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105-2

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of $\mathcal{B}$.
Solution: The dual basis $\mathcal{B}^*$ of $\mathcal{B}$ is $\mathcal{B}^* = \{f_1, f_2, f_3\}$, where $f_1(x, y, z) = y - x$, $f_2(x, y, z) = x - y + z$, and $f_3(x, y, z) = \frac{2y-x-z}{2}$.

105-5

If $A, B \in \mathbb{C}^{n\times n}$, show that $AB - BA = I$ is impossible.
Proof: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ but $\mathrm{tr}(I) = 1 \neq 0$.

106-12

Let V be a finite-dimensional vector space over F, and W a subspace of V. If $f \in \mathcal{L}(W, F)$, prove that there exists $g \in \mathcal{L}(V, F)$ such that $g(\alpha) = f(\alpha)$ for any $\alpha \in W$.
Proof: Take a base $\{\alpha_1, \cdots, \alpha_k\}$ of W and extend it to a base $\{\alpha_1, \cdots, \alpha_n\}$ of V. For any $f \in \mathcal{L}(W, F)$, let $g \in \mathcal{L}(V, F) : \alpha = \sum_{i=1}^n c_i\alpha_i \mapsto \sum_{i=1}^k c_i f(\alpha_i)$, then $g|_W = f$.

107-16

Show that the trace functional on $n \times n$ matrices is unique in the following sense. If $W = F^{n\times n}$ and $f \in \mathcal{L}(W, F)$ such that $f(AB) = f(BA)$ for any $A, B \in W$, then f is a scalar multiple of tr. If in addition, $f(I) = n$, then $f = \mathrm{tr}$.
Proof: It suffices to prove the case $f(I) = n$. Let $E_i \in F^{n\times n}$ be the matrix where all entries except (i, i) are 0 and (i, i) is 1. Consider $P \in F^{n\times n}$ that interchanges rows 1 and i, then $PE_iP^{-1} = E_1$ so $f(E_i) = f(P^{-1}(PE_i)) = f(PE_iP^{-1}) = f(E_1)$. Since $n = f(I) = \sum_{k=1}^n f(E_k) = nf(E_1)$ we obtain $f(E_i) = f(E_1) = 1$.
For $M_{i,j} \in F^{n\times n}$ where all entries except (i, j) are 0 and (i, j) is 1, likewise consider $P \in F^{n\times n} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)$, then $PM_{i,j} = M_{i,j}$ but $M_{i,j}P = O$ so $0 = f(M_{i,j}P) = f(PM_{i,j}) = f(M_{i,j})$. Hence $f = \mathrm{tr}$.

107-17

Let $W = F^{n\times n}$, and W_0 be the sub-space spanned by the matrices C of commutator $C = [A, B] = AB - BA$. Prove that $W_0 = \{A : \mathrm{tr}(A) = 0\}$.
Proof: First, if $C = [A, B]$, then $\mathrm{tr}(C) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$.
Now we show that if $\mathrm{tr}C = 0$ then C is a commutator. Note that the if $P^{-1}CP = [A, B]$ is a commutator, then $C = [PAP^{-1}, PBP^{-1}]$ is a commutator. Since $\mathrm{tr}(C) = 0$, C is similar to a matrix D whose diagonal contains only zeros. Let $A = \mathrm{diag}(1, 2, \cdots, n)$, we find B such that $D = [A, B]$. Note that

$$[A, B]_{i,j} = \sum_{k=1}^n A_{ik}B_{kj} - B_{ik}A_{kj} = B_{ij}(i - j),$$

hence we only need to define $B_{ij} = \frac{D_{ij}}{i-j}$, then $D = [A, B]$.
(Prove that C is similar to a matrix D whose diagonal is all zero: $\mathtt{tr}(C) = 0$ implies the sum of its eigenvectors are zero, hence $0 \in \{x^* Cx : |x| = 1\}$. Let $u_1^* C u_1 = 0$ and extend it to $u_1, \cdots, u_n$ a orthogonal base of $F^{n\times 1}$. Under this base, D is a matrix with $D_{11} = 0$. Then use induction.)