2025/10/15

作业五：【尤】**2.3**：12；**2.4**：11；**2.5**：14（6）、17（3）；**2.6**：6；**2.7**：6（说明如何选取适当的仿射标架，使坐标方程具有马鞍面的标准形式）。**回收日期：10/22/三**

2.3.12

Find the shape formed by all lines that intersect

$$l_1 : \frac{x-6}{3} = \frac{y}{2} = \frac{z-1}{2},\, l_2 : \frac{x}{3} = \frac{y}{2} = \frac{z+4}{-2}$$

and is parallel to the plane $2x+3y-5=0$.
Solution: Suppose the line is

$$l : \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

and it is on the plane $\pi : 2x+3y-t=0$. Then the intersection of π and l_1 is point $P = \left(\frac{t}{4}+3, \frac{t}{6}-2, \frac{t}{6}-1\right)$, and l is the intersection of π and the plane passing l_1 and P. Hence l is the line

$$\begin{cases} 2x+3y-t=0, \\ \left(\frac{2}{3}t+2\right)x-(t+15)y-12z-48=0. \end{cases}$$

So $t=2x+3y$, and

$$\frac{4}{3}x^2+2x-3y^2-15y-12z-48=0.$$

2.4.11

Find the curve formed by all points in $\mathbb{R}^3$ that have the same distance to planes $3x-2y-6z-4=0$ and $2x+2y-z+5=0$.
Solution: For any point (x_0, y_0, z_0), the two distances are

$$d_1 = \frac{|3x_0-2y_0-6z_0-4|}{\sqrt{3^2+2^2+6^2}} = d_2 = \frac{|2x_0+2y_0-z_0+5|}{\sqrt{2^2+2^2+1^2}}$$

i.e.

$$3|3x_0-2y_0-6z_0-4| = 7|2x_0+2y_0-z_0+5|.$$

So the curve consists of two planes $9x_0-6y_0-18z_0-12 = 14x_0+14y_0-7z_0+35$ and $9x_0-6y_0-18z_0-12+14x_0+14y_0-7z_0+35=0$, which are $5x_0+20y_0+11z_0+47=0$ and $23x_0+8y_0-25z_0+23$.

2.5.14(6)

Determine the equation of the cylinder that is parallel with $u=(-1,0,1)$ and one directrix

$$\begin{cases} x^2+y^2=z, \\ z=2x. \end{cases}$$

Proof: For any point (x,y,z) on the cylinder, there exists t such that

$$\begin{cases} (x-t)^2+y^2 & =(z+t), \\ (z+t) & =2(x-t). \end{cases}$$

Hence $t = \frac{2x-z}{3}$ so

$$\frac{(x+z)^2}{3^2} + y^2 = \frac{2(x+z)}{3} \implies x^2 + 2xz + z^2 + 9y^2 - 6x - 6z = 0.$$

2.5.17(3)

Determine the equation of the cone that has center $(0, 1, 2)$ and with one directrix

$$\begin{cases} 2x^2 + 3y^2 = 4, \\ z = 0. \end{cases}$$

Solution: For any point (x, y, z) on the cone, there exists t such that

$$\begin{cases} 2(tx)^2 + 3(t(y-1)+1)^2 & = 4, \\ t(z-2) + 2 & = 0. \end{cases}$$

Hence $t = \frac{2}{2-z}$ and we obtain

$$\frac{8x^2}{(2-z)^2} + \frac{3(2y-z)^2}{(2-z)^2} = 4 \implies 8x^2 + 3(4y^2 - 4yz + z^2) = 4(4 - 4z + z^2)$$

so the equation of the cone is $8x^2 + 12y^2 - 12yz - z^2 + 16z - 16 = 0$. (If $z = 2$ then $x = 2y - z = 0$ so $(x, y, z) = (0, 1, 2)$ is the center.)

2.6.6

Prove that of all curves that are the intersection of planes through $(0, 0, 0)$ and the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ (a > b > c > 0)$$

only two are circles, and point out their position.
Proof: Suppose the plane is $Ax + By + Cz = 0$. If $ABC \neq 0$, then the intersection is

$$\begin{cases} Ax + By + Cz = 0, \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \end{cases}$$

and every point (x, y, z) satisfy $x^2 + y^2 + z^2 = r^2$. Then $r^2 = c^2 + \left(1 - \frac{c^2}{a^2}\right)x^2 + \left(1 - \frac{c^2}{b^2}\right)y^2$, while $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(Ax+By)^2}{C^2c^2} = 1$. If (x, y, z) is a solution then $\left(-\frac{2By}{A} - x, y, z\right)$ is also a solution, but clearly $x^2 \neq (2By/A - x)^2$ for all but 2 solutions.
Clearly the circle has radius $r \in (c, a)$, so $A, C \neq 0$ (otherwise $(a, 0, 0)$ or $(0, 0, c)$ is on the circle), hence $B = 0$. Suppose $C = 1$, then $\left(1 - \frac{c^2}{a^2}\right)x^2 + \left(1 - \frac{c^2}{b^2}\right)y^2 = r^2 - c^2$ while $\left(\frac{1}{a^2} + \frac{A^2}{c^2}\right)x^2 + \frac{y^2}{b^2} = 1$, hence

$$\left(1 - \frac{c^2}{a^2}\right) \cdot \frac{1}{b^2} = \left(1 - \frac{c^2}{b^2}\right) \cdot \left(\frac{1}{a^2} + \frac{A^2}{c^2}\right) \implies \frac{1}{b^2} - \frac{1}{a^2} = A^2 \left(\frac{1}{c^2} - \frac{1}{b^2}\right)$$

So there are only two such planes $Ax + z = 0$ where

$$A = \pm\sqrt{\frac{b^{-2} - a^{-2}}{c^{-2} - b^{-2}}}.$$

2.7.6

Suppose l_1, l_2 are disjoint lines, that are not parallel to π. Prove that all lines that intersect with l_1, l_2 and is parallel to π forms a saddle surface.
Proof: Suppose $\pi : z = 0$ and $l_1 : x = y = z$, $l_2 : \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, then for any line l on the surface, it is contained in a plane $\pi' : z = ct + z_0$. The intersection point P of l_2 and π' is $(x_0 + at, y_0 + bt, z_0 + ct)$, so the equation of l is

$$\begin{cases} z = ct + z_0, \\ (z_0 - y_0 + (c-b)t)x + (x_0 - z_0 + (a-c)t)y + (y_0 - x_0 + (b-a)t)z = 0. \end{cases}$$

Hence

$$\left(z_0 - y_0 + \frac{(c-b)(z-z_0)}{c}\right)x + \left(x_0 - z_0 + \frac{(a-c)(z-z_0)}{c}\right)y + \left(y_0 - x_0 + \frac{(b-a)(z-z_0)}{b}\right)z = 0,$$

which is

$$\left(\frac{x}{c} - \frac{y_0 - x_0}{c-b}\right)((c-b)z + (bz_0 - cy_0)) = \left(\frac{y}{c} - \frac{z_0 - x_0}{c-a}\right)((c-a)(z-z_0))$$

in the form of $PQ = RS$ where P, Q, R, S are four planes, so it is a doubly ruled surface. Since all lines in the same family are parallel to a plane π, it must be a saddle surface.

Another proof: Suppose $l_1 = P \cap Q, l_2 = R \cap S$. For any line l on the surface, there is a plane π_1 containing l and l_1, and λ such that $\pi_1 = P - \lambda Q$. Likewise there exists $\pi_2 = R - \mu S$. Hence for any point $M(x, y, z)$ on line l, $P = \lambda Q$ and $R = \mu S$.
Suppose the normal vectors of P, Q, R, S are n_P, n_Q, n_R, n_S, then $n_1 = n_P - \lambda n_Q$ and $n_2 = n_R - \mu n_S$, so $g \cdot n_\pi = 0$ implies $(n_1 \times n_2) \cdot n_\pi = 0$. Hence there exists constants A, B, C, D such that $A + B\lambda + C\mu + D\lambda\mu = 0$. Then $A + \frac{BP}{Q} + \frac{CR}{S} + \frac{DPR}{QS} = 0$ so $AQS + BPS + CRQ + DPR = 0$, which is $P(BS + DR) = -Q(AS + CR)$. Therefore it is a saddle surface.