2025/9/15

## 1.

Prove that for any $a < b$, the interval $(a, b)$ contains a rational number and an irrational number.
Proof: Let $q > 2/(b-a)$ and $p = \lfloor qa \rfloor + 1$, then $bq - aq > 2$ so $p \in (aq, bq)$ i.e. $p/q \in (a, b)$. Hence $(a, b)$ contains a rational number. Apply this to the interval $(a/\sqrt{2}, b/\sqrt{2})$, then $(a, b)$ contains an irrational number of the form $\sqrt{2}p/q$.

## 2.

Given an ellipsis $\Gamma$ with foci $F_1, F_2$.
(1) Suppose a light ray is emitted from point $A_1$ on $\Gamma$, that intersects $F_1F_2$ and then $\Gamma$ at $A_2$. After reflecting on $\Gamma$, it again intersects $\Gamma$ at points $A_3, A_4, \cdots$, prove that $A_nA_{n+1}$ always intersect $F_1F_2$.
Proof: It suffices to show that $A_2A_3$ intersects $F_1F_2$. Note that the line $A_2A_2$ bisects both angles $\angle F_1A_2F_2$ and $\angle A_1A_2A_3$, hence $A_2A_3$ intersects $F_1F_2$ iff $A_1A_2$ intersects $F_1F_2$.
(2) is similar

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2025/9/17

## 1.

Prove that the Cantor set is uncountable but has measure $0$.
Proof: The Cantor set

$$\mathcal{C} = \left\{ \sum_{i=1}^{\infty} a_n 3^{-n} : a_n \in \{0, 2\} \right\}$$

is uncountable since it can be mapped bijectively to $(0, 1)$ by $\sum_{i=1}^{\infty} a_n 3^{-n} \mapsto \sum_{i=1}^{\infty} b_n 2^{-n}$ where $b_n = a_n/2$. By the construction of $\mathcal{C}$ we know $m(\mathcal{C}) = 0$, where $m$ is the Lebesgue measure.

## 2.

Prove that any infinite set contains a countable subset.
Proof: For any infinite set $X$, let $x_1$ be an arbitrary element of $X$, and define $x_n$ inductively by $x_n \in X \backslash \{x_1, x_2, \cdots, x_{n-1}\}$. $x_n$ always exist since $X$ is infinite, so $X$ has a countable subset $\{x_1, x_2, \cdots\}$.

## 3.

Prove that the union of a countable set and an uncountable set is uncountable.
Proof: Let $A$ be countable and $B$ uncountable. If there is an injective map $f : A \cup B \to \mathbb{N}$, then let $g : B \to \mathbb{N}$ , $b \mapsto f(b)$ is injective, leading to contradiction.

## 4.

Suppose $S$ is countable, and $f : S \to X$ is surjective, then $X$ is countable.
Proof: Consider the injective mapping $g : S \to \mathbb{N}$, then $\varphi : X \to \mathbb{N}$, $x \mapsto \min g(f^{-1}(\{x\}))$ is injective, so $X$ is countable.

## 5.

Prove that any interval of positive length on $\mathbb{R}$ is uncountable.
Proof: An unbounded interval contains a bounded interval. Any bounded open interval $(a, b)$ can be mapped bijectively to $(0,1)$ by $x \mapsto (x-a)/(b-a)$, hence it is uncountable. By problem3, any interval is uncountable.

## 6.

6.7(i): Consider the discrete metric $d(x, y)=\begin{cases}1, & x \neq y \\ 0, & x=y\end{cases}$ on $\mathbb{Z}$.

6.7(ii): Consider the metric $d(x, y)=\begin{cases}1, & x y<0 \\ 0, & x y>0\end{cases}$ on $\mathbb{Z} \backslash\{0\}$.