2025/10/22

第四次作业 （2025-10-29 课间交 4：10之前）
周民强《实变函数论(第三版)》
P88 思考题 3, 5, P107 思考题 3,P125-127 习题3：2,3,6,7,15

88-3

Find an unmeasurable set $W \subset [0,1]$ such that $W - W$ has no interior.
Proof: Consider the unmeasurable set W obtained by choosing a representative of each element of $[0,1]/Q$. If $B(x,\varepsilon) \subset W - W$ for some $x \in \mathbb{R}$ and $\varepsilon > 0$, then there exists $q \in B(x,\varepsilon) \cap \mathbb{Q}$, and $w_1, w_2 \in W$ such that $q = w_1 - w_2$. By the construction of W, $w_1 = w_2$, so $B(x,\varepsilon) \cap \mathbb{Q} \subset \{0\}$ which is clearly false.

88-5

Suppose $E \subset \mathbb{R}^n$, and for any $F \subset E \subset G$, where F closed and G open, we have

$$\sup_F \{m(F)\} < \inf_G \{m(G)\},$$

prove that E is not measurable.
Proof: If E is measurable, then $\sup_F \{m(F)\} = m(E) = \inf_G \{m(G)\}$, so E is not measurable.

107-3

If $\{f_k\}$ is a sequence of measurable functions on $E \subset \mathbb{R}^n$, then the points x such that $f_k(x)$ converges form a measurable set.
Proof: Let $A = \{x \in E : f_k(x) \text{ is Cauchy}\}$. Then

$$A = \bigcap_{p \geqslant 1} \bigcup_{N \geqslant 1} \bigcap_{n \geqslant N} \bigcap_{m \geqslant n} \{x : |f_n(x) - f_m(x)| < 1/p\}$$

clearly every set $\{x : |f_n(x) - f_m(x)| < 1/p\}$ is measurable, so A is measurable.

126-2

Suppose $z = f(x,y)$ is continuous on $\mathbb{R}^2$, $g_1(x)$, $g_2(x)$ are real-valued measurable functions on $[a,b] \subset \mathbb{R}$. Prove that $F(x) = f(g_1(x), g_2(x))$ is a measurable function on $[a,b]$.
Proof: For any t, $\Omega = f^{-1}([t,\infty))$ is closed, so for any $n \geqslant 1$, there exists a countable subset $J_n \subset \Omega$ such that for $z = (x,y)$, $\Omega \subset \bigcup_{z \in J_n} (x - 1/n, x + 1/n) \times (y - 1/n, y + 1/n) \subset \Omega_n$, where $\Omega_n = \{z : d(z,\Omega) < 2/n\}$. Hence

$$F^{-1}([t,\infty)) = \bigcap_{n \geqslant 1} \bigcup_{z \in J_n} g_1^{-1}((x - 1/n, x + 1/n)) \cap g_2^{-1}((y - 1/n, y + 1/n)).$$

Since $g_1^{-1}((x - 1/n, x + 1/n))$ and $g_2^{-1}((y - 1/n, y + 1/n))$ are measurable, so is $F^{-1}([t,\infty))$.

126-3

Suppose $f'_+(x)$ exists on $[a,b)$, prove that $f'_+(x)$ is a measurable function on $[a,b)$.
Proof: Since $f'_+(x)$ exists, f is right-continuous which implies it is measurable, so every function $n(f(x+1/n) - f(x))$ is measurable, hence $f'_+(x) = \lim_{n\to\infty} n(f(x+1/n) - f(x))$ is measurable.

126-6

Suppose $\{f_k(x)\}$ is a sequence of real-valued measurable functions on $E \subset \mathbb{R}^n$, and $m(E) < \infty$. Prove that $\lim_{n\to\infty} f_n(x) = 0$ a.e $x \in E$ iff for any $\varepsilon > 0$

$$\lim_{N\to\infty} m(\{x \in E : \sup_{n\geqslant N} |f_n(x)| \geqslant \varepsilon\}) = 0.$$

Proof: Let $B_{n,p} = \left\{x \in E : |f_n(x)| < \frac{1}{p}\right\}$, and $C_{n,p} = B_{n,p}^C$, then

$$\{x : f_n(x) \to 0\} = \bigcap_{p\geqslant 1} \bigcup_{N\geqslant 1} \bigcap_{n\geqslant N} B_{n,p}$$

so $\lim_{n\to\infty} f_n(x) = 0$ a.e. iff

$$0 = m\left(\bigcup_{p\geqslant 1} \bigcap_{N\geqslant 1} \bigcup_{n\geqslant N} C_{n,p}\right) \iff \forall p \geqslant 1, m\left(\bigcap_{N\geqslant 1} \bigcup_{n\geqslant N} C_{n,p}\right) = 0.$$

Since $m(E) < \infty$,

$$m\left(\bigcap_{N\geqslant 1} \bigcup_{n\geqslant N} C_{n,p}\right) = \lim_{N\to\infty} m\left(\bigcup_{n\geqslant N} C_{n,p}\right) = \lim_{N\to\infty} m(\{x \in E : \sup_{n\geqslant N} |f_n(x)| \geqslant 1/p\})$$

hence it is equivalent to $\lim_{N\to\infty} m(\{x \in E : \sup_{n\geqslant N} |f_n(x)| \geqslant \varepsilon\}) = 0$ for any $\varepsilon > 0$.

126-7

Suppose $f(x), f_1(x), \cdots, f_n(x), \cdots$ are measurable functions on $[a, b]$ that are almost everywhere finite, and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., prove that there exists $E_n \subset [a, b]$, such that

$$m\left([a, b] \backslash \bigcup_{n\geqslant 1} E_n\right) = 0,$$

and $\{f_k\}$ converges uniformly to f on every subset E_n.
Proof: By Egorov theorem, for any n, we can find a compact subset $E_n \subset E = [a, b]$ such that $m([a, b]\backslash E_n) < 1/n$ and $\{f_k\} \to f$ uniformly on K. Clearly $m\left([a, b]\backslash \bigcup_{n\geqslant 1} E_n\right) = 0$.

127-15

Suppose $\{f_n(x)\}$ are measurable functions on $[a, b]$, and $f : [a, b] \to \mathbb{R}$. If for any $\varepsilon > 0$,

$$\lim_{n\to\infty} m^*(\{x \in [a, b] : |f_n(x) - f(x)| > \varepsilon\}) = 0,$$

is f a measurable function?
Proof: $\{f_n\}$ converges in measure to f, so by Riestz Lemma, there is a sub sequence $\{f_{n_k}\}$ that converges to f almost everywhere. Hence f is measurable.
Riestz Lemma: We still let $B_{n,p} = \{x \in E : |f_n(x) - f(x)| < 1/p\}$ and $C_{n,p} = B_{n,p}^C$, then $\{f_n\} \to f$ in measure m iff $\lim_{n\to\infty} m(C_{n,p}) = 0$ for every p. And likewise $\{f_{n_k}\} \to f$ a.e. if $\lim_{N\to\infty} m\left(\bigcup_{k\geqslant N} C_{n_k,p}\right) = 0$ for every p. So there is a sub sequence n_k such that $m(C_{n_k,k}) < 2^{-k}$. Note that $C_{n,p} \subset C_{n,p+1}$, then for $N > p$,

$$m\left(\bigcup_{k\geqslant N} C_{n_k,p}\right) < \sum_{k=N}^{\infty} 2^{-k} < 2^{1-N} \to 0.$$

Therefore $\{f_{n_k}\} \to f$ almost everywhere.