2025/9/17

作业二：【尤】**1.3**：5、13；**1.4**：7；**1.5**：1、4、7

1.3.5

$\alpha = (1,0,2)$, $\beta = (1,1,1)$, $\gamma = (1,0,-2)$. Try to write α in the form $\alpha_1 + \alpha_2$ where $\alpha_1 \in \langle \beta, \gamma \rangle$, and $\alpha_2 \in \langle \beta, \gamma \rangle^\perp$.
Solution: Note that $\langle \beta, \gamma \rangle^\perp = \langle \beta \times \gamma \rangle$ since β, γ are linearly independent. $\beta \times \gamma = (-2, 3, -1)$, so $\alpha_2 = \langle \alpha, \beta \times \gamma \rangle (\beta \times \gamma)/|\beta \times \gamma|^2 = (4, -6, 2)/7$, and $\alpha_1 = \alpha - \alpha_2$.

1.3.13

For $A, B, C, D \in \mathbb{R}^3$, prove that
(1) $\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CA} \cdot \overrightarrow{BD} = 0$;
Proof: Let $\alpha = \overrightarrow{AB}$, $\beta = \overrightarrow{AC}$, $\gamma = \overrightarrow{AD}$, then the left hand side becomes

$$\alpha \cdot (\gamma - \beta) + (\beta - \alpha) \cdot \gamma + (-\beta) \cdot (\gamma - \alpha) = 0,$$

which is trivial.
(2) $\overrightarrow{AB^2} + \overrightarrow{CD^2} = \overrightarrow{AC^2} + \overrightarrow{BD^2} \iff \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$.
Likewise the first identity becomes

$$\alpha^2 + (\beta - \gamma)^2 = \beta^2 + (\alpha - \gamma)^2 \iff \beta\gamma = \alpha\gamma,$$

and the second one becomes $\gamma(\alpha - \beta) = 0$.

1.4.7

Suppose $A, B, C \in \mathbb{R}^2$ with coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ respectively. Prove that the area of $\triangle ABC$ is

$$S = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

Proof: Note that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \overrightarrow{AB} \times \overrightarrow{AC}$$

Hence $S = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$, implying the givin identity.

1.5.1

Prove that for any $\alpha, \beta, \gamma \in \mathbb{R}^3$,

$$\alpha \times (\beta \times \gamma) + \beta \times (\gamma \times \alpha) + \gamma \times (\alpha \times \beta) = \mathbf{0}.$$

Proof: We know that $\alpha \times (\beta \times \gamma) = (\alpha \cdot \gamma)\beta - (\alpha \cdot \beta)\gamma$, so adding up all three terms we obtain this identity.

1.5.4

Given points $A, B, C, D \in \mathbb{R}^3$ with coordinates $(1,0,1), (-1,1,5), (-1,-3,-3), (0,3,4)$ respectively. Determine this tetrahedral's volume.

Solution: Note that $\alpha = \overrightarrow{AB} = (-2,1,4)$, $\beta = \overrightarrow{AC} = (-2,-3,-4)$, $\gamma = \overrightarrow{AD} = (-1,3,3)$.
The volume should be

$$V = \frac{1}{6}|\alpha \cdot (\beta \times \gamma)| = \frac{1}{6}|(-2,1,4) \cdot (3,10,-9)| = \frac{1}{6} \cdot |-6+10-36| = \frac{16}{3}.$$

1.5.7

Prove that

$$\alpha \times (\beta \times (\gamma \times \delta)) = (\beta \cdot \delta)(\alpha \times \gamma) - (\beta \cdot \gamma)(\alpha \times \delta).$$

Proof:

$$\alpha \times (\beta \times (\gamma \times \delta)) = \alpha \times ((\beta \cdot \delta)\gamma - (\beta \cdot \gamma)\delta) = (\beta \cdot \delta)(\alpha \times \gamma) - (\beta \cdot \gamma)(\alpha \times \delta),$$

where the first equality is Lagrange's identity and the second from linearity of $\times$.