2025/9/16
讲义习题2.1:1,2; 教材33-34页2,4,5,7; 教材39页1.

2.1

Suppose F is a field and V is a linear space on F, prove that
(1) For $c \in F \backslash \{0\}$, $\alpha, \beta \in V$, if $c\alpha = c\beta$ then $\alpha = \beta$.
Proof: Note that

$$\alpha = c^{-1} \cdot c\alpha = c^{-1} \cdot c\beta = \beta.$$

(2) For $c_1, c_2 \in F$, and $\alpha \in V \backslash \{0\}$, if $c_1\alpha = c_2\alpha$ then $c_1 = c_2$.
Proof: Note that

$$0 = c_1\alpha - c_2\alpha = (c_1 - c_2)\alpha \implies c_1 = c_2.$$

33-2

If V is a vector space over the field F, verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in V.
Proof:

$$\begin{aligned}(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) &= [(\alpha_1 + \alpha_2) + \alpha_3] + \alpha_4 \\ &= [(\alpha_2 + \alpha_1) + \alpha_3] + \alpha_4 \\ &= [\alpha_2 + (\alpha_1 + \alpha_3)] + \alpha_4 \\ &= [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4.\end{aligned}$$

34-4

Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$\begin{gathered}(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \\ c(x, y) = (cx, y).\end{gathered}$$

Is V, with these operations, a vector space over the field of real numbers?
Proof: Yes. Clearly addition and scalar multiplication are both associative and communicative. The zero element is $(0, 0)$, and for any $(x, y) \in V$, $(x, y) + (-x, -y) = (0, 0)$ so there is an additive inverse. For any c_1, c_2 and (x, y), $(c_1 + c_2)(x, y) = ((c_1 + c_2)x, y) = c_1(x, y) + c_2(x, y)$. For any $c \in F$ and $(x, y), (z, w) \in V$, $c((x, y) + (z, w)) = c(x + z, y + w) = (cx + cz, y + w) = c(x, y) + c(z, w)$.
Therefore V is a linear space.

34-5

On $\mathbb{R}^n$, define two operations

$$\begin{gathered}\alpha \oplus \beta = \alpha - \beta \\ c \cdot \alpha = -c\alpha.\end{gathered}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?
Solution:

- For $\alpha = (1, 0, \cdots, 0)$ and $\beta = (0, 0, \cdots, 0)$, $\alpha \oplus \beta \neq \beta \oplus \alpha$.
- Let $\alpha = \beta = \gamma = (1, 0, \cdots, 0)$, then $(\alpha \oplus \beta) \oplus \gamma = (-1, 0, \cdots, 0) \neq \alpha = \alpha \oplus (\beta \oplus \gamma)$.
- For $c_1, c_2 \in F$, $(c_1 \cdot c_2)\alpha = -c_1c_2\alpha \neq c_1c_2\alpha = c_1 \cdot (c_2 \cdot \alpha)$,
- while $(-1) \cdot \alpha = \alpha$ for any $\alpha \in V$.
- There exists a right identity $0 = (0, \cdots, 0)$ such that $\alpha + 0 = \alpha$, but no left identity.
- There exists an inverse $\alpha \oplus \alpha = 0$.
- For any c_1, c_2 and $\alpha \in V$, $(c_1 + c_2) \cdot \alpha = -c_1\alpha - c_2\alpha = c_1 \cdot \alpha + c_2 \cdot \alpha$.
- For any $c \in F$ and $\alpha, \beta \in V$, $c \cdot (\alpha \oplus \beta) = c \cdot (\alpha - \beta) = c\beta - c\alpha = c \cdot \alpha - c \cdot \beta = (c \cdot \alpha) \oplus (c \cdot \beta)$.

34-7

Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$

Is V, with these operations, a vector space?
Solution: No, since the additive identity does not exists: for any $(a, b) \in V$, $(0, 1) + (a, b) = (a, 0) \neq (0, 1)$.

39-1

Which of the following sets of vectors $\alpha = (a_1, \cdots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$, ($n \geqslant 3$)
(a) all α such that $a_1 \geqslant 0$;
No, consider the inverse of $(1, 0, \cdots, 0)$.
(b) all α such that $a_1 + 3a_2 = a_3$;
Yes, this is a linear equation.
(c) all α such that $a_2 = a_1^2$;
No, consider $2 \cdot (1, 1, 0, \cdots, 0) = (2, 2, 0, \cdots, 0)$.
(d) all α such that $a_1a_2 = 0$;
No, consider $(1, 0, \cdots, 0) + (0, 1, 0, \cdots, 0) = (1, 1, 0, \cdots, 0)$.
(e) all α such that a_2 is rational.
No, consider $\sqrt{2} \cdot (0, 1, \cdots, 0)$.