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21-3

Find two different 2×2 matrices A such that $A^2 = 0$ but $A \neq 0$.
Solution:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

21-5

Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}.$$

Is there a matrix C such that $CA = B$?
Solution: Yes, for example

$$C = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{5}{2} & -\frac{3}{4} & 0 \end{pmatrix}.$$

21-6

Let A be an $m \times n$ matrix and B an $n \times k$ matrix. Show that the columns of $C = AB$ are the linear combinations of the columns of A. If $\alpha_1, \cdots, \alpha_n$ are the columns of A and $\gamma_1, \cdots, \gamma_k$ are the columns of C, then

$$\gamma_j = \sum_{r=1}^{n} B_{r,j} \alpha_r.$$

Proof: Since $C^T = B^T A^T$, and γ_j is the j-th row of C^T, α_r is the r-th row of A^T.

21-7

Let A, B be 2×2 matrices such that $AB = I$. Prove that $BA = I$.
Proof: If $AB = I$ then A is invertible so $B = A^{-1}$ and $BA = I$.

21-8

Let

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices A, B such that $C = AB - BA$. Prove that such matrices can be found iff $C_{11} + C_{22} = 0$.
Proof: Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then $\mathrm{tr}(AB) = ax + bz + cy + dw = \mathrm{tr}(BA)$ so $\mathrm{tr}(C) = 0$.

Let $[A, B] = AB - BA$. We show that if $\mathrm{tr} C = 0$ then C is a commutator. Note that the if $P^{-1}CP = [A, B]$ is a commutator, then $C = [PAP^{-1}, PBP^{-1}]$ is a commutator. Since $\mathrm{tr}(C) = 0$, C is similar to a matrix D whose diagonal contains only zeros. Let $A = \mathrm{diag}(1, 2, \cdots, n)$, we find B such that $D = [A, B]$. Note that

$$[A, B]_{i,j} = \sum_{k=1}^{n} A_{ik}B_{kj} - B_{ik}A_{kj} = B_{ij}(i - j),$$

hence we only need to define $B_{ij} = \frac{D_{ij}}{i-j}$, then $D = [A, B]$.
(Prove that C is similar to a matrix D whose diagonal is all zero: $\mathrm{tr}(C) = 0$ implies the sum of its eigenvectors are zero, hence $0 \in \{x^*Cx : |x| = 1\}$. Let $u_1^*Cu_1 = 0$ and extend it to $u_1, \cdots, u_n$ a orthogonal base of $F^{n\times 1}$. Under this base, D is a matrix with $D_{11} = 0$. Then use induction.)

73-9

Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If $T(A) = AB - BA$, verify that T is a linear transformation from V into V.
Proof: Clearly AB and BA are both linear, so T is linear.

73-11

Let $V = F^{n\times 1}, W = F^{m\times 1}$. Let A be a fixed $m \times n$ matrix over F and let T be the linear transformation from V into W defined by $T(A) = AX$. Prove that T is the zero transformation iff A is the zero matrix.
Proof: If $T = 0$, then $T(e_j) = 0$ so every row of X is 0, hence $X = 0$.