2025/9/25

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2.3.7

Suppose V is a finite dimensional linear space, W_1, W_2, W_3 are linear subspaces. Prove that

$$\begin{aligned}\dim(W_1+W_2+W_3) \leqslant & \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1 \cap W_2) \\ & - \dim(W_2 \cap W_3) - \dim(W_3 \cap W_1) + \dim(W_1 \cap W_2 \cap W_3).\end{aligned}$$

Proof: Note that

$$\begin{aligned}\dim(W_1+W_2+W_3) &= \dim(W_1+W_2) + \dim W_3 - \dim(W_1+W_2)\cap W_3 \\ &= \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1\cap W_2) - \dim(W_1+W_2)\cap W_3,\end{aligned}$$

and $\dim(W_2\cap W_3) + \dim(W_3 \cap W_1) - \dim(W_1\cap W_2 \cap W_3) = \dim((W_1\cap W_3) + (W_2 \cap W_3))$. Also, $((W_1\cap W_3)+(W_2\cap W_3)) \subset ((W_1+W_2)\cap W_3)$, hence the desired inequality.

49-10

Let V be a vector space over the field F. Suppose there are a finite number of vectors $\alpha_1, \cdots, \alpha_r$ in V which span V. Prove that V is finite dimensional.

Proof: Consider an algorithm where every step we remove the first α_k such that $\alpha_k \in \mathrm{Span}(\alpha_1, \cdots, \alpha_{k-1})$. Then after each step the list of vectors still span V, and it must stop after finite steps. We obtain finally a subset $\{\alpha_1, \cdots, \alpha_n\}$ which spans V and is linearly independent, hence $\dim V = n$ is finite.

49-14

Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite dimensional.

Proof: Otherwise if $\mathbb{R}$ has a finite base $\{a_1, \cdots, a_n\}$, then every element of $\mathbb{R}$ can be uniquely written as $q_1 a_1 + \cdots + q_n a_n$, hence $|\mathbb{R}| = |\mathbb{Q}^n|$ which is clearly false.