2025/9/29
讲义习题2.3:1-6; 教材48-49页4,5,9.

2.3.1

Let V be a linear space on F, $S \subset V$ is linearly independent, $\alpha \in V \backslash S$. Prove that $S \cup \{a\}$ is linearly independent iff $\alpha \notin \operatorname{Span}(S)$.
Proof: $S \cup \{\alpha\}$ is linearly dependent iff there α can be written as the linear combination of S (since S is linearly independent), iff $\alpha \in \operatorname{Span}(S)$.

2.3.2

Suppose $S_1, S_2, S_3 \subset V$, $W_i = \operatorname{Span}(S_i)$, and $S_1 \cup S_2 \cup S_3$ is linearly independent, prove that

$$W_1 \cap (W_2 + W_3) = (W_1 \cap W_2) + (W_1 \cap W_3).$$

Proof: Since $S_1 \cup S_2 \cup S_3$ is linearly independent, both sides are $\{0\}$.

2.3.3&2.3.4

Give another proof of the theorem: if $S \subset V$ is linearly independent, $T \subset V$ generates V and both are finite, then $|S| \leqslant |T|$.
Proof: Let $S = \{u_1, u_2, \cdots, u_n\}$ and $T = \{v_1, v_2, \cdots, v_m\}$. Consider the following algorithm that maintains a list that generates V: Step1: Note that $u_1 \in \operatorname{Span}\langle v_1, \cdots, v_m\rangle$ so $u_1, v_1, \cdots, v_m$ are linearly dependent, and generates V. Take the first $v_k \in \operatorname{Span}\langle u_1, v_1, \cdots, v_{k-1}\rangle$, and remove it from the list. Rename the elements of the list as $u_1, v_1, \cdots, v_{m-1}$.
Step$k$: Suppose the list is in the form $u_1, \cdots, u_{k-1}, v_1, \cdots, v_{m-k+1}$, then add u_k to the list. Since $u_1, \cdots, u_{k-1}, v_1, \cdots, v_{m-k+1}$ generates V, the new list $u_1, \cdots, u_k, v_1, \cdots, v_{m-k+1}$ is linearly dependent, and for any $i \leqslant k$, $u_i \notin \operatorname{Span}\langle u_1, \cdots, u_{i-1}\rangle$. Hence there exists $v_j \in \operatorname{Span}\langle u_1, \cdots, u_k, v_1, \cdots, v_{j-1}\rangle$ and we remove it from the list.
If $n > m$ then after step$m$ the list becomes $u_1, \cdots, u_m$ which generates V, but $u_1, \cdots, u_n$ is linearly independent, leading to contradiction.

2.3.5

Suppose $m, n \in \mathbb{N}$, $a_{ij}, b_i \in F$. Consider the equation

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ \cdots \\ a_{m_1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

(1) Suppose $m < n$, and $b_1 = \cdots = b_m = 0$. Prove that there is a non-trivial solution.
Proof: Consider the linear map $f : F^n \to F^m$, $(x_i)_{1 \leqslant i \leqslant n} \mapsto \left(y_j = \sum_{i=1}^n a_{ij}x_i\right)_{1 \leqslant j \leqslant m}$, then $\operatorname{dim}\operatorname{Ker} f = \operatorname{dim} F^n - \operatorname{dim}\operatorname{Im} f > 0$, so $\operatorname{Ker} f$ contains a non-trivial solution.
(2) Suppose $m > n$. Prove that there exists $b_1, \cdots, b_m$ such that the equation has no solutions.
Proof: Consider again the linear map in (1), since $\operatorname{dim}\operatorname{Im} f \leqslant \operatorname{dim} F^n < m$, f can't be surjective.

48-4

Show that the vectors

$$\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$$

form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$.
Proof: Note that

$$e_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3, e_2 = \frac{-\alpha_1 + \alpha_2 + \alpha_3}{5}, e_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.$$

Hence $\mathrm{Span}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{R}^3$ which is of dimension 3, so they are a basis of $\mathbb{R}^3$.

48-5

Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.
Solution: Consider $(1,0,0), (1,1,0), (0,1,0)$. Then $(1,0,0) + (-1) \cdot (1,1,0) + (0,1,0) = 0$, but each pair are linearly independent.

49-9

Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, γ are linearly independent, prove that $(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are linearly independent.
Proof: If there exists $c_1, c_2, c_3 \in F$ not identically zero, such that $c_1(\alpha+\beta) + c_2(\beta+\gamma) + c_3(\gamma+\alpha) = 0$, then $(c_1+c_3)\alpha + (c_1+c_2)\beta + (c_2+c_3)\gamma = 0$. Since α, β, γ are linearly independent, $c_1 + c_2 = c_2 + c_3 = c_3 + c_1 = 0$, hence $c_1 = c_2 = c_3 = 0$, a contradiction.