of $(V/W)^*$. Let $f_k = g_k \circ \pi$ where $\pi : V \to V/W$, then $f_k \in V^*$ and $\mathrm{Ker} f_k = W + \mathrm{Span}(\alpha_1, \cdots, \alpha_{k-1}, \alpha_{k+1}, \cdots, \alpha_m)$. Hence $W = \bigcap_{i=1}^m \mathrm{Ker} f_i$.

4.

Suppose V, W are linear spaces on the field F, $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\mathrm{Ker}T_1 \subset \mathrm{Ker}T_2$ iff there exists $U \in \mathcal{L}(W)$ such that $T_2 = UT_1$.
Proof: If $T_2 = UT_1$, clearly $\mathrm{Ker}T_1 \subset \mathrm{Ker}T_2$.
If $\mathrm{Ker}T_1 \subset \mathrm{Ker}T_2$, we construct a linear map $U_0 : \mathrm{Im}T_1 \to W, y \mapsto T_2(x)$ for any $y = T_1(x)$. U_0 is well-defined, since if $T_1(x) = T_1(x')$, then $x - x' \in \mathrm{Ker}T_1 \subset \mathrm{Ker}T_2$ so $T_2(x) = T_2(x')$. Now extend $U_0 : \mathrm{Im}T_1 \to W$ to $U \in \mathcal{L}(W)$, we have $T_2 = UT_1$.