2025/10/29

第四章作业: 习题 4.1 全部，其中题 2 见勘误表;习题 4.2 2,4

第五章作业: 习题 5.1 1-6,9-11,14 选做; 习题 5.2 全部; 习题 5.3 任选不少于十道题目; 习题 5.4 任选 4 道题

4.1

4.1.1

For $a > 0$, prove that $\lim_{x\to a}\sqrt{x} = \sqrt{a}$.
Proof: For any $\varepsilon > 0$ there exists $\delta = \varepsilon\sqrt{a} > 0$ such that for any $|x-a| < \delta$, $|\sqrt{x}-\sqrt{a}| = |x-a|/(\sqrt{x}+\sqrt{a}) < \delta/\sqrt{a} = \varepsilon$. Hence $\lim_{x\to a}\sqrt{x} = \sqrt{a}$.

4.1.2

Calculate

$$\lim_{x\to 2}\left(\frac{7x^3-2x^2-17x-19}{2x^3+3x^2-11x-6}+\frac{1}{2x^2-3x-2}\right)=\frac{12}{5}.$$

4.1.3

Suppose ψ, φ are periodic functions on $(0,\infty)$, such that $\lim_{x\to\infty}(\psi(x)-\varphi(x)) = 0$. Prove that $\psi = \varphi$.
Proof: Let T, T' be the periods of ψ, φ, $f(x) = \psi(x) - \varphi(x)$ and $h(x) = f(x+T) - f(x)$, then $h(x) = \varphi(x+T) - \varphi(x)$ so h has period T'. Note that $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} f(x+T) - f(x) = 0 - 0 = 0$ and h is periodic so $h = 0$. Hence f is periodic and $\lim_{x\to\infty} f(x) = 0$, which implies $f = 0$ and $\varphi = \psi$.

4.1.5

Suppose $a_0, a_1, \ell, \alpha > 0$, $a_1 \neq a_0$, $a_{n+1} = \frac{(\ell+n^\alpha)a_n^2}{\ell a_n + n^\alpha a_{n-1}}$. Prove that:
(1) If $\alpha < 1$, then $\lim_{n\to\infty} a_n > 0$;
(2) If $\alpha = 1, \ell > 1$, then $\lim_{n\to\infty} a_n > 0$;
(3) If $\alpha > 1$, then a_n diverges or $\lim_{n\to\infty} a_n = 0$.
Proof: Let $b_n = \frac{a_n}{a_{n+1}} - 1$, then

$$\frac{a_{n+1}}{a_n} = \frac{\ell+n^\alpha}{\ell+n^\alpha\cdot a_{n-1}/a_n} \implies b_n = \frac{n^\alpha}{\ell+n^\alpha}b_{n-1}.$$

Note that $a_n = a_0\prod_{k=0}^{n-1}(1+b_n)^{-1}$, so $\lim_{n\to\infty} a_n > 0$ iff $\prod_{k=0}^{\infty}(1+b_k)$ converges, iff $\sum_{k=0}^{\infty}|b_k|$ converges (since b_n have the same sign). $b_n = b_0\prod_{k=1}^{n}\frac{n^\alpha}{\ell+n^\alpha}$ so we can denote $c_n = \prod_{k=1}^{n}\frac{n^\alpha}{\ell+n^\alpha}$. Consider Raabe's test: $R = \lim_{n\to\infty} n\left(\frac{c_n}{c_{n+1}}-1\right)$, then

$$n\left(\frac{c_n}{c_{n+1}}-1\right) = \frac{\ell}{n^{\alpha-1}} \implies R = \ell\lim_{n\to\infty} n^{1-\alpha}.$$

If $\alpha < 1$ or $\alpha = 1, \ell > 1$, then $R > 1$ so the limit converges. If $\alpha > 1$ then $R = 0 < 1$ so it diverges. Furthermore if $\alpha \leqslant 0$, then it converges; if $\alpha = 1, \ell \leqslant 1$ then it diverges.

4.2

4.2.2

Suppose $\{x_n\}$, $\{y_n\}$ satisfy

$$\begin{cases} x_n^2 + y_n^2 + 2y_n = 1 + \ln \frac{n+1}{n}, \\ x_n + \left(1 + \frac{1}{3n}\right) y_n = n^{-1/n}. \end{cases}$$

Prove that $\{x_n\}$, $\{y_n\}$ converges and determine the limit.
Proof: $y_n^2 + 2y_n + \left(n^{-1/n} - \left(1 + \frac{1}{3n}\right) y_n\right)^2 = 1 + \ln \frac{n+1}{n}$, so $\left(2 + \frac{2}{3n} + \frac{1}{9n^2}\right) y_n^2 - 2\left(n^{-1/n}\left(1 + \frac{1}{3n}\right) - 1\right) y_n + n^{-2/n} - 1 - \ln \frac{n+1}{n} = 0$. Then $\lim_{n\to\infty} n^{-1/n}\left(1 + \frac{1}{3n}\right) - 1 = 0$ and $\lim_{n\to\infty} n^{-2/n} - 1 - \ln \frac{n+1}{n} = 0$ so $\lim_{n\to\infty} y_n = 0$ ($y_n \sim \sqrt{\frac{\log n}{n}}$).
Likewise we know $\lim_{n\to\infty} x_n = 1$.

4.2.4

$$\begin{aligned} \sum_{k=1}^{n} \sqrt{1 + \frac{2k}{n^3}} - 1 &= \sum_{k=1}^{n} \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} \left(\frac{2k}{n^3}\right)^j = \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} 2^j n^{-3j} \sum_{k=1}^{n} k^j \\ &= \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} \frac{2^j n^{-3j}}{j+1} \left(\sum_{k=1}^{j+1} \binom{j+1}{k} n^k B_{p+1-k} + n^j\right) = \frac{1}{2n} + \frac{1}{2n^2} - \frac{1}{6n^3} + O(n^{-4}). \end{aligned}$$

5.1

5.1.1

Prove that $D(x) = \mathbf{1}_{\mathbb{Q}}$ is nowhere continuous on $\mathbb{R}$.
Proof: Note that $\mathbb{Q}$ and $\mathbb{Q}^C$ are both dense on $\mathbb{R}$. For $x \in \mathbb{Q}$, and any $\delta > 0$, there exists $y \in \mathbb{Q}^C$ such that $|x - y| < \delta$ but $|D(x) - D(y)| = 1$, so D is not continuous at x. Likewise for $x \in \mathbb{Q}^C$, and any $\delta > 0$, there exists $y \in \mathbb{Q}$ such that $|x - y| < \delta$ but $|D(x) - D(y)| = 1$, so D is not continuous at x.

5.1.2

Consider

$$R(x) = \begin{cases} 1, & x = 0 \\ q^{-1}, & x = p/q, \gcd(p, q) = 1, \\ 0, & x \in \mathbb{Q}^C. \end{cases}$$

Prove that R is only continuous on $\mathbb{Q}^C$.
Proof: If $x \in \mathbb{Q}$, then likewise for any $\delta > 0$ there exists $y \in \mathbb{Q}^C$ such that $|x - y| < \delta$ but $|R(x) - R(y)| = R(x)$, so R is not continuous at x. If $x \in \mathbb{Q}^C$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta < \min\{\|qx\|/q : 1 \leqslant q \leqslant \lfloor \varepsilon^{-1} \rfloor\}$, then $|x - y| < \delta$ implies either $x \in \mathbb{Q}^C$, or $x = p/q \in \mathbb{Q}$, and $q > 1/\varepsilon$, so $|f(x) - f(y)| < \varepsilon$.

5.1.3

Let $S_n = \frac{1}{n^2}\sum_{k=0}^n \log\binom{n}{k}$. Calculate $\lim_{n\to\infty} S_n$.
Solution: By Stolz,

$$\lim_{n\to\infty}\frac{\sum_{k=0}^n \log\binom{n}{k}}{n^2} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\log\binom{n}{k}/\binom{n-1}{k}}{2n} = \lim_{n\to\infty}\frac{\log n^n/n!}{2n} = \lim_{n\to\infty}\frac{\log\left(n/(n-1)\right)^{n-1}}{2} = \frac{1}{2}.$$

5.1.4

Suppose $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0}\frac{f(2x)-f(x)}{x} = 0$, prove that $\lim_{x\to 0} f(x)/x = 0$.
Proof: Let $f(2x) - f(x) = xh(x)$, then $h(x)\to 0$, and

$$f(x) = -f(2^{-n}x) + \sum_{k=0}^{n-1} 2^{-k-1} x h(2^{-k-1}x)$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $x < \delta$ implies $-\varepsilon < h(x), f(x) < \varepsilon$, then

$$\left|\frac{f(x)}{x}\right| \leqslant \left|\frac{f(2^{-n}x)}{x}\right| + \sum_{k=0}^{n-1} 2^{-k-1}|h(2^{-k-1}x)| \leqslant \left|\frac{f(2^{-n}x)}{x}\right| + 4\varepsilon.$$

Let $n\to\infty$ then $|f(x)/x| < 4\varepsilon$.

5.1.5

If f is locally Lipschitz on $\mathbb{R}$, then f is Lipschitz on any compact subset $[-A, A]$.
Proof: For any $x\in\mathbb{R}$, let O_x be the neighborhood of x such that $|f(y) - f(z)| \leqslant M_x|y - z|$. For any compact subset $K\subset\mathbb{R}$, $K\subset\bigcup_{x\in K} O_x$ so there is a finite subset $J\subset K$ such that $K\subset\bigcup_{x\in J} O_x$. Let $M = \max\{M_x : x\in J\}$, then $|f(y) - f(z)| \leqslant M|y - z|$ for any $y, z\in K$. (Let $E = \{x\in[-A, A], |f(y) - f(z)| \leqslant M|y - z|, \forall y, z\in[-A, x]\}$, then clearly $x = \sup E > -A$ and $x\in E, x = A$).

5.1.6

If $f:\mathbb{R}\to\mathbb{R}$ satisfy for any $x\in\mathbb{R}$, exists $\delta > 0$ and $M > 0$ such that $|f(x) - f(y)| \leqslant M|x - y|$ for all $y\in(x-\delta, x+\delta)$. Must f be locally Lipschitz?
Solution: Consider $f(x) = x\sin x^{-1}$, then for any $x\in\mathbb{R}\backslash\{0\}$, f is locally C^1 so it is locally Lipschitz. For $x = 0$, $|f(y)| \leqslant |y|$ for all $y\in(-1, 1)$. But f is not locally Lipschitz at 0, since $x_n = (2\pi n)^{-1}$ and $y_n = \left(2\pi\left(n+\frac{1}{2}\right)\right)^{-1}$ satisfy $|f(x_n) - f(y_n)| = x_n + y_n$.

5.1.9

Prove that if $f: I\to\mathbb{R}$ where I is an interval and f monotonic, then $A = \{x : f \text{ not continuous at } x\}$ is countable.
Proof: Consider the map $\varphi: A\to\mathbb{Q}, x\mapsto q_x$ where q_x is an arbitrary element of $\mathbb{Q}\cap(\sup_{y<x} f(y), \inf_{y>x} f(y))$, then φ is an injection so A is countable.

5.1.10

Suppose $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and let $f(x) = \sum_{k=1}^{\infty} 2^{-k} \chi_{(x_k,\infty)}$, prove that all discontinuities of f are $\{x_k : k \geqslant 1\}$. Furthermore, if $\{x_k\}$ is dense in $\mathbb{R}$, then f is strictly increasing.
Proof: If $x \notin \{x_k\}$, then $2^{-k}\chi_{(x_k,\infty)}$ are continuous at x, and the series converges uniformly, so f is continuous at x.
For $k \geqslant 1$, $2^{-l}\chi_{(x_l,\infty)}$ is only discontinuous at x_k when $k = l$, so f is not continuous at x_k.
If $\{x_k\}$ is dense in $\mathbb{R}$, then for any $x < y$, there exists k such that $x_k \in (x, y)$, so $\chi_{(x_k,\infty)}(x) = 0 < 1 = \chi_{(x_k,\infty)}(y)$. Hence $f(x) < f(y)$ and f is strictly increasing.

5.1.11

Write the real numbers in $(0, 1)$ into the decimal form $0.a_1a_2 \cdots a_n \cdots$, (there does not exists N such that $n \geqslant N \implies a_n = 9$). Define

$$f(0.a_1a_2 \cdots a_n \cdots) = 0.a_10a_20 \cdots a_n0 \cdots,$$

determine at which points is f continuous.
Solution: If $a_n = 0$ for all $n \geqslant N$ while $a_{N-1} = 1$, then for any $\varepsilon > 0$, there exists $y = x - 10^{-M}$ where $x = 0.a_1a_2 \cdots a_{N-1}$, such that $|y - x| < \varepsilon$, and $|f(x) - f(y)| \geqslant 8 \cdot 10^{-2N}$, so f is not continuous at $x = 0.a_1a_2 \cdots a_{N-1}$.
If infinitely many $a_n \neq 0$, then for any $\varepsilon > 0$ there exists $N \geqslant -10 \log_{10} \varepsilon + 10$, and $\delta < 10^{-10N}$, such that $\forall |y - x| < \delta$, the first N digits of x, y are the same, then $|f(x) - f(y)| < 10^{-2N+2} < \varepsilon$. Hence f is continuous at x.

5.1.14

The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfy the following property: for any $\delta > 0$,

$$\lim_{n\to\infty} f(n\delta) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.
Proof: Consider any $\varepsilon > 0$. For any $N \in \mathbb{N}$,

$$A_N = \{\delta > 0 : \forall n \geqslant N, |f(n\delta)| \leqslant \varepsilon\}.$$

Then since f is continuous, A_N is closed, and by $\lim_{n\to\infty} f(n\delta) = 0$ for any $\delta > 0$, $\bigcup_{N\geqslant 1} A_N = \mathbb{R}_{>0}$. Hence by Baire Category Theorem, there exists an $N > 0$ such that $(a, b) \subset A_N$ for some interval (a, b). Let $X = \{x \in \mathbb{R}_{>0} : |f(x)| \leqslant \varepsilon\}$, then since $(a, b) \subset A_N$, for any $n \geqslant N$, $(na, nb) \subset X$. Note that when $n > b/(b - a)$, $nb > (n + 1)a$, hence there exists $M > 0$ such that $(M, \infty) \subset X$. Therefore $\lim_{x\to\infty} f(x) = 0$.

5.2

5.2.1

Calculate

$$\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.$$

5.2.2

For $a, b > 0$, prove that

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \sqrt{ab}.$$

Proof: $f(x) = \left(\frac{a^x+b^x}{2}\right)^{1/x}$, then f is monotonically decreasing on $(0, \infty)$, so using L'Hopital, $\lim_{n\to\infty} f(1/n) = \lim_{x\to 0} f(x) = \exp\lim_{x\to 0}\frac{\log(a^x+b^x)-\log 2}{x} = \sqrt{ab}$.

5.2.3

Calculate

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \frac{1}{2}.$$

5.2.4

Calculate

$$\lim_{x\to 0}\frac{\log(1+x)-x}{x^2} = -\frac{1}{2}.$$

5.2.5

For $\alpha \in \mathbb{R}$, calculate

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1-x}{x^2} = \frac{\alpha(\alpha-1)}{2}.$$

5.3

5.3.8

Prove that $f(x) = \sin^2 x + \sin x^2$ is not periodic.
Proof: If f is periodic, then f is continuous so f is uniformly continuous. Since $\sin^2 x$ is uniformly continuous, so is $\sin x^2$. But for $\varepsilon = 1/2$, and any $\delta > 0$, consider $x = \sqrt{2\pi N}$ and $y = \sqrt{2\pi(N+1/2)}$, then $|x-y| < 1/2N < \delta$ when $N > 2/\delta$, but $|\sin x^2 - \sin y^2| = 1 > \varepsilon$, leading to contradiction.

5.3.9

Suppose f is uniformly continuous on $\mathbb{R}$, prove that there exists a, b such that for any $x \in \mathbb{R}$, $|f(x)| \leqslant a + b|x|$.
Proof: For $\varepsilon = 1$ there exists δ such that $|x-y| \leqslant \delta$ implies $|f(x)-f(y)| \leqslant 1$. Let $M = \sup_{x\in[0,\delta]}|f(x)|$, then for any $x \in \mathbb{R}$, $|f(x)| \leqslant \delta^{-1}|x| + M$.

5.3.10

Suppose $a > 0, a^2 + 4b < 0$. Prove that there does not exist $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = af(x) + bx$ and for any $a < b$ and $r \in (f(a), f(b))$ there exists $c \in (a. b)$ such that $f(c) = r$.
Proof: For any such f, clearly f is injective, and unbounded. If for some $a < b < c$ we have $(f(a) - f(b))(f(c) - f(b)) \geqslant 0$, then we can find $u \in (a, b)$ and $v \in (b, c)$ such that $f(u) = f(v)$, a contradiction. Hence f is strictly monotonic, and so f is continuous.
If f has a fixed point $f(t) = t$, then $t = at + bt$ so $t = 0$ or $a + b = 1$ which is impossible.
Consider any $x_0 \in \mathbb{R}\backslash\{0\}$, and $x_{n+1} = f(x_n)$, then $x_n = A\alpha^n + B\beta^n$, where $\alpha, \beta = (-a \pm \sqrt{a^2 + 4b})/2$ so $\alpha, \beta \in \mathbb{C}\backslash\mathbb{R}$. It is well-known that x_n change signs infinitely often, so f is monotonically decreasing. Since $f(x_n) - x_n = x_{n+1} - x_n$ takes both positive and negative values, f has a fixed point so $f(0) = 0$. Let $x_1 > 0$, then $x_2 = f(x_1) < f(0) = 0$, so $x_{2k+1} > 0$ and $x_{2k} < 0$. But $x_3 = ax_2 + bx_1 < 0$, leading to contradiction.

5.3.11

If for a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$, $\{f_n(x)\}_{n \geqslant 1}$ is bounded for any $x \in \mathbb{R}$, prove that there is an interval (a, b) and $M > 0$ such that $|f_n(x)| \leqslant M$ for all $n \geqslant 1, x \in (a, b)$.
Proof (Osgood): For any n, M let $F_{n,M} = \{x \in \mathbb{R} : |f_n(x)| \leqslant M\}$. Then $F_{n,M}$ is closed, so $\tilde{F}_M = \bigcap_{n \geqslant 1} F_{n,M}$ is also closed. We want to find M such that $\tilde{F}_M$ has an interior (a, b). Since $\mathbb{R} = \bigcup_{M \geqslant 1} \tilde{F}_M$, this is a simple application of Baire category theorem.

5.3.17

Suppose $f, g \in C([a, b])$, and exists $x_n \in [a, b]$ such that $f(x_n) = g(x_{n+1})$, for all $n \geqslant 1$. Prove that $\exists \xi \in [a, b]$, $f(\xi) = g(\xi)$.
Proof: Otherwise suppose $f > g$, so $g(x_{n+1}) = f(x_n) > g(x_n)$ and $f(x_n) < f(x_{n+1})$. Take a sub-sequence $\{x_{n_k}\} \to u$, and suppose $\{x_{n_k+1}\} \to v$ (otherwise choose a sub-sequence of $\{x_{n_k}\}$). Then $f(x_{k_n}) = g(x_{k_n+1})$, so $f(u) = \lim_{n\to\infty} f(x_{k_n}) = g(v)$. Note that $g(x_{k_n}) < g(x_{k_n+1}) < \cdots < g(x_{k_{n+1}})$, so $g(u) = \lim_{n\to\infty} g(x_{k_n}) = \lim_{n\to\infty} g(x_{k_{n+1}}) = g(v)$, hence $f(u) = g(v) = g(u)$, leading to contradiction.

5.3.18

Prove that there does not exists $f \in C(\mathbb{R})$ such that for any $\alpha \in \mathbb{R}$, $f(x) = \alpha$ has exactly two roots.
Proof: Let $f(a) = f(b) = 0$ where $a < b$, and suppose $f(x) > 0$ when $x \in (a, b)$ (otherwise consider $-f$).
Suppose $M = \sup_{x\in[a,b]} f(x) > 0$. If there exists $a < c < d < b$ such that $f(c) = f(d) = M$, then take $e \in (c, d)$ and $r = f(e) \in (0, M)$. There exists $u \in (a, c)$ and $v \in (d, b)$ such that $f(u) = f(v) = r = f(e)$, leading to contradiction. Otherwise suppose there exists $a < c < b < d$ such that $f(c) = f(d) = M$, then there exists $u \in (a, c)$, $v \in (c, b)$, $w \in (b, d)$ such that $f(u) = f(v) = f(w) = M/2$, a contradiction.

5.3.19

Suppose $n \in \mathbb{Z}$, $f \in C([0, n])$ and $f(0) = f(n)$. Prove that there are n distinct sets $\{x, y\}$ such that $f(x) = f(y)$ and $x - y \in \mathbb{Z}\backslash\{0\}$.
Proof: For any $k \in \{1, \cdots, n\}$, let $g(x) = f(x + k) - f(x)$ where $g : [0, n - k] \to \mathbb{R}$. Then $g(0) = -g(n - k)$ so there exists $x \in [0, n - k]$ such that $g(x) = 0$ so $f(x + k) = f(x)$, and we obtain the n distinct sets $\{x, x + k\}$.

5.3.20

Suppose for any $a < b$, and $y \in (f(a), f(b))$ (or $y \in (f(b), f(a))$), there exists $c \in (a, b)$ such that $f(c) = y$. If for any $r \in \mathbb{Q}$, $\{x \in \mathbb{R} : f(x) = r\}$ is closed, prove that $f \in C(\mathbb{R})$.
Proof: Otherwise if f is discontinuous at x_0, i.e. there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $y \in (x_0 - \delta, x_0 + \delta)$ such that $|f(x_0) - f(y)| > \varepsilon$. Either $f(y) > f(x_0) + \varepsilon$ or $f(y) < f(x_0) - \varepsilon$ so we can assume that there is a sequence $x_n \to x_0$ such that $f(x_n) > f(x_0) + \varepsilon$. Take $r \in \mathbb{Q} \cap (f(x_0), f(x_0) + \varepsilon)$, then there is a sequence $y_m \to x_0$ such that $f(y_m) = r$. Since $f^{-1}(\{r\})$ is closed, $x_0 \in f^{-1}(\{r\})$, leading to contradiction.

5.3.21

Suppose f, g, xf are uniformly continuous on $\mathbb{R}$, prove that fg is uniformly continuous on $\mathbb{R}$.
Proof: Since xf, g are uniformly continuous, there exists A, B such that $|xf(x)| \leqslant A + B|x|$, and C, D such that $|g(x)| \leqslant C + D|x|$.
For any $x < y$, such that $|x - y| < \delta$,

$$|f(x)g(x) - f(y)g(y)| \leqslant |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|$$

and

$$|f(x)g(x) - f(y)g(y)| \leqslant |xf(x) - yf(y)| \cdot |g(x)/x| + |yf(y)| \cdot \left|\frac{g(x)}{x} - \frac{g(y)}{y}\right|.$$

Note that

$$\left|\frac{g(x)}{x} - \frac{g(y)}{y}\right| \leqslant \frac{|y| \cdot |g(x) - g(y)| + |g(y)| \cdot |x - y|}{|xy|}.$$

For any $\varepsilon > 0$, take $M > 0$, and suppose $m = \sup_{x \in [-M-1, M+1]} |f(x)| + |g(x)|$, then there exists $\delta > 0$ such that $|x - y| < \delta$ implies
$|xf(x) - yf(y)|, |f(x) - f(y)|, |g(x) - g(y)| < \varepsilon' = \varepsilon/(m \cdot (10 + 10\max\{A, B, C, D\}))$. For any $|x - y| < \delta$, if $x \in [-M - 1/2, M + 1/2]$, then

$$|f(x)g(x) - f(y)g(y)| \leqslant |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \leqslant 2m\varepsilon' < \varepsilon.$$

Otherwise $|x|, |y| > M$, so

$$\left|\frac{g(x)}{x} - \frac{g(y)}{y}\right| \leqslant \frac{|g(x) - g(y)|}{|y|} + \frac{|g(x)| \cdot |x - y|}{|xy|} < \frac{\varepsilon'}{|y|} + (D + C \cdot M^{-1})\frac{\delta}{|y|},$$

then

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\leqslant |xf(x) - yf(y)| \cdot |g(x)/x| + |yf(y)| \cdot |g(x)/x - g(y)/y| \\ &\leqslant \varepsilon' \cdot (D + CM^{-1}) + (A + B|y|) \cdot \left(\frac{\varepsilon' + \delta(D + CM^{-1})}{|y|}\right) < \varepsilon.\end{aligned}$$

So in both cases $|f(x)g(x) - f(y)g(y)| < \varepsilon$, then fg is uniformly continuous.

5.3.23

Suppose $A, B \in M_n(\mathbb{C})$ prove that

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A+B)\det(A-B).$$

Proof: Note that

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det\begin{pmatrix} A & B \\ B-A & A-B \end{pmatrix} = \det\begin{pmatrix} A+B & B \\ 0 & A-B \end{pmatrix} = \det(A+B)\det(A-B).$$

5.4

5.4.1

Consider

$$f(x) = \begin{cases} x\sin x^{-2}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Prove that $f \in C([0,1])$ and determine whether f is Hölder on $[0,1]$.
Proof: Clearly f is continuous on $(0,1]$, and $|x\sin x^{-2}| \leqslant x$ so $\lim_{x\to 0} x\sin x^{-2} = 0$ hence $f \in C([0,1])$.
We prove that for $M = 100$ and $\alpha = 1/3$, we have $|f(x) - f(y)| \leqslant M|x-y|^{\alpha}$. The case $x = 0$ is trivial, now suppose $0 < y < x$.
Note that $f'(x) = \sin x^{-2} - 2x^{-2}\cos x^{-2}$ so $|f'(t)| \leqslant 1 + 2t^{-2}$.
Case1: $x - y < x^3$, then $|f(x) - f(y)| \leqslant (x-y)(1+2x^{-2})$, and $(x-y)^{2/3}(1+2x^{-2}) < x^2 + 2 \leqslant 3$ is bounded, so $|f(x) - f(y)| \leqslant 3|x-y|^{1/3}$.
Case2: $x - y > x^3$, then $|f(x) - f(y)| \leqslant |f(x)| + |f(y)| \leqslant x + y \leqslant 2x \leqslant 2|x-y|^{1/3}$.
Hence $|f(x) - f(y)| \leqslant 3|x-y|^{1/3}$.

5.4.2

Consider

$$f(x) = \begin{cases} x\sin e^{1/x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Prove that $f \in C([0,1])$ and determine whether f is Hölder on $[0,1]$.
Proof: $|f(x)| \leqslant |x|$ so $\lim_{x\to 0} f(x) = 0$ and clearly f is continuous on $(0,1]$.
Let $x_n = 1/\log\left(2\pi n - \frac{\pi}{2}\right)$, $y_n = 1/\log\left(2\pi n + \frac{\pi}{2}\right)$, then
$|x_n - y_n| = \frac{\log(1+1/2n)}{\log(2n\pi)\log(2\pi(n+1/2))} = O\left(\frac{1}{n\log^2 n}\right)$, and $|f(x_n) - f(y_n)| = |x_n + y_n| = O\left(\frac{1}{\log n}\right)$, hence for any $\alpha \in (0,1)$ and $M > 0$, there exists n such that $\frac{1}{\log n} > CM\left(\frac{1}{n\log^2 n}\right)^{\alpha}$ so f is not Hölder.

5.4.3

Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x-y|$ for any $x \neq y$ but f has no fixed points.
Solution: Consider $f(x) = \sqrt{x^2+1}$, then $f'(x) = \frac{x}{\sqrt{x^2+1}} < 1$ so $|f(x) - f(y)| < |x-y|$.

5.4.5

Suppose f has period 1, and $|f(x)-f(y)| \leqslant |x-y|$. Consider $g(x)=x+f(x)$. For any $x_0 \in \mathbb{R}$, let $x_{n+1}=g(x_n)$, prove that $\lim_{n\to\infty} x_n/n$ exists and its value is independent of x_0.
Proof: Clearly f, g are continuous, and g is monotonically increasing: if $x>y$ then $0 \leqslant g(x)-g(y) \leqslant 2(x-y)$. Note that $g(x+k)=g(x)+k$ so $g^{(n)}(x+k)=g^{(n)}(x)+k$. If $u<v<u+1$, then $|g(u)-g(v)|=|v+f(v)-u-f(u)|<|v-u|+|u+1-v|=1$, so $|u-v|<1$ implies $|g^{(n)}(u)-g^{(n)}(v)|<1$.
For any two sequences $\{x_n\}$ and $\{y_n\}$, suppose $x_0+k-1 \leqslant y_0 < x_0+k$, then $x_n+k-1=g^{(n)}(x_0+k-1) \leqslant y_n = g^{(n)}(y_0) < g^{(n)}(x_0+k) = x_n+k$. So if $\lim_{n\to\infty} x_n/n$ exists, then $\lim_{n\to\infty} y_n/n = \lim_{n\to\infty} x_n/n$ so the limit is independent of x_0.
Let $h(n,x)=g^{(n)}(x)-x$, then $h(n+m,x)=h(n,g^m(x))+h(m,x)$ and $h(n,x)=h(n,\{x\})$. Also, $|h(n,x)-h(n,y)|=|h(n,\{x\})-h(n,\{y\})| \leqslant |\{x\}-\{y\}|+|g(\{x\})-g(\{y\})| \leqslant 2$.
Hence $h(n,x)+h(m,x)-2 \leqslant h(n+m,x) \leqslant h(n,x)+h(m,x)$, so $\lim_{n\to\infty} h(n,x)/n$ exists (recall problem 3.4.4).