2025/9/18

讲义习题2.2:1,2; 教材40页6-9.

2.2.1

Suppose W_1, W_2 are non-trivial subspaces of V, prove that $W_1 \cup W_2 \neq V$.

Proof: The cases $W_1 \subset W_2$ and $W_2 \subset W_1$ is trivial. Otherwise take $a \in W_1 \backslash W_2$ and $b \in W_2 \backslash W_1$, then $a + b \in V$, but $a + b \notin W_1$ (otherwise $b = (a+b) + (-1) \cdot a \in W_1$) and $a + b \notin W_2$. Hence $W_1 \cup W_2 \neq V$.

2.2.2

Suppose $N \subset M \subset V$ are linear spaces. Prove that for any subspace $W \subset V$,

$$M \cap (N + W) = N + (M \cap W).$$

Proof: If $a \in M \cap (N + W)$ then write $a = n + w$ where $n \in N \subset M, w \in W$. $a, n \in M$ so $w \in M \cap W$, and $a = n + w \in N + (M \cap W)$. If $a \in N + (M \cap W)$ then write $a = n + w$ where $n \in N$ and $w \in M \cap W$, then $n, w \in M$ implies $a = n + w \in M$, and $n \in N, w \in W$ implies $a = n + w \in N + W$, hence $a \in M \cap (N + W)$. Therefore $M \cap (N + W) = N + (M \cap W)$.

40-6

(a) Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

(b) Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$ or $\{0\}$ or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$.

(c) Describe the subspaces of $\mathbb{R}^3$.

Proof: For any subspace W of $\mathbb{R}^n$, let $m = \dim W$ and take a base $a_1, a_2, \cdots, a_m$ of W, then $W = \operatorname{Span}(a_1, \cdots, a_m)$.

When $n = 1$, $m \in \{0, 1\}$ leading to $\{0\}$ and $\mathbb{R}^1$.

When $n = 2$, $m = 0, 2$ leads to $\{0\}$ and $\mathbb{R}^2$, and when $m = 1$, $W = \operatorname{Span}(v) = \{c \cdot v : c \in \mathbb{R}\}$.

40-7

Let W_1, W_2 be subspaces of a vector space V such that the set-theoretic union of W_1, W_2 is also a subspace. Prove that $W_1 \subset W_2$ or vice versa.

Proof: It is already shown in Exercise 2.1.1 that for $a \in W_1 \backslash W_2$ and $b \in W_2 \backslash W_1$, $a + b \notin W_1 \cup W_2$ so $W_1 \cup W_2$ is not a subspace.

40-8

Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$; let V_e be the subset of even functions and V_o that of odd functions.

(a) Prove that V_e, V_o are subspaces of V.

(b) Prove that $V_e + V_o = V$.

(c) Prove that $V_e \cap V_o = \{0\}$.

Proof: (a) is trivial, since $f, g \in V_e$ implies $f + g, cf \in V$. (b) For any $f \in V$, $f = g + h$ where $g(x) = (f(x) + f(-x))/2 \in V_e$ and $h(x) = (f(x) - f(-x))/2 \in V_o$. (c) If $f \in V_e \cap V_o$ then $f(x) = f(-x) = -f(-x)$ so $f(x) = 0$ for any $x \in \mathbb{R}$.

40-9

Let $W_1, W_2 \subset V$ such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$, there are unique vectors $\alpha_1 \in W_1, \alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.
Proof: Existence comes from $V = W_1 + W_2$. Uniqueness: If $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ where $\alpha_i, \beta_i \in W_i$, then $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 \in W_1 \cap W_2 = \{0\}$ hence $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.