2025/10/10

作业四：【尤】**3.4**：1（1）（4）（7）、14、18、24；**3.5**：4、5。**回收日期：10/15/三**

3.4.1

Determine the characteristics of the following curves:
(1) $x^2-2xy-4y^2+6x+2y+3=0$.
Combine $x-y+3=0$ and $x+4y-1=0$ then the center is: $(-11/4,4/5)$.
$I_2=1\cdot(-4)-(-1)^2=-5<0$ so it is a hyperbola.
Asymptotes: $X^2-2XY-4Y^2=0 \iff X/Y=1\pm\sqrt{5}$. So they are $\left(x+\frac{11}{4}\right)=(1\pm\sqrt{5})\left(y-\frac{4}{5}\right)$.
Axis of symmetry: $\cot(2\theta)=\frac{A-C}{2B}=-\frac{5}{2}$ so they are $y-\frac{4}{5}=\frac{5\pm\sqrt{29}}{2}\left(x+\frac{11}{4}\right)$.
(4) $x^2+2xy+y^2+2x+2y-3=0$.
This is a degenerate parabola: $(x+y-1)(x+y+3)=0$.
(7) $5x^2+4xy+y^2-6x+4y-6=0$.
Combine $5x+2y-3=0$ and $2x+y+2=0$ then the center is $(7,-16)$.
$I_2=5-2^2=1>0$ so it is an ellipsis.
Axis of symmetry: $\cot(2\theta)=\frac{A-C}{2B}=1$ so they are $y+16=(-1\pm\sqrt{2})(x-7)$.

3.4.14

Prove that for every pair of conjugate diameters of an ellipsis, the sum of squares of the two radii are constant.
Proof: Assume the ellipsis is centered at $(0,0)$. We can show that for any two pairs of conjugate radii OA,OB where $A=(x_1,y_1),B=(x_2,y_2)$, $x_1^2+x_2^2=a^2$ and $y_1^2+y_2^2=b^2$ where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipsis. Note that the value $\frac{x_1^2+x_2^2}{a^2}$, $\frac{y_1^2+y_2^2}{b^2}$ and conjugateness are invariant under dilation, hence we only need to prove the case of a circle, which is trivial.

3.4.18

Suppose $\Gamma: a_{11}x^2+a_{22}y^2+2a_{12}xy+2b_1x+2b_2y+c=0$. Prove that if there are two perpendicular tangents through (x',y'), then

$$F_1^2(x',y')+F_2^2(x',y')=(a_{11}+a_{22})F(x',y').$$

where $F_1(u,v)=a_{11}u+a_{12}v+b_1, F_2(u,v)=a_{12}u+a_{22}v+b_2$.
Proof: There exists m,n such that $\varphi(m,n)\neq 0$ and

$$\varphi(m,n)F(x',y')=[mF_1(x',y')+nF_2(x',y')]^2,\varphi(-n,m)F(x',y')=[nF_1(x',y')-mF_2(x',y')]^2.$$

Hence $F_1^2(x',y')+F_2^2(x',y')=(a_{11}+a_{22})F(x',y')$.

3.4.24

Assume the conic curve Γ has conjugate diameters $x-y-10=0$ and $x+y+6=0$, and passes $(3,-3),(3,-7)$. Determine the equation of Γ, and the tangent of Γ at $(3,-3)$.
Solution: By problem3.5.4 (after dilation), $\Gamma: x^2+y^2-6xy-52x+28y+168=0$, and the tangent of Γ at $(-3,3)$ is $7x-y-24=0$.

3.5.4

The axis of symmetry of Γ is $x-y-5=0$ and $x+y+3=0$, and Γ passes points $(3/2,-3/2)$, $(3/2,-7/2)$. Determine the equation of Γ.
Solution: The center is $(1,-4)$. Let $x'=(x-1)/\sqrt{2}-(y+4)/\sqrt{2}$, $y'=(x-1)/\sqrt{2}+(y+4)/\sqrt{2}$, then the axis of symmetry of Γ' is $x=0$ and $y=0$, and Γ' passes the points $(-\sqrt{2},3/\sqrt{3})$ and $(0,1/\sqrt{2})$, hence $\Gamma':2y^2-4x^2=1$, so $\Gamma:x^2+y^2-6xy-26x+14y+42=0$.

3.5.5

The equation of a parabola in I is $ax^2+4ay^2+4xy+10x-20y-1=0$ where $a>0$. Determine the value of a, and construct another coordinate system I' such that this parabola is in the then $y'=c(x')^2$ ($c>0$).
Solution: $I_2=2^2-a\cdot 4a=0$ so $a=1$. $\cot(2\theta)=\frac{A-C}{2B}=-\frac{3}{4}$ and $c=\frac{\sqrt{5}}{8}$.