讲义习题5.4:1,2,4,6,7.

162-3

An $n \times n$ matrix A over a field F is skew-symmetric if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det A = 0$.
Proof: Note that $\det A = \det A^t = \det(-A) = (-1)^n \det A = -\det A$, so $\det A = 0$.

162-4

An $n \times n$ matrix A over a field F is called orthogonal if $AA^t = I$. If A is orthogonal, show that $\det A = \pm 1$. Give an example of an orthogonal matrix for which $\det A = -1$.
Proof: $A = \mathrm{diag}(1, -1)$ is orthogonal, but $\det A = -1$.
If A is orthogonal, then $1 = \det I = \det AA^t = \det A \det A^t = (\det A)^2$, so $\det A = \pm 1$.

162-5

An $n \times n$ matrix A over $\mathbb{C}$ is said to be unitary if $AA^* = I$. If A is unitary, show that $|\det A| = 1$.
Proof: Note that $|Av|^2 = \langle Av, Av \rangle = \langle v, A^*Av \rangle = \langle v, v \rangle = |v|^2$, so any eigenvalue λ of A satisfy $|\lambda| = 1$, so $|\det A| = 1$.

5.4.2

丘维生（第二版）26-1(2);35-1(3),2(1),3(1),4(2)

26-1(2)

Calculate the determinant
$$\begin{vmatrix}
0 & 0 & \cdots & 0 & a{1} \
0 & 0 & \cdots & a{2} & 0 \
 & & \cdots \
0 & a{n-1} & \cdots & 0 & 0 \
a{n} & 0 & \cdots & 0 & 0
\end{vmatrix}

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\begin{vmatrix}
1 & 0 & -3 & 2 \
-4 & -1 & 0 & -5 \
2 & 3 & -1 & -6 \
3 & 3 & -4 & 1
\end{vmatrix}

$$Solution:$$

\begin{vmatrix}
1 & 0 & -3 & 2\-4 & -1 & 0 & -5\2 & 3 & -1 & -6\3 & 3 & -4 & 1
\end{vmatrix}=\begin{vmatrix}
1 & 0 & -3 & 2 \

0 & 5 & -2 & -17 \
0 & 3 & 5 & -10 \
0 & 0 & 0 & 5
\end{vmatrix}=1\cdot(5\cdot 5+2\cdot 3)\cdot 5=155.

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\begin{vmatrix}
a & 1 & 1 & \cdots & 1 \
1 & a & 1 & \cdots & 1 \
 & & & \cdots \
1 & 1 & 1 & \cdots & a
\end{vmatrix}

$$Solution: Notethat$$

\begin{vmatrix}
a & 1 & 1 & \cdots & 1 \
1 & a & 1 & \cdots & 1 \
 & & & \cdots & \
1 & 1 & 1 & \cdots & a
\end{vmatrix}=\begin{vmatrix}
a+(n-1) & 1 & \cdots & 1 \
a+(n-1) & a & \cdots & 1 \
 & & \cdots \
a+(n-1) & 1 & \cdots & a
\end{vmatrix}=(a+n-1)\begin{vmatrix}
1 & 1 & \cdots & 1 \
0 & a-1 & \cdots & 0 \
 & & \cdots \
0 & 0 & \cdots & a-1
\end{vmatrix}

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\begin{vmatrix}
a*{1}-b*{1} & b*{1}-c*{1} & c*{1}-a*{1} \
a*{2}-b*{2} & b*{2}-c*{2} & c*{2}-a*{2} \
a*{3}-b*{3} & b*{3}-c*{3} & c*{3}-a*{3}
\end{vmatrix}=0.

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\begin{vmatrix}
a*{1}+b*{1} & a*{1}+b*{2} & \cdots & a*{1}+b*{n} \
a*{2}+b*{1} & a*{2}+b*{2} & \cdots & a*{2}+b*{n} \
 & & \cdots & \
a*{n}+b*{1} & a*{n}+b*{2} & \cdots & a*{n}+b*{n}
\end{vmatrix}

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T*{x}(u+v\omega+w\omega*{2})=(au+2bw+2cv)+(av+bu+2cw)\omega+(aw+bv+cu)\omega*{2},*
$$
and the matrix of $T{x}*is*\begin{pmatrix}a & 2c & 2b\b & a & 2c\c & b & a\end{pmatrix}*anditsdeterminantis*
a{3}+2b{3}+4c^{3}-6abc$.

5.4.7

Suppose V is a finite-dimensional complex linear space, $T \in \mathcal{L}(V)$. Applying the forgetful functor to T, we obtain $T_{\mathbb{R}} \in \mathcal{L}(V_{\mathbb{R}})$. Prove that $\det T_{\mathbb{R}} = |\det T|^2$.
Proof: For any $\omega \in \Lambda^n(V)$ where $n = \dim V_{\mathbb{C}}$, $T^*\omega = (\det T)\omega$. If $e_1, \cdots, e_n$ form a basis of $V_{\mathbb{C}}$, then $e_1, ie_1, \cdots, e_n, ie_n$ form a basis of $V_{\mathbb{R}}$. Let $z_1, \cdots, z_n$ be the dual basis, then $z_k = x_k + iy_k$ so $x_1, y_1, \cdots, x_n, y_n$ is the dual basis of $V_{\mathbb{R}}^*$.
Consider $\overline{\omega}(x_1, \cdots, x_n) = \overline{\omega(x_1, \cdots, x_n)}$, notice that $T^*\overline{\omega}(x_1, \cdots, x_n) = \overline{\omega}(Tx_1, \cdots, Tx_n) = \overline{T^*\omega(x_1, \cdots, x_n)} = \overline{\det T\omega(x_1, \cdots, x_n)}$ so $T^*\overline{\omega} = \overline{(\det T)}\omega$.
Let $\Omega = \omega \wedge \overline{\omega}$, note that $z_k \wedge \overline{z_k} = (x_k + iy_k) \wedge (x_k - iy_k) = -2ix_k \wedge y_k$, so $\Omega \in \Lambda^{2n}(V_{\mathbb{R}})$ and $T^*\Omega = \det T_{\mathbb{R}} \cdot \Omega$. By $T^*\Omega = (T^*\omega) \wedge (T^*\overline{\omega}) = ((\det T)\omega) \wedge (\overline{\det T} \cdot \overline{\omega}) = |\det T|^2\Omega$ we obtain $\det T_{\mathbb{R}} = |\det T|^2$.