

习题8.1--习题8.4, 每节习题中任选 5 道题(每道小题也算一道题).

## 8.1

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### 8.1.1

Prove that if  $\int f(x) dx = F(x) + C$ , then  $\int f(ax + b) dx = \frac{1}{a}F(ax + b) + C, a \neq 0$ .

Proof:  $F' = f$  so  $(\frac{1}{a}F(ax + b))' = f(ax + b)$ .

### 8.1.2(1)

$$\int (2^x + 3^x) dx = \frac{2^x}{\log 2} + \frac{3^x}{\log 3} + C.$$

### 8.1.2(2)

$$\int \tan^2 x dx = -x + \tan x + C.$$

### 8.1.2(3)

$$\int \frac{2x^2}{1+x^2} dx = 2x - 2 \arctan x + C.$$

### 8.1.2(4)

$$\int \cos^2 x dx = \int \frac{\cos 2x + 1}{2} dx = \frac{2x + \sin x}{4} + C.$$

## 8.2

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### 8.2.2

Suppose  $a < c < b$ ,  $f \in C((a, b))$ , and  $F'(x) = f(x), \forall x \in (a, c) \cup (c, b)$  where  $F \in C((a, b))$ . Prove that  $F$  is the primitive of  $f$  on  $(a, b)$ .

Proof: By L'Hopital rule,

$$\lim_{h \rightarrow 0^+} \frac{F(c+h) - F(c)}{h} = \lim_{h \rightarrow 0^+} f(c+h) = f(c) = \lim_{h \rightarrow 0^-} \frac{F(c+h) - F(c)}{h},$$

hence  $F'(c) = f(c)$ , so  $F' = f$  on  $(a, b)$ .

### 8.2.4

Suppose  $f$  is bounded and has a primitive on  $[a, b]$ ,  $g \in C([a, b])$ , prove that  $fg$  has a primitive on  $[a, b]$ .

Proof: Suppose  $|f| \leq M, F' = f$ . Take polynomials  $P_n \rightarrow g$  uniformly, then  $fP_n = (FP_n)' - FP_n'$  where  $FP_n'$  is continuous, so  $fP_n$  has a primitive  $G_n$ , and we can let  $G_n(a) = 0$ .

Since  $P_n \rightarrow g$  uniformly and  $f$  is bounded,  $G_n' \rightarrow fg$  uniformly. By mean value theorem,

$|(G_n - G_m)(x)| = (x-a)|(G_n' - G_m')(\xi)| < (b-a)\varepsilon_m$ . Hence  $G_n(x)$  is uniformly Cauchy for ever  $x \in [a, b]$ , so  $G_n \rightarrow G$  uniformly. Therefore  $G' = \lim_{n \rightarrow \infty} G_n' = fg$ .

## 8.2.5

Suppose  $f$  is differentiable on  $\mathbb{R}$ ,  $\lim_{x \rightarrow \infty} f(x)/x = 0$ , prove that  $g(x) = \begin{cases} f'(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$  has a primitive on  $\mathbb{R}$ .

Proof: Consider  $F(x) = \begin{cases} x^2 f(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ , then  $F'(x) = -f'(1/x) + 2xf(1/x)$  for  $x \neq 0$ , and

$F'(0) = \lim_{x \rightarrow 0} xf(1/x) = 0$ . Let  $G(x) = \begin{cases} 2xf(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ , then  $G \in C(\mathbb{R})$  so  $G$  has a primitive.

Hence  $g = G - F'$  has a primitive on  $\mathbb{R}$ .

## 8.2.8

Suppose  $F$  is a primitive of  $f$  on  $\mathbb{R}$  with a lower bound, prove that  $\inf_{x \in \mathbb{R}} |f(x)| = 0$ .

Proof: Otherwise, by Darboux's theorem, suppose  $f(x) > \varepsilon > 0, \forall x \in \mathbb{R}$ , then

$F(0) - F(-n) = nF'(\xi) = nf(\xi) > n\varepsilon$  so  $F(-n) < F(0) - n\varepsilon$  has no lower bound.

## 8.2.9

Suppose  $f, g$  both have primitives on the interval  $I$ , is it necessary that  $fg$  also has a primitive on  $I$ ?

Solution: No, consider  $f(x) = \begin{cases} \cos 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$  for  $x \in [-1, 1]$ , then

$(x^2 \sin 1/x)' = 2x \sin 1/x - \cos 1/x$  so  $f$  has a primitive on  $[-1, 1]$ . However,

$f^2 = \begin{cases} \frac{1}{2} + \frac{1}{2} \cos 2/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$  so  $f^2(x) - \frac{1}{2}f(x/2)$  does not satisfy the Darboux property. Hence  $f^2$  has

no primitive on  $I$ .

## 8.3

### 8.3.1(6)

$$\int \sin mx \sin nx \, dx = \int \frac{\cos(m-n)x - \cos(m+n)x}{2} \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C.$$

### 8.3.2(10)

$$\begin{aligned} \int \frac{dx}{a \sin x + b \cos x} &= \int \frac{2dt/(1+t^2)}{2at/(1+t^2) + b(1-t^2)/(1+t^2)} = 2 \int \frac{dt}{2at + b - bt^2} \\ &= \frac{1}{\sqrt{a^2 + b^2}} \log \frac{bt - a + \sqrt{a^2 + b^2}}{bt - a - \sqrt{a^2 + b^2}} + C. \end{aligned}$$

where  $t = \tan \frac{x}{2}$ .

### 8.3.3(3)

$$\begin{aligned} \int \log(x + \sqrt{1+x^2}) \, dx &= x \log(x + \sqrt{1+x^2}) - \int \frac{x}{x + \sqrt{1+x^2}} \left(1 + \frac{x}{\sqrt{1+x^2}}\right) dx \\ &= x \log(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C. \end{aligned}$$

### 8.3.4(9)

$$I_n = \int x^n (1+x)^{-1} dx = \int x^{n-1} - \frac{x^{n-1}}{1+x} dx = \frac{x^n}{n} - I_{n-1}.$$

### 8.3.5

$$\int x f''(x) dx = \int x df' = x f' - \int f' dx = x f' - f + C.$$

## 8.4

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### 8.4.1(6)

$$\int \frac{x^4 + 1}{(x^2 + 1)^2} dx = x - \int \frac{2x^2}{(x^2 + 1)^2} dx = x + \int x d\frac{1}{x^2 + 1} = x + \frac{x}{x^2 + 1} - \arctan x + C.$$

### 8.4.2(6)

$$\int \frac{dx}{\sin(x+a)\cos(x+b)} = \frac{\log|\sin(x+a)| - \log|\cos(x+b)|}{\cos(b-a)} + C.$$

### 8.4.3

$$\begin{aligned} \int \frac{x^2}{(x \sin x + \cos x)^2} dx &= - \int \frac{x}{\cos x} d\frac{1}{x \sin x + \cos x} = - \frac{x}{\cos x (x \sin x + \cos x)} + \int \frac{dx}{\cos^2 x} \\ &= - \frac{x}{\cos x (x \sin x + \cos x)} + \tan x + C. \end{aligned}$$

### 8.4.4

Suppose  $x^3 + y^3 = x^2 + y^2$ , calculate  $\int y^{-3} dx$ .

Solution: Let  $t = y/x$ , then  $x(1+t^3) = 1+t^2$  so  $x = (1+t^2)/(1+t^3)$ .

$$\begin{aligned} \int \frac{1}{y^3} dx &= \int \frac{1}{x^3 t^3} d\frac{1+t^2}{1+t^3} = \int \frac{1}{x^3 t^3} \frac{2t(1+t^3) - (1+t^2)3t^2}{(1+t^3)^2} dt \\ &= \int \frac{(1+t^3)(2-3t-t^3)}{t^2(1+t^2)^3} dt \\ &= -\frac{2}{t} - \frac{9}{2} \arctan t + 3 \log \left( \frac{\sqrt{1+t^2}}{|t|} \right) - \frac{3t+3}{2(1+t^2)} - \frac{1}{(1+t^2)^2} + C. \end{aligned}$$

### 8.4.5

Suppose  $\int \frac{e^x}{x}$  is not an elementary function, when is  $\int e^x P_n(1/x)$  an elementary function, where  $P_n$  is a polynomial of degree  $n$ .

Solution: Note that

$$\int e^x x^{-n} dx = \frac{1}{1-n} \int e^x dx^{-n+1} = \frac{e^x x^{1-n}}{1-n} + \frac{1}{n-1} \int e^x x^{n-1} dx.$$

So for  $P_n(x) = \sum_{k=0}^n a_k x^k$ ,

$$\int e^x P_n(1/x) \, dx = F + \sum_{k=1}^n \frac{a_k}{(k-1)!} \int e^x/x \, dx.$$

Therefore the condition is  $\sum_{k=1}^n a_k/(k-1)! = 0$ .