

6.1: 2, 3-8 >= 5道; 6.2: 1的两小题, 2, 3, 4-9 >= 3道;

6.3: 1-5, 7-9 每题 >= 2 小题. 其余选 >= 6 道.

6.4: >= 3 题. 6.5: 1-9 >= 8 题, 10, 11

## 6.1

### 6.1.2

尺规作图画出  $f(x)$  在  $(t, f(t))$  处的切线:

(1)  $f(x) = x^k$  (2)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  (3)  $f(x) = \sin x$  (4)  $f(x) = \tan x$  (5)  $f(x) = e^x$  (6)  $f(x) = \log x$ .

引理: 尺规作图可以做加法、数乘有理数, 且有单位长度时, 可以做乘法、倒数、开根. 且做切线只需作出斜率的长度.

证明: 加法: 拼接两条线段即可. 数乘正整数: 若干次加法.

除正整数: 对于线段  $AB$ , 过  $A$  作射线  $AC_1$ , 并在射线上取  $C_2, \dots, C_n$  使得  $AC_k = kAC_1$ . 过  $C_1$  做  $BC_n$  的平行线交  $AB$  于  $D$  则  $AD = AB/n$ . 故可以数乘任意有理数.

下面设已有单位长度:

倒数: 作以  $x, 1$  为直角边的直角三角形  $ABC$ ,  $AC = 1, BC = x$ , 作  $AD \perp AB$  交  $BC$  于  $D$  则  $CD = 1/x$ .

乘法: 做  $OA = a, OB = b$ , 在  $OB$  上取  $E$  使得  $OE = 1$ , 过  $B$  作  $AE$  平行线交  $OA$  于  $D$ , 则  $OD = ab$ .

开根: 设射线  $OA$  上  $OA = a, OB = 1$ , 以  $OA$  为直径作圆, 过  $B$  作  $OA$  垂线交圆于  $C$ , 则  $BC = \sqrt{a}$ .

若要作切线,  $l: y = f'(t)(x - t) + f(t)$ , 只需画出  $f'(t)$ , 则会有  $(t, f(t))$  和  $(0, -tf'(t) + f(t))$ , 连接可得到  $l$ .

原题:

(1)  $f'(x) = kx^{k-1} = k \cdot x \cdots x, -tf'(t) + f(t) = (1 - k)t^k$  可以作出.

(2) 用 Pascal 定理.

(3) 找到正半轴首个根  $\pi$ , 取中点得到  $\pi/2$ , 再做坐标轴垂线得到单位长度.  $f'(t) = \cos t = \sqrt{1 - \sin^2 t}$  可以作出.

(4) 同样通过根找到  $\pi$ , 再用  $\tan \frac{\pi}{4} = 1$  找到单位长度.  $f'(t) = \sec^2 t = 1 + \tan^2 t$  可以作出.

(5)  $f(0) = 1$  是单位长度,  $f'(t) = e^t$  可以作出.

(6) 唯一一个根是单位长度 1,  $f'(x) = 1/x$  可以作出.

### 6.1.3

Suppose  $f$  is differentiable at  $x_0$ , calculate  $\lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0 - h))/h$ .

Solution: Note that

$$\frac{f(x_0 + h) - f(x_0 - h)}{h} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f(x_0) - f(x_0 - h)}{h}$$

hence the limit is  $2f'(x_0)$ .

### 6.1.4

Suppose  $f$  is continuous at 0 and  $\lim_{x \rightarrow 0} (f(2x) - f(x))/x = k$ , prove that  $f'(0) = k$ .

Proof: Let  $f(2x) - f(x) = xk + xh(x)$ , then  $h(x) \rightarrow 0$ , and

$$f(x) = f(2^{-n}x) + \sum_{k=0}^{n-1} 2^{-k-1} x h(2^{-k-1}x)$$

For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $x < \delta$  implies  $-\varepsilon < h(x), f(x) < \varepsilon$ , then

$$\left| \frac{f(x) - f(0)}{x} - k \right| \leq \left| \frac{f(2^{-n}x) - f(0)}{x} - k \right| + \sum_{k=0}^{n-1} 2^{-k-1} |h(2^{-k-1}x)| \leq \left| \frac{f(2^{-n}x) - f(0)}{x} - k \right| + 4\varepsilon.$$

Let  $n \rightarrow \infty$  then  $|(f(x) - f(0))/x - k| < 4\varepsilon$ . Hence  $f'(0) = k$ .

## 6.1.5

Prove that if  $f(x)$  is differentiable and even/odd/periodic, then  $f'(x)$  is odd/even/periodic.

Proof: If  $f(x) = -f(-x)$ , then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} = f'(x).$$

If  $f(x) = f(-x)$ , then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} -\frac{f(x-h) - f(x)}{h} = -f'(x).$$

If  $f(x) = f(x+T)$  then

$$f'(x+T) = \lim_{h \rightarrow 0} \frac{f(x+T+h) - f(x+T)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

## 6.1.7

Suppose  $g(0) = g'(0) = 0$ , and

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine the value of  $f'(0)$ .

Solution:

$$f'(0) = \lim_{h \rightarrow 0} \frac{g(h)}{h} \sin \frac{1}{h} = 0,$$

since  $g(h)/h \rightarrow 0$  and  $\sin \frac{1}{h}$  is bounded.

## 6.1.8

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfy  $f(x+y) = f(x)f(y)$ ,  $\forall x, y \in \mathbb{R}$ . If  $f'(0) = 1$ , prove that  $f'(x) = f(x)$  for any  $x \in \mathbb{R}$ .

Proof:  $f'(0) = 1$  so  $f$  is not constant. Hence let  $x = 0$  we have  $f(y) = f(0)f(y)$  so  $f(0) = 1$ . Note that

$f(x) = f(x/2)^2 \geq 0$ , so we can let  $g(x) = \log f(x)$ , then  $g'(x) = \frac{f'(x)}{f(x)}$  and  $g'(0) = 1$ ,  $g(0) = 0$ ,

$g(x+y) = g(x) + g(y)$ . Since  $g$  is continuous at 0,  $g(x) = cx$  so  $g(x) = x \forall x \in \mathbb{R}$ , hence  $g'(x) = 1$  and  $f'(x) = f(x)$ .

## 6.2

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### 6.2.1

(1)  $f(x) = x^{1/3}$ ,  $f'_+(0)$ :

$$f'_0(x) = \lim_{h \rightarrow 0^+} \frac{h^{1/3}}{h} = \infty.$$

(2)  $f(x) = |x-2|^3$ ,  $f'_-(2)$ :

$$f'_-(2) = \lim_{h \rightarrow 0^+} \frac{-h^3}{h} = 0.$$

## 6.2.2

Determine the values of  $a, b$  such that

$$f(x) = \begin{cases} x^2, & x \geq 1, \\ ax + b, & x < 1 \end{cases}$$

is differentiable at  $x = 1$ .

Solution: Clearly  $f'_+(1) = 2$ , while

$$f'_-(1) = \lim_{h \rightarrow 0^+} \frac{f(1) - f(1-h)}{h} = \lim_{h \rightarrow 0^+} \frac{1 - a(1-h) - b}{h} = 2$$

Hence  $a = 2$  and  $1 - a - b = 0$ , so  $(a, b) = (2, -1)$ .

## 6.2.3

Suppose  $f$  is continuous at 0. Prove that  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x}$  exists if  $\lim_{x \rightarrow 0} \frac{f(x) - f(x-x^3)}{x^3}$  exists.

Proof: Let  $f(x) - f(0) = xg(x)$ , and  $f(x) - f(x-x^3) = x^3h(x)$ .

If  $\lim_{x \rightarrow 0} \frac{f(x) - f(x-x^3)}{x^3} = L$  exists, then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x| < \delta$  implies  $|h(x) - L| < \varepsilon$ . Consider  $x = x_0 \in (0, \delta)$ , and  $x_{k+1} = x_k - x_k^3$ , then clearly  $\lim_{n \rightarrow \infty} x_n = 0$ . Note that

$$xg(x) = \sum_{k=0}^{\infty} f(x_k) - f(x_{k+1}) = \sum_{k=0}^{\infty} x_k^3 S(x_k) < (L + \varepsilon) \sum_{k=0}^{\infty} x_k - x_{k+1} = (L + \varepsilon)x$$

and likewise  $xg(x) > (L - \varepsilon)x$ , hence  $|g(x) - L| < \varepsilon$ . Therefore  $\lim_{x \rightarrow 0} g(x) = L$ .

(The reverse is incorrect, for example  $x^2 \sin x^{-1}$ ).

## 6.2.4

Calculate the Dini derivatives of  $f$  at  $x = 0$ :

$$(1) f(x) = x^2 \sin x^{-1}; (2) f(x) = |x|^{3/2}; (3) f(x) = e^{-|x|^2}.$$

Solution: (1)  $|f(x)| \leq |x|^2$ , so all four Dini derivatives are 0.

(2)(3) Likewise all four Dini derivatives are 0.

## 6.2.5

Calculate the Dini derivatives of  $f$ :

$$(1) f(x) = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0 \end{cases} \text{ at } x = 0;$$

(2)  $f(x) = \sin x \cos x$  at  $x = \pi$ :

(3)  $f(x) = x^2 \log|x|$  at  $x = 0$ :

Solution:

(1)  $|f(x)/x| \leq |x^2|$  so all Dini derivatives are 0.

(2)  $f'(\pi) = \cos 2\pi = 1$  so all Dini derivatives are 1.

(3)  $|f(x)/x| = |x| \log|x| \rightarrow 0$  so all Dini derivatives are 0.

## 6.2.6

Calculate the symmetric derivatives of  $f$ :

(1)  $f(x) = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0 \end{cases}$  at  $x = 0$ :

(2)  $f(x) = \sin x^2$  at  $x = \pi$ :

(3)  $f(x) = \log|x^2 - 1|$  at  $x = 1$ :

(4)  $f(x) = x^2 e^{-|x|}$  at  $x = 0$ :

Solution: (1)  $f(x) - f(-x) = 0$  so the derivative is 0.

(2)  $f$  is differentiable so the symmetric derivative is  $f'(\pi) = 2x \cos x^2|_{\pi} = 2\pi \cos \pi^2$ .

(3)  $f(1+t) - f(1-t) = \log|2t + t^2| - \log|2t - t^2| = \log(2+t)/(2-t)$ , and

$$\lim_{h \rightarrow 0} \frac{\log|2+h| - \log|2-h|}{2h} = \frac{1}{2}.$$

So the derivative is  $1/2$ .

(4)  $f$  is even so the derivative is clearly 0.

## 6.3

Calculate derivatives:

### 6.3.1

If  $f = \prod_{k=1}^n g_k$ , then

$$\frac{f'}{f} = \sum_{k=1}^n \frac{g'_k}{g_k}.$$

(1)  $f(x) = (x+1)(x+2)^2(x+3)^3$

$$f'(x) = f(x) \left( \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} \right).$$

(2)  $f(x) = (1+nx^m)(1+mx^n)$

$$f'(x) = f(x)nm \left( \frac{x^{m-1}}{1+nx^m} + \frac{x^{n-1}}{1+mx^n} \right).$$

(3)  $f(x) = (ax^m + b)^n(cx^n + d)^m$

$$f'(x) = nmf(x) \left( \frac{ax^{m-1}}{ax^m + b} + \frac{cx^{n-1}}{cx^n + d} \right).$$

### 6.3.2

(1)  $f(x) = \sqrt{\sin^2 x}$

$f'(x) = \text{sign}(\sin x) \cos x$  when  $x \neq k\pi$ .

(2)  $f(x) = \frac{x}{(1-x^2)(1+x)}$ .

$$f'(x) = \frac{1+x^2+2x^3}{(1-x)^2(1+x)^4}.$$

(3)  $f(x) = x^p(1-x)^q/(1+x)$

$$f'(x) = -\frac{x^{p-1}(1-x)^{q-1}((p+q-1)x^2 + (q+1)x - p)}{(x+1)^2}.$$

$$(4) f(x) = \frac{x \sin x + \cos x}{x \sin x - \cos x}.$$

$$f'(x) = -\frac{2x + \sin 2x}{(x \sin x - \cos x)^2}.$$

### 6.3.3

$$(1) f(x) = \frac{x}{\sqrt{a^2+x^2}}: f'(x) = a^2(a^2+x^2)^{-3/2}.$$

$$(2) f(x) = \arcsin(\cos^2 x): f'(x) = -\sin 2x / \sqrt{1 - \cos^4 x}.$$

$$(3) f(x) = \arcsin x \sqrt{1-x^2}: f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}\sqrt{x^4-x^2+1}}.$$

$$(4) f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}: f'(x) = \frac{1+2\sqrt{x}+4\sqrt{x+\sqrt{x}}\sqrt{x}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}.$$

$$(5) f(x) = e^{ax} \sin bx: f'(x) = e^{ax}(a \sin bx + b \cos bx).$$

### 6.3.4

$$(1) f(x) = (x \sin x)^x: f'(x) = (x \sin x)^x(x \cot x + \log(x \sin x) + 1).$$

$$(2) f(x) = x \sqrt{(1-x)/(1+x)}: f'(x) = \frac{1-x-x^2}{(1+x)^{3/2}\sqrt{1-x}}.$$

$$(3) f(x) = (x)^{a^x}: f'(x) = a^x x^{a^x-1}(x \log a \log x + 1).$$

$$(4) f(x) = a^{x^n}: f'(a^{x^n} n x^{n-1} \log a).$$

$$(5) f(x) = (\cos x)^{\sin x} + (\sin x)^{\cos x}:$$

$$f'(x) = (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x) + (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x).$$

$$(6) f(x) = (x + \sqrt{1+x^2})^n: f'(x) = \frac{n(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}}.$$

### 6.3.5

Suppose  $f(x) = \cos(2 \arctan(\sin(\arccot \sqrt{(1-x)/x})))$ , prove that  $\frac{f'(x)}{f(x)^2} = \pm \frac{2}{(1 \pm x)^2}$ .

Proof:  $f(x) = \frac{1-x}{1+x}$  so  $\frac{f'}{f^2} = -\frac{2}{(1-x)^2}$ .

### 6.3.6

Suppose  $y = \frac{1}{4\sqrt{2}} \log \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} - \frac{1}{2\sqrt{2}} \arctan \frac{\sqrt{2}x}{x^2 - 1}$ , calculate  $y'(x)$ .

Solution:

$$y'(x) = \frac{1}{x^4 + 1}.$$

### 6.3.7

Determine  $dy/dx$ :

$$(1) \begin{cases} x = \cos^4 t, \\ y = \sin^4 t \end{cases}, \text{ at } t = \pi/3;$$

$$dx/dt = 4 \cos^3 t (-\sin t) = -\frac{\sqrt{3}}{4}, dy/dt = 4 \sin^3 t \cos t, \text{ so } dy/dx = -3.$$

$$(2) \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \text{ at } t = \pi/2, \pi:$$

$$\begin{aligned}\frac{dx}{dt} &= a(1 - \cos t), \frac{dy}{dt} = a \sin t. \\ t = \frac{\pi}{2} : \frac{dy}{dx} &= 1. t = \pi. \frac{dy}{dx} = 0.\end{aligned}$$

### 6.3.8

Suppose  $f$  is differentiable, calculate  $y'$ :

- (1)  $y = f(x \sin x)$ :  $y' = (\sin x + x \cos x)f'(x \sin x)$ .
- (2)  $y = f(e^x)e^{f(x)}$ :  $y' = f'(e^x)e^{x+f(x)} + f(e^x)e^{f(x)}f'(x)$ .

### 6.3.9

Suppose  $u, v$  differentiable, calculate  $y'$ :

$$\begin{aligned}(1) y &= \sqrt{u^2 + v^2}: y' = \frac{uu' + vv'}{\sqrt{u^2 + v^2}}. \\ (5) y &= 1/\sqrt{u^2 + v^2}: y' = -\frac{u'u + v'v}{(u^2 + v^2)^{3/2}}.\end{aligned}$$

### 6.3.10

Suppose  $a_{ij}(x)$  are differentiable, determine the derivative of  $\det(a_{ij}(x))_{n \times n}$ .

Solution: Let  $\varphi_i(x) = (a_{i1}(x), \dots, a_{in}(x))$ , then  $\det(\varphi_1, \dots, \varphi_n)$  is a multi-linear form, so

$$\frac{d}{dx} \Big|_{x=x_0} \det(\varphi_1, \dots, \varphi_n) = \sum_{k=1}^n \det(\varphi_1, \dots, \varphi'_k, \dots, \varphi_n).$$

(If  $A = (a_{ij})$  is invertible, then suppose  $A' = CA$ , we have  $\varphi'_k = \sum_j C_{k,j} \varphi_j$ . Since  $\det$  is alternating,  $\det(\varphi_1, \dots, \varphi'_k, \dots, \varphi_n) = C_{k,k} \det A$ , hence  $(\det A)' = \det A \sum C_{k,k} = \det A \cdot \text{tr}(A'A^{-1})$ .)

### 6.3.14

Suppose  $x_1, \dots, x_n$  are distinct, and  $y(x) = \prod_{k=1}^n (x - x_k)$ . Calculate  $\sum_{k=1}^n 1/y'(x_k)$ . What if  $x_k$  are not distinct?

Solution: Note that

$$\frac{y'}{y} = \sum \frac{1}{x - x_k},$$

hence  $y'(x_k) = \prod_{j \neq k} (x_k - x_j)$ .

By Lagrange interpolation,

$$1 = \sum_{k=1}^n \prod_{j \neq k} \frac{x - x_j}{x_k - x_j},$$

so by considering the coefficient of  $x^{n-1}$ , we obtain

$$\sum_{k=1}^n 1/y'(x_k) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

If  $f(x) = \prod_{k=1}^n (x - x_k)^{\alpha_k}$ , then  $f'(x_k) = \alpha_k \prod_{j \neq k} (x_k - x_j)^{\alpha_j}$ , hence

$$\sum \frac{1}{f'(x_k)} = \sum_{k=1}^n \frac{\alpha_k}{f'(x_k)} = \sum_{k=1}^n \frac{1}{\prod_{j \neq k} (x_k - x_j)^{\alpha_j}} = \begin{cases} 1, & \deg f = 1, \\ 0, & \deg f > 1. \end{cases}$$

### 6.3.19

Suppose  $f(x_0) = g(x_0)$ ,  $f, g$  are differentiable at  $x_0$ , and  $f'(x_0) = g'(x_0) = k$ . Prove that for any  $h$  such that  $\min\{f, g\} \leq h \leq \max\{f, g\}$ ,  $h$  is differentiable at  $x_0$  and  $h'(x_0) = k$ .

Proof: Clearly  $h(x_0) = f(x_0)$  and we can assume it to be 0. For any  $t$ ,

$$\min \left\{ \frac{f(x_0 + t)}{t}, \frac{g(x_0 + t)}{t} \right\} \leq \frac{h(x_0 + t) - h(x_0)}{t} = \frac{h(x_0 + t)}{t} \leq \max \left\{ \frac{g(x_0 + t)}{t}, \frac{f(x_0 + t)}{t} \right\}.$$

Since  $\lim_{t \rightarrow 0} f(x_0 + t)/t = \lim_{t \rightarrow 0} g(x_0 + t)/t$ , the two sides of the inequality both tend to  $k$ , therefore  $\lim_{t \rightarrow 0} (h(x_0 + t) - h(x_0))/t = k$ . Hence  $h'(x_0) = k$ .

### 6.3.22

Prove that there is no  $f : \mathbb{R} \rightarrow \mathbb{R}$  differentiable such that  $f \circ f(x) = -x^3 + x^2 + 1$ .

Proof:  $f'(f(x))f'(x) = -3x^2 + 2x$ , and  $f(f(x)) = x \iff x = 1$ . If  $a = f(1)$  then  $f(f(a)) = f(f(f(1))) = f(1) = a$ , so by uniqueness  $a = 1$ . However,  $-1 = f'(f(1))f'(1) = (f'(1))^2$ , leading to contradiction.

### 6.3.23

Prove that there is no  $f : \mathbb{R} \rightarrow \mathbb{R}$  differentiable such that  $f \circ f(x) = x^2 - 3x + 3$ .

Proof: Suppose  $f(f(x)) = x^2 - 3x + 3$ , then  $f'(f(x))f'(x) = 2x - 3$ , but  $f(f(x)) = x \iff x = 1, 3$ .

Suppose  $a = f(1)$ , then  $f(a) = 1$ , so  $-1 = f'(f(1))f'(1) = f'(a)f'(1)$ , and

$2a - 3 = f'(f(a))f'(a) = f'(a)f'(1)$ , leading to contradiction.

(Note:  $x^2 - 3x + 3$  is not the iteration of any mapping  $f$  and any order  $r$ , so the differentiable condition is not needed.)

## 6.4

### 6.4.1

If  $x^y = y^x$ , prove that  $x(y \log x - x)dy = y(x \log y - y)dx$ .

Proof:  $x \log y = y \log x$ , so  $y' \log x + y/x = \log y + xy'/y$ . Hence  $dy(\log x - x/y) = dx(\log y - y/x)$ .

Therefore  $x(y \log x - x)dy = y(x \log y - y)dx$ .

### 6.4.2

If  $y = \frac{x-a}{1+ax}$ , then  $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$ .

Proof: Note that  $y = \tan(\arctan x - \arctan a)$ , so  $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$  since  $d\arctan y = d\arctan x$ .

( $ds = \frac{dx}{1+x^2}$  is the angular distance in spherical geometry, and clearly rotation on the sphere is an isometry. We can do likewise for hyperbolas and the Blaschke factors.)

### 6.4.3-6.3.6

If  $xy = -1$  prove that  $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$ .

Proof: Clearly  $\frac{dy}{dx} = x^{-2}$ , so  $\frac{dy}{\sqrt{1+y^4}} = \frac{dx}{\sqrt{1+x^4}}$ .

( $ds = \frac{dx}{\sqrt{1+x^4}}$  is the parametrization of the hyperbolic lemniscate, and  $y = -1/x$  is the inversion which is an isometry).

Note: For the next 3 problems, they are all about preserving the arc length of the lemniscate (the  $\infty$  shaped

curve). (The hyperbolic lemniscate  $\frac{dx}{\sqrt{1+x^4}}$  is just the lemniscate  $\frac{dx}{\sqrt{1-x^4}}$  rotated  $\frac{\pi}{4}$ .)  
6.4.4 comes from  $(1 + \text{sl}^2(u))(1 + \text{cl}^2(u)) = 2$ , since  $x^2 + y^2 + x^2y^2 = 1$  implies  
 $x = \text{sl}(u)$ ,  $y = \pm \text{cl}(u) = \pm \text{sl}(\varpi/2 - u)$ , and  $\frac{dx}{\sqrt{1-x^4}} = du = \pm \frac{dy}{\sqrt{1-y^4}}$ .  
6.4.5 comes from the rotation  $\text{sl}(u+v) = \frac{\text{sl}(u)\text{cl}(v)+\text{cl}(u)\text{sl}(v)}{1+\text{sl}^2(u)\text{sl}^2(v)}$ , since the equation implies  
 $x = \text{sl}(u)$ ,  $c = \text{sl}(v)$ ,  $y = \text{sl}(u+v)$ .  
6.4.6 comes from duplication  $\text{sl}(2u) = \frac{2\text{sl}(u)\text{cl}(u)}{1+\text{sl}^4(u)}$ , since  $(y^4 - 2y^2 - 1)x + y^4 + 2y^2 - 1 = 0$  implies  
 $x = \text{sl}(2u)$ ,  $y = \text{sl}(u)$ .

We can unify the problems above using elliptic integrals:

Consider the elliptic metric  $\omega_J = \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ , and the isometry  $x = \text{sn}(u) \mapsto y = \text{sn}(u+v)$ , where  
 $t = \int_0^{\text{sn}(t)} \omega$  is the Jacobi elliptic integral (of the first kind). Since it is an isometry, we have

$$(x^2 + y^2 + a^2) - 2(x^2y^2 + y^2a^2 + a^2x^2) + 4k^2x^2y^2a^2 - k^4(x^2y^2a^2)^2 = 0$$

implies  $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$ .

For  $k = 0, 1, i$  we obtain all the results mentioned above (spherical, hyperbolic, lemniscate).

## 6.5

### 6.5.1

- (1)  $f(x) = e^x \cos x$ ,  $f^{(5)}(x) =$   
 $f(x) = \text{Re}(e^{(1+i)x})$ , then  $f^{(5)}(x) = \text{Re}((-4 - 4i)e^{(1+i)x}) = 4e^x(\sin x - \cos x)$ .  
(2)  $f(x) = x^2 \log x + x \log^2 x$ ,  $f'' =$   
 $f''(x) = 2 \log x + 3 + 2 \frac{\log x + 1}{x}$ .  
(3)  $f(x) = x^2 e^x$ ,  $f^{(10)} =$   
 $f^{(10)}(x) = \sum_{k=0}^{10} \binom{10}{k} (x^2)^{(k)} (e^x)^{(10-k)} = e^x(x^2 + 20x + 90)$ .  
(4)  $f(x) = x^5 \cos x$ ,  $f^{(50)} =$   
Likewise  $f^{(50)}(x) = (-x^5 + 20 \binom{50}{2} x^3 - 120 \binom{50}{4}) \cos x + (-250x^4 + 60 \binom{50}{3} x^2 - 120 \binom{50}{5}) \sin x$ .

### 6.5.2

Prove that

$$(1) \sum_{k=0}^n \binom{n}{k} k = n2^{n-1} \quad (2) \sum_{k=0}^n \binom{n}{k} k^2 = n(n+1)2^{n-2}.$$

Proof:  $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ , hence

$$\sum_{k=0}^n \binom{n}{k} kx^k = xn(1+x)^{n-1}, \sum_{k=0}^n \binom{n}{k} k^2 x^{k-1} = n(1+x)^{n-1} + xn(n-1)(1+x)^{n-2}.$$

Let  $x = 1$  we obtain the desired identities.

### 6.5.3

Prove that  $y = \cos(n \arccos x)$  satisfy

$$\frac{ndx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}}, (1-x^2)y'' - xy' + n^2y = 0.$$

Proof:  $y = P_n$  is the  $n^{th}$  Chebyshev polynomial.

Note that  $\arccos y = n \arccos x$ , hence  $\frac{dy}{\sqrt{1-y^2}} = d \arccos y = n \frac{dx}{\sqrt{1-x^2}}$ .

Let  $x = \cos \theta$ , and  $y = \cos(n\theta)$ , then  $y' = -\sin(n\theta)n\theta' = n \sin(n\theta)/\sin \theta$ . Differentiate  $y' \sin \theta = n \sin n\theta$ , we obtain

$$y'' \sin \theta + y'\theta' \cos \theta = n^2 \theta' \cos(n\theta) \implies y'' \sqrt{1-x^2} - \frac{y'x}{\sqrt{1-x^2}} = -\frac{n^2 y}{\sqrt{1-x^2}}.$$

Hence  $(1-x^2)y'' - xy' + n^2 y = 0$ .

(We can use the fact the  $P_n$  are orthogonal under the weight  $w = \frac{1}{\sqrt{1-x^2}}$ , and solve the Pearson differential

equation  $\frac{d}{dx}(\sigma w) = \tau w$  to get  $\sigma = 1-x^2$ ,  $\tau = -x$  and directly obtain this differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0.)$$

## 6.5.4

Prove that: For points  $(x, y)$  on the circle  $(x-a)^2 + (y-b)^2 = c^2$ ,  $y \neq b$ ,

$$(1) y''/(1+(y')^2)^{3/2} = \pm 1/c; (2) y'''(1+(y')^2) = 3y'(y'')^2.$$

Proof: (1)  $1+(y')^2 = 1 + \frac{(x-a)^2}{(y-b)^2} = \frac{c^2}{(y-b)^2}$ , and  $y'' = -\frac{c^2}{(y-b)^3}$ , hence  $y''/(1+(y')^2)^{3/2} = \pm 1/c$ . Its absolute value is the curvature at  $(x, y)$ .

$$(2) y''' = 3c^2 y'/(y-b)^4$$
, so  $y'''(1+(y')^2) = 3c^4 y'/(y-b)^6 = 3y'(y'')^2$ .

## 6.5.5

Given the polar coordinate form  $r^2 = \cos 2\theta$  of  $y = f(x)$ , calculate  $\frac{d^2y}{dx^2}$ .

Solution:  $r^2 = \cos 2\theta \implies x^2 + y^2 = \frac{1-y^2/x^2}{1+y^2/x^2}$ . Hence  $(x^2 + y^2)^2 = x^2 - y^2$ . Therefore  $\frac{d^2y}{dx^2} = -\frac{3r}{\sin^3(3\theta)}$ .

## 6.5.6

Suppose  $f(x) = x \log(2^{1/x} + 3^{1/x})$ ,  $\forall x > 0$  prove that  $f'(x), f''(x) > 0$ .

Proof: Let  $g(x) = \log(2^x + 3^x)/x$  and  $h(x) = \log(2^x + 3^x)$ . Note that by Cauchy-Schwarz,

$$h''(x) = \frac{(2^x \log^2 2 + 3^x \log^2 3)(2^x + 3^x) - (2^x \log 2 + 3^x \log 3)^2}{(2^x + 3^x)^2} > 0.$$

Since  $f''(x) = x^{-3} h''(x^{-1})$ , we infer  $f''(x) > 0$ .

Let  $F(t) = f'(t^{-1}) = h(t) - th'(t)$ , then  $F'(t) = -th''(t) < 0$ , hence  $F$  is strictly decreasing. Clearly  $\lim_{t \rightarrow \infty} F(t) = 0$ , so  $f'(x) = F(x^{-1}) > 0$ .

## 6.5.7

Suppose  $P$  is a polynomial of degree  $n$  with  $n$  distinct roots. Prove that  $P'(x)^2 \geq P(x)P''(x)$ .

Proof: Suppose  $P(x) = \prod_{k=1}^n (x - x_k)$ , and  $P(t) \neq 0$ . Then

$$\frac{PP'' - P'(t)^2}{P(t)^2} = \frac{d}{dt} \frac{P'(t)}{P(t)} = \frac{d}{dt} \sum_{k=1}^n \frac{1}{t - x_k} = \sum_{k=1}^n -(t - x_k)^{-2} < 0.$$

Therefore  $P'(t)^2 \geq P(t)P''(t)$ .

## 6.5.9

Suppose  $f'(x) \neq 0$  and  $f^{-1}$  exists. Express  $(f^{-1})^{(3)}$  in terms of  $f'$ ,  $f''$ ,  $f^{(3)}$ .

Solution:

$$(f^{-1})^{(3)}(y) = \frac{3[f''(f^{-1}(y))]^2 - f'(f^{-1}(y))f^{(3)}(f^{-1}(y))}{[f'(f^{-1}(y))]^5}$$

## 6.5.10

$a, b \in \mathbb{R}$ ,  $b > 0$ . Consider  $f : [-1, 1] \rightarrow \mathbb{R}$ , where

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

1.  $f \in C([-1, 1])$  iff  $a > 0$ ;

Proof:  $f \in C([-1, 1])$  iff  $\lim_{x \rightarrow 0} x^a \sin(x^{-b}) = 0$ . If  $a > 0$  then  $|x^a \sin(x^{-b})| \leq |x|^a \rightarrow 0$ . If  $a < 0$  then let  $x = ((2n + 1/2)\pi)^{-1/b}$ , when  $n \rightarrow \infty$ ,  $x \rightarrow 0$  but  $|x^a \sin(x^{-b})| \rightarrow \infty$ . If  $a = 0$ , then let  $x = ((2n + 1/2)\pi)^{-1/b}$ ,  $|x^a \sin(x^{-b})| = 1$ . Therefore  $f \in C([-1, 1])$  iff  $a > 0$ .

2.  $f$  is differentiable at 0 iff  $a > 1$ ;

Proof:  $f$  is differentiable at 0 iff  $\lim_{x \rightarrow 0} x^{-a} \sin(x^{-b})$  exists. By 1 we know that  $a > 1$ . ( $a = 1$  is invalid since  $x = (2n\pi)^{-1/b}$  and  $x = ((2n + 1/2)\pi)^{-1/b}$  converge to different values.)

3.  $f'$  is bounded on  $[-1, 1]$  ( $\iff f$  is Lipschitz) iff  $a \geq 1 + b$ ;

Proof:  $f'(x) = ax^{a-1} \sin(x^{-b}) + x^a \cos(x^{-b})(-b)x^{-b-1}$  is bounded iff  $x^{a-1}$  and  $x^{a-b-1}$  are bounded, i.e.  $a \geq 1 + b$ .

4.  $f \in C^1([-1, 1])$  iff  $a > 1 + b$ ;

Proof:  $f \in C^1([-1, 1])$  iff  $f'(0) = 0 = \lim_{x \rightarrow 0} f'(x)$ . By 1 we know it is equivalent to  $a > 1 + b$ .

5.  $f'$  is differentiable at 0 iff  $a > 2 + b$ ;

6.  $f''$  is bounded on  $[-1, 1]$  iff  $a \geq 2 + 2b$ ;

7.  $f \in C^2([-1, 1])$  iff  $a > 2 + 2b$ .

Proof: 5, 6, 7 are exactly the same as 2, 3, 4.

Likewise  $f \in C^n([-1, 1])$  iff  $a > n(1 + b)$ ,  $f^{(n-1)}$  differentiable at 0 iff  $a > n + (n - 1)b$ , and  $f^{(n)}$  is bounded iff  $a \geq n + nb$ .

## 6.5.11

Suppose  $f(x) = \prod_{k=1}^n (x - x_k)$  where  $x_k$  are distinct, and  $\xi_1, \dots, \xi_{n-1}$  are the roots of  $f'(x)$ .

Calculate  $\sum_{k=1}^{n-1} 1/f''(\xi_k)$ .

Solution: Same as 6.3.14, use Lagrange interpolation, the sum for the  $k^{th}$  derivative is  $\frac{\delta_{k,n}}{k!}$ .