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106-8

Let W be the subspace of $\mathbb{R}^5$ which is spanned by the vectors $\alpha_1 = e_1 + 2e_2 + e_3$, $\alpha_2 = e_2 + 3e_3 + 3e_4 + e_5$, $\alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5$. Find a basis for W^0.
Solution: $\mathrm{Span}\{\alpha_1, \alpha_2, \alpha_3\} = \mathrm{Span}\{(1,2,1,0,0), (0,1,2,1,0), (0,0,1,2,1)\}$, hence a basis for W^0 is $f_1 : (x_1, x_2, x_3, x_4, x_5) \mapsto x_1 - x_3 + 2x_4 - 3x_5$, and $f_2 : (x_1, x_2, x_3, x_4, x_5) \mapsto x_2 - 2x_3 + 3x_4 - 4x_5$.

106-9

Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$ and let $B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$.
Let W be the subspace of V consisting of all A such that $AB = 0$. Let $f \in W^0$, suppose that $f(I) = 0$ and $f(C) = 3$, where $C = \mathrm{diag}(0, 1)$. Find $f(B)$.
Solution: Note that $W = \{\begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\}$, and I, C spans all diagonal matrices, hence $B = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} + \mathrm{diag}(3, 3)$, so $f(B) = 3f(I) = 0$.

106-10

Let F be a subfield of the complex numbers. We define n linear functionals on F^n by $f_k(x_1, \cdots, x_n) = \sum_{j=1}^{n} (k - j)x_j$, $1 \leqslant k \leqslant n$. What is the dimension of the subspace annihilated by $f_1, \cdots, f_n$?
Solution: $(x_1, \cdots, x_n) \in \mathrm{Ker} f_k \iff \sum_{j=1}^{n} jx_j = k \sum_{j=1}^{n} x_j$, so $\bigcap_{k=1}^{n} \mathrm{Ker} f_k = \left\{(x_1, \cdots, x_n) : \sum_{j=1}^{n} jx_j = \sum_{j=1}^{n} x_j = 0\right\}$. Therefore $\mathrm{dim} \mathrm{Span}(f_1, \cdots, f_n)^0 = n - 2$.
(It is 0 when $n = 1$.)

106-11

Let W_1, W_2 be subspaces of a finite-dimensional vector space V.
(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.
(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.
Proof: (a) If $f \in (W_1 + W_2)^0$ then $\forall x \in W_1 \cup W_2$, $f(x) = 0$ so $f \in W_1^0 \cap W_2^0$.
If $f \in W_1^0 \cap W_2^0$ then for any $x \in W_1 + W_2$ let $x = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$, then $f(x) = f(w_1) + f(w_2) = 0$ so $f \in (W_1 + W_2)^0$.
(b) If $f \in W_1^0 + W_2^0$ let $f = f_1 + f_2$ where $f_1 \in W_1^0, f_2 \in W_2^0$ then $f_1, f_2 \in (W_1 \cap W_2)^0$, so $f = f_1 + f_2 \in (W_1 \cap W_2)^0$.
If $f \in (W_1 \cap W_2)^0$, take a base $\{\alpha_1, \cdots, \alpha_k\}$ of $W_1 \cap W_2$, and extend it to a base $\{\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_m\}$ of W_1, and a base $\{\alpha_1, \cdots, \alpha_k, \gamma_1, \cdots, \gamma_n\}$ of W_2. Then $\{\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_m, \gamma_1, \cdots, \gamma_n\}$ are linearly independent, so we can extend it to a base $\{\alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_m, \gamma_1, \cdots, \gamma_n\}$ of V. Let $f_1 \in \mathcal{L}(V, F)$ such that $f_1(\gamma_1) = f(\gamma_1), \cdots, f_1(\gamma_n) = f(\gamma_n)$, and $f_1(\alpha_i) = f_1(\beta_j) = 0$ for all i, j. Then $f_1 \in W_1^0$ and $f_2 = f - f_1 \in W_2^0$, so $f = f_1 + f_2 \in W_1^0 + W_2^0$.

106-13

Let F be a subfield of $\mathbb{C}$ and V a vector space over F. Suppose that $f, g \in \mathcal{L}(V, F)$ such that $h(\alpha) = f(\alpha)g(\alpha) \in \mathcal{L}(V, F)$. Prove that $f = 0$ or $g = 0$.
Solution: Note that $h(-\alpha) = (-f(\alpha))(-g(\alpha)) = h(\alpha)$ so $h = 0$. $V = \mathrm{Ker}h = \mathrm{Ker}f \cup \mathrm{Ker}g$ so either $\mathrm{Ker}f = V$ or $\mathrm{Ker}g = V$, which implies $f = 0$ or $g = 0$.

106-14

Let F be a field with characteristic zero and V a finite-dimensional vector space over F. If $\alpha_1, \cdots, \alpha_m$ are finitely many non-zero vectors in V, prove that there is a linear functional f on V such that $f(\alpha_i) \neq 0$.
Proof: Let $W_k = \{\alpha_k\}^0$, then $W_k \subsetneq V^*$, so $\bigcup_{k=1}^n W_k \neq V^*$ (which was proved in previous problems). Therefore there exists $f \in V^* \backslash \bigcup_{k=1}^n W_k$, then $f(\alpha_k) \neq 0$.

111-1

Let $n \in \mathbb{N}$ and F be a field. Let $W = \left\{(x_1, \cdots, x_n) \in F^n : \sum_{k=1}^n x_k = 0\right\}$.
(a) Prove that W^0 consist of linear functionals of the form

$$f(x_1, \cdots, x_n) = c\sum_{j=1}^n x_j.$$

(b) If $char(F) = 0$, show that the dual space W^* of W can be 'naturally' identified with the linear functionals

$$f(x_1, \cdots, x_n) = c_1x_1 + \cdots + c_nx_n$$

on F^n which satisfy $c_1 + \cdots + c_n = 0$.
Proof: (1) Clearly $f(x_1, \cdots, x_n) = c\sum_{j=1}^n x_j \implies f \in W^0$. Since $\dim W = n - 1$, $\dim W^0 = 1$ so $W^0 = \left\{f(x_1, \cdots, x_n) = c\sum_{j=1}^n x_j\right\}$.
(2) Consider $\pi : F^n \to W, (x_1, \cdots, x_n) \mapsto (x_1 - c, \cdots, x_n - c), c = \sum_{j=1}^n x_j/n$. For any $f \in W^*$, let $\tilde{f} = f \circ \pi \in (F^n)^*$, then we can let $c_k = \tilde{f}(e_k)$, such that $f(x_1, \cdots, x_n) = \sum_{k=1}^n c_kx_k$.

111-3

Let S be a set, F a field, and $V(S; F)$ the space of all functions from S into F. Let W be any $n-$dimensional subspace of $V(S; F)$. Show that there exists points $x_1, \cdots, x_n$ in S and functionals $f_1, \cdots, f_n$ in W such that $f_i(x_j) = \delta_{ij}$.
Proof: Consider the evaluation functions $E_x \in W^*, f \mapsto f(x)$, we need to find $x_1, \cdots, x_n$ such that E_{x_k} form a basis of W^*, then $f_1, \cdots, f_n$ will be the dual basis (viewing $W^{**} = W$).
Since $W \neq \{0\}$, take $g \neq 0$ and $g(x_1) \neq 0$. Define $W_1 = \{x_1\}^0$, then $\dim W_1 = n - 1$. Likewise take $g \in W_1 \backslash \{0\}$ and $g(x_2) \neq 0$, and define $W_2 = \{x_1, x_2\}^0$. Continue the process to obtain $x_1, \cdots, x_n$, and $W_1, \cdots, W_n$ such that $\dim W_k = n - k$, then the map $\Phi : V(S; F) \to F^n, f \mapsto (f(x_1), \cdots, f(x_n))$ is injective, since $\mathrm{Ker}\Phi = \{x_1, \cdots, x_n\}^0 = \{0\}$. Hence E_{x_k} form a basis of W^*.