PSA: Riemann Integral

A1) $f \in C([a,b]), g \in \mathcal{R}([a,b])$, where g is positive. Prove that there exists $\xi \in (a,b)$, such that

$$\int_a^b fg = f(\xi) \int_a^b g.$$

Proof: Since g is positive on $[a,b]$,

$$\inf_{x\in[a,b]} f(x) \int_a^b g \leqslant \int_a^b fg \leqslant \sup_{x\in[a,b]} f(x) \int_a^b g.$$

By $f \in C([a,b])$, there exists such an $\xi \in (a,b)$.

A2) Prove without using Lebesgue theorem: if f is monotonously increasing on $[a,b]$, then $f \in \mathcal{R}([a,b])$.

Proof: For any $\varepsilon > 0$ let $n = [1/\varepsilon] + 1$, and

$$\mathcal{C} = \left\{ x_k = a + (b-a)\frac{k}{n} : k = 0, 1, \cdots, n \right\}.$$

Then

$$g(x) = \max_{x_k \leqslant x}\{f(x_k)\} \leqslant f, h(x) = \min_{x_k \geqslant x}\{f(x_k)\} \geqslant f.$$

and both are monotonous simple functions.
Therefore

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leqslant \overline{S}(f;\mathcal{C}) - \underline{S}(f;\mathcal{C}) = \frac{1}{n}(f(b) - f(a)) \to 0.$$

Hence f is Riemann integrable.

A3) Prove that $1_{\mathbb{Q}}$ is not Riemann integrable on $[0,1]$.

Proof: Let $\varepsilon = \frac{1}{2}$. For any $\mathcal{C} = \{0 = x_0 < \cdots < x_n = 1\}$, $\omega(x_{k-1}, x_k) = 1$, hence

$$\sum_{k=1}^n \omega(x_{k-1}, x_k)(x_k - x_{k-1}) = 1 > \varepsilon.$$

Therefore $1_{\mathbb{Q}}$ is not Riemann integrable.

A4) Prove that if $f \in \mathcal{R}([a,b])$, then $|f|^p \in \mathcal{R}([a,b])$, where $p \geqslant 0$.

Proof: Since $x \mapsto |x|^p$ is continuous, $|f|^p$ is continuous as x whenever f is continuous at x.
Hence

$$f \in \mathcal{R}([a,b]) \implies |f|^p \in \mathcal{R}([a,b]).$$

A5) Prove Hölder's Inequality: if $f, g \in \mathcal{R}([a,b]), p, q > 0, 1/p + 1/q = 1$, then

$$\left|\int_a^b fg\right| \leqslant \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}.$$

Proof: By A4) the functions are integrable. We can assume that

$$\int_a^b |f|^p = \int_a^b |g|^q = 1.$$

Then by Young's inequality,

$$\left|\int_a^b fg\right| \leqslant \int_a^b |f| \cdot |g| \leqslant \int_a^b \frac{1}{p}|f|^p + \frac{1}{q}|g|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

A6) Prove Minkowski's inequality: if $f, g \in \mathcal{R}([a,b]), p \geqslant 1$, then

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

Proof: Note that if $1/p + 1/q = 1$, then

$$\begin{aligned}\int_a^b |f+g|^p &= \int_a^b |f| \cdot |f+g|^{1-p} + \int_a^b |g| \cdot |f+g|^{1-p} \\ &\leqslant \left(\left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}\right)\left(\int_a^b |f+g|^{(1-p)q}\right)^{1/q}\end{aligned}$$

Hence

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

The equality holds, when $|f|/|f+g|^{1-p}, |g|/|f+g|^{1-p}$ are both constant, which is equivalent to $|f|/|g|$ is constant.

PSB: Convex Functions

B1) Assume $f \in \mathcal{R}([a,b])$ and f is convex, prove that

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

Proof: Note that $f\left(\frac{a+b}{2}\right) \leqslant \frac{f(x)+f(a+b-x)}{2} \leqslant \frac{f(a)+f(b)}{2}$, and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \frac{f(x)+f(a+b-x)}{2}\,\mathrm{d}x.$$

Hence

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

B2) Assume f is twice differentiable on $[a, b]$ and for any $x, f''(x) > 0, f(x) \leqslant 0$. Prove that for any x,

$$f(x) \geqslant \frac{2}{b-a} \int_a^b f(y) \, \mathrm{d}y.$$

Proof: For any $x \leqslant y \leqslant b$,

$$f(y) \leqslant \frac{b-y}{b-x} f(x) + \frac{y-x}{b-x} f(b) \leqslant \frac{b-y}{b-x} f(x),$$

hence

$$\int_x^b f(y) \, \mathrm{d}y \leqslant f(x) \int_x^b \frac{b-y}{b-x} \, \mathrm{d}y = \frac{b-x}{2} f(x).$$

Likewise,

$$\int_a^x f(y) \, \mathrm{d}y \leqslant f(x) \int_a^x \frac{y-a}{x-a} \, \mathrm{d}y = \frac{x-a}{2} f(x).$$

Therefore

$$f(x) \geqslant \frac{2}{b-a} \int_a^b f(y) \, \mathrm{d}y.$$

B3) Assume f is twice differentiable on $\mathbb{R}$ and $f''(x) \geqslant 0, \varphi \in C([a, b])$. Prove that

$$\frac{1}{b-a} \int_a^b (f \circ \varphi)(t) \, \mathrm{d}t \geqslant f\left(\frac{1}{b-a} \int_a^b \varphi(t) \, \mathrm{d}t \right).$$

Proof: We prove the proposition for any convex function f and φ on the set X.
Let

$$\langle g \rangle = \frac{1}{\mu(X)} \int_X g \, \mathrm{d}\mu.$$

Then since f is convex, there is a constant K such that $f(y) - f(\langle \varphi \rangle) \geqslant K(y - \langle \varphi \rangle)$. Hence

$$\begin{aligned}
\langle f(\varphi) \rangle &= \frac{1}{\mu(X)} \int_X f(\varphi(t)) \, \mathrm{d}\mu \\
&\geqslant \frac{1}{\mu(x)} \int_X f(\langle \varphi \rangle) \, \mathrm{d}\mu + \frac{1}{\mu(X)} \int_X K(\varphi(t) - \langle \varphi \rangle) \, \mathrm{d}\mu \\
&= f(\langle \varphi \rangle).
\end{aligned}$$

B4) Assume $f \in C([a, b])$ and for any x, $f(x) > 0$. Prove that

$$\log \left(\frac{1}{b-a} \int_a^b f \right) \geqslant \frac{1}{b-a} \int_a^b \log f.$$

Proof: Since $-\log x$ is convex, we can use B3).

B5) Prove that if f is convex on $\mathbb{R}$, $\varphi \in C([0, 1])$, then

$$f\left(\int_0^1 \varphi \right) \leqslant \int_0^1 f \circ \varphi.$$

Proof: A special case of what we proved in B3).

PSC: Integrals and Derivatives

C1) Assume $f \in C^1([0,2])$, $|f'| \leqslant 1$, $f(0)=f(2)=1$. Prove that

$$1 \leqslant \int_0^2 f \leqslant 3.$$

Proof: Note that for $0 \leqslant x \leqslant 1$,

$$|f(x)-1| = x|f'(\xi)| \leqslant x.$$

and for $1 \leqslant x \leqslant 2$,

$$|f(x)-1| = (2-x)|f'(\xi)| \leqslant 2-x.$$

Hence

$$\int_0^2 |f(x)-1| \,\mathrm{d}x \leqslant \int_0^1 x \,\mathrm{d}x + \int_1^2 (2-x)\,\mathrm{d}x = 1.$$

C2) Assume that $f \in C^2([0,1])$. Prove that $\exists \xi \in [0,1]$, such that

$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$

Proof: Let $g(x) = f(x) + f(1-x)$, then

$$\begin{aligned}
\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) &= \int_0^{1/2} g(x) - 2f\left(\frac{1}{2}\right)\mathrm{d}x \\
\text{(integration by parts)} &= -\int_0^{1/2} x g'(x)\,\mathrm{d}x = -\frac{1}{2}\int_0^{1/2} g'(x)\,\mathrm{d}x^2 \\
\text{(integration by parts)} &= \frac{1}{2}\int_0^{1/2} x^2 g''(x)\,\mathrm{d}x.
\end{aligned}$$

Note that $g'' \in C([0,1])$ hence by A1), $\exists \eta \in (0, \frac{1}{2})$,

$$\int_0^1 f(x)\,\mathrm{d}x - f\left(\frac{1}{2}\right) = g''(\eta)\frac{1}{2}\int_0^{1/2} x^2\,\mathrm{d}x = \frac{1}{48} g''(\eta).$$

Since $f'' \in C([0,1])$, there exists $\xi \in (\eta, 1-\eta)$, such that

$$f''(\xi) = \frac{f''(\eta) + f''(1-\eta)}{2} = \frac{g''(\eta)}{2}.$$

Therefore

$$\int_0^1 f(x)\,\mathrm{d}x = f\left(\frac{1}{2}\right) + \frac{1}{24} f''(\xi).$$

C3) Assume $f \in C^1([0,1])$. Prove that

$$\max_{x\in[a,b]} |f(x)| \leqslant \frac{1}{b-a}\left|\int_a^b f(x)\,\mathrm{d}x\right| + \int_a^b |f'(x)|\,\mathrm{d}x.$$

Proof: For any $t \in [a,b]$,

$$(b-a)|f(t)| \leqslant \left|\int_a^b f(x)\,\mathrm{d}x\right| + \left|\int_a^b f(x) - f(t)\,\mathrm{d}x\right|$$

where

$$\begin{aligned}\left|\int_a^b f(x)-f(t)\,\mathrm{d}x\right| &= \left|\int_a^b \left(\int_t^x f'(u)\,\mathrm{d}u\right)\mathrm{d}x\right| \\ &\leqslant \int_a^b\int_t^x |f'(u)|\,\mathrm{d}u\,\mathrm{d}x \\ &\leqslant (b-a)\int_a^b |f'(u)|\,\mathrm{d}u.\end{aligned}$$

C4) Suppose $f \in C([0,1])$ and for any $g \in C([0,1]), g(0)=g(1)=0$, we have

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise assume $f(t) > 0$ for some $t \in (0,1)$, then there exists an $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon) \subset [0,1]$ and $\forall x \in (t-\varepsilon, t+\varepsilon), f(x) > f(t)/2$.
Let

$$g(x) = \begin{cases} 0, & x \notin (t-\varepsilon, t+\varepsilon), \\ 1 - \frac{|x-t|}{\varepsilon}, & x \in (t-\varepsilon, t+\varepsilon). \end{cases}$$

Then

$$\int_0^1 f(x)g(x)\,\mathrm{d}x > \int_{t-\varepsilon}^{t+\varepsilon} \frac{f(t)}{2} g(x)\,\mathrm{d}x > 0,$$

leading to contradiction. Hence $f(x) \equiv 0$.

C5) Suppose $f \in C([0,1])$ and for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$\int_0^1 f(x)x^n\,\mathrm{d}x = 0.$$

Prove that $f(x) \equiv 0$.
Proof: Otherwise, $\int_0^1 f^2 > 0$. By Stone-Weierstrass theorem, for any $\varepsilon > 0$, there is a polynomial P such that $\sup_{x\in[0,1]} |f(x) - P(x)| < \varepsilon$. Hence

$$0 = \int_0^1 f(x)P(x)\,\mathrm{d}x = \int_0^1 f^2 - \int_0^1 f(x)(f(x)-P(x))\,\mathrm{d}x \geqslant \int_0^1 f^2 - \sup_{x\in[0,1]} |f(x)|\varepsilon > 0$$

when $\varepsilon \to 0$, leading to contradiction.

C6) (Gronwall's Inequality) Suppose $\varphi \in C([0,T])$ and for any $t \in [0,T]$, $|\varphi(t)| \leqslant M + k\int_0^t |\varphi(s)|\,\mathrm{d}s$, where M, k are positive real numbers. Prove that $\forall t \in [0,T]$, $|\varphi(t)| \leqslant Me^{kt}$.

Proof: Let
$f : [0, \frac{T}{k}] \to \mathbb{R}, t \mapsto \frac{e^{-t}|\varphi(t/k)|}{M}$,
then for any $t \in [0, T/k]$,

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(s)e^s\,\mathrm{d}s.$$

Let $f(t) = \sup_{s\in[0,T/k]}\{f(s)\}$ then

$$f(t) \leqslant e^{-t} + e^{-t}\int_0^t f(t)e^s \,\mathrm{d}x = e^{-t} + f(t)(1-e^{-t}).$$

Hence $f(s) \leqslant f(t) \leqslant 1$, $\implies |\varphi(t)| \leqslant Me^{kt}$.

C7) Assume $a, b > 0$, $f \in C([-a,b])$. If for any $x \in (-a,b)$, $f(x) > 0$ and $\int_{-a}^b xf(x)\,\mathrm{d}x = 0$. Prove that

$$\int_{-a}^b x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^b f(x)\,\mathrm{d}x.$$

Proof: Note that

$$\int_{-a}^b (x+a)(x-b)f(x)\,\mathrm{d}x \leqslant 0.$$

Combined with $\int_{-a}^b xf(x)\,\mathrm{d}x = 0$ we get

$$\int_{-a}^b x^2 f(x)\,\mathrm{d}x \leqslant ab\int_{-a}^b f(x)\,\mathrm{d}x.$$

C8) Assume $f \in C([-1,1])$. Prove that

$$\lim_{\lambda\to 0^+}\int_{-1}^1 \frac{\lambda}{\lambda^2+x^2}f(x)\,\mathrm{d}x = \pi f(0).$$

Proof: Let $M = \sup_{|x|\leqslant 1}|f(x)|$ and

$$g(x) = \frac{\lambda}{\lambda^2+x^2},$$

then (g is sort of a good kernel)

$$\int_{-1}^1 g(x)\,\mathrm{d}x = 2\arctan\frac{1}{\lambda}.$$

Hence

$$\begin{aligned}
&\left|\int_{-1}^1 f(x)g(x)\,\mathrm{d}x - \pi f(0)\right| \\
\leqslant&\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \int_{-\varepsilon}^{\varepsilon}|f(x)-f(0)|g(x)\,\mathrm{d}x + \int_{\varepsilon\leqslant x\leqslant 1} Mg(x)\,\mathrm{d}x \\
\leqslant&\left|\pi - 2\arctan\frac{1}{\lambda}\right| f(0) + \sup_{|x|\leqslant\varepsilon}|f(x)-f(0)|\pi + 2M\left|\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda}\right| \\
&\to 0
\end{aligned}$$

since

$$\arctan\frac{1}{\lambda} - \arctan\frac{\varepsilon}{\lambda} = \arctan\frac{\lambda(1-\varepsilon)}{\lambda^2+\varepsilon} \to 0, \text{when } \lambda \to 0^+.$$

and $\sup_{|x|\leqslant\varepsilon}|f(x)-f(0)| \to 0$ when $\varepsilon \to 0$.

C9) Assume f is differentiable on $[1, \infty)$ and both $\int_1^\infty f(x)\,\mathrm{d}x$ and $\int_1^\infty f'(x)\,\mathrm{d}x$ converges. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

Proof: For any $\varepsilon > 0$, there exists $N > 1$, such that $\forall u, v > N$,

$$\left|\int_u^v f'(x)\,\mathrm{d}x\right| < \varepsilon, \text{ i.e. } |f(u) - f(v)| < \varepsilon$$

Hence for any $u > N$, if $|f(u)| > \varepsilon$,

$$\left|\int_u^M f(x)\,\mathrm{d}x\right| \geqslant (M-u)(|f(u) - \varepsilon|) \to \infty, \text{ as } M \to \infty,$$

which contradicts the fact that $\int_1^\infty f(x)\,\mathrm{d}x$ converges. Therefore $|f(u)| < \varepsilon$ for any $u > N$, which implies $\lim_{x\to\infty} f(x) = 0$.

C10) Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x, \int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

Proof:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = -\int_0^\infty \sin^2 x\,\mathrm{d}\frac{1}{x} = \int_0^\infty \frac{\sin 2x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$
$$\int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{1}{1+x}\,\mathrm{d}\sin x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$

PSD: Calculation of improper integrals

D1)

$$\int_0^1 \log x\,\mathrm{d}x = (x\log x - x)\Big|_0^1 = -1.$$

D2)

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x\Big|_{-\infty}^\infty = \pi.$$

D3)

Calculating residues, we get

$$\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^4+1} = 2\pi i \cdot (Res(f; e^{i\pi/4}) + Res(f; e^{3i\pi/4})) = \frac{\pi}{\sqrt{2}}.$$

Hence

$$\int_0^\infty \frac{\mathrm{d}x}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

D4)

Same as D3)

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \sqrt{2}\pi.$$

Hence

$$\int_{0}^{\infty} \frac{1+x^2}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

D5)

$$\int_{-\infty}^{0} xe^x\,\mathrm{d}x = \int_{-\infty}^{0} x\,\mathrm{d}e^x = -\int_{-\infty}^{0} e^x\,\mathrm{d}x = -1.$$

D6)

$$\int_{0}^{\infty} e^{-\sqrt{x}}\,\mathrm{d}x = 2\int_{0}^{\infty} ye^{-y}\,\mathrm{d}y = 2\int_{0}^{\infty} e^{-y}\,\mathrm{d}y = 2.$$

D7)

$$\int_{0}^{\infty} \frac{\mathrm{d}x}{(a^2+x^2)^{3/2}} = \frac{1}{a^2}\int_{0}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{3/2}} = \frac{1}{2a^2}B\left(\frac{1}{2},1\right) = \frac{1}{a^2}.$$

(We can also substitute $x = a\tan\theta$).

D8)

$$\int_{2}^{\infty} \frac{\mathrm{d}x}{x^2+x-2} = \frac{1}{3}\log\frac{x-1}{x+2}\Big|_2^\infty = \frac{\log 3}{3}.$$

D9)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+x+1)^2} &= \frac{8}{3\sqrt{3}}\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1+u^2)^2} \\ (u=\tan\theta) &= \frac{8}{3\sqrt{3}}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \frac{4\sqrt{3}\pi}{9}.\end{aligned}$$

D10)

$$\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} 1\,\mathrm{d}\theta = \pi.$$

D11)

$$\int_{-1}^{1} \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \theta\,\mathrm{d}\theta = 0.$$

D12)

Let γ be the unit circle, then

$$
\begin{aligned}
\int_{-1}^{1} \frac{\mathrm{d}x}{(2-x)^2\sqrt{1-x^2}} &= \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{(2-\sin\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{(2-\sin\theta)^2} \\
&= \frac{1}{2}\int_{\gamma} -\frac{4}{i}\frac{z\mathrm{d}z}{(z^2-4iz-1)^2} \\
&= -4\pi \mathrm{Res}\left(\frac{z}{(z^2-4iz-1)^2}; (2-\sqrt{3})i\right) \\
&= \frac{2\pi}{3\sqrt{3}}.
\end{aligned}
$$

D13)

$$
\int_0^1 \frac{\arcsin\sqrt{x}}{x(1-x)}\,\mathrm{d}x > \int_{1/4}^1 \frac{\pi}{6}\frac{1}{1-x}\,\mathrm{d}x \text{ which diverges.}
$$

D14)

$$
\int_0^1 (1-x)^n x^{1/2-1}\,\mathrm{d}x = B\left(n+1, \frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{n!2^{n+1}}{(2n+1)!!}.
$$

D15)

$$
\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x = \begin{cases} \frac{(n-1)!!}{n!!}, n \text{ is odd,} \\ \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, n \text{ is even.} \end{cases}
$$

D16)

Using integration by parts, and substitute $x = e^{-y}$,

$$
\begin{aligned}
\int_0^1 x^m(\log x)^n\,\mathrm{d}x &= (-1)^n \int_0^\infty e^{-(m+1)y}y^n\,\mathrm{d}y \\
&= (-1)^n \frac{n!}{(m+1)^n}\int_0^\infty e^{-(m+1)y}\,\mathrm{d}y = \frac{(-1)^n n!}{(m+1)^{n+1}}.
\end{aligned}
$$

D17)

$$
\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p} = \int_{\log 2}^\infty \frac{\mathrm{d}y}{y^p} = \frac{(\log 2)^{1-p}}{p-1}.
$$

D18)

Substitute $x = ay$, then

$$
\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\log a}{2a} + \frac{1}{a}\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = \frac{\pi\log a}{2a}.
$$

since by substituting $y = 1/z$,

$$
\int_0^\infty \frac{\log y}{1+y^2}\,\mathrm{d}y = -\int_0^\infty \frac{\log z}{1+z^2}\,\mathrm{d}z = 0.
$$

D19)

$$
\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \Gamma(n) = (n-1)!.
$$

D20)

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(ax^2+2bx+c)^n}=\frac{1}{d^n}\sqrt{\frac{d}{a}}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{(1+u^2)^n}=\frac{1}{d^n}\sqrt{\frac{d}{a}}\pi\frac{(2n-3)!!}{(2n-2)!!}.$$

where $d=\frac{ac-b^2}{a}$

D21)

$$\int_0^{\infty}x^{2n-1}e^{-x^2}\,\mathrm{d}x=\frac{1}{2}\int_0^{\infty}y^{n-1}e^{-y}\,\mathrm{d}y=\frac{(n-1)!}{2}.$$

D22)

The Poisson kernel

$$\begin{aligned}\frac{1-r^2}{1-2r\cos x+r^2}&=\frac{1-r^2}{(1-re^{ix})(1-re^{-ix})}\\&=(1-r^2)\sum_{n=0}^{\infty}r^ne^{inx}\sum_{m=0}^{\infty}r^me^{-imx}\\&=\sum_{k=-\infty}^{\infty}r^{|k|}e^{ikx}.\end{aligned}$$

Hence

$$\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos x+r^2}\,\mathrm{d}x=2\pi.$$

D23)

$$\begin{aligned}\int_0^{\infty}e^{-ax}\cos bx\,\mathrm{d}x&=\frac{1}{b}\int_0^{\infty}e^{-ax}\,\mathrm{d}\sin bx=\frac{a}{b}\int_0^{\infty}e^{-ax}\sin bx\,\mathrm{d}x\\&=-\frac{a}{b^2}\int_0^{\infty}e^{-ax}\,\mathrm{d}\cos bx=\frac{a}{b^2}-\frac{a^2}{b^2}\int_0^{\infty}e^{-ax}\cos bx\,\mathrm{d}x\\&=\frac{a}{a^2+b^2}.\end{aligned}$$

D24)

Same as (23),

$$\int_0^{\infty}e^{-ax}\sin bx\,\mathrm{d}x=\frac{b}{a^2+b^2}.$$

D25)

$$\begin{aligned}&\int_1^{\infty}\frac{\mathrm{d}x}{x(x+1)\cdots(x+n)}=\lim_{N\to\infty}\int_0^N\sum_{k=0}^{n}\frac{(-1)^k}{x+k}\binom{n}{k}\,\mathrm{d}x\\&=\lim_{N\to\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(\frac{N+k}{(k+1)}\right)\\&=-\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(k+1)+\lim_{N\to\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(1+\frac{k}{N}\right)\\&=-\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(k+1).\end{aligned}$$

D26)

$$\begin{aligned}&\int_0^{\pi}\log\sin x\,\mathrm{d}x=2\int_0^{\pi/2}\log\sin x\,\mathrm{d}x=2\int_0^{\pi/2}\log\cos x\,\mathrm{d}x\\=&\int_0^{\pi/2}\log\sin 2x-\log 2\,\mathrm{d}x=\frac{1}{2}\int_0^{\pi}\log\sin x\,\mathrm{d}x-\frac{\pi}{2}\log 2\\=&-\pi\log 2.\end{aligned}$$

D27)

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}.$$

Note that

$$\max\{0,1-x^2\}<e^{-x^2}<\frac{1}{1+x^2}.$$

Hence

$$\frac{(2n)!!}{(2n+1)!!}<\int_0^\infty e^{-nx^2}\,\mathrm{d}x<\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

Therefore

$$\sqrt{n}\frac{(2n)!!}{(2n+1)!!}<\int_0^\infty e^{-x^2}\,\mathrm{d}x<\sqrt{n}\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

By Wallis's formula,

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}.$$

PSE: Density of sum of squares

Let $I=(0,\infty)$.

Part 1

E1) Prove that $e^{-u}/\sqrt{u}$ is integrable on I, and let $K=\int_0^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u$.

Proof:

$$\begin{aligned}\int_1^\infty e^{-u}/\sqrt{u}\,\mathrm{d}u<\int_1^\infty e^{-u}\,\mathrm{d}u=\frac{1}{e}.\\\int_0^1 e^{-u}/\sqrt{u}\,\mathrm{d}u<\int_0^1 u^{-1/2}\,\mathrm{d}u=\frac{1}{2}.\end{aligned}$$

E2) Prove that for any $x\in I$,

$F(x)=\int_0^\infty\frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u$
is well-defined.
Proof:

$$F(x)<\int_0^\infty\frac{e^{-u}}{x\sqrt{u}}\,\mathrm{d}u\text{ converges.}$$

E3) Prove that $F \in C^1(I)$ and calculate $F'(x)$.

Solution: Let $f(x,u) = \frac{e^{-u}}{\sqrt{u}(u+x)}$, then f is uniformly continuous on any closed subinterval of I, and so is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x,u) = -\frac{e^{-u}}{\sqrt{u}(u+x)^2}.$$

Also,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} f(x,u)\,\mathrm{d}u$$

converges uniformly.
Hence F is continuously differentiable and

$$F'(x) = -\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u.$$

E4) Prove that for any $x \in I$,

$$xF'(x) - \left(x - \frac{1}{2}\right)F(x) = -K.$$

Proof: We show that

$$x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u + \left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u = \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u.$$

Note that, by substituting $u \to ux$,

$$\begin{aligned}
x\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)^2}\,\mathrm{d}u &= \frac{1}{\sqrt{x}}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u,\\
\left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-u}}{\sqrt{u}(u+x)}\,\mathrm{d}u &= \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+x)}\,\mathrm{d}u,\\
\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u &= \sqrt{x}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.
\end{aligned}$$

Hence it is equivalent to

$$x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u = \int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)^2}\,\mathrm{d}u + \left(x - \frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u.$$

Note that $\mathrm{d}e^{-ux}\sqrt{u} = -e^{-ux}\left(x\sqrt{u} - \frac{1}{2\sqrt{u}}\right)\mathrm{d}u$, hence

$$\begin{aligned}
&x\int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u - \left(x + \frac{1}{2}\right)\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u\\
&= \int_0^\infty e^{-ux}\left(x\sqrt{u} - \frac{1}{2\sqrt{u}}\right)\frac{\mathrm{d}u}{1+u}\\
&= -\int_0^\infty \frac{\mathrm{d}e^{-ux}\sqrt{u}}{1+u} = -\int_0^\infty e^{-ux}\sqrt{u}\,\frac{\mathrm{d}u}{(1+u)^2}\\
&= \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{(1+u)^2} - \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\frac{\mathrm{d}u}{1+u}.
\end{aligned}$$

($\sqrt{u} = \frac{1}{\sqrt{u}}((1+u) - 1)$)

E5) Define $G : I \to \mathbb{R}, x \mapsto \sqrt{x}e^{-x}F(x)$. Prove that $\exists C \in \mathbb{R}$ such that

$$G(x) = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$

Proof: By B4)

$$G'(x) = \sqrt{x}e^{-x}F'(x) + \left(\frac{1}{2\sqrt{x}} - \sqrt{x}\right)e^{-x}F(x) = -K\frac{e^{-x}}{\sqrt{x}}.$$

Hence let $C = G(0)$, then

$$G(x) = C + \int_0^x G'(x)\,\mathrm{d}x = C - K\int_0^x \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t.$$

E6) Calculate the value of K.

Solution: Note that when $x \to \infty$, $F(x) \to 0$ hence $G(x) \to 0$. Therefore

$$0 = \lim_{x\to\infty} G(x) = G(0) - K\int_0^\infty \frac{e^{-t}}{t}\,\mathrm{d}t = G(0) - K^2.$$

Where

$$\begin{aligned} G(0) &= \lim_{x\to 0^+} \frac{\sqrt{x}}{e^x}\int_0^\infty \frac{e^{-u}}{\sqrt{u}(x+u)}\,\mathrm{d}u = \lim_{x\to 0}\int_0^\infty \frac{e^{-ux}}{\sqrt{u}(1+u)}\,\mathrm{d}u \\ &= \int_0^\infty \frac{1}{\sqrt{u}(1+u)}\,\mathrm{d}u = \int_0^\infty \frac{2\mathrm{d}t}{1+t^2} = \pi. \end{aligned}$$

Hence $K = \sqrt{\pi}$.

Part 2

Define

$$f(x) = \sum_{n=1}^\infty \frac{e^{-nx}}{\sqrt{n}}, g(x) = \sum_{n=0}^\infty \sqrt{n}e^{-nx}.$$

E7) Prove that f, g are well-defined on I and are both continuous on I.

Proof: Let $C = \sup_{x\geqslant 0} x^3e^{-x}$, then

$$\sum_{n=1}^\infty \frac{e^{-nx}}{\sqrt{n}} < \sum_{n=0}^\infty \sqrt{n}e^{-nx} \leqslant \sum_{n=1}^\infty \frac{C}{(nx)^2\sqrt{x}} \text{ converges.}$$

On any closed sub-interval of I, the two series both converge uniformly, and e^{-nx} is continuous, hence f, g are both continuous on I.

E8) Prove that $\forall x \in I$,

$$\int_1^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u \leqslant f(x) \leqslant \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.$$

Proof: The function $e^{-ux}/\sqrt{u}$ is monotonously decreasing by u, hence

$$\int_1^N \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u \leqslant \sum_{n=1}^{N-1} \frac{e^{-nx}}{\sqrt{n}} \leqslant f(x).$$

$$\sum_{n=1}^{N} \frac{e^{-nx}}{\sqrt{n}} \leqslant \int_0^N \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u \leqslant \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.$$

Therefore

$$\int_1^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u \leqslant f(x) \leqslant \int_0^\infty \frac{e^{-ux}}{\sqrt{u}}\,\mathrm{d}u.$$

E9) Prove that $\exists C_0$ such that

$$\lim_{x\to 0^+} \sqrt{x} f(x) = C_0.$$

Proof: By E8)

$$\sqrt{x} f(x) \leqslant \int_0^\infty \frac{e^{-ux}}{\sqrt{ux}}\,\mathrm{d}ux = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t = \sqrt{\pi}.$$

$$\sqrt{x} f(x) \geqslant \int_1^\infty \frac{e^{-ux}}{\sqrt{ux}}\,\mathrm{d}ux = \int_x^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t \to \sqrt{\pi}.$$

Hence

$$\lim_{x\to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E10) Define the sequence $\{a_n\}_{n\geqslant 1}$ as follows:

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) - 2\sqrt{n}.$$

Prove that $\{a_n\}$ converges.
Proof: By Euler-Maclaurin formula, for $f(x) = 1/\sqrt{x}$,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &= \frac{f(1)+f(n)}{2} + \int_1^n \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x \\ &= 2\sqrt{n} - \frac{3}{2} + \frac{1}{2\sqrt{n}} + \int_1^n \widetilde{B}_1(x) f'(x)\,\mathrm{d}x\end{aligned}$$

Hence

$$\lim_{n\to\infty} a_n = -\int_1^\infty \frac{\widetilde{B}_1(x)}{2x^{3/2}}\,\mathrm{d}x - \frac{3}{2}.$$

E11) Prove that for any $x \in I$, the function

$$h(x) = \sum_{n\geqslant 1} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) e^{-nx}$$

is well-defined.
Proof: By E10), $|a_n|$ is bounded, hence

$$h(x) = \sum_{n\geqslant 1} 2\sqrt{n} e^{-nx} + a_n e^{-nx} = 2g(x) + \sum_{n\geqslant 1} a_n e^{-nx} \leqslant 2g(x) + \sup_n |a_n| \cdot \frac{1}{e^x - 1}.$$

E12) Express $h(x)$ using $f(x)$ and find a constant C_1 such that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} h(x) = C_1.$$

Proof: Since $e^{-nx}/k > 0$, we can interchange the sums

$$h(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{n=k}^{\infty} e^{-nx} = \sum_{k=1}^{\infty} \frac{e^{-kx}}{\sqrt{k}} \frac{1}{1-e^{-x}} = \frac{1}{1-e^{-x}} f(x).$$

Therefore

$$\lim_{x\to 0^+} x^{3/2} h(x) = \lim_{x\to 0^+} \sqrt{x} f(x) = \sqrt{\pi}.$$

E13) Prove that

$$\lim_{x\to 0^+} x^{\frac{3}{2}} g(x) = \frac{\sqrt{\pi}}{2}.$$

Proof:

$$\lim_{x\to 0^+} x^{3/2} |h(x) - 2g(x)| \leqslant \lim_{x\to 0^+} \sup_n |a_n| \cdot \frac{x^{3/2}}{e^x - 1} = 0.$$

Hence

$$\lim_{x\to 0^+} x^{3/2} g(x) = \frac{1}{2} \lim_{x\to 0^+} x^{3/2} h(x) = \frac{\sqrt{\pi}}{2}.$$

Part 3

Given $A \subset \mathbb{Z}_{\geqslant 0}$, we can define a sequence $\{a_n\}_{n\geqslant 0}$:

$$a_n = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$$

Define the set $I_A \subset \mathbb{R}_{\geqslant 0}$ as follows:

$$I_A = \left\{ x \geqslant 0 : \text{the series } \sum_{n\geqslant 0} a_n e^{-nx} \text{ converges} \right\}.$$

Define the function $f_A : I_A \to \mathbb{R}$ as follows:

$$f_A(x) = \sum_{n\geqslant 0} a_n e^{-nx}.$$

Let $\Phi(A) = \lim_{x\to 0} x f_A(x)$ (if the limit exists) and let

$$\mathcal{S} = \{A \subset \mathbb{Z}_{\geqslant 0} : \lim_{x\to 0^+} x f_A(x) \text{ exists}\}.$$

For example, let

$$A_1 = \{n^2 : n \in \mathbb{Z}_{\geqslant 1}\}, A_2 = \{p^2 + q^2 : p, q \in \mathbb{Z}_{\geqslant 1}\}.$$

E14) Determine the set I_A.

Solution: If A is finite, then $I_A = \mathbb{R}_{\geqslant 0}$. Otherwise $I_A = \mathbb{R}_{>0} = I$.

E15) Given $A \subset \mathbb{Z}_{\geqslant 0}$, for any $n \geqslant 0$, define the set $A_{\leqslant n}$:

$$A_{\leqslant n} = \{\ell \in A : \ell \leqslant n\}.$$

Prove that for any $x > 0$, the series

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx}$$

converges, and satisfy

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \frac{f_A(x)}{1 - e^{-x}}.$$

Proof: $|A_{\leqslant n}| \leqslant n + 1$, hence

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} \text{ converges.}$$

Therefore

$$\sum_{n=0}^{\infty} |A_{\leqslant n}| \cdot e^{-nx} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \cdot e^{-nx} = \sum_{k=0}^{\infty} a_k \cdot \frac{e^{-kx}}{1 - e^{-x}} = \frac{f_A(x)}{1 - e^{-x}}.$$

E16) Prove that for any $x > 0$

$$\frac{f_{A_1}(x)}{1 - e^{-x}} = \sum_{n=0}^{\infty} \lfloor \sqrt{n} \rfloor e^{-nx}.$$

Proof: By E15),

$$|A_{1\leqslant n}| = \sum_{k=0}^{n} [\sqrt{k} \in \mathbb{Z}_{\geqslant 1}] = \lfloor \sqrt{n} \rfloor.$$

E17) Prove that

$$\lim_{x \to 0^+} \sqrt{x} f_{A_1}(x)$$

exists and calculate the value of $\Phi(A_1)$.
Proof:

$$\lim_{x \to 0^+} \sqrt{x} f_{A_1}(x) = \lim_{x \to 0^+} \sqrt{x}(1 - e^{-x}) \left(g(x) - \sum_{n=0}^{\infty} \{\sqrt{n}\} e^{-nx} \right).$$

Since $1 - e^{-x} \sim x$, $g(x) \sim \frac{\sqrt{\pi}}{2} x^{-3/2}$, and

$$\left| \sum_{n=0}^{\infty} \{\sqrt{n}\} e^{-nx} \right| \leqslant \frac{1}{1 - e^{-x}}.$$

Hence

$$\lim_{x \to 0^+} \sqrt{x} f_{A_1}(x) = \frac{\sqrt{\pi}}{2}.$$

and

$$\Phi(A_1) = \lim_{x \to 0^+} x f_{A_1}(x) = 0.$$

E18) Let $v(n) = \#\{(p,q) \in \mathbb{Z}^2_{\geqslant 1} : p^2 + q^2 = n\}$. Prove that for any $x > 0$, the series

$$\sum_{n \geqslant 1} v(n) e^{-nx}$$

converges and

$$\sum_{n \geqslant 1} v(n) e^{-nx} = (f_{A_1}(x))^2.$$

Proof: Since $v(n) \leqslant n$, $\sum_{n \geqslant 1} v(n) e^{-nx}$ converges.

$$\sum_{n \geqslant 1} v(n) e^{-nx} = \sum_{n \geqslant 1} \sum_{k=0}^{n} a_k a_{n-k} e^{-nx} = \sum_{n \geqslant 1} \sum_{k=0}^{n} a_k e^{-kx} \cdot a_{n-k} e^{-(n-k)x} = (f_{A_1}(x))^2.$$

E19) Prove that for any $x > 0$

$$f_{A_2}(x) \leqslant (f_{A_1}(x))^2$$

and give an upper-bound of $\Phi(A_2)$ (assuming it exists).
Proof:

$$f_{A_2}(x) = \sum_{n \geqslant 1} [v(n) \geqslant 1] \cdot e^{-nx} \leqslant \sum_{n \geqslant 1} v(n) e^{-nx} = (f_{A_1}(x))^2.$$

Hence

$$\Phi(A_2) = \lim_{x \to 0^+} x f_{A_2}(x) \leqslant \lim_{x \to 0^+} (\sqrt{x} f_{A_1}(x))^2 = \frac{\pi}{4}.$$

Part 4

Assume $\{a_n\}_{n \geqslant 0}$ is a sequence of non-negative numbers, such that for any $x > 0$ the series

$$S(x) = \sum_{n \geqslant 0} a_n e^{-nx}$$

converges. Moreover, assume that the limit below exists:

$$\lim_{x \to 0^+} x S(x) = \lim_{x \to 0^+} x \sum_{n \geqslant 0} a_n e^{-nx} = \ell \in [0, +\infty).$$

Let $F = \{f : [0,1] \to \mathbb{R}\}$, $E_0 = C([0,1])$. Let E be the space of piecewise continuous functions, and define the norm on E:

$$\|\psi\|_\infty = \sup_{x \in [0,1]} |\psi(x)|.$$

E20) Define $L : E \to F$ as follows:

$$(L(\psi))(x) = \sum_{n=0}^{\infty} a_n e^{-nx} \psi(e^{-nx}),\ \psi \in E.$$

Prove that L is well-defined and is linear. Moreover, if for any $x \in [0,1]$, $\psi_1(x) \leqslant \psi_2(x)$, the for any $x \in [0,1]$,

$$(L(\psi_1))(x) \leqslant (L(\psi_2))(x).$$

Proof: Since $\psi \in E$, ψ is bounded, hence L is well-defined and is clearly linear. The inequality holds since a_n are non-negative.

E21) Define the subspace of E

$$E_1 = \{\psi \in E : \lim_{x \to 0^+} x(L(\psi))(x) \text{ exists}\}.$$

Define the linear map $\Delta : E_1 \to \mathbb{R}$ as follows:

$$\Delta(\psi) = \lim_{x \to 0^+} x(L(\psi))(x),\ \psi \in E_1.$$

Prove that E_1 is a subspace of E and there is a constant $M > 0$ such that for any $\psi \in E_1$,

$$|\Delta(\psi)| \leqslant M \|\psi\|_\infty.$$

Proof: Since L is linear, so is Δ, thus E_1 is clearly a subspace of E.

$$|\Delta(\psi)| = \left| \lim_{x \to 0^+} x \sum_{n=0}^{\infty} a_n e^{-nx} \psi(e^{-nx}) \right| \leqslant \|\psi\|_\infty \cdot |\lim_{x \to 0^+} x S(x)| = \ell \|\psi\|_\infty.$$

E22) For the polynomial $P_n(x) = x^n$, prove that $P_n \in E_1$ and calculate $\Delta(P_n)$.

Proof:

$$\Delta(P_n) = \lim_{x \to 0^+} x \sum_{k=0}^{\infty} a_k e^{-kx} e^{-nkx} = \frac{1}{n+1} \ell.$$

E23) Prove that $E_0 \subset E_1$ and for every $\psi \in E_0$ calculate $\Delta(\psi)$.

Proof: Since Δ is linear, by E22) we know that for any polynomial P,

$$\Delta(P) = \int_0^1 P(x)\, \mathrm{d}x.$$

By Stone-Weierstraß theorem, any continuous function on $[0,1]$ can be uniformly approximated with polynomials, hence (same as E24)

$$\Delta(\psi) = \int_0^1 \psi(x)\, \mathrm{d}x,\ \forall \psi \in E_0.$$

E24) For $a \in (0,1)$, $\varepsilon \in (0, \min(a, 1-a))$, define the functions

$$g_-(x) = \begin{cases} 1, & x \in [0, a-\varepsilon]; \\ \frac{a-x}{\varepsilon}, & x \in (a-\varepsilon, a) \\ 0, & x \in [a, 1] \end{cases}, g_+(x) = \begin{cases} 1, & x \in [0, a]; \\ \frac{a+\varepsilon-x}{\varepsilon}, & x \in (a, a+\varepsilon). \\ 0, & x \in [a+\varepsilon, 1] \end{cases}$$

Prove that $g_\pm \in E_0$ and calculate $\Delta(g_\pm)$. Further prove that $\mathbf{1}_{[0,a]} \in E_1$ and calculate $\Delta(\mathbf{1}_{[0,a]})$.
Proof: $g_\pm \in E_0$ is trivial, and $\Delta(g_\pm) = \ell \int_0^1 g_\pm = \ell(a \pm \varepsilon/2)$. Since $g_- \leqslant \mathbf{1}_{[0,a]} \leqslant g_+$,

$$x(L(g_-))(x) \leqslant x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x)$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < x < \delta$,

$$|x(L(g_-))(x) - \Delta(g_-)|, |x(L(g_+))(x) - \Delta(g_+)| < \frac{\ell\varepsilon}{2}.$$

Hence for any $0 < x < \delta$,

$$x(L(\mathbf{1}_{[0,a]}))(x) \leqslant x(L(g_+))(x) \leqslant \Delta(g_+) + \ell\frac{\varepsilon}{2} = a + \ell\varepsilon.$$
$$x(L(\mathbf{1}_{[0,a]}))(x) \geqslant x(L(g_-))(x) \geqslant \Delta(g_-) - \ell\frac{\varepsilon}{2} = a - \ell\varepsilon.$$

Therefore

$$\Delta(\mathbf{1}_{[0,a]}) = \lim_{x\to 0^+} x(L(\mathbf{1}_{[0,a]}))(x) = a.$$

E25) Prove that $E_1 = E$ and for $\psi \in E$ determine the formula of $\Delta(\psi)$.

Proof: Use the same method as E24) applied to Darboux's sum. Hence

$$E_1 = E, \text{ and } \Delta(\psi) = \ell \int_0^1 \psi(x)\,\mathrm{d}x.$$

E26) Define the function

$$\psi(x) = \begin{cases} 0, & x \in [0, e^{-1}); \\ \frac{1}{x}, & x \in [e^{-1}, 1]. \end{cases}$$

Prove the following equation by calculating $L(\psi)\left(\frac{1}{N}\right)$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} a_k = \ell.$$

Proof:

$$(L(\psi))\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} a_n e^{-n/N} \psi(e^{-n/N}) = \sum_{n=0}^{N} a_n.$$

Hence by E25),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} a_n = \Delta(\psi) = \ell \int_0^1 \psi(x)\,\mathrm{d}x = \ell.$$

E27) Consider $A \in \mathcal{S}$, and calculate

$$\lim_{n\to\infty} \frac{|A_{\leqslant n}|}{n}.$$

which is called the asymptomatic density of A on $\mathbb{Z}_{\geqslant 0}$.
Solution:

$$\lim_{n\to\infty} \frac{|A_{\leqslant n}|}{n} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} a_n = \lim_{x\to 0^+} x \sum_{n=0}^{\infty} a_n e^{-nx} = \Phi(A).$$

E28) Calculate

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}v(k)}{n},$$

and give an upper-bound of the asymptomatic density of A_2.
Solution:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}v(k)}{n}=\lim_{x\to0^+}x\sum_{n=0}^{\infty}v(n)e^{-nx}=\lim_{x\to0^+}x(f_{A_1}(x))^2=\frac{\pi}{4}.$$

From E19) $\Phi(A_2)\leqslant\frac{\pi}{4}$.

> Quote:
> God does not care about our mathematical difficulties. He integrates empirically.
> ——Albert Einstein