BMF 1-1

Prove the correctness of the Horner's rule:

- Show that

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \iff (\otimes c) \cdot \oplus / = \oplus / \cdot (\otimes c) * \cdot$$

- Show that

$$f = \oplus / \cdot \otimes / * \cdot \text{tails}$$

satisfy the equations $f\,[] = e$ and $f(x \mathbin{+\!\!+} [a]) = f\,x \odot a$.
Proof: (a) $\Longleftarrow$ is trivial, by applying both sides to $[a, b]$ we obtain

$$(a \oplus b) \otimes c = (\otimes c)(a \oplus b) = (\otimes c) \cdot \oplus / [a, b] = \oplus / \cdot (\otimes c) * [a, b] = \oplus / [a \otimes c, b \otimes c] = (a \otimes c) \oplus (b \otimes c).$$

$\Longrightarrow$ can be proved by induction: The base $n = 1$ is trivial. If it holds for arrays of length n, then for any array $[a_1, \cdots, a_{n+1}]$,

$$\begin{aligned} &(\otimes c) \cdot \oplus / [a_1, \cdots, a_{n+1}] = (a_1 \oplus \cdots \oplus a_{n+1}) \otimes c = ((a_1 \oplus \cdots \oplus a_n) \otimes c) \oplus (a_{n+1} \otimes c) \\ =&((\otimes c) \cdot \oplus / [a_1, \cdots, a_n]) \oplus (a_{n+1} \otimes c) = (\oplus / \cdot (\otimes c) * [a_1, \cdots, a_n]) \oplus ((\otimes c) * [a_{n+1}]) \\ =& \oplus / \cdot (\otimes c) * [a_1, \cdots, a_{n+1}]. \end{aligned}$$

Hence it holds for any array.
(b) $f\,[] = \oplus / \cdot \otimes / * \cdot \text{tails}[] = \oplus / \cdot \otimes / * [\,[\,]\,] = \oplus / [e] = e$, and

$$\begin{aligned} f(x \mathbin{+\!\!+} [a]) &= \oplus / \cdot \otimes / * \cdot \text{tails}(x \mathbin{+\!\!+} [a]) = \oplus / \cdot \otimes / * \cdot ((\mathbin{+\!\!+}[a]) * \cdot \text{tails} x \mathbin{+\!\!+} [[]]) \\ &= \oplus / \cdot ((\otimes / * \cdot (+ + [a]) * \cdot \text{tails} x) \mathbin{+\!\!+} (\otimes / * [[]])) \\ &= \oplus / \cdot ((\otimes a) * \cdot \otimes / * \cdot \text{tails} x \mathbin{+\!\!+} ([])) \\ &= ((\otimes a) \cdot (\oplus / \cdot \otimes / * \cdot \text{tails} x)) \oplus e \\ &= (fx \otimes a) \oplus e = fx \odot a \end{aligned}$$

BMF 1-2

Code the derived linear algorithm for mss in Haskell:

```haskell
mss :: [Integer] -> Integer
mss xs = snd $ foldl f (0,0) xs
         where f = \(s,ms) x-> let s' = if s+x > 0 then s+x else 0
                                   ms' = max s' ms in (s',ms')
```

BMF 1-3

Prove the Segment Decomposition Theorem:
Suppose S, T are defined by $S = \oplus / \cdot f * \cdot \text{segs}$, $T = \oplus / \cdot f * \cdot \text{tails}$. If T can be expressed in the form $T = h \cdot \odot \not\to_e$, then we have $S = \oplus / \cdot h * \cdot \odot \not\!\!\!\to_e$.
Proof:

$$
\begin{aligned}
S &= \otimes/ \cdot f* \cdot \mathrm{segs} = \otimes/ \cdot f* \cdot +\!\!\!+/ \cdot\ \mathrm{tails}* \cdot \mathrm{inits} \\
(\text{map promotion}) &= \otimes/ \cdot (+\!\!\!+/ \cdot f**) \cdot \mathrm{tails}* \cdot \mathrm{inits} \\
(\text{reduce promotion}) &= (\otimes/ \cdot \otimes/*) \cdot f** \cdot \mathrm{tails}* \cdot \mathrm{inits} \\
&= \otimes/ \cdot \otimes/* \cdot f** \cdot \mathrm{tails}* \cdot \mathrm{inits} \\
(\text{map distribution}) &= \otimes/ \cdot (\otimes/ \cdot f* \cdot \mathrm{tails})* \cdot \mathrm{inits} \\
(T = h \cdot \odot\!\not\rightarrow_e) &= \otimes/ \cdot (h \cdot \odot\!\not\rightarrow_e)* \cdot \mathrm{inits} \\
(\text{map distribution}) &= \otimes/ \cdot h* \cdot (\odot\!\not\rightarrow_e)* \cdot \mathrm{inits} \\
&= \otimes/ \cdot h* \cdot \odot\ /\!\!\!\not\rightarrow_e
\end{aligned}
$$