PSA

f is a function on the interval I.

A1) Suppose f is twice-differentiable at x, prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

Proof: For any $h>0$, consider the function $g(t)=f(t)-f(t-h)$, then there exists $\xi\in[0,h]$ such that $g(x+h)=g(x)+hg'(\xi)$, and there exists $\eta\in[\xi-h,\xi]\subset[-h,h]$ such that $f'(\xi)-f'(\xi-h)=hf''(\eta)$

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \frac{f'(\xi)-f'(\xi-h)}{h} = f''(\eta),$$

therefore

$$f''(x) = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

A2) Suppose $x_0\in I$, and

$$\begin{aligned} f(x) &= a_0 + a_1(x-x_0)+\cdots+a_n(x-x_0)^n + o(|x-x_0|^n) \\ &= b_0 + b_1(x-x_0)+\cdots+b_n(x-x_0)^n + o(|x-x_0|^n). \end{aligned}$$

when $x\to x_0$, then for any $i=0,1,\cdots,n$, $a_i=b_i$.
Proof: Otherwise let $c_i=a_i-b_i$ and take the least k such that $c_k\neq 0$, then

$$c_k(x-x_0)^k+\cdots+c_n(x-x_0)^n+o(|x-x_0|^n)=0 \implies c_k = -c_{k+1}(x-x_0)-\cdots-c_n(x-x_0)^{n-k}+o(|x-x_0|^{n-k}),$$

which leads to contradiction when $x\to x_0$.

A3) Suppose f is n-times differentiable at 0. Prove that if f is an even (odd) function, then the Taylor expansion of f at 0 has only even (odd) terms.

Proof: Use the fact that if f is even (odd) then f' is odd (even).

A4) If f is differentiable on (a,b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ prove that exists $x_0\in(a,b)$ such that $f'(x_0)=0$.

Proof: Otherwise if $f'(x)\neq 0$ for all $x\in(a,b)$, by Darboux's theorem $f'(x)$ have the same sign over (a,b), hence f is monotonic and non-constant on (a,b), contradicting $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$.

A5) Suppose $f\in C([a,b])$ and is differentiable on (a,b). Prove that f is strictly increasing on $[a,b]$ iff for any $x\in(a,b)$, $f'(x)\geqslant 0$ and on any sub-interval $(c,d)\subset(a,b)$, $f'(x)$ does not vanish.

Proof: ==> For any $x\in(a,b)$, $(f(x+h)-f(x))/h\geqslant 0$ so

$$f'(x) = \lim_{h\to\infty}\frac{f(x+h)-f(x)}{h}\geqslant 0.$$

If $f'(x)$ vanish on some sub-interval (c,d) then $f(c)=f(d)$, a contradiction.
<== For any $a\leqslant x<y\leqslant b$, there exists $\xi\in(a,b)$ such that $f(y)-f(x)=(y-x)f'(\xi)$, hence $f(y)\geqslant f(x)$ and f is increasing. If $f(x)=f(y)$ for some $x<y$ then $f(t)$ is constant on $[x,y]$ and hence f' vanish on (x,y), a contradiction.

PSB

Use L'Hôpital theorem to calculate limits:

B1) $a>0$, then

$$\lim_{x\to\infty}\frac{\log x}{x^a} = \lim_{x\to\infty}\frac{x^{-1}}{ax^{a-1}} = 0.$$

B2) $a > 0, b > 1$ **then**

$$\lim_{x\to\infty} \frac{x^a}{b^x} = \lim_{x\to\infty} \frac{ax^{a-1}}{b^x \ln b} = \cdots = \lim_{x\to\infty} \frac{a(a-1)\cdots\{a\}}{b^x (\ln b)^{\lfloor a \rfloor} x^{1-\{a\}}} = 0.$$

B3)

$$\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx} = \lim_{x\to 0} \frac{ae^{ax} - be^{bx}}{a\cos ax - b\cos bx} = 1.$$

B4)

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x\to 0} \frac{1+\cos x}{\cos^2 x} = 2.$$

B5)

$$\lim_{x\to 0} \frac{1-\cos x^2}{x^2 \sin x^2} = \lim_{x\to 0} \frac{1-\cos x}{x\sin x} = \lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\cos x}{2\cos x - x\sin x} = \frac{1}{2}.$$

B6)

$$\lim_{x\to 1} \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - x^{4/3}} = \lim_{x\to 1} \frac{(2x - x^4)^{-1/2}(1 - 2x^3) - x^{-2/3}/3}{-\frac{4}{3}x^{1/3}} = 1.$$

B7)

$$\lim_{x\to 1^-} (\log x)(\log(1-x)) = \lim_{x\to 1^-} \frac{\log(1-x)}{1/\log x} = \lim_{x\to 1^-} \frac{x\log^2 x}{1-x} = 0.$$

B8)

$$\lim_{x\to 0^+} \frac{\log\sin ax}{\log\sin bx} = \lim_{x\to 0^+} \frac{\sin bx}{\sin ax} \cdot \frac{a\cos ax}{b\cos bx} = 1.$$

B9)

$$\lim_{x\to 0^+} x^x = \exp \lim_{x\to 0^+} \frac{\log x}{x^{-1}} = \exp \lim_{x\to 0^+} -x = 1.$$

B10)

$$\lim_{x\to 1} x^{1/(1-x)} = \exp \lim_{x\to 1} \frac{\log x}{1-x} = e^{-1}.$$

B11)

$$\begin{aligned}\lim_{x\to 1} \left(\frac{1}{\log x} - \frac{1}{x-1}\right) &= \lim_{x\to 1} \frac{x - 1 - \log x}{(x-1)\log x} = \lim_{x\to 1} \frac{1 - x^{-1}}{1 - x^{-1} + \log x} \\ &= \lim_{x\to 1} \frac{x-1}{x - 1 + x\log x} = \frac{1}{2}.\end{aligned}$$

B12)

$$\lim_{x\to 0^+} (\sin x)^x = \exp \lim_{x\to 0^+} \frac{\log\sin x}{x^{-1}} = \exp \lim_{x\to 0^+} -\frac{x^2}{\tan x} = 1.$$

B13)

$$\begin{aligned}\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/(1-\cos x)} &= \exp \lim_{x\to 0} \frac{\log\sin x - \log x}{1 - \cos x} = \exp \lim_{x\to 0} \frac{\cot x - x^{-1}}{\sin x} \\ &= \exp \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin^2 x} = \exp \lim_{x\to 0} \frac{-x\sin x}{\sin^2 x + x\sin 2x} \\ &= e^{-1/3}.\end{aligned}$$

B14)

$$\lim_{x\to a}\frac{a^x-x^a}{x-a}=\lim_{x\to a}\frac{a^x\log a-ax^{a-1}}{1}=a^a(\log a-1).$$

B15)

$$\lim_{x\to\infty}\frac{(1+1/x)^x-e}{1/x}=\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}(1+x)^{1/x}\cdot\frac{x/(x+1)-\log(1+x)}{x^2}$$
$$=e\lim_{x\to 0}\frac{(x+1)^{-2}-(x+1)^{-1}}{2x}=\frac{e}{2}.$$

B16)

$$\lim_{x\to\infty}\frac{x^{\log x}}{(\log x)^x}=\exp\lim_{x\to\infty}(\log x)^2-x\log\log x=0.$$

B17)

$$\lim_{x\to\infty}(x+a)^{1+1/x}-x^{1+1/(x+a)}=\lim_{x\to\infty}\frac{(x+a)^{1+1/x}x^{-1}-x^{1/(x+a)}}{x^{-1}}$$
$$=a.$$

B18)

$$\lim_{x\to\infty}\sqrt[3]{x^3+x^2+x+1}-\sqrt{x^2+x+1}\cdot\frac{\log(e^x+x)}{x}=-\frac{1}{6}.$$

(Using WolframAlpha)

PSC

Calculate the maximum and minimum values of the following functions:
1.$f(x)=x^4-2x^2+5, x\in[-2,2]$.
$f(x)=(x^2-1)^2+4\in[4,13]$.
2.$f(x)=\frac{2x}{1+x^2}$, $x\in\mathbb{R}$
$1-f(x)=(1+x^2)^{-1}(x-1)^2\geqslant 0$, $f(x)+1=(1+x^2)^{-1}(x+1)^2\geqslant 0$, therefore $f(x)\in[-1,1]$.
3.$f(x)=\arctan x-\frac{1}{2}\log(1+x^2)$, $x\in\mathbb{R}$.
$f'(x)=\frac{1-x}{x^2+1}$, hence $\sup_{x\in\mathbb{R}}f(x)=f(1)=\frac{\pi}{4}-\frac{\log 2}{2}$, and f has no minimum.
4.$f(x)=x\log x$, $x\in(0,\infty)$.
$f'(x)=\log x+1$, hence $\inf_{x\in(0,\infty)}f(x)=f(e^{-1})=-e^{-1}$, and f has no maximum.
5.$f(x)=\sqrt{x}\log x$,$x\in(0,\infty)$.
$f'(x)=x^{-1/2}\left(1+\frac{\log x}{2}\right)$, hence $\inf_{x\in(0,\infty)}f(x)=f(e^{-2})=-2e^{-1}$.
6.$f(x)=2\tan x-\tan^2 x$, $x\in[0,\pi/2)$.
$f(x)=1-(1-\tan x)^2\in(-\infty,1]$.

PSD

f is differentiable on (a,b). Suppose $x_0\in(a,b)$ and $f'(x_0)=0$.

D1) Prove that $f(x_0)$ is a local maximum if there exists $(x_0-\delta,x_0+\delta)\subset(a,b)$ such that

$$f'(x)=\begin{cases}>0, & \forall x\in(x_0-\delta,x_0),\\ <0, & \forall x\in(x_0,x_0+\delta).\end{cases}$$

Proof: Trivial by Lagrange mean-value theorem.

D2) Prove that if $f''(x_0)$ exists and $f''(x_0)<0$ then $f(x_0)$ is a local maximum.

Proof: $f''(x_0)<0$ and $f'(x_0)=0$ implies for some $\delta>0$, $f'(x)<0$ for $x\in(x_0,x_0+\delta)$ and $f'(x)>0$ for $x\in(x_0-\delta,x_0)$. Hence by D1), $f(x_0)$ is a local maximum.

D3) Suppose f is n-times differentiable at x_0, $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Determine the conditions that $f(x_0)$ is a local maximum.

Solution: n is even and $f^{(n)}(x_0) < 0$.

PSE: Roots of Polynomials

E1) Prove that if all the roots of the polynomial $P_n(x) \in \mathbb{R}[x]$ are real numbers, then so are the polynomials $P_n'(x), P_n''(x), \cdots, P_n^{n-1}(x)$, where $n = \deg P_n$.

Proof: We only need to prove that P_n' has $n-1$ real roots. By Rolle's mean-value theorem, between any two roots of P_n there is a root of P_n' hence P_n' has $n-1$ real roots.

E2) Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$ has n different roots in the interval $(-1, 1)$.

Proof: We know that the polynomials $\sqrt{(2n+1)/2}P_n(x)$ form a set of orthogonal base on the space $L^2([-1,1])$, hence it must have n different roots in the interval $(-1, 1)$.

E3) Prove that the Laguerre polynomial $L_n(x) = \frac{e^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^n)$ has n different real roots.

Proof: We know that the Laguerre polynomials are orthogonal on the space $L^2([0,\infty))$ with weight e^{-x}, hence it must have n distinct roots.
Or note that $f(x) = x^n e^{-x}$ has a root with multiplicity n at 0 and it vanishes at ∞, hence use Rolle's theorem and induction we can show that $f^{(k)}(x)$ has a root with multiplicity $n-k$ at 0 and k roots between 0 and ∞.

E4) Prove that the Hermite polynomial $H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$ has n different real roots.

Proof: We know that the polynomials $H_n(x)/\sqrt{2^n n! \sqrt{n}}$ form a set of orthogonal base on the Hilbert space $L^2(\mu)$ where $\mu(\mathrm{d}x) = e^{-x^2}\mathrm{d}x$, hence it must have n distinct real roots.

PSF: Émile Borel's Lemma

Part 1:

F1) Define $\phi : \mathbb{R} \to \mathbb{R}$:

$$\phi(x) = \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Prove that $\phi \in C^\infty(\mathbb{R})$.
Proof: We prove by induction that for any $n \in \mathbb{Z}_{\geqslant 0}$, there is a polynomial $P_n \in \mathbb{R}[x]$ such that

$$\phi^{(n)}(x) = \begin{cases} P_n(1/x) \cdot e^{-1/x^2}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

(Which implies $\phi^{(n)}$ is continuous.)
The case $n = 0$ is trivial. Suppose it holds for n, then for any $x > 0$,

$$\phi^{(n+1)}(x) = e^{-1/x^2}\left(P_n(1/x)\frac{2}{x^3} - P_n'(1/x)\frac{1}{x^2}\right),$$

for any $x < 0$, $\phi^{(n+1)}(x) = 0$, and for $x = 0$,

$$\phi_+^{(n+1)}(0) = \lim_{x\to 0^+} e^{-1/x^2} P_n(1/x)\frac{1}{x} = 0.$$

Hence the claim holds for $n+1$ too.
Therefore $\phi \in C^\infty(\mathbb{R})$.

F2) Define $\chi : \mathbb{R} \to \mathbb{R}$:

$$\chi(x) = \frac{\phi(2-|x|)}{\phi(2-|x|)+\phi(|x|-1)}.$$

Prove that $\chi(x) \in C^\infty(\mathbb{R})$ and $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $0 \leqslant \chi(x) \leqslant 1$ and χ is an even function.
Proof: $2-|x|$ and $|x|-1$ cannot be both negative, hence the denominator is always positive, so $\chi \in C^\infty(\mathbb{R})$. The fact that $\chi|_{[-1,1]} \equiv 1$, $\chi|_{(-\infty,-2]\cup[2,\infty)} \equiv 0$, $\chi(x) \in [0,1]$ and χ is even is trivial.

F3) Prove that for any $0 < a < b$, there exists a smooth function $\rho(x) \in C^\infty(\mathbb{R})$ such that $\rho|_{[-a,a]} \equiv 1$, $\rho|_{(-\infty,-b]\cup[b,\infty)} \equiv 0$, and $0 \leqslant \rho(x) \leqslant 1$.

Proof: Same as F2), define

$$\rho(x) = \frac{\phi(b-|x|)}{\phi(b-|x|)+\phi(|x|-a)}.$$

F4) Prove that there exists an even function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi|_{\{x:|x|\leqslant 1\}} \equiv 1$, $\psi|_{\{x:|x|\geqslant 2\}} \equiv 0$, and $0 \leqslant \psi(x) \leqslant 1$.

Proof: (A special case of Urysohn's lemma)
Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(\mathbf{x}) = \phi(1-|\mathbf{x}|^2)$ and $g : \mathbb{R}^n \to \mathbb{R}$ as $g(\mathbf{x}) = \phi(|x^2|/4-1)$, then f vanishes on $B(0,1)^C$ and g vanishes on $\bar{B}(0,2)$. Therefore

$$\psi(\mathbf{x}) = \frac{f(\mathbf{x})}{f(\mathbf{x})+g(\mathbf{x})}$$

satisfy the requirements.

Part 2: Interchanging $\sum$ and $\frac{\mathrm{d}}{\mathrm{d}x}$

$I = [a,b]$ is a closed interval, $\{f_k\}_{k\geqslant 0}$ is a sequence of functions in $C^1(I)$. Assume $\sum_{k=0}^\infty f_k$ converges point-wise on I, and let $f(x) = \sum_{k=0}^\infty f_k(x)$.

F5) Assume the series $\sum_{k=0}^\infty f_k'(x)$ converges absolutely on I, i.e. $\sum_{k=0}^\infty \|f_k'\|_\infty$ converges. Prove that f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Note that

$$\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty \frac{f_k(x+h)-f_k(x)}{h} = \sum_{k=0}^\infty f_k'(x+\xi_k).$$

Hence

$$\left|\frac{f(x+h)-f(x)}{h} - \sum_{k=0}^\infty f_k'(x)\right| \leqslant \sum_{n=0}^N |f_k'(x+\xi_n) - f_k'(x)| + 2\sum_{n=N+1}^\infty \|f_k'\|$$

Note that f_k' is uniformly continuous, so

$$\lim_{h\to 0}\sum_{n=0}^N |f_k'(x+\xi_k) - f_k'(x)| = 0,\ \lim_{N\to\infty} 2\sum_{n=N+1}^\infty \|f_k'\| = 0.$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \sum_{k=0}^\infty f_k'(x).$$

F6) Assume $\sum_{k=0}^\infty f_k'(x)$ converges uniformly on I, then f is differentiable and $f'(x) = \sum_{k=0}^\infty f_k'(x)$.

Proof: Let $g(x) = \sum_{k=0}^\infty f_k'(x)$, since the series converges uniformly, $g(x)$ is continuous on I. By Lebesgue's Dominated Convergence Theorem,

$$\int_{x_0}^x g(t)\,\mathrm{d}t = \sum_{k=0}^\infty f_k(t)\Big|_{x_0}^x = f(x) - f(x_0).$$

Hence $f'(x) = g(x) = \sum_{k=0}^{\infty} f_k'(x)$.

F7) Calculate the derivative of e^x using F6).

Solution: On any closed interval $[-M, M]$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

converges uniformly. Hence

$$(e^x)' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Part 3: Borel's Lemma

Given an arbitrary sequence $\{a_k\}_{k \geqslant 0}$.

F8) For any $t_k > 0$, let $f_k(x) = \frac{a_k}{k!} x^k \chi(t_k x)$, determine the derivatives of any order of f_k at $x = 0$.

Solution: Note that when $x = 0$, $\chi^{(m)}(t_k x) = 0$ for any $m \geqslant 1$ and $\chi(t_k x) = 1$. Hence

$$f_k^{(n)}(0) = \frac{a_k}{k!} \sum_{j=0}^{n} \binom{n}{j} (x^k)^{(j)} \chi^{(n-j)}(t_k x)\Big|_{x=0} = \frac{a_k}{k!} (x^k)^{(n)}\Big|_{x=0} = a_k \delta_{n,k}.$$

F9) Prove that when $k \geqslant 2n$,

$$f_k^{(n)}(x) = a_k \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{t_k^{n-\ell}}{(k-\ell)!} x^{k-\ell} \chi^{(n-\ell)}(t_k x).$$

Proof: Leibniz's Formula.

F10) (Borel's lemma) Prove that for any sequence $\{a_k\}_{k \geqslant 0}$, there exists a smooth function f on $\mathbb{R}$, such that for any $k \geqslant 0$, $f^{(k)}(0) = a_k$.

Proof: Let $f_k(x) = \frac{a_k}{k!} x^k \chi(t_k x)$ where t_k is yet to be determined, and

$$f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \chi(t_k x).$$

For any $n \geqslant 0$, we want to show that $\sum_{k=0}^{\infty} f_k^{(n)}(x)$ converges uniformly on $\mathbb{R}$.
Suppose $M_n = \sup_{x \in \mathbb{R}, m \leqslant n} |\chi^{(m)}(x)|$, and

$$C_k = \sup_{n < k/2} \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{(k-\ell)!},$$

then for any $x \in \mathbb{R}$,

$$|f_k^{(n)}(x)| \leqslant |a_k| C_k M_k t_k^{-k/2}.$$

Hence if we choose t_k such that

$$|a_k| C_k M_k t_k^{-k/2} < 2^{-k},$$

then the series

$$\sum_{k=0}^{\infty} f_k^{(n)}(x)$$

converges uniformly on $\mathbb{R}$. By F6) we know that $f(x) = \sum_{k=0}^{\infty} f_k(x)$ is smooth, and by F8) we obtain $f^{(n)}(0) = a_n$ for any $n \geqslant 0$,

Part 4: Peano's Proof

F11) $\{c_k\}$ and $\{b_k\}$ are two sequences, and $b_k > 0$. Prove that

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = \begin{cases} n!(-1)^j c_{n-2j} b_{n-2j}^j, & \text{if } k = n-2j, j \in \mathbb{Z}_{\geqslant 0}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: For $x \to 0$,

$$\frac{c_k x^k}{1+b_k x^2} = c_k \sum_{n=0}^{\infty} (-1)^n x^{2n+k} b_k^n.$$

Which converges absolutely on the interval $[-b_k^{-1/2}/2, b_k^{-1/2}/2]$, and so are its n-times derivations, hence by F5)

$$\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(0) = c_k \sum_{j=0}^{\infty} (-1)^j \frac{(2j+k)!}{(2j+k-n)!} x^{2j+k-n} b_k^j \bigg|_{x=0} = \begin{cases} n!(-1)^j c_k b_k^j, & k = n-2j, \\ 0, & \text{otherwise.} \end{cases}$$

F12) Prove that there is a constant C such that for any $k \geqslant n+2$, and any x,

$$\left|\left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}(x)\right| \leqslant C(n+1)! \frac{|c_k| k!}{b_k} |x|^{k-n-2}.$$

Proof: Use du Bois-Reymond, we can let $C = 1$.

F13) Prove that for a given $\{c_k\}$, we can choose $\{b_k\}$ such that b_k depends only on the value of c_k, and the function

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$$

is infinitely differentiable.
Proof: Let $b_k = (k!)^2 c_k$, then by F12),

$$\left|\sum_{k \geqslant n+2} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}\right| \leqslant (n+1)! \sum_{k \geqslant n+2} \frac{|x|^{k-n-2}}{k!}$$

hence the series

$$\sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}$$

converges uniformly for any $n \geqslant 1$. By F6) the function $f(x)$ is infinitely differentiable, and

$$f^{(n)}(x) = \sum_{k=0}^{\infty} \left(\frac{c_k x^k}{1+b_k x^2}\right)^{(n)}.$$

F14) Prove that $f(0) = c_0, f'(0) = c_1$ and when $n \geqslant 2$,

$$\frac{f^{(n)}(0)}{n!} = c_n + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j.$$

Proof: Combine F11) and F13).

F15) Prove that by carefully choosing $\{c_k\}$ and $\{b_k\}$, we can prove Borel's lemma.

Proof: Let $b_k = (k!)^2 c_k$ and define c_k inductively such that

$$c_n = \frac{a_n}{n!} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j c_{n-2j} b_{n-2j}^j$$

Then let $f(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{1+b_k x^2}$

PSG: Midterm Test Part B

Consider $f : \mathbb{R} \to \mathbb{R}$.

- Let $\mathcal{B}$ be all bounded function on $\mathbb{R}$.
- Let $\mathcal{L}$ be all Lipschitz functions on $\mathbb{R}$.
  Suppose $a, \lambda \in \mathbb{R}$, $f \in \mathcal{B} \cap \mathcal{L}$, the goal is to find a function $F \in \mathcal{L}$ to solve:

$$F(x) - \lambda F(x+a) = f(x),\ x \in \mathbb{R}. \tag{$\star$}$$

Part 1: Basic Properties of Lipschitz Functions

B1) Prove that if $f, g \in \mathcal{B} \cap \mathcal{L}$, then $fg \in \mathcal{L}$.

Proof: Suppose $|f(x) - f(y)|, |g(x) - g(y)| \leqslant A|x-y|$, and $|f(x)|, |g(x)| \leqslant C$, then for any $x, y \in \mathbb{R}$,

$$|f(x)g(x) - f(y)g(y)| \leqslant 2MA|x-y|.$$

Hence $fg \in \mathcal{L}$.

B2) Prove that if f is differentiable and $f \in \mathcal{L}$ then $f' \in \mathcal{B}$.

Proof: If $|f(x) - f(y)| \leqslant C|x-y|$ then for any $x \in \mathbb{R}$,

$$|f'(x)| = \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leqslant C.$$

Hence $f' \in \mathcal{B}$.

B3) Prove that if f is differentiable and $f' \in \mathcal{B}$ then $f \in \mathcal{L}$.

Proof: For any $x, y \in \mathbb{R}$, there exists $\xi \in (x, y)$ such that

$$|f(x) - f(y)| = |x-y| \cdot |f'(\xi)| \leqslant \sup_{t \in \mathbb{R}} |f'(t)| \cdot |x-y|.$$

Hence $f \in \mathcal{L}$.

B4) If $f \in \mathcal{B}$ and there exists $B > 0$ such that for any $x, y \in \mathbb{R}$, $|x-y| \leqslant 1$ implies $|f(x) - f(y)| \leqslant B|x-y|$. Prove that $f \in \mathcal{L}$.

Proof: Suppose $M = \sup_{x \in \mathbb{R}} |f(x)|$, then for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant \max\{B, 2M\}|x-y|.$$

Hence $f \in \mathcal{L}$.

Part 2: Solution of $(\star)$ when $|\lambda| < 1$.

Suppose $f \in \mathcal{B} \cap \mathcal{L}$ and $|\lambda| < 1$.

B5) Suppose F satisfy $(\star)$. Prove that for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geqslant 1}$,

$$\begin{aligned} F(x) &= \lambda^n F(x+na) + \sum_{k=0}^{n-1} \lambda^k f(x+ka), \\ F(x) &= \lambda^{-n} F(x-na) - \sum_{k=1}^{n} \lambda^{-k} f(x-ka). \end{aligned}$$

Proof: Use induction and apply $(\star)$.
(Let $n \to \infty$ and we can obtain F formally.)

B6) Prove that for any $x \in \mathbb{R}$, $\sum_{k \geqslant 0} \lambda^k f(x+ka)$ converges.

Proof: Since f is bounded,

$$\left| \sum_{k=n}^{n+p} \lambda^k f(x+ka) \right| \leqslant \frac{M\lambda^n}{1-\lambda}.$$

Hence the series converges.

B7-8) Let $F(x)=\sum_{k\geqslant 0}\lambda^k f(x+ka)$. Prove that $F\in\mathcal{L}$ and solve $(\star)$.

Proof: For any $x,y\in\mathbb{R}$,

$$|F(x)-F(y)|\leqslant\sum_{k=0}^{\infty}\lambda^k|f(x+ka)-f(y+ka)|\leqslant\sum_{k=0}^{\infty}\lambda^k C|x-y|=\frac{C}{1-\lambda}|x-y|.$$

Hence $F\in\mathcal{L}$. For any $x\in\mathbb{R}$,

$$F(x)-\lambda F(x+a)=\sum_{k\geqslant 0}\lambda^k f(x+ka)-\sum_{k\geqslant 1}\lambda^k f(x+ka)=f(x).$$

Therefore F solves $(\star)$.
If F' also solves $(\star)$, let $G=F-F'$, then G is bounded and

$$G(x)=\lambda G(x+a),\ x\in\mathbb{R}.$$

Therefore for any $x\in\mathbb{R}$,

$$|G(x)|=\lambda^n|G(x+na)|\leqslant M\lambda^n\to 0.$$

Hence $G\equiv 0$ and $F\equiv F'$, so the solution to $(\star)$ is F.

B9) Solve $(\star)$ when $f(x)\equiv 1$ and $f(x)=\cos x$.

Solution: When $f(x)\equiv 1$,

$$F(x)=\sum_{k=0}^{\infty}\lambda^k f(x+ka)=\frac{1}{1-\lambda}.$$

When $f(x)=\cos x$,

$$\begin{aligned}F(x)&=\sum_{k=0}^{\infty}\lambda^k\cos{(x+ka)}=\sum_{k=0}^{\infty}\lambda^k\frac{e^{i(x+ka)}+e^{-i(x+ka)}}{2}=\frac{1}{2}\left(\frac{e^{ix}}{1-\lambda e^{ia}}+\frac{e^{-ix}}{1-\lambda e^{-ia}}\right)\\&=\frac{\cos x-\lambda\cos{(x-a)}}{1-2\lambda\cos a+\lambda^2}.\end{aligned}$$

Part 3: Solution of $(\star)$ when $|\lambda|>1$.

B10) Solve $(\star)$ as in Part 2.

Solution: By B5), the solution should be

$$F(x)=-\sum_{k=1}^{\infty}\lambda^{-k}f(x-ka).$$

$f\in\mathcal{B}$ implies the series converges. Same as B8) we can show that the solution to $(\star)$ is unique, and like B7) we can show that $F\in\mathcal{L}$ and F satisfy $(\star)$.

B11) Solve $(\star)$ for $f(x)\equiv 1$ and $f(x)=\cos x$.

Solution: When $f(x)\equiv 1$,

$$F(x)=-\sum_{k=1}^{\infty}\lambda^{-k}f(x-ka)=\frac{1}{1-\lambda}.$$

When $f(x)=\cos x$, same as B9) we have

$$F(x)=-\sum_{k=1}^{\infty}\lambda^{-k}f(x-ka)=\frac{\cos x-\lambda\cos{(x-a)}}{1-2\lambda\cos a+\lambda^2}.$$

Part 4: The Case when $|\lambda|=1$.

B12) Suppose $\lambda = 1$. Prove that there exists $F \in \mathcal{L}$ not identically zero, such that for any x, $F(x) - F(x+a) = 0$.

Proof: Let $F(x) = |\{x/a\} - 1/2|$, then $F(x) = F(x+a)$, and $F \in \mathcal{L} \cap \mathcal{B}$.

B13) Let $f(x) = \cos x$ in $(\star)$. Prove that if $\cos a \neq 1$, then there exists $F \in \mathcal{L}$ that solves $(\star)$. Determine whether the solution is unique.

Proof: The equation $(\star)$ becomes $F(x) = F(x+a) + \cos x$. Let

$$F(x) = \{x/a\} - \sum_{k=0}^{\lfloor x/a \rfloor - 1} \cos{(k + \{x/a\})a},$$

(if $x < 0$ the sum is viewed as from $\lfloor x/a \rfloor - 1$ to 0) then clearly $F(x) = F(x+a) + \cos x$, and F is bounded since $\cos a \neq 1$.
For any $x, y \in \mathbb{R}$, if $|x - y| < a/2$, then suppose $na \leqslant x < y < (n+1)a$,

$$\begin{aligned}|F(x) - F(y)| &\leqslant \left|\left\{\frac{x}{a}\right\} - \left\{\frac{y}{a}\right\}\right| + 2\left|\sin\frac{\{x/a\} - \{y/a\}}{2}a\right| \cdot \left|\sum_{k=0}^{n-1} \sin{(k + (\{x/a\} + \{y/a\})/2)a}\right| \\ &\leqslant \frac{|x-y|}{a} + \frac{|x-y|}{|\sin a|}.\end{aligned}$$

Hence $F \in \mathcal{L}$ by B4), so F solves $(\star)$.
The solution is clearly not unique since we can add any factor of the F in B12) to the solution.

B14) Following B13), if $a = 2\pi$, then $(\star)$ has no solution in $\mathcal{L}$.

Proof: If $a = 2\pi$ and F is a solution to $(\star)$, then for any $x, y \in \mathbb{R}$,

$$|F(x + 2\pi n) - F(y + 2\pi n)| = n|\cos x - \cos y| \to \infty.$$

Hence $F \notin \mathcal{L}$.

B15) Suppose $\lambda = -1$, Prove that there exists $F \in \mathcal{L}$ not identically zero, such that for any x, $F(x) + F(x+a) = 0$.

Proof: Let $F(x) = |2\{x/2a\} - 1| - 1/2$, then $F \in \mathcal{L}$ and $F(x) + F(x+a) = 0$.

B16) Suppose $\lambda = -1$, $a = 1$, $f \in \mathcal{L}$ is monotonically decreasing and $\lim_{x\to\infty} f(x) = 0$, f is differentiable and f' is increasing. Prove that there exists $F \in \mathcal{L}$ such that

$$F(x) + F(x+1) = f(x),\ x \in \mathbb{R}.$$

Further show that if we require $F \in \mathcal{L}$ and $\lim_{x\to\infty} F(x) = 0$, then the solution is unique.
Proof: Since f is monotonically decreasing, for any $x \in \mathbb{R}$, the series

$$F(x) = \sum_{n=0}^{\infty} (-1)^n f(x+n)$$

converges.
For any $x, y \in \mathbb{R}$, $|x - y| < 1$, there exists $\xi_n \in (x+n, y+n)$ such that $f(y+n) - f(x+n) = (y-x)f'(\xi_n)$, hence (by B3) f' is bounded)

$$|F(x) - F(y)| = |y - x| \cdot \left|\sum_{n=0}^{\infty} (-1)^n f'(\xi_n)\right| \leqslant \sup_{t\in\mathbb{R}} |f'(t)| \cdot |y - x|.$$

so $F \in \mathcal{L}$. Clearly $F(x) + F(x+1) = f(x)$, so F solves $(\star)$, and $0 < F(x) < f(x)$ so $\lim_{x\to\infty} F(x) = 0$.
If $F' \in \mathcal{L}$ also satisfy $(\star)$ and $\lim_{x\to\infty} F(x) = 0$, let $G = F - F'$, then $G(x) + G(x+1) = 0$ and $\lim_{x\to\infty} G(x) = 0$. Hence $G(x) = \lim_{n\to\infty}(-1)^n G(x+n) = 0$ for any $x \in \mathbb{R}$, so $G \equiv 0$. Therefore F is the unique solution.