2025/9/29

讲义习题2.3:1-6; 教材48-49页4,5,9.

2.3.1

Let V be a linear space on F, $S \subset V$ is linearly independent, $\alpha \in V \backslash S$. Prove that $S \cup \{a\}$ is linearly independent iff $\alpha \notin \mathrm{Span}(S)$.

Proof: $S \cup \{\alpha\}$ is linearly dependent iff there α can be written as the linear combination of S (since S is linearly independent), iff $\alpha \in \mathrm{Span}(S)$.

2.3.2

Suppose $S_1, S_2, S_3 \subset V$, $W_i = \mathrm{Span}(S_i)$, and $S_1 \cup S_2 \cup S_3$ is linearly independent, prove that

$$W_1 \cap (W_2 + W_3) = (W_1 \cap W_2) + (W_1 \cap W_3).$$

Proof: Since $S_1 \cup S_2 \cup S_3$ is linearly independent, both sides are $\{0\}$.

2.3.3&2.3.4

Give another proof of the theorem: if $S \subset V$ is linearly independent, $T \subset V$ generates V and both are finite, then $|S| \leqslant |T|$.

Proof: Let $S = \{u_1, u_2, \cdots, u_n\}$ and $T = \{v_1, v_2, \cdots, v_m\}$. Consider the following algorithm that maintains a list that generates V: Step1: Note that $u_1 \in \mathrm{Span}\langle v_1, \cdots, v_m \rangle$ so $u_1, v_1, \cdots, v_m$ are linearly dependent, and generates V. Take the first $v_k \in \mathrm{Span}\langle u_1, v_1, \cdots, v_{k-1} \rangle$, and remove it from the list. Rename the elements of the list as $u_1, v_1, \cdots, v_{m-1}$.

Step$k$: Suppose the list is in the form $u_1, \cdots, u_{k-1}, v_1, \cdots, v_{m-k+1}$, then add u_k to the list. Since $u_1, \cdots, u_{k-1}, v_1, \cdots, v_{m-k+1}$ generates V, the new list $u_1, \cdots, u_k, v_1, \cdots, v_{m-k+1}$ is linearly dependent, and for any $i \leqslant k$, $u_i \notin \mathrm{Span}\langle u_1, \cdots, u_{i-1} \rangle$. Hence there exists $v_j \in \mathrm{Span}\langle u_1, \cdots, u_k, v_1, \cdots, v_{j-1} \rangle$ and we remove it from the list.

If $n > m$ then after step$m$ the list becomes $u_1, \cdots, u_m$ which generates V, but $u_1, \cdots, u_n$ is linearly independent, leading to contradiction.

2.3.5

Suppose $m, n \in \mathbb{N}$, $a_{ij}, b_i \in F$. Consider the equation

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ \cdots \\ a_{m_1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

(1) Suppose $m < n$, and $b_1 = \cdots = b_m = 0$. Prove that there is a non-trivial solution.

Proof: Consider the linear map $f : F^n \to F^m$, $(x_i)_{1 \leqslant i \leqslant n} \mapsto \left(y_j = \sum_{i=1}^n a_{ij}x_i\right)_{1 \leqslant j \leqslant m}$, then $\mathrm{dimKer} f = \mathrm{dim} F^n - \mathrm{dimIm} f > 0$, so $\mathrm{Ker} f$ contains a non-trivial solution.

(2) Suppose $m > n$. Prove that there exists $b_1, \cdots, b_m$ such that the equation has no solutions.

Proof: Consider again the linear map in (1), since $\mathrm{dimIm} f \leqslant \mathrm{dim} F^n < m$, f can't be surjective.

2.3.6

Suppose V is a finite dimensional linear space and $n = \mathrm{dim}V$. Let $M_1, \cdots, M_{n-1}, N_1, \cdots, N_{n-1}$ be subspaces of V, such that $\mathrm{dim}M_i = \mathrm{dim}N_i = i$ for $1 \leqslant i \leqslant n-1$, and $M_1 \subset M_2 \subset \cdots \subset M_{n-1}$, $N_1 \subset \cdots \subset N_{n-1}$. Prove that there is a base S of V such that every one of these $2n-2$ subspaces is spanned by a subset of S.
Proof: Use induction on n. The case $n=2$ is trivial (take $v_1 \in M_1 \backslash \{0\}, v_2 \in N_2 \backslash \{0\}$ then $\{v_1, v_2\}$ is suitable). Suppose the proposition holds for $n-1$.
Case1: $M_{n-1} = N_{n-1}$. Then use the induction hypothesis on the linear space M_{n-1} and $M_1 \subset M_2 \subset \cdots \subset M_{n-2}$, $N_1 \subset \cdots \subset N_{n-2}$, we obtain a base $\{v_1, \cdots, v_{n-1}\}$ of M_{n-1}. Add $v_n \in V \backslash M_{n-1}$ then the base $\{v_1, \cdots, v_n\}$ is suitable.
Case2: $M_{n-1} \neq N_{n-1}$. Then

$$\mathrm{dim}(M_{n-1} \cap N_{n-1}) = \mathrm{dim}M_{n-1} + \mathrm{dim}N_{n-1} - \mathrm{dim}(M_{n-1} + N_{n-1}) = n-2.$$

Take $v \in V \backslash (M_{n-1} \cup N_{n-1})$ (the existence of v was proved in previous exercises), and consider the quotient space $V' = V/\mathrm{Span}(v)$ with the canonical projection $\pi : V \to V'$. Since $v \notin M_{n-1}$, $M_j \cap \mathrm{Span}(v) = \emptyset$ so π is an isomorphism on M_j, N_j. Let $M_j' = \pi(M_j)$ then $\mathrm{dim}M_j' = \mathrm{dim}M_j = j$, so in the $n-1$ dimensional linear space V', we have $M_1' \subset \cdots \subset M_{n-2}'$, $N_1' \subset \cdots \subset N_{n-2}'$ and $\mathrm{dim}M_j' = \mathrm{dim}N_j' = j$. Apply the induction hypothesis to get a base $\{\overline{v_1}, \cdots, \overline{v_{n-1}}\}$ of V' such that every M_j', N_j' is spanned by some of them. Take any $v_j \in \pi^{-1}(\overline{v_j})$, we show that $\{v_1, \cdots, v_{n-1}, v\}$ is the desired base (it is clearly a base of V). Suppose $M_j' = \mathrm{Span}(\overline{v_1}, \cdots, \overline{v_j})$, then for any $u \in M_j$, $\pi(u) = \sum_{t=1}^{j} c_t \overline{v_t} = \sum_{t=1}^{j} c_t \pi(v_t)$ so $u - \sum_{t=1}^{j} c_t v_t \in \mathrm{Ker}\pi$ which implies $u \in \mathrm{Span}(v_1, \cdots, v_j, v)$. $u \in M_j$ implies $v \notin \mathrm{Span}(u, v_1, \cdots, v_j)$ hence $u \in \mathrm{Span}(v_1, \cdots, v_j)$. Since M_j and $\mathrm{Span}(v_1, \cdots, v_j)$ both have dimension j, $M_j = \mathrm{Span}(v_1, \cdots, v_j)$. Therefore the proposition holds for n, and by induction it is true for any V having finite dimension.

48-4

Show that the vectors

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$$

form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$.
Proof: Note that

$$e_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3, e_2 = \frac{-\alpha_1 + \alpha_2 + \alpha_3}{5}, e_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.$$

Hence $\mathrm{Span}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{R}^3$ which is of dimension 3, so they are a basis of $\mathbb{R}^3$.

48-5

Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.
Solution: Consider $(1, 0, 0), (1, 1, 0), (0, 1, 0)$. Then $(1, 0, 0) + (-1) \cdot (1, 1, 0) + (0, 1, 0) = 0$, but each pair are linearly independent.

49-9

Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, γ are linearly independent, prove that $(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are linearly independent.
Proof: If there exists $c_1, c_2, c_3 \in F$ not identically zero, such that $c_1(\alpha+\beta)+c_2(\beta+\gamma)+c_3(\gamma+\alpha)=0$, then $(c_1+c_3)\alpha+(c_1+c_2)\beta+(c_2+c_3)\gamma=0$. Since α, β, γ are linearly independent, $c_1+c_2=c_2+c_3=c_3+c_1=0$, hence $c_1=c_2=c_3=0$, a contradiction.