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66-3

Consider the vectors in $\mathbb{R}^4$ defined by $\alpha_1 = (-1, 0, 1, 2)$, $\alpha_2 = (3, 4, -2, 5)$, $\alpha_3 = (1, 4, 0, 9)$. Find a system of homogenous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.
Solution: Let $V = \langle \alpha_1, \alpha_2, \alpha_3 \rangle^\perp$, and take $\beta_1 = \left(1, -\frac{1}{4}, 1, 0\right), \beta_2 = \left(0, -\frac{9}{4}, -2, 1\right) \in V$ which forms a base of V, then we obtain the equation

$$\begin{cases} a - \frac{1}{4}b + c = 0, \\ -\frac{9}{4}b - 2c + d = 0. \end{cases}$$

66-5

Give an explicit description of the type (2-25) for the vectors $\beta = (b_1, b_2, b_3, b_4, b_5)$ in $\mathbb{R}^5$ which are linear combinations of the vectors $\alpha_1 = (1, 0, 2, 1, -1)$, $\alpha_2 = (-1, 2, -4, 2, 0)$, $\alpha_3 = (2, -1, 5, 2, 1)$, $\alpha_4 = (2, 1, 3, 5, 2)$.
Solution: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ can be transformed into the following row-reduced echelon matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Then $\mathrm{Span}\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \left\{(x_1, x_2, x_3, x_4, x_5) : x_5 = x_4 + 2x_3 - \frac{2}{3}x_2 - \frac{1}{3}x_1\right\}$.

66-6

Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

(a) Find a basis for V.
(b) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.
(c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?
Solution: Note that $(3, 21, 0, 9, 0) + (1, 7, -1, -2, -1) + (2, 14, 0, 6, 1) = (6, 42, -1, 13, 0)$, and $(1, 7, 0, 3, 0), (1, 7, -1, -2, -1), (2, 14, 0, 6, 1)$ can be transformed into $(1, 7, 0, 3, 0), (0, 0, 0, 0, 1), (0, 0, 1, 5, 0)$, clearly they are linearly independent so they form a basis of V.
(b) All $(x_1, x_2, x_3, x_4, x_5)$ such that $x_1 = -3x_4 - 7x_2$ and $x_3 = -5x_4$.
(c) Note that $(x_1, x_2, x_3, x_4, x_5) = x_5(0, 0, 0, 0, 1) + x_1(1, 7, 0, 3, 0) + x_3(0, 0, 1, 5, 0)$, so its coordinates are (x_1, x_5, x_3).

66-7

Let A be an $m \times n$ matrix over the field F, and consider the system of equations $AX = Y$. Prove that this system of equations has a solution iff the row rank of A is equal to the row rank of the augmented matrix of the system.

Proof: $AX = Y$ has a solution iff there exists $a_1, \cdots, a_n$ such that $A\left(\sum_{i=1}^{n} a_i e_i\right) = Y$. Denote the k-th row of A by C_k, then $AX = Y$ has a solution iff $\sum_{k=1}^{n} a_k C_k = Y$ for some $a_k \in F$, which is equivalent to $Y \in col(A)$ which means column rank of A is equal to column rank of the augmented matrix. Since row rank equals column rank, we obtain the desired proposition.