2025/10/9
教材10-11页2,7,8; 16页7,9. 求F_p^n中二维子空间的个数.

10-2

If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

find all solutions of $AX = 0$ by row-reducing A.
Solution:

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 8 & 2 \\ 0 & 7 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

hence the three rows are linearly independent ($8 \cdot 1 - 7 \cdot 2 \neq 0$), and the only solution is $(0, 0, 0)$.

11-7

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.
Solution: To interchange rows i, j, denote the $k-$th row by r_k. Let $r_i \to r_i + r_j$, $r_j \to r_j - r_i$, $r_j \to -r_j$, $r_i \to r_i - r_j$, then these operations interchange r_i, r_j.

11-8

Consider the system of equations $AX = 0$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following
(a) If every entry if A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.
(b) If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.
(c) If $ad - bc = 0$ and some entry of A is not 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution iff there is a scalar y such that $x_1 = yx_1^0, x_2 = yx_2^0$.
Proof: (a) is trivial. (b): If $ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$, then $0 = c(ax_1 + bx_2) - a(cx_1 + dx_2) = -(ad - bc)x_2$ so $x_2 = 0$. Likewise $x_1 = 0$ so the only solution is $(0, 0)$.
(c) Proof: Suppose $b \neq 0$, let $(x_0, y_0) = (b, -a)$, then the solutions of $AX = 0$ are exactly $\mathrm{Span}(x_0, y_0)$.

16-7

Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_6 &= -2 \\ 2x_1 + 0x_2 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

Solution: Apply elementary row operations to the matrix:

$$\begin{pmatrix}2 & -3 & -7 & 5 & 2 & -2\\ 1 & -2 & -4 & 3 & 1 & -2\\ 2 & 0 & -4 & 2 & 1 & 3\\ 1 & -5 & -7 & 6 & 2 & -7\end{pmatrix} \to \begin{pmatrix}0 & 1 & 1 & -1 & 0 & 2\\ 1 & -2 & -4 & 3 & 1 & -2\\ 0 & 4 & 4 & -4 & -1 & 7\\ 0 & -3 & -3 & 3 & 1 & -5\end{pmatrix}$$
$$\to \begin{pmatrix}0 & 1 & 1 & -1 & 0 & 2\\ 1 & 0 & -2 & 1 & 1 & 2\\ 0 & 0 & 0 & 0 & -1 & -1\\ 0 & 0 & 0 & 0 & 1 & 1\end{pmatrix} \to \begin{pmatrix}1 & 0 & -2 & 1 & 0 & 1\\ 0 & 1 & 1 & -1 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

Hence all solutions are:

$$\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 1 + 2x_3 - x_4, x_2 = 2 - x_3 + x_4, x_5 = 1\}.$$

16-9

Let

$$A = \begin{pmatrix}3 & -6 & 2 & -1\\ -2 & 4 & 1 & 3\\ 0 & 0 & 1 & 1\\ 1 & -2 & 1 & 0\end{pmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?
Proof: Since A map each base $(0, \cdots, 1, 0, \cdots, 0)$ to its corresponding column, the answer is $\mathrm{Span}\langle(3, -2, 0, 1), (-6, 4, 0, 2), (2, 1, 1, 1), (-1, 3, 1, 0)\rangle$ which is $\mathrm{Span}\langle(3, -2, 0, 1), (-1, 3, 1, 0)\rangle$.

Ex1

求 F_p^n 中二维子空间的个数.
Solution: Any subspace $W \subset F_p^n$ such that $\dim W = 2$ corresponds to a $2 \times n$ non-zero row-reduced echelon matrix. Hence there are

$$\sum_{1\leqslant i<j\leqslant n} p^{n-i-1} \cdot p^{n-j} = \sum_{i=1}^{n-1} p^{n-i-1} \sum_{j=i+1}^{n} p^{n-j} = \sum_{i=1}^{n-1} p^{n-i-1} \cdot \frac{p^{n-i}-1}{p-1} = \frac{1}{p-1}\left(\sum_{i=1}^{n-1} p^{2(n-i)-1} - p^{n-i-1}\right)$$
$$= \frac{1}{p-1}\left(p\frac{p^{2(n-1)}-1}{p^2-1} - \frac{p^{n-1}-1}{p-1}\right) = \frac{(p^n-1)(p^{n-1}-1)}{(p-1)(p^2-1)}$$

distinct subspaces.
Actually, change 2 to m, and the answer becomes the Gauss binomial coefficient $\begin{bmatrix}n\\ m\end{bmatrix}_p$.